7 Geometric Algebra

7.1 Introduction

In 1844 when Hamilton published his invention of quaternions, the German mathematician and schoolteacher, Hermann Gunther Grassmann [1809–1877], published the first edition of his geometric calculus *Lineale Ausdehnungslehre*, which offered an original algebra for describing geometric operations. The word '*Ausdehnungslehre*' translates as '*theory of extension*' and the principal algebraic product of the theory was the *exterior product*. The notion of *extension* is something that Euclid was aware of, in that the product of two lengths create an area, and the product of a length and an area create a volume. Grassmann discovered an algebra where his exterior product of vectors created areas, volumes and higher-dimensional objects.

Being *just* a schoolteacher and standing in the shadow of Hamilton, who was knighted, and a

> Fellow of the American Society of Arts and Sciences,
> Fellow of the Society of Arts for Scotland,
> Fellow of the Royal Astronomical Society of London,
> Fellow of the Royal Northern Society of Antiquaries at Copenhagen,
> Honorary Member of the Institute of France,
> Honorary Member of the Imperial or Royal Academies of St. Petersburgh, Berlin and Turin,
> Honorary Member of the Royal Societies of Edinburgh and Dublin,
> Honorary Member of the Cambridge Philosophical Society,
> Honorary Member of the New York Historical Society,
> Honorary Member of the Society of Natural Sciences at Lausanne,
> Honorary Member of other Scientific Societies in British and Foreign Countries,
> Andrews' Professor of Astronomy in the University of Dublin, and
> Royal Astronomer of Ireland,

it is not surprising that few people bothered to buy or read Grassmann's book!

Grassmann had not helped matters by writing a rather dense description of his geometric calculus. For in his book he presented new ideas on vector analysis, vector addition and subtraction, two vector products and vector differentiation, all interwoven with his philosophy on pure thought and existence. Not only that, it also applied to any number of dimensions [6].

In 1861 Grassmann published an updated version of his book with the title *Die Ausdehnungslehre: Vollstanding und in streger Form bearbeitet,* by which time he was Professor at the Stettin Gymnasium. But in spite of this academic promotion, Grassmann had to pay for the publishing costs, which covered a run of three-hundred books, and he died a few years later before mathematicians realized that he had been a genius of the first order.

Eventually, the English mathematician, William Kingdom Clifford [1845–1879], recognized the brilliance of Grassmann's ideas and formalized what today has become known as *geometric algebra*.

7.2 Foundations of geometric algebra

Basically, there are three ways authors approach an introduction to GA: The first group adopt an abstract algebraic approach where axioms give rise to an algebra — a Clifford algebra — which describe and resolve geometric problems in any number of dimensions. The second group starts with some simple algebraic axioms and show how GA flows naturally from these axioms. The third group take a vectorial approach and show how existing vector products lead to the principles of GA. Either approach is valid, but the author's personal preference is to support the second and third approaches, which are explored in this chapter, and we begin by reviewing some important ideas that should have emerged from the previous chapters. But one more point before we start. To distinguish vectors from scalars it is common practice to embolden vector names. Indeed, this convention was employed in the previous chapters. But in GA virtually everything is a vector of some sort and some authors have abandoned this convention and identify vectorial quantities using an italic font. Obviously, one has to be careful to distinguish scalars from vectorial quantities, which is accomplished by using letters from the Greek alphabet for scalars.

In the every-day algebra of real numbers we are familiar with its associative, distributive and commutative properties. In the algebra of complex numbers we make allowances for the fact that $i^2 = -1$ and that multiplying a complex number by i effectively rotates it anticlockwise 90° on the complex plane. In vector algebra we discover that the vector product creates a third vector perpendicular to the plane containing the original vectors, and is antisymmetric. Well, it just so happens that GA is associative, distributive and involves an antisymmetric product, therefore we should not be surprised that it also has imaginary properties.

7.3 Introduction to geometric algebra

7.3.1 Length, area and volume

In the physical world of 3D space we measure the linear extension or something, i.e. its length; its planar coverage – its area; and its space filling capacity – its volume. This enables us to describe a room as being 3 meters high, a floor as being 16 square meters, and a room's volume being 48 cubic meters. It is difficult to think of a situation when in every-day parlance we would describe something as having a negative length, area or volume, but in mathematics, such entities do exist, and GA provides a framework for their description.

Primarily, GA manipulates vectors, although scalar quantities are easily integrated into the equations, but, for the moment, we will concentrate on the role vectors play within the algebra.

A single vector, independent of its spatial dimension, has two qualities: orientation and magnitude. Its orientation is determined by the sign of its components, whilst its magnitude is represented by its length, which in turn is derived from its components. A vector's orientation is reversed, simply by switching the signs of its components.

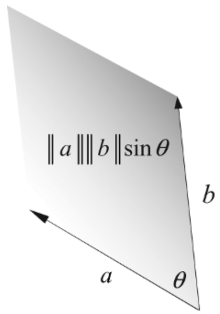

Figure 7.1.

The product of two vectors can be used to represent the area of a parallelogram as shown in Fig. 7.1, where the area is given by

$$area = \|a\|\|b\|\sin\theta. \tag{7.1}$$

Because $\|a\|$ and $\|b\|$ are scalars, their order is immaterial. Furthermore, we have assumed that the angle θ is always positive, hence its sign is always positive, which is why area is normally regarded as a positive quantity.

Grassmann was aware that mathematics, especially determinants, supported positive *and* negative areas and volumes, and wanted to exploit this feature. His solution was to create a vector product that he called the *outer product* and written $a \wedge b$. The wedge symbol "$\wedge$" is why the product is also known as the *wedge product*, and it is worth noting that this symbol is also used by French mathematicians for the vector (cross) product. The outer product is sensitive to the order of the vectors it manipulates, and permits us to distinguish between $a \wedge b$ and $b \wedge a$. In fact, the algebra ensures that

$$a \wedge b = -b \wedge a. \tag{7.2}$$

Therefore, when using the outer product we must think carefully about their order, which is why in chapter 6 we discussed the order of axial systems. This idea is developed in Fig. 7.2 where we see the graphical difference between the two products.

Figure 7.2a shows that $a \wedge b$ creates an area from vectors a and b forming an anticlockwise rotation, whereas Fig. 7.2b shows that $b \wedge a$ creates an area from vectors b and a forming a clockwise rotation. The directed circle is included to remind us of the area's orientation.

From vector algebra we know that there are two important products: the scalar and the vector product. The scalar product creates a non-zero scalar value when the associated vectors are not perpendicular, and tells us something about the mutual alignment of the two vectors. Whereas,

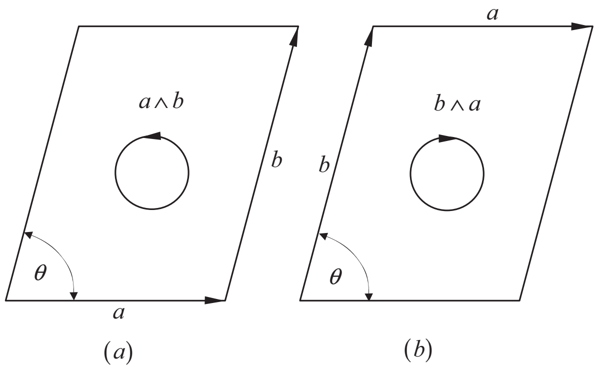

Figure 7.2.

the vector product creates a non-zero vector when the associated vectors are not parallel, and tells us something about the area of the parallelogram formed by the two vectors.

GA adopts these two products but changes the interpretation of the vector product. Hamilton interpreted the result $a \times b$ as a third vector c perpendicular to the plane containing a and b. Although this interpretation works in three dimensions, it is ambiguous in higher dimensions. Grassmann interpreted the result of the vector product in terms of its capacity to compute a signed area, which is why he created the *outer product.*

7.4 The outer product

Now we already know that the magnitude of the vector product is given by

$$\|a \times b\| = \|a\|\|b\| \sin\theta \tag{7.3}$$

where θ is the angle between the two vectors. The outer product preserves this value but abandons the concept of a perpendicular vector. Instead, the value $\|a\|\|b\| \sin\theta$ is retained as the signed area formed by the two vectors.

Now although $\|a \wedge b\| = \|a\|\|b\| \sin\theta$, we must pose the question: What sort of object is $a \wedge b$? Well, for a start, it is not a vector, nor is it a simple scalar. In fact, we have to invent a new name, which is always unsettling as it is difficult to relate it to things with which we are familiar. Where the cross product $a \times b$ creates a vector, the outer product $a \wedge b$ is called a *bivector*, which is a totally new concept to grasp.

A bivector describes the orientation of a plane in terms of two vectors, and its magnitude is the area of the parallelogram formed by the vectors. Reversing the vector sequence in the product flips the sign of the area. The outer product has the same components as the cross product, but instead of using the components to form a vector, they become the projective characteristics of a planar surface.

We are very familiar with the concept of a vector and accept that it has magnitude and orientation, where its components are expressed using orthogonal basis vectors. Reversing the direction of the vector reverses its components without changing its magnitude. Similarly, a bivector has magnitude and orientation, where its components are expressed in terms of areas projected onto the bivectors formed by the orthogonal unit basis vectors. Reversing the direction of the bivector reverses its components without changing its magnitude. This is illustrated later in this chapter. For the moment, a bivector is just a name used to orient a planar area.

7.4.1 Some algebraic properties

Even with our sketchy knowledge of a bivector, it is possible to describe how the outer product responds to parallel vectors. For example

$$\|a \wedge a\| = \|a\|\|a\| \sin 0^\circ = 0. \tag{7.4}$$

Although the outer product is antisymmetric, it behaves just like the scalar product when multiplying a group of vectors:

$$\text{scalar:} \quad a \cdot (b + c) = a \cdot b + a \cdot c \tag{7.5}$$

similarly

$$\text{outer:} \quad a \wedge (b + c) = a \wedge b + a \wedge c. \tag{7.6}$$

7.4.2 Visualizing the outer product

The cross product is easy to visualize: $a \times b = c$, where c is orthogonal to the plane containing a and b. The relative direction of c is determined by the right-hand rule where using one's right hand, where the thumb aligns with a, the first finger with b, and the middle finger aligns with c. The magnitude of c equals $\|a\|\|b\| \sin\theta$, where θ is the angle between a and b, and equals the area of the parallelogram formed by a and b. This relationship is shown in Fig. 7.3.

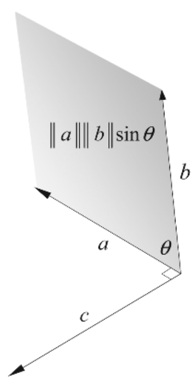

Figure 7.3.

Visualizing the outer product is slightly different. It is true that the magnitude $\|a \wedge b\|$ is $\|a\|\|b\| \sin\theta$ which represents the area of the parallelogram formed by a and b, but consider what happens if we form the product $a' \wedge b$ where $a' = a + \lambda b$:

$$\begin{aligned} a' \wedge b &= (a + \lambda b) \wedge b \\ &= a \wedge b + \lambda b \wedge b \\ a' \wedge b &= a \wedge b. \end{aligned} \tag{7.7}$$

Two other vectors generate the same bivector! Figure 7.4 illustrates what is happening.

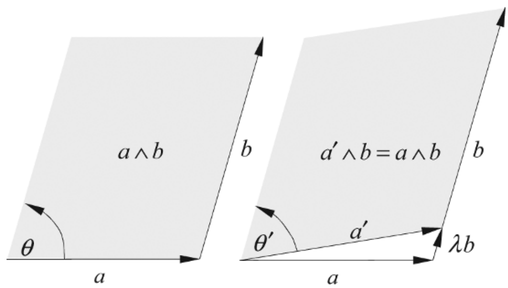

Figure 7.4.

The area created by $a' \wedge b$ is identical to that created by $a \wedge b$, so there is no single parallelogram that represents $a \wedge b$ — there are an infinite number! So why bother trying to represent $a \wedge b$ as a parallelogram in the first place? Well, it was a starting point, but now that we have discovered this feature of the outer product, why not substitute another shape such as a circle instead of a parallelogram, and make the area of the circle equal to $\|a\|\|b\| \sin\theta$? That was a rhetorical question, but a useful suggestion, and Fig. 7.5 shows what is implied.

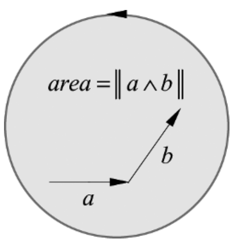

Figure 7.5.

7.4.3 Orthogonal bases

GA works in any number of dimensions, and anticipating the need to embrace a large number of dimensions we require a notation for the extended orthogonal axial systems. Conventionally, i and j represent the unit basis vectors for $\mathbb{R}^2$, and i, j and k represent the unit basis vectors for $\mathbb{R}^3$.

If we continue with this notation the alphabet cannot support very high-dimensional spaces. An alternative convention is to use $e_1, e_2, e_3, \ldots e_n$ to represent the orthogonal unit basis vectors.

Using this notation we define two vectors in $\mathbb{R}^2$ as

$$a = a_1 e_1 + a_2 e_2 \tag{7.8}$$

$$b = b_1 e_1 + b_2 e_2. \tag{7.9}$$

We can now state the outer product as

$$a \wedge b = (a_1 e_1 + a_2 e_2) \wedge (b_1 e_1 + b_2 e_2) \tag{7.10}$$

which expands to

$$a \wedge b = a_1 b_1 (e_1 \wedge e_1) + a_1 b_2 (e_1 \wedge e_2) + a_2 b_1 (e_2 \wedge e_1) + a_2 b_2 (e_2 \wedge e_2). \tag{7.11}$$

Substituting the following observations

$$e_1 \wedge e_1 = e_2 \wedge e_2 = 0 \text{ and } e_2 \wedge e_1 = -e_1 \wedge e_2 \tag{7.12}$$

we obtain

$$a \wedge b = a_1 b_2 (e_1 \wedge e_2) - a_2 b_1 (e_1 \wedge e_2) \tag{7.13}$$

simplifying, we obtain

$$a \wedge b = (a_1 b_2 - a_2 b_1)(e_1 \wedge e_2). \tag{7.14}$$

The scalar term $a_1 b_2 - a_2 b_1$ in Eq. (7.14) looks familiar — in fact, it is the magnitude of the imaginary term of Eq. (3.17), the value of which equals $\|a\|\|b\| \sin\theta$, which is the area of the parallelogram formed by a and b. So in this context, the outer product $a \wedge b$ is a scalar area multiplying the unit bivector $e_1 \wedge e_2$, which just means that the area is associated with the plane defined by $e_1 \wedge e_2$. Figure 7.6 illustrates this relationship.

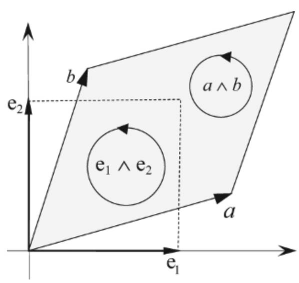

Figure 7.6.

Now let's compute $b \wedge a$:

$$b \wedge a = (b_1 e_1 + b_2 e_2) \wedge (a_1 e_1 + a_2 e_2)$$

which expands to

$$b \wedge a = a_1 b_1 (e_1 \wedge e_1) + a_2 b_1 (e_1 \wedge e_2) + a_1 b_2 (e_2 \wedge e_1) + a_2 b_2 (e_2 \wedge e_2). \tag{7.15}$$

Substituting the following observations

$$e_1 \wedge e_1 = e_2 \wedge e_2 = 0 \text{ and } e_2 \wedge e_1 = -e_1 \wedge e_2 \tag{7.16}$$

we obtain

$$b \wedge a = a_2 b_1 (e_1 \wedge e_2) - a_1 b_2 (e_1 \wedge e_2). \tag{7.17}$$

Simplifying, we obtain

$$b \wedge a = -(a_1 b_2 - a_2 b_1)(e_1 \wedge e_2) \tag{7.18}$$

which confirms that $b \wedge a = -a \wedge b$.

Now let's consider the outer product in $\mathbb{R}^3$:

$$a = a_1 e_1 + a_2 e_2 + a_3 e_3 \tag{7.19}$$

$$b = b_1 e_1 + b_2 e_2 + b_3 e_3. \tag{7.20}$$

The outer product is

$$a \wedge b = (a_1 e_1 + a_2 e_2 + a_3 e_3) \wedge (b_1 e_1 + b_2 e_2 + b_3 e_3) \tag{7.21}$$

which expands to

$$\begin{aligned} a \wedge b = {} & a_1 b_1 (e_1 \wedge e_1) + a_1 b_2 (e_1 \wedge e_2) + a_1 b_3 (e_1 \wedge e_3) + a_2 b_1 (e_2 \wedge e_1) + a_2 b_2 (e_2 \wedge e_2) \\ & + a_2 b_3 (e_2 \wedge e_3) + a_3 b_1 (e_3 \wedge e_1) + a_3 b_2 (e_3 \wedge e_2) + a_3 b_3 (e_3 \wedge e_3). \end{aligned} \tag{7.22}$$

Substituting

$$e_1 \wedge e_1 = e_2 \wedge e_2 = e_3 \wedge e_3 = 0 \tag{7.23}$$

and

$$e_2 \wedge e_1 = -e_1 \wedge e_2 \quad e_1 \wedge e_3 = -e_3 \wedge e_1 \quad e_3 \wedge e_2 = -e_2 \wedge e_3 \tag{7.24}$$

we obtain

$$\begin{aligned} a \wedge b = {} & a_1 b_2 (e_1 \wedge e_2) - a_1 b_3 (e_3 \wedge e_1) - a_2 b_1 (e_1 \wedge e_2) \\ & + a_2 b_3 (e_2 \wedge e_3) + a_3 b_1 (e_3 \wedge e_1) - a_3 b_2 (e_2 \wedge e_3). \end{aligned} \tag{7.25}$$

Simplifying, we obtain

$$a \wedge b = (a_1 b_2 - a_2 b_1) e_1 \wedge e_2 + (a_2 b_3 - a_3 b_2) e_2 \wedge e_3 + (a_3 b_1 - a_1 b_3) e_3 \wedge e_1. \tag{7.26}$$

You may be wondering why the unit basis bivectors in Eq. (7.26) have been chosen in this way, especially $e_3 \wedge e_1$. This could easily be $e_1 \wedge e_3$. To understand why, refer to Fig. 7.7, which shows a right-handed axial system and where each orthogonal plane is defined by its associated unit basis bivectors.

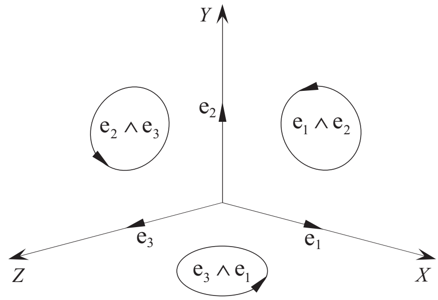

FIGURE 7.7.

Figure 7.7 also shows the orthogonal alignment of the Cartesian axes with the unit basis bivectors:

$$\text{the } x\text{-axis is orthogonal to } e_2 \wedge e_3$$

$$\text{the } y\text{-axis is orthogonal to } e_3 \wedge e_1$$

$$\text{the } z\text{-axis is orthogonal to } e_1 \wedge e_2$$

and if Eq. (7.26) is rearranged in this sequence we obtain

$$a \wedge b = (a_2 b_3 - a_3 b_2) e_2 \wedge e_3 + (a_3 b_1 - a_1 b_3) e_3 \wedge e_1 + (a_1 b_2 - a_2 b_1) e_1 \wedge e_2. \tag{7.27}$$

Now let's look at a definition of the cross product. We begin by declaring two vectors using the conventional orthogonal unit basis vectors i, j and k:

$$a = a_1 \mathrm{i} + a_2 \mathrm{j} + a_3 \mathrm{k} \tag{7.28}$$

$$b = b_1 \mathrm{i} + b_2 \mathrm{j} + b_3 \mathrm{k}. \tag{7.29}$$

The cross product is

$$a \times b = (a_1 \mathrm{i} + a_2 \mathrm{j} + a_3 \mathrm{k}) \times (b_1 \mathrm{i} + b_2 \mathrm{j} + b_3 \mathrm{k}) \tag{7.30}$$

which expands to

$$\begin{aligned} a \times b = {} & a_1 b_1 (\mathrm{i} \times \mathrm{i}) + a_1 b_2 (\mathrm{i} \times \mathrm{j}) + a_1 b_3 (\mathrm{i} \times \mathrm{k}) + a_2 b_1 (\mathrm{j} \times \mathrm{i}) + a_2 b_2 (\mathrm{j} \times \mathrm{j}) \\ & + a_2 b_3 (\mathrm{j} \times \mathrm{k}) + a_3 b_1 (\mathrm{k} \times \mathrm{i}) + a_3 b_2 (\mathrm{k} \times \mathrm{j}) + a_3 b_3 (\mathrm{k} \times \mathrm{k}). \end{aligned} \tag{7.31}$$

The magnitude of the cross product is $\|a\| \|b\| \sin\theta$, which means that

$$\mathrm{i} \times \mathrm{i} = \mathrm{j} \times \mathrm{j} = \mathrm{k} \times \mathrm{k} = 0. \tag{7.32}$$

Therefore,

$$\begin{aligned} a \times b = {} & a_1 b_2 (\mathrm{i} \times \mathrm{j}) + a_1 b_3 (\mathrm{i} \times \mathrm{k}) + a_2 b_1 (\mathrm{j} \times \mathrm{i}) \\ & + a_2 b_3 (\mathrm{j} \times \mathrm{k}) + a_3 b_1 (\mathrm{k} \times \mathrm{i}) + a_3 b_2 (\mathrm{k} \times \mathrm{j}). \end{aligned} \tag{7.33}$$

Because the cross product is antisymmetric

$$\mathrm{j} \times \mathrm{i} = -\mathrm{i} \times \mathrm{j} \quad \mathrm{k} \times \mathrm{j} = -\mathrm{j} \times \mathrm{k} \quad \mathrm{i} \times \mathrm{k} = -\mathrm{k} \times \mathrm{i}. \tag{7.34}$$

Substituting these relationships:

$$\begin{aligned} a \times b = a_1 b_2(\mathrm{i} \times \mathrm{j}) - a_1 b_3(\mathrm{k} \times \mathrm{i}) - a_2 b_1(\mathrm{i} \times \mathrm{j}) \\ + a_2 b_3(\mathrm{j} \times \mathrm{k}) + a_3 b_1(\mathrm{k} \times \mathrm{i}) - a_3 b_2(\mathrm{j} \times \mathrm{k}). \end{aligned} \tag{7.35}$$

Collecting up like terms:

$$a \times b = (a_2 b_3 - a_3 b_2)\mathrm{j} \times \mathrm{k} + (a_3 b_1 - a_1 b_3)\mathrm{k} \times \mathrm{i} + (a_1 b_2 - a_2 b_1)\mathrm{i} \times \mathrm{j}. \tag{7.36}$$

If we place Eqs. (7.27) and (7.36) together and substitute the e notation for i, j and k, we obtain

$$a \wedge b = (a_2 b_3 - a_3 b_2)\mathrm{e}_2 \wedge \mathrm{e}_3 + (a_3 b_1 - a_1 b_3)\mathrm{e}_3 \wedge \mathrm{e}_1 + (a_1 b_2 - a_2 b_1)\mathrm{e}_1 \wedge \mathrm{e}_2 \tag{7.37}$$

$$a \times b = (a_2 b_3 - a_3 b_2)\mathrm{e}_2 \times \mathrm{e}_3 + (a_3 b_1 - a_1 b_3)\mathrm{e}_3 \times \mathrm{e}_1 + (a_1 b_2 - a_2 b_1)\mathrm{e}_1 \times \mathrm{e}_2. \tag{7.38}$$

In the cross product, the terms $(a_2 b_3 - a_3 b_2)$, $(a_3 b_1 - a_1 b_3)$ and $(a_1 b_2 - a_2 b_1)$ are the components of an orthogonal vector, whereas in the outer product they become signed areas projected onto the planes defined by the unit bivectors $\mathrm{e}_2 \wedge \mathrm{e}_3$, $\mathrm{e}_3 \wedge \mathrm{e}_1$ and $\mathrm{e}_1 \wedge \mathrm{e}_2$. And in spite of there being such similarity between the two equations, it would be dangerous to conclude that $a \wedge b \equiv a \times b$.

What Hamilton had proposed was that

$$\mathrm{e}_2 \times \mathrm{e}_3 = \mathrm{e}_1 \quad \mathrm{e}_3 \times \mathrm{e}_1 = \mathrm{e}_2 \quad \mathrm{e}_1 \times \mathrm{e}_2 = \mathrm{e}_3 \tag{7.39}$$

which is fine for $\mathbb{R}^3$, but is ambiguous for higher dimensions. So, in GA we substitute the outer product for the cross product and introduce the concept of a directed area, which holds for any number of dimensions.

Before we reveal the imaginary nature of the outer product in the next chapter, consider the scenario shown in Fig. 7.8. Two vectors a and b are shown forming a parallelogram created by their outer product $a \wedge b$ with parallel projections of the parallelogram projected onto the three orthogonal planes. The projections will normally be parallelograms, but under some conditions they could collapse to a line. Whatever happens, at least one will be a parallelogram.

We define two vectors as

$$a = a_1\mathrm{e}_1 + a_2\mathrm{e}_2 + a_3\mathrm{e}_3 \tag{7.40}$$

$$b = b_1\mathrm{e}_1 + b_2\mathrm{e}_2 + b_3\mathrm{e}_3. \tag{7.41}$$

Starting with the plane containing e_1 and e_2, which is defined by $\mathrm{e}_1 \wedge \mathrm{e}_2$, the projections of a and b are a''' and b''', respectively, where

$$a''' = a_1\mathrm{e}_1 + a_2\mathrm{e}_2 \tag{7.42}$$

$$b''' = b_1\mathrm{e}_1 + b_2\mathrm{e}_2. \tag{7.43}$$

Therefore,

$$\begin{aligned} a''' \wedge b''' &= (a_1\mathrm{e}_1 + a_2\mathrm{e}_2) \wedge (b_1\mathrm{e}_1 + b_2\mathrm{e}_2) \\ &= a_1 b_1(\mathrm{e}_1 \wedge \mathrm{e}_1) + a_1 b_2(\mathrm{e}_1 \wedge \mathrm{e}_2) + a_2 b_1(\mathrm{e}_2 \wedge \mathrm{e}_1) + a_2 b_2(\mathrm{e}_2 \wedge \mathrm{e}_2) \\ a''' \wedge b''' &= (a_1 b_2 - a_2 b_1)\mathrm{e}_1 \wedge \mathrm{e}_2 \end{aligned} \tag{7.44}$$

which is the last term in Eq. (7.27).

Similarly, we can show that

$$a' \wedge b' = (a_2b_3 - a_3b_2)e_2 \wedge e_3 \tag{7.45}$$

$$a'' \wedge b'' = (a_3b_1 - a_1b_3)e_3 \wedge e_1. \tag{7.46}$$

Thus we see that instead of creating a new vector, the outer product projects the parallelogram onto the three orthogonal planes to create three new bivectors, whose area is positive or negative. The cross product, however, takes these areas and uses them to form a vector, which happens to be orthogonal to the original parallelogram.

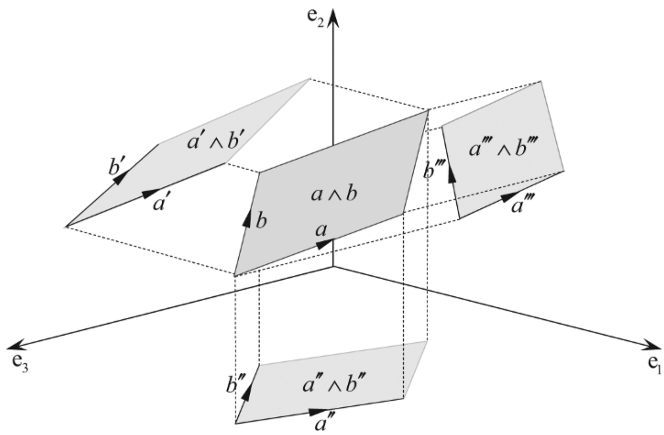

Figure 7.8.

To illustrate this concept, consider two vectors a and b

$$a = a_1e_1 + a_2e_2 + a_3e_3 \tag{7.47}$$

$$b = b_1e_1 + b_2e_2 + b_3e_3 \tag{7.48}$$

where

$$a_1 = 1 \quad a_2 = 0 \quad a_3 = 1$$

$$b_1 = 1 \quad b_2 = 1 \quad b_3 = 0 \tag{7.49}$$

which makes

$$a = e_1 + e_3 \quad b = e_1 + e_2. \tag{7.50}$$

Using Eq. (7.26)

$$a \wedge b = (a_1b_2 - a_2b_1)e_1 \wedge e_2 + (a_2b_3 - a_3b_2)e_2 \wedge e_3 + (a_3b_1 - a_1b_3)e_3 \wedge e_1$$

$$a \wedge b = (1)e_1 \wedge e_2 + (-1)e_2 \wedge e_3 + (1)e_3 \wedge e_1. \tag{7.51}$$

The signed area on the plane $e_1 \wedge e_2$ is $+1$ and is shown in Fig. 7.9. The projected area is shown crosshatched.

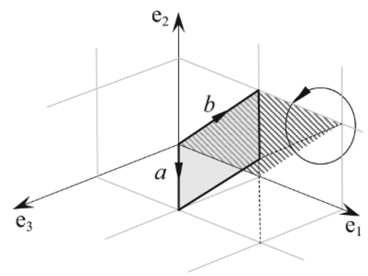

FIGURE 7.9.

Similarly, the signed area on the plane $e_2 \wedge e_3$ is -1 and is shown in Fig. 7.10. Note that the direction of the projected area opposes the direction of $e_2 \wedge e_3$.

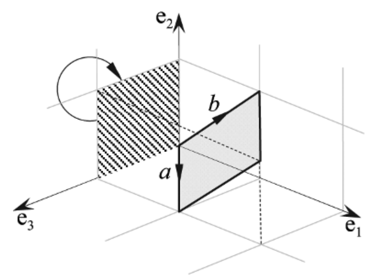

FIGURE 7.10.

And the signed area on the plane $e_3 \wedge e_1$ is $+1$, and is shown in Fig. 7.11.

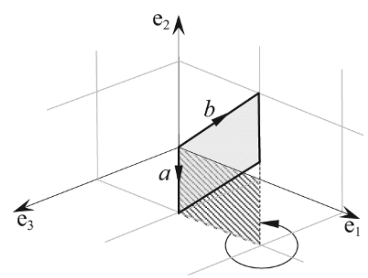

FIGURE 7.11.

Now let's compute the magnitude of the bivector $a \wedge b$.

To begin with, we need to know the angle between a and b, which is revealed using the dot product:

$$\theta = \cos^{-1}\left(\frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{\|a\|\|b\|}\right)$$

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{2}\sqrt{2}}\right) = 60^\circ. \tag{7.52}$$

Therefore,

$$\|a \wedge b\| = \|a\|\|b\| \sin 60^\circ$$
$$\|a \wedge b\| = \sqrt{2}\sqrt{2}\frac{\sqrt{3}}{2} = \sqrt{3}. \tag{7.53}$$

The next question to pose is whether this value is related to the other three areas? Well the answer is “yes”, and for a very good reason:

$$\|a \wedge b\|^2 = (a_1b_2 - a_2b_1)^2 + (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 \tag{7.54}$$

therefore,

$$\sqrt{3}^2 = (1)^2 + (-1)^2 + (1)^2 = 3. \tag{7.55}$$

Remember, that the cross product uses these coefficients as Cartesian components of the axial vector and satisfy the Pythagorean rule:

$$\|a\|^2 = a_1^2 + a_2^2 + a_3^2. \tag{7.56}$$

To prove that this holds, we need to show that Eq. (7.54) is correct.

Expanding the LHS of Eq. (7.54):

$$\|a \wedge b\|^2 = \|a\|^2\|b\|^2 \sin^2\theta = \|a\|^2\|b\|^2(1 - \cos^2\theta)$$
$$\|a \wedge b\|^2 = \|a\|^2\|b\|^2 - \|a\|^2\|b\|^2 \cos^2\theta. \tag{7.57}$$

From the dot product

$$\cos^2\theta = \frac{(a_1b_1 + a_2b_2 + a_3b_3)^2}{\|a\|^2\|b\|^2}. \tag{7.58}$$

Therefore,

$$\|a \wedge b\|^2 = \|a\|^2\|b\|^2 - (a_1b_1 - a_2b_2 - a_3b_3)^2$$
$$\|a \wedge b\|^2 = (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 - a_2b_2 - a_3b_3)^2$$

and we obtain

$$\|a \wedge b\|^2 = (a_1^2b_2^2 - 2a_1a_2b_1b_2 + a_2^2b_1^2) + (a_2^2b_3^2 - 2a_2a_3b_2b_3 + a_3^2b_2^2)$$
$$+ (a_3^2b_1^2 - 2a_3a_1b_3b_1 + a_1^2b_3^2)$$
$$\|a \wedge b\|^2 = (a_1b_2 - a_2b_1)^2 + (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2. \tag{7.59}$$

Therefore, Eq. (7.54) is correct.

Now, as

$$\|a \wedge b\| = \|a\|\|b\| \sin\theta \tag{7.60}$$
$$\|a \wedge b\|^2 = \|a\|^2\|b\|^2 \sin^2\theta \tag{7.61}$$

and

$$\|a\|\|b\|\sin^2\theta = (a_1b_2 - a_2b_1)^2 + (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 \tag{7.62}$$

therefore

$$\theta = \sin^{-1}\left(\frac{\sqrt{(a_1b_2 - a_2b_1)^2 + (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2}}{\|a\|\|b\|}\right). \tag{7.63}$$

Substituting the values for the above example:

$$\theta = \sin^{-1}\left(\frac{\sqrt{3}}{\sqrt{2}\sqrt{2}}\right) = 60^\circ. \tag{7.64}$$

The beauty of the outer product is that it works in any number of dimensions. For example, we can create two vectors in $\mathbb{R}^4$ as follows:

$$a = a_1\mathrm{e}_1 + a_2\mathrm{e}_2 + a_3\mathrm{e}_3 + a_4\mathrm{e}_4 \tag{7.65}$$

$$b = b_1\mathrm{e}_1 + b_2\mathrm{e}_2 + b_3\mathrm{e}_3 + b_4\mathrm{e}_4 \tag{7.66}$$

and form their outer product:

$$a \wedge b = (a_1\mathrm{e}_1 + a_2\mathrm{e}_2 + a_3\mathrm{e}_3 + a_4\mathrm{e}_4) \wedge (b_1\mathrm{e}_1 + b_2\mathrm{e}_2 + b_3\mathrm{e}_3 + b_4\mathrm{e}_4). \tag{7.67}$$

This explodes into

$$\begin{aligned} a \wedge b = {} & a_1b_1(\mathrm{e}_1 \wedge \mathrm{e}_1) + a_1b_2(\mathrm{e}_1 \wedge \mathrm{e}_2) + a_1b_3(\mathrm{e}_1 \wedge \mathrm{e}_3) + a_1b_4(\mathrm{e}_1 \wedge \mathrm{e}_4) \\ & + a_2b_1(\mathrm{e}_2 \wedge \mathrm{e}_1) + a_2b_2(\mathrm{e}_2 \wedge \mathrm{e}_2) + a_2b_3(\mathrm{e}_2 \wedge \mathrm{e}_3) + a_2b_4(\mathrm{e}_2 \wedge \mathrm{e}_4) \\ & + a_3b_1(\mathrm{e}_3 \wedge \mathrm{e}_1) + a_3b_2(\mathrm{e}_3 \wedge \mathrm{e}_2) + a_3b_3(\mathrm{e}_3 \wedge \mathrm{e}_3) + a_3b_4(\mathrm{e}_3 \wedge \mathrm{e}_4) \\ & + a_4b_1(\mathrm{e}_4 \wedge \mathrm{e}_1) + a_4b_2(\mathrm{e}_4 \wedge \mathrm{e}_2) + a_4b_3(\mathrm{e}_4 \wedge \mathrm{e}_3) + a_4b_4(\mathrm{e}_4 \wedge \mathrm{e}_4) \end{aligned}$$

and collapses to

$$\begin{aligned} a \wedge b = {} & (a_1b_2 - a_2b_1)(\mathrm{e}_1 \wedge \mathrm{e}_2) + (a_2b_3 - a_3b_2)(\mathrm{e}_2 \wedge \mathrm{e}_3) + (a_3b_1 - a_1b_3)(\mathrm{e}_3 \wedge \mathrm{e}_1) \\ & + (a_1b_4 - a_4b_1)(\mathrm{e}_1 \wedge \mathrm{e}_4) + (a_2b_4 - a_4b_2)(\mathrm{e}_2 \wedge \mathrm{e}_4) + (a_3b_4 - a_4b_3)(\mathrm{e}_3 \wedge \mathrm{e}_4) \end{aligned} \tag{7.68}$$

which resolves the outer product into six bivectors.

These bivectors arise because there are six ways of making 2-tuples from four axes:

$$ {}_4C_2 = \frac{4!}{(4-2)!2!} = 6. \tag{7.69} $$

In five dimensions there are 10 bivectors.

$$ {}_5C_2 = \frac{5!}{(5-2)!2!} = 10. \tag{7.70} $$

As a final example, let's consider two vectors in $\mathbb{R}^4$ and compute their outer product. The vectors are

$$ a = \mathrm{e}_1 + \mathrm{e}_3 + \mathrm{e}_4 \tag{7.71} $$

$$ b = \mathrm{e}_1 + \mathrm{e}_2 + \mathrm{e}_4. \tag{7.72} $$

Then

$$ \|a\| = \sqrt{3} \quad \|b\| = \sqrt{3} \tag{7.73} $$

and the separating angle θ is

$$ \theta = \cos^{-1}\left(\frac{2}{3}\right) \simeq 48.19°. \tag{7.74} $$

Similarly,

$$ \theta = \sin^{-1}\left(\frac{\sqrt{5}}{3}\right) \simeq 48.19°. \tag{7.75} $$

Substituting the vectors into Eq. (7.68):

$$ a \wedge b = (1)(\mathrm{e}_1 \wedge \mathrm{e}_2) + (-1)(\mathrm{e}_2 \wedge \mathrm{e}_3) + (1)(\mathrm{e}_3 \wedge \mathrm{e}_1) + (-1)(\mathrm{e}_2 \wedge \mathrm{e}_4) + (1)(\mathrm{e}_3 \wedge \mathrm{e}_4). \tag{7.76} $$

Therefore, $\|a \wedge b\|$ is given by

$$ \|a \wedge b\| = \|a\|\|b\| \sin\theta = \sqrt{3}\sqrt{3} \sin 48.19° \simeq 2.2361. \tag{7.77} $$

Finally, let's show that the $\mathbb{R}^4$ equivalent of Eq. (7.54) still holds:

$$ \begin{aligned} \|a \wedge b\|^2 &= |a_1 b_2 - a_2 b_1|^2 + |a_2 b_3 - a_3 b_2|^2 + |a_3 b_1 - a_1 b_3|^2 \\ &\quad + |a_1 b_4 - a_4 b_1|^2 + |a_2 b_4 - a_4 b_2|^2 + |a_3 b_4 - a_4 b_3|^2 \\ 2.2361^2 &= (1)^2 + (-1)^2 + (1)^2 + (0)^2 + (-1)^2 + (1)^2 = 5. \end{aligned} \tag{7.78} $$

7.5 The outer product in action

Later, we investigate a number of scenarios where the outer product is used to solve problems in computer graphics, but at this point it is worth looking at three problems where it seems that we have been using the outer product without knowing.

7.5.1 Area of a triangle

There are many ways to find the area of a triangle, but the one proposed here uses a triangle's vertex coordinates, as shown in Fig. 7.12a. The triangle has vertices A, B, C defined in an anticlockwise order, and its area is given by

$$area = \frac{1}{2}\begin{vmatrix} x_A & y_A & 1 \\ x_B & y_B & 1 \\ x_C & y_C & 1 \end{vmatrix}. \tag{7.79}$$

Using the coordinates from Fig. 7.12a we have

$$area = \frac{1}{2}\begin{vmatrix} 0 & 2 & 1 \\ 3 & 1 & 1 \\ 3 & 3 & 1 \end{vmatrix} \tag{7.80}$$

$$area = \frac{1}{2}(9 + 6 - 6 - 3) = +3 \tag{7.81}$$

which is correct. Note that reversing the triangle's vertex sequence creates a negative area:

$$area = \frac{1}{2}\begin{vmatrix} 0 & 2 & 1 \\ 3 & 3 & 1 \\ 3 & 1 & 1 \end{vmatrix}$$

$$area = \frac{1}{2}(3 + 6 - 6 - 9) = -3. \tag{7.82}$$

We can prove Eq. (7.79) algebraically, and if we create the right diagram, outer products come to our rescue. Figure 7.12b shows three position vectors a, b, c locating the vertices, which we use to form three outer products. The first product $a \wedge b$ computes the area of the parallelogram $OBCA$, and $\frac{1}{2}(a \wedge b)$ computes the area of the triangle ΔOBA. The sequence of the vertices O, A, B create a clockwise outer product, which accounts for the negative signs in ΔOBA.

The second product $b \wedge c$ computes the area of the parallelogram $OBEC$, and $\frac{1}{2}(b \wedge c)$ computes the area of the triangle ΔOBC. The sequence of the vertices O, B, C create an anticlockwise outer product, which accounts for the positive signs in ΔOBC.

The third product $c \wedge a$ computes the area of the parallelogram $OCFA$, and $\frac{1}{2}(c \wedge a)$ computes the area of the triangle ΔOCA. The sequence of the vertices O, C, A create an anticlockwise outer product, which accounts for the positive signs in ΔOCA.

The sum of the three outer products is

$$\frac{1}{2}(a \wedge b) + \frac{1}{2}(b \wedge c) + \frac{1}{2}(c \wedge a)$$

and creates three areas: two of the areas contribute toward the triangles ΔABC and ΔOBA, whilst the third area cancels the area of triangle ΔOBA, leaving behind the area of ΔABC.

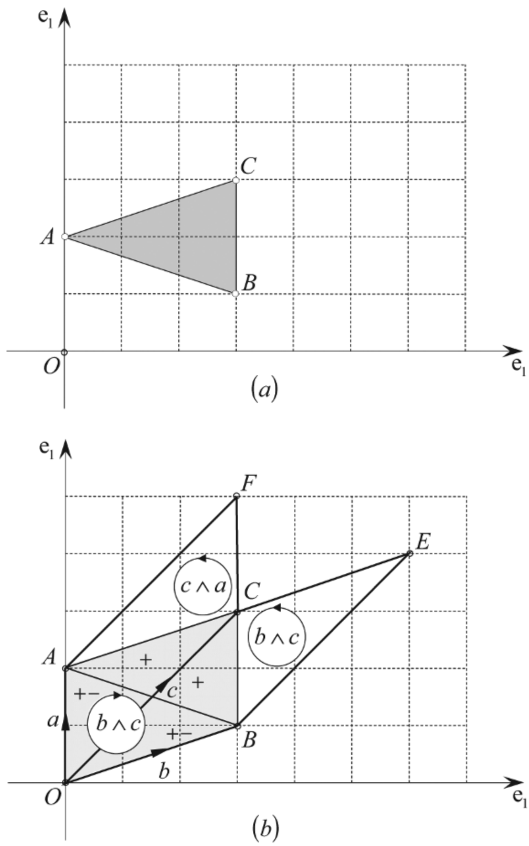

FIGURE 7.12.

The sum of the outer products become

$$\text{area } \Delta ABC = \frac{1}{2}[(a \wedge b) + (b \wedge c) + (c \wedge a)] \tag{7.83}$$

which expand to

$$\text{area } \Delta ABC = \frac{1}{2}(x_A y_B - y_A x_B + x_B y_C - y_B x_C + x_C y_A - y_C x_A)$$

and

$$\text{area } \Delta ABC = \frac{1}{2}\begin{vmatrix} x_A & y_A & 1 \\ x_B & y_B & 1 \\ x_C & y_C & 1 \end{vmatrix}. \tag{7.84}$$

What is useful about summing these outer products is that it works for any irregular shape.

7.5.2 The sine rule

The traditional way of proving the sine rule is to take a triangle and drop a perpendicular from one of its vertices onto the opposite side to form two right-angled triangles, from which we define the sine ratio of two angles. Using Fig. 7.13 we can state that

$$\frac{H}{B} = \sin\alpha \quad \text{and} \quad \frac{H}{A} = \sin\beta \tag{7.85}$$

from which we can write

$$B\sin\alpha = A\sin\beta \tag{7.86}$$

or

$$\frac{A}{\sin\alpha} = \frac{B}{\sin\beta}. \tag{7.87}$$

Using another vertex and an associated perpendicular we can show that

$$\frac{A}{\sin\alpha} = \frac{B}{\sin\beta} = \frac{C}{\sin\chi}. \tag{7.88}$$

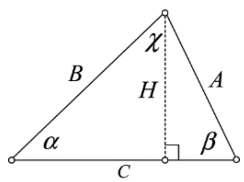

FIGURE 7.13.

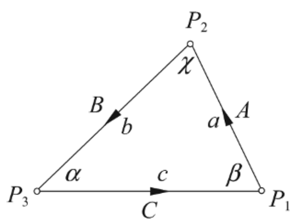

FIGURE 7.14.

The GA approach is to remember that the outer product includes a sine function and computes an area. Therefore, we develop Fig. 7.13 to include three vectors as shown in Fig. 7.14 where

$$A = \|a\| \quad B = \|b\| \quad C = \|c\| \tag{7.89}$$

From the figure we observe that

$$\text{area of } \Delta P_1P_2P_3 = \frac{1}{2}\|a \wedge -c\| = \frac{1}{2}AC \sin\beta \tag{7.90}$$

$$\text{area of } \Delta P_2P_3P_1 = \frac{1}{2}\|b \wedge -a\| = \frac{1}{2}BA \sin\chi \tag{7.91}$$

$$\text{area of } \Delta P_3P_1P_2 = \frac{1}{2}\|c \wedge -b\| = \frac{1}{2}CB \sin\alpha. \tag{7.92}$$

Therefore,

$$AC \sin\beta = CB \sin\alpha = BA \sin\chi \tag{7.93}$$

and

$$\frac{A}{\sin\alpha} = \frac{B}{\sin\beta} = \frac{C}{\sin\chi}. \tag{7.94}$$

7.5.3 Intersection of two lines

The traditional way of calculating the intersection point of two lines in a plane is to define two vectors as shown in Fig. 7.15, where

$$p = r + \lambda a \quad \lambda \in \mathbb{R} \tag{7.95}$$

$$p = s + \varepsilon b \quad \varepsilon \in \mathbb{R}. \tag{7.96}$$

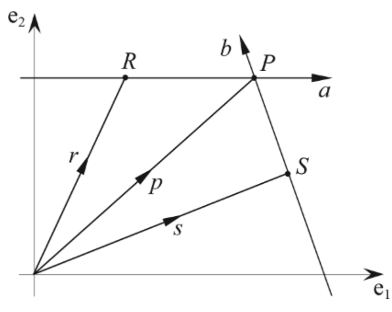

FIGURE 7.15.

Therefore,

$$r + \lambda a = s + \varepsilon b. \tag{7.97}$$

From Eq. (7.95) we can write

$$x_r + \lambda x_a = x_s + \varepsilon x_b \tag{7.98}$$

$$y_r + \lambda y_a = y_s + \varepsilon y_b. \tag{7.99}$$

To find λ we eliminate ε by multiplying Eq. (7.98) by y_b and Eq. (7.99) by x_b:

$$x_r y_b + \lambda x_a y_b = x_s y_b + \varepsilon x_b y_b \tag{7.100}$$

$$x_b y_r + \lambda x_b y_a = x_b y_s + \varepsilon x_b y_b. \tag{7.101}$$

Subtracting Eq. (7.101) from Eq. (7.100):

$$x_r y_b - x_b y_r + \lambda(x_a y_b - x_b y_a) = x_s y_b - x_b y_s \tag{7.102}$$

where

$$\lambda = \frac{x_b(y_r - y_s) - y_b(x_r - x_s)}{x_a y_b - x_b y_a}. \tag{7.103}$$

Let's test this with the following vectors

$$r = \mathrm{j} \quad a = 2\mathrm{i} - \mathrm{j} \tag{7.104}$$

$$s = 2\mathrm{j} \quad b = 2\mathrm{i} - 2\mathrm{j} \tag{7.105}$$

$$\lambda = \frac{2(1-2) + 2(0-0)}{-4+2} = \frac{-2}{-2} = 1 \tag{7.106}$$

therefore,

$$p = \mathrm{j} + 2\mathrm{i} - \mathrm{j} = 2\mathrm{i} \tag{7.107}$$

and the point of intersection is $(2, 0)$.

Another approach is to reason that

$$p = \alpha a + \beta b \tag{7.108}$$

therefore, we can write

$$x_p = \alpha x_a + \beta x_b \tag{7.109}$$

$$y_p = \alpha y_a + \beta y_b. \tag{7.110}$$

To find α we eliminate β by multiplying Eq. (7.109) by y_b and Eq. (7.110) by x_b:

$$x_p y_b = \alpha x_a y_b + \beta x_b y_b \tag{7.111}$$

$$x_b y_p = \alpha x_b y_a + \beta x_b y_b. \tag{7.112}$$

Subtracting Eq. (7.112) from Eq. (7.111) we obtain

$$x_p y_b - x_b y_p = \alpha x_a y_b - \alpha x_b y_a = \alpha(x_a y_b - x_b y_a) \tag{7.113}$$

where

$$\alpha = \frac{x_p y_b - x_b y_p}{x_a y_b - x_b y_a} = \frac{\begin{vmatrix} x_p & y_p \\ x_b & y_b \end{vmatrix}}{\begin{vmatrix} x_a & y_a \\ x_b & y_b \end{vmatrix}}. \tag{7.114}$$

To find β we eliminate α by multiplying Eq. (7.109) by y_a and Eq. (7.110) by x_a:

$$x_p y_a = \alpha x_a y_a + \beta x_b y_a \tag{7.115}$$

$$x_a y_p = \alpha x_a y_a + \beta x_a y_b. \tag{7.116}$$

Subtracting Eq. (7.116) from Eq. (7.115) we obtain

$$x_p y_a - x_a y_p = \beta x_b y_a - \beta x_a y_b = \beta(x_b y_a - x_a y_b) \tag{7.117}$$

where

$$\beta = \frac{x_p y_a - x_a y_p}{x_b y_a - x_a y_b} = \frac{\begin{vmatrix} x_p & y_p \\ x_a & y_a \end{vmatrix}}{\begin{vmatrix} x_b & y_b \\ x_a & y_a \end{vmatrix}}. \tag{7.118}$$

Using Eq. (7.114) and Eq. (7.118) we can rewrite Eq. (7.108) as

$$p = \frac{\begin{vmatrix} x_p & y_p \\ x_b & y_b \end{vmatrix}}{\begin{vmatrix} x_a & y_a \\ x_b & y_b \end{vmatrix}} a + \frac{\begin{vmatrix} x_p & y_p \\ x_a & y_a \end{vmatrix}}{\begin{vmatrix} x_b & y_b \\ x_a & y_a \end{vmatrix}} b. \tag{7.119}$$

The problem with Eq. (7.119) is that the determinants reference the coordinates of the point we are trying to discover. Nevertheless, let's continue and write Eq. (7.119) using outer products

$$p = \frac{p \wedge b}{a \wedge b} a + \frac{p \wedge a}{b \wedge a} b. \tag{7.120}$$

Figure 7.16a provides a graphical interpretation of part of Eq. (7.120) where the parallelogram formed by the outer product $p \wedge a$ is identical to the outer product formed by $r \wedge a$. Which means that we can substitute $r \wedge a$ for $p \wedge a$ in Eq. (7.120):

$$p = \frac{p \wedge b}{a \wedge b} a + \frac{r \wedge a}{b \wedge a} b. \tag{7.121}$$

Similarly, in Fig. 7.16b the parallelogram formed by the outer product $p \wedge b$ is identical to the outer product formed by $s \wedge b$. Which means that we can substitute $s \wedge b$ for $p \wedge b$ in Eq. (7.121):

$$p = \frac{s \wedge b}{a \wedge b} a + \frac{r \wedge a}{b \wedge a} b. \tag{7.122}$$

The positions of R and S are not very important, as they could be anywhere along the two vectors, even positioned as shown in Fig. 7.17:

In Fig. 7.17 the three parallelograms: $OSTU$, $OVWR$ and $OVXU$ have areas:

$$area\ OSTU = s \wedge b \tag{7.123}$$

$$area\ OVWR = r \wedge a \tag{7.124}$$

$$area\ OVXU = a \wedge b. \tag{7.125}$$

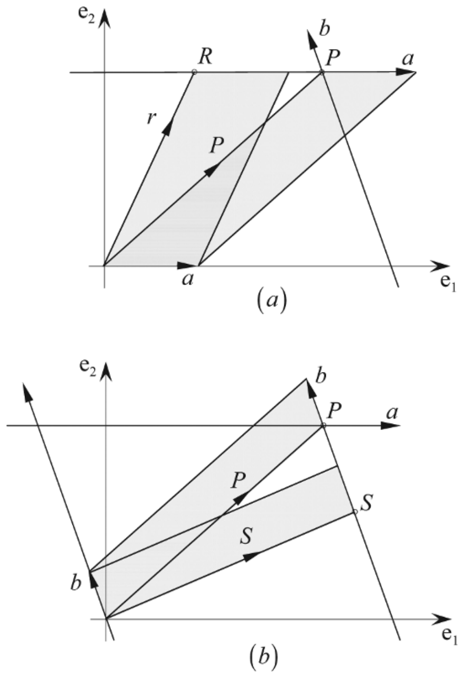

FIGURE 7.16.

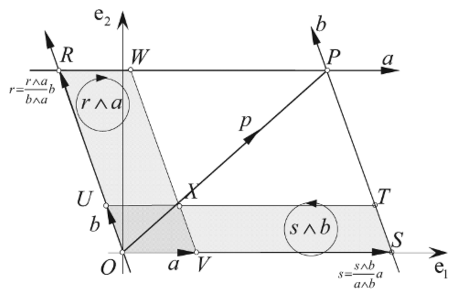

FIGURE 7.17.

Simply by relocating S and R, we have created a convenient visual symmetry where

$$s = \frac{s \wedge b}{a \wedge b} a \tag{7.126}$$

and

$$r = \frac{r \wedge a}{b \wedge a} b. \tag{7.127}$$

Note how $s \wedge b$ and $a \wedge b$ are in the same sense, whilst $r \wedge a$ and $b \wedge a$ are in the opposite sense. Observe, also, from Fig. (7.17) why

$$\frac{s}{a} = \frac{s \wedge b}{a \wedge b} \tag{7.128}$$

and

$$\frac{r}{b} = \frac{r \wedge a}{b \wedge a}. \tag{7.129}$$

It now becomes obvious that

$$p = s + r = \frac{s \wedge b}{a \wedge b} a + \frac{r \wedge a}{b \wedge a} b \tag{7.130}$$

where the solution to the problem is based upon the ratios of areas of parallelograms!

Let's test Eq. (7.130) using the same vectors above:

$$r = \mathrm{e}_2 \qquad a = 2\mathrm{e}_1 - \mathrm{e}_2 \tag{7.131}$$

$$s = 2\mathrm{e}_2 \quad b = 2\mathrm{e}_1 - 2\mathrm{e}_2 \tag{7.132}$$

$$\begin{aligned}
p &= \frac{(2\mathrm{e}_2) \wedge (2\mathrm{e}_1 - 2\mathrm{e}_2)}{(2\mathrm{e}_1 - \mathrm{e}_2) \wedge (2\mathrm{e}_1 - 2\mathrm{e}_2)}(2\mathrm{e}_1 - \mathrm{e}_2) + \frac{\mathrm{e}_2 \wedge (2\mathrm{e}_1 - \mathrm{e}_2)}{(2\mathrm{e}_1 - 2\mathrm{e}_2) \wedge (2\mathrm{e}_1 - \mathrm{e}_2)}(2\mathrm{e}_1 - 2\mathrm{e}_2) \\
p &= \frac{-4(\mathrm{e}_1 \wedge \mathrm{e}_2)}{-4(\mathrm{e}_1 \wedge \mathrm{e}_2) + 2(\mathrm{e}_1 \wedge \mathrm{e}_2)}(2\mathrm{e}_1 - \mathrm{e}_2) + \frac{-2(\mathrm{e}_1 \wedge \mathrm{e}_2)}{-2(\mathrm{e}_1 \wedge \mathrm{e}_2) + 4(\mathrm{e}_1 \wedge \mathrm{e}_2)}(2\mathrm{e}_1 - 2\mathrm{e}_2) \\
p &= 2(2\mathrm{e}_1 - \mathrm{e}_2) - (2\mathrm{e}_1 - 2\mathrm{e}_2) = 2\mathrm{e}_1.
\end{aligned} \tag{7.133}$$

Therefore, the point of intersection is $(2, 0)$. Which is the same as the previous result.

We have spent some time exploring the above techniques, which in some cases are quite tedious. However, the conformal model, which is explored in chapter 11, simplifies the whole process.

7.6 Summary

It seems that the outer product is a very natural way of describing the orientation of two vectors, and has immediate applications in a variety of geometric problems. Let's now examine the properties of another product—the *geometric product*.