6

Independent Case Analysis

In this chapter, we exploit the independence between traffic processes and service processes to improve performance bounds. Two approaches will be introduced. One is based on the concept of a stochastic strict service process due to impairment, introduced in Section 4.3. Another is based on the concepts of traffic and service envelope processes introduced in Section 5.7 and applies moment generating functions (MGF) for the analysis.

In Chapter 5, various results for stochastic network calculus were presented. These results were obtained without considering the dependence condition between flows and servers. In this chapter, we focus on independent case analysis and introduce the five basic properties (P.1)–(P.5) when flows and servers are independent.

6.1 Introduction

We start with a lemma, which is followed by a simple example to demonstrate the importance of independent case analysis.

Lemma 6.1. *Consider non-negative random variables X and Y. Suppose they are independent and $\bar{F}_X(x) \leq f(x)$ and $\bar{F}_Y(x) \leq g(x)$, where $\bar{F}_X(x)$ and $\bar{F}_Y(x)$, respectively, denote their complementary cummulative distribution functions (CCDF), and $f, g \in \bar{\mathcal{F}}$. Then, for all $x \geq 0$, there holds*

$$P\{X + Y > x\} \leq 1 - (\bar{f} * \bar{g})(x), \tag{6.1}$$

where $\bar{f}(x) = 1 - [f(x)]_1$ and $\bar{g}(x) = 1 - [g(x)]_1$.

Proof. For independent random variables X and Y, it is well known that $F_{X+Y} = F_X * F_Y \equiv \int_{-\infty}^{+\infty} F_X(x-y)dF_Y(y)$. Since X and Y are non-negative, $F_X(x) = 0$ and $F_Y(x) = 0$ for all $x < 0$. Hence, $F_{X+Y} = \int_0^x F_X(x-y)dF_Y(y)$. Notice that F_X, F_Y, $\bar{f}$, and $\bar{g}$ are wide-sense increasing, $\bar{f} \leq F_X$ and $\bar{g} \leq F_Y$, and the Stieltjes convolution operation is commutative. Then

Y. Jiang, Y. Liu, *Stochastic Network Calculus*,
DOI: 10.1007/978-1-84800-127-5_6,

$$\begin{aligned} F_X * F_Y(x) &= \int_0^x F_X(x-y) dF_Y(y) \\ &\geq \int_0^x \bar{f}(x-y) dF_Y(y) = \int_0^x F_Y(x-y) d\bar{f}(y) \\ &\geq \int_0^x \bar{g}(x-y) d\bar{f}(y) = (\bar{f} * \bar{g})(x), \end{aligned}$$

and with this and $P\{X+Y > x\} = \bar{F}_{X+Y} = 1 - F_X * F_Y$, the lemma is proved. □

Example 6.2. In Lemma 1.5, it was proved that $P\{X+Y > x\} \leq (f \otimes g)(x)$. If X and Y are independent, we then have two bounds for $P\{X+Y > x\}$, which are (1.12) and (6.1). Suppose $f(x) = g(x) = e^{-x}$. With Lemma 1.5, we obtain

$$P\{X+Y > x\} \leq 2e^{-x/2},$$

and with Lemma 6.1, we get

$$P\{X+Y > x\} \leq (1+x)e^{-x}.$$

These two bounds are plotted in Figure 6.1. The figure clearly shows that the bound obtained from Lemma 6.1 is much better than the bound from Lemma 1.5.

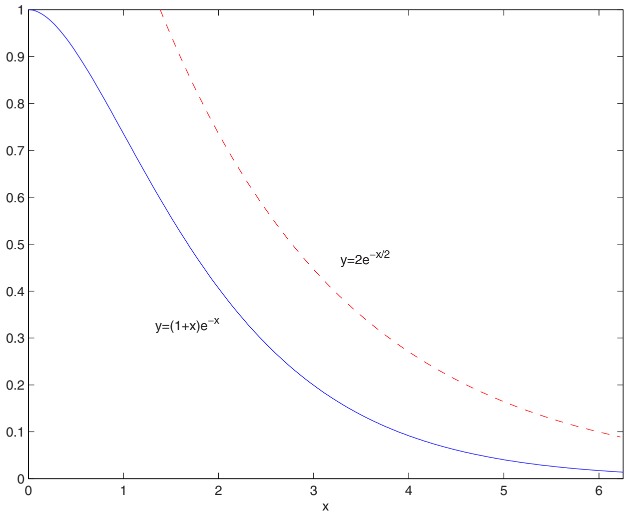

Fig. 6.1. Comparison of Lemmas 6.1 and 1.5

This example implies that by considering the independence condition, significant improvement may be obtained for the result.

From the example above, we expect that when flows and servers are independent in a network, much better results or tighter bounds can be obtained for properties (P.1)–(P.5). However, except for the superposition property (P.5), it is not straightforward to obtain properties (P.1)–(P.4) for the independent case.

The difficulty relates to the dependences implied in the definitions of the various stochastic service curve server models introduced in Chapter 4. For example, the weak stochastic service curve model is defined on the following inequality that duplicates (4.1):

$$P\{A \otimes \beta(t) - A^*(t) > x\} \leq g(x). \tag{6.2}$$

The definition of the weak stochastic service curve model implies that a weak stochastic service curve $\beta(t)$ is generally dependent on the arrival process $A(t)$ and the output process $A^*(t)$. Similar dependence can be found in the stochastic service curve model and the θ-the stochastic service curve model, as well as in the stochastic service envelope process definition.

The difficulty also relates to the inherent dependences found in intermediate results obtained by using the analysis approach in the previous chapter. For example, in (5.26), we obtain

$$\begin{aligned} &\sup_{0\leq s\leq t} \sup_{0\leq u\leq s} [A^*(u,s) - \alpha^*(s-u)] \\ \leq &\sup_{0\leq s\leq t} \sup_{0\leq u\leq s} [A(u,s) - \alpha(s-u)] + \sup_{0\leq s\leq t} [A \otimes \beta(s) - A^*(s)] \end{aligned} \tag{6.3}$$

where both $\sup_{0\leq s\leq t} \sup_{0\leq u\leq s}[A(u,s) - \alpha(s-u)]$ and $\sup_{0\leq s\leq t}[A \otimes \beta(s) - A^*(s)]$ are defined to depend on the arrival process A, which further makes them dependent on each other.

In deterministic network calculus, the dependences mentioned above do not cause any difficulties in the analysis since only deterministic worst case scenarios are considered and the dependences need not be taken into account.

In stochastic network calculus, however, the dependences make it difficult to obtain independent case results directly. For example, even when the bounds on the complementary probability distribution functions (CPDF) of the two terms on the right-hand side of (6.3) are given, we cannot apply Lemma 6.1 to (6.3) since these two terms are inherently dependent, as discussed above.

In the following section, the concept of a *stochastic strict server,* which was introduced earlier in Chapter 4, is used to help decouple the dependences discussed above. As a result, a further independent case analysis on properties (P.1)–(P.4) can be conducted.

6.2 Analysis Based on Stochastic Strict Server

In Section 4.3, we introduced the concept of a *stochastic strict server*. In addition, we defined a special type of stochastic strict server. In such a stochastic server, the stochastic nature of service is due to some random impairment processes. Particularly, a system is said to be a *stochastic strict server* providing *strict service curve* $\hat{\beta} \in \mathcal{F}$ with *impairment process* I if, during any period $(s,t]$, the actual service $S(s,t)$ provided by the system satisfies

$$S(s,t) \geq \hat{\beta}(t-s) - I(s,t). \tag{6.4}$$

Note that in defining *stochastic strict server due to impairment* only one impairment process I is used, which can actually be the superposition of multiple constituent processes that cause the system to be unable to deliver the corresponding service to the input considered. Two important types of such processes are worth highlighting. One is the process describing the actually impaired service. For example, due to random errors, a wireless channel fails to deliver the corresponding service to its users. In this case, the error process can be considered an impairment process. Another important type of processes that can be viewed as an impairment processes to the flow considered is due to cross traffic or flows competing service with the flow considered.

Also in Section 4.3, it has been shown that when the stochastic arrival curve characterization of the impairment process is known, the stochastic service curve characterization of the stochastic strict server can be found as shown by Theorems 4.12 and 4.13.

In the rest of this section, we further exploit the concept of a stochastic strict server due to impairment and present results under independent case analysis. The focus is on the five basic properties introduced in Chapter 1.

6.2.1 Backlog and Delay Bounds

We start with the backlog bound and delay bound. We proved in (5.5) that

$$\begin{aligned} &B(t) \\ &\leq \sup_{0\leq s\leq t}\{A(s,t)-\alpha(t-s)\} + \sup_{t\geq 0}\{\alpha(t)-\beta(t)\} + A\otimes\beta(t) - A^*(t). \end{aligned} \tag{6.5}$$

In addition, assuming the server is a stochastic strict server providing strict service curve $\hat{\beta}$ with impairment process $I \sim_{ta} \langle g, \gamma \rangle$, we have from (4.12) that

$$A\otimes\beta(t) - A^*(t) \leq \left(\sup_{0\leq s\leq t}[I(s,t)-\gamma(t-s)]\right)^+, \tag{6.6}$$

where $\beta(t) = \hat{\beta}(t) - \gamma(t)$. Applying (6.6) to (6.5), we get

$$\begin{aligned} B(t) \leq &\sup_{0\leq s\leq t}\{A(s,t)-\alpha(t-s)\} + \left(\sup_{0\leq s\leq t}[I(s,t)-\gamma(t-s)]\right)^+ \\ &+\alpha \oslash \beta(0). \end{aligned} \tag{6.7}$$

If A and I are independent random processes, since α, β, and γ are non-random functions, the first two terms on the right-hand side of (6.7) are also independent. Then, together with the fact the m.b.c. stochastic arrival curve and θ-m.b.c stochastic arrival curve imply a v.b.c. stochastic arrival curve, we have the following theorem.

Theorem 6.3. *Consider a system S with input A. Let $\sim_{sac}$ be either $\sim_{vb}$, $\sim_{mb}$, or $\sim_{\theta-mb}$. Suppose the input has a stochastic arrival curve $\alpha \in \mathcal{F}$ with bounding function $f \in \bar{\mathcal{F}}$; i.e., $A \sim_{sac} \langle f, \alpha \rangle$. Also suppose the server is a stochastic strict server providing strict service curve $\hat{\beta}$ with impairment process $I \sim_{sac} \langle g, \gamma \rangle$. If A and I are independent, the backlog $B(t)$ is guaranteed such that, for all $x \geq 0$,*

$$P\{B(t) > x\} \leq 1 - \bar{f} * \bar{g}\left(x + \inf_{s \geq 0}[\beta(s) - \alpha(s)]\right)$$

where $\beta(t) = \hat{\beta}(t) - \gamma(t)$, $\bar{f}(x) = 1 - [f(x)]_1$, and $\bar{g}(x) = 1 - [g(x)]_1$.

If the input process and/or the impairment process is characterized by a t.a.c. stochastic arrival curve, the corresponding results of Theorem 6.3 easily follow from the relationship between the t.a.c. stochastic arrival curve and v.b.c. stochastic arrival curve introduced in Theorem 3.13.

For the delay $D(t)$, under the same assumption as for (6.6), we proved in (5.13) that

$$\begin{aligned} P\{D(t) > x\} &\leq P\{X_1 + X_2 > \inf_{s \geq 0}[\beta(s) - \alpha(s - x)]\} \\ &\leq P\{X_1 + X_3 > \inf_{s \geq 0}[\beta(s) - \alpha(s - x)]\} \end{aligned} \tag{6.8}$$

with

$$\begin{aligned} X_1 &= \sup_{0 \leq s \leq t}[A(s,t) - \alpha(t - s)], \\ X_2 &= A \otimes \beta(t + x) - A^*(t + x), \\ X_3 &= \left(\sup_{0 \leq s \leq t+x}[I(s, t + x) - \gamma(t + x - s)]\right)^+, \end{aligned}$$

where we have used $X_2 \leq X_3$ based on (4.12).

If A and I are independent, X_1 and X_3 are also independent. Then, together with the fact that the m.b.c. stochastic arrival curve and θ-m.b.c. stochastic arrival curve imply the v.b.c. stochastic arrival curve, we have the following theorem.

Theorem 6.4. *Consider a system S with input A. Let $\sim_{sac}$ be either $\sim_{vb}$, $\sim_{mb}$, or $\sim_{\theta-mb}$. Suppose the input has a stochastic arrival curve $\alpha \in \mathcal{F}$ with bounding function $f \in \bar{\mathcal{F}}$; i.e., $A \sim_{sac} \langle f, \alpha \rangle$. Also suppose the server is a stochastic strict server providing strict service curve $\hat{\beta}$ with impairment process $I \sim_{sac} \langle g, \gamma \rangle$. If A and I are independent, the delay $D(t)$ is guaranteed such that, for all $x \geq 0$,*

$$P\{D(t) > h(\alpha + x, \beta)\} \leq 1 - \bar{f} * \bar{g}(x),$$

where $\beta(t) = \hat{\beta}(t) - \gamma(t)$, $\bar{f}(x) = 1 - [f(x)]_1$ *and* $\bar{g}(x) = 1 - [g(x)]_1$.

If the input process and/or the impairment process are characterized by a t.a.c. stochastic arrival curve, the corresponding results of Theorem 6.4 can be obtained from Theorem 6.3 and based on the relationship between the t.a.c. stochastic arrival curve and v.b.c. stochastic arrival curve introduced in Theorem 3.13.

6.2.2 Output Characterization

First, we consider the output t.a.c. stochastic arrival curve characterization. Assuming the server is a stochastic strict server providing strict service curve $\hat{\beta}$ with impairment process $I \sim_{ta} \langle g, \gamma \rangle$, we get from (5.20) and (4.12)

$$\begin{aligned} & A^*(s,t) - \alpha \oslash \beta(t-s) \\ \leq & \sup_{0 \leq u \leq t} \{A(u,t) - \alpha(t-u)\} + [A \otimes \beta(s) - A^*(s)] \\ \leq & \sup_{0 \leq u \leq t} \{A(u,t) - \alpha(t-u)\} + \left(\sup_{0 \leq u \leq s} [I(u,s) - \gamma(s-u)] \right)^+ . \end{aligned} \tag{6.9}$$

If A and I are independent, the two terms on the right-hand side of (6.9) are also independent. Then, together with the relationship between the t.a.c. stochastic arrival curve and v.b.c. stochastic arrival curve introduced in Theorem 3.13, we have the following result on output traffic characterization from (6.9).

Theorem 6.5. *Consider a system* S *with input* A. *Let* $\sim_{sac}$ *be either* $\sim_{vb}$, $\sim_{mb}$, *or* $\sim_{\theta-mb}$. *Suppose the input has a stochastic arrival curve* $\alpha \in \mathcal{F}$ *with bounding function* $f \in \bar{\mathcal{F}}$; *i.e.,* $A \sim_{sac} \langle f, \alpha \rangle$. *Also suppose the server is a stochastic strict server providing strict service curve* $\hat{\beta}$ *with impairment process* $I \sim_{sac} \langle g, \gamma \rangle$. *If* A *and* I *are independent, the output has a t.a.c. stochastic arrival curve* $A^* \sim_{ta} \langle f^*, \alpha^* \rangle$ *with*

$$\alpha^*(t) = \alpha \oslash \beta(t),$$
$$f^*(x) = 1 - \bar{f} * \bar{g}(x),$$

where $\beta(t) = \hat{\beta}(t) - \gamma(t)$, $\bar{f}(x) = 1 - [f(x)]_1$, *and* $\bar{g}(x) = 1 - [g(x)]_1$.

If the input process and/or the impairment process are characterized by a t.a.c. stochastic arrival curve, the output t.a.c. stochastic arrival curve characterization can be derived from Theorem 6.5 based on the relationship between the t.a.c. stochastic arrival curve and v.b.c. stochastic arrival curve introduced in Theorem 3.13.

Let us now consider the output t.a.c. stochastic arrival curve characterization. Under the same conditions as in Theorem 6.5, if $f^* \in \bar{\mathcal{G}}$, the output v.b.c. stochastic arrival curve characterization can also be obtained from Theorem 6.5 based on the relationship between the t.a.c. stochastic arrival curve and v.b.c. stochastic arrival curve. Specifically, we have the following corollary.

Corollary 6.6. *Under the same conditions as Theorem 6.5, if $f^* \in \bar{\mathcal{G}}$, the output has a v.b.c stochastic arrival curve $A^* \sim_{vb} \langle f^{*,\theta}, \alpha^*_\theta \rangle$ with*

$$\alpha^*_\theta(t) = \alpha \oslash \beta(t) + \theta \cdot t,$$
$$f^{*,\theta}(x) = f^*(x) + \frac{1}{\theta} \int_x^\infty f^*(y)dy,$$

where $\beta(t) = \hat{\beta}(t) - \gamma(t)$, $f^*(x) = 1 - \bar{f} * \bar{g}(x)$, $\bar{f}(x) = 1 - [f(x)]_1$ and $\bar{g}(x) = 1 - [g(x)]_1$, for any $\theta > 0$.

Alternatively, for the output v.b.c. stochastic arrival curve characterization, we get from (6.9) that

$$\begin{aligned}
&\sup_{0 \le s \le t} \{A^*(s,t) - \alpha \oslash \beta(t-s)\} \\
\le &\sup_{0 \le s \le t} \{A(s,t) - \alpha(t-s)\} \\
&+ \left(\sup_{0 \le s \le t} \sup_{0 \le u \le s} [I(u,s) - \gamma(s-u)] \right)^+
\end{aligned} \tag{6.10}$$

and

$$\begin{aligned}
&\sup_{0 \le s \le t} \{A^*(s,t) - \alpha \oslash \beta(t-s) - \theta \cdot (t-s)\} \\
\le &\sup_{0 \le s \le t} \{A(s,t) - \alpha(t-s)\} \\
&+ \sup_{0 \le s \le t} \left[\left(\sup_{0 \le u \le s} [I(u,s) - \gamma(s-u)] \right)^+ - \theta \cdot (t-s) \right],
\end{aligned} \tag{6.11}$$

and with this we can conclude the following theorem.

Theorem 6.7. *Consider a system S with input A. Suppose the input has a stochastic arrival curve $\alpha \in \mathcal{F}$ with bounding function $f \in \bar{\mathcal{F}}$; i.e., $A \sim_{sac} \langle f, \alpha \rangle$, where $\sim_{sac}$ is either $\sim_{vb}$, $\sim_{mb}$, or $\sim_{\theta-mb}$. Also suppose the server is a stochastic strict server providing strict service curve $\hat{\beta}$ with impairment process I. Assume A and I are independent.*

- *If $I \sim_{mb} \langle g, \gamma \rangle$, the output has a v.b.c stochastic arrival curve $A^* \sim_{vb} \langle f^*, \alpha^* \rangle$ with $\alpha^*(t) = \alpha \oslash \beta(t)$ and $f^*(x) = 1 - \bar{f} * \bar{g}(x)$;*

- *If $I \sim_{\theta-mb} \langle g^{\theta}, \gamma \rangle$, the output has a v.b.c stochastic arrival curve $A^* \sim_{vb} \langle f^{*,\theta}, \alpha^*_{\theta} \rangle$ with $\alpha^*_{\theta}(t) = \alpha \oslash \beta(t) + \theta \cdot t$ and $f^{*,\theta}(x) = 1 - \bar{f} * \bar{g^{\theta}}(x)$, where $\beta(t) = \hat{\beta}(t) - \gamma(t)$, $\bar{f}(x) = 1 - [f(x)]_1$, $\bar{g}(x) = 1 - [g(x)]_1$ and $\bar{g^{\theta}}(x) = 1 - [g^{\theta}(x)]_1$, for any $\theta > 0$.*

Under the same conditions as in Theorem 6.5, if the input is characterized by a t.a.c. stochastic arrival curve and/or the impairment process is by other types of stochastic arrival curves, the output v.b.c. stochastic arrival curve characterization can also be obtained from Theorem 6.5 based on its relationship with the v.b.c. stochastic arrival curve for the input, m.b.c. stochastic arrival curve, or θ-m.b.c. stochastic arrival curve for the impairment process.

We now consider the output m.b.c. stochastic arrival curve characterization. Under the same conditions as in Theorem 6.5, if $f^* \in \bar{\mathcal{G}}$, the output m.b.c. stochastic arrival curve characterization can also be obtained from Theorem 6.5 based on the relationship between the m.b.c. stochastic arrival curve and v.b.c. stochastic arrival curve. Specifically, we have the following corollary.

Corollary 6.8. *Under the same conditions as Theorem 6.5, if $f^* \in \bar{\mathcal{G}}$, the output has an m.b.c. stochastic arrival curve $A^* \sim_{mb} \langle f_t^{*,\theta}, \alpha^*_{\theta} \rangle$ with $\alpha^*_{\theta}(t) = \alpha \oslash \beta(t) + \theta \cdot t$ and $f_t^{*,\theta}(x) = \frac{1}{\theta} \int_{x-\theta t}^{\infty} f^*(y)dy$, where $\beta(t) = \hat{\beta}(t) - \gamma(t)$, $f^*(x) = 1 - \bar{f} * \bar{g}(x)$, $\bar{f}(x) = 1 - [f(x)]_1$, and $\bar{g}(x) = 1 - [g(x)]_1$ for any $\theta > 0$.*

Alternatively, from (5.26), it is known that

$$\begin{aligned}
&\sup_{0\le s\le t} \sup_{0\le u\le s} [A^*(u,s) - \alpha^*(s-u)] \\
&\le \sup_{0\le s\le t} \sup_{0\le u\le s} [A(u,s) - \alpha(s-u)] + \sup_{0\le s\le t} [A \otimes \beta(s) - A^*(s)]. \quad (6.12)
\end{aligned}$$

In addition, with the strict stochastic server assumption, it has been shown in (4.13) that

$$\sup_{0\le s\le t} [A \otimes \beta(s) - A^*(s)] \le \left(\sup_{0\le s\le t} \sup_{0\le u\le s} [I(u,s) - \gamma(s-u)] \right)^+. \quad (6.13)$$

Applying (6.13) to (6.12), we get

$$\begin{aligned}
&\sup_{0\le s\le t} \sup_{0\le u\le s} [A^*(u,s) - \alpha^*(s-u)] \\
&\le \sup_{0\le s\le t} \sup_{0\le u\le s} [A(u,s) - \alpha(s-u)] + \quad (6.14) \\
&\left(\sup_{0\le u\le t} \sup_{u\le s\le t} [I(u,s) - \gamma(s-u)] \right)^+. \quad (6.15)
\end{aligned}$$

Since A and I are independent and so are the first two terms on the right-hand side of (6.14), the following theorem follows easily.

Theorem 6.9. *Consider a system S with input A. Suppose the input has an m.b.c. stochastic arrival curve $\alpha \in \mathcal{F}$ with bounding function $f \in \bar{\mathcal{F}}$; i.e., $A \sim_{mb} \langle f, \alpha \rangle$. Also suppose the server is a stochastic strict server providing strict service curve $\hat{\beta}$ with impairment process I and the impairment process has an m.b.c. stochastic arrival curve $I \sim_{mb} \langle g, \gamma \rangle$. If A and I are independent, the output has an m.b.c. stochastic arrival curve $A^* \sim_{mb} \langle f^*, \alpha^* \rangle$ with $\alpha^*(t) = \alpha \oslash \beta(t)$ and $f^*(x) = 1 - \bar{f} * \bar{g}(x)$, where $\beta(t) = \hat{\beta}(t) - \gamma(t)$, $\bar{f}(x) = 1 - [f(x)]_1$, and $\bar{g}(x) = 1 - [g(x)]_1$.*

Under other types of traffic arrival curves for the input and the impairment process, the corresponding output m.b.c. stochastic arrival curve can be derived from Corollary 6.8 and Theorem 6.9 based on the relationships among the various types of traffic arrival curve characterizations presented in Chapter 3.

Finally, we consider the output θ–m.b.c. stochastic arrival curve characterization. Under the same conditions as in Theorem 6.5, if $f^* \in \bar{\mathcal{G}}$, the output m.b.c. stochastic arrival curve characterization can also be obtained from Theorem 6.5 based on the relationship between the t.a.c. stochastic arrival curve and m.b.c. stochastic arrival curve. Specifically, we have the following corollary.

Corollary 6.10. *Under the same conditions as in Theorem 6.5, if $f^* \in \bar{\mathcal{G}}$, the output has a v.b.c. stochastic arrival curve $A^* \sim_{vb} \langle f^{*,\theta}, \alpha^*_\theta \rangle$ with*

$$\alpha^*_\theta(t) = \alpha \oslash \beta(t) + (\theta_1 + \theta_2) \cdot t,$$
$$f^{*,\theta}(x) = \hat{f}^*(x) + \frac{1}{\theta_2} \int_x^\infty \hat{f}^*(y)dy,$$

where $\beta(t) = \hat{\beta}(t) - \gamma(t)$, $\hat{f}^*(x) = f^*(x) + \frac{1}{\theta_1} \int_x^\infty f^*(y)dy$, $f^*(x) = 1 - \bar{f} * \bar{g}(x)$, $\bar{f}(x) = 1 - [f(x)]_1$, and $\bar{g}(x) = 1 - [g(x)]_1$ for any $\theta_1, \theta_2 > 0$.

Alternatively, from (5.27), it is known that

$$\begin{aligned} &\sup_{0 \le s \le t} \left[\sup_{0 \le u \le s} [A^*(u,s) - \alpha \oslash \beta(s-u)] - \theta(t-s) \right] \\ \le &\sup_{0 \le s \le t} \left[\sup_{0 \le u \le s} [A(u,s) - \alpha(s-u)] - \theta(t-s) \right] \\ &+ \sup_{0 \le s \le t} [A \otimes \beta(s) - A^*(s)], \end{aligned} \tag{6.16}$$

which, with (6.13) applied, results in

$$\begin{aligned} &\sup_{0 \le s \le t} \left[\sup_{0 \le u \le s} [A^*(u,s) - \alpha \oslash \beta(s-u)] - \theta(t-s) \right] \\ \le &\sup_{0 \le s \le t} \left[\sup_{0 \le u \le s} [A(u,s) - \alpha(s-u)] - \theta(t-s) \right] \\ &+ \left(\sup_{0 \le s \le t} \sup_{0 \le u \le s} [I(u,s) - \gamma(s-u)] \right)^+. \end{aligned} \tag{6.17}$$

Then, we similarly have the following result.

Theorem 6.11. *Consider a system S with input A. Suppose the input has a θ-m.b.c. stochastic arrival curve $\alpha \in \mathcal{F}$ with bounding function $f \in \bar{\mathcal{F}}$; i.e., $A \sim_{\theta-mb} \langle f, \alpha \rangle$. Also suppose the server is a stochastic strict server providing strict service curve $\hat{\beta}$ with impairment process I, and the impairment process has an m.b.c. stochastic arrival curve $I \sim_{mb} \langle g, \gamma \rangle$. If A and I are independent, the output has a θ-m.b.c stochastic arrival curve $A^* \sim_{mb} \langle f^*, \alpha^* \rangle$ with $\alpha^*(t) = \alpha \oslash \beta(t)$ and $f^*(x) = 1 - \bar{f} * \bar{g}(x)$, where $\beta(t) = \hat{\beta}(t) - \gamma(t)$, $\bar{f}(x) = 1 - [f(x)]_1$ and $\bar{g}(x) = 1 - [g(x)]_1$.*

Under other types of traffic arrival curves for the input and the impairment process, the corresponding output θ-m.b.c stochastic arrival curve can be derived from Corollary 6.10 and Theorem 6.11 based on the relationships among the various types of traffic arrival curve characterizations presented in Chapter 3.

6.2.3 Concatenation Property

Consider two servers in tandem. If each server provides a stochastic service curve β^n, $n = 1, 2$, we have shown in (5.32) that

$$\begin{aligned} &\sup_{0 \le s \le t} [A \otimes \beta^1 \otimes \beta^2(s) - A^*(s)] \\ &\le \sup_{0 \le s \le t} [A^1 \otimes \beta^1(s) - A^{1*}(s)] + \sup_{0 \le s \le t} [A^2 \otimes \beta^2(s) - A^{2*}(s)]. \end{aligned} \tag{6.18}$$

Assume each server is a stochastic strict server providing strict service curve $\hat{\beta}^n$, $n = 1, 2$, with impairment process $I^n \sim_{mb} \langle g^n, \gamma^n \rangle$. Let $\beta^n(t) = \hat{\beta}^n(t) - \gamma^n(t)$. We then have (6.13), and applying it to (6.18), we obtain

$$\begin{aligned} &\sup_{0 \le s \le t} [A \otimes \beta^1 \otimes \beta^2(s) - A^*(s)] \\ &\le \left(\sup_{0 \le s \le t} \sup_{0 \le u \le s} [I^1(u, s) - \gamma^1(s - u)] \right)^+ \\ &\quad + \left(\sup_{0 \le s \le t} \sup_{0 \le u \le s} [I^2(u, s) - \gamma^2(s - u)] \right)^+ . \end{aligned} \tag{6.19}$$

If I^1 and I^2 are independent, so are the two terms of the right-hand side of (6.19). The discussion above can be easily extended to more than two nodes, and the following theorem is obtained that corresponds to the concatenation property of the stochastic service curve.

Theorem 6.12. *Consider a flow passing through a network of N nodes in tandem, and assume each node is a stochastic strict server providing stochastic strict service curve $\hat{\beta}^n$ with impairment process $I^n \sim_{mb} \langle g^n, \gamma^n \rangle$. If I^n are* *independent and* $\beta^n \in \mathcal{F}$, $(n = 1, 2, \dots, N)$, *then the network guarantees to the flow a stochastic service curve* $S \sim_{sc} \langle g, \beta \rangle$ *with*

$$\beta(t) = \beta^1 \otimes \beta^2 \otimes \cdots \otimes \beta^N(t),$$
$$g(x) = 1 - \bar{g}^1 * \bar{g}^2 * \cdots * \bar{g}^N(x),$$

where $\beta^n(t) = \hat{\beta}^n(t) - \gamma^n(t)$, $\bar{g}^n(x) = 1 - [g^n(x)]_1$, $n = 1, 2, \dots, N$.

By iteratively applying Lemma 5.39, we have in (5.39) that

$$\begin{aligned}
& A \otimes \beta^1 \otimes \beta^2_{-\theta} \otimes \cdots \otimes \beta^N_{-(N-1)\theta}(t) - A^*(t) \\
\leq & \sup_{0 \leq s \leq t} \left[A^1 \otimes \beta^1(s) - A^{1*}(s) - \theta \cdot (t-s)\right] \\
& + \sup_{0 \leq s \leq t} \left[A^2 \otimes \beta^2(s) - A^{2*}(s) - \theta \cdot (t-s)\right] + \cdots + \\
& + \sup_{0 \leq s \leq t} \left[A^{N-1} \otimes \beta^{N-1}(s) - A^{(N-1)*}(s) - \theta \cdot (t-s)\right] \\
& + A^N \otimes \beta\,(t) - A^*(t).
\end{aligned} \tag{6.20}$$

Assume each server is a stochastic strict server providing strict service curve $\hat{\beta}^n$, $n = 1, 2, \dots, N$, with impairment process $I^n \sim_{\theta-mb} \langle g^n, \gamma^n \rangle$. Let $\beta^n(t) = \hat{\beta}^n(t) - \gamma^n(t)$. We then have (4.14), and applying it to (6.20), we obtain

$$\begin{aligned}
& A \otimes \beta^1 \otimes \beta^2_{-\theta} \otimes \cdots \otimes \beta^N_{-(N-1)\theta}(t) - A^*(t) \\
\leq & \left(\sup_{0 \leq s \leq t} \left[\sup_{0 \leq u \leq s} [I^1(u,s) - \gamma^1(s-u)] - \theta \cdot (t-s) \right] \right)^+ \\
& + \left(\sup_{0 \leq s \leq t} \left[\sup_{0 \leq u \leq s} [I^2(u,s) - \gamma^2(s-u)] - \theta \cdot (t-s) \right] \right)^+ + \cdots + \\
& + \left(\sup_{0 \leq s \leq t} \left[\sup_{0 \leq u \leq s} [I^{N-1}(u,s) - \gamma^{N-1}(s-u)] - \theta \cdot (t-s) \right] \right)^+ \\
& + \left(\sup_{0 \leq s \leq t} [I^N(s,t) - \gamma^N(t-s)] \right)^+ .
\end{aligned} \tag{6.21}$$

If I^n, $n = 1, 2, \dots, N$, are independent, so are the terms on the right-hand side of (6.21), and hence the following result is obtained.

Theorem 6.13. *Consider a flow passing through a network of* N *nodes in tandem, and assume each node is a stochastic strict server providing stochastic strict service curve* $\hat{\beta}^n$ *with impairment process* $I^n \sim_{\theta-mb} \langle g^n, \gamma^n \rangle$. *If* I^n *are independent,* $\beta^n_{-(n-1)\theta} \in \mathcal{F}$, *and* $g^n \in \bar{\mathcal{F}}$, $(n = 1, 2, \dots, N)$, *then the network guarantees to the flow a weak stochastic service curve* $S \sim_{ws} \langle g, \beta \rangle$ *with*

$$\beta(t) = \beta^1 \otimes \beta^2_{-\theta} \otimes \cdots \otimes \beta^N_{-(N-1)\theta}(t), \tag{6.22}$$

$$g(x) = 1 - \bar{g}^1 * \bar{g}^2 * \cdots * \bar{g}^N(x), \tag{6.23}$$

where

$$\beta^n_{-(n-1)\theta}(t) = \hat{\beta}^n(t) - \gamma^n(t) - (n-1)\theta \cdot t, \qquad n = 1, 2, \ldots, N,$$
$$\bar{g}^n(x) = 1 - [g^n(x)]_1, \quad n = 1, 2, \ldots, N,$$

for any $\theta > 0$.

Based on the relationship between the weak stochastic service curve and θ-stochastic service curve, the following result corresponds to the concatenation property of the θ-stochastic service curve.

Corollary 6.14. *Under the same conditions as in Theorem 6.13, if* $g \in \bar{\mathcal{G}}$, *the network guarantees to the flow a* θ*-stochastic service curve* $S \sim_{\theta-sc} \langle g^\theta, \beta\rangle$ *with* $g^\theta(x) = g(x) + \frac{1}{\theta}\int_x^y g(y)dy$, *where* $\beta(t)$ *and* $g(x)$ *are as shown in (6.22) and (6.23), respectively.*

Based on the relationship between the v.b.c. stochastic arrival curve and θ-m.b.c. stochastic arrival curve, the following result corresponds to the concatenation property of the weak stochastic service curve.

Corollary 6.15. *Consider a flow passing through a network of* N *nodes in tandem, and assume each node is a stochastic strict server providing stochastic strict service curve* $\hat{\beta}^n$ *with impairment process* $I^n \sim_{vb} \langle g^n, \gamma^n\rangle$. *If* I^n *are independent,* $\beta^n_{-(n-1)\theta} \in \mathcal{G}$, *and* $g^n \in \bar{\mathcal{G}}$, $(n = 1, 2, \ldots, N)$, *then the network guarantees to the flow a weak stochastic service curve* $S \sim_{ws} \langle g, \beta\rangle$ *with*

$$\beta(t) = \beta^1 \otimes \beta^2_{-\theta} \otimes \cdots \otimes \beta^N_{-(N-1)\theta}(t), \tag{6.24}$$
$$g(x) = 1 - \bar{g}^{1,\theta_1} * \bar{g}^{2,\theta_2} * \cdots * \bar{g}^{N,\theta_N}(x), \tag{6.25}$$

where

$$\beta^n_{-(n-1)\theta}(t) = \hat{\beta}^n(t) - \gamma^n(t) - (n-1)\theta \cdot t, \qquad n = 1, 2, \ldots, N,$$
$$\bar{g}^{n,\theta_n}(x) = 1 - \left[g^n(x) + \frac{1}{\theta_n}\int_x^\infty g^n(y)dy\right]_1, \quad n = 1, 2, \ldots, N-1,$$
$$\bar{g}^{N,\theta_N}(x) = 1 - \left[g^N(x)\right]_1,$$

for any $\theta, \theta_1, \ldots, \theta_{N-1} > 0$.

6.2.4 Leftover Service Characterization

Consider a system fed with a flow A that is the aggregation of two constituent flows, A_1 and A_2. For the output, there holds $A^*(t) = A_1^*(t) + A_2^*(t)$. In addition, we have $A^*(t) \le A(t)$, $A_1^*(t) \le A_1(t)$, and $A_2^*(t) \le A_2(t)$. As in (5.40), we have for functions β, α_2 and any $t \ge 0$

$$\begin{aligned} &A_1 \otimes (\beta - \alpha_2)(t) - A_1^*(t) \\ &\leq [A \otimes \beta(t) - A^*(t)] + \sup_{0\leq s\leq t} [A_2(s,t) - \alpha_2(t-s)], \end{aligned} \tag{6.26}$$

from which we also have as in (5.41) and (5.42),

$$\begin{aligned} &\sup_{0\leq s\leq t} [A_1 \otimes (\beta - \alpha_2)(s) - A_1^*(s)] \\ &\leq \sup_{0\leq s\leq t} [A \otimes \beta(s) - A^*(s)] + \sup_{0\leq s\leq t} \sup_{0\leq u\leq s} [A_2(u,s) - \alpha_2(s-u)] \end{aligned} \tag{6.27}$$

and

$$\begin{aligned} &\sup_{0\leq s\leq t} [A_1 \otimes (\beta - \alpha_2)(s) - A_1^*(s) - \theta(t-s)] \\ &\leq \sup_{0\leq s\leq t} [A \otimes \beta(s) - A^*(s) - \theta_1(t-s)] \\ &\quad + \sup_{0\leq s\leq t} \left[\sup_{0\leq u\leq s} [A_2(u,s) - \alpha_2(s-u)] - \theta_2(t-s) \right] \end{aligned} \tag{6.28}$$

for any $\theta_1, \theta_2 > 0$ and $\theta = \theta_1 + \theta_2$.

Assume the system is a stochastic strict server providing strict service curve $\hat{\beta}$ with impairment process $I \sim_{sac} \langle g, \gamma \rangle$, where $\sim_{sac}$ may be $\sim_{vb}$, $\sim_{mb}$, or $\sim_{\theta-mb}$. Let $\beta(t) = \hat{\beta}(t) - \gamma(t)$. We then have (4.12), (4.13) and (4.14), and applying them respectively to (6.26), (6.27), and (6.28), we obtain the following theorems.

Theorem 6.16 (Leftover Weak Stochastic Service Curve). *Consider a server fed with a flow A that is the aggregation of two constituent flows A_1 and A_2. Assume the server is a stochastic strict server to the aggregate, providing stochastic strict service curve $\hat{\beta}$ with impairment process $I \sim_{vb} \langle g, \gamma \rangle$.*

(i) The server guarantees that

$$\begin{aligned} &A_1 \otimes (\beta - \alpha_2)(t) - A_1^*(t) \\ &\leq \left(\sup_{0\leq s\leq t} [I(s,t) - \gamma(t-s)] \right)^+ + \sup_{0\leq s\leq t} [A_2(s,t) - \alpha_2(t-s)]. \end{aligned} \tag{6.29}$$

(ii) If A_2 and I are independent, $A_2 \sim_{mb} \langle f_2, \alpha_2 \rangle$, and $\beta_1' \in \mathcal{F}$, then the server guarantees to flow A_1 a weak stochastic service curve $S_1 \sim_{ws} \langle g_1', \beta_1' \rangle$, where

$$g_1'(x) = 1 - \bar{g} * \bar{f}_2(x), \quad \beta_1'(t) = \beta(t) - \alpha_2(t),$$

with $\beta(t) = \hat{\beta}(t) - \gamma(t)$, $\bar{g}(x) = 1 - [g(x)]_1$, and $\bar{f}_2(x) = 1 - [f_2(x)]_1$.

Theorem 6.17 (Leftover Stochastic Service Curve). *Consider a server fed with a flow A that is the aggregation of two constituent flows A_1 and A_2. Assume the server is a stochastic strict server to the aggregate, providing stochastic strict service curve $\hat{\beta}$ with impairment process $I \sim_{mb} \langle g, \gamma \rangle$.*

(i) The server guarantees that

$$\begin{aligned}&\sup_{0\le s\le t}[A_1\otimes(\beta-\alpha_2)(s)-A_1^*(s)]\\ \le&\left(\sup_{0\le s\le t}\sup_{0\le u\le s}[I(u,s)-\gamma(s-u)]\right)^+\\ &+\sup_{0\le s\le t}\sup_{0\le u\le s}[A_2(u,s)-\alpha_2(s-u)].\end{aligned}\tag{6.30}$$

(ii) If A_2 and I are independent, $A_2\sim_{mb}\langle f_2,\alpha_2\rangle$, and $\beta_1'\in\mathcal{F}$, then the server guarantees to flow A_1 a stochastic service curve $S_1\sim_{sc}\langle g_1',\beta_1'\rangle$, where

$$g_1'(x)=1-\bar{g}*\bar{f}_2(x),\quad \beta_1'(t)=\beta(t)-\alpha_2(t),$$

with $\beta(t)=\hat{\beta}(t)-\gamma(t)$, $\bar{g}(x)=1-[g(x)]_1$, and $\bar{f}_2(x)=1-[f_2(x)]_1$.

Theorem 6.18 (Leftover θ-Stochastic Service Curve). *Consider a server fed with a flow A that is the aggregation of two constituent flows A_1 and A_2. Assume the server is a stochastic strict server to the aggregate, providing stochastic strict service curve $\hat{\beta}$ with impairment process $I\sim_{\theta-mb}\langle g,\gamma\rangle$.*

(i) The server guarantees that

$$\begin{aligned}&\sup_{0\le s\le t}[A_1\otimes(\beta-\alpha_2)(s)-A_1^*(s)-\theta(t-s)]\\ \le&\left(\sup_{0\le s\le t}\left[\sup_{0\le u\le s}[I(u,s)-\gamma(s-u)]-\theta_1\cdot(t-s)\right]\right)^+\\ &+\sup_{0\le s\le t}\left[\sup_{0\le u\le s}[A_2(u,s)-\alpha_2(s-u)]-\theta_2(t-s)\right]\end{aligned}\tag{6.31}$$

for any $\theta_1,\theta_2>0$ and $\theta=\theta_1+\theta_2$.

(ii) If A_2 and I are independent, $A_2\sim_{\theta-mb}\langle f_2,\alpha_2\rangle$, and $\beta_1'\in\mathcal{F}$, then the server guarantees to flow A_1 a θ-stochastic service curve $S_1\sim_{\theta-sc}\langle g_1',\beta_1'\rangle$, where

$$g_1'(x)=1-\bar{g}*\bar{f}_2(x),\quad \beta_1'(t)=\beta(t)-\alpha_2(t)$$

with $\beta(t)=\hat{\beta}(t)-\gamma(t)$, $\bar{g}(x)=1-[g(x)]_1$, and $\bar{f}_2(x)=1-[f_2(x)]_1$.

Note that in Theorems 6.16 to 6.18, the first part is an intermediate step for getting the second part. The intention of including the first part is as follows: When the leftover service property is used to derive other results, such as the concatenation property, the first part can be applied to their derivations. Then, if flows and the impairment processes of servers are independent, Lemma 6.1 can be used to derive the corresponding independent case bounds. However, if we were only given the second part, such an independent case analysis could not be applied and the general case (min, +) analysis in Chapter 5 would have to be used. As a result, looser bounds may be obtained.

Also note that from the viewpoint of the service provided to flow A_1, $A_2(t)$ can be considered as an impairment process. In other words, for flow A_1, the server has two independent impairment processes $I(t)$ and $A_2(t)$. From this viewpoint, Theorems 6.16 to 6.18 can also be proved based on the independent case superposition property in the next subsection and the results for the stochastic strict server due to impairment in Section 4.3.1.

Based on the relationships between the stochastic arrival curve models and between the stochastic service curve models, the corresponding results of Section 5.4 can be derived from Theorems 6.16 to 6.18 for the independent case.

6.2.5 Superposition Property

The superposition property means that the superposition of flows can be represented using the same traffic model. With this property, the aggregate of (possibly many) individual flows may be considered as a single aggregate flow, so that the QoS performance for the aggregate can be derived in the same way as for a single flow. This section discusses the superposition property for the various stochastic traffic models introduced in Chapter 2.

Consider N flows with arrival processes $A_i(t)$, $i = 1, \dots, N$. Let $A(t)$ be the superposition of the N flows. In other words, we have for any $s, t \geq 0$,

$$A(s, s+t) = A_1(s, s+t) + \cdots + A_N(s, s+t).$$

It has been shown in (5.43), (5.44), (5.45), and (5.46) that, for any functions $\alpha_i(t)$, $i = 1, \dots, N$, we have

$$\begin{aligned} & A(s, s+t) - [\alpha_1(t) + \cdots + \alpha_N(t)] \\ &= [A_1(s, s+t) - \alpha_1(t)] + \cdots + [A_N(s, s+t) - \alpha_N(t)], \end{aligned} \tag{6.32}$$

$$\begin{aligned} & \sup_{0 \leq s \leq t} [A(s,t) - [\alpha_1(t-s) + \cdots + \alpha_N(t-s)]] \\ \leq & \sup_{0 \leq s \leq t} [A_1(s,t) - \alpha_1(t-s)] + \cdots + \sup_{0 \leq s \leq t} [A_N(s,t) - \alpha_N(t-s)], \end{aligned} \tag{6.33}$$

$$\begin{aligned} & \sup_{0 \leq s \leq t} \sup_{0 \leq u \leq s} [A(u,s) - [\alpha_1(s-u) + \cdots + \alpha_N(s-u)]] \\ \leq & \sup_{0 \leq s \leq t} \sup_{0 \leq u \leq s} [A_1(u,s) - \alpha_1(s-u)] + \cdots \\ & + \sup_{0 \leq s \leq t} \sup_{0 \leq u \leq s} [A_N(u,s) - \alpha_N(s-u)] \end{aligned} \tag{6.34}$$

$$\sup_{0\le s\le t}\left[\sup_{0\le u\le s}\{A(u,s)-[\alpha_1(s-u)+\cdots+\alpha_N(s-u)]\}-\theta\cdot(t-s)\right]$$
$$\le \sup_{0\le s\le t}\left[\sup_{0\le u\le s}[A_1(u,s)-\alpha_1(s-u)]-\theta_1\cdot(t-s)\right]+\cdots$$
$$+\sup_{0\le s\le t}\left[\sup_{0\le u\le s}[A_N(u,s)-\alpha_N(s-u)]-\theta_N\cdot(t-s)\right]. \tag{6.35}$$

Assume $A_i(t)$, $i=1,\ldots,N$, are independent. Then, the independent case superposition properties in Theorems 6.19 to 6.22 follow from (6.32) to (6.35), respectively.

Theorem 6.19. *Consider N flows with arrival processes $A_i(t)$, $i=1,\ldots,N$. Let $A(t)$ denote the aggregate arrival process. If $A_i(t)$ are independent processes and $\forall i, A_i \sim_{\underline{ta}} \langle f_i,\alpha_i\rangle$, then $A \sim_{\underline{ta}} \langle f,\alpha\rangle$ with $\alpha(t)=\sum_{i=1}^N \alpha_i(t)$ and $f(x)=1-\bar{f}_1*\cdots*\bar{f}_N(x)$, where $\bar{f}_i=1-f_i$ and $*$ denotes the Stieltjes convolution.*

Theorem 6.20. *Consider N flows with arrival processes $A_i(t)$, $i=1,\ldots,N$. Let $A(t)$ denote the aggregate arrival process. If $A_i(t)$ are independent processes and $\forall i, A_i \sim_{vb} \langle f_i,\alpha_i\rangle$, then $A \sim_{vb} \langle f,\alpha\rangle$ with $\alpha(t)=\sum_{i=1}^N \alpha_i(t)$ and $f(x)=1-\bar{f}_1*\cdots*\bar{f}_N(x)$, where $\bar{f}_i=1-f_i$ and $*$ denotes the Stieltjes convolution.*

Theorem 6.21. *Consider N flows with arrival processes $A_i(t)$, $i=1,\ldots,N$. Let $A(t)$ denote the aggregate arrival process. If $A_i(t)$ are independent processes and $\forall i, A_i \sim_{mb} \langle f_i,\alpha_i\rangle$, then $A \sim_{mb} \langle f,\alpha\rangle$ with $\alpha(t)=\sum_{i=1}^N \alpha_i(t)$ and $f(x)=1-\bar{f}_1*\cdots*\bar{f}_N(x)$, where $\bar{f}_i=1-f_i$ and $*$ denotes the Stieltjes convolution.*

Theorem 6.22. *Consider N flows with arrival processes $A_i(t)$, $i=1,\ldots,N$. Let $A(t)$ denote the aggregate arrival process. If $A_i(t)$ are independent processes and $\forall i, A_i \sim_{\theta-mb} \langle f_i,\alpha_i\rangle$, then $A \sim_{\theta-mb} \langle f^\theta,\alpha\rangle$ with $\alpha(t)=\sum_{i=1}^N \alpha_i(t)$ and $f^\theta(x)=1-\bar{f}_1^{\theta_1}*\cdots*\bar{f}_N^{\theta_N}(x)$, where $\bar{f}_i^\theta=1-f_i^\theta$ and $*$ denotes the Stieltjes convolution for any $\theta_1,\ldots,\theta_N>0$ and $\theta=\theta_1+\cdots+\theta_N$.*

6.2.6 Scaling of End-to-End Delay Bound

In Section 5.6, it was introduced that the end-to-end delay bound is a scaling in $\mathcal{O}\left(n^2\log n\right)$ from the node-by-node analysis approach and a scaling in $\mathcal{O}\left(n\log n\right)$ from the concatenation property of the stochastic service curve. In Section 5.6, the possible independence between flows and servers is not taken into account. To demonstrate the use of independent case analysis results, we consider the same network as studied in Section 5.6 and show that the end-to-end delay bound is a scaling in $\mathcal{O}(n)$ when some independence conditions are satisfied.

Specifically, we consider a network of n servers in tandem through which flow A passes. Each server is a constant-rate server with capacity C. At each server, there is a cross-flow that joins and leaves. Assume the considered flow A and all the cross flows are independent. For ease of expression, we also assume that flow considered and all cross-flows have the same m.b.c. stochastic arrival curve (SAC) $r \cdot t$ with bounding function $f(x) = e^{-x}$ and $2r < C$.

As discussed in Section 4.3.1, each server along the end-to-end path of the flow F considered can be viewed as a stochastic strict server with impairment process. Particularly, it is a stochastic strict server S providing strict service curve $\hat{\beta}(t) = Ct$ with impairment process $I^i \sim_{mb} \langle f, r \rangle$, $i = 1, \ldots, n$. Then, it is known from Theorem 6.12 that the network provides to the flow an end-to-end stochastic service curve $\beta(t)$. More specifically, iteratively applying (6.18) and (6.13), we can obtain

$$\begin{aligned} &\sup_{0 \le s \le t} [A \otimes \beta(t) - A^*(s)] \\ &\le \left(\sup_{0 \le s \le t} \left[\sup_{0 \le u \le s} [I^1(u,s) - r(s-u)] \right] \right)^+ + \cdots \\ &\quad + \left(\sup_{0 \le s \le t} \left[\sup_{0 \le u \le s} [I^n(u,s) - r(s-u)] \right] \right)^+ . \end{aligned} \tag{6.36}$$

For the end-to-end delay $D(t)$, (6.36) can be applied to Theorem 6.4 and particularly (6.8). Then, one easily obtains

$$P\{D(t) > x\} \le P\{X + Y_1 + \cdots + Y_n > rx\}$$

with

$$\begin{aligned} X &= \sup_{0 \le s \le t} \sup_{0 \le u \le s} [A(u,s) - r(s-u)], \\ Y_i &= \left(\sup_{0 \le s \le t} \left[\sup_{0 \le u \le s} [I^i(u,s) - r(s-u)] \right] \right)^+, \quad i = 1, \ldots, n. \end{aligned}$$

Since A and I^i, $i = 1, \ldots, n$ are independent, so are X and Y_i, $i = 1, \ldots, n$. In addition, we have simply assumed the same bounding function e^{-x} for X and Y_i. So, $X + Y_1 + \cdots + Y_n$ is Gamma-distributed with parameters $\Gamma(n+1, 1)$. Then, we get for the end-to-end delay bound

$$P\{D(t) > x\} \le 1 - \frac{\gamma(n+1, rx)}{(n+1)!}, \tag{6.37}$$

where the function $\gamma(n, x)$ is defined as

$$\gamma(n, x) = \int_0^x y^{n-1} e^{-y} dy.$$

While (6.37) provides a good delay bound, it is difficult to see how it scales with respect to the number of servers in the network. In the following, we consider a possibly looser bound, but it is easy to see its scaling. From the Chernoff bound, we get

$$
\begin{aligned}
P\{D(t) > x\} &\le e^{-\theta r x} M_{X+Y_1+\cdots+Y_n}(\theta) \\
&= e^{-\theta r x} \left[M_X(\theta)\right]^{n+1} &(6.38)\\
&= \frac{1}{(1-\theta)^{n+1}} e^{-\theta r x}. &(6.39)
\end{aligned}
$$

Suppose ϵ is the allowed delay violation probability. Letting the right-hand side of (6.38) equal ϵ, we then have the corresponding delay bound

$$d = \frac{1}{\theta r}\left[\log\frac{1}{\epsilon} + (n+1)\log\frac{1}{(1-\theta)}\right],$$

which clearly scales in $\mathcal{O}(n)$.

Note that the right-hand side of (6.37) is obtained directly from the distribution function of $X + Y_1 + \cdots + Y_n$, while the right-hand side of (6.38) is an upper bound on the distribution function. It can hence be concluded that the end-to-end delay bound under the independent case is a scaling of $\mathcal{O}(n)$.

6.3 Calculus with Moment Generating Functions

This section presents stochastic network calculus results based on moment generating functions (MGFs). In Chapters 3 and 4, respectively we introduced the concepts of the traffic envelope process and service envelope process. In Chapter 5, we showed that the five basic properties can be represented using traffic and service envelope processes. In this section, we further present the corresponding results using the moment generating functions of these processes.

6.3.1 Moment Generating Function Basics

As introduced in Chapter 1, the moment generating function of a random variable X is defined, for any $\theta \ge 0$

$$M_X(\theta) = Ee^{\theta X}, \tag{6.40}$$

where E is the expectation of its argument.

Let $M_X(-\theta) = Ee^{-\theta X}$. It can be easily verified that

$$
\begin{aligned}
M_{\min[X,Y]}(\theta) &\le \min\left[M_X(\theta), M_Y(\theta)\right], &(6.41)\\
M_{\max[X,Y]}(-\theta) &\le \min\left[M_X(-\theta), M_Y(-\theta)\right]. &(6.42)
\end{aligned}
$$

For two independent variables, it is known that

$$M_{X+Y}(\theta) = M_X(\theta) M_Y(\theta), \tag{6.43}$$

$$M_{X-Y}(\theta) = M_X(\theta) M_Y(-\theta), \tag{6.44}$$

and

$$M_{X+Y}(-\theta) = M_X(-\theta) M_Y(-\theta), \tag{6.45}$$

$$M_{X-Y}(-\theta) = M_X(-\theta) M_Y(\theta). \tag{6.46}$$

Once the MGF is obtained for a random variable X, the complementary cumulative distribution function (CCDF) of X is bounded by the well-known Chernoff bound as follows:

$$P\{X \geq x\} \leq e^{-\theta x} E e^{\theta X} = e^{-\theta x} M_X(\theta). \tag{6.47}$$

Throughout this book, we often deal with min-plus convolutions or deconvolutions of functions or random processes. To deal with them using moment generating functions, we define the operators $\star$ and $\circ$ as

$$X \star Y(t) = \sum_{s=0}^{t} X(s)Y(t-s), \tag{6.48}$$

$$X \circ Y(\tau, t) = \sum_{s=0}^{\tau} X(s+t)Y(s), \tag{6.49}$$

where $X(t)$ and $Y(t)$ are two processes. The operator $\star$ indeed defines the discrete convolution operation. When $\tau \to \infty$ in (6.49), we denote

$$X \circ Y(t) \equiv \sum_{s=0}^{\infty} X(s+t)Y(s).$$

We then have the following result for min-plus convolution $X \otimes Y(t)$.

Lemma 6.23. *Let $X(t)$ and $Y(t)$ be independent random processes. The moment generating function of their min-plus convolution is upper-bounded:*

$$M_{X\otimes Y(t)}(-\theta) \leq [M_X(-\theta) \star M_Y(-\theta)](t).$$

Proof. We have from the definition

$$M_{X\otimes Y(t)}(-\theta) = E e^{-\theta \inf_{0\leq s\leq t}[X(s)+Y(t-s)]}.$$

An upper bound on $M_{X\otimes Y}(-\theta, t)$ for any $\theta \geq 0$ is

$$\begin{aligned} M_{X\otimes Y(t)}(-\theta) &\leq E \sup_{0\leq s\leq t} [e^{-\theta[X(s)+Y(t-s)]}] \\ &\leq E \sum_{s=0}^{t} e^{-\theta[X(s)+Y(s-t)]} \\ &= \sum_{s=0}^{t} E\left[e^{-\theta X(s)}\right] \cdot E\left[e^{-\theta Y(t-s)}\right]. \end{aligned}$$

□

An important property of Lemma 6.23 is that it can be easily extended to the min-plus convolution of multiple random processes,

$$M_{X_1\otimes X_2\otimes\cdots\otimes X_n(t)}(-\theta) \le \left[M_{X_1}(-\theta)\star M_{X_2}(-\theta)\star\cdots\star M_{X_n}(-\theta)\right](t).$$

For ease of expression, we define a generalized version of the min-plus de-convolution as

$$(x\oslash y)\,(\tau,t) = \sup_{0\le s\le\tau}\left[x\,(s+t) - y\,(s)\right],$$

which reduces to the normal min-plus deconvolution definition when $\tau\to\infty$. We now have the following result for the generalized min-plus deconvolution.

Lemma 6.24. *Let $X(t)$ and $Y(t)$ be independent random processes. The moment generating function of their min-plus deconvolution is upper-bounded:*

$$M_{X\oslash Y(\tau,t)}(\theta) \le \left[M_X(\theta)\circ M_Y(-\theta)\right](\tau,t).$$

Proof. We have from the definition

$$\begin{aligned}
M_{X\oslash Y(\tau,t)}(\theta) &= Ee^{\theta\sup_{0\le s\le\tau}[X(s+t)-Y(s)]}\\
&\le E\left[\sum_{s0}^{\tau} e^{\theta[X(s+t)-Y(s)]}\right]\\
&\le \sum_{s=0}^{\tau} E\left[e^{\theta X(s+t)}\right]E\left[e^{-\theta Y(s)}\right].
\end{aligned}$$

□

6.3.2 Basic Properties and Performance Bounds

In Section 5.7, the basis network calculus properties have been introduced based on the concepts of the traffic envelope process and service envelope process. In the rest of this section, these results are reproduced by applying the corresponding moment generating functions, Lemma 6.23 and the Chernoff bound. We shall only present in detail the delay analysis using moment generating functions. For other properties, they follow similarly based on the results in Section 5.7.

By definition, the delay in a system at time t is

$$D(t) = \inf\{\tau : A(t) \le A^*(t+\tau)\}.$$

Suppose A has a traffic envelope process $\hat{A}$ and the system provides a service envelope process $\hat{S}(t)$. Then, we have

$$
\begin{aligned}
& A(t) - A^*(t+\tau) \\
= & \sup_{0 \le s \le t+\tau} [A(t) - A(s) - \hat{A}(t-s) + \hat{A}(t-s) - \hat{S}(t+\tau-s)] \\
& + A \otimes \hat{S}(t+\tau) - A^*(t+\tau) \\
\le & \sup_{0 \le s \le t+\tau} [A(t) - A(s) - \hat{A}(t-s)] + A \otimes \hat{S}(t+\tau) - A^*(t+\tau) \\
& + \sup_{0 \le s \le t+\tau} [\hat{A}(t-s) - \hat{S}(t+\tau-s)]. \qquad (6.50)
\end{aligned}
$$

For the first term on the right-hand side of (6.50), when $0 \le s \le t$, $A(t) - A(s) - \hat{A}(t-s) \le 0$ by the definition of a traffic envelope process, and when $t < s \le t+\tau$, we also have $A(t) - A(s) - \hat{A}(t-s) \le 0$ because $\hat{A} \ge 0$ and A is a non-decreasing function. For the second term, we have $A \otimes \hat{S}(t+\tau) - A^*(t+\tau) \le 0$ from the definition of a service envelope process. Applying both to (6.50), we obtain

$$A(t) - A^*(t+\tau) \le \sup_{0 \le s \le t+\tau} [\hat{A}(t-s) - \hat{S}(t+\tau-s)],$$

where we always have $\hat{A}(t-s) - \hat{S}(t+\tau-s) \le 0$ when $t < s \le t+\tau$. It is hence sufficient to consider only $0 \le s \le t$:

$$D(t) \le \inf \left\{ \tau : \sup_{0 \le s \le t} [\hat{A}(s) - \hat{S}(s+\tau)] \le 0 \right\}.$$

In addition, as shown by (5.11) in Section 5.1, we have, for all $x \ge 0$,

$$P\{D(t) > x\} \le P\{A(t) > A^*(t+x)\}.$$

Following the discussion above, we easily get from the Chernoff bound

$$
\begin{aligned}
P\{D(t) > x\} &\le P\left\{ \sup_{0 \le s \le t} [\hat{A}(s) - \hat{S}(s+x)] > 0 \right\} \\
&\le E e^{\theta \sup_{0 \le s \le t} [\hat{A}(s) - \hat{S}(s+x)]},
\end{aligned}
$$

and if $A(t)$ and $\hat{S}(t)$ are independent,

$$
\begin{aligned}
P\{D(t) > x\} &\le \sum_{s=0}^{t} M_{\hat{A}(s)}(\theta) M_{\hat{S}(s+x)}(-\theta) \\
&= \left[M_{\hat{S}}(-\theta) \circ M_{\hat{A}}(\theta) \right](t, x)
\end{aligned}
$$

for any $\theta \ge 0$.

Formally, we have derived the following result.

Corollary 6.25 (Delay Bound). *Consider a system that provides a* **strict** *service envelope process $\hat{S}(t)$ to the input flow $A(t)$. Suppose A has a stochastic envelope process $\hat{A}$. Then, the delay $D(t)$ of the flow at time t satisfies*

$$D(t) \leq \inf\left\{\tau : \sup_{0\leq s\leq t}[\hat{A}(s) - \hat{S}(s+\tau)] \leq 0\right\},$$

and if $\hat{A}$ *and* $\hat{S}(t)$ *are independent, there holds*

$$P\{D(t) > x\} \leq \left[M_{\hat{S}}(-\theta) \circ M_{\hat{A}}(\theta)\right](t, x)$$

for any $\theta \geq 0$.

Corollary 6.26 (Backlog Bound). *Consider a system that provides a* **strict** *service envelope process* $\hat{S}(t)$ *to the input flow* $A(t)$*. Suppose* A *has a stochastic envelope process* $\hat{A}$*. Then the backlog* $B(t)$ *of the flow at time* t *satisfies*

$$B(t) \leq \hat{A} \oslash \hat{S}(0),$$

and particularly if $\hat{A}$ *and* $\hat{S}$ *are independent, there holds*

$$M_{B(t)}(\theta) \leq \left[M_{\hat{A}}(\theta) \circ M_{\hat{S}}(-\theta)\right](t, 0)$$

and

$$P\{B(t) > x\} \leq e^{-\theta x}\left[M_{\hat{A}}(\theta) \circ M_{\hat{S}}(-\theta)\right](t, 0)$$

for any $\theta \geq 0$.

Corollary 6.27 (Output Characterization). *Consider a system that provides a* **strict** *service envelope process* $\hat{S}(t)$ *to the input flow* $A(t)$*. Suppose* A *has a stochastic envelope process* $\hat{A}$*. Then, the output* A^* *has a stochastic envelope process*

$$\hat{A} = \hat{A} \oslash \hat{S}(t),$$

and particularly, if $\hat{A}$ *and* $\hat{S}$ *are independent, there holds*

$$M_{\hat{A}^*(t)}(\theta) \leq \left[M_{\hat{A}}(\theta) \circ M_{\hat{S}}(-\theta)\right](t)$$

and, for any $s, t \geq 0$,

$$P\{\hat{A}^*(s, s+t) > x\} \leq e^{-\theta x}\left[M_{\hat{A}}(\theta) \circ M_{\hat{S}}(-\theta)\right](t)$$

for any $\theta \geq 0$.

Corollary 6.28 (Concatenation Property). *Consider a flow passing through systems* S^h, $h = 1, \ldots, H$, *in sequence. Suppose each system* S^h *provides a* **strict** *service envelope process* $\hat{S}^h(t)$ *to the input, and* $\hat{S}^h(t)$, $h = 1, \ldots, H$ *are independent. Then, the concatenation of these systems offers to the flow a service envelope process*

$$\hat{S}(t) = \hat{S}^1 \otimes \hat{S}^2 \cdots \otimes \hat{S}^H(t),$$

and particularly, if S^h, $h = 1, \ldots, H$, *are independent, there holds*

$$M_{\hat{S}(t)}(-\theta) \leq M_{\hat{S}^1(t)}(-\theta) \star \cdots \star M_{\hat{S}^H(t)}(-\theta). \tag{6.51}$$

Corollary 6.29 (Leftover Service). *Consider a system serving an aggregate of two (possibly aggregate) flows A_1 and A_2. Assume the system provides a* **strict** *service envelope process $\hat{S}$ to the aggregate, and A_2 has a stochastic envelope process $\hat{A}_2$. Then, the system offers to the flow A_1 a service envelope process*

$$\hat{S}_1(t) = (\hat{S} - \hat{A}_2)(t),$$

and particularly, if $\hat{S}$ and $\hat{A}_2$ are independent, there holds

$$M_{\hat{S}_1(t)}(\theta) = M_{\hat{S}(t)}(\theta) \cdot M_{\hat{A}_2(t)}(-\theta). \tag{6.52}$$

Corollary 6.30 (Superposition). *Consider the superposition of n flows A_i, $i = 1, \ldots, n$. If each flow A_i has a stochastic envelope process $\hat{A}_i(t)$, then the aggregate flow $A = \sum_{i=1}^{n} A_i$ has a stochastic envelope process*

$$\hat{A}(t) = \sum_{i=1}^{n} \hat{A}_i(t),$$

and particularly, if A_i, $i = 1, \ldots, n$, are independent, there holds

$$M_{\hat{A}(t)}(\theta) = M_{\hat{A_1}(t)} \cdots M_{\hat{A_n}(t)}(\theta).$$

It is worth highlighting that in the results above, strict service envelope processes are required instead of service envelope processes. This is because by definition the service envelope process of a server is coupled with both its arrival process and departure process; i.e., the stochastic envelope process, the arrival process, and the departure process are dependent. If we had only assumed service envelope processes, the independence analysis would not have been applicable.

Note that, based on Corollary 6.25 for delay and Corollary 6.28 for concatenation, it is easily seen that the end-to-end delay in a tandem network satisfies

$$P\{D^{e2e}(t) > x\} \leq \left[\left(M_{\hat{S}^1(t)}(-\theta) \star \cdots \star M_{\hat{S}^H(t)}(-\theta) \right) \circ M_{\hat{A}}(\theta) \right](t, x)$$

for any $\theta \geq 0$.

For the tandem network considered in Sections 5.6 and 6.2.6, if the cross traffic is $(\sigma(\theta), \rho(\theta))$ constrained, it is shown in [44] that the end-to-end delay scales in $\mathcal{O}(n)$, which is consistent with the finding in Section 6.2.6, where a different approach is used for independent case analysis.

6.4 Summary and Bibliographic Comments

We began this chapter with a simple example demonstrating the performance improvement when the independence condition is taken into account. We then introduced two approaches to independent case analysis. One is based on the concept of a stochastic strict server. This approach is the focus of this chapter. We showed that the five basic properties can be proved for the independent case. As an example, we considered the scaling issue of the end-to-end delay bound of a tandem network that was also studied in Chapter 5. It was shown in Section 5.6 that while the end-to-end delay bound obtained from node-by-node analysis scales in $\mathcal{O}\left(n^2 \log n\right)$, it has a scaling in $\mathcal{O}\left(n \log n\right)$ by utilizing the concatenation property. In this chapter, we further showed in Section 6.2.6 that, by exploiting the independence condition, the end-to-end delay bound has a scaling in $\mathcal{O}(n)$.

In Section 6.3, we introduced another approach that can be used for the independent case analysis. In this approach, moment generating functions are applied to the traffic and service envelope processes and the five basic properties based on these processes introduced in Section 5.7. Comparing this with the approach introduced in Section 6.2, the approach based on the moment generating function is perhaps conceptually easier to adopt since the moment generating function is a well-known concept used in analyzing stochastic processes. However, when it comes to deriving closed-form bounds, the approach based on the moment generating function may need some hard work. In addition, the bounds obtained may be looser than those from the approach based on the stochastic strict server.

In the stochastic network calculus literature, independence has long been considered in the analysis. Particularly, independence is often assumed between flows in the vast effective bandwidth literature (e.g., [36] [81] [80]) and early stochastic network calculus works (e.g., [138] [15]). However, these works mainly focused on the superposition property and the single-node deterministic server case. The independent case analysis approach introduced in Section 6.2 was initially proposed by Jiang [69]. The paper [69] provides the first full analysis of the five basic properties for the independent case. The concept of a stochastic strict server due to impairment, an important concept for independent case analysis, was initially proposed by Jiang and Emstad [73]. Applying moment generating functions to the independent case analysis of the full five basic properties was first made by Fidler [44]. Also in [44], it was reported that the end-to-end delay bound for the tandem network as studied in Section 6.2.6 has a scaling in $\mathcal{O}(n)$. While this conclusion comes after some complex analysis in [44], it can be easily obtained from the approach based on the stochastic strict server as shown in Section 6.2.6.

Problems

6.1. Consider a server fed with a flow A that is the aggregation of two constituent independent flows A_f and A_h. Suppose the server provides a deterministic strict service curve β to the aggregate flow A. Flow A_h has m.b.c. stochastic arrival curve $A_h \sim_{mb} \langle f^h, r^h \rangle$ and $\beta^f \in \mathcal{F}$.

(i) Prove that flow A_f receives a stochastic strict service curve β with *impairment process* $I = A_h(t) - A_h(t-s)$.
(ii) Derive the per-flow service curve received by A_f.

6.2. Consider a constant-rate server with link capacity C fed with N input flows with maximum packet size M. All flows are independent of each other and all are $(\sigma(\theta), \rho(\theta))$ upper constrained with the same parameters. The buffer size is B.

(i) How many such flows can be admitted into the system such that the buffer overflow probability is less than P_{loss}?
(ii) How many such flows can be admitted into the system such that the probability that the delay experienced by a packet in this system is greater than D is less than P_{delay}?

6.3. What is the MGF of the service process for a constant-rate server with link capacity C?

6.4. What is the MGF of a Poisson process with mean arrival rate λ and mean packet size μ?

6.5. Consider a constant-rate server with link capacity C fed with a Poisson input flow with arrival rate λ. The packet size is exponentially distributed with mean μ but limited by a maximum packet size M. Analyze the delay distribution using the MGF-based approach and compare it with the results obtained by queuing theory and the approach based on stochastic network calculus.

6.6. Prove Theorem 6.18.

6.7. Prove Corollary 6.26.

6.8. Prove Corollary 6.27.

6.9. Prove Corollary 6.28.

6.10. Prove Corollary 6.29.