Chapter 5
Systems Satisfying the Conditions of LaSalle

Abstract The LaSalle Invariance Principle uses non-strict Lyapunov functions to show asymptotic stability. However, even when a system is known to be asymptotically stable, it is still desirable to be able to construct a strict Lyapunov function for the system, e.g., for robustness analysis and feedback design. In this chapter, we give two more methods for constructing strict Lyapunov functions, which apply to cases where asymptotic stability is already known from the LaSalle Invariance Principle.

The first imposes simple algebraic conditions on the higher order Lie derivatives of the non-strict Lyapunov functions, in the directions of the vector fields that define the systems. Our second method uses our continuous time Matrosov Theorem from Chap. 3. We illustrate our approach by constructing a strict Lyapunov function for an appropriate error dynamics involving the Lotka-Volterra Predator-Prey System.

5.1 Background and Motivation

As we noted in preceding chapters, Lyapunov functions are a vital tool for the analysis of, and controller design for, nonlinear systems. The two main types of Lyapunov functions are *strict* Lyapunov functions (also known as *strong* Lyapunov functions, having negative definite time derivatives along the trajectories of the system) and *non-strict* Lyapunov functions (whose time derivatives along the trajectories are negative *semi*-definite, and which are also called *weak* Lyapunov functions).

Strict Lyapunov functions are typically far more useful than non-strict ones. The key point is that in general, non-strict Lyapunov functions can only be used to prove stability, via the LaSalle Invariance Principle, while *strict* Lyapunov functions can be used to show robustness properties, such as ISS to actuator errors. Robustness is an essential feature in engineering applications, largely due to the uncertainty in dynamical models and noise entering into controllers. Many controller design methods, e.g., backstepping [75], forwarding [113, 149] and universal stabilizing controllers [158], are based on strict Lyapunov functions. In particular, if V is a global strict Lyapunov function for $\dot{x} = f(t,x)$ for which $\alpha(x) = \inf_t\{-[V_t(t,x) + V_x(t,x)f(t,x)]\}$ is radially unbounded, with f and g both locally Lipschitz, and with V, f, and g all periodic in t with the same period T, then $\dot{x} = f(t,x) + g(t,x)[K(t,x) + d]$ is ISS if we take the feedback $K(t,x) = -V_x(t,x)g(t,x)$. Consequently, when an explicit strict Lyapunov function is known, many important stabilization problems can be solved almost immediately.

In general, it is much easier to obtain non-strict Lyapunov functions than strict ones, owing to the more restrictive decay condition in the strict Lyapunov function definition. For instance, when a passive nonlinear system is stabilized by linear output feedback, the energy (i.e., storage) function can typically be used as the weak Lyapunov function. This fact is useful for electro-mechanical systems. Also, when a system is stabilized via the Jurdjevic-Quinn Theorem, non-strict Lyapunov functions are typically available, e.g., by taking the Hamiltonian for Euler-Lagrange systems; see Chap. 4 or [41, 68, 102, 127]. This has motivated a significant literature devoted to transforming non-strict Lyapunov functions into strict Lyapunov functions.

In this chapter, we present two more strict Lyapunov function constructions, both based on transforming non-strict Lyapunov functions into strict ones under suitable Lie derivative conditions. The assumptions in our first construction agree with those of [110], but they lead to simpler designs than the one in [110]. Our second result uses the Matrosov approach in Theorem 3.1. In general, Matrosov's Method can be difficult to apply because one has to find suitable auxiliary functions. Here we give simple sufficient conditions leading to a systematic design of auxiliary functions. This makes it possible to construct strict Lyapunov functions via Theorem 3.1. We illustrate our approach by constructing a strict Lyapunov function for an error dynamics involving the celebrated Lotka-Volterra System, which plays a fundamental role in bioengineering. Throughout the chapter, all (in)equalities should be understood to hold globally unless otherwise indicated, and we omit the arguments of our functions when they are clear from the context.

5.2 First Method: Iterated Lie Derivatives

Recall that if $f : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth (i.e., C^∞) vector field and $V : \mathbb{R}^n \to \mathbb{R}$ is a smooth scalar function, the *Lie derivatives* of V in the direction of f are defined recursively by

$$L_f^1 V(x) \doteq L_f V(x) \doteq \tfrac{\partial V}{\partial x}(x) f(x) \ \text{ and}$$
$$L_f^k V(x) \doteq L_f\big(L_f^{k-1} V\big)(x) \ \text{ for } k \geq 2\ .$$

We refer to the functions $L_f^k V$ as *iterated Lie derivatives*. We next construct a strict Lyapunov function for the system

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n \tag{5.1}$$

with f smooth and $f(0) = 0$, under appropriate Lie derivative assumptions. Specifically, assume that (5.1) admits a global non-strict Lyapunov function such that for each $p \in \mathbb{R}^n \setminus \{0\}$, there is an $i \in \mathbb{N}$ such that $L_f^i V(p) \neq 0$. If $L_f V(\phi(t, x_0)) \equiv 0$ along some trajectory $\phi(\cdot, x_0)$ of (5.1), then we can differentiate repeatedly to get

$$L_f^k V(\phi(t, x_0)) \equiv 0 \ \ \forall t \geq 0 \text{ and } k \in \mathbb{N},$$

so $x_0 = 0$. Hence, GAS follows from the LaSalle Invariance Principle. On the other hand, it is not obvious how to construct a *strict* Lyapunov function in this situation. This motivates our hypotheses in the following theorem, in which $a_i(x) \doteq (-1)^i L_f^i V(x)$ for all i:

Theorem 5.1. *Assume that there exists a smooth function $V : \mathbb{R}^n \to [0, \infty)$ such that the following conditions hold:*

1. *$V(\cdot)$ is a non-strict Lyapunov function for the system (5.1); and*
2. *there exists a positive integer $\ell \in \mathbb{N}$ such that for each $x \neq 0$, there exists an integer $i \in [1, \ell]$ (possibly depending on x) such that $L_f^i V(x) \neq 0$.*

Then we can construct explicit expressions for functions $\mathcal{F}_j$ and $\mathcal{G}$ so that

$$\begin{aligned} V^\sharp(x) &= \sum_{j=1}^{\ell-1} \mathcal{F}_j\big(V(x)\big) A_j(x) + \mathcal{G}\big(V(x)\big), \quad \textit{where} \\ A_j(x) &= \sum_{m=1}^{j} a_{m+1}(x) a_m(x) \end{aligned} \tag{5.2}$$

is a strict Lyapunov function for (5.1).

Proof. Since Condition 2. in Theorem 5.1 is satisfied for some $\ell \geq 1$, it holds for all larger integers as well, so we assume without loss of generality (to simplify the proof) that $\ell \geq 3$. Note for later use that $a_{i+1} \equiv -\dot{a}_i$ for all i, along the trajectories of (5.1).

Condition 2. from Theorem 5.1 guarantees that we can construct a positive definite continuous function ρ such that

$$a_1(x) + \sum_{m=2}^{\ell} a_m^2(x) \ \geq \ \rho(V(x)) \ \ \forall x \in \mathbb{R}^n , \tag{5.3}$$

e.g.,

$$\rho(r) = \min\left\{ a_1(x) + \sum_{m=2}^{\ell} a_m^2(x) : V(x) = r \right\}.$$

By minorizing ρ as necessary without relabeling and using Lemma A.7, we can assume that

$$\rho(r) = \frac{\omega(r)}{K(r)} \tag{5.4}$$

for some function $\omega \in \mathcal{K}_\infty \cap C^1$ and some increasing everywhere positive function $K \in C^1$. We can also determine an everywhere positive increasing function $\Gamma \in C^1$ such that

$$\Gamma\big(V(x)\big) \geq (\ell+2)|a_m(x)| + 1 \ \ \forall m \in \{1, ..., \ell+1\} \tag{5.5}$$

holds for all $x \in \mathbb{R}^n$. For example, take

$$\Gamma_0(r) = (\ell+2) \max\left\{ \sum_{m=1}^{\ell+1} |a_m(x)| + 1 : V(x) \leq r \right\},$$

and then majorize by an increasing C^1 function.

Let us introduce the following functions:

$$M_j(x) = \sum_{m=1}^{j} a_{m+1}(x) a_m(x) + \int_0^{V(x)} \Gamma(r)\mathrm{d}r, \quad j = 1, 2, \ldots, \ell - 1\,; \tag{5.6}$$

$$N_0(x) = a_1(x), \ \text{ and } \ N_j(x) = \sum_{m=2}^{j+1} a_m^2(x) + a_1(x), \quad j = 1, 2, \ldots \ell - 1. \tag{5.7}$$

Since $a_1(x) \geq 0$ everywhere, (5.5) gives

$$\begin{aligned} \dot{M}_1(x) &= \dot{a}_2(x) a_1(x) - a_2^2(x) - \Gamma\big(V(x)\big) a_1(x) \\ &\leq -a_2^2(x) - a_1(x) \\ &= -N_1(x)\ . \end{aligned} \tag{5.8}$$

Also, for each $j \in \{2, ..., \ell - 1\}$, we get

$$\begin{aligned} \dot{M}_j(x) &= -\sum_{m=1}^{j} a_{m+1}^2(x) + \sum_{m=1}^{j} \dot{a}_{m+1}(x) a_m(x) - \Gamma\big(V(x)\big) a_1(x) \\ &\leq -\sum_{m=1}^{j} a_{m+1}^2(x) + \sum_{m=2}^{j} |a_{m+2}(x)||a_m(x)| + |a_3(x)| a_1(x) \\ &\quad -\Gamma\big(V(x)\big) a_1(x) \\ &\leq -\sum_{m=1}^{j} a_{m+1}^2(x) + \sum_{m=2}^{j} |a_{m+2}(x)||a_m(x)| + |a_3(x)| a_1(x) \\ &\quad -\big[(\ell+2)|a_3(x)| + 1\big] a_1(x). \end{aligned} \tag{5.9}$$

From this inequality and (5.5), we deduce that for all $j \in \{2, ..., \ell - 1\}$,

$$\begin{aligned}\dot{M}_j(x) \leq & -\sum_{m=1}^{j} a_{m+1}^2(x) \\ & + \frac{\Gamma\big(V(x)\big)}{\ell+2} \sum_{m=2}^{j} |a_m(x)| - \big[(\ell+1)|a_3(x)| + 1\big] a_1(x).\end{aligned} \tag{5.10}$$

It follows from the Cauchy Inequality that for all $j \in \{2, ..., \ell - 1\}$,

$$\begin{aligned}\dot{M}_j(x) \leq & -\sum_{m=1}^{j} a_{m+1}^2(x) + \Gamma(V(x)) \sqrt{\sum_{m=2}^{j} a_m^2(x)} \\ & -\big[(\ell+1)|a_3(x)| + 1\big] a_1(x) \\ = & -\sum_{m=2}^{j+1} a_m^2(x) - a_1(x) + \Gamma(V(x)) \sqrt{\sum_{m=2}^{j} a_m^2(x)} \\ & -(\ell+1)|a_3(x)| a_1(x).\end{aligned} \tag{5.11}$$

From the definitions of the functions N_j, we deduce that for all $j \in \{2, ..., \ell - 1\}$,

$$\dot{M}_j(x) \leq -N_j(x) + \Gamma(V(x)) \sqrt{N_{j-1}(x)}. \tag{5.12}$$

Set

$$\Omega(v) \; = \; \frac{2\omega(v)}{\ell \Gamma^2(v) K(v)} \tag{5.13}$$

and define the positive definite functions $k_1, k_2, \ldots, k_{\ell-1} \in C^1$ by

$$k_{\ell-1}(v) = 2K(v)\omega^{2^{\ell-1}}(v) \tag{5.14}$$

and

$$k_p(v) = k_{\ell-1}(v)\Omega^{1-2^{\ell-p-1}}(v) \tag{5.15}$$

for $p = 1, 2, \ldots, \ell - 2$.

Pick a C^1 everywhere positive increasing function k_0 such that

$$k_0(V(x)) + k_0'(V(x))V(x) \geq \sum_{p=1}^{\ell-1} \big|k_p'(V(x))M_p(x)\big| + 1. \tag{5.16}$$

Let

$$S_1(x) \; \doteq \; \sum_{p=1}^{\ell-1} k_p\big(V(x)\big) M_p(x) + k_0\big(V(x)\big) V(x). \tag{5.17}$$

Then

$$\dot{S}_1(x) = \sum_{p=1}^{\ell-1} k_p\big(V(x)\big)\dot{M}_p(x) + \left[\sum_{p=1}^{\ell-1} k_p'\big(V(x)\big)M_p(x)\right]\dot{V}(x)$$
$$+\big[k_0\big(V(x)\big) + k_0'\big(V(x)\big)V(x)\big]\dot{V}(x). \tag{5.18}$$

It follows from (5.16) and the fact that $\dot{V}$ is non-positive everywhere that

$$\dot{S}_1(x) \;\le\; \sum_{p=1}^{\ell-1} k_p(V(x))\dot{M}_p(x). \tag{5.19}$$

Using (5.8) and (5.12), we deduce that

$$\begin{aligned}\dot{S}_1(x) &\le -k_1\big(V(x)\big)N_1(x)\\ &\quad+\sum_{p=2}^{\ell-1} k_p\big(V(x)\big)\left[-N_p(x) + \varGamma\big(V(x)\big)\sqrt{N_{p-1}(x)}\right]\\ &= -\sum_{p=1}^{\ell-1} k_p\big(V(x)\big)N_p(x) + \sum_{p=2}^{\ell-1} k_p\big(V(x)\big)\varGamma\big(V(x)\big)\sqrt{N_{p-1}(x)}.\end{aligned} \tag{5.20}$$

By (5.3) and (5.4), we deduce that

$$N_{\ell-1}(x) \ge \frac{\omega(V(x))}{K(V(x))}. \tag{5.21}$$

Therefore,

$$\begin{aligned}\dot{S}_1(x) \le -k_{\ell-1}(V(x))\frac{\omega(V(x))}{K(V(x))} - \sum_{p=1}^{\ell-2} k_p\big(V(x)\big)N_p(x)\\ +\sum_{p=1}^{\ell-2} k_{p+1}\big(V(x)\big)\varGamma\big(V(x)\big)\sqrt{N_p(x)}.\end{aligned} \tag{5.22}$$

From the triangular inequality $c_1c_2 \le c_1^2 + \frac{1}{4}c_2^2$ for non-negative values c_1 and c_2, we deduce that

$$\begin{aligned}&\left\{\sqrt{k_p(V(x))N_p(x)}\right\}\left\{\frac{\varGamma\big(V(x)\big)k_{p+1}\big(V(x)\big)}{\sqrt{k_p(V(x))}}\right\}\\ &\le \;\; k_p(V(x))N_p(x) + \frac{\varGamma^2(V(x))k_{p+1}^2(V(x))}{4k_p(V(x))}\end{aligned} \tag{5.23}$$

for $p = 1, 2, \ldots, \ell - 2$ when $x \neq 0$. Summing the inequalities in (5.23) over $p = 1, 2, \ldots, \ell - 2$ and combining with (5.22), we deduce that for $x \neq 0$,

$$\dot{S}_1(x) \leq -k_{\ell-1}\big(V(x)\big)\frac{\omega(V(x))}{K(V(x))} + \sum_{p=1}^{\ell-2} \frac{\varGamma^2(V(x))k_{p+1}^2(V(x))}{4k_p(V(x))}. \tag{5.24}$$

By our choices of the k_p's, we get

$$\begin{aligned} \dot{S}_1(x) &\leq -k_{\ell-1}\big(V(x)\big)\frac{\omega(V(x))}{K(V(x))} \\ &\quad + \sum_{p=1}^{\ell-2} \frac{\varGamma^2(V(x))k_{\ell-1}^2(V(x))\varOmega^{2(1-2^{\ell-p-2})}(V(x))}{4k_{\ell-1}(V(x))\varOmega^{1-2^{\ell-p-1}}(V(x))} \\ &= -k_{\ell-1}\big(V(x)\big)\frac{\omega(V(x))}{K(V(x))} \\ &\quad + (\ell-2)\frac{\varGamma^2(V(x))k_{\ell-1}(V(x))\varOmega(V(x))}{4}, \quad x \neq 0 . \end{aligned} \tag{5.25}$$

Our choice of $\varOmega$ in (5.13) now gives

$$\dot{S}_1(x) \leq -k_{\ell-1}\big(V(x)\big)\frac{\omega(V(x))}{2K(V(x))} \quad \forall x \in \mathbb{R}^n . \tag{5.26}$$

Recalling our choice (5.14) of $k_{\ell-1}$ now gives

$$\dot{S}_1(x) \leq -\omega^{2^{\ell-1}+1}\big(V(x)\big). \tag{5.27}$$

All of the functions k_p are C^1 and the right hand side of (5.27) is negative definite. However, S_1 is not necessarily a strict Lyapunov function because S_1 is not necessarily positive definite and radially unbounded. To obtain a strict Lyapunov function, consider

$$V^\sharp(x) = V(x)S_1(x) + \kappa\big(V(x)\big)V(x) , \tag{5.28}$$

where $\kappa \in C^1$ is an everywhere positive function with an everywhere positive first derivative such that $\kappa(V(x)) \geq |S_1(x)| + 1$ for all $x \in \mathbb{R}^n$. Then $V^\sharp$ is positive definite and radially unbounded because $V^\sharp(x) \geq V(x)$ and

$$\begin{aligned} \dot{V}^\sharp(x) &= V(x)\dot{S}_1(x) + \dot{V}(x)S_1(x) + \big[\kappa'\big(V(x)\big)V(x) + \kappa\big(V(x)\big)\big]\dot{V}(x) \\ &\leq -\omega^{2^{\ell-1}+1}\big(V(x)\big)V(x). \end{aligned} \tag{5.29}$$

The result readily follows from the formula (5.17) for S_1, by collecting the functions involving V to form the expression for $V^\sharp$. □

5.3 Discussion and Extensions of First Method

5.3.1 Local vs. Global

While stated for systems on $\mathbb{R}^n$, we can also prove the following local version of Theorem 5.1 [110]: *Suppose that all conditions of Theorem 5.1 hold on a given neighborhood of the origin $E \subseteq \mathbb{R}^n$. Then, there exists a neighborhood of the origin E_1 with $E_1 \subseteq E$ and functions $\mathcal{F}_i$ and $\mathcal{G}$ such that (5.2) is a strict Lyapunov function for the system (3.17) on the set E_1.* The proof is similar to that of Theorem 5.1, by taking E_1 to be a suitable open sublevel set of V. Alternatively, we can prove the local version by using the construction from [110], which in general leads to a strict Lyapunov function that differs from the one we gave in Theorem 5.1.

5.3.2 Real Analytic Case

When V and f are real analytic, Theorem 5.1 remains true if its Condition 2. is replaced by the assumption that there exist positive constants $\underline{B}$ and $\bar{B}$ such that: There is an integer $\ell \in \mathbb{N}$ such that for each $x \in \{p \in \mathbb{R}^n : 0 < |p| < \underline{B}$ or $|p| > \bar{B}\}$, there is an integer $i \in [1, \ell]$ such that $L_f^i V(x) \neq 0$. This follows from the following simple observation from [110]:

Proposition 5.1. *Assume that (5.1) is GAS, f is real analytic, and Condition 1. of Theorem 5.1 holds with a real analytic function V. Then, for each compact set $E \subseteq \mathbb{R}^n$ that does not contain the origin, there exists $\ell \in \mathbb{N}$ such that each point $x \in E$ admits an index $i \in [1, \ell]$ such that $L_f^i V(x) \neq 0$.*

Hence, to apply the local version of Theorem 5.1, it suffices to check its Condition 1., and then check its Condition 2. on a set of the form $\underline{B}\mathcal{B}_n \setminus \{0\}$ for some constant $\underline{B} > 0$. Let us sketch the proof of Proposition 5.1.

Proof. We proceed in two steps.

Step 1. Fix any $x_0 \in E$. Since the system is assumed to be GAS, there must be a time $t_c > 0$ at which $L_f V(x(t_c, x_0)) \neq 0$. (This is because if no such t_c existed, then

$$V\big(x(t, x_0)\big) = V(x_0) + \int_0^t L_f V\big(x(r, x_0)\big)\mathrm{d}r \equiv V(x_0)$$

for all $t \geq 0$ would contradict the GAS property.) Since V and f are real analytic functions, so is $t \mapsto L_f V(\phi(t, x_0))$. Consider its expansion

$$L_f V\big(\phi(t, x_0)\big) = \sum_{i=0}^{\infty} L_f^{i+1} V(x_0) \frac{t^i}{i!} \tag{5.30}$$

around $t = 0$. Since $t \mapsto L_f V(\phi(t, x_0))$ is not the zero function, there must exist an integer $i = i(x_0)$ such that $L_f^i V(x_0) \neq 0$.

Step 2. Suppose that the statement of the proposition were false. Then there would exist a sequence $x_p \in E$ and a strictly increasing sequence of positive integers n_p such that

$$L_f^i V(x_p) = 0 \ \ \forall i \in [1, n_p - 1] \ , \ \ \text{but} \ \ \ L_f^{n_p} V(x_p) \neq 0 \ . \tag{5.31}$$

Since E is compact, we can assume that $x_p \to x^*$ for some non-zero $x^* \in E$. (Otherwise, we can pass to a subsequence without relabeling.) By Step 1 of the proof applied with $x_0 = x^*$, we can find an integer $J = J(x^*)$ such that $L_f^J V(x^*) \neq 0$. Since $L_f^J V$ is continuous, there exists a constant $\bar{p} \in \mathbb{N}$ such that for each $p \geq \bar{p}$, we have

$$L_f^J V(x_p) \neq 0.$$

This contradicts (5.31) once we pick p so that $n_p > J$. The result follows. □

5.3.3 Necessity vs. Sufficiency

Conditions 1. and 2. from Theorem 5.1 are not necessary for GAS of the system (5.1) [110]. To see why, consider the following example from [127]:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 - x_2 B(x_2), \end{cases} \tag{5.32}$$

where B is the smooth function

$$B(s) = \begin{cases} \exp\left(-\frac{1}{(s-1)^2}\right), & s \neq 1 \\ 0, & s = 1 \end{cases} .$$

Then Condition 1. of Theorem 5.1 is satisfied with $V(x_1, x_2) = x_1^2 + x_2^2$ since $\dot{V} = -2x_2^2 B(x_2)$, and the LaSalle Invariance Principle implies that (5.32) is GAS to zero. However, Condition 2. of Theorem 5.1 does not hold since for $x^* = (0\ 1)^\top$, we have $L_f^i V(x^*) = 0$ for all $i \in \mathbb{N}$.

5.3.4 Recovering Exponential Stability

When (5.1) is locally exponentially stable, the time derivative of (5.2) along the trajectories of (5.1) will not in general be upper bounded by a negative definite quadratic function. Moreover, it is not clear how to use (5.2) to verify local or global exponential stability. However, we can use $V^\sharp$ to get another strong Lyapunov function W that can be used to verify exponential stability. For example, if Conditions 1.-2. of Theorem 5.1 hold and (5.1) has an exponentially stable linearization, then one can construct a Lyapunov function $V^\star$ and $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ such that

$$\alpha_1(|x|) \leq V^\star(x) \leq \alpha_2(|x|) \quad \text{and} \quad \dot{V}^\star \leq -\alpha_3(|x|)$$

hold for all $x \in \mathbb{R}^n$ and, moreover, there exist positive constants δ, a, and b such that $\alpha_1(s) = as^2$ and $\alpha_3(s) = bs^2$ for all $s \in [0, \delta]$ [60, Lemma 10.1.5].

5.4 Second Method: Matrosov Conditions

We again consider a general nonlinear system

$$\dot{x} = f(x), \quad x \in \mathcal{X} \tag{5.33}$$

evolving on an open positively invariant set $\mathcal{X} \subseteq \mathbb{R}^n$ that contains the origin, where $f(0) = 0$. We use Theorem 3.1 in Chap. 3 to construct strict Lyapunov functions for (5.33). Recall that Theorem 3.1 is a continuous time Matrosov Theorem, which requires auxiliary functions, in addition to a non-strict Lyapunov function. In general, it can be difficult to find appropriate auxiliary functions to apply the Matrosov Theorem. Hence, our work sheds light on the Matrosov Theorems as well, because it gives a new mechanism for choosing auxiliary functions.

However, the most important features of our second method are that (a) the result applies to systems for which the state space is only a proper subset of $\mathbb{R}^n$ and (b) it may yield Lyapunov functions that are simpler than the ones obtained from Theorem 5.1, and that also have desirable local properties such as local boundedness from below by positive definite quadratic functions; see Sect. 5.5.

To account for the restricted state space for (5.33), we use the following definitions. A C^1 function $V : \mathcal{X} \to \mathbb{R}$ on a general open set $\mathcal{X} \subseteq \mathbb{R}^n$ containing the origin is called a *storage function* provided there exist continuous positive definite functions $\alpha_1, \alpha_2 : \mathcal{X} \to [0, \infty)$ such that the following hold:

1. for each i, $\alpha_i(x) \to +\infty$ whenever $|x| \to +\infty$ with x remaining in $\mathcal{X}$; and
2. $\alpha_1(x) \leq V(x) \leq \alpha_2(x)$ for all $x \in \mathcal{X}$.

Condition 1. holds vacuously when $\mathcal{X}$ is bounded. A storage function V is called a *non-strict* (resp., *strict*) *Lyapunov-like function* for (5.33) provided it is C^1 and $L_f V(x)$ is negative semi-definite (resp., negative definite). If, in addition, for each i and each $\bar{q} \in \partial\mathcal{X}$, $\alpha_i(q) \to +\infty$ when $q \to \bar{q}$ then a non-strict (resp., strict) Lyapunov-like function is called a *non-strict* (resp., *strict*) *Lyapunov function.* In the rest of this subsection, we assume:

Assumption 5.1 *There exist a smooth storage function $V_1 : \mathcal{X} \to [0,\infty)$; functions $h_1,\dots,h_m \in C^\infty(\mathbb{R}^n)$ such that $h_j(0) = 0$ for all j; everywhere positive functions $r_1,\dots,r_m \in C^\infty(\mathbb{R}^n)$ and $\rho \in C^\infty(\mathbb{R})$; and an integer $N > 0$ for which*

$$\nabla V_1(x) f(x) \leq -r_1(x)h_1^2(x) - \dots - r_m(x)h_m^2(x) \tag{5.34}$$

$$\text{and} \quad \sum_{l=0}^{N-1}\sum_{j=1}^{m}\left[L_f^l h_j(x)\right]^2 \geq \rho(V_1(x))V_1(x) \tag{5.35}$$

hold for all $x \in \mathcal{X}$. Moreover, f is defined on $\mathbb{R}^n$ and there is a function $\overline{\Gamma} \in \mathcal{K}_\infty$ such that

$$|f(x)| \leq \overline{\Gamma}(|x|) \ \ \forall x \in \mathbb{R}^n. \tag{5.36}$$

Also, V_1 has a positive definite quadratic lower bound near the origin.

To simplify our notation, we introduce the functions

$$\begin{aligned} \mathcal{N}_1(x) &= R(x)\sum_{l=1}^{m} h_l^2(x) \\ \text{and} \quad \mathcal{N}_i(x) &= \sum_{l=1}^{m}\left[L_f^{i-1}h_l(x)\right]^2 \end{aligned} \tag{5.37}$$

for all $i \geq 2$, where

$$R(x) = \frac{\prod_{i=1}^{m} r_i(x)}{\prod_{i=1}^{m}[r_i(x)+1]}$$

for all $i \geq 2$. We assume that f is sufficiently smooth.

The following is shown in [105]:

Theorem 5.2. *If (5.33) satisfies Assumption 5.1, then one can determine explicit functions $k_l, \Omega_l \in \mathcal{K}_\infty \cap C^1$ and an everywhere positive continuous function ρ_0 such that*

$$S(x) = \sum_{l=1}^{N} \Omega_l\big(k_l\big(V_1(x)\big) + V_l(x)\big) \tag{5.38}$$

with the choices

$$V_i(x) = -\sum_{l=1}^{m} L_f^{i-2}h_l(x)L_f^{i-1}h_l(x)\ , \quad i = 2,\dots,N \tag{5.39}$$

satisfies $S(x) \geq V_1(x)$ and $\nabla S(x)f(x) \leq -\rho_0(x)V_1(x)$ for all $x \in \mathcal{X}$. If, in addition, $\mathcal{X} = \mathbb{R}^n$, then the system (5.33) is GAS.

Proof. Sketch. Since R is everywhere positive and satisfies $R(x) \le r_i(x)$ for all $x \in \mathbb{R}^n$ and all $i \in \{1, ..., m\}$, we get

$$\begin{aligned} &\nabla V_1(x) f(x) \le -\mathcal{N}_1 \quad \text{by (5.34), and} \\ &\nabla V_i(x) f(x) \le -\mathcal{N}_i + \textstyle\sum_{l=1}^m |L_f^{i-2} h_l||L_f^i h_l| \end{aligned} \tag{5.40}$$

for $i = 2, \dots, N$ and $x \in \mathcal{X}$. In particular, we have:

$$\begin{aligned} \nabla V_2(x) f(x) &\le -\mathcal{N}_2(x) + \sum_{l=1}^m \frac{|L_f^2 h_l(x)|}{\sqrt{R(x)}} \sqrt{\mathcal{N}_1(x)}; \\ \nabla V_i(x) f(x) &\le -\mathcal{N}_i(x) + \left[\sum_{l=1}^m |L_f^i h_l(x)| \right] \sqrt{\mathcal{N}_{i-1}(x)} \end{aligned}$$

for $i = 3, 4, \dots, N$. Moreover, the fact that V_1 is a storage function implies that there exists a function $\underline{\alpha} \in \mathcal{K}_\infty$ such that $V_1(x) \ge \underline{\alpha}(|x|)$ for all $x \in \mathcal{X}$.

Therefore, we can use (5.36) to determine a continuous everywhere positive function ϕ_1 such that

$$\sum_{l=1}^m \frac{|L_f^2 h_l(x)|}{\sqrt{R(x)}} \;\le\; \phi_1\big(V_1(x)\big) \sqrt{V_1(x)} \tag{5.41}$$

and

$$\sum_{l=1}^m |L_f^i h_l(x)| \;\le\; \phi_1\big(V_1(x)\big) \sqrt{V_1(x)} \tag{5.42}$$

for all $x \in \mathcal{X}$ and $i = 3, \dots, N$. The construction of ϕ_1 satisfying (5.42) is as follows; the requirement (5.41) is handled in a similar way. Since $L_f^i h_l$ is sufficiently smooth for each i and l and zero at the origin, we have

$$\sum_{l=1}^m |L_f^i h_l(x)| \;\le\; |x| \mathcal{G}_1(|x|) \;\le\; \bar{\kappa} \sqrt{V_1(x)} \mathcal{G}_1\big(\underline{\alpha}^{-1}\big(V_1(x)\big)\big)$$

for some increasing everywhere positive function $\mathcal{G}_1$ and constant $\bar{\kappa} > 0$ in some neighborhood $\mathcal{O}$ of the origin. We can also find a function $\mathcal{G}_2 \in \mathcal{K}_\infty$ such that $\sum_{l=1}^m |L_f^i h_l(x)| / (\underline{\alpha}(|x|))^{1/2} \le \mathcal{G}_2(|x|)$ on $\mathbb{R}^n \setminus \mathcal{O}$. Hence, we can take $\phi_1(r) = 1 + \bar{\kappa} \mathcal{G}_1(\underline{\alpha}^{-1}(r)) + \mathcal{G}_2(\underline{\alpha}^{-1}(r))$.

It follows that

$$\nabla V_i(x) f(x) \le -\mathcal{N}_i(x) + \phi_1(V_1(x)) \sqrt{\mathcal{N}_{i-1}(x)} \sqrt{V_1(x)} \tag{5.43}$$

for $i = 2, \dots, N$. We can determine an everywhere non-negative function p_1 such that $|V_i(x)| \le p_1(V_1(x)) V_1(x)$ for $i = 1, \dots, N$ for all $x \in \mathcal{X}$. Hence, Theorem 3.1 constructs the necessary strict Lyapunov-like function. □

5.5 Application: Lotka-Volterra Model

5.5.1 Strict Lyapunov Function Construction

We illustrate Theorem 5.2 using the celebrated Lotka-Volterra Predator-Prey System

$$\begin{cases} \dot{\chi} = \gamma\chi\left(1 - \frac{\chi}{L}\right) - a\chi\zeta \\ \dot{\zeta} = \beta\chi\zeta - \Delta\zeta \end{cases} \tag{5.44}$$

with positive constants a, β γ, Δ, and L. System (5.44) is a simple model of one predator feeding on one prey. The population of the predator is ζ, χ is the population of the prey, and the constants are related to the birth and death rates of the predator and prey. We assume that the population levels are positive.

The time scaling, change of coordinates, and constants

$$\begin{array}{l} x(t) = \frac{1}{L}\chi\left(\frac{t}{\gamma}\right), \quad y(t) = \frac{a}{\beta L}\zeta\left(\frac{t}{\gamma}\right), \\ \alpha = \frac{\beta L}{\gamma} \ \text{ and } \ d = \frac{\Delta}{\gamma} \end{array} \tag{5.45}$$

give the simpler Lotka-Volterra system

$$\begin{cases} \dot{x} = x(1 - x) - \alpha xy \\ \dot{y} = \alpha xy - dy. \end{cases} \tag{5.46}$$

We assume that $\alpha > d$, and we set

$$x_* = \frac{d}{\alpha} \ \text{ and } \ y_* = \frac{1}{\alpha} - \frac{d}{\alpha^2}. \tag{5.47}$$

Then $x_* \in (0, 1)$ and $y_* > 0$. Also, the new variables $\tilde{x} = x - x_*$ and $\tilde{y} = y - y_*$ have the dynamics

$$\begin{cases} \dot{\tilde{x}} = -[\tilde{x} + \alpha\tilde{y}](\tilde{x} + x_*) \\ \dot{\tilde{y}} = \alpha\tilde{x}(\tilde{y} + y_*)\ , \end{cases} \tag{5.48}$$

with state space $\mathcal{X} = (-x_*, \infty) \times (-y_*, \infty)$. We do our Lyapunov function construction for (5.48), so we set

$$f(\tilde{x}, \tilde{y}) = \begin{bmatrix} -[\tilde{x} + \alpha\tilde{y}](\tilde{x} + x_*) \\ \alpha\tilde{x}(\tilde{y} + y_*) \end{bmatrix}. \tag{5.49}$$

Let us check that the assumptions from Theorem 5.2 are satisfied with $m = 1$, $N = 2$, $r_1 \equiv 1$, $h_1(\tilde{x}, \tilde{y}) \doteq \tilde{x}$, and

$$V_1(\tilde{x}, \tilde{y}) = \tilde{x} - x_* \ln\left(1 + \frac{\tilde{x}}{x_*}\right) + \tilde{y} - y_* \ln\left(1 + \frac{\tilde{y}}{y_*}\right). \tag{5.50}$$

One easily checks that $V_1 : \mathcal{X} \to [0, \infty)$ is a storage function. Along the trajectories of (5.48), it has the time derivative

$$\begin{aligned} \dot{V}_1 &= \frac{\tilde{x}}{x_* + \tilde{x}}\dot{\tilde{x}} + \frac{\tilde{y}}{y_* + \tilde{y}}\dot{\tilde{y}} \\ &= -\frac{\tilde{x}}{x_* + \tilde{x}}[\tilde{x} + \alpha\tilde{y}](\tilde{x} + x_*) + \frac{\alpha\tilde{y}}{y_* + \tilde{y}}\tilde{x}(\tilde{y} + y_*) \\ &= -\tilde{x}[\tilde{x} + \alpha\tilde{y}] + \alpha\tilde{y}\tilde{x} \ = \ -\tilde{x}^2 \, . \end{aligned} \tag{5.51}$$

Also,

$$L_f h_1(\tilde{x}, \tilde{y}) = -[\tilde{x} + \alpha\tilde{y}](\tilde{x} + x_*).$$

Defining the $\mathcal{N}_i$'s as in (5.37), a simple argument based on the fact that V_1 becomes unbounded as $\tilde{x}$ approaches $-x_*$ or $\tilde{y}$ approaches $-y_*$ provides a constant $\underline{d} > 0$ such that

$$\sum_{i=1}^{2} \mathcal{N}_i(\tilde{x}, \tilde{y}) \geq \underline{d}\frac{V_1(\tilde{x}, \tilde{y})}{1 + V_1^2(\tilde{x}, \tilde{y})} \tag{5.52}$$

on $\mathcal{X}$; see Appendix A.3. Also, Lemma A.8 provides a positive definite quadratic lower bound for V_1 near 0. Hence, Theorem 5.2 provides the necessary strict Lyapunov function for (5.48).

We now construct the strict Lyapunov function of the type provided by the theorem. Notice that

$$\begin{aligned} &\mathcal{N}_1(\tilde{x}, \tilde{y}) = \tfrac{1}{2}h_1^2(\tilde{x}, \tilde{y}), \;\; \mathcal{N}_2(\tilde{x}, \tilde{y}) = \big(L_f h_1(\tilde{x}, \tilde{y})\big)^2, \\ &V_2(\tilde{x}, \tilde{y}) = -h_1(\tilde{x}, \tilde{y})L_f h_1(\tilde{x}, \tilde{y}), \;\; L_f V_1(\tilde{x}, \tilde{y}) \leq -\mathcal{N}_1(\tilde{x}, \tilde{y}), \end{aligned}$$

and

$$\begin{aligned} L_f V_2(\tilde{x}, \tilde{y}) &= -\big(L_f h_1(\tilde{x}, \tilde{y})\big)^2 - h_1(\tilde{x}, \tilde{y})L_f^2 h_1(\tilde{x}, \tilde{y}) \\ &= -\mathcal{N}_2(\tilde{x}, \tilde{y}) - h_1(\tilde{x}, \tilde{y})L_f^2 h_1(\tilde{x}, \tilde{y}). \end{aligned} \tag{5.53}$$

Simple calculations yield

$$\begin{aligned} L_f^2 h_1(\tilde{x}, \tilde{y}) &= -\big(\dot{\tilde{x}} + \alpha\dot{\tilde{y}}\big)\big(\tilde{x} + x_*\big) - \big[\tilde{x} + \alpha\tilde{y}\big]\dot{\tilde{x}} \\ &= -\big(x_* + 2\tilde{x} + \alpha\tilde{y}\big)\dot{\tilde{x}} - \big(x_* + \tilde{x}\big)\alpha\dot{\tilde{y}} \\ &= -\big(x_* + 2\tilde{x} + \alpha\tilde{y}\big)L_f h_1(\tilde{x}, \tilde{y}) \\ &\quad\; -\alpha^2\big(x_* + h_1(\tilde{x}, \tilde{y})\big)h_1\big(\tilde{x}, \tilde{y}\big)\big(\tilde{y} + y_*\big). \end{aligned} \tag{5.54}$$

Substituting (5.54) into (5.53) gives

$$\begin{aligned} L_f V_2(\tilde{x},\tilde{y}) &\le -\mathcal{N}_2(\tilde{x},\tilde{y}) + \big(x_* + 2|\tilde{x}| + \alpha|\tilde{y}|\big)\big|h_1(\tilde{x},\tilde{y})\big|\big|L_f h_1(\tilde{x},\tilde{y})\big| \\ &\quad + \alpha^2\big(x_* + |\tilde{x}|\big)\big(|\tilde{y}| + y_*\big)h_1^2(\tilde{x},\tilde{y}) \\ &\le -\mathcal{N}_2(\tilde{x},\tilde{y}) + \big(x_* + 2|\tilde{x}| + \alpha|\tilde{y}|\big)\big|h_1(\tilde{x},\tilde{y})\big|\big|L_f h_1(\tilde{x},\tilde{y})\big| \\ &\quad + \alpha^2 x_* y_* \left(1 + \tfrac{|\tilde{x}|}{x_*}\right)\left(1 + \tfrac{|\tilde{y}|}{y_*}\right) h_1^2(\tilde{x},\tilde{y}). \end{aligned}$$

Next, observe that

$$\begin{aligned} &\left(\tfrac{1}{x_*} + \tfrac{1}{y_*}\right) V_1(\tilde{x},\tilde{y}) \ \ge \\ &\tfrac{\tilde{x}}{x_*} - \ln\left(1 + \tfrac{\tilde{x}}{x_*}\right) + \tfrac{\tilde{y}}{y_*} - \ln\left(1 + \tfrac{\tilde{y}}{y_*}\right). \end{aligned} \tag{5.55}$$

This, Lemma A.8, and the relation $1 + A^2 \ge \frac{1}{2}(1 + |A|)$ give

$$\begin{aligned} e^{\left(\frac{1}{x_*} + \frac{1}{y_*}\right)V_1(\tilde{x},\tilde{y})} &\ge \left(\frac{e^{\frac{\tilde{x}}{x_*}}}{1+\frac{\tilde{x}}{x_*}}\right)\left(\frac{e^{\frac{\tilde{y}}{y_*}}}{1+\frac{\tilde{y}}{y_*}}\right) \\ &\ge \tfrac{1}{36}\left(1 + \tfrac{\tilde{x}^2}{x_*^2}\right)\left(1 + \tfrac{\tilde{y}^2}{y_*^2}\right) \\ &\ge \tfrac{1}{144}\left(1 + \tfrac{|\tilde{x}|}{x_*}\right)\left(1 + \tfrac{|\tilde{y}|}{y_*}\right). \end{aligned} \tag{5.56}$$

Hence,

$$\begin{aligned} &|\tilde{x}| \le 144 x_* e^{\left(\frac{1}{x_*} + \frac{1}{y_*}\right)V_1(\tilde{x},\tilde{y})} \text{ and} \\ &|\tilde{y}| \le 144 y_* e^{\left(\frac{1}{x_*} + \frac{1}{y_*}\right)V_1(\tilde{x},\tilde{y})}. \end{aligned}$$

Setting $\mathcal{M}(r) = (289x_* + 144\alpha y_*)\, e^{\left(\frac{1}{x_*} + \frac{1}{y_*}\right)r}$ therefore gives

$$\begin{aligned} L_f V_2(\tilde{x},\tilde{y}) &\le -\mathcal{N}_2(\tilde{x},\tilde{y}) \\ &\quad + 2\mathcal{M}\big(V_1(\tilde{x},\tilde{y})\big)\sqrt{\mathcal{N}_1(\tilde{x},\tilde{y})}\sqrt{\mathcal{N}_2(\tilde{x},\tilde{y})} \\ &\quad + 288\alpha^2 x_* y_* e^{\left(\frac{1}{x_*} + \frac{1}{y_*}\right)V_1(\tilde{x},\tilde{y})}\mathcal{N}_1(\tilde{x},\tilde{y}). \end{aligned}$$

Using the triangular inequality, we have

$$\begin{aligned} &\mathcal{M}(V_1)\sqrt{\mathcal{N}_1}\sqrt{\mathcal{N}_2} \\ &\le \ \tfrac{1}{4}\mathcal{N}_2 + (289x_* + 144\alpha y_*)^2\, e^{2\left(\frac{1}{x_*} + \frac{1}{y_*}\right)V_1}\mathcal{N}_1 \end{aligned} \tag{5.57}$$

where we omit the dependencies on $(\tilde{x},\tilde{y})$. Therefore,

$$L_f V_2(\tilde{x},\tilde{y}) \ \le\ -\frac{1}{2}\mathcal{N}_2(\tilde{x},\tilde{y}) + \phi_1(V_1(\tilde{x},\tilde{y}))\mathcal{N}_1(\tilde{x},\tilde{y}), \tag{5.58}$$

where

$$\phi_1(r) = 2\left[(289x_* + 144\alpha y_*)^2 + 144\alpha^2 x_* y_*\right] e^{2\left(\frac{1}{x_*} + \frac{1}{y_*}\right)r}.$$

Since $V_2(\tilde{x},\tilde{y}) = \tilde{x}[\tilde{x} + \alpha\tilde{y}](\tilde{x} + x_*)$, we easily get

$$|V_2(\tilde{x},\tilde{y})| \leq 2(x_*+1)(1+\alpha)\left[\tilde{y}^4+|\tilde{x}|^3+\tilde{x}^2+\tilde{y}^2\right], \tag{5.59}$$

and Lemma A.8 applied with $A = \tilde{x}/x_*$ gives

$$\left|\frac{\tilde{x}}{x_*}\right| \leq 2\left\{\left[\frac{V_1}{x_*}\right]+\left[\frac{V_1}{x_*}\right]^2\right\}^{1/2} \leq 2\left[\max\left\{\frac{1}{x_*},\frac{1}{x_*^2}\right\}\left\{V_1+V_1^2\right\}\right]^{1/2}$$

and similarly for y, where we omit the dependence of V_1 on $(\tilde{x},\tilde{y})$. Combining these estimates with (5.59) and setting $\bar{d} = 1 + x_* + y_*$, simple algebra gives

$$|V_2(\tilde{x},\tilde{y})| \leq 4(x_*+1)(1+\alpha)\sum_{i=2}^{4}\left\{2\bar{d}\sqrt{V_1+V_1^2}\right\}^i \leq p_1\big(V_1(\tilde{x},\tilde{y})\big)V_1(\tilde{x},\tilde{y}),$$

where $p_1(r) = 640(x_*+1)(\alpha+1)\bar{d}^4(1+r)^3$, by separately considering points where $V_1 \geq 1$ and $V_1 \leq 1$.

Then the strict Lyapunov function we get is

$$S(\tilde{x},\tilde{y}) = V_2(\tilde{x},\tilde{y}) + \left[p_1(V_1(\tilde{x},\tilde{y}))+1\right]V_1(\tilde{x},\tilde{y}) + \int_0^{V_1(\tilde{x},\tilde{y})}\phi_1(r)\,\mathrm{d}r. \tag{5.60}$$

In fact, $S(\tilde{x},\tilde{y}) \geq V_1(\tilde{x},\tilde{y})$ and $L_f S(\tilde{x},\tilde{y}) \leq -\frac{1}{2}[\mathcal{N}_1(\tilde{x},\tilde{y}) + \mathcal{N}_2(\tilde{x},\tilde{y})]$ are satisfied everywhere.

5.5.2 Robustness to Uncertainty

We can use our strict Lyapunov function constructions to quantify the effects of uncertainty in the Lotka-Volterra dynamics. For simplicity, we only consider additive uncertainty in the death rate Δ for the predator. Using the coordinate change and constants (5.45), this means that we replace the constant d with $d + \mathbf{u}$ in the dynamics (5.46), where $\mathbf{u} : [0,\infty) \to \mathbb{R}$ is a measurable essentially bounded uncertainty, and where the constant $d > 0$ now represents the nominal value of the parameter. Later, we impose bounds on the allowable values for $|\mathbf{u}|_\infty$. We continue to use d in the formulas (5.47) for x_* and y_*; we do not introduce uncertainty in the equilibrium values.

We first define an appropriately restricted state space for the dynamics. Along the trajectories of (5.46), with d replaced by $d + \mathbf{u}$, we have $\dot{x} + \dot{y} = x(1-x) - (d+\mathbf{u})y$. Hence, if $|\mathbf{u}|_\infty \leq d/2$, then we get $\dot{x} + \dot{y} < 0$ when $x + y > 1 + \frac{2}{d}$ (by separately considering the cases $x > 1$ and $x \leq 1$). Therefore, we restrict to disturbances satisfying $|\mathbf{u}|_\infty \leq d/2$ and the forward invariant set $\mathcal{S} = \{(x,y) \in (0,\infty)^2 : x + y \leq \mathcal{B}\}$ containing (x_*, y_*), where

$$\mathcal{B} = 1 + \frac{2}{d} + y_* \ . \tag{5.61}$$

The corresponding perturbed error dynamics is

$$\begin{cases} \dot{\tilde{x}} = -[\tilde{x} + \alpha\tilde{y}](\tilde{x} + x_*) \\ \dot{\tilde{y}} = \alpha\tilde{x}(\tilde{y} + y_*) - \mathbf{u}y \end{cases} \tag{5.62}$$

which we view as having the state space $\mathcal{X}^\flat = \{(\tilde{x}, \tilde{y}) : (x, y) \in \mathcal{S}\}$ and a control set U we will specify.

To account for the restricted state space, we use the following definitions. Given an open subset $\mathcal{D}$ of a Euclidean space that contains the origin, we say that a positive definite function $\bar{\alpha} : \mathcal{D} \to [0, \infty)$ is a *modulus with respect to* $\mathcal{D}$ provided $\bar{\alpha}(p) \to +\infty$ as $|p| \to +\infty$ or as $\text{dist}(p, \partial\mathcal{D}) \to 0$ (with p remaining in $\mathcal{D}$). We say that (5.62) is ISS with respect to $\mathbf{u}$ provided there exist functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$, and a modulus with respect to $\mathcal{X}^0 \doteq (-x_*, \infty) \times (-y_*, \infty)$, such that for each disturbance $\mathbf{u} : [0, \infty) \to U$ and each trajectory $(\tilde{x}, \tilde{y}) : [0, \infty) \to \mathcal{X}^\flat$ of (5.62) corresponding to $\mathbf{u}$, we have

$$|(\tilde{x}, \tilde{y})(t)| \leq \beta\big(\bar{\alpha}((\tilde{x}, \tilde{y})(0)), t\big) + \gamma(|\mathbf{u}|_\infty) \;\; \forall t \geq 0. \tag{5.63}$$

We define iISS for (5.62) in an analogous way; see Remark 5.1 below.

To simplify the statements of our results, we use the constants

$$K_0 = 2\left[\frac{(3+\alpha)^2}{2} + \alpha^2\right]\mathcal{B}^2, \quad \theta = \min\left\{\frac{K_0 x_*^2}{8}, \frac{K_0 x_*^2 y_*^2 \alpha^2}{8(x_* + 2\sqrt{K_0})^2}\right\},$$

$$K = \mathcal{B}^2 \max\left\{(3+\alpha)^2 + 2\alpha^2, 2\max\{9, 3\alpha^2\}\right\},$$

$$\hat{K} = \frac{\min\left\{32x_*, x_*^2\alpha^2 y_*\right\}}{16[K + \mathcal{B}^2\max\{9, 3\alpha^2\}]}, \;\; \text{and} \;\; \bar{U} = \frac{\min\{\hat{K}, \theta\}}{4(\alpha\mathcal{B}^3 + K\mathcal{B})}.$$

We continue to use the functions V_1 and V_2 from the preceding subsection. The following is shown in [105] (but see Sect. 5.5.3 for a specific numerical example):

Theorem 5.3. *The system (5.62) is ISS with respect to disturbances* $\mathbf{u}$ *valued in the control set* $\bar{U}\mathcal{B}_1$*, and iISS with respect to disturbances* $\mathbf{u}$ *valued in* $\frac{d}{2}\mathcal{B}_1$*.*

The proof of Theorem 5.3 entails showing that

$$\mathcal{U}_K(\tilde{x}, \tilde{y}) = V_2(\tilde{x}, \tilde{y}) + KV_1(\tilde{x}, \tilde{y}) \tag{5.64}$$

is an iISS Lyapunov function for (5.62) when the disturbance $\mathbf{u}$ is valued in $\frac{d}{2}\mathcal{B}_1$, and that

$$\overline{\mathcal{U}_K}(\tilde{x}, \tilde{y}) = \mathcal{U}_K(\tilde{x}, \tilde{y})e^{\mathcal{U}_K(\tilde{x}, \tilde{y})} \tag{5.65}$$

is an ISS Lyapunov function for (5.62) when $\mathbf{u}$ is valued in $\bar{U}\mathcal{B}_1$, where V_1 and V_2 are as defined in Sect. 5.5. It leads to the decay estimates

$$\dot{\mathcal{U}}_K \leq -\mho\frac{\mathcal{U}_K(\tilde{x}, \tilde{y})}{1 + \mathcal{U}_K(\tilde{x}, \tilde{y})} + \overline{\mathcal{B}}|\mathbf{u}|\,, \tag{5.66}$$

where

$$\mho = \min\left\{\widehat{K}, \theta\right\}$$

(which implies that $\mathcal{U}_K$ is an iISS Lyapunov function for the Lotka-Volterra error dynamics (5.62)) when the disturbance $\mathbf{u}$ satisfies the less stringent bound $|\mathbf{u}|_\infty \leq \frac{d}{2}$ and then

$$\dot{\overline{\mathcal{U}_K}} \leq -\tfrac{\mho}{4}\overline{\mathcal{U}_K}(\tilde{x}, \tilde{y}) + \overline{\mathcal{B}}|\mathbf{u}| \ . \tag{5.67}$$

along the trajectories of (5.62) when $\mathbf{u}$ is valued in $\bar{U}\mathcal{B}_1$, which gives the ISS estimate. For a summary of the robustness analysis, see Appendix A.4.

Remark 5.1. A slight variant of the iISS arguments from [8] in conjunction with (5.66) and the growth properties of $\mathcal{U}_K$ can be used to show that there exist functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$, a constant $\bar{G} > 0$, and a modulus with respect to $\mathcal{X}^0$, such that for each disturbance $\mathbf{u} : [0, \infty) \to [-d/2, d/2]$ and each trajectory $(\tilde{x}, \tilde{y}) : [0, \infty) \to \mathcal{X}^\flat$ of (5.62) corresponding to $\mathbf{u}$, we have

$$\gamma(|(\tilde{x}, \tilde{y})(t)|) \leq \beta\big(\bar{\alpha}((\tilde{x}, \tilde{y})(0)), t\big) + \bar{G}\int_0^t |\mathbf{u}(r)|\mathrm{d}r \ \ \forall t \geq 0. \tag{5.68}$$

This is less stringent than the ISS condition (5.63) because it allows the possibility that a bounded (but non-integrable) disturbance $\mathbf{u}$ could give rise to an unbounded trajectory. However, if $\mathbf{u}$ is integrable, then (5.68) guarantees boundedness of the trajectories, and it also quantifies the effects of the disturbance. We next illustrate these ideas in simulations.

5.5.3 Numerical Validation

To illustrate our findings, we simulated the dynamics (5.62) using the parameter values

$$\alpha = 2, \ \ d = 1, \ \ x_* = 0.5, \ \text{ and } \ y_* = 0.25\ , \tag{5.69}$$

corresponding to the parameter choices

$$a = \gamma = \beta = \Delta = 0.5 \ \text{ and } \ L = 2 \tag{5.70}$$

in the original model. Hence, the dynamics are iISS with respect to disturbances that are bounded by 0.5. We chose the disturbance $\mathbf{u}(t) = 0.49e^{-t}$. In Figs. 5.1 and 5.2, we plot the corresponding levels of predator population ζ and the prey population χ, which are related to x and y in terms of the coordinate changes (5.45).

If

$$x(t) \to x_* = 0.5 \ \text{ and } \ y(t) \to y_* = 0.25,$$

then the coordinate changes (5.45) give

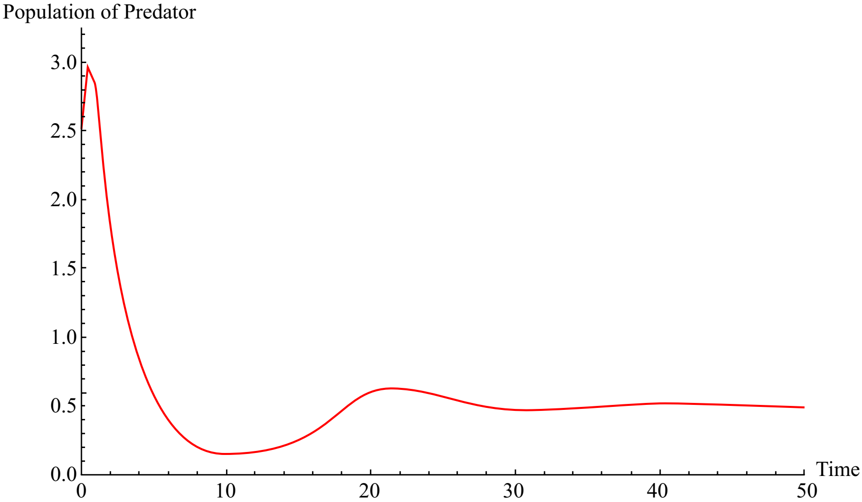

Fig. 5.1 Population of predator ζ with parameters (5.70) and $\mathbf{u}(t) = 0.49e^{-t}$

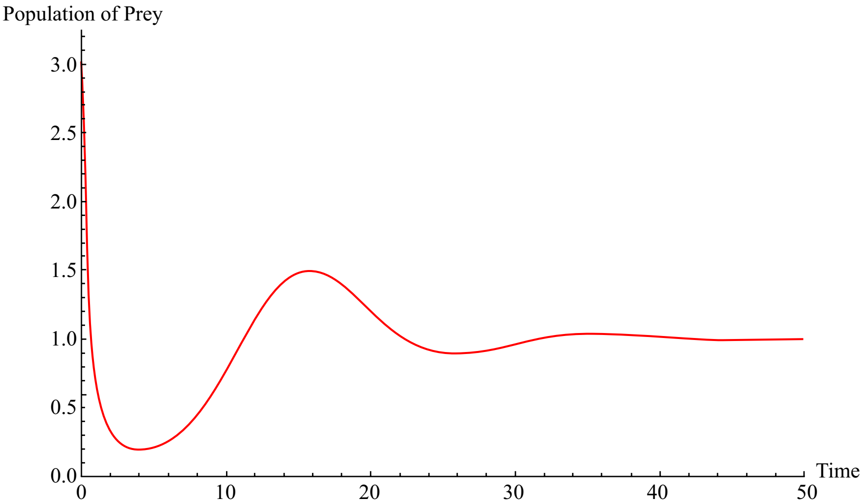

Fig. 5.2 Population of prey χ with parameters (5.70) and $\mathbf{u}(t) = 0.49e^{-t}$

$$\zeta(t) \to 0.25\frac{\beta L}{a} = 0.5 \quad \text{and} \quad \chi(t) \to 0.5L = 1, \tag{5.71}$$

which is in fact the behavior we see in the figures. This shows the robustness of the convergence in the face of the disturbance $\mathbf{u}$.

5.6 Comments

Several authors have studied ways to construct strict Lyapunov under appropriate conditions on the iterated Lie derivatives, or using non-strict Lyapunov functions. Two significant results in this direction are [5, 41]. The results of [5] deal with ISS, and [41] with controller design by using CLFs for systems that satisfy Jurdjevic-Quinn Conditions. The construction in [5] uses a weak Lyapunov function and an auxiliary Lyapunov function V_2 that satisfies certain detectability properties of the system with respect to an appropriate output $h(x)$.

More precisely, [5] assumes that there are two positive definite radially unbounded functions V_1 and V_2 and functions $\alpha_1, \alpha_2, \gamma \in \mathcal{K}_\infty$ satisfying

$$\dot{V}_1 \leq -\alpha_1(|y|) \text{ and } \dot{V}_2 \leq -\alpha_2(|x|) + \gamma(|y|) \ , \tag{5.72}$$

for all $x \in \mathbb{R}^n$, where $y = h(x)$. Note that V_1 in (5.72) is typically a weak Lyapunov function since $|h(x)|$ is often positive semi-definite. The function V_2 in (5.72) is an output-to-state Lyapunov function [73] that characterizes a particular form of detectability of x from the output y. The strong Lyapunov function in [5] then takes the form

$$U(x) = V_1(x) + \rho(V_2(x)),$$

where ρ is a suitable $\mathcal{K}_\infty$ function.

The main difference between our approach from Theorem 5.1 and [5] is that our conditions appear to be stronger but easier to check than those in [5]. While very general, the challenge in applying [5] stems from the need to find V_2. The auxiliary function can be found in certain useful cases, but to our knowledge there is no general procedure for finding V_2 in the context of [5]. This gives a possible advantage in checking the iterated Lie derivative condition from Theorem 5.1 and then using our construction (5.2). Another difference between [5] and our methods is that our auxiliary functions are not required to be radially unbounded or everywhere positive.

By contrast, the strict Lyapunov construction of [41] only uses the given non-strict Lyapunov function V_1 and the iterated Lie derivatives of V_1 along solutions of an auxiliary system with a scaled vector field. The results in [41] seem more direct than those of [5], but the method of [41] is in general only applicable to homogenous systems. (The translational oscillator with rotating actuator or TORA example in [41] is inhomogeneous, but [41] does not give a systematic method for inhomogenous systems.) To our knowledge, [102] provides the first general construction for CLFs for general classes of Jurdjevic Quinn systems that do not necessarily satisfy the homogeneity conditions from [41].

Conditions 1. and 2. from Theorem 5.1 agree with the assumptions from the strict Lyapunov function construction in [110, Theorem 3.1]. However, our proof of Theorem 5.1 is simpler than the arguments used in [110]. The construction in [110, Theorem 3.1] proceeds by finding a non-increasing function $\lambda : [0, \infty) \to (0, \infty)$ such that the function

$$U(x) = V(x)\left[1 + V(x) - \sum_{i=1}^{\ell-1} L^i_{f_\lambda} V(x) \cdot \left(L^{i+1}_{f_\lambda} V(x)\right)^{3^i}\right] \tag{5.73}$$

is a strict Lyapunov function for the system (5.1), where

$$f_\lambda(x) \doteq \lambda(V(x))f(x).$$

In [86, Sect. 3.3], conditions similar to Assumption 5.1 were used to conclude asymptotic stability of systems which admit a non-strict Lyapunov function, via an extension of Matrosov's Theorem. However, no strict Lyapunov functions were constructed in this earlier work.

It is possible to extend Theorem 5.1 to periodic time-varying systems, in which case we instead take

$$a_1(t, x) = -[V_t(t, x) + V_x(t, x)f(t, x)]$$

and $a_i = -\dot{a}_{i-1}$ for all $i \geq 2$ and consider the non-negative function

$$\sum_{i=2}^{\ell} a_i^2(t, x) + a_1(t, x),$$

which is allowed to be zero for some $x \neq 0$ on some intervals of positive length; see [104]. Section 5.4 is based on [104].

Our strict Lyapunov function construction for the Lotka-Volterra system is based on [104]. The Lotka-Volterra model is used extensively in mathematical biology. See [58, 79] for an extensive analysis of this model and generalizations to several predators. While there are many Lyapunov constructions for Lotka-Volterra models available (based on computing the LaSalle Invariant Set), to the best of our knowledge, the result we gave in this chapter is original and significant because we provide a *global strict Lyapunov function.*