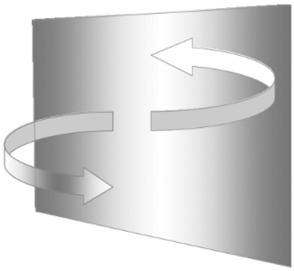

9 Reflections and Rotations

9.1 Introduction

Rotating objects and virtual cameras are central to computer animation and computer games and are traditionally effected using matrix transforms representing Euler angle rotations. For example, to rotate a 2D point about the origin we use

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \tag{9.1}$$

To rotate a 3D point about the origin we use one transform for each axis:
to rotate about the x-axis

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(pitch) & -\sin(pitch) \\ 0 & \sin(pitch) & \cos(pitch) \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{9.2}$$

to rotate about the y-axis

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos(yaw) & 0 & \sin(yaw) \\ 0 & 1 & 0 \\ -\sin(yaw) & 0 & \cos(yaw) \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{9.3}$$

and to rotate about the z-axis

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos(roll) & -\sin(roll) & 0 \\ \sin(roll) & \cos(roll) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}. \tag{9.4}$$

These rotations are not very intuitive to use, especially when we need to rotate points about an arbitrary axis, for which the following transform is used:

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} a^2K + \cos\theta & abK - c\sin\theta & acK + b\sin\theta \\ abK + c\sin\theta & b^2K + \cos\theta & bcK - a\sin\theta \\ acK - b\sin\theta & bcK + a\sin\theta & c^2K + \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{9.5}$$

where

$$K = 1 - \cos\theta \tag{9.6}$$

and the axis of rotation is defined by

$$\hat{v} = a\mathrm{i} + b\mathrm{j} + c\mathrm{k}. \tag{9.7}$$

In recent years, Hamilton's quaternions have been embraced by the computer animation and games communities where a point P is rotated to P', through an angle θ, about an axis $\hat{v}$ using the pure quaternion q. The process involves the following steps:

1. Convert the point $P(x, y, z)$ to a pure quaternion p:

$$p = [0 + x\mathrm{i} + y\mathrm{j} + z\mathrm{k}]. \tag{9.8}$$

2. Define the axis of rotation as a unit vector $\hat{v}$:

$$\hat{v} = [x_v\mathrm{i} + y_v\mathrm{j} + z_v\mathrm{k}]. \tag{9.9}$$

3. Define the transforming quaternion q:

$$q = [\cos(\theta/2) + \sin(\theta/2)\hat{v}]. \tag{9.10}$$

4. Define the inverse of the transforming quaternion q^{-1}:

$$q^{-1} = [\cos(\theta/2) - \sin(\theta/2)\hat{v}]. \tag{9.11}$$

5. Compute p':

$$p' = qpq^{-1}. \tag{9.12}$$

6. Unpack (x', y', z') from p':

$$p' = [0 + x'\mathrm{i} + y'\mathrm{j} + z'\mathrm{k}]. \tag{9.13}$$

Given a quaternion $[w + x\mathrm{i} + y\mathrm{j} + z\mathrm{k}]$, its equivalent matrix is

$$\begin{bmatrix} 1 - 2(y^2 + z^2) & 2(xy - wz) & 2(xz + wy) \\ 2(xy + wz) & 1 - 2(x^2 + z^2) & 2(yz - wx) \\ 2(xz - wy) & 2(yz + wx) & 1 - 2(x^2 + y^2) \end{bmatrix} \tag{9.14}$$

where

$$w^2 + x^2 + y^2 + z^2 = 1. \tag{9.15}$$

We saw in the previous chapter that there is a strong relationship between quaternions and GA, and in this chapter this relationship is further strengthened when we examine how GA implements reflections and rotations.

9.2 Reflections

9.2.1 Reflecting vectors

Solving problems using vector algebra is not always straight forward, for much depends upon the nature of the diagram used to annotate the relevant vectors. For instance, say we are given a mirror whose orientation is defined by an orthogonal unit vector $\hat{n}$, and the task is to find the reflection of the vector a in the mirror. Figure 9.1 shows the advantages of placing the vector so that its tail touches the mirror. It then becomes obvious that the vector's head is a distance $a \cdot \hat{n}$ in front of the mirror, which means that the head of the reflected vector a' is an equal distance behind the mirror. Thus, we can write the following vector equation for a' as

$$a' = a - (2a \cdot \hat{n})\hat{n} \tag{9.16}$$

which is rather succinct.

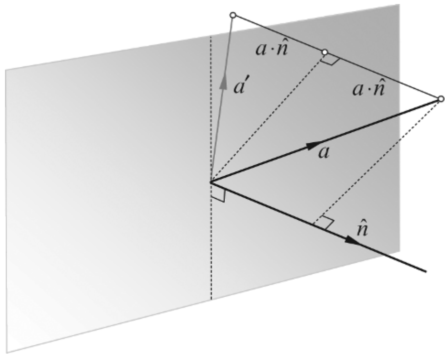

Figure 9.1.

Now let's solve the problem using GA.

Figure 9.2 shows the same diagram, but annotated differently.

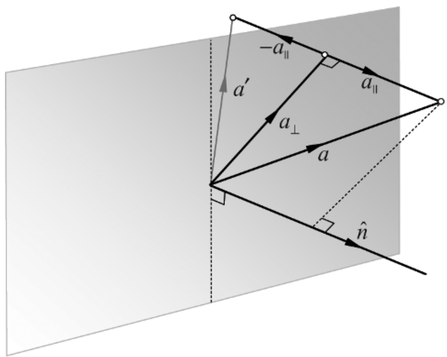

Figure 9.2.

As the mirror's surface normal is defined as a unit vector, then $\hat{n}^2 = 1$, which permits us to write

$$a = \hat{n}^2 a = \hat{n}(\hat{n}a) \tag{9.17}$$

and substituting the geometric product we have

$$a = \hat{n}(\hat{n} \cdot a + \hat{n} \wedge a). \tag{9.18}$$

From Fig. 9.2 it is obvious that

$$a = a_\perp + a_\parallel \tag{9.19}$$

and

$$a_\parallel = (\hat{n} \cdot a)\hat{n}. \tag{9.20}$$

Therefore,

$$a = \hat{n}(\hat{n} \wedge a) + a_\parallel$$

which means that

$$a_\perp = \hat{n}(\hat{n} \wedge a). \tag{9.21}$$

From Fig. 9.2 it is also obvious that

$$a' = a_\perp - a_\parallel. \tag{9.22}$$

Substituting Eqs. (9.20) and (9.21) in Eq. (9.22) we have

$$a' = \hat{n}(\hat{n} \wedge a) - (\hat{n} \cdot a)\hat{n}.$$

Reordering the products we get

$$a' = -(\hat{n} \cdot a)\hat{n} + \hat{n}(\hat{n} \wedge a). \tag{9.23}$$

Recalling from the previous chapter that vectors and bivectors anticommute, i.e. $aB = -Ba$, therefore,

$$\hat{n}(\hat{n} \wedge a) = -(\hat{n} \wedge a)\hat{n} \tag{9.24}$$

which means that we can write Eq. (9.23) as

$$a' = -(\hat{n} \cdot a)\hat{n} - (\hat{n} \wedge a)\hat{n} \tag{9.25}$$

which simplifies to

$$a' = -((\hat{n} \cdot a) + (\hat{n} \wedge a))\hat{n}. \tag{9.26}$$

The reason behind the above strategy was to create the geometric product $\hat{n}a$ within Eq. (9.26), which now simplifies to

$$a' = -\hat{n}a\hat{n}. \tag{9.27}$$

This sandwiching effect is reminiscent of the structure for using quaternions to rotate vectors, and will become even more obvious when we consider rotations. For the moment, let's test Eq. (9.27) with an example.

Figure 9.3 shows a vector a with reflection a' in the plane defined by e_{12}, with surface normal $\hat{n} = \mathrm{e}_3$.

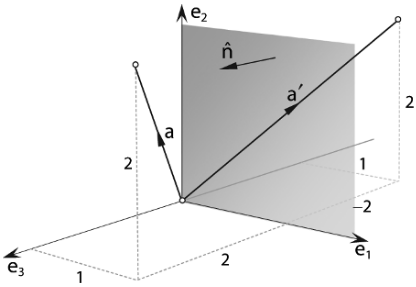

FIGURE 9.3.

If, for example,

$$a = e_1 + 2e_2 + 2e_3 \tag{9.28}$$

then, from Fig. 9.3 it is obvious that its reflection is

$$a' = e_1 + 2e_2 - 2e_3. \tag{9.29}$$

We can confirm this result using Eq. (9.27):

$$\begin{aligned} a' &= -\hat{n}a\hat{n} \\ &= -e_3(e_1 + 2e_2 + 2e_3)e_3 \\ &= -(e_{313} + 2e_{323} + 2e_{333}) \\ a' &= e_1 + 2e_2 - 2e_3. \end{aligned} \tag{9.30}$$

It is important to note that this reflection formula assumes that the line and plane intersect at the origin. Now let's investigate what happens when a bivector is reflected.

9.2.2 Reflecting bivectors

As a bivector is formed from a pair of vectors, its reflection must be formed from the reflections of its vectors, as shown in Fig. 9.4.

If $B = a \wedge b$ then its reflection is $B' = a' \wedge b'$ where a' and b' are the reflections of the original vectors. Algebraically, we proceed as follows using Eq. (9.27):

$$a' = -\hat{n}a\hat{n} \tag{9.31}$$

and

$$b' = -\hat{n}b\hat{n}. \tag{9.32}$$

Therefore,

$$\begin{aligned} B' &= (-\hat{n}a\hat{n}) \wedge (-\hat{n}b\hat{n}) \\ B' &= (\hat{n}a\hat{n}) \wedge (\hat{n}b\hat{n}). \end{aligned} \tag{9.33}$$

But we know that

$$B = a \wedge b = \frac{1}{2}(ab - ba) \tag{9.34}$$

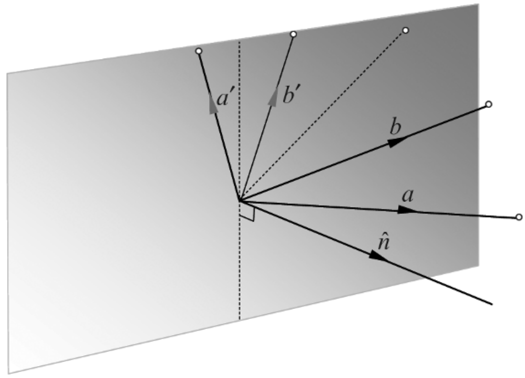

Figure 9.4.

therefore,

$$
\begin{aligned}
B' &= \frac{1}{2}(\hat{n}a\hat{n}\hat{n}b\hat{n} - \hat{n}b\hat{n}\hat{n}a\hat{n}) \\
&= \frac{1}{2}(\hat{n}ab\hat{n} - \hat{n}ba\hat{n}) \\
&= \frac{1}{2}\hat{n}(ab - ba)\hat{n} \\
B' &= \hat{n}B\hat{n}
\end{aligned}
\tag{9.35}
$$

which, apart from the minus sign, is identical to the equation for reflecting a vector.

Again, it's worth testing the action of Eq. (9.35) with an example.

Figure 9.5 shows a bivector $B = a \wedge b$ reflected in the plane defined by $\hat{n} = \mathrm{e}_3$.

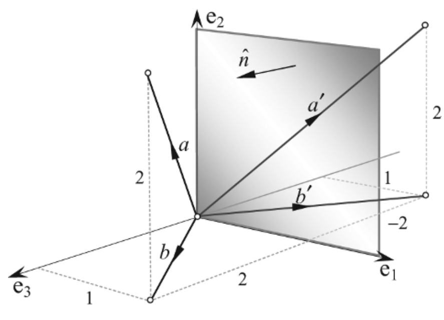

Figure 9.5.

If

$$a = \mathrm{e}_1 + 2\mathrm{e}_2 + 2\mathrm{e}_3 \tag{9.36}$$

and

$$b = \mathrm{e}_1 + 2\mathrm{e}_3 \tag{9.37}$$

then

$$B = (\mathrm{e}_1 + 2\mathrm{e}_2 + 2\mathrm{e}_3) \wedge (\mathrm{e}_1 + 2\mathrm{e}_3). \tag{9.38}$$

To save time evaluating outer products, the following *aide-mémoire* reminds us how to calculate the coefficients of the bivector terms.

	e_1	e_2	e_3
m	m_1	m_2	m_3
n	n_1	n_2	n_3
$m \wedge n$	$x = m_2n_3 - m_3n_2$	$y = m_3n_1 - m_1n_3$	$z = m_1n_2 - m_2n_1$
	e_{23}	e_{31}	e_{12}

therefore,

	e_1	e_2	e_3
a	1	2	2
b	1	0	2
B	4	0	-2
	e_{23}	e_{31}	e_{12}

and

$$B = -2e_{12} + 4e_{23}. \tag{9.39}$$

Now let's calculate the reflections of a and b.

From Eq. (9.30)

$$a' = e_1 + 2e_2 - 2e_3. \tag{9.40}$$

From Eq. (9.27)

$$\begin{aligned} b' &= -\hat{n}b\hat{n} \\ &= -e_3(e_1 + 2e_3)e_3 \\ b' &= e_1 - 2e_3. \end{aligned} \tag{9.41}$$

Therefore,

$$\begin{aligned} B' &= a' \wedge b' \\ &= (e_1 + 2e_2 - 2e_3) \wedge (e_1 - 2e_3) \end{aligned}$$

	e_1	e_2	e_3
a'	1	2	-2
b'	1	0	-2
B'	-4	0	-2
	e_{23}	e_{31}	e_{12}

and

$$B' = -2e_{12} - 4e_{23}. \tag{9.42}$$

Comparing Eq. (9.39) and Eq. (9.42) we see that the sign of the unit basis bivector e_{23} coefficient has flipped.

Alternatively, we can calculate B' using Eq. (9.35):

$$\begin{aligned} B' &= \hat{n}B\hat{n} \\ &= e_3(-2e_1 \wedge e_2 + 4e_2 \wedge e_3)e_3 \\ &= e_3(-2e_{12} + 4e_{23})e_3 \\ &= -2e_{3123} + 4e_{3233} \\ B' &= -2e_{12} - 4e_{23}. \end{aligned} \tag{9.43}$$

It should be obvious from Fig. 9.5 why the coefficients of e_{12} and e_{23} are negative, and why the coefficient of e_{31} is zero.

9.2.3 Reflecting trivectors

Finally, let's examine how trivectors behave when reflected in a mirror. Experience confirms that when we hold up our right hand in front of a mirror, we see a reflection identical to an image of our left hand – and vice versa. Therefore, a set of right-handed orthogonal axes should appear reflected as a left-handed set, as shown in Fig. 9.6. We can confirm this algebraically as follows:

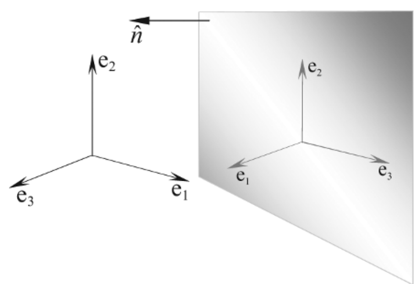

Figure 9.6.

Starting with the unit trivector, which is also a pseudoscalar:

$$I = e_1 \wedge e_2 \wedge e_3 \tag{9.44}$$

its reflection consists of three reflected unit vectors:

$$-\hat{n}e_1\hat{n} \quad -\hat{n}e_2\hat{n} \quad -\hat{n}e_3\hat{n} \tag{9.45}$$

which form the reflected unit trivector

$$(-\hat{n}e_1\hat{n}) \wedge (-\hat{n}e_2\hat{n}) \wedge (-\hat{n}e_3\hat{n}). \tag{9.46}$$

Expanding the first two terms of Eq. (9.46) using $a \wedge b = \frac{1}{2}(ab - ba)$ we have:

$$\begin{aligned}(-\hat{n}e_1\hat{n}) \wedge (-\hat{n}e_2\hat{n}) &= \frac{1}{2}((-\hat{n}e_1\hat{n})(-\hat{n}e_2\hat{n}) - (-\hat{n}e_2\hat{n})(-\hat{n}e_1\hat{n}))\\ &= \frac{1}{2}(\hat{n}e_{12}\hat{n} - \hat{n}e_{21}\hat{n})\\ &= \frac{1}{2}(\hat{n}e_{12}\hat{n} + \hat{n}e_{12}\hat{n})\end{aligned}$$

therefore,

$$(-\hat{n}e_1\hat{n}) \wedge (-\hat{n}e_2\hat{n}) = \hat{n}e_{12}\hat{n}. \tag{9.47}$$

Expanding the rest of Eq. (9.46) we have

$$(\hat{n}e_{12}\hat{n}) \wedge (-\hat{n}e_3\hat{n}) \tag{9.48}$$

and using $B \wedge a = \frac{1}{2}(Ba + aB)$ we have:

$$\begin{aligned}\hat{n}e_{12}\hat{n} \wedge (-\hat{n}e_3\hat{n}) &= \frac{1}{2}((\hat{n}e_{12}\hat{n}) \wedge (-\hat{n}e_3\hat{n}) + (-\hat{n}e_3\hat{n}) \wedge (\hat{n}e_{12}\hat{n}))\\ &= \frac{1}{2}((\hat{n}e_{12}\hat{n})(-\hat{n}e_3\hat{n}) + (-\hat{n}e_3\hat{n})(\hat{n}e_{12}\hat{n}))\\ &= \frac{1}{2}(-\hat{n}e_{123}\hat{n} - \hat{n}e_{312}\hat{n})\\ &= -\hat{n}e_{123}\hat{n}\end{aligned}$$

therefore,

$$(-\hat{n}e_1\hat{n}) \wedge (-\hat{n}e_2\hat{n}) \wedge (-\hat{n}e_3\hat{n}) = -\hat{n}I\hat{n} \tag{9.49}$$

where I is the pseudoscalar, which commutes with vectors. Therefore,

$$(-\hat{n}e_1\hat{n}) \wedge (-\hat{n}e_2\hat{n}) \wedge (-\hat{n}e_3\hat{n}) = -\hat{n}\hat{n}I = -I. \tag{9.50}$$

Equation (9.50) confirms that the sign of the trivector's reflection has switched from positive to negative, as predicted.

It is possible to show that the reflection of a general trivector behaves in exactly the same way.

9.3 Rotations

9.3.1 Rotating by double reflecting

The reason why we started with reflections is that they provide a way to rotate vectors. To illustrate this, consider Fig. 9.7(a) showing a mirror m and a vector a forming an angle α with the mirror. By the laws of reflection, a's reflection is b forming an equal angle α on the other side of the mirror. Now consider Fig. 9.7(b) which shows two superimposed mirrors m and n, where a's reflection in m is b, and b's reflection in n is c, which must coincide with a. We can reason that as the separating angle between the mirrors is 0°, the separating angle between a and its double reflection c is also 0°.

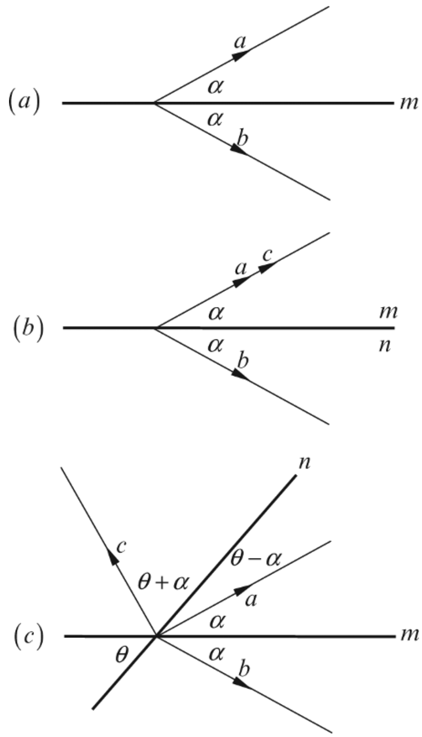

FIGURE 9.7.

Now consider Fig. 9.7(c) where the two mirrors m and n are separated by an angle θ. Vector a's reflection is still b, whilst b's reflection in n has rotated anticlockwise to c. By inspection, the angle of rotation between b and n is $\theta + \alpha$, which places c at an angle $\theta + \alpha$ on the opposite side of n. The interesting result about this double mirror arrangement is that the angle between a and c is 2θ, exactly double the angle between the mirrors. Now let's make a subtle substitution by representing the mirror m by a perpendicular unit vector $\hat{m}$, and mirror n by a perpendicular unit vector $\hat{n}$. This in no way changes the geometry, but allows us to describe the double reflection using GA. We will also define the plane supporting the mirrors by the outer product $\hat{m} \wedge \hat{n}$, as this represents the order of the mirrors.

Vector a's reflection b is given by

$$b = -\hat{m}a\hat{m} \tag{9.51}$$

which, in turn, is reflected in n to create c:

$$\begin{aligned} c &= -\hat{n}b\hat{n} \\ &= -\hat{n}(-\hat{m}a\hat{m})\hat{n} \\ c &= \hat{n}\hat{m}a\hat{m}\hat{n}. \end{aligned} \tag{9.52}$$

Substituting $R = \hat{n}\hat{m}$ in Eq. (9.52) we have

$$c = Ra\tilde{R} \tag{9.53}$$

where $\tilde{R} = \hat{m}\hat{n}$, the reverse product of R.

Although this description is based on an imaginary 2D scenario, it works in any number of dimensions, however, we are particularly interested in $\mathbb{R}^3$.

Figure 9.8 shows two mirrors m and n represented by their normal vectors $\hat{m}$ and $\hat{n}$, separated by an angle θ. Vector a's reflection is still b, and b's reflection in n is still c, and effectively, a has been rotated 2θ to c.

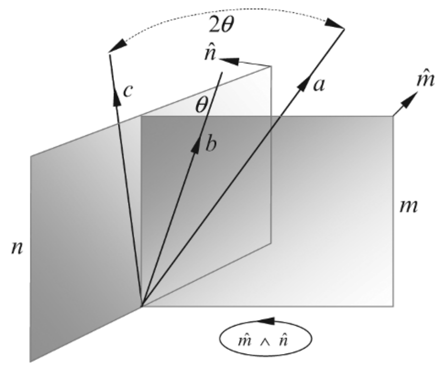

Figure 9.8.

To illustrate this double reflection, consider the two mirrors shown in Fig. 9.9 with normal vectors

$$\hat{m} = -e_3 \tag{9.54}$$

$$\hat{n} = -e_1. \tag{9.55}$$

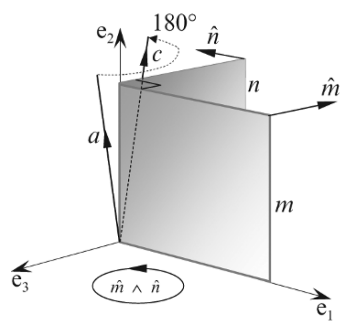

Figure 9.9.

If the vector a is

$$a = e_1 + e_2 + e_3 \tag{9.56}$$

then

$$
\begin{aligned}
R &= \hat{n}\hat{m} \\
R &= (-\mathrm{e}_1)(-\mathrm{e}_3) = \mathrm{e}_{13}
\end{aligned}
\tag{9.57}
$$

and

$$
\begin{aligned}
\tilde{R} &= \hat{m}\hat{n} \\
\tilde{R} &= (-\mathrm{e}_3)(-\mathrm{e}_1) = \mathrm{e}_{31}.
\end{aligned}
\tag{9.58}
$$

Therefore,

$$
\begin{aligned}
c &= Ra\tilde{R} \\
&= \mathrm{e}_{13}(\mathrm{e}_1 + \mathrm{e}_2 + \mathrm{e}_3)\mathrm{e}_{31} \\
&= (-\mathrm{e}_3 + \mathrm{e}_{132} + \mathrm{e}_1)\mathrm{e}_{31} \\
&= -\mathrm{e}_1 + \mathrm{e}_2 - \mathrm{e}_3 \\
c &= -\mathrm{e}_1 + \mathrm{e}_2 - \mathrm{e}_3
\end{aligned}
\tag{9.59}
$$

which is as expected.

We have seen from the previous examples that the mirrors and the angle of rotation are controlled by the bivector associated with the plane perpendicular to the mirrors, so let's drop the idea of mirrors and reflections and adopt the idea of rotating vectors using a bivector.

Figure 9.10 shows two vectors m and n forming the bivector $m \wedge n$ directed anticlockwise. As the internal angle of the bivector is 60°, vector a will be rotated 120° anticlockwise, which we can predict will be $\mathrm{e}_1 + \mathrm{e}_3$. Now let's construct the geometric products to perform the rotation.

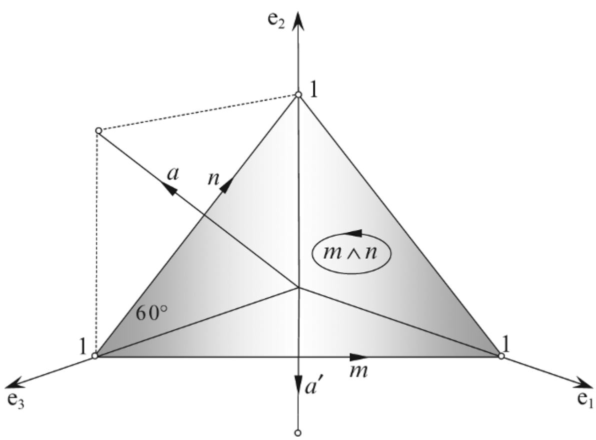

Figure 9.10.

First we define the unit vectors $\hat{m}$ and $\hat{n}$:

$$\hat{m} = \frac{1}{\sqrt{2}}(\mathrm{e}_1 - \mathrm{e}_3) \tag{9.60}$$

$$\hat{n} = \frac{1}{\sqrt{2}}(\mathrm{e}_2 - \mathrm{e}_3) \tag{9.61}$$

and

$$a = \mathrm{e}_2 + \mathrm{e}_3. \tag{9.62}$$

Therefore,

$$\begin{aligned} R &= \hat{n}\hat{m} \\ &= \frac{1}{2}(\mathrm{e}_2 - \mathrm{e}_3)(\mathrm{e}_1 - \mathrm{e}_3) \\ R &= \frac{1}{2}(\mathrm{e}_{21} - \mathrm{e}_{23} - \mathrm{e}_{31} + 1) \end{aligned} \tag{9.63}$$

and

$$\begin{aligned} \hat{n}\hat{m}a &= \frac{1}{2}(\mathrm{e}_{21} - \mathrm{e}_{23} - \mathrm{e}_{31} + 1)(\mathrm{e}_2 + \mathrm{e}_3) \\ &= \frac{1}{2}(\mathrm{e}_{212} + \mathrm{e}_{213} - \mathrm{e}_{232} - \mathrm{e}_{233} - \mathrm{e}_{312} - \mathrm{e}_{313} + \mathrm{e}_2 + \mathrm{e}_3) \\ &= \frac{1}{2}(-\mathrm{e}_1 - \mathrm{e}_{123} + \mathrm{e}_3 - \mathrm{e}_2 - \mathrm{e}_{123} + \mathrm{e}_1 + \mathrm{e}_2 + \mathrm{e}_3) \\ \hat{n}\hat{m}a &= \mathrm{e}_3 - \mathrm{e}_{123}. \end{aligned} \tag{9.64}$$

Now we compute the reverse product

$$\begin{aligned} \tilde{R} &= \hat{m}\hat{n} \\ &= \frac{1}{2}(\mathrm{e}_1 - \mathrm{e}_3)(\mathrm{e}_2 - \mathrm{e}_3) \\ \tilde{R} &= \frac{1}{2}(\mathrm{e}_{12} - \mathrm{e}_{13} - \mathrm{e}_{32} + 1) \end{aligned} \tag{9.65}$$

and

$$\begin{aligned} Ra\tilde{R} &= \frac{1}{2}(\mathrm{e}_3 - \mathrm{e}_{123})(\mathrm{e}_{12} - \mathrm{e}_{13} - \mathrm{e}_{32} + 1) \\ &= \frac{1}{2}(\mathrm{e}_{312} - \mathrm{e}_{313} - \mathrm{e}_{332} + \mathrm{e}_3 - \mathrm{e}_{12312} + \mathrm{e}_{12313} + \mathrm{e}_{12332} - \mathrm{e}_{123}) \\ Ra\tilde{R} &= \mathrm{e}_1 + \mathrm{e}_3 \end{aligned} \tag{9.66}$$

which confirms our prediction.

9.3.2 Rotors

Much of mathematics is about patterns, especially in formulae. One such pattern is about to emerge, and whoever discovered it deserves some sort of recognition. In chapter 3 we saw that a complex number is rotated in the complex plane by multiplying it by $e^{i\theta}$, which is equivalent to $\cos\theta + i\sin\theta$. We are about to discover that a multivector can also be rotated in a plane defined by a unit bivector, which plays a similar role to the imaginary i.

As sandwiching a multivector between R and $\tilde{R}$ results in a rotation, R is called a *rotor*, much like $e^{i\theta}$. What is strange, though, is that the bivector defining the plane is $\hat{m} \wedge \hat{n}$, whilst the rotor sequence is $R = \hat{n}\hat{m}$. The vectors are switched, and we will have to watch out for this.

We start the process as follows:

$$R = \hat{n}\hat{m} \tag{9.67}$$

which, using the geometric product, expands to

$$R = \hat{n} \cdot \hat{m} + \hat{n} \wedge \hat{m}. \tag{9.68}$$

But

$$\hat{n} \cdot \hat{m} = \|\hat{n}\|\|\hat{m}\|\cos\theta = \cos\theta \tag{9.69}$$

therefore,

$$R = \cos\theta + \hat{n} \wedge \hat{m}. \tag{9.70}$$

This is where we begin looking for a pattern. We already know that

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{9.71}$$

so could it be that R has a similar structure? To find the answer to this question consider the following expansion $(\hat{m} \wedge \hat{n})^2$ using

$$\hat{m} \wedge \hat{n} = \hat{m}\hat{n} - \hat{m} \cdot \hat{n}$$

and

$$\hat{m} \wedge \hat{n} = \hat{m} \cdot \hat{n} - \hat{n}\hat{m}. \tag{9.72}$$

Therefore,

$$\begin{aligned}
(\hat{m} \wedge \hat{n})^2 &= (\hat{m}\hat{n} - \hat{m} \cdot \hat{n})(\hat{m} \cdot \hat{n} - \hat{n}\hat{m}) \\
&= \hat{m}\hat{n}(\hat{m} \cdot \hat{n}) - \hat{m}\hat{n}^2\hat{m} - (\hat{m} \cdot \hat{n})^2 + \hat{n}\hat{m}(\hat{m} \cdot \hat{n}) \\
&= \hat{m} \cdot \hat{n}(\hat{m}\hat{n} + \hat{n}\hat{m}) - \hat{m}\hat{n}^2\hat{m} - (\hat{m} \cdot \hat{n})^2 \\
(\hat{m} \wedge \hat{n})^2 &= -\hat{m}^2\hat{n}^2 - (\hat{m} \cdot \hat{n})^2.
\end{aligned} \tag{9.73}$$

But as

$$(\hat{m} \cdot \hat{n})^2 = \|\hat{m}\|^2\|\hat{n}\|^2\cos^2\theta \tag{9.74}$$

and

$$\hat{m}^2\hat{n}^2 = \|\hat{m}\|^2\|\hat{n}\|^2 \tag{9.75}$$

then

$$\begin{aligned}(\hat{m} \wedge \hat{n})^2 &= -\|\hat{m}\|^2\|\hat{n}\|^2 - \|\hat{m}\|^2\|\hat{n}\|^2\cos^2\theta \\ &= -1 - \cos^2\theta \\ (\hat{m} \wedge \hat{n})^2 &= -\sin^2\theta. \end{aligned} \tag{9.76}$$

Note the imaginary feature of this result, which can be interpreted as follows:

$$\hat{m} \wedge \hat{n} = \hat{B}\sin\theta \tag{9.77}$$

where $\hat{B}$ is the unit bivector in the $\hat{m} \wedge \hat{n}$ plane where $\hat{B}^2 = -1$.

Similarly,

$$\hat{n} \wedge \hat{m} = -\hat{B}\sin\theta \tag{9.78}$$

which can be substituted in Eq. (9.70)

$$R = \cos\theta - \hat{B}\sin\theta \tag{9.79}$$

which has a similar structure to Eq. (9.71) apart from a negative imaginary component.

We can convert Eq. (9.79) to its exponential form as follows:

$$R = \exp(-\hat{B}\theta). \tag{9.80}$$

Remembering that the double reflection technique doubles the angle of rotation, we must compensate for this by halving the original angle:

$$R = \exp(-\hat{B}\theta/2). \tag{9.81}$$

Similarly,

$$\tilde{R} = \exp(\hat{B}\theta/2) \tag{9.82}$$

which enables us to write the result

$$c = e^{-\hat{B}\theta/2} a e^{\hat{B}\theta/2}. \tag{9.83}$$

More generally, the vector a is rotated through an angle θ in the plane defined by the unit bivector $\hat{B}$ using

$$a' = e^{-\hat{B}\theta/2} a e^{\hat{B}\theta/2}. \tag{9.84}$$

So now we have two ways of visualizing a rotor: either as a bivector or as an exponential, which is readily represented as

$$\exp(-\hat{B}\theta/2) = \cos(\theta/2) - \hat{B}\sin(\theta/2) \tag{9.85}$$

Therefore, we can rewrite Eq. (9.84) as

$$a' = (\cos(\theta/2) - \hat{B}\sin(\theta/2))a(\cos(\theta/2) + \hat{B}\sin(\theta/2)). \tag{9.86}$$

Let's test Eq. (9.86) with an example.

Figure 9.11 shows two vectors m and n forming a bivector $m \wedge n$. The angle of rotation is 120°, which means that the vector $a = e_2 + e_3$ will be rotated to $a' = e_1 + e_3$ as shown in Fig. 9.12.

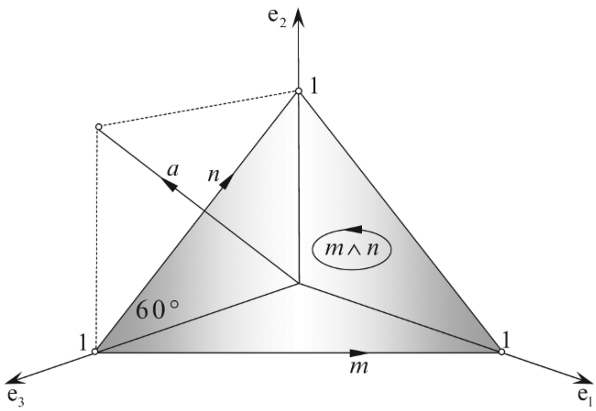

Figure 9.11.

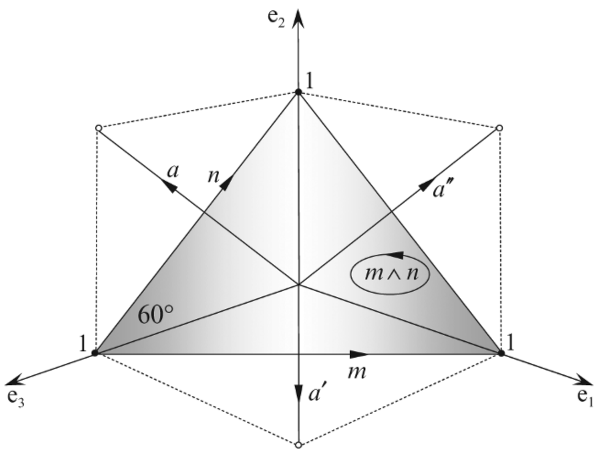

Figure 9.12.

Using Eq. (9.86) we have

$$a' = (\cos 60^\circ - \hat{B}\sin 60^\circ)(e_2 + e_3)(\cos 60^\circ + \hat{B}\sin 60^\circ)$$
$$= \left(\frac{1}{2} - \hat{B}\sqrt{3}/2\right)(e_2 + e_3)\left(\frac{1}{2} + \hat{B}\sqrt{3}/2\right)$$
$$a' = \frac{1}{4}\big(1 - \hat{B}\sqrt{3}\big)(e_2 + e_3)\big(1 + \hat{B}\sqrt{3}\big). \tag{9.87}$$

Given that

$$m = e_1 - e_3 \tag{9.88}$$

and

$$n = e_2 - e_3 \tag{9.89}$$

we evaluate the outer product using our *aide-mémoire*.

	e_1	e_2	e_3
m	1	0	-1
n	0	1	-1
$m \wedge n$	$x = 1$	$y = 1$	$z = 1$
	e_{23}	e_{31}	e_{12}

where

$$m \wedge n = e_{23} + e_{31} + e_{12}. \tag{9.90}$$

But we require a unit bivector, which makes

$$\hat{B} = \frac{1}{\sqrt{3}}(e_{23} + e_{31} + e_{12}). \tag{9.91}$$

Therefore,

$$\begin{aligned}
a' &= \frac{1}{4}(1 - e_{23} - e_{31} - e_{12})(e_2 + e_3)(1 + e_{23} + e_{31} + e_{12}) \\
&= \frac{1}{4}(e_2 + e_3 - e_{232} - e_{233} - e_{312} - e_{313} - e_{122} - e_{123})(1 + e_{23} + e_{31} + e_{12}) \\
&= \frac{1}{2}(e_3 - e_{123})(1 + e_{23} + e_{31} + e_{12}) \\
&= \frac{1}{2}(e_3 + e_{323} + e_{331} + e_{312} - e_{123} - e_{12323} - e_{12331} - e_{12312}) \\
a' &= e_1 + e_3
\end{aligned} \tag{9.92}$$

which is what we predicted.

9.3.3 Rotor matrix

Another way of implementing a rotor is using a matrix, which is created as follows. We begin with the bivector defining the plane $m \wedge n$, about which the rotation is effected, where

$$m = m_1 e_1 + m_2 e_2 + m_3 e_3 \tag{9.93}$$

and

$$n = n_1 e_1 + n_2 e_2 + n_3 e_3. \tag{9.94}$$

Notice in the following how the bivectors are associated with their perpendicular axes.

Therefore,

$$\begin{aligned}
R &= mn \\
R &= w + x e_{23} + y e_{31} + z e_{12}
\end{aligned} \tag{9.95}$$

and

$$\tilde{R} = nm$$
$$\tilde{R} = w - x\mathrm{e}_{23} - y\mathrm{e}_{31} - z\mathrm{e}_{12} \tag{9.96}$$

where

$$w^2 + x^2 + y^2 + z^2 = 1 \tag{9.97}$$

and x, y and z are computed using the outer product *aide-mémoire*.

We derive the matrix $[\![R]\!]$ representing $Ra\tilde{R}$ by expanding the individual elements for $R\mathrm{e}_1\tilde{R}, R\mathrm{e}_2\tilde{R}$ and $R\mathrm{e}_3\tilde{R}$:

$$\begin{aligned} R\mathrm{e}_1\tilde{R} &= (w + x\mathrm{e}_{23} + y\mathrm{e}_{31} + z\mathrm{e}_{12})\mathrm{e}_1(w - x\mathrm{e}_{23} - y\mathrm{e}_{31} - z\mathrm{e}_{12}) \\ &= (w\mathrm{e}_1 + x\mathrm{e}_{123} + y\mathrm{e}_3 - z\mathrm{e}_2)(w - x\mathrm{e}_{23} - y\mathrm{e}_{31} - z\mathrm{e}_{12}) \\ R\mathrm{e}_1\tilde{R} &= (w^2 + x^2 - y^2 - z^2)\mathrm{e}_1 + 2(-wz + xy)\mathrm{e}_2 + 2(wy + xz)\mathrm{e}_3. \end{aligned} \tag{9.98}$$

The first term is simplified by substituting

$$w^2 + x^2 = 1 - y^2 - z^2 \tag{9.99}$$
$$R\mathrm{e}_1\tilde{R} = (1 - 2(y^2 + z^2))\mathrm{e}_1 + 2(xy - wz)\mathrm{e}_2 + 2(xz + wy)\mathrm{e}_3. \tag{9.100}$$

Next is $R\mathrm{e}_2\tilde{R}$

$$\begin{aligned} R\mathrm{e}_2\tilde{R} &= (w + x\mathrm{e}_{23} + y\mathrm{e}_{31} + z\mathrm{e}_{12})\mathrm{e}_2(w - x\mathrm{e}_{23} - y\mathrm{e}_{31} - z\mathrm{e}_{12}) \\ &= (w\mathrm{e}_2 - x\mathrm{e}_3 + y\mathrm{e}_{123} + z\mathrm{e}_1)(w - x\mathrm{e}_{23} - y\mathrm{e}_{31} - z\mathrm{e}_{12}) \\ R\mathrm{e}_2\tilde{R} &= 2(xy + wz)\mathrm{e}_1 + (w^2 - x^2 + y^2 - z^2)\mathrm{e}_2 + 2(yz - wx)\mathrm{e}_3. \end{aligned} \tag{9.101}$$

Substituting

$$w^2 + y^2 = 1 - x^2 - z^2 \tag{9.102}$$
$$R\mathrm{e}_2\tilde{R} = 2(xy + wz)\mathrm{e}_1 + (1 - 2(x^2 + z^2))\mathrm{e}_2 + 2(yz - wx)\mathrm{e}_3. \tag{9.103}$$

Finally $R\mathrm{e}_3\tilde{R}$

$$\begin{aligned} R\mathrm{e}_3\tilde{R} &= (w + x\mathrm{e}_{23} + y\mathrm{e}_{31} + z\mathrm{e}_{12})\mathrm{e}_3(w - x\mathrm{e}_{23} - y\mathrm{e}_{31} - z\mathrm{e}_{12}) \\ &= (w\mathrm{e}_3 + x\mathrm{e}_2 - y\mathrm{e}_1 + z\mathrm{e}_{123})(w - x\mathrm{e}_{23} - y\mathrm{e}_{31} - z\mathrm{e}_{12}) \\ R\mathrm{e}_3\tilde{R} &= 2(xz - wy)\mathrm{e}_1 + 2(yz + wx)\mathrm{e}_2 + (w^2 - x^2 - y^2 + z^2)\mathrm{e}_3. \end{aligned} \tag{9.104}$$

Substituting

$$w^2 + z^2 = 1 - x^2 - y^2 \tag{9.105}$$
$$R\mathrm{e}_3\tilde{R} = 2(xz - wy)\mathrm{e}_1 + 2(yz + wx)\mathrm{e}_2 + (1 - 2(x^2 + y^2))\mathrm{e}_3. \tag{9.106}$$

Therefore, the final matrix is

$$[\![R]\!] = \begin{bmatrix} 1-2(y^2+z^2) & 2(xy-wz) & 2(xz+wy) \\ 2(xy+wz) & 1-2(x^2+z^2) & 2(yz-wx) \\ 2(xz-wy) & 2(yz+wx) & 1-2(x^2+y^2) \end{bmatrix} \tag{9.107}$$

which is also used in its transposed form. Notice that it is identical to the matrix representing a quaternion. (Eq. (9.52)).

Let's illustrate this matrix using the previous example.

Figure 9.11 shows the bivector $m \wedge n$ which will be used to rotate the vector a through an angle 120°.

Given the following vectors:

$$m = \mathrm{e}_1 - \mathrm{e}_3 \tag{9.108}$$

$$n = \mathrm{e}_2 - \mathrm{e}_3 \tag{9.109}$$

$$a = \mathrm{e}_2 + \mathrm{e}_3 \tag{9.110}$$

then

$$m \cdot n = (\mathrm{e}_1 - \mathrm{e}_3) \cdot (\mathrm{e}_2 - \mathrm{e}_3) = 1 \tag{9.111}$$

and

$$m \wedge n = \mathrm{e}_{23} + \mathrm{e}_{31} + \mathrm{e}_{12}. \tag{9.112}$$

Therefore,

$$\begin{aligned} R &= mn \\ &= m \cdot n + m \wedge n \\ R &= 1 + \mathrm{e}_{23} + \mathrm{e}_{31} + \mathrm{e}_{12}. \end{aligned} \tag{9.113}$$

But this has to be normalized, which makes the scaling factor $\frac{1}{2}$ and in matrix form using Eq. (9.107) becomes

$$[\![R]\!] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \tag{9.114}$$

Multiplying vector a by this matrix we have

$$a' = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \tag{9.115}$$

which shows that a' is now pointing to $(1, 0, 1)$.

If a' is subjected to the same rotation we obtain

$$a'' = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \tag{9.116}$$

which shows that a'' is now pointing to $(1, 1, 0)$.

If a'' is subjected to the same rotation we should return to the original vector:

$$a = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \tag{9.117}$$

which does reassuringly, brings us back to the original vector a. These rotations are shown in Fig. 9.12.

This seems too good to be true! So let's test it with another example. This time, let's reverse the bivector as shown in Fig. 9.13, where the bivector creates a clockwise rotation of 45°.

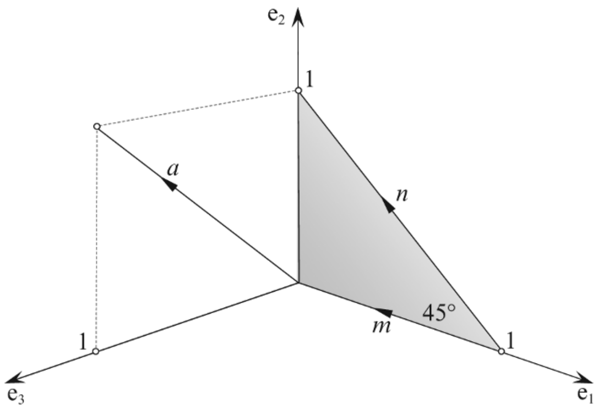

Figure 9.13.

Given the following vectors:

$$m = -e_1 \tag{9.118}$$

$$n = -e_1 + e_2 \tag{9.119}$$

$$a = e_2 + e_3. \tag{9.120}$$

	e_1	e_2	e_3
m	-1	0	0
n	-1	1	0
$m \wedge n$	$x = 0$	$y = 0$	$z = -1$
	e_{23}	e_{31}	e_{12}

Therefore,

$$m \cdot n = (-e_1) \cdot (-e_1 + e_2) = 1 \tag{9.121}$$

and

$$m \wedge n = -e_{12}. \tag{9.122}$$

Therefore,

$$R = mn = m \cdot n + m \wedge n$$
$$R = 1 - \mathrm{e}_{12}. \tag{9.123}$$

But this has to be normalized, which makes the scaling factor $1/\sqrt{2}$ and in matrix form using Eq. (9.107) becomes

$$[\![R]\!] = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{9.124}$$

Multiplying vector a by this matrix we have

$$a' = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \tag{9.125}$$

which shows that a' is now pointing to $(1, 0, 1)$.

If a' is subjected to the same rotation we obtain

$$a'' = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \tag{9.126}$$

which shows that a'' is now pointing to $(0, -1, 1)$.

If a'' is subjected to the same rotation we obtain

$$a''' = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}. \tag{9.127}$$

Finally, If a''' is subjected to the same rotation we obtain

$$a = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}. \tag{9.128}$$

which again reassuringly, brings us back to the original vector a. These rotations are shown in Fig. 9.14.

So once again, we see how close GA is to the algebra discovered by Hamilton. And even though Grassmann had discovered many of the ideas outlined above, he was unable to persuade mathematicians of the day to adopt his algebra, and it was left to Clifford to unify both men's work. Furthermore, it has taken several decades for GA to be applied seriously to science and physics, and only during the past decade has GA found application within computer graphics.

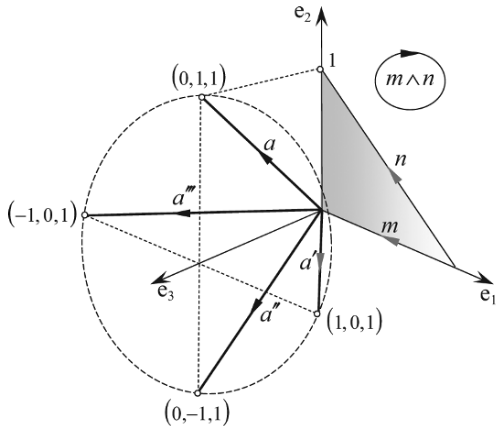

FIGURE 9.14.

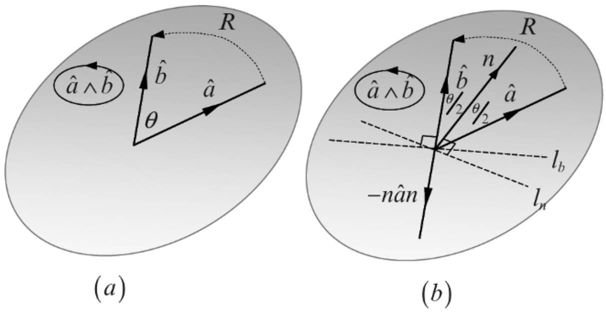

FIGURE 9.15.

9.3.4 Building rotors

In the previous section we discovered how to rotate a vector using a bivector. In this section we investigate how to derive the rotor that rotates one vector into another vector. Figure 9.15(a) illustrates the problem, where we see two vectors a and b in the plane defined by $a \wedge b$, and the objective is to find a rotor R that rotates a into b.

To do this, we bisect the angle θ between the two vectors and create a mid-vector n using

$$n = \frac{\hat{a} + \hat{b}}{\|\hat{a} + \hat{b}\|}. \tag{9.129}$$

Perpendicular to vector n is a reflector l_n which is used to create a reflection of $\hat{a}$: $-n\hat{a}n$, as shown in Fig. 9.15(b), which must equal $-\hat{b}$. But rather than use

$$\hat{b} = n\hat{a}n \tag{9.130}$$

we create a reflection about the line l_b perpendicular to $\hat{b}$:

$$\begin{aligned}
\hat{b} &= -\hat{b}(-n\hat{a}n)\hat{b} \\
\hat{b} &= \hat{b}n\hat{a}n\hat{b}
\end{aligned} \tag{9.131}$$

which enables us to define $\hat{b}n$ as the rotor:

$$R = \hat{b}n \tag{9.132}$$

and

$$\hat{b} = R\hat{a}\tilde{R}. \tag{9.133}$$

We now have a geometric product which expands to

$$\begin{aligned}
R &= \hat{b}n \\
&= \hat{b}\left(\frac{\hat{a}+\hat{b}}{\|\hat{a}+\hat{b}\|}\right) \\
R &= \frac{1+\hat{b}\hat{a}}{\|\hat{a}+\hat{b}\|}.
\end{aligned} \tag{9.134}$$

We can simply the denominator to avoid unnecessary arithmetic by the following subterfuge. Figure 9.16 shows part of the geometry associated with vectors $\hat{a}$ and $\hat{b}$, where we see that $d = \cos(\theta/2)$, which means that using the half-angle identity

$$\cos(\theta/2) = \sqrt{\frac{1+\cos\theta}{2}} \tag{9.135}$$

we have

$$\begin{aligned}
\|\hat{a}+\hat{b}\| &= 2d \\
&= 2\cos(\theta/2) \\
\|\hat{a}+\hat{b}\| &= \sqrt{2(1+\cos\theta)}.
\end{aligned} \tag{9.136}$$

This permits us to substitute

$$\cos\theta = \hat{a}\cdot\hat{b} \tag{9.137}$$

and

$$R = \frac{1+\hat{b}\hat{a}}{\sqrt{2(1+\hat{b}\cdot\hat{a})}} \tag{9.138}$$

which has the effect of rotating $\hat{a}$ to $\hat{b}$.

In chapter 3 we showed that a complex number is rotated through an angle θ in the complex plane using

$$z' = ze^{i\theta}. \tag{9.139}$$

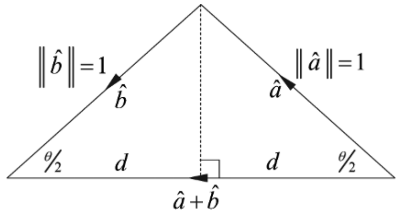

FIGURE 9.16.

Could it be that in the above scenario that $\hat{a}$ can be rotated into $\hat{b}$ using a similar formula? In fact, the answer is "yes", and we can prove it as follows. Using Eq. (9.138) we have

$$R\hat{a} = \frac{(1+\hat{b}\hat{a})\hat{a}}{\sqrt{2(1+\hat{b}\cdot\hat{a})}}$$

$$R\hat{a} = \frac{\hat{a}+\hat{b}}{\sqrt{2(1+\hat{b}\cdot\hat{a})}} \tag{9.140}$$

and

$$\tilde{R} = \frac{1+\hat{a}\hat{b}}{\sqrt{2(1+\hat{b}\cdot\hat{a})}} \tag{9.141}$$

therefore

$$\hat{a}\tilde{R} = \frac{\hat{a}(1+\hat{a}\hat{b})}{\sqrt{2(1+\hat{b}\cdot\hat{a})}}$$

$$\hat{a}\tilde{R} = \frac{\hat{a}+\hat{b}}{\sqrt{2(1+\hat{b}\cdot\hat{a})}} \tag{9.142}$$

and

$$R\hat{a} = \hat{a}\tilde{R}. \tag{9.143}$$

Equation (9.143) confirms that pre-multiplying a vector by a rotor is equivalent to post-multiplying it by the rotor's inverse, which leads to

$$\begin{aligned} R^2\hat{a} &= \hat{a}\tilde{R}^2 \\ &= R\hat{a}\tilde{R} \\ R^2\hat{a} &= \hat{b}. \end{aligned} \tag{9.144}$$

But we showed above in Eq. (9.81) that

$$R = \exp(-\hat{B}\theta/2) \tag{9.145}$$

where $\hat{B}$ is the unit bivector representing the plane of rotation. Therefore, applying the rules of exponentiation to Eq. (9.144) we have

$$\begin{aligned} R^2 &= \left(e^{-\hat{B}\theta/2}\right)^2 \\ &= e^{-\hat{B}\theta} \\ R^2 &= \exp(-\hat{B}\theta). \end{aligned} \tag{9.146}$$

From Eq. (9.144) we have

$$\begin{aligned} \hat{b} &= R^2\hat{a} \\ &= \exp(-\hat{B}\theta)\hat{a} \\ &= \hat{a}e^{\hat{B}\theta} \\ \hat{b} &= \hat{a}(\cos\theta + \hat{B}\sin\theta). \end{aligned} \tag{9.147}$$

Let's illustrate this process with an example.

Figure 9.17 shows two vectors

$$\hat{a} = \mathrm{e}_1 \quad \hat{b} = \mathrm{e}_2 \tag{9.148}$$

which belong to the plane defined by $\hat{B} = \mathrm{e}_{12}$. The separating angle is $\pi/2$ radians. Using Eq. (9.147) we have

$$\begin{aligned} \hat{b} &= \mathrm{e}_1 e^{\mathrm{e}_{12}\pi/2} \\ &= \mathrm{e}_1(\cos(\pi/2) + \mathrm{e}_{12}\sin(\pi/2)) \\ \hat{b} &= \mathrm{e}_1\mathrm{e}_{12} = \mathrm{e}_2 \end{aligned} \tag{9.149}$$

which is correct.

Now let's try another combination, as shown in Figure 9.18 using vectors

$$\hat{a} = \mathrm{e}_1 \quad \hat{b} = -\mathrm{e}_1. \tag{9.150}$$

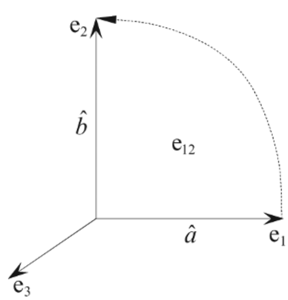

Figure 9.17.

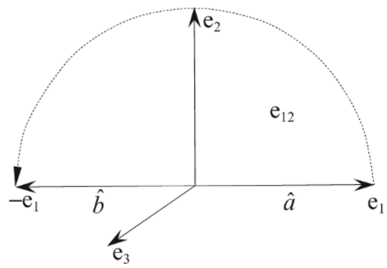

FIGURE 9.18.

Using Eq. (9.147) we have

$$\begin{aligned}\hat{b} &= e_1 e^{e_{12}\pi}\\ &= e_1(\cos\pi + e_{12}\sin\pi)\\ \hat{b} &= -e_1\end{aligned} \tag{9.151}$$

which is correct.

9.3.5 Interpolating rotors

Interpolating scalars is a trivial exercise and is readily implemented using the linear interpolant

$$s = s_1(1-\lambda) + s_2\lambda \quad 0 \le \lambda \le 1. \tag{9.152}$$

And there is no reason why we cannot use the same equation for interpolating two vectors:

$$v = v_1(1-\lambda) + v_2\lambda \quad 0 \le \lambda \le 1 \tag{9.153}$$

apart from the fact that the magnitude of the interpolated vector is not preserved, and could collapse to zero under some conditions. To overcome this problem a *slerp* (spherical linear interpolant) [8] is used

$$v = \frac{\sin((1-\lambda)\theta)}{\sin\theta}v_1 + \frac{\sin(\lambda\theta)}{\sin\theta}v_2 \qquad 0 \le \lambda \le 1 \tag{9.154}$$

where θ is the angle between two vectors or quaternions, which preserves the integrity of their magnitude during the interpolation.

Fortunately, this slerp can also be used to interpolate between two rotors as follows:

$$R = \frac{\sin((1-\lambda)\theta/2)}{\sin(\theta/2)}R_1 + \frac{\sin(\lambda\theta/2)}{\sin(\theta/2)}R_2 \quad 0 \le \lambda \le 1 \tag{9.155}$$

where θ is the angle of rotation. An example will quickly reveal the action of Eq. (9.155).

Figure 9.19 shows a vector $a = e_1$ and a plane of rotation defined by the bivector e_{12}. We will now design an interpolant that will interpolate between two rotors using the scalar λ, where $0 \le \lambda \le 1$.

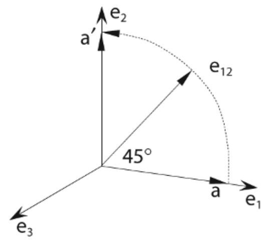

FIGURE 9.19.

We begin by defining the two rotors R_1 and R_2, where R_1 is the rotor locating a and R_2 rotates a to a'. Using Eq. (9.85)

$$R = \cos(\theta/2) - \hat{B}\sin(\theta/2) \tag{9.156}$$

then

$$\begin{aligned} R_1 &= \cos 0^\circ - \mathrm{e}_{12}\sin 0^\circ \\ R_1 &= 1 \end{aligned} \tag{9.157}$$

and

$$\begin{aligned} R_2 &= \cos 45^\circ - \mathrm{e}_{12}\sin 45^\circ \\ R_2 &= \sqrt{2}/2(1 - \mathrm{e}_{12}). \end{aligned} \tag{9.158}$$

Therefore, R_1 and R_2 can be substituted in Eq. (9.155) to produce

$$R = \frac{\sin((1-\lambda)45^\circ)}{\sin(45^\circ)} + \frac{\sin(\lambda 45^\circ)}{\sin(45^\circ)}\frac{\sqrt{2}(1-\mathrm{e}_{12})}{2}. \tag{9.159}$$

We can see from Eq. (9.159) that when $\lambda = 0$, $R_0 = 1$, and when $\lambda = 1$,

$$R_1 = \sqrt{2}/2(1 - \mathrm{e}_{12}). \tag{9.160}$$

Using R_0 to rotate vector a we have

$$a' = R_0 a \tilde{R}_0 = a \tag{9.161}$$

which is expected.

Using R_1 to rotate a we have

$$\begin{aligned} a' &= R_1 a \tilde{R}_1 \\ &= \sqrt{2}/2(1-\mathrm{e}_{12})a\sqrt{2}/2(1+\mathrm{e}_{12}) \\ &= \frac{1}{2}(1-\mathrm{e}_{12})\mathrm{e}_1(1+\mathrm{e}_{12}) \\ a' &= \frac{1}{2}(\mathrm{e}_1+\mathrm{e}_2)(1+\mathrm{e}_{12}) \end{aligned} \tag{9.162}$$

and

$$\begin{aligned} a' &= \frac{1}{2}(\mathrm{e}_1+\mathrm{e}_2+\mathrm{e}_2-\mathrm{e}_1) \\ a' &= \mathrm{e}_2 \end{aligned} \tag{9.163}$$

which is also correct.

Now let's compute a half-way rotor when $\lambda = \frac{1}{2}$.

$$\begin{aligned} R_{\frac{1}{2}} &= \frac{\sin(45^\circ/2)}{\sin(45^\circ)} + \frac{\sin(45^\circ/2)}{\sin(45^\circ)}\frac{\sqrt{2}(1-\mathrm{e}_{12})}{2} \\ &= \frac{\sin(45^\circ/2)}{\sin(45^\circ)}\left(1+\frac{\sqrt{2}(1-\mathrm{e}_{12})}{2}\right) \\ R_{\frac{1}{2}} &\simeq 0.9238 - 0.3827\mathrm{e}_{12}. \end{aligned} \tag{9.164}$$

Using $R_{\frac{1}{2}}$ to rotate a we have

$$a' \simeq (0.9239 - 0.3827\mathrm{e}_{12})\mathrm{e}_1(0.9239 + 0.3827\mathrm{e}_{12})$$

Table 9.1

Action	Algebra
Reflecting a vector	$a' = -\hat{n}a\hat{n}$
Reflecting a bivector	$B' = \hat{n}B\hat{n}$
Reflecting a trivector	$T' = -\hat{n}T\hat{n}$
Rotating a vector	$a' = Ra\tilde{R}$
Rotor	$R = \exp(-\hat{B}\theta/2) = \cos(\theta/2) - \hat{B}\sin(\theta/2)$
Rotor matrix	$[\![R]\!] = \begin{bmatrix} 1-2(y^2+z^2) & 2(xy-wz) & 2(xz+wy) \\ 2(xy+wz) & 1-2(x^2+z^2) & 2(yz-wx) \\ 2(xz-wy) & 2(yz+wx) & 1-2(x^2+y^2) \end{bmatrix}$ where $w^2+z^2 = 1-x^2-y^2$.
Interpolating between two rotors	$R = \frac{\sin((1-\lambda)\theta/2)}{\sin(\theta/2)}R_1 + \frac{\sin(\lambda\theta/2)}{\sin(\theta/2)}R_2 \quad 0 \le \lambda \le 1$

and

$$a' \simeq 0.7071\mathrm{e}_1 + 0.7071\mathrm{e}_2 \tag{9.165}$$

which shows that a has been rotated 45° anticlockwise.

Hopefully, the reader is convinced that the interpolant works for all other values of λ!

9.4 Summary

Reflections and rotations are one of GA's strengths and it is interesting to discover a notation that does not require an explicit matrix transform, even though one is lurking just beneath the surface. Finally, one must be extremely careful to ensure that the correct sign is used for the different blades. Table 9.1 summarizes most of the important formulae associated with reflections and rotations.