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Networked Boundary Control of Damped Wave Equations

YangQuan Chen

Utah State University, Logan, UT 84322, USA yqchen@ece.usu.edu

Abstract. This chapter considers the boundary control of damped wave equations using a boundary measurement in a networked control system (NCS) setting. In this networked boundary control system, the induced delays can be lumped as the boundary measurement delay. The Smith predictor is applied to the networked boundary control problem and the instability problem due to large delays is solved and the scheme is proved to be robust against a small difference between the assumed delay and the actual delay. In addition, we analyze the robustness of the time-fractional order wave equation with a fractional order boundary controller subject to delayed boundary measurement. Conditions are given to guarantee stability when the delay is small. For large delays, again the Smith predictor is applied to solve the instability problem and the scheme is proved to be robust against a small difference between the assumed delay and the actual delay. The analysis shows that fractional order controllers are better than integer order controllers in the robustness against delays in the boundary measurement.

Keywords. Boundary control, distributed parameter system, fractional order calculus, robustness, wave equation.

9.1 Introduction

In recent years, boundary control of flexible systems has become an active research area, due to the increasing demand on high precision control of many mechanical systems, such as spacecraft with flexible attachments or robots with flexible links, which are governed by PDEs (partial differential equations) rather than ODEs (ordinary differential equations) [2, 3, 4, 7, 8, 18, 19, 20, 21]. In this research area, the robustness of controllers against delays is an important topic and has been studied by many researchers [5, 6, 14, 15, 17], due to the fact that delays are unavoidable in practical engineering. All the available publications focus on the analysis of systems against a small delay, i.e., under what conditions a very small delay will not cause instability problems and can therefore be neglected. An equally important and very practical issue is, how to synthesize a boundary controller when the delay is large and makes the system unstable. To the best of our knowledge, publications studying this problem are very few. In this chapter, we solve the instability problem caused by large delays by applying the Smith predictor to the boundary control of the damped wave equation. The control scheme is shown to be stable and robust against a small difference between the actual delay and the assumed delay.

Fractional diffusion and wave equations are obtained from the classical diffusion and wave equations by replacing the first and second order time derivative term by a fractional derivative of an order satisfying $0 < \alpha \le 1$ and $1 < \alpha \le 2$, respectively. Since many of the universal phenomena can be modeled accurately using the fractional diffusion and wave equations (see [22]), there has been growing interest in investigating the solutions and properties of these evolution equations. Compared with the publications on control of integer order PDEs, results on control of fractional wave equations are relatively few [10, 11, 16].

In this chapter, we will also investigate two robust stabilization problems of the fractional wave equations subject to delayed boundary measurement. First, under what conditions a very small delay in boundary measurement will not cause instability problems. Second, how to stabilize the system when the delay is large and makes the system unstable.

9.2 A Brief Introduction to the Smith Predictor

The Smith predictor was proposed by Smith in [24] and is probably the most famous method for control of systems with time delays; see [9] and [25]. Consider a typical feedback control system with a time delay in Fig. 9.1, where $C(s)$ is the controller and $P(s)e^{-\theta s}$ is the plant with a time delay θ.

With the presence of the time delay, the transfer function of the closed-loop system relating the output $y(s)$ to the reference $r(s)$ becomes

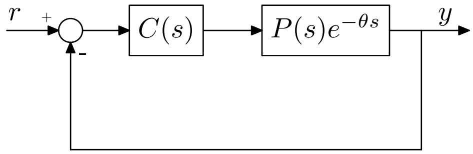

Fig. 9.1. A feedback control system with a time delay

$$\frac{y(s)}{r(s)} = \frac{C(s)P(s)e^{-\theta s}}{1 + C(s)P(s)e^{-\theta s}}. \tag{9.1}$$

Obviously, the time delay θ directly changes the closed-loop poles. Usually, the time delay reduces the stability margin of the control system, or more seriously, destabilizes the system.

The classical configuration of a system containing a Smith predictor is depicted in Fig. 9.2, where $\hat{P}(s)$ is the assumed model of $P(s)$ and $\hat{\theta}$ is the assumed delay. The block $C(s)$ combined with the block $\hat{P}(s) - \hat{P}(s)e^{-\hat{\theta}s}$ is called "the Smith predictor". If we assume perfect model matching, i.e., $\hat{P}(s) = P(s)$ and $\theta = \hat{\theta}$, the closed-loop transfer function becomes

$$\frac{y(s)}{r(s)} = \frac{C(s)P(s)e^{-\theta s}}{1 + C(s)P(s)}. \tag{9.2}$$

Now, it is clear what the underlying idea of the Smith predictor is. With perfect model matching, the time delay can be removed from the denominator of the transfer function, making the closed-loop stability independent of the time delay.

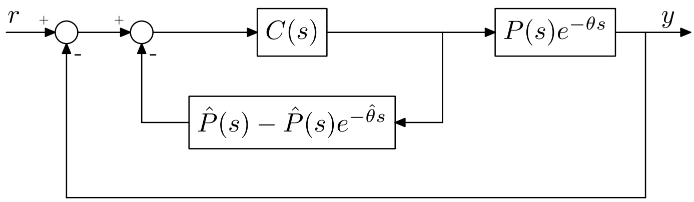

Fig. 9.2. The Smith predictor

9.3 Boundary Control of Damped Wave Equations with Large Delays

Consider a string clamped at one end and free at the other end. We denote the displacement of the string by $u(x,t)$, where $x \in [0,1]$ and $t \geq 0$. The string is controlled by a boundary control force at the free end. The governing equations are given as

$$u_{tt}(x,t) - u_{xx}(x,t) + 2au_t(x,t) + a^2u(x,t) = 0, \tag{9.3}$$

$$u(0,t) = 0, \tag{9.4}$$

$$u_x(1,t) = f(t), \tag{9.5}$$

where $a > 0$ is the damping constant and $f(t)$ is the boundary control force applied at the free end of the string.

It is known that the following boundary feedback controller stabilizes the system [2],

$$f(t) = -ku_t(1,t), \tag{9.6}$$

where $k > 0$ is the constant boundary control gain.

Now, we consider the presence of a time delay in the feedback loop, which is shown as follows:

$$f(t) = -ku_t(1,t-\theta), \tag{9.7}$$

where θ is the time delay.

In [6] and [15], it was shown that if k and a satisfy

$$k\frac{e^{2a}+1}{e^{2a}-1} < 1, \tag{9.8}$$

then the delayed feedback systems is stable for all sufficiently small delays.

In this chapter, we will solve the following problem: what if the time delay θ is large enough to make the system unstable? We will apply the Smith predictor to solve this problem.

Comparing Equation (9.7) with Fig. 9.2, we can see that in our case, the plant output y is the tip end displacement $u(1,t)$; the controller $C(s)$ is a derivative controller with the transfer function ks; and $P(s)$ is the transfer function from the control force $f(t)$ to the undelayed displacement of the tip end. If we assume $\hat{P}(s) = P(s)$ and the time delay θ is known, the remaining problem is how to get $P(s)$, which is shown as follows.

Assuming zero initial conditions of $u(x,0)$ and $u_t(x,0)$, take the Laplace transform of (9.3), (9.4), and (9.5) with respect to t, the original PDE of $u(x,t)$ with initial and boundary conditions can be transformed into the following ODE of $U(x,s)$ with boundary conditions:

$$\frac{d^2U(x,s)}{dx^2} - (s+a)^2U(x,s) = 0, \tag{9.9}$$

$$U(0,s) = 0, \tag{9.10}$$

$$U_x(1,s) = F(s), \tag{9.11}$$

where $U(x,s)$ is the Laplace transform of $u(x,t)$ and $F(s)$ is the Laplace transform of $f(t)$.

Solving the ODE (9.9), we have the following solution of $U(x,s)$ with two arbitrary constants C_1 and C_2 (s can be treated as a constant in this step),

$$U(x,s) = C_1e^{-(s+a)x} + C_2e^{(s+a)x}. \tag{9.12}$$

Substitute (9.12) into (9.10) and (9.11), we have the following two equations:

$$C_1 + C_2 = 0, \tag{9.13}$$

$$(-C_1 e^{-(s+a)} + C_2 e^{s+a})(s+a) = F(s). \tag{9.14}$$

Solving (9.13) and (9.14) simultaneously, we can obtain the exact values of C_1 and C_2

$$C_1 = \frac{-F(s)}{(s+a)(e^{-(s+a)} + e^{s+a})}, \tag{9.15}$$

$$C_2 = \frac{F(s)}{(s+a)(e^{-(s+a)} + e^{s+a})}. \tag{9.16}$$

Now we have obtained the solution of $U(x,s)$. Substituting $x = 1$ into $U(x,s)$, we obtain the following Laplace transform of the tip end displacement.

$$U(1,s) = \frac{F(s)(1 - e^{-2(s+a)})}{(s+a)\left(1 + e^{-2(s+a)}\right)}. \tag{9.17}$$

So the transfer function of the plant, which is $P(s)$ in Fig. 9.2, is obtained as

$$P(s) = \frac{U(1,s)}{F(s)} = \frac{1 - e^{-2(s+a)}}{(s+a)\left(1 + e^{-2(s+a)}\right)}. \tag{9.18}$$

Finally, we have the following expression for the boundary controller (the Smith predictor), denoted as $C_{sp}(s)$:

$$C_{sp}(s) = \frac{ks}{1 + ksP(s)(1 - e^{-\hat{\theta}s})}. \tag{9.19}$$

Notice that the controller (9.19) is physically implementable.

9.4 Stability and Robustness Analysis

In [2], the stability of the controller (9.6) was proved for the boundary control of the damped wave equation without delays. If the assumed delay is equal to the actual delay, the Smith predictor removes the delay term completely from the denominator of the closed-loop transfer function. This means the stability of the controller (9.19) is already proved.

Since the actual delay θ and the assumed delay $\hat{\theta}$ cannot be exactly the same, another important issue is the robustness of the controller (9.19), i.e., what if an unknown small difference ϵ between the assumed delay and the actual delay is introduced to the system, as shown in Fig. 9.3.

To study the robustness of the controller (9.19), we will first introduce a theorem presented in [14, 15].

Theorem 9.1. Let $H(s)$ be the open-loop transfer function as illustrated in Fig. 9.4, and $\mathfrak{D}_H$ the set of all its poles. Define two closed-loop transfer functions $G_0(s)$ and $G_\epsilon(s)$ as

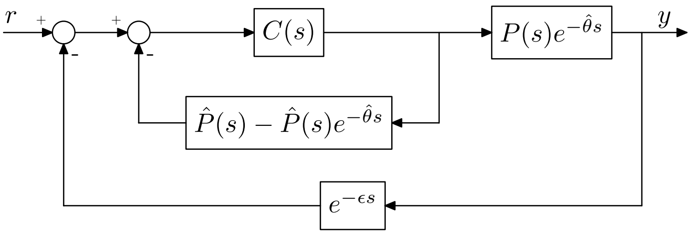

Fig. 9.3. System with mis-matched delays

$$G_0(s) = \frac{H(s)}{1 + H(s)},$$

and

$$G_\epsilon(s) = \frac{H(s)}{1 + e^{-\epsilon s} H(s)}.$$

Define again

$$\mathbb{C}_0 = \{s \in \mathbb{C} | \Re(s) > 0\},$$

and

$$\gamma(H(s)) = \limsup_{|s| \to \infty, s \in \mathbb{C}_0 \setminus \mathfrak{D}_H} |H(s)|.$$

Suppose G_0 is L^2-stable. If $\gamma(H) < 1$, then there exists ϵ^* such that G_ϵ is L^2-stable for all $\epsilon \in (0, \epsilon^*)$. □

The underlying idea of the above theorem is that the robustness of the closed-loop transfer function $G_0(s)$ against a small unknown delay can be determined by studying the open-loop transfer function $H(s)$. Now we can prove the robustness of the controller (9.19).

Claim. If $\hat{\theta}$ is chosen as the minimum value of the possible delay and k is chosen to satisfy

$$k \frac{e^{2a} + 1}{e^{2a} - 1} \leq \frac{1}{3}, \tag{9.20}$$

then the controller (9.19) is robust against a small difference ϵ between the assumed delay $\hat{\theta}$ and the actual delay $\theta = \hat{\theta} + \epsilon$.

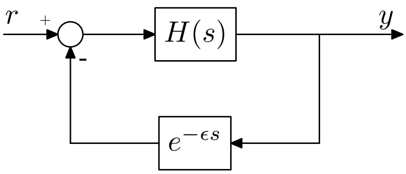

Fig. 9.4. Feedback system with delay

Proof. For

$$H(s) = C_{sp}(s)P(s)e^{-\hat{\theta}s} = \frac{ksP(s)e^{-\theta s}}{1 + ksP(s)(1 - e^{-\theta s})}.$$

Let $T(s) = ksP(s)$. Then

$$|H(s)| = \frac{1}{\left|\left(\frac{1}{T(s)} + 1\right) e^{\hat{\theta}s} - 1\right|}. \tag{9.21}$$

Let $Q(s) = \left(\frac{1}{T(s)} + 1\right) e^{\hat{\theta}s} - 1$. Then

$$\begin{aligned} |Q(s)| &= \left|\left(\frac{1}{T(s)} + 1\right) e^{\hat{\theta}s} - 1\right| \\ &\geq \left|\left|\left(\frac{1}{T(s)} + 1\right) e^{\hat{\theta}s}\right| - 1\right| \\ &\geq \left|\left|\frac{1}{T(s)} + 1\right| \left|e^{\hat{\theta}s}\right| - 1\right|. \end{aligned} \tag{9.22}$$

In [15], it was proved that

$$\limsup_{|s|\to\infty, s\in\mathbb{C}_0} |T(s)| = k\frac{e^{2a} + 1}{e^{2a} - 1}.$$

So, if

$$k\frac{e^{2a} + 1}{e^{2a} - 1} \leq \frac{1}{3},$$

for $|s|$ large enough,

$$\left|\frac{1}{T(s)} + 1\right| \geq \left|\left|\frac{1}{T(s)}\right| - 1\right| \geq 2. \tag{9.23}$$

Considering $|e^{\hat{\theta}s}| > 1$, we have

$$|Q(s)| > 1. \tag{9.24}$$

So

$$\limsup_{|s|\to\infty, s\in\mathbb{C}_0} |H(s)| < 1. \tag{9.25}$$

□

Remark 9.1. In Theorem 9.1, ϵ is positive. To satisfy this condition, $\hat{\theta}$ should be chosen as the minimal value of the possible delay.

The damping constant a plays a key role in making the controllers (both the original derivative controller ks and the Smith predictor) robust. if $a =$

0, the damped wave equation becomes the conservative wave equation, the transfer function of which is

$$P(s) = \frac{1 - e^{-2s}}{s(1 + e^{-2s})}. \tag{9.26}$$

We can see that $P(s)$ has an infinite number of poles on the imaginary axis. In order to make $\gamma(H(s)) < 1$, controllers must cancel these poles completely, which is impossible due to the uncertainty of the plant parameters. This means both the original derivative controller ks and the Smith predictor are not robust when applied to the boundary control of conservative wave equation. □

9.5 Fractional Order Case – Problem Formulation

We consider a cable made with special smart materials governed by the fractional wave equation, fixed at one end, and stabilized by a boundary controller at the other end. Omitting the mass of the cable, the system can be represented by

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2}, \quad 1 < \alpha \le 2, \quad x \in [0,1], \quad t \ge 0 \tag{9.27}$$

$$u(0,t) = 0, \tag{9.28}$$

$$u_x(1,t) = f(t), \tag{9.29}$$

$$u(x,0) = u_0(x), \tag{9.30}$$

$$u_t(x,0) = v_0(x), \tag{9.31}$$

where $u(x,t)$ is the displacement of the cable at $x \in [0,1]$ and $t \ge 0$, $f(t)$ is the boundary control force at the free end of the cable, $u_0(x)$ and $v_0(x)$ are the initial conditions of displacement and velocity, respectively.

The control objective is to stabilize $u(x,t)$, given the initial conditions (9.30) and (9.31).

We adopt the following Caputo definition for fractional derivative of order α of any function $f(t)$, because the Laplace transform of the Caputo derivative allows utilization of initial values of classical integer-order derivatives with known physical interpretations [1, 23]

$$\frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(\alpha - n)} \int_0^t \frac{f^{(n)}(\tau) d\tau}{(t-\tau)^{\alpha+1-n}}, \tag{9.32}$$

where n is an integer satisfying $n - 1 < \alpha \le n$ and Γ is Euler's Gamma function.

In this chapter, we study the robustness of the controllers in the following format:

$$f(t) = -k \frac{\mathrm{d}^\mu u(1,t)}{\mathrm{d}t^\mu}, \quad 0 < \mu \le 1 \tag{9.33}$$

where k is the controller gain, μ is the order of fractional derivative of the displacement at the free end of the cable.

Based on the definition in (9.32), the Laplace transform of the fractional derivative is [1, 23]:

$$\mathcal{L}\left\{\frac{\mathrm{d}^\alpha f}{\mathrm{d}t^\alpha}\right\} = s^\alpha F(s) - \sum_{k=0}^{n-1} f^k(0^+)s^{\alpha-1-k}. \tag{9.34}$$

In the following, the transfer function from the boundary controller $f(t)$ to the tip end displacement will be derived for later use.

Assuming zero initial conditions of $u(x,0)$ and $u_t(x,0)$, take the Laplace transform of (9.27), (9.28), and (9.29) with respect to t, making use of (9.34), the original PDE of $u(x,t)$ with initial and boundary conditions can be transformed into the following ODE of $U(x,s)$ with boundary conditions,

$$\frac{\mathrm{d}^2U(x,s)}{\mathrm{d}x^2} - s^\alpha U(x,s) = 0, \tag{9.35}$$

$$U(0,s) = 0, \tag{9.36}$$

$$U_x(1,s) = F(s), \tag{9.37}$$

where $U(x,s)$ is the Laplace transform of $u(x,t)$ and $F(s)$ is the Laplace transform of $f(t)$.

Solving the ODE (9.35), we have the following solution of $U(x,s)$ with two arbitrary constants C_1 and C_2 (s can be treated as a constant in this step),

$$U(x,s) = C_1 e^{xs^{\alpha/2}} + C_2 e^{-xs^{\alpha/2}}. \tag{9.38}$$

Substituting (9.38) into (9.36) and (9.37), we have the following two equations,

$$C_1 + C_2 = 0, \tag{9.39}$$

$$s^{\alpha/2}(C_1 e^{s^{\alpha/2}} - C_2 e^{-s^{\alpha/2}}) = F(s). \tag{9.40}$$

Solving (9.39) and (9.40) simultaneously, we can obtain the exact value of C_1 and C_2

$$C_1 = -C_2 = \frac{F(s)e^{s^{\alpha/2}}}{s^{\alpha/2}(e^{2s^{\alpha/2}} + 1)}. \tag{9.41}$$

Now we have obtained the solution of $U(x,s)$. Substituting $x = 1$ into $U(x,s)$ and dividing $U(x,s)$ by $F(s)$, we obtain the following transfer function of the fractional wave equation $P(s)$:

$$P(s) = \frac{U(1,s)}{F(s)} = \frac{1 - e^{-2s^{\alpha/2}}}{s^{\alpha/2}\left(1 + e^{-2s^{\alpha/2}}\right)}. \tag{9.42}$$

9.6 Fractional Order Case – Robustness of Boundary Stabilization

We consider the presence of a very small time delay θ in boundary measurement, shown as follows

$$f(t) = -ku_t^{(\mu)}(1, t-\theta), \tag{9.43}$$

where θ is the time delay.

The situation is also illustrated in Fig.9.1, where $P(s)$ is the transfer function of the plant and $C(s)$ is the Laplace transform of the controller. In our case, $P(s)$ is (9.42) and $C(s)$ is

$$C(s) = k\, s^{\mu}. \tag{9.44}$$

In [5, 6, 14, 15], it was shown that an arbitrarily small delay in boundary measurement causes the instability problem in boundary control of wave equations using integer order controllers $f(t) = -ku_t(1,t)$. Does this problem exist in boundary control of the fractional wave equation? Since fractional order controllers are chosen in this chapter, will this additional tuning knob bring us any benefits of robustness against the small delay? To answer these questions, we will use Theorem 9.1 in Section 9.4 [14, 15].

Again, the underlying idea of the above theorem is that the robustness of the closed-loop transfer function $G_0(s)$ against a small unknown delay can be determined by studying the open-loop transfer function $H(s)$. Notice that $H(s) = C(s)P(s)$ in our case.

Claim. If the derivative order μ of controller (9.33) and the fractional order α in the fractional wave equation (9.27) satisfy

$$\mu < \frac{\alpha}{2}, \tag{9.45}$$

then the system is stable for a delay θ small enough in boundary measurement.

Proof. For $s \in \mathbb{C}_0$,

$$\begin{aligned} |H(s)| &= |C(s)P(s)| \\ &= \left| \frac{ks^{\mu}(1-e^{-2s^{\alpha/2}})}{s^{\alpha/2}\left(1+e^{-2s^{\alpha/2}}\right)} \right| \\ &= \left| \frac{k(1-e^{-2s^{\alpha/2}})}{s^{(\alpha/2-\mu)}\left(1+e^{-2s^{\alpha/2}}\right)} \right| \\ &\le \frac{k|1-e^{-2s^{\alpha/2}}|}{|s^{(\alpha/2-\mu)}||1+e^{-2s^{\alpha/2}}|}. \end{aligned} \tag{9.46}$$

Since $\frac{\alpha}{2} > \mu$, $|s^{(\alpha/2-\mu)}| \to \infty$ for $|s| \to \infty$.

Since $\frac{1}{2} < \frac{\alpha}{2} < 1$, for $|s|$ large enough, $|1 - e^{-2s^{\alpha/2}}|$ is bounded and

$$|1 - e^{-2s^{\alpha/2}}| > \eta > 0,$$

where η is a positive number.

So

$$\limsup_{|s|\to\infty, s\in\mathbb{C}_0} |H(s)| = 0 < 1. \qquad \square$$

Following the above proof, it can easily be proved that an integer order controller $f(t) = -ku_t(1,t)$ is not robust against an arbitrarily small delay.

9.7 Fractional Order Case – Compensation of Large Delays in Boundary Measurement

In the last section, it is shown that a fractional order controller is robust against a small delay under the condition (9.45). In this section, we investigate the problem that the delay is large and makes the system unstable. We will apply the Smith predictor to solve this problem.

In Section 9.2, it is shown that if the assumed delay is equal to the actual delay, the Smith predictor removes the delay term completely from the denominator of the closed-loop. However, the actual delay is not exactly known. In this section, we will investigate what happens if an unknown small difference ϵ between the assumed delay and the actual delay is introduced to the system, as shown in Fig. 9.3.

Claim. If $\hat{\theta}$ is chosen as the minimum value of the possible delay and μ is chosen to satisfy (9.45), then the controller (9.19) is robust against a small difference ϵ between the assumed delay $\hat{\theta}$ and the actual delay $\theta = \hat{\theta} + \epsilon$.

Proof. For $s \in \mathbb{C}_0$,

$$\begin{aligned}|H(s)| &= \left| \frac{ks^{\mu}P(s)e^{-\hat{\theta}s}}{1 + ks^{\mu}P(s)(1 - e^{-\hat{\theta}s})} \right| \\ &\le \frac{k|1 - e^{-2s^{\alpha/2}}||e^{-\theta s}|}{|s^{(\alpha/2-\mu)}(1 + e^{-2s^{\alpha/2}}) + k(1 - e^{-2s^{\alpha/2}})(1 - e^{-\theta s})|} \\ &< \frac{k|1 - e^{-2s^{\alpha/2}}|}{||s^{(\alpha/2-\mu)}(1 + e^{-2s^{\alpha/2}})| - k|(1 - e^{-2s^{\alpha/2}})(1 - e^{-\theta s})||}.\end{aligned}$$

When $|s| \to \infty$,

$$|s^{(\alpha/2-\mu)}(1 + e^{-2s^{\alpha/2}})| \to \infty,$$

while both $|1 - e^{-2s^{\alpha/2}}|$ and $|(1 - e^{-2s^{\alpha/2}})(1 - e^{-\theta s})|$ are bounded.

So

$$\limsup_{|s|\to\infty, s\in\mathbb{C}_0} |H(s)| = 0 < 1. \qquad \square$$

Remark 9.2. In Theorem 9.1, ϵ is positive. To satisfy this condition, $\hat{\theta}$ should be chosen as the minimal value of the possible delay. □

9.8 Conclusions

For both integer order and fractional order cases, this chapter considers the boundary control of damped wave equations using a boundary measurement in a networked control system (NCS) setting. In this networked boundary control system, the induced delays can be lumped as the boundary measurement delay. The Smith predictor is applied to the networked boundary control problem and the instability problem due to large delays is solved and the scheme is proved to be robust against a small difference between the assumed delay and the actual delay. Our analysis shows that fractional order boundary controllers are better than integer order boundary controllers in terms of robustness against delays in the boundary measurement.

Future work includes studying the robustness of the controller against plant modeling errors and the controller performance of the Smith predictor.

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