9

Connectivity

Contents

9.1 Vertex Connectivity

In Section 3.1, we discussed the concept of connection in graphs. Consider, now, the four connected graphs in Figure 9.1.

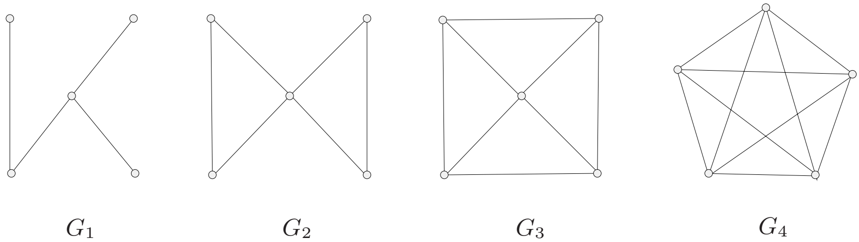


Fig. 9.1. Four connected graphs

G_1 is a tree, a minimal connected graph; deleting any edge disconnects it. G_2 cannot be disconnected by the deletion of a single edge, but can be disconnected by the deletion of one vertex, its cut vertex. There are no cut edges or cut vertices in G_3, but even so G_3 is clearly not as well connected as G_4, the complete graph on five vertices. Thus, intuitively, each successive graph is better connected than the previous one. We now introduce two parameters of a graph, its connectivity and edge connectivity, which measure the extent to which it is connected. We first define these parameters in terms of numbers of disjoint paths connecting pairs of vertices, and then relate those definitions to sizes of vertex and edge cuts, as suggested by the above examples.

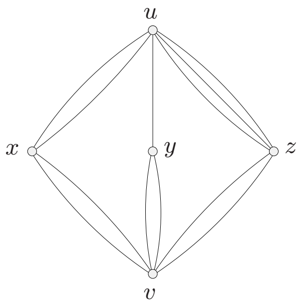
Connectivity and Local Connectivity

We begin by discussing the notion of vertex connectivity, commonly referred to simply as connectivity. Recall that xy-paths P and Q in G are *internally disjoint* if they have no internal vertices in common, that is, if $V(P) \cap V(Q) = \{x, y\}$. The *local connectivity* between distinct vertices x and y is the maximum number of pairwise internally disjoint xy-paths, denoted $p(x, y)$; the local connectivity is undefined when $x = y$. The matrix in Figure 9.2b displays the local connectivities between all pairs of vertices of the graph shown in Figure 9.2a. (The function shown in Figure 9.2c will be defined shortly.)

A nontrivial graph G is *k-connected* if $p(u, v) \geq k$ for any two distinct vertices u and v. By convention, a trivial graph is 0-connected and 1-connected, but is not k-connected for any $k > 1$. The *connectivity* $\kappa(G)$ of G is the maximum value of k for which G is k-connected. Thus, for a nontrivial graph G,

$$\kappa(G) := \min\{p(u, v) : \ u, v \in V, \ u \neq v\} \tag{9.1}$$

A graph is 1-connected if and only if it is connected; equivalently, a graph has connectivity zero if and only if it is disconnected. Nonseparable graphs on at least three vertices are 2-connected; conversely, every 2-connected loopless graph is nonseparable. For the four graphs shown in Figure 9.1, $\kappa(G_1) = 1$, $\kappa(G_2) = 1$, $\kappa(G_3) = 3$, and $\kappa(G_4) = 4$. Thus, of these four graphs, the only graph that is 4-connected is G_4. Graphs G_3 and G_4 are 2-connected and 3-connected, whereas



(a)

	u	x	y	z	v
u	$*$	3	2	4	3
x	3	$*$	2	2	3
y	2	2	$*$	2	3
z	4	2	2	$*$	3
v	3	3	3	3	$*$

(b)

	u	x	y	z	v
u	$*$	$*$	$*$	$*$	3
x	$*$	$*$	2	2	$*$
y	$*$	2	$*$	2	$*$
z	$*$	2	2	$*$	$*$
v	3	$*$	$*$	$*$	$*$

(c)

Fig. 9.2. (a) A graph, (b) its local connectivity function, and (c) its local cut function

G_1 and G_2 are not. And, because all four graphs are connected, they are all 1-connected. The graph in Figure 9.2a is 1-connected and 2-connected, but is not 3-connected. Thus, the connectivity of this graph is two.

Vertex Cuts and Menger's Theorem

We now rephrase the definition of connectivity in terms of 'vertex cuts'. This is not totally straightforward because complete graphs (and, more generally, graphs in which any two vertices are adjacent) have no such cuts. For this reason, we first determine the connectivities of these graphs.

Distinct vertices x and y of K_n are connected by one path of length one and $n-2$ internally disjoint paths of length two. It follows that $p(x,y) = n-1$ and that $\kappa(K_n) = n-1$ for $n \geq 2$. More generally, if the underlying simple graph of a graph G is complete, and x and y are joined by $\mu(x,y)$ links, there are $\mu(x,y)$ paths of length one, and $n-2$ internally disjoint paths of length two connecting x and y; thus $p(x,y) = n-2+\mu(x,y)$. Hence the connectivity of a nontrivial graph G in which any two vertices are adjacent is $n-2+\mu$, where μ is the minimum edge multiplicity in the graph. On the other hand, if x and y are nonadjacent, there are at most $n-2$ internally disjoint paths connecting x and y. Thus, if the underlying simple graph of a graph G is not complete, its connectivity κ cannot exceed $n-2$. For such a graph, the connectivity is equal to the minimum number of vertices whose deletion results in a disconnected graph, as we now explain.

Let x and y be distinct nonadjacent vertices of G. An *xy-vertex-cut* is a subset S of $V \setminus \{x,y\}$ such that x and y belong to different components of $G-S$. We also say that such a subset S *separates x and y*. The minimum size of a vertex cut separating x and y is denoted by $c(x,y)$. This function, the *local cut function* of G, is not defined if either $x=y$ or x and y are adjacent. The matrix displayed in Figure 9.2c gives the values of the local cut function of the graph shown in Figure 9.2a.

A vertex cut separating some pair of nonadjacent vertices of G is a *vertex cut* of G, and one with k elements is a *k-vertex cut*. A complete graph has no vertex cuts; moreover, the only graphs which do not have vertex cuts are those whose underlying simple graphs are complete. We now show that, if G has at least one pair of nonadjacent vertices, the size of a minimum vertex cut of G is equal to the connectivity of G. The main ingredient required is a version of Menger's Theorem which relates the two functions p and c.

Finding the maximum number of internally disjoint xy-paths in a graph $G := G(x, y)$ amounts to finding the maximum number of internally disjoint directed (x, y)-paths in the associated digraph $D(x, y) := D(G)$. In turn, as noted in Section 8.3, the latter problem may be reduced to one of finding the maximum number of arc-disjoint directed (x, y)-paths in a related digraph $D'(x, y)$ (of order $2n-2$), and this number can be determined by the Max-Flow Min-Cut Algorithm (7.9). Thus the Max-Flow Min-Cut Algorithm may be adapted to find, in polynomial time, the maximum number of internally disjoint xy-paths in G. The same algorithm will also return an xy-vertex-cut whose cardinality is equal to the maximum number of internally disjoint xy-paths, implying the validity of the following fundamental theorem of Menger (1927). We include here a simple inductive proof of this theorem due to Göring (2000), as an alternative to the above-mentioned constructive proof.

For this purpose, we need the operation of shrinking a set of vertices in a graph. Let G be a graph and let X be a proper subset of V. To *shrink* X is to delete all edges between vertices of X and then identify the vertices of X into a single vertex. We denote the resulting graph by $G\,/\,X$.

Theorem 9.1 Menger's Theorem (Undirected Vertex Version)
In any graph $G(x, y)$, where x and y are nonadjacent, the maximum number of pairwise internally disjoint xy-paths is equal to the minimum number of vertices in an xy-vertex-cut, that is,

$$p(x, y) = c(x, y)$$

Proof By induction on $e(G)$. For convenience, let us set $k := c_G(x, y)$. Note that $p_G(x, y) \leq k$, because any family $\mathcal{P}$ of internally disjoint xy-paths meets any xy-vertex-cut in at least $|\mathcal{P}|$ distinct vertices. Thus it suffices to show that $p_G(x, y) \geq k$. We may assume that there is an edge $e = uv$ incident neither with x nor with y; otherwise, every xy-path is of length two, and the conclusion follows easily.

Set $H := G \setminus e$. Because H is a subgraph of G, $p_G(x, y) \geq p_H(x, y)$. Also, by induction, $p_H(x, y) = c_H(x, y)$. Furthermore, $c_G(x, y) \leq c_H(x, y) + 1$ because any xy-vertex-cut in H, together with either end of e, is an xy-vertex-cut in G. We therefore have:

$$p_G(x, y) \geq p_H(x, y) = c_H(x, y) \geq c_G(x, y) - 1 = k - 1$$

We may assume that equality holds throughout; if not, $p_G(x, y) \geq k$ and there is nothing more to prove. Thus, in particular, $c_H(x, y) = k - 1$. Let $S := \{v_1, \ldots, v_{k-1}\}$ be a minimum xy-vertex-cut in H, let X be the set of vertices reachable from x in $H - S$, and let Y be the set of vertices reachable from y in

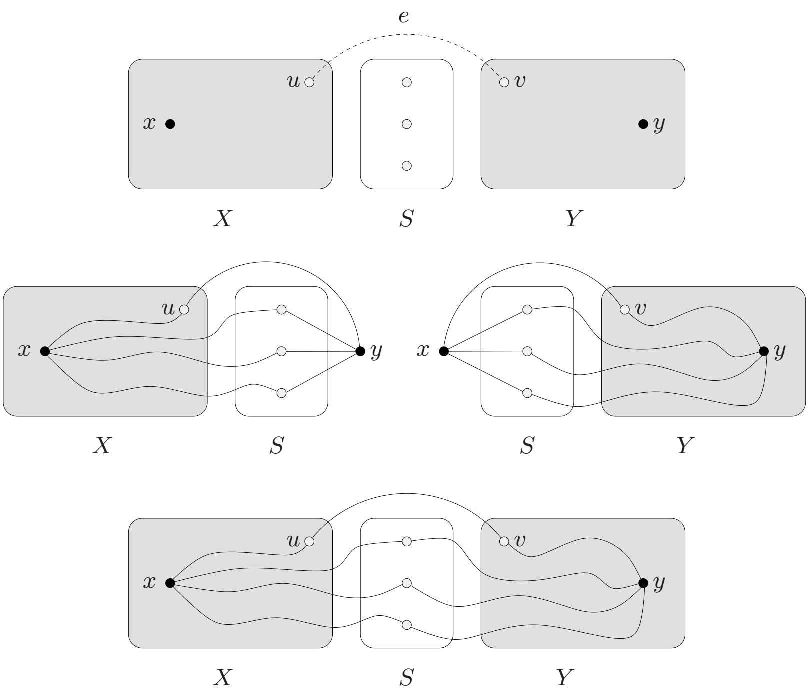


Fig. 9.3. Proof of Menger's Theorem (9.1)

$H - S$. Because $|S| = k - 1$, the set S is not an xy-vertex-cut of G, so there is an xy-path in $G - S$. This path necessarily includes the edge e. We may thus assume, without loss of generality, that $u \in X$ and $v \in Y$.

Now consider the graph $G\,/\,Y$ obtained from G by shrinking Y to a single vertex y. Every xy-vertex-cut T in $G\,/\,Y$ is necessarily an xy-vertex-cut in G, because if P were an xy-path in G avoiding T, the subgraph $P\,/\,Y$ of $G\,/\,Y$ would contain an xy-path in $G\,/\,Y$ avoiding T, too. Therefore $c_{G\,/\,Y}(x, y) \geq k$. On the other hand, $c_{G\,/\,Y}(x, y) \leq k$ because $S \cup \{u\}$ is an xy-vertex-cut of $G\,/\,Y$. It follows that $S \cup \{u\}$ is a minimum xy-vertex-cut of $G\,/\,Y$. By induction, there are k internally disjoint xy-paths $P_1, \ldots, P_k$ in $G\,/\,Y$, and each vertex of $S \cup \{u\}$ lies on one of them. Without loss of generality, we may assume that $v_i \in V(P_i)$, $1 \leq i \leq k - 1$, and $u \in V(P_k)$. Likewise, there are k internally disjoint xy-paths $Q_1, \ldots, Q_k$ in the graph $G\,/\,X$ obtained by shrinking X to x with $v_i \in V(Q_i)$, $1 \leq i \leq k - 1$, and $v \in Q_k$. It follows that there are k internally disjoint xy-paths in G, namely $xP_iv_iQ_iy$, $1 \leq i \leq k - 1$, and xP_kuvQ_ky (see Figure 9.3, where the vertices not in $X \cup S \cup Y$ are omitted, as they play no role in the proof.). □

As a consequence of Theorem 9.1 we have:

$$\min\{p(u, v) :\ u, v \in V,\ u \neq v, uv \notin E\} = \min\{c(u, v) :\ u, v \in V,\ u \neq v, uv \notin E\} \tag{9.2}$$

Suppose that G is a graph that has at least one pair of nonadjacent vertices. In this case, the right-hand side of equation (9.2) is the size of a minimum vertex cut of G. But we cannot immediately conclude from (9.2) that the connectivity of G is equal to the size of a minimum vertex cut of G because, according to our definition, κ is the minimum value of $p(u, v)$ taken over all pairs of distinct vertices u, v (whether adjacent or not). However, the following theorem, due to Whitney (1932a), shows that the minimum local connectivity taken over all pairs of distinct vertices, is indeed the same as the minimum taken over all pairs of distinct nonadjacent vertices.

Theorem 9.2 *If G has at least one pair of nonadjacent vertices,*

$$\kappa(G) = \min\{p(u, v) : \ u, v \in V, \ u \neq v, uv \notin E\} \tag{9.3}$$

Proof If G has an edge e which is either a loop or one of a set of parallel edges, we can establish the theorem by deleting e and applying induction. So we may assume that G is simple.

By (9.1), $\kappa(G) = \min\{p(u, v) : \ u, v \in V, \ u \neq v\}$. Let this minimum be attained for the pair x, y, so that $\kappa(G) = p(x, y)$. If x and y are nonadjacent, there is nothing to prove. So suppose that x and y are adjacent.

Consider the graph $H := G \setminus xy$, obtained by deleting the edge xy from G. Clearly, $p_G(x, y) = p_H(x, y) + 1$. Furthermore, by Menger's Theorem, $p_H(x, y) = c_H(x, y)$. Let X be a minimum vertex cut in H separating x and y, so that $p_H(x, y) = c_H(x, y) = |X|$, and $p_G(x, y) = |X| + 1$. If $V \setminus X = \{x, y\}$, then

$$\kappa(G) = p_G(x, y) = |X| + 1 = (n - 2) + 1 = n - 1$$

But this implies that G is complete, which is contrary to the hypothesis. So we may assume that $V \setminus X$ has at least three vertices, x, y, z. We may also assume, interchanging the roles of x and y if necessary, that x and z belong to different components of $H - X$. Then x and z are nonadjacent in G and $X \cup \{y\}$ is a vertex cut of G separating x and z. Therefore,

$$c(x, z) \leq |X \cup \{y\}| = p(x, y)$$

On the other hand, by Menger's Theorem, $p(x, z) = c(x, z)$. Hence $p(x, z) \leq p(x, y)$. By the choice of $\{x, y\}$, we have $p(x, z) = p(x, y) = \kappa(G)$. Because x and z are nonadjacent,

$$\kappa(G) = p(x, z) = \min\{p(u, v) : u, v \in V, \ u \neq v, uv \notin E\} \qquad \square$$

It follows from Theorems 9.1 and 9.2 that the connectivity of a graph G which has at least one pair of nonadjacent vertices is equal to the size of a minimum vertex cut of G. In symbols,

$$\kappa(G) = \min\{c(u, v) : \ u, v \in V, \ u \neq v, uv \notin E\} \tag{9.4}$$

The vertex cuts of a graph are the same as those of its underlying simple graph, thus (9.4) implies that the connectivity of a graph which has at least one pair of nonadjacent vertices is the same as the connectivity of its underlying simple graph.

As noted in Section 8.2, for every nonadjacent pair x, y of vertices of G, the value of $c(x, y)$ may be computed by running the Max-Flow Min-Cut Algorithm (7.9) on an auxiliary digraph of order $2n - 2$ with unit capacities. It follows that the connectivity of any graph may be computed in polynomial time.

Exercises

9.1.1 Consider the vertices $x = (0, 0, \ldots, 0)$ and $y = (1, 1, \ldots, 1)$ of the n-cube Q_n. Describe a maximum collection of edge-disjoint xy-paths in Q_n and a minimum vertex cut of Q_n separating x and y.

9.1.2 Let G and H be simple graphs. Show that $\kappa(G \vee H) = \min\{v(G) + \kappa(H),\ v(H) + \kappa(G)\}$.

9.1.3

a) Show that if G is simple and $\delta \geq n - 2$, then $\kappa = \delta$.
b) For each $n \geq 4$, find a simple graph G with $\delta = n - 3$ and $\kappa < \delta$.

9.1.4 Show that if G is simple, with $n \geq k + 1$ and $\delta \geq (n + k - 2)/2$, then G is k-connected.

9.1.5 An edge e of a 2-connected graph G is called *contractible* if $G\,/\,e$ is 2-connected also. (The analogous concept, for nonseparable graphs, was defined in Exercise 5.3.2.) Show that every 2-connected graph on three or more vertices has a contractible edge.

9.1.6 An edge of a graph G is *deletable* (with respect to connectivity) if $\kappa(G \setminus e) = \kappa(G)$. Show that each edge of a 2-connected graph on at least four vertices is either deletable or contractible.

9.1.7 A k-connected graph G is *minimally k-connected* if the graph $G \setminus e$ is not k-connected for any edge e (that is, if no edge is deletable).

a) Let G be a minimally 2-connected graph. Show that:
   i) $\delta = 2$,
   ii) if $n \geq 4$, then $m \leq 2n - 4$.
b) For all $n \geq 4$, find a minimally 2-connected graph with n vertices and $2n - 4$ edges.

9.1.8 Let G be a connected graph which is not complete. Show that G is k-connected if and only if any two vertices at distance two are connected by k internally disjoint paths.

9.1.9 Consider the following statement, which resembles Menger's Theorem. *Let $G(x, y)$ be a graph of diameter d, where x and y are nonadjacent vertices. Then the maximum number of internally disjoint xy-paths of length d or less is equal to the minimum number of vertices whose deletion destroys all xy-paths of length d or less.*

a) Prove this statement for $d = 2$.
b) Verify that the graph shown in Figure 9.4 is a counterexample to the statement in general. (J.A. Bondy and P. Hell)

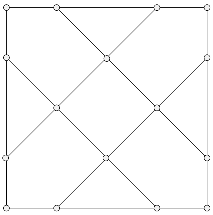


Fig. 9.4. A counterexample to Menger's Theorem for short paths (Exercise 9.1.9)

9.1.10

a) Show that if G is a k-connected graph and e is any edge of G, then $G\,/\,e$ is $(k-1)$-connected.
b) For each $k \geq 4$, find a k-connected graph $G \neq K_{k+1}$ such that $\kappa(G\,/\,e) = k-1$, for every edge e of G.

9.1.11

a) Let $D := D(X, Y)$ be a directed graph, where X and Y are disjoint subsets of V. Obtain an undirected graph G from D as follows.
   ▷ For each vertex v of D, replace v by two adjacent vertices, v^- and v^+.
   ▷ For each arc (u, v) of D, join u^+ and v^- by an edge.
   ▷ Delete the set of vertices $\{x^- : x \in X\} \cup \{y^+ : y \in Y\}$.

   Observe that G is a bipartite graph with bipartition

$$\big(\{v^- : v \in V(D)\} \setminus \{x^- : x \in X\}, \{v^+ : v \in V(D)\} \setminus \{y^+ : y \in Y\}\big)$$

   Show that:
   i) $\alpha'(G) = |V(D)| - |X \cup Y| + p_D(X, Y)$, where $p_D(X, Y)$ denotes the maximum number of disjoint directed (X, Y)-paths in D,
   ii) $\beta(G) = |V(D)| - |X \cup Y| + c_D(X, Y)$, where $c_D(X, Y)$ denotes the minimum number of vertices whose deletion destroys all directed (X, Y)-paths in D. (A. Schrijver)

b) Derive Menger's Theorem (9.8) from the König–Egerváry Theorem (8.32).

9.1.12 Let xPy be a path in a graph G. Two vines on P (defined in Exercise 5.3.12) are *disjoint* if:

▷ their constituent paths are internally disjoint,
▷ x is the only common initial vertex of two paths in these vines,
▷ y is the only common terminal vertex of two paths in these vines.

If G is k-connected, where $k \geq 2$, show that there are $k-1$ pairwise disjoint vines on P. (S.C. Locke)

9.1.13 Let P be a path in a 3-connected cubic graph G.

a) Consider two disjoint vines on P. Denote by F the union of P and the constituent paths of these two vines. Show that F admits a double cover by three cycles.
b) Deduce that if P is of length l, then G has a cycle of length greater than $2l/3$. (Compare Exercise 5.3.12.) (J.A. Bondy and S.C. Locke)

9.1.14 Let G be a 3-connected graph, and let e and f be two edges of G. Show that:

a) the subspace generated by the cycles through e and f has dimension $\dim(\mathcal{C})-1$, (C. Thomassen)
b) G has an odd cycle through e and f unless $G \setminus \{e, f\}$ is bipartite. (W.D. McCuaig and M. Rosenfeld)

9.2 The Fan Lemma

One can deduce many properties of a graph merely from a knowledge of its connectivity. In this context, Menger's Theorem, or a derivative of it, invariably plays a principal role. We describe here a very useful consequence of Menger's Theorem known as the *Fan Lemma*, and apply it to deduce a theorem of Dirac (1952b) about cycles in k-connected graphs.

The following lemma establishes a simple but important property of k-connected graphs.

Lemma 9.3 *Let G be a k-connected graph and let H be a graph obtained from G by adding a new vertex y and joining it to at least k vertices of G. Then H is also k-connected.*

Proof The conclusion clearly holds if any two vertices of H are adjacent, because $v(H) \geq k+1$. Let S be a subset of $V(H)$ with $|S| = k-1$. To complete the proof, it suffices to show that $H - S$ is connected.

Suppose first that $y \in S$. Then $H - S = G - (S \setminus \{y\})$. By hypothesis, G is k-connected and $|S \setminus \{y\}| = k-2$. We deduce that $H - S$ is connected.

Now suppose that $y \notin S$. Since, by hypothesis, y has at least k neighbours in $V(G)$ and $|S| = k - 1$, there is a neighbour z of y which does not belong to S. Because G is k-connected, $G - S$ is connected. Furthermore, z is a vertex of $G - S$, and hence yz is an edge of $H - S$. It follows that $(G - S) + yz$ is a spanning connected subgraph of $H - S$. Hence $H - S$ is connected. □

The following useful property of k-connected graphs can be deduced from Lemma 9.3.

Proposition 9.4 *Let G be a k-connected graph, and let X and Y be subsets of V of cardinality at least k. Then there exists in G a family of k pairwise disjoint (X, Y)-paths.*

Proof Obtain a new graph H from G by adding vertices x and y and joining x to each vertex of X and y to each vertex of Y. By Lemma 9.3, H is k-connected. Therefore, by Menger's Theorem, there exist k internally disjoint xy-paths in H. Deleting x and y from each of these paths, we obtain k disjoint paths $Q_1, Q_2, \ldots, Q_k$ in G, each of which has its initial vertex in X and its terminal vertex in Y. Every path Q_i necessarily contains a segment P_i with initial vertex in X, terminal vertex in Y, and no internal vertex in $X \cup Y$, that is, an (X, Y)-path. The paths $P_1, P_2, \ldots, P_k$ are pairwise disjoint (X, Y)-paths. □

A family of k internally disjoint (x, Y)-paths whose terminal vertices are distinct is referred to as a *k-fan* from x to Y. The following assertion is another very useful consequence of Menger's Theorem. Its proof is similar to the proof of Proposition 9.4 (Exercise 9.2.1).

Proposition 9.5 THE FAN LEMMA
Let G be a k-connected graph, let x be a vertex of G, and let $Y \subseteq V \setminus \{x\}$ be a set of at least k vertices of G. Then there exists a k-fan in G from x to Y. □

We now give the promised application of the Fan Lemma. By Theorem 5.1, in a 2-connected graph any two vertices are connected by two internally disjoint paths; equivalently, any two vertices in a 2-connected graph lie on a common cycle. Dirac (1952b) generalized this latter statement to k-connected graphs.

Theorem 9.6 *Let S be a set of k vertices in a k-connected graph G, where $k \geq 2$. Then there is a cycle in G which includes all the vertices of S.*

Proof By induction on k. We have already observed that the assertion holds for $k = 2$, so assume that $k \geq 3$. Let $x \in S$, and set $T := S \setminus x$. Because G is k-connected, it is $(k - 1)$-connected. Therefore, by the induction hypothesis, there is a cycle C in G which includes T. Set $Y := V(C)$. If $x \in Y$, then C includes all the vertices of S. Thus we may assume that $x \notin Y$. If $|Y| \geq k$, the Fan Lemma (Proposition 9.5) ensures the existence of a k-fan in G from x to Y. Because $|T| = k - 1$, the set T divides C into $k - 1$ edge-disjoint segments. By the Pigeonhole Principle, some two paths of the fan, P and Q, end in the same

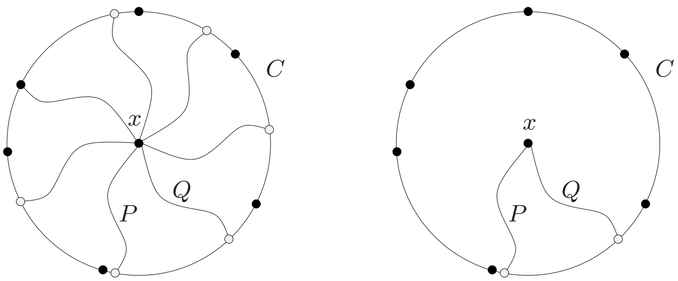


Fig. 9.5. Proof of Theorem 9.6

segment. The subgraph $C \cup P \cup Q$ contains three cycles, one of which includes $S = T \cup \{x\}$ (see Figure 9.5). If $|Y| = k - 1$, the Fan Lemma yields a $(k-1)$-fan from x to Y in which each vertex of Y is the terminus of one path, and we conclude as before. □

It should be pointed out that the order in which the vertices of S occur on the cycle whose existence is established in Theorem 9.6 cannot be specified in advance. For example, the 4-connected graph shown in Figure 9.6 has no cycle including the four vertices x_1, y_1, x_2, y_2 in this exact order, because every x_1y_1-path intersects every x_2y_2-path.

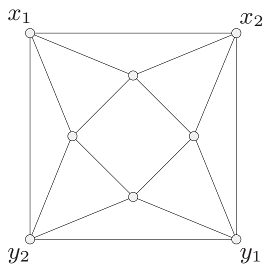


Fig. 9.6. No cycle includes x_1, y_1, x_2, y_2 in this order

Exercises

⋆**9.2.1** Give a proof of the Fan Lemma (Proposition 9.5).

9.2.2 Show that a 3-connected nonbipartite graph contains at least four odd cycles.

⋆**9.2.3** Let C be a cycle of length at least three in a nonseparable graph G, and let S be a set of three vertices of C. Suppose that some component H of $G - V(C)$

is adjacent to all three vertices of S. Show that there is a 3-fan in G from some vertex v of H to S.

9.2.4 Find a 5-connected graph G and a set $\{x_1, y_1, x_2, y_2\}$ of four vertices in G, such that no cycle of G contains all four vertices in the given order. (In a 6-connected graph, it can be shown that there is a cycle containing any four vertices in any prescribed order.)

9.2.5 Let G be a graph, x a vertex of G, and Y and Z subsets of $V \setminus \{x\}$, where $|Y| < |Z|$. Suppose that there are fans from x to Y and from x to Z. Show that there is a fan from x to $Y \cup \{z\}$ for some $z \in Z \setminus Y$. (H. Perfect)

9.3 Edge Connectivity

We now turn to the notion of edge connectivity. The *local edge connectivity* between distinct vertices x and y is the maximum number of pairwise edge-disjoint xy-paths, denoted $p'(x, y)$; the local edge connectivity is undefined when $x = y$. A nontrivial graph G is *k-edge-connected* if $p'(u, v) \geq k$ for any two distinct vertices u and v of G. By convention, a trivial graph is both 0-edge-connected and 1-edge-connected, but is not k-edge-connected for any $k > 1$. The *edge connectivity* $\kappa'(G)$ of a graph G is the maximum value of k for which G is k-edge-connected.

A graph is 1-edge-connected if and only if it is connected; equivalently, the edge connectivity of a graph is zero if and only if it is disconnected. For the four graphs shown in Figure 9.1, $\kappa'(G_1) = 1$, $\kappa'(G_2) = 2$, $\kappa'(G_3) = 3$, and $\kappa'(G_4) = 4$. Thus, of these four graphs, the only graph that is 4-edge-connected is G_4. Graphs G_3 and G_4 are 3-edge-connected, but G_1 and G_2 are not. Graphs G_2, G_3, and G_4 are 2-edge-connected, but G_1 is not. And, because all four graphs are connected, they are all 1-edge-connected.

For distinct vertices x and y of a graph G, recall that an edge cut $\partial(X)$ *separates* x and y if $x \in X$ and $y \in V \setminus X$. We denote by $c'(x, y)$ the minimum cardinality of such an edge cut. With this notation, we may now restate the edge version of Menger's Theorem (7.17).

Theorem 9.7 Menger's Theorem (Edge Version)
For any graph $G(x, y)$,

$$p'(x, y) = c'(x, y)$$

This theorem was proved in Chapter 7 using flows. It may also be deduced from Theorem 9.1 by considering a suitable line graph (see Exercise 9.3.11).

A *k-edge cut* is an edge cut $\partial(X)$, where $\emptyset \subset X \subset V$ and $|\partial(X)| = k$, that is, an edge cut of k elements which separates some pair of vertices. Because every nontrivial graph has such edge cuts, it follows from Theorem 9.7 that the edge connectivity $\kappa'(G)$ of a nontrivial graph G is equal to the least integer k for which

G has a k-edge cut. For any particular pair x, y of vertices of G, the value of $c'(x, y)$ can be determined by an application of the Max-Flow Min-Cut Algorithm (7.9). Therefore the parameter κ' can obviously be determined by $\binom{n}{2}$ applications of that algorithm. However, the function c' takes at most $n - 1$ distinct values (Exercise 9.3.13b). Moreover, Gomory and Hu (1961) have shown that κ' can be computed by just $n - 1$ applications of the Max-Flow Min-Cut Algorithm. A description of their approach is given in Section 9.6.

Essential Edge Connectivity

The vertex and edge connectivities of a graph G, and its minimum degree, are related by the following basic inequalities (Exercise 9.3.2).

$$\kappa \leq \kappa' \leq \delta$$

Thus, for 3-regular graphs, the connectivity and edge connectivity do not exceed three. They are, moreover, always equal for such graphs (Exercise 9.3.5). These two measures of connectivity therefore fail to distinguish between the triangular prism $K_3 \,\square\, K_2$ and the complete bipartite graph $K_{3,3}$, both of which are 3-regular graphs with connectivity and edge connectivity equal to three. Nonetheless, one has the distinct impression that $K_{3,3}$ is better connected than $K_3 \,\square\, K_2$. Indeed, $K_3 \,\square\, K_2$ has a 3-edge cut which separates the graph into two nontrivial subgraphs, whereas $K_{3,3}$ has no such cut.

Recall that a *trivial* edge cut is one associated with a single vertex. A k-edge-connected graph is termed *essentially* $(k+1)$*-edge-connected* if all of its k-edge cuts are trivial. For example, $K_{3,3}$ is essentially 4-edge-connected whereas $K_3 \,\square\, K_2$ is not. If a k-edge-connected graph has a k-edge cut $\partial(X)$, the graphs $G \,/\, X$ and $G \,/\, \overline{X}$ (obtained by shrinking X to a single vertex x and $\overline{X} := V \setminus X$ to a single vertex $\overline{x}$, respectively) are also k-edge-connected (Exercise 9.3.8). By iterating this shrinking procedure, any k-edge-connected graph with $k \geq 1$, can be 'decomposed' into a set of essentially $(k + 1)$-edge-connected graphs. For many problems, it is enough to treat each of these 'components' separately. (When $k = 0$ — that is, when the graph is disconnected — this procedure corresponds to considering each of its components individually.)

The notion of essential edge connectivity is particularly useful for 3-regular graphs. For instance, to show that a 3-connected 3-regular graph has a cycle double cover, it suffices to verify that each of its essentially 4-edge-connected components has one; the individual cycle double covers can then be spliced together to yield a cycle double cover of the entire graph (Exercise 9.3.9).

Connectivity in Digraphs

The definitions of connectivity and edge connectivity have straightforward extensions to digraphs. It suffices to replace 'path' by 'directed path' throughout. We have already seen three versions of Menger's Theorem, namely the arc version

(Theorem 7.16), and the edge and vertex versions for undirected graphs (Theorems 7.17 and 9.1). Not surprisingly, there is also a vertex version for directed graphs. It can be deduced easily from the reduction of IDDP to ADDP described in Section 8.3. An (x, y)-*vertex-cut* is a subset S of $V \setminus \{x, y\}$ whose deletion destroys all directed (x, y)-paths.

Theorem 9.8 MENGER'S THEOREM (DIRECTED VERTEX VERSION)
In any digraph $D(x, y)$, where $(x, y) \notin A(D)$, the maximum number of pairwise internally disjoint directed (x, y)-paths is equal to the minimum number of vertices in an (x, y)-vertex-cut. □

As already noted, of the four versions of Menger's Theorem, Theorem 7.16 implies the other three. Also, Theorem 9.8 clearly implies Theorem 9.1. Although less obvious, the converse implication holds too (see Exercise 9.1.11). By using a suitable line graph, Theorem 9.7 may be derived from Theorem 9.1 (see Exercise 9.3.11).

Exercises

9.3.1 Determine the connectivity and the edge connectivity of the Kneser graph $KG_{m,n}$.

⋆**9.3.2**

a) Show that every graph G satisfies the inequalities $\kappa \le \kappa' \le \delta$.
b) Find a graph G with $\kappa = 3$, $\kappa' = 4$, and $\delta = 5$.

9.3.3 Let G be a simple graph of diameter two. Show that $\kappa' = \delta$. (J. PLESNÍK)

9.3.4

a) Show that if G is simple and $\delta \ge (n-1)/2$, then $\kappa' = \delta$.
b) For each even $n \ge 2$, find a simple graph G with $\delta = (n/2) - 1$ and $\kappa' < \delta$.

⋆**9.3.5** Show that if G is cubic, then $\kappa = \kappa'$.

9.3.6

a) Show that if G is k-edge-connected, where $k > 0$, and if S is a set of k edges of G, then $c(G \setminus S) \le 2$.
b) For $k > 0$, find a k-connected graph G and a set S of k vertices of G such that $c(G - S) > 2$.

9.3.7 Show that if G is a k-edge-connected graph on at least three vertices, and e is any edge of G, then G / e is k-edge-connected.

⋆**9.3.8** Show that if $\partial(X)$ is a k-edge cut of a k-edge-connected graph G, the graphs G / X and $G / \overline{X}$ are also k-edge-connected, where $\overline{X} := V \setminus X$.

$\star$**9.3.9** Let $\partial(X)$ be a 3-edge cut of a cubic graph G. Show that G has a cycle double cover if and only if both $G\,/\,X$ and $G\,/\,\overline{X}$ have cycle double covers, where $\overline{X} := V \setminus X$.

9.3.10 Show that in a nontrivial connected graph, any minimal edge cut separating two of its vertices is a bond.

9.3.11 Deduce Theorem 9.7 from Theorem 9.1. (F. Harary)

9.3.12 Let S be a set of three pairwise-nonadjacent edges in a simple 3-connected graph G. Show that there is a cycle in G containing all three edges of S unless S is an edge cut of G. (L. Lovász, N. Robertson)

$\star$**9.3.13**

a) Show that, for any three vertices x, y, and z of a graph G:

$$c'(x,z) \geq \min\{c'(x,y), c'(y,z)\}$$

and that at least two of the values $c'(x,y)$, $c'(x,z)$, and $c'(y,z)$ are equal.

b) Deduce from (a) that:
   i) the function c' takes on at most $n-1$ distinct values,
   ii) for any sequence $(v_1, v_2, \ldots, v_k)$ of vertices of a graph G,

$$c'(v_1, v_k) \geq \min\{c'(v_1, v_2), c'(v_2, v_3), \ldots, c'(v_{k-1}, v_k)\}$$

9.3.14 A k-edge-connected graph G is *minimally k-edge-connected* if, for any edge e of G, the graph $G \setminus e$ is not k-edge-connected.

a) Let G be a minimally k-edge-connected graph. Prove that:
   i) every edge e of G is contained in a k-edge cut of G,
   ii) G has a vertex of degree k,
   iii) $m \leq k(n-1)$.

b) Deduce that every k-edge-connected graph G contains a spanning k-edge-connected subgraph with at most $k(n-1)$ edges. (W. Mader)

(Halin (1969) and Mader (1971b) found analogues of the above statements for vertex connectivity.)

9.4 Three-Connected Graphs

As we observed in Chapter 5, in most instances it is possible to draw conclusions about a graph by examining each of its blocks individually. For example, a graph has a cycle double cover if and only if each of its blocks has a cycle double cover. Because blocks on more than two vertices are 2-connected, the question of the existence of a cycle double cover can therefore be restricted, or 'reduced', to the study of 2-connected graphs. A similar reduction applies to the problem of deciding whether a given graph is planar, as we show in Chapter 10.

In many cases, further reductions can be applied, allowing one to restrict the analysis to 3-connected graphs, or even to 3-connected essentially 4-edge-connected graphs. The basic idea is to decompose a 2-connected graph which has a 2-vertex cut into smaller 2-connected graphs. Loops do not play a significant role in this context. For clarity, we therefore assume that all graphs considered in this section are loopless.

Let G be a connected graph which is not complete, let S be a vertex cut of G, and let X be the vertex set of a component of $G - S$. The subgraph H of G induced by $S \cup X$ is called an *S-component* of G. In the case where G is 2-connected and $S := \{x, y\}$ is a 2-vertex cut of G, we find it convenient to modify each S-component by adding a new edge between x and y. We refer to this edge as a *marker edge* and the modified S-components as *marked S-components*. The set of marked S-components constitutes the *marked S-decomposition* of G. The graph G can be recovered from its marked S-decomposition by taking the union of its marked S-components and deleting the marker edge. This procedure is illustrated in Figure 9.7, the cut S and the marker edge being indicated by solid dots and lines.

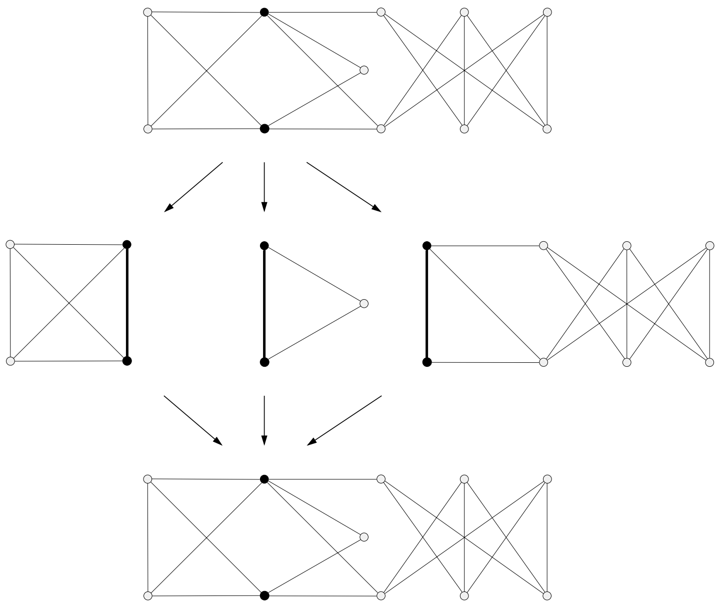


Fig. 9.7. A marked decomposition of a 2-connected graph and its recomposition

Theorem 9.9 *Let G be a 2-connected graph and let S be a 2-vertex cut of G. Then the marked S-components of G are also 2-connected.*

Proof Let H be a marked S-component of G, with vertex set $S \cup X$. Then $|V(H)| = |S| + |X| \geq 3$. Thus if H is complete, it is 2-connected. On the other hand, if H is not complete, every vertex cut of H is also a vertex cut of G, hence of cardinality at least two. □

Decomposition Trees

By Theorem 9.9, a 2-connected graph G with a 2-vertex cut S has a marked S-decomposition into 2-connected graphs. If any one of these marked S-components itself has a 2-vertex cut, it in turn can be decomposed into still smaller marked 2-connected graphs. This decomposition process may be iterated until G has been decomposed into 2-connected graphs without 2-vertex cuts. The marked S-components which arise during the entire procedure form the vertices of a *decomposition tree* of G, as illustrated in Figure 9.8.

The root of this decomposition tree is G, and its leaves are either 3-connected graphs or else 2-connected graphs whose underlying graphs are complete (and which therefore have at most three vertices). We refer to the 3-connected graphs in such a decomposition as the *3-connected components* of G. The 3-connected components of the root graph in Figure 9.8 are K_3 (both with and without multiple edges), K_4, and $K_{3,3}$.

At any stage, there may be a choice of cuts along which to decompose a graph. Consequently, two separate applications of this decomposition procedure may well result in different sets of graphs (Exercise 9.4.1). However, it was shown by Cunningham and Edmonds (1980) that any two applications of the procedure always result in the same set of 3-connected components (possibly with different edge multiplicities).

To observe the relevance of the above decomposition to cycle double covers, let G be a 2-connected graph with a 2-vertex cut S. If each marked S-component of G has a cycle double cover, one can show that G also has a cycle double cover (Exercise 9.4.2). Because 2-connected graphs on at most three vertices clearly have cycle double covers, we conclude that if the Cycle Double Cover Conjecture is true for all 3-connected graphs, it is true for all 2-connected graphs. This fact may be expressed more strikingly in terms of potential counterexamples to the conjecture: if the Cycle Double Cover Conjecture is false, a smallest counterexample to it (that is, one with the minimum possible number of vertices) must be 3-connected. Jaeger (1976) and Kilpatrick (1975) proved that every 4-edge-connected graph has a cycle double cover (see Theorem 21.24). Thus, if the Cycle Double Cover Conjecture happens to be false, a minimum counterexample must have connectivity precisely three.

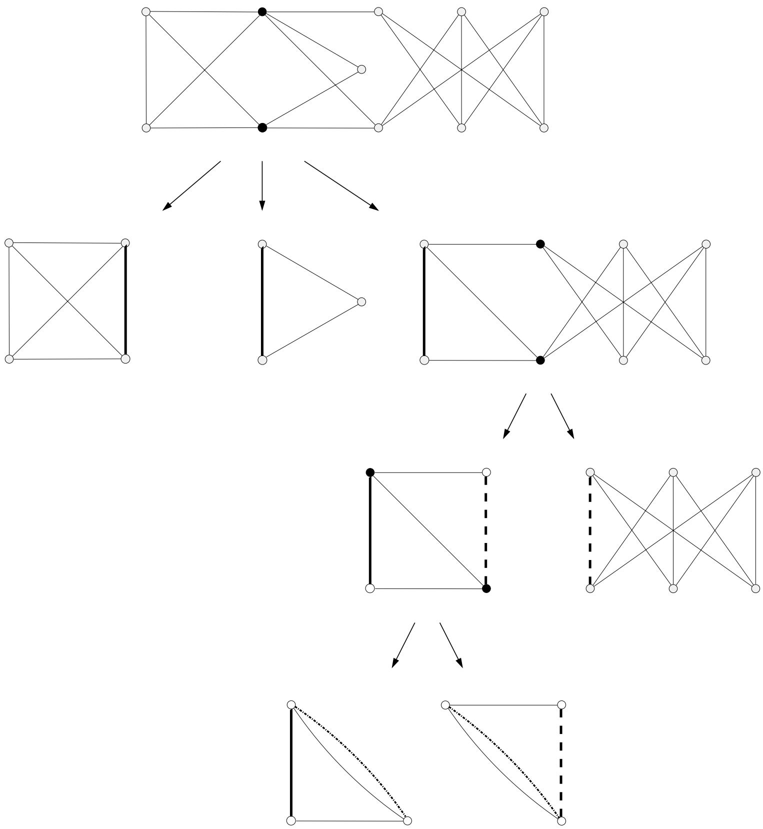


Fig. 9.8. A decomposition tree of a 2-connected graph

Contractions of Three-Connected Graphs

The relevance of 3-connectivity to the study of planar graphs is discussed in Section 10.5. In this context, the following property of 3-connected graphs, established by Thomassen (1981), plays an extremely useful role.

Theorem 9.10 *Let G be a 3-connected graph on at least five vertices. Then G contains an edge e such that $G\,/\,e$ is 3-connected.*

The proof of Theorem 9.10 requires the following lemma.

Lemma 9.11 *Let G be a 3-connected graph on at least five vertices, and let $e = xy$ be an edge of G such that $G\,/\,e$ is not 3-connected. Then there exists a vertex z such that $\{x, y, z\}$ is a 3-vertex cut of G.*

Proof Let $\{z, w\}$ be a 2-vertex cut of $G\,/\,e$. At least one of these two vertices, say z, is not the vertex resulting from the contraction of e. Set $F := G - z$.

Because G is 3-connected, F is certainly 2-connected. However $F \,/\, e = (G-z) \,/\, e = (G \,/\, e) - z$ has a cut vertex, namely w. Hence w must be the vertex resulting from the contraction of e (Exercise 9.4.3). Therefore $G - \{x, y, z\} = (G \,/\, e) - \{z, w\}$ is disconnected, in other words, $\{x, y, z\}$ is a 3-vertex cut of G. $\square$

Proof of Theorem 9.10. Suppose that the theorem is false. Then, for any edge $e = xy$ of G, the contraction $G \,/\, e$ is not 3-connected. By Lemma 9.11, there exists a vertex z such that $\{x, y, z\}$ is a 3-vertex cut of G (see Figure 9.9).

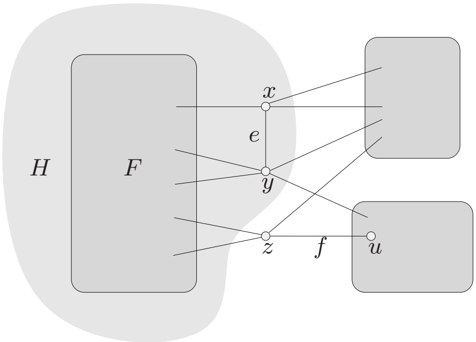


Fig. 9.9. Proof of Theorem 9.10

Choose the edge e and the vertex z in such a way that $G - \{x, y, z\}$ has a component F with as many vertices as possible. Consider the graph $G-z$. Because G is 3-connected, $G-z$ is 2-connected. Moreover $G-z$ has the 2-vertex cut $\{x, y\}$. It follows that the $\{x, y\}$-component $H := G[V(F) \cup \{x, y\}]$ is 2-connected.

Let u be a neighbour of z in a component of $G - \{x, y, z\}$ different from F. Since $f := zu$ is an edge of G, and G is a counterexample to Theorem 9.10, there is a vertex v such that $\{z, u, v\}$ is a 3-vertex cut of G, too. (The vertex v is not shown in Figure 9.9; it might or might not lie in H.) Moreover, because H is 2-connected, $H - v$ is connected (where, if $v \notin V(H)$, we set $H - v := H$), and thus is contained in a component of $G - \{z, u, v\}$. But this component has more vertices than F (because H has two more vertices than F), contradicting the choice of the edge e and the vertex v. $\square$

Although the proof of Theorem 9.10 proceeds by way of contradiction, the underlying idea can be used to devise a polynomial-time algorithm for finding an edge e in a 3-connected graph G such that $G \,/\, e$ is 3-connected (Exercise 9.4.4).

Expansions of Three-Connected Graphs

We now define an operation on 3-connected graphs which may be thought of as an inverse to contraction. Let G be a 3-connected graph and let v be a vertex of

G of degree at least four. Split v into two vertices, v_1 and v_2, add a new edge e between v_1 and v_2, and distribute the edges of G incident to v among v_1 and v_2 in such a way that v_1 and v_2 each have at least three neighbours in the resulting graph H. This graph H is called an *expansion* of G at v (see Figure 9.10).

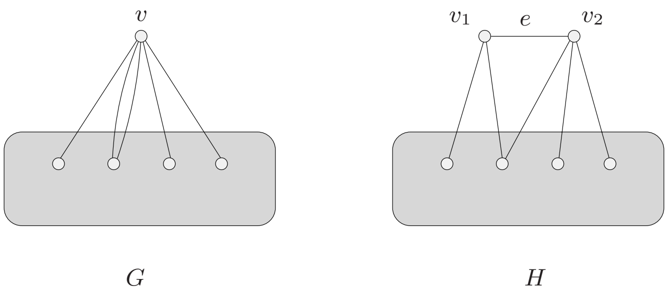


Fig. 9.10. An expansion of a graph at a vertex

Note that there is usually some freedom as to how the edges of G incident with v are distributed between v_1 and v_2, so expansions are not in general uniquely defined. On the other hand, the contraction $H\,/\,e$ is clearly isomorphic to G.

The following theorem may be regarded as a kind of converse of Theorem 9.10.

Theorem 9.12 *Let G be a 3-connected graph, let v be a vertex of G of degree at least four, and let H be an expansion of G at v. Then H is 3-connected.*

Proof Because $G - v$ is 2-connected and v_1 and v_2 each have at least two neighbours in $G - v$, the graph $H \setminus e$ is 2-connected, by Lemma 9.3. Using the fact that any two vertices of G are connected by three internally disjoint paths, it is now easily seen that any two vertices of H are also connected by three internally disjoint paths. $\square$

In light of Theorems 9.10 and 9.12, every 3-connected graph G can be obtained from K_4 by means of edge additions and vertex expansions. More precisely, given any 3-connected graph G, there exists a sequence $G_1, G_2, \ldots, G_k$ of graphs such that (i) $G_1 = K_4$, (ii) $G_k = G$, and (iii) for $1 \le i \le k-1$, G_{i+1} is obtained by adding an edge to G_i or by expanding G_i at a vertex of degree at least four.

It is not possible to obtain a simple 3-connected graph, different from K_4, by means of the above construction if we wish to stay within the realm of simple graphs. However, Tutte (1961b) has shown that, by starting with the class of all wheels, all simple 3-connected graphs may be constructed by means of the two above-mentioned operations without ever creating parallel edges. This result may be deduced from Theorem 9.10 (see Exercise 9.4.8).

Recursive constructions of 3-connected graphs have been used to prove many interesting theorems in graph theory; see, for example, Exercise 9.4.10. For further examples, see Tutte (1966a).

For $k \geq 4$, no recursive procedure for generating all k-connected graphs is known. This is in striking contrast with the situation for k-edge-connected graphs (see Exercise 9.5.5). We refer the reader to Frank (1995) for a survey of recursive procedures for generating k-connected and k-edge-connected graphs.

Exercises

9.4.1 Find a 2-connected graph and two decomposition trees of your graph which result in different collections of leaves.

9.4.2 Let G be a 2-connected graph with a 2-vertex cut $S := \{u, v\}$. Prove that if each marked S-component of G has a cycle double cover then so has G.

$\star$**9.4.3** Let G be a 2-connected graph and let e be an edge of G such that $G\,/\,e$ is not 2-connected. Prove that $G\,/\,e$ has exactly one cut vertex, namely the vertex resulting from the contraction of the edge e.

9.4.4 Describe a polynomial-time algorithm to find, in a 3-connected graph G on five or more vertices, an edge e such that $G\,/\,e$ is 3-connected.

9.4.5

a) Let G be a 4-regular 4-connected graph each edge of which lies in a triangle. Show that no edge-contraction of G is 4-connected.
b) For each integer $k \geq 4$, find a k-connected graph G on at least $k + 2$ vertices, none of whose edge contractions is k-connected.

9.4.6 Show how the Petersen graph can be obtained from the wheel W_6 by means of vertex expansions.

9.4.7 Let G be a graph and let $e = xy$ and $e' = x'y'$ be two distinct (but possibly adjacent) edges of G. The operation which consists of subdividing e by inserting a new vertex v between x and y, subdividing e' by inserting a new vertex v' between x' and y', and joining v and v' by a new edge, is referred to as an *edge-extension* of G. Show that:

a) any edge-extension of a 3-connected cubic graph is also 3-connected and cubic,
b) every 3-connected cubic graph can be obtained from K_4 by means of a sequence of edge-extensions,
c) an edge-extension of an essentially 4-edge-connected cubic graph G is also essentially 4-edge-connected provided that the two edges e and e' of G involved in the extension are nonadjacent in G.

(Wormald (1979) has shown that all essentially 4-edge-connected cubic graphs may be obtained from K_4 and the cube by means of edge-extensions involving nonadjacent pairs of edges.)

9.4.8 Let G be a 3-connected graph with $n \geq 5$. Show that, for any edge e, either $G\,/\,e$ is 3-connected or $G \setminus e$ can be obtained from a 3-connected graph by subdividing at most two edges. (W.T. Tutte)

9.4.9 Let G be a simple 3-connected graph different from a wheel. Show that, for any edge e, either $G\,/\,e$ or $G \setminus e$ is also a 3-connected simple graph.

(W.T. Tutte)

$\star$**9.4.10**

a) Let $\mathcal{G}$ be the family of graphs consisting of K_5, the wheels W_n, $n \geq 3$, and all graphs of the form $H \vee \overline{K}_n$, where H is a spanning subgraph of K_3 and $\overline{K}_n$ is the complement of K_n, $n \geq 3$. Show that a 3-connected simple graph G does not contain two disjoint cycles if and only if $G \in \mathcal{G}$.

(W.G. Brown; L. Lovász)

b) Deduce from (a) that any simple graph not containing two disjoint cycles has three vertices whose deletion results in an acyclic graph.
(The same result holds for directed cycles in digraphs, although the proof, due to McCuaig (1993), is very much harder. For undirected graphs, Erdős and Pósa (1965) showed that there exists a constant c such that any graph either contains k disjoint cycles or has $ck \log k$ vertices whose deletion results in an acyclic graph. This is discussed in Section 19.1.)

9.5 Submodularity

A real-valued function f defined on the set of subsets of a set S is *submodular* if, for any two subsets X and Y of S,

$$f(X \cup Y) + f(X \cap Y) \leq f(X) + f(Y)$$

The degree function d defined on the set of subsets of the vertex set of a graph G by $d(X) := |\partial(X)|$ for all $X \subseteq V$ is a typical example of a submodular function associated with a graph (see Exercise 2.5.4). Another example is described in Exercise 9.5.7.

Submodular functions play an important role in combinatorial optimization (see Fujishige (2005)). Here, we describe three interesting consequences of the submodularity of the degree function. One of these is Theorem 9.16, which has many applications, including a theorem on orientations of graphs due to Nash-Williams (1960). A second use of submodularity is described below, and a third is given in Section 9.6.

It is convenient both here and in the next section to denote the complement $V \setminus X$ of a set X by $\overline{X}$.

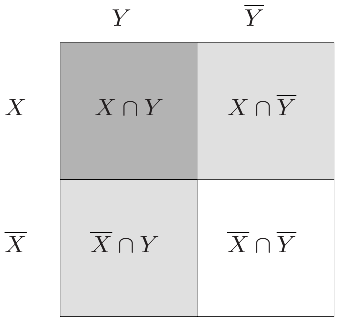


Fig. 9.11. Crossing sets X and Y

Edge Connectivity of Vertex-Transitive Graphs

Two subsets X and Y of a set V are said to *cross* if the subsets $X \cap Y$, $X \cap \overline{Y}$, $\overline{X} \cap Y$, and $\overline{X} \cap \overline{Y}$ (shown in the Venn diagram of Figure 9.11) are all nonempty. When V is the vertex set of a graph G, we say that the edge cuts $\partial(X)$ and $\partial(Y)$ *cross* if the sets X and Y cross. In such cases, it is often fruitful to consider the edge cuts $\partial(X \cup Y)$ and $\partial(X \cap Y)$ and invoke the submodularity of the degree function. Here, we apply this idea to show that the edge connectivity of a nontrivial connected vertex-transitive graph is always equal to its degree, a result due independently to Mader (1971a) and Watkins (1970). Its proof relies on the concept of an atom.

An *atom* of a graph G is a minimal subset X of V such that $d(X) = \kappa'$ and $|X| \leq n/2$. Thus if $\kappa' = \delta$, then any vertex of minimum degree is a singleton atom. On the other hand, if $\kappa' < \delta$, then G has no singleton atoms.

Proposition 9.13 *The atoms of a graph are pairwise disjoint.*

Proof Let X and Y be two distinct atoms of a graph G. Suppose that $X \cap Y \neq \emptyset$. Because X and Y are atoms, neither is properly contained in the other, so $X \cap \overline{Y}$ and $\overline{X} \cap Y$ are both nonempty. We show that $\overline{X} \cap \overline{Y}$ is nonempty, too, and thus that X and Y cross.

Noting that $\overline{X} \cup Y$ and $X \cap \overline{Y}$ are complementary sets, and that $X \cap \overline{Y}$ is a nonempty proper subset of the atom X, we have

$$d(\overline{X} \cup Y) = d(X \cap \overline{Y}) > d(X) = d(Y)$$

It follows that $\overline{X} \cup Y \neq Y$ or, equivalently, $\overline{X} \cap \overline{Y} \neq \emptyset$. So X and Y do indeed cross.

Because $\partial(X)$ and $\partial(Y)$ are minimum edge cuts,

$$d(X \cup Y) \geq d(X) \quad \text{and} \quad d(X \cap Y) \geq d(Y)$$

Therefore

$$d(X \cup Y) + d(X \cap Y) \geq d(X) + d(Y)$$

On the other hand, because d is a submodular function,

$$d(X \cup Y) + d(X \cap Y) \leq d(X) + d(Y)$$

These inequalities thus all hold with equality. In particular, $d(X \cap Y) = d(Y)$. But this contradicts the minimality of the atom Y. We conclude that X and Y are disjoint. □

Theorem 9.14 *Let G be a simple connected vertex-transitive graph of positive degree d. Then $\kappa' = d$.*

Proof Let X be an atom of G, and let u and v be two vertices in X. Because G is vertex-transitive, it has an automorphism θ such that $\theta(u) = v$. Being the image of an atom under an automorphism, the set $\theta(X)$ is also an atom of G. As v belongs to both X and $\theta(X)$, it follows from Proposition 9.13 that $\theta(X) = X$, which implies that $\theta|_X$ is an automorphism of the graph $G[X]$, with $\theta|_X(u) = v$. This being so for any two vertices u, v in X, we deduce that $G[X]$ is vertex-transitive.

Suppose that $G[X]$ is k-regular. Because G is simple, $|X| \geq k+1$, and because G is connected, $\partial(X) \neq \emptyset$. Therefore $d \geq k+1$, and we have:

$$\kappa' = d(X) = |X|(d-k) \geq (k+1)(d-k) = d + k(d-k-1) \geq d$$

Since κ' cannot exceed d, we conclude that $\kappa' = d$. □

Nash-Williams' Orientation Theorem

By Theorem 5.10 every 2-edge-connected graph admits a strongly connected orientation. Nash-Williams (1960) established the following beautiful generalization of this result. (In the remainder of this section, k denotes a positive integer.)

Theorem 9.15 *Every $2k$-edge-connected graph has a k-arc-connected orientation.*

Mader (1978) proved an elegant theorem concerning the splitting off of edges (an operation introduced in Chapter 5) and deduced Theorem 9.15 from it. We present here a special case of Mader's result which is adequate for proving Theorem 9.15. The proof is due to Frank (1992).

Let v be a vertex of a graph G. We say that G is *locally $2k$-edge-connected modulo v* if the local edge connectivity between any two vertices different from v is at least $2k$. Using Menger's theorem and the fact that $d(X) = d(\overline{X})$, it can be seen that a graph G on at least three vertices is locally $2k$-edge-connected modulo v if and only if:

$$d(X) \geq 2k, \quad \text{for all } X, \quad \emptyset \subset X \subset V \setminus \{v\}$$

Theorem 9.16 *Let G be a graph which is locally $2k$-edge-connected modulo v, where v is a vertex of even degree in G. Given any link uv incident with v, there exists a second link vw incident with v such that the graph G' obtained by splitting off uv and vw at v is also locally $2k$-edge-connected modulo v.*

Proof We may assume that $n \geq 3$ as the statement holds trivially when $n = 2$. We may also assume that G is loopless. Consider all nonempty proper subsets X of $V \setminus \{v\}$. Splitting off uv and another link vw incident with v preserves the degree of X if at most one of u and w belongs to X, and reduces it by two if both u and w belong to X. Thus if all such sets either do not contain u or have degree at least $2k + 2$, any link vw may be chosen as the companion of uv. Suppose that this is not the case and that there is a proper subset X of $V \setminus \{v\}$ with $u \in X$ and $d(X) \leq 2k + 1$. Call such a set *tight*. We show that the union of two tight sets X and Y is also tight. We may assume that X and Y cross; otherwise, $X \cup Y$ is equal either to X or to Y. Therefore $X \cap \overline{Y}$ and $\overline{X} \cap Y$ are nonempty subsets of $V \setminus \{v\}$. Note, also, that $uv \in E[X \cap Y, \overline{X} \cap \overline{Y}]$. We thus have (using Exercise 2.5.4):

$$\begin{aligned}(2k+1) + (2k+1) &\geq d(X) + d(Y)\\ &= d(X \cap \overline{Y}) + d(\overline{X} \cap Y) + 2e(X \cap Y, \overline{X} \cap \overline{Y}) \geq 2k + 2k + 2\end{aligned}$$

so

$$d(X) = d(Y) = 2k+1, \quad d(X \cap \overline{Y}) = d(\overline{X} \cap Y) = 2k, \quad \text{and} \quad e(X \cap Y, \overline{X} \cap \overline{Y}) = 1$$

One may now deduce that $e(X \cap \overline{Y}, \overline{X} \cap \overline{Y}) = e(\overline{X} \cap Y, \overline{X} \cap \overline{Y})$ (Exercise 9.5.4). Thus $d(\overline{X} \cap \overline{Y})$ is odd. Because the degree of v is even, by hypothesis, $X \cup Y \neq V \setminus \{v\}$. Therefore $\emptyset \subset X \cup Y \subset V \setminus \{v\}$. Moreover, by submodularity,

$$d(X \cup Y) \leq d(X) + d(Y) - d(X \cap Y) \leq (2k+1) + (2k+1) - 2k = 2k + 2$$

Since $d(X \cup Y) = d(\overline{X} \cap \overline{Y})$ is odd, we may conclude that $d(X \cup Y) \leq 2k + 1$. Thus the union of any two tight sets is tight, as claimed. Now let S denote the union of all tight sets and let w be an element of $V \setminus S$ distinct from v. Because w belongs to no tight set in G, the graph G' obtained from G by splitting off uv and vw is locally $2k$-edge-connected modulo v. $\square$

Proof of Theorem 9.15. By induction on the number of edges. Let G be a $2k$-edge-connected graph. Suppose first that G has an edge e such that $G \setminus e$ is also $2k$-edge-connected. Then, by induction, $G \setminus e$ has an orientation such that the resulting digraph is k-arc-connected. That orientation of $G \setminus e$ may be extended to a k-arc-connected orientation of G itself by orienting e arbitrarily. Thus, we may assume that G is minimally $2k$-edge-connected and so has a vertex of degree $2k$ (Exercise 9.3.14). Let v be such a vertex.

By Theorem 9.16, the $2k$ edges incident with v may be divided into k pairs and each of these pairs may be split off, one by one, to obtain k new edges $e_1, e_2, \ldots, e_k$ and a $2k$-edge-connected graph H. By the induction hypothesis, there is an orientation $\overrightarrow{H}$ of H which is k-arc-connected. Let $a_1, a_2, \ldots, a_k$, respectively, be the k arcs of $\overrightarrow{H}$ corresponding to the edges $e_1, e_2, \ldots, e_k$ of H. By subdividing, for $1 \leq i \leq k$, the arc a_i by a vertex v_i, and then identifying the k vertices $v_1, v_2, \ldots, v_k$ to form vertex v, we obtain an orientation $\overrightarrow{G}$ of G. Using the fact that $\overrightarrow{H}$ is k-arc-connected, one may easily verify that $\overrightarrow{G}$ is also k-arc-connected. We leave the details to the reader as Exercise 9.5.5. $\square$

Nash-Williams (1960) in fact proved a far stronger result than Theorem 9.15. He showed that any graph G admits an orientation $\overrightarrow{G}$ such that, for any two vertices u and v, the size of a minimum outcut in $\overrightarrow{G}$ separating v from u is at least $\lfloor \frac{1}{2}c'(u,v) \rfloor$. We refer the reader to Schrijver (2003) for further details.

Exercises

9.5.1 Let X be an atom of a graph G. Show that the induced subgraph $G[X]$ is connected.

9.5.2 Give an example of a connected cubic vertex-transitive graph that is not 3-edge-connected.
(This shows that Theorem 9.14 is not valid for graphs with multiple edges.)

9.5.3 Give an example of a simple connected vertex-transitive k-regular graph whose connectivity is strictly less than k.
(Watkins (1970) showed that the connectivity of any such graph exceeds $2k/3$.)

$\star$**9.5.4** In the proof of Theorem 9.16, show that $e(X \cap \overline{Y}, \overline{X} \cap \overline{Y}) = e(\overline{X} \cap Y, \overline{X} \cap \overline{Y})$.

$\star$**9.5.5** Let $\overrightarrow{G}$ and $\overrightarrow{H}$ be the digraphs described in the proof of Theorem 9.15. Deduce that $\overrightarrow{G}$ is k-arc-connected from the fact that $\overrightarrow{H}$ is k-arc-connected.

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9.5.6 Let G be a $2k$-edge-connected graph with an Euler trail. Show that G has an orientation in which any two vertices u and v are connected by at least k arc-disjoint directed (u,v)-paths.

9.5.7 Let G be a graph. For a subset S of E, denote by $c(S)$ the number of components of the spanning subgraph of G with edge set S.

a) Show that the function $c : 2^E \to \mathbb{N}$ is *supermodular*: for any two subsets X and Y of E,
$$c(X \cup Y) + c(X \cap Y) \geq c(X) + c(Y)$$
b) Deduce that the function $r : 2^E \to \mathbb{N}$ defined by $r(S) := n - c(S)$ for all $S \subseteq E$ is submodular. (This function r is the rank function of a certain matroid associated with G.)

9.5.8 Given any graph G and k distinct edges $e_1, e_2, \ldots, e_k$ (loops or links) of G, the operation of *pinching together* those k edges consists of subdividing, for $1 \leq i \leq k$, the edge e_i by a vertex v_i, and then identifying the k vertices $v_1, v_2, \ldots, v_k$ to form a new vertex of degree $2k$.

a) Show that if G is $2k$-edge-connected, then the graph G' obtained from G by pinching together any k edges of G is also $2k$-edge-connected.

b) Using Theorem 9.16, show that, given any $2k$-edge-connected graph G, there exists a sequence $(G_1, G_2, \ldots, G_r)$ of graphs such that (i) $G_1 = K_1$, (ii) $G_r = G$, and (iii) for $1 \le i \le r-1$, G_{i+1} is obtained from G_i either by adding an edge (a loop or a link) or by pinching together k of its edges.
(Mader (1978) found an analogous construction for $(2k+1)$-edge-connected graphs.)

9.6 Gomory–Hu Trees

As mentioned earlier, Gomory and Hu (1961) showed that only $n-1$ applications of the Max-Flow Min-Cut Algorithm (7.9) are needed in order to determine the edge connectivity of a graph G. The following theorem, in which two edge cuts $\partial(X)$ and $\partial(Y)$ that cross are replaced by two, $\partial(X)$ and $\partial(X \cap Y)$, that do not cross, is the basis of their approach. This procedure is referred to as *uncrossing*. We leave the proof of the theorem, which makes use of submodularity, as an exercise (9.6.1).

Theorem 9.17 *Let $\partial(X)$ be a minimum edge cut in a graph G separating two vertices x and y, where $x \in X$, and let $\partial(Y)$ be a minimum edge cut in G separating two vertices u and v of X, where $y \notin Y$. Then $\partial(X \cap Y)$ is a minimum edge cut in G separating u and v.* □

A consequence of Theorem 9.17 is that, given a minimum edge cut $\partial(X)$ in G separating vertices x and y, in order to find a minimum edge cut in G separating u and v, where $\{u, v\} \subset X$, it suffices to consider the graph $G / \overline{X}$ obtained from G by shrinking $\overline{X} := V \setminus X$ to a single vertex. Using this idea, Gomory and Hu showed how to find all the $\binom{n}{2}$ values of the function c' by just $n-1$ applications of the Max-Flow Min-Cut Algorithm (7.9). They also showed that the $n-1$ cuts found by their procedure have certain special properties which may be conveniently visualized in terms of an appropriately weighted tree associated with G. We first describe the characteristics of this weighted tree and then explain how to construct it.

Given any tree T with vertex set V, and an edge e of T, there is a unique edge cut $B_e := \partial(X)$ of G associated with e, where X is the vertex set of one component of $T \setminus e$. (This is akin to the notion of a fundamental bond, introduced in Chapter 4, except that here we do not insist on T being a spanning tree of G.) A weighted tree (T, w) on V is a *Gomory–Hu tree* of G if, for each edge $e = xy$ of T,

i) $w(e) = c'(x, y)$,
ii) the cut B_e associated with e is a minimum edge cut in G separating x and y.

As an example, consider the graph G on five vertices shown in Figure 9.12a, where the weights indicate edge multiplicities. Figure 9.12b is a Gomory–Hu tree T of G. The four edge cuts of G corresponding to the four edges of T are indicated

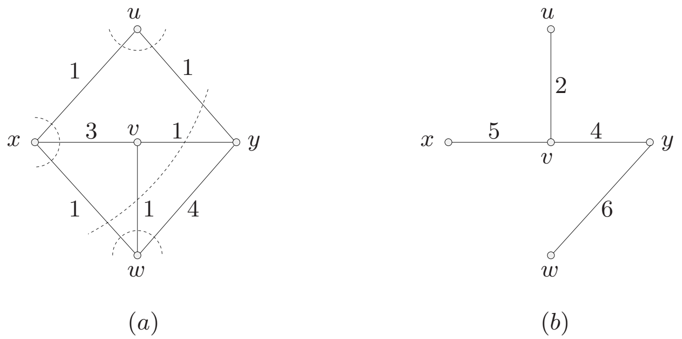

Fig. 9.12. (a) A graph G, and (b) a Gomory–Hu tree T of G

by dashed lines in Figure 9.12a. Note that this particular tree T is not a spanning tree of G.

The $n-1$ edge cuts associated with a Gomory–Hu tree are pairwise noncrossing. As a consequence of the following proposition, these $n-1$ cuts are sufficient for determining $\kappa'(G)$.

Proposition 9.18 *Let (T, w) be a Gomory–Hu tree of a graph G. For any two vertices x and y of G, $c'(x, y)$ is the minimum of the weights of the edges on the unique xy-path in T.*

Proof Clearly, for every edge e on the xy-path in T, the edge cut B_e associated with e separates x and y. If $v_1, v_2, \ldots, v_k$ is the xy-path in T, where $x = v_1$ and $y = v_k$, it follows that

$$c'(x, y) \leq \min\{c'(v_1, v_2), c'(v_2, v_3), \ldots, c'(v_{k-1}v_k)\}$$

On the other hand, by Exercise 9.3.13b,

$$c'(x, y) \geq \min\{c'(v_1, v_2), c'(v_2, v_3), \ldots, c'(v_{k-1}v_k)\}$$

The required equality now follows. □

Determining Edge Connectivity

We conclude this section with a brief description of the Gomory–Hu Algorithm. For this purpose, we consider trees whose vertices are the parts in a partition of V; every edge of such a tree determines a unique edge cut of G. A weighted tree (T, w) whose vertex set is a partition $\mathcal{P}$ of V is a *Gomory–Hu tree* of G *relative to* $\mathcal{P}$ if, for any edge $e := XY$ of T (where $X, Y \in \mathcal{P}$), there is an element x of X and an element y of Y such that $c'(x, y) = w(e)$ and the edge cut B_e associated with e is a minimum edge cut in G separating x and y. For example, if $\partial(X)$ is a minimum edge cut in G separating x and y, the tree consisting of two vertices X

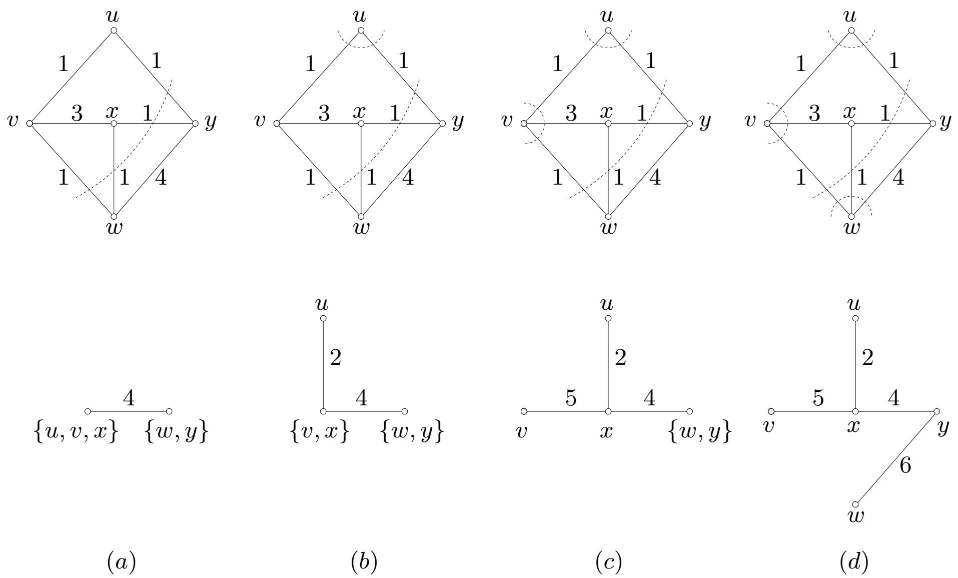

Fig. 9.13. Growing a Gomory–Hu tree

and $\overline{X} := V \setminus X$ joined by an edge with weight $c'(x, y) = d(X)$ is the Gomory–Hu tree relative to the partition $\{X, \overline{X}\}$ (see Figure 9.13a).

Suppose that we are given a Gomory–Hu tree (T, w) relative to a certain partition $\mathcal{P}$. If each part is a singleton, then (T, w) is already a Gomory–Hu tree of G. Thus, suppose that there is a vertex X of T (that is, a part X in $\mathcal{P}$) which contains two distinct elements u and v. It may be deduced from Theorem 9.17 that, in order to find a minimum edge cut in G separating u and v, it suffices to consider the graph G' obtained from G by shrinking, for each component of $T - X$, the union of the vertices (parts) in that component to a single vertex. Let $\partial(S)$ be a minimum edge cut separating u and v in G', and suppose that $u \in S$ and $v \in \overline{S}$, where $\overline{S} := V(G') \setminus S$. Now let $X_1 := X \cap S$ and $X_2 := X \cap \overline{S}$ and let $\mathcal{P}'$ be the partition obtained from $\mathcal{P}$ by replacing X by X_1 and X_2 and leaving all other parts as they are. A weighted tree T' with vertex set $\mathcal{P}'$ may now be obtained from T by:

i) splitting the vertex X into X_1 and X_2, and joining them by an edge of weight $c'(u, v) = d(S)$,
ii) joining a neighbour Y of X in T either to X_1 or to X_2 in T' (depending on whether the vertex of G' corresponding to the component of $T - X$ containing Y is in S or in $\overline{S}$).

It may be shown that T' is a Gomory–Hu tree relative to $\mathcal{P}'$ (Exercise 9.6.2). Proceeding in this manner, one may refine $\mathcal{P}$ into a partition in which each part is a singleton and thereby find a Gomory–Hu tree of G. This process is illustrated in Figure 9.13.

For a detailed description of the Gomory–Hu Algorithm, see Ford and Fulkerson (1962). Padberg and Rao (1982) showed how this algorithm may be adapted to find minimum odd cuts in graphs (see Exercise 9.6.3). Nagamochi and Ibaraki (1992) discovered a simple procedure for determining $\kappa'(G)$ that does not rely on the Max-Flow Min-Cut Algorithm (7.9) (see Exercise 9.6.4).

Exercises

$\star$**9.6.1** Prove Theorem 9.17 by proceeding as follows.

a) Show that $\partial(X \cup Y)$ is an edge cut separating x and y, and that $\partial(X \cap Y)$ is an edge cut separating u and v.
b) Deduce that $d(X \cup Y) \geq d(X)$ and $d(X \cap Y) \geq d(Y)$.
c) Apply the submodularity inequality.

9.6.2 Show that the weighted tree T' obtained from T in the Gomory–Hu Algorithm is a Gomory–Hu tree of G relative to $\mathcal{P}'$.

9.6.3 Let G be a graph with at least two vertices of odd degree.

a) Suppose that $\partial(X)$ is an edge cut of smallest size among those separating pairs of vertices of odd degree in G. Show that:
   i) if $d(X)$ is odd, it is a smallest odd edge cut of G,
   ii) if $d(X)$ is even, a smallest odd edge cut of G is an edge cut of either G / X or $G / \overline{X}$, where $\overline{X} := V \setminus X$.
b) Using (a), show how to find a smallest odd edge cut of a graph by applying the Gomory–Hu Algorithm. (M.W. Padberg and M.R. Rao)

9.6.4 Call an ordering $(v_1, v_2, \ldots, v_n)$ of the vertices of a connected graph G a *cut-greedy order* if, for $2 \leq i \leq n$,

$$d(v_i, \{v_1, v_2, \ldots, v_{i-1}\}) \geq d(v_j, \{v_1, v_2, \ldots, v_{i-1}\}), \text{ for all } j \geq i$$

a) Show that one can find, starting with any vertex of G, a cut-greedy order of the vertices of G in time $O(m)$.
b) If $(v_1, v_2, \ldots, v_n)$ is a cut-greedy order of the vertices of G, show that

$$c'(v_{n-1}, v_n) = d(v_n)$$

c) Describe a polynomial-time algorithm for finding $\kappa'(G)$ based on part (b). (H. Nagamochi and T. Ibaraki)
d) Find the edge connectivity of the graph in Figure 9.12 by applying the above algorithm.

9.6.5 Well-balanced Orientation
An orientation D of a graph G is *well-balanced* if its local arc connectivities $p'_D(u, v)$ satisfy $p'_D(u, v) \geq \lfloor p'_G(u, v)/2 \rfloor$ for all ordered pairs (u, v) of vertices. Show that every well-balanced orientation of an eulerian graph is eulerian. (Z. Szigeti)

9.7 Chordal Graphs

A *chordal graph* is a simple graph in which every cycle of length greater than three has a chord. Equivalently, the graph contains no induced cycle of length four or more. Thus every induced subgraph of a chordal graph is chordal. An example of a chordal graph is shown in Figure 9.14.

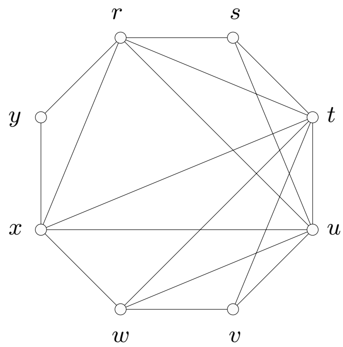

Fig. 9.14. A chordal graph

Complete graphs and trees are simple instances of chordal graphs. Moreover, as we now show, all chordal graphs have a treelike structure composed of complete graphs (just as trees are composed of copies of K_2). In consequence, many $\mathcal{NP}$-hard problems become polynomial when restricted to chordal graphs.

Clique Cuts

A *clique cut* is a vertex cut which is also a clique. In a chordal graph, every minimal vertex cut is a clique cut.

Theorem 9.19 *Let G be a connected chordal graph which is not complete, and let S be a minimal vertex cut of G. Then S is a clique cut of G.*

Proof Suppose that S contains two nonadjacent vertices x and y. Let G_1 and G_2 be two components of $G - S$. Because S is a minimal cut, both x and y are joined to vertices in both G_1 and G_2. Let P_i be a shortest xy-path all of whose internal vertices lie in G_i, $i = 1, 2$. Then $P_1 \cup P_2$ is an induced cycle of length at least four, a contradiction. □

From Theorem 9.19, one may deduce that every connected chordal graph can be built by pasting together complete graphs in a treelike fashion.

Theorem 9.20 *Let G be a connected chordal graph, and let V_1 be a maximal clique of G. Then the maximal cliques of G can be arranged in a sequence $(V_1, V_2, \ldots, V_k)$ such that $V_j \cap (\cup_{i=1}^{j-1} V_i)$ is a clique of G, $2 \leq j \leq k$.*

Proof There is nothing to prove if G is complete, so we may assume that G has a minimal vertex cut S. By Theorem 9.19, S is a clique of G. Let H_i, $1 \leq i \leq p$, be the S-components of G, and let Y_i be a maximal clique of H_i containing S, $1 \leq i \leq p$. Observe that the maximal cliques of $H_1, H_2 \ldots, H_p$ are also maximal cliques of G, and that every maximal clique of G is a maximal clique of some H_i (Exercise 9.7.1). Without loss of generality, suppose that V_1 is a maximal clique of H_1. By induction, the maximal cliques of H_1 can be arranged in a sequence starting with V_1 and having the stated property. Likewise, for $2 \leq i \leq p$, the maximal cliques of H_i can be arranged in a suitable sequence starting with Y_i. The concatenation of these sequences is a sequence of the maximal cliques of G satisfying the stated property. $\square$

A sequence $(V_1, V_2, \ldots, V_k)$ of maximal cliques as described in Theorem 9.20 is called a *simplicial decomposition* of the chordal graph G. The graph in Figure 9.14 has the simplicial decomposition shown in Figure 9.15. Dirac (1961) proved that a graph is chordal if and only if it has such a decomposition (see Exercise 9.7.2).

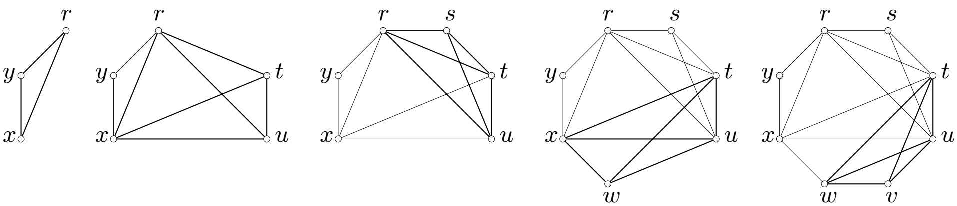

Fig. 9.15. A simplicial decomposition of the chordal graph of Figure 9.14

Simplicial Vertices

A *simplicial vertex* of a graph is a vertex whose neighbours induce a clique. Dirac (1961) showed that every noncomplete chordal graph has at least two such vertices (just as every nontrivial tree has at least two vertices of degree one). The graph in Figure 9.14, for example, has three simplicial vertices, namely s, v, and y.

Theorem 9.21 *Every chordal graph which is not complete has two nonadjacent simplicial vertices.*

Proof Let $(V_1, V_2, \ldots, V_k)$ be a simplicial decomposition of a chordal graph, and let $x \in V_k \setminus (\cup_{i=1}^{k-1} V_i)$. Then x is a simplicial vertex. Now consider a simplicial decomposition $(V_{\pi(1)}, V_{\pi(2)}, \ldots, V_{\pi(k)})$, where π is a permutation of $\{1, 2, \ldots, k\}$ such that $\pi(1) = k$. Let $y \in V_{\pi(k)} \setminus (\cup_{i=1}^{k-1} V_{\pi(i)})$. Then y is a simplicial vertex nonadjacent to x. $\square$

A *simplicial order* of a graph G is an enumeration $v_1, v_2, \ldots, v_n$ of its vertices such that v_i is a simplicial vertex of $G[\{v_i, v_{i+1}, \ldots, v_n\}]$, $1 \leq i \leq n$. Because induced subgraphs of chordal graphs are chordal, it follows directly from Theorem 9.21 that every chordal graph has a simplicial order. Conversely, if a graph has a simplicial order, it is necessarily chordal (Exercise 9.7.3).

Corollary 9.22 *A graph is chordal if and only if it has a simplicial order.* □

There is a linear-time algorithm due to Rose et al. (1976), and known as *lexicographic breadth-first search*, for finding a simplicial order of a graph if one exists. A brief description is given in Section 9.8.

Tree Representations

Besides the characterizations of chordal graphs given above in terms of simplicial decompositions and simplicial orders, chordal graphs may also be viewed as intersection graphs of subtrees of a tree.

Theorem 9.23 *A graph is chordal if and only if it is the intersection graph of a family of subtrees of a tree.*

Proof Let G be a chordal graph. By Theorem 9.20, G has a simplicial decomposition $(V_1, V_2, \ldots, V_k)$. We prove by induction on k that G is the intersection graph of a family of subtrees $\mathcal{T} = \{T_v : v \in V\}$ of a tree T with vertex set $\{x_1, x_2, \ldots, x_k\}$ such that $x_i \in T_v$ for all $v \in V_i$. If $k = 1$, then G is complete and we set $T_v := T$ for all $v \in V$. If $k \geq 2$, let $G' = (V', E')$ be the chordal graph with simplicial decomposition $(V_1, V_2, \ldots, V_{k-1})$. By induction, G' is the intersection graph of a family of subtrees $\mathcal{T}' = \{T'_v : v \in V'\}$ of a tree T' with vertex set $\{x_1, x_2, \ldots, x_{k-1}\}$. Let V_j be a maximal clique of G' such that $V_j \cap V_k \neq \emptyset$. We form the tree T by adding a new vertex x_k adjacent to x_j. For $v \in V_j$, we form the tree T_v by adding x_k to T'_v and joining it to x_j. For $v \in V' \setminus V_j$, we set $T_v := T'_v$. Finally, for $v \in V_k \setminus V'$, we set $T_v := x_k$. It can be checked that G is the intersection graph of $\{T_v : v \in V\}$.

We leave the proof of the converse statement as an exercise (9.7.4). □

We refer to the pair $(T, \mathcal{T})$ described in the proof of Theorem 9.23 as a *tree representation* of the chordal graph G.

Exercises

$\star$**9.7.1** Let G be a connected chordal graph which is not complete, and let S be a clique cut of G. Show that the maximal cliques of the S-components of G are also maximal cliques of G, and that every maximal clique of G is a maximal clique of some S-component of G.

$\star$**9.7.2** Show that a graph is chordal if it has a simplicial decomposition.

$\star$**9.7.3** Show that a graph is chordal if it has a simplicial order.

⋆9.7.4

a) Show that the intersection graph of a family of subtrees of a tree is a chordal graph.
b) Represent the chordal graph of Figure 9.14 as the intersection graph of a family of subtrees of a tree.

9.7.5

a) Let G be a chordal graph and v a simplicial vertex of G. Set $X := N(v) \cup \{v\}$ and $G' := G - X$, and let S' be a maximum stable set and $\mathcal{K}'$ a minimum clique covering of G'. Show that:
   i) $S := S' \cup \{v\}$ is a maximum stable set of G,
   ii) $\mathcal{K} := \mathcal{K}' \cup \{X\}$ is a minimum clique covering of G,
   iii) $|S| = |\mathcal{K}|$.
b) Describe a linear-time algorithm which accepts as input a simplicial order of a chordal graph G and returns a maximum stable set and a minimum clique covering of G.

9.8 Related Reading

LEXICOGRAPHIC BREADTH-FIRST SEARCH

By Exercise 9.7.3b, a graph is chordal if and only if it has a simplicial order. Breadth-first search, with a special rule for determining the head of the queue, may be used to find a simplicial order of an input graph, if one exists. The rule, which gives the procedure its name, involves assigning sequences of integers to vertices and comparing them lexicographically to break ties. (Sequences of integers from the set $\{1.2, \ldots, n\}$ may be thought of as words of a language whose alphabet consists of n letters $1, 2, \ldots, n$, the first letter being 1, the second letter 2, and so on. A sequence S is *lexicographically smaller* than another sequence S' if S appears before S' in a dictionary of that language.) If G happens to be chordal, the sequence of vertices generated by this tree-search will be the converse of a simplicial order.

We choose an arbitrary vertex of the input graph G as root, and denote the vertex incorporated into the tree at time t by v_t, the root being v_1. Each vertex v of the graph is assigned a sequence $S(v)$ of integers, initially the empty sequence. When vertex v_t enters the tree, for each $v \in N(v_t) \setminus \{v_1, v_2, \ldots, v_{t-1}\}$, we modify $S(v)$ by appending to it the integer $n - t + 1$. The next vertex selected to enter the tree is any vertex in the queue whose label is lexicographically largest.

Rose et al. (1976), who introduced lexicographic breadth-first search (Lex BFS), showed that it will find a simplicial order of the input graph if there is one. A very readable account of chordal graphs, including a proof of the validity of Lex BFS, can be found in Golumbic (2004). In recent years, Lex BFS has been used extensively in algorithms for recognizing various other classes of graphs (see, for example, Corneil (2004)).

Tree-Decompositions

Due to their rather simple structure, chordal graphs can be recognized in polynomial time, as outlined above. Moreover, many $\mathcal{NP}$-hard problems, such as Max Stable Set, can be solved in polynomial time when restricted to chordal graphs (see Exercise 9.7.5). A more general class of graphs for which polynomial-time algorithms exist for such $\mathcal{NP}$-hard problems was introduced by Robertson and Seymour (1986).

Recall that by Theorem 9.23 every chordal graph G has a tree representation, that is, an ordered pair $(T, \mathcal{T})$, where T is a tree and $\mathcal{T} := \{T_v : v \in V\}$ is a family of subtrees of T such that $T_u \cap T_v \neq \emptyset$ if and only if $uv \in E$. For an arbitrary simple graph G, a *tree-decomposition* of G is an ordered pair $(T, \mathcal{T})$, where T is a tree and $\mathcal{T} := \{T_v : v \in V\}$ is a family of subtrees of T such that $T_u \cap T_v \neq \emptyset$ if (but not necessarily only if) $uv \in E$. Equivalently, $(T, \mathcal{T})$ is a tree-decomposition of a simple graph G if and only if G is a spanning subgraph of the chordal graph with tree representation $(T, \mathcal{T})$.

Every simple graph G has the *trivial* tree-decomposition $(T, \mathcal{T})$, where T is an arbitrary tree and $T_v = T$ for all $v \in V$ (the corresponding chordal graph being K_n). For algorithmic purposes, one is interested in finer tree-decompositions, as measured by a parameter called the width of the decomposition. A nontrivial tree-decomposition of $K_{2,3}$ is shown in Figure 9.16.

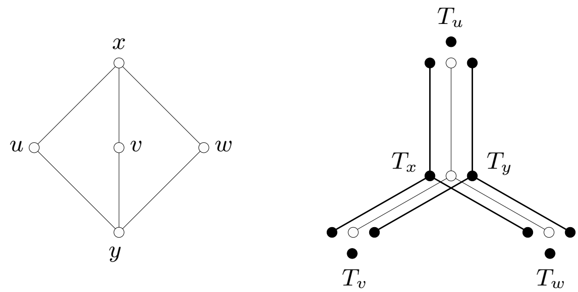

Fig. 9.16. A tree-decomposition of $K_{2,3}$, of width three

Let $(T, \{T_v : v \in V\})$ be a tree-decomposition of a graph G, where $V(T) = X$ and $V(T_v) = X_v$, $v \in V$. The dual of the hypergraph $(X, \{X_v : v \in V\})$ is the hypergraph $(V, \{V_x : x \in X\})$, where $V_x := \{v \in V : x \in X_v\}$. For instance, if G is a chordal graph, the sets V_x, $x \in X$, are the cliques in its simplicial decomposition. The greatest cardinality of an edge of this dual hypergraph, $\max\{|V_x| : x \in X\}$, is called the *width* of the decomposition.[1] The tree-decomposition of $K_{2,3}$ shown in Figure 9.16 has width three, the sets V_x, $x \in X$, being $\{u, x, y\}$, $\{v, x, y\}$, $\{w, x, y\}$, and $\{x, y\}$.

As another example, consider the tree-decomposition of the (3×3)-grid $P_3 \,\square\, P_3$ with vertex set $\{(i, j) : 1 \leq i, j \leq 3\}$ shown in Figure 9.17. This tree-decomposition has width four, all six sets V_x (the horizontal sets) being of cardinality four.

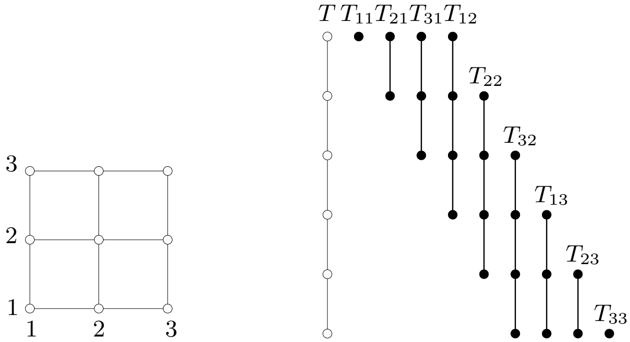

Fig. 9.17. A tree-decomposition of the (3×3)-grid, of width four

In general, a graph may have many different tree-decompositions. The *tree-width* of the graph is the minimum width among all tree-decompositions. Thus the tree-width of a chordal graph is its clique number; in particular, every nontrivial tree has tree-width two. Cycles also have tree-width two. More generally, one can show that every series-parallel graph (defined in Exercise 10.5.11) has tree-width at most three. The $(n \times n)$-grid has tree-width $n + 1$; that this is an upper bound follows from a generalization of the tree-decomposition given in Figure 9.17, but establishing the lower bound is more difficult (see Section 10.7). For graphs in general, Arnborg et al. (1987) showed that computing the tree-width is an $\mathcal{NP}$-hard problem. On the other hand, there exists polynomial-time algorithm for deciding whether a graph has tree-width at most k, where k is a fixed integer (Robertson and Seymour (1986)).

If a graph has a small tree-width, then it has a treelike structure, resembling a 'thickened' tree, and this structure has enabled the development of polynomial-time algorithms for many $\mathcal{NP}$-hard problems (see, for example, Arnborg and Proskurowski (1989)). More significantly, tree-decompositions have proved to be a fundamental tool in the work of Robertson and Seymour on linkages and graph minors (see Section 10.7).

[1] WARNING: the value of the width as defined here is *one greater than the standard definition.* This difference has no bearing on qualitative statements about tree-width, many of which are of great significance. On the other hand, as regards quantitative statements, this is certainly the right definition from an aesthetic viewpoint.

A number of other width parameters have been studied, including the path-width (where the tree T is constrained to be a path), the branch-width, and the cut-width. We refer the reader to one of the many surveys on this topic; for example, Bienstock and Langston (1995), Reed (2003), or Bodlaender (2006).