# Chapter 4
# Jurdjevic-Quinn Conditions

**Abstract** The Jurdjevic-Quinn Theorem provides a powerful framework for guaranteeing globally asymptotic stability, using a smooth feedback of arbitrarily small amplitude. It requires certain algebraic conditions on the Lie derivatives of a suitable non-strict Lyapunov function, in the directions of the vector fields that define the system. The non-strictness of the Lyapunov function is an obstacle to proving robustness, since robustness analysis typically requires *strict* Lyapunov functions.

In this chapter, we provide a method for overcoming this obstacle. It involves transforming the non-strict Lyapunov function into an explicit global CLF. This gives a strict Lyapunov function construction for closed-loop Jurdjevic-Quinn systems with feedbacks of arbitrarily small magnitude. This is valuable because (a) the non-strict Lyapunov function from the Jurdjevic-Quinn Theorem is often known explicitly and (b) our methods apply to Hamiltonian systems, which commonly arise in mechanical engineering. We illustrate our work using a two-link manipulator model, as well as an integral input-to-state stability result.

## 4.1 Motivation

Consider the two-link manipulator system from [5]. This is a fully actuated system obtained by viewing the robot arm as a segment with length $L$ and mass $M$. Letting $m$ denote the mass of the hand, $r$ the position of the hand, and $\theta$ the angle of the arm, we get the Euler-Lagrange equations

$$\left\{ \begin{array}{r} \left( mr^2 + M\dfrac{L^2}{3} \right) \ddot{\theta} + 2Mr\dot{r}\dot{\theta} = \tau \\ m\ddot{r} - mr\dot{\theta}^2 = F \ , \end{array} \right. \tag{4.1}$$

where $\tau$ and $F$ are forces acting on the system. See Fig. 4.1.

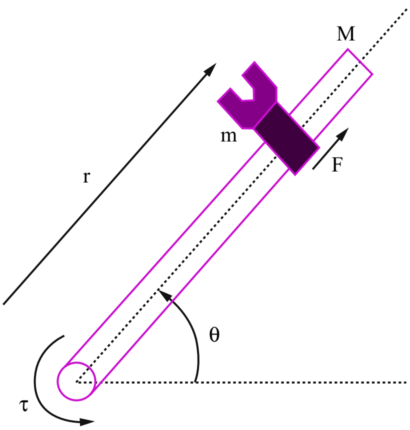


**Fig. 4.1** Linear rotational actuated arm modeled by Euler-Lagrange Eq. (4.1)

It is well-known that (4.1) can be stabilized by bounded control laws. However, it is not clear how to construct a CLF for the system whose time derivative along the trajectory is made negative definite by an appropriate choice of bounded feedback. Let us show how such a CLF can be constructed.

For simplicity, we take

$$m = M = 1,\ L = \sqrt{3},\ x_1 \doteq \theta,\ x_2 \doteq \dot{\theta},\ x_3 \doteq r,\ \text{ and }\ x_4 \doteq \dot{r}.$$

The system (4.1) becomes

$$\begin{cases} \dot{x}_1 = x_2, \quad \dot{x}_2 = -\dfrac{2x_3x_2x_4}{x_3^2+1} + \dfrac{\tau}{x_3^2+1}, \\ \dot{x}_3 = x_4, \quad \dot{x}_4 = x_3x_2^2 + F. \end{cases} \tag{4.2}$$

We construct a globally asymptotically stabilizing feedback that is bounded by 2, and an associated CLF for (4.2). We set

$$\langle p \rangle = \frac{1}{2\sqrt{1+p^2}}$$

for all $p \in \mathbb{R}$ throughout the sequel.

Consider the positive definite and radially unbounded function

$$V(x) = \frac{1}{2}\left[(x_3^2+1)x_2^2 + x_4^2 + \sqrt{1+x_1^2} + \sqrt{1+x_3^2} - 2\right], \tag{4.3}$$

which is composed of the kinetic energy of the system with additional terms. With the change of feedback

$$\tau = -x_1\langle x_1\rangle + \tau_b \ , \ F = -x_3\langle x_3\rangle + F_b, \tag{4.4}$$

the system (4.2) takes the control affine form

$$\dot{x} = f(x) + g(x)u, \quad \text{where}$$
$$f(x) = \begin{bmatrix} x_2 \\ \frac{-2x_3x_2x_4 - x_1\langle x_1\rangle}{x_3^2+1} \\ x_4 \\ x_2^2x_3 - x_3\langle x_3\rangle \end{bmatrix}, \quad g(x) = \begin{bmatrix} 0 & 0 \\ \frac{1}{x_3^2+1} & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \text{ and } u = \begin{bmatrix} \tau_b \\ F_b \end{bmatrix}. \tag{4.5}$$

Next consider the vector field

$$G(x) = (0, x_1, 0, x_3)^\top$$

and the function

$$V^\sharp(x) \ = \ 40[2 + 2V(x)]^6 + L_G V(x) - 40(2^6) \ . \tag{4.6}$$

One can show that when we choose the feedbacks

$$\tau_b = -x_2\langle x_2\rangle \ \text{ and } \ F_b = -x_4\langle x_4\rangle, \tag{4.7}$$

the time derivative of $V^\sharp$ along the trajectories of the closed-loop system (4.5) satisfies

$$\dot{V}^\sharp(x) \leq -\frac{1}{2}\left[x_1^2\langle x_1\rangle + x_2^2\langle x_2\rangle + x_3^2\langle x_3\rangle + x_4^2\langle x_4\rangle\right] \ , \tag{4.8}$$

and that $V^\sharp$ is proper and positive definite; see Sect. 4.7.2 for details. The right hand side of this inequality is negative definite, and the feedback $(\tau, F)$ given by (4.4) and (4.7) is bounded in norm by 2, as desired. Also, since the feedback is 0 and continuous at the origin, the CLF (4.6) satisfies the small control property. We turn next to a general construction that leads to $V^\sharp$ as a special case.

## 4.2 Control Affine Case

### *4.2.1 Assumptions and Statement of Result*

We first consider control affine systems

$$\dot{x} = f(x) + g(x)u \tag{4.9}$$

with state space $\mathcal{X} = \mathbb{R}^n$ and control set $U = \mathbb{R}^m$, where $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are assumed to be smooth, i.e., $C^\infty$, and $f(0) = 0$.[1] In Sect. 4.3, we use our arguments for the control affine case to extend our results to general nonlinear systems $\dot{x} = \mathcal{F}(x, u)$. We assume the following:

**Assumption 4.1** *There is a storage function* $V : \mathbb{R}^n \to [0, \infty)$ *such that* $L_fV(x) \le 0$ *everywhere. Moreover, there is a smooth scalar function* $\psi$ *such that if* $x \neq 0$ *is such that* $L_fV(x) = 0$ *and* $L_gV(x) = 0$ *both hold, then* $L_f\psi(x) < 0$.

We refer to $\psi$ as the *auxiliary scalar field*; see [41] or Sect. 4.4 below for general methods for constructing $\psi$ when the Weak Jurdjevic-Quinn Conditions are satisfied. Our main result for (4.9) is:

**Theorem 4.1.** *Let (4.9) be such that Assumption 4.1 is satisfied. Then we can explicitly construct* $C^1$ *functions* $\lambda$ *and* $\Gamma$ *such that*

$$\mathcal{U}(x) = \lambda\big(V(x)\big)\psi(x) + \Gamma\big(V(x)\big) \tag{4.10}$$

*is a CLF for (4.9) that satisfies the small control property. In fact, for any smooth everywhere positive function* $\xi : \mathbb{R}^n \to (0, \infty)$*, we can construct* $\lambda$ *and* $\Gamma$ *in such a way that (4.10) is a strict Lyapunov function for (4.9) in closed-loop with the feedback* $u(x) = -\xi(x)L_gV(x)^\top$.

In particular, we get stabilizing feedbacks of arbitrarily small amplitude.

## *4.2.2 Main Lemmas*

We use the following lemmas to prove Theorem 4.1. We use the notation

$$\begin{aligned} \mathcal{N}(x) &= -\min\{0, L_f\psi(x)\}\ , \\ H(x) &= -L_fV(x) + |L_gV(x)|^2\ , \text{ and} \\ S(x) &= H(x) + \mathcal{N}(x)\ . \end{aligned} \tag{4.11}$$

**Lemma 4.1.** *The function* $S(x)$ *is continuous and positive definite.*

*Proof.* By Assumption 4.1, both $H$ and $\mathcal{N}$ are non-negative, so $S$ is non-negative. On the other hand, if $S(x) = 0$, then $L_fV(x) = 0$, $L_gV(x) = 0$ and $L_f\psi(x) \ge 0$. Then Assumption 4.1 implies that $x = 0$. □

A key feature of our proof of Theorem 4.1 is that it provides explicit formulas for functions $\lambda$ and $\Gamma$ such that (4.10) is a CLF for (4.9). In fact, we will prove later that (4.10) is a CLF for (4.9) when

[1] The smoothness assumptions in this section can be replaced by the assumption that the relevant functions are $C^k$ where $k$ is large enough to make the CLF and feedback we construct $C^1$.

$$\begin{aligned} &\Gamma(r) = \textstyle\int_0^r \gamma(s)\mathrm{d}s, \\ &\text{where } \; \gamma(s) = 1 + K_1'(s)s + 3\left[K_1(s) + K_1^{3/2}(s)\right] \end{aligned} \tag{4.12}$$

and $K_1$ and $\lambda$ satisfy the requirements of the following lemma:

**Lemma 4.2.** *Let Assumption 4.1 hold. Then we can construct a function $\lambda \in \mathcal{K}_\infty \cap C^1$ and a $C^1$ increasing function $K_1 : [0,\infty) \to (0,\infty)$ such that $\lambda(s) \le K_1(s)$ everywhere,*

$$\lambda(v) \le v \;\; \forall v \ge 0\,, \quad \textit{and} \tag{4.13}$$

$$\lambda'(V(x))|\psi(x)| + \lambda(V(x)) \le K_1(V(x))\,, \tag{4.14}$$

$$|\psi(x)| \le K_1(V(x))\,, \tag{4.15}$$

$$|L_g\psi(x)|^2 \le K_1(V(x))\,, \tag{4.16}$$

*and*

$$\lambda(V(x))\big[1 + \max\{0, L_f\psi(x)\}\big] \le S(x)K_1(V(x)) \tag{4.17}$$

*hold for all $x \in \mathbb{R}^n$.*

*Proof.* Since $S$ is positive definite and $V$ is proper and positive definite, we can find a continuous positive definite function $\rho_0$ so that $S(x) \ge \rho_0(V(x))$ for all $x \in \mathbb{R}^n$ (by first finding a positive definite function $\tilde{\rho}$ that is increasing on $[0,1]$ and decreasing on $[1,\infty)$ such that $S(x) \ge \tilde{\rho}(|x|)$ everywhere). Hence, Lemma A.7 provides $\lambda \in \mathcal{K}_\infty \cap C^1$ such that (4.13) is satisfied and an everywhere positive increasing function $\bar{K} \in C^1$ such that

$$\lambda(V(x)) \le S(x)\bar{K}(V(x)) \;\; \forall x \in \mathbb{R}^n. \tag{4.18}$$

We can also find an increasing function $\bar{\kappa} \in C^1$ such that

$$1 + \max\{0, L_f\psi(x)\} \le \bar{\kappa}\big(V(x)\big) \;\; \forall x \in \mathbb{R}^n. \tag{4.19}$$

Combining (4.18) and (4.19) provides an increasing function $\kappa_1 \in C^1$ such that

$$\lambda(V(x))\big[1 + \max\{0, L_f\psi(x)\}\big] \le S(x)\kappa_1\big(V(x)\big) \;\; \forall x \in \mathbb{R}^n. \tag{4.20}$$

Next, one can determine everywhere positive increasing functions $\kappa_i \in C^1$ for $i = 2, 3, 4$ such that

$$\lambda'(V(x))|\psi(x)| + \lambda\big(V(x)\big) \le \kappa_2\big(V(x)\big) \tag{4.21}$$

and

$$|\psi(x)| \le \kappa_3\big(V(x)\big) \;\text{ and }\; |L_g\psi(x)|^2 \le \kappa_4\big(V(x)\big) \tag{4.22}$$

hold for all $x \in \mathbb{R}^n$. Since $\lambda' \ge 0$, the inequality $\lambda(s) \le \kappa_2(s)$ is satisfied everywhere. It follows that

$$K_1(v) = \sum_{i=1}^{4} \kappa_i(v), \tag{4.23}$$

is such that the inequalities (4.14)-(4.17) are all satisfied. □

In the sequel, all (in)equalities should be understood to hold for all $x \in \mathbb{R}^n$ unless otherwise indicated. The following lemma is a key ingredient in our proof of the Lyapunov decay condition for (4.10):

**Lemma 4.3.** *Let the functions $\lambda$ and $K_1$ satisfy the requirements of Lemma 4.2. Then for all $x \in \mathbb{R}^n$, the inequality*

$$\lambda\big(V(x)\big)L_f\psi(x) \le -\lambda\big(V(x)\big)S(x) + 2K_1\big(V(x)\big)H(x) \tag{4.24}$$

*is satisfied.*

*Proof.* According to the definition of $\mathcal{N}$, we get

$$L_f\psi(x) \;=\; -\mathcal{N}(x) + \max\{0, L_f\psi(x)\}. \tag{4.25}$$

Therefore, (4.17) from Lemma 4.2 gives

$$\begin{aligned}\lambda(V(x))L_f\psi(x) &= -\lambda(V(x))\mathcal{N}(x) + \lambda(V(x))\max\{0, L_f\psi(x)\}\\ &\le -\lambda(V(x))\mathcal{N}(x) + S(x)K_1(V(x)).\end{aligned} \tag{4.26}$$

We consider two cases.

*Case 1.* $L_f\psi(x) \le 0$. Then, (4.26) gives

$$\begin{aligned}\lambda(V(x))L_f\psi(x) &= -\lambda(V(x))\mathcal{N}(x)\\ &= -\lambda(V(x))S(x) + H(x)\lambda(V(x)).\end{aligned} \tag{4.27}$$

*Case 2.* $L_f\psi(x) > 0$. Then, the definition of $\mathcal{N}$ in (4.11) gives $\mathcal{N}(x) = 0$, which implies that $S(x) = H(x)$. This combined with (4.26) yields

$$\begin{aligned}\lambda(V(x))L_f\psi(x) &\le H(x)K_1(V(x))\\ &= -\lambda(V(x))S(x) + H(x)[K_1(V(x)) + \lambda(V(x))].\end{aligned} \tag{4.28}$$

We deduce that in both cases,

$$\lambda(V(x))L_f\psi(x) \le -\lambda(V(x))S(x) + H(x)\left[K_1(V(x)) + \lambda(V(x))\right]. \tag{4.29}$$

The result follows because $\lambda(s) \le K_1(s)$ everywhere, by (4.14). □

### *4.2.3 Checking the CLF Properties*

Returning to the proof of Theorem 4.1, let $\Gamma$ be the function defined in (4.12), and let $K_1$ and $\lambda$ be the functions provided by Lemma 4.2. We check that the resulting function $\mathcal{U}$ from (4.10) satisfies the required CLF properties. Notice that $\Gamma(v) \geq v + \int_0^v [K_1'(s)s + K_1(s)]\mathrm{d}s = v + K_1(v)v$ everywhere, so

$$\mathcal{U}(x) \geq \lambda\big(V(x)\big)\psi(x) + V(x) + K_1\big(V(x)\big)V(x) \tag{4.30}$$

holds for all $x \in \mathbb{R}^n$. From (4.13) and (4.15), we deduce that

$$\mathcal{U}(x) \geq V(x) \tag{4.31}$$

so $\mathcal{U}$ is positive definite and radially unbounded.

The time derivative of $\mathcal{U}$ along the trajectories of (4.9) is

$$\begin{aligned} \dot{\mathcal{U}}(x) &= \lambda\big(V(x)\big)\big[L_f\psi(x) + L_g\psi(x)u\big] \\ &\quad + \big[\lambda'(V(x))\psi(x) + \Gamma'\big(V(x)\big)\big]\big[L_fV(x) + L_gV(x)u\big] \\ &= \lambda\big(V(x)\big)L_f\psi(x) + \big[\lambda'(V(x))\psi(x) + \Gamma'(V(x))\big]L_fV(x) \\ &\quad + \Theta(x)u\ , \end{aligned} \tag{4.32}$$

where

$$\Theta(x) = \lambda(V(x))L_g\psi(x) + \Big\{\lambda'(V(x))\psi(x) + \gamma(V(x))\Big\}L_gV(x). \tag{4.33}$$

Using Lemma 4.3 and the definition of $H$ in (4.11), we deduce that

$$\begin{aligned} \dot{\mathcal{U}}(x) &\leq -\lambda\big(V(x)\big)S(x) + 2K_1\big(V(x)\big[-L_fV(x) + |L_gV(x)|^2\big] \\ &\quad + \big[\lambda'\big(V(x)\big)\psi(x) + \Gamma'\big(V(x)\big)\big]L_fV(x) + \Theta(x)u \\ &= -\lambda\big(V(x)\big)S(x) + 2K_1\big(V(x)\big)|L_gV(x)|^2 + \Theta(x)u \\ &\quad + \big[\lambda'\big(V(x)\big)\psi(x) - 2K_1\big(V(x)\big) + \gamma\big(V(x)\big)\big]L_fV(x)\ . \end{aligned} \tag{4.34}$$

Recalling (4.14) and the facts that $\gamma(\ell) \geq 3K_1(\ell)$ for all $\ell \geq 0$ and $L_fV(x) \leq 0$ everywhere, we obtain

$$\dot{\mathcal{U}}(x) \leq \mho(x) + \Theta(x)u\ , \tag{4.35}$$

where

$$\mho(x) = -\lambda\big(V(x)\big)S(x) + 2K_1\big(V(x)\big)|L_gV(x)|^2\ . \tag{4.36}$$

Consider the control

$$u = -50\Theta(x)\ . \tag{4.37}$$

It suffices to show that $\mho(x) - 50\Theta^2(x)$ is negative definite because (4.35) and (4.37) combine to give $\dot{\mathcal{U}}(x) \leq \mho(x) - 50\Theta^2(x)$. To prove that $\mho(x) - 50\Theta^2(x)$ is negative definite, we proceed by contradiction. Suppose that there exists $x \neq 0$ such that $\mho(x) - 50\Theta^2(x) \geq 0$, or equivalently,

$$-\lambda\big(V(x)\big)S(x) + 2K_1\big(V(x)\big)|L_gV(x)|^2 - 50\Big[\lambda\big(V(x)\big)L_g\psi(x) + \left\{\lambda'\big(V(x)\big)\psi(x) + \gamma\big(V(x)\big)\right\}L_gV(x)\Big]^2 \geq 0. \tag{4.38}$$

Then,

$$2K_1\big(V(x)\big)|L_gV(x)|^2 \geq 50\Big[\lambda\big(V(x)\big)L_g\psi(x) + \left\{\lambda'\big(V(x)\big)\psi(x) + \gamma\big(V(x)\big)\right\}L_gV(x)\Big]^2. \tag{4.39}$$

Therefore, the general relation $|a+b+c| \geq |a| - |b| - |c|$ for any real numbers $a$, $b$, and $c$ gives

$$\begin{aligned}\sqrt{K_1(V(x))}|L_gV(x)| &\geq 5\Big|\lambda\big(V(x)\big)L_g\psi(x) \\ &\quad + \left\{\lambda'\big(V(x)\big)\psi(x) + \gamma\big(V(x)\big)\right\}L_gV(x)\Big| \\ &\geq -5\big|\lambda\big(V(x)\big)L_g\psi(x)\big| - 5\big|\lambda'\big(V(x)\big)\psi(x)\big||L_gV(x)| \\ &\quad +5\gamma(V(x))|L_gV(x)|\ .\end{aligned}$$

It follows that

$$5|\lambda\big(V(x)\big)L_g\psi(x)| \geq \Big[-\sqrt{K_1\big(V(x)\big)} - 5\big|\lambda'\big(V(x)\big)\psi(x)\big| + 5\gamma\big(V(x)\big)\Big]|L_gV(x)|\ . \tag{4.40}$$

From (4.14), we deduce that

$$\begin{aligned}5|\lambda(V(x))L_g\psi(x)| &\geq \Big[-\sqrt{K_1\big(V(x)\big)} - 5K_1(V(x)) + 5\gamma\big(V(x)\big)\Big]|L_gV(x)| \\ &\geq \big[-1 - 6K_1\big(V(x)\big) + 5\gamma(V(x))\big]|L_gV(x)|\ .\end{aligned}$$

Our choice of $\gamma$ in (4.12) then gives$5|\lambda V(x))L_g\psi(x)| \geq 3\gamma(V(x))|L_gV(x)|$ and therefore

$$50K_1(V(x))\frac{|\lambda(V(x))L_g\psi(x)|^2}{9\gamma^2\big(V(x)\big)} \ \geq\ 2K_1\big(V(x)\big)|L_gV(x)|^2\ . \tag{4.41}$$

Since (4.38) implies that $\lambda(V(x))S(x) \leq 2K_1(V(x))|L_gV(x)|^2$, we get

$$50K_1(V(x))\frac{|\lambda(V(x))L_g\psi(x)|^2}{9\gamma^2(V(x))} \ \geq\ \lambda\big(V(x)\big)S(x) \tag{4.42}$$

and then

$$50K_1^2(V(x))\frac{|L_g\psi(x)|^2}{9\gamma^2(V(x))}\lambda(V(x)) \ \geq\ S(x)K_1(V(x))\ . \tag{4.43}$$

From (4.16), we deduce that

$$\tfrac{50K_1^3(V(x))}{9\gamma^2(V(x))}\lambda(V(x)) \;\geq\; S(x)K_1(V(x)) \; . \tag{4.44}$$

Recalling our choice of $\gamma$ in (4.12), we have

$$\tfrac{50K_1^3(V(x))}{9\gamma^2(V(x))} \leq \tfrac{50K_1^3(V(x))}{9[1+3K_1^{3/2}(V(x))]^2} \;\; \leq \;\; \tfrac{50}{81} \; . \tag{4.45}$$

It follows that $\frac{50}{81}\lambda(V(x)) \geq S(x)K_1(V(x))$. This contradicts (4.17). We conclude that $\mathcal{U}$ is a CLF for (4.9) that satisfies the small control property, when $\Gamma$ is defined by (4.12).

### *4.2.4 Arbitrarily Small Stabilizing Feedbacks*

In the previous section, we constructed a family of CLFs of the form

$$\mathcal{U}(x) = \lambda(V(x))\psi(x) + \int_0^{V(x)} \gamma(r)\mathrm{d}r \tag{4.46}$$

for the control affine system (4.9). We now show that for any smooth function $\xi : \mathbb{R}^n \to (0, \infty)$, we can choose $\gamma$ in such a way that (4.46) is a strict Lyapunov function for (4.9) in closed-loop with

$$u(x) = -\xi(x)L_gV(x)^\top \; . \tag{4.47}$$

This will prove that for any control set $U \subseteq \mathbb{R}^m$ containing a neighborhood of the origin, (4.9) is $C^1$ globally asymptotically stabilizable by a feedback that takes all of its values in $U$.

To this end, pick any $C^1$ function $\gamma$ such that

$$\gamma(s) \geq 1 + K_1'(s)s + 3\left[K_1(s) + K_1^{3/2}(s)\right] \;\; \forall s \geq 0 \tag{4.48}$$

and

$$\gamma(V(x)) \geq \frac{4K_1(V(x))}{\xi(x)} + 2K_1(V(x)) + 2\xi(x)K_1^2(V(x)) \;\; \forall x \in \mathbb{R}^n \; , \tag{4.49}$$

where $K_1$ is the function provided by Lemma 4.2. We show that the time derivative of (4.46) along the solutions of (4.9) in closed-loop with (4.47) satisfies

$$\dot{\mathcal{U}}(x) \;\leq\; -W(x), \tag{4.50}$$

where

$$W(x) = \frac{3}{4}\lambda\big(V(x)\big)S(x) + \frac{1}{2}\xi(x)\gamma\big(V(x)\big)\big|L_gV(x)\big|^2. \tag{4.51}$$

Here $\lambda$ is the function constructed in Lemma 4.2. The result is then immediate from the smoothness of $u$ and the fact that $u(0) = 0$, because $V$, $\lambda$, and $S$ are all positive definite.

To prove the estimate (4.50), first notice that (4.35) and (4.48) give

$$\begin{aligned} \dot{\mathcal{U}}(x) &\leq \mho(x) - \Theta(x)\xi(x)L_gV(x)^\top \\ &= \mho(x) - \xi(x)\Big[\lambda\big(V(x)\big)L_g\psi(x)L_gV(x)^\top \\ &\quad +\big\{\lambda'\big(V(x)\big)\psi(x) + \gamma\big(V(x)\big)\big\}\big|L_gV(x)\big|^2\Big]. \end{aligned} \tag{4.52}$$

Therefore, our choice of $\mho$ in (4.36) gives

$$\begin{aligned} \dot{\mathcal{U}}(x) \;&\leq\; -\lambda\big(V(x)\big)S(x) + \xi(x)\lambda\big(V(x)\big)\big|L_g\psi(x)\big|\big|L_gV(x)\big| \\ &+\Big[2K_1\big(V(x)\big) + \xi(x)\big\{|\lambda'\big(V(x)\big)\psi(x)| - \gamma\big(V(x)\big)\big\}\Big]\big|L_gV(x)\big|^2. \end{aligned} \tag{4.53}$$

Recalling (4.14) and (4.16) gives

$$\begin{aligned} \dot{\mathcal{U}}(x) &\leq -\lambda\big(V(x)\big)S(x) + \xi(x)\lambda\big(V(x)\big)\sqrt{K_1(V(x))}|L_gV(x)| \\ &\quad +\Big[2K_1\big(V(x)\big) + \xi(x)\big\{K_1\big(V(x)\big) - \gamma\big(V(x)\big)\big\}\Big]\big|L_gV(x)\big|^2. \end{aligned} \tag{4.54}$$

The triangle inequality $c_1c_2 \leq c_1^2 + \frac{1}{4}c_2^2$ for non-negative $c_1$ and $c_2$ gives

$$\begin{aligned} &\xi(x)\lambda\big(V(x)\big)\sqrt{K_1(V(x))}\big|L_gV(x)\big| \\ &\leq\; \xi^2(x)K_1^2\big(V(x)\big)\big|L_gV(x)\big|^2 + \tfrac{1}{4K_1(V(x))}\lambda^2\big(V(x)\big). \end{aligned}$$

Combining with (4.54), we get

$$\begin{aligned} \dot{\mathcal{U}}(x) &\leq -\lambda\big(V(x)\big)S(x) + \tfrac{1}{4K_1(V(x))}\lambda^2\big(V(x)\big) \\ &\quad +\Big[2K_1\big(V(x)\big) + \xi(x)\big\{K_1\big(V(x)\big) + \xi(x)K_1^2\big(V(x)\big) \\ &\quad -\gamma\big(V(x)\big)\big\}\Big]\big|L_gV(x)\big|^2. \end{aligned} \tag{4.55}$$

Property (4.17) from Lemma 4.2 now gives

$$\frac{1}{4K_1(V(x))}\lambda^2\big(V(x)\big) \;\leq\; \frac{1}{4}\lambda\big(V(x)\big)S(x). \tag{4.56}$$

Hence, (4.49) and (4.55) give

$$\dot{\mathcal{U}}(x) \leq -\tfrac{3}{4}\lambda\big(V(x)\big)S(x) - \tfrac{1}{2}\xi(x)\gamma\big(V(x)\big)\big|L_gV(x)\big|^2 \tag{4.57}$$

for all $x \in \mathbb{R}^n$, which is the desired Lyapunov decay condition. This completes the proof of Theorem 4.1. □

*Remark 4.1.* The simplicity of the formula for $\mathcal{U}$ depends on the choice for $\xi$. For example, if we pick

$$\xi(x) = \frac{1}{\sqrt{K_1(V(x))}},$$

then (4.49) becomes

$$\gamma\big(V(x)\big) \;\geq\; 2K_1\big(V(x)\big) + 6K_1^{3/2}\big(V(x)\big),$$

so we can satisfy (4.48)-(4.49) by taking

$$\gamma(s) \;=\; 1 + 3K_1(s) + K_1'(s)s + 6K_1^{3/2}(s)$$

to obtain our strict Lyapunov function for the corresponding closed-loop system.

## 4.3 General Case

We now use our results for the control affine system (4.9) to get analogous constructions for general nonlinear systems

$$\dot{x} = \mathcal{F}(x, u) \tag{4.58}$$

evolving on $\mathbb{R}^n$ with controls in $\mathbb{R}^m$, where $\mathcal{F}$ is assumed to be smooth. We also assume $\mathcal{F}(0,0) = 0$.

We can write

$$\begin{aligned} \mathcal{F}(x,u) &= f(x) + g(x)u + h(x,u)u, \;\; \text{where} \\ f(x) = \mathcal{F}(x,0), \;\; g(x) \;&=\; \frac{\partial \mathcal{F}}{\partial u}(x,0), \;\; \text{and} \\ h(x,u) &= \int_0^1 \left[\frac{\partial \mathcal{F}}{\partial u}(x,\lambda u) - \frac{\partial \mathcal{F}}{\partial u}(x,0)\right] \mathrm{d}\lambda. \end{aligned} \tag{4.59}$$

Since $\mathcal{F}$ is $C^2$ in $u$, we can find a continuous function $R : [0,\infty) \times [0,\infty) \to (0,\infty)$ that is non-decreasing in both variables such that

$$|h(x,u)u| \;\leq\; R\big(|x|,|u|\big)|u|^2$$

for all $x$ and $u$. Hence, we assume in the rest of the subsection that our system has the form

$$\dot{x} = f(x) + g(x)u + r(x,u) \tag{4.60}$$

with $f(0)=0$ and with $r$ admitting an everywhere positive continuous function $R$ that is non-decreasing in both variables so that

$$\big|r(x,u)\big| \leq R\big(|x|,|u|\big)|u|^2 \tag{4.61}$$

everywhere. Let Assumption 4.1 hold for the functions $f$ and $g$ in the system (4.60) and some functions $V$ and $\psi$. Fix $C^1$ functions $\lambda$ and $K_1$ satisfying the requirements of Lemma 4.2 as well as

$$\max\left\{\left|\frac{\partial V}{\partial x}(x)\right|, \left|\frac{\partial \psi}{\partial x}(x)\right|, \big|L_gV(x)\big|\right\} \;\leq\; K_1\big(V(x)\big) \tag{4.62}$$

for all $x \in \mathbb{R}^n$. We prove the following:

**Theorem 4.2.** *Let Assumption 4.1 hold. Let $\xi : \mathbb{R}^n \to (0,\infty)$ be any smooth function such that*

$$\xi(x) \;\leq\; \min\left\{\frac{1}{K_1(V(x))}, \frac{1}{4K_1(V(x))R(|x|,1)}\right\} \;\forall x \in \mathbb{R}^n. \tag{4.63}$$

*Let $\gamma$ be any continuous everywhere positive function satisfying (4.48) and (4.49) and*

$$\gamma(V(x)) \;\geq\; \frac{\xi(x)R\big(|x|,1\big)K_1\big(V(x)\big)\left[V(x)+K_1(V(x))\right]}{0.5-\xi(x)K_1(V(x))R(|x|,1)} \;\;\forall x \in \mathbb{R}^n\,. \tag{4.64}$$

*Set*

$$\Gamma(r) \doteq \int_0^r \gamma(s)\mathrm{d}s,$$

*and let $S(x)$ be as in Lemma 4.1. Then*

$$\mathcal{U}(x) = \lambda(V(x))\psi(x) + \Gamma(V(x)) \tag{4.65}$$

*is a CLF for (4.60) whose time derivative along trajectories of (4.60) in closed-loop with*

$$u(x) \;=\; -\xi(x)L_gV(x)^\top \tag{4.66}$$

*satisfies*

$$\dot{\mathcal{U}}(x) \leq -\frac{3}{4}\lambda\big(V(x)\big)S(x) \;\;\forall x \in \mathbb{R}^n\,. \tag{4.67}$$

*In particular, $\mathcal{U}$ satisfies the small control property, and (4.60) can be rendered GAS to 0 with a smooth feedback $u(x)$ of arbitrary small amplitude.*

*Proof.* Since the requirements from (4.48) and (4.49) are satisfied, we deduce from (4.57) and (4.61) that for all smooth everywhere positive functions $\xi$,

$$\begin{aligned}\dot{\mathcal{U}}(x) &\le -\frac{3}{4}\lambda\big(V(x)\big)S(x) - \frac{1}{2}\xi(x)\gamma\big(V(x)\big)\big|L_gV(x)\big|^2 \\ &\quad + \frac{\partial \mathcal{U}}{\partial x}(x) r\left(x, -\xi(x)L_gV(x)^\top\right) \\ &\le -\frac{3}{4}\lambda\big(V(x)\big)S(x) - \frac{1}{2}\xi(x)\gamma\big(V(x)\big)\big|L_gV(x)\big|^2 \\ &\quad + \left|\frac{\partial \mathcal{U}}{\partial x}(x)\right| R\big(|x|, |\xi(x)L_gV(x)|\big)\big|\xi(x)L_gV(x)\big|^2\end{aligned} \tag{4.68}$$

along all trajectories (4.60) when the controller $u$ is from (4.47).

Next, observe that

$$\frac{\partial \mathcal{U}}{\partial x}(x) \;=\; \lambda'\big(V(x)\big)\psi(x)\frac{\partial V}{\partial x}(x) + \lambda\big(V(x)\big)\frac{\partial \psi}{\partial x}(x) + \gamma\big(V(x)\big)\frac{\partial V}{\partial x}(x). \tag{4.69}$$

Recalling (4.13) and (4.14) from Lemma 4.2, as well as the bounds (4.62), we deduce that

$$\left|\frac{\partial \mathcal{U}}{\partial x}(x)\right| \;\le\; K_1^2\big(V(x)\big) + V(x)K_1\big(V(x)\big) + \gamma\big(V(x)\big)K_1\big(V(x)\big)\;. \tag{4.70}$$

Therefore, (4.68) gives

$$\begin{aligned}\dot{\mathcal{U}}(x) &\le -\tfrac{3}{4}\lambda\big(V(x)\big)S(x) - \tfrac{1}{2}\xi(x)\gamma\big(V(x)\big)\big|L_gV(x)\big|^2 \\ &\quad + \Big[K_1\big(V(x)\big) + V(x) + \gamma\big(V(x)\big)\Big] R\big(|x|, \xi(x)K_1(V(x))\big) \\ &\quad \times \xi^2(x)K_1\big(V(x)\big)\big|L_gV(x)\big|^2\;.\end{aligned} \tag{4.71}$$

By (4.63),

$$\xi(x) \le \frac{1}{K_1(V(x))}. \tag{4.72}$$

Hence,

$$\begin{aligned}\dot{\mathcal{U}}(x) &\le -\tfrac{3}{4}\lambda\big(V(x)\big)S(x) - \tfrac{1}{2}\xi(x)\gamma\big(V(x)\big)\big|L_gV(x)\big|^2 \\ &\quad + R(|x|, 1)K_1\big(V(x)\big)\Big[K_1\big(V(x)\big) + V(x) + \gamma\big(V(x)\big)\Big] \\ &\quad \times \xi^2(x)\big|L_gV(x)\big|^2\;.\end{aligned} \tag{4.73}$$

Finally, our requirement (4.64) on $\gamma$ gives

$$\dot{\mathcal{U}} \le -\tfrac{3}{4}\lambda\big(V(x)\big)S(x)\;. \tag{4.74}$$

This concludes the proof. □

## 4.4 Construction of the Auxiliary Scalar Field

Recall that Assumption 4.1 requires an auxiliary scalar field $\psi$ with the following property: If $x \neq 0$ is such that $L_f V(x) = 0$ and $L_g V(x) = 0$ both hold, then $L_f \psi(x) < 0$. There are several methods for constructing $\psi$. In the next section, we discuss a method for Hamiltonian systems. Here we present a more general construction that applies to any control affine system

$$\dot{x} = f_0(x) + \sum_{i=1}^{m} f_i(x) u_i \tag{4.75}$$

with smooth functions $f_i : \mathbb{R}^n \to \mathbb{R}^n$ for $i = 0, 1, \ldots, m$ that satisfies the[2]

*Weak Jurdjevic Quinn Conditions:* There exists a smooth function $V : \mathbb{R}^n \to \mathbb{R}$ satisfying:

1. $V$ is positive definite and radially unbounded;
2. for all $x \in \mathbb{R}^n$, $L_{f_0} V(x) \leq 0$; and
3. there exists an integer $l \geq 2$ such that the set

$$W(V) = \left\{ \begin{array}{l} x \in \mathbb{R}^n : \forall k \in \{1, \ldots, m\} \text{ and } \forall i \in \{0, \ldots, l\}, \\ L_{f_0} V(x) = L_{ad^i_{f_0}(f_k)} V(x) = 0 \end{array} \right\}$$

   equals $\{0\}$.

We construct $\psi$ as follows, where we omit the arguments of our functions when they are clear from the context:

**Proposition 4.1.** *If (4.75) satisfies the Weak Jurdjevic-Quinn Conditions for some integer $l$ and some storage function $V$, and if we define $G$ by*

$$G = \sum_{i=0}^{l-1} \sum_{k=1}^{m} \lambda_{i,k} \mathrm{ad}^i_{f_0}(f_k), \tag{4.76}$$

*where*

$$\lambda_{i,k} = \sum_{j=i}^{l-1} (-1)^{j-i+1} L_{\mathrm{ad}^{(2j-i+1)}_{f_0}(f_k)} V \quad \forall i, k, \tag{4.77}$$

*then the scalar field $\psi(x) = L_G V(x)$ satisfies the following property: If $x \in \mathbb{R}^n \setminus \{0\}$, and if $L_{f_i} V(x) = 0$ for $i = 0, 1, \ldots, m$, then $L_{f_0} \psi(x) < 0$.*

---

[2] We are using slightly different notation for our control affine systems, to simplify the statement of the next proposition. Recall that for smooth vector fields $f, g : \mathbb{R}^n \to \mathbb{R}^n$, we use the notation

$$\mathrm{ad}^0_f(g) = g, \ \mathrm{ad}_f(g) = [f, g] = g_* f - f_* g, \ \text{and } \mathrm{ad}^k_f(g) = \mathrm{ad}_f\left(\mathrm{ad}^{k-1}_f(g)\right),$$

where the $*$ subscripts indicate gradients.

*Proof.* The proof closely follows that of [41, Theorem 4.3]. The fact that

$$[f_0, G] = \sum_{i=0}^{l-1} \sum_{k=1}^{m} \left( \mathrm{ad}^i_{f_0}(f_k) L_{f_0} \lambda_{i,k} + \lambda_{i,k} \mathrm{ad}^{i+1}_{f_0}(f_k) \right) \tag{4.78}$$

gives

$$\begin{aligned} L_{[f_0,G]}V = & \sum_{k=1}^{m} L_{f_0}\lambda_{0,k} L_{f_k} V + \sum_{k=1}^{m} \lambda_{l-1,k} L_{\mathrm{ad}^l_{f_0}(f_k)} V \\ & + \sum_{i=0}^{l-2} \sum_{k=1}^{m} \left( L_{f_0}\lambda_{i+1,k} + \lambda_{i,k} \right) L_{\mathrm{ad}^{i+1}_{f_0}(f_k)} V. \end{aligned}$$

Recalling our choices (4.77) of the $\lambda_{i,k}$'s gives

$$\begin{aligned} L_{f_0}\lambda_{i+1,k} + \lambda_{i,k} = & \sum_{j=i+1}^{l-1} (-1)^{j-i} L_{f_0} L_{\mathrm{ad}^{2j-i}_{f_0}(f_k)} V \\ & + \sum_{j=i}^{l-1} (-1)^{j-i+1} L_{\mathrm{ad}^{2j-i+1}_{f_0}(f_k)} V \\ = & \sum_{j=i+1}^{l-1} (-1)^{j-i} \left[ L_{f_0} L_{\mathrm{ad}^{2j-i}_{f_0}(f_k)} V - L_{\mathrm{ad}^{2j-i+1}_{f_0}(f_k)} V \right] \\ & - L_{\mathrm{ad}^{i+1}_{f_0}(f_k)} V, \quad i \le l-2\ . \end{aligned}$$

For any smooth vector field $X$ and any point $x$ where $L_{f_0}V(x) = 0$,

$$L_{[f_0,X]}V(x) \ = \ L_{f_0} L_X V(x), \tag{4.79}$$

since $\nabla L_{f_0}V(x) = 0$ at points where the non-positive function $L_{f_0}V$ is maximized. (We are using the fact that $L_{[f,g]} = L_f L_g - L_g L_f$ for smooth vector fields $f$ and $g$.) Taking $X = G$ and then

$$X = \mathrm{ad}^{2j-i}_{f_0}(f_k)$$

in (4.79), we conclude that at all points $x$ where $L_{f_0}V(x) = 0$, we have

$$\begin{aligned} L_{f_0}\psi(x) \ &= \ L_{f_0} L_G V(x) \ = \ L_{[f_0,G]}V(x) \ \text{ and} \\ L_{f_0}\lambda_{i+1,k} + \lambda_{i,k} \ &= \ -L_{\mathrm{ad}^{i+1}_{f_0}(f_k)} V \end{aligned}$$

if $i \le l-2$. By our choice of $\lambda_{l-1,k}$ from (4.77), we conclude that

$$[L_{f_i}V(x) = 0 \ \forall i = 0, 1, \ldots, m] \ \Rightarrow \ L_{f_0}\psi(x) = -\sum_{i=1}^{l} \sum_{k=1}^{m} \left[ L_{\mathrm{ad}^i_{f_0}(f_k)} V(x) \right]^2.$$

The result is now immediate from the Weak Jurdjevic-Quinn Conditions. □

## 4.5 Hamiltonian Systems

Theorem 4.1 covers an important class of dynamics that are governed by the *Euler-Lagrange equations*

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}(q,\dot{q})\right) - \frac{\partial L}{\partial q}(q,\dot{q}) = \tau \tag{4.80}$$

for the motion of mechanical systems. Here $q \in \mathbb{R}^n$ represents the generalized configuration coordinates, $L = K - P$ is the difference between the kinetic energy $K$ and the potential energy $P(q) \geq 0$, and $\tau$ is the control [183]. In many applications,

$$K(q,\dot{q}) = \frac{1}{2}\dot{q}^\top M(q)\dot{q}$$

where the inertia matrix $M(q)$ is $C^1$ and symmetric and positive definite for all $q \in \mathbb{R}^n$. The generalized momenta $\partial L/\partial \dot{q}$ are then given by

$$p = M(q)\dot{q}.$$

Hence, using the state $x = (q,p) \in \mathbb{R}^n \times \mathbb{R}^n$ leads to the system

$$\dot{q} = \frac{\partial H}{\partial p}(q,p)^\top = M^{-1}(q)p, \quad \dot{p} = -\frac{\partial H}{\partial q}(q,p)^\top + \tau, \tag{4.81}$$

where

$$H(q,p) = \frac{1}{2}p^\top M^{-1}(q)p + P(q) \tag{4.82}$$

is the *total energy* of the system. We refer to (4.81) as the *Hamiltonian system.* We assume that $P$ is $C^1$ and positive definite.

The Hamiltonian system can be written as the control affine dynamics

$$\dot{x} = f(x) + g(x)u$$

with state space $\mathcal{X} = \mathbb{R}^{2n}$, control set $U = \mathbb{R}^n$,

$$f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}, \text{ where } f_1(x) = \frac{\partial H}{\partial p}(q,p)^\top \text{ and } f_2(x) = -\frac{\partial H}{\partial q}(q,p)^\top,$$

and

$$g(x) = \begin{pmatrix} O \\ I_n \end{pmatrix} \in \mathbb{R}^{2n\times n}, \tag{4.83}$$

where $O \in \mathbb{R}^{2n\times n}$ denotes the matrix whose entries are all 0. One readily checks that the time derivative of $H$ along the trajectories of (4.81) satisfies

$$\dot{H}(q,p) = \frac{\partial H}{\partial p}(q,p)\tau \tag{4.84}$$

Therefore, if $P(q)$ is positive definite and radially unbounded, then $H$ is a non-strict Lyapunov function. The radial unboundedness of $H$ would follow from the continuity of the (positive) eigenvalues of the positive definite matrix $M^{-1}(q)$ as functions of $q$ [161, Appendix A4], which implies that each compact set $\mathcal{S}$ of $q$ values admits a constant $c_{\mathcal{S}} > 0$ such that

$$p^\top M^{-1}(q)p \geq c_{\mathcal{S}}|p|^2$$

for all $q \in \mathcal{S}$ and all $p \in \mathbb{R}^n$. However, $P$ is not necessarily positive definite and radially unbounded. Fortunately, one can determine a real-valued function $\Lambda \in C^1$ satisfying $\Lambda(0) = 0$ such that the function

$$V(x) = H(q,p) + \Lambda(q) = \frac{1}{2}p^\top M^{-1}(q)p + P_n(q) \tag{4.85}$$

with

$$P_n(q) = P(q) + \Lambda(q) \tag{4.86}$$

is positive definite, radially unbounded and $C^1$. In fact, we can assume that $\Lambda$ is such that

$$\left|\frac{\partial P_n}{\partial q}(q)\right| \geq |q|. \tag{4.87}$$

For simplicity, we take $\Lambda(q) = \frac{1}{2}|q|^2 - P(q)$.

Using the change of feedback

$$\tau = \tau_n - \frac{\partial \Lambda}{\partial q}(q)^\top, \tag{4.88}$$

one can then check readily that the time derivative of $V$ along the trajectories of (4.81) satisfies

$$\begin{aligned} \dot{V}(q,p) &= \tfrac{\partial H}{\partial p}(q,p)\tau + \tfrac{\partial \Lambda}{\partial q}(q)\tfrac{\partial H}{\partial p}(q,p)^\top \\ &= p^\top M^{-1}(q)\tau + \tfrac{\partial \Lambda}{\partial q}(q)M^{-1}(q)p \\ &= p^\top M^{-1}(q)\tau_n\ . \end{aligned} \tag{4.89}$$

After the change of feedback (4.88), the system (4.81) can be rewritten as

$$\begin{cases} \dot{q} = \frac{\partial V}{\partial p}(q,p)^\top, \\ \dot{p} = -\frac{\partial V}{\partial q}(q,p)^\top + \tau_n. \end{cases} \tag{4.90}$$

Let

$$\begin{aligned} &f_n(x) = \begin{pmatrix} f_{1n}(x) \\ f_{2n}(x) \end{pmatrix}, \quad \text{where} \\ &f_{1n}(x) = \tfrac{\partial V}{\partial p}(q,p)^\top \ \text{ and } \ f_{2n}(x) = -\tfrac{\partial V}{\partial q}(q,p)^\top\ . \end{aligned} \tag{4.91}$$

We now show that Assumption 4.1 is satisfied by (4.90) with the choice

$$\psi(x) = q^\top p. \tag{4.92}$$

We have

$$L_{f_n} V(x) = 0\ , \tag{4.93}$$

$$L_g V(x) = p^\top M^{-1}(q)\ ,\ \text{and} \tag{4.94}$$

$$L_{f_n}\psi(x) \ = \ q^\top f_{2n}(x) + p^\top f_{1n}(x). \tag{4.95}$$

Therefore, if $L_{f_n} V(x) = 0$ and $L_g V(x) = 0$, then $p = 0$ and therefore $L_{f_n}\psi(x) = q^\top f_{2n}(x) = -|q|^2$, so Assumption 4.1 is satisfied.

Since Assumption 4.1 is satisfied, we can construct a CLF that satisfies the small control property for the system (4.90) and therefore also for the system (4.81). In the particular case we consider, it turns out that we can determine a function $\Gamma \in C^1 \cap \mathcal{K}_\infty$ such that

$$\mathcal{U}(x) = \psi(x) + \Gamma(V(x)) \tag{4.96}$$

is a CLF for the system (4.90) that satisfies the small control property. To stipulate $\Gamma$, we first let $m_{i,j}(q)$ denote the $(i,j)$ entry of $M^{-1}(q)$ for all $q \in \mathbb{R}^n$. The construction is as follows:

**Proposition 4.2.** *Fix any non-decreasing everywhere positive $C^1$ function $\Upsilon$ such that*

$$1 + ||M(q)||^4 \ \leq \ \Upsilon(V(x)) \tag{4.97}$$

*and*

$$\frac{n^2}{2}|q| \sup\left\{\left|\frac{\partial m_{i,j}}{\partial q_k}(q)\right| : (i,j,k) \in \{1,...,n\}^3\right\} \leq \sqrt{\Upsilon(V(x))} \tag{4.98}$$

*hold for all $x = (q,p) \in \mathbb{R}^{2n}$. Choose a function $\alpha \in \mathcal{K}_\infty \cap C^1$ with $\alpha'(0) > 0$ such that*

$$V(x) \ \geq \ \alpha(|p|^2 + |q|^2)$$

*everywhere.*[3] *Then with the choice*

$$\Gamma(\ell) = \frac{3}{2}\ell + 2\int_0^\ell \Upsilon(r)\mathrm{d}r + \frac{1}{2}\Upsilon(\ell)\alpha^{-1}(\ell)\ , \tag{4.99}$$

*the function (4.96) is a CLF for the system (4.90) that satisfies the small control property.*

[3] Such a function $\alpha$ exists because the positive definiteness of $M^{-1}$ provides a constant $c_0 > 0$ such that $V(x) \geq c_0|x|^2$ on $\mathcal{B}_{2n}$. To construct $\alpha$, first find a function $\underline{\alpha} \in \mathcal{K}_\infty \cap C^1$ such that $V(x) \geq \underline{\alpha}(|x|)$ for all $x \in \mathbb{R}^{2n}$. By reducing $c_0$ as needed without relabeling, we can assume that $c_0 r \leq \underline{\alpha}(\sqrt{r})$ on $[0.5, 1]$. Choose a non-decreasing $C^1$ function $p : \mathbb{R} \to [0,1]$ such that $p(r) \equiv 0$ on $[0, 0.5]$ and $p(r) \equiv 1$ on $[1,\infty)$. We can then take $\alpha(r) = [1 - p(r)]c_0 r + p(r)\underline{\alpha}(\sqrt{r})$. In fact, $\alpha'(0) = c_0$.

*Proof.* Choose

$$\tau_n \;=\; -M^{-1}(q)p. \tag{4.100}$$

Then, along the trajectories of (4.90), we get

$$\begin{aligned}\dot{V}(x) &= -|p^\top M^{-1}(q)|^2 \;\text{ and}\\ \dot{\psi}(x) &= L_{f_n}\psi(x) - L_g\psi(x)M^{-1}(q)p.\end{aligned} \tag{4.101}$$

Therefore,

$$\begin{aligned}\dot{\psi}(x) &= q^\top f_{2n}(x) + p^\top f_{1n}(x) - L_g\psi(x)M^{-1}(q)p\\ &= -q^\top \frac{\partial V}{\partial q}(q,p)^\top + p^\top \frac{\partial V}{\partial p}(q,p)^\top - q^\top M^{-1}(q)p\\ &= -|q|^2 - \frac{1}{2}q^\top\left(p^\top \tfrac{\partial(M^{-1}(q)p)}{\partial q}\right)^\top + p^\top M^{-1}(q)p\\ &\quad -q^\top M^{-1}(q)p.\end{aligned} \tag{4.102}$$

On the other hand,

$$-q^\top M^{-1}(q)p \le \frac{1}{2}|q|^2 + \frac{1}{2}|M^{-1}(q)p|^2, \tag{4.103}$$

and (4.98) gives

$$\left|\frac{1}{2}q^\top\left(p^\top \frac{\partial(M^{-1}(q)p)}{\partial q}\right)^\top\right| \le \sqrt{\Upsilon(V(x))}|p|^2. \tag{4.104}$$

Therefore,

$$\begin{aligned}\dot{\psi}(x) &\le -\tfrac{1}{2}|q|^2 + \sqrt{\Upsilon(V(x))}|p|^2\\ &\quad +p^\top M^{-1}(q)p + \tfrac{1}{2}|M^{-1}(q)p|^2\\ &\le -\tfrac{1}{2}|q|^2 + \sqrt{\Upsilon(V(x))}|p|^2 + ||M(q)||\,|M^{-1}(q)p|^2\\ &\quad +\tfrac{1}{2}|M^{-1}(q)p|^2\,.\end{aligned} \tag{4.105}$$

Hence, (4.97) gives

$$\begin{aligned}\dot{\psi}(x) &\le -\tfrac{1}{2}|q|^2 + \sqrt{\Upsilon(V(x))}\sqrt{\Upsilon(V(x))}|M^{-1}(q)p|^2\\ &\quad + \left[\Upsilon(V(x)) + \tfrac{1}{2}\right]|M^{-1}(q)p|^2\\ &\le -\tfrac{1}{2}|q|^2 + \left(2\Upsilon(V(x)) + \tfrac{1}{2}\right)|M^{-1}(q)p|^2\,.\end{aligned} \tag{4.106}$$

We deduce easily that the derivative of $\mathcal{U}$ defined in (4.96) along the trajectories of (4.90), in closed-loop with $\tau_n$ defined in (4.100), satisfies

$$\dot{\mathcal{U}}(x) \leq -\tfrac{1}{2}|q|^2 - |M^{-1}(q)p|^2, \tag{4.107}$$

using the fact that

$$\varGamma'(\ell) \geq \frac{3}{2} + 2\varUpsilon(l). \tag{4.108}$$

Next, observe that

$$\begin{aligned} \mathcal{U}(x) &\geq -|\psi(x)| + \varGamma\big(V(x)\big) \;\geq\; -\varUpsilon\big(V(x)\big)|q||p| + \varGamma\big(V(x)\big) \\ &\geq -\tfrac{1}{2}\varUpsilon\big(V(x)\big)\alpha^{-1}\big(V(x)\big) + \varGamma\big(V(x)\big)\ , \end{aligned} \tag{4.109}$$

by our choice of $\alpha$ and the relation $|q||p| \leq \frac{1}{2}|p|^2 + \frac{1}{2}|q|^2$. Using the fact that

$$\varGamma(v) \geq v + \frac{1}{2}\varUpsilon(v)\alpha^{-1}(v), \tag{4.110}$$

we get

$$\mathcal{U}(x) \geq V(x)\ , \tag{4.111}$$

so $\mathcal{U}$ is positive definite and radially unbounded. Moreover,

$$\dot{\mathcal{U}}(x) \leq -W(x) < 0 \;\; \forall x \neq 0\ , \tag{4.112}$$

where

$$W(x) = \frac{1}{2}|q|^2 + \left|M^{-1}(q)p\right|^2 . \tag{4.113}$$

We conclude that we have determined a CLF for the system (4.81) that satisfies the small control property. Moreover, both $\mathcal{U}(x)$ and $W(x)$ are lower bounded in a neighborhood of the origin by a positive definite quadratic function. □

*Remark 4.2.* Systems of the form (4.90) can be globally asymptotically stabilized by using backstepping to design the controls. Therefore, backstepping provides an alternative construction of CLFs satisfying the small control property. However, this technique provides control laws that remove the term $-\frac{\partial H}{\partial q}(q,p)^\top$, which may lead to more complicated control laws with large nonlinearities when the system can be stabilized through arbitrarily small control laws.

## 4.6 Robustness

We saw in Theorem 4.1 how to construct a CLF $\mathcal{U}$ for

$$\dot{x} = f(x) + g(x)u \tag{4.114}$$

that has the small control property, provided Assumption 4.1 is satisfied. In fact, for each $\varepsilon > 0$, we can choose a $C^1$ function $K_1$ satisfying $|K_1(x)| \le \varepsilon$ for all $x \in \mathbb{R}^n$ such that $\mathcal{U}$ is a strict Lyapunov function for

$$\dot{x} = f(x) + g(x)K_1(x),$$

which is therefore GAS to $x = 0$.

As we saw in previous chapters, ISS is a significant generalization of the GAS [157]. Recall that for a nonlinear system $\dot{x} = F(x, d)$ with state space $\mathcal{X} = \mathbb{R}^n$ and control set $U = \mathbb{R}^m$, the ISS property says that there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that for all measurable essentially bounded functions $\mathbf{d} : [0, \infty) \to \mathbb{R}^m$, the corresponding trajectories $x(t)$ for

$$\dot{x}(t) \;=\; F(x(t), \mathbf{d}(t)) \tag{4.115}$$

satisfy

$$|x(t)| \le \beta\big(|x(0)|, t\big) + \gamma(|\mathbf{d}|_\infty) \;\; \forall t \ge 0. \tag{ISS}$$

Here $\mathbf{d}$ represents a disturbance, and $|\cdot|_\infty$ is the essential supremum. The ISS property includes GAS to 0 for the system $\dot{x} = f(x)$, because in that case the term $\gamma(|\mathbf{d}|_\infty)$ in the ISS decay condition is not present. Therefore, given any constant $\varepsilon > 0$, it may seem reasonable to search for a feedback $K(x)$ for (4.114) (which could in principle differ from $K_1$) for which

$$\dot{x} = F(x, d) \doteq f(x) + g(x)[K(x) + d] \tag{4.116}$$

is ISS with respect to the disturbance $d$, and for which $|K(x)| \le \varepsilon$ for all $x \in \mathbb{R}^n$. Hence, we would want an arbitrarily small feedback $K$ that renders (4.114) GAS to $x = 0$ and that has the additional property that (4.116) is ISS with respect to the disturbance $d$.

This objective cannot be met in general, since there is no *bounded* feedback $K(x)$ such that the one-dimensional system $\dot{x} = K(x) + d$ is ISS. Therefore, instead of using ISS to analyze Jurdjevic-Quinn systems, we use iISS [160]. Recall from Chap. 1 that for a general nonlinear system $\dot{x} = F(x, d)$ evolving on $\mathbb{R}^n \times \mathbb{R}^m$, the iISS condition says: There exist $\beta \in \mathcal{KL}$ and $\alpha, \gamma \in \mathcal{K}_\infty$ such that for all measurable essentially bounded functions $\mathbf{d} : [0, \infty) \to \mathbb{R}^m$ and corresponding trajectories $x(t)$ for (4.115), we have

$$\alpha(|x(t)|) \;\le\; \beta\big(|x(0)|, t\big) + \int_0^t \gamma(|\mathbf{d}(s)|)\mathrm{d}s \quad \forall t \ge 0. \tag{iISS}$$

See Chap. 1 or [8, 9] for the background and further motivation for iISS. To get our iISS result, we add the following assumption to our system (4.114), which we assume in addition to Assumption 4.1:

**Assumption 4.2** *An everywhere positive non-decreasing smooth function* $\mathcal{D}$ *such that*

1. $\int_0^{+\infty} \frac{1}{\mathcal{D}(s)}\,\mathrm{d}s = +\infty$*; and*
2. $|L_g V(x)| \leq \mathcal{D}(V(x))$ *for all* $x \in \mathbb{R}^n$

*is known.*

Assumption 4.2 holds for the two-link manipulator example we introduced in Sect. 4.1, because in that case,

$$|L_g V(x)| \;\leq\; 2(V(x)+1)$$

for all $x \in \mathbb{R}^n$, so we can take

$$\mathcal{D}(s) \;=\; 2(s+1).$$

In fact, our assumptions hold for a broad class of Hamiltonian systems as well; see Remark 4.3. We claim that if Assumptions 4.1 and 4.2 both hold, then for any constant $\varepsilon > 0$ and any $C^\infty$ function $\xi : \mathbb{R}^n \to (0,\infty)$ such that

$$|\xi(x) L_g V(x)| \;\leq\; \varepsilon \quad \forall x \in \mathbb{R}^n\,, \tag{4.117}$$

the system

$$\dot{x} = f(x) + g(x)\big[K(x) + \mathbf{d}(t)\big]$$

is iISS with the choice $K(x) = -\xi(x) L_g V(x)^\top$.

To prove this claim, we begin by applying Theorem 4.1 to $\dot{x} = f(x) + g(x)u$, with $\xi : \mathbb{R}^n \to (0,\infty)$ satisfying (4.117) for an arbitrary prescribed constant $\varepsilon > 0$. This provides a CLF $\mathcal{U}$ satisfying the small control property for (4.114), which is also a strict Lyapunov function for (4.114) in closed-loop with

$$K(x) \;=\; -\xi(x) L_g V(x)^\top.$$

Setting

$$\mathcal{F}(x) = f(x) - g(x)\xi(x) L_g V(x)^\top, \tag{4.118}$$

it follows that $W(x) \doteq -L_{\mathcal{F}}\mathcal{U}(x)$ is positive definite.

We can determine a non-decreasing everywhere positive function $A \in C^1$ such that

$$|L_g \mathcal{U}(x)| \leq A\big(V(x)\big) \;\; \forall x \in \mathbb{R}^n\,. \tag{4.119}$$

Since $\mathcal{D}$ in Assumption 4.2 is a positive non-decreasing smooth function, we can easily construct a function $\Gamma_u \in \mathcal{K}_\infty \cap C^1$ such that $\Gamma_u'$ is everywhere positive and increasing and

$$A\big(V(x)\big) \leq \Gamma_u'\big(V(x)\big)\mathcal{D}\big(V(x)\big) \;\; \forall x \in \mathbb{R}^n. \tag{4.120}$$

Therefore, for all $x \in \mathbb{R}^n$,

$$|L_g\mathcal{U}(x)| \leq \Gamma_u'\big(V(x)\big)\mathcal{D}\big(V(x)\big) \ . \tag{4.121}$$

Next, consider

$$\mathcal{U}_a(x) = \mathcal{U}(x) + \Gamma_u\big(V(x)\big) \ . \tag{4.122}$$

Then

$$\big|L_g\mathcal{U}_a(x)\big| \ \leq \ \big|L_g\mathcal{U}(x)\big| + \Gamma_u'\big(V(x)\big)\big|L_gV(x)\big|.$$

Using Assumption 4.2 and (4.121), we obtain

$$\big|L_g\mathcal{U}_a(x)\big| \ \leq \ 2\Gamma_u'\big(V(x)\big)\mathcal{D}\big(V(x)\big) \ . \tag{4.123}$$

Let

$$\mathcal{U}_*(x) = \Gamma_u^{-1}\big(\mathcal{U}_a(x)\big) \ . \tag{4.124}$$

Since $\Gamma_u^{-1} \in C^1$ and $\Gamma_u^{-1}$ is increasing, we have

$$\begin{aligned} L_g\mathcal{U}_*(x) &= \{\Gamma_u^{-1}\}'\big(\mathcal{U}_a(x)\big)L_g\mathcal{U}_a(x) \\ &= \frac{1}{\Gamma_u'(\Gamma_u^{-1}(\mathcal{U}_a(x)))}L_g\mathcal{U}_a(x) \ . \end{aligned} \tag{4.125}$$

In combination with (4.123), we obtain

$$|L_g\mathcal{U}_*(x)| \ \leq \ 2\frac{\Gamma_u'(V(x))\mathcal{D}(V(x))}{\Gamma_u'(\Gamma_u^{-1}(\mathcal{U}_a(x)))} \ . \tag{4.126}$$

By the definition (4.122) of $\mathcal{U}_a$, we get

$$\Gamma_u^{-1}\big(\mathcal{U}_a(x)\big) \geq V(x) \ . \tag{4.127}$$

Since $\Gamma_u'$ is non-decreasing, we obtain

$$\Gamma_u'\big(\Gamma_u^{-1}\big(\mathcal{U}_a(x)\big)\big) \ \geq \ \Gamma_u'(V(x)) \ , \tag{4.128}$$

so (4.126) gives

$$|L_g\mathcal{U}_*(x)| \ \leq \ 2\mathcal{D}\big(V(x)\big) \ . \tag{4.129}$$

Since $\mathcal{D}$ is non-decreasing, (4.127) gives

$$\mathcal{D}\big(V(x)\big) \ \leq \ \mathcal{D}\Bigg(\Gamma_u^{-1}\big(\mathcal{U}_a(x)\big)\Bigg) = \mathcal{D}\big(\mathcal{U}_*(x)\big)$$

and therefore

$$|L_g\mathcal{U}_*(x)| \ \leq \ 2\mathcal{D}\big(\mathcal{U}_*(x)\big) \ . \tag{4.130}$$

Then

$$\tilde{U}(x) \ = \ \frac{1}{2}\int_0^{\mathcal{U}_*(x)} \frac{\mathrm{d}p}{\mathcal{D}(p)} \tag{4.131}$$

satisfies

$$\left|L_g\tilde{U}(x)\right| \le 1 . \tag{4.132}$$

The function $\tilde{U}$ is again a CLF for our dynamics (4.114) that satisfies the small control property. Moreover, (4.132) ensures that we can determine a positive definite function $\tilde{W}(x)$ such that the time derivative of $\tilde{U}$ along the trajectories of

$$\dot{x} = f(x) + g(x)[-\xi(x)L_gV(x)^\top + d] \tag{4.133}$$

satisfies

$$\dot{\tilde{U}}(x) \le -\tilde{W}(x) + |d| \tag{4.134}$$

for all $x$ and $d$. Inequality (4.134) says (see [8]) that the positive definite radially unbounded $C^1$ function $\tilde{U}$ is an iISS Lyapunov function for (4.133). The fact that (4.133) is iISS now follows from the standard iISS Lyapunov characterization; see Lemma 2.3 or [8, Theorem 1]. We conclude as follows:

**Corollary 4.1.** *Assume that the system (4.114) satisfies Assumptions 4.1 and 4.2 for some auxiliary scalar field* $\psi : \mathbb{R}^n \to \mathbb{R}$ *and some storage function* $V : \mathbb{R}^n \to \mathbb{R}$, *and let* $\varepsilon > 0$ *be given. Then there exists an everywhere positive function* $\xi$ *such that (a) the system*

$$\dot{x} \;=\; f(x) + g(x)[K(x) + \mathbf{d}] \tag{4.135}$$

*with the feedback*

$$K(x) \;\doteq\; -\xi(V(x))L_gV(x)^\top \tag{4.136}$$

*is iISS and (b)* $|K(x)| \le \varepsilon$ *for all* $x \in \mathbb{R}^n$. *Moreover, if* $\mathcal{U}$ *is a CLF satisfying the requirements of Theorem 4.1, and if* $\Gamma_u \in \mathcal{K}_\infty \cap C^1$ *is such that* $\Gamma_u'$ *is increasing and everywhere positive and satisfies* $|L_g\mathcal{U}(x)| \le \Gamma_u'(V(x))\mathcal{D}(V(x))$ *everywhere, then*

$$\tilde{U}(x) = \frac{1}{2}\int_0^{\Gamma_u^{-1}\left(\mathcal{U}(x)+\Gamma_u(V(x))\right)} \frac{\mathrm{d}p}{\mathcal{D}(p)} \tag{4.137}$$

*is an iISS Lyapunov function for (4.135).*

*Remark 4.3.* Assumptions 4.1 and 4.2 are satisfied by a broad class of important systems. For example, assume that the Hamiltonian system (4.90) satisfies the conditions from Sect. 4.5 and the following additional condition:

R. There exist $\underline{\lambda}, \bar{\lambda} > 0$ such that

$$\text{spectrum}\{M^{-1}(q)\} \subseteq [\underline{\lambda}, \bar{\lambda}]$$

for all $q$.

Condition R. means that there are positive constants $\underline{c}$ and $\bar{c}$ such that

$$\underline{c}|p|^2 \;\le\; p^\top M(q)p \;\le\; \bar{c}|p|^2$$

for all $q$ and $p$. This is more restrictive than merely saying that $M^{-1}$ is everywhere positive definite, since the smallest eigenvalue $\lambda_{\min}(q)$ of $M^{-1}(q)$ could in principle be such that

$$\liminf_{|q|\to+\infty} \lambda_{\min}(q) = 0.$$

Then (4.81) satisfies our Assumptions 4.1-4.2 and so is covered by the preceding corollary. In fact, we saw in Sect. 4.5 that Assumption 4.1 holds with $x = (q, p)$ and

$$V(x) = H(q,p) + \frac{1}{2}|q|^2 - P(q),$$

and then Assumption 4.2 follows from Condition R. because

$$\begin{aligned} |L_g V(x)|^2 \;&=\; \left|\tfrac{\partial H}{\partial p}(x)\right|^2 = |p^\top M^{-1}(q)|^2 \\ &\le \bar{\lambda}^2 |p|^2 \\ &\le \tfrac{\bar{\lambda}^2}{\underline{\lambda}} p^\top M^{-1}(q) p \;\le\; 2\tfrac{\bar{\lambda}^2}{\underline{\lambda}} V(x) \end{aligned}$$

for all $x = (q, p)$. Therefore, we can take

$$\mathcal{D}(s) \doteq \sqrt{2\frac{\bar{\lambda}^2}{\underline{\lambda}}(s+1)}$$

to satisfy Assumption 4.2.

## 4.7 Illustrations

We showed how to construct CLFs for systems

$$\dot{x} = f(x) + g(x)u \tag{4.138}$$

that have the form

$$\mathcal{U}(x) = \lambda\big(V(x)\big)\psi(x) + \Gamma\big(V(x)\big) \tag{4.139}$$

for suitable $C^1$ functions $\lambda$ and $\Gamma$, under the Jurdjevic-Quinn Conditions. In many cases, the construction is simplified because we can take either $\Gamma(v) \equiv v$ or $\lambda \equiv 1$. For example, the Hamiltonian systems in Sect. 4.5 lead to $\lambda \equiv 1$. We now further illustrate this point in two examples. In the first example, $\Gamma$ can be taken to be the identity, so we get a simple weighted sum of $V$ and $\psi(x) = L_G V(x)$. Then we revisit the two-link manipulator, which requires a more complicated $\Gamma$ but has $\lambda \equiv 1$.

### *4.7.1 Two-Dimensional Example*

We illustrate Theorem 4.1 using the two-dimensional system

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1^3 + u \; . \end{cases} \tag{4.140}$$

In this case, we have

$$f(x_1, x_2) = \begin{pmatrix} x_2 \\ -x_1^3 \end{pmatrix} \text{ and } g(x_1, x_2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \; . \tag{4.141}$$

Let us check that (4.140) satisfies Assumption 4.1.

1. The positive definite radially unbounded function

$$V(x_1, x_2) \;=\; \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2 \tag{4.142}$$

   is not a CLF for (4.140), but it satisfies $L_f V(x) = 0$ on $\mathbb{R}^2$.
2. Choosing the vector field

$$G(x_1, x_2) = \begin{pmatrix} 0 \\ x_1 \end{pmatrix} \tag{4.143}$$

   gives

$$\begin{aligned} L_g V(x_1, x_2) &= x_2 \; , \; L_G V(x_1, x_2) = x_1 x_2, \;\text{ and} \\ L_f L_G V(x_1, x_2) &= x_2^2 - x_1^4 \; . \end{aligned} \tag{4.144}$$

   If $L_g V(x_1, x_2) = 0$ and $(x_1, x_2) \neq (0, 0)$, then $x_2 = 0$ and $x_1 \neq 0$, so $L_f L_G V(x_1, 0) = -x_1^4 < 0$.

Therefore Assumption 4.1 is satisfied with (4.142) and $\psi(x) = L_G V(x)$, so Theorem 4.1 applies to the system (4.140). Let us show that with the choice

$$\delta(v) \;=\; \frac{v^2}{8(1+v)^2} \; , \tag{4.145}$$

the function

$$\begin{aligned} U(x) &= V(x) + \delta\big(V(x)\big) L_G V(x) \\ &= \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2 + \delta\left(\frac{1}{4}x_1^4 + \frac{1}{2}x_2^2\right) x_1 x_2 \end{aligned} \tag{4.146}$$

is a CLF for the system (4.140) whose time derivative along the trajectories of (4.140) in closed-loop with

$$u \;=\; -L_gV(x)^\top \;=\; -x_2 \tag{4.147}$$

is negative definite.

To this end, we first observe that

$$\frac{1}{2}x_1^2 \le 1 + \frac{1}{4}x_1^4, \;\text{ so }\; |x_1x_2| \le 1 + V(x) \;\; \forall x \in \mathbb{R}^2 \,. \tag{4.148}$$

Therefore

$$\begin{aligned} U(x) &\ge \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2 \\ &\quad - \frac{\left(\frac{1}{4}x_1^4 + \frac{1}{2}x_2^2\right)^2}{8\left(1 + \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2\right)^2}\left(1 + \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2\right) \\ &= \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2 - \frac{\left(\frac{1}{4}x_1^4 + \frac{1}{2}x_2^2\right)^2}{8\left(1 + \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2\right)} \\ &\ge \frac{1}{8}x_1^4 + \frac{1}{4}x_2^2 \,. \end{aligned}$$

The time derivative of $U(x)$ along the trajectories of (4.140) in closed-loop with the feedback (4.147) is

$$\begin{aligned} \dot{U} &= -x_2^2\left[1 + \frac{V(x)}{4(1+V(x))^3}x_1x_2\right] \\ &\quad + \delta\big(V(x)\big)[-x_1^4 - x_1x_2 + x_2^2] \\ &\le -\frac{5}{8}x_2^2 - \delta\big(V(x)\big)x_1^4 - \delta\big(V(x)\big)x_1x_2 \\ &\le -\frac{3}{8}x_2^2 - \delta\big(V(x)\big)x_1^4 + \delta^2(V(x))x_1^2 \\ &\le -\frac{3}{8}x_2^2 - \delta\big(V(x)\big)x_1^4 + \delta\big(V(x)\big)\frac{\left(\frac{1}{4}x_1^4 + \frac{1}{2}x_2^2\right)x_1^2}{8\left(1 + \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2\right)} \\ &\le -\frac{1}{4}x_2^2 - \frac{1}{4}\delta\big(V(x)\big)x_1^4, \end{aligned} \tag{4.149}$$

where we used (4.148) to get the first and last inequalities, and the second inequality used $(\delta(V(x))x_1 + \frac{1}{2}x_2)^2 \ge 0$. Since the right hand side of this inequality is negative definite, the result follows.

### *4.7.2 Two-Link Manipulator Revisited*

We show how the CLF (4.6) for the two-link manipulator dynamics follows as a special case of the construction from Theorem 4.1. Recall that the dynamics is the control affine system $\dot{x} = f(x) + g(x)u$ where

$$f(x) = \begin{bmatrix} x_2 \\ \frac{-2x_3x_2x_4 - x_1\langle x_1\rangle}{x_3^2+1} \\ x_4 \\ x_2^2x_3 - x_3\langle x_3\rangle \end{bmatrix}, \quad g(x) = \begin{bmatrix} 0 & 0 \\ \frac{1}{x_3^2+1} & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \text{ and } u = \begin{bmatrix} \tau_b \\ F_b \end{bmatrix}. \quad (4.150)$$

We show that Assumption 4.1 is satisfied for the system with the choices

$$V(x) = \frac{1}{2}\left[(x_3^2+1)x_2^2 + x_4^2 + \sqrt{1+x_1^2} + \sqrt{1+x_3^2} - 2\right] \quad (4.151)$$

and $\psi = L_G V$, where

$$G(x) = (0, x_1, 0, x_3)^\top. \quad (4.152)$$

Setting

$$\langle p\rangle \; = \; \frac{1}{2\sqrt{1+p^2}}$$

for all $p \in \mathbb{R}$, simple calculations show that

$$\nabla V(x) = \Big(x_1\langle x_1\rangle, x_2\left[x_3^2+1\right], x_3\langle x_3\rangle + x_2^2x_3, x_4\Big).$$

Hence, along the trajectories of the system, we have

$$\dot{V}(x) = x_2\tau_b + x_4F_b,$$

and

$$L_G V(x) = \frac{\partial V}{\partial x_2}(x)x_1 + \frac{\partial V}{\partial x_4}(x)x_3 = (x_3^2+1)x_2x_1 + x_4x_3. \quad (4.153)$$

Since

$$\nabla(L_G V(x)) \; = \; \big(x_2(x_3^2+1), x_1(x_3^2+1), x_4 + 2x_1x_2x_3, x_3\big),$$

we have

$$L_f L_G V(x) = x_2^2(2x_3^2+1) + x_4^2 - x_1^2\langle x_1\rangle - x_3^2\langle x_3\rangle. \quad (4.154)$$

Notice that

$$L_f V(x) = 0 \;\text{ and }\; L_g V(x) = [x_2 \;\; x_4]$$

everywhere. Also, if $L_g V(x) = 0$, then $x_2 = x_4 = 0$, in which case we get

$$L_f L_G V(x) = -x_1^2\langle x_1\rangle - x_3^2\langle x_3\rangle.$$

It follows that if $x \neq 0$ and $L_gV(x) = 0$, then $L_fL_GV(x) < 0$. Therefore Assumption 4.1 is satisfied with

$$\psi(x) = L_GV(x), \tag{4.155}$$

so Theorem 4.1 applies.

We now derive the CLF whose existence is guaranteed by the theorem. To this end, first note that

$$a \;\le\; 3\Big\{\big(\sqrt{1+a}-1\big) + \big(\sqrt{1+a}-1\big)^2\Big\} \quad \forall a \ge 0. \tag{4.156}$$

It follows from the formula for $V$ that

$$\begin{aligned} \max\{x_1^2, x_3^2\} \;&\le\; 3\{2V(x) + 4V^2(x)\} \;\text{ and} \\ \max\{x_2^2, x_4^2\} \;&\le\; 2V(x) \quad \forall x \in \mathbb{R}^4. \end{aligned} \tag{4.157}$$

Combining the triangle inequality, (4.153), and (4.157) gives

$$\begin{aligned} |L_GV(x)| &\le \tfrac{1}{2}x_3^4 + \tfrac{1}{4}x_1^4 + \tfrac{1}{4}x_2^4 + \tfrac{1}{2}|x|^2 \\ &\le 288V^4(x) + 85V^2(x) + 8V(x). \end{aligned} \tag{4.158}$$

We readily conclude that the function

$$V^\sharp(x) \;=\; 40\big[2 + 2V(x)\big]^6 + L_GV(x) - 40(2^6) \tag{4.159}$$

is such that

$$V^\sharp(x) \;\ge\; 3\big(x_1^2 + x_2^2 + x_3^2 + x_4^2\big)$$

for all $x \in \mathbb{R}^4$, so $V^\sharp$ is positive definite and radially unbounded.

Moreover, its time derivative along trajectories of the system satisfies

$$\begin{aligned} \dot{V}^\sharp(x) = 480\big[2 + 2V(x)\big]^5\big(x_2\tau_b + x_4F_b\big) + x_2^2\big(2x_3^2 + 1\big) \\ +x_4^2 - x_1^2\langle x_1\rangle - x_3^2\langle x_3\rangle + x_1\tau_b + x_3F_b\ , \end{aligned} \tag{4.160}$$

since $\dot{V}(x) = x_2\tau_b + x_4F_b$. Hence, the triangle inequality gives

$$\begin{aligned} \dot{V}^\sharp(x) \le \sqrt{1+x_1^2}\tau_b^2 + 480\big[2 + 2V(x)\big]^5 x_2\tau_b + x_2^2\big(2x_3^2 + 1\big) \\ +\sqrt{1+x_3^2}F_b^2 + 480\big[2 + 2V(x)\big]^5 x_4F_b \\ +x_4^2 - \tfrac{1}{2}x_1^2\langle x_1\rangle - \tfrac{1}{2}x_3^2\langle x_3\rangle\ . \end{aligned} \tag{4.161}$$

To show that $V^\sharp$ is a CLF for the system, we show that the right side of (4.161) is negative definite when we take

$$\tau_b = -x_2\langle x_2\rangle \;\text{ and }\; F_b = -x_4\langle x_4\rangle. \tag{4.162}$$

This will also show that $V^\sharp$ has the small control property.

To this end, we first note that with the choices (4.162), we have

$$\begin{aligned} \dot{V}^\sharp(x) \;\leq\; & T_1(x)x_2^2\langle x_2\rangle + T_2(x)x_4^2\langle x_4\rangle \\ & -\tfrac{1}{2}\left[x_1^2\langle x_1\rangle + x_2^2\langle x_2\rangle + x_3^2\langle x_3\rangle + x_4^2\langle x_4\rangle\right], \end{aligned} \tag{4.163}$$

where we define the $T_i$'s by

$$T_1(x) = \sqrt{1+x_1^2} - 480(2+2V(x))^5 + 2\sqrt{1+x_2^2}(2x_3^2+1) + \frac{1}{2}$$

and

$$T_2(x) = \sqrt{1+x_3^2} - 480(2+2V(x))^5 + 2\sqrt{1+x_4^2} + \frac{1}{2}.$$

We deduce from (4.157) that $T_1$ and $T_2$ are non-positive and therefore

$$\dot{V}^\sharp(x) \leq -\frac{1}{2}\left[x_1^2\langle x_1\rangle + x_2^2\langle x_2\rangle + x_3^2\langle x_3\rangle + x_4^2\langle x_4\rangle\right]. \tag{4.164}$$

The right hand side of this inequality is negative definite and the feedbacks resulting from (4.4) and (4.162) give the small control property.

In fact, our analysis from Sect. 4.6 shows that for any positive constant $c > 0$, the scaled feedback

$$K(x) = -c\begin{pmatrix} x_1\langle x_1\rangle + x_2\langle x_2\rangle \\ x_3\langle x_3\rangle + x_4\langle x_4\rangle \end{pmatrix} \tag{4.165}$$

renders the system iISS to actuator errors, meaning

$$\dot{x} = f(x) + g(x)[K(x) + \mathbf{d}(t)]$$

is iISS. We illustrate this point in the simulation below, where we took the feedback

$$K^\flat(x) \;=\; -0.005\begin{pmatrix} x_1\langle x_1\rangle + x_2\langle x_2\rangle \\ x_3\langle x_3\rangle + x_4\langle x_4\rangle \end{pmatrix}, \tag{4.166}$$

the disturbance

$$\mathbf{d}(t) = \begin{pmatrix} \dfrac{1}{1+0.25t^2} \\ e^{-0.25t} \end{pmatrix} \tag{4.167}$$

and the initial state $x(0) = (1,1,1,1)$. While the feedback (4.166) renders the closed-loop system GAS to 0 when the disturbance is set to 0, the state components may or may not be driven to zero when there are disturbances present. In our simulation, the angle of the link $x_1$ converges to zero by time $t = 1000$. However, the gripper position $x_3$ has an overshoot caused by the disturbance that keeps this component from converging to zero. See Figs. 4.2 and 4.3.

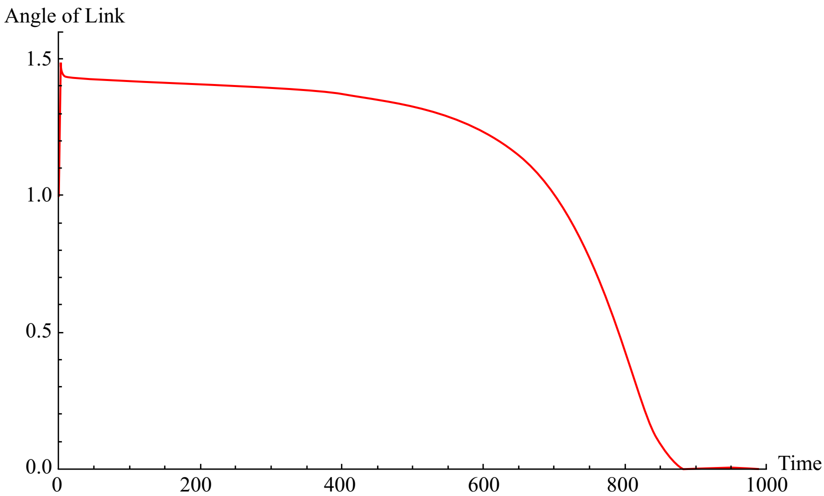


**Fig. 4.2** Angle of link $x_1$ using feedback (4.166) and disturbance (4.167)

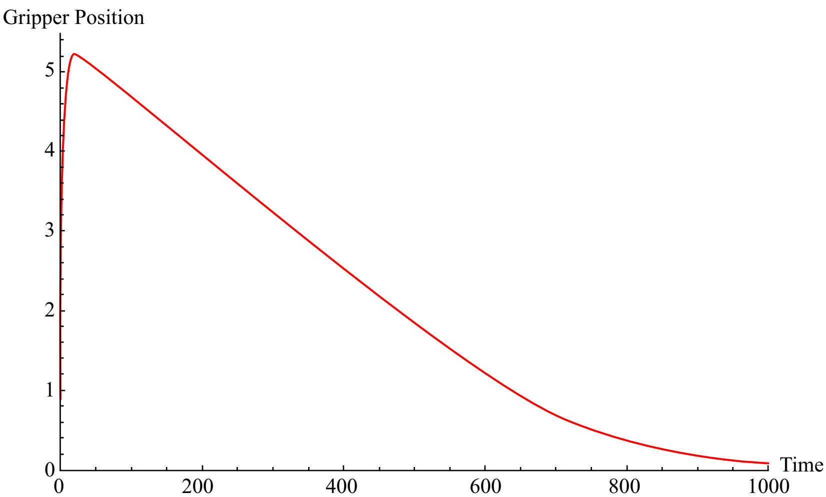


**Fig. 4.3** Gripper position $x_3$ using feedback (4.166) and disturbance (4.167)

*Remark 4.4.* An important feature of the preceding analysis is that the strict Lyapunov function $V^\sharp$ has a negative definite time derivative along the closed-loop trajectories, using a bounded feedback. In fact, for each constant $\varepsilon > 0$, our constructions from the preceding sections provide a strict Lyapunov function whose time derivative is negative definite using a feedback stabilizer $K^\flat : \mathbb{R}^n \to \varepsilon\mathcal{B}_2$ that is bounded by $\varepsilon$. This is done by choosing the function $\Gamma$ in our strict Lyapunov function construction appropriately. Moreover, we see from (4.164) that $-\dot{V}^\sharp$ is *proper* along the closed trajectories, and the dynamics are control affine, so we can immediately use control redesign to get ISS to actuator errors, if we allow unbounded feedbacks. For example, the combined feedback

$$K^\sharp(x) = -(x_1\langle x_1\rangle + x_2\langle x_2\rangle, x_3\langle x_3\rangle + x_4\langle x_4\rangle)^\top - L_g V^\sharp(x)$$

renders the system ISS with respect to actuator errors, so we recover the ISS results for the two-link manipulator from [5].

The properness of $\dot{V}^\sharp$ is essential for the preceding control redesign argument. In general, if the time derivative of a strict Lyapunov function $\mathcal{V}$ is merely negative definite along the closed-loop trajectories of a given control affine system, then adding $-L_g\mathcal{V}$ to the feedback will not necessarily give ISS. On the other hand, we can always transform $\mathcal{V}$ into a new strict Lyapunov function $\mathcal{V}_a$ for which $-\dot{\mathcal{V}}_a$ is proper along the closed-loop trajectories (e.g., by arguing as in [157, p.440]), and then we can generate ISS with respect to actuator errors by subtracting $L_g\mathcal{V}_a$ as above.

## 4.8 Comments

The Jurdjevic-Quinn Method can be summarized by saying that appropriate controllability conditions and a first integral of the drift vector can be used to design smooth asymptotically stabilizing control laws. Since Jurdjevic and Quinn's original paper [68], the method has been extended in several directions [11, 41, 45, 126]. The first general result on global explicit strict Lyapunov function constructions under the Weak Jurdjevic-Quinn Conditions appears to be [40], whose results are limited to homogenous systems. Our construction of the auxiliary scalar field in Sect. 4.4 is similar to, but somewhat simpler than, the one in [41, Theorem 4.3]. This is because [41] uses a more complicated construction that guarantees that $G$ is homogenous of degree zero, assuming the original dynamics and given non-strict Lyapunov function are both homogeneous.

The model (4.1) and accompanying figure are from [165]. There it is shown that if one takes closed-loop controllers of the form

$$\tau = -k_1\dot{\theta} - k_2(\theta - q_d) \text{ and } F = -k_3\dot{r} - k_4(r - r_d), \tag{4.168}$$

then (4.1) in closed-loop with (4.168) is not ISS with respect to $(q_d, r_d)$. In particular, bounded signals can destabilize the system, which is called a nonlinear resonance effect. Our treatment of the two-link manipulator is based on [102], which provides an alternative CLF construction under the

Jurdjevic-Quinn conditions that differs from the one we presented in this chapter.