8 The Geometric Product

8.1 Introduction

Whenever we attempt to learn something completely new, all sorts of mental barriers are raised, especially if the subject matter appears foreign or irrational. This often happens when we learn a new language and wonder why its syntax differs from our own native language. Mathematics is a minefield for such experiences, and as we explore the world of GA, don't be surprised if you feel uncomfortable or bewildered by its structure and ideas.

If this is the first time you have studied GA this chapter will be both challenging and exciting. It will be challenging not because it is difficult, but because there is so much to remember as the algebra unfolds. For example, some elements of the algebra commute, whilst others anti-commute. Some functions are symmetric, whilst others are antisymmetric. Certain conditions arise with orthogonal elements, and others arise with parallel elements, and there is a sense of being overwhelmed by a world of axioms, rules and special conditions. It will be exciting as GA is extremely rich in new concepts that will draw you into its strange world of symbol manipulation that correspond with the world of space.

Basically, GA allows us manipulate scalars, vectors, areas and volumes using a simple and consistent notation. Combinations of such elements are called *multivectors*, which may be added, subtracted and multiplied. Adding or subtracting multivectors create no problems, as we simply add or subtract like elements. What is strange, however, are the products. In vector algebra there are only two products to consider: the inner product and the outer vector product. The inner product creates a scalar, whilst the vector product creates another vector normal to the original vectors. From these products arise all sorts of triple products such as:

$$(a \cdot b)c \quad a \cdot (b \times c) \quad (a \times b) \times c \quad a \times (b \times c) \tag{8.1}$$

which are easy to interpret and visualize. On the other hand, GA employs a new product called the *geometric product*, which operates upon multivectors containing scalars, vectors, areas and volumes. Visualizing these products can be difficult. For example, how should we visualize the product of two areas, or the product of a vector and a volume, or even the product of two volumes? These are new concepts and take some getting used to. What is even more strange is that the algebra involves imaginary elements, which, unlike the reasonably familiar i, do not always commute with their neighbor. An unusual, but useful, feature of GA is that multivectors can be divided by vectors, which is something conventional vector algebra is unable to do.

The problem now is how best to reveal this algebra? The approach taken in this chapter is to split GA's features in two: the first part explores GA in 2D space followed by 3D space. But remember, the algebra can be applied to space of any number of dimensions. In the first part we look at vectors, bivectors (areas), pseudoscalars, multivectors and their products in an $\mathbb{R}^2$ context. We also discover how these products give rise to rotations, much in the same way that complex numbers can be rotated. And because there is a close relationship between GA and complex numbers, we look at how it is possible to move between the two systems. In the second part we look at vectors, bivectors, trivectors (volumes), pseudoscalars, multivectors and their products in an $\mathbb{R}^3$ context. We also discover how simple rotations arise from these products, and the close relationship between GA and quaternions.

This said, let's begin this journey with a description of Clifford's geometric product.

8.2 Clifford's definition of the geometric product

Clifford defined the geometric product of two vectors a and b as

$$ab = a \cdot b + a \wedge b \tag{8.2}$$

which is the sum of a scalar and a bivector. Now there is always a good reason why such definitions are made, and it is far from arbitrary. In order to develop this new product we start by defining the axioms associated with the algebra. These comprise an associative axiom, distributive axiom, and a definition of a modulus.

For the moment, let's put to one side what we have discovered about the outer product, and see whether its properties emerge from the following axioms.

Associative axiom

$$a(bc) = (ab)c = abc \tag{8.3}$$

$$(\lambda a)b = \lambda(ab) = \lambda ab \quad [\lambda \in \mathbb{R}]. \tag{8.4}$$

Distributive axiom

$$a(b + c) = ab + ac \tag{8.5}$$

and

$$(b + c)a = ba + ca. \tag{8.6}$$

Modulus

$$a^2 = \pm\|a\|^2. \tag{8.7}$$

From these axioms we can derive the meaning of the product ab. Just in case the product is antisymmetric, we pay particular attention to the order of vectors.

We begin with two vectors a and b and represent their sum as

$$c = a + b. \tag{8.8}$$

Therefore,

$$c^2 = (a + b)^2 \tag{8.9}$$

and

$$c^2 = a^2 + b^2 + ab + ba. \tag{8.10}$$

To simplify this relationship we investigate how Eq. (8.10) behaves when vectors a and b are orthogonal, linearly dependent and linearly independent.

8.2.1 Orthogonal vectors

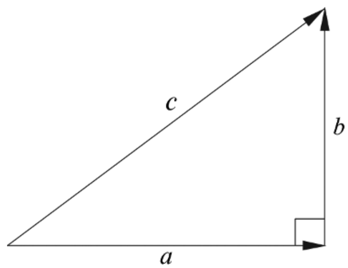

FIGURE 8.1.

With reference to Fig. 8.1, when

$$b \perp a$$

then

$$\|c\|^2 = \|a\|^2 + \|b\|^2. \tag{8.11}$$

Invoking the modulus axiom, we have

$$c^2 = a^2 + b^2 \tag{8.12}$$

which implies that in Eq. (8.10)

$$ab + ba = 0 \tag{8.13}$$

or

$$ab = -ba \tag{8.14}$$

which confirms that orthogonal vectors anticommute.

8.2.2 Linearly dependent vectors

With reference to Fig. 8.2, when

$$b \parallel a \quad \text{and} \quad b = \lambda a \quad \text{where} \quad [\lambda \in \mathbb{R}] \tag{8.15}$$

$$ab = a\lambda a = \lambda aa = ba \tag{8.16}$$

which confirms that linearly dependent vectors commute.

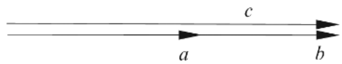

Figure 8.2.

Invoking the modulus axiom we have

$$\lambda aa = \lambda a^2 = \lambda \|a\|^2 \tag{8.17}$$

which is a scalar.

8.2.3 Linearly independent vectors

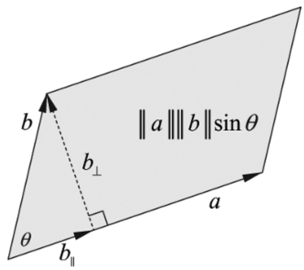

Figure 8.3.

With reference to Fig. 8.3

$$b = b_\parallel + b_\perp. \tag{8.18}$$

Therefore, we can write

$$ab = a(b_\parallel + b_\perp) \tag{8.19}$$

and

$$ab = ab_\parallel + ab_\perp. \tag{8.20}$$

Let's examine the RHS products of Eq. (8.20):
$ab_\parallel$: As a and $b_\parallel$ are linearly dependent, $ab_\parallel$ is a scalar. Furthermore,

$$ab_\parallel = \|a\|\|b\|\cos\theta = a \cdot b \tag{8.21}$$

which is defined as the inner product, or the inner product, and is symmetric.

$ab_{\perp}$: As a and $b_{\perp}$ are orthogonal

$$ab_{\perp} = \|a\|\|b\| \sin\theta = a \wedge b \tag{8.22}$$

which is defined as the outer product and is antisymmetric; i.e.

$$a \wedge b = -b \wedge a. \tag{8.23}$$

The area of the parallelogram formed by a and b in Fig. 8.20 is

$$\|a\|\|b\| \sin\theta. \tag{8.24}$$

Therefore,

$$\|a \wedge b\| = \|a\|\|b\| \sin\theta \tag{8.25}$$

which enables us to write Eq. (8.20) as

$$ab = a \cdot b + a \wedge b. \tag{8.26}$$

The parallel and orthogonal components created by $a \cdot b$ and $a \wedge b$ describe everything about the vectors a and b, which is why Clifford combined them into his geometric product. Furthermore, because these product components are linearly independent, the modulus of ab is computed using the Pythagorean rule:

$$\|ab\|^2 = \|a \cdot b\|^2 + \|a \wedge b\|^2$$

$$\|ab\|^2 = \|a\|^2\|b\|^2 \cos^2\theta + \|a\|^2\|b\|^2 \sin^2\theta$$

$$\|ab\|^2 = \|a\|^2\|b\|^2(\cos^2\theta + \sin^2\theta) \tag{8.27}$$

$$\|ab\| = \|a\|\|b\|. \tag{8.28}$$

Now we already know that $a \cdot b$ is a pure scalar and $a \wedge b$ is a directed area, which we suspect has an imaginary flavor. So it may seem strange adding two different mathematical objects together, but no stranger than a complex number. Nevertheless, we still require a name for this new object, which is a *multivector* and is described in section 8.5.

If we reverse the product to ba we have

$$ba = b \cdot a + b \wedge a = a \cdot b - a \wedge b. \tag{8.29}$$

Note how the antisymmetry of the outer product introduces the negative sign.

Knowing that the geometric product is the sum of the inner and outer products, it is possible to define the inner and outer products in terms of the geometric product as follows.

Subtracting Eq. (8.29) from Eq. (8.26) we obtain

$$ab - ba = (a \cdot b + a \wedge b) - (a \cdot b - a \wedge b) = 2(a \wedge b) \tag{8.30}$$

therefore,

$$a \wedge b = \frac{1}{2}(ab - ba). \tag{8.31}$$

Similarly, adding Eq. (8.29) to Eq. (8.26) we obtain

$$ab + ba = 2a \cdot b \tag{8.32}$$

therefore,

$$a \cdot b = \frac{1}{2}(ab + ba). \tag{8.33}$$

These are important relationships and will be called upon frequently.

Now let's explore the geometric product further using the unit basis vectors for $\mathbb{R}^2$.

8.2.4 The product of identical basis vectors

Before we begin exploring this product, it is worth introducing a shorthand notation that simplifies our equations. Very often we have to write down a string of basis vectors such as $e_1e_2e_1$ which can also be written as e_{121}, and saves space on the printed page. In general this is expressed as:

$$e_ie_je_k \equiv e_{ijk}. \tag{8.34}$$

So let's start with the product e_1e_1:

$$e_1e_1 = e_1 \cdot e_1 + e_1 \wedge e_1. \tag{8.35}$$

Now we already know that $e_1 \wedge e_1 = 0$ and $e_1 \cdot e_1 = 1$, which means that

$$e_1e_1 = e_1^2 = 1. \tag{8.36}$$

Similarly,

$$e_2^2 = 1. \tag{8.37}$$

8.2.5 The product of orthogonal basis vectors

Next, the product e_1e_2:

$$e_1e_2 = e_1 \cdot e_2 + e_1 \wedge e_2. \tag{8.38}$$

Again, we know that $e_1 \cdot e_2 = 0$, which means that

$$e_1e_2 = e_1 \wedge e_2. \tag{8.39}$$

So, whenever we find the unit bivector $e_1 \wedge e_2$ we can substitute e_1e_2 or e_{12}.

Now let's compute the product e_2e_1:

$$e_2e_1 = e_2 \cdot e_1 + e_2 \wedge e_1 = e_2 \cdot e_1 - e_1 \wedge e_2. \tag{8.40}$$

But $e_2 \cdot e_1 = 0$, therefore,

$$e_2e_1 = -e_1 \wedge e_2 = -e_{12}. \tag{8.41}$$

8.2.6 The imaginary properties of the outer product

The imaginary properties of the outer product are revealed by evaluating the product $(e_1 \wedge e_2)^2$:

$$(e_1 \wedge e_2)^2 = (e_1 \wedge e_2)(e_1 \wedge e_2) = e_1e_2e_1e_2. \tag{8.42}$$

But as

$$e_2e_1 = -e_1e_2 \tag{8.43}$$

then

$$(e_1 \wedge e_2)^2 = -e_1e_1e_2e_2 = -e_1^2e_2^2. \tag{8.44}$$

But as

$$e_1^2 = e_2^2 = 1 \tag{8.45}$$

then

$$(e_1 \wedge e_2)^2 = -1. \tag{8.46}$$

So the unit bivector possess the same qualities as imaginary i in that it squares to -1.

Now this has all sorts of ramifications as it suggests that GA is related to complex numbers and possibly, quaternions, and could perform rotations in n-dimensions. At this point, the algebra explodes into many paths, which will have to be explored in turn.

8.3 The unit bivector pseudoscalar

GA uses the term *grade* to distinguish its algebraic elements. For example, a scalar is grade-0, a vector grade-1 and a bivector grade-2, etc. In each algebra, the highest grade element is called the *pseudoscalar* and its grade equals the dimension of the associated space, which in $\mathbb{R}^2$ is the bivector $e_1 \wedge e_2$ and is a two-dimensional element. Later on, we discover that the trivector in $\mathbb{R}^3$ is also called a pseudoscalar.

Because the pseudoscalar has imaginary properties, some authors use the lowercase i to represent it, whilst others opt for the uppercase I. The reason for this is that i is normally associated with scalars, where there are no commuting problems. On the other hand, we will soon discover that the pseudoscalar anticommutes with vectors in $\mathbb{R}^2$, and it is safer to employ the symbol I so that its anticommuting properties do not get confused with those of i.

8.3.1 The rotational properties of the pseudoscalar

Now that we know that the unit bivector possesses imaginary properties, let's confirm that it rotates vectors in the same way we saw in section 7.2. We begin with the product e_1I:

$$e_1I = e_1e_1e_2 = e_1^2e_2 = e_2. \tag{8.47}$$

Taking the result e_2 and post-multiplying this by I:

$$e_2I = e_2e_1e_2 = e_2(-e_2e_1) = -e_2^2e_1 = -e_1. \tag{8.48}$$

Taking the result $-e_1$ and post-multiplying this by I:

$$-e_1 I = -e_1 e_1 e_2 = -e_1^2 e_2 = -e_2. \tag{8.49}$$

Taking the result $-e_2$ and multiplying this by I:

$$-e_2 I = -e_2 e_1 e_2 = -e_2(-e_2 e_2) = e_2^2 e_1 = e_1 \tag{8.50}$$

which brings us back to the starting point. Similarly, when the product is reversed, the direction of rotation is reversed.

As a simple example of the algebra in action, consider post-multiplying a vector a by the pseudoscalar I where

$$a = a_1 e_1 + a_2 e_2. \tag{8.51}$$

Then

$$aI = a e_1 e_2 = (a_1 e_1 + a_2 e_2) e_1 e_2 = a_1 e_1^2 e_2 + a_2 e_2 e_1 e_2 \tag{8.52}$$

and

$$aI = a_1 e_2 - a_2 e_2^2 e_1 = -a_2 e_1 + a_1 e_2 \tag{8.53}$$

which has clearly rotated the vector 90° anticlockwise.

Pre-multiplying the vector a by I produces:

$$Ia = e_1 e_2 a = e_1 e_2 (a_1 e_1 + a_2 e_2) = a_1 e_1 e_2 e_1 + a_2 e_1 e_2^2 \tag{8.54}$$

and

$$Ia = -a_1 e_2 + a_2 e_1 = a_2 e_1 - a_1 e_2 \tag{8.55}$$

which has rotated the vector 90° clockwise.

Therefore,

$$aI = -Ia \tag{8.56}$$

and confirms that in $\mathbb{R}^2$, the pseudoscalar and vectors anticommute.

These rotations are illustrated in Fig. 8.4.

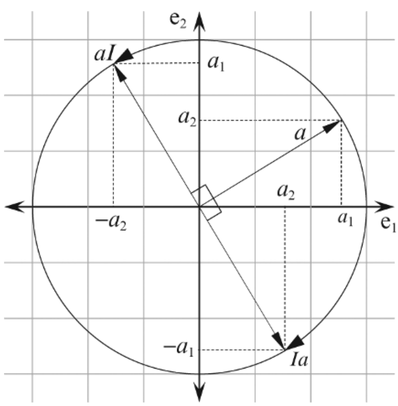

FIGURE 8.4.

8.4 Summary of the products

Table 8.1 summarizes the products we have encountered so far.

TABLE 8.1

Products in $\mathbb{R}^2$			
Type	Product	Absolute Value	Notes
inner	$e_1 \cdot e_1$	1	$e_2 \cdot e_2 = e_1 \cdot e_1$
outer	$e_1 \wedge e_1$	0	$e_2 \wedge e_2 = e_1 \wedge e_1$
geometric	e_1^2	1	$e_2^2 = e_1^2$
			$e_1 I = -I e_1$
inner	$e_1 \cdot e_2$	0	$e_2 \cdot e_1 = e_1 \cdot e_2$
outer	$e_1 \wedge e_2$	1	$e_1 \wedge e_2 = -(e_2 \wedge e_1)$
geometric	$e_1 e_2$	1	$e_{12} = -e_{21}$
			$e_{12} = I$
			$I^2 = -1$
inner	$a \cdot a$	$\Vert a \Vert^2$	
outer	$a \wedge a$	0	
geometric	a^2	$\Vert a \Vert^2$	
inner	$a \cdot b$	$\Vert a \Vert \Vert b \Vert \cos\theta$	$a \cdot b = \frac{1}{2}(ab + ba)$
		$a_1 b_1 + a_2 b_2$	
outer	$a \wedge b$	$\Vert a \Vert \Vert b \Vert \sin\theta$	$a \wedge b = \frac{1}{2}(ab - ba)$
		$a_1 b_2 - a_2 b_1$	$a \wedge b = (a_1 b_2 - a_2 b_1) e_1 \wedge e_2$
geometric	ab	$\Vert a \Vert \Vert b \Vert$	$ab = a \cdot b + a \wedge b$
			$aI = -Ia$

8.5 Multivectors in $\mathbb{R}^2$

In Chapters 2, 3, 4 and 5 we reviewed four algebraic systems with their axioms and elements and saw that elementary algebra supports scalars; complex algebra supports complex numbers (a scalar and an imaginary); vector algebra supports vectors (n-tuples); and quaternion algebra supports quaternions (a scalar and a vector). Clifford required that geometric algebra should support an element containing scalars, vectors, bivectors and any other object that could be created using the geometric product, which seems to be an impossible task. But his deep understanding of algebra and geometry resulted in an object he called a *multivector* which can be added and multiplied together just like any other element. For example, a multivector in $\mathbb{R}^2$ contains a scalar, vectors and a bivector, whereas in $\mathbb{R}^3$ a multivector contains a scalar, vectors, bivectors and a *trivector*. Higher-dimensional spaces contain similar combinations of scalar and vector-based objects.

The multivector elements that exist in $\mathbb{R}^2$ are scalars, vectors and bivectors, which are summarized in Table 8.2.

TABLE 8.2

Element	Symbol	Grade
1 scalar	λ	0
2 vectors	$\{e_1, e_2\}$	1
1 unit bivector	$e_1 \wedge e_2 = e_{12}$	2

A multivector is defined as a linear combination of the graded elements associated with the size of the linear space, which, in the case of $\mathbb{R}^2$ are scalars, vectors and bivectors. Therefore a multivector A is defined as follows:

$$A = \lambda_0 + \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_{12} \qquad [\lambda_i \in \mathbb{R}] \tag{8.57}$$

Note that we have substituted the geometric product for the outer product, as this is much more convenient. Using arbitrary values, the following are possible multivectors:

$$A = 4 + 3e_1 + 4e_2 + 5e_{12} \tag{8.58}$$

$$B = 3 + 2e_1 + 3e_2 + 4e_{12} \tag{8.59}$$

which, allows us to write:

$$A + B = 7 + 5e_1 + 7e_2 + 9e_{12} \tag{8.60}$$

and

$$A - B = 1 + e_1 + e_2 + e_{12}. \tag{8.61}$$

But what about the product AB? To answer this question, let's define B in general terms and form the product AB:

$$B = \beta_0 + \beta_1 e_1 + \beta_2 e_2 + \beta_3 e_{12}. \tag{8.62}$$

Therefore,

$$AB = (\lambda_0 + \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_{12})(\beta_0 + \beta_1 e_1 + \beta_2 e_2 + \beta_3 e_{12}). \tag{8.63}$$

Expanding, we obtain

$$\begin{aligned} AB = {} & \lambda_0\beta_0 + \lambda_0\beta_1 e_1 + \lambda_0\beta_2 e_2 + \lambda_0\beta e_{12} + \lambda_1\beta_0 e_1 + \lambda_1\beta_1 e_1^2 \\ & + \lambda_1\beta_2 e_{12} + \lambda_1\beta e_{112} + \lambda_2\beta_0 e_2 + \lambda_2\beta_1 e_{21} + \lambda_2\beta_2 e_2^2 \\ & + \lambda_2\beta e_{212} + \lambda_3\beta_0 e_{12} + \lambda_3\beta_1 e_{121} + \lambda_3\beta_2 e_{122} + \lambda_3\beta e_{12}^2. \end{aligned} \tag{8.64}$$

Substituting

$$e_1^2 = e_2^2 = 1 \qquad e_{21} = -e_{12} \qquad e_{12}^2 = -1 \tag{8.65}$$

and collecting up like terms:

$$\begin{aligned} AB = {} & (\lambda_0\beta_0 + \lambda_1\beta_1 + \lambda_2\beta_2 - \lambda_3\beta_3) + (\lambda_0\beta_1 + \lambda_1\beta_0 + \lambda_3\beta_2 - \lambda_2\beta_3)e_1 \\ & + (\lambda_0\beta_2 + \lambda_1\beta_3 + \lambda_2\beta_0 - \lambda_3\beta_1)e_2 + (\lambda_0\beta_3 + \lambda_1\beta_2 + \lambda_3\beta_0 - \lambda_2\beta_1)e_{12}. \end{aligned} \tag{8.66}$$

Which confirms that the multivector product AB creates another multivector and consequently forms a closed algebra.

Using the above multivectors

$$AB = (4 + 3e_1 + 4e_2 + 5e_{12})(3 + 2e_1 + 3e_2 + 4e_{12}) \tag{8.67}$$

then

$$AB = 10 + 16e_1 + 26e_2 + 32e_{12}. \tag{8.68}$$

8.6 The relationship between bivectors, complex numbers and vectors

The geometric product reveals the relationship between bivectors and complex numbers, and is demonstrated by computing the product of two vectors in $\mathbb{R}^2$.

Given two vectors a and b where

$$a = a_1e_1 + a_2e_2 \tag{8.69}$$

$$b = b_1e_1 + b_2e_2 \tag{8.70}$$

then

$$\begin{aligned} ab &= (a_1e_1 + a_2e_2)(b_1e_1 + b_2e_2) \\ &= a_1b_1e_1^2 + a_1b_2e_{12} + a_2b_1e_{21} + a_2b_2e_2^2 \\ &= a_1b_1 + a_2b_2 + a_1b_2e_{12} - a_2b_1e_{12} \\ &= (a_1b_1 + a_2b_2) + (a_1b_2 - a_2b_1)e_{12} \\ ab &= (a_1b_1 + a_2b_2) + (a_1b_2 - a_2b_1)I \end{aligned} \tag{8.71}$$

which is a complex number! Note that $(a_1b_1 + a_2b_2)$ is a scalar whilst $(a_1b_2 - a_2b_1)I$ is a bivector, which means that we can form the equivalent of a complex number Z by combining a scalar and a unit bivector as follows:

$$Z = a_1 + a_2e_{12} = a_1 + a_2I \tag{8.72}$$

where a_1 is the real part, and a_2 is the imaginary part.

Furthermore, we can convert a vector a into a complex number Z as follows.

Given a vector a:

$$a = a_1e_1 + a_2e_2 \tag{8.73}$$

then pre-multiplying a by e_1 we obtain:

$$e_1a = e_1(a_1e_1 + a_2e_2) = a_1e_1^2 + a_2e_1e_2 = a_1 + a_2I. \tag{8.74}$$

Therefore,

$$e_1a = Z. \tag{8.75}$$

But what happens if we reverse e_1 and a?

$$ae_1 = (a_1e_1 + a_2e_2)e_1 = a_1e_1^2 + a_2e_2e_1 = a_1 - a_2I \tag{8.76}$$

which we recognize as the complex conjugate. Therefore,

$$ae_1 = Z^{\dagger}. \tag{8.77}$$

(The dagger symbol † is sometimes used to represent the complex conjugate of a multivector).

8.7 Reversion

Reversing sequences of symbols happens to be a useful operation in GA. For instance, we may wish to reverse the sequence of three vectors abc to cba, or swap two bivectors AB to BA. Whatever the elements may be, the reversion operator performs this task and is used as follows:

$$(abc \cdots d)^{\sim} = (d \cdots cba). \tag{8.78}$$

The tilde superscript reminds us of this action, but other authors may employ the dagger symbol.

For any vectors a and b

$$\begin{aligned}(ab)^{\sim} &= (a \cdot b + a \wedge b)^{\sim} \\ &= a \cdot b - b \wedge a \\ (ab)^{\sim} &= ba.\end{aligned} \tag{8.79}$$

Similarly, for any multivectors A and B

$$(AB)^{\sim} = \tilde{B}\tilde{A}. \tag{8.80}$$

Unfortunately, some reversions involve a sign change, and are summarized in Table 8.3.

Table 8.3

Blade	k	Sign
scalar	0	+
vector	1	+
bivector	2	−
trivector	3	−
4-vector	4	+
5-vector	5	+
6-vector	6	−
7-vector	7	−
etc.		

This sign switching pattern is accommodated by the following formula:

$$\tilde{A}_k = (-1)^{\frac{k(k-1)}{2}} A_k. \tag{8.81}$$

8.8 Rotations in $\mathbb{R}^2$

In chapter 3 we saw that a complex number z is rotated through an angle ϕ using

$$z' = ze^{i\phi} \tag{8.82}$$

where

$$e^{i\phi} = \cos\phi + i\sin\phi. \tag{8.83}$$

But as $i^2 = I^2$

$$e^{I\phi} = \cos\phi + I\sin\phi. \tag{8.84}$$

Therefore,

$$z' = ze^{I\phi}. \tag{8.85}$$

But a multivector consisting of a scalar and a bivector is identical to a complex number, which means that we can write Eq. (8.85) as

$$Z' = Ze^{I\phi}. \tag{8.86}$$

So now let's see how a vector is rotated using a similar operation.

Pre-multiplying Eq. (8.75) by e_1 we obtain

$$e_1 e_1 v = e_1 Z \tag{8.87}$$

and

$$v = e_1 Z. \tag{8.88}$$

Let's assume that there exists another vector v' with an associated multivector Z' such that

$$v' = e_1 Z'. \tag{8.89}$$

Substituting Eq. (8.86) we obtain

$$v' = e_1 Z e^{I\phi}. \tag{8.90}$$

Substituting Eq. (8.75) we obtain

$$v' = e_1 e_1 v e^{I\phi} = v e^{I\phi} \tag{8.91}$$

which rotates the vector v through an angle ϕ to v'.

Let's illustrate Eq. (8.91) with an example.

Rotate $v = e_1$ anticlockwise 90° in the plane e_{12}:

$$\begin{aligned} v' &= ve^{I\phi} = e_1(\cos 90^\circ + e_{12}\sin 90^\circ) \\ v' &= e_1 e_{12} = e_2. \end{aligned} \tag{8.92}$$

Which is correct.

8.9 The vector-bivector product in $\mathbb{R}^2$

In section 8.3.1 we saw that a pseudoscalar rotates a vector 90° in the plane without scaling the vector. Now let's see what happens when we form the geometric product of a vector and a bivector. For example, given a vector a and a bivector B where

$$a = a_1 e_1 + a_2 e_2 \tag{8.93}$$

$$B = (b_1 e_1 + b_2 e_2) \wedge (c_1 e_1 + c_2 e_2) \tag{8.94}$$

then a' is the product aB

$$
\begin{aligned}
a' &= aB \\
&= (a_1\mathrm{e}_1 + a_2\mathrm{e}_2)((b_1\mathrm{e}_1 + b_2\mathrm{e}_2) \wedge (c_1\mathrm{e}_1 + c_2\mathrm{e}_2)) \\
&= (a_1\mathrm{e}_1 + a_2\mathrm{e}_2)(b_1c_1\mathrm{e}_1 \wedge \mathrm{e}_1 + b_1c_2\mathrm{e}_1 \wedge \mathrm{e}_2 + b_2c_1\mathrm{e}_2 \wedge \mathrm{e}_1 + b_2c_2\mathrm{e}_2 \wedge \mathrm{e}_2) \\
&= (a_1\mathrm{e}_1 + a_2\mathrm{e}_2)(b_1c_2 - b_2c_1)\mathrm{e}_{12} \\
&= a_1(b_1c_2 - b_2c_1)\mathrm{e}_1^2\mathrm{e}_2 + a_2(b_1c_2 - b_2c_1)\mathrm{e}_{212} \\
&= a_1(b_1c_2 - b_2c_1)\mathrm{e}_2 - a_2(b_1c_2 - b_2c_1)\mathrm{e}_1 \\
a' &= -a_2(b_1c_2 - b_2c_1)\mathrm{e}_1 + a_1(b_1c_2 - b_2c_1)\mathrm{e}_2. \qquad (8.95)
\end{aligned}
$$

But

$$\|B\| = b_1c_2 - b_2c_1. \qquad (8.96)$$

Therefore,

$$a' = \|B\|(-a_2\mathrm{e}_1 + a_1\mathrm{e}_2). \qquad (8.97)$$

It is clear from Eq. (8.97) that vector a has been rotated anticlockwise 90° and scaled by the magnitude of the bivector B. Reversing the product reverses the direction of rotation:

$$
\begin{aligned}
a' &= Ba \\
&= ((b_1\mathrm{e}_1 + b_2\mathrm{e}_2) \wedge (c_1\mathrm{e}_1 + c_2\mathrm{e}_2))(a_1\mathrm{e}_1 + a_2\mathrm{e}_2) \\
&= (b_1c_2 - b_2c_1)\mathrm{e}_{12}(a_1\mathrm{e}_1 + a_2\mathrm{e}_2) \\
&= a_1(b_1c_2 - b_2c_1)\mathrm{e}_{121} + a_2(b_1c_2 - b_2c_1)\mathrm{e}_{122} \\
&= -a_1(b_1c_2 - b_2c_1)\mathrm{e}_2 + a_2(b_1c_2 - b_2c_1)\mathrm{e}_1 \\
&= a_2(b_1c_2 - b_2c_1)\mathrm{e}_1 - a_1(b_1c_2 - b_2c_1)\mathrm{e}_2 \\
a' &= \|B\|(a_2\mathrm{e}_1 - a_1\mathrm{e}_2). \qquad (8.98)
\end{aligned}
$$

Equation (8.98) confirms that a has been rotated clockwise 90° and scaled by the magnitude of the bivector B.

8.10 Volumes and the trivector

By now you will have observed that geometric algebra is highly structured. We start with scalars, which in various tuples create vectors, which in turn create bivectors and ultimately lead to multivectors. The next element after the bivector is the *trivector* and is used to represent a directed volume. Starting with a bivector $a \wedge b$, which represents a directed area, we can imagine that this is moved along a third vector c to sweep out a parallelpiped as shown in Fig. 8.5 (a).

Remember that we are working with a right-handed axial system, and the bivector $a \wedge b$ is anticlockwise as viewed from inside the volume and moves along the direction of vector c to create the trivector $(a \wedge b) \wedge c$. In Fig. 8.5 (b) the bivector $b \wedge c$ is still anticlockwise as viewed from inside the volume and moves along the direction of vector a to create the trivector $(b \wedge c) \wedge a$.

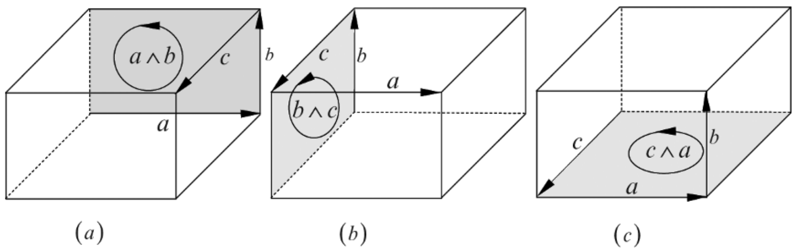

FIGURE 8.5.

Finally, in Fig. 8.5 (c), the bivector $c \wedge a$ is still anticlockwise as viewed from inside the volume and moves along the direction of vector b to create the trivector $(c \wedge a) \wedge b$. It is obvious that the three volumes are identical, which allows us to state

$$(a \wedge b) \wedge c = (b \wedge c) \wedge a = (c \wedge a) \wedge b. \tag{8.99}$$

Although the volumes in Fig. 8.5 are rectangular parallelpipeds, the above reasoning still holds for general parallelpipeds. In fact, just as the parallelogram helped us visualize the area computing powers of the bivector, the parallelpiped is just a useful object to illustrate the volumetric computing powers of the trivector. However, any volume can be used to visualize a trivector.

When vectors a, b, c are described in terms of the unit basis vectors:

$$a = a_1 e_1 + a_2 e_2 + a_3 e_3 \tag{8.100}$$

$$b = b_1 e_1 + b_2 e_2 + b_3 e_3 \tag{8.101}$$

$$c = c_1 e_1 + c_2 e_2 + c_3 e_3 \tag{8.102}$$

and multiplied together using $a \wedge b \wedge c$, it is obvious that this will give rise to terms such as:

$$e_1 \wedge e_1 \wedge e_1 = 0 \tag{8.103}$$

$$e_2 \wedge e_2 \wedge e_2 = 0 \tag{8.104}$$

$$e_3 \wedge e_3 \wedge e_3 = 0, \text{ etc.} \tag{8.105}$$

Furthermore, a variety of new terms arise involving triple outer products such as:

$$e_1 \wedge e_2 \wedge e_3 \tag{8.106}$$

$$e_1 \wedge e_2 \wedge e_1 \tag{8.107}$$

$$e_1 \wedge e_2 \wedge e_2, \text{ etc.} \tag{8.108}$$

The product $e_1 \wedge e_2 \wedge e_3$ is interpreted as:

'sweep the unit bivector $e_1 \wedge e_2$ along the orthogonal vector e_3 creating a volume represented by the trivector $e_1 \wedge e_2 \wedge e_3$.'

The product $e_1 \wedge e_2 \wedge e_1$ is interpreted as:

'sweep the unit bivector $e_1 \wedge e_2$ along one of its vectors: e_1, which does not create a volume.'

The product $e_1 \wedge e_2 \wedge e_2$ is interpreted as:

'sweep the unit bivector $e_1 \wedge e_2$ along one of its vectors: e_2, which also does not create a volume.'

Although these interpretations are correct, we require an algebraic explanation, which is provided as follows.

For completeness, let's expand the triple outer product:

$$\begin{aligned}
a \wedge b \wedge c &= (a_1 e_1 + a_2 e_2 + a_3 e_3) \wedge (b_1 e_1 + b_2 e_2 + b_3 e_3) \wedge (c_1 e_1 + c_2 e_2 + c_3 e_3) \\
&= \begin{pmatrix} a_1 b_1 e_1 \wedge e_1 + a_1 b_2 e_1 \wedge e_2 + a_1 b_3 e_1 \wedge e_3 + \\ a_2 b_1 e_2 \wedge e_1 + a_2 b_2 e_2 \wedge e_2 + a_2 b_3 e_2 \wedge e_3 + \\ a_3 b_1 e_3 \wedge e_1 + a_3 b_2 e_3 \wedge e_2 + a_3 b_3 e_3 \wedge e_3 \end{pmatrix} \wedge (c_1 e_1 + c_2 e_2 + c_3 e_3) \\
&= \begin{pmatrix} a_1 b_2 e_1 \wedge e_2 - a_1 b_3 e_3 \wedge e_1 - a_2 b_1 e_1 \wedge e_2 + \\ a_2 b_3 e_2 \wedge e_3 + a_3 b_1 e_3 \wedge e_1 - a_3 b_2 e_2 \wedge e_3 \end{pmatrix} \wedge (c_1 e_1 + c_2 e_2 + c_3 e_3)
\end{aligned}$$

$$a \wedge b \wedge c = \begin{pmatrix} (a_1 b_2 - a_2 b_1) e_1 \wedge e_2 + (a_2 b_3 - a_3 b_2) e_2 \wedge e_3 \\ +(a_3 b_1 - a_1 b_3) e_3 \wedge e_1 \end{pmatrix} \wedge (c_1 e_1 + c_2 e_2 + c_3 e_3) \tag{8.109}$$

At this point we can reject terms such as

$$e_1 \wedge e_2 \wedge e_1, \quad e_1 \wedge e_2 \wedge e_2, \quad e_2 \wedge e_3 \wedge e_2 \tag{8.110}$$

as they are zero volume elements, and means that we are left with the following trivector coefficients:

$$\begin{aligned}
a \wedge b \wedge c &= (a_1 b_2 - a_2 b_1) c_3 e_{123} + (a_2 b_3 - a_3 b_2) c_1 e_{123} + (a_3 b_1 - a_1 b_3) c_2 e_{123} \\
&= ((a_2 b_3 - a_3 b_2) c_1 + (a_3 b_1 - a_1 b_3) c_2 + (a_1 b_2 - a_2 b_1) c_3) e_{123}
\end{aligned}$$

and

$$a \wedge b \wedge c = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} e_{123} \tag{8.111}$$

which we recognize as the volume of a parallelpiped. Thus a trivector computes a directed volume.

8.11 The unit trivector pseudoscalar

Earlier in this chapter we discovered that

$$(e_1 \wedge e_2)^2 = -1 \tag{8.112}$$

and the name pseudoscalar was given to this product. Now let's do the same for the trivector:

$$\begin{aligned}
(e_1 \wedge e_2 \wedge e_3)^2 &= (e_1 e_2 e_3)^2 \\
&= e_1 e_2 e_3 e_1 e_2 e_3 = e_1 e_2 e_1 e_3 e_3 e_2 \\
(e_1 \wedge e_2 \wedge e_3)^2 &= e_1 e_2 e_1 e_2 = -1
\end{aligned} \tag{8.113}$$

which shows that the unit trivector also possesses imaginary properties.

With this new-found knowledge, let's compute the volume of a rectangular parallelpiped and a general parallelpiped.

We start by defining the edges of a box using the following vectors as shown in Fig. 8.6:

$$a = 2e_1 \quad b = 3e_2 \quad c = 4e_3. \tag{8.114}$$

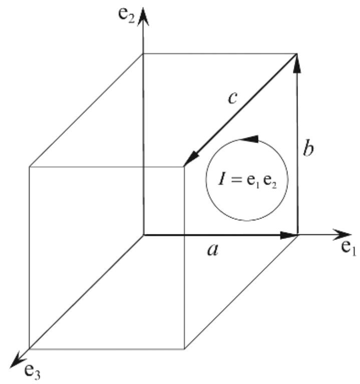

FIGURE 8.6.

Its volume V is defined as

$$\begin{aligned} V &= \|a \wedge b \wedge c\| \\ &= \|2e_1 \wedge 3e_2 \wedge 4e_3\| \\ &= \|24e_1 \wedge e_2 \wedge e_3\| \\ &= \|24e_{123}\| \\ V &= \|24I\|. \end{aligned} \tag{8.115}$$

Although the volume is represented as $24I$, we are only interested in its magnitude, which is 24. Hopefully, it is obvious that by reversing one of the vectors reverses the sign of the volume.

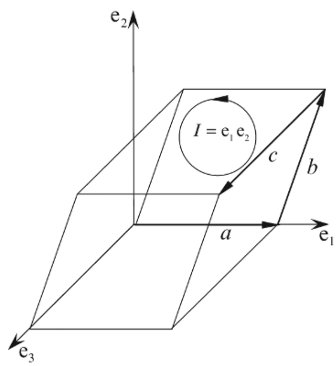

FIGURE 8.7.

For a second example, Fig. 8.7 illustrates a general parallelpiped where

$$a = 2e_1 \quad b = 0.5e_1 + 2e_2 \quad c = 3e_3. \tag{8.116}$$

Therefore, its volume V is given by

$$\begin{aligned} V &= \|a \wedge b \wedge c\| \\ &= \|2\mathrm{e}_1 \wedge (0.5\mathrm{e}_1 + 2\mathrm{e}_2) \wedge 3\mathrm{e}_3\| \\ &= \|4\mathrm{e}_{12} \wedge 3\mathrm{e}_3\| \\ &= \|12\mathrm{e}_{123}\| \\ V &= 12. \end{aligned} \tag{8.117}$$

At this point it is worth summarizing a pseudoscalar's features. To begin with, the pseudoscalar squares to -1:

$$I^2 = -1 \tag{8.118}$$

which guarantees

$$\|I\|^2 = 1. \tag{8.119}$$

Secondly, the pseudoscalar defines orientation. For instance, the 2D unit bivector is defined by $\mathrm{e}_1 \wedge \mathrm{e}_2$, and if any other bivector has the same sign as $\mathrm{e}_1 \wedge \mathrm{e}_2$ it shares the same orientation. Similarly, the 3D unit trivector is defined by $\mathrm{e}_1 \wedge \mathrm{e}_2 \wedge \mathrm{e}_3$, and if any other trivector has the same sign as $\mathrm{e}_1 \wedge \mathrm{e}_2 \wedge \mathrm{e}_3$ it shares the same orientation. Convention dictates that $\mathrm{e}_1 \wedge \mathrm{e}_2 \wedge \mathrm{e}_3$ describes a right-handed system of axes.

8.12 The product of the unit basis vectors in $\mathbb{R}^3$

8.12.1 The product of identical basis vectors

The three unit basis vectors in $\mathbb{R}^3$ are $\mathrm{e}_1, \mathrm{e}_2$ and e_3, and although it is self-evident, for the sake of completeness, we will record that

$$\mathrm{e}_1^2 = \mathrm{e}_2^2 = \mathrm{e}_3^2 = 1. \tag{8.120}$$

8.12.2 The product of orthogonal basis vectors

The third unit basis vector e_3 gives rise to three orthogonal unit basis bivector combinations:

$$\mathrm{e}_{12}, \mathrm{e}_{23} \text{ and } \mathrm{e}_{31}. \tag{8.121}$$

We already know that

$$\mathrm{e}_{12} = \mathrm{e}_1 \wedge \mathrm{e}_2 \tag{8.122}$$

and it should come as no surprise that

$$\mathrm{e}_{23} = \mathrm{e}_2 \wedge \mathrm{e}_3 \tag{8.123}$$

and

$$\mathrm{e}_{31} = \mathrm{e}_3 \wedge \mathrm{e}_1. \tag{8.124}$$

8.12.3 The imaginary properties of the unit bivectors

In section 8.2.6 we discovered that

$$e_{12}^2 = (e_1 \wedge e_2)^2 = -1 \tag{8.125}$$

and the same pattern is repeated for $\mathbb{R}^3$:

$$e_{23}^2 = (e_2 \wedge e_3)^2 = -1 \tag{8.126}$$

and

$$e_{31}^2 = (e_3 \wedge e_1)^2 = -1. \tag{8.127}$$

8.13 The vector-unit bivector product in $\mathbb{R}^3$

In section 8.3.1 we discovered that pre-multiplying a vector in $\mathbb{R}^2$ by the pseudoscalar $I = e_{12}$ rotates the vector clockwise 90° and post-multiplying rotates the vector anticlockwise 90°. Let's see what happens when we multiply a vector in $\mathbb{R}^3$ by a unit bivector. We begin by defining a vector and a unit bivector $e_{12} = e_1 \wedge e_2$

$$a = a_1 e_1 + a_2 e_2 + a_3 e_3. \tag{8.128}$$

Therefore,

$$\begin{aligned} e_{12}a &= a_1 e_{12} e_1 + a_2 e_{12} e_2 + a_3 e_{12} e_3 \\ &= -a_1 e_2 + a_2 e_1 + a_3 e_{123} \\ e_{12}a &= a_2 e_1 - a_1 e_2 + a_3 e_{123}. \end{aligned} \tag{8.129}$$

Equation (8.129) contains two elements:
a vector

$$a_2 e_1 - a_1 e_2 \tag{8.130}$$

and a volume

$$a_3 e_{123}. \tag{8.131}$$

What has happened is this. The multiplier e_{12} has:

1. Rotated the projection of a on the bivector $e_1 \wedge e_2$, clockwise 90°. (Fig. 8.8a)
2. Formed a volume of a_3 by sweeping $e_1 \wedge e_2$ along e_3. (Fig. 8.8b)

Reversing the product to ae_{12} produces

$$\begin{aligned} ae_{12} &= a_1 e_1 e_{12} + a_2 e_2 e_{12} + a_3 e_3 e_{12} \\ &= a_1 e_2 - a_2 e_1 + a_3 e_{123} \\ ae_{12} &= -a_2 e_1 + a_1 e_2 + a_3 e_{123}. \end{aligned} \tag{8.132}$$

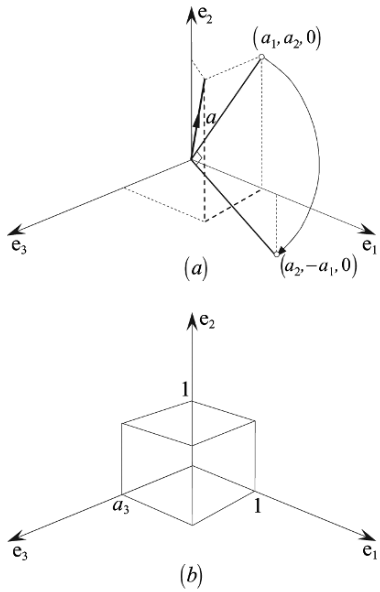

FIGURE 8.8.

Equation (8.13) confirms that the direction of rotation has been reversed to anticlockwise, whilst the sign of the volume remains unchanged.

Similar results are obtained with the products with $e_{23}a$ and $e_{31}a$:

$$\begin{aligned} e_{23}a &= a_1 e_{23} e_1 + a_2 e_{23} e_2 + a_3 e_{23} e_3 \\ &= a_1 e_{123} - a_2 e_3 + a_3 e_2 \\ e_{23}a &= a_3 e_2 - a_2 e_3 + a_1 e_{123} \text{ (Figs. 8.9(a) and (b))} \end{aligned}$$

and

$$a e_{23} = -a_3 e_2 + a_2 e_3 + a_1 e_{123}. \tag{8.133}$$

and

$$\begin{aligned} e_{31}a &= a_1 e_{31} e_1 + a_2 e_{31} e_2 + a_3 e_{31} e_3 \\ &= a_1 e_3 + a_2 e_{123} - a_3 e_1 \\ e_{31}a &= a_1 e_3 - a_3 e_1 + a_2 e_{123} \text{ (Figs. 8.10(a) and (b))} \end{aligned}$$

and

$$a e_{31} = -a_1 e_3 + a_3 e_1 + a_2 e_{123}. \tag{8.134}$$

These are interesting patterns, so let's see what happens when the multiplying bivector is not a unit bivector.

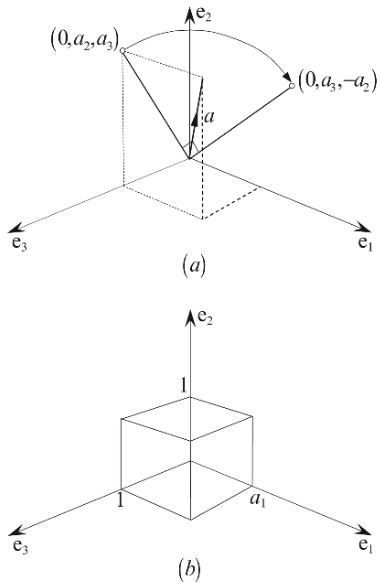

FIGURE 8.9.

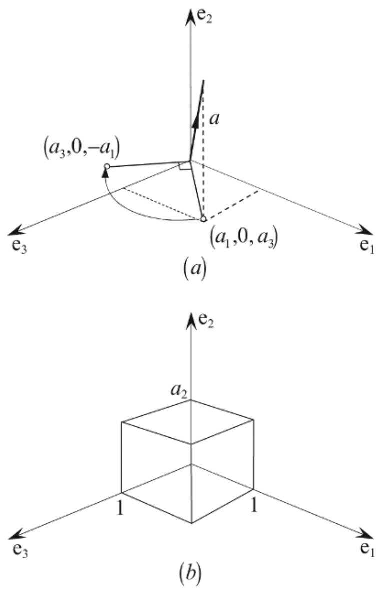

FIGURE 8.10.

8.14 The vector-bivector product in $\mathbb{R}^3$

Our bivector B is defined by the outer product of two vectors, whose precise values are not important, as any relevant combination will do. The vector a will have the form

$$a = a_1 e_1 + a_2 e_2 + a_3 e_3. \tag{8.135}$$

However, it is much more useful to express it in terms of two orthogonal components:

$$a = a_\parallel + a_\perp \tag{8.136}$$

where $a_\parallel$ is parallel with B and $a_\perp$ is perpendicular to B. This scenario is shown in Fig. 8.11.

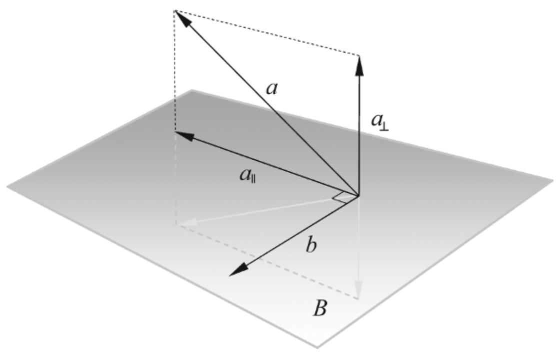

FIGURE 8.11.

Figure 8.11 shows an extra vector b which is orthogonal to $a_\parallel$ and also lies in the plane B. The value of b is chosen such that

$$a_\parallel \wedge b = B. \tag{8.137}$$

Therefore, the geometric product $a_\parallel b$ is

$$a_\parallel b = a_\parallel \cdot b + a_\parallel \wedge b = a_\parallel \wedge b = B \tag{8.138}$$

therefore,

$$B = a_\parallel b. \tag{8.139}$$

Now we explore the product of a vector and a bivector using $a_\parallel$, $a_\perp$ and B.

Starting with $a_\parallel B$ we obtain

$$a_\parallel B = a_\parallel (a_\parallel b) = a_\parallel^2 b. \tag{8.140}$$

But

$$a_\parallel^2 = \|a_\parallel\|^2. \tag{8.141}$$

Therefore,

$$a_\parallel B = \|a_\parallel\|^2 b \tag{8.142}$$

which is a vector, and must lie in the plane B. In fact, Eq. (8.142) shows that when a vector and a bivector are coplanar, their product rotates the vector 90° and scales it.

Next, we investigate the product $a_\perp B$:

$$a_\perp B = a_\perp(a_\parallel b) = a_\perp a_\parallel b. \tag{8.143}$$

With reference to Fig. 8.11, $a_\perp$, $a_\parallel$ and b are three orthogonal vectors, and can be visualized as sweeping the bivector $a_\perp \wedge a_\parallel$ along vector b creating a volume represented by the trivector $a_\perp a_\parallel b$.

Using Eqs. (8.142) and (8.143) we can express the product aB as

$$\begin{aligned} aB &= (a_\parallel + a_\perp)B \\ &= a_\parallel B + a_\perp B \\ aB &= \|a_\parallel\|^2 b + a_\perp a_\parallel b \end{aligned} \tag{8.144}$$

which is the sum of a vector and a trivector. Although this will not always be the case, because vector a could be orthogonal to bivector B, which only creates a trivector, in general, we can predict that the product aB will contain two terms: a vector and a trivector.

We are now in a position to define the product aB in terms of the inner and outer products, where there is a temptation to assume that it obeys the same rule for the geometric product of two vectors:

$$aB = a \cdot B + a \wedge B, \tag{8.145}$$

which although is true, has to be proved.

We begin by declaring B as the outer product

$$B = b \wedge c. \tag{8.146}$$

Therefore,

$$aB = a(b \wedge c). \tag{8.147}$$

We now have to find a way of converting Eq. (8.147) into Eq. (8.145), which is achieved as follows:

Using the geometric product

$$b \wedge c = \frac{1}{2}(bc - cb) \tag{8.148}$$

then

$$a(b \wedge c) = a\frac{1}{2}(bc - cb) = \frac{1}{2}(abc - acb). \tag{8.149}$$

Similarly,

$$a \cdot b = \frac{1}{2}(ab + ba) \quad \Rightarrow \quad ab = 2a \cdot b - ba \tag{8.150}$$

and

$$a \cdot c = \frac{1}{2}(ac + ca) \quad \Rightarrow \quad ac = 2a \cdot c - ca. \tag{8.151}$$

Substituting Eqs. (8.150) and (8.151) in Eq. (8.149)

$$
\begin{aligned}
a(b \wedge c) &= \frac{1}{2}((2a \cdot b - ba)c - (2a \cdot c - ca)b) \\
&= \frac{1}{2}(2(a \cdot b)c - bac - 2(a \cdot c)b + cab) \\
&= (a \cdot b)c - (a \cdot c)b + \frac{1}{2}(cab - bac) \\
&= (a \cdot b)c - (a \cdot c)b + \frac{1}{2}(abc + abc) \\
a(b \wedge c) &= (a \cdot b)c - (a \cdot c)b + abc. \qquad (8.152)
\end{aligned}
$$

Equation (8.152) shows that the product of a vector and a bivector (aB) creates two components: a vector

$$(a \cdot b)c - (a \cdot c)b \qquad (8.153)$$

and a trivector

$$abc. \qquad (8.154)$$

Next, we will show that reversing the product (Ba) creates a vector

$$-(a \cdot b)c + (a \cdot c)b \qquad (8.155)$$

and a trivector

$$abc \qquad (8.156)$$

where the vector is reversed, and the trivector remains unaltered.

What we want to do now is arrange that some combination of aB and Ba forms $a \cdot B$ to create the vector component, and another combination forms $a \wedge B$ to create the trivector component.

We can isolate each part using the following subterfuge:

Reversing the product $a(b \wedge c)$ to $(b \wedge c)a$

$$(b \wedge c)a = \frac{1}{2}(bc - cb)a = \frac{1}{2}(bca - cba). \qquad (8.157)$$

Now

$$c \cdot a = \frac{1}{2}(ca + ac) \quad \Rightarrow \quad ca = 2a \cdot c - ac \qquad (8.158)$$

and

$$b \cdot a = \frac{1}{2}(ba + ab) \quad \Rightarrow \quad ba = 2a \cdot b - ab. \qquad (8.159)$$

Substituting Eqs. (8.158) and (8.159) in Eq. (8.157)

$$\begin{aligned}(b \wedge c)a &= \frac{1}{2}(b(2a \cdot c - ac) - c(2a \cdot b - ab)) \\ &= \frac{1}{2}(2b(a \cdot c) - bac - 2c(a \cdot b) + cab) \\ &= (a \cdot c)b - (a \cdot b)c + \frac{1}{2}(cab - bac) \\ &= (a \cdot c)b - (a \cdot b)c + \frac{1}{2}(abc + abc) \\ (b \wedge c)a &= (a \cdot c)b - (a \cdot b)c + abc. \end{aligned} \tag{8.160}$$

Subtracting Eq. (8.160) from Eq. (8.152)

$$a(b \wedge c) - (b \wedge c)a = 2(a \cdot b)c - 2(a \cdot c)b. \tag{8.161}$$

Therefore,

$$(a \cdot b)c - (a \cdot c)b = \frac{1}{2}(aB - Ba). \tag{8.162}$$

As $\frac{1}{2}(aB - Ba)$ creates a vector, which is a lower grade object compared to a bivector, it is defined using the dot symbol as:

$$a \cdot B = \frac{1}{2}(aB - Ba) = (a \cdot b)c - (a \cdot c)b. \tag{8.163}$$

Adding Eq. (8.160) and (8.152) together, we obtain

$$a(b \wedge c) + (b \wedge c)a = 2abc. \tag{8.164}$$

Therefore,

$$\frac{1}{2}(aB + Ba) = abc. \tag{8.165}$$

As $\frac{1}{2}(aB + Ba)$ creates a higher grade object compared to a bivector, it is defined using the outer symbol as:

$$a \wedge B = \frac{1}{2}(aB + Ba) = abc. \tag{8.166}$$

Combining Eqs. (8.163) and (8.166) we define the geometric product aB as

$$aB = a \cdot B + a \wedge B \tag{8.167}$$

$$aB = (a \cdot b)c - (a \cdot c)b + abc \tag{8.168}$$

where

$$B = b \wedge c. \tag{8.169}$$

Now let's derive formulae for the reverse product Ba.

Subtracting Eq. (8.168) from Eq. (8.160) we have

$$(b \wedge c)a - a(b \wedge c) = 2(a \cdot c)b - 2(a \cdot b)c \tag{8.170}$$

therefore,

$$\frac{1}{2}(Ba - aB) = (a \cdot c)b - (a \cdot b)c. \tag{8.171}$$

As this is a grade lowering operation, it is defined as an inner product:

$$B \cdot a = \frac{1}{2}(Ba - aB) = (a \cdot c)b - (a \cdot b)c. \tag{8.172}$$

Adding Eq. (8.168) to Eq. (8.160) we have

$$(b \wedge c)a + a(b \wedge c) = 2abc \tag{8.173}$$

therefore,

$$\frac{1}{2}(Ba + aB) = abc. \tag{8.174}$$

As this is a grade raising operation, it is defined as an outer product:

$$B \wedge a = \frac{1}{2}(Ba + aB) = abc. \tag{8.175}$$

Therefore,

$$B \cdot a + B \wedge a = \frac{1}{2}(Ba - aB) + \frac{1}{2}(Ba + aB) = Ba \tag{8.176}$$

and

$$Ba = B \cdot a + B \wedge a. \tag{8.177}$$

Let's bring these results to life with two simple examples.

We start with three vectors

$$a = 2e_1 + e_2 - e_3 \tag{8.178}$$

$$b = e_1 - e_2 + e_3 \tag{8.179}$$

$$c = 2e_1 + 2e_2 + e_3. \tag{8.180}$$

We now construct a bivector B:

$$\begin{aligned} B = b \wedge c &= (e_1 - e_2 + e_3) \wedge (2e_1 + 2e_2 + e_3) \\ &= 2e_{12} - e_{31} + 2e_{12} - e_{23} + 2e_{31} - 2e_{23} \\ B &= 4e_{12} - 3e_{23} + e_{31}. \end{aligned} \tag{8.181}$$

Therefore,

$$\begin{aligned} aB &= (2e_1 + e_2 - 2e_3)(4e_{12} - 3e_{23} + e_{31}) \\ &= 8e_2 - 6e_{123} - 2e_3 - 4e_1 - 3e_3 + e_{123} - 8e_{123} - 6e_2 - 2e_1 \\ aB &= -6e_1 + 2e_2 - 5e_3 - 13e_{123}. \end{aligned} \tag{8.182}$$

Similarly,

$$\begin{aligned} Ba &= (4e_{12} - 3e_{23} + e_{31})(2e_1 + e_2 - 2e_3) \\ &= -8e_2 + 4e_1 - 8e_{123} - 6e_{123} + 3e_3 + 6e_2 + 2e_3 + e_{123} + 2e_1 \\ Ba &= 6e_1 - 2e_2 + 5e_3 - 13e_{123}. \end{aligned} \tag{8.183}$$

Now let's calculate the inner and outer products.

The inner product:

$$\begin{aligned} a \cdot B &= \frac{1}{2}(aB - Ba) \\ &= \frac{1}{2}(-6e_1 + 2e_2 - 5e_3 - 13e_{123} - 6e_1 + 2e_2 - 5e_3 + 13e_{123}) \\ &= \frac{1}{2}(-12e_1 + 4e_2 - 10e_3) \\ a \cdot B &= -6e_1 + 2e_2 - 5e_3. \end{aligned} \tag{8.184}$$

The outer product:

$$\begin{aligned} a \wedge B &= \frac{1}{2}(aB + Ba) \\ &= \frac{1}{2}(-6e_1 + 2e_2 - 5e_3 - 13e_{123} + 6e_1 - 2e_2 + 5e_3 - 13e_{123}) \\ a \wedge B &= -13e_{123}. \end{aligned} \tag{8.185}$$

Thus

$$\begin{aligned} aB &= a \cdot B + a \wedge B \\ aB &= -6e_1 + 2e_2 - 5e_3 - 13e_{123}. \end{aligned} \tag{8.186}$$

It is clear from these examples how the inner and outer products identify the two parts of the geometric product.

In vector algebra the inner product is also known as the scalar product or dot product. Conversely, in geometric algebra we can create products between vectors, bivectors, trivectors, etc, and any combination such as a vector and a bivector, or a bivector and trivector. Because these objects have different grades, we require a new interpretation of the dot symbol, which also embraces the original definition. Thus the dot product in Eq. (8.167) means the "*lowest grade part of the product.*" Similarly, the outer product in Eq. (8.167) means the "*highest grade part of the product.*"

If you believe that you have seen the RHS of Eq. (8.163) before, you may recall from vector algebra that

$$(b \times c) \times a = (a \cdot b)c - (a \cdot c)b \tag{8.187}$$

and

$$a \times (b \times c) = (a \cdot c)b - (a \cdot b)c. \tag{8.188}$$

Later in this chapter we show how GA can be used to derive these relationships.

8.15 Unit bivector-bivector products in $\mathbb{R}^3$

In $\mathbb{R}^3$ we have to consider the possible products that exist between e_{12}, e_{23} and e_{31}. We already know that

$$e_{12}^2 = e_{23}^2 = e_{31}^2 = -1 \tag{8.189}$$

and it is easy to show that

$$e_{12}e_{23} = e_{13} = -e_{31} \tag{8.190}$$

$$e_{23}e_{31} = e_{21} = -e_{12} \tag{8.191}$$

$$e_{31}e_{12} = e_{32} = -e_{23} \tag{8.192}$$

$$e_{12}e_{31} = e_{23} \tag{8.193}$$

$$e_{23}e_{12} = e_{31} \tag{8.194}$$

$$e_{31}e_{23} = e_{12}. \tag{8.195}$$

Thus we see that unit bivectors anticommute.

These results are summarized in Table 8.4

Table 8.4

GP	e_{12}	e_{23}	e_{31}
e_{12}	-1	$-e_{31}$	e_{23}
e_{23}	e_{31}	-1	$-e_{12}$
e_{31}	$-e_{23}$	e_{12}	-1

Accordingly, when we encounter an expression such as $\alpha e_{12}\beta e_{23}$ we can rewrite it as

$$\alpha\beta e_{12}e_{23} = -\alpha\beta e_{31}. \tag{8.196}$$

8.16 Unit vector-trivector product in $\mathbb{R}^3$

To begin with, let's consider the products e_1e_{123}, e_2e_{123} and e_3e_{123}:

$$e_1e_{123} = e_{23} \tag{8.197}$$

$$e_2e_{123} = e_{31} \tag{8.198}$$

$$e_3e_{123} = e_{12}. \tag{8.199}$$

Similarly,

$$e_{123}e_1 = e_{23} \tag{8.200}$$

$$e_{123}e_2 = e_{31} \tag{8.201}$$

$$e_{123}e_3 = e_{12}. \tag{8.202}$$

Thus we see that vectors and trivectors commute.

Let's illustrate this with a simple example $a5e_{123}$ where

$$a = 2e_1 + 3e_2 + 4e_3. \tag{8.203}$$

Therefore

$$\begin{aligned} a5e_{123} &= (2e_1 + 3e_2 + 4e_3)5e_{123} \\ a5e_{123} &= 20e_{12} + 10e_{23} + 15e_{31}. \end{aligned} \tag{8.204}$$

The volumetric element has been reduced to three bivector terms.

8.17 Unit bivector-trivector product in $\mathbb{R}^3$

To begin, let's consider the products $e_{12}e_{123}$, $e_{23}e_{123}$ and $e_{31}e_{123}$:

$$e_{12}e_{123} = -e_3 \tag{8.205}$$

$$e_{23}e_{123} = -e_1 \tag{8.206}$$

$$e_{31}e_{123} = -e_2. \tag{8.207}$$

Similarly,

$$e_{123}e_{12} = -e_3 \tag{8.208}$$

$$e_{123}e_{23} = -e_1 \tag{8.209}$$

$$e_{123}e_{31} = -e_2. \tag{8.210}$$

Thus we see that bivectors and trivectors commute.

Again, let's illustrate this product with an example $B5e_{123}$ where

$$B = 2e_{12} + 3e_{23} + 4e_{31}. \tag{8.211}$$

Therefore,

$$\begin{aligned} B5e_{123} &= (2e_{12} + 3e_{23} + 4e_{31})5e_{123} \\ B5e_{123} &= -15e_1 - 20e_2 - 10e_3. \end{aligned} \tag{8.212}$$

The volumetric element has been reduced to three vector terms.

8.18 Unit trivector-trivector product in $\mathbb{R}^3$

We have already discovered in section 8.10 that the square of the pseudoscalar equals -1.

8.19 Higher products in $\mathbb{R}^3$

Having considered the product of two trivectors in $\mathbb{R}^3$, it is worth exploring the concept of expanded outer products. For example, in $\mathbb{R}^3$, what is meant by

$$a \wedge b \wedge c \wedge d? \tag{8.213}$$

We can resolve this question by reasoning that if a, b, c are not coplanar, then d must be a linear combination of a, b, c:

$$d = \lambda_a a + \lambda_b b + \lambda_c c \quad [\lambda_a, \lambda_b, \lambda_c \in \mathbb{R}] \tag{8.214}$$

therefore,

$$a \wedge b \wedge c \wedge d = a \wedge b \wedge c \wedge (\lambda_a a + \lambda_b b + \lambda_c c)$$

and

$$a \wedge b \wedge c \wedge d = \lambda_a a \wedge b \wedge c \wedge a + \lambda_b a \wedge b \wedge c \wedge b + \lambda_c a \wedge b \wedge c \wedge c. \tag{8.215}$$

Recall that

$$a \wedge b = -b \wedge a \tag{8.216}$$

therefore,

$$a \wedge b \wedge c \wedge a = b \wedge a \wedge a \wedge c. \tag{8.217}$$

But

$$a \wedge a = 0 \tag{8.218}$$

therefore,

$$a \wedge b \wedge c \wedge a = 0. \tag{8.219}$$

Similarly,

$$a \wedge b \wedge c \wedge b = a \wedge b \wedge c \wedge c = 0 \tag{8.220}$$

therefore,

$$a \wedge b \wedge c \wedge d = 0. \tag{8.221}$$

8.20 Blades

Now that we covered bivectors and trivectors and tentatively explored higher dimensions, it appears that each space possesses a unique element created by the outer product. Starting with vectors, the outer product produces bivectors, trivectors, and even quadvectors, and there is no reason why higher n-vector elements cannot exist. Such a pattern was recognized by Hestenes who proposed the name *blade* for these elements [14]. Thus a blade is any multivector that can be formed as the outer product of a set of vectors. However, this definition has been widened by some authors to embrace scalars as 0-blades, vectors as 1-blades, bivectors as 2-blades, etc.

8.21 Duality transformation

An interesting relationship exists between blades and the pseudoscalar that is referred to as the *duality transformation*. Consider the following products involving the pseudoscalar $I = \mathrm{e}_1 \wedge \mathrm{e}_2$ and 2D basis vectors:

$$I\mathrm{e}_1 = \mathrm{e}_1\mathrm{e}_2\mathrm{e}_1 = -\mathrm{e}_2 \tag{8.222}$$

$$I\mathrm{e}_2 = \mathrm{e}_1\mathrm{e}_2\mathrm{e}_2 = \mathrm{e}_1 \tag{8.223}$$

and

$$\mathrm{e}_1 I = \mathrm{e}_1\mathrm{e}_1\mathrm{e}_2 = \mathrm{e}_2 \tag{8.224}$$

$$\mathrm{e}_2 I = \mathrm{e}_2\mathrm{e}_1\mathrm{e}_2 = -\mathrm{e}_1. \tag{8.225}$$

Note that they anticommute. Now consider the following products involving the pseudoscalar $I = \mathrm{e}_1 \wedge \mathrm{e}_2 \wedge \mathrm{e}_3$ and the 3D basis vectors:

$$I\mathrm{e}_1 = \mathrm{e}_1\mathrm{e}_2\mathrm{e}_3\mathrm{e}_1 = \mathrm{e}_2\mathrm{e}_3 \tag{8.226}$$

$$I\mathrm{e}_2 = \mathrm{e}_1\mathrm{e}_2\mathrm{e}_3\mathrm{e}_2 = \mathrm{e}_3\mathrm{e}_1 \tag{8.227}$$

$$I\mathrm{e}_3 = \mathrm{e}_1\mathrm{e}_2\mathrm{e}_3\mathrm{e}_3 = \mathrm{e}_1\mathrm{e}_2 \tag{8.228}$$

and

$$\mathrm{e}_1 I = \mathrm{e}_1\mathrm{e}_1\mathrm{e}_2\mathrm{e}_3 = \mathrm{e}_2\mathrm{e}_3 \tag{8.229}$$

$$\mathrm{e}_2 I = \mathrm{e}_2\mathrm{e}_1\mathrm{e}_2\mathrm{e}_3 = \mathrm{e}_3\mathrm{e}_1 \tag{8.230}$$

$$\mathrm{e}_3 I = \mathrm{e}_3\mathrm{e}_1\mathrm{e}_2\mathrm{e}_3 = \mathrm{e}_1\mathrm{e}_2. \tag{8.231}$$

Note that they commute, and further analysis shows that spaces with an odd number of dimensions, the pseudoscalar commutes will all vectors and multivectors, whereas spaces with an even number of dimensions they anticommute. This relationship is summarized by the relationship:

$$I_n A_r = (-1)^{r(n-1)} A_r I_n. \tag{8.232}$$

Doran and Lasenby [15] show how this relationship can be used to relate the inner and outer products:

$$\begin{aligned}
a \cdot (A_r I) &= \frac{1}{2}(aA_r I - (-1)^{n-r} A_r I a) \\
&= \frac{1}{2}(aA_r I - (-1)^{n-r}(-1)^{n-1} A_r a I) \\
&= \frac{1}{2}(aA_r + (-1)^r A_r a) I \\
a \cdot (A_r I) &= a \wedge A_r I.
\end{aligned} \tag{8.233}$$

The product aI is a grade lowering operation as a volumetric element is reduced into bivector elements, and consequently is denoted using the dot product:

$$aI = a \cdot I. \tag{8.234}$$

Figure 8.12 shows the duality relationship between bivectors and vectors in 3D space.

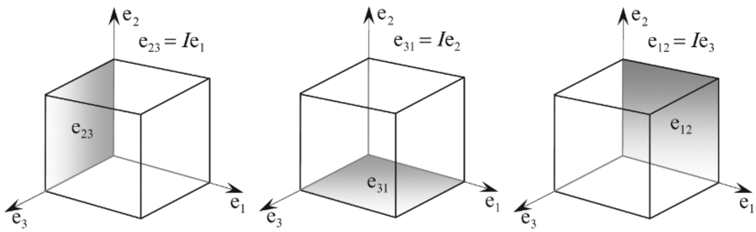

FIGURE 8.12.

A convenient notation used to represent the dual of A is A^*.

8.22 Summary of products in $\mathbb{R}^3$

We are now in a position to summarize the above products in tabular form as shown in Table 8.5.

TABLE 8.5

Inner product	
Vectors commute	$a \cdot b = b \cdot a$
Vectors and bivectors anticommute	$a \cdot B = -B \cdot a$
	$a \cdot B = \frac{1}{2}(aB - Ba)$
	$a \cdot B = (a \cdot b)c - (a \cdot c)b$
	$B \cdot a = \frac{1}{2}(Ba - aB)$
	$B \cdot a = (a \cdot c)b - (a \cdot b)c$
Outer product	
Vectors anticommute	$a \wedge b = -b \wedge a$
Vectors and bivectors commute	$a \wedge B = B \wedge a$
	$a \wedge B = \frac{1}{2}(aB + Ba)$
	$a \wedge B = abc$
	$B \wedge a = \frac{1}{2}(Ba + aB)$
	$B \wedge a = abc$

Table 8.5 (continued)

Geometric product	
Orthogonal vectors anticommute	$e_{12} = -e_{21}$
Orthogonal bivectors anticommute	$e_{12}e_{23} = -e_{23}e_{12}$
Bivectors square to -1	$e_{12}^2 = e_{23}^2 = e_{31}^2 = -1$
Definition	$ab = a \cdot b + a \wedge b$
Vectors and bivectors anticommute	$aB = -Ba$
	$aB = a \cdot B + a \wedge B$
	$aB = (a \cdot b)c - (a \cdot c)b + abc$
	$Ba = B \cdot a + B \wedge a$
	$Ba = (a \cdot c)b - (a \cdot b)c + abc$
Trivector commutes with all multivectors in the space	$aT = Ta \quad BT = TB$
The pseudoscalar	$e_{123} = I$
Vectors and the pseudoscalar commute	$aI = Ia$
	$aI = a \cdot I$
Duality transformation	$e_{23} = Ie_1$
	$e_{31} = Ie_2$
	$e_{12} = Ie_3$
The trivector squares to -1	$I^2 = -1$

Where a and b are vectors, B is a bivector, and T is a trivector.

Table 8.6 summarizes the commutative rules that exist between vectors, bivectors and trivectors when using the inner, outer and geometric products. The fact that every product is resolved in terms of the table's indices means that the product of two multivectors forms a closed algebra.

Table 8.6

GP	λ	e_1	e_2	e_3	e_{12}	e_{23}	e_{31}	e_{123}
λ	λ^2	λe_1	λe_2	λe_3	λe_{12}	λe_{23}	λe_{31}	λe_{123}
e_1	λe_1	1	e_{12}	$-e_{31}$	e_2	e_{123}	$-e_3$	e_{23}
e_2	λe_2	$-e_{12}$	1	e_{23}	$-e_1$	e_3	e_{123}	e_{31}
e_3	λe_3	e_{31}	$-e_{23}$	1	e_{123}	$-e_2$	e_1	e_{12}
e_{12}	λe_{12}	$-e_2$	e_1	e_{123}	-1	$-e_{31}$	e_{23}	$-e_3$
e_{23}	λe_{23}	e_{123}	$-e_3$	e_2	e_{31}	-1	$-e_{12}$	$-e_1$
e_{31}	λe_{31}	e_3	e_{123}	$-e_1$	$-e_{23}$	e_{12}	-1	$-e_2$
e_{123}	λe_{123}	e_{23}	e_{31}	e_{12}	$-e_3$	$-e_1$	$-e_2$	-1

8.23 Multivectors in $\mathbb{R}^3$

In section 8.5 we defined a multivector in $\mathbb{R}^2$ as a linear combination of scalars, vectors and bivectors. We now extend this definition to include trivectors. Table 8.7 summarizes the elements and confirms that we have 1 scalar, 3 vectors, 3 bivectors and 1 trivector.

Table 8.7

Element	Symbol	Grade
1 scalar	λ	0
3 vectors	$\{e_1, e_2, e_3\}$	1
3 bivectors	$e_1 \wedge e_2 = e_{12}$	2
	$e_2 \wedge e_3 = e_{23}$	
	$e_3 \wedge e_1 = e_{31}$	
1 trivector	e_{123}	3

For completeness, let's form the product of two multivectors to demonstrate that we have a closed algebra.

We begin by defining two multivectors A and B:

$$A = \lambda_0 + \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_{12} + \lambda_4 e_{23} + \lambda_5 e_{31} + \lambda_6 e_{123} \quad [\lambda_i \in \mathbb{R}] \tag{8.235}$$

$$B = \beta_0 + \beta_1 e_1 + \beta_2 e_2 + \beta_3 e_{12} + \beta_4 e_{23} + \beta_5 e_{31} + \beta_6 e_{123} \quad [\beta_i \in \mathbb{R}] \tag{8.236}$$

Therefore,

$$\begin{aligned} AB = {} & \lambda_0(\beta_0 + \beta_1 e_1 + \beta_2 e_2 + \beta_3 e_{12} + \beta_4 e_{23} + \beta_5 e_{31} + \beta_6 e_{123}) \\ & + \lambda_1 e_1(\beta_0 + \beta_1 e_1 + \beta_2 e_2 + \beta_3 e_{12} + \beta_4 e_{23} + \beta_5 e_{31} + \beta_6 e_{123}) \\ & + \lambda_2 e_2(\beta_0 + \beta_1 e_1 + \beta_2 e_2 + \beta_3 e_{12} + \beta_4 e_{23} + \beta_5 e_{31} + \beta_6 e_{123}) \\ & + \lambda_3 e_{12}(\beta_0 + \beta_1 e_1 + \beta_2 e_2 + \beta_3 e_{12} + \beta_4 e_{23} + \beta_5 e_{31} + \beta_6 e_{123}) \\ & + \lambda_4 e_{23}(\beta_0 + \beta_1 e_1 + \beta_2 e_2 + \beta_3 e_{12} + \beta_4 e_{23} + \beta_5 e_{31} + \beta_6 e_{123}) \\ & + \lambda_5 e_{31}(\beta_0 + \beta_1 e_1 + \beta_2 e_2 + \beta_3 e_{12} + \beta_4 e_{23} + \beta_5 e_{31} + \beta_6 e_{123}) \\ & + \lambda_6 e_{123}(\beta_0 + \beta_1 e_1 + \beta_2 e_2 + \beta_3 e_{12} + \beta_4 e_{23} + \beta_5 e_{31} + \beta_6 e_{123}) \end{aligned} \tag{8.237}$$

expanding

$$\begin{aligned} AB = {} & \lambda_0\beta_0 + \lambda_0\beta_1 e_1 + \lambda_0\beta_2 e_2 + \lambda_0\beta_3 e_{12} + \lambda_0\beta_4 e_{23} + \lambda_0\beta_5 e_{31} + \lambda_0\beta_6 e_{123} + \lambda_1\beta_0 e_1 + \lambda_1\beta_1 \\ & + \lambda_1\beta_2 e_{12} + \lambda_1\beta_3 e_2 + \lambda_1\beta_4 e_{123} - \lambda_1\beta_5 e_3 + \lambda_1\beta_6 e_{23} + \lambda_2\beta_0 e_2 - \lambda_2\beta_1 e_{12} + \lambda_2\beta_2 \\ & - \lambda_2\beta_3 e_1 + \lambda_2\beta_4 e_3 + \lambda_2\beta_5 e_{123} + \lambda_2\beta_6 e_{31} + \lambda_3\beta_0 e_{12} - \lambda_3\beta_1 e_2 + \lambda_3\beta_2 e_1 - \lambda_3\beta_3 \\ & - \lambda_3\beta_4 e_{31} + \lambda_3\beta_5 e_{23} - \lambda_3\beta_6 e_3 + \lambda_4\beta_0 e_{23} + \lambda_4\beta_1 e_{123} - \lambda_4\beta_2 e_3 + \lambda_4\beta_3 e_{31} - \lambda_4\beta_4 \\ & - \lambda_4\beta_5 e_{12} - \lambda_4\beta_6 e_1 + \lambda_5\beta_0 e_{31} + \lambda_5\beta_1 e_3 + \lambda_5\beta_2 e_{123} - \lambda_5\beta_3 e_{23} + \lambda_5\beta_4 e_{12} - \lambda_5\beta_5 \\ & - \lambda_5\beta_6 e_2 + \lambda_6\beta_0 e_{123} + \lambda_6\beta_1 e_{23} + \lambda_6\beta_2 e_{31} - \lambda_6\beta_3 e_3 - \lambda_6\beta_4 e_1 - \lambda_6\beta_5 e_2 - \lambda_6\beta_6 \end{aligned} \tag{8.238}$$

simplifying and collecting up like terms

$$\begin{aligned} AB = {} & \lambda_0\beta_0 + \lambda_1\beta_1 + \lambda_2\beta_2 - \lambda_3\beta_3 - \lambda_4\beta_4 - \lambda_5\beta_5 - \lambda_6\beta_6 \\ & + (\lambda_0\beta_1 + \lambda_1\beta_0 - \lambda_2\beta_3 + \lambda_3\beta_2 - \lambda_4\beta_6 - \lambda_6\beta_4)e_1 \\ & + (\lambda_0\beta_2 + \lambda_1\beta_3 + \lambda_2\beta_0 - \lambda_3\beta_1 - \lambda_5\beta_6 - \lambda_6\beta_5)e_2 \end{aligned}$$

$$
\begin{aligned}
&+ (-\lambda_1\beta_5 + \lambda_2\beta_4 - \lambda_3\beta_6 - \lambda_4\beta_2 + \lambda_5\beta_1 - \lambda_6\beta_3)\mathrm{e}_3 \\
&+ (\lambda_0\beta_3 + \lambda_1\beta_2 - \lambda_2\beta_1 + \lambda_3\beta_0 - \lambda_4\beta_5 + \lambda_5\beta_4)\mathrm{e}_{12} \\
&+ (\lambda_0\beta_4 + \lambda_1\beta_6 + \lambda_3\beta_5 + \lambda_4\beta_0 - \lambda_5\beta_3 + \lambda_6\beta_1)\mathrm{e}_{23} \\
&+ (\lambda_0\beta_5 + \lambda_2\beta_6 - \lambda_3\beta_4 + \lambda_4\beta_3 + \lambda_5\beta_0 + \lambda_6\beta_2)\mathrm{e}_{31} \\
&+ (\lambda_0\beta_6 + \lambda_1\beta_4 + \lambda_2\beta_5 + \lambda_4\beta_1 + \lambda_5\beta_2 + \lambda_6\beta_0)\mathrm{e}_{123}
\end{aligned} \tag{8.239}
$$

which is another multivector and forms a closed algebra.

A multivector that contains terms of only a single grade is said to be *homogeneous*.

You may have noticed an obvious pattern associated with the number of elements in each multivector. Table 8.8 summarizes the number of elements for $\mathbb{R}^2$, $\mathbb{R}^3$ and $\mathbb{R}^4$ where it is obvious that Pascal's numbers are in control.

Table 8.8

	Scalar	Vector	Bivector	Trivector	Quadvector
$\mathbb{R}^2$	1	2	1		
$\mathbb{R}^3$	1	3	3	1	
$\mathbb{R}^4$	1	4	6	4	1

As a multivector contains elements with a variety of grades, it is useful to isolate each grade using the following notation: $\langle A\rangle_n$, where n is the required grade. For example, given

$$A = 3 + 2\mathrm{e}_1 + \mathrm{e}_2 - 3\mathrm{e}_3 + 5\mathrm{e}_1 \wedge 2\mathrm{e}_2 + 7\mathrm{e}_{123} \tag{8.240}$$

then

$$\langle A\rangle_0 = 3 \tag{8.241}$$

$$\langle A\rangle_1 = 2\mathrm{e}_1 + \mathrm{e}_2 - 3\mathrm{e}_3 \tag{8.242}$$

$$\langle A\rangle_2 = 5\mathrm{e}_1 \wedge 2\mathrm{e}_2 \tag{8.243}$$

$$\langle A\rangle_3 = 7\mathrm{e}_{123}. \tag{8.244}$$

In the case of the geometric product

$$ab = a \cdot b + a \wedge b \tag{8.245}$$

$$\langle ab\rangle_0 = a \cdot b \tag{8.246}$$

$$\langle ab\rangle_2 = a \wedge b. \tag{8.247}$$

8.24 Relationship between vector algebra and geometric algebra

We are now in a position to compare vector algebra with geometric algebra, especially with the way vectors relate to complex numbers, and how rotations in the plane are effected.

Table 8.9 summarizes the two algebras beginning with a vector, the mapping from a vector into a complex number, and the technique for rotating.

Table 8.9

Vector Algebra		Geometric Algebra	
vector	$v = a_1\mathrm{i} + a_2\mathrm{j}$	vector	$v = a_1\mathrm{e}_1 + a_2\mathrm{e}_2$
map	$a = a_1 \quad b = a_2$	map	$Z = \mathrm{e}_1 v$
complex number	$z = a + bi$	multivector	$Z = a_1 + a_2 I$
rotor	$z' = ze^{i\phi}$	rotor	$Z' = Ze^{I\phi}$
			$v' = ve^{I\phi}$
90° rotor	$v' = -a_2\mathrm{i} + a_1\mathrm{j}$	90° rotor	$v' = vI$

8.25 Relationship between the outer product and the cross product

In chapter 7 we discovered the close similarity between the outer product and the cross product and saw that the bivector coefficients for the outer product are identical to the coefficients for the axial vector resulting from the cross product. We are now in a position to discover the algebraic relationship between the two products.

Starting with two vectors a and b, their cross and outer products are

$$a \times b = \begin{vmatrix} \mathrm{e}_1 & \mathrm{e}_2 & \mathrm{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$a \times b = (a_2b_3 - a_3b_2)\mathrm{e}_1 + (a_3b_1 - a_1b_3)\mathrm{e}_2 + (a_1b_2 - a_2b_1)\mathrm{e}_3 \tag{8.248}$$

and

$$a \wedge b = (a_2b_3 - a_3b_2)\mathrm{e}_{23} + (a_3b_1 - a_1b_3)\mathrm{e}_{31} + (a_1b_2 - a_2b_1)\mathrm{e}_{12}. \tag{8.249}$$

Let's see what happens when we pre-multiply Eq. (8.249) by the pseudoscalar e_{123}:

$$\mathrm{e}_{123}(a \wedge b) = (a_2b_3 - a_3b_2)\mathrm{e}_{123}\mathrm{e}_{23} + (a_3b_1 - a_1b_3)\mathrm{e}_{123}\mathrm{e}_{31} + (a_1b_2 - a_2b_1)\mathrm{e}_{123}\mathrm{e}_{12}$$

$$\mathrm{e}_{123}(a \wedge b) = -(a_2b_3 - a_3b_2)\mathrm{e}_1 - (a_3b_1 - a_1b_3)\mathrm{e}_2 - (a_1b_2 - a_2b_1)\mathrm{e}_3. \tag{8.250}$$

This has almost created the cross product in Eq. (8.248) apart from the inverted sign. So let's multiply Eq. (8.250) by -1:

$$-\mathrm{e}_{123}(a \wedge b) = (a_2b_3 - a_3b_2)\mathrm{e}_1 + (a_3b_1 - a_1b_3)\mathrm{e}_2 + (a_1b_2 - a_2b_1)\mathrm{e}_3 \tag{8.251}$$

which creates the cross product in Eq. (8.248). Thus we can state that

$$a \times b = -\mathrm{e}_{123}(a \wedge b) \text{ or } -I(a \wedge b). \tag{8.252}$$

Therefore, given a bivector B, the vector v orthogonal to the planar surface is given by

$$v = -IB \tag{8.253}$$

which is determined algebraically.

For example, Fig. 8.13 shows two vectors a and b where

$$a = -\mathrm{e}_2 + \mathrm{e}_3 \tag{8.254}$$

$$b = \mathrm{e}_1 - \mathrm{e}_2. \tag{8.255}$$

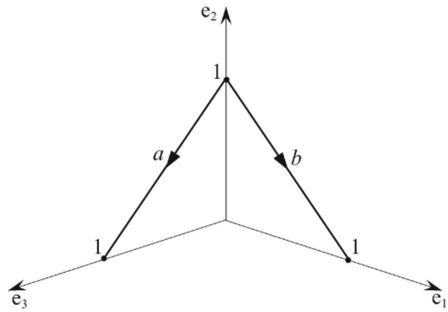

FIGURE 8.13.

The cross product is given by

$$a \times b = c = \begin{vmatrix} e_1 & e_2 & e_3 \\ 0 & -1 & 1 \\ 1 & -1 & 0 \end{vmatrix}$$
$$= e_1 + e_2 + e_3 \tag{8.256}$$

which is expected from the symmetry of the vectors.

Now let's compute c using Eq. (8.253):

$$\begin{aligned} B &= a \wedge b \\ &= (-e_2 + e_3) \wedge (e_1 - e_2) \\ &= -e_2 \wedge e_1 + e_2 \wedge e_2 + e_3 \wedge e_1 - e_3 \wedge e_2 \\ B &= e_{12} + e_{31} + e_{23}. \end{aligned} \tag{8.257}$$

Therefore,

$$\begin{aligned} c &= -IB \\ &= -e_{123}(e_{12} + e_{31} + e_{23}) \\ &= e_3 + e_2 + e_1 \\ c &= e_1 + e_2 + e_3 \end{aligned} \tag{8.258}$$

which is identical to the previous result.

Now that we have a mechanism to move between GA and the cross product, we can prove various identities in vector analysis using GA. For example, let's expand the vector triple product $(a \times b) \times c$.

Starting with $a \times b$:

$$\begin{aligned} a \times b &= -I(a \wedge b) \\ &= -e_{123}(a_1e_1 + a_2e_2 + a_3e_3) \wedge (b_1e_1 + b_2e_2 + b_3e_3) \\ &= -e_{123}((a_2b_3 - a_3b_2)e_{23} + (a_3b_1 - a_1b_3)e_{31} + (a_1b_2 - a_2b_1)e_{12}) \\ a \times b &= (a_2b_3 - a_3b_2)e_1 + (a_3b_1 - a_1b_3)e_2 + (a_1b_2 - a_2b_1)e_3 \end{aligned} \tag{8.259}$$

substitute λ_i for each coefficient

$$a \times b = \lambda_1e_1 + \lambda_2e_2 + \lambda_3e_3. \tag{8.260}$$

Expand $(a \times b) \times c$:

$$\begin{aligned} (a \times b) \times c &= -I(\lambda_1e_1 + \lambda_2e_2 + \lambda_3e_3) \wedge (c_1e_1 + c_2e_2 + c_3e_3) \\ &= -e_{123}((\lambda_2c_3 - \lambda_3c_2)e_{23} + (\lambda_3c_1 - \lambda_1c_3)e_{31} + (\lambda_1c_2 - \lambda_2c_1)e_{12}) \\ (a \times b) \times c &= (\lambda_2c_3 - \lambda_3c_2)e_1 + (\lambda_3c_1 - \lambda_1c_3)e_2 + (\lambda_1c_2 - \lambda_2c_1)e_3. \end{aligned}$$

Re-substitute for each λ_i

$$\begin{aligned} (a \times b) \times c = {} & ((a_3b_1 - a_1b_3)c_3 - (a_1b_2 - a_2b_1)c_2)e_1 \\ & + ((a_1b_2 - a_2b_1)c_1 - (a_2b_3 - a_3b_2)c_3)e_2 \\ & + ((a_2b_3 - a_3b_2)c_2 - (a_3b_1 - a_1b_3)c_1)e_3. \end{aligned} \tag{8.261}$$

Rearrange the order

$$\begin{aligned} (a \times b) \times c = {} & (a_2c_2 + a_3c_3)b_1e_1 + (a_1c_1 + a_3c_3)b_2e_2(a_1c_1 + a_2c_2)b_3e_3 \\ & - ((b_2c_2 + b_3c_3)a_1e_1 + (b_1c_1 + b_3c_3)a_2e_2 + (b_1c_1 + b_2c_2)a_3e_3) \end{aligned} \tag{8.262}$$

Now we add the following zero term to complete the inner products:

$$(a_1c_1)b_1e_1 + (a_2c_2)b_2e_2 + (a_3c_3)b_3e_3 - (b_1c_1)a_1e_1 - (b_2c_2)a_2e_2 - (b_3c_3)a_3e_3 = 0 \tag{8.263}$$

$$\begin{aligned} (a \times b) \times c = {} & (a_1c_1 + a_2c_2 + a_3c_3)b_1e_1 + (a_1c_1 + a_2c_2 + a_3c_3)b_2e_2(a_1c_1 + a_2c_2 + a_3c_3)b_3e_3 \\ & - ((b_1c_1 + b_2c_2 + b_3c_3)a_1e_1 + (b_1c_1 + b_2c_2 + b_3c_3)a_2e_2 + (b_1c_1 + b_2c_2 + b_3c_3)a_3e_3) \end{aligned} \tag{8.264}$$

therefore,

$$(a \times b) \times c = (a \cdot c)b - (b \cdot c)a. \tag{8.265}$$

8.26 Relationship between geometric algebra and quaternions

In chapter 6 we reviewed the ideas behind quaternions and saw that a quaternion is defined as the sum of a scalar and a vector, where Hamilton's imaginaries i, j and k obey the product rules shown in Table 8.10.

TABLE 8.10

	i	j	k
i	-1	k	$-j$
j	$-k$	-1	i
k	j	$-i$	-1

and $ijk = -1$.

If we let

$$B_1 = e_2 \wedge e_3 \tag{8.266}$$

$$B_2 = e_3 \wedge e_1 \tag{8.267}$$

$$B_3 = e_1 \wedge e_2 \tag{8.268}$$

the bivector products obey the rules shown in Table 8.11.

TABLE 8.11

	B_1	B_2	B_3
B_1	-1	$-B_3$	B_2
B_2	B_3	-1	$-B_1$
B_3	$-B_2$	B_1	-1

and

$$B_1 B_2 B_3 = +1. \tag{8.269}$$

The subtle difference between Table 8.10 and Table 8.11 is that, apart from the diagonal, the signs are reversed, which suggests that there is a difference in the handedness of the axial systems. To confirm this, Fig. 8.14 shows a left-handed set of bivectors, which obey the rules shown in Table 8.12.

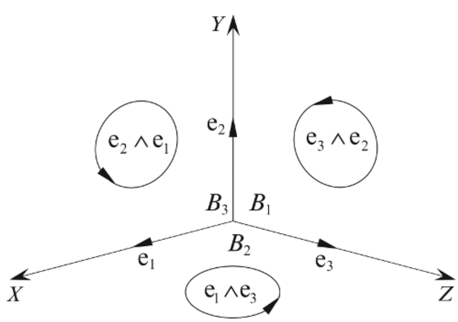

FIGURE 8.14.

Table 8.12

	B_1	B_2	B_3
B_1	-1	B_3	$-B_2$
B_2	$-B_3$	-1	B_1
B_3	B_2	$-B_1$	-1

We can see from Tables 8.10 and 8.12 the intimate relationship between Hamilton's imaginaries and a left-handed set of bivectors, which is elegantly described by Chris Doran and Anthony Lasenby in their book *Geometric Algebra for Physicists* [14]. What is strange, is that even the greatest mathematicians can misinterpret their discoveries, and what is so ironic is that Grassmann's algebra embraced vectors, bivectors and quaternions and would have changed the path of mathematics had it been adopted at the time.

8.27 Inverse of a vector

Associative algebras such as the algebra of real numbers and complex numbers permit division. For example, if

$$\alpha\beta = \delta \tag{8.270}$$

then

$$\alpha = \delta\beta^{-1}. \tag{8.271}$$

Similarly, given these complex numbers

$$(a + ib)(c + id) = e + if \tag{8.272}$$

then we can state that

$$(a + ib) = (e + if)(c + id)^{-1}. \tag{8.273}$$

And as geometric algebra is associative, we can divide by vectors. For example, given that a multivector $B = ab$, then we can multiply throughout by b and state that

$$Bb = (ab)b = ab^2 \tag{8.274}$$

which means that

$$B\frac{b}{b^2} = a \tag{8.275}$$

or

$$Bb^{-1} = a \tag{8.276}$$

where

$$b^{-1} = \frac{b}{b^2} = \frac{b}{\|b\|^2}. \tag{8.277}$$

We can illustrate this with an example.

Two vectors a and b are given by

$$a = 3e_1 + 4e_2 \tag{8.278}$$

$$b = e_1 + e_2 \tag{8.279}$$

and a multivector B is given by

$$\begin{aligned} B &= ab \\ &= (3e_1 + 4e_2)(e_1 + e_2) \\ &= 3 + 3e_{12} - 4e_{12} + 4 \\ B &= 7 - e_{12}. \end{aligned} \tag{8.280}$$

Now let's compute b^{-1}

$$\begin{aligned} b^{-1} &= \frac{b}{b^2} \\ &= \frac{e_1 + e_2}{\|\sqrt{e_1^2 + e_2^2}\|^2} \\ b^{-1} &= \frac{1}{2}(e_1 + e_2). \end{aligned} \tag{8.281}$$

Therefore, we can recover a from B as follows:

$$\begin{aligned} a &= Bb^{-1} \\ &= \frac{1}{2}(7 - e_{12})(e_1 + e_2) \\ &= \frac{1}{2}(7e_1 + 7e_2 - e_{12}e_1 - e_{12}e_2) \\ &= \frac{1}{2}(7e_1 + 7e_2 + e_2 - e_1) \\ a &= 3e_1 + 4e_2 \end{aligned} \tag{8.282}$$

which is correct.

Next, we consider a multivector $C = abc$, where

$$a = 3e_1 + 4e_2 \tag{8.283}$$

$$b = e_1 + e_2 \tag{8.284}$$

$$c = e_3. \tag{8.285}$$

Therefore,

$$\begin{aligned} C &= (3e_1 + 4e_2)(e_1 + e_2)e_3 \\ &= (3e_1^2 + 3e_{12} - 4e_{12} + 4e_2^2)e_3 \\ &= (7 - e_{12})e_3 \\ C &= 7e_3 - e_{123}. \end{aligned} \tag{8.286}$$

Now, if we are given c, we can find the bivector term as follows:

$$\begin{aligned} ab &= Cc^{-1} \\ &= \frac{Cc}{\|a\|^2} \\ &= (7e_3 - e_{123})e_3 \\ ab &= 7 - e_{12} \end{aligned} \tag{8.287}$$

which is correct.

8.28 The meet operation

For all sorts of reasons we are always interested in the intersections of lines, planes, spheres, cylinders, etc., and GA's *meet* operation provides a way of calculating such intersections. For example, the meet of A and B is written $A \vee B$, and without proof, is defined as

$$A \vee B = A^* \cdot B. \tag{8.288}$$

To illustrate how this operation works we first examine the intersections of the three basis 2-blades, followed by the intersection of two arbitrary blades.

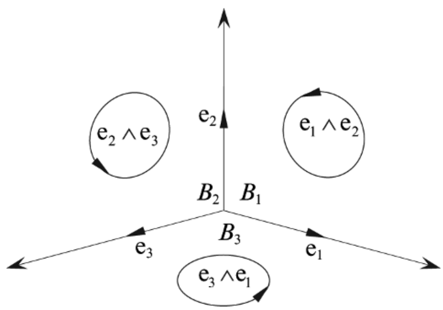

FIGURE 8.15.

Figure 8.15 shows the three basis 2-blades B_1, B_2, B_3, and it is obvious that

$$B_1 \vee B_2 = e_2 \quad B_2 \vee B_3 = e_3 \quad B_3 \vee B_1 = e_1. \tag{8.289}$$

Now let's demonstrate how the meet operation confirms this result.

$$B_1 = e_1 \wedge e_2 \quad B_2 = e_2 \wedge e_3 \quad B_3 = e_3 \wedge e_1. \tag{8.290}$$

Therefore,

$$\begin{aligned} B_1 \vee B_2 &= B_1^* \cdot B_2 \\ &= (e_{123}e_{12}) \cdot e_{23} \\ B_1 \vee B_2 &= -e_3 \cdot e_{23}. \end{aligned} \tag{8.291}$$

Using the identity $a \cdot B = \frac{1}{2}(aB - Ba)$

$$B_1 \vee B_2 = \frac{1}{2}(-e_{323} + e_{233})$$

$$B_1 \vee B_2 = e_2. \tag{8.292}$$

Similarly,

$$\begin{aligned} B_2 \vee B_3 &= B_2^* \cdot B_3 \\ &= (e_{123}e_{23}) \cdot e_{31} \\ &= -e_1 \cdot e_{31} \\ &= \frac{1}{2}(-e_{131} + e_{311}) \end{aligned}$$

$$B_2 \vee B_3 = e_3 \tag{8.293}$$

and

$$\begin{aligned} B_3 \vee B_1 &= B_3^* \cdot B_1 \\ &= (e_{123}e_{31}) \cdot e_{12} \\ &= -e_2 \cdot e_{12} \\ &= \frac{1}{2}(-e_{212} + e_{122}) \end{aligned}$$

$$B_3 \vee B_1 = e_1. \tag{8.294}$$

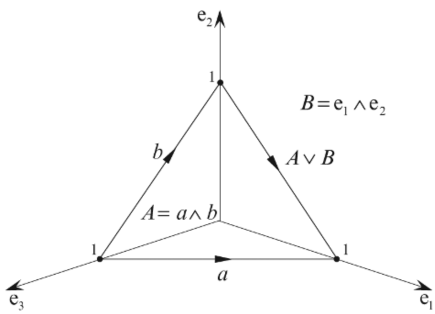

FIGURE 8.16.

The next example is shown in Fig. 8.16 where one of the planes is away from the origin. The meet of the two blades A and B is a line passing through the two points $(1,0,0)$ and $(0,1,0)$, whose direction vector is given by $\pm(e_1 - e_2)$. Let's compute the product $A^* \cdot B$ to confirm this prediction.

Given

$$a = e_1 - e_3 \tag{8.295}$$

$$b = e_2 - e_3 \tag{8.296}$$

$$A = a \wedge b$$
$$= (e_1 - e_3) \wedge (e_2 - e_3)$$
$$A = e_{12} - e_{13} - e_{32} \quad (8.297)$$
$$B = e_{12} \quad (8.298)$$

then

$$A \vee B = A^* \cdot B$$
$$= e_{123}(e_{12} - e_{13} - e_{32}) \cdot e_{12}$$
$$A \vee B = (-e_3 - e_2 - e_1) \cdot e_{12}. \quad (8.299)$$

Expand using $a \cdot B = \langle aB \rangle_1$

$$A \vee B = \langle(-e_3 - e_2 - e_1)e_{12}\rangle_1$$
$$A \vee B = e_1 - e_2 \quad (8.300)$$

which is correct.

We explore other applications of the meet operation in the following chapters.

8.29 Summary

This chapter has covered a large number of topics, which, if understood completely, can be summarized as follows:

Geometric algebra provides a coordinate free, algebraic framework for describing geometry in any number of dimensions. At the heart of the algebra is an associative, geometric product which has real and imaginary parts and is defined as the sum of the inner and outer products. It is also invertible. The inner product is the familiar inner product $a \cdot b$ whereas the outer product is defined as the outer product $a \wedge b$. Thus the geometric product of two vectors is defined as

$$ab = a \cdot b + a \wedge b. \quad (8.301)$$

The outer product defines a directed area, which, unlike the cross product, exists in space of any number of dimensions. However, like the cross product, it is antisymmetric:

$$a \wedge b = -b \wedge a. \quad (8.302)$$

The outer product creates a new entity called a bivector, which is a directed area defined by a pair of vectors. In $\mathbb{R}^2$ there is only one unit bivector: $e_1 \wedge e_2 = e_{12}$, whereas in $\mathbb{R}^3$ there are three: $e_1 \wedge e_2, e_2 \wedge e_3$ and $e_3 \wedge e_1$. Thus, the outer product of two vectors in $\mathbb{R}^2$ is represented as

$$a \wedge b = \lambda_1(e_1 \wedge e_2) \quad \{\lambda_1 \in \mathbb{R}\} \quad (8.303)$$

and in $\mathbb{R}^3$ it is represented as

$$a \wedge b = \lambda_1(e_1 \wedge e_2) + \lambda_2(e_2 \wedge e_3) + \lambda_3(e_3 \wedge e_1) \quad \{\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}\}. \quad (8.304)$$

In $\mathbb{R}^3$ the outer product of three vectors $a \wedge b \wedge c$ (i.e. a trivector) represents a directed volume.

It is possible to linearly combine scalars, vectors, bivectors and trivectors to create multivectors that form a closed algebra. These elements possess a natural hierarchy which is described in terms of their grade where scalars are grade 0, vectors are grade 1, bivectors are grade 2, and trivectors are grade 3. The inner product has grade reducing qualities as it transforms two vectors into a scalar, whereas the outer product has grade raising qualities as it transforms two vectors into a bivector.

The inner and outer products can be defined in terms of the geometric product using

$$a \cdot b = \frac{1}{2}(ab + ba) \tag{8.305}$$

and

$$a \wedge b = \frac{1}{2}(ab - ba). \tag{8.306}$$

The axioms defining the algebra are

$$a(bc) = (ab)c \tag{8.307}$$

$$a(b + c) = ab + ac \tag{8.308}$$

$$(b + c)a = ba + ca \tag{8.309}$$

$$\lambda a = a\lambda \tag{8.310}$$

$$a^2 = \pm\|a\|^2. \tag{8.311}$$

An unusual feature of geometric algebra is that the highest graded element for any space (bivector for $\mathbb{R}^2$, trivector for $\mathbb{R}^3$) squares to -1, which introduces imaginary features to multivectors. These elements are called pseudoscalars.

Multivectors can be added, subtracted, multiplied together and even divided by a vector. When adding or subtracting multivectors, like elements are combined individually. However, the product of two multivectors is computed using the rules summarized in Table 8.5.

The number of elements belonging to a multivector is determined by the number of combinations of n elements selected p at a time ${}_nC_p$. For example, in $\mathbb{R}^2$ we have 1 scalar, 2 unit basis vectors and 1 unit bivector. Whereas in $\mathbb{R}^3$, we have 1 scalar, three unit basis vectors, three unit bivectors and 1 unit trivector. In $\mathbb{R}^4$ we have 1 scalar, 4 unit basis vectors, 6 unit bivectors, 4 unit trivectors and 1 unit quadvector.

In $\mathbb{R}^2$ the product of a unit bivector (pseudoscalar) I and a vector rotate the vector 90°. For example

$$\mathrm{e}_1 I = \mathrm{e}_2 \tag{8.312}$$

whereas

$$I\mathrm{e}_1 = -\mathrm{e}_2. \tag{8.313}$$

In $\mathbb{R}^3$ premultiplying a vector by a bivector performs two operations:

- first, it rotates the projection of the vector on the bivector clockwise 90°
- second, it creates a volume by sweeping the bivector along the perpendicular component of the vector.

In $\mathbb{R}^3$ premultiplying a vector by a trivector creates a multivector consisting of bivector terms. In the case of the unit basis vectors we have

$$e_{123}e_1 = e_{23} \tag{8.314}$$

$$e_{123}e_2 = e_{31} \tag{8.315}$$

$$e_{123}e_3 = e_{12}. \tag{8.316}$$

Apart from the rotations described above, GA contains some powerful 3D rotation features that are described in the following chapter.