5

Basic Properties of Stochastic Network Calculus

This chapter introduces the basic results of stochastic network calculus under the various traffic models and server models discussed earlier in Chapters 3 and 4. We focus particularly on the five basic properties introduced in Chapter 1 that are essential for network service guarantee analysis.

5.1 Service Guarantees

We start by deriving probabilistic bounds on the backlog and delay under different combinations of traffic and server models.

5.1.1 Backlog Bound

Consider a system with arrival process $A(t)$, service process $S(t)$, and departure process $A^*(t)$. By definition, the backlog in the system at time $t \geq 0$ is

$$B(t) = A(t) - A^*(t), \tag{5.1}$$

which implies that if both $A(t)$ and $A^*(t)$ were known, $B(t)$ would be derived. However, in most cases, $A^*(t)$ needs to be derived from $B(t)$; i.e., $A^*(t) = A(t) - B(t)$, which causes the chicken–egg problem.

The Lindley equation can be used to derive $B(t)$:

$$B(t) = \max\{0, B(t-1) + A(t-1,t) - S(t-1,t)\}. \tag{5.2}$$

By applying (5.2) iteratively to its right-hand side, the Lindley equation results in

$$B(t) = \sup_{0 \leq s \leq t} \{A(s,t) - S(s,t)\}, \tag{5.3}$$

and consequently

$$A^*(t) = A(t) - B(t) = \inf_{0 \leq s \leq t} \{A(s) + S(s,t)\}. \tag{5.4}$$

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In the simple case where the system provides a constant service rate c to the input, (5.2) becomes

$$B(t) = \sup_{0 \le s \le t} \{A(s,t) - c \cdot (t-s)\}.$$

Comparing the right-hand side of the equation above with the definitions of the various stochastic traffic models defined in Chapter 3, we find that the probabilistic bound of $B(t)$ is easily derived if the input has a v.b.c. stochastic arrival curve (SAC) or a m.b.c. SAC, or an θ-m.b.c. SAC. However, if the input is known only with its $(\sigma(\theta), \rho(\theta))$ or t.a.c. SAC characterization, additional effort is needed to derive $B(t)$ from such input traffic characterizations.

For the more general case where the system provides stochastic service to the input, the following method can be used to derive $B(t)$. Specifically, (5.1) can be rewritten for any functions $\alpha(t)$ and $\beta(t)$ in $\mathcal{F}$, as

$$\begin{aligned} B(t) &= A(t) - A^*(t) = [A(t) - A \otimes \beta(t)] + [A \otimes \beta(t) - A^*(t)] \\ &= \sup_{0 \le s \le t} \{A(s,t) - \alpha(t-s) + \alpha(t-s) - \beta(t-s)\} + [A \otimes \beta(t) - A^*(t)] \\ &\le \sup_{0 \le s \le t} \{A(s,t) - \alpha(t-s)\} + \sup_{0 \le s \le t} \{\alpha(s) - \beta(s)\} + [A \otimes \beta(t) - A^*(t)] \\ &\le \sup_{0 \le s \le t} \{A(s,t) - \alpha(t-s)\} + \sup_{t \ge 0} \{\alpha(t) - \beta(t)\} + [A \otimes \beta(t) - A^*(t)]. \end{aligned} \tag{5.5}$$

The right-hand side of (5.5) implies a sufficient condition to obtain $P\{B(t) > x\}$; that is, $P\{\sup_{0 \le s \le t}\{A(s,t) - \alpha(t-s)\} > x\}$ and $P\{A \otimes \beta(t) - A^*(t) > x\}$ are known and

$$\lim_{t \to \infty} \frac{1}{t} [\alpha(t) - \beta(t)] \le 0. \tag{5.6}$$

In the rest of the book, unless explicitly stated, we shall assume inequality (5.6) holds.

Based on the analysis above, we present results for a probabilistic backlog bound under different combinations of the traffic models and server models introduced in Chapters 3 and 4.

For backlog, if the input has a v.b.c. SAC and the system provides a weak stochastic service curve (SSC), we can conclude immediately from (5.5) that $P\{B(t) > x\} \le f \otimes g(x - \alpha \oslash \beta(0))$. Since both the m.b.c. SAC and θ-m.b.c SAC imply a v.b.c. SAC, and both SSC and θ–SSC imply weak SSC, the conclusion is readily extended to cases where the input has either an m.b.c. SAC or a θ–m.b.c SAC and/or the system provides either an SSC or a θ-SSC. Formally, we have Theorem 5.1 under these combinations.

Theorem 5.1. *Consider a system S with input A. If the input has a v.b.c. (or m.b.c. or θ-m.b.c.) stochastic arrival curve $\alpha \in \mathcal{F}$ with bounding function $f \in \hat{\mathcal{F}}$, (i.e., $A \sim_{sac} \langle f, \alpha \rangle$), where $\sim_{sac}$ is one of either $\sim_{vb}$, $\sim_{mb}$, or $\sim_{\theta-mb}$ and the system provides to the input a weak stochastic service curve (or stochastic* *service curve, or θ–stochastic service curve) $\beta \in \mathcal{F}$ with bounding function $g \in \hat{\mathcal{F}}$, i.e. $S \sim_{ssc} \langle g, \beta \rangle$ where $\sim_{ssc}$ is either one of $\sim_{ws}$, $\sim_{sc}$, $\sim_{\theta-sc}$, then for all $t \geq 0$ and $x \geq 0$, the backlog $B(t)$ is bounded by*

$$P\{B(t) > x\} \leq f \otimes g(x - \alpha \oslash \beta(0)). \tag{5.7}$$

Based on the relationship between t.a.c SAC and v.b.c SAC in Theorem 3.13 which is also shown in Figure 3.1, the following result is obtained.

Corollary 5.2. *Consider a system S with input A. Suppose the input has a t.a.c stochastic arrival curve $\alpha \in \mathcal{F}$ with bounding function $f \in \bar{\mathcal{F}}$ (i.e., $A \sim_{ta} \langle f, \alpha \rangle$) and the system provides to the input a weak stochastic service curve (or stochastic service curve or θ-stochastic service curve) $\beta \in \mathcal{F}$ with bounding function $g \in \bar{\mathcal{F}}$ (i.e., $S \sim_{ssc} \langle g, \beta \rangle$, where $\sim_{ssc}$is one of either $\sim_{ws}$, $\sim_{sc}$, or $\sim_{\theta-sc}$). Then, for all $t \geq 0$ and $x \geq 0$, the backlog $B(t)$ is bounded by*

$$P\{B(t) > x\} \leq f^{\theta} \otimes g(x - \alpha \oslash \beta(0)), \tag{5.8}$$

where $f^{\theta}(x) = f(x) + \frac{1}{\theta} \int_x^{\infty} f(y)dy$ for any $\theta > 0$.

Actually, Theorems 5.1 and 5.2 show the backlog bounds under all combinations of the various SAC and SSC models defined in Chapters 3 and 4.

Similarly, the following theorem can be derived according to the mapping between the $(\sigma(\theta), \rho(\theta))$ upper constrained traffic characterization and the t.a.c. SAC model shown in Theorem 3.4 and the mapping between the $(b(\theta), r(\theta))$ lower constrained service characterization and the weak SSC model as shown in Theorem 4.19.

Corollary 5.3. *Consider a system S with input A. Suppose the input is $(\sigma(\theta), \rho(\theta))$ upper constrained, and the service provided by the system is $(b(\theta), r(\theta))$ lower constrained. Then, the backlog $B(t)$ is bounded by*

$$P\{B(t) > x\} \leq f^{\theta} \otimes g^{\theta}(x - \alpha \oslash \beta(0)), \tag{5.9}$$

where $f^{\theta}(x) = e^{-\theta x}$, $\alpha(t) = \rho(\theta) \cdot t + \sigma(\theta)$, $\beta(t) = r(\theta) \cdot (t - b(\theta))$, and $g^{\theta}(x) = e^{-\theta x}$ for any $\theta > 0$.

5.1.2 Delay Bound

Now we discuss the probabilistic delay bounds under different combinations of traffic models and server models. For a delay in the system at time $t \geq 0$, by definition, it is

$$D(t) = \inf\{\tau : A(t) \leq A^*(t + \tau)\}, \tag{5.10}$$

which implies that, for any $x \geq 0$, if $D(t) > x$, there must be $A(t) > A^*(t+x)$ since otherwise if $A(t) \leq A^*(t + x)$ and $D(t) \leq x$, that would contradict the condition $D(t) > x$. In other words, event $\{D(t) > x\}$ implies event $\{A(t) > A^*(t+x)\}$, or

$$\{D(t) > x\} \subset \{A(t) > A^*(t+x)\},$$

and hence

$$P\{D(t) > x\} \le P\{A(t) > A^*(t+x)\}. \tag{5.11}$$

Following similar steps in (5.5), we can get

$$\begin{aligned}
& A(t) - A^*(t+x) \\
= & \sup_{0\le s\le t+x} [A(t) - A(s) - \alpha(t-s) + \alpha(t-s) - \beta(t+x-s)] \\
& + A \otimes \beta(t+x) - A^*(t+x) \\
\le & \sup_{0\le s\le t+x} [A(t) - A(s) - \alpha(t-s)] \\
& + A \otimes \beta(t+x) - A^*(t+x) \\
& + \sup_{0\le s\le t+x} [\alpha(t-s) - \beta(t+x-s)] \qquad (5.12) \\
\le & \sup_{0\le s\le t} [A(t) - A(s) - \alpha(t-s)] \\
& + A \otimes \beta(t+x) - A^*(t+x) \\
& + \sup_{0\le s\le t+x} [\alpha(t-s) - \beta(t+x-s)], \qquad (5.13)
\end{aligned}$$

where the step from (5.12) to (5.13) holds because by default $A(t) \le A(t+y)$ for any $y > 0$, $\alpha(y) = 0$ for any $y < 0$.

Under the same sufficient condition as for analyzing the backlog, the complementary cumulative distribution function of the right-hand side of (5.13) is bounded and so are the left-hand side of (5.13) and the delay.

With simple manipulation, we have from (5.13)

$$\begin{aligned}
& A(t) - A^*\left(t + h(\alpha + y, \beta)\right) \\
\le & \sup_{0\le s\le t} [A(t) - A(s) - \alpha(t-s)] \\
& + A \otimes \beta\left(t + h(\alpha + y, \beta)\right) - A^*\left(t + h(\alpha + y, \beta)\right) - y, \qquad (5.14)
\end{aligned}$$

where $h\,(\alpha + y, \beta)$ is the maximum horizontal distance between functions $\alpha(t) + y$ and $\beta(t)$ for $y \ge 0$. This is obtained by simply replacing $x = h\,(\alpha + y, \beta)$ in (5.13), and with the definition of maximum horizontal distance function $h(\cdot,\cdot)$, that implies $\alpha(u) + y \le \beta(u + h\,(\alpha + y, \beta))$ for any u.

Similar to the backlog, if the input has a v.b.c SAC and the system provides a weak SSC, we immediately conclude from (5.14) and (5.11) that for any $t \ge 0$ and $y \ge 0$, the delay $D(t)$ is bounded by $P\{D(t) > h\,(\alpha + y, \beta)\} \le f \otimes g(y)$.

Since both the m.b.c. SAC and θ-m.b.c. SAC imply a v.b.c. SAC, and both the SSC and θ-SSC imply a weak SSC, the conclusion is also readily extended to cases where the input has either an m.b.c. SAC or a θ-m.b.c. SAC, and/or the system provides either an SSC or a θ-SSC. Similar to the backlog analysis, we have the following result for delay.

Theorem 5.4. *Consider a system S with input A. Suppose that the input has a v.b.c. (or m.b.c. or θ-m.b.c.) stochastic arrival curve $\alpha \in \mathcal{F}$ with bounding function $f \in \bar{\mathcal{F}}$ (i.e., $A \sim_{sac} \langle f, \alpha \rangle$ where $\sim_{sac}$ is one of either $\sim_{vb}$, $\sim_{mb}$, or $\sim_{\theta-mb}$) and the system provides to the input a weak stochastic service curve (or stochastic service curve or θ-stochastic service curve) $\beta \in \mathcal{F}$ with bounding function $g \in \bar{\mathcal{F}}$ (i.e., $S \sim_{ssc} \langle g, \beta \rangle$, where $\sim_{ssc}$ is one of either $\sim_{ws}$, $\sim_{sc}$, or $\sim_{\theta-sc}$). Then, for all $t \geq 0$ and $x \geq 0$, the delay $D(t)$ is bounded by*

$$P\{D(t) > h\left(\alpha + x, \beta\right)\} \leq f \otimes g(x). \tag{5.15}$$

Based on the relationship between the t.a.c. SAC and v.b.c. SAC in Theorem 3.13, the following result is obtained.

Corollary 5.5. *Consider a system S with input A. Suppose the input has a t.a.c. stochastic arrival curve $\alpha \in \mathcal{F}$ with bounding function $f \in \hat{\mathcal{G}}$ (i.e., $A \sim_{ta} \langle f, \alpha \rangle$) and the system provides to the input a weak stochastic service curve (or stochastic service curve, or θ–stochastic service curve) $\beta \in \mathcal{F}$ with bounding function $g \in \bar{\mathcal{F}}$ (i.e., $S \sim_{ssc} \langle g, \beta \rangle$ where $\sim_{ssc}$ is one of either $\sim_{ws}$, $\sim_{sc}$, or $\sim_{\theta-sc}$). Then, for all $t \geq 0$ and $x \geq 0$, the delay $D(t)$ is bounded by*

$$P\{D(t) > h\left(\alpha + x, \beta\right)\} \leq f^{\theta} \otimes g(x), \tag{5.16}$$

where $f^{\theta}(x) = f\left(x\right) + \frac{1}{\theta} \int_{x}^{\infty} f(y)dy$ for any $\theta > 0$.

Actually, Theorems 5.4 and 5.5 show the stochastic delay bounds under all combinations of the various SAC and SSC models defined in Chapters 3 and 4.

Similarly, the following theorem can be derived according to the mapping between the $(\sigma(\theta), \rho(\theta))$ upper constrained traffic characterization and t.a.c. SAC shown in Theorem 3.4 and the mapping between the $(b(\theta), r(\theta))$ lower constrained service characterization and weak SSC as shown in Theorem 4.19.

Corollary 5.6. *Consider a system S with input A. Suppose the input is $(\sigma(\theta), \rho(\theta))$ upper constrained and the service provided by the system is $(b(\theta), r(\theta))$ lower constrained. The delay $D(t)$ is bounded by*

$$P\{D(t) > h\left(\alpha + x, \beta\right)\} \leq f^{\theta} \otimes g^{\theta}(x), \tag{5.17}$$

where $f^{\theta}(x) = e^{-\theta x}$, $\alpha(t) = \rho(\theta) \cdot t + \sigma(\theta)$, $\beta(t) = r(\theta) \cdot (t - \sigma(\theta))$, and $g^{\theta}(x) = e^{-\theta x}$ for any $\theta > 0$.

Example 5.7. Consider a server with constant service rate C. If the input is an EBB (exponentially bounded burstiness) process, i.e.

$$P\left\{A\left(s, t\right) - \alpha\left(t - s\right) > x\right\} \leq f\left(x\right),$$

where $\alpha = \rho \cdot t$, and $f(x) = ae^{-bx}$. As shown in Chapter 2, EBB is a special case of t.a.c. stochastic arrival curve. In addition, the constant-rate server provides a deterministic service curve $\beta(t) = Ct$, which is a special case of weak stochastic service curve. Then, according to Theorem 5.2, for all $t \geq 0$ and $x \geq 0$, the backlog $B(t)$ of this system is bounded by

$$\begin{aligned} P\{B(t) > x\} &\leq f^{\theta} \otimes g(x - \alpha \oslash \beta(0)) \\ &= f^{\theta}(x - \alpha \oslash \beta(0)) = f^{\theta}(x) \\ &= ae^{-bx} + \frac{ae^{-bx}}{\theta b} \end{aligned}$$

for any $\theta > 0$.

5.2 Output Characterization

This section presents results for characterizing the output traffic. The focus is on using the same traffic model as the input for the characterization.

Equation (5.4) implies the following: for any $t \geq s \geq 0$,

$$A^*(t) - A^*(s) = A(t) - A(s) - (B(t) - B(s)).$$

If the backlog has an upper bound b (i.e., $B(t) \leq b$ for all $t \geq 0$), we immediately get

$$A^*(t) - A^*(s) \leq A(t) - A(s) - b,$$

and in this case it is easy to show that the output has the same characterization as the input. Specifically, we have the following result.

Theorem 5.8. *Consider a system S with input A. Suppose the backlog of A in the system is upper-bounded by b for all times. If the input has a stochastic arrival curve $\alpha(t)$ with bounding function $f(x)$, denoted by $A \sim_{sac} \langle f(x), \alpha(t)\rangle$, where $\sim_{sac}$ can be one of either $\sim_{tac}$, $\sim_{vbc}$, $\sim_{mbc}$ or $\sim_{\theta-mbc}$, then the output also has a stochastic arrival curve $\alpha(t)$ with bounding function $f(x+b)$; i.e., $A^* \sim_{sac} \langle f(x+b), \alpha(t)\rangle$.*

In general, the backlog may not be deterministically upper-bounded. In such cases, to characterize the output traffic requires some effort.

5.2.1 Output t.a.c Stochastic Arrival Curve

First, we focus on characterizing the output traffic with the t.a.c. stochastic arrival curve model. For any $t \geq s \geq 0$ and any functions $\alpha, \beta \in \mathcal{F}$, there holds

$$
\begin{aligned}
&A^*(t) - A^*(s) \le A(t) - A \otimes \beta(s) + [A \otimes \beta(s) - A^*(s)] \\
&= \sup_{0 \le u \le s} \{A(u,t) - \alpha(t-u) + \alpha(t-u) - \beta(s-u)\} + [A \otimes \beta(s) - A^*(s)] \\
&\le \sup_{0 \le u \le t} \{A(u,t) - \alpha(t-u)\} + \sup_{0 \le v \le s} \{\alpha(t-s+v) - \beta(v)\} \\
&\quad + [A \otimes \beta(t) - A^*(t)] \qquad (5.18) \\
&\le \sup_{0 \le u \le t} \{A(u,t) - \alpha(t-u)\} + \alpha \oslash \beta(t-s) + [A \otimes \beta(s) - A^*(s)], \qquad (5.19)
\end{aligned}
$$

where $\alpha \oslash \beta(t) = \sup_{u \ge 0}\{\alpha(t+u) - \beta(u)\}$.

Rewriting (5.19), we get

$$
\begin{aligned}
&A^*(s,t) - \alpha \oslash \beta(t-s) \\
&\le \sup_{0 \le u \le t} \{A(u,t) - \alpha(t-u)\} + [A \otimes \beta(s) - A^*(s)], \qquad (5.20)
\end{aligned}
$$

which implies that if the input m.b.c SAC and the system's weak SSC are known, the output t.a.c SAC characterization is easily derived.

Since the m.b.c. SAC and θ-m.b.c. SAC imply the v.b.c. SAC and the SSC and θ-SSC imply the weak SSC, we have the following theorem.

Theorem 5.9. *Consider a system S with input A. If the input has a v.b.c. (or m.b.c or θ–m.b.c.) stochastic arrival curve $\alpha \in \mathcal{F}$ with bounding function $f \in \bar{\mathcal{F}}$ (i.e., $A \sim_{sac} \langle f, \alpha \rangle$ where $\sim_{sac}$ is one of either $\sim_{vb}$, $\sim_{mb}$, or $\sim_{\theta-mb}$) and the system provides to the input a weak stochastic service curve (or stochastic service curve or θ-stochastic service curve) $\beta \in \mathcal{F}$ with bounding function $g \in \bar{\mathcal{F}}$ (i.e., $S \sim_{ssc} \langle g, \beta \rangle$ where $\sim_{ssc}$ is one of either $\sim_{ws}$, $\sim_{sc}$, or $\sim_{\theta-sc}$) then the output has a t.a.c. stochastic arrival curve $\alpha \oslash \beta$ with bounding function $f \otimes g$; i.e., $A^* \sim_{ta} \langle f \otimes g, \alpha \oslash \beta \rangle$.*

Based on the relationship between the t.a.c. SAC and v.b.c. SAC as shown in Theorem 3.13, the following result is obtained.

Corollary 5.10. *Consider a system S with input A. If the input has a t.a.c. stochastic arrival curve $\alpha \in \mathcal{F}$ with bounding function $f \in \bar{\mathcal{G}}$ (i.e., $A \sim_{ta} \langle f, \alpha \rangle$) and the system provides to the input a weak stochastic service curve (or stochastic service curve or θ–stochastic service curve) $\beta \in \mathcal{F}$ with bounding function $g \in \bar{\mathcal{F}}$ (i.e., $S \sim_{ssc} \langle g, \beta \rangle$ where $\sim_{ssc}$ is one of either $\sim_{ws}$, $\sim_{sc}$, or $\sim_{\theta-sc}$), then the output has a t.a.c. stochastic arrival curve $\alpha_\theta \oslash \beta$ with bounding function $f \otimes g^\theta$, i.e., $A^* \sim_{ta} \langle f^\theta \otimes g, \alpha_\theta \oslash \beta \rangle$, where $\alpha_\theta(t) = \alpha(t) + \theta \cdot t$ and $f^\theta(x) = f(x) + \frac{1}{\theta}\int_x^\infty f(x)$ for any $\theta > 0$.*

In addition, the following theorem follows from Theorem 5.9, the mapping between the $(\sigma(\theta), \rho(\theta))$ upper constrained traffic characterization and v.b.c. SAC shown in Example 3.18, and the mapping between $(b(\theta), r(\theta))$–lower constrained service characterization and weak SSC shown in Theorem 4.19.

Corollary 5.11. *Consider a system S with input A. Suppose the input is $(\sigma(\theta), \rho(\theta))$ upper constrained, $a(t) \equiv A(t-1, t), t = 1, 2, \ldots,$ are i.i.d., and the service provided by the system is $(b(\theta), r(\theta))$ lower constrained. Then, the output has a t.a.c stochastic arrival curve $\alpha \oslash \beta$ with bounding function $f \otimes g$; i.e., $A^* \sim_{ta} \langle f \otimes g, \alpha \oslash \beta \rangle$, where $f(x) = \frac{e^{\theta\sigma(\theta)}}{1-e^{\theta(\rho(\theta)-r)}} e^{-\theta x}$, $\alpha(t) = r \cdot t$, $\beta(t) = r(\theta) \cdot (t - b(\theta))$, and $g(x) = e^{-\theta x}$ for any $\theta > 0, r < \rho(\theta)$.*

5.2.2 Output v.b.c. Stochastic Arrival Curve

We now characterize the output traffic with the v.b.c. stochastic arrival curve model.

Based on (5.20), we can get

$$\begin{aligned} &\sup_{0 \le s \le t} \{A^*(s,t) - \alpha \oslash \beta(t-s)\} \\ &\le \sup_{0 \le u \le t} \{A(u,t) - \alpha(t-u)\} + \sup_{0 \le s \le t} \{A \otimes \beta(s) - A^*(s)\}, \end{aligned} \tag{5.21}$$

from this and the fact that the m.b.c. SAC and θ-m.b.c. SAC imply a v.b.c. SAC, the following theorem can be easily verified.

Theorem 5.12. *Consider a system S with input A. If the input has a v.b.c. (or m.b.c. or θ-m.b.c.) stochastic arrival curve $\alpha \in \mathcal{F}$ with bounding function $f \in \bar{\mathcal{F}}$ (i.e., $A \sim_{sac} \langle f, \alpha \rangle$, where $\sim_{sac}$ is either one of $\sim_{vb}$, $\sim_{mb}$, or $\sim_{\theta-mb}$) and the system provides to the input a stochastic service curve $\beta \in \mathcal{F}$ with bounding function $g \in \bar{\mathcal{F}}$ (i.e., $S \sim_{sc} \langle g, \beta \rangle$), then the output has a v.b.c. stochastic arrival curve $\alpha \oslash \beta$ with bounding function $f \otimes g$, i.e. $A^* \sim_{vb} \langle f \otimes g, \alpha \oslash \beta \rangle$.*

Based on the relationship between the stochastic service curve and θ-stochastic service curve shown in Theorem 4.6, the following result is obtained.

Corollary 5.13. *Consider a system S with input A. If the input has a v.b.c. (or m.b.c. or θ-m.b.c.) stochastic arrival curve $\alpha \in \mathcal{F}$ with bounding function $f \in \bar{\mathcal{F}}$ (i.e., $A \sim_{sac} \langle f, \alpha \rangle$, where $\sim_{sac}$ is one of either $\sim_{vb}$, $\sim_{mb}$, or $\sim_{\theta-mb}$) and the system provides to the input a θ–stochastic service curve $\beta \in \mathcal{F}$ with bounding function $g^\theta \in \bar{\mathcal{F}}$ (i.e. $S \sim_{\theta-sc} \langle g^\theta, \beta \rangle$), then the output has a v.b.c. stochastic arrival curve $\alpha \oslash \beta$ with bounding function $f \otimes g_t$; i.e., $A^* \sim_{vb} \langle f \otimes g_t, \alpha \oslash \beta \rangle$, where $g_t(x) = g^\theta(x - \theta \cdot t)$.*

In addition, based on the relationship between the stochastic service curve and weak stochastic service curve shown in Theorem 4.4, the following result can be easily verified.

Corollary 5.14. *Consider a system S with input A. If the input has a v.b.c. (or m.b.c. or θ-m.b.c.) stochastic arrival curve $\alpha \in \mathcal{F}$ with bounding function $f \in \bar{\mathcal{F}}$ (i.e., $A \sim_{sac} \langle f, \alpha \rangle$ where $\sim_{sac}$ is one of either $\sim_{vb}$, $\sim_{mb}$, or $\sim_{\theta-mb}$)*

and the system provides to the input a weak stochastic service curve $\beta \in \mathcal{F}$ *with bounding function* $g \in \bar{\mathcal{G}}$ *(i.e.,* $S \sim_{ws} \langle g, \beta \rangle$*), then the output has a v.b.c. stochastic arrival curve* $\alpha \oslash \beta_{-\theta}$ *with bounding function* $f \otimes g_t^\theta$*; i.e.,* $A^* \sim_{vb} \langle f \otimes g_t^\theta, \alpha \oslash \beta_{-\theta} \rangle$*, where* $\beta_{-\theta}(t) = \beta(t) - \theta \cdot t$ *and* $g_t^\theta(x) = \left[\frac{1}{\theta} \int_{x-\theta t}^{\infty} g(y)\, dy \right]_1$.

Corollaries 5.13 and 5.14 are obtained directly from the relationship of a θ-stochastic service curve or weak stochastic service curve with a stochastic service curve. The resulting bounding functions for the output are time-dependent. In the following, we present results for the output characterization where the bounding function does not rely on time.

Let $\alpha_\theta(t) = \alpha(t) + \theta \cdot t$. Similar to (5.19), we get, for any $\theta > 0$,

$$\begin{aligned}
&A^*(t) - A^*(s) - \alpha_\theta \oslash \beta(t-s) \\
&\le A(t) - A \otimes \beta(s) + A \otimes \beta(s) - A^*(s) - \alpha_\theta \oslash \beta(t-s) \\
&\le A(t) - A \otimes \beta(s) - \alpha \oslash \beta(t-s) + A \otimes \beta(s) - A^*(s) - \theta \cdot (t-s)
\end{aligned}$$

since $\sup_{w \ge 0}[\alpha(t-s+w] + (t-s+w)\theta - \beta(w)] \ge \sup_{w \ge 0}[\alpha(t-s+w] - \beta(w)] + (t-s)\theta$. Then, there holds:

$$\begin{aligned}
&\sup_{0 \le s \le t} \{A^*(t) - A^*(s) - \alpha_\theta \oslash \beta(t-s)\} \\
\le &\sup_{0 \le s \le t} [A(t) - A \otimes \beta(s) - \alpha \oslash \beta(t-s)] \\
&+ \sup_{0 \le s \le t} [A \otimes \beta(s) - A^*(s) - \theta \cdot (t-s)] \qquad (5.22) \\
\le &\sup_{0 \le s \le t} \left[\sup_{0 \le u \le s} \{A(t) - A(u) - \beta(s-u) - \sup_{w \ge 0}\{\alpha(t-s+w) - \beta(w)\}\} \right] \\
&+ \sup_{0 \le s \le t} [A \otimes \beta(s) - A^*(s) - \theta \cdot (t-s)] \\
\le &\sup_{0 \le s \le t} \sup_{0 \le u \le s} \{A(t) - A(u) - \alpha(t-u)\} \\
&+ \sup_{0 \le s \le t} [A \otimes \beta(s) - A^*(s) - \theta \cdot (t-s)] \qquad (5.23) \\
= &\sup_{0 \le u \le t} \{A(t) - A(u) - \alpha(t-u)\} \qquad (5.24) \\
&+ \sup_{0 \le s \le t} [A \otimes \beta(s) - A^*(s) - \theta \cdot (t-s)]\,. \qquad (5.25)
\end{aligned}$$

From (5.24), we can conclude the following theorem.

Theorem 5.15. *Consider a system* S *with input* A*. If the input has a v.b.c. (or m.b.c. or* θ*–m.b.c.) stochastic arrival curve* $\alpha \in \mathcal{F}$ *with bounding function* $f \in \bar{\mathcal{F}}$ *(i.e.,* $A \sim_{sac} \langle f, \alpha \rangle$ *where* $\sim_{sac}$ *is one of either* $\sim_{vb}$*,* $\sim_{mb}$*, or* $\sim_{\theta-mb}$*) and the system provides to the input a* θ*–stochastic service curve* $\beta \in \mathcal{F}$ *with bounding function* $g^\theta \in \bar{\mathcal{F}}$ *(i.e.,* $S \sim_{\theta-sc} \langle g^\theta, \beta \rangle$*), then the output has a v.b.c. stochastic arrival curve* $\alpha_\theta \oslash \beta$ *with bounding function* $f \otimes g^\theta$*; i.e.,* $A^* \sim_{vb} \langle f \otimes g^\theta, \alpha_\theta \oslash \beta \rangle$*, where* $\alpha_\theta(t) = \alpha(t) + \theta \cdot t$.

Then, based on the relationship between the θ–stochastic service curve and weak stochastic service curve, the following result is easily verified.

Corollary 5.16. *Consider a system S with input A. If the input has a v.b.c. (or m.b.c. or θ–m.b.c.) stochastic arrival curve $\alpha \in \mathcal{F}$ with bounding function $f \in \bar{\mathcal{F}}$ (i.e., $A \sim_{sac} \langle f, \alpha \rangle$ where $\sim_{sac}$ is one of either $\sim_{vb}$, $\sim_{mb}$, or $\sim_{\theta-mb}$) and the system provides to the input a weak stochastic service curve $\beta \in \mathcal{F}$ with bounding function $g \in \bar{\mathcal{G}}$ (i.e., $S \sim_{ws} \langle g, \beta \rangle$), then the output has a v.b.c. stochastic arrival curve $\alpha_\theta \oslash \beta_{-\theta}$ with bounding function $f \otimes g^\theta$; i.e., $A^* \sim_{vb} \langle f \otimes g^\theta, \alpha_\theta \oslash \beta_{-\theta} \rangle$, where $\alpha_\theta(t) = \alpha(t) + \theta \cdot t$, $\beta_{-\theta}(t) = \beta(t) - \theta \cdot t$, and $g^\theta(x) = g(x) + \frac{1}{\theta} \int_x^\infty g(y)\, dy$ for any $\theta > 0$.*

If the input has a t.a.c. SAC, we can use the relationship between the t.a.c. SAC and v.b.c. SAC to represent the input with a v.b.c. SAC and consequently get the following results under the stochastic service curve, θ-stochastic service curve, and weak stochastic service curve.

Corollary 5.17. *Consider a system S with input A. If the input has a t.a.c. stochastic arrival curve $\alpha \in \mathcal{F}$ with bounding function $f \in \bar{\mathcal{G}}$ (i.e., $A \sim_{ta} \langle f, \alpha \rangle$) and the system provides to the input a stochastic service curve $\beta \in \mathcal{F}$ with bounding function $g \in \bar{\mathcal{F}}$ (i.e., $S \sim_{sc} \langle g, \beta \rangle$), then the output has a v.b.c. stochastic arrival curve $\alpha_\theta \oslash \beta$ with bounding function $f^\theta \otimes g$; i.e., $A^* \sim_{vb} \langle f^\theta \otimes g, \alpha_\theta \oslash \beta \rangle$, where $\alpha_\theta(t) = \alpha(t) + \theta \cdot t$ and $f^\theta(x) = f(x) + \frac{1}{\theta} \int_x^\infty f(y)dy$ for any $\theta > 0$.*

Corollary 5.18. *Consider a system S with input A. If the input has a t.a.c. stochastic arrival curve $\alpha \in \mathcal{F}$ with bounding function $f \in \bar{\mathcal{G}}$ (i.e., $A \sim_{ta} \langle f, \alpha \rangle$) and the system provides to the input a θ-stochastic service curve $\beta \in \mathcal{F}$ with bounding function $g^{\theta_2} \in \bar{\mathcal{F}}$ (i.e. $S \sim_{\theta-sc} \langle g^{\theta_2}, \beta \rangle$), then*

- *the output has a v.b.c. stochastic arrival curve $\alpha \oslash \beta$ with bounding function $f^{\theta_1} \otimes g_t$, (i.e., $A^* \sim_{vb} \langle f^{\theta_1} \otimes g_t, \alpha_{\theta_1} \oslash \beta \rangle$), where $\alpha_{\theta_1}(t) = \alpha(t) + \theta_1 \cdot t$, $f^{\theta_1}(x) = f(x) + \frac{1}{\theta_1} \int_x^\infty f(y)dy$, and $g_t(x) = g^{\theta_2}(x - \theta_2 \cdot t)$ for any $\theta_1, \theta_2 > 0$) or*
- *the output has a v.b.c. stochastic arrival curve $\alpha_\theta \oslash \beta$ with bounding function $f^{\theta_1} \otimes g^{\theta_2}$; i.e., $A^* \sim_{vb} \langle f^{\theta_1} \otimes g^{\theta_2}, \alpha_\theta \oslash \beta \rangle$, where $\alpha_\theta(t) = \alpha(t) + (\theta_1 + \theta_2) \cdot t$ and $f^{\theta_1}(x) = f(x) + \frac{1}{\theta_1} \int_x^\infty f(y)dy$ for any $\theta_1, \theta_2 > 0$.*

Corollary 5.19. *Consider a system S with input A. If the input has a t.a.c. stochastic arrival curve $\alpha \in \mathcal{F}$ with bounding function $f \in \bar{\mathcal{G}}$ (i.e., $A \sim_{ta} \langle f, \alpha \rangle$) and the system provides to the input a weak stochastic service curve $\beta \in \mathcal{F}$ with bounding function $g \in \bar{\mathcal{G}}$ (i.e., $S \sim_{ws} \langle g, \beta \rangle$), then*

- *the output has a v.b.c. stochastic arrival curve $\alpha_{\theta_1} \oslash \beta_{-\theta_2}$ with bounding function $f^{\theta_1} \otimes g_t^{\theta_2}$ (i.e., $A^* \sim_{vb} \langle f^{\theta_1} \otimes g_t^\theta, \alpha_{\theta_1} \oslash \beta_{-\theta_2} \rangle$, where $\alpha_{\theta_1}(t) = \alpha(t) + \theta_1 \cdot t$, $\beta_{-\theta_2}(t) = \beta(t) - \theta_2 \cdot t$, $f^{\theta_1}(x) = f(x) + \frac{1}{\theta_1} \int_x^\infty f(y)dy$ and $g_t^\theta(x) = \frac{1}{\theta_2} \int_{x-\theta_2 t}^\infty g(y)\, dy$, for any $\theta_1, \theta_2 > 0$) or*

- *the output has a v.b.c stochastic arrival curve* $\alpha_\theta \oslash \beta$ *with bounding function* $f^{\theta_1} \otimes g^{\theta_2}$*, i.e.,* $A^* \sim_{vb} \langle f^{\theta_1} \otimes g^{\theta_2}, \alpha_\theta \oslash \beta \rangle$*, where* $\alpha_\theta(t) = \alpha(t) + (\theta_1 + \theta_2) \cdot t$*,* $f^{\theta_1}(x) = f(x) + \frac{1}{\theta_1} \int_x^\infty f(y)dy$*, and* $g^{\theta_2}(x) = g(x) + \frac{1}{\theta_2} \int_x^\infty g(y)\, dy$ *for any* $\theta_1, \theta_2 > 0$*.*

As a special case, the following result follows from Corollary 5.16, the mapping between the $(\sigma(\theta), \rho(\theta))$ upper constrained traffic characterization and v.b.c. SAC shown in Example 3.18, and the mapping between the $(b(\theta), r(\theta))$ lower constrained service characterization and weak SSC shown in Theorem 4.19.

Corollary 5.20. *Consider a system* S *with input* A*. Suppose the input is* $(\sigma(\theta), \rho(\theta))$ *upper constrained,* $a(t) \equiv A(t-1, t), t = 1, 2, \ldots,$ *are i.i.d., and the service provided by the system is* $(b(\theta), r(\theta))$ *lower constrained. Then, the output has a v.b.c. stochastic arrival curve* $\alpha_\theta \oslash \beta$ *with bounding function* $f^\theta \otimes g^\theta$*; i.e.,* $A^* \sim_{vb} \langle f^\theta \otimes g^\theta, \alpha_\theta \oslash \beta \rangle$*, where* $f(x) = \frac{e^{\theta\sigma(\theta)}}{1 - e^{\theta(\rho(\theta) - r)}} e^{-\theta x}$*,* $\alpha_\theta(t) = (r + \theta) \cdot t$*, and* $g^\theta(x) = e^{-\theta x} + \frac{1}{\theta} \int_x^\infty e^{-\theta y} dy$ *for any* $\theta > 0, r < \rho(\theta)$*.*

5.2.3 Output m.b.c Stochastic Arrival Curve

We now characterize the output traffic with the m.b.c. stochastic arrival curve model.

Based on (5.21), the following is obtained:

$$\begin{aligned} &\sup_{0 \le s \le t} \sup_{0 \le u \le s} [A^*(u, s) - \alpha \oslash \beta(s - u)] \\ &\le \sup_{0 \le s \le t} \sup_{0 \le u \le s} [A(u, s) - \alpha(s - u)] + \sup_{0 \le s \le t} [A \otimes \beta(s) - A^*(s)]. \end{aligned} \tag{5.26}$$

Inequality (5.26) implies that the output m.b.c. stochastic arrival curve characterization is easily derived if the input's m.b.c. stochastic arrival curve characterization and the system's stochastic service curve characterization are known. Specifically, we have the following result.

Theorem 5.21. *Consider a system* S *with input* A*. If the input has an m.b.c. stochastic arrival curve* $\alpha \in \mathcal{F}$ *with bounding function* $f \in \bar{\mathcal{F}}$ *(i.e.,* $A \sim_{mb} \langle f, \alpha \rangle$*) and the system provides to the input a stochastic service curve* $\beta \in \mathcal{F}$ *with bounding function* $g \in \bar{\mathcal{F}}$ *(i.e.,* $S \sim_{sc} \langle g, \beta \rangle$*), then the output has an m.b.c. stochastic arrival curve* $\alpha \oslash \beta$ *with bounding function* $f \otimes g$*; i.e.,* $A^* \sim_{mb} \langle f \otimes g, \alpha \oslash \beta \rangle$*.*

Based on the relationship between the weak stochastic service curve and stochastic service curve shown in Theorem 4.4, we have the following.

Corollary 5.22. *Consider a system* S *with input* A*. If the input has an m.b.c stochastic arrival curve* $\alpha \in \mathcal{F}$ *with bounding function* $f \in \bar{\mathcal{F}}$ *(i.e.,* $A \sim_{mb} \langle f, \alpha \rangle$*) and the system provides to the input a weak stochastic service curve*

$\beta \in \mathcal{F}$ with bounding function $g \in \bar{\mathcal{G}}$ (i.e., $S \sim_{ws} \langle g, \beta \rangle$), then the output has an m.b.c. stochastic arrival curve $\alpha \oslash \beta_{-\theta}$ with bounding function $f \otimes g_t^{\theta}$; i.e., $A^* \sim_{mb} \langle f \otimes g_t^{\theta}, \alpha \oslash \beta_{-\theta} \rangle$, where $g_t^{\theta}(x) = \frac{1}{\theta} \int_{x-\theta t}^{\infty} g(y)\, dy$ and $\beta_{-\theta}(t) = \beta(t) - \theta \cdot t$ for any $\theta > 0$.

Based on the relationship between the stochastic service curve and θ-stochastic service curve, we have the following.

Corollary 5.23. *Consider a system S with input A. If the input has an m.b.c. stochastic arrival curve $\alpha \in \mathcal{F}$ with bounding function $f \in \bar{\mathcal{F}}$ (i.e., $A \sim_{mb} \langle f, \alpha \rangle$) and the system provides to the input a θ-stochastic service curve $\beta \in \mathcal{F}$ with bounding function $g^{\theta} \in \bar{\mathcal{F}}$ (i.e., $S \sim_{sc} \langle g^{\theta}, \beta \rangle$), then the output has an m.b.c. stochastic arrival curve $\alpha \oslash \beta$ with bounding function $f \otimes g_t(x)$; i.e., $A^* \sim_{mb} \langle f \otimes g_t, \alpha \oslash \beta \rangle$, where $g_t(x) = g^{\theta}(x - \theta t)$.*

Corresponding to Theorem 5.21, Corollary 5.22 and Corollary 5.23, where the input is modeled with m.b.c stochastic arrival curve, Corollaries 5.24 to Corollary 5.26 have the input modeled with a θ-m.b.c. stochastic arrival curve.

Corollary 5.24. *Consider a system S with input A. If the input has a θ-m.b.c. stochastic arrival curve $\alpha \in \mathcal{F}$ with respect to $\theta(> 0)$ with bounding function $f^{\theta} \in \bar{\mathcal{F}}$ (i.e., $A \sim_{\theta-mb} \langle f^{\theta}, \alpha \rangle$) and the system provides to the input a stochastic service curve $\beta \in \mathcal{F}$ with bounding function $g \in \bar{\mathcal{F}}$ (i.e., $S \sim_{sc} \langle g, \beta \rangle$), then the output has an m.b.c. stochastic arrival curve $\alpha \oslash \beta$ with bounding function $f_t \otimes g(x)$; i.e., $A^* \sim_{mb} \langle f_t \otimes g, \alpha \oslash \beta \rangle$, where $f_t(x) = f^{\theta}(x - \theta t)$ for any $\theta > 0$.*

Corollary 5.25. *Consider a system S with input A. If the input has a θ-m.b.c. stochastic arrival curve $\alpha \in \mathcal{F}$ with bounding function $f^{\theta_1} \in \bar{\mathcal{F}}$ (i.e., $A \sim_{\theta-mb} \langle f^{\theta_1}, \alpha \rangle$) and the system provides to the input a weak stochastic service curve $\beta \in \mathcal{F}$ with bounding function $g \in \bar{\mathcal{G}}$ (i.e., $S \sim_{ws} \langle g, \beta \rangle$), then the output has an m.b.c. stochastic arrival curve $\alpha \oslash \beta$ with bounding function $f_t \otimes g^{\theta_2}$; i.e., $A^* \sim_{mb} \langle f_t \otimes g^{\theta_2}, \alpha \oslash \beta \rangle$, where $f_t(x) = f^{\theta_1}(x - \theta_1 t)$ and $g^{\theta_2}(x) = \frac{1}{\theta_2} \int_{x-\theta t}^{\infty} g(y)\, dy$ for any $\theta_1, \theta_2 > 0$.*

Corollary 5.26. *Consider a system S with input A. If the input has a θ-m.b.c. stochastic arrival curve $\alpha \in \mathcal{F}$ with bounding function $f^{\theta_1} \in \bar{\mathcal{F}}$ (i.e., $A \sim_{\theta-mb} \langle f^{\theta_1}, \alpha \rangle$) and the system provides to the input a θ-stochastic service curve $\beta \in \mathcal{F}$ with bounding function $g^{\theta_2} \in \bar{\mathcal{F}}$ (i.e., $S \sim_{sc} \langle g^{\theta_2}, \beta \rangle$), then the output has an m.b.c. stochastic arrival curve $\alpha \oslash \beta$ with bounding function $f_t \otimes g_t$; i.e., $A^* \sim_{mb} \langle f_t \otimes g_t, \alpha \oslash \beta \rangle$, where $f_t(x) = f^{\theta_1}(x - \theta_1 t)$ and $g_t(x) = g^{\theta_2}(x - \theta_2 t)$ for any $\theta_1, \theta_2 > 0$.*

We now consider that the input is modeled with a v.b.c. stochastic arrival curve. Corresponding to Theorem 5.21, Corollary 5.22 and Corollary 5.23, Corollaries 5.27 to 5.29 are easily obtained based on the relationship between the v.b.c. stochastic arrival curve and m.b.c. stochastic arrival curve shown in Theorem 3.24.

Corollary 5.27. *Consider a system S with input A. If the input has a v.b.c. stochastic arrival curve $\alpha \in \mathcal{G}$ with bounding function $f \in \bar{\mathcal{G}}$ (i.e., $A \sim_{vb} \langle f, \alpha \rangle$) and the system provides to the input a stochastic service curve $\beta \in \mathcal{F}$ with bounding function $g \in \bar{\mathcal{F}}$ (i.e. $S \sim_{sc} \langle g, \beta \rangle$), then the output has an m.b.c. stochastic arrival curve $\alpha \oslash \beta$ with bounding function $f_t \otimes g$; i.e., $A^* \sim_{mb} \langle f_t \otimes g, \alpha \oslash \beta \rangle$, where $f_t(x) = \frac{1}{\theta} \int_{x-\theta t}^{\infty} f(y)\, dy$ for any $\theta > 0$.*

Corollary 5.28. *Consider a system S with input A. If the input has a v.b.c. stochastic arrival curve $\alpha \in \mathcal{F}$ with bounding function $f \in \bar{\mathcal{G}}$ (i.e., $A \sim_{vb} \langle f, \alpha \rangle$) and the system provides to the input a weak stochastic service curve $\beta \in \mathcal{F}$ with bounding function $g \in \bar{\mathcal{G}}$ (i.e., $S \sim_{ws} \langle g, \beta \rangle$), then the output has an m.b.c. stochastic arrival curve $\alpha \oslash \beta_{-\theta_2}$ with bounding function $f_t \otimes g^{\theta_2}$; i.e., $A^* \sim_{mb} \langle f_t \otimes g^{\theta_2}, \alpha \oslash \beta_{-\theta_2} \rangle$, where $f_t = \frac{1}{\theta_1} \int_{x-\theta_1 t}^{\infty} f(y)dy$, $g^{\theta_2}(x) = \frac{1}{\theta_2} \int_{x-\theta_2 t}^{\infty} g(y)\, dy$ and $\beta_{-\theta_2}(t) = \beta(t) - \theta_2 \cdot t$ for any $\theta_1, \theta_2 > 0$.*

Corollary 5.29. *Consider a system S with input A. If the input has a v.b.c. stochastic arrival curve $\alpha \in \mathcal{F}$ with bounding function $f \in \bar{\mathcal{G}}$ (i.e., $A \sim_{vb} \langle f, \alpha \rangle$) and the system provides to the input a θ-stochastic service curve $\beta \in \mathcal{F}$ with bounding function $g^{\theta_2} \in \bar{\mathcal{F}}$ (i.e., $S \sim_{\theta-sc} \langle g^{\theta_2}, \beta \rangle$), then the output has a m.b.c stochastic arrival curve $\alpha \oslash \beta$ with bounding function $f_t \otimes g_t^{\theta_2}(x)$; i.e., $A^* \sim_{mb} \langle f_t \otimes g_t^{\theta_2}, \alpha \oslash \beta \rangle$, where $f_t = \frac{1}{\theta_1} \int_{x-\theta_1 t}^{\infty} f(y)dy$, and $g_t^{\theta_2}(x) = g^{\theta_2}(x - \theta_2 t)$ for any $\theta_1, \theta_2 > 0$.*

We then consider that the input is initially modeled with a v.b.c. stochastic arrival curve. In this case, we can first convert it into a v.b.c. stochastic arrival curve and then into an m.b.c. stochastic arrival curve. Afterwards, we can apply Theorem 5.21, Corollary 5.22, and Corollary 5.23 and obtain Corollaries 5.30, 5.31, and 5.32, respectively.

Corollary 5.30. *Consider a system S with input A. If the input has a t.a.c. stochastic arrival curve $\alpha \in \mathcal{F}$ with bounding function $f \in \bar{\mathcal{G}}$ (i.e., $A \sim_{ta} \langle f, \alpha \rangle$) and the system provides to the input a stochastic service curve $\beta \in \mathcal{F}$ with the bounding function $g \in \bar{\mathcal{F}}$ (i.e., $S \sim_{sc} \langle g, \beta \rangle$), then the output has an m.b.c. stochastic arrival curve $\alpha_{\theta_1} \oslash \beta$ with bounding function $f_t \otimes g$; i.e., $A^* \sim_{mb} \langle f_t \otimes g, \alpha_{\theta_1} \oslash \beta \rangle$ where $f_t(x) = \frac{1}{\theta_2} \int_{x-\theta_2 t}^{\infty} \hat{f}(y)\, dy$, $\hat{f}(y) = f(y) + \frac{1}{\theta_1} \int_{y}^{\infty} f(z)\, dz$, and $\alpha_{\theta_1}(t) = \alpha(t) + \theta_1 \cdot t$ for any $\theta_1, \theta_2 > 0$.*

Corollary 5.31. *Consider a system S with input A. If the input has a t.a.c. stochastic arrival curve $\alpha \in \mathcal{F}$ with bounding function $f \in \bar{\mathcal{G}}$ (i.e., $A \sim_{ta} \langle f, \alpha \rangle$) and the system provides to the input a weak stochastic service curve $\beta \in \mathcal{F}$ with bounding function $g \in \bar{\mathcal{F}}$ (i.e., $S \sim_{ws} \langle g, \beta \rangle$), then the output has a m.b.c stochastic arrival curve $\alpha_{\theta_1} \oslash \beta$ with bounding function $f_t \otimes g^{\theta_2}$; i.e., $A^* \sim_{mb} \langle f_t \otimes g^{\theta_2}, \alpha_{\theta_1} \oslash \beta \rangle$, where $f_t(x) = \frac{1}{\theta_3} \int_{x-\theta_3 t}^{\infty} \hat{f}(y)\, dy$ and $\hat{f}(y) = f(y) + \frac{1}{\theta_1} \int_{y}^{\infty} f(z)\, dz$, $g^{\theta_2}(x) = \frac{1}{\theta_2} \int_{x-\theta_2 t}^{\infty} g(y)\, dy$, and $\alpha_{\theta_1}(t) = \alpha(t) + \theta_1 \cdot t$ for any $\theta_1, \theta_2, \theta_3 > 0$.*

Corollary 5.32. *Consider a system S with input A. If the input has a t.a.c stochastic arrival curve $\alpha \in \mathcal{F}$ with bounding function $f \in \bar{\mathcal{G}}$ (i.e., $A \sim_{ta} \langle f, \alpha \rangle$) and the system provides to the input a θ-stochastic service curve $\beta \in \mathcal{F}$ with bounding function $g^{\theta_2} \in \bar{\mathcal{F}}$ (i.e., $S \sim_{\theta-sc} \langle g^{\theta_2}, \beta \rangle$), then the output has an m.b.c stochastic arrival curve $\alpha_{\theta_1} \oslash \beta$ with bounding function $f_t \otimes g^{\theta_2}$; i.e, $A^* \sim_{mb} \langle f_t \otimes g_t^{\theta_2}, \alpha_{\theta_1} \oslash \beta \rangle$, where $f_t(x) = \frac{1}{\theta_3} \int_{x-\theta_3 t}^{\infty} \hat{f}(y)\, dy$ and $\hat{f}(y) = f(y) + \frac{1}{\theta_1} \int_y^{\infty} f(z)\, dz$, $g_t^{\theta_2}(x) = g^{\theta_2}(x - \theta_2 t)$, and $\alpha_{\theta_1}(t) = \alpha(t) + \theta_1 \cdot t$ for any $\theta_1, \theta_2, \theta_3 > 0$.*

5.2.4 Output θ-m.b.c. Stochastic Arrival Curve

Based on the relationships of the θ-m.b.c. stochastic arrival curve with the m.b.c., v.b.c. and t.a.c. stochastic arrival curves, the output θ-m.b.c. stochastic arrival curve characterization can be readily obtained from results in the previous subsections, when the input is characterized using an m.b.c., v.b.c. or t.a.c. stochastic arrival curve. We leave this to the reader to investigate further.

In addition, when the input is characterized using the θ-m.b.c. stochastic arrival curve model, we easily obtain from (5.21)

$$
\begin{aligned}
&\sup_{0 \le s \le t} \left[\sup_{0 \le u \le s} [A^*(u,s) - \alpha \oslash \beta(s-u)] - \theta(t-s) \right] \\
\le &\sup_{0 \le s \le t} \left[\sup_{0 \le u \le s} [A(u,s) - \alpha(s-u)] - \theta(t-s) \right] \\
&+ \sup_{0 \le s \le t} [A \otimes \beta(s) - A^*(s)], \qquad (5.27)
\end{aligned}
$$

from which we have the following result.

Theorem 5.33. *Consider a system S with input A. If the input has an m.b.c. or θ-m.b.c. stochastic arrival curve $\alpha \in \mathcal{F}$, with bounding function $f \in \bar{\mathcal{F}}$ (i.e., $A \sim_{sac} \langle f, \alpha \rangle$, where $\sim_{sac}$ can be either $\sim_{mb}$ or $\sim_{\theta-mb}$), and the system provides to the input a stochastic service curve $\beta \in \mathcal{F}$ with bounding function $g \in \bar{\mathcal{F}}$, (i.e. $S \sim_{sc} \langle g, \beta \rangle$), then the output has a θ-m.b.c. stochastic arrival curve $\alpha \oslash \beta$ with bounding function $f \otimes g$, i.e., $A^* \sim_{\theta-mb} \langle f \otimes g, \alpha \oslash \beta \rangle$.*

Then, based on the relationship between the stochastic service curve and θ-stochastic service curve, the following result is obtained.

Corollary 5.34. *Consider a system S with input A. If the input has a θ-m.b.c. or m.b.c. stochastic arrival curve $\alpha \in \mathcal{F}$ with bounding function $f \in \bar{\mathcal{F}}$ (i.e., $A \sim_{sac} \langle f, \alpha \rangle$, where $\sim_{sac}$can be either $\sim_{\theta-mb}$ or $\sim_{mb}$), and the system provides to the input a θ-stochastic service curve $\beta \in \mathcal{F}$ with bounding function $g^{\theta} \in \bar{\mathcal{F}}$ (i.e. $S \sim_{\theta-sc} \langle g, \beta \rangle$), then the output has a θ-m.b.c. stochastic arrival curve $\alpha \oslash \beta$ with bounding function $f \otimes g_t^{\theta}(x)$; i.e., $A^* \sim_{\theta-mb} \langle f \otimes g_t^{\theta}, \alpha \oslash \beta \rangle$, where $g_t^{\theta}(x) = g^{\theta}(x - \theta \cdot t)$.*

The following corollary is based on Theorem 5.33 and the relationship between the stochastic service curve and weak stochastic service curve.

Corollary 5.35. *Consider a system S with input A. If the input has a θ-m.b.c. or m.b.c. stochastic arrival curve $\alpha \in \mathcal{F}$ with bounding function $f \in \bar{\mathcal{F}}$ (i.e., $A \sim_{sac} \langle f, \alpha \rangle$, where $\sim_{sac}$ can be either $\sim_{\theta-mb}$ or $\sim_{mb}$) and the system provides to the input a weak stochastic service curve $\beta \in \mathcal{F}$ with bounding function $g \in \hat{\mathcal{G}}$ (i.e., $S \sim_{ws} \langle g, \beta \rangle$), then the output has a θ-m.b.c. stochastic arrival curve $\alpha \oslash \beta_{-\theta}$ with bounding function $f \otimes g_t^\theta(x)$; i.e., $A^* \sim_{\theta-mb} \langle f \otimes g_t^\theta, \alpha \oslash \beta_{-\theta} \rangle$, where $\beta_{-\theta}(t) = \beta(t) - \theta \cdot t$ and $g_t^\theta(x) = \frac{1}{\theta} \int_{x-\theta \cdot t} g(y) dy$ for any $\theta > 0$.*

5.3 Concatenation Property

This section presents the concatenation property of the stochastic service curve, θ-stochastic service curve, and weak stochastic service curve for stochastic network calculus.

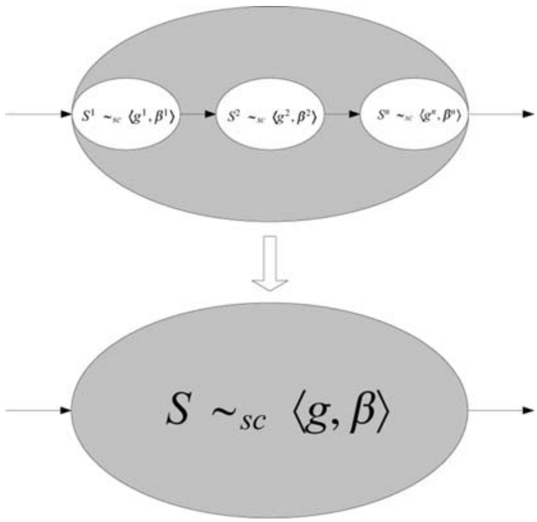

Fig. 5.1. Concatenation property of stochastic service curve

As illustrated in Figure 5.1, it can be proved that multiple systems in tandem, each of which provides a stochastic service curve to the input, can be concatenated and viewed as one system characterized by a stochastic service curve. Particularly, we have the following concatenation property of a stochastic service curve.

Theorem 5.36. *Consider a flow passing through a network of N systems in tandem. If each system $n(= 1, 2, \ldots, N)$ provides a stochastic service curve $S^n \sim_{sc} \langle g^n, \beta^n \rangle$ to its input, then the network guarantees to the flow a stochastic service curve $S \sim_{sc} \langle g, \beta \rangle$ with*

$$\beta(t) = \beta^1 \otimes \beta^2 \otimes \cdots \otimes \beta^N(t) \tag{5.28}$$

$$g(x) = g^1 \otimes g^2 \otimes \cdots \otimes g^N(x). \tag{5.29}$$

Proof. We shall only prove the two-node case, from which the proof can be easily extended to the N-node case. For the two-node case, the departure of the first node is the arrival at the second node, so $A^{1*}(t) = A^2(t)$. In addition, the arrival at the network is the arrival to the first node, or $A(t) = A^1(t)$, and the departure from the network is the departure from the second node, or $A^*(t) = A^{2*}(t)$, where $A(t)$ and $A^*(t)$ denote the arrival process at and departure process from the network, respectively. We then have

$$\begin{aligned}
&\sup_{0\le s\le t} [A \otimes \beta^1 \otimes \beta^2(s) - A^*(s)] \\
&= \sup_{0\le s\le t} [(A^1 \otimes \beta^1) \otimes \beta^2(s) - A^{2*}(s)].
\end{aligned} \tag{5.30}$$

Now let us consider any s, $(0 \le s \le t)$, for which we get

$$\begin{aligned}
&(A^1 \otimes \beta^1) \otimes \beta^2(s) - A^{2*}(s) \\
&= (A^1 \otimes \beta^1) \otimes \beta^2(s) - A^{2*}(s) \\
&= \inf_{0\le u\le s} [A^1 \otimes \beta^1(u) + \beta^2(s-u) - A^{1*}(u) + A^2(u)] - A^{2*}(s) \\
&\le \sup_{0\le u\le t} \left[A^1 \otimes \beta^1(u) - A^{1*}(u)\right] + \inf_{0\le u\le s} [A^2(u) + \beta^2(s-u)] - A^{2*}(s) \\
&= \sup_{0\le u\le t} \left[A^1 \otimes \beta^1(u) - A^{1*}(u)\right] + A^2 \otimes \beta(s) - A^{2*}(s).
\end{aligned} \tag{5.31}$$

Applying (5.31) to (5.30), we obtain

$$\begin{aligned}
&\sup_{0\le s\le t} [A \otimes \beta^1 \otimes \beta^2(s) - A^*(s)] \\
&\le \sup_{0\le u\le t} [A^1 \otimes \beta^1(u) - A^{1*}(u)] + \sup_{0\le u\le t} [A^2 \otimes \beta^2(u) - A^{2*}(u)],
\end{aligned} \tag{5.32}$$

and with this, since both nodes provide a stochastic service curve to their input, the theorem follows from Lemma 1.5 and the definition of a stochastic service curve. □

In deriving (5.31), we have proved $[(A^1 \otimes \beta^1) \otimes \beta^2(s) - A^{2*}(s)] \le \sup_{0\le u\le s}[A^1 \otimes \beta^1(u) - A^{1*}(u)] + \sup_{0\le u\le s}[A^2 \otimes \beta^2(u) - A^{2*}(u)]$ for all $s \ge 0$. However, if we want to prove the concatenation property for a weak stochastic service curve, we need to prove $[(A^1 \otimes \beta^1) \otimes \beta^2(s) - A^{2*}(s)] \le [A^1 \otimes \beta^1(s) - A^{1*}(s)] + [A^2 \otimes \beta^2(s) - A^{2*}(s)]$ for all $s \ge 0$, which is difficult to obtain and does not hold in general. This explains why a weak stochastic service curve does not have property (P.2) when servers only provide weak stochastic service curves.

Since a stochastic service curve implies a weak stochastic service curve, the following result follows immediately from Theorem 5.36, particularly (5.31).

Corollary 5.37. *Consider a flow passing through a network of N systems in tandem. If each system $n(= 1, 2, \ldots, N-1)$ provides a stochastic service curve $S^n \sim_{sc} \langle g^n, \beta^n \rangle$, and system N provides a weak stochastic service curve $S^N \sim_{ws} \langle g^N, \beta^N \rangle$ to their input, then the network guarantees to the flow a weak stochastic service curve $S \sim_{ws} \langle g, \beta \rangle$ with*

$$\beta(t) = \beta^1 \otimes \beta^2 \otimes \cdots \otimes \beta^N(t), \tag{5.33}$$

$$g(x) = g^1 \otimes g^2 \otimes \cdots \otimes g^N(x). \tag{5.34}$$

In the network of tandem systems, if each system provides a θ-stochastic service curve, the following theorem holds.

Theorem 5.38. *Consider a flow passing through a network of N systems in tandem. If each system $n(= 1, 2, \ldots, N)$ provides a θ-stochastic service curve $S^n \sim_{\theta^n - sc} \langle g^n, \beta^n \rangle$ to its input, then, if $\beta \in \mathcal{F}$, the network guarantees to the flow a weak stochastic service curve $S \sim_{ws} \langle g, \beta \rangle$ with*

$$\beta(t) = \beta^1 \otimes \beta^2_{-\theta} \otimes \cdots \otimes \beta^N_{-(N-1)\theta}(t),$$

$$g(x) = g^1 \otimes g^2 \otimes \cdots \otimes g^N(x),$$

where $\beta^n_{-(n-1)\theta}(t) = \beta^n(t) - (n-1)\theta$, $n = 1, \ldots, N$, for any $\theta > 0$.

Theorem 5.38 is proved by iteratively applying the following result.

Lemma 5.39. *Consider any functions $A(t), A^*(t), b(t), c(t), d(t), e(t)$. The following relationships hold:*

$$A \otimes b \otimes c(t) \le \sup_{0\le s\le t}[A \otimes b(s) - A^*(s) - \theta \cdot (t-s)] + A^* \otimes c_\theta(t), \tag{5.35}$$

$$[d \otimes e]_\theta(t) = d_\theta \otimes e_\theta(t), \tag{5.36}$$

$$[d \otimes e]_{-\theta}(t) = d_{-\theta} \otimes e_{-\theta}(t), \tag{5.37}$$

for any $\theta \ge 0$, where $\alpha_\theta(t) = \alpha(t) + \theta \cdot t$, $\alpha_{-\theta}(t) = \alpha(t) - \theta \cdot t$.

Proof. For (5.35), we have

$$
\begin{aligned}
& A \otimes b \otimes c(t) \\
= & \inf_{0 \le s \le t} \{A \otimes b(s) - A^*(s) - \theta \cdot (t-s) + A^*(s) + c(t-s) + \theta \cdot (t-s)\} \\
\le & \sup_{0 \le s \le t} [A \otimes b(s) - A^*(s) - \theta \cdot (t-s)] + A^* \otimes c_\theta(t).
\end{aligned}
$$

For (5.36), we have

$$
\begin{aligned}
[d \otimes e]_\theta (t) &= d \otimes e(t) + \theta \cdot (t) \\
&= \inf_{0 \le s \le t} \{d(s) + \theta \cdot s + e(t-s) + \theta \cdot (t-s)\} \\
&= \inf_{0 \le s \le t} \{d_\theta(s) + e_\theta(t-s)\} = d_\theta \otimes e_\theta(t)
\end{aligned}
$$

and (5.37) can be verified similarly. □

Lemma 5.39 may be used iteratively. For example, letting $c(t) = d \otimes e(t)$ in (5.35), we immediately obtain from (5.36)

$$
A \otimes b \otimes (d \otimes e)(t) \le \sup_{0 \le s \le t} [A \otimes b(s) - A^*(s) - \theta \cdot (t-s)] + A^* \otimes d_\theta \otimes e_\theta(t). \quad (5.38)
$$

By iteratively applying Lemma 5.39, we can get

$$
\begin{aligned}
& A \otimes \beta^1 \otimes \beta^2_{-\theta} \otimes \cdots \otimes \beta^N_{-(N-1)\theta}(t) - A^*(t) \\
\le & \sup_{0 \le s \le t} \left[A^1 \otimes \beta^1(s) - A^{1*}(s) - \theta \cdot (t-s)\right] \\
& + \sup_{0 \le s \le t} \left[A^2 \otimes \beta^2(s) - A^{2*}(s) - \theta \cdot (t-s)\right] + \cdots + \\
& + \sup_{0 \le s \le t} \left[A^{N-1} \otimes \beta^{N-1}(s) - A^{(N-1)*}(s) - \theta \cdot (t-s)\right] \\
& + A^N \otimes \beta(t) - A^*(t),
\end{aligned} \quad (5.39)
$$

and with this, Theorem 5.38 can be easily verified since $A^N \otimes \beta(t) - A^*(t) \le \sup_{0 \le s \le t} \left[A^{N-1} \otimes \beta^2(s) - A^{(N-1)*}(s) - \theta \cdot (t-s)\right]$.

Based on the relationship between the weak stochastic service curve and θ-stochastic service curve shown in Theorem 4.7, the following corollary, which presents the concatenation property for the θ-stochastic service curve model, immediately follows from Theorem 5.38.

Corollary 5.40. *Consider a flow passing through a network of N systems in tandem. If each system $n (= 1, 2, \ldots, N)$ provides a θ-stochastic service curve $S^n \sim_{\theta^n-ss} \langle g^n, \beta^n \rangle$ to its input and $g \in \bar{\mathcal{G}}$, then the network guarantees to the flow a θ-stochastic service curve $S \sim_{\theta-sc} \langle g^\theta, \beta \rangle$, where $g^\theta(x) = g(x) + \frac{1}{\theta} g^{(1)}(x)$ and*

$$
\begin{aligned}
\beta(t) &= \beta^1 \otimes \beta^2_{-\theta} \otimes \cdots \otimes \beta^N_{-(N-1)\theta}(t), \\
g(x) &= g^1 \otimes g^2 \otimes \cdots \otimes g^N(x),
\end{aligned}
$$

with $\beta^n_{-(n-1)\theta}(t) = \beta^n(t) - (n-1)\theta$, $n = 1, \ldots, N$.

Also based on the relationship between the weak stochastic service curve and θ-stochastic service curve shown in Theorem 4.7, the following corollary presents the concatenation property for the weak stochastic service curve model, obtained particularly from (5.39).

Corollary 5.41. *Consider a flow passing through a network of N systems in tandem. If each system $n(=1, 2, \ldots, N)$ provides weak stochastic service curve $S^n \sim_{ws} \langle g^n, \beta^n \rangle$ to its input and $g \in \bar{\mathcal{G}}$, then the network guarantees to the flow a weak stochastic service curve $S \sim_{ws} \langle g, \beta \rangle$, where*

$$\beta(t) = \beta^1 \otimes \beta^2_{-\theta} \otimes \cdots \otimes \beta^N_{-(N-1)\theta}(t),$$
$$g(x) = g^{1,\theta_1} \otimes g^{2,\theta_2} \otimes \cdots \otimes g^{N,\theta_N}(x),$$

with $\beta^n_{-(n-1)\theta}(t) = \beta^n(t) - (n-1)\theta$ for $n = 1, \ldots, N$, $g^{n,\theta_n}(x) = g(x) + \frac{1}{\theta_n}\int_x^\infty g(y)dy$ for $n{=}1, \ldots, N-1$, and $g^{N,\theta}(x){=}g^N(x)$ for any $\theta, \theta_1, \cdots, \theta_N{>}0$.

5.4 Leftover Service Characterization

This section presents results for characterizing the leftover service under aggregate scheduling. To ease the expression, we consider the case where there are two flows competing for resources in a system under aggregate scheduling. Consider a system fed with a flow A that is the aggregation of two constituent flows A_1 and A_2. Suppose both the service characterization from the server and traffic characterization from A_2 are given, and we are interested in characterizing the service received by A_1, with which per-flow bounds for A_1 can then be easily obtained using earlier results.

For the output, there holds $A^*(t) = A_1^*(t) + A_2^*(t)$. In addition, we have $A^*(t) \le A(t)$, $A_1^*(t) \le A_1(t)$, and $A_2^*(t) \le A_2(t)$. We now have for any $s \ge 0$

$$\begin{aligned}
&A_1 \otimes (\beta - \alpha_2)(s) - A_1^*(s) \\
&= \inf_{0 \le u \le s}[A(u) + \beta(s-u) - \alpha_2(s-u) - A_2(u)] - A^*(s) + A_2^*(s) \\
&\le [A \otimes \beta(s) - A^*(s)] + A_2(s) - \inf_{0 \le u \le s}[A_2(u) + \alpha_2(s-u)] \\
&= [A \otimes \beta(s) - A^*(s)] + \sup_{0 \le u \le s}[A_2(u,s) - \alpha_2(s-u)]. \qquad (5.40)
\end{aligned}$$

5.4.1 Leftover Weak Stochastic Service Curve

From (5.40), together with the fact that both the m.b.c SAC and θ-m.b.c SAC imply v.b.c. SAC and both the SSC and θ-SSC imply a weak SSC, the following theorem can be easily verified.

Theorem 5.42. *Consider a system S with input A that is the aggregation of two constituent flows A_1 and A_2. Suppose A_2 has a v.b.c. (or m.b.c. or θ-m.b.c.) stochastic arrival curve $\alpha \in \mathcal{F}$ with bounding function $f \in \bar{\mathcal{F}}$ (i.e., $A_2 \sim_{sac} \langle f_2, \alpha_2 \rangle$), where $\sim_{sac}$ is one of either $\sim_{vb}$, $\sim_{mb}$, or $\sim_{\theta-mb}$) and the system provides to the input a weak stochastic service curve (or a stochastic service curve or a θ-stochastic service curve) $\beta \in \mathcal{F}$ with bounding function $g \in \bar{\mathcal{F}}$; i.e., $S \sim_{ssc} \langle g, \beta \rangle$ where $\sim_{ssc}$ is one of either $\sim_{ws}$ or $\sim_{sc}$ and $\sim_{\theta-sc}$. Then, if $\beta - \alpha_2 \in \mathcal{F}$, A_1 receives a weak stochastic service curve $\beta - \alpha_2$ with bounding function $f_2 \otimes g(x)$; i.e., $S_1 \sim_{ws} \langle f_2 \otimes g(x), \beta - \alpha_2 \rangle$.*

Based on the relationship between the t.a.c. SAC and v.b.c. SAC, we can obtain the following result from Theorem 5.42.

Corollary 5.43. *Consider a system S with input A that is the aggregation of two constituent flows A_1 and A_2. Suppose A_2 has a t.a.c. stochastic arrival curve $\alpha \in \mathcal{F}$ with bounding function $f \in \bar{\mathcal{G}}$ (i.e., $A_2 \sim_{ta} \langle f_2, \alpha_2 \rangle$) and the system provides to the input a weak stochastic service curve (or a stochastic service curve or a θ-stochastic service curve) $\beta \in \mathcal{F}$ with bounding function $g \in \hat{\mathcal{F}}$; i.e., $S \sim_{ssc} \langle g, \beta \rangle$, where $\sim_{ssc}$ is one of either $\sim_{ws}$ or $\sim_{sc}$ and $\sim_{\theta-sc}$. Then, if $\beta - \alpha_{2,\theta} \in \mathcal{F}$, A_1 receives a weak stochastic service curve $\beta - \alpha_{2,\theta}$ with bounding function $f_2^\theta \otimes g(x)$; i.e., $S_1 \sim_{ws} \langle f_2^\theta \otimes g(x), \beta - \alpha_{2,\theta} \rangle$, where $f_2^\theta = f_2(x) + \frac{1}{\theta}\int_x^\infty f_2(y)dy$ and $\alpha_{2,\theta}(t) = \alpha(t) + \theta \cdot t$ for any $\theta > 0$.*

Since a deterministic service curve is a special case of a stochastic service curve, we have the following result.

Corollary 5.44. *Consider a system S with input A that is the aggregation of two constituent flows A_1 and A_2. Suppose A_2 has a v.b.c. (or m.b.c., or θ-m.b.c.) stochastic arrival curve $\alpha \in \mathcal{F}$ with bounding function $f \in \bar{\mathcal{F}}$ (i.e., $A_2 \sim_{sac} \langle f_2, \alpha_2 \rangle$, where $\sim_{sac}$ is one of either $\sim_{vb}$, $\sim_{mb}$, or $\sim_{\theta-mb}$. In addition, the system provides to the input a deterministic service curve $\beta \in \mathcal{F}$. Then, if $\beta - \alpha_2 \in \mathcal{F}$, A_1 receives a weak stochastic service curve $\beta - \alpha_2$ with bounding function f_2; i.e., $S_1 \sim_{ws} \langle f_2, \beta - \alpha_2 \rangle$.*

Corollary 5.44 can be easily verified since $0 \otimes f_2(x) = \min_{0 \le u \le x}[0 + f_2(u)] \le f_2(x)$. An important implication of this corollary is that a deterministic server with a deterministic service curve can be considered as a stochastic server with weak stochastic service curve for each input flow. This property is very useful for deriving stochastic QoS bounds per-flow under aggregate scheduling since there are many types of servers that provide a deterministic service curve as, introduced in Chapter 2.

5.4.2 Leftover Stochastic Service Curve

From (5.40), we easily get

$$
\begin{aligned}
&\sup_{0\le s\le t} \left[A_1 \otimes (\beta - \alpha_2)(s) - A_1^*(s)\right] \\
&= A_1 \otimes (\beta - \alpha_2)(s_0) - A_1^*(s_0) \\
&\le \sup_{0\le s\le t} \left[A \otimes \beta(s) - A^*(s)\right] + \sup_{0\le s\le t} \sup_{0\le u\le s} \left[A_2(u,s) - \alpha_2(s-u)\right], \quad (5.41)
\end{aligned}
$$

and with this, the following theorem can be verified.

Theorem 5.45. *Consider a system S with input A that is the aggregation of two constituent flows, A_1 and A_2. Suppose A_2 has an m.b.c stochastic arrival curve $\alpha \in \mathcal{F}$ with bounding function $f \in \bar{\mathcal{F}}$ (i.e., $A_2 \sim_{mb} \langle f_2, \alpha_2\rangle$) and the system provides to the input a stochastic service curve $\beta \in \mathcal{F}$ with bounding function $g \in \bar{\mathcal{F}}$; (i.e., $S \sim_{sc} \langle g, \beta\rangle$). Then, if $\beta - \alpha_2 \in \mathcal{F}$, A_1 receives a stochastic service curve $\beta - \alpha_2$ with bounding function $f_2 \otimes g$; i.e., $S_1 \sim_{sc} \langle f_2 \otimes g, \beta - \alpha_2\rangle$.*

With Theorem 5.45 and based on the relationship between the weak stochastic service curve and stochastic service curve, Corollary 5.46 is obtained.

Corollary 5.46. *Consider a system S with input A that is the aggregation of two constituent flows A_1 and A_2. Suppose A_2 has an m.b.c. stochastic arrival curve $\alpha \in \mathcal{F}$ with bounding function $f \in \bar{\mathcal{F}}$ (i.e., $A_2 \sim_{mb} \langle f_2, \alpha_2\rangle$) and the system provides to the input a weak stochastic service curve $\beta \in \mathcal{F}$ with bounding function $g \in \bar{\mathcal{G}}$ (i.e., $S \sim_{ws} \langle g, \beta\rangle$). Then, if $\beta - \alpha_2 \in \mathcal{F}$, A_1 receives a stochastic service curve $\beta - \alpha_2$ with bounding function $f_2 \otimes g_t^\theta$; (i.e., $S_1 \sim_{sc} \langle f_2 \otimes g_t^\theta, \beta - \alpha_2\rangle$), where $g_t^\theta = \frac{1}{\theta}\int_{x-\theta t}^{\infty} g(y)\,dy$ for any $\theta > 0$.*

In addition, based on the relationship between the θ-stochastic service curve and stochastic service curve, the following result is obtained.

Corollary 5.47. *Consider a system S with input A that is the aggregation of two constituent flows A_1 and A_2. Suppose A_2 has an m.b.c. stochastic arrival curve $\alpha \in \mathcal{F}$ with bounding function $f \in \bar{\mathcal{F}}$ (i.e., $A_2 \sim_{mb} \langle f_2, \alpha_2\rangle$) and the system provides to the input a θ–stochastic service curve $\beta \in \mathcal{F}$ with bounding function $g^\theta \in \bar{\mathcal{F}}$ (i.e., $S \sim_{\theta-sc} \langle g^\theta, \beta\rangle$). Then, if $\beta - \alpha_2 \in \mathcal{F}$, A_1 receives a stochastic service curve $\beta - \alpha_2$ with bounding function $f_2 \otimes g_t^\theta$ (i.e., $S_1 \sim_{sc} \langle f_2 \otimes g_t(x), \beta - \alpha_2\rangle$), where $g_t^\theta = g^\theta(x - \theta \cdot t)$ for any $\theta > 0$.*

Corresponding to Theorem 5.45, Corollary 5.46, and Corollary 5.47, the following results are obtained based on the relationship between the v.b.c stochastic arrival curve and m.b.c stochastic arrival curve.

Corollary 5.48. *Consider a system S with input A that is the aggregation of two constituent flows A_1 and A_2. Suppose A_2 has a v.b.c. stochastic arrival curve $\alpha \in \mathcal{F}$ with bounding function $f \in \bar{\mathcal{G}}$ (i.e., $A_2 \sim_{vb} \langle f_2, \alpha_2\rangle$) and the system provides to the input a stochastic service curve $\beta \in \mathcal{F}$ with bounding function $g \in \bar{\mathcal{F}}$ (i.e., $S \sim_{sc} \langle g, \beta\rangle$). Then, if $\beta - \alpha_2 \in \mathcal{F}$, A_1 receives a* *stochastic service curve* $\beta - \alpha_{2,\theta}$ *with bounding function* $f_{2,t}^{\theta} \otimes g(x)$ *(i.e.,* $S_1 \sim_{sc} \langle f_{2,t}^{\theta} \otimes g(x), \beta - \alpha_{2,\theta} \rangle$*), where* $f_{2,t}^{\theta}(x) = \frac{1}{\theta} \int_{x-\theta t}^{\infty} f_2(y)\, dy$*, and* $\alpha_{2,\theta}(t) = \alpha(t) + \theta \cdot t$ *for any* $\theta > 0$*.*

Corollary 5.49. *Consider a system* S *with input* A *that is the aggregation of two constituent flows* A_1 *and* A_2*. Suppose* A_2 *has a v.b.c. stochastic arrival curve* $\alpha \in \mathcal{F}$ *with bounding function* $f \in \bar{\mathcal{G}}$ *(i.e.,* $A_2 \sim_{vb} \langle f_2, \alpha_2 \rangle$*) and the system provides to the input a* θ*-stochastic service curve* $\beta \in \mathcal{F}$ *with bounding function* $g^{\theta} \in \bar{\mathcal{F}}$*; i.e.,* $S \sim_{\theta-sc} \langle g^{\theta}, \beta \rangle$*. Then, if* $\beta - \alpha_2 \in \mathcal{F}$*,* A_1 *receives a stochastic service curve* $\beta - \alpha_{2,\theta}$ *with bounding function* $f_{2,t}^{\theta} \otimes g_t(x)$*; i.e.,* $S_1 \sim_{sc} \langle f_{2,t}^{\theta} \otimes g_t^{\theta}, \beta - \alpha_2 \rangle$*, where* $f_{2,t}^{\theta}(x) = \frac{1}{\theta_2} \int_{x-\theta_2 t}^{\infty} f_2(y)\, dy$*,* $\alpha_{2,\theta}(t) = \alpha(t) + \theta_2 \cdot t$*, and* $g_t^{\theta} = g^{\theta}(x - \theta_1 \cdot t)$ *for any* $\theta_1, \theta_2 > 0$*.*

Corollary 5.50. *Consider a system* S *with input* A *that is the aggregation of two constituent flows* A_1 *and* A_2*. Suppose* A_2 *has a v.b.c. stochastic arrival curve* $\alpha \in \mathcal{F}$ *with bounding function* $f \in \bar{\mathcal{G}}$ *(i.e.,* $A_2 \sim_{vb} \langle f_2, \alpha_2 \rangle$*) and the system provides to the input a weak stochastic service curve* $\beta \in \mathcal{F}$ *with bounding function* $g \in \bar{\mathcal{G}}$*; i.e.,* $S \sim_{ws} \langle g, \beta \rangle$*. Then, if* $\beta - \alpha_2 \in \mathcal{F}$*,* A_1 *receives a stochastic service curve* $\beta - \alpha_{2,\theta}$ *with bounding function* $f_{2,t}^{\theta} \otimes g_t^{\theta}(x)$*, i.e.* $S_1 \sim_{sc} \langle f_{2,t}^{\theta} \otimes g_t, \beta - \alpha_2 \rangle$*, where* $f_{2,t}^{\theta}(x) = \frac{1}{\theta_2} \int_{x-\theta_2 t}^{\infty} f_2(y)\, dy$*,* $\alpha_{2,\theta}(t) = \alpha(t) + \theta_2 \cdot t$*, and* $g_t^{\theta} = \frac{1}{\theta_1} \int_{x-\theta_1 t}^{\infty} g(y)\, dy$ *for any* $\theta_1, \theta_2 > 0$*.*

Similarly, the following results correspond to Theorem 5.45, Corollaries 5.46 and 5.47 and are obtained based on the relationship between the θ-m.b.c. stochastic arrival curve and m.b.c. stochastic arrival curve.

Corollary 5.51. *Consider a system* S *with input* A *that is the aggregation of two constituent flows* A_1 *and* A_2*. Suppose* A_2 *has a* θ*-m.b.c stochastic arrival curve* $\alpha \in \mathcal{F}$ *with bounding function* $f \in \bar{\mathcal{F}}$ *(i.e.,* $A_2 \sim_{\theta-mb} \langle f_2^{\theta}, \alpha_2 \rangle$*) and the system provides to the input a stochastic service curve* $\beta \in \mathcal{F}$ *with bounding function* $g \in \bar{\mathcal{F}}$*; i.e.,* $S \sim_{sc} \langle g, \beta \rangle$*. Then, if* $\beta - \alpha_2 \in \mathcal{F}$*,* A_1 *receives a stochastic service curve* $\beta - \alpha_2$ *with bounding function* $f_{2,t}^{\theta} \otimes g$*; i.e.,* $S_1 \sim_{sc} \langle f_{2,t}^{\theta} \otimes g, \beta - \alpha_2 \rangle$*, where* $f_{2,t}^{\theta}(x) = f_2^{\theta}(x - \theta t)$ *for any* $\theta > 0$*.*

Corollary 5.52. *Consider a system* S *with input* A *that is the aggregation of two constituent flows* A_1 *and* A_2*. Suppose* A_2 *has a* θ*-m.b.c. stochastic arrival curve* $\alpha \in \mathcal{F}$ *with bounding function* $f \in \bar{\mathcal{F}}$*; (i.e.,* $A_2 \sim_{\theta-mb} \langle f_2^{\theta}, \alpha_2 \rangle$*) and the system provides to the input a* θ*–stochastic service curve* $\beta \in \mathcal{F}$ *with bounding function* $g^{\theta} \in \bar{\mathcal{F}}$*; i.e.,* $S \sim_{sc} \langle g^{\theta}, \beta \rangle$*. Then, if* $\beta - \alpha_2 \in \mathcal{F}$*,* A_1 *receives a stochastic service curve* $\beta - \alpha_2$ *with bounding function* $f_{2,t}^{\theta} \otimes g_t^{\theta}$*; i.e.,* $S_1 \sim_{sc} \langle f_{2,t}^{\theta} \otimes g_t^{\theta}, \beta - \alpha_2 \rangle$*, where* $f_{2,t}^{\theta}(x) = f_2^{\theta_2}(x - \theta_2 t)$ *and* $g_t^{\theta} = g^{\theta}(x - \theta_1 t)$ *for any* $\theta_1, \theta_2 > 0$*.*

Corollary 5.53. *Consider a system* S *with input* A *that is the aggregation of two constituent flows* A_1 *and* A_2*. Suppose* A_2 *has a* θ*-m.b.c stochastic arrival curve* $\alpha \in \mathcal{F}$ *with bounding function* $f \in \bar{\mathcal{F}}$*, i.e.* $A_2 \sim_{\theta-mb} \langle f_2^{\theta}, \alpha_2 \rangle$*, and* *the system provides to the input a weak stochastic service curve $\beta \in \mathcal{F}$ with bounding function $g \in \bar{\mathcal{F}}$, i.e. $S \sim_{ws} \langle g, \beta \rangle$. Then, if $\beta - \alpha_2 \in \mathcal{F}$, A_1 receives a stochastic service curve $\beta - \alpha_2$ with bounding function $f_{2,t}^{\theta} \otimes g_t^{\theta}$, i.e. $S_1 \sim_{sc} \langle f_{2,t}^{\theta} \otimes g_t^{\theta}, \beta - \alpha_2 \rangle$, where $f_{2,t}^{\theta}(x) = f_2^{\theta}(x - \theta_2 t)$ and $g_t^{\theta} = \frac{1}{\theta_1} \int_{x-\theta_1 t}^{\infty} g^{\theta}(y) dy$ for any $\theta_1, \theta_2 > 0$.*

Finally, we suppose the input A_2 is characterized using a t.a.c. stochastic arrival curve. The following results correspond to Theorem 5.45, Corollary 5.46, and Corollary 5.47 and are similarly obtained based on the relationship between the t.a.c. stochastic arrival curve and m.b.c. stochastic arrival curve.

Corollary 5.54. *Consider a system S with input A that is the aggregation of two constituent flows A_1 and A_2. Suppose A_2 has a t.a.c stochastic arrival curve $\alpha \in \mathcal{F}$ with bounding function $f \in \bar{\mathcal{G}}$ (i.e., $A_2 \sim_{ta} \langle f_2, \alpha_2 \rangle$) and the system provides to the input a stochastic service curve $\beta \in \mathcal{F}$ with bounding function $g \in \bar{\mathcal{F}}$; i.e., $S \sim_{sc} \langle g, \beta \rangle$. Then, if $\beta - \alpha_2 \in \mathcal{F}$, A_1 receives a stochastic service curve $\beta - \alpha_{2,\theta}$ with bounding function $f_{2,t}^{\theta} \otimes g$; i.e., $S_1 \sim_{sc} \langle f_{2,t}^{\theta} \otimes g, \beta - \alpha_{2,\theta} \rangle$, where $f_{2,t}^{\theta}(x) = \frac{1}{\theta_2} \int_{x-\theta_2 t}^{\infty} \hat{f}_2(y)\, dy$, $\hat{f}_2(y) = f(y) + \frac{1}{\theta_1} \int_y^{\infty} f(z)\, dz$, and $\alpha_{2,\theta}(t) = \alpha_2(t) + \theta_1 \cdot t$ for any $\theta_1, \theta_2 > 0$.*

Corollary 5.55. *Consider a system S with input A that is the aggregation of two constituent flows A_1 and A_2. Suppose A_2 has a t.a.c stochastic arrival curve $\alpha \in \mathcal{F}$ with bounding function $f \in \bar{\mathcal{G}}$ (i.e., $A_2 \sim_{ta} \langle f_2, \alpha_2 \rangle$) and the system provides to the input a θ-stochastic service curve $\beta \in \mathcal{F}$ with bounding function $g^{\theta} \in \bar{\mathcal{F}}$; (i.e., $S \sim_{\theta-sc} \langle g^{\theta}, \beta \rangle$). Then, if $\beta - \alpha_2 \in \mathcal{F}$, A_1 receives a stochastic service curve $\beta - \alpha_{2,\theta}$ with bounding function $f_{2,t}^{\theta} \otimes g_t^{\theta}$; (i.e., $S_1 \sim_{sc} \langle f_{2,t}^{\theta} \otimes g_t^{\theta}, \beta - \alpha_{2,\theta} \rangle$), where $f_{2,t}^{\theta}(x) = \frac{1}{\theta_2} \int_{x-\theta_2 t}^{\infty} \hat{f}_2(y)\, dy$, $\hat{f}_2(y) = f(y) + \frac{1}{\theta_1} \int_y^{\infty} f(z)\, dz$, $g_t^{\theta} = g^{\theta}(x - \theta_3 t)$, and $\alpha_{2,\theta}(t) = \alpha_2(t) + \theta_1 \cdot t$ for any $\theta_1, \theta_2, \theta_3 > 0$.*

Corollary 5.56. *Consider a system S with input A that is the aggregation of two constituent flows A_1 and A_2. Suppose A_2 has a t.a.c. stochastic arrival curve $\alpha \in \mathcal{F}$ with bounding function $f \in \bar{\mathcal{G}}$ (i.e., $A_2 \sim_{ta} \langle f_2, \alpha_2 \rangle$) and the system provides to the input a weak stochastic service curve $\beta \in \mathcal{F}$ with bounding function $g \in \bar{\mathcal{G}}$ (i.e., $S \sim_{ws} \langle g, \beta \rangle$). Then, if $\beta - \alpha_2 \in \mathcal{F}$, A_1 receives a stochastic service curve $\beta - \alpha_{2,\theta}$ with bounding function $f_{2,t}^{\theta} \otimes g_t^{\theta}$; (i.e., $S_1 \sim_{sc} \langle f_{2,t}^{\theta} \otimes g_t^{\theta}, \beta - \alpha_{2,\theta} \rangle$), where $f_{2,t}^{\theta}(x) = \frac{1}{\theta_2} \int_{x-\theta_2 t}^{\infty} \hat{f}_2(y)\, dy$, $\hat{f}_2(y) = f(y) + \frac{1}{\theta_1} \int_y^{\infty} f(z)\, dz$, $g_t^{\theta} = \frac{1}{\theta_3} \int_{x-\theta_3 t}^{\infty} g(y)\, dy$, and $\alpha_{2,\theta}(t) = \alpha_2(t) + \theta_1 \cdot t$ for any $\theta_1, \theta_2, \theta_3 > 0$.*

5.4.3 Leftover θ-Stochastic Service Curve

From (5.40), we also obtain, for any $\theta_1, \theta_2 > 0$ and $\theta = \theta_1 + \theta_2$,

$$\begin{aligned}
&\sup_{0\le s\le t}\left[A_1\otimes(\beta-\alpha_2)(s)-A_1^*(s)-\theta(t-s)\right]\\
&\le \sup_{0\le s\le t}\left[A\otimes\beta(s)-A^*(s)-\theta_1(t-s)\right]\\
&\quad+\sup_{0\le s\le t}\left[\sup_{0\le u\le s}\left[A_2(u,s)-\alpha_2(s-u)\right]-\theta_2(t-s)\right]
\end{aligned}\tag{5.42}$$

and with this and the relationship between the m.b.c. SAC and θ-m.b.c. SAC and the relationship between the stochastic service curve and θ-stochastic service curve, the following theorem can be easily verified.

Theorem 5.57. *Consider a system S with input A that is the aggregation of two constituent flows A_1 and A_2. Suppose A_2 has an m.b.c. (or θ-m.b.c.) stochastic arrival curve $\alpha\in\mathcal{F}$ with bounding function $f\in\bar{\mathcal{F}}$ (i.e., $A_2\sim_{mb}\langle f_2,\alpha_2\rangle$ (or $A_2\sim_{\theta-mb}\langle f_2,\alpha_2\rangle$)) and the system provides to the input a stochastic service curve (or θ-stochastic service curve) $\beta\in\mathcal{F}$ with bounding function $g\in\bar{\mathcal{F}}$ (i.e., $S\sim_{sc}\langle g,\beta\rangle$ (or $S\sim_{\theta-sc}\langle g,\beta\rangle$)). Then, if $\beta-\alpha_2\in\mathcal{F}$, A_1 receives a θ-stochastic service curve $\beta-\alpha_2$ with bounding function $f_2\otimes g(x)$; i.e., $S_1\sim_{\theta-sc}\langle f_2\otimes g(x),\beta-\alpha_2\rangle$.*

Based on the relationship between the weak stochastic service curve and θ-stochastic service curve, the following corollary is obtained.

Corollary 5.58. *Consider a system S with input A that is the aggregation of two constituent flows A_1 and A_2. Suppose A_2 has an m.b.c. (or θ-m.b.c.) stochastic arrival curve $\alpha\in\mathcal{F}$ with bounding function $f\in\bar{\mathcal{F}}$ (i.e., $A_2\sim_{mb}\langle f_2,\alpha_2\rangle$ (or $A_2\sim_{\theta-mb}\langle f_2,\alpha_2\rangle$)) and the system provides to the input a weak stochastic service curve $\beta\in\mathcal{F}$ with bounding function $g\in\bar{\mathcal{G}}$, (i.e., $S\sim_{ws}\langle g,\beta\rangle$). Then, if $\beta-\alpha_2\in\mathcal{F}$, A_1 receives a θ-stochastic service curve $\beta-\alpha_2$ with bounding function $f_2\otimes g^\theta(x)$; i.e., $S_1\sim_{\theta-sc}\langle f_2\otimes g^\theta(x),\beta-\alpha_2\rangle$, where $g^\theta=g(x)+\frac{1}{\theta}\int_x^\infty g(y)dy$ for any $\theta>0$.*

Corresponding to Theorem 5.57 and Corollary 5.58, Corollaries 5.59 and 5.60 are obtained based on the relationship between the v.b.c. stochastic arrival curve and θ–m.b.c. stochastic arrival curve.

Corollary 5.59. *Consider a system S with input A that is the aggregation of two constituent flows A_1 and A_2. Suppose A_2 has a v.b.c. stochastic arrival curve $\alpha\in\mathcal{F}$ with bounding function $f\in\bar{\mathcal{G}}$, i.e. $A_2\sim_{vb}\langle f_2,\alpha_2\rangle$ and the system provides to the input a stochastic service curve (or θ-stochastic service curve) $\beta\in\mathcal{F}$ with bounding function $g\in\bar{\mathcal{F}}$ (i.e., $S\sim_{sc}\langle g,\beta\rangle$ (or $S\sim_{\theta-sc}\langle g,\beta\rangle$)). Then, if $\beta-\alpha_{2,\theta}\in\mathcal{F}$, A_1 receives a θ-stochastic service curve $\beta-\alpha_{2,\theta}$ with bounding function $f_2^\theta\otimes g$; i.e., $S_1\sim_{\theta-sc}\langle f_2^\theta\otimes g(x),\beta-\alpha_{2,\theta}\rangle$, where $f_2^\theta=f_2(x)+\frac{1}{\theta}\int_x^\infty f_2(y)dy$ and $\alpha_{2,\theta}(t)=\alpha_2(t)+\theta t$ for any $\theta>0$.*

Corollary 5.60. *Consider a system S with input A that is the aggregation of two constituent flows A_1 and A_2. Suppose A_2 has a v.b.c. stochastic arrival curve $\alpha\in\mathcal{F}$ with bounding function $f\in\bar{\mathcal{G}}$; (i.e., $A_2\sim_{vb}\langle f_2,\alpha_2\rangle$)*

and the system provides to the input a weak stochastic service curve $\beta \in \mathcal{F}$ *with bounding function* $g \in \bar{\mathcal{G}}$*; (i.e.,* $S \sim_{ws} \langle g, \beta \rangle$*). Then, if* $\beta - \alpha_{2,\theta} \in \mathcal{F}$*,* A_1 *receives a* θ*-stochastic service curve* $\beta - \alpha_{2,\theta}$ *with bounding function* $f_2^{\theta} \otimes g^{\theta}$*; i.e.,* $S_1 \sim_{\theta-sc} \langle f_2^{\theta} \otimes g^{\theta}, \beta - \alpha_{2,\theta} \rangle$*, where* $f_2^{\theta} = f_2(x) + \frac{1}{\theta_2} \int_x^{\infty} f_2(y)dy$*,* $g^{\theta_1} = g(x) + \frac{1}{\theta_1} \int_x^{\infty} g(y)dy$*, and* $\alpha_{2,\theta}(t) = \alpha_2(t) + \theta_2 t$ *for any* $\theta_1, \theta_2 > 0$*.*

Finally, based on the relationship between the t.a.c. stochastic arrival curve and θ-m.b.c. stochastic arrival curve, we can have Corollaries 5.61 and 5.62, which correspond to Theorem 5.57 and Corollary 5.58.

Corollary 5.61. *Consider a system* S *with input* A *that is the aggregation of two constituent flows* A_1 *and* A_2*. Suppose* A_2 *has a t.a.c stochastic arrival curve* $\alpha \in \mathcal{F}$ *with bounding function* $f \in \bar{\mathcal{G}}$*; (i.e.,* $A_2 \sim_{ta} \langle f_2, \alpha_2 \rangle$*) and the system provides to the input a stochastic service curve (or* θ*-stochastic service curve)* $\beta \in \mathcal{F}$ *with bounding function* $g \in \bar{\mathcal{F}}$*; i.e.,* $S \sim_{sc} \langle g, \beta \rangle$ *(or* $S \sim_{\theta-sc} \langle g, \beta \rangle$*). Then, if* $\beta - \alpha_{2,\theta} \in \mathcal{F}$*,* A_1 *receives a* θ*-stochastic service curve* $\beta - \alpha_{2,\theta}$ *with bounding function* $f_2^{\theta} \otimes g$*, i.e.,* $S_1 \sim_{\theta-sc} \langle f_2^{\theta} \otimes g, \beta - \alpha_{2,\theta} \rangle$*, where* $f_2^{\theta} = \hat{f}_2(x) + \frac{1}{\theta_2} \int_x^{\infty} \hat{f}_2(y)dy$*,* $\hat{f}_2(y) = f(y) + \frac{1}{\theta_1} \int_y^{\infty} f(z)\,dz$*, and* $\alpha_{2,\theta_2}(t) = \alpha_2(t) + \theta_2 t$*, for any* $\theta_1, \theta_2 > 0$*.*

Corollary 5.62. *Consider a system* S *with input* A *that is the aggregation of two constituent flows* A_1 *and* A_2*. Suppose* A_2 *has a t.a.c stochastic arrival curve* $\alpha \in \mathcal{F}$ *with bounding function* $f \in \bar{\mathcal{G}}$*; (i.e.,* $A_2 \sim_{ta} \langle f_2, \alpha_2 \rangle$*) and the system provides to the input a weak stochastic service curve* $\beta \in \mathcal{F}$ *with bounding function* $g \in \bar{\mathcal{G}}$*; i.e.,* $S \sim_{ws} \langle g, \beta \rangle$*. Then, if* $\beta - \alpha_{2,\theta} \in \mathcal{F}$*,* A_1 *receives a* θ*-stochastic service curve* $\beta - \alpha_{2,\theta}$ *with bounding function* $f_2^{\theta} \otimes g^{\theta}$*; i.e.,* $S_1 \sim_{\theta-sc} \langle f_2^{\theta} \otimes g^{\theta}, \beta - \alpha_{2,\theta} \rangle$*, where* $f_2^{\theta_2} = \hat{f}_2(x) + \frac{1}{\theta_2} \int_x^{\infty} \hat{f}_2(y)dy$*,* $\hat{f}_2(y) = f(y) + \frac{1}{\theta_3} \int_y^{\infty} f(z)\,dz$*, and* $g^{\theta_1} = g(x) + \frac{1}{\theta_1} \int_x^{\infty} g(y)dy$ *and* $\alpha_{2,\theta}(t) = \alpha_2(t) + \theta_2 t$ *for any* $\theta_1, \theta_2, \theta_3 > 0$*.*

5.5 Superposition Property

The superposition property means that the superposition of flows can be represented using the same traffic model. With this property, the aggregate of (possibly many) individual flows may be considered as a single aggregate flow, so that the QoS performance for the aggregate can be derived in the same way as for a single flow. This section discusses the superposition property for the various stochastic traffic models introduced in Chapter 2.

Consider N flows with arrival processes $A_i(t)$, $i = 1, \ldots, N$. Let $A(t)$ be the superposition of the N flows. In other words, we have for any $s, t \geq 0$,

$$A(s, s+t) = A_1(s, s+t) + \cdots + A_N(s, s+t).$$

Then, for any functions $\alpha_i(t)$, $i = 1, \ldots, N$, we have

$$
\begin{aligned}
& A(s, s+t) - [\alpha_1(t) + \cdots + \alpha_N(t)] \\
= & [A_1(s, s+t) - \alpha_1(t)] + \cdots + [A_N(s, s+t) - \alpha_N(t)].
\end{aligned} \tag{5.43}
$$

With (5.43), the superposition property of the t.a.c. stochastic arrival curve can be easily verified.

Theorem 5.63. *Consider N flows with arrival processes $A_i(t)$, $i = 1, \ldots, N$, respectively. Let $A(t)$ denote the aggregate arrival process. If $\forall i, A_i \sim_{ta} \langle f_i, \alpha_i \rangle$, then $A \sim_{ta} \langle f, \alpha \rangle$ with $\alpha(t) = \sum_{i=1}^{N} \alpha_i(t)$ and $f(x) = f_1 \otimes \cdots \otimes f_N(x)$.*

From (5.43), we can also obtain

$$
\begin{aligned}
& \sup_{0 \le s \le t} [A(s,t) - [\alpha_1(t-s) + \cdots + \alpha_N(t-s)]] \\
\le & \sup_{0 \le s \le t} [A_1(s,t) - \alpha_1(t-s)] + \cdots + \sup_{0 \le s \le t} [A_N(s,t) - \alpha_N(t-s)],
\end{aligned} \tag{5.44}
$$

and with this, the superposition property of the v.b.c. stochastic arrival curve can be derived.

Theorem 5.64. *Consider N flows with arrival processes $A_i(t)$, $i = 1, \ldots, N$, respectively. Let $A(t)$ denote the aggregate arrival process. If $\forall i, A_i \sim_{vb} \langle f_i, \alpha_i \rangle$, then $A \sim_{vb} \langle f, \alpha \rangle$ with $\alpha(t) = \sum_{i=1}^{N} \alpha_i(t)$ and $f(x) = f_1 \otimes \cdots \otimes f_N(x)$.*

Further, from (5.44), we get

$$
\begin{aligned}
& \sup_{0 \le s \le t} \sup_{0 \le u \le s} [A(u,s) - [\alpha_1(s-u) + \cdots + \alpha_N(s-u)]] \\
\le & \sup_{0 \le s \le t} \sup_{0 \le u \le s} [A_1(u,s) - \alpha_1(s-u)] + \cdots \\
& + \sup_{0 \le s \le t} \sup_{0 \le u \le s} [A_N(u,s) - \alpha_N(s-u)]
\end{aligned} \tag{5.45}
$$

with which, the superposition property of the m.b.c. stochastic arrival curve is obtained.

Theorem 5.65. *Consider N flows with arrival processes $A_i(t)$, $i = 1, \ldots, N$, respectively. Let $A(t)$ denote the aggregate arrival process. If $\forall i, A_i \sim_{mb} \langle f_i, \alpha_i \rangle$, then $A \sim_{mb} \langle f, \alpha \rangle$ with $\alpha(t) = \sum_{i=1}^{N} \alpha_i(t)$ and $f(x) = f_1 \otimes \cdots \otimes f_N(x)$.*

Also from (5.44), we get for any $\theta_1, \ldots, \theta_N > 0$ and $\theta = \theta_1 + \cdots + \theta_N$

$$
\begin{aligned}
& \sup_{0 \le s \le t} \left[\sup_{0 \le u \le s} \{A(u,s) - [\alpha_1(s-u) + \cdots + \alpha_N(s-u)]\} - \theta \cdot (t-s) \right] \\
\le & \sup_{0 \le s \le t} \left[\sup_{0 \le u \le s} [A_1(u,s) - \alpha_1(s-u)] - \theta_1 \cdot (t-s) \right] + \cdots \\
& + \sup_{0 \le s \le t} \left[\sup_{0 \le u \le s} [A_N(u,s) - \alpha_N(s-u)] - \theta_N \cdot (t-s) \right],
\end{aligned} \tag{5.46}
$$

and with this, the superposition property of the m.b.c. stochastic arrival curve is obtained.

Theorem 5.66. *Consider N flows with arrival processes $A_i(t)$, $i = 1, \ldots, N$, respectively. Let $A(t)$ denote the aggregate arrival process. If $\forall i, A_i \sim_{\theta-mb} \langle f_i, \alpha_i \rangle$, then $A \sim_{\theta-mb} \langle f^\theta, \alpha \rangle$ with $\alpha(t) = \sum_{i=1}^{N} \alpha_i(t)$, and $f^\theta(x) = f_1^{\theta_1} \otimes \cdots \otimes f_N^{\theta_N}(x)$ for any $\theta_1, \ldots, \theta_N > 0$ and $\theta = \theta_1 + \cdots + \theta_N$.*

5.6 Scaling of End-to-End Delay Bound

As discussed earlier in Chapter 2, when we consider the deterministic network with n nodes in tandem, we see that the end-to-end delay bound obtained is a scaling in $\mathcal{O}\left(n^2\right)$ from the node-by-node analysis approach. However, with the concatenation property of the service curve, the end-to-end delay bound is a scaling in $\mathcal{O}\left(n\right)$, which gives a much tighter bound. The *scaling property* provides us an important metric for evaluating the tightness and scalability of performance bounds under different approaches.

In this section, we investigate the scaling property of end-to-end delay bounds under a stochastic setting to demonstrate the use of stochastic network calculus results introduced in this chapter.

Consider the scenario shown in Figure 5.2.A. Flow F passes n servers in tandem. Each server is a constant-rate server with capacity C. At each server, a cross-flow joins and leaves. Assume flow F and all cross-flows have the same m.b.c. stochastic arrival curve (SAC) $A \sim_{mb} \langle r, f \rangle$ with $f\left(x\right) = ae^{-bx}$. To ensure the stability of the system, we also assume $2r < C$. We are interested in deriving the stochastic end-to-end delay bound for flow F and investigate how the delay bound increases as the number of servers increases.

To facilitate the explanation, we introduce a useful lemma as follows, which can also be found from [24].

Lemma 5.67. *For any positive numbers a_k, b_k, $k = 1, \cdots, K$ and any $x \geq 0$, we have*

$$\inf_{x_1+\cdots+x_K=x} \sum_{k=1}^{K} a_k e^{-b_k x_k} = e^{\frac{-x}{w}} \prod_{k=1}^{K} \left(a_k b_k w\right)^{\frac{1}{b_k w}},$$

where $w = \sum_{k=1}^{K} \frac{1}{b_k}$.

Proof. Let

$$a_k b_k e^{-b_k x_k} = \lambda, \ \ k = 1, \cdots, K.$$

Then,

$$\sum_{k=1}^{K} a_k e^{-b_k x_k} = \sum_{k=1}^{K} \frac{\lambda}{b_k}. \tag{5.47}$$

Let $w = \sum_{k=1}^{K} \frac{1}{b_k}$ and $p_k = \frac{1}{b_k w}$. Since $\sum_{i=1}^{K} p_i = 1$, we have

$$\lambda = \prod_{k=1}^{K} \lambda^{p_k} = e^{-x/w} \prod_{k=1}^{K} \left(\frac{a_k}{w p_k}\right)^{p_k}. \tag{5.48}$$

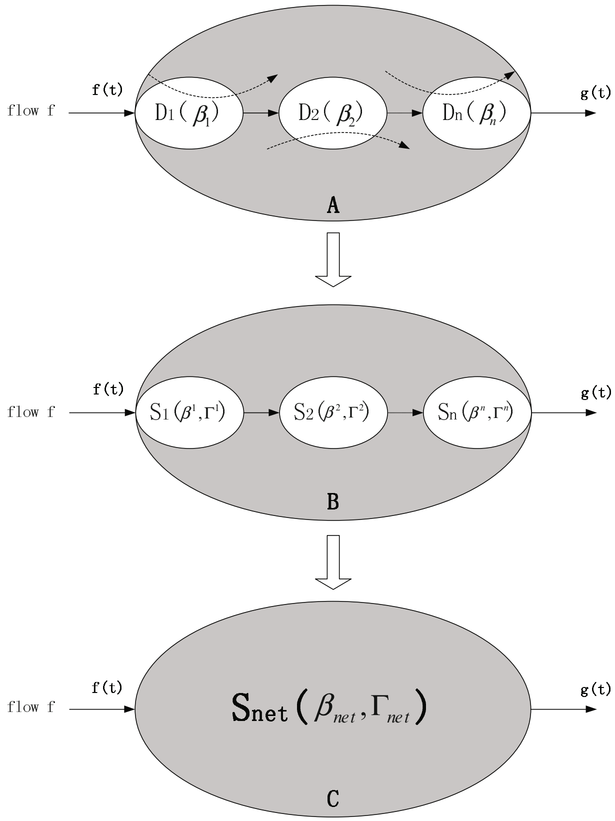

Fig. 5.2. Stochastic servers in tandem

Combining (5.47) and (5.48), this lemma follows. □

We now derive end-to-end stochastic delay bounds for the flow F. We first base the derivation on the concatenation property and then derive if using the node-by-node analysis approach.

5.6.1 Delay Bound From the Concatenation Property

As shown in Figure 5.2.B, according to Theorem 5.45, each node provides a leftover stochastic service curve

$$S^i \sim_{sc} \left\langle \beta^i, \Gamma^i \right\rangle,$$

where

$$\beta_i\left(t\right) = \left(C - r\right)t$$

and

$$\Gamma^i\left(x\right) = f\left(x\right).$$

Then, as shown in Figure 5.2 C, according to the concatenation property of the stochastic service curve, we have

$$S_{net} \sim_{sc} (\beta_{net}, \Gamma_{net}),$$

where

$$\beta_{net} = \beta^1 \otimes \cdots \otimes \beta^n = (C - r)\, t$$

and

$$f_{net} = \Gamma^1 \otimes \cdots \otimes \Gamma^n.$$

In addition, according to Theorem 5.4, we can have the stochastic end-to-end delay bound for flow F

$$P\{D > h(\alpha + x, \beta_{net})\} \leq f \otimes \Gamma^1 \otimes \cdots \otimes \Gamma^n(x),$$

and with this and Lemma 5.67, we have

$$P\left\{D > \frac{x}{C - r}\right\} \leq e^{-\frac{xb}{n+1}}\,(a(n+1)). \tag{5.49}$$

Then, we determine the delay bound d such that $P\{D > d\} \leq \varepsilon$, where ε is a small delay bound violation probability.

Let $d = \frac{x}{C-r}$ and set the right side of (5.49) equal to ε. We have for the delay bound d

$$d = \frac{n+1}{(C-r)\,b} \log \frac{(a(n+1))}{\varepsilon}. \tag{5.50}$$

It is found from (5.50) that the delay bound derived from the concatenation property scales in $\mathcal{O}(n \log n)$, where n is the number of nodes the flow passes through.

5.6.2 Delay Bound from Node-by-Node Analysis

Now we derive the stochastic end-to-end delay bound by using the node-by-node analysis approach. As shown in Figure 5.2.B, for the first node, according to Theorem 5.4, we can have the following delay bound at the first node

$$P\left\{D_1 > \frac{x}{C - r}\right\} \leq f \otimes \Gamma^1(x).$$

For the second node, we need to have the input burstiness of flow F at the second node, which is the output burstiness of flow F at the first node. According to Theorem 5.21, the input of flow F has an m.b.c. SAC $\left\langle f \otimes \Gamma^1(x), r\right\rangle$. Then we have the following delay bound at the second node:

$$P\left\{D_2 > \frac{x}{C - r}\right\} \leq f \otimes \Gamma^1 \otimes \Gamma^2(x).$$

Similarly, we have the following delay bound at each node i on the path

$$P\left\{D_i > \frac{x}{C-r}\right\} \leq f \otimes \Gamma^1 \otimes \cdots \otimes \Gamma^i,$$

and with this, following the same approach as in getting (5.49), we obtain

$$P\left\{D_i > \frac{x}{C-r}\right\} \leq e^{-\frac{xb}{i+1}} \left(a(i+1)\right). \tag{5.51}$$

Now we consider the distribution of $D_1 + D_2 + \cdots + D_n$. According to Lemma 5.67, we have

$$P\left\{D > \frac{x}{C-r}\right\} \leq e^{-\frac{2xb}{(n+1)(n+3)}} \left(a\frac{(n+1)(n+3)}{2}\right) \tag{5.52}$$

Then, we determine the delay bound d such that $P\{D > d\} \leq \varepsilon$, where ε is a small delay bound violation probability.

Letting $d = \frac{x}{C-r}$ and setting the right side of (5.49) equal to ε, we have

$$d = \frac{(n+1)(n+3)}{2\,(C-r)\,b} \log \frac{\left(a\frac{(n+1)(n+3)}{2}\right)}{\varepsilon}. \tag{5.53}$$

It is found from (5.53) that the delay bound derived through node-by-node analysis scales in $\mathcal{O}\left(n^2 \log n\right)$. Comparing this with the stochastic delay bound obtained in (5.50) by using the concatenation property, it is clear that the one in (5.50) is much better than the one obtained in (5.53) through node-by-node analysis.

5.7 Calculus on Traffic and Service Envelope Processes

In Chapters 2 and 3, the traffic envelope process and service envelope process respectively were introduced. This section presents results based on these processes. Theorems 5.68 to 5.73 correspond to the five basic properties. Their proofs follow similarly from their deterministic counterpart theorems and the definitions of stochastic envelope process in Definition 3.28, service envelope process in Definition 4.15, and strict service envelope process in Definition 4.16.

Theorem 5.68 (Delay Bound). *Consider a system that provides a service envelope process $\hat{S}(t)$ to the input flow $A(t)$. Suppose A has a stochastic envelope process $\hat{A}$. Then, the delay $D(t)$ of the flow at time t satisfies*

$$D(t) \leq h\left(\hat{A}(t), \hat{S}(t)\right).$$

Theorem 5.69 (Backlog Bound). *Consider a system that provides a service envelope process $\hat{S}(t)$ to the input flow $A(t)$. Suppose A has a stochastic envelope process $\hat{A}$. Then, the backlog $B(t)$ of the flow at time t satisfies*

$$B(t) \leq \hat{A} \oslash \hat{S}(0).$$

Theorem 5.70 (Output Characterization). *Consider a system that provides a service envelope process $\hat{S}(t)$ to the input flow $A(t)$. Suppose A has a stochastic envelope process $\hat{A}$. Then, the output A^* has a stochastic envelope process $\hat{A}^*$, i.e., for all $s, t \geq 0$, $A^*(s, s+t) \leq \hat{A}^*(t)$, where*

$$\hat{A}^*(t) = \hat{A} \oslash \hat{S}(t).$$

Theorem 5.71 (Concatenation Property). *Consider a flow passing through systems S^h, $h = 1, \ldots, H$, in sequence. Suppose each system S^h provides a service envelope process $\hat{S}^h(t)$ to the input. Then the concatenation of these systems offers a service envelope process $\hat{S}$ to the flow, where*

$$\hat{S}(t) = \hat{S}^1 \otimes \hat{S}^2 \cdots \otimes \hat{S}^H(t). \tag{5.54}$$

Theorem 5.72 (Leftover Service). *Consider a system serving an aggregate of two (possibly aggregate) flows A_1 and A_2. Assume the system provides a service envelope process $\hat{S}$ to the aggregate, and A_2 has a stochastic envelope process $\hat{A}_2$. Then, the system offers to the flow A_1 a service envelope process $\hat{S}_1(t)$, where*

$$\hat{S}_1(t) = (\hat{S} - \hat{A}_2)(t). \tag{5.55}$$

Theorem 5.73 (Superposition). *Consider the superposition of n flows A_i, $i = 1, \ldots, n$. If each flow A_i has a stochastic envelope process $\hat{A}_i(t)$, then the aggregate flow $A = \sum_{i=1}^n A_i$ has a stochastic envelope process $\hat{A}(t) = \sum_{i=1}^n \hat{A}_i(t)$.*

While looking similar to the corresponding deterministic results, Theorems 5.68 to 5.73 indeed have critical differences from their deterministic counterparts. One is that all envelope processes in Theorems 5.68 to 5.73 are random processes. Due to this, another difference is that in order to use Theorems 5.68 to 5.73 in real network analysis, the statistical properties of the various envelope processes have to be known and explored. The third difference is that to apply Theorems 5.68 to 5.73, strict service envelope process is often needed instead of service envelope process. This is because, as shown by its definition, the service envelope process model is dependent on the input process, which complicates finding the service envelope process. In Chapter 6, moment generating functions of the various envelope processes will be used to derive the five basic properties, where strict service envelope processes are implicitly required and the analysis generally assumes independence between the envelope processes considered.

Table 5.1. Properties provided by a combination of traffic model and server model

	weak SSC	SSC	θ-SSC
t.a.c. SAC	(P.1), (P.5)	(P.1), (P.3), (P.5)	(P.1), (P.3), (P.5)
v.b.c. SAC	(P.1), (P.4), (P.5)	(P.1)–(P.3), (P.5)	(P.1), (P.3), (P.5)
m.b.c. SAC	(P.1), (P.4), (P.5)	(P.1)–(P.5)	(P.1)–(P.5)
θ-m.b.c. SAC	(P.1), (P.4), (P.5)	(P.1), (P.5)	(P.1)–(P.5)

5.8 Summary and Bibliographic Comments

In this chapter, we presented the five basic properties of stochastic network calculus under the various traffic models and server models introduced in Chapters 3 and 4.

Table 5.1 summarizes the properties that are provided by the combination of a traffic model, chosen from t.a.c., v.b.c., m.b.c., and θ-m.b.c. stochastic arrival curve (SAC), and a server model, chosen from weak stochastic service curve, stochastic service curve (SSC), and θ–stochastic service curve, without any additional constraints on the traffic model or the server model, where, as introduced in Chapter 1, (P.1)–(P.5) denote the following properties:

- (P.1) – Service Guarantees
- (P.2) – Output Characterization
- (P.3) – Concatenation Property
- (P.4) – Leftover Service
- (P.5) – Superposition Property

In Chapter 3, we discussed that under the context of network calculus, many (if not most) traffic models used in the literature [138] [31] [128] [140] [14] [95] [98] [5] [74] [73] [24] belong to the t.a.c. and/or v.b.c. stochastic arrival curve. In Chapter 4, many (if not most) server models [93][31][14][95][98][5][24] were shown to belong to the weak stochastic service curve. Table 5.1 shows that, without additional constraints, these works can only support part of the five required properties for stochastic network calculus. In contrast, under the combination of the m.b.c. stochastic arrival curve and stochastic service curve, all five basic properties have been proved in this chapter without additional constraints added to these two models. While appealing, this combination has a potential problem in the bounding function under the m.b.c. stochastic arrival curve or stochastic service curve may be dependent on time.

Note that with some additional constraints on the bounding functions in the models discussed in Table 5.1, one combination may have more properties among (P.1)–(P.5) than those listed in the table. The most frequently used constraint in this book is the bounding function belonging to $\bar{\mathcal{G}}$. This constraint, initially suggested by Starobinski and Sidi [128], is that the bounding function belongs to a specific subset of $\bar{\mathcal{F}}$, denoted by $\bar{\mathcal{G}}$, which consists of all functions in $\bar{\mathcal{F}}$ whose nth-fold integration still belongs to the subset for any $n \geq 1$. Under this constraint, as presented in this chapter, the unlisted properties among (P.1)–(P.5) can be proved for the combination of the t.a.c. or v.b.c. stochastic arrival curve and weak stochastic service curve, and particularly for the combination of a θ-stochastic arrival curve and θ-stochastic service curve. It is worth highlighting that if the bounding functions considered are in $\bar{\mathcal{G}}$, the set of results for the combination of a θ-stochastic arrival curve and θ-stochastic service curve can be used as the basis for deriving the basic properties for all other combinations. Except for combinations where an m.b.c. stochastic arrival curve and/or stochastic service curve is used, the results can have bounding functions independent on time. This makes the θ-stochastic arrival curve and θ-stochastic service curve models attractive.

Another constraint, which was recently proposed by Li, et al. [96], assumes that there is a *timescale* T that bounds the convolution in the definition of a weak stochastic service curve. In [96], Li et al. also discussed network cases where such timescales exist. As an analogy, we may assume the existence of a timescale T that bounds the convolution in the m.b.c. stochastic arrival curve model and stochastic service curve model. Consequently, we conjecture that all results presented in this chapter under combinations where the m.b.c. stochastic arrival curve and/or stochastic service curve are used will be bounded by such timescales, and this solves the possibly time-dependent bounding function problem with the m.b.c. stochastic arrival curve and stochastic service curve.

Also note that Table 5.1 only provides a comparison of the basic properties supported by a combination of the four types of stochastic arrival curves and the three types of stochastic service curves. While we believe they cover a wide range of traffic models and server models proposed and studied in the literature as discussed in Chapters 3 and 4, there are other types of traffic and server models that are not covered by them.

One type uses a sequence of random variables to stochastically bound the arrival process [87] or the service process [115]. Properties similar to (P.1), (P.3), (P.4), and (P.5) have been studied [87][115]. These studies generally need the independence assumption. Under these types of traffic and service models, several problems remain open. One is the concatenation property (P.2), another is the general case analysis, and the third is researching/designing approaches to map known traffic and service characterizations to the required sequences of random variables.

Another type is built upon moments or moment generating functions. This type was initially used for traffic (see e.g. Chang [15] and Knightly [85]) and has also been extended to service (see, e.g., Chang [18], Wu and Negt [133], and Fidler [44]). The independence assumption is generally required between arrival and service processes. Extensive studies have been conducted for deriving the characteristics of a process under this type of model from some known characterization of the process [15][16][18]. The main challenges for this type are the concatenation property and the general case analysis. For these, we have presented results in Section 3.5 in Chapter 3 that allow us to further relate known traffic/service characterizations to the traffic and service models discussed in this book.

Scaling of end-to-end performance bounds has recently attracted research interest in the context of stochastic network calculus. The purpose is to study similar scaling properties found in deterministic network calculus. Essentially the study is related to investigating the concatenation property under stochastic settings. In Section 5.6, it is shown that with the concatenation property, the end-to-end stochastic delay bound obtained scales in $\mathcal{O}(n \log n)$. However, if the node-by-node analysis is used, the bound scales in $\mathcal{O}\left(n^2 \log n\right)$. Similar observations were made by Fidler [44] and Ciucu et al. [25]. It should be noted that the scalings from the analysis in Section 5.6 and [25] do not assume the independence between arrival processes and service processes. With the independence assumption, a scaling of $O(n)$ can indeed be obtained for the end-to-end stochastic delay bound as discussed by Fidler [44] and as will also be shown in the next chapter.

All the results in this chapter are proved for the general case where flows and servers could be dependent. In the next chapter, the independent case will be investigated, and the investigation can help improve performance bounds significantly.

Problems

5.1. Consider a server fed with a flow A that is the aggregation of two constituent flows A_1 and A_2. Suppose the server provides a deterministic service curve β to the aggregate flow A. Also suppose flows A_1 and A_2 have v.b.c stochastic arrival curve $A_i \sim_{vb} \langle f_i, \alpha_i \rangle$, $i = 1, 2$. Derive the leftover service curve received by A_1 and stochastic delay bound for A_1.

5.2. Consider a server fed with a flow A that is the aggregation of two constituent flows A_1 and A_2. Suppose that the server provides a deterministic service curve β to the aggregate flow A, and flows A_1 and A_2 have m.b.c. stochastic arrival curve $A_i \sim_{mb} \langle f_i, \alpha_i \rangle$, $i = 1, 2$. Derive the leftover service curve received by A_1.

5.3. Consider a system with three servers S_1, S_2, and S_3 in tandem, where S_1 provides a deterministic service curve β_1, $S_2 \sim_{wc} \langle g_2, \beta_2 \rangle$, and $S_3 \sim_{sc} \langle g_3, \beta_3 \rangle$. Derive an end-to-end service curve for this system.

5.4. Consider a constant-rate server with capacity C fed with a Poisson input flow with average arrival rate λ. The packet size is exponentially distributed with mean value μ but limited by a maximum packet size M.

(i) Derive a probabilistic delay bound for the flow using the methods discussed in this chapter.

(ii) Derive a delay distribution for the flow using queuing theory, and explain the difference with the results obtained in (i).

5.5. A server is called a fluctuation constrained server if [93]

$$\int_a^b C(t)\,dt \geq (\mu(b-a) - \delta)^+ ,$$

where $C(t)$ is the instantaneous output capacity of a server. The server is fed by constant input traffic with rate ρ.

(i) Derive the backlog bound for the system.
(ii) Derive the delay bound for the system.

5.6. A stochastic process A is called an exponentially bounded bursty (EBB) process if for any $x \geq 0$, [138]

$$P\{A(t) \geq x\} \leq ae^{-bx}.$$

Consider a system with an EBB input and a constant-rate server with capacity C.

(i) Derive the backlog bound for the system.
(ii) Derive the delay bound for the system.

5.7. Prove Theorem 5.68.

5.8. Prove Theorem 5.69.

5.9. Prove Theorem 5.70.

5.10. Prove Theorem 5.71.

5.11. Prove Theorem 5.72.

5.12. Prove Theorem 5.73.

5.13. Suppose traffic is characterized by

$$E[A(s, s+t) - \alpha^{\epsilon}(t)] \leq \epsilon(t)$$

and service by [58]

$$E[A \otimes \beta^{\xi}(t) - A^*(t)] \leq \xi(t).$$

Derive the five basic properties under this combination of traffic model and server model directly from the definitions of these two models, and discuss what additional constraints are needed to allow the derivation.

5.14. Based on the basic properties of the various stochastic arrival curve and stochastic service curve models presented in this chapter, find the five basic properties for the combination of traffic model and server model introduced in the previous problem. Compare them with the results obtained from the previous problem.