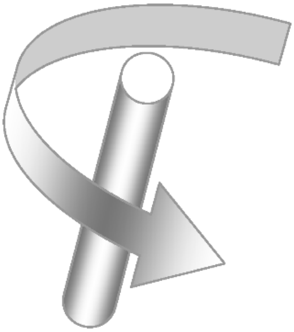

5 Quaternion Algebra

5.1 Introduction

Quaternions are the result of one man's determination to find the 3D equivalent of complex numbers. Sir William Rowan Hamilton was the man, and in 1843 he revealed to the world his discovery which had taken him over a decade to resolve.

Knowing that a complex number in $\mathbb{R}^2$ has the form

$$z = a + ib \tag{5.1}$$

it is reasonable to presume that a complex number in $\mathbb{R}^3$ should take the form

$$z = a + ib + jc \tag{5.2}$$

where i and j are unit imaginaries: $i^2 = j^2 = -1$. However, when two such objects are multiplied together we have

$$z_1 z_2 = (a_1 + ib_1 + jc_1)(a_2 + ib_2 + jc_2) \tag{5.3}$$

which expands to

$$z_1 z_2 = a_1 a_2 + ia_1 b_2 + ja_1 c_2 + ib_1 a_2 + i^2 b_1 b_2 + ijb_1 c_2 + jc_1 a_2 + jic_1 b_2 + j^2 c_1 c_2. \tag{5.4}$$

Substituting $i^2 = j^2 = -1$ into Eq. (5.4) and collecting up like terms we obtain

$$z_1 z_2 = (a_1 a_2 - b_1 b_2 - c_1 c_2) + i(a_1 b_2 + b_1 a_2) + j(a_1 c_2 + c_1 a_2) + ijb_1 c_2 + jic_1 b_2 \tag{5.5}$$

which leaves the terms ij and ji undefined. These stumped Hamilton for many years, but his tenacity won the day, and he eventually came up with an incredible idea which involved extending the triple into a 4-tuple:

$$z = a + ib + jc + kd. \tag{5.6}$$

When two such objects are multiplied together we have

$$z_1 z_2 = (a_1 + ib_1 + jc_1 + kd_1)(a_2 + ib_2 + jc_2 + kd_2) \tag{5.7}$$

which expands to

$$\begin{aligned} z_1 z_2 = {} & a_1 a_2 + i a_1 b_2 + j a_1 c_2 + k a_1 d_2 \\ & + i b_1 a_2 + i^2 b_1 b_2 + ij b_1 c_2 + ik b_1 d_2 \\ & + j c_1 a_2 + ji c_1 b_2 + j^2 c_1 c_2 + jk c_1 d_2 \\ & + k d_1 a_2 + ki d_1 b_2 + kj d_1 c_2 + k^2 d_1 d_2. \end{aligned} \tag{5.8}$$

Substituting $i^2 = j^2 = k^2 = -1$ in Eq. (5.8) and collecting up like terms we obtain

$$\begin{aligned} z_1 z_2 = {} & a_1 a_2 - b_1 b_2 - c_1 c_2 - d_1 d_2 \\ & + i(a_1 b_2 + b_1 a_2) + j(a_1 c_2 + c_1 a_2) + k(a_1 d_2 + d_1 a_2) \\ & + ij b_1 c_2 + ik b_1 d_2 + ji c_1 b_2 + jk c_1 d_2 + ki d_1 b_2 + kj d_1 c_2. \end{aligned} \tag{5.9}$$

But this, too, has some undefined terms: ij, ik, ji, jk, ki, kj. However, Hamilton was a genius and he resolved the problem by proposing the following rules:

$$ij = k \quad jk = i \quad ki = j \quad ji = -k \quad kj = -i \quad ik = -j \tag{5.10}$$

which when substituted into Eq. (5.9) produces

$$\begin{aligned} z_1 z_2 = {} & a_1 a_2 - b_1 b_2 - c_1 c_2 - d_1 d_2 \\ & + i(a_1 b_2 + b_1 a_2) + j(a_1 c_2 + c_1 a_2) + k(a_1 d_2 + d_1 a_2) \\ & + k b_1 c_2 - j b_1 d_2 - k c_1 b_2 + i c_1 d_2 + j d_1 b_2 - i d_1 c_2. \end{aligned} \tag{5.11}$$

Collecting up like terms we obtain

$$\begin{aligned} z_1 z_2 = {} & a_1 a_2 - (b_1 b_2 + c_1 c_2 + d_1 d_2) \\ & + i(a_1 b_2 + b_1 a_2 + c_1 d_2 - d_1 c_2) \\ & + j(a_1 c_2 + c_1 a_2 + d_1 b_2 - b_1 d_2) \\ & + k(a_1 d_2 + d_1 a_2 + b_1 c_2 - c_1 b_2). \end{aligned} \tag{5.12}$$

Although this does not have any undefined terms it can be tidied up as follows:

$$\begin{aligned} z_1 z_2 = {} & a_1 a_2 - (b_1 b_2 + c_1 c_2 + d_1 d_2) \\ & + a_1(i b_2 + j c_2 + k d_2) + a_2(i b_1 + j c_1 + k d_1) \\ & + i(c_1 d_2 - d_1 c_2) + j(d_1 b_2 - b_1 d_2) + k(b_1 c_2 - c_1 b_2) \end{aligned} \tag{5.13}$$

The last step is to write the original object as the sum of a scalar and a vector starting with:

$$z_1 = s_1 + \mathbf{v}_1 \quad z_2 = s_2 + \mathbf{v}_2 \tag{5.14}$$

and the following symmetry emerges:

$$z_1 z_2 = s_1 s_2 - \mathbf{v}_1 \cdot \mathbf{v}_2 + s_1\, \mathbf{v}_2 + s_2\, \mathbf{v}_1 + \mathbf{v}_1 \times \mathbf{v}_2. \tag{5.15}$$

Hamilton called this object a '*quaternion*' and gave the name '*vector*' to the imaginary portion.

The product $\mathbf{v}_1 \cdot \mathbf{v}_2$ is equivalent to

$$b_1 b_2 + c_1 c_2 + d_1 d_2 \tag{5.16}$$

and became the *scalar* or *dot product*, whilst $\mathbf{v}_1 \times \mathbf{v}_2$, which is equivalent to

$$i(c_1 d_2 - d_1 c_2) + j(d_1 b_2 - b_1 d_2) + k(b_1 c_2 - c_1 b_2) \tag{5.17}$$

became the *vector* or *cross product* and led to the definitions:

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = \|\mathbf{v}_1\| \|\mathbf{v}_2\| \cos\theta \tag{5.18}$$

and

$$\mathbf{v}_1 \times \mathbf{v}_2 = \mathbf{v}_3 \tag{5.19}$$

where

$$\mathbf{v}_3 = i(c_1 d_2 - d_1 c_2) + j(d_1 b_2 - b_1 d_2) + k(b_1 c_2 - c_1 b_2) \tag{5.20}$$

and

$$\|\mathbf{v}_3\| = \|\mathbf{v}_1\| \|\mathbf{v}_2\| \sin\theta \tag{5.21}$$

where

θ is the angle between $\mathbf{v}_1$ and $\mathbf{v}_2$.

Strictly speaking, the i, j and k are unit imaginaries which obey Hamilton's rules where

$$i^2 = j^2 = k^2 = ijk = -1 \tag{5.22}$$

$$ij = k \quad jk = i \quad ki = j \quad ji = -k \quad kj = -i \quad ik = -j. \tag{5.23}$$

However, when vector algebra became the preferred system over quaternion algebra, the i, j and k terms became the Cartesian unit vectors i, j and k.

One very important feature of quaternion algebra is its anticommuting rules. Maintaining order between the unit imaginaries is vital for the algebra to remain consistent, which is also a feature of GA.

5.2 Adding quaternions

Two quaternions q_1 and q_2

$$q_1 = s_1 + ix_1 + jy_1 + kz_1 \tag{5.24}$$

$$q_2 = s_2 + ix_2 + jy_2 + kz_2 \tag{5.25}$$

are equal if, and only if, their corresponding terms are equal. Furthermore, like vectors, they can be added or subtracted as follows:

$$q_1 \pm q_2 = [(s_1 \pm s_2) + i(x_1 \pm x_2) + j(y_1 \pm y_2) + k(z_1 \pm z_2)]. \tag{5.26}$$

For example, given two quaternions

$$q_1 = 1 + i2 + j3 + k4 \tag{5.27}$$

$$q_2 = 2 - i + j5 - k2 \tag{5.28}$$

their sum is given by

$$q_1 + q_2 = 3 + i + j8 + k2. \tag{5.29}$$

5.3 The quaternion product

Given two quaternions

$$q_1 = s_1 + \mathbf{v}_1 = s_1 + ix_1 + jy_1 + kz_1 \tag{5.30}$$

$$q_2 = s_2 + \mathbf{v}_2 = s_2 + ix_2 + jy_2 + kz_2 \tag{5.31}$$

their product is given by

$$q_1q_2 = s_1s_2 - \mathbf{v}_1 \cdot \mathbf{v}_2 + s_1\mathbf{v}_2 + s_2\mathbf{v}_1 + \mathbf{v}_1 \times \mathbf{v}_2 \tag{5.32}$$

which is still a quaternion and ensures closure. However, the quaternion product anticommutes, which we can prove by computing q_2q_1:

$$q_2q_1 = s_2s_1 - \mathbf{v}_2 \cdot \mathbf{v}_1 + s_2\mathbf{v}_1 + s_1\mathbf{v}_2 + \mathbf{v}_2 \times \mathbf{v}_1. \tag{5.33}$$

The pure scalar terms $s_2s_1, \mathbf{v}_2 \cdot \mathbf{v}_1$ and the products $s_2\mathbf{v}_1$ and $s_1\mathbf{v}_2$ commute, but the cross product $\mathbf{v}_2 \times \mathbf{v}_1$ anticommutes, therefore $q_1q_2 \neq q_2q_1$.

For example, given the quaternions

$$q_1 = 1 + i2 + j3 + k4 \tag{5.34}$$

$$q_2 = 2 - i + j5 - k2 \tag{5.35}$$

their product q_1q_2 is

$$\begin{aligned} q_1q_2 &= (1 + i2 + j3 + k4)(2 - i + j5 - k2) \qquad (5.36)\\ &= [1 \times 2 - (2 \times (-1) + 3 \times 5 + 4 \times (-2))\\ &\quad + 1(-i + j5 - k2) + 2(i2 + j3 + k4)\\ &\quad + i(3 \times (-2) - 4 \times 5) + j(4 \times (-1) - (-2) \times 2) + k(2 \times 5 - (-1) \times 3)]\\ &= -3 + i3 + j11 + k6 - i26 + k13\\ q_1q_2 &= -3 - i23 + j11 + k19 \qquad (5.37) \end{aligned}$$

which is a quaternion.

Whereas the product $q_2 q_1$ is

$$\begin{aligned} q_2 q_1 &= (2 - i + j5 - k2)(1 + i2 + j3 + k4) \\ &= [2 - ((-1) \times 2 + 5 \times 3 + (-2) \times 4) \\ &\quad + 2(i2 + j3 + k4) + 1(-i + j5 - k2) \\ &\quad + i(5 \times 4 - 3 \times (-2)) + j((-2) \times 2 - 4 \times (-1)) + k((-1) \times 3 - 2 \times 5)] \\ q_2 q_1 &= -3 + i29 + j11 - k7 \end{aligned} \tag{5.38}$$

which is also a quaternion, but $q_2 q_1 \neq q_1 q_2$.

5.4 The magnitude of a quaternion

Given the quaternion

$$q = s + ix + jy + kz \tag{5.39}$$

its magnitude is defined as

$$\|q\| = \sqrt{s^2 + x^2 + y^2 + z^2}. \tag{5.40}$$

For example, given the quaternion

$$q = 1 + i2 + j3 + k4 \tag{5.41}$$

$$\|q\| = \sqrt{1^2 + 2^2 + 3^2 + 4^2} = \sqrt{30}. \tag{5.42}$$

5.5 The unit quaternion

Like vectors, quaternions have a unit form where the magnitude equals unity. For example, the magnitude of the quaternion

$$q = 1 + i2 + j3 + k4 \tag{5.43}$$

is

$$\|q\| = \sqrt{1^2 + 2^2 + 3^2 + 4^2} = \sqrt{30} \tag{5.44}$$

therefore, the unit quaternion $\hat{q}$ equals

$$\hat{q} = \frac{1}{30}(1 + i2 + j3 + k4). \tag{5.45}$$

5.6 The pure quaternion

Hamilton named a quaternion with a zero scalar term a *pure quaternion*. For example,

$$q_1 = ix_1 + jy_1 + kz_1 \text{ and } q_2 = ix_2 + jy_2 + kz_2 \tag{5.46}$$

are pure quaternions. Let's see what happen when we multiply them together:

$$q_1 q_2 = (ix_1 + jy_1 + kz_1)(ix_2 + jy_2 + kz_2)$$
$$q_1 q_2 = [-(x_1x_2 + y_1y_2 + z_1z_2) + i(y_1z_2 - y_2z_1) + j(z_1x_2 - z_2x_1) + k(x_1y_2 - x_2y_1)] \quad (5.47)$$

which is no longer a pure quaternion, as a negative scalar term has emerged. Thus the algebra of pure quaternions is not closed.

5.7 The conjugate of a quaternion

Given the quaternion

$$q = s + \mathbf{v}$$
$$q = s + ix + jy + kz \quad (5.48)$$

by definition, its conjugate is

$$\overline{q} = s - \mathbf{v} = s - (ix + jy + kz). \quad (5.49)$$

For example, the quaternion

$$q = 1 + i2 + j3 + k4 \quad (5.50)$$

its conjugate is

$$\overline{q} = 1 - i2 - j3 - k4. \quad (5.51)$$

5.8 The inverse quaternion

Given the quaternion

$$q = s + ix + jy + kz \quad (5.52)$$

the inverse quaternion q^{-1} is

$$q^{-1} = \frac{s - ix - jy - kz}{\|q\|^2} \quad (5.53)$$

because this satisfies the product

$$qq^{-1} = \frac{(s + ix + jy + kz)(s - ix - jy - kz)}{\|q\|^2} = 1. \quad (5.54)$$

We can show that this is true by expanding the product as follows:

$$qq^{-1} = \begin{pmatrix} s^2 - isx - jsy - ksz + isx + x^2 - ijxy - ikxz + \\ jsy - jixy + y^2 - jkyz + ksz - kixz - kjyz + z^2 \end{pmatrix} / \|q\|^2$$
$$= \frac{s^2 + x^2 + y^2 + z^2 - ijxy - ikxz - jixy - jkyz - kixz - kjyz}{\|q\|^2}$$
$$qq^{-1} = \frac{s^2 + x^2 + y^2 + z^2}{\|q\|^2} = 1 \quad (5.55)$$

and confirms that the inverse quaternion q^{-1} is

$$q^{-1} = \frac{\overline{q}}{\|q\|^2}. \tag{5.56}$$

Because the unit imaginaries do not commute, we need to discover whether

$$qq^{-1} = q^{-1}q. \tag{5.57}$$

Expanding this product

$$\begin{aligned}
q^{-1}q &= \frac{(s - ix - jy - kz)(s + ix + jy + kz)}{\|q\|^2} \\
&= \begin{pmatrix} s^2 + isx + jsy + ksz - isx + x^2 - ijxy - ikxz- \\ jsy - jixy + y^2 - jkyz - ksz - kixz - kjyz + z^2 \end{pmatrix} / \|q\|^2 \\
&= \frac{s^2 + x^2 + y^2 + z^2 - ijxy - ikxz - jixy - jkyz - kixz - kjyz}{\|q\|^2} \\
q^{-1}q &= \frac{s^2 + x^2 + y^2 + z^2}{\|q\|^2} = 1
\end{aligned}$$

therefore,

$$qq^{-1} = q^{-1}q. \tag{5.58}$$

5.9 Quaternion algebra

The axioms associated with quaternions are as follows:

Given $$q, q_1, q_2, q_3 \in \mathbb{C}: \tag{5.59}$$

Closure

For all q_1 and q_2

$$\text{addition} \quad q_1 + q_2 \in \mathbb{C} \tag{5.60}$$

$$\text{multiplication} \quad q_1 q_2 \in \mathbb{C}. \tag{5.61}$$

Identity

For each q there is an identity element $\mathbf{0}$ and $\mathbf{1}$ such that:

$$\text{addition} \quad q + \mathbf{0} = \mathbf{0} + q = q \ (\mathbf{0} = 0 + i0 + j0 + k0) \tag{5.62}$$

$$\text{multiplication} \quad q(\mathbf{1}) = (\mathbf{1})q = q \ (\mathbf{1} = 1 + i0 + j0 + k0). \tag{5.63}$$

Inverse

For each q there is an inverse element $-q$ and q^{-1} such that:

$$\text{addition} \quad q + (-q) = -q + q = 0 \tag{5.64}$$

$$\text{multiplication} \quad qq^{-1} = q^{-1}q = 1 \ (q \neq 0). \tag{5.65}$$

Associativity

For all q_1, q_2 and q_3

$$\text{addition} \quad q_1 + (q_2 + q_3) = (q_1 + q_2) + q_3 \tag{5.66}$$

$$\text{multiplication} \quad q_1(q_2 q_3) = (q_1 q_2) q_3. \tag{5.67}$$

Commutativity

For all q_1 and q_2

$$\text{addition} \quad q_1 + q_2 = q_2 + q_1 \tag{5.68}$$

$$\text{multiplication} \quad q_1 q_2 \neq q_2 q_1. \tag{5.69}$$

Distributivity

For all q_1, q_2 and q_3

$$q_1(q_2 + q_3) = q_1 q_2 + q_1 q_3 \tag{5.70}$$

$$(q_1 + q_2) q_3 = q_1 q_3 + q_2 q_3. \tag{5.71}$$

5.10 Rotating vectors using quaternions

One excellent application for quaternions is rotating vectors, and readers requiring an introduction to this topic are directed to the author's book *Mathematics for Computer Graphics* [8].

It can be shown that a position vector $\mathbf{p}$ can be rotated about an axis $\hat{\mathbf{u}}$ by an angle θ to $\mathbf{p}'$ using the following operation:

$$\mathbf{p}' = q\mathbf{p}q^{-1} \tag{5.72}$$

where

$$\mathbf{p} = x\mathrm{i} + y\mathrm{j} + z\mathrm{k} \tag{5.73}$$

$$p = 0 + ix + jy + kz \tag{5.74}$$

$$q = \cos(\theta/2) + \sin(\theta/2)\hat{\mathbf{u}} \tag{5.75}$$

$$q^{-1} = \cos(\theta/2) - \sin(\theta/2)\hat{\mathbf{u}} \tag{5.76}$$

and the axis of rotation is

$$\hat{\mathbf{u}} = [x_u\mathrm{i} + y_u\mathrm{j} + z_u\mathrm{k}] \quad (\|\hat{\mathbf{u}}\| = 1). \tag{5.77}$$

This is best demonstrated through an example.

Let the point to be rotated be

$$P(0, 1, 1). \tag{5.78}$$

Let the axis of rotation be

$$\hat{\mathbf{u}} = \mathrm{j}. \tag{5.79}$$

Let the angle of rotation be

$$\theta = 90°. \tag{5.80}$$

Therefore,

$$p = 0 + i0 + j + k \tag{5.81}$$

$$q = \cos 45° + \sin 45(i0 + j + k0)$$

$$q = \frac{\sqrt{2}}{2}(1 + i0 + j + k0) \tag{5.82}$$

$$q^{-1} = \cos 45° - \sin 45(i0 + j + k0)$$

$$q^{-1} = \frac{\sqrt{2}}{2}(1 - i0 - j - k0). \tag{5.83}$$

The rotated point is given by

$$p' = qpq^{-1}$$

$$p' = \frac{\sqrt{2}}{2}(1 + i0 + j + k0)(0 + i0 + j + k)\frac{\sqrt{2}}{2}(1 - i0 - j - k0). \tag{5.84}$$

This is best expanded in two steps, and zero imaginary terms are included for clarity.

qp followed by $(qp)q^{-1}$.

Step 1

$$qp = \frac{\sqrt{2}}{2}(1 + i0 + j + k0)(0 + i0 + j + k)$$

$$qp = \frac{\sqrt{2}}{2}(-1 + i + j + k). \tag{5.85}$$

Step 2

$$\begin{aligned}(qp)q^{-1} &= \frac{\sqrt{2}}{2}(-1 + i + j + k)\frac{\sqrt{2}}{2}(1 - i0 - j - k0)\\ &= \frac{1}{2}(-1 + 1 + j + i + j + k + i - k)\\ &= \frac{1}{2}(0 + i2 + j2 + k0)\\ (qp)q^{-1} &= 0 + i + j + k0.\end{aligned} \tag{5.86}$$

The coordinates of the rotated point are stored in the pure part of the quaternion: (1, 1, 0).

5.11 Summary

Out of all the algebras we have so far considered, quaternion algebra paves the way to geometric algebra. In fact, as we will soon discover, GA shows that quaternions are a left-handed system and employ the concepts of GA. The good news is that if you understand quaternions, you will find it much easier to understand GA.