1

Mathematical Fundamentals

This chapter deals with mathematical fundamentals of curves and surfaces, and more generally manifolds and varieties.[1] For that, we will pay particular attention to their smoothness or, putting it differently, to their singularities (i.e. lack of smoothness). As will be seen later on, these shape particularities are important in the design and implementation of rendering algorithms for implicit curves and surfaces. Therefore, although the context is the differential topology and geometry, we are interested in their applications in geometric modelling and computer graphics.

1.1 Introduction

The rationale behind the writing of this chapter was to better understand the subtleties of the manifolds, in particular to exploit the smooth structure of manifolds (e.g. Euclidean spaces) through the study of the *intrinsic* properties of their subsets or subspaces, i.e. independently of any choice of local coordinates (e.g. spherical coordinates, Cartesian coordinates, etc.). As known, manifolds provide us with the proper category in which most efficiently one can develop a coordinate-free approach to the study of the intrinsic geometry of point sets. It is obvious that the explicit formulas for a subset may change when one goes from one set of coordinates to another. This means that any geometric equivalence problem can be viewed as the problem of determining whether two different local coordinate expressions define the same intrinsic subset of a manifold. Such coordinate expressions (or change of coordinates) are defined by mappings between manifolds.

Thus, by defining mappings between manifolds such as Euclidean spaces, we are able to uncover the local properties of their subspaces. In geometric

[1] A real, algebraic or analytic variety is a point set defined by a system of equations $f_1 = \cdots = f_k = 0$, where the functions f_i $(0 \leq i \leq k)$ are real, algebraic or analytic, respectively.

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modelling, we are particularly interested in properties such as, for example, local smoothness, i.e. to know whether the neighbourhood of a point in a submanifold is (visually) smooth, or the point is a singularity. In other words, we intend to study the relationship between smoothness of mappings and smoothness of manifolds. The idea is to show that a mathematical theory exists to describe manifolds and varieties (e.g. curves and surfaces), regardless of whether they are defined explicitly, implicitly, or parametrically.

1.2 Functions and Mappings

In simple terms, a function is a relationship between two variables, typically x and y, so it often denoted by $f(x) = y$. The variable x is the independent variable (also called primary variable, function argument, or function input), while the variable y is the dependent variable (secondary variable, value of the function, function output, or the image of x under f). Therefore, a function allows us to associate a unique output for each input of a given type (e.g. a real number).

In more formal terms, a function is a particular type of binary relation between two sets, say X and Y. The set X of input values is said to be the *domain* of f, while the set Y of output values is known as the *codomain* of f. The *range* of f is the set $\{f(x) : x \in X\}$, i.e. the subset of Y which contains all output values of f. The usual definition of a function satisfies the condition that for each $x \in X$, there is at most one $y \in Y$ such that x is related to y. This definition is valid for most elementary functions, as well as maps between algebraic structures, and more importantly between geometric objects, such as manifolds.

There are three major types of functions, namely, injections, surjections and bijections. An *injection* (or one-to-one function) has the property that if $f(a) = f(b)$, then a and b must be identical. A *surjection* (or onto function) has the property that for every y in the codomain there is an x in the domain such that $f(x) = y$. Finally, a *bijection* is both one-to-one and onto.

The notion of a function can be extended to several input variables. That is, a single output is obtained by combining two (or more) input values. In this case, the domain of a function is the Cartesian product of two or more sets. For example, $f(x, y, z) = x^2 + y^2 + z^2 = 0$ is a trivariate function (or a function of three variables) that outputs the single value 0; the domain of this function is the Cartesian product $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ or, simply, $\mathbb{R}^3$. In geometric terms, this function defines an implicit sphere in $\mathbb{R}^3$.

Functions can be even further extended in order to have several outputs. In this case, we have a component function for each output. Functions with several outputs or component functions are here called *mappings*. For example, the mapping $f : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $f(x, y, z) = (x^2+y^2+z^2-1, 2x^2+2y^2-1)$ has two component functions $f_1(x, y, z) = x^2 + y^2 + z^2 - 1$ and $f_2(x, y, z) = 2x^2 + 2y^2 - 1$. These components represent a sphere and a cylinder in $\mathbb{R}^3$, respectively, so that, intuitively, we can say that f represents the point set that results from the intersection between the sphere and the cylinder.

Before proceeding any further, it is also useful to review how functions are classified in respect to the properties of their derivatives. Let $f : X \to Y$ be a mapping of X into Y, where X, Y are open subsets of $\mathbb{R}^m, \mathbb{R}^n$, respectively. If $n = 1$, we say that the function f is C^r (or C^r *differentiable* or *differentiable of class* C^r, or C^r *smooth* or *smooth of class* C^r) on X, for $r \in \mathbb{N}$, if the partial derivatives of f exist and are continuous on X, that is, at each point $\mathbf{x} \in X$. In particular, f is C^0 if f is continuous. If $n > 1$, the mapping f is C^r if each of the *component functions* f_i $(1 \leq i \leq n)$ of f is C^r. We say that f is C^∞ (or just *differentiable* or *smooth*) if it is C^r for all $r \geq 0$. Moreover, f is called a C^r *diffeomorphism* if: (i) f is a homeomorphism[2] and (ii) both f and f^{-1} are C^r differentiable, $r \geq 1$ (when $r = \infty$ we simply say *diffeomorphism*). For further details about smooth mappings, the reader is referred to, for example, Helgason [182, p. 2].

1.3 Differential of a Smooth Mapping

Let U, V be open sets in $\mathbb{R}^m, \mathbb{R}^n$, respectively. Let $f : U \to V$ be a mapping with component functions $f_1, \ldots, f_n$. Note that f is defined on every point $\mathbf{p}$ of U in the coordinate system $x_1, \ldots x_m$. We call f smooth provided that all derivatives of the f_i of all orders exist and are continuous in U. Thus for f smooth, $\partial^2 f_i/\partial x_1 \partial x_2$, $\partial^3 f_i/\partial x_1^3$, etc., and $\partial^2 f_i/\partial x_1 \partial x_2 = \partial^2 f_i/\partial x_2 \partial x_1$, etc., all exist and are continuous. Therefore, a mapping $f : U \to V$ is *smooth* (or *differentiable*) if f has continuous partial derivatives of all orders. And we call f a *diffeomorphism* of U onto V when it is a bijection, and both f, f^{-1} are smooth.

Let $f : U \to V$ be a smooth (or differentiable or C^∞) and let $\mathbf{p} \in U$. The matrix

$$\mathrm{J}f(\mathbf{p}) = \begin{bmatrix} \partial f_1(\mathbf{p})/\partial x_1 & \partial f_1(\mathbf{p})/\partial x_2 & \cdots & \partial f_1(\mathbf{p})/\partial x_m \\ \vdots & \vdots & & \vdots \\ \partial f_n(\mathbf{p})/\partial x_1 & \partial f_n(\mathbf{p})/\partial x_2 & \cdots & \partial f_n(\mathbf{p})/\partial x_m \end{bmatrix}$$

where the partial derivatives are evaluated at $\mathbf{p}$, is called **Jacobian matrix** of f at $\mathbf{p}$ [68, p. 51]. The linear mapping $\mathrm{D}f(\mathbf{p}) : \mathbb{R}^m \to \mathbb{R}^n$ whose matrix is the Jacobian is called the **derivative** or **differential** of f at $\mathbf{p}$; the Jacobian $\mathrm{J}f(\mathbf{p})$ is also denoted by $[\mathrm{D}f(\mathbf{p})]$. It is known in mathematics and geometric design that every polynomial mapping f (i.e. mappings whose component functions

[2] In topology, two topological spaces are said to be *equivalent* if it is possible to transform one to the other by continuous deformation. Intuitively speaking, these topological spaces are seen as being made out of ideal rubber which can be deformed somehow. However, such a continuous deformation is constrained by the fact that the dimension is unchanged. This kind of transformation is mathematically called *homeomorphism*.

f_i are all polynomial functions) is smooth. If the components are rational functions, then the mapping is smooth provided none of the denominators vanish anywhere.

Besides, the composite of two smooth mappings, possibly restricted to a smaller domain, is smooth [68, p. 51]. It is worth noting that the chain rule holds not only for smooth mappings, but also for differentials. This fact provides us with a simple proof of the following theorem.

Theorem 1.1. *Let* U, V *be open sets in* $\mathbb{R}^m, \mathbb{R}^n$*, respectively. If* $f : U \to V$ *is a diffeomorphism, at each point* $\mathbf{p} \in U$ *the differential* $\mathrm{D}f(\mathbf{p})$ *is invertible, so that necessarily* $m = n$.

Proof. See Gibson [159, p. 9].

The justification for $m = n$ is that it is not possible to have a diffeomorphism between open subspaces of Euclidean spaces of different dimensions [58, p. 41]. In fact, a famous theorem of algebraic topology (Brouwer's invariance of dimension) asserts that even a homeomorphism between open subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$, $m \neq n$, is impossible. This means that, for example, a point and a line cannot be homeomorphic (i.e. topologically equivalent) to each other because they have distinct dimensions.

Theorem 1.1 is very important not only to distinguish between two manifolds in the sense of differential geometry, but also to relate the invertibility of a diffeomorphism to the invertibility of the associated differential. More subtle is the hidden relationship between singularities and noninvertibility of the Jacobian. We should emphasise here that the direct inverse of Theorem 1.1 does not hold. However, there is a partial or local inverse, called the inverse mapping theorem, possibly one of the most important theorems in calculus. It is introduced in the next section, where we discuss the relationship between invertibility of mappings and smoothness of manifolds.

1.4 Invertibility and Smoothness

The smoothness of a submanifold that is the image of a mapping depends not only on smoothness but also the invertibility of its associated mapping. This section generalises such a relationship between smoothness and *invertibility* to mappings of several variables. This generalisation is known in mathematics as the *inverse mapping theorem*. This leads to a general mathematical theory for geometric continuity in geometric modelling, which encompasses not only parametric objects but also implicit ones. Therefore, this generalisation is *representation-independent*, i.e. no matter whether a submanifold is parametrically or implicitly represented.

Before proceeding, let us then briefly review the invertibility of mappings in the linear case.

Definition 1.2. *Let X, Y be Euclidean spaces, and $f : X \to Y$ a continuous linear mapping. One says that f is* ***invertible*** *if there exists a continuous linear mapping $g : Y \to X$ such that $g \circ f = id_X$ and $f \circ g = id_Y$ where id_X and id_Y denote the identity mappings of X and Y, respectively. Thus, by definition, we have:*

$$g(f(x)) = x \quad \text{and} \quad f(g(y)) = y$$

for every $x \in X$ and $y \in Y$. We write f^{-1} for the inverse of f.

But, unless we have an algorithm to evaluate whether or not a mapping is invertible, smoothness analysis of a point set is useless from the geometric modelling point of view. Fortunately, linear algebra can help us at this point. Consider the particular case $f : \mathbb{R}^n \to \mathbb{R}^n$. The linear mapping f is represented by a matrix $A = [a_{ij}]$. It is known that f is invertible iff A is invertible (as a matrix), and the inverse of A, if it exists, is given by

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A$$

where $\operatorname{adj} A$ is a matrix whose components are polynomial functions of the components of A. In fact, the components of $\operatorname{adj} A$ are subdeterminants of A. Thus, A is invertible iff its determinant $\det A$ is not zero.

Now, we are in position to define invertibility for differential mappings.

Definition 1.3. *Let U be an open subset of X and $f : U \to Y$ be a C^1 mapping, where X, Y are Euclidean spaces. We say that f is* ***C^1-invertible on*** *U if the image of f is an open set V in Y, and if there is a C^1 mapping $g : V \to U$ such that f and g are inverse to each other, i.e.*

$$g(f(x)) = x \quad \text{and} \quad f(g(y)) = y$$

for all $x \in U$ and $y \in V$.

It is clear that f is C^0-invertible if the inverse mapping exists and is continuous. One says that f is C^r-invertible if f is itself C^r and its inverse mapping g is also C^r. In the linear case, we are interested in linear invertibility, which basically is the strongest requirement that we can make. From the theorem that states that a C^r mapping that is a C^1 diffeomorphism is also a C^r diffeomorphism (see Hirsch [190]), it turns out that if f is a C^1-invertible, and if f happens to be C^r, then its inverse mapping is also C^r. This is the reason why we emphasise C^1 at this point. However, a C^1 mapping with a continuous inverse is not necessarily C^1-invertible, as illustrated in the following example:

Example 1.4. Let $f : \mathbb{R} \to \mathbb{R}$ be the mapping $f(x) = x^3$. It is clear that f is infinitely differentiable. Besides, f is strictly increasing, and hence has an inverse mapping $g : \mathbb{R} \to \mathbb{R}$ given by $g(y) = y^{1/3}$. The inverse mapping g is continuous, but not differentiable, at 0.

Let us now see the behaviour of invertibility under composition. Let $f : U \to V$ and $g : V \to W$ be invertible C^r mappings, where V is the image of f and W is the image of g. It follows that $g \circ f$ and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$ are C^r-invertible, because we know that a composite of C^r mappings is also C^r.

Definition 1.5. *Let $f : X \to Y$ be a C^r mapping, and let $\mathbf{p} \in X$. One says that f is **locally C^r-invertible** at $\mathbf{p}$ if there exists an open subset U of X containing $\mathbf{p}$ such that f is C^r-invertible on U.*

This means that there is an open set V of Y and a C^r mapping $g : V \to U$ such that $f \circ g$ and $g \circ f$ are the corresponding identity mappings of V and U, respectively. Clearly, a composite of locally invertible mappings is locally invertible. Putting this differently, if $f : X \to Y$ and $g : Y \to Z$ are C^r mappings, with $f(\mathbf{p}) = \mathbf{q}$ for $\mathbf{p} \in U$, and f, g are locally C^r-invertible at $\mathbf{p}$, $\mathbf{q}$, respectively, then $g \circ f$ is locally C^r-invertible at $\mathbf{p}$.

In Example 1.4, we used the derivative as a test for invertibility of a real-valued function of one variable. That is, if the derivative does not vanish at a given point, then the inverse function exists, and we have a formula for its derivative. The inverse mapping theorem generalises this result to mappings, not just functions.

Theorem 1.6. (**Inverse Mapping Theorem**) *Let U be an open subset of $\mathbb{R}^m$, let $\mathbf{p} \in U$, and let $f : U \to \mathbb{R}^n$ be a C^1 mapping. If the derivative $\mathrm{D}f$ is invertible, f is* locally C^1-invertible *at $\mathbf{p}$. If f^{-1} is its local inverse, and $\mathbf{y} = f(\mathbf{x})$, then $\mathrm{J}f^{-1}(\mathbf{y}) = [\mathrm{J}f(\mathbf{x})]^{-1}$.*

Proof. See Boothby [58, p. 43].

This is equivalent to saying that there exists open neighbourhoods U, V of $\mathbf{p}, f(\mathbf{p})$, respectively, such that f maps U diffeomorphically onto V. Note that, by Theorem 1.1, $\mathbb{R}^m$ has the same dimension as the Euclidean space $\mathbb{R}^n$, that is, $m = n$.

Example 1.7. Let U be an open subset of $\mathbb{R}^2$ consisting of all pairs (r, θ), with $r > 0$ and arbitrary θ. Let $f : U \to V \subset \mathbb{R}^2$ be defined by $f(r, \theta) = (r\cos\theta, r\sin\theta)$, i.e. V represents a circle of radius r in $\mathbb{R}^2$. Then

$$\mathrm{J}f(r, \theta) = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

and

$$\det \mathrm{J}f(r, \theta) = r\cos^2\theta + r\sin^2\theta = r.$$

Thus, $\mathrm{J}f$ is invertible at every point, so that f is locally invertible at every point. The local coordinates f_1, f_2 are usually denoted by x, y so that we usually write

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta.$$

The local inverse can be defined for certain regions of Y. In fact, let V be the set of all pairs (x, y) such that $x > 0$ and $y > 0$. Then the inverse on V is given by

$$r = \sqrt{x^2 + y^2} \quad \text{and} \quad \theta = \arcsin \frac{y}{\sqrt{x^2 + y^2}}.$$

As an immediate consequence of the inverse mapping theorem, we have:

Corollary 1.8. *Let U be an open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}^n$. A necessary and sufficient condition for the C^r mapping f to be a C^r diffeomorphism from U to $f(U)$ is that it be one-to-one and $\mathrm{J}f$ be nonsingular at every point of U.*

Proof. Boothby [58, p. 46].

Thus, diffeomorphisms have nonsingular Jacobians. This parallel between differential geometry and linear algebra makes us to think of an algorithm to check whether or not a C^r mapping is a C^r diffeomorphism. So, using computational differentiation techniques and matrix calculus, we are able to establish smoothness conditions on a submanifold of $\mathbb{R}^n$.

Note that the domain and codomain of the mappings used in Theorem 1.1, Theorem 1.6 and its Corollary 1.8 have the same dimension. This may suggest that only smooth mappings between spaces of the same dimension are C^r invertible. This is not the case. Otherwise, this would be useless, at least for geometric modelling. For example, a parametrised k-manifold in $\mathbb{R}^n$ is defined by the *image* of a parametrisation $f : \mathbb{R}^k \to \mathbb{R}^n$, with $k < n$. On the other hand, an implicit k-manifold is defined by the *level set* of a function $f : \mathbb{R}^k \to \mathbb{R}$, i.e. by an equation $f(\mathbf{x}) = c$, where c is a real constant.

1.5 Level Set, Image, and Graph of a Mapping

Let us then review the essential point sets associated with a mapping. This will help us to understand how a manifold or even a variety is defined, either implicitly, explicitly, or parametrically. Basically, we have three types of sets associated with any mapping $f : U \subset \mathbb{R}^m \to \mathbb{R}^n$ which play an important role in the study of manifolds and varieties: *level sets*, *images*, and *graphs*.

1.5.1 Mapping as a Parametrisation of Its Image

Definition 1.9. (Baxandall and Liebeck [35, p. 26]) *Let U be open in $\mathbb{R}^m$. The **image** of a mapping $f : U \subset \mathbb{R}^m \to \mathbb{R}^n$ is the subset of $\mathbb{R}^n$ given by*

$$\mathit{Image}\, f = \{\mathbf{y} \in \mathbb{R}^n \,|\, \mathbf{y} = f(\mathbf{x}),\ \forall \mathbf{x} \in U\},$$

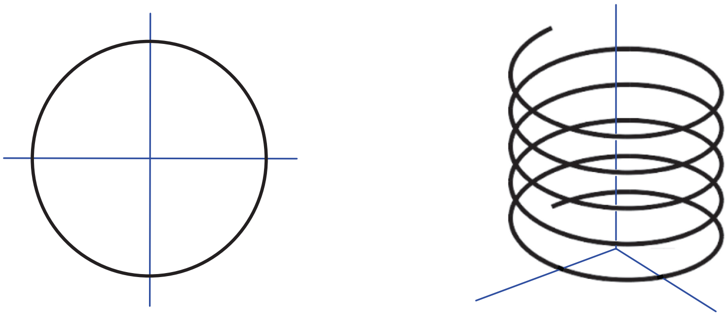
*being f a **parametrisation** of its image with parameters $(x_1, \ldots, x_m)$.*

This definition suggests that *practically any mapping is a "parametrisation" of something* [197, p. 263].

Example 1.10. The mapping $f : \mathbb{R} \to \mathbb{R}^2$ defined by $f(t) = (\cos t, \sin t)$, $t \in \mathbb{R}$, has an image that is the unit circle $x^2+y^2 = 1$ in $\mathbb{R}^2$ (Figure 1.1(a)). A distinct function with the same image as f is the mapping $g(t) = (\cos 2t, \sin 2t)$.

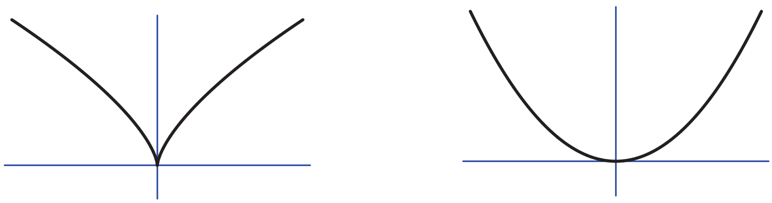
Example 1.10 suggests that two or more distinct mappings can have the same image. In fact, it can be proven that there is an infinity of different parametrisations of any nonempty subset of $\mathbb{R}^n$ [35, p. 29]. Free-form curves and surfaces used in geometric design are just images in $\mathbb{R}^3$ of some parametrisation $\mathbb{R}^1 \to \mathbb{R}^3$ or $\mathbb{R}^2 \to \mathbb{R}^3$, respectively. The fact that an image can be parametrised by several mappings poses some problems to meet smoothness conditions when we patch together distinct parametrised curves or surfaces, simply because it is not easy to find a global reparametrisation for a compound curve or surface. Besides, the *smoothness of the component functions that describe the image of a mapping does not guarantee smoothness for its image.*

Example 1.11. A typical example is the cuspidal cubic curve that is the image of a smooth mapping $f : \mathbb{R}^1 \to \mathbb{R}^2$ defined by $t \mapsto (t^3, t^2)$ which presents a cusp at $t = 0$, Figure 1.2(a). Thus, the cuspidal cubic is not a smooth curve.



(a) (b)

Fig. 1.1. (a) Image and (b) graph of $f(t) = (\cos t, \sin t)$.



(a) (b)

Fig. 1.2. (a) Cuspidal cubic $x^3 = y^2$ and (b) parabola $y = x^2$ as *images* of different parametrisations.

Conversely, the *smoothness of the image of a mapping does not imply that such a mapping is smooth.* The following example illustrates this situation.

Example 1.12. Let f, g and h be continuous mappings from $\mathbb{R}$ into $\mathbb{R}^2$ defined by the following rules:

$$f(t) = (t, t^2), \qquad g(t) = (t^3, t^6), \qquad \text{and} \qquad h(t) = \begin{cases} f(t), & t \geq 0, \\ g(t), & t < 0. \end{cases}$$

All three mappings have the same image, the parabola $y = x^2$ in $\mathbb{R}^2$, Figure 1.2(b). Their Jacobians are however distinct,

$$\mathrm{J}f(t) = [1 \quad 2t], \qquad \mathrm{J}g(t) = [3t^2 \quad 6t^5], \qquad \text{and} \qquad \mathrm{J}h(t) = \begin{cases} \mathrm{J}f(t), & t \geq 0, \\ \mathrm{J}g(t), & t < 0. \end{cases}$$

As polynomials, f, g are differentiable or smooth everywhere. Furthermore, because of $\mathrm{J}f(t) \neq [0 \quad 0]$ for any $t \in \mathbb{R}$, f is C^1-invertible everywhere. Consequently, its image is surely smooth. The function g is also smooth, but its Jacobian is null at $t = 0$, i.e. $\mathrm{J}g(0) = [0 \quad 0]$. This means that g is not C^1-invertible, or, equivalently, g has a singularity at $t = 0$, even though its image is smooth. Thus, a singularity of a mapping does not necessarily determine a singularity on its image. Even more striking is the fact that h is not differentiable at $t = 0$ (the left and right derivatives have different values at $t = 0$). This is so despite the smoothness of the image of h. This kind of situation where a smooth curve is formed by piecing together smooth curve patches is common in geometric design of free-form curves and surfaces used in industry.

The discussion above shows that every parametric smooth curve (in general, a manifold) can be described by several mappings, and that at least one of them is surely smooth and invertible, i.e. a diffeomorphism (see Corollary 1.8).

1.5.2 Level Set of a Mapping

Level sets of a mapping are varieties in some Euclidean space. That is, they are defined by equalities. Obviously, they are not necessarily smooth.

Definition 1.13. (Dineen [112, p. 6]) *Let U be open in $\mathbb{R}^m$. Let $f : U \subset \mathbb{R}^m \to \mathbb{R}^n$ and $\mathbf{c} = (c_1, \ldots, c_n)$ a point in $\mathbb{R}^n$. A* ***level set*** *of f, denoted by $f^{-1}(\mathbf{c})$, is defined by the formula*

$$f^{-1}(\mathbf{c}) = \{\mathbf{x} \in U \,|\, f(\mathbf{x}) = \mathbf{c}\}$$

In terms of coordinate functions $f_1, \ldots, f_n$ of f, we write

$$f(\mathbf{x}) = \mathbf{c} \Longleftrightarrow f_i(\mathbf{x}) = c_i \quad \text{for } i = 1, \ldots, n$$

and thus

$$f^{-1}(\mathbf{c}) = \bigcap_{i=1}^{n} \{\mathbf{x} \in U \mid f_i(\mathbf{x}) = c_i\} = \bigcap_{i=1}^{n} f_i^{-1}(c_i).$$

The smoothness criterion for a variety defined as a level set of a vector-valued function is given by the following theorem.

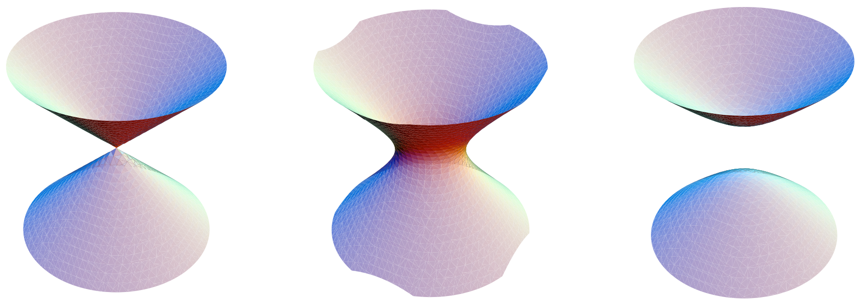
Theorem 1.14. (**Implicit Function Theorem**, Baxandall [35, p. 145]) *A set $X \subseteq \mathbb{R}^m$ is a smooth variety if it is a level set of a C^1 function $f : \mathbb{R}^m \to \mathbb{R}$ such that $\mathrm{J}f(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in X$.*

This theorem is a particular case of the implicit mapping theorem (IMT) for mappings which are functions. The IMT will be discussed later.

Example 1.15. The circle $x^2 + y^2 = 4$ is a variety in $\mathbb{R}^2$ that is a level set corresponding to the value 4 (i.e. point 4 in $\mathbb{R}$) of a function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = x^2 + y^2$. Its Jacobian is given by $\mathrm{J}f(x, y) = [2x \quad 2y]$ which is null at (0,0). However, the point $(0, 0)$ is not on the circle $x^2 + y^2 = 4$; hence the circle is a smooth curve.

Example 1.16. The sphere $x^2 + y^2 + z^2 = 9$ is a smooth surface in $\mathbb{R}^3$. It is the level set for the value 9 of a C^1 function $f : \mathbb{R}^3 \to \mathbb{R}$ defined by $f(x, y, z) = x^2 + y^2 + z^2$, and $\mathrm{J}f(x, y, z) \neq [0 \quad 0 \quad 0]$ at points on the sphere.

Example 1.17. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a function given by $f(x, y, z) = x^2 + y^2 - z^2$. Its level set corresponding to 0 is the right circular cone $z = \pm\sqrt{x^2 + y^2}$, whose apex is the point $(0, 0, 0)$ as illustrated in Figure 1.3(a). The Jacobian $\mathrm{J}f(x, y, z) = [2x \quad 2y \quad -2z]$ is null at the apex. Hence, the cone is not smooth at the apex, and the apex is said to be a singularity. Nevertheless, the level sets of the same function for which $x^2 + y^2 - z^2 = c \neq 0$ are smooth surfaces everywhere because the point $(0, 0, 0)$ is not on them. We have a hyperboloid of one sheet for $c > 0$ and a hyperboloid of two sheets for $c < 0$, as illustrated in Figure 1.3(b) and (c), respectively.



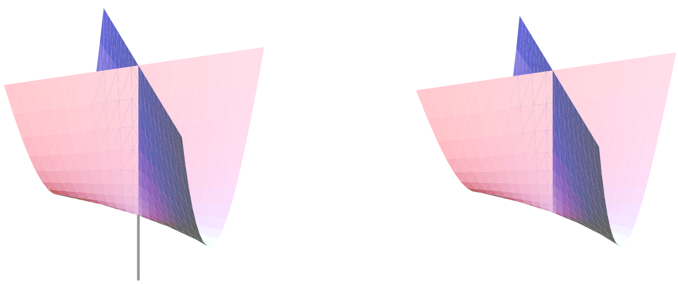
(a) (b) (c)

Fig. 1.3. (a) Cone $x^2 + y^2 - z^2 = 0$; (b) hyperboloid of one sheet $x^2 + y^2 - z^2 = a^2$; (c) hyperboloid of two sheets $x^2 + y^2 - z^2 = -a^2$.

Example 1.18. The Whitney umbrella with-handle $x^2 - zy^2 = 0$ in $\mathbb{R}^3$ (Figure 1.4) is not smooth. It is defined as the zero set of the function $f(x, y, z) = x^2 - zy^2$ whose Jacobian is $\mathrm{J}f(x, y, z) = [2x \quad -2yz \quad -y^2]$. It is easy to see that the Whitney umbrella is not smooth along the z-axis, i.e. the singular point set $\{(0, 0, z)\}$ where the Jacobian is zero. This singular point set is given by the intersection $\{2x = 0\} \cap \{-2yz = 0\} \cap \{-y^2 = 0\}$, which basically is the intersection of two planes, $\{x = 0\}$ and $\{y = 0\}$, i.e. the z-axis.

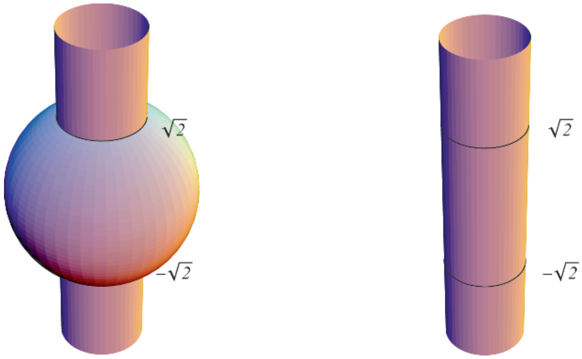
The smoothness criterion based on the Jacobian is valid for functions and can be generalised to mappings. In this case, we have to use the implicit mapping theorem given further on. Even so, let us see an example of a level set for a general mapping, not a function.

Example 1.19. Let $f(x, y, z) = (x^2 + y^2 + z^2 - 1, 2x^2 + 2y^2 - 1)$ a mapping $f : \mathbb{R}^3 \to \mathbb{R}^2$ with component functions $f_1(x, y, z) = x^2 + y^2 + z^2 - 1$ and $f_2(x, y, z) = 2x^2 + 2y^2 - 1$. The set $f_1^{-1}(0)$ is a sphere of radius 1 in $\mathbb{R}^3$ while $f_2^{-1}(0)$ is a cylinder parallel to the z-axis in $\mathbb{R}^3$ (Figure 1.5). If $\mathbf{0} = (0, 0)$ is



(a) (b)

Fig. 1.4. (a) Whitney umbrella with-handle $x^2 - zy^2 = 0$; (b) Whitney umbrella without-handle $\{x^2 - zy^2 = 0\} - \{z < 0\}$.



(a) (b)

Fig. 1.5. Two circles as the intersection of a cylinder and sphere in $\mathbb{R}^3$.

the origin in $\mathbb{R}^2$, the level set

$$f^{-1}(\mathbf{0}) = f^{-1}(0,0) = f_1^{-1}(0) \cap f_2^{-1}(0)$$

is the intersection of a sphere and a cylinder in $\mathbb{R}^3$. This intersection consists of two circles that can be obtained by solving the equations $f_1(x,y,z) = f_2(x,y,z) = 0$. Such circles are in the planes $z = \sqrt{2}$ and $z = -\sqrt{2}$.

Let us see now the role of the differentiability in the local structure of level sets defined by general mappings as in Example 1.19. As noted in [112, p. 11], by taking into account the linear approximation of differentiable functions and standard results on solving systems of linear equations, we start to recognise and accept that level sets are locally graphs.

Let $f : U \subset \mathbb{R}^m \to \mathbb{R}^n$, U an open subset of $\mathbb{R}^m$, $f = (f_1, \ldots, f_n)$, $\mathbf{c} = (c_1, \ldots, c_n)$. We assume that f is differentiable. Let us consider the level set $f^{-1}(\mathbf{c}) = \bigcap_{i=1}^n f_i^{-1}(c_i)$, i.e. the set whose points $(x_1, \ldots, x_m) \in U$ satisfy the equations

$$\begin{aligned} f_1(x_1, \ldots, x_m) &= c_1 \\ &\vdots \\ f_n(x_1, \ldots, x_m) &= c_n. \end{aligned} \tag{1.1}$$

We have m unknowns $(x_1, \ldots, x_m)$ and n equations. If each component function f_i is linear, we have a system of linear equations and the rank of the matrix gives us the number of linearly independent solutions, and information enough to identify a complete set of independent variables. The Implicit Mapping Theorem states that all this information can be *locally* obtained for differentiable mappings. This is due to the fact that differentiable mappings, by definition, enjoy a good local linear approximation.

If $\mathbf{p} \in f^{-1}(\mathbf{c})$, then $f(\mathbf{p}) = \mathbf{c}$. If $\mathbf{x} \in \mathbb{R}^n$ is close to zero, then, since f is differentiable, we have

$$f(\mathbf{p} + \mathbf{x}) = f(\mathbf{p}) + f'(\mathbf{p}).\mathbf{x} + \epsilon(\mathbf{x})$$

where $\epsilon(\mathbf{x}) \to 0$ when $\mathbf{x} \to 0$ (see Dineen [112, p. 3, p. 12]). Because we wish to find $\mathbf{x}$ close to $\mathbf{0}$ such that $f(\mathbf{p} + \mathbf{x}) = \mathbf{c}$, we are considering points such that

$$f'(\mathbf{p}).\mathbf{x} + \epsilon(\mathbf{x}) = \mathbf{0}$$

and thus $f'(\mathbf{p}).\mathbf{x} \approx \mathbf{0}$ (where $\approx$ means approximately equal). Let us assume that $m \geq n$. Therefore, not surprisingly, we have something very close to the following system of linear equations

$$\begin{aligned} \frac{\partial f_1}{\partial x_1}(\mathbf{p})x_1 + \cdots + \frac{\partial f_1}{\partial x_m}(\mathbf{p})x_m &= 0 \\ &\vdots \\ \frac{\partial f_n}{\partial x_1}(\mathbf{p})x_1 + \cdots + \frac{\partial f_n}{\partial x_m}(\mathbf{p})x_m &= 0, \end{aligned} \tag{1.2}$$

whose matrix is the Jacobian $\mathrm{J}f$.

From linear algebra we know that

$$\begin{aligned}\operatorname{rank}\mathrm{J}f = n &\Longleftrightarrow n \text{ rows of } \mathrm{J}f \text{ are linearly independent}\\ &\Longleftrightarrow n \text{ columns of } \mathrm{J}f \text{ are linearly independent}\\ &\Longleftrightarrow \mathrm{J}f \text{ contains } n \text{ columns, and the associated}\\ &\qquad n\times n \text{ matrix has nonzero determinant}\\ &\Longleftrightarrow \text{the space of solutions of the system (1.2)}\\ &\qquad \text{is } (m-n)\text{-dimensional.}\end{aligned} \tag{1.3}$$

Besides, if any of the conditions (1.3) are satisfied, and we select n columns that are linearly independent, then the variables concerning the remaining columns can be taken as a complete set of independent variables. If the conditions (1.3) are satisfied, we say that f has full or maximum rank at $\mathbf{p}$.

Example 1.20. Let us consider the following system of equations

$$\begin{aligned} 2x - y + z \qquad &= 0\\ y \qquad - w &= 0,\end{aligned}$$

whose matrix of coefficients is

$$\mathrm{A} = \begin{bmatrix} 2 & -1 & 1 & 0\\ 0 & 1 & 0 & -1\end{bmatrix}.$$

The submatrix

$$\begin{bmatrix} 2 & -1\\ 0 & 1\end{bmatrix}$$

is obtained by taking the first two columns from A, and has determinant $2 \neq 0$. Thus, A has rank 2, or, equivalently, the two rows are linearly independent. So, the two variables z, w in the remaining two columns can be taken as the independent variables. In other words, $y = w$, $2\,x = y - z = w - z$, and hence $\{(\frac{w-z}{2}, w, z, w) : z \in \mathbb{R}, w \in \mathbb{R}\}$ is the solution set. Alternatively, the solution set can be written in the following form

$$\{(g(z,w), z, w) : (z,w) \in \mathbb{R}^2\}$$

where $g(z,w) = (\frac{w-z}{2}, w)$ is a mapping $g : \mathbb{R}^2 \to \mathbb{R}^2$. In this format, the solution space is the graph of g (defined in the next subsection).

Assuming that the rows of $\mathrm{J}f(\mathbf{p})$ are linearly independent is equivalent to supposing that the gradient vectors $\{\nabla f_1(\mathbf{p}), \ldots, \nabla f_n(\mathbf{p})\}$ are linearly independent in $\mathbb{R}^m$. The implicit mapping theorem states that with this condition we can solve the nonlinear system of equations (1.1) near $\mathbf{p}$ and apply the same approach to identify a set of independent variables. The hypothesis of a good linear approximation in the definition of differentiable functions implies that the equation systems (1.1) and (1.2) are very close to one another [112, p. 13]. Roughly speaking, this linear approximation is the tangent space to the solution set defined by the at $\mathbf{p}$.

Theorem 1.21. (**Implicit Mapping Theorem**, Munkres [292]) *Let $f : U \subset \mathbb{R}^m \to \mathbb{R}^n$ $(m \geq n)$ be a differentiable mapping, let $\mathbf{p} \in U$ and assume that $f(\mathbf{p}) = \mathbf{c}$ and* $\operatorname{rank} \mathrm{J}f(\mathbf{p}) = n$. *For convenience, we also assume that the last n columns of the Jacobian are linearly independent. If $\mathbf{p} = (p_1, \ldots, p_m)$, let $\mathbf{p}_1 = (p_1, \ldots, p_{m-n})$ and $\mathbf{p}_2 = (p_{m-n+1}, \ldots, p_m)$ so that $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2)$. Then, there exists an open set $V \subset \mathbb{R}^{m-n}$ containing $\mathbf{p}_1$, a differentiable mapping $g : V \to \mathbb{R}^n$, an open subset $U' \subset U$ containing $\mathbf{p}$ such that $g(\mathbf{p}_1) = \mathbf{p}_2$ and*

$$f^{-1}(\mathbf{c}) \cap U' = \{(\mathbf{x}, g(\mathbf{x})) : \mathbf{x} \in V\} = \operatorname{graph} g.$$

Therefore, locally every level set is a graph.

1.5.3 Graph of a Mapping

Definition 1.22. (Dineen [112, p. 6]) *Let U be open in $\mathbb{R}^m$. The **graph** of a mapping $f : U \subset \mathbb{R}^m \to \mathbb{R}^n$ is the subset of the product space $\mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n$ defined by*

$$\operatorname{graph} f = \{(\mathbf{x}, \mathbf{y}) \,|\, \mathbf{x} \in U \textit{ and } \mathbf{y} = f(\mathbf{x})\}$$

or

$$\operatorname{graph} f = \{(\mathbf{x}, f(\mathbf{x})) \,|\, \mathbf{x} \in U\}.$$

Example 1.23. Let us consider both mappings $f(t) = (\cos t, \sin t)$ and $g(t) = (\cos 2t, \sin 2t)$ of Example 1.10. They have the same image in $\mathbb{R}^2$, say a unit circle. However, their graphs are distinct point sets in $\mathbb{R}^3$. The graph of f is a circular helix $(t, \cos t, \sin t)$ in $\mathbb{R}^3$, Figure 1.1(b). But, although the graph of g is a circular helix with windings being around the same circular cylinder, those windings have half the pitch.

This suggests that there is a one-to-one correspondence between a mapping and its graph, that different mappings have distinct graphs. This leads us to think of a possible relationship between the smoothness of a mapping and the smoothness of its graph. In other words, the smoothness of a mapping determines the smoothness of its graph. This is corroborated by the following theorem.

Theorem 1.24. (Baxandall [35, p. 147]) *The* graph *of a C^1 mapping $f : U \subseteq \mathbb{R}^m \to \mathbb{R}^n$ is a smooth variety in $\mathbb{R}^m \times \mathbb{R}^n$.*

Proof. Consider the mapping $F : U \times \mathbb{R}^n \subseteq \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$F(\mathbf{x}, \mathbf{y}) = f(\mathbf{x}) - \mathbf{y}, \quad \mathbf{x} \in U, \, \mathbf{y} \in \mathbb{R}^n.$$

The graph of f is the level set of F corresponding to the value $\mathbf{0}$, that is

$$\operatorname{graph} f = \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^m \times \mathbb{R}^n \,|\, f(\mathbf{x}) - \mathbf{y} = \mathbf{0}\}.$$

To prove that $\operatorname{graph} f$ is a smooth variety in $\mathbb{R}^m \times \mathbb{R}^n$ we show that:

(i) F is a C^1 mapping.
(ii) $J_F(\mathbf{x}, \mathbf{y}) \neq (\mathbf{0}, \mathbf{0})$ for all $\mathbf{x} \in U$, $\mathbf{y} \in \mathbb{R}^n$.

It follows from the definition of F above that for each $i = 1, \dots, m$, $j = m+1, \dots, m+n$ and each $\mathbf{x} \in U$, $\mathbf{y} \in \mathbb{R}^n$

$$\frac{\partial F}{\partial x_i}(\mathbf{x}, \mathbf{y}) = \frac{\partial f}{\partial x_i}(\mathbf{x}) \quad \text{and} \quad \frac{\partial F}{\partial y_j}(\mathbf{x}, \mathbf{y}) = -1.$$

Therefore the partial derivatives of F are continuous and so F is a C^1 mapping.

Also, for any $\mathbf{x} \in U$, $\mathbf{y} \in \mathbb{R}^n$

$$\mathrm{J}F(\mathbf{x}, \mathbf{y}) = (\mathrm{J}f(\mathbf{x}), -\mathbf{1}) \neq (\mathbf{0}, \mathbf{0}).$$

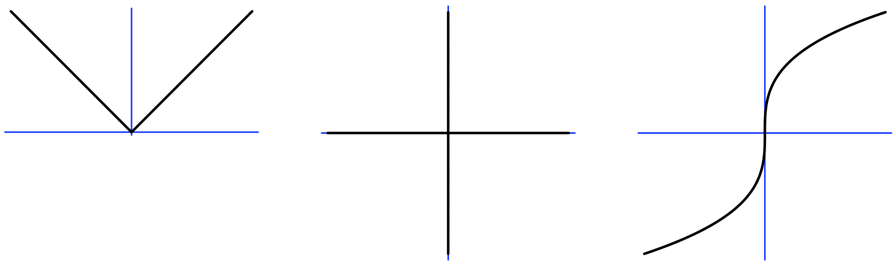
This completes the proof.

Example 1.25. Let us consider the curves sketched in Figure 1.6. Figure 1.6(a) shows the curve $y = |x|$ in $\mathbb{R}^2$ that is not smooth. It is the graph of the function $f : \mathbb{R} \to \mathbb{R}$ that explicitly expresses y as a function of x, but f is not differentiable at $x = 0$. Nor is it the graph of (an inverse) function g expressing x as a function of y, because in the neighbourhood of $(0, 0)$ the same value of y corresponds to two values of x.

Figure 1.6(b) shows another nonsmooth curve $xy = 0$ in $\mathbb{R}^2$, which is the union of the two coordinate axes, x and y. Any neighbourhood of $(0, 0)$ contains infinitely many y values corresponding to $x = 0$, and infinitely many x values corresponding to $y = 0$. This means that the curve is not a graph of an explicit function $y = f(x)$, nor of a function $x = g(y)$. Incidentally, this curve can be regarded as a slice at $z = 0$ through the graph of $h : \mathbb{R}^2 \to \mathbb{R}$ where $h(x, y) = xy$, which defines the implicit curve $h(x, y)$ in $\mathbb{R}^2$.

Finally, the graph of the function $f(x) = x^{1/3}$, depicted in Figure 1.6(c), is a smooth curve. Note that the curve is smooth despite the function being not differentiable at $x = 0$. This happens because the curve is the graph of the function $x = f(y) = y^3$ that is differentiable.

From these examples, we come to the following conclusions:



(a) (b) (c)

Fig. 1.6. Not all point sets in $\mathbb{R}^2$ are graphs of a mapping.

- Rewording Theorem 1.24, every point set that is the graph of a differentiable mapping is smooth.
- The fact that a mapping is not differentiable does not imply that its graph is not smooth; but if the graph is smooth, then it is necessarily the graph of a related function by changing the roles of the variables, possibly the inverse function. This is the case for the curve $x = y^3$ in Figure 1.6(c).
- The graph of a mapping that is not differentiable is possibly nonsmooth. This happens because of the differentiable singularities such as the cusp point in $y = |x|$, Figure 1.6.
- There are point sets in $\mathbb{R}^n$ that cannot be described as graphs of mappings, unless we break them up into pieces. For example, with appropriate constraints we can split $xy = 0$ (the union of axes in $\mathbb{R}^2$) into the origin and four half-axes, each piece described by a function. The origin is a cut point of $xy = 0$, that is, a topological singularity. The idea of partitioning a point set into smaller point sets by its topological singularities leads to a particular sort of stratification as briefly detailed in the next chapter. Another alternative to describe a point set that is not describable by a graph of a function is to describe it as a level set of a function.

The relationship between graphs and level sets plays an important role in the study of varieties. It is easy to see that *every graph is a level set.* Let us consider a mapping $f : U \subseteq \mathbb{R}^m \to \mathbb{R}^n$. We define $F : U \times \mathbb{R}^m \to \mathbb{R}^n$ by $F(\mathbf{x}, \mathbf{y}) = f(\mathbf{x}) - \mathbf{y}$. If $\mathbf{0}$ is the origin in $\mathbb{R}^n$, we have

$$\begin{aligned}(\mathbf{x}, \mathbf{y}) \in F^{-1}(\mathbf{0}) &\Longleftrightarrow F(\mathbf{x}, \mathbf{y}) = \mathbf{0} \\ &\Longleftrightarrow f(\mathbf{x}) - \mathbf{y} = \mathbf{0} \\ &\Longleftrightarrow (\mathbf{x}, \mathbf{y}) \in \operatorname{graph} f.\end{aligned}$$

Thus, $F^{-1}(\mathbf{0}) = \operatorname{graph} f$ and every graph is a level set. This fact has been used to prove the Theorem 1.24. As a summary, we can say that:

- Not all varieties in some Euclidean space are graphs of a mapping.
- Every variety as a graph of a mapping is a level set.
- Every variety is a level set of a mapping.

This shows us why the study of algebraic and analytic varieties in geometry is carried out using level sets of mappings, i.e. point sets defined implicitly. The reason is a bigger geometric coverage of point sets in some Euclidean space. In addition to this, many (not necessarily smooth) varieties admit a global parametrisation, whilst others can only be partially (locally) and piecewise parametrised.

Example 1.26. Let $z = x^2 - y^2$ be a level set of a function $F : \mathbb{R}^3 \to \mathbb{R}$ defined by $F(x, y, z) = x^2 - y^2 - z$ corresponding to the value 0. It is observed that $\mathrm{J}F(x, y, z) = [2x \quad -2y \quad -1]$ is not zero everywhere. So $z = x^2 - y^2$ in $\mathbb{R}^3$ is smooth everywhere. It is a variety known as a *saddle surface*. Note that z is explicitly defined in terms of x and y. So, the saddle surface can be viewed as the graph of the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = x^2 - y^2$. Consequently, the saddle surface can be given a global parametrisation $g : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $g(x, y) = (x, y, x^2 - y^2)$.

Not all varieties can be *globally* parametrised, even when they are smooth. But, as proved later, every smooth level set can be always *locally* parametrised, i.e. every smooth level set is locally a graph. This fact is proved by the implicit mapping theorem.

Level sets correspond to implicit representations, say functions, on some Euclidean space, while graphs correspond to explicit representations. In fact, we have from calculus that

Definition 1.27. (Baxandall and Liebeck [35, p. 226]) *Let $f : X \subseteq \mathbb{R}^m \to \mathbb{R}$ be a function, where $m \geq 2$. If there exists a function $g : Y \subseteq \mathbb{R}^{m-1} \to \mathbb{R}$ such that for all $(x_1, \ldots, x_{m-1}) \in Y$,*

$$f(x_1, \ldots, x_{m-1}, g(x_1, \ldots, x_{m-1})) = 0,$$

*then the function g is said to be defined **implicitly** on Y by the equation*

$$f(x_1, \ldots, x_m) = 0.$$

Likewise, the graph of $g : Y \subseteq \mathbb{R}^{m-1} \to \mathbb{R}$ is the subset of $\mathbb{R}^m$ given by

$$\{(x_1, \ldots, x_{m-1}, x_m) \in \mathbb{R}^m | \, x_m = g(x_1, \ldots, x_{m-1})\}.$$

The expression $x_m = g(\mathbf{x})$ is called the equation of the graph [35, p.100]. *Hence, g is said to be **explicitly** defined on Y by the equation $x_m = g(x_1, \ldots, x_{m-1})$.*

Example 1.28. The graph of the function $f(x, y) = -x^2 - y^2$ has equation $-z = x^2 + y^2$. This graph is a 2-manifold in $\mathbb{R}^3$ called a paraboloid (Figure 1.7). The equation $-z = x^2 + y^2$ *explicitly* defines the paraboloid in $\mathbb{R}^3$.

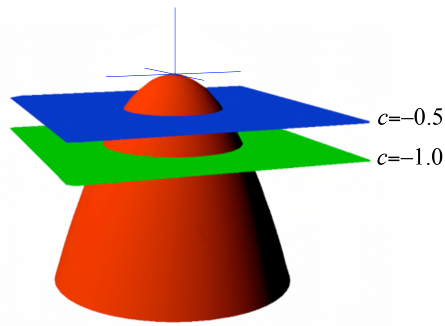


Fig. 1.7. The paraboloid $-z = x^2 + y^2$ in $\mathbb{R}^3$.

For $c < 0$ the plane $z = c$ intersects the graph in a circle lying below the level set $x^2 + y^2 = -c$ in the (x, y)-plane. The equation $x^2 + y^2 = -c$ of a circle (i.e. a 1-manifold) in $\mathbb{R}^2$ is said to define y *implicitly* in terms of x. This circle is said to be an implicit 1-manifold.

1.6 Rank-based Smoothness

Now, we are in position to show that the rank of a mapping gives us a general approach to check the C^r invertibility or C^r smoothness of a mapping, and whether or not a variety is smooth. This smoothness test is carried out independently of how a variety is defined, implicitly, explicitly or parametrically, i.e. no matter whether a variety is considered a level set, a graph, or an image of a mapping, respectively.

Definition 1.29. (Olver [313, p. 11]) *The **rank** of a mapping $f : \mathbb{R}^m \to \mathbb{R}^n$ at a point $\mathbf{p} \in \mathbb{R}^m$ is defined to be the rank of the $n \times m$ Jacobian matrix Jf of any local coordinate expression for f at the point $\mathbf{p}$. The mapping f is called **regular** if its rank is constant.*

Standard transformation properties of the $\mathrm{J}f$ imply that the definition of rank is independent of the choice of local coordinates [313, p. 11] (see [58, p. 110] for a proof). Moreover, the rank of the Jacobian matrix (shortly $\operatorname{rank} \mathrm{J}f$) provides us with a general algebraic procedure to check the smoothness of a submanifold or, putting it differently, to determine its singularities. It is proved in differential geometry that the set of points where the rank of f is maximal is an open submanifold of the manifold $\mathbb{R}^m$ (which is dense if f is analytic), and the restriction of f to this subset is regular. The subsets where the rank of a mapping decreases are *singularities* [313, p. 11]. The types and properties of such singularities are studied in singularity theory.

From linear algebra we have

$$\begin{aligned} \operatorname{rank} \mathrm{J}f = k &\Longleftrightarrow k \text{ rows of } \mathrm{J}f \text{ are linearly independent} \\ &\Longleftrightarrow k \text{ columns of } \mathrm{J}f \text{ are linearly independent} \\ &\Longleftrightarrow \mathrm{J}f \text{ has a } k \times k \text{ submatrix that has nonzero determinant.} \end{aligned}$$

The fact that the $n \times m$ Jacobian matrix $\mathrm{J}f$ has rank k means that it includes a $k \times k$ submatrix that is invertible. Thus, a necessary and sufficient condition for a k-variety to be smooth is that $\operatorname{rank} \mathrm{J}f = k$ at every point of it, no matter whether it is defined parametrically or implicitly by f. This is clearly a generalisation of Corollary 1.8, and is a consequence of a generalisation of the inverse mapping theorem, called the rank theorem:

Theorem 1.30. (**Rank Theorem**) *Let $U \subset \mathbb{R}^m$, $V \subset \mathbb{R}^n$ be open sets, $f : U \to V$ be a C^r mapping, and suppose that* $\operatorname{rank} \mathrm{J}f = k$. *If $\mathbf{p} \in U$ and*

$\mathbf{q} = f(\mathbf{p})$, *there exists open sets* $U_0 \subset U$ *and* $V_0 \subset V$ *with* $\mathbf{p} \in U_0$ *and* $\mathbf{q} \in V_0$, *and there exists* C^r *diffeomorphisms*

$$\phi : U_0 \to X \subset \mathbb{R}^m,$$

$$\psi : V_0 \to Y \subset \mathbb{R}^n$$

with X, Y *open in* $\mathbb{R}^m, \mathbb{R}^n$, *respectively, such that*

$$\psi \circ f \circ \phi^{-1}(X) \subset Y$$

and such that this mapping has the simple form

$$\psi \circ f \circ \phi^{-1}(p_1, \ldots, p_m) = (p_1, \ldots, p_k, 0, \ldots, 0).$$

Proof. See Boothby [58, p. 47].

This is a very important theorem because it states that a mapping of constant rank k behaves *locally* as a projection of $\mathbb{R}^m = \mathbb{R}^k \times \mathbb{R}^{m-k}$ to $\mathbb{R}^k$ followed by injection of $\mathbb{R}^k$ onto $\mathbb{R}^k \times \{0\} \subset \mathbb{R}^k \times \mathbb{R}^{n-k} = \mathbb{R}^n$.

1.6.1 Rank-based Smoothness for Parametrisations

The rank theorem for parametrisations is as follows:

Theorem 1.31. (**Rank Theorem for Parametrisations**) *Let* U *be an open set in* $\mathbb{R}^m$ *and* $f : U \to \mathbb{R}^n$. *A necessary and sufficient condition for the* C^∞ *mapping* f *to be a diffeomorphism from* U *to* $f(U)$ *is that it be one-to-one and the Jacobian* $\mathrm{J}f$ *have rank* m *at every point of* U.

Proof. See Boothby [58, p. 46].

This is a generalisation of Corollary 1.8, with $m \leq n$. It means that the kernel[3] of the linear mapping represented by $\mathrm{J}f$ is 0 precisely when the Jacobian matrix has rank m.

Let us review some simple examples of parametrised curves.

Example 1.32. We know that the bent curve in $\mathbb{R}^2$ depicted in Figure 1.6 and defined by the parametrisation $f(t) = (t, |t|)$ is not differentiable at $t = 0$, even though its rank is 1 everywhere.

Example 1.32 shows that the differentiability test should always precede the rank test in order to detect differentiable singularities.

[3] Let $F : X \to Y$ be a linear mapping of vector spaces. By the *kernel* of F, denoted by $\operatorname{kernel} F$, is meant the set of all those vectors $\mathbf{x} \in X$such that $F(V) = \mathbf{0} \in Y$, i.e. $\operatorname{kernel} F = \{\mathbf{x} \in X : F(\mathbf{x}) = \mathbf{0}\}$ (see Edwards [128, p. 29]). In other words, the kernel of a linear mapping corresponds to the level set of a mapping.

Example 1.33. A parametrised curve that passes the differentiability test, but not the rank test, is the cuspidal cubic in $\mathbb{R}^2$ given by $f(t) = (t^3, t^2)$ (Figure 1.2(a)). The component functions are polynomials and therefore differentiable. However, the rank $\mathrm{J}f(t) = [3t^2 \quad 2t]$ is not 1 (i.e. its maximal value) at $t = 0$; in fact it is zero. This means that the parametrised cuspidal cubic is not smooth at $t = 0$, that is, it possesses a singularity at $t = 0$.

Example 1.34. Let us take the parametrised parabola in $\mathbb{R}^2$ given by $f(t) = (t, t^2)$ (Figure 1.2(b)). f is obviously differentiable, and its rank is 1 everywhere, so it is globally smooth.

Nevertheless, algorithmic detection of singularities of a parametrised variety fails for self-intersections, i.e. topological singularities. Let us see some examples.

Example 1.35. The curve parametrised by the differentiable mapping $f(t) = (t^3 - 3t - 2, t^2 - t - 2)$ is not smooth at $(0, 0)$, despite the differentiability of f and its maximal rank. In fact, we get the same point $(0, 0)$ on the curve for two distinct points $t = -1$ and $t = 2$ of the domain, that is, $f(-1) = f(2) = (0, 0)$, and thus f is not one-to-one. These singularities are known as *self-intersections* in geometry or *topological singularities* in topology.

The problem with a parametrised self-intersecting variety is that its self-intersections are topological singularities for the corresponding underlying topological space, but not for the parametrisation. However, it is an easy task to check whether a non-self-intersecting point in a parametrised variety is singular or not. A non-self-intersecting point is singular if the rank of Jacobian at this point is not maximal.

Example 1.36. Let us consider a parametrisation $f(u, v) = (uv, u, v^2)$ of the Whitney umbrella without-handle (the negative z-axis) (Figure 1.4(b)). The effect of this parametrisation on $\mathbb{R}^2$ can be described as the 'fold' of the v-axis at the origin $(0, 0)$ in order to superimpose negative v-axis and positive v-axis. The 'fold' is identified by the exponent 2 of the third component coordinate function. Thus, all points $(0, 0, v^2)$ along v-axis are double points and determine that all points on the positive z-axis are singularities or self-intersecting points in $\mathbb{R}^3$. However, this is not so apparent if we restrict the discussion to the Jacobian and try to determine where the rank drops below 2. In fact,

$$\mathrm{J}f(u, v) = \begin{bmatrix} v & u \\ 1 & 0 \\ 0 & 2v \end{bmatrix}$$

and we observe that the rank drops below 2 only at $(0, 0)$. This happens because only $(0, 0)$ is a differential singularity, that is, the tangent plane is not defined at $(0, 0)$. Any other point on the positive z-axis has a parametrised neighbourhood that can be approximated by a tangent plane in relation to the parametrisation.

Example 1.37. Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be the mapping given by

$$f(x, y) = (\sin x,\ e^x \cos y,\ \sin y).$$

Then

$$\mathrm{J}f(x,y) = \begin{bmatrix} \cos x & 0 \\ e^x \cos y & -e^x \sin y \\ 0 & \cos y \end{bmatrix}$$

and hence

$$\mathrm{J}f(0,0) = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

has rank 2, so that in a neighbourhood of $(0, 0)$, the mapping f parametrises a subset of $\mathbb{R}^3$.

1.6.2 Rank-based Smoothness for Implicitations

The implicit function theorem is particularly useful for geometric modelling because it provides us with a computational tool to test whether an implicit manifold, and more generally a variety, is smooth in the neighbourhood of a point. Specifically, it gives us a local parametrisation for which it is possible to check the local C^r-invertibility by means of its Jacobian.

Before proceeding, let us see how C^r-invertibility and smoothness is defined for implicit manifolds and varieties.

Theorem 1.38. (**Rank Theorem for Implicitations**) *Let U be open in $\mathbb{R}^m$ and let $f : U \to \mathbb{R}$ be a C^r function on U. Let $(\mathbf{p}, q) = (p_1, \ldots, p_{m-1}, q) \in U$ and assume that $f(\mathbf{p}, q) = 0$ but $\frac{\partial f}{\partial x_m}(\mathbf{p}, q) \neq 0$. Then the mapping*

$$F : U \to \mathbb{R}^{m-1} \times \mathbb{R} = \mathbb{R}^m$$

given by

$$(\mathbf{x}, y) \mapsto (\mathbf{x}, f(\mathbf{x}, y))$$

is locally C^r-invertible at $(\mathbf{p}, q)$.

Proof. (See Lang [223, p.523]). All we need to do is to compute the derivative of F at $(\mathbf{p}, q)$. We write F in terms of its coordinates, $F = (F_1, \ldots, F_{m-1}, F_m) = (x_1, \ldots, x_{m-1}, f)$. Its Jacobian matrix is therefore

$$\mathrm{J}F(\mathbf{x}) = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & \ldots 1 & 0 \\ \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \ldots & \frac{\partial f}{\partial x_m} \end{bmatrix}$$

and is invertible since its determinant is equal to $\frac{\partial f}{\partial x_m} \neq 0$ at $(\mathbf{p}, q)$. The inverse function theorem guarantees that F is locally C^r-invertible at $(\mathbf{p}, q)$.

As a corollary of this Theorem, we have the implicit function theorem for functions of several variables, which can be reworded as follows:

Theorem 1.39. (**Multivariate Implicit Function Theorem**) *Let U be open in $\mathbb{R}^m$ and let $f : U \to \mathbb{R}$ be a C^r function on U. Let $(\mathbf{p}, q) = (p_1, \ldots, p_{m-1}, q) \in U$ and assume that $f(\mathbf{p}, q) = 0$ but $\frac{\partial f}{\partial x_m}(\mathbf{p}, q) \neq 0$. Then there exists an open ball V in $\mathbb{R}^{m-1}$ centred at $\mathbf{p}$ and a C^r function*

$$g : V \to \mathbb{R}$$

such that $g(\mathbf{p}) = q$ and

$$f(\mathbf{x}, g(\mathbf{x})) = 0$$

for all $\mathbf{x} \in V$.

Proof. (See Lang [223, p. 524]). By Theorem 1.38 we know that the mapping

$$F : U \to \mathbb{R}^{m-1} \times \mathbb{R} = \mathbb{R}^m$$

given by

$$(\mathbf{x}, y) \mapsto (\mathbf{x}, f(\mathbf{x}, y))$$

is locally C^r-invertible at $(\mathbf{p}, q)$. Let $F^{-1} = (F_1^{-1}, \ldots, F_m^{-1})$ be the local inverse of F such that

$$F^{-1}(\mathbf{x}, z) = (\mathbf{x}, F_m^{-1}(\mathbf{x}, z)) \quad \text{for} \quad \mathbf{x} \in \mathbb{R}^{m-1},\ z \in \mathbb{R}.$$

We let $g(\mathbf{x}) = F_m^{-1}(\mathbf{x}, 0)$. Since $F(\mathbf{p}, q) = (\mathbf{p}, 0)$ it follows that $F_m^{-1}(\mathbf{p}, 0) = q$ so that $g(\mathbf{p}) = q$. Furthermore, since F, F^{-1} are inverse mappings, we obtain

$$(\mathbf{x}, 0) = F(F^{-1}(\mathbf{x}, 0)) = F(\mathbf{x}, g(\mathbf{x})) = (\mathbf{x}, f(\mathbf{x}, g(\mathbf{x}))).$$

This proves that $f(\mathbf{x}, g(\mathbf{x})) = 0$, as shown by previous equality.

Note that we have expressed y as a function of $\mathbf{x}$ explicitly by means of g, starting with what is regarded as an implicit relation $f(\mathbf{x}, y) = 0$. Besides, from the implicit function theorem, we see that the mapping G given by

$$\mathbf{x} \mapsto (\mathbf{x}, g(\mathbf{x})) = G(\mathbf{x})$$

or writing down the coordinates

$$(x_1, \ldots, x_{m-1}) \mapsto (x_1, \ldots, x_{m-1}, g(x_1, \ldots, x_{m-1}))$$

provides a *parametrisation* of the variety defined by $f(x_1, \ldots, x_{m-1}, y) = 0$ in the neighbourhood of a given point $(\mathbf{p}, q)$. This is illustrated in Figure 1.8 for convenience. On the right, we have the surface $f(\mathbf{x}) = 0$, and we have also pictured the gradient $\operatorname{grad} f(\mathbf{p}, q)$ at the point $(\mathbf{p}, q)$ as in Theorem 1.39. Note that the condition $\frac{\partial f}{\partial x_m}(\mathbf{p}, q) \neq 0$ in Theorem 1.39 implies that the $\operatorname{grad} f(\mathbf{p}, q) = [\frac{\partial f}{\partial x_1} \quad \frac{\partial f}{\partial x_2} \quad \ldots \quad \frac{\partial f}{\partial x_m}] \neq 0$.

An example follows to illustrate the implicit function theorem at work.

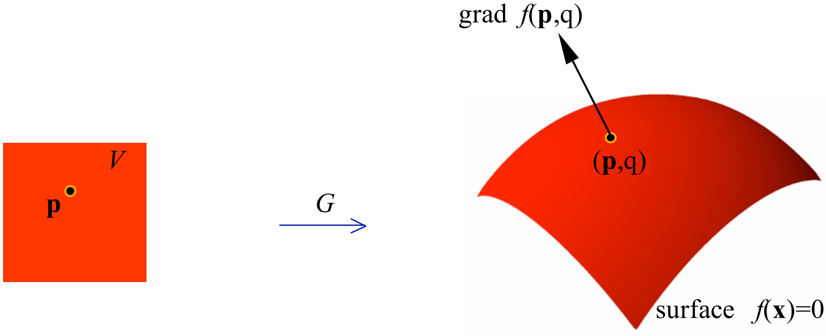


Fig. 1.8. Local parametrisation of an implicitly defined variety.

Example 1.40. The Whitney umbrella $x^2 - zy^2 = 0$ in $\mathbb{R}^3$ is the level set for the value 0 of the function $f : \mathbb{R}^3 \to \mathbb{R}$ given by $f(x, y, z) = x^2 - zy^2$. According to the Theorem 1.39, we have only to make sure that $\frac{\partial f}{\partial z} \neq 0$ in order to guarantee a regular neighbourhood for a point. But

$$\frac{\partial f}{\partial z} = -y^2 = 0 \quad \Rightarrow \quad y = 0$$

i.e. all points of $x^2 - zy^2 = 0$ with $y = 0$ are singular points. These singular points are then given by

$$\begin{cases} y = 0 \\ x^2 - zy^2 = 0 \end{cases} \Leftrightarrow \begin{cases} y = 0 \\ x = 0 \end{cases} \Leftrightarrow \{x = 0\} \cap \{y = 0\}$$

or, equivalently, the point set $\{(x, y, z) \in \mathbb{R}^3 : x = 0, y = 0\}$. That is, the singular set of the Whitney umbrella is the z-axis $0 \times 0 \times z$.

This result agrees with the fact that the Jacobian $\mathrm{J}\, f = [2x \quad 2yz \quad y^2]$ has maximal rank 1 for $(x, y, z) \neq (0, 0, z)$. However, because the rank cannot fall below zero, we have no way to algorithmically detect via rank criterion any possible singularities in the z-axis. In fact, the z-axis is a smooth line, but we know that the origin is a special singularity of the Whitney umbrella provided that, unlike the points of the positive z-axis, it is a cut-point.[4]

The question now is whether or not there is any method to compute such singularities. An algorithm to determine the singularities of a variety is useful for many geometry software packages. For example, the graphical visualisation of the Whitney umbrella with-handle $x^2 - zy^2 = 0$ in $\mathbb{R}^3$ requires the detection of its singular set along the z-axis. Therefore, unless we use a parametric Whitney umbrella without-handle, such a point set cannot be visualised on

[4] In topology, a point of a connected space is a cut-point if its removal makes its space disconnected. For example, every point of a straight line is a cut-point because it splits the line into two; the same is not true for any circle point.

a display screen. This is an example amongst others that shows how much a stratification algorithm of varieties can be useful.

Amongst other applications of implicit function theorem, we can mention two:

- To prove the existence of smooth curves passing through a point on a surface [223, p. 525].
- To state the smoothness conditions when an implicit surface and a parametric surface are stitched along an edge.

The first refers a theorem of major importance because it allows the study of smoothness of higher-dimensional submanifolds via, for example, Taylor or Frénet approximations. The second is also important because it makes it possible to avoid the conversion of an implicit surface patch to its parametric representation, or vice-versa. So, in principle, it is possible to design a smooth surface composed of parametric and implicit patches.

1.7 Submanifolds

By definition, a submanifold is a subset of a manifold that is a manifold in its own right. In geometric modelling, manifolds are usually Euclidean spaces, and submanifolds are points, curves, surfaces, etc. in some Euclidean space of equal or higher dimension. Manifolds and varieties in an Euclidean space are usually defined by either the image, level set or graph associated with a mapping.

1.7.1 Parametric Submanifolds

As shown in previous sections, the smoothness characterisation of a submanifold clearly depends on its defining smooth mapping and its rank. We have seen that the notion of smooth mapping of constant rank leads to the definition of *smooth* submanifolds. In this respect, the rank theorem, and ultimately, the inverse function theorem, can be considered as the major milestones in the theory of smooth submanifolds. Notably, the smoothness of a mapping does not ensure the smoothness of a submanifold. In fact, not all smooth submanifolds, say parametric smooth submanifolds, can be considered as topological submanifolds, i.e. submanifolds equipped with the submanifold topology.

Extreme cases of mappings $f : M \to N$ of constant rank are those corresponding to maximal rank, that is, the rank is the same as the dimension of M or N.

Definition 1.41. *Let $f : M \to N$ be a smooth mapping with constant rank. Then, for all $\mathbf{p} \in M$, f is called:*

$$\begin{aligned} &\textit{an } \textbf{immersion} \quad \textit{if} \quad \textit{rank}\, f = \textit{dim}\, M, \\ &\textit{a } \textbf{submersion} \quad \textit{if} \quad \textit{rank}\, f = \textit{dim}\, N. \end{aligned}$$

Let us now concentrate on immersions, that is, mappings whose images are parametric submanifolds. To say that $f : M \to N$ is an immersion means that the differential $\mathrm{D}\, f(\mathbf{p})$ is injective at every point $\mathbf{p} \in M$. This is the same as saying that the Jacobian matrix of f has rank equal to $\dim M$ (which is only possible if $\dim M \leq \dim N$). Then by the rank theorem, we have

Corollary 1.42. *Let M, N be two manifolds of dimensions m, n, respectively, and $f : M \to N$ a smooth mapping. The mapping f is an* immersion *if and only if for each point $\mathbf{p} \in M$ there are coordinate systems (U, φ), (V, ψ) about $\mathbf{p}$ and $f(\mathbf{p})$, respectively, such that the composite $\psi\, f\, \varphi^{-1}$ is a restriction of the coordinate inclusion $\iota : \mathbb{R}^m \to \mathbb{R}^m \times \mathbb{R}^{n-m}$.*

Proof. See Sharpe [360, p. 15].

This corollary provides the canonical form for immersed submanifolds:

$$(x_1, \ldots, x_m) \mapsto (x_1, \ldots, x_m, 0, \ldots, 0).$$

Definition 1.43. *A smooth (analytic) m-dimensional* ***immersed submanifold*** *of a manifold N is a subset $M' \subset N$ parametrised by a smooth (analytic), one-to-one mapping $f : M \to M' \subset N$, whose domain M, the parameter space, is a smooth (analytic) m-dimensional manifold, and such that f is everywhere regular, of maximal rank m.*

Thus, an m-dimensional immersed submanifold M' is the *image* of an immersion $f : M \to M' = f(M)$. To verify that f is an immersion it is necessary to check that the Jacobian has rank m at every point. Observe that an immersed submanifold is defined by a *parametrisation*. Thus, an immersed submanifold is nothing more than a *parametrically defined submanifold*, or simply a **parametric submanifold**. Despite its smoothness, an immersed or parametric submanifold may include self-intersections. A submanifold with self-intersections is the image $M' = f(M)$ of an arbitrary regular mapping $f : M \to M' \subset N$ of maximal rank m, which is the dimension of the parameter space M. Examples of parametric submanifolds with self-intersections such as Bézier curves and surfaces are often found in geometric design activities. Immersed submanifolds constitute the largest family of parametric submanifolds. It includes the subfamily of parametric submanifolds without self-intersections, also known as *parametric embedded submanifolds.*

Definition 1.44. *An* ***embedding*** *is a one-to-one immersion $f : M \to N$ such that the mapping $f : M \to f(M)$ is a homeomorphism (where the topology on $f(M)$ is the subspace topology inherited from N). The image of an embedding is called an* ***embedded submanifold****.*

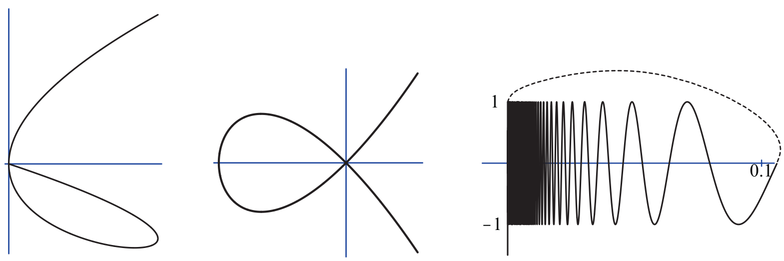
In other words, the topological type is invariant for any point of an embedded submanifold. This is why embedded submanifolds are often called simply submanifolds. Obviously, $f : M \to N$ considered as a smooth mapping is called an embedding if $f(M) \subset N$ is a smooth manifold and $f : M \to f(M)$ is a diffeomorphism [65, p. 10].

Parametric immersed submanifolds have been mainly used in computer-aided geometric design (CAGD) of parametric curves and surfaces, while embedded submanifolds are preferably used as "building blocks" of solids in solid geometric modelling, which usually embody mechanical parts and other engineering artifacts. This means that an eventual computational integration of these two research areas of geometric modelling becomes mandatory to reconcile immersed and embedded submanifolds.

Let us see first some examples of 1-dimensional immersed submanifolds that are not embedded manifolds.

Example 1.45. Let $f : \mathbb{R} \to \mathbb{R}^2$ an immersion given by $f(t) = (\cos 2\pi t, \sin 2\pi t)$. Its image $f(\mathbb{R})$ is the unit circle $\mathbb{S}^1 = \{(x, y) \,|\, x^2 + y^2 = 1\}$ in $\mathbb{R}^2$. This shows that an immersion need not be one-to-one into (injective) in the large, even though it is one-to-one locally. In fact, for example, all the points $t = 0, \pm 1, \pm 2, \ldots$ have the same image point $(0, 1)$ in $\mathbb{R}^2$. Moreover, the circle intersects itself for consecutive unit intervals in $\mathbb{R}$, even though its self-intersections are not "visually" apparent. Thus, this circle is an immersed submanifold, but not an embedded submanifold in $\mathbb{R}^2$. The same holds if we consider the immersion $f : [0, 1] \to \mathbb{R}^2$ because $f(0) = f(1)$. But, if we take the immersion $f :]0, 1[\to \mathbb{R}^2$, its image is an embedded manifold, that is, a unit circle minus one of its points.

Example 1.46. Let $f :]-\infty, 2[\to \mathbb{R}^2$ be an immersion given by $f(t) = (-t^3 + 3t + 2, t^2 - t - 2)$. Its image $f(]-\infty, 2[)$ is an immersed 6-shaped submanifold of dimension 1 (Figure 1.9(a)). Although f is injective (say, injective globally, and consequently injective locally), that is, without self-intersections, its image is not an embedded manifold. This is so because $]-\infty, 2[$ and its image $f(]-\infty, 2[)$ are not homeomorphic. In fact the point $(0, 0)$ in $f(]-\infty, 2[)$ is a cut point of $f(]-\infty, 2[)$, and hence the local topological type of such a



(a) (b) (c)

Fig. 1.9. Examples of immersed, but not embedded, submanifolds.

6-shaped submanifold is not constant. Note that the curve intersects itself at $t = -1$ and $t = 2$, but because $t = 2$ is not part of the domain, one says that the curve touches itself at the origin $(0, 0)$.

Example 1.47. $f : \mathbb{R} \to \mathbb{R}^2$ defined by $f(t) = (t^2 - 1, t^3 - t)$ is an immersion (Figure 1.9(b)). It is not injective. However, it is injective when restricted to, say, the range $-1 < t < \infty$.

Example 1.48. A more striking example of a self-touching submanifold is given by the image of the mapping $f : \mathbb{R} \to \mathbb{R}^2$ so that

$$f(t) = \begin{cases} (\frac{1}{t}, \sin \pi t) & \text{for} \quad 1 \leq t < \infty, \\ (0, t + 2) & \text{for} \quad -\infty < t \leq -1. \end{cases}$$

The result is a curve with a gap (Figure 1.9(c)). Let us connect the two pieces together smoothly by a dotted line as pictured in Figure 1.9(c). Then we get a smooth submanifold that results from the immersion of all of $\mathbb{R}$ in $\mathbb{R}^2$. This submanifold is not embedded because near $t = \infty$ the curve converges to the segment line $0 \times [-1, 1]$ in y-axis. In fact, while t converges to a point near ∞, its image converges to a line segment. Thus, the submanifold is not embedded because f is not a homeomorphism.

Embedded submanifolds are a subclass of immersed submanifolds that exclude self-intersecting submanifolds and self-touching submanifolds, that is, submanifolds that corrupt the local topological type invariance. Any other submanifold that keeps the same topological type everywhere in it is an embedded submanifold. Equivalently, a subset $f(M) \in N$ of a manifold N is called a smooth m-dimensional embedded submanifold if there is a covering $\{U_i\}$ of $f(M)$ by open sets (i.e. arbitrarily small neighbourhoods) of the ambient smooth manifold N such that the components of $U_i \cap f(M)$ are all connected open subsets of $f(M)$ of dimension m. Thus, there is no limitation on the number of components of an embedded submanifold in a chart of the ambient manifold; it may even be infinite [360, p. 19]. This means that, even with differential and topological singularities removed, a smooth embedded submanifold may be nonregular. Regular submanifolds intersect more neatly with coordinate charts of the ambient manifold; in particular, the family of components of this intersection do not pile up.

Definition 1.49. *An m-dimensional smooth submanifold $M \subset N$ is **regular** if, in addition to the regularity of the parametrising mapping, there is a covering $\{U_i\}$ of M by open sets of N such that, for each i, $U_i \cap M$ is a single open connected subset of M.*

By this definition, smooth regular submanifolds constitute a subclass of smooth embedded submanifolds. Let us see three counterexamples of regular submanifolds.

Example 1.50. Let $f :]1, \infty[\to \mathbb{R}^2$ be a mapping given by

$$f(t) = \left(\frac{1}{t} \cos 2\pi t, \frac{1}{t} \sin 2\pi t \right).$$

Its image (Figure 1.10(a)) in $\mathbb{R}^2$ is an embedded curve because the image of every point $t \in]1, \infty[$ is a point in $\mathbb{R}^2$; hence, f is a homeomorphism. Note that even near $t = \infty$, f is still a homeomorphism because its image is a point, the origin $(0, 0)$. That is, a point and its image have the same dimension. (This is not true in Example 1.48.) However, the image of $]1, \infty[$ is not a regular curve because it spirals to $(0, 0)$ as $t \to \infty$ and tends to $(1, 0)$ as $t \to 1$, Figure 1.10(a). This happens because near (in a neighbourhood of) $t = \infty$ the relative neighbourhood in the image curve has several (possibly an infinite number of) components.

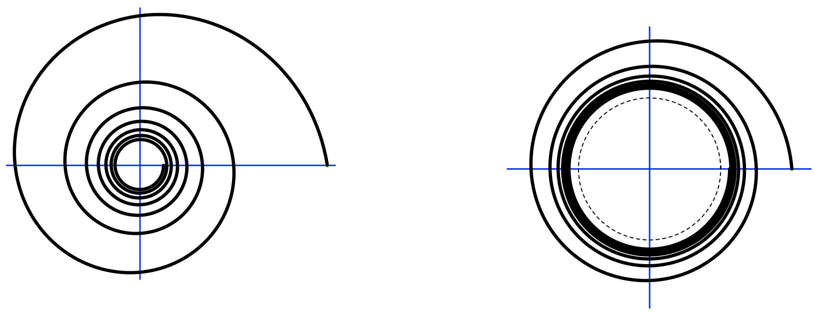
Example 1.51. Let us slightly change the previous mapping $f :]1, \infty[\to \mathbb{R}^2$ to be a mapping given by

$$f(t) = \left(\frac{t+1}{2t} \cos 2\pi t, \frac{t+1}{2t} \sin 2\pi t \right).$$

Its image (Figure 1.10(b)) in $\mathbb{R}^2$ is a nonregular embedded curve, now spiralling to the circle with centre at $(0, 0)$ and radius $1/2$ as $t \to \infty$, Figure 1.10(b). It is quite straightforward to check that the Jacobian is always 1. In fact, it could be 0 if both derivatives of the component functions could vanish simultaneously on $]1, \infty[$; this would happen if and only if $\cos 2\pi t = -\tan 2\pi t$, an impossible equality.

Thus, every regular m-dimensional submanifold of an n-dimensional manifold locally looks like an m-dimensional subspace of $\mathbb{R}^n$. A trickier, but very important counterexample is as follows.

Example 1.52. Let us consider a torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ with angular coordinates (θ, γ), $0 \leq \theta, \gamma < 2\pi$. The curve $f(t) = (t, \mathrm{k}t) \bmod 2\pi$ is closed if k/t is a



(a) (b)

Fig. 1.10. Counterexamples of regular submanifolds.

rational number, and hence a regular submanifold of $\mathbb{T}^2$, being $\mathbb{S}^1$ the parameter space. But, if k/t is irrational, the curve forms a dense subset of $\mathbb{T}^2$ and, consequently, is not a regular submanifold.

This example shows us that a regular submanifold such as a torus in $\mathbb{R}^3$ may include nonregular submanifolds. One should be careful to avoid irrational numbers in the representation and construction of submanifolds in a geometric kernel.

1.7.2 Implicit Submanifolds and Varieties

An alternative to the parametric approach for submanifolds is to define them *implicitly* as a common or intersecting level set of a collection of functions [313, p. 16]. We have seen this in Subsection 1.5.2, where the implicit mapping theorem was introduced. This theorem provides an immediate canonical form for regular manifolds as follows:

Theorem 1.53. (Olver [313, p. 14]) *A n-dimensional submanifold $N \subset \mathbb{R}^m$ is **regular** if and only if for each point $\mathbf{p} \in N$ there exist local coordinates $\mathbf{x} = (x_1, \ldots, x_m)$ defined on a neighbourhood U of $\mathbf{p}$ such that $U \cap N = \{\mathbf{x} : x_1 = \cdots = x_{m-n} = 0\}$.*

Therefore, every regular n-dimensional submanifold of an m-dimensional manifold locally looks like a n-dimensional subspace of $\mathbb{R}^m$. This means that all regular n-dimensional submanifolds are locally equivalent. They are the basic constituents of some space decompositions introduced in Chapter 2.

Let us now see how all this works for varieties. They are generalisations of implicit submanifolds, and thus they are defined by submersions. In general, the variety $V_{\mathcal{F}}$ determined by a family of real-valued functions $\mathcal{F}$ is defined by the subset where they simultaneously vanish, that is,

$$V_{\mathcal{F}} = \{\mathbf{x} \,|\, f_i(\mathbf{x}) = 0 \text{ for all } f_i \in \mathcal{F}\}.$$

In particular, when these functions $\{f_i\}$ are components of a mapping $f : \mathbb{R}^m \to \mathbb{R}^n$, the variety $V_f = \{f(\mathbf{x}) = 0\}$ is just the set of solutions to the simultaneous system of equations $f_1(\mathbf{x}) = \cdots = f_n(\mathbf{x}) = 0$.

It is clear that the notion of rank has a natural generalisation to (infinite) families of smooth functions.

Definition 1.54. *Let $\mathcal{F}$ be a family of smooth real-valued functions $f_i : M \to \mathbb{R}$, with $M, \mathbb{R}$ smooth manifolds. The **rank** of $\mathcal{F}$ at a point $\mathbf{p} \in M$ is the dimension of the space spanned by their differentials. The family is **regular** if its rank is constant on M.*

Definition 1.55. *A set $\{f_1, \ldots, f_k\}$ of smooth real-valued functions on a manifold M with a common domain of definition is called **functionally dependent** if, for each $\mathbf{p} \in M$, there is a neighbourhood U and a smooth* *function* $H(y_1, \ldots, y_k)$*, not identically zero on any subset of* $\mathbb{R}^k$*, such that* $H(f_1(\mathbf{x}), \ldots, f_k(\mathbf{x})) = 0$ *for all* $\mathbf{x} \in U$*. The functions are called* ***functionally independent*** *if they are not functionally dependent when restricted to any open subset of* M*.*

Example 1.56. The functions $f_1(x, y) = x/y$ and $f_2(x, y) = xy/(x^2 + y^2)$ are functionally dependent on the upper half-plane $\{y > 0\}$ because the second can be written as a function of the first, $f_2 = f_1/(1 + f_1^2)$.

Thus, for a regular family of functions, the rank gives us the number of functionally independent functions it contains. So, we obtain an implicit function family theorem generalising the implicit mapping theorem as follows.

Theorem 1.57. (**Implicit Function Family Theorem**) *If a family of functions* $\mathcal{F}$ *is regular of rank* n*, there exists* n *functionally independent functions* $f_1, \ldots, f_n \in \mathcal{F}$ *in the neighbourhood of any point, with the property that any other function* $g \in \mathcal{F}$ *can be expressed as a function thereof,* $g = H(f_1, \ldots, f_n)$*.*

Proof. See Olver [313, p.13].

Thus, if $f_1, \ldots, f_r$ is a set of functions whose $m \times r$ Jacobian matrix has maximal rank r at $\mathbf{p} \in M$, they also have, by continuity, the same rank r in a neighbourhood of $U \subset M$ of $\mathbf{p}$, and hence are functionally independent near $\mathbf{p}$. As expected, Theorem 1.57 also implies that, locally, there are at most m functionally independent functions on any m-dimensional manifold M.

Definition 1.58. *A variety (or system of equations)* $V_{\mathcal{F}}$ *is* ***regular*** *if it is not empty and the rank of* $\mathcal{F}$ *is constant.*

Clearly, the rank of $\mathcal{F}$ is constant if $\mathcal{F}$ itself is a regular family. In particular, regularity holds if the variety is defined by the vanishing of a mapping $f : N \to \mathbb{R}^r$ which has maximal rank r at each point $\mathbf{x} \in V_{\mathcal{F}}$, or equivalently, at each solution $\mathbf{x}$ to the system of equations $f(\mathbf{x}) = 0$ [313, p. 16]. The implicit function family theorem 1.57, together with Theorem 1.53, shows that a regular variety is a regular submanifold, as stated by the following theorem.

Theorem 1.59. *Let* $\mathcal{F}$ *be a family of functions defined on an* m*-dimensional manifold* M*. If the associated variety* $V_{\mathcal{F}} \subset M$ *is regular, it defines a regular submanifold of dimension* $m - r$*.*

Proof. See Olver [313, p. 17].

As for parametric submanifolds, to say that an implicit submanifold is regular means that it is smooth. However, a smooth parametric submanifold is not necessarily regular. But, for implicit submanifolds, regularity and smoothness coincide. This is so because, unlike a parametric submanifold, regularity of an implicit submanifold is completely determined by the regularity of its defining family of functions.

Thus, Theorem 1.59 gives us a simple criterion for the smoothness of a submanifold described implicitly.

Example 1.60. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a function given by $f(x, y, z) = x^2 + y^2 + z^2 - 1$. Its Jacobian matrix $[2x \quad 2y \quad 2z]$ has rank 1 everywhere except at the origin, and hence its variety (the unit sphere) is a regular 2-dimensional submanifold of $\mathbb{R}^3$.

Example 1.61. The function $f : \mathbb{R}^3 \to \mathbb{R}$ given by $f(x, y, z) = xyz$ is not regular, and its variety (the union of the three coordinate planes) is not a submanifold.

The fact that regularity and smoothness coincide for implicit submanifolds suggests that we may have an algorithm to determine singularities on a variety via the Jacobian matrix. Let us define regular points and singular points before providing some examples that illustrate the computation of such singularities.

Definition 1.62. *Let $f : U \subset \mathbb{R}^m \to \mathbb{R}^r$ be a smooth mapping. A point $\mathbf{p} \in \mathbb{R}^m$ is a **regular point** of f, and f is called a submersion at $\mathbf{p}$, if the differential $Df(\mathbf{p})$ is surjective. This is the same as saying that the Jacobian matrix of f at $\mathbf{p}$ has rank r (which is only possible if $r \leq m$). A point $\mathbf{q} \in \mathbb{R}^r$ is a **regular value** of f if every point of $f^{-1}(\mathbf{q})$ is regular.*

Instead of 'nonregular' we can also say *singular* or *critical*. In general, we have:

Definition 1.63. *Let $f : U \subset \mathbb{R}^m \to \mathbb{R}^r$ be a smooth mapping. A point $\mathbf{p} \in \mathbb{R}^m$ is a **singular point** of f if the rank of its Jacobian matrix falls below its largest possible value $min(m, r)$. Likewise, a **singular value** is any $f(\mathbf{p}) \in \mathbb{R}^r$ where $\mathbf{p}$ is a singular point.*

Recall that a singular point of an immersion determines a singular point in a parametric submanifold, but its self-intersections are not determined by the singular points of its associated function. This happens because the regularity of an immersion at a given point is necessary but not sufficient to guarantee the regularity of its image. But, for implicit submanifolds and varieties, the regularity of functions is necessary and sufficient to ensure their regularity.

Example 1.64. Let $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$. Then any $c \neq 0$ is a regular value of f. Its Jacobian $[2x]$ has rank 1 iff $x \neq 0$; hence $x = 0$ is the only singular point of f. This corresponds to the minimum point of the graph of f (i.e. the vertex of a parabola), but here we are concerned with implicit submanifolds that are defined by level sets, not graphs.

Example 1.65. Let $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = 2x^2 + 3y^2$. Its Jacobian $[4x \quad 6y]$ has rank 1 unless $x = y = 0$. So any $c \neq 0$ is a regular value of f. For $c > 0$, $f^{-1}(c)$ is an ellipse in the plane.

Example 1.66. Let $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = x^3 + y^3 - xy$. The maximal possible rank for its Jacobian $[3x^2 - y \quad 3y^2 - x]$ is 1, and we can find all points where this fails, i.e. all singular points, by solving the system $\partial f/\partial x = \partial f/\partial y = 0$, that is,

$$\begin{cases} 3x^2 - y &= 0 \\ 3y^2 - x &= 0 \, . \end{cases}$$

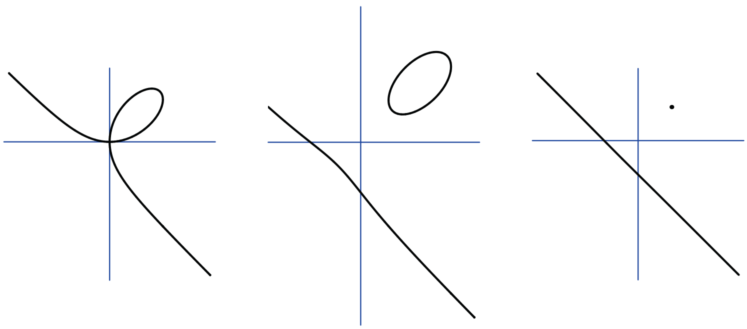
This yields the points $(0,0)$ and $(\frac{1}{3}, \frac{1}{3})$ as the only singular points of f. Since $f(0,0) = 0$ and $f(\frac{1}{3}, \frac{1}{3}) = -\frac{1}{27}$ it follows that any c other than 0 or $-\frac{1}{27}$ is a regular value of f. Also, 0 is a regular value of restrictions $f|(\mathbb{R}^2 - \{(0,0)\})$ and $-\frac{1}{27}$ is a regular value of $f|(\mathbb{R}^2 - \{(\frac{1}{3}, \frac{1}{3})\})$. This is because the singular points $(0,0)$, $(\frac{1}{3}, \frac{1}{3})$ do not belong to the domain of the restrictions of f, say $f|(\mathbb{R}^2 - \{(0,0)\})$, $f|(\mathbb{R}^2 - \{(\frac{1}{3}, \frac{1}{3})\})$, respectively.

Figure 1.11 illustrates $f^{-1}(c)$ for some values of c. For $c = 0$ we have the well-known folium of Descartes (Figure 1.11(a)). The folium of Descartes is the variety $x^3 + y^3 - xy = 0$ which self-intersects at the singular point $(0,0)$, i.e. the level set defined by $f(x,y) = 0$. The level set defined by $f(x,y) = -\frac{1}{27}$ is the variety $x^3 + y^3 - xy = -\frac{1}{27}$ (Figure 1.11(c)) whose singular point is the isolated point $(\frac{1}{3}, \frac{1}{3})$. For $c = -\frac{1}{54}$, we have the regular variety $x^3 + y^3 - xy = -\frac{1}{54}$ (Figure 1.11(b)).

Example 1.67. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be given by $f(x,y,z) = x^2 - zy^2$. The associated variety has dimension $m - r = 3 - 1 = 2$, but the maximal possible rank of its Jacobian $[2x \;\; -2zy \;\; -y^2]$ is 1. Its singular points are the solutions of the following system of equations:

$$\begin{cases} 2x = 0 \\ -2zy = 0 \\ -y^2 = 0 \end{cases} \iff \begin{cases} x = 0 \\ zy = 0 \\ y = 0 \end{cases} .$$

The expressions $x = 0$ and $y = 0$ denote the two coordinate planes in $\mathbb{R}^3$, whose intersection is the z-axis. That is, the Jacobian vanishes along the



(a) (b) (c)

Fig. 1.11. Varieties as level sets $x^3 + y^3 - xy = c$.

z-axis, or, equivalently, Each point in the z-axis is a singular point. Since $f(0,0,z) = 0$ it follows that any c other than 0 is a regular value of f. Also, 0 is a regular value of $f|(\mathbb{R}^3 - \{(0,0,z)\})$. Figure 1.12(a) illustrates $f^{-1}(0)$, the Whitney umbrella with-handle (already seen in Figure 1.4(a)).

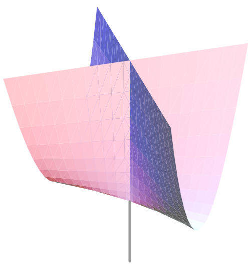
Example 1.68. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be given by $f(x,y,z) = y^2 - z^2x^2 + x^3$. As for the previous example, the Jacobian $(-2z^2x + 3x^2 \quad 2y \quad -2zx^2)$ vanishes precisely on the z-axis. The z-axis is the line of "double points" where the surface intersects itself at $c = 0$. This surface is depicted in Figure 1.12(b).

Example 1.69. Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be the mapping given by $f(x,y,z) = (xy, xz)$. The Jacobian of f is

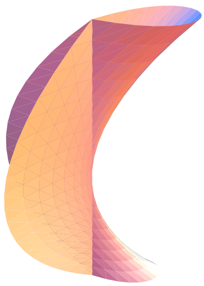
$$\begin{pmatrix} y & x & 0 \\ z & 0 & x \end{pmatrix}$$

which has rank 2 unless all 2×2 minors are zero, i.e. unless $xz = xy = x^2 = 0$, which is equivalent to $x = 0$. Since $f(0,y,z) = (0,0)$, any point of $\mathbb{R}^2$ other than $(0,0)$ is a regular value. This variety (the union of the x-axis and the plane $x = 0$) has dimension 2 and is the intersection of two 2-dimensional varieties defined by the levels sets of the components functions of f. The first level set is the union of the planes $x = 0$ and $y = 0$, while the second level set is the union of the the planes $x = 0$ and $z = 0$ in $\mathbb{R}^3$.

In short, the implicit function theorem and its generalisations allow us to determine the singular set of an implicit variety. In the particular case of an implicit surface $f(x,y,z) = 0$, the singular set is a 0- or 1-dimensional set at which all the partial derivatives simultaneously vanish. Therefore, in essence, a k-dimensional smooth (or differentiable) submanifold can be approximated by a k-dimensional subspace of $\mathbb{R}^n$ at each of its points. In particular, this the same as saying that a smooth curve in $\mathbb{R}^2$ can be approximated by a tangent line at each one of its points, a smooth surface by its tangent plane, etc. It is



(a)



(b)

Fig. 1.12. (a) Whitney umbrella with-handle as a level set $x^2 - zy^2 = 0$; (b) the surface $y^2 - z^2x^2 + x^3 = 0$.

clear that such an approximation is not possible at (differential) singularities; for example, a tangent plane flips at any corner and along any edge of the surface of cube.

1.8 Final Remarks

In this chapter, we have seen that manifolds can be either smooth or nonsmooth. Nonsmooth manifolds are in principle piecewise smooth manifolds. This leads us to the idea of partitioning a n-dimensional manifold into smooth k-dimensional submanifolds ($k \leq n$). The family of smooth submanifolds of dimension less than n are singularities of such a n-dimensional manifold. This simple idea is based on the pioneering work of two mathematicians, Whitney and Thom, nowadays known as Thom-Whitney stratification theory. They shows us that there is a close relationship between the concepts of differentiability and stratificability of manifolds. Notably, both concepts are related even when they are applied to more general geometric point sets such as algebraic, analytic or even semianalytic varieties.

The essential key for having a smooth manifold is the concept of diffeomorphism, that is, a differentiable mapping with a differentiable inverse. The differentiability of a mapping is not enough to guarantee the smoothness of a manifold; its inverse must be also differentiable. As noted in [132, p. 106], smoothness and differentiability do not agree. Smoothness means that the mapping which defines a submanifold is a diffeomorphism.

Only a diffeomorphism (i.e. a smooth mapping with smooth inverse) ensures the smoothness of a parametric curve or surface. Thus, the smoothness of a submanifold depends more on the properties of the mapping used to define it than on its associated geometric invariants (e.g. curvature and torsion). The use of a geometric invariant may be not conclusive to ensure smoothness on a submanifold, as a topological invariant (e.g. Betti numbers) is not sufficient to characterise the continuity of a subspace.

The relationship between the invertibility and smoothness of a mapping has led us to its algebraic counterpart, that is, the relationship between the invertibility of the Jacobian and smoothness of a submanifold. We have shown that this relationship is independent of whether we treat submanifolds as level sets, images, or graphs of mappings, i.e. it is representation-independent. So, we have shown that C^1 smoothness can be determined by the rank-based criterion. This suggests that we can determine the singularities of a submanifold by observing where the rank is not constant.