7

Flows in Networks

Contents

7.1 Transportation Networks

Transportation networks that are used to ship commodities from their production centres to their markets can be most effectively analysed when viewed as digraphs that possess additional structure. The resulting theory has a wide range of interesting applications and ramifications. We present here the basic elements of this important topic.

A *network* $N := N(x, y)$ is a digraph D (the *underlying digraph* of N) with two distinguished vertices, a *source* x and a *sink* y, together with a nonnegative real-valued function c defined on its arc set A. The vertex x corresponds to a production centre, and the vertex y to a market. The remaining vertices are called *intermediate vertices*, and the set of these vertices is denoted by I. The function c is the *capacity function* of N and its value on an arc a the *capacity* of a. The capacity of an arc may be thought of as representing the maximum rate at which a commodity can be transported along it. It is convenient to allow arcs of infinite capacity, along which commodities can be transported at any desired rate. Of course, in practice, one is likely to encounter transportation networks with several production centres and markets, rather than just one. However, this more general situation can be reduced to the case of networks that have just one source and one sink by means of a simple device (see Exercise 7.1.3).

We find the following notation useful. If f is a real-valued function defined on a set A, and if $S \subseteq A$, we denote the sum $\sum_{a \in S} f(a)$ by $f(S)$. Furthermore, when A is the arc set of a digraph D, and $X \subseteq V$, we set

$$f^+(X) := f(\partial^+(X)) \quad \text{and} \quad f^-(X) := f(\partial^-(X))$$

Flows

An (x,y)-*flow* (or simply a *flow*) in N is a real-valued function f defined on A satisfying the condition:

$$f^+(v) = f^-(v) \quad \text{for all} \quad v \in I \tag{7.1}$$

The value $f(a)$ of f on an arc a can be likened to the rate at which material is transported along a by the flow f. Condition (7.1) requires that, for any intermediate vertex v, the rate at which material is transported into v is equal to the rate at which it is transported out of v. For this reason, it is known as the *conservation condition.*

A flow f is *feasible* if it satisfies, in addition, the *capacity constraint*:

$$0 \leq f(a) \leq c(a) \quad \text{for all} \quad a \in A \tag{7.2}$$

The upper bound in condition (7.2) imposes the natural restriction that the rate of flow along an arc cannot exceed the capacity of the arc. Throughout this chapter, the term flow always refers to one that is feasible.

Every network has at least one flow, because the function f defined by $f(a) := 0$, for all $a \in A$, clearly satisfies both (7.1) and (7.2); it is called the *zero flow.* A less trivial example of a flow is given in Figure 7.1. The flow along each arc is indicated in bold face, along with the capacity of the arc.

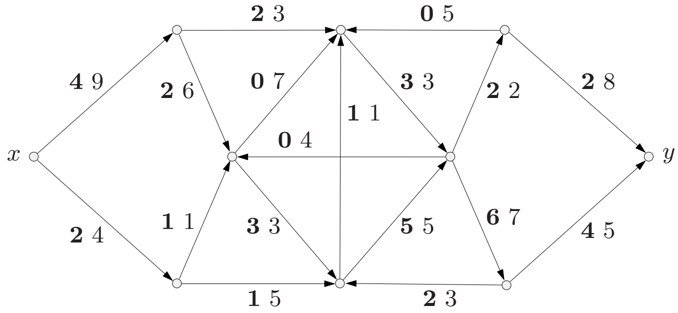


Fig. 7.1. A flow in a network

If X is a set of vertices in a network N and f is a flow in N, then $f^+(X) - f^-(X)$ is called the *net flow* out of X, and $f^-(X) - f^+(X)$ the *net flow* into X, relative to f. The conservation condition (7.1) requires that the net flow $f^+(v) - f^-(v)$ out of any intermediate vertex be zero, thus it is intuitively clear and not difficult to show that, relative to any (x, y)-flow f, the net flow $f^+(x) - f^-(x)$ out of x is equal to the net flow $f^-(y) - f^+(y)$ into y (Exercise 7.1.1b). This common quantity is called the *value* of f, denoted $\mathrm{val}\,(f)$. For example, the value of the flow indicated in Figure 7.1 is $2 + 4 = 6$. The value of a flow f may, in fact, be expressed as the net flow out of any subset X of V such that $x \in X$ and $y \in V \setminus X$, as we now show.

Proposition 7.1 *For any flow f in a network $N(x, y)$ and any subset X of V such that $x \in X$ and $y \in V \setminus X$,*

$$\mathrm{val}\,(f) = f^+(X) - f^-(X) \tag{7.3}$$

Proof From the definition of a flow and its value, we have

$$f^+(v) - f^-(v) = \begin{cases} \mathrm{val}\,(f) & \text{if } v = x \\ 0 & \text{if } v \in X \setminus \{x\} \end{cases}$$

Summing these equations over X and simplifying (Exercise 7.1.2), we obtain

$$\mathrm{val}\,(f) = \sum_{v \in X} (f^+(v) - f^-(v)) = f^+(X) - f^-(X) \qquad \square$$

A flow in a network N is a *maximum flow* if there is no flow in N of greater value. Maximum flows are of obvious importance in the context of transportation networks. A network $N(x, y)$ which has a directed (x, y)-path all of whose arcs are of infinite capacity evidently admits flows of arbitrarily large value. However, such networks do not arise in practice, and we assume that all the networks discussed here have maximum flows. We study the problem of finding such flows efficiently.

Problem 7.2 MAXIMUM FLOW
GIVEN: *a network $N(x, y)$,*
FIND: *a maximum flow from x to y in N.*

CUTS

It is convenient to denote a digraph D with two distinguished vertices x and y by $D(x, y)$. An (x, y)-*cut* in a digraph $D(x, y)$ is an outcut $\partial^+(X)$ such that $x \in X$ and $y \in V \setminus X$, and a *cut* in a network $N(x, y)$ is an (x, y)-cut in its underlying digraph. We also say that such a cut *separates y from x*. In the network of Figure 7.2, the heavy lines indicate a cut $\partial^+(X)$, where X is the set of solid vertices. The *capacity* of a cut $K := \partial^+(X)$ is the sum of the capacities of its arcs, $c^+(X)$. We denote the capacity of K by $\mathrm{cap}\,(K)$. The cut indicated in Figure 7.2 has capacity $3 + 7 + 1 + 5 = 16$.

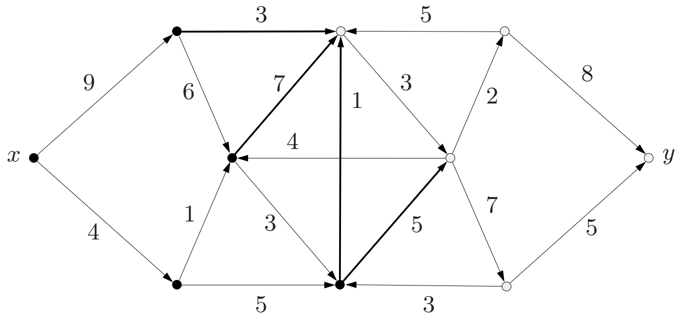


Fig. 7.2. A cut in a network

Flows and cuts are related in a simple fashion: the value of any (x, y)-flow is bounded above by the capacity of any cut separating y from x. In proving this inequality, it is convenient to call an arc a *f-zero* if $f(a) = 0$, *f-positive* if $f(a) > 0$, *f-unsaturated* if $f(a) < c(a)$, and *f-saturated* if $f(a) = c(a)$.

Theorem 7.3 *For any flow f and any cut $K := \partial^+(X)$ in a network N,*

$$\mathrm{val}\,(f) \leq \mathrm{cap}\,(K)$$

Furthermore, equality holds in this inequality if and only if each arc in $\partial^+(X)$ is f-saturated and each arc in $\partial^-(X)$ is f-zero.

Proof By (7.2),

$$f^+(X) \leq c^+(X) \quad \text{and} \quad f^-(X) \geq 0 \tag{7.4}$$

Thus, applying Proposition 7.1,

$$\mathrm{val}\,(f) = f^+(X) - f^-(X) \leq c^+(X) = \mathrm{cap}\,(K)$$

We have $\mathrm{val}\,(f) = \mathrm{cap}\,(K)$ if and only if equality holds in (7.4), that is, if and only if each arc of $\partial^+(X)$ is f-saturated and each arc of $\partial^-(X)$ is f-zero. $\square$

A cut K in a network N is a *minimum cut* if no cut in N has a smaller capacity.

Corollary 7.4 *Let f be a flow and K a cut. If $\mathrm{val}\,(f) = \mathrm{cap}\,(K)$, then f is a maximum flow and K is a minimum cut.*

Proof Let f^* be a maximum flow and K^* a minimum cut. By Theorem 7.3,

$$\mathrm{val}\,(f) \leq \mathrm{val}\,(f^*) \leq \mathrm{cap}\,(K^*) \leq \mathrm{cap}\,(K)$$

But, by hypothesis, $\mathrm{val}\,(f) = \mathrm{cap}\,(K)$. It follows that $\mathrm{val}\,(f) = \mathrm{val}\,(f^*)$ and $\mathrm{cap}\,(K^*) = \mathrm{cap}\,(K)$. Thus f is a maximum flow and K is a minimum cut. $\square$

Exercises

$\star$**7.1.1** Let $D = (V, A)$ be a digraph and f a real-valued function on A. Show that:

a) $\sum\{f^+(v) : v \in V\} = \sum\{f^-(v) : v \in V\}$,
b) if f is an (x, y)-flow, the net flow $f^+(x) - f^-(x)$ out of x is equal to the net flow $f^-(y) - f^+(y)$ into y.

$\star$**7.1.2**

a) Show that, for any flow f in a network N and any set $X \subseteq V$,

$$\sum_{v \in X} (f^+(v) - f^-(v)) = f^+(X) - f^-(X)$$

b) Give an example of a flow f in a network such that $\sum_{v \in X} f^+(v) \neq f^+(X)$ and $\sum_{v \in X} f^-(v) \neq f^-(X)$.

$\star$**7.1.3** Let $N := N(X, Y)$ be a network with source set X and sink set Y. Construct a new network $N' := N'(x, y)$ as follows.

- ▷ Adjoin two new vertices x and y.
- ▷ Join x to each source by an arc of infinite capacity.
- ▷ Join each sink to y by an arc of infinite capacity.

For any flow f in N, consider the function f' defined on the arc set of N' by:

$$f'(a) := \begin{cases} f(a) & \text{if } a \text{ is an arc of } N \\ f^+(v) & \text{if } a = (x, v) \\ f^-(v) & \text{if } a = (v, y) \end{cases}$$

a) Show that f' is a flow in N' with the same value as f.
b) Show, conversely, that the restriction of a flow in N' to the arc set of N is a flow in N of the same value.

7.1.4 Let $N(x, y)$ be a network which contains no directed (x, y)-path. Show that the value of a maximum flow and the capacity of a minimum cut in N are both zero.

7.2 The Max-Flow Min-Cut Theorem

We establish here the converse of Corollary 7.4, namely that the value of a maximum flow is always equal to the capacity of a minimum cut.

Let f be a flow in a network $N := N(x, y)$. With each x-path P in N (not necessarily a directed path), we associate a nonnegative integer $\epsilon(P)$ defined by:

$$\epsilon(P) := \min\{\epsilon(a) : a \in A(P)\}$$

where

$$\epsilon(a) := \begin{cases} c(a) - f(a) & \text{if } a \text{ is a forward arc of } P \\ f(a) & \text{if } a \text{ is a reverse arc of } P \end{cases}$$

As we now explain, $\epsilon(P)$ is the largest amount by which the flow f can be increased along P without violating the constraints (7.2). The path P is said to be f-*saturated* if $\epsilon(P) = 0$ and f-*unsaturated* if $\epsilon(P) > 0$ (that is, if each forward arc of P is f-unsaturated and each reverse arc of P is f-positive). Put simply, an f-unsaturated path is one that is not being used to its full capacity. An f-*incrementing* path is an f-unsaturated (x, y)-path. For example, in the network of Figure 7.3a, the path $P := xv_1v_2v_3y$ is such a path. The forward arcs of P are (x, v_1) and (v_3, y), and $\epsilon(P) = \min\{5, 2, 5, 4\} = 2$.

The existence of an f-incrementing path P is significant because it implies that f is not a maximum flow. By sending an additional flow of $\epsilon(P)$ along P, one obtains a new flow f' of greater value. More precisely, define $f' : A \to \mathbb{R}$ by:

$$f'(a) := \begin{cases} f(a) + \epsilon(P) & \text{if } a \text{ is a forward arc of } P \\ f(a) - \epsilon(P) & \text{if } a \text{ is a reverse arc of } P \\ f(a) & \text{otherwise} \end{cases} \tag{7.5}$$

We then have the following proposition, whose proof is left as an exercise (7.2.1).

Proposition 7.5 *Let f be a flow in a network N. If there is an f-incrementing path P, then f is not a maximum flow. More precisely, the function f' defined by (7.5) is a flow in N of value* $\mathrm{val}(f') = \mathrm{val}(f) + \epsilon(P)$. □

We refer to the flow f' defined by (7.5) as the *incremented flow* based on P. Figure 7.3b shows the incremented flow in the network of Figure 7.3a based on the f-incrementing path $xv_1v_2v_3y$.

What if there is no f-incrementing path? The following proposition addresses this eventuality.

Proposition 7.6 *Let f be a flow in a network $N := N(x, y)$. Suppose that there is no f-incrementing path in N. Let X be the set of all vertices reachable from x by f-unsaturated paths, and set $K := \partial^+(X)$. Then f is a maximum flow in N and K is a minimum cut.*

Proof Clearly $x \in X$. Also, $y \in V \setminus X$ because there is no f-incrementing path. Therefore K is a cut in N.

Consider an arc $a \in \partial^+(X)$, with tail u and head v. Because $u \in X$, there exists an f-unsaturated (x, u)-path Q. If a were f-unsaturated, Q could be extended by the arc a to yield an f-unsaturated (x, v)-path. But $v \in V \setminus X$, so there is no such path. Therefore a must be f-saturated. Similar reasoning shows that if $a \in \partial^-(X)$,

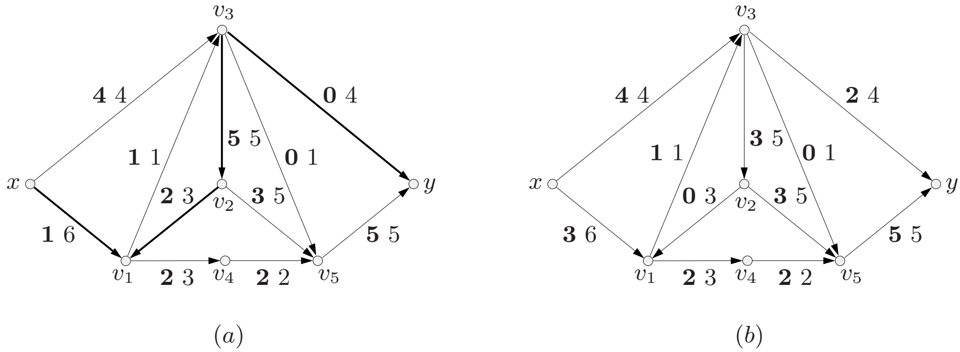


Fig. 7.3. (a) An f-incrementing path P, and (b) the incremented flow based on P

then a must be f-zero. By Theorem 7.3, we have $\mathrm{val}\,(f) = \mathrm{cap}\,(K)$. Corollary 7.4 now implies that f is a maximum flow in N and that K is a minimum cut. □

A far-reaching consequence of Propositions 7.5 and 7.6 is the following theorem, due independently to Elias et al. (1956) and Ford and Fulkerson (1956).

Theorem 7.7 The Max-Flow Min-Cut Theorem
In any network, the value of a maximum flow is equal to the capacity of a minimum cut.

Proof Let f be a maximum flow. By Proposition 7.5, there can be no f-incrementing path. The theorem now follows from Proposition 7.6. □

The Max-Flow Min-Cut Theorem (7.7) shows that one can always demonstrate the optimality of a maximum flow simply by exhibiting a cut whose capacity is equal to the value of the flow. Many results in graph theory are straightforward consequences of this theorem, as applied to suitably chosen networks. Among these are two fundamental theorems due to K. Menger, discussed at the end of this chapter (Theorems 7.16 and 7.17). Other important applications of network flows are given in Chapter 16.

The Ford–Fulkerson Algorithm

The following theorem is a direct consequence of Propositions 7.5 and 7.6.

Theorem 7.8 *A flow f in a network is a maximum flow if and only if there is no f-incrementing path.* □

This theorem is the basis of an algorithm for finding a maximum flow in a network. Starting with a known flow f, for instance the zero flow, we search for an f-incrementing path by means of a tree-search algorithm. An x-tree T is *f-unsaturated* if, for every vertex v of T, the path xTv is f-unsaturated. An example

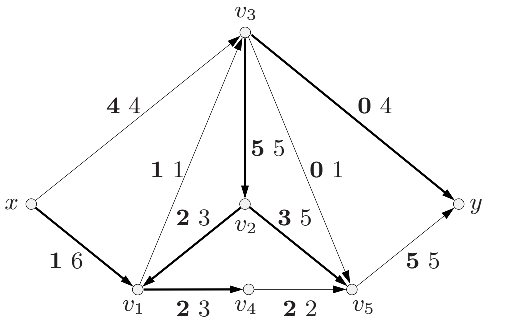


Fig. 7.4. An f-unsaturated tree

is shown in the network of Figure 7.4. It is a tree T of this type that we grow in searching for an f-incrementing path.

Initially, the tree T consists of just the source x. At any stage, there are two ways in which the tree may be grown. If there exists an f-unsaturated arc a in $\partial^+(X)$, where $X = V(T)$, both a and its head are adjoined to T. Similarly, if there exists an f-positive arc a in $\partial^-(X)$, both a and its tail are adjoined to T. If the tree T reaches the sink y, the path xTy is an f-incrementing path, and we replace f by the flow f' defined in (7.5). If T fails to reach the sink, and is a maximal f-unsaturated tree, each arc in $\partial^+(X)$ is f-saturated and each arc in $\partial^-(X)$ is f-zero. We may then conclude, by virtue of Theorem 7.3, that the flow f is a maximum flow and the cut $\partial^+(X)$ a minimum cut. We refer to this tree-search algorithm as *Incrementing Path Search* (IPS) and to a maximal f-unsaturated tree which does not include the sink as an *IPS-tree*.

Algorithm 7.9 MAX-FLOW MIN-CUT (MFMC)

INPUT*:* a network $N := N(x, y)$ and a feasible flow f in N
OUTPUT*:* a maximum flow f and a minimum cut $\partial^+(X)$ in N

1: *set* $X := \{x\}$, $p(v) := \emptyset$, $v \in V$
2: ***while*** *there is either an* f*-unsaturated arc* $a := (u, v)$ *or an* f*-positive arc* $a := (v, u)$ *with* $u \in X$ *and* $v \in V \setminus X$ ***do***
3: *replace* X *by* $X \cup \{v\}$
4: *replace* $p(v)$ *by* u
5: ***end while***
6: ***if*** $y \in X$ ***then***
7: *compute* $\epsilon(P) := \min\{\epsilon(a) : a \in A(P)\}$, *where* P *is the* xy*-path in the tree whose predecessor function is* p
8: *for each forward arc* a *of* P, *replace* $f(a)$ *by* $f(a) + \epsilon(P)$
9: *for each reverse arc* a *of* P, *replace* $f(a)$ *by* $f(a) - \epsilon(P)$
10: *return to 1*
11: ***end if***
12: *return* $(f, \partial^+(X))$

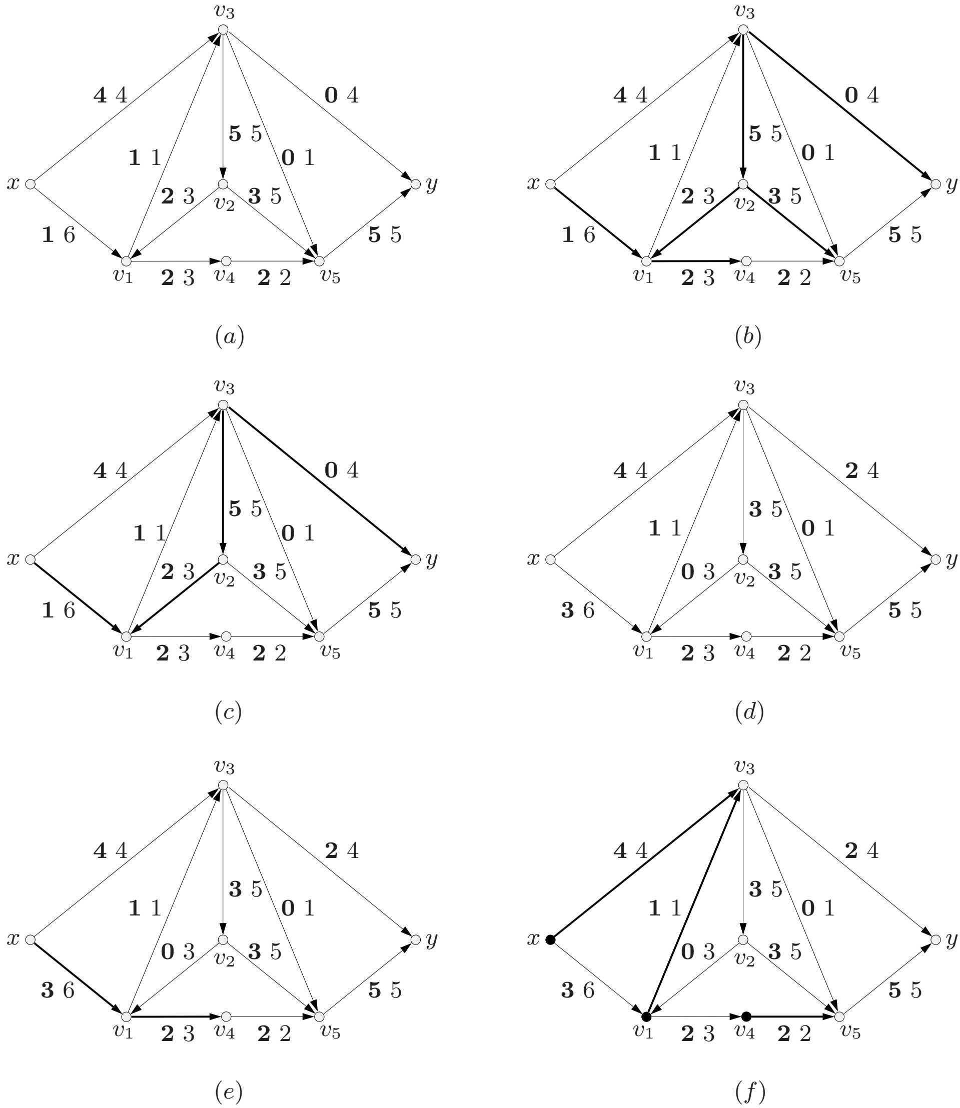


Fig. 7.5. (a) A flow f, (b) an f-unsaturated tree, (c) the f-incrementing path, (d) the f-incremented flow, (e) an IPS-tree, and (f) a minimum cut

As an example, consider the network shown in Figure 7.5a, with the indicated flow. Applying IPS, we obtain the f-unsaturated tree shown in Figure 7.5b. Because this tree includes the sink y, the xy-path contained in it, namely $xv_1v_2v_3y$, is an f-incrementing path (see Figure 7.5c). By incrementing f along this path, we obtain the incremented flow shown in Figure 7.5d.

Now, an application of IPS to the network with this new flow results in the IPS-tree shown in Figure 7.5e. We conclude that the flow shown in Figure 7.5d is a maximum flow. The minimum cut $\partial^+(X)$, where X is the set of vertices reached by the IPS-tree, is indicated in Figure 7.5f.

When all the capacities are integers, the value of the flow increases by at least one at each iteration of the Max-Flow Min-Cut Algorithm, so the algorithm will certainly terminate after a finite number of iterations. A similar conclusion applies to the case in which all capacities are rational numbers (Exercise 7.2.3). On the other hand, the algorithm will not necessarily terminate if irrational capacities are allowed. An example of such a network was constructed by Ford and Fulkerson (1962).

In applications of the theory of network flows, one is often required to find flows that satisfy additional restrictions, such as supply and demand constraints at the sources and sinks, respectively, or specified positive lower bounds on flows in individual arcs. Most such problems can be reduced to the problem of finding maximum flows in associated networks. Examples may be found in the books by Bondy and Murty (1976), Chvátal (1983), Ford and Fulkerson (1962), Lovász and Plummer (1986), and Schrijver (2003).

Exercises

$\star$**7.2.1** Give a proof of Proposition 7.5.

7.2.2 If all the capacities in a network are integer-valued, show that the maximum flow returned by the Max-Flow Min-Cut Algorithm is integer-valued.

$\star$**7.2.3** Show that the Max-Flow Min-Cut Algorithm terminates after a finite number of incrementing path iterations when all the capacities are rational numbers.

7.2.4 Let f be a function on the arc set A of a network $N := N(x, y)$ such that $0 \leq f(a) \leq c(a)$ for all $a \in A$. Show that f is a flow in N if and only if f is a nonnegative linear combination of incidence vectors of directed (x, y)-paths.

7.2.5 Degree Sequences of Bipartite Graphs
Let $\mathbf{p}{:=}(p_1, p_2, \ldots, p_m)$ and $\mathbf{q}{:=}(q_1, q_2, \ldots, q_n)$ be two sequences of nonnegative integers. The pair $(\mathbf{p}, \mathbf{q})$ is said to be *realizable* by a simple bipartite graph if there exists a simple bipartite graph G with bipartition $(\{x_1, x_2, \ldots, x_m\}, \{y_1, y_2, \ldots, y_n\})$, such that $d(x_i) = p_i$, for $1 \leq i \leq m$, and $d(y_j) = q_j$, for $1 \leq j \leq n$.

a) Formulate as a network flow problem the problem of determining whether a given pair $(\mathbf{p}, \mathbf{q})$ is realizable by a simple bipartite graph.
b) Suppose that $q_1 \geq q_2 \geq \cdots \geq q_n$. Deduce from the Max-Flow Min-Cut Theorem that $(\mathbf{p}, \mathbf{q})$ is realizable by a simple bipartite graph if and only if:

$$\sum_{i=1}^{m} p_i = \sum_{j=1}^{n} q_j \quad \text{and} \quad \sum_{i=1}^{m} \min\{p_i, k\} \geq \sum_{j=1}^{k} q_j \quad \text{for} \quad 1 \leq k \leq n$$

(D. Gale and H.J. Ryser)

7.2.6 Degree Sequences of Directed Graphs
Let D be a strict digraph and let p and q be two nonnegative integer-valued functions on V.

a) Consider the problem of determining whether D has a spanning subdigraph H such that:

$$d_H^-(v) = p(v) \quad \text{and} \quad d_H^+(v) = q(v) \quad \text{for all} \quad v \in V$$

Formulate this as a network flow problem.

b) Deduce from the Max-Flow Min-Cut Theorem that D has a subdigraph H satisfying the condition in (a) if and only if:
   i) $\sum_{v \in V} p(v) = \sum_{v \in V} q(v)$,
   ii) $\sum_{v \in S} q(v) \leq \sum_{v \in T} p(v) + a(S, V \setminus T)$ for all $S, T \subseteq V$.

c) Taking D to be the complete directed graph on n vertices and applying (b), find necessary and sufficient conditions for two sequences $\mathbf{p} := (p_1, p_2, \ldots, p_n)$ and $\mathbf{q} := (q_1, q_2, \ldots, q_n)$ to be realizable as the in- and outdegree sequences of a strict digraph on n vertices.

7.3 Arc-Disjoint Directed Paths

A communications network N with one-way communication links may be modelled by a directed graph D whose vertices correspond to the stations of N and whose arcs correspond to its links. In order to be able to relay information in N from station x to station y, the digraph D must clearly contain a directed (x, y)-path. In practice, however, the possible failure of communication links (either by accident or by sabotage) must also be taken into account. For example, if all the directed (x, y)-paths in D should happen to contain one particular arc, and if the communication link corresponding to that arc should fail or be destroyed, it would no longer be possible to relay information from x to y. This situation would not arise if D contained two arc-disjoint directed (x, y)-paths. More generally, if D had k arc-disjoint directed (x, y)-paths, x would still be able to send messages to y even if $k - 1$ links should fail. The maximum number of arc-disjoint directed (x, y)-paths is therefore a relevant parameter in this context, and we are led to the following problem.

Problem 7.10 Arc-Disjoint Directed Paths (ADDP)
Given*: a digraph* $D := D(x, y)$*,*
Find*: a maximum family of arc-disjoint directed* (x, y)*-paths in* D*.*

Let us now look at the network from the viewpoint of a saboteur who wishes to disrupt communications from x to y. The saboteur will seek to eliminate all directed (x, y)-paths in D by destroying arcs, preferably as few as possible. Now, a minimal set of arcs whose deletion destroys all directed (x, y)-paths is nothing but an (x, y)-cut. The saboteur's problem can thus be stated as follows.

Problem 7.11 MINIMUM ARC CUT
GIVEN: *a digraph* $D := D(x, y)$,
FIND: *a minimum* (x, y)*-cut in* D.

As the reader might have guessed, these problems can be solved by applying network flow theory. The concept of a circulation provides the essential link.

CIRCULATIONS

A *circulation* in a digraph D is a function $f : A \to \mathbb{R}$ which satisfies the conservation condition at every vertex:

$$f^+(v) = f^-(v), \quad \text{for all } v \in V \tag{7.6}$$

Figure 7.6a shows a circulation in a digraph.

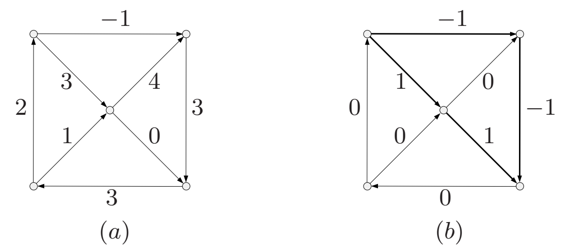


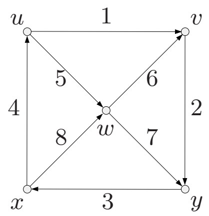
Fig. 7.6. (a) A circulation in a digraph, and (b) a circulation associated with a cycle

Circulations in a digraph D can be expressed very simply in terms of the incidence matrix of D. Recall that this is the matrix $\mathbf{M} = (m_{va})$ whose rows and columns are indexed by the vertices and arcs of D, respectively, where, for a vertex v and arc a,

$$m_{va} := \begin{cases} 1 & \text{if } a \text{ is a link and } v \text{ is the tail of } a \\ -1 & \text{if } a \text{ is a link and } v \text{ is the head of } a \\ 0 & \text{otherwise} \end{cases}$$

The incidence matrix of a digraph is shown in Figure 7.7.

We frequently identify a real-valued function f defined on a set S with the vector $\mathbf{f} := (f(a) : a \in S)$. With this convention, the conservation condition (7.6) for a function f to be a circulation in D may be expressed in matrix notation as:



D

	1	2	3	4	5	6	7	8
u	1	0	0	-1	1	0	0	0
v	-1	1	0	0	0	-1	0	0
w	0	0	0	0	-1	1	1	-1
x	0	0	-1	1	0	0	0	1
y	0	-1	1	0	0	0	-1	0

$\mathbf{M}$

Fig. 7.7. A digraph and its incidence matrix

$$\mathbf{Mf} = \mathbf{0} \tag{7.7}$$

where $\mathbf{M}$ is the $n \times m$ incidence matrix of D and $\mathbf{0}$ the $n \times 1$ zero-vector.

Circulations and flows can be readily transformed into one another. If f is a circulation in a digraph $D := (V, A)$, and if $a = (y, x)$ is an arc of D, the restriction f' of f to $A \setminus a$ is an (x, y)-flow of value $f(a)$ in the digraph $D' := D \setminus a$ (Exercise 7.3.2). Conversely, if f is an (x, y)-flow in a digraph $D := (V, A)$, and if D' is the digraph obtained from D by adding a new arc a' from y to x, the extension f' of f to $A \cup \{a'\}$ defined by $f'(a') := \mathrm{val}\,(f)$ is a circulation in D'. By virtue of these transformations, results on flows and circulations go hand in hand. Often, it is more convenient to study circulations rather than flows because the conservation condition (7.6) is then satisfied uniformly, at all vertices.

The *support* of a real-valued function is the set of elements at which its value is nonzero.

Lemma 7.12 *Let f be a nonzero circulation in a digraph. Then the support of f contains a cycle. Moreover, if f is nonnegative, then the support of f contains a directed cycle.*

Proof The first assertion follows directly from Theorem 2.1, because the support of a nonzero circulation can contain no vertex of degree less than two. Likewise, the second assertion follows from Exercise 2.1.11a. □

Certain circulations are of particular interest, namely those associated with cycles. Let C be a cycle, together with a given sense of traversal. An arc of C is a *forward arc* if its direction agrees with the sense of traversal of C, and a *reverse arc* otherwise. We denote the sets of forward and reverse arcs of C by C^+ and C^-, respectively, and associate with C the circulation f_C defined by:

$$f_C(a) := \begin{cases} 1 & \text{if } a \in C^+ \\ -1 & \text{if } a \in C^- \\ 0 & \text{if } a \notin C \end{cases}$$

It can be seen that f_C is indeed a circulation. Figure 7.6b depicts a circulation associated with a cycle (the sense of traversal being counterclockwise).

Proposition 7.13 *Every circulation in a digraph is a linear combination of the circulations associated with its cycles.*

Proof Let f be a circulation, with support S. We proceed by induction on $|S|$. There is nothing to prove if $S = \emptyset$. If S is nonempty, then S contains a cycle C by Lemma 7.12. Let a be any arc of C, and choose the sense of traversal of C so that $f_C(a) = 1$. Then $f' := f - f(a)f_C$ is a circulation whose support is a proper subset of S. By induction, f' is a linear combination of circulations associated with cycles, so $f = f' + f(a)f_C$ is too. $\square$

There is an analogous statement to Proposition 7.13 in the case where the circulation is nonnegative. The proof is essentially the same (Exercise 7.3.4).

Proposition 7.14 *Every nonnegative circulation in a digraph is a nonnegative linear combination of the circulations associated with its directed cycles. Moreover, if the circulation is integer-valued, the coefficients of the linear combination may be chosen to be nonnegative integers.* $\square$

The relationship between circulations and flows described above implies the following corollary.

Corollary 7.15 *Let $N := N(x, y)$ be a network in which each arc is of unit capacity. Then N has an (x, y)-flow of value k if and only if its underlying digraph $D(x, y)$ has k arc-disjoint directed (x, y)-paths.* $\square$

Menger's Theorem

In view of Corollary 7.15, Problems 7.10 and 7.11 can both be solved by the Max-Flow Min-Cut Algorithm. Moreover, the Max-Flow Min-Cut Theorem in this special context becomes a fundamental min–max theorem on digraphs, due to Menger (1927).

Theorem 7.16 Menger's Theorem (Arc Version)
In any digraph $D(x, y)$, the maximum number of pairwise arc-disjoint directed (x, y)-paths is equal to the minimum number of arcs in an (x, y)-cut. $\square$

There is a corresponding version of Menger's Theorem for undirected graphs. As with networks and digraphs, it is convenient to adopt the notation $G(x, y)$ to signify a graph G with two distinguished vertices x and y. By an *xy-cut* in a graph $G(x, y)$, we mean an edge cut $\partial(X)$ such that $x \in X$ and $y \in V \setminus X$. We say that such an edge cut *separates x and y*.

Theorem 7.17 Menger's Theorem (Edge Version)
In any graph $G(x, y)$, the maximum number of pairwise edge-disjoint xy-paths is equal to the minimum number of edges in an xy-cut. $\square$

Theorem 7.17 can be derived quite easily from Theorem 7.16. Likewise, the undirected version of Problem 7.10 can be solved by applying the Max-Flow Min-Cut Algorithm to an appropriate network (Exercise 7.3.5). In Chapter 8, we explain how vertex versions of Menger's Theorems (7.16 and 7.17) can be derived from Theorem 7.16. These theorems play a central role in graph theory, as is shown in Chapter 9.

Exercises

⋆7.3.1

a) Let $D = (V, A)$ be a digraph, and let f be a real-valued function on A. Show that f is a circulation in D if and only if $f^+(X) = f^-(X)$ for all $X \subseteq V$.
b) Let f be a circulation in a digraph D, with support S. Deduce that:
   i) $D[S]$ has no cut edges,
   ii) if f is nonnegative, then $D[S]$ has no directed bonds.

⋆7.3.2 Let f be a circulation in a digraph $D := (V, A)$, and let $a = (y, x)$ be an arc of D. Show that the restriction f' of f to $A' := A \setminus a$ is an (x, y)-flow in $D' := (V, A')$ of value $f(a)$.

7.3.3 Let f and f' be two flows of equal value in a network N. Show that $f - f'$ is a circulation in N.

⋆7.3.4 Prove Proposition 7.14.

⋆7.3.5

a) Deduce Theorem 7.17 from Theorem 7.16.
b) The undirected version of Problem 7.10 may be expressed as follows.

Problem 7.18 Edge-Disjoint Paths (EDP)
Given: *a graph $G := G(x, y)$,*
Find: *a maximum family of edge-disjoint xy-paths in G.*

Explain how this problem can be solved by applying the Max-Flow Min-Cut Algorithm to an appropriate network.

7.4 Related Reading

Multicommodity Flows

In this chapter, we have dealt with the problem of transporting a single commodity along the arcs of a network. In practice, transportation networks are generally shared by many users, each wishing to transport a different commodity from one location to another. This gives rise to the notion of a *multicommodity flow*. Let N be a network with k source-sink pairs (x_i, y_i), $1 \le i \le k$, and let d_i denote the demand at y_i for commodity i, $1 \le i \le k$. The *k-commodity flow problem* consists of finding functions $f_i : A \to \mathbb{R}$, $1 \le i \le k$, such that:

(i) f_i is a flow in N of value d_i from x_i to y_i, $1 \le i \le k$,
(ii) for each arc a of D, $\sum_{i=1}^{k} f_i(a) \le c(a)$.

For a subset X of V, let $d(X)$ denote the quantity $\sum\{d_i : x_i \in X, y_i \in V \setminus X\}$. If there is a solution to the k-commodity flow problem, the inequality $d(X) \le c^+(X)$, known as the *cut condition*, must hold for all subsets X of V. For $k = 1$, this cut condition is equivalent to the condition $\mathrm{val}\,(f) \le \mathrm{cap}\,(K)$ of Theorem 7.3. By the Max-Flow Min-Cut Theorem (7.7), this condition is sufficient for the existence of a flow of value d_1. However, even for $k = 2$, the cut condition is not sufficient for the 2-commodity flow problem to have a solution, as is shown by the network with unit capacities and demands depicted in Figure 7.8a.

There is another noteworthy distinction between the single commodity and the multicommodity flow problems. Suppose that all capacities and demands are integers and that there is a k-commodity flow meeting all the requirements. When $k = 1$, this implies the existence of such a flow which is integer-valued (Exercise 7.2.2). The same is not true for $k \ge 2$. Consider, for example, the network in Figure 7.8b, again with unit capacities and demands. This network has the 2-commodity flow (f_1, f_2), where $f_1(a) = f_2(a) = 1/2$ for all $a \in A$, but it has no 2-commodity flow which takes on only integer values.

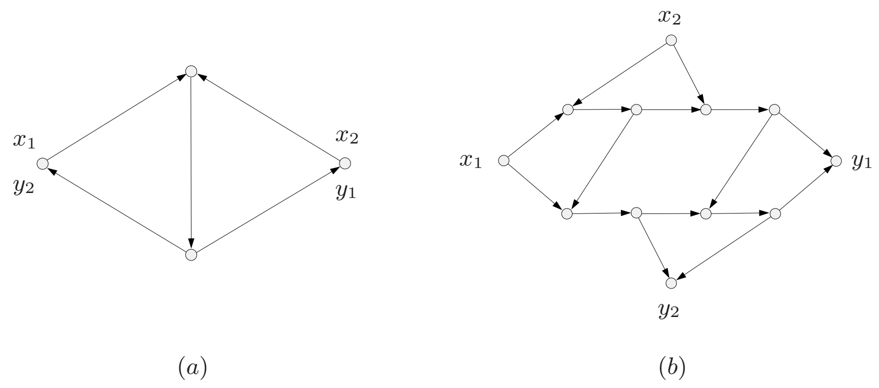


Fig. 7.8. Examples of networks: (a) satisfies the cut condition but has no 2-commodity flow, (b) has a fractional 2-commodity flow, but not one which is integral