8

Continuous-time MPC with Prescribed Degree of Stability

8.1 Introduction

This chapter will propose a set of continuous-time model predictive control algorithms that are numerically stable and have a prescribed degree of stability. Section 8.2 begins with an example of the control of an unstable system, demonstrating that when the prediction horizon increases, the original approaches to continuous-time MPC design described in Chapter 6 will lead to an ill-conditioned Hessian matrix. This problem is caused by the open-loop prediction using the unstable model in addition to the embedded integrator(s) in the system matrix A for integral action. In Section 8.3, we show a strategy to overcome this by using a stable matrix A for the design, which is achieved by using an exponential weight in the cost function. This essentially transforms the original state and derivative of the control variables into exponentially weighted variables for the optimization procedure. In Section 8.4, we move on to the next step that produces a model predictive control system with infinite prediction horizon with asymptotic stability. With a slight modification on the weight matrices, a prescribed degree of stability can be achieved in the design of model predictive control (see Section 8.5). The final section discusses how constraints are introduced in the design (see Section 8.6). The stability results in this chapter are all based on the assumption of a sufficiently large prediction horizon T_p used in the design.

8.2 Motivating Example

This section examines an example based on the design algorithms introduced in Chapter 6. It emphasizes that because the design model is unstable, the algorithms are numerically ill-conditioned for a large prediction horizon T_p. This problem is particularly severe for systems that contain unstable poles, as illustrated in the example below.

Example 8.1. A dynamic system is described by the state-space model given as

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \frac{\alpha_1}{V} & \frac{\beta_1}{V} & 0 \\ 0 & 0 & -a & a \\ 0 & 0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} 1 \; 0 \; 0 \; 0 \end{bmatrix} x(t), \tag{8.1}$$

where $\alpha_1 = 10.2$, $\beta_1 = 0.32$,$a = 72$ and $V = 60$. This system has four open-loop eigenvalues as

$$\begin{bmatrix} 0 \; 0 \; 0.17 \; -72 \end{bmatrix}.$$

Design a continuous-time model predictive control for this system using the algorithm presented in Chapter 6. The design parameters are $p = 0.8$, $N = 4$, and $R = 0.1$, however, the prediction horizon T_p should be used as a tuning parameter. The design objective is for reference following of a step input signal.

Solution. We change the prediction horizon T_p and observe what happens with respect to the closed-loop performance and numerical condition of the algorithm.

Case A. The case of a short prediction horizon $T_p = 10$ is examined. With this choice the closed-loop control system is unstable, which is indicated by the location of the closed-loop eigenvalues

$$\begin{bmatrix} -71.99 \; -2.883 \; 0.0554 \pm j0.166 \; -0.088 \end{bmatrix},$$

where the pair of complex poles are on the right half of the complex plane. The condition number of the Hessian matrix is 146.68, which is irrelevant because the predictive control system is unstable.

Case B. The prediction horizon T_p is selected to be 13, which is increased on the one used in Case A. The Hessian matrix is

$$\Omega = \begin{bmatrix} 85.9886 & -57.8910 & 37.0031 & -21.7936 \\ -57.8910 & 47.8025 & -35.1036 & 21.6875 \\ 37.0031 & -35.1036 & 31.2090 & -21.6632 \\ -21.7936 & 21.6875 & -21.6632 & 18.8841 \end{bmatrix}.$$

The closed-loop control system is stable, which is seen from the location of the closed-loop eigenvalues

$$\begin{bmatrix} -71.99 \; -2.7355 \; -0.0334 \pm j0.1568 \; -0.1394 \end{bmatrix}.$$

The state feedback control gain obtained from the predictive control is

$$K_{mpc} = \begin{bmatrix} 18.6 \; 212.24 \; 0.0164 \; 3.11 \; 1.837 \end{bmatrix}.$$

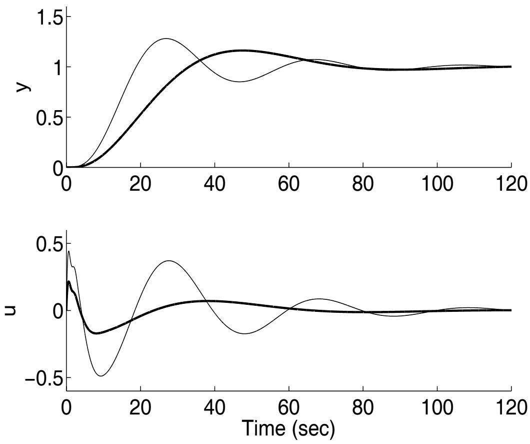


Fig. 8.1. Comparison of closed-loop responses for Case B (solid-line) and Case C (darker-solid-line)

For this choice of prediction horizon, the condition number for the Hessian matrix is 240.8.

Case C. The case of long prediction horizon T_p is examined. Let us choose $T_p = 50$. The Hessian matrix is

$$\Omega = 10^8 \times \begin{bmatrix} 1.4102 & -0.9097 & 0.5854 & -0.3754 \\ -0.9097 & 0.5869 & -0.3776 & 0.2422 \\ 0.5854 & -0.3776 & 0.2430 & -0.1558 \\ -0.3754 & 0.2422 & -0.1558 & 0.0999 \end{bmatrix}.$$

The closed-loop control system is stable with the location of the closed-loop eigenvalues as

$$\left[-71.99 \; -2.6016 \; -0.0412 \pm j0.0737 \; -0.2616\right],$$

and the state feedback control gain obtained from the predictive control scheme is

$$K_{mpc} = \left[14.355 \; 273.33 \; 0.0208 \; 3.115 \; 0.909\right].$$

Figure 8.1 shows the comparison results of the closed- loop responses for Case B and Case C. It is seen that the closed-loop output response from Case C is less oscillatory than the one from Case B. Although the closed-loop response is satisfactory when $T_p = 50$, the condition number for the Hessian has increased from 240.8 to 3.11×10^8. It is clear that the predictive control scheme is numerically ill-conditioned for Case C.

This example shows that the predictive control algorithm is sensitive to the choice of prediction horizon. If the prediction horizon is short, the closed-loop control system could become unstable; however, if the prediction horizon is long, then the algorithm would become numerically ill-conditioned. Despite these sensitivities in the predictive control algorithms, they have still gained acceptance by the process industry. This is mainly due to their simplicity and easy-to-implement features.

For the rest of the chapter, the continuous-time predictive control algorithms presented in Chapter 6 will be modified to achieve the three objectives: (1) removing the numerical ill-condition problem from the design when the prediction horizon T_p is large; (2) deriving a design that will lead to asymptotic closed-loop stability for a large prediction horizon; (3) providing a solution that will have a prescribed degree of stability. Perhaps, above all, the key features of the model predictive control algorithms will be maintained to be simple and easy-to-implement.

8.3 CMPC Design Using Exponential Data Weighting

From the analysis, we can see that the model predictive control algorithm became numerically ill-conditioned when the prediction horizon T_p became large. The reason for this is that the system matrix used for prediction contains eigenvalues with positive real parts and an integrator, which leads to

$$||e^{At}|| \rightarrow \infty,$$

as $t \rightarrow \infty$.

In this section, we explore the exponential data weighting strategy used in Anderson and Moore (1971) to produce a predictive control algorithm that is numerically well-conditioned. To begin, let us define a cost function to be optimized by the predictive control as

$$J = \int_0^{T_p} [e^{-2\alpha\tau} x(t_i + \tau \mid t_i)^T Q x(t_i + \tau \mid t_i) + e^{-2\alpha\tau} \dot{u}(\tau)^T R \dot{u}(\tau)] d\tau, \quad (8.2)$$

subject to

$$\dot{x}(t_i + \tau \mid t_i) = Ax(t_i + \tau \mid t_i) + B\dot{u}(\tau),$$

with initial condition $x(t_i)$. If the set-point signal is non-zero, as before, the last block variables in $x(t_i + \tau \mid t_i)$ correspond to the error signals between the output and set-point, which is translated to the difference in initial condition $x(t_i)$, while the rest of the formulations remain unchanged.

As before, Q is a symmetric nonnegative definite matrix, and R is a symmetric positive definite matrix. The constant α can be either positive or negative or equal to zero, depending on the application. The exponential weight used by Anderson and Moore was $e^{-2\alpha\tau}$ with α negative, which effectively produces an exponentially increasing weight. Their results were to produce an optimal regulator with prescribed degree of stability (see Section 8.5). Our interest here is to use a positive α that effectively produces an exponentially decreasing weight.

Minimization of Exponentially Weighted Cost

The results of $\alpha \geq 0$ are summarized in the theorem as follows.

Theorem 8.1. *For a given $\alpha \geq 0$, $T_p > 0$, $Q \geq 0$, and $R > 0$, minimization of the cost function*

$$J_1 = \int_0^{T_p} \left[e^{-2\alpha\tau} x(t_i + \tau \mid t_i)^T Q x(t_i + \tau \mid t_i) + e^{-2\alpha\tau} \dot{u}(\tau)^T R \dot{u}(\tau)\right] d\tau, \quad (8.3)$$

subject to

$$\dot{x}(t_i + \tau \mid t_i) = Ax(t_i + \tau \mid t_i) + B\dot{u}(\tau); \quad x(t_i \mid t_i) = x(t_i),$$

is equivalent to minimization of

$$J = \int_0^{T_p} \left[x_\alpha(t_i + \tau \mid t_i)^T Q x_\alpha(t_i + \tau \mid t_i) + \dot{u}_\alpha(\tau)^T R \dot{u}_\alpha(\tau)\right] d\tau, \quad (8.4)$$

subject to

$$\dot{x}_\alpha(t_i+\tau \mid t_i) = (A-\alpha I)x_\alpha(t_i+\tau \mid t_i)+B\dot{u}_\alpha(\tau);\ x_\alpha(t_i \mid t_i) = x(t_i \mid t_i) = x(t_i),$$

where $x_\alpha(.)$ and $\dot{u}_\alpha(.)$ are the exponentially weighted variables of $x(.)$ and $\dot{u}(.)$

$$x_\alpha(t_i + \tau \mid t_i) = e^{-\alpha\tau} x(t_i + \tau \mid t_i); \quad \dot{u}_\alpha(\tau) = e^{-\alpha\tau}\dot{u}(\tau).$$

Proof. The cost function J_1 (8.3) equals the cost function J (8.4) with the transformed variables $x_\alpha(.)$ and $\dot{u}_\alpha(.)$. In addition

$$\begin{aligned} \dot{x}(t_i + \tau \mid t_i) &= \frac{de^{\alpha\tau} x_\alpha(t_i + \tau \mid t_i)}{d\tau} \\ &= \alpha e^{\alpha\tau} x_\alpha(t_i + \tau \mid t_i) + e^{\alpha\tau}\dot{x}_\alpha(t_i + \tau \mid t_i) \quad (8.5) \\ &= Ax(t_i + \tau \mid t_i) + B\dot{u}(\tau). \quad (8.6) \end{aligned}$$

Therefore, by multiplying both (8.5) and (8.6) with $e^{-\alpha\tau}$, and re-arranging, we obtain the following equation

$$\dot{x}_\alpha(t_i + \tau \mid t_i) = (A - \alpha I)x_\alpha(t_i + \tau \mid t_i) + B\dot{u}_\alpha(\tau),$$

with the identical initial condition at $\tau = 0$,

$$x_\alpha(t_i \mid t_i) = x(t_i \mid t_i).$$

Use of the Results in CMPC Design

Theorem 8.1 shows that if we use a cost function that contains exponential decay weight, which is a time-varying weight, then the optimal solution is found by minimizing a cost function that has eliminated the time-varying weight. However, the state-space system matrix A is shifted by a $-\alpha I$ matrix. If α is positive, then all eigenvalues of original A matrix are shifted by the scalar $-\alpha$ to yield the eigenvalues of the matrix $A - \alpha I$, which effectively changes the real part of all eigenvalues. In this transformed formulation, the α value is selected such that the eigenvalues of $A - \alpha I$ lie strictly on the left-half of the complex plane, *i.e.*, $real(\lambda_i(A - \alpha I)) < -\epsilon$ for $\epsilon > 0$ and all i. Thus, the continuous-time model used for the design then becomes a stable model, instead of the unstable model in the original formulation. As a result, the numerical ill-conditioning problem is overcome.

Denote $A_\alpha = A - \alpha I$. At time t_i, $x_\alpha(t_i \mid t_i) = x(t_i)$ and the transformed derivative of the control signal is

$$\dot{u}_\alpha(\tau) = \left[L_1(\tau)^T \; L_2(\tau)^T \; \ldots \; L_m(\tau)^T \right] \eta.$$

The predicted, transformed state variable $x_\alpha(\tau \mid t_i)$ at time τ is

$$\begin{aligned} &x_\alpha(t_i + \tau \mid t_i) = e^{A_\alpha \tau} x(t_i) \\ &+ \int_0^\tau e^{A_\alpha(\tau - \gamma)} \left[B_1 L_1(\gamma)^T \; B_2 L_2(\gamma)^T \; ... \; B_m L_m(\gamma)^T \right] d\gamma \eta, \end{aligned} \tag{8.7}$$

where we assume that the number of inputs is m and Laguerre functions are used in the parameterization of the derivative of the control signal. Namely, we optimize the transformed control variable $\dot{u}_\alpha(\tau)$, instead of the original variable $\dot{u}(\tau)$. Introducing

$$\phi_i(\tau)^T = \int_0^\tau e^{A_\alpha(\tau - \gamma)} B_i L_i(\gamma)^T d\gamma, \tag{8.8}$$

and

$$\phi(\tau)^T = \left[\phi_1(\tau)^T \; \phi_2(\tau)^T \; \ldots \; \phi_m(\tau)^T \right].$$

Equation (8.7) is simplified into

$$x_\alpha(t_i + \tau \mid t_i) = e^{A_\alpha \tau} x(t_i) + \phi(\tau)^T \eta. \tag{8.9}$$

Substituting (8.9) into the cost function (8.4), we obtain

$$J = \int_0^{T_p} (\eta^T \phi(\tau) Q \phi(\tau)^T \eta + 2\eta^T \phi(\tau) Q e^{A_\alpha \tau} x(t_i)) d\tau + \eta^T R_L \eta + constant, \tag{8.10}$$

where R_L is a block diagonal matrix with the kth block being R_k, and $R_k = r_{wk} I_{N_k \times N_k}$ (where $I_{N_k \times N_k}$ is a unit matrix with dimension N_k). Here, for simplicity of the solution, we have assumed that the weight matrix R is a diagonal matrix.

Defining the data matrices

$$\Omega = \int_0^{T_p} \phi(\tau) Q \phi(\tau)^T d\tau + R_L \tag{8.11}$$

$$\Psi = \int_0^{T_p} \phi(\tau) Q \mathrm{e}^{A_\alpha \tau} d\tau, \tag{8.12}$$

the quadratic cost function (8.10) becomes

$$J = \eta^T \Omega \eta + 2\eta^T \Psi x(t_i) + constant. \tag{8.13}$$

The optimal solution that minimizes the above quadratic cost function is

$$\eta = -\Omega^{-1} \Psi x(t_i). \tag{8.14}$$

Upon obtaining η, the exponentially weighted derivative of the control signal $\dot{u}_\alpha(\tau)$ is constructed through

$$\dot{u}_\alpha(\tau) = \left[\, L_1(\tau)^T \; L_2(\tau)^T \; \ldots \; L_m(\tau)^T \,\right] \eta. \tag{8.15}$$

From the receding horizon control, the optimal solution for the actual $\dot{u}(0)$ is

$$\dot{u}(0) = \dot{u}_\alpha(0) = \left[\, L_1(0)^T \; L_2(0)^T \; \ldots \; L_m(0)^T \,\right] \eta. \tag{8.16}$$

Because the optimization is performed on the transformed variables $x_\alpha(.)$ and $\dot{u}_\alpha(.)$, when constraints are introduced, all the original constraints are required to be transformed from the variables $x(.)$ and $\dot{u}(.)$ to $x_\alpha(.)$ and $\dot{u}_\alpha(.)$. Constrained control will be discussed further in the later sections of the chapter.

8.4 CMPC with Asymptotic Stability

This section establishes equivalent results with LQR when exponential weighting is used. The results are investigated through two different cost functions, and we then establish that the optimal control results are identical. The results are summarized in the theorem as follows.

Case A

Suppose that the optimal control $\dot{u}_1(\tau)$ is obtained by minimizing cost function J_1 with $Q \geq 0$, and $R > 0$

$$J_1 = \int_0^{\infty} \left[x(t_i + \tau \mid t_i)^T Q x(t_i + \tau \mid t_i) + \dot{u}(\tau)^T R \dot{u}(\tau) \right] d\tau, \tag{8.17}$$

subject to

$$\dot{x}(t_i + \tau \mid t_i) = Ax(t_i + \tau \mid t_i) + B\dot{u}(\tau),$$

where the initial state is $x(t_i)$. The optimal solution of the derivative of the control $\dot{u}(\tau)$ is obtained through the state feedback control law

$$\dot{u}_1(\tau) = -R^{-1}BPx(t_i + \tau \mid t_i), \tag{8.18}$$

and P is the solution of the Riccati equation

$$PA + A^T P - PBR^{-1}B^T P + Q = 0. \tag{8.19}$$

Case B

Choosing $Q_\alpha = Q + 2\alpha P$, $\alpha > 0$, R unchanged, the optimal control $\dot{u}_2(\tau)$ is obtained by minimizing

$$J_2 = \int_0^\infty e^{-2\alpha\tau} \left(x(t_i + \tau \mid t_i)^T Q_\alpha x(t_i + \tau \mid t_i) + \dot{u}(\tau)^T R\dot{u}(\tau) \right) d\tau, \tag{8.20}$$

subject to

$$\dot{x}(t_i + \tau \mid t_i) = Ax(t_i + \tau \mid t_i) + B\dot{u}(\tau),$$

with the initial condition $x(t_i)$.

Theorem 8.2. *The optimal control solutions stated in Case A and Case B have the following relation:*

$$\dot{u}_2(\tau) = \dot{u}_1(\tau);\ min(J_2) = min(J_1).$$

Proof. The optimal solution for Case A is found through the algebraic Riccati equation (Kailath, 1980, Bay, 1999)

$$PA + A^T P - PBR^{-1}B^T P + Q = 0, \tag{8.21}$$

with $\dot{u}_1(\tau) = -R^{-1}BPx(t_i + \tau \mid t_i)$ and $min(J_1) = x(t_i)^T Px(t_i)$.

By adding and subtracting the term $2\alpha P$, (8.21) becomes

$$PA + A^T P - PBR^{-1}B^T P + Q + 2\alpha P - 2\alpha P = 0, \tag{8.22}$$

which is

$$P(A - \alpha I) + (A - \alpha I)^T P - PBR^{-1}B^T P + Q + 2\alpha P = 0. \tag{8.23}$$

With $Q_\alpha = Q + 2\alpha P$, the Riccati equation (8.23) becomes identical to

$$P(A - \alpha I) + (A - \alpha I)^T P - PBR^{-1}B^T P + Q_\alpha = 0. \tag{8.24}$$

Relating these back to the exponential data weighting results in Theorem 8.1, (8.24) is the Riccati equation for the optimization of Case B. Since (8.24) is identical to (8.21), therefore, the Riccati solution P from (8.23) remains unchanged, and hence

$$\dot{u}_2(\tau) = \dot{u}_1(\tau);\ min(J_1) = min(J_2).$$

The original Case A is not solvable in the context of predictive control for a sufficiently large prediction horizon T_p, when the design model contains eigenvalues on the imaginary axis or on the right half of the complex plane, and the prediction becomes numerically ill-conditioned. In contrast, the Case B is solvable in the context of predictive control because of the choice of the exponential weight $\alpha > 0$ that will lead to the system matrix $A - \alpha I$ becoming stable.

The following list summarizes the relationship between the design parameters and variables in LQR and the continuous-time MPC with exponential data weighting for sufficiently large N and T_p.

Model (LQR)	$\dot{x}(t_i + \tau \mid t_i) = Ax(t_i + \tau \mid t_i) + B\dot{u}(\tau)$
Model (CMPC)	$\dot{x}_\alpha(t_i + \tau \mid t_i) = (A - \alpha I)x_\alpha(t_i + \tau \mid t_i) + B\dot{u}_\alpha(\tau)$
Weight matrices (LQR)	Q, R
Weight matrices (CMPC)	$Q_\alpha = Q + 2\alpha P$, R unchanged
Cost (LQR)	$J = \int_0^\infty (x(\cdot)^T Q x(\cdot) + \dot{u}(\tau)^T R \dot{u}(\tau))d\tau$
Cost (CMPC)	$J = \int_0^{T_p} (x_\alpha(\cdot)^T Q_\alpha x_\alpha(\cdot) + \dot{u}_\alpha(\tau)^T R \dot{u}_\alpha(\tau))d\tau$
Optimal control (LQR)	$\dot{u}(\tau) = -R^{-1}BPx(t_i + \tau \mid t_i)$
Optimal control (CMPC)	$\dot{u}(\tau) = -L(\tau)^T \Omega^{-1} \Psi x(t_i)$
$0 \leq \tau \leq T_p$	$\dot{u}_\alpha(\tau) = \dot{u}(\tau)\mathrm{e}^{-\alpha\tau}$
$0 \leq \tau \leq T_p$	$x_\alpha(t_i + \tau \mid t_i) = x(t_i + \tau \mid t_i)\mathrm{e}^{-\alpha\tau}$
Feedback gain	$K_{mpc} = K_{lqr}$
Closed-loop Eigenvalues.	$\lambda_i(A - BK_{mpc}) = \lambda_i(A - BK_{lqr})$, for all i.

Tutorial 8.1. *We consider the same system as in Example 8.1, where the augmented dynamic system with an integrator is described by the state-space model given as below*

$$\begin{aligned}\dot{x}(t) &= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{\alpha_1}{V} & \frac{\beta_1}{V} & 0 & 0 \\ 0 & 0 & -a & a & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \dot{u}(t) \\ y(t) &= \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix} x(t), \end{aligned} \tag{8.25}$$

where $\alpha_1 = 10.2$, $\beta_1 = 0.32$,$a = 72$ and $V = 60$. The parameters in the Laguerre functions are selected as $p = 0.8$ and $N = 4$.

1. *Choosing $Q = C^T C$ and $R = 0.1$, design the LQR control system with the weight matrices Q and R, and find the Riccati equation solution P, the feedback gain matrix K and the closed-loop eigenvalues.*

2. *Using exponential data weighting in the cost function of predictive control with* $\alpha = 0.18$ *in the modified* Q_α*, compute the data matrices* Ω *and* Ψ *and verify the convergence of* Ω *and* Ψ *with respect to a large prediction horizon* T_p*.*
3. *Compare the exponentially weighted predictive control system with the LQR system.*

Step by Step

1. *Create a program called exptut.m.*
2. *We will first set-up the state-space model and the augmented state-space model. Enter the following the program into the file:*

```
alpha1=10.2;
beta1=0.32;
a=72;
v=60;
Ap=[0 1 0 0; 0 alpha1/v beta1/v
0; 0 0 -a a; 0 0 0 0] ;
Bp=[0;0;0;1];
Cp=[1 0 0 0];
Dp=0;
[m1,n1]=size(Cp);
[n1,n_in]=size(Bp);
A=zeros(n1+m1,n1+m1);
A(1:n1,1:n1)=Ap;
A(n1+1:n1+m1,1:n1)=Cp;
B=zeros(n1+m1,n_in);
B(1:n1,:)=Bp;
C=zeros(m1,n1+m1);
C(:,n1+1:n1+m1)=eye(m1,m1);
```

3. *Compute the LQR solution using the MATLAB 'lqr' function. K is the feedback gain, P is the Riccati equation solution and E is the set of closed-loop eigenvalues. Continue entering the following program into the file:*

```
Q=C'*C;
R=0.1*eye(m1,m1);
[K,P,E]=lqr(A,B,Q,R);
```

4. *Compute* Q_α *and specify the design parameters for the continuous-time predictive control system. We also modify the unstable system matrix A to stable* A_α*. A large N is used in this design to demonstrate that the results converge to the LQR system. You can choose a smaller N and discover the difference is small.*

```
alpha=0.18;
Q_alpha=Q+2*alpha*P;
A_alpha=A-alpha*eye(n1+m1,n1+m1);
```

```
p=0.6;
N=10;
Tp=35;
[Omega,Psi]=cmpc(A_alpha,B,p,N,Tp,Q_alpha,R);
```

5. *With Ω and Ψ matrices, the cost function for the on-line optimization is determined. With receding horizon control, the feedback control gain matrix K_{mpc} is calculated. Continue entering the following program into the file:*

```
[Al,L0]=lagc(p,N);
K_mpc=L0'*(Omega\Psi);
A_cl=A-B*K_mpc;
E_mpc=eig(A_cl);
```

6. *We need to verify the relationship between $u_\alpha(\tau)$ and $u(\tau)$; and the relationship between $x_\alpha(t_i+\tau \mid t_i)$ and $x(t_i+\tau \mid t_i)$. We solve for the Laguerre parameter vector η using an initial state variable, and construct the whole control trajectory $\dot{u}_\alpha(.)$ using Laguerre functions. The trajectory of $x_\alpha(.)$ is calculated by solving the differential equation. Continue entering the following program into the file:*

```
N_sim=22000;
h=0.001;
X0=[0.1;0.2;0.3;0.4;0.5];
eta=-Omega\Psi*X0;
x=X0;
t=0:h:(N_sim-1)*h;
for kk=1:N_sim
u_dot(kk)=(expm(Al*t(kk))*L0)'*eta;
xs(:,kk)=x;
x=x+(A_alpha*x+B*u_dot(kk))*h;
end
```

7. *To compare the results, we also compute the LQR control trajectory and the trajectory of $x(.)$. Continue entering the following program into the file:*

```
A_lqr=A-B*K;
A_lqr_alpha=A_lqr-alpha*eye(n1+m1,n1+m1);
for kk=1:N_sim
x_lqr(:,kk)=expm(A_lqr*t(kk))*X0;
u_dot_lqr(kk)=-K*x_lqr(:,kk);
x_lqr_alpha(:,kk)=expm(A_lqr_alpha*t(kk))*X0;
end
```

8. *Run this program, then you will have the numerical results for the exponentially weighted continuous-time MPC system.*

9. *Increasing prediction horizon T_p, you will notice this parameter does not affect the control results after a certain large number.*

Figure 8.2 shows the comparison results within one optimization window, between the variables in the original LQR systems and the transformed predictive control system using exponential weighting. For simplicity, instead of examining all components in x, we examine the last component in x, which is the output y. By visual inspection, we can see that the exponentially weighted variables decay faster. To show that indeed there is the factor of $e^{-\alpha\tau}$ difference between x_α and x, and u_α and u, we calculate the errors

$$e_y = \int_0^{22} (y_\alpha(\tau) - y(\tau)e^{-\alpha\tau})^2 d\tau = 5.5529 \times 10^{-4}, \tag{8.26}$$

$$e_u = \int_0^{22} (\dot{u}_\alpha(\tau) - \dot{u}(\tau)e^{-\alpha\tau})^2 d\tau = 6.1761 \times 10^{-6}. \tag{8.27}$$

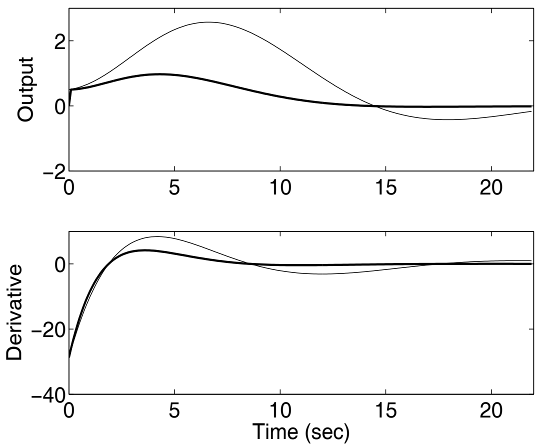


Fig. 8.2. Top figure: comparison between $y(.)$ (solid-line) and $y_\alpha(.)$ (darker-solid-line); bottom figure: comparison between $\dot{u}(.)$ (solid-line) and $\dot{u}_\alpha(.)$ (darker-solid-line)

Furthermore, the gain and closed-loop eigenvalues of LQR and continuous-time MPC systems are compared as below:

K_{mpc}	$[23.2346 \;\; 121.8435 \;\; 0.0090 \;\; 1.1371 \;\; 3.1639]$
K_{lqr}	$[23.2281 \;\; 121.8430 \;\; 0.0090 \;\; 1.1372 \;\; 3.1623]$
$eigenvalues(mpc)$	$-72.0000 \quad -0.1411 \pm j0.3241 \quad -0.3424 \pm j0.1334$
$eigenvalues(lqr)$	$-72.0000 \quad -0.1411 \pm j0.3240 \quad -0.3425 \pm j0.1334$

8.5 Continuous-time MPC with Prescribed Degree of Stability

The term 'prescribed degree of stability of β' means that the closed-loop eigenvalues of the predictive control system reside to the left of the line $s = -\beta$ on the complex plane. This is very practical in the design of continuous-time predictive control systems. For instance, the value of β becomes part of the closed-loop performance specification. For a multi-input and multi-output system, tuning the predictive control system can be very time consuming. By specifying a degree of stability, the tuning process for a complex system can be simplified.

8.5.1 The Original Anderson and Moore's Results

We resort to the wealth of literature on the linear quadratic regulator (LQR). In Anderson and Moore (1971), the cost function with ($\beta > 0$) is

$$J_1 = \int_0^\infty e^{2\beta t} \left[x(t)^T Q x(t) + \dot{u}(t)^T R \dot{u}(t)\right] dt, \tag{8.28}$$

and it is minimized subject to

$$\dot{x}(t) = Ax(t) + B\dot{u}(t). \tag{8.29}$$

The minimization of the cost function (8.28) produces a closed-loop system with a prescribed degree of stability determined by the value of β. Note that the choice of the weight exponent, β, has an opposite sign to what we proposed earlier, and let us call this an exponentially increasing weight.

To proceed further, denote

$$x_\beta(t) = e^{\beta t} x(t);\ u_\beta(t) = e^{\beta t} u(t).$$

Then, the problem of minimizing (8.28) is equivalent to the minimization of the cost function:

$$J_2 = \int_0^\infty \left[x_\beta(t)^T Q x_\beta(t) + \dot{u}_\beta(t)^T R \dot{u}_\beta(t)\right] dt, \tag{8.30}$$

subject to

$$\dot{x}_\beta(t) = (A + \beta I)x_\beta(t) + B\dot{u}_\beta(t). \tag{8.31}$$

The optimal control is obtained through the solution of the algebraic Riccati equation to solve for the transformed system $(A + \beta I, B)$

$$P(A + \beta I) + (A + \beta I)^T P - PBR^{-1}B^T P + Q = 0, \tag{8.32}$$

$$\dot{u}_\beta(t) = -R^{-1}BPx_\beta(t). \tag{8.33}$$

However, the original control signal is

$$\dot{u}(t) = \dot{u}_\beta(t)e^{-\beta t} = -R^{-1}BPx(t). \tag{8.34}$$

Therefore, the feedback controller gain has the identical formula, except that the Riccati matrix P is solved using (8.32), where $(A + \beta I)$ is used to replace the original system matrix A.

The following points are given to establish that the closed-loop system has a prescribed degree of stability β.

1. If the pair (A, D) is observable where $Q = D^T D$, then $(A + \beta I, D)$ is observable; if the pair (A, B) is controllable, then $(A+\beta I, B)$ is controllable. The solution of the Riccati equation (8.32) leads to asymptotic stability of the closed-loop system for the pair $(A + \beta I, B)$. Namely
$$||x_\beta(t)|| \rightarrow 0$$
as $t \rightarrow \infty$.
2. Note that
$$x(t) = e^{-\beta t}x_\beta(t).$$
This means that $x(t)$ decays at least as fast as the rate of $e^{-\beta t}$.
3. This establishes that the exponentially weighted cost function produces a closed-loop system with a prescribed degree of stability β.
4. The asymptotic stability of the closed-loop system for the pair $(A+\beta I, B)$ ensures that the closed-loop eigenvalues, for all k
$$real\{\lambda_k(A + \beta I - BK)\} < 0,$$
where $K = R^{-1}BP$. This means that the closed-loop eigenvalues for the pair (A, B) must be at least, for all k
$$real\{\lambda_k(A - BK)\} < -\beta.$$
5. This establishes that the closed-loop eigenvalues are on the left of the $s = -\beta$ line in the complex plane.

8.5.2 CMPC with a Prescribed Degree of Stability

Although the exponentially increasing weight proposed by Anderson and Moore produces a closed-loop system with a prescribed degree of stability, their solution was obtained through the Riccati equation (see (8.32)). If their approach was used in predictive control design, then numerical problems would arise. This is because the system matrix $A + \beta I$ (design model) has eigenvalues shifted further towards the right-half of the complex plane by a distance of β, and the prediction that uses this model will exponentially grow at least at a rate of β. This approach to obtain a prescribed degree of stability is re-developed in the context of predictive control design. The results are summarized as below.

Case A

Suppose that the optimal control $\dot{u}_1(\tau)$ is obtained by minimizing with $Q \geq 0$, $R > 0$, $\beta > 0$,

$$J_1 = \int_0^\infty e^{2\beta\tau} \left[x(t_i + \tau \mid t_i)^T Q x(t_i + \tau \mid t_i) + \dot{u}(\tau)^T R \dot{u}(\tau) \right] d\tau, \qquad (8.35)$$

subject to

$$\dot{x}(t_i + \tau \mid t_i) = Ax(t_i + \tau \mid t_i) + B\dot{u}(\tau);\; x(t_i \mid t_i) = x(t_i),$$

where A may contain eigenvalues that are either on the jw axis or on the right-half of the complex plane. The optimal solution of the derivative of the control $\dot{u}(\tau)$ is obtained through the state feedback law

$$\dot{u}_1(\tau) = -R^{-1}BPx(t_i + \tau \mid t_i), \qquad (8.36)$$

and P is the solution of the Riccati equation

$$P(A + \beta I) + (A + \beta I)^T P - PBR^{-1}B^T P + Q = 0. \qquad (8.37)$$

Case B

Choosing $\alpha > 0$, R unchanged, and

$$Q_\alpha = Q + 2(\alpha + \beta)P,$$

the optimal control $\dot{u}_2(\tau)$ is obtained by minimizing

$$J_2 = \int_0^\infty e^{-2\alpha\tau} \left[x(t_i + \tau \mid t_i)^T Q_\alpha x(t_i + \tau \mid t_i) + \dot{u}(\tau)^T R \dot{u}(\tau) \right] d\tau, \qquad (8.38)$$

subject to

$$\dot{x}(t_i + \tau \mid t_i) = Ax(t_i + \tau \mid t_i) + B\dot{u}(\tau);\; x(t_i \mid t_i) = x(t_i).$$

Theorem 8.3. *The optimal solutions given in Case A and Case B satisfy the following relation:*

$$\dot{u}_2(\tau) = \dot{u}_1(\tau);\; min(J_2) = min(J_1).$$

Proof. The proof follows a similar procedure to that in the proof of Theorem 8.2.

From the Anderson and Moore's results, the optimal solution for Case A is found through the algebraic Riccati equation

$$P(A + \beta I) + (A + \beta I)^T P - PBR^{-1}B^T P + Q = 0, \qquad (8.39)$$

with $\dot{u}_1(\tau) = -R^{-1}BPx(t_i + \tau \mid t_i)$ and $min(J_1) = x(t_i)^T Px(t_i)$.
By adding and subtracting the term $2\alpha P$, (8.39) becomes

$$P(A+\beta I) + (A+\beta I)^T P - PBR^{-1}B^T P + Q + 2\alpha P - 2\alpha P = 0, \quad (8.40)$$

which is

$$P(A-\alpha I) + (A-\alpha I)^T P - PBR^{-1}B^T P + Q + 2\alpha P + 2\beta P = 0. \quad (8.41)$$

With $Q_\alpha = Q + 2(\alpha+\beta)P$, the Riccati equation (8.41) becomes identical to

$$P(A-\alpha I) + (A-\alpha I)^T P - PBR^{-1}B^T P + Q_\alpha = 0. \quad (8.42)$$

Comparing the Riccati equation (8.42) to the exponential data weighting results in Theorem 8.1, (8.42) is the Riccati equation for the optimization Case B. Since (8.42) is identical to (8.39), therefore, the Riccati solution P from (8.42) remains unchanged, and hence

$$\dot{u}_2(\tau) = \dot{u}_1(\tau); \; min(J_1) = min(J_2).$$

8.5.3 Tuning Parameters and Design Procedure

Selection of the Exponential Weighting Factor

From a given augmented state-space model (A, B), the eigenvalues of A are determined. If the plant is stable, then the unstable eigenvalues of A come from the integrators that have been embedded in the model. In this case, any $\alpha > 0$ will serve the purpose of exponential data weighting. However, if the plant is unstable with all its eigenvalues lying on the left of the ϵ line of the complex plane where $\epsilon > 0$, the parameter α is required to be at least greater than ϵ. In summary, the idea behind the selection of α is to make sure that the design model with $(A - \alpha I)$ is stable with all eigenvalues on the left-half of the complex plane.

Selection of Prediction Horizon

Once the exponential weight factor α is selected, the eigenvalues of the matrix $A - \alpha I$ are fixed. Since this matrix is stable with an appropriate choice of α, the prediction of the state variables is numerically sound. Thus, the prediction horizon T_p is selected sufficiently large to capture the transformed state variable response. In general, if the eigenvalues of $A - \alpha I$ were further away from the imaginary axis on the complex plane, then a smaller T_p would be required. However, some attention needs to be paid to the computation of Ω and Ψ matrices when discretization is used to recursively evaluate the integrals given in Section 6.3.4. In general, the discretization interval (h) for the computation should be smaller if the exponential weight factor α is used.

Choice of Weight Matrices in the Cost Functions

From the model formulation, the Q matrix is usually selected as $Q = C^T C$, which corresponds to minimization of integral squared output errors. This choice has been found to produce satisfactory closed-loop performance for set-point tracking of a reference signal. Weight matrix R is selected as a diagonal matrix, with each element weighting the corresponding control signal. For instance, if the influence of a particular control is to be reduced, then the corresponding diagonal element will be increased to reflect this intention.

Selection of Degree of Stability β

The closed-loop performance of a predictive control system so far is determined by the choice of Q and R matrices. The tuning could be very time consuming as it often requires finding the off-diagonal elements in Q and R to achieve satisfactory performance. This is often carried out in a trial-and-error manner. Now, with the additional parameter β that dictates the degree of stability, the closed-loop eigenvalues of the predictive control system are effectively positioned to some desired regions on the complex plane. This parameter is very useful in the closed-loop performance specification. For instance, β is related to the minimal decay rate of the closed-loop system. So we can use this parameter to specify the closed-loop response speed.

The Parameters in Laguerre Functions

When N increases, the predictive control trajectory converges to the underlying optimal control trajectory of the linear quadratic regulator. However, with a small N, the pole location p will affect the closed-loop response. The pair of parameters (p, N) can be used as a pair of fine tuning parameters for the closed-loop performance. Detailed discussion on the Laguerre parameters for the discrete-time counterpart is given in Chapter 4.

The Q_α Matrix

With the choice of β, which is the degree of stability, the Riccati equation is solved for the P matrix:

$$P(A + \beta I) + (A + \beta I)^T P - PBR^{-1}B^T P + Q = 0. \tag{8.43}$$

MATLAB script can be used for this solution:

```
[K,P,E]= lqr(A+beta*eye(n,n), B, Q, R);
```

Matrix Q_α is determined, with the values of α, β and P, using

$$Q_\alpha = Q + 2(\alpha + \beta)P.$$

The Modified Design Model

The augmented state-space model (A, B) is modified for use in the design. The matrix B is unchanged, however, the matrix A is modified to become

$$A - \alpha I$$

With this set of performance parameters (Q_α, R) and the design model $(A - \alpha I, B)$, the predictive control problem is converted back to the original problem stated in Chapter 6, thus the cost function for the predictive control system is expressed as a function of η

$$J = \eta^T \Omega \eta + 2\eta^T \Psi x(t_i) + constant. \tag{8.44}$$

8.6 Constrained Control with Exponential Data Weighting

The design of continuous-time predictive control using exponential data weighting is based on the transformed variables $x_\alpha(.)$ and $\dot{u}_\alpha(.)$. By using the transformed variables, a large prediction horizon is used to approximate the infinity horizon case so as to guarantee asymptotic stability or to achieve a prescribed degree of stability, and the numerical ill-conditioning problem is overcome. In the constraints handling, the design specification of the constraints is given for the original variables $x(.)$ and $\dot{u}(.)$, and these are required to be mapped into constraints with respect to the transformed variables $x_\alpha(.)$ and $\dot{u}_\alpha(.)$.

Constraints at $\tau = 0$

Since within one optimization window, at $\tau = 0$ the transformed variables $x_\alpha(t_i \mid t_i)$ and $\dot{u}_\alpha(0)$ are identical to the original variables $x(t_i \mid t_i)$ and $\dot{u}(0)$, there is no change for the constraints at the time $\tau = 0$.

Constraints at $\tau > 0$

As we know, the relationship between the transformed variables and the original variables is given as

$$\dot{u}_\alpha(\tau) = \dot{u}(\tau)\mathrm{e}^{-\alpha\tau}; \; x_\alpha(t_i + \tau \mid t_i) = x(t_i + \tau \mid t_i)\mathrm{e}^{-\alpha\tau}.$$

Thus, supposing that the upper and lower limits of $\dot{u}(\tau)$ are specified as

$$du^{min} \le \dot{u}(\tau) \le du^{max},$$

with respect the transformed variable $\dot{u}_\alpha(t)$ within one optimization window, the constraints are mapped into the relation:

$$e^{-\alpha\tau} du^{min} \leq \dot{u}_\alpha(\tau) \leq du^{max} e^{-\alpha\tau}, \tag{8.45}$$

which is expressed, in terms of the decision variable η, as

$$e^{-\alpha\tau} du^{min} \leq L(\tau)\eta \leq du^{max} e^{-\alpha\tau}. \tag{8.46}$$

Similarly, the original constraints on the state variables, x^{min} and x^{max}, are transformed into

$$e^{-\alpha\tau} x^{min} \leq \overbrace{e^{A_\alpha \tau} x(t_i) + \phi(\tau)^T \eta}^{x_\alpha(t_i+\tau|t_i)} \leq x^{max} e^{-\alpha\tau}. \tag{8.47}$$

Since the transformed variables exponentially decay in a faster rate, the original constant bounds become exponentially decaying with respect to $\dot{u}_\alpha$ and x_α to form tighter bounds.

What we will do next is to transform the bounds on the original control signal to the bounds on the transformed variables. Note that the control signal with assumed zero initial condition is expressed as

$$u(\tau) = \int_0^\tau \dot{u}(\gamma) d\gamma. \tag{8.48}$$

By substituting $\dot{u}(\gamma) = \dot{u}_\alpha(\gamma) e^{\alpha\gamma} = L(\gamma)^T e^{\alpha\gamma} \eta$ into (8.48), we obtain

$$\begin{aligned} u(\tau) &= \int_0^\tau L(0)^T e^{(A_p+\alpha I)^T \gamma} \eta d\gamma & (8.49) \\ &= L(0)^T \left(e^{(A_p+\alpha I)^T \tau} - I \right) (A_p + \alpha I)^{-T} \eta. & (8.50) \end{aligned}$$

Adding the first sample of the control signal, the bounds are expressed in terms of the decision variable η as

$$u^{min} \leq u(t_i - \Delta t) + L(0)^T \eta \Delta t + L(0)^T \left(e^{(A_p+\alpha I)^T \tau} - I \right) (A_p + \alpha I)^{-T} \eta \leq u^{max}. \tag{8.51}$$

Upon setting up the constraints with respect to the transformed variables, the remaining procedures to the solution of the constrained control problem are identical to those without exponential data weighting, as discussed in Chapter 7, namely, the inequality constraints are used in the optimization of η by minimizing the cost function:

$$J = \eta^T \Omega \eta + 2\eta^T \Psi x(t_i).$$

Example 8.2. Consider a two-input and two-output system described by the transfer function:

$$\begin{bmatrix} y_1(s) \\ y_2(s) \end{bmatrix} = \begin{bmatrix} \frac{12.8(-s+4)^2}{(40s+1)(s+4)^2} & \frac{-10.9(-3s+4)^2}{(21.0s+1)(3s+4)^2} \\ \frac{12.8(-7s+4)^2}{(10.9s+1)(7s+4)^2} & \frac{-19.4(-3s+4)^2}{(20s+1)(3s+4)^2} \end{bmatrix} \begin{bmatrix} u_1(s) \\ u_2(s) \end{bmatrix}. \tag{8.52}$$

This system has complex unstable zeros and strong couplings as shown by the off diagonal elements of the transfer function. Choose the weight matrices $Q = C^T C$, $R = I$, and the prediction horizon $T_p = 50$, $N_1 = N_2 = 6$, and $p_1 = p_2 = 1$. An observer is needed in the implementation of the predictive control system. MATLAB function lqr is used with $Q_{ob} = I$ and $R = 0.2I$. With zero initial conditions on the state variables, for a unit set-point change, the operational constraints on the control signals are specified as

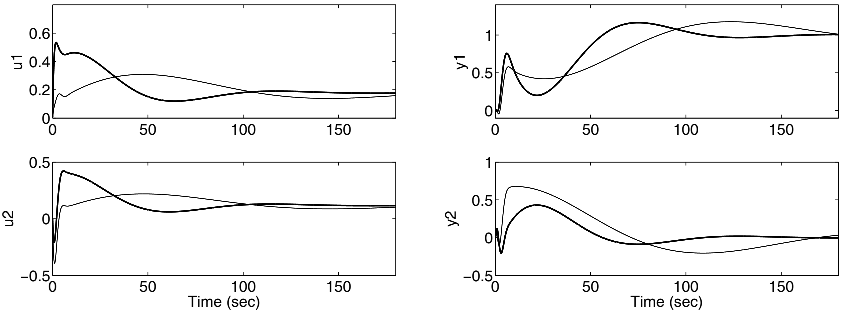
$$0 \leq u_1(t),\ u_2(t) \leq 0.3;\ \ -0.2 \leq \dot{u}_1(t),\ \dot{u}_2(t) \leq 0.4.$$

Design and simulate a continuous-time predictive control system with constraints using exponential data weighting, where $\alpha = 0.18$, and compare the results with the case when $\alpha = 0$. The sampling interval is selected as $\Delta t = 0.009$ sec.

Solution. The condition number of the Hessian matrix with exponential data weighting ($\alpha = 0.18$) is $\kappa(\Omega) = 720$. In contrast, the condition number without exponential data weighting ($\alpha = 0$) is $\kappa(\Omega) = 1.474 \times 10^5$, which clearly indicates that the Hessian matrix is ill-conditioned. Table 8.3 shows the comparison between the elements of the first row in the state feedback gain matrix with three different approaches. It is seen that with exponential weighting ($\alpha = 0.18$), the elements of the predictive feedback control gain K_{mpc} are very close to the elements of feedback gain from LQR design. However, without exponential data weighting, there are large differences between the elements of the predictive controller gain and those from LQR design. We also confirm the large differences between the closed-loop responses from using exponential data weighting and not using exponential weighting (see Figure 8.3). In the simulation, we introduce a unit set-point change for output y_1 and zero set-point signal for output y_2. Without constraints, the responses when using exponential data weighting are almost identical to those from LQR design (not shown here), and exhibit faster set-point responses than those from not using exponential weighting. We also compare the constrained control results with and without exponential data weighting, where we only impose the constraints on the first sample of the control signals. All constraints are satisfied for both cases. Figure 8.4 shows the comparative results. It is seen that the responses are quite different. Again, the output responses from using exponential data weighting are faster than those without exponential data weighting.

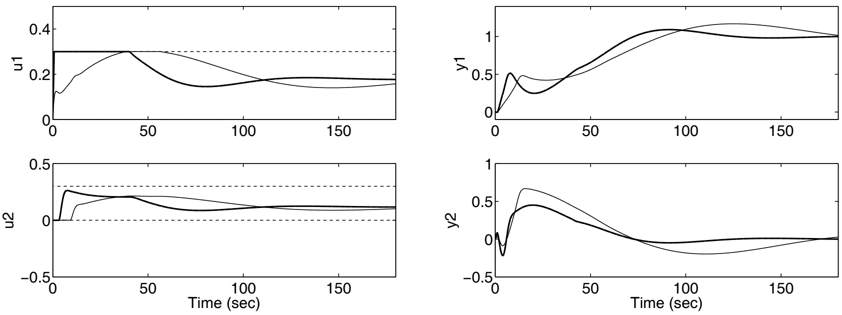
Table 8.3. Elements of the first row in K_{lqr}, K_{mpc} with and without exponential data weighting

K_{lqr}	2.7	−11.2	3.9	−3.0	3.1	1.2	−2.9	9.0	3.5	1.1	0.8	−0.6
$K_{mpc}(\alpha = 0.18)$	2.7	−11.2	3.9	−3.0	3.1	1.2	−2.8	8.9	3.5	1.1	0.8	−0.6
$K_{mpc}(\alpha = 0)$	0.4	−1.7	0.6	−0.4	0.5	0.4	0.6	0.7	−0.1	0.3	0.2	−0.2



(a) Control Signal (b) Output

Fig. 8.3. Comparison of CMPC with and without exponential data weighting. Key: solid-line without exponential data weighting $\alpha = 0$; darker-solid-line with exponential data weighting $\alpha = 0.18$



(a) Control Signal (b) Output

Fig. 8.4. Comparison of CMPC with and without exponential data weighting, in the presence of constraints. Key: solid-line without exponential data weighting $\alpha = 0$; darker-solid-line with exponential data weighting $\alpha = 0.18$

8.7 Summary

This chapter has discussed continuous-time model predictive control with exponential data weighting. In the original design of a continuous-time predictive control system, because of embedded integrator(s) in the model, the prediction horizon is limited to a finite value, and a numerical ill-conditioning problem occurs when the prediction horizon is large. These problems are resolved in this chapter by choosing a cost function with an exponential weight factor $e^{-2\alpha t}$, where $\alpha > 0$:

$$J = \int_0^{T_p} [e^{-2\alpha\tau} x(t_i + \tau \mid t_i)^T Q x(t_i + \tau \mid t_i) + e^{-2\alpha\tau} \dot{u}(\tau)^T R \dot{u}(\tau)] d\tau, \quad (8.53)$$

subject to

$$\dot{x}(t_i + \tau \mid t_i) = Ax(t_i + \tau \mid t_i) + B\dot{u}(\tau).$$

With the exponential weight, the optimization problem at time t_i is solved based on a pair of transformed variables $x_\alpha(t_i+\tau \mid t_i)$ and $u_\alpha(\tau)$ by minimizing

$$J = \int_0^{T_p} \left[x_\alpha(t_i + \tau \mid t_i)^T Q x_\alpha(t_i + \tau \mid t_i) + \dot{u}_\alpha(\tau)^T R \dot{u}_\alpha(\tau)\right] d\tau, \tag{8.54}$$

subject to

$$\dot{x}_\alpha(t_i + \tau \mid t_i) = (A - \alpha I)x_\alpha(t_i + \tau \mid t_i) + B\dot{u}_\alpha(\tau),$$

where the transformed variables are defined by

$$x_\alpha(t_i + \tau \mid t_i) = e^{-\alpha\tau}x(t_i + \tau \mid t_i); \quad u_\alpha(\tau) = e^{-\alpha\tau}u(\tau).$$

The initial conditions are identical when $\tau = 0$. The central idea is that when the system matrix A contains eigenvalues on the imaginary axis or on the right-half of the complex plane, by choosing a suitable $\alpha > 0$ such that the eigenvalues of the modified system matrix $A - \alpha I$ are all strictly on the left-half complex plane, then the model used for prediction is stable, and a sufficiently large prediction horizon can be used in the design. As a consequence, the numerical conditioning problem is overcome. Without any modification on the pair of weight matrices Q, R, the solution does not guarantee exponential decay of the original variable $x(t_i + \tau \mid t_i)$ within the optimization window. To resolve this issue, a simple modification of the weight matrix Q is proposed. Choosing $Q_\alpha = Q + 2\alpha P$, $\alpha > 0$, R unchanged, the optimal control $\dot{u}(\tau)$ is obtained by minimizing

$$J = \int_0^{T_p} \left[x_\alpha(t_i + \tau \mid t_i)^T Q_\alpha x_\alpha(t_i + \tau \mid t_i) + \dot{u}_\alpha(\tau)^T R \dot{u}_\alpha(\tau)\right] d\tau, \tag{8.55}$$

subject to

$$\dot{x}_\alpha(t_i + \tau \mid t_i) = (A - \alpha I)x_\alpha(t_i + \tau \mid t_i) + B\dot{u}_\alpha(\tau),$$

where P is the solution of the steady-state Riccati equation:

$$PA + A^T P - PBR^{-1}B^T P + Q = 0.$$

In fact, the optimal solution with the exponentially weighted cost function is identical to the original optimal control solution without exponential weighting, when the prediction horizon is sufficiently large. The proposed approach is not only numerically sound, but also allows the use of a sufficiently large prediction horizon to guarantee asymptotic stability.

To introduce a prescribed degree of stability β in the predictive control system such that all the closed-loop eigenvalues are on the left of the line $s = -\beta$ in the complex plane, we only need to choose Q_α as

$$Q_\alpha = Q + 2(\alpha + \beta)P,$$

and R unchanged and minimize the cost function J given by (8.55) with transformed variables x_α and u_α, where P is the solution of the steady-state Riccati equation:

$$P(A + \beta I) + (A + \beta I)^T P - PBR^{-1}B^T P + Q = 0.$$

When imposing constraints, all the constraints are transformed and expressed using the exponentially weighted variables.

Problems

8.1. A mathematical model for an inverted pendulum is described by the Laplace transfer function:

$$G(s) = \frac{-K_i}{s^2 - a^2}, \tag{8.56}$$

where the input to the inverted pendulum is external force, and output is angle θ (rad) (see Figure 8.5). The parameters in the model are $K_i = 0.01$ and $a = 3$.

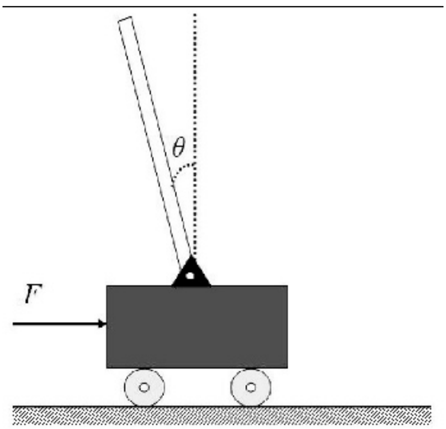


Fig. 8.5. Schematic diagram for an inverted pendulum

Since the inverted pendulum is an unstable system with one pole on the right-half of the complex plane, the choice of the prediction horizon needs careful consideration without using exponential data weighting. The design parameters are $N = 4$, $p = 3.3$, $Q = C^T C$ (C is the output matrix of the augmented model) and $R = 0.1$. Design a continuous-time predictive control system with a final prediction horizon that will bring the angle θ as close as possible to 0^o in the presence of input step disturbance.

1. Show that the Hessian matrix Ω in the cost function

$$J = \eta^T \Omega \eta + 2\eta^T \Psi$$

is numerically ill-conditioned by examining its condition number with respect to an increasing prediction horizon T_p.
2. Demonstrate that this numerical sensitivity causes the variations of the closed-loop feedback control gain K_{mpc} and the closed-loop pole locations.

8.2. Continue from Problem 8.1 and use exponential data weighting in the design of predictive control for this inverted pendulum.

1. Choose the exponential weight factor $\alpha = 3.8$ that is greater than the unstable pole, and modify the weight matrix Q_α according to

$$Q_\alpha = Q + 2\alpha P;\ PA + A^T P - PBR^{-1}B^T P + Q = 0,$$

where A and B are the matrices in the augmented state-space model.
2. With exponential data weighting, examine the elements of Ω and Ψ matrices in the cost function

$$J = \eta^T \Omega \eta + 2\eta^T \Psi,$$

as functions of prediction horizon T_p and demonstrate graphically that the diagonal elements in Ω converge to constants as T_p increases.
3. For a large T_p, compute the closed-loop feedback control gain K_{mpc} and the closed-loop poles with the predictive control system. Compare them with K_{lqr} and the LQR closed-loop poles (use the MATLAB lqr function for this computation).
4. If K_{mpc} and K_{lqr} are not sufficiently close to your expectation, increase the number of terms N in the Laguerre functions to improve the accuracy.

8.3. A continuous-time system has three inputs and two outputs described by the Laplace transfer function

$$G(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) & G_{13}(s) \\ G_{21}(s) & G_{22}(s) & G_{23}(s) \end{bmatrix}, \tag{8.57}$$

where $G_{11}(s) = \frac{1}{(s+1)^3}$, $G_{12}(s) = \frac{0.1}{0.1s+1}$, $G_{13}(s) = \frac{-0.8}{s+4}$, $G_{21}(s) = \frac{0.01}{s+1}$, $G_{22}(s) = \frac{(-3s+1)}{(10s+1)(3s+1)}$, $G_{23}(s) = \frac{-0.4}{0.3s+1}$.

1. Find the state-space model and augment it with integrators.
2. Choose $Q = C^T C$ and $R = I$, $p_1 = p_2 = p_3 = 0.8$, and $N_1 = N_2 = N_3 = 3$ as the design parameters. Find the matrices Ω and Ψ in the cost function of the predictive control J, where J is expressed as

$$J = \eta^T \Omega \eta + 2\eta^T \Psi,$$

such that the closed-loop eigenvalues of the predictive control system are positioned on the left of a line $s = -1$ in the complex plane. Hint: you need to solve the following Riccati equation to find P matrix

$$P(A + I) + (A + I)^T P - PBR^{-1}B^T P + Q = 0,$$

with exponential weight factor $\alpha > 0$ (say $\alpha = 0.5$), Q is modified to $Q_\alpha = Q + 2(\alpha + 1)P$. The solution of the Riccati equation is performed using MATLAB lqr function.

3. We can also design an observer with the observer poles to be constrained. For instance,if we want to position the observer poles on the left of a line $s = -\gamma$ ($\gamma > 0$) in the complex plane, we choose $Qob = I$ and $Rob = 0.1I$, and then modify A with $A + \gamma I$. The MATLAB script for doing this is

```
K_ob=lqr((A+gamma*eye(n,n))',C',Qob,Rob)';
```

where n is the dimension of A matrix. Find the observer K_{ob} such that the closed-loop observer poles are on the left of a line $s = -2$ in the complex plane.

8.4. Consider the problems of using a prescribed degree of stability to improve the robustness of a predictive control system. Assume that the open-loop system is described by a transfer function

$$G(s) = \frac{K}{(10s + 1)^2(s - 0.3)}.$$

1. Assuming $K = 1$, design a predictive control system with prescribed degree of stability $\beta = 0.4$ (*i.e.*, all closed-loop eigenvalues are on the left of a line $s = -0.4$ in the complex plane). The remaining design parameters are specified as $N = 6$, $p = 0.6$, $Q = C^T C$, $R = 1$ and $T_p = 35$. Parameter α is chosen as $\alpha = 0.38$ to be greater than the magnitude of the unstable pole. An observer is used in the implementation. The weight matrices for the observer are $Qob = I$ and $Rob = 0.0001$.
2. Construct the closed-loop predictive control system, respectively, with $K = 0.8, 0.9, 1, 1.2, 1.4$ and 1.6, and calculate the closed-loop eigenvalues with the variations of parameter K, show that the closed-loop system is stable for this range of parameters.
3. Repeat the design with $\beta = 0$, but the other design parameters remaining the same. Show that the closed-loop system is only stable with respect to the changes of K between 0.9 and 1.1. Present your comparative results using a tabulation of closed-loop eigenvalues, and comment on your findings.