Chapter 3
Matrosov Conditions: Simple Case

Abstract In the preceding chapter, we saw two ways to use non-strict Lyapunov functions to prove asymptotic stability. The first was the LaSalle Invariance Principle. A second involved Matrosov Theorems, which require a non-strict Lyapunov function and auxiliary functions that satisfy appropriate decay conditions. In general, the decay conditions in Matrosov type theorems are less restrictive than those in the strict Lyapunov function definition. Hence, the Matrosov method can be regarded as a way to prove stability without having to find strict Lyapunov functions.

On the other hand, it is very desirable to have explicit strict Lyapunov functions, even when the Matrosov Conditions are satisfied, because, e.g., strict Lyapunov functions make it possible to quantify the effects of uncertainty using the ISS paradigm. In this chapter, we discuss several methods for constructing strict Lyapunov functions for time-invariant systems that satisfy appropriate Matrosov Conditions. In Chapters 8 and 12, we generalize to much more complex time-varying systems, including Matrosov type theorems for hybrid systems.

3.1 Motivation

To motivate our constructions, let us return to the experimental anaerobic digester model

$$\begin{cases} \dot{s} = u(s_{in} - s) - kr(s,x) \\ \dot{x} = r(s,x) - \alpha ux \\ y = (\lambda r(s,x), s) \end{cases} \tag{3.1}$$

we considered in the preceding chapter, where the biomass growth rate r is any non-negative C^1 function that admits everywhere positive functions $\underline{\Delta}$ and $\bar{\Delta}$ such that

$$s\bar{\Delta}(s,x) \;\geq\; r(s,x) \;\geq\; xs\underline{\Delta}(s,x) \tag{3.2}$$

for all $s \geq 0$ and $x \geq 0$; u is the non-negative input (i.e., dilution rate); and the positive constants α, λ, k, and s_{in} are as defined in Sect. 2.4.6. This includes the one species chemostat model with a Monod growth rate, as a special case [107]. This time our objective is to construct a strict Lyapunov-like function for an appropriate adaptively controlled error dynamics for (3.1).

Arguing as in the previous chapter, we introduce the dynamics $\dot{\gamma} = y_1(\gamma - \gamma_m)(\gamma_M - \gamma)\nu$ evolving on (γ_m, γ_M), where ν is a function to be selected that is independent of x and the γ_i's are prescribed positive constants. With $u = \gamma y_1$, the system (3.1) with its dynamic extension becomes

$$\begin{cases} \dot{\tilde{s}} = -\gamma\tilde{s} + \tilde{\gamma}v_* \\ \dot{\tilde{x}} = \alpha\left[-\gamma\tilde{x} - \tilde{\gamma}x_*\right] \\ \dot{\tilde{\gamma}} = (\gamma - \gamma_m)(\gamma_M - \gamma)\nu. \end{cases} \tag{3.3}$$

Here $\tilde{s} = s - s_*$, $\tilde{x} = x - x_*$, $\tilde{\gamma} = \gamma - \gamma_*$, $s_* \in (0, s_{in})$ is the desired equilibrium substrate level, and

$$\gamma_* \doteq \frac{k}{\lambda(s_{in} - s_*)} \in (\gamma_m, \gamma_M) \quad \text{and} \quad \frac{k}{\lambda s_{in}} < \gamma_m, \tag{3.4}$$

where $v_* = s_{in} - s_*$ and $x_* = \frac{v_*}{k\alpha}$. The dynamics (3.3) follow by applying the Erdmann transformation

$$\tau = \int_{t_0}^{t} y_1(l)\mathrm{d}l,$$

and the state space for (3.3) is $D = (-s_*, \infty) \times (-x_*, \infty) \times (\gamma_m - \gamma_*, \gamma_M - \gamma_*)$. We first consider the subsystem

$$\begin{cases} \dot{\tilde{s}} = -\gamma\tilde{s} + \tilde{\gamma}v_* \\ \dot{\tilde{\gamma}} = (\gamma - \gamma_m)(\gamma_M - \gamma)\nu \end{cases} \tag{3.5}$$

with state space $\mathcal{X} = (-s_*, \infty) \times (\gamma_m - \gamma_*, \gamma_M - \gamma_*)$.

Let us transform the non-strict Lyapunov-like function

$$V_1(\tilde{s}, \tilde{\gamma}) = \frac{1}{2\gamma_m}\tilde{s}^2 + \frac{v_*}{K\gamma_m}\int_0^{\tilde{\gamma}} \frac{l}{(l + \gamma_* - \gamma_m)(\gamma_M - \gamma_* - l)}\,\mathrm{d}l \tag{3.6}$$

from [89] into a *strict* Lyapunov-like function for (3.5), where $K > 1$ is a tuning parameter. Later in this chapter, we will see how this transformation process is a special case of a general method for constructing strict Lyapunov-like functions.

Choosing

$$\nu(\tilde{s}) = -K\tilde{s} \tag{3.7}$$

as before gives

$$\dot{V}_1 = -\frac{\gamma}{\gamma_m}\tilde{s}^2 \;\leq\; -\mathcal{N}_1(\tilde{s}, \tilde{\gamma}), \quad \text{where } \mathcal{N}_1(\tilde{s}, \tilde{\gamma}) = \tilde{s}^2. \tag{3.8}$$

Set

$$V_2(\tilde{s}, \tilde{\gamma}) \;=\; -\tilde{s}\tilde{\gamma}. \tag{3.9}$$

Along the trajectories of (3.5), in closed-loop with (3.7), simple calculations yield

$$\dot{V}_2 = \gamma\tilde{s}\tilde{\gamma} - \tilde{\gamma}^2 v_* + (\gamma - \gamma_m)(\gamma_M - \gamma)K\tilde{s}^2.$$

From the relation

$$\gamma\tilde{s}\tilde{\gamma} \le \frac{v_\star\tilde{\gamma}^2}{2} + \frac{\gamma^2\tilde{s}^2}{2v_\star}$$

and the fact that the maximum value of $(\gamma - \gamma_m)(\gamma_M - \gamma)$ over $\gamma \in [\gamma_m, \gamma_M]$ is $(\gamma_M - \gamma_m)^2/4$, we get

$$\dot{V}_2 \;\le\; -\mathcal{N}_2(\tilde{s}, \tilde{\gamma}) + \left[\frac{\gamma_M^2}{2v_\star} + \frac{K(\gamma_M - \gamma_m)^2}{4}\right]\mathcal{N}_1(\tilde{s}, \tilde{\gamma}), \tag{3.10}$$

where $\mathcal{N}_2(\tilde{s}, \tilde{\gamma}) = \frac{v_*}{2}\tilde{\gamma}^2$.

Setting

$$\Upsilon_1 \;=\; 1 + \left[\min\left\{\frac{1}{\gamma_m}, \frac{4v_*}{K\gamma_m(\gamma_M - \gamma_m)^2}\right\}\right]^{-1}, \tag{3.11}$$

we can use the decay conditions (3.8)-(3.10) to check that

$$S(\tilde{s}, \tilde{\gamma}) \;=\; V_2(\tilde{s}, \tilde{\gamma}) + \left[\Upsilon_1 + \frac{2\gamma_M^2}{v_*} + K(\gamma_M - \gamma_m)^2\right] V_1(\tilde{s}, \tilde{\gamma}) \tag{3.12}$$

is a strict Lyapunov-like function for (3.5) in closed-loop with (3.7). In fact,

$$\begin{aligned} &\dot{S} \;\le\; -W(\tilde{s}, \tilde{\gamma}), \quad \text{where} \\ &W(\tilde{s}, \tilde{\gamma}) = \mathcal{N}_2(\tilde{s}, \tilde{\gamma}) + \Upsilon_1\mathcal{N}_1(\tilde{s}, \tilde{\gamma}) = \tfrac{v_\star}{2}\tilde{\gamma}^2 + \Upsilon_1\tilde{s}^2 \end{aligned} \tag{3.13}$$

along the closed-loop trajectories of (3.5), and S is also positive definite; see Sect. 3.6.1 for details and our reasoning behind the choice (3.12) of S.

The fact that the *full system* (3.3) in closed-loop with (3.7) is GAS to the origin now follows because (a) its $\tilde{x}$ sub-dynamics is ISS with respect to $(\tilde{s}, \tilde{\gamma})$ and (b) the asymptotically stable $(\tilde{s}, \tilde{\gamma})$ sub-dynamics does not depend on $\tilde{x}$. Let us now construct a strict Lyapunov-like function for the full closed-loop system. We claim that

$$M(\tilde{x}, \tilde{s}, \tilde{\gamma}) = \bar{\omega}S(\tilde{s}, \tilde{\gamma}) + \tilde{x}^2, \;\text{ where }\; \bar{\omega} = \frac{4\alpha x_\star^2}{\gamma_m v_\star} \tag{3.14}$$

is a strict Lyapunov-like function for the system (3.3), in closed-loop with (3.7), for which

$$\dot{M} \;\le\; -\alpha\gamma_m\tilde{x}^2 - \frac{\alpha x_\star^2}{\gamma_m}\tilde{\gamma}^2 - \bar{\omega}\Upsilon_1\tilde{s}^2 \tag{3.15}$$

along the trajectories of (3.3). To see why, first notice that the relation

$$2\alpha x_* |\tilde{x}\tilde{\gamma}| \ \leq \ \alpha\gamma_m \tilde{x}^2 + \alpha x_\star^2 \frac{\tilde{\gamma}^2}{\gamma_m}$$

implies that

$$\begin{aligned} \tfrac{d}{dt}\tilde{x}^2 &= -2\alpha\gamma\tilde{x}^2 - 2\tilde{x}\tilde{\gamma}\alpha x_* \ \leq \ -2\alpha\gamma_m\tilde{x}^2 + 2\alpha x_*|\tilde{x}\tilde{\gamma}| \\ &\leq -\alpha\gamma_m\tilde{x}^2 + \frac{\alpha x_\star^2}{\gamma_m}\tilde{\gamma}^2 \end{aligned} \tag{3.16}$$

along the trajectories of (3.3). Then (3.15) follows by adding the inequality

$$\bar{\omega}\dot{S} \ \leq \ -\bar{\omega}\frac{v_\star}{2}\tilde{\gamma}^2 - \bar{\omega}\Upsilon_1\tilde{s}^2$$

to (3.16). We turn next to a general theory that leads to the preceding analysis as a special case.

3.2 Continuous Time Theorem

For simplicity, we first state our main result for time-invariant systems

$$\dot{x} = f(x) \tag{3.17}$$

evolving on an open set $\mathcal{X} \subseteq \mathbb{R}^n$. Later we generalize to time-varying systems. In the rest of this section, we assume that the relevant functions are sufficiently smooth. We also assume:

Assumption 3.1 *There exist an integer $j \geq 2$; known functions*

$$\begin{aligned} &V_i : \mathcal{X} \to \mathbb{R}, \\ &\mathcal{N}_i : \mathcal{X} \to [0,\infty), \ \text{ and} \\ &\phi_i : [0,\infty) \to (0,\infty); \end{aligned}$$

and real numbers $a_i \in (0,1]$ such that $V_i(0) = 0$ and $\mathcal{N}_i(0) = 0$ for all i;

$$\nabla V_1(x) f(x) \ \leq \ -\mathcal{N}_1(x) \quad \forall x \in \mathcal{X} \ ; \text{ and} \tag{3.18}$$

$$\nabla V_i(x) f(x) \ \leq \ -\mathcal{N}_i(x) + \phi_i\big(V_1(x)\big) \sum_{l=1}^{i-1} \mathcal{N}_l^{a_i}(x) V_1^{1-a_i}(x) \tag{3.19}$$

for $i = 2, \ldots, j$ and all $x \in \mathcal{X}$. The function V_1 is also assumed to be positive definite on $\mathcal{X}$.

Assumption 3.2 *The following conditions hold:*

1. *there exists a function* $\rho : [0, \infty) \to (0, \infty)$ *such that*

$$\sum_{l=1}^{j} \mathcal{N}_l(x) \;\geq\; \rho\big(V_1(x)\big)V_1(x) \quad \forall x \in \mathcal{X}; \quad \textit{and} \tag{3.20}$$

2. *there exist functions* $p_2, \ldots, p_j : [0, \infty) \to [0, \infty)$ *and a positive definite function* $\bar{p}$ *such that for each* $i \in \{2, \ldots, j\}$*, the following hold: (a) If* V_i *is positive definite, then*

$$p_i(r) = 0 \;\; \textit{and} \;\; |V_i(x)| \leq \bar{p}\big(V_1(x)\big) \tag{3.21}$$

   for all $r \geq 0$ *and* $x \in \mathcal{X}$*. (b) If* V_i *is not positive definite, then*

$$|V_i(x)| \;\leq\; p_i\big(V_1(x)\big)V_1(x) \tag{3.22}$$

   holds for all $x \in \mathcal{X}$*.*

Assumptions 3.1 and 3.2 agree with the ones in [106], except that [106] requires the functions p_i to satisfy (3.22) for all i and all $x \in \mathcal{X}$ (instead of Condition 2. from Assumption 3.2). We refer to Assumptions 3.1 and 3.2 as our Matrosov Conditions, owing to their use of multiple functions V_i. However, there are several different sets of conditions that are referred to as Matrosov Conditions in the control literature. We prove:

Theorem 3.1. *Let Assumptions 3.1 and 3.2 be satisfied. Then one can explicitly determine* C^1 *functions* $k_l, \Omega_l \in \mathcal{K}_\infty$ *such that the function*

$$S(x) \;=\; \sum_{l=1}^{j} \Omega_l\Big(k_l\big(V_1(x)\big) + V_l(x)\Big) \tag{3.23}$$

satisfies

$$S(x) \;\geq\; V_1(x) \tag{3.24}$$

and

$$\nabla S(x) f(x) \;\leq\; -\frac{1}{4}\rho\big(V_1(x)\big)V_1(x) \tag{3.25}$$

for all $x \in \mathcal{X}$*.*

Remarks on Assumptions

Remark 3.1. If $\mathcal{X} = \mathbb{R}^n$ and V_1 is radially unbounded, then (3.24) implies that S is a strict Lyapunov function for (3.17). If V_1 is not radially unbounded, then S is not necessarily radially unbounded and therefore one cannot conclude from standard Lyapunov theory that the origin is GAS. However, in many cases, GAS can be proved through a Lyapunov-like function and extra arguments, e.g., by proving that any trajectory belongs to a compact set included in $\mathcal{X}$. This is often true in biological models that are based on mass conservation properties, such as the one we discussed in Sect. 3.1.

Remark 3.2. If V_1 is also lower bounded by a positive definite quadratic form in a neighborhood of 0, then (3.25) implies that the time derivative of S along the trajectories of (3.17) is upper bounded in a neighborhood of 0 by a negative definite quadratic function. Also, (3.24) gives a positive definite quadratic lower bound on S near the origin.

3.3 Proof of Continuous Time Theorem

Throughout the sequel, all inequalities should be understood to hold globally unless otherwise indicated, and we omit the arguments of our functions when they are clear from the context.

Construction of the k_i's and Ω_i's

Fix $j \geq 2$ and functions satisfying Assumptions 3.1 and 3.2. Fix $k_2, \ldots, k_j \in C^1 \cap \mathcal{K}_\infty$ such that

$$k_i(s) \;\geq\; s + p_i(s)s \quad \text{and} \quad k_i'(s) \;\geq\; 1 \tag{3.26}$$

for all $s \geq 0$ for $i = 2, 3, \ldots, j$. The following simple lemma is key:

Lemma 3.1. *The functions* $\{U_i\}$ *defined by*

$$U_1(x) = V_1(x) \quad \textit{and} \quad U_i(x) = k_i\big(V_1(x)\big) + V_i(x) \;\text{ for }\; i \geq 2$$

satisfy $2k_i\big(V_1(x)\big) + \bar{p}(V_1(x)) \geq U_i(x) \geq V_1(x)$ *for* $i = 2, \ldots, j$ *and all* $x \in \mathcal{X}$.

Proof. Assumption 3.2 and our choices of the k_i's give

$$\begin{aligned} &U_i(x) \geq V_1(x) + p_i\big(V_1(x)\big)V_1(x) - p_i\big(V_1(x)\big)V_1(x) = V_1(x) \;\text{ and} \\ &U_i(x) \leq k_i\big(V_1(x)\big) + p_i\big(V_1(x)\big)V_1(x) \leq 2k_i\big(V_1(x)\big) \end{aligned}$$

for all indices $i \geq 2$ for which V_i is not positive definite. For the other indices, the desired inequalities follow from (3.21) and the non-negativity of the corresponding functions V_i. This proves the lemma. □

Returning to the proof of the theorem, set $k_1(s) \equiv s$, and define the functions U_i according to Lemma 3.1. We can recursively define continuous non-decreasing functions $\mu_i : [0, \infty) \to [1, \infty)$ such that

$$\mu_i(V_1) \geq 2\Phi(V_1) \sum_{l=1+i}^{j} \mu_l^{\frac{1}{a_l}} (2k_l(V_1) + \bar{p}(V_1)) \tag{3.27}$$

everywhere, where

$$\Phi(V_1) = \max_{i=2,..,j} \left\{ \phi_i^{\frac{1}{a_i}}(V_1) \left[\frac{4(j-1)(i-1)}{\rho(V_1)} \right]^{(1-a_i)/a_i} \right\} \tag{3.28}$$

for $i = 1, 2, \ldots, j$. For convenience, we set $\mu_j(v) \equiv 1$, and we introduce the functions

$$\Omega_i(p) = \int_0^p \mu_i(r) dr.$$

Then $\Omega_i'(s) \geq 1$ for all $s \geq 0$ and i, and Lemma 3.1 gives

$$\Omega_i'(U_i) \ \geq \ 2\Phi(V_1) \sum_{l=1+i}^{j} \Omega_l'(U_l)^{\frac{1}{a_l}} \tag{3.29}$$

for all i and $x \in \mathcal{X}$. In particular, we have $\Omega_j(p) \equiv p$.

Stability Analysis

Since $\Omega_1'(s) \geq 1$ everywhere, we get $\Omega_1(U_1(x)) \geq U_1(x) = V_1(x)$ everywhere. Hence,

$$S(x) \ = \ \Omega_1(2V_1(x)) + \sum_{i=2}^{j} \Omega_i(U_i(x)) \tag{3.30}$$

satisfies (3.24). To check the decay estimate (3.25), first note that Assumption 3.1 and our choices of the k_i's give

$$\begin{aligned} \nabla S(x) f(x) &= 2\Omega_1'(2U_1)\dot{V}_1 + \sum_{i=2}^{j} \Omega_i'(U_i) \left[k_i'(V_1)\dot{V}_1 + \dot{V}_i \right] \\ &\leq \sum_{i=1}^{j} \Omega_i'(U_i)\dot{V}_i \\ &\leq -\sum_{i=1}^{j} \Omega_i'(U_i)\mathcal{N}_i \\ &\quad + \sum_{i=2}^{j} \Omega_i'(U_i) \left(\phi_i(V_1) \sum_{l=1}^{i-1} \mathcal{N}_l^{a_i} V_1^{1-a_i} \right) \end{aligned} \tag{3.31}$$

along the trajectories of $\dot{x} = f(x)$. Define the everywhere positive functions $\Gamma_2, \ldots, \Gamma_j$ by

$$\Gamma_i(x) = \frac{4(j-1)(i-1)\Omega_i'(U_i(x))\phi_i(V_1(x))}{\rho(V_1(x))}.$$

For any $i \geq 2$ for which $0 < a_i < 1$, we can apply Young's Inequality

$$v_1 v_2 \leq v_1^p + v_2^q, \quad \text{with} \quad p = \frac{1}{a_i}, \quad q = \frac{1}{1-a_i},$$
$$v_1 = \Gamma_i^{1-a_i}(x)\mathcal{N}_l^{a_i}(x), \quad \text{and} \quad v_2 = \left\{\frac{V_1(x)}{\Gamma_i(x)}\right\}^{1-a_i}$$

to get

$$\mathcal{N}_l^{a_i}(x)V_1^{1-a_i}(x) \ \leq \ \Gamma_i^{(1-a_i)/a_i}(x)\mathcal{N}_l(x) + \frac{V_1(x)}{\Gamma_i(x)}$$

for all $x \in \mathcal{X}$. The preceding inequality also holds when $a_i = 1$, so we can substitute it into (3.31) to get

$$\begin{aligned}
\nabla S(x)f(x) &\leq -\sum_{i=1}^{j} \Omega_i'(U_i)\mathcal{N}_i \\
&\quad + \sum_{i=2}^{j}\left(\Omega_i'(U_i)\phi_i(V_1)\Gamma_i^{\frac{1-a_i}{a_i}}\sum_{l=1}^{i-1}\mathcal{N}_l\right) \\
&\quad + \left(\sum_{i=2}^{j}\Omega_i'(U_i)\frac{\phi_i(V_1)(i-1)}{\Gamma_i}\right)V_1 \\
&\leq -\sum_{i=1}^{j}\Omega_i'(U_i)\mathcal{N}_i + \frac{1}{4}\rho(V_1)V_1 \qquad (3.32)\\
&\quad + \sum_{i=2}^{j}\left(\Omega_i'(U_i)\phi_i(V_1)\Gamma_i^{\frac{1-a_i}{a_i}}\sum_{l=1}^{i-1}\mathcal{N}_l\right) \\
&\leq -\sum_{i=1}^{j}\Omega_i'(U_i)\mathcal{N}_i + \frac{1}{4}\rho(V_1)V_1 \\
&\quad + \Phi(V_1)\sum_{i=2}^{j}\left(\Omega_i'(U_i)^{\frac{1}{a_i}}\sum_{l=1}^{i-1}\mathcal{N}_l\right),
\end{aligned}$$

by our choices of the Γ_i's and the formula for Φ in (3.29).

Since $\Omega_i' \geq 1$ for all i, Assumption 3.2 gives

$$\sum_{i=1}^{j}\Omega_i'(U_i)\mathcal{N}_i \ \geq \ \rho(V_1)V_1.$$

Hence, (3.32) gives

$$\nabla S(x)f(x) \leq -\tfrac{1}{4}\rho(V_1)V_1 - \tfrac{1}{2}\sum_{i=1}^{j}\Omega_i'(U_i)\mathcal{N}_i$$
$$+\Phi(V_1)\sum_{i=2}^{j}\left(\Omega_i'(U_i)^{\frac{1}{a_i}}\sum_{l=1}^{i-1}\mathcal{N}_l\right).$$

By reorganizing terms, one can prove that

$$\sum_{i=2}^{j}\left(\Omega_i'(U_i)^{\frac{1}{a_i}}\sum_{l=1}^{i-1}\mathcal{N}_l\right) \;=\; \sum_{i=1}^{j-1}\left(\sum_{l=1+i}^{j}\Omega_l'(U_l)^{\frac{1}{a_l}}\right)\mathcal{N}_i\;. \tag{3.33}$$

It follows that

$$\nabla S(x)f(x) \leq -\frac{1}{4}\rho(V_1)V_1$$
$$+\sum_{i=1}^{j-1}\left[-\frac{1}{2}\Omega_i'(U_i) + \Phi(V_1)\sum_{l=1+i}^{j}\Omega_l'(U_l)^{\frac{1}{a_l}}\right]\mathcal{N}_i\;.$$

Since the $\mathcal{N}_i$'s are non-negative, (3.25) now readily follows from (3.29). □

Remark 3.3. When $a_2 = \ldots = a_j = 1$, Assumption 3.2 can be relaxed by replacing (3.20) by the assumption that

$$x \mapsto \sum_{l=1}^{j}\mathcal{N}_l(x) \tag{3.34}$$

is positive definite, in which case we instead conclude that $\nabla S(x)f(x)$ is negative definite. The proof proceeds as in the proof of Theorem 3.1 through (3.31). Then we can directly apply (3.29) and (3.33) to get

$$\nabla S(x)f(x) \;\leq\; -\frac{1}{2}\sum_{i=1}^{j}\Omega_i'(U_i(x))\mathcal{N}_i(x)$$

everywhere. The result follows because $\Omega_i' \geq 1$ everywhere for all i.

3.4 Discrete Time Theorem

We turn next to an analog of Theorem 3.1 for the discrete time system

$$x_{k+1} = f(x_k), \quad x_k \in \mathcal{X} \tag{3.35}$$

where $\mathcal{X} \subseteq \mathbb{R}^n$ is open and contains the origin. Throughout this subsection, we make these two assumptions:

Assumption 3.3 *There exist a constant $a \in (0, 1]$; an integer $j \geq 2$; continuous functions $\nu_1, \mathcal{M}_i, \phi_i : \mathcal{X} \to [0, \infty)$ for $i = 1, 2, \ldots, j$; and continuous functions $\nu_i : \mathcal{X} \to \mathbb{R}$ for $i = 2, \ldots, j$ such that*

$$\nu_1(f(x)) - \nu_1(x) \leq -\mathcal{M}_1(x) \tag{3.36}$$

for all $x \in \mathcal{X}$ and

$$\nu_i(f(x)) - \nu_i(x) \leq -\mathcal{M}_i(x) + \phi_i(\nu_1(x))\nu_1(x)^{1-a} \sum_{l=1}^{i-1} \mathcal{M}_l^a(x) \tag{3.37}$$

for all $x \in \mathcal{X}$ and $i = 2, 3, \ldots, j$.

Assumption 3.4 *There are continuous functions $C_k : [0, \infty) \to (0, \infty)$ for $k = 1, 2, 3, 4$ such that the functions from Assumption 3.3 satisfy*

$$\sum_{l=1}^{j} \mathcal{M}_l(x) \; \geq \; C_1(\nu_1(x))|x|^2 \tag{3.38}$$

and

$$C_2(\nu_1(x))|x|^2 \; \leq \; \nu_1(x) \leq C_3(\nu_1(x))|x|^2 \tag{3.39}$$

for all $x \in \mathcal{X}$ and

$$|\nu_i(x)| \; \leq \; C_4(\nu_1(x))|x|^2 \tag{3.40}$$

for all $x \in \mathcal{X}$ and $i = 2, 3, \ldots, j$.

Assumption 3.3 is the discrete time analog of the continuous time Matrosov Condition in Assumption 3.1 except for simplicity, we took all of the a_i's to be equal. Notice that we are not requiring the auxiliary functions ν_i to be non-negative for $i \geq 2$, although ν_1 is non-negative.

Theorem 3.2. *Assume that the system (3.35) satisfies Assumptions 3.3 and 3.4. Then we can explicitly determine non-decreasing everywhere positive C^1 functions κ_l such that the function*

$$S(x) \; = \; \sum_{l=1}^{j} \kappa_l\big(\nu_1(x)\big)\nu_l(x) \tag{3.41}$$

satisfies

$$S(x) \; \geq \; |x|^2 \tag{3.42}$$

and

$$S(f(x)) - S(x) \; \leq \; -\nu_1(x) \tag{3.43}$$

for all $x \in \mathcal{X}$. Therefore, S is a strict Lyapunov function when $\mathcal{X} = \mathbb{R}^n$.

Remark 3.4. We have chosen to study the case where the auxiliary functions $\nu_2, \nu_3, \ldots, \nu_j$ are bounded from above by a positive definite quadratic function in a neighborhood of the origin. We made this choice because it leads to reasonably simple calculations. We strongly conjecture that a strict Lyapunov function construction can also be carried out without making this local quadratic upper bound assumption.

3.5 Proof of Discrete Time Theorem

Throughout our proof, all inequalities should be understood to hold for all $x \in \mathcal{X}$ unless otherwise indicated. We can easily find a C^1 non-decreasing function $\Gamma : [0, \infty) \to [1, \infty)$ such that

$$\frac{C_4(r)}{C_2(r)} + 1 \leq \Gamma(r) \;\; \forall r \geq 0. \tag{3.44}$$

Hence, (3.36) and the non-negativity of ν_1 give

$$\Gamma(\nu_1(f(x)))\nu_1(f(x)) - \Gamma(\nu_1(x))\nu_1(x) \;\leq\; -\Gamma(\nu_1(x))\mathcal{M}_1(x) \tag{3.45}$$

for all $x \in \mathcal{X}$. Also, (3.39) and (3.40) give

$$\frac{C_4(\nu_1(x))}{C_2(\nu_1(x))}\nu_1(x) + \nu_i(x) \geq 0. \tag{3.46}$$

We introduce the functions

$$\bar{\nu}_1 \doteq \nu_1, \;\; \text{and} \;\; \overline{\nu}_i(x) \doteq \Gamma(\nu_1(x))\nu_1(x) + \nu_i(x) \;\; \text{for } i = 2, 3, \ldots, j. \tag{3.47}$$

Then

$$\overline{\nu}_i(x) \geq \nu_1(x) \;\; \forall i. \tag{3.48}$$

Also, (3.37) and (3.45) give

$$\overline{\nu}_i(f(x)) - \overline{\nu}_i(x) \leq -\mathcal{M}_i(x) + \phi_i(\nu_1(x)) \sum_{l=1}^{i-1} \mathcal{M}_l^a(x)\nu_1^{1-a}(x) \tag{3.49}$$

for $i \geq 2$. We define the functions $V_1, V_2, \ldots, V_j$ by

$$V_1(x) = \nu_1(x) \;\; \text{and} \;\; V_l(x) = \sum_{r=1}^{l} \overline{\nu}_r(x) \;\; \text{for} \;\; l \geq 2\,. \tag{3.50}$$

Each function V_l is positive definite, because ν_1 is positive definite. Moreover, a simple calculation yields

$$V_i(f(x)) - V_i(x) \ \le\ -\mathcal{N}_i(x) + \psi_i(V_1(x)) V_1^{1-a}(x) \mathcal{N}_{i-1}^a(x) \tag{3.51}$$

for all $i \geq 2$, where

$$\mathcal{N}_i(x) \ =\ \sum_{r=1}^{i} \mathcal{M}_r(x) \tag{3.52}$$

and

$$\psi_i(V_1(x)) \ =\ i \sum_{r=2}^{i} \phi_r(V_1(x)) \tag{3.53}$$

everywhere for $i = 1, \ldots, j$. We also set

$$\psi_1(m) \ =\ 0 \ \ \forall m \ . \tag{3.54}$$

We can recursively define everywhere positive non-decreasing C^1 functions $\alpha_j, \alpha_{j-1}, \ldots, \alpha_1$ that satisfy

$$\frac{\alpha_j(r) C_1(r)}{2C_3(r)} \geq 1 \tag{3.55}$$

and

$$(2j)^{\frac{1-a}{a}} \frac{C_3^{\frac{1-a}{a}}(s)}{C_1^{\frac{1-a}{a}}(s)} \frac{\alpha_{i+1}^{\frac{1}{a}}(s)}{\alpha_j^{\frac{1-a}{a}}(s)} \psi_{i+1}^{\frac{1}{a}}(s) \ \le\ \frac{1}{2} \alpha_i(s) \tag{3.56}$$

for $i = 1, 2, \ldots, j-1$.

Consider the functions

$$\begin{aligned} U_i(x) &= \alpha_i\big(V_1(x)\big) V_i(x) \ \text{ for } i = 1, 2, ..., j \quad \text{and} \\ \mathcal{U}(x) \ &=\ \sum_{r=1}^{j} U_r(x). \end{aligned} \tag{3.57}$$

Notice that for all $i \in \{1, \ldots, j\}$, we have

$$U_i\big(f(x)\big) - U_i(x) = \alpha_i\Bigg(V_1\big(f(x)\big)\Bigg) V_i\big(f(x)\big) - \alpha_i\big(V_1(x)\big) V_i(x).$$

Since $V_1\big(f(x)\big) \leq V_1(x)$ and each α_i is non-decreasing, and since each V_i is positive definite, we deduce that

$$U_i(f(x)) - U_i(x) \ \le\ \alpha_i\big(V_1(x)\big)\big[V_i\big(f(x)\big) - V_i(x)\big]. \tag{3.58}$$

It follows from (3.51) that

$$\begin{aligned} &U_i(f(x)) - U_i(x) \\ &\le \alpha_i\big(V_1(x)\big)\Big[- \mathcal{N}_i(x) + \psi_i\big(V_1(x)\big) V_1^{1-a}(x) \mathcal{N}_{i-1}^a(x)\Big]. \end{aligned} \tag{3.59}$$

Therefore,

$$\begin{aligned}\mathcal{U}(f(x)) - \mathcal{U}(x) \leq & \sum_{r=1}^{j} \Big[- \alpha_r\big(V_1(x)\big)\mathcal{N}_r(x) \\ & + \alpha_r\big(V_1(x)\big)\psi_r\big(V_1(x)\big)V_1^{1-a}(x)\mathcal{N}_{r-1}^a(x)\Big] \\ \leq & -\sum_{r=1}^{j} \alpha_r\big(V_1(x)\big)\mathcal{N}_r(x) \\ & + \sum_{r=2}^{j} \Big[\alpha_r(V_1(x))\psi_r(V_1(x))V_1^{1-a}(x)\mathcal{N}_{r-1}^a(x)\Big] ,\end{aligned} \tag{3.60}$$

where the last inequality is from (3.54). We deduce that

$$\begin{aligned}\mathcal{U}(f(x)) - \mathcal{U}(x) \leq & -\sum_{r=1}^{j} \alpha_r\big(V_1(x)\big)\mathcal{N}_r(x) \\ & + \sum_{r=1}^{j-1} \alpha_{r+1}\big(V_1(x)\big)\psi_{r+1}\big(V_1(x)\big)V_1^{1-a}(x)\mathcal{N}_r^a(x).\end{aligned} \tag{3.61}$$

Using the fact that

$$\begin{aligned}\sum_{r=1}^{j} \alpha_r(V_1(x))\mathcal{N}_r(x) &= \alpha_j\big(V_1(x)\big)\mathcal{N}_j(x) + \sum_{r=1}^{j-1} \alpha_r\big(V_1(x)\big)\mathcal{N}_r(x) \\ &\geq \alpha_j\big(V_1(x)\big)C_1(\nu_1(x))|x|^2 + \sum_{r=1}^{j-1} \alpha_r\big(V_1(x)\big)\mathcal{N}_r(x)\end{aligned}$$

and therefore also

$$\begin{aligned}\sum_{r=1}^{j} \alpha_r(V_1(x))\mathcal{N}_r(x) \geq & \frac{\alpha_j(V_1(x))C_1(V_1(x))}{C_3(V_1(x))}V_1(x) \\ & + \sum_{r=1}^{j-1} \alpha_r\big(V_1(x)\big)\mathcal{N}_r(x),\end{aligned} \tag{3.62}$$

we deduce that

$$\begin{aligned}\mathcal{U}(f(x)) - \mathcal{U}(x) \leq & -\frac{\alpha_j\big(V_1(x)\big)C_1(V_1(x))}{C_3(V_1(x))}V_1(x) \\ & + \sum_{r=1}^{j-1} \Big[- \alpha_r(V_1(x))\mathcal{N}_r(x) \\ & + \alpha_{r+1}\big(V_1(x)\big)\psi_{r+1}\big(V_1(x)\big)V_1^{1-a}(x)\mathcal{N}_r^a(x)\Big].\end{aligned} \tag{3.63}$$

Setting

$$\Gamma_r(s) = \frac{C_3^{\frac{1-a}{a}}(s)\, \alpha_{r+1}^{\frac{1}{a}}(s)}{C_1^{\frac{1-a}{a}}(s)\, \alpha_j^{\frac{1-a}{a}}(s)}$$

for $r = 1, 2, \ldots, j-1$, Young's Inequality $pq \leq p^{1/(1-a)} + q^{1/a}$ applied with

$$p = \frac{\alpha_j^{1-a}(V_1(x)) C_1^{1-a}(V_1(x)) V_1^{1-a}(x)}{(2j)^{1-a} C_3^{1-a}(V_1(x))} \quad \text{and}$$
$$q = (2j)^{1-a} \psi_{r+1}(V_1(x)) \Gamma_r^a(V_1(x)) \mathcal{N}_r^a(x)$$

for $a \neq 1$ gives

$$\begin{aligned} &\alpha_{r+1}(V_1(x))\psi_{r+1}(V_1(x))V_1^{1-a}(x)\mathcal{N}_r^a(x) \\ &\leq \frac{\alpha_j(V_1(x))C_1(V_1(x))}{2jC_3(V_1(x))} V_1(x) \\ &\quad + (2j)^{\frac{1-a}{a}} \Gamma_r(V_1(x)) \psi_{r+1}^{\frac{1}{a}}(V_1(x)) \mathcal{N}_r(x) \end{aligned} \tag{3.64}$$

for all possible $a \in (0, 1]$.

Combined with (3.63), this gives

$$\begin{aligned} &\mathcal{U}(f(x)) - \mathcal{U}(x) \\ &\leq -\frac{\alpha_j(V_1(x))C_1(V_1(x))}{2C_3(V_1(x))} V_1(x) \\ &\quad + \sum_{r=1}^{j-1} \left[-\alpha_r(V_1(x)) + (2j)^{\frac{1-a}{a}} \Gamma_r(V_1(x)) \psi_{r+1}^{\frac{1}{a}}(V_1(x)) \right] \mathcal{N}_r(x), \end{aligned} \tag{3.65}$$

for all possible $a \in (0, 1]$. Since our functions α_i satisfy (3.56), we get

$$\mathcal{U}(f(x)) - \mathcal{U}(x) \leq -\frac{\alpha_j(V_1(x))C_1(V_1(x))}{2C_3(V_1(x))} V_1(x). \tag{3.66}$$

Using (3.40), we can determine an increasing C^1 function $\Lambda : [0, \infty) \to [1, \infty)$ such that

$$\begin{aligned} &|\mathcal{U}(x)| \leq \Lambda(V_1(x))V_1(x) \quad \forall x \in \mathcal{X} \quad \text{and} \\ &\Lambda(r) \geq \tfrac{1}{C_2(r)} \quad \forall r \geq 0. \end{aligned} \tag{3.67}$$

We easily deduce that

$$S(x) \;=\; \mathcal{U}(x) + 2\Lambda\big(V_1(x)\big)V_1(x) \tag{3.68}$$

satisfies

$$S(x) \;\geq\; \Lambda(V_1(x))V_1(x) \;\geq\; \frac{V_1(x)}{C_2(V_1(x))} \;\geq\; |x|^2 \tag{3.69}$$

and

$$S(f(x)) - S(x) \;\leq\; -\frac{\alpha_j(V_1(x))C_1(V_1(x))}{2C_3(V_1(x))}V_1(x). \tag{3.70}$$

Combined with our condition (3.55) on α_j, this proves the theorem. □

3.6 Illustrations

3.6.1 Continuous Time: One Auxiliary Function

Let us show how the strict Lyapunov-like function (3.12) we constructed in Sect. 3.1 follows as a special case of Theorem 3.1. Choose V_1 and V_2 as defined in (3.6) and (3.9), respectively. Then our decay conditions (3.8) and (3.10) imply that Assumption 3.1 is satisfied with $j = 2$, $\mathcal{N}_1(\tilde{s}) = \tilde{s}^2$, $\mathcal{N}_2(\tilde{\gamma}) = \frac{v_*}{2}\tilde{\gamma}^2$, $a_2 = 1$, and the constant function

$$\phi_2(s) \;\equiv\; \frac{\gamma_M^2}{2v_\star} + \frac{K(\gamma_M - \gamma_m)^2}{4}.$$

Moreover, since V_1 is bounded from above by a positive definite quadratic function near 0, we can find an everywhere positive function ρ so that

$$\rho(V_1)V_1 \;\leq\; \min\left\{1, \frac{v_*}{2}\right\}(\tilde{s}^2 + \tilde{\gamma}^2) \;\leq\; \sum_{i=1}^{2}\mathcal{N}_i(\tilde{s}, \tilde{\gamma})$$

on $\mathcal{X}$. (In fact, we can choose ρ so that outside a neighborhood of zero,

$$\rho(v) = \frac{c}{1+v}$$

for a suitable constant c.) Thus, the first part of Assumption 3.2 is also satisfied.

Next note that because $\max\{(\gamma_M - \gamma)(\gamma - \gamma_m) : \gamma \in [\gamma_m, \gamma_M]\} = \frac{1}{4}(\gamma_M - \gamma_m)^2$, we know that

$$V_1(\tilde{s}, \tilde{\gamma}) \;\geq\; \frac{1}{2\gamma_m}\tilde{s}^2 + \frac{2v_*}{K\gamma_m(\gamma_M - \gamma_m)^2}\tilde{\gamma}^2 \;\geq\; \frac{1}{2}\underline{v}(\tilde{s}^2 + \tilde{\gamma}^2),$$

where

$$\underline{v} \;=\; \min\left\{\frac{1}{\gamma_m}, \frac{4v_*}{K\gamma_m(\gamma_M - \gamma_m)^2}\right\}, \tag{3.71}$$

holds on $\mathcal{X}$. This and the triangle inequality $|\tilde{s}\tilde{\gamma}| \leq \frac{1}{2}\tilde{s}^2 + \frac{1}{2}\tilde{\gamma}^2$ give

$$|V_2(\tilde{s}, \tilde{\gamma})| \;=\; |\tilde{s}\tilde{\gamma}| \;\leq\; \frac{V_1(\tilde{s}, \tilde{\gamma})}{\underline{v}}.$$

(Our choice of V_2 was motivated by our desire to have the preceding estimate.) Hence, the second part of Assumption 3.2 is satisfied as well, so Theorem 3.1 applies with the constant function

$$p_2(s) \;\equiv\; \frac{1}{\underline{v}}.$$

We now explicitly build the strict Lyapunov-like function from Theorem 3.1.

Since $j = 2$ and $a_2 = 1$, we get

$$k_2(s) \;=\; \left(\frac{1}{\underline{v}} + 1\right) s,$$

hence

$$U_2(\tilde{s}, \tilde{\gamma}) = \Upsilon_1 V_1(\tilde{s}, \tilde{\gamma}) + V_2(\tilde{s}, \tilde{\gamma}),$$

where Υ_1 is the constant we defined in (3.11). Also, we can take Φ from (3.28) to be ϕ_2 to get

$$\Omega_1(s) = \left[\frac{\gamma_M^2}{v_\star} + \frac{K(\gamma_M - \gamma_m)^2}{2}\right] s \;\text{ and}$$
$$\Omega_2(s) \equiv s.$$

Hence the formula (3.30) for S becomes

$$\begin{aligned} S(\tilde{s}, \tilde{\gamma}) &= U_2(\tilde{s}, \tilde{\gamma}) + 2\left[\frac{\gamma_M^2}{v_\star} + \frac{K(\gamma_M - \gamma_m)^2}{2}\right] V_1(\tilde{s}, \tilde{\gamma}) \\ &= V_2(\tilde{s}, \tilde{\gamma}) + \left[\Upsilon_1 + \frac{2\gamma_M^2}{v_*} + K(\gamma_M - \gamma_m)^2\right] V_1(\tilde{s}, \tilde{\gamma}) \end{aligned} \tag{3.72}$$

which agrees with (3.12).

3.6.2 Continuous Time: Two Auxiliary Functions

We next consider a case where the function (3.23) constructed in Theorem 3.1 is radially unbounded and therefore is a strict Lyapunov function. We consider the system

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 - x_2^3. \end{cases} \tag{3.73}$$

We use the functions

$$\begin{aligned} V_1(x) &= \tfrac{1}{4}(x_1^2+x_2^2)^2\ , \quad \mathcal{N}_1(x) = (x_1^2+x_2^2)x_2^4\ , \\ V_2(x) &= \tfrac{1}{2}(x_1^2+x_2^2)\ , \quad \mathcal{N}_2(x) = x_2^4\ , \\ V_3(x) &= \tfrac{1}{2}(x_1^2+x_2^2)x_1x_2\ , \quad \text{and} \quad \mathcal{N}_3(x) = \tfrac{1}{2}[x_1^2+x_2^2]x_1^2\ . \end{aligned} \tag{3.74}$$

Along the trajectories of (3.73), the functions V_i satisfy

$$\begin{aligned} \dot{V}_1(x) &= -\mathcal{N}_1(x), \\ \dot{V}_2(x) &= -\mathcal{N}_2(x), \quad \text{and} \\ \dot{V}_3(x) &= \tfrac{1}{2}[x_1^2+x_2^2][x_2^2-x_1^2-x_1x_2^3] - \mathcal{N}_2(x)x_1x_2. \end{aligned}$$

Therefore, the inequality $x_1x_2 \le \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$ gives

$$\dot{V}_3(x) \le -\mathcal{N}_3(x) + \phi_3(V_1(x))\sqrt{\mathcal{N}_2(x)}\sqrt{V_1(x)}, \tag{3.75}$$

where

$$\phi_3(r) = 1 + 3\sqrt{r}. \tag{3.76}$$

One can easily check that

$$\begin{aligned} \sum_{r=1}^{3}\mathcal{N}_r(x) &= (x_1^2+x_2^2)x_2^4 + x_2^4 + \tfrac{1}{2}[x_1^2+x_2^2]x_1^2 \\ &\ge \rho(V_1(x))V_1(x), \end{aligned} \tag{3.77}$$

where $\rho(r) \equiv 1$. Also, V_1 and V_2 are positive definite, and

$$|V_3(x)| \le p_3(V_1(x))V_1(x)\ , \tag{3.78}$$

where $p_3(r) \equiv 1$. Therefore, Assumptions 3.1 and 3.2 are satisfied with $j = 3$, $\bar{\rho}(s) = \sqrt{s}$, $\phi_2 \equiv 1$, $a_2 = 1$, and $a_3 = 1/2$. Hence, Theorem 3.1 provides a strict Lyapunov-like function for (3.73), which turns out to be a strict Lyapunov function.

Let us construct the strict Lyapunov-like function from the theorem. Since $p_2(s) \equiv 0$, we can satisfy the conditions (3.26) on the k_i's by taking

$$k_1(s) = k_2(s) = s \quad \text{and} \quad k_3(s) = 2s.$$

The functions U_i from Lemma 3.1 are therefore

$$\begin{aligned} &U_1(x) = V_1(x)\ , \quad U_2(x) = V_1(x) + V_2(x)\ , \text{ and} \\ &U_3(x) = 2V_1(x) + V_3(x). \end{aligned} \tag{3.79}$$

Since $a_2 = 1$ and $a_3 = 1/2$, the function Φ from (3.28) is $\Phi(s) = 16(1+3\sqrt{s})^2$. Therefore, we can satisfy the conditions on the Ω_i's in (3.29) by taking

$$\begin{aligned} \Omega_3(s) &= s\ , \quad \Omega_2(s) = 32s + 128s^{3/2} + 144s^2\ , \\ \Omega_1(s) &= \Omega_2(s) + 32^2\left(49s + 105s^2 + 80s^3\right)\ , \text{ and} \\ S(x) &= \Omega_1(2U_1(x)) + \Omega_2(U_2(x)) + U_3(x). \end{aligned} \tag{3.80}$$

With these choices, we obtain

$$\dot{S}(x) \leq -\tfrac{1}{4}V_1(x). \tag{3.81}$$

In conjunction with the properness and positive definiteness of S, this shows that S is a strict Lyapunov function for (3.73).

Remark 3.5. The parameters in the functions Ω_1 and Ω_2 in (3.80) are large. However, we can construct a global strict Lyapunov function for (3.73) with smaller parameters, by the following direct construction.

We have

$$\begin{aligned} \dot{U}_1(x) &= -\mathcal{N}_1(x)\ , \\ \dot{U}_2(x) &= -\big(\mathcal{N}_1(x) + \mathcal{N}_2(x)\big)\ , \text{ and} \\ \dot{U}_3(x) &\leq -2\mathcal{N}_1(x) - \mathcal{N}_3(x) + \phi_3\big(V_1(x)\big)\sqrt{\mathcal{N}_2(x)}\sqrt{V_1(x)}\ . \end{aligned} \tag{3.82}$$

Therefore,

$$\begin{aligned} \dot{U}_3(x) + \dot{U}_2(x) &\leq -\mathcal{N}_1(x) - \mathcal{N}_2(x) - \mathcal{N}_3(x) \\ &\quad + \phi_3(V_1(x))\sqrt{\mathcal{N}_2(x)}\sqrt{V_1(x)} \\ &\leq -V_1(x) + \phi_3(V_1(x))\sqrt{\mathcal{N}_2(x)}\sqrt{V_1(x)} \\ &\leq -\tfrac{1}{2}V_1(x) + \tfrac{1}{2}\phi_3^2(V_1(x))\mathcal{N}_2(x) \\ &\leq -\tfrac{1}{2}V_1(x) + (1 + 9V_1(x))\mathcal{N}_2(x)\ , \end{aligned} \tag{3.83}$$

where the second inequality is by (3.77). Let

$$\overline{S}(x) = 2U_2(x) + 8U_2^2(x) + U_3(x). \tag{3.84}$$

This function satisfies

$$\dot{\overline{S}}(x) \leq -\frac{1}{2}V_1(x)\ . \tag{3.85}$$

Moreover, $\overline{S}$ is positive definite and radially unbounded, because the U_i's are bounded below by V_1. Therefore $\overline{S}$ is a strict Lyapunov function for (3.73).

3.6.3 Discrete Time Context

We illustrate our discrete time Lyapunov function construction from Theorem 3.2 using the system

$$\begin{cases} p_{k+1} = q_k \\ q_{k+1} = r_k \\ r_{k+1} = p_k - \frac{3}{4}\frac{p_k}{1+p_k^2}. \end{cases} \tag{3.86}$$

Let $x = (p, q, r)$. We check the assumptions of the theorem using

$$\begin{aligned} &\nu_1(x) = \tfrac{1}{2}\left[p^2 + q^2 + r^2\right], \quad \nu_2(x) = r^2, \quad \nu_3(x) = q^2, \\ &\mathcal{M}_1(x) = \tfrac{15}{32}\tfrac{p^2}{1+p^2}, \quad \mathcal{M}_2(x) = r^2, \quad \text{and} \quad \mathcal{M}_3(x) = q^2 . \end{aligned} \tag{3.87}$$

Notice that

$$\begin{aligned} \nu_1(x_{k+1}) - \nu_1(x_k) &= \tfrac{1}{2}[p_{k+1}^2 + q_{k+1}^2 + r_{k+1}^2] - \tfrac{1}{2}[p_k^2 + q_k^2 + r_k^2] \\ &= \tfrac{1}{2}\left[\left(p_k - \tfrac{3}{4}\tfrac{p_k}{1+p_k^2}\right)^2 - p_k^2\right] \\ &= \tfrac{1}{2}\left[-\tfrac{3}{2}\tfrac{p_k^2}{1+p_k^2} + \tfrac{9}{16}\tfrac{p_k^2}{(1+p_k^2)^2}\right] \\ &\le -\mathcal{M}_1(x_k). \end{aligned} \tag{3.88}$$

Also,

$$\begin{aligned} \nu_2(x_{k+1}) - \nu_2(x_k) &= r_{k+1}^2 - r_k^2 \\ &= -\mathcal{M}_2(x_k) + \left(1 - \tfrac{3}{4}\tfrac{1}{1+p_k^2}\right)^2 p_k^2 \\ &\le -\mathcal{M}_2(x_k) + \tfrac{32}{15}\left(1 - \tfrac{3}{4}\tfrac{1}{1+p_k^2}\right)^2 (1+p_k^2)\mathcal{M}_1(x_k) \end{aligned} \tag{3.89}$$

and

$$\nu_3(x_{k+1}) - \nu_3(x_k) = q_{k+1}^2 - q_k^2 \ = \ -\mathcal{M}_3(x_k) + \mathcal{M}_2(x_k). \tag{3.90}$$

In summary,

$$\begin{aligned} &\nu_1(x_{k+1}) - \nu_1(x_k) \le -\mathcal{M}_1(x_k) \\ &\nu_2(x_{k+1}) - \nu_2(x_k) \le -\mathcal{M}_2(x_k) + \phi_2(\nu_1(x_k))\mathcal{M}_1(x_k) \\ &\nu_3(x_{k+1}) - \nu_3(x_k) = -\mathcal{M}_3(x_k) + \phi_3(\nu_1(x_k))\mathcal{M}_2(x_k) , \end{aligned} \tag{3.91}$$

where

$$\phi_2(l) \ = \ \frac{32}{15}(1+2l) \ \text{and} \ \phi_3(l) = l . \tag{3.92}$$

It follows that Assumption 3.3 is satisfied. Moreover, for all choices of x,

$$\begin{gathered}\sum_{l=1}^{3} \mathcal{M}_l(x) = \frac{15}{32}\frac{p^2}{1+p^2} + q^2 + r^2 \ \geq \ C_1(\nu_1(x))|x|^2 \ , \\ C_2(\nu_1(x))|x|^2 \leq \nu_1(x) \ \leq \ C_3(\nu_1(x))|x|^2, \ \text{ and} \\ |\nu_i(x)| \leq C_4(\nu_1(x))|x|^2 \ \text{ for } i = 2, 3\end{gathered} \tag{3.93}$$

where

$$C_1(l) = \frac{15}{32(1+2l)}, \ \ C_2(l) = C_3(l) = \frac{1}{2}, \ \text{ and } \ C_4(l) = 1$$

for all $l \geq 0$. Therefore, Assumption 3.4 is satisfied as well, so Theorem 3.2 applies. Hence, we can construct a strict Lyapunov function for the system (3.86) by arguing as in the proof of Theorem 3.2.

Let us construct a strict Lyapunov function for (3.86) of the type guaranteed by the theorem. Since

$$\begin{aligned}&2\nu_2(x_{k+1}) - 2\nu_2(x_k) + \nu_3(x_{k+1}) - \nu_3(x_k) \\ &= -\mathcal{M}_3(x_k) - \mathcal{M}_2(x_k) + \frac{64}{15}[1 + 2\nu_1(x_k)]\mathcal{M}_1(x_k),\end{aligned} \tag{3.94}$$

the radially unbounded positive definite function

$$\overline{S}(x) = \frac{94}{15}\big[1 + 2\nu_1(x)\big]\nu_1(x) + 2\nu_2(x) + \nu_3(x). \tag{3.95}$$

satisfies

$$\begin{aligned}\overline{S}(x_{k+1}) - \overline{S}(x_k) &\leq -2\big[1 + 2\nu_1(x_k)\big]\mathcal{M}_1(x_k) - \mathcal{M}_2(x_k) - \mathcal{M}_3(x_k) \\ &\leq -\nu_1(x_k) \ ,\end{aligned} \tag{3.96}$$

which is the desired decay condition.

3.7 Comments

The recent paper [111] provides an alternative and very general Matrosov approach for constructing strict Lyapunov-like functions. However the Lyapunov functions provided by [111] are not in general locally bounded from below by positive definite quadratic functions, even for globally asymptotically linear systems, which admit a quadratic strict Lyapunov function. The shape of Lyapunov functions, their local properties and their simplicity matter when they are used to investigate robustness properties and construct feedbacks and gains.

The differences between Assumptions 3.1 and 3.2 and the assumptions from [111] are as follows. First, while our Assumption 3.1 ensures that V_1 is positive definite but not necessarily proper, [111] assumes that a radially unbounded non-strict Lyapunov function is known. Second, our Assumption 3.1 is a restrictive version of Assumption 2 from [111]. More precisely, our Assumption 3.1 specifies the local properties of the functions that correspond to the χ_i's of Assumption 2 in [111]. Finally, our Assumption 3.2 imposes relations between the functions $\mathcal{N}_i$ and V_1, which are not required in [111]. An important feature is that we do not require the functions $V_2, \ldots, V_j$ to be non-negative.

Our treatment of (3.3) is based on [106]. Since the Matrosov constructions in [111] assume that the given non-strict Lyapunov function is globally proper on the whole Euclidean space, and since (3.6) does not satisfy this requirement, we cannot construct the required explicit strong Lyapunov function for (3.3) using the results of [111]. Notice that the strict Lyapunov-like function (3.72) that we constructed for the anaerobic digester is a simple linear combination of V_1 and V_2. By contrast, the strong Lyapunov functions provided by [111, Theorem 3] for the $j = 2$ time-invariant case have the form

$$S(x) \;=\; Q_1\big(V_1(x)\big)V_1(x) + Q_2\big(V_1(x)\big)V_2(x),$$

where Q_1 is non-negative, and where the positive definite function Q_2 needs to globally satisfy

$$Q_2(V_1) \leq \phi^{-1}\left(\frac{\omega(x)}{2\rho(|x|)}\right),$$

where

$$\nabla V_2(x)f(x) \;\leq\; -\mathcal{N}_2(x) + \phi\big(\mathcal{N}_1(x)\big)\rho(|x|)$$

for some $\phi \in \mathcal{K}_\infty$ and some everywhere positive non-decreasing function ρ and the positive definite function ω needs to satisfy $\mathcal{N}_1(x) + \mathcal{N}_2(x) \geq \omega(x)$ everywhere. In particular, we cannot take Q_2 to be constant to get a linear combination of the V_i's, so the construction of [111] is more complicated than the one we provide here. Similar remarks apply to the other constructions in [111]. See Chap. 8 for strict Lyapunov function constructions under more general Matrosov type conditions.