Reliability of Semi-Markov Systems in Discrete Time: Modeling and Estimation

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Abstract: This chapter presents the reliability of discrete-time semi-Markov systems. After some basic definitions and notation, we obtain explicit forms for reliability indicators. We propose non-parametric estimators for reliability, availability, failure rate, mean hitting times and we study their asymptotic properties. Finally, we present a three state example with detailed calculations and numerical evaluations.

24.1 Introduction

In the last 50 years, a lot of work has been carried out in the field of probabilistic and statistical methods in reliability. We do not intend to provide here an overview of the field, but only to point out some bibliographical references that are close to the work presented in this chapter. More precisely, we are interested in discrete-time models for reliability and in models based on semi-Markov processes that extend the classical *i.i.d.* or Markovian approaches. The generality is important, because we pass from a geometric distributed sojourn time in the Markov case, to a general distribution on the set of non-negative integers $\mathbb{N}$, like the discrete-time Weibull distribution.

It is worth noticing here that most mathematical models for reliability consider time to be continuous. However, there are real situations when systems have natural discrete lifetimes. We can cite here those systems working on demand, those working on cycles or those monitored only at certain discrete times (once a month, say). In such situations, the lifetimes are expressed in terms of the number of working periods, the number of working cycles or the number of months before failure. In other words, all these lifetimes are intrinsically discrete. However, even in the continuous-time modeling case, we pass to the numerical calculus by first discetizing the concerned model. A good overview of discrete probability distributions used in reliability theory can be found in [1].

Several authors have studied discrete-time models for reliability in a general *i.i.d.* setting (see [1–4]). The discrete-time reliability modeling via homogeneous and non-homogeneous Markov chains can be found in [5, 6]. Statistical estimations and asymptotic properties for reliability metrics, using discrete-time homogeneous Markov chains, are presented in [7]. The continuous-time semi-Markov model in reliability can be found in [8–10].

As compared to the attention given to the continuous-time semi-Markov processes and related inference problems, the discrete-time semi-Markov processes (DTSMP) are less studied. For an introduction to discrete-time renewal processes, see, for instance, [11]; an introduction to DTSMP can be found in [12–14]. The reliability of discrete-time semi Markov systems is investigated in [14–18] and in [22].

We present here a detailed modeling of reliability, availability, failure rate and mean times, with closed form solutions and statistical estimation based on a censured trajectory in the time interval $[0, M]$. The discrete time modeling presented here is more adapted to applications and is numerically easy to implement using computer software, in order to compute and estimate the above metrics.

The present chapter is structured as follows. In Section 24.2, we define homogeneous discrete-time Markov renewal processes, homogeneous semi-Markov chains and we establish some basic notation. In Section 24.3, we consider a repairable discrete-time semi-Markov system and obtain explicit forms for reliability measures: reliability, availability, failure rate and mean hitting times. Section 24.4 is devoted to the non-parametric estimation. We first obtain estimators for the characteristics of a semi-Markov system. Then, we propose estimators for measures of the reliability and we present their asymptotic properties. We end this chapter by a numerical application.

24.2 The Semi-Markov Setting

In this section we define the discrete-time semi-Markov model, introduce the basic notation and definitions and present some probabilistic results on semi-Markov chains.

Consider a random system with finite state space $E = \{1, \ldots, s\}$. We denote by $\mathcal{M}_E$ the set of matrices on $E \times E$ and by $\mathcal{M}_E(\mathbb{N})$ the set of matrix-valued functions defined on the set of non-negative integers $\mathbb{N}$, with values in $\mathcal{M}_E$. For $A \in \mathcal{M}_E(\mathbb{N})$, we write $A = (A(k); k \in \mathbb{N})$, where, for $k \in \mathbb{N}$ fixed, $A(k) = (A_{ij}(k); k \in E) \in \mathcal{M}_E$. Put $\mathbf{I}_E \in \mathcal{M}_E$ for the identity matrix and $\mathbf{0}_E \in \mathcal{M}_E$ for the null matrix.

We suppose that the evolution in time of the system is described by the following chains (see Figure 24.1.):

- The chain $J = (J_n)_{n \in \mathbb{N}}$ with state space E, where J_n is the system state at the n th jump time.
- The chain $S = (S_n)_{n \in \mathbb{N}}$ with state space $\mathbb{N}$, where S_n is the n th jump time. We suppose that $S_0 = 0$ and $0 < S_1 < S_2 < \ldots < S_n < S_{n+1} < \ldots$.
- The chain $X = (X_n)_{n \in \mathbb{N}^*}$ with state space $\mathbb{N}^*$, where X_n is the sojourn time in state J_{n-1} before the n th jump. Thus, for all $n \in \mathbb{N}^*$, we have $X_n = S_n - S_{n-1}$.

A fundamental notion for semi-Markov systems is that of semi-Markov kernel in discrete time.

Definition 1: A matrix-valued function $\mathbf{q} \in \mathcal{M}_E(\mathbb{N})$ is said to be *a discrete-time semi-Markov kernel* if it satisfies the following three properties:

1. $0 \le q_{ij}(k) \le 1, i, j \in E, k \in \mathbb{N}$;
2. $q_{ij}(0) = 0$ and $\sum_{k=0}^{\infty} q_{ij}(k) \le 1, i, j \in E$;
3. $\sum_{k=0}^{\infty} \sum_{j \in E} q_{ij}(k) = 1, i \in E.$

Definition 2: The chain $(J, S) = (J_n, S_n)_{n \in \mathbb{N}}$ is said to be *a Markov renewal chain* (MRC) if for all $n \in \mathbb{N}$, for all $i, j \in E$ and for all $k \in \mathbb{N}$ it satisfies almost surely

$$\begin{aligned} &\mathbb{P}(J_{n+1} = j, S_{n+1} - S_{n+1} = k \mid J_0, \ldots J_n, S_0, \ldots S_n) \\ &= \mathbb{P}(J_{n+1} = j, S_{n+1} - S_{n+1} = k \mid J_n). \end{aligned}$$

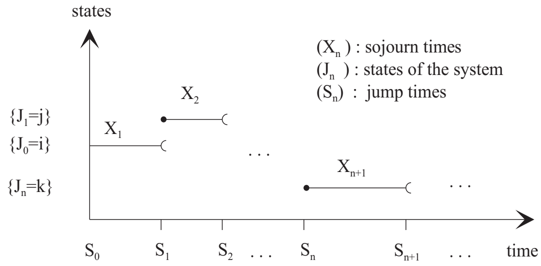

Figure 24.1. A typical sample path of a Markov renewal chain

Moreover, if the previous equation is independent of n, (J,S) is said to be homogeneous and the discrete-time semi-Markov kernel $\mathbf{q}$ is defined by

$$q_{ij}(k) := \mathbb{P}(J_{n+1} = j, X_{n+1} = k | J_n = i).$$

Figure 24.1 provides a representation of the evolution of the system.

We also introduce the cumulative semi-Markov kernel as the matrix-valued function $\mathbf{Q} = (\mathbf{Q}(k); k \in \mathbb{N}) \in \mathcal{M}_E(\mathbb{N})$ defined by

$$Q_{ij}(k) := \mathbb{P}(J_{n+1} = j, X_{n+1} \le k | J_n = i) = \sum_{l=0}^{k} q_{ij}(l), i, j \in E, k \in \mathbb{N}. \quad (24.1)$$

Note that for (J,S) a Markov renewal chain, we can easily see that $(J_n)_{n\in\mathbb{N}}$ is a Markov chain, called *the embedded Markov chain associated to MRC* (J,S). We denote by $\mathbf{p} = (p_{ij})_{i,j\in E} \in \mathcal{M}_E$ the transition matrix of $(J_n)_{n\in\mathbb{N}}$ defined by

$$p_{ij} = \mathbb{P}(J_{n+1} = j | J_n = i), i, j \in E, n \in \mathbb{N}.$$

We also assume that $p_{ii} = 0$, $q_{ii}(k) = 0$, $i \in E$, $k \in \mathbb{N}$, *i.e.,* we do not allow transitions to the same state. Let us define now the conditional sojourn time distributions depending on the next state to be visited and the sojourn time distributions in a given state.

Definition 3: For all $i, j \in E$, let us define:

1. $f_{ij}(\cdot)$, the conditional distribution of $X_{n+1}, n \in \mathbb{N}$,
$$f_{ij}(k) := \mathbb{P}(X_{n+1} = k | J_n = i, J_{n+1} = j), k \in \mathbb{N}.$$
2. $F_{ij}(\cdot)$, the conditional cumulative distribution of $X_{n+1}, n \in \mathbf{N}$,
$$F_{ij}(k) := \mathbb{P}(X_{n+1} \le k | J_n = i, J_{n+1} = j) = \sum_{l=0}^{k} f_{ij}(l), k \in \mathbb{N}.$$

Obviously, for all $i, j \in E$ and for all $k \in \mathbb{N} \cup \{\infty\}$, we have

$$f_{ij}(k) = \begin{cases} q_{ij}(k)/p_{ij}, \text{if } \mathrm{p}_{ij} \neq 0, \\ \mathbf{1}_{\{k=\infty\}}, \text{if } \mathrm{p}_{ij} = 0. \end{cases}$$

Definition 4: For all $i \in E$, let us define:

1. $h_i(\cdot)$, the sojourn time distribution in state i:
$$h_i(k) := \mathbb{P}(X_{n+1} = k | J_n = i) = \sum_{j \in E} q_{ij}(k), k \in \mathbb{N}.$$
2. $H_i(\cdot)$, the sojourn time cumulative distribution function in state i:
$$H_i(k) := \mathbb{P}(X_{n+1} \le k | J_n = i) = \sum_{l=0}^{k} h_i(l), k \in \mathbb{N}.$$

We consider that in each state i the chain stays at least one time unit, *i.e.*, for any state j we have

$$f_{ij}(0) = q_{ij}(0) = h_i(0) = 0.$$

Let us also denote by m_i the mean sojourn time in a state $i \in E$,

$$m_i = \mathbb{E}(S_1 | J_0 = i) = \sum_{k \ge 0} (1 - H_i(k)).$$

For G the cumulative distribution function of a certain *r.v.* X, we denote its *survival function* by $\overline{G}(k) = 1 - G(k) = \mathbf{P}(X > k), k \in \mathbb{N}$. Thus, for all states $i, j \in E$, we put $\overline{F_{ij}}$ and $\overline{H_i}$ for the corresponding survival functions.

The operation which will be commonly used when working on the space $\mathcal{M}_E(\mathbb{N})$ of matrix-valued functions will be the discrete-time matrix convolution product. In the sequel we recall its definition, we see that there exists an identity element, we define recursively the *n*-fold convolution and we introduce the notion of the inverse in the convolution sense.

Definition 5: Let $A, B \in \mathcal{M}_E(\mathbb{N})$ be two matrix-valued functions. *The matrix convolution product* $A * B$ is a matrix-valued function $C \in \mathcal{M}_E(\mathbb{N})$ defined by

$$C_{ij}(k) := \sum_{r \in E} \sum_{l=0}^{k} A_{ir}(k-l) B_{rj}(l), i, j \in E, k \in \mathbb{N}.$$

The following result concerns the existence of the identity element for the matrix convolution product in discrete time.

Lemma 1: Let $\delta I = (d_{ij}(k); i, j \in E) \in \mathcal{M}_E(\mathbb{N})$ be the matrix-valued function defined by

$$d_{ij}(k) := \begin{cases} 1 & \text{if } i = j \text{ and } k = 0, \\ 0 & \text{elsewhere.} \end{cases}$$

Then, δI satisfies

$$\delta I * A = A * \delta I = A, A \in \mathcal{M}_E(\mathbb{N}),$$

i.e., δI is *the identity element* for the discrete-time matrix convolution product.

The power in the sense of convolution is defined straightforwardly, using Definition 5.

Definition 6: Let $A \in \mathcal{M}_E(\mathbb{N})$ be a matrix-valued function and $n \in \mathbb{N}$. *The n-fold convolution* $A^{(n)}$ is a matrix-valued function in $\mathcal{M}_E(\mathbb{N})$ defined recursively by:

$$A_{ij}^{(0)}(k) := d_{ij}(k),$$

$$A_{ij}^{(1)}(k) := A_{ij}(k),$$

$$A_{ij}^{(n)}(k) := \sum_{r \in E} \sum_{l=0}^{k} A_{ir}(k-l) A_{rj}^{(n-1)}(l), n \geq 2, k \in \mathbb{N}.$$

For a MRC (J, S) the n-fold convolution of the semi-Markov kernel has the property expressed in the following result.

Lemma 2: Let $(J, S) = (J_n, S_n)_{n \in \mathbb{N}}$ be a Markov renewal chain and $\mathbf{q} = \left(q_{ij}(k); i, j \in E, k \in \mathbb{N}\right)$ be its associated semi-Markov kernel. Then, for all $n, k \in \mathbb{N}$ such that $n \geq k+1$, we have $\mathbf{q}^{(n)}(k) = 0$.

This property of the discrete-time semi-Markov kernel convolution is essential for the simplicity and the numerical exactitude of the results obtained in discrete time. We need to stress the fact that this property is intrinsic to the work in discrete time and it is no longer valid for a continuous-time Markov renewal process.

Definition 7: Let $A \in \mathcal{M}_E(\mathbb{N})$ be a matrix-valued function. If there exists a $B \in \mathcal{M}_E(\mathbb{N})$ such that $B * A = \delta I$, then B is called *the left inverse of* A *in the convolution sense* and it is denoted by $A^{(-1)}$.

It can be shown that given a matrix-valued function $A \in \mathcal{M}_E(\mathbb{N})$ such that $\det A(0) \neq 0$, then the left inverse B of A exists and is unique (see [14] for the proof).

Let us now introduce the notion of the semi-Markov chain, strictly related to that of the Markov renewal chain.

Definition 8: Let (J, S) be a Markov renewal chain. The chain $Z = (Z_k)_{k \in \mathbb{N}}$ is said to be *a semi-Markov chain* associated to the MRC (J, S), if

$$Z_k := J_{N(k)}, k \in \mathbb{N},$$

where

$$N(k) := \{n \in \mathbb{N} | S_n \leq k\} \qquad (24.2)$$

is the discrete-time counting process of the number of jumps in $[1, k] \subset \mathbb{N}$. Thus, Z_k gives the system state at time k. We also have $J_n = Z_{S_n}, n \in \mathbb{N}$. Let the row vector $\boldsymbol{\alpha} = (\alpha(1), \ldots, \alpha(s))$ denote *the initial distribution* of the semi-Markov chain $Z = (Z_k)_{k \in \mathbb{N}}$, where $\alpha(i) := \mathbf{P}(Z_0 = i) = \mathbf{P}(J_0 = i)$, $i \in E$.

Definition 9: The transition function of the semi-Markov chain Z is the matrix-valued function $\mathbf{P} \in \mathcal{M}_E(\mathbb{N})$ defined by

$$P_{ij}(k) := \mathbb{P}(Z_k = j | Z_0 = i), i, j \in E, k \in \mathbb{N}.$$

The following result consists in a recursive formula for computing the transition function $\mathbf{P}$ of the semi-Markov chain Z.

Proposition 1: For all $i, j \in E$ and for all $k \in \mathbb{N}$, we have

$$P_{ij}(k) = \mathbf{1}_{\{i=j\}}\left(1 - H_i(k)\right) + \sum_{r \in E} \sum_{l=0}^{k} q_{ir}(l) P_{rj}(k-l),$$

where

$$\mathbf{1}_{\{i=j\}} := \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{elsewhere.} \end{cases}$$

Let us define for all $k \in \mathbb{N}$:

- $I(k) := \mathbf{I}_E$ for $k \in \mathbb{N}$, $\mathbf{I} := (I(k); k \in \mathbb{N})$;
- $\mathbf{H}(k) := \operatorname{diag}(H_i(k); i \in E)$,
  $\mathbf{H} := (H(k); k \in \mathbf{N})$.

In matrix-valued function notation, the transition function $\mathbf{P}$ of the semi-Markov chain verifies the equation

$$\mathbf{P} = \mathbf{I} - \mathbf{H} + \mathbf{q} * \mathbf{P}.$$

This is an example of what is called the discrete-time Markov renewal equation. We know that the solution of this equation exists, is unique (see [14]) and, for all $k \in \mathbb{N}$, has the following form:

$$\begin{aligned} \mathbf{P}(k) &= (\delta I - \mathbf{q})^{(-1)} * (\mathbf{I} - \mathbf{H})(k) \\ &= (\delta I - \mathbf{q})^{(-1)} * (\mathbf{I} - \operatorname{diag}(\mathbf{Q} \cdot \mathbf{1}))(k). \end{aligned} \tag{24.3}$$

24.3 Reliability Modeling

In this section we consider a reparable discrete-time semi-Markov system and we obtain closed form solutions for reliability measures: reliability, availability, failure rate, mean time to failure, mean time to repair.

24.3.1 State Space Split

Consider a system (or a component) $\mathcal{S}$ whose possible states during its evolution in time are $E = \{1, \ldots, s\}$. Denote by $U = \{1, \ldots, s_1\}$ the subset of working states of the system (the up-states) and by $D = \{s_1 + 1, \ldots, s\}$ the subset of failure states (the down-states), with $0 < s_1 < s$ (obviously, $E = U \bigcup D$ and $U \bigcap D = \emptyset$, $U \neq \emptyset$, $D \neq \emptyset$). One can think of the states of U as different operating modes or performance levels of the system, whereas the states of D can be seen as failures of the system with different modes. According to the partition of the state space in up-states and down-states, we will partition the vectors, matrices or matrix functions we are working with.

Firstly, for $\boldsymbol{\alpha}, \mathbf{p}, \mathbf{q}(k), \mathbf{f}(k), \mathbf{F}(k), \mathbf{H}(k), \mathbf{Q}(k)$, we consider the natural matrix partition corresponding to the state space partition U and D. For example, we have

$$\mathbf{p} = \begin{pmatrix} \mathbf{p}_{11} & \mathbf{p}_{12} \\ \mathbf{p}_{21} & \mathbf{p}_{22} \end{pmatrix} \text{ and } \mathbf{q}(k) = \begin{pmatrix} \mathbf{q}_{11}(k) & \mathbf{q}_{12}(k) \\ \mathbf{q}_{21}(k) & \mathbf{q}_{22}(k) \end{pmatrix}.$$

Secondly, for $\mathbf{P}(k)$ we consider the restrictions to $U \times U$ and $D \times D$ induced by the corresponding restrictions of the semi-Markov kernel $\mathbf{q}(k)$. To be more specific, using the partition given above for the kernel $\mathbf{q}(k)$, we note that:

$$\mathbf{P}_{11}(k) := (\delta I - \mathbf{q}_{11})^{(-1)} * (\mathbf{I} - \operatorname{diag}(\mathbf{Q} \cdot \mathbf{1})_{11})(k),$$

$$\mathbf{P}_{22}(k) := (\delta I - \mathbf{q}_{22})^{(-1)} * (\mathbf{I} - \operatorname{diag}(\mathbf{Q} \cdot \mathbf{1})_{22})(k).$$

The reasons fort taking this partition for $\mathbf{P}(k)$ can be found in [19].

For $m, n \in \mathbb{N}^*$ such that $m > n$, let $\mathbf{1}_{m,n}$ denote the m-dimensional column vector whose n first elements are 1 and last $m - n$ elements are 0; for $m \in \mathbb{N}^*$ let $\mathbf{1}_m$ denote the m-column vector whose elements are all 1, that is, $\mathbf{1}_m = \mathbf{1}_{m,m}$.

24.3.2 Reliability

Consider a system $\mathcal{S}$ starting to function at time $k = 0$ and let T_D denote the first passage time in subset D, called *the lifetime of the system*, *i.e.*,

$$T_D := \inf\{k \in \mathbb{N} | Z_k \in D\} \text{ and } \inf \emptyset := \infty.$$

The reliability of a discrete-time semi-Markov system $\mathcal{S}$ at time $k \in \mathbb{N}$, that is the probability that the system has functioned without failure in the period $[0, k]$, is

$$R(k) := \mathbb{P}(T_D > k) = \mathbb{P}(Z_n \in U, n = 0, \ldots, k).$$

The following result gives the reliability of the system in terms of the basic quantities of the semi-Markov chain.

Proposition 2: The reliability of a discrete-time semi-Markov system at time $k \in \mathbb{N}$ is given by

$$\begin{aligned} R(k) &= \boldsymbol{\alpha}_1 \mathbf{P}_{11}(k) \mathbf{1}_{s_1} \\ &= \boldsymbol{\alpha}_1 (\delta I - \mathbf{q}_{11})^{(-1)} * (\mathbf{I} - \operatorname{diag}(\mathbf{Q} \cdot \mathbf{1})_{11})(k) \mathbf{1}_{s_1}. \end{aligned}$$

24.3.3 Availability

The point-wise (or instantaneous) availability of a system $\mathcal{S}$ at time $k \in \mathbb{N}$ is the probability that the system is operational at time k (independently of the fact that the system has failed or not in $[0, k)$). So, the point-wise availability of a semi-Markov system at time $k \in \mathbb{N}$ is

$$A(k) := \mathbb{P}(Z_k \in U) = \sum_{i \in E} \alpha(i) A_i(k),$$

where we have denoted by $A_i(k)$ the system's availability at time $k \in \mathbb{N}$, given that it starts in state $i \in E$,

$$A_i(k) = \mathbb{P}(Z_k \in U \mid Z_0 = i).$$

The following result gives an explicit form of the availability of a discrete-time semi-Markov system.

Proposition 3: The point-wise availability of a discrete-time semi-Markov system at time $k \in \mathbb{N}$ is given by

$$\begin{aligned} A(k) &= \boldsymbol{\alpha} \mathbf{P}(k) \mathbf{1}_{s,s_1} \\ &= \boldsymbol{\alpha}(\delta I - \mathbf{q})^{(-1)} * (\mathbf{I} - \operatorname{diag}(\mathbf{Q} \cdot \mathbf{1}))(k) \mathbf{1}_{s,s_1}. \end{aligned}$$

24.3.4 The Failure Rate

We consider here the classical failure rate, introduced by Barlow, Marshall and Proschan in 1963 (see [20]). We call it *the BMP-failure rate* and denote it by $\lambda(k), k \in \mathbb{N}$.

Let $\mathcal{S}$ be a system starting to function at time $k = 0$. The BMP-failure rate at time $k \in \mathbb{N}$ is the conditional probability that the failure of the system occurs at time k, given that the system has worked until time $k-1$.

For a discrete-time semi-Markov system, the failure rate at time $k \geq 1$ has the expression

$$\begin{aligned} \lambda(k) &:= \mathbb{P}(T_D = k \mid T_D \geq k), \\ &= \begin{cases} 1 - \dfrac{R(k)}{R(k-1)}, & R(k-1) \neq 0, \\ 0, & \text{otherwise,} \end{cases} \\ &= \begin{cases} 1 - \dfrac{\boldsymbol{\alpha}_1 \mathbf{P}_{11}(k) \mathbf{1}_{s_1}}{\boldsymbol{\alpha}_1 \mathbf{P}_{11}(k-1) \mathbf{1}_{s_1}}, & R(k-1) \neq 0, \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \quad (24.4)$$

The failure rate at time $k = 0$ is defined by $\lambda(0) := 1 - R(0)$.

It is worth noticing that the failure rate $\lambda(k)$ in discrete-time case is a probability function and not a general positive function as in the continuous-time case. There exists a more recent failure rate, proposed in [2] as being adapted to reliability studies carried out in discrete time. Discussions justifying the use of this discrete-time adapted failure rate can also be found in [3, 4]. In this chapter we do not present this alternative failure rate. Its use for discrete-time semi-Markov systems can be found in [18, 19].

24.3.5 Mean Hitting Times

There are various mean times which are interesting for the reliability analysis of a system. We will be concerned here only with the mean time to failure and the mean time to repair.

We suppose that $\boldsymbol{\alpha}_2 = 0$, *i.e.,* the system starts in a working state. The *mean time to failure* (MTTF) is defined as the mean lifetime, *i.e.*, the expectation of the hitting time to down-set D, $MTTF := \mathbb{E}(T_D)$.

Symmetrically, consider now that $\boldsymbol{\alpha}_1 = 0$, *i.e.* the system fails at the time $t = 0$. Denote by T_U the first hitting time of the up-set U, called *the repair duration*, *i.e.*,

$$T_U := \inf\{k \in \mathbb{N} \mid Z_k \in U\}.$$

The *mean time to repair* (MTTR) is defined as the mean of the repair duration, *i.e.*, $MTTR := \mathbb{E}(T_U)$. The following result gives expressions for the MTTF and the MTTR of a discrete-time semi-Markov system.

Proposition 4: If the matrices $I-\mathbf{p}_{11}$ and $I-\mathbf{p}_{22}$ are non-singular, then

$$MTTF = \boldsymbol{\alpha}_1 (I-\mathbf{p}_{11})^{-1}\mathbf{m}_1,$$
$$MTTR = \boldsymbol{\alpha}_2 (I-\mathbf{p}_{22})^{-1}\mathbf{m}_2,$$

where $\mathbf{m}=(\mathbf{m}_1\ \mathbf{m}_2)^{\mathrm{T}}$ is the partition of the mean sojourn times vector corresponding to the partition of the state space E in up-states U and down-states D. If the matrices are singular, we put $MTTF=\infty$ or $MTTF=\infty$.

24.4 Reliability Estimation

The objective of this section is to provide estimators for reliability indicators of a system and to present their asymptotic properties. In order to achieve this purpose, we firstly show how estimators of the basic quantities of a discrete-time semi-Markov system are obtained.

24.4.1 Semi-Markov Estimation

Let us consider a sample path of a Markov renewal chain $(J_n, S_n)_{n\in\mathbb{N}}$, censored at fixed arbitrary time $M\in\mathbb{N}^*$,

$$\mathcal{H}(M) = (J_0, X_1 \ldots, J_{N(M)-1}, X_{N(M)}, J_{N(M)}, u_M),$$

where $N(M)$ is the discrete-time counting process of the number of jumps in (see (24.2)) and $u_M := M - S_{N(M)}$ is the censored sojourn time in the last visited state $J_{N(M)}$.

Starting from the sample path $\mathcal{H}(M)$, we will propose empirical estimators for the quantities of interest. Let us firstly define the number of visits to a certain state, the number of transitions between two states and so on.

Definition 10: For all states $i,j\in E$ and positive integer $k\le M$, define:

1. $N_i(M) := \sum_{n=0}^{N(M)-1} \mathbf{1}_{\{J_n=i\}}$ - the number of visits to state i, up to time M;
2. $N_{ij}(M) := \sum_{n=1}^{N(M)} \mathbf{1}_{\{J_{n-1}=i, J_n=j\}}$ - the number of transitions from i to j, up to time M;
3. $N_{ij}(k,M) := \sum_{n=1}^{N(M)} \mathbf{1}_{\{J_{n-1}=i, J_n=j, X_n=k\}}$ - the number of transitions from i to j, up to time M, with sojourn time in state i equal to $k, 1\le k\le M$.

For a sample path of length M of a semi-Markov chain, for any states $i,j\in E$ and positive integer $k\in\mathbb{N}, k\le M$, we define the empirical estimators of the transition matrix of the embedded Markov chain p_{ij}, of the conditional distributions of the sojourn times $f_{ij}(k)$ and of the discrete-time semi-Markov kernel $q_{ij}(k)$ by:

$$\begin{aligned}\widehat{p}_{ij}(M) &:= N_{ij}(M)/N_i(M),\\ \widehat{f}_{ij}(k,M) &:= N_{ij}(k,M)/N_{ij}(M),\\ \widehat{q}_{ij}(k,M) &:= N_{ij}(k,M)/N_i(M).\end{aligned} \tag{24.5}$$

Note that the proposed estimators are natural estimators. For instance, the probability p_{ij} that the system goes from state i to state j is estimated by the number of transitions from i to j, devised by the number of visits to state i. As can be seen in [17] or [19], the empirical estimators proposed in (24.5) have good asymptotic properties. Moreover, they are in fact approached maximum likelihood estimators (Theorem 1). In order to see this, consider the likelihood function corresponding to the history $\mathcal{H}(M)$

$$L(M) = \prod_{k=1}^{N(M)} p_{J_{k-1}J_k} f_{J_{k-1}J_k}(X_k)\overline{H}_{J_{N(M)}}(u_M),$$

where $\overline{H}_i(\cdot)$ is the survival function in state i. We have the following result concerning the asymptotic behavior of u_M (see [19] for a proof).

Lemma 3: For a semi-Markov chain $(Z_n)_{n\in\mathbb{N}}$ we have $u_M/M \xrightarrow{a.s.} 0$, as $M\to\infty$.

Let us consider the approached likelihood function

$$L_1(M) = \prod_{k=1}^{N(M)} p_{J_{k-1}J_k} f_{J_{k-1}J_k}(X_k), \qquad (24.6)$$

obtained by neglecting the last term in the expression of $L(M)$. Using Lemma 3, we see that the maximum likelihood function $L(M)$ and the approached maximum likelihood function $L_1(M)$ are asymptotically equivalent, as M tends to infinity. Consequently, the estimators obtained by estimating $L(M)$ or $L_1(M)$ are asymptotically equivalent, as M tends to infinity.

The following result shows that $\widehat{p}_{ij}(M)$, $\widehat{f}_{ij}(k,M)$ and $\widehat{q}_{ij}(k,M)$ defined in (24.5) are obtained in fact by maximizing $L_1(M)$ (a proof can be found in [17]).

Theorem 1: For a sample path of a semi-Markov chain $(Z_n)_{n\in\mathbb{N}}$, of arbitrary fixed length $M \in \mathbb{N}$, the empirical estimators of the transition matrix of the embedded Markov chain $(J_n)_{n\in\mathbb{N}}$, of the conditional distributions of the sojourn times and of the discrete-time semi-Markov kernel, proposed in (24.5), are approached nonparametric maximum likelihood estimators, *i.e.*, they maximize the approached likelihood function $L_1(M)$ given in (24.6).

As any quantity of interest of a semi-Markov system can be written in terms of the semi-Markov kernel, we can now use the kernel estimator $\widehat{q}_{ij}(k,M)$ in order to obtain plug-in estimators for any functional of the kernel. For instance, the cumulative semi-Markov kernel $\mathbf{Q} = (\mathbf{Q}(k); k \in \mathbb{N})$ defined in (24.1) has the estimator

$$\widehat{\mathbf{Q}}(k,M) := \sum_{l=1}^{k} \widehat{\mathbf{q}}(l,M).$$

Similarly, using the expression of the transition function of the semi-Markov chain Z given in (24.3), we get its estimator

$$\widehat{\mathbf{P}}(k,M) = (\delta I - \widehat{\mathbf{q}})^{(-1)}(\cdot,M) * (\mathbf{I} - \operatorname{diag}(\widehat{\mathbf{Q}}(\cdot,M)\cdot\mathbf{1}))(k).$$

Proofs of the consistency and of the asymptotic normality of the estimators defined up to now can be found in [16, 17, 19].

We are able now to construct estimators of the reliability indicators of a semi-Markov system and to present their asymptotic properties.

24.4.2 Reliability Estimation

The expression of the reliability given in Proposition 2, together with the estimators of the semi-Markov transition function and of the cumulative semi-Markov kernel given above, allow us to obtain the estimator of the system's reliability at time k given by

$$\widehat{R}(k,M) = \boldsymbol{\alpha}_1 \widehat{\mathbf{P}}_{11}(k,M)\mathbf{1}_{s_1}. \qquad (24.7)$$

Let us give now the result concerning the consistency and the asymptotic normality of the reliability estimator. A proof of the asymptotic normality of reliability estimator, based on CLT for Markov renewal processes (see [21]) can be found in [17]. An alternative proof based on CLT for martingales, can be found in [19].

Theorem 2: For any fixed arbitrary positive integer $k \in \mathbb{N}$, the estimator of the reliability of a discrete-time semi-Markov system at instant k is strongly consistent, *i.e.,*

$$\left|\widehat{R}(k,M) - R(k)\right| \xrightarrow{a.s.} 0, \text{ as } M \to \infty,$$

and asymptotically normal, *i.e.*, we have

$$\sqrt{M}\left[\widehat{R}(k,M) - R(k)\right] \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \sigma_R^2(k)\right), \text{ as } M \to \infty,$$

with the asymptotic variance

$$\sigma_R^2(k) = \sum_{i=1}^{s} \mu_{ii} \left\{ \sum_{j=1}^{s} \left[D_{ij}^U - \mathbf{1}_{\{i\in U\}} \sum_{t\in U} \alpha(t)\Psi_{ti} \right]^2 * q_{ij}(k) - \left[\sum_{j=1}^{s} \left(D_{ij}^U * q_{ij} - \mathbf{1}_{\{i\in U\}} \sum_{t\in U} \alpha(t)\psi_{ti} * Q_{ij} \right) \right]^2 (k) \right\},$$

where

$$D_{ij}^U := \sum_{n\in U} \sum_{r\in U} \alpha(n)\psi_{ni} * \psi_{jr} * \left(\mathbf{I} - \operatorname{diag}(\mathbf{Q}\cdot\mathbf{1})\right)_{rr},$$

$$\boldsymbol{\psi}(k) := \sum_{n=0}^{k} \mathbf{q}^{(n)}(k), \quad \boldsymbol{\Psi}(k) := \sum_{n=0}^{k} \mathbf{Q}^{(n)}(k),$$

and μ_{ii} is the mean recurrence time of the state i for the chain Z.

24.4.3 Availability Estimation

Taking into account the expression of the availability given in Proposition 3, we propose the following estimator for the availability of a discrete-time semi-Markov system:

$$\widehat{A}(k,M) = \boldsymbol{\alpha}\widehat{\mathbf{P}}(k,M)\mathbf{1}_{s,s_1}.$$

The following result concerns the consistency and the asymptotic normality of the reliability estimator. A proof can be found in [19].

Theorem 3: For any fixed arbitrary positive integer $k \in \mathbb{N}$, the estimator of the availability of a discrete-time semi-Markov system at instant k is strongly consistent and asymptotically normal, in the sense that

$$\left|\widehat{A}(k,M) - A(k)\right| \xrightarrow{a.s.} 0, \text{ as } M \to \infty,$$

and

$$\sqrt{M}\left[\widehat{A}(k,M) - A(k)\right] \xrightarrow{\mathcal{D}} \mathcal{N}\left(0,\sigma_A^2(k)\right), \text{ as } M \to \infty,$$

with the asymptotic variance

$$\sigma_A^2(k) = \sum_{i=1}^{s}\mu_{ii}\left\{\sum_{j=1}^{s}\left[D_{ij} - \mathbf{1}_{\{i\in U\}}\sum_{t=1}^{s}\alpha(t)\Psi_{ti}\right]^2 * q_{ij}(k) - \left[\sum_{j=1}^{s}\left(D_{ij} * q_{ij} - \mathbf{1}_{\{i\in U\}}\sum_{t=1}^{s}\alpha(t)\psi_{ti} * Q_{ij}\right)\right]^2(k)\right\},$$

where $D_{ij} := \sum_{n=1}^{s}\sum_{r\in U}\alpha(n)\psi_{ni} * \psi_{jr} * \left(\mathbf{I} - \operatorname{diag}(\mathbf{Q}\cdot\mathbf{1})\right)_{rr}$.

24.4.4 Failure Rate Estimation

For a matrix function $A \in \mathcal{M}_E(\mathbb{N})$, we denote by $A^+ \in \mathcal{M}_E(\mathbb{N})$ the matrix function defined by $A^+(k) := A(k+1), k \in \mathbb{N}$. Using the expression of the failure rate obtained in (24.4), we obtain the following estimator:

$$\widehat{\lambda}(k,M) := \begin{cases} 1 - \dfrac{\widehat{R}(k,M)}{\widehat{R}(k-1,M)}, & \widehat{R}(k-1,M) \neq 0, \mathrm{k} \geq 1, \\ 0, & \widehat{R}(k-1,M) = 0, \mathrm{k} \geq 1, \end{cases}$$

$$\widehat{\lambda}(0,M) := 1 - \widehat{R}(0,M).$$

For the failure rate estimator we have a similar result as for reliability and availability estimators. A proof can be found in [18, 19].

Theorem 4: For any fixed arbitrary positive integer $k \in \mathbb{N}$, the estimator of the failure rate of a discrete-time semi-Markov system at instant k is strongly consistent and asymptotically normal, *i.e.*,

$$\left|\widehat{\lambda}(k,M) - \lambda(k)\right| \xrightarrow{a.s.} 0, \text{ as } M \to \infty,$$

and

$$\sqrt{M}\left[\widehat{\lambda}(k,M) - \lambda(k)\right] \xrightarrow{\mathcal{D}} \mathcal{N}\left(0,\sigma_\lambda^2(k)\right), \text{ as } M \to \infty,$$

swith the asymptotic variance

$$\sigma_\lambda^2(k) = \sigma_1^2(k) / R^4(k-1),$$

where $\sigma_1^2(k)$ is given by

$$\sigma_1^2(k) = \sum_{i=1}^{s}\mu_{ii}\left\{R^2(k)\sum_{j=1}^{s}\left[D_{ij}^U - \mathbf{1}_{\{i\in U\}}\sum_{t\in U}\alpha(t)\Psi_{ti}\right]^2 * q_{ij}(k-1) + R^2(k-1)\sum_{j=1}^{s}\left[D_{ij}^U - \mathbf{1}_{\{i\in U\}}\sum_{t\in U}\alpha(t)\Psi_{ti}\right]^2 * q_{ij}(k) - T_i^2(k) + 2R(k-1)R(k)\sum_{j=1}^{s}\left[\mathbf{1}_{\{i\in U\}}D_{ij}^U\sum_{t\in U}\alpha(t)\Psi_{ti}^+ + \mathbf{1}_{\{i\in U\}}\left(D_{ij}^U\right)^+\sum_{t\in U}\alpha(t)\Psi_{ti} - \left(D_{ij}^U\right)^+ D_{ij}^U - \mathbf{1}_{\{i\in U\}}\left(\sum_{t\in U}\alpha(t)\Psi_{ti}\right)\left(\sum_{t\in U}\alpha(t)\Psi_{ti}^+\right)\right] * q_{ij}(k-1)\right\},$$

where

$$T_i(k) := \sum_{j=1}^{s}\left[R(k)D_{ij}^U * q_{ij}(k-1) - R(k-1)D_{ij}^U * q_{ij}(k) - R(k)\mathbf{1}_{\{i\in U\}}\sum_{t\in U}\alpha(t)\psi_{ti} * Q_{ij}(k-1) + R(k-1)\mathbf{1}_{\{i\in U\}}\sum_{t\in U}\alpha(t)\psi_{ti} * Q_{ij}(k)\right]$$

and D_{ij}^U is given in Theorem 2.

24.4.5 Asymptotic Confidence Intervals

The previously obtained asymptotic results allow one to construct asymptotic confidence intervals for reliability, availability and failure rate. For this purpose, we need to construct a consistent estimator of the asymptotic variances.

Firstly, using the definitions of $\boldsymbol{\psi}(k)$ and of $\boldsymbol{\Psi}(k)$ given in Theorem 2, we can construct the corresponding estimators $\widehat{\boldsymbol{\psi}}(k,M)$ and $\widehat{\boldsymbol{\Psi}}(k,M)$. One can check that these estimators are strongly consistent. Secondly, for $k \le M$, replacing $\mathbf{q}(k)$, $\mathbf{Q}(k)$, $\boldsymbol{\psi}(k)$ and $\boldsymbol{\Psi}(k)$ by the corresponding estimators in the asymptotic variance of the reliability given in Theorem 2, we obtain an estimator $\widehat{\sigma}_R^2(k,M)$ of the asymptotic variance $\sigma_R^2(k)$. From the strong consistency of the estimators $\widehat{\mathbf{q}}(k,M)$, $\widehat{\mathbf{Q}}(k,M)$, $\widehat{\boldsymbol{\psi}}(k,M)$ and $\widehat{\boldsymbol{\Psi}}(k,M)$ (see [17, 19]), we obtain that $\widehat{\sigma}_R^2(k,M)$ converges almost surely to $\sigma_R^2(k)$, as M tends to infinity. Finally, the asymptotic confidence interval of $R(k)$ at level $100(1-\gamma)\%$, $\gamma \in (0,1)$, is:

$$\left[\widehat{R}(k,M) - u_{1-\gamma/2}\frac{\widehat{\sigma}_R(k,M)}{\sqrt{M}}, \widehat{R}(k,M) + u_{1-\gamma/2}\frac{\widehat{\sigma}_R(k,M)}{\sqrt{M}}\right],$$

where $u_{1-\gamma/2}$ is the $(1-\gamma/2)$ fractile of $\mathcal{N}(0,1)$. In the same way, we can obtain the other confidence intervals.

24.5 A Numerical Example

Let us consider the three-state discrete-time semi-Markov system described in Figure 24.2.

The state space $E = \{1,2,3\}$ is partitioned into the up-state set $U = \{1,2\}$ and the down-state set $D = \{3\}$.

The system is defined by the initial distribution $\boldsymbol{\alpha} = (1\ 0\ 0)$, by the transition probability matrix $\mathbf{p}$ of the embedded Markov chain $(J_n)_{n\in\mathbb{N}}$ and by the conditional distributions of the sojourn times:

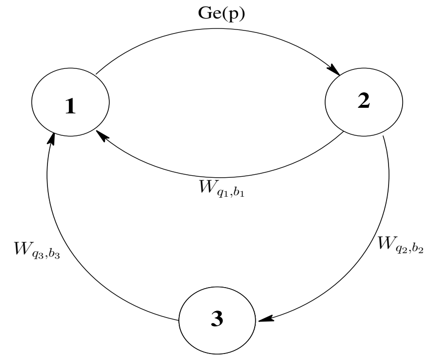

Figure 24.2. A three-state semi-Markov system

$$\mathbf{p} = \begin{pmatrix} 0 & 1 & 0 \\ 0.95 & 0 & 0.05 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\mathbf{f}(k) = \begin{pmatrix} 0 & f_{12}(k) & 0 \\ f_{21}(k) & 0 & f_{23}(k) \\ f_{31}(k) & 0 & 0 \end{pmatrix}, k \in \mathbb{N}.$$

We consider the following distributions for the conditional sojourn time:

$-f_{12}$ is a geometric distribution on $\mathbb{N}^*$, of parameter $p = 0.8$.

$-f_{21} := W_{q_1,b_1}$, $f_{23} := W_{q_2,b_2}$, $f_{31} := W_{q_3,b_3}$ are discrete-time, first type Weibull distributions (see [1]), defined by $W_{q,b}(0) := 0$, $W_{q,b}(k) := q^{(k-1)^b} - q^{k^b}$, $k \ge 1$, where we take $q_1 = 0.3$, $b_1 = 0.5$, $q_2 = 0.5$, $b_2 = 0.7$, $q_3 = 0.6$, $b_3 = 0.9$. Note that we study here a strictly semi-Markov system, which cannot be reduced to a Markov one.

Using the transition probability matrix and the sojourn time distributions given above, we have simulated a sample path of the three state semi-Markov chain, of length M. This sample path allows us to compute $N_i(M)$, $N_{ij}(M)$ and $N_{ij}(k,M)$, using Definition 10, and to obtain the empirical estimators $\widehat{p}_{ij}(M)$, $\widehat{f}_{ij}(k,M)$ and $\widehat{q}_{ij}(k,M)$ from (24.5). Consequently, we can obtain the estimators $\widehat{\mathbf{Q}}(k,M)$, $\widehat{\boldsymbol{\psi}}(k,M)$ and $\widehat{\boldsymbol{\Psi}}(k,M)$. Thus, from (24.7), we obtain the estimator of the reliability. In Theorem 2, we have obtained the expression of the asymptotic variance of reliability. Replacing $\mathbf{q}(k)$, $\mathbf{Q}(k)$, $\boldsymbol{\psi}(k)$ and $\boldsymbol{\Psi}(k)$ by the corresponding estimators, we get the estimator $\widehat{\sigma}_R^2(k,M)$ of the asymptotic variance $\sigma_R^2(k)$. This estimator will allow us to have the asymptotic confidence interval for reliability, as shown in Section 24.4.5.

The consistency of the reliability estimator is illustrated in Figure 24.3, where reliability estimators obtained for several values of the sample size M are drawn. We can note that the estimator approaches the true value, as the sample size M increases. Figure 24.4 presents the estimators of the asymptotic variance of the reliability $\sigma_R^2(k)$, obtained for different sample sizes. Note also that the estimator approaches the true value, as M increases. In Figure 24.5, we present the confidence interval of the reliability. Note that the confidence interval covers the true value of the reliability.

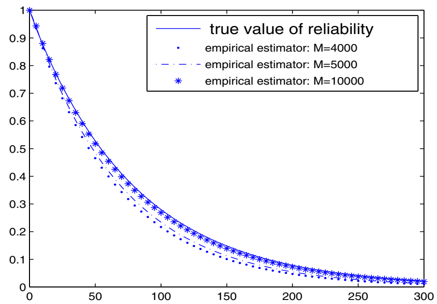

Figure 24.3. Consistency of reliability estimator

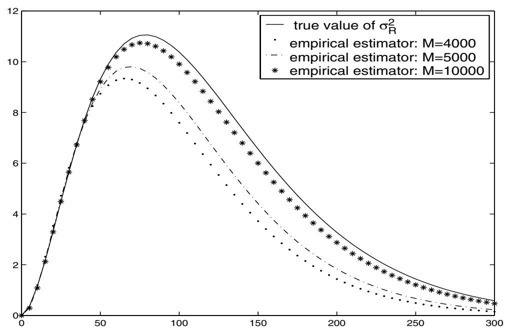

Figure 24.4. Consistency of $\widehat{\sigma}_R^2(k,M)$

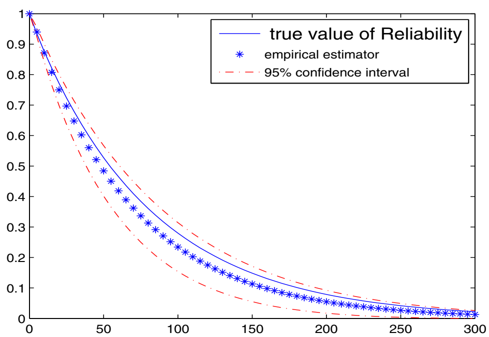

Figure 24.5. Confidence interval of reliability

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