2

Subgraphs

Contents

2.1 Subgraphs and Supergraphs

Edge and Vertex Deletion

Given a graph G, there are two natural ways of deriving smaller graphs from G. If e is an edge of G, we may obtain a graph on $m-1$ edges by deleting e from G but leaving the vertices and the remaining edges intact. The resulting graph is denoted by $G \setminus e$. Similarly, if v is a vertex of G, we may obtain a graph on $n-1$ vertices by deleting from G the vertex v together with all the edges incident with v. The resulting graph is denoted by $G - v$. These operations of *edge deletion* and *vertex deletion* are illustrated in Figure 2.1.

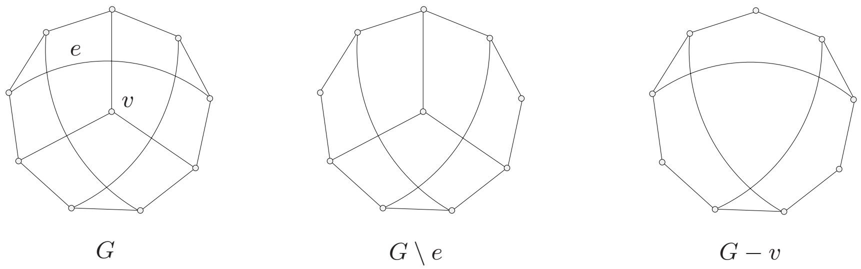

Fig. 2.1. Edge-deleted and vertex-deleted subgraphs of the Petersen graph

The graphs $G \setminus e$ and $G - v$ defined above are examples of subgraphs of G. We call $G \setminus e$ an *edge-deleted subgraph*, and $G - v$ a *vertex-deleted subgraph*. More generally, a graph F is called a *subgraph* of a graph G if $V(F) \subseteq V(G)$, $E(F) \subseteq E(G)$, and ψ_F is the restriction of ψ_G to $E(F)$. We then say that G *contains* F or that F is *contained in* G, and write $G \supseteq F$ or $F \subseteq G$, respectively. Any subgraph F of G can be obtained by repeated applications of the basic operations of edge and vertex deletion; for instance, by first deleting the edges of G not in F and then deleting the vertices of G not in F. Note that the null graph is a subgraph of every graph.

We remark in passing that in the special case where G is vertex-transitive, all vertex-deleted subgraphs of G are isomorphic. In this case, the notation $G - v$ is used to denote any vertex-deleted subgraph. Likewise, we write $G \setminus e$ to denote any edge-deleted subgraph of an edge-transitive graph G.

A *copy* of a graph F in a graph G is a subgraph of G which is isomorphic to F. Such a subgraph is also referred to as an *F-subgraph* of G; for instance, a K_3-subgraph is a triangle in the graph. An *embedding* of a graph F in a graph G is an isomorphism between F and a subgraph of G. For each copy of F in G, there are $\mathrm{aut}(F)$ embeddings of F in G.

A *supergraph* of a graph G is a graph H which contains G as a subgraph, that is, $H \supseteq G$. Note that any graph is both a subgraph and a supergraph of itself.

All other subgraphs F and supergraphs H are referred to as *proper*; we then write $F \subset G$ or $H \supset G$, respectively.

The above definitions apply also to digraphs, with the obvious modifications.

In many applications of graph theory, one is interested in determining if a given graph has a subgraph or supergraph with prescribed properties. The theorem below provides a sufficient condition for a graph to contain a cycle. In later chapters, we study conditions under which a graph contains a long path or cycle, or a complete subgraph of given order. Although supergraphs with prescribed properties are encountered less often, they do arise naturally in the context of certain applications. One such is discussed in Chapter 16 (see also Exercises 2.2.17 and 2.2.24).

Theorem 2.1 *Let G be a graph in which all vertices have degree at least two. Then G contains a cycle.*

Proof If G has a loop, it contains a cycle of length one, and if G has parallel edges, it contains a cycle of length two. So we may assume that G is simple.

Let $P := v_0 v_1 \ldots v_{k-1} v_k$ be a longest path in G. Because the degree of v_k is at least two, it has a neighbour v different from v_{k-1}. If v is not on P, the path $v_0 v_1 \ldots v_{k-1} v_k v$ contradicts the choice of P as a longest path. Therefore, $v = v_i$, for some i, $0 \leq i \leq k-2$, and $v_i v_{i+1} \ldots v_k v_i$ is a cycle in G. $\square$

Maximality and Minimality

The proof of Theorem 2.1 proceeded by first selecting a longest path in the graph, and then finding a cycle based on this path. Of course, from a purely mathematical point of view, this is a perfectly sound approach. The graph, being finite, must certainly have a longest path. However, if we wished to actually find a cycle in our graph by tracing through the steps of the proof, we would first have to find such a path, and this turns out to be a very hard task in general (in a sense to be made precise in Chapter 8). Fortunately, the very same proof remains valid if 'longest path' is replaced by 'maximal path', a maximal path being one that cannot be extended to a longer path from either end. Moreover, a maximal path is easily found: one simply starts at any vertex and grows a path until it can no longer be extended either way. For reasons such as this, the concepts of maximality and minimality (of subgraphs) turn out to be rather important.

Let $\mathcal{F}$ be a family of subgraphs of a graph G. A member F of $\mathcal{F}$ is *maximal* in $\mathcal{F}$ if no member of $\mathcal{F}$ properly contains F; likewise, F is *minimal* in $\mathcal{F}$ if no member of $\mathcal{F}$ is properly contained in F. When $\mathcal{F}$ consists of the set of all paths of G, we simply refer to a maximal member of $\mathcal{F}$ as a *maximal path* of G. We use similar terminology for describing maximal and minimal members of other special families of subgraphs. For instance, when $\mathcal{F}$ is the set of all connected subgraphs of G, the maximal members of $\mathcal{F}$ are simply its components (Exercise 2.1.1). Similarly, because an odd cycle is not bipartite, but each of its proper subgraphs is bipartite (Exercise 1.1.3), the odd cycles of a graph are its minimal nonbipartite subgraphs (see Figure 2.2b). Indeed, as we shall see, the odd cycles are the only minimal nonbipartite subgraphs.

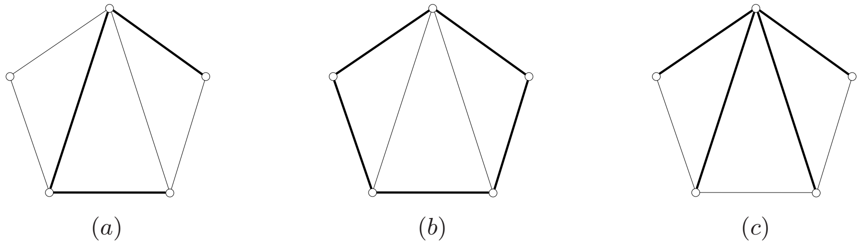

Fig. 2.2. (a) A maximal path, (b) a minimal nonbipartite subgraph, and (c) a maximal bipartite subgraph

The notions of maximality and minimality should not be confused with those of maximum and minimum cardinality. Every cycle in a graph is a maximal cycle, because no cycle is contained in another; by the same token, every cycle is a minimal cycle. On the other hand, by a maximum cycle of a graph we mean one of maximum length, that is, a longest cycle, and by a minimum cycle we mean one of minimum length. In a graph G which has at least one cycle, the length of a longest cycle is called its *circumference* and the length of a shortest cycle its *girth*.

Acyclic Graphs and Digraphs

A graph is *acyclic* if does not contain a cycle. It follows from Theorem 2.1 that an acyclic graph must have a vertex of degree less than two. In fact, every nontrivial acyclic graph has at least two vertices of degree less than two (Exercise 2.1.2).

Analogously, a digraph is *acyclic* if it has no directed cycle. One particularly interesting class of acyclic digraphs are those associated with partially ordered sets. A *partially ordered set*, or for short *poset*, is an ordered pair $P = (X, \prec)$, where X is a set and $\prec$ is a *partial order* on X, that is, an irreflexive, antisymmetric, and transitive binary relation. Two elements u and v of X are *comparable* if either $u \prec v$ or $v \prec u$, and *incomparable* otherwise. A set of pairwise comparable elements in P is a *chain*, a set of pairwise incomparable elements an *antichain*.

One can form a digraph $D := D(P)$ from a poset $P = (X, \prec)$ by taking X as the set of vertices, (u, v) being an arc of D if and only if $u \prec v$. This digraph is acyclic and transitive, where *transitive* here means that (u, w) is an arc whenever both (u, v) and (v, w) are arcs. (It should be emphasized that, despite its name, this notion of transitivity in digraphs has no connection whatsoever with the group-theoretic notions of vertex-transitivity and edge-transitivity defined earlier.) Conversely, to every strict acyclic transitive digraph D there corresponds a poset P on the vertex set of D. An acyclic tournament is frequently referred to as a *transitive tournament*. It can be seen that chains in P correspond to transitive subtournaments of D.

PROOF TECHNIQUE: THE PIGEONHOLE PRINCIPLE

If $n+1$ letters are distributed among n pigeonholes, at least two of them will end up in the same pigeonhole. This is known as the *Pigeonhole Principle*, and is a special case of a simple statement concerning multisets (sets with repetitions allowed) of real numbers.

Let $S = (a_1, a_2, \ldots, a_n)$ be a multiset of real numbers and let a denote their average. Clearly, the minimum of the a_i is no larger than a, and the maximum of the a_i is at least as large as a. Thus, if all the elements of S are integers, we may assert that there is an element that is no larger than $\lfloor a \rfloor$, and also one that is at least as large as $\lceil a \rceil$. The Pigeonhole Principle merely amounts to saying that if n integers sum to $n+1$ or more, one of them is at least $\lceil (n+1)/n \rceil = 2$.

Exercise 1.1.6a is a simple example of a statement that can be proved by applying this principle. As a second application, we establish a sufficient condition for the existence of a quadrilateral in a graph, due to Reiman (1958).

Theorem 2.2 *Any simple graph G with $\sum_{v \in V} \binom{d(v)}{2} > \binom{n}{2}$ contains a quadrilateral.*

Proof Denote by p_2 the number of paths of length two in G, and by $p_2(v)$ the number of such paths whose central vertex is v. Clearly, $p_2(v) = \binom{d(v)}{2}$. As each path of length two has a unique central vertex, $p_2 = \sum_{v \in V} p_2(v) = \sum_{v \in V} \binom{d(v)}{2}$. On the other hand, each such path also has a unique pair of ends. Therefore the set of all paths of length two can be partitioned into $\binom{n}{2}$ subsets according to their ends. The hypothesis $\sum_{v \in V} \binom{d(v)}{2} > \binom{n}{2}$ now implies, by virtue of the Pigeonhole Principle, that one of these subsets contains two or more paths; that is, there exist two paths of length two with the same pair of ends. The union of these paths is a quadrilateral. $\square$

Exercises

$\star$**2.1.1** Show that the maximal connected subgraphs of a graph are its components.

$\star$**2.1.2**

a) Show that every nontrivial acyclic graph has at least two vertices of degree less than two.
b) Deduce that every nontrivial connected acyclic graph has at least two vertices of degree one. When does equality hold?

2.1.3

a) Show that if $m \geq n$, then G contains a cycle.
b) For each positive integer n, find an acyclic graph with n vertices and $n-1$ edges.

2.1.4

a) Show that every simple graph G contains a path of length δ.
b) For each $k \geq 0$, find a simple graph G with $\delta = k$ which contains no path of length greater than k.

2.1.5

a) Show that every simple graph G with $\delta \geq 2$ contains a cycle of length at least $\delta + 1$.
b) For each $k \geq 2$, find a simple graph G with $\delta = k$ which contains no cycle of length greater than $k + 1$.

2.1.6 Show that every simple graph has a vertex x and a family of $\lfloor \frac{1}{2} d(x) \rfloor$ cycles any two of which meet only in the vertex x.

2.1.7

a) Show that the Petersen graph has girth five and circumference nine.
b) How many cycles are there of length k in this graph, for $5 \leq k \leq 9$?

2.1.8

a) Show that a k-regular graph of girth four has at least $2k$ vertices.
b) For $k \geq 2$, determine all k-regular graphs of girth four on exactly $2k$ vertices.

2.1.9

a) Show that a k-regular graph of girth five has at least $k^2 + 1$ vertices.
b) Determine all k-regular graphs of girth five on exactly $k^2 + 1$ vertices, $k = 2, 3$.

2.1.10 Show that the incidence graph of a finite projective plane has girth six.

$\star$**2.1.11** A *topological sort* of a digraph D is an linear ordering of its vertices such that, for every arc a of D, the tail of a precedes its head in the ordering.

a) Show that every acyclic digraph has at least one source and at least one sink.
b) Deduce that a digraph admits a topological sort if and only if it is acyclic.

2.1.12 Show that every strict acyclic digraph contains an arc whose reversal results in an acyclic digraph.

2.1.13 Let D be a strict digraph. Setting $k := \max\{\delta^-, \delta^+\}$, show that:

a) D contains a directed path of length at least k,
b) if $k > 0$, then D contains a directed cycle of length at least $k + 1$.

2.1.14

a) Let G be a graph all of whose vertex-deleted subgraphs are isomorphic. Show that G is vertex-transitive.
b) Let G be a graph all of whose edge-deleted subgraphs are isomorphic. Is G necessarily edge-transitive?

2.1.15 Using Theorem 2.2 and the Cauchy–Schwarz Inequality[1], show that a simple graph G contains a quadrilateral if $m > \frac{1}{4}n(\sqrt{4n-3}+1)$. (I. REIMAN)

2.1.16

a) Show that if $m \geq n+4$, then G contains two edge-disjoint cycles. (L. PÓSA)
b) For each integer $n \geq 5$, find a graph with n vertices and $n+3$ edges which does not contain two edge-disjoint cycles.

2.1.17 TRIANGLE-FREE GRAPH
A *triangle-free* graph is one which contains no triangles. Let G be a simple triangle-free graph.

a) Show that $d(x) + d(y) \leq n$ for all $xy \in E$.
b) Deduce that $\sum_{v \in V} d(v)^2 \leq mn$.
c) Applying the Cauchy–Schwarz Inequality[1], deduce that $m \leq n^2/4$.
(W. MANTEL)
d) For each positive integer n, find a simple triangle-free graph G with $m = \lfloor n^2/4 \rfloor$.

2.1.18

a) Let G be a triangle-free graph with $\delta > 2n/5$. Show that G is bipartite.
b) For $n \equiv 0 \pmod 5$, find a nonbipartite triangle-free graph with $\delta = 2n/5$.
(B. ANDRÁSFAI, P. ERDŐS, AND V.T. SÓS)

2.1.19 Let G be a simple graph with $v(G) = kp$ and $\delta(G) \geq kq$. Show that G has a subgraph F with $v(F) = p$ and $\delta(F) \geq q$. (C.ST.J.A. NASH-WILLIAMS)

2.1.20 Show that the Kneser graph $KG_{m,n}$ has no odd cycle of length less than $n/(n-2m)$.

⋆**2.1.21** Let K_n be a complete graph whose edges are coloured red or blue. Call a subgraph of this graph *monochromatic* if all of its edges have the same colour, and *bichromatic* if edges of both colours are present.

a) Let v be a vertex of K_n. Show that the number of bichromatic 2-paths in K_n whose central vertex is v is at most $(n-1)^2/4$. When does equality hold?

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[1] $\sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2 \geq \left(\sum_{i=1}^n a_i b_i\right)^2$ for real numbers a_i, b_i, $1 \leq i \leq n$.

b) Deduce that the total number of bichromatic 2-paths in K_n is at most $n(n-1)^2/4$.
c) Observing that each bichromatic triangle contains exactly two bichromatic 2-paths, deduce that the number of monochromatic triangles in K_n is at least $n(n-1)(n-5)/24$. When does equality hold? (A.W. Goodman)
d) How many monochromatic triangles must there be, at least, when $n = 5$ and when $n = 6$?

2.1.22 Let T be a tournament on n vertices, and let v be a vertex of T.

a) Show that the number of directed 2-paths in T whose central vertex is v is at most $(n-1)^2/4$. When does equality hold?
b) Deduce that the total number of directed 2-paths in T is at most $n(n-1)^2/4$.
c) Observing that each transitive triangle contains exactly one directed 2-path and that each directed triangle contains exactly three directed 2-paths, deduce that the number of directed triangles in T is at most $\frac{1}{4}\binom{n+1}{3}$. When does equality hold?

$\star$**2.1.23** Let $P = (X, \prec)$ be a poset. Show that the maximum number of elements in a chain of P is equal to the minimum number of antichains into which X can be partitioned. (L. Mirsky)

2.1.24 Geometric Graph
A *geometric graph* is a graph embedded in the plane in such a way that each edge is a line segment. Let G be a geometric graph in which any two edges intersect (possibly at an end).

a) Show that G has at most n edges.
b) For each $n \geq 3$, find an example of such a graph G with n edges.
(H. Hopf and E. Pannwitz)

2.2 Spanning and Induced Subgraphs

Spanning Subgraphs

A *spanning subgraph* of a graph G is a subgraph obtained by edge deletions only, in other words, a subgraph whose vertex set is the entire vertex set of G. If S is the set of deleted edges, this subgraph of G is denoted $G \setminus S$. Observe that every simple graph is a spanning subgraph of a complete graph.

Spanning supergraphs are defined analogously. The inverse operation to edge deletion is *edge addition*. Adding a set S of edges to a graph G yields a *spanning supergraph* of G, denoted $G + S$. By starting with a disjoint union of two graphs G and H and adding edges joining every vertex of G to every vertex of H, one obtains the *join* of G and H, denoted $G \vee H$. The join $C_n \vee K_1$ of a cycle C_n and a single vertex is referred to as a *wheel* with n *spokes* and denoted W_n. (The graph

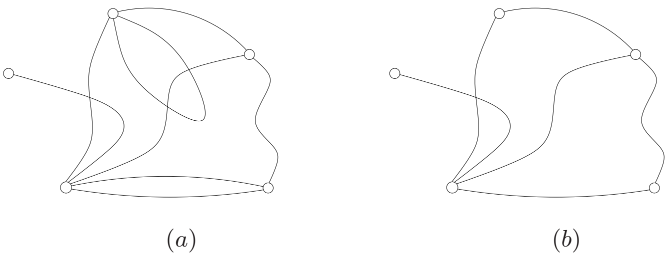


Fig. 2.3. (a) A graph and (b) its underlying simple graph

H of Figure 1.1 is the wheel W_5.) One may also add a set X of vertices to a graph, resulting in a supergraph of G denoted $G + X$.

Certain types of spanning subgraph occur frequently in applications of graph theory and, for historical reasons, have acquired special names. For example, spanning paths and cycles are called *Hamilton paths* and *Hamilton cycles*, respectively, and spanning k-regular subgraphs are referred to as *k-factors*. Rédei's Theorem (Theorem 2.3, see inset) tells us that every tournament has a directed Hamilton path. Not every tournament (on three or more vertices) has a directed Hamilton cycle, however; for instance, the transitive tournament has no directed cycles at all. Nonetheless, Camion (1959) proved that every tournament in which any vertex can be reached from any other vertex by means of a directed path does indeed have a directed Hamilton cycle (Exercise 3.4.12a).

By deleting from a graph G all loops and, for every pair of adjacent vertices, all but one link joining them, we obtain a simple spanning subgraph called the *underlying simple graph* of G. Up to isomorphism, each graph has a unique underlying simple graph. Figure 2.3 shows a graph and its underlying simple graph.

Given spanning subgraphs $F_1 = (V, E_1)$ and $F_2 = (V, E_2)$ of a graph $G = (V, E)$, we may form the spanning subgraph of G whose edge set is the symmetric difference $E_1 \triangle E_2$ of E_1 and E_2. This graph is called the *symmetric difference* of F_1 and F_2, and denoted $F_1 \triangle F_2$. Figure 2.4 shows the symmetric difference of two spanning subgraphs of a graph on five vertices.

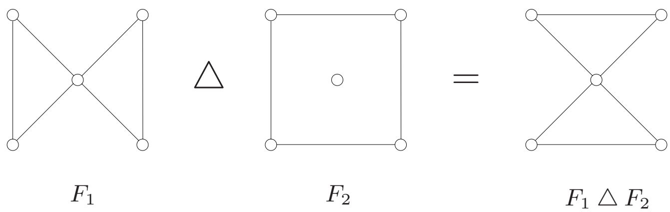


Fig. 2.4. The symmetric difference of two graphs

Proof Technique: Induction

One of the most widely used proof techniques in mathematics is the *Principle of Mathematical Induction*. Suppose that, for each nonnegative integer i, we have a mathematical statement S_i. One may prove that all assertions in the sequence $(S_0, S_1, \ldots)$ are true by:

- ▷ directly verifying S_0 (the *basis* of the induction),
- ▷ for each integer $n \geq 1$, deducing that S_n is true (the *inductive step*) from the assumption that S_{n-1} is true (the *inductive hypothesis*).

The justification for this technique is provided by the principle that each nonempty subset of $\mathbb{N}$ has a minimal element: if not all S_i were true, the set $\{i \in \mathbb{N} : S_i \text{ is false}\}$ would be a nonempty subset of $\mathbb{N}$, and would therefore have a minimal element n. Thus S_{n-1} would be true and S_n false.

We shall come across many examples of inductive proofs throughout the book. Here, as a simple illustration of the technique, we prove a basic theorem on tournaments due to Rédei (1934).

Theorem 2.3 Rédei's Theorem
Every tournament has a directed Hamilton path.

Proof Clearly, the trivial tournament (on one vertex) has a directed Hamilton path. Assume that, for some integer $n \geq 2$, every tournament on $n - 1$ vertices has a directed Hamilton path. Let T be a tournament on n vertices and let $v \in V(T)$. The digraph $T' := T - v$ is a tournament on $n-1$ vertices. By the inductive hypothesis, T' has a directed Hamilton path $P' := (v_1, v_2, \ldots, v_{n-1})$. If (v, v_1) is an arc of T, the path $(v, v_1, v_2, \ldots, v_{n-1})$ is a directed Hamilton path of T. Similarly, if (v_{n-1}, v) is an arc of T, the path $(v_1, v_2, \ldots, v_{n-1}, v)$ is a directed Hamilton path of T. Because T is a tournament, v is adjacent to each vertex of P', so we may assume that both (v_1, v) and (v, v_{n-1}) are arcs of T. It follows that there exists an integer i, $1 \leq i < n - 1$, such that both (v_i, v) and (v, v_{i+1}) are arcs of T. But now $P := (v_1, \ldots, v_i, v, v_{i+1}, \ldots, v_{n-1})$ is a directed Hamilton path of T. □

Inductive proofs may be presented in a variety of ways. The above proof, for example, may be recast as a 'longest path' proof. We take P to be a longest directed path in the tournament T. Assuming that P is not a directed Hamilton path, we then obtain a contradiction by showing that T has a directed path longer than P (Exercise 2.2.4).

Graph-theoretical statements generally assert that all graphs belonging to some well-defined class possess a certain property. Any 'proof' that fails to cover all cases is false. This is a common mistake in attempts to prove statements of this sort by induction. Another common error is neglecting to verify the basis of the induction. For an example of how not to use induction, see Exercise 2.2.19.

Proof Technique: Contradiction

A common approach to proving graph-theoretical statements is to proceed by assuming that the stated assertion is false and analyse the consequences of that assumption so as to arrive at a contradiction. As a simple illustration of this method, we present an interesting and very useful result due to Erdős (1965).

Theorem 2.4 *Every loopless graph G contains a spanning bipartite subgraph F such that $d_F(v) \geq \frac{1}{2}d_G(v)$ for all $v \in V$.*

Proof Let G be a loopless graph. Certainly, G has spanning bipartite subgraphs, one such being the empty spanning subgraph. Let $F := F[X, Y]$ be a spanning bipartite subgraph of G with the greatest possible number of edges. We claim that F satisfies the required property. Suppose not. Then there is some vertex v for which

$$d_F(v) < \frac{1}{2}d_G(v) \tag{2.1}$$

Without loss of generality, we may suppose that $v \in X$. Consider the spanning bipartite subgraph F' whose edge set consists of all edges of G with one end in $X \setminus \{v\}$ and the other in $Y \cup \{v\}$. The edge set of F' is the same as that of F except for the edges of G incident to v; those which were in F are not in F', and those which were not in F are in F'. We thus have:

$$e(F') = e(F) - d_F(v) + (d_G(v) - d_F(v)) = e(F) + (d_G(v) - 2d_F(v)) > e(F)$$

the inequality following from (2.1). But this contradicts the choice of F. It follows that F does indeed have the required property. $\square$

The method of contradiction is merely a convenient way of presenting the idea underlying the above proof. Implicit in the proof is an algorithm which finds, in any graph, a spanning bipartite subgraph with the stated property: one starts with any spanning bipartite subgraph and simply moves vertices between parts so as to achieve the desired objective (see also Exercises 2.2.2 and 2.2.18).

Induced Subgraphs

A subgraph obtained by vertex deletions only is called an *induced subgraph*. If X is the set of vertices deleted, the resulting subgraph is denoted by $G - X$. Frequently, it is the set $Y := V \setminus X$ of vertices which remain that is the focus of interest. In such cases, the subgraph is denoted by $G[Y]$ and referred to as the subgraph of G *induced by* Y. Thus $G[Y]$ is the subgraph of G whose vertex set is Y and whose edge set consists of all edges of G which have both ends in Y.

The following theorem, due to Erdős (1964/1965), tells us that every graph has a induced subgraph whose minimum degree is relatively large.

Theorem 2.5 *Every graph with average degree at least $2k$, where k is a positive integer, has an induced subgraph with minimum degree at least $k + 1$.*

Proof Let G be a graph with average degree $d(G) \geq 2k$, and let F be an induced subgraph of G with the largest possible average degree and, subject to this, the smallest number of vertices. We show that $\delta(F) \geq k + 1$. This is clearly true if $v(F) = 1$, since then $\delta(F) = d(F) \geq d(G)$, by the choice of F. We may therefore assume that $v(F) > 1$.

Suppose, by way of contradiction, that $d_F(v) \leq k$ for some vertex v of F. Consider the vertex-deleted subgraph $F' := F - v$. Note that F' is also an induced subgraph of G. Moreover

$$d(F') = \frac{2e(F')}{v(F')} \geq \frac{2(e(F) - k)}{v(F) - 1} \geq \frac{2e(F) - d(G)}{v(F) - 1} \geq \frac{2e(F) - d(F)}{v(F) - 1} = d(F)$$

Because $v(F') < v(F)$, this contradicts the choice of F. Therefore $\delta(F) \geq k + 1$. $\square$

The bound on the minimum degree given in Theorem 2.5 is sharp (Exercise 3.1.6).

Subgraphs may also be induced by sets of edges. If S is a set of edges, the *edge-induced subgraph* $G[S]$ is the subgraph of G whose edge set is S and whose vertex set consists of all ends of edges of S. Any edge-induced subgraph $G[S]$ can be obtained by first deleting the edges in $E \setminus S$ and then deleting all resulting isolated vertices; indeed, an edge-induced subgraph is simply a subgraph without isolated vertices.

Weighted Graphs and Subgraphs

When graphs are used to model practical problems, one often needs to take into account additional factors, such as costs associated with edges. In a communications network, for example, relevant factors might be the cost of transmitting data along a link, or of constructing a new link between communication centres. Such situations are modelled by weighted graphs.

With each edge e of G, let there be associated a real number $w(e)$, called its *weight*. Then G, together with these weights on its edges, is called a *weighted graph*, and denoted (G, w). One can regard a weighting $w : E \to \mathbb{R}$ as a vector whose coordinates are indexed by the edge set E of G; the set of all such vectors is denoted by $\mathbb{R}^E$ or, when the weights are rational numbers, by $\mathbb{Q}^E$.

If F is a subgraph of a weighted graph, the *weight* $w(F)$ of F is the sum of the weights on its edges, $\sum_{e \in E(F)} w(e)$. Many optimization problems amount to finding, in a weighted graph, a subgraph of a certain type with minimum or maximum weight. Perhaps the best known problem of this kind is the following one.

A travelling salesman wishes to visit a number of towns and then return to his starting point. Given the travelling times between towns, how should he plan his itinerary so that he visits each town exactly once and minimizes his total travelling time? This is known as the *Travelling Salesman Problem*. In graph-theoretic terms, it can be phrased as follows.

Problem 2.6 THE TRAVELLING SALESMAN PROBLEM (TSP)
GIVEN: *a weighted complete graph* (G, w),
FIND: *a minimum-weight Hamilton cycle of* G.

Note that it suffices to consider the TSP for complete graphs because nonadjacent vertices can be joined by edges whose weights are prohibitively high. We discuss this problem, and others of a similar flavour, in Chapters 6 and 8, as well as in later chapters.

Exercises

2.2.1 Let G be a graph on n vertices and m edges and c components.

a) How many spanning subgraphs has G?
b) How many edges need to be added to G to obtain a connected spanning supergraph?

$\star$**2.2.2**

a) Deduce from Theorem 2.4 that every loopless graph G contains a spanning bipartite subgraph F with $e(F) \geq \frac{1}{2}e(G)$.
b) Describe an algorithm for finding such a subgraph by first arranging the vertices in a linear order and then assigning them, one by one, to either X or Y, using a simple rule.

2.2.3 Determine the number of 1-factors in each of the following graphs: (a) the Petersen graph, (b) the pentagonal prism, (c) K_{2n}, (d) $K_{n,n}$.

2.2.4 Give a proof of Theorem 2.3 by means of a longest path argument.
(D. KÖNIG AND P. VERESS)

2.2.5

a) Show that every Hamilton cycle of the k-prism uses either exactly two consecutive edges linking the two k-cycles or else all of them.
b) How many Hamilton cycles are there in the pentagonal prism?

2.2.6 Show that there is a Hamilton path between two vertices in the Petersen graph if and only if these vertices are nonadjacent.

2.2.7
Which grids have Hamilton paths, and which have Hamilton cycles?

2.2.8 Give an example to show that the following simple procedure, known as the *Greedy Heuristic*, is not guaranteed to solve the Travelling Salesman Problem.

- ▷ Select an arbitrary vertex v.
- ▷ Starting with the trivial path v, grow a Hamilton path one edge at a time, choosing at each iteration an edge of minimum weight between the terminal vertex of the current path and a vertex not on this path.
- ▷ Form a Hamilton cycle by adding the edge joining the two ends of the Hamilton path.

2.2.9 Let G be a graph on n vertices and m edges.

a) How many induced subgraphs has G?
b) How many edge-induced subgraphs has G?

2.2.10 Show that every shortest cycle in a simple graph is an induced subgraph.

$\star$**2.2.11** Show that if G is simple and connected, but not complete, then G contains an induced path of length two.

$\star$**2.2.12** Let P and Q be distinct paths in a graph G with the same initial and terminal vertices. Show that $P \cup Q$ contains a cycle by considering the subgraph $G[E(P) \triangle E(Q)]$ and appealing to Theorem 2.1.

2.2.13

a) Show that any two longest paths in a connected graph have a vertex in common.
b) Deduce that if P is a longest path in a connected graph G, then no path in $G - V(P)$ is as long as P.

2.2.14 Give a constructive proof of Theorem 2.5.

2.2.15

a) Show that an induced subgraph of a line graph is itself a line graph.
b) Deduce that no line graph can contain either of the graphs in Figure 1.19 as an induced subgraph.
c) Show that these two graphs are minimal with respect to the above property. Can you find other such graphs? (There are nine in all.)

2.2.16

a) Show that an induced subgraph of an interval graph is itself an interval graph.
b) Deduce that no interval graph can contain the graph in Figure 1.20 as an induced subgraph.
c) Show that this graph is minimal with respect to the above property.

2.2.17 Let G be a bipartite graph of maximum degree k.

a) Show that there is a k-regular bipartite graph H which contains G as an induced subgraph.
b) Show, moreover, that if G is simple, then there exists such a graph H which is simple.

2.2.18 Let G be a simple connected graph.

a) Show that there is an ordering $v_1, v_2, \ldots, v_n$ of V such that at least $\frac{1}{2}(n-1)$ vertices v_j are adjacent to an odd number of vertices v_i with $i < j$.
b) By starting with such an ordering and adopting the approach outlined in Exercise 2.2.2b, deduce that G has a bipartite subgraph with at least $\frac{1}{2}m + \frac{1}{4}(n-1)$ edges. (C. Edwards; P. Erdős)

2.2.19 Read the 'Theorem' and 'Proof' given below, and then answer the questions which follow.

'Theorem'. Let G be a simple graph with $\delta \geq n/2$, where $n \geq 3$. Then G has a Hamilton cycle.

'Proof'. By induction on n. The 'Theorem' is true for $n = 3$, because $G = K_3$ in this case. Suppose that it holds for $n = k$, where $k \geq 3$. Let G' be a simple graph on k vertices in which $\delta \geq k/2$, and let C' be a Hamilton cycle of G'. Form a graph G on $k+1$ vertices in which $\delta \geq (k+1)/2$ by adding a new vertex v and joining v to at least $(k+1)/2$ vertices of G'. Note that v must be adjacent to two consecutive vertices, u and w, of C'. Replacing the edge uw of C' by the path uvw, we obtain a Hamilton cycle C of G. Thus the 'Theorem' is true for $n = k+1$. By the Principle of Mathematical Induction, it is true for all $n \geq 3$. □

a) Is the 'Proof' correct?
b) If you claim that the 'Proof' is incorrect, give reasons to support your claim.
c) Can you find any graphs for which the 'Theorem' fails? Does the existence or nonexistence of such graphs have any relationship to the correctness or incorrectness of the 'Proof'? (D.R. Woodall)

2.2.20

a) Let D be an oriented graph with minimum outdegree k, where $k \geq 1$.
   i) Show that D has a vertex x whose indegree and outdegree are both at least k.
   ii) Let D' be the digraph obtained from D by deleting $N^-(x) \cup \{x\}$ and adding an arc (u, v) from each vertex u of the set $N^{--}(x)$ of in-neighbours of $N^-(x)$ to each vertex v of $N^+(x)$, if there was no such arc in D. Show that D' is a strict digraph with minimum outdegree k.
b) Deduce, by induction on n, that every strict digraph D with minimum outdegree k, where $k \geq 1$, contains a directed cycle of length at most $2n/k$. (V. Chvátal and E. Szemerédi)

2.2.21 The *complement* $\overline{D}$ of a strict digraph D is its complement in $D(K_n)$. Let $D = (V, A)$ be a strict digraph and let P be a directed Hamilton path of D. Form a bipartite graph $B[\mathcal{F}, S_n]$, where $\mathcal{F}$ is the family of spanning subgraphs of D each component of which is a directed path and S_n is the set of permutations of V, a subgraph $F \in \mathcal{F}$ being adjacent in B to a permutation $\sigma \in S_n$ if and only if $\sigma(F) \subseteq \sigma(D) \cap P$.

a) Which vertices $F \in \mathcal{F}$ are of odd degree in B?
b) Describe a bijection between the vertices $\sigma \in S_n$ of odd degree in B and the directed Hamilton paths of $\overline{D}$.
c) Deduce that $h(D) \equiv h(\overline{D}) \pmod 2$, where $h(D)$ denotes the number of directed Hamilton paths in D.

2.2.22 Let D be a tournament, and let (x, y) be an arc of D. Set $D^- := D \setminus (x, y)$ and $D^+ := D + (y, x)$.

a) Describe a bijection between the directed Hamilton paths of D^- and those of $\overline{D^+}$.
b) Deduce from Exercise 2.2.21 that $h(D^-) \equiv h(D^+) \pmod 2$.
c) Consider the tournament D' obtained from D on reversing the arc (x, y). Show that $h(D') = h(D^+) - h(D) + h(D^-)$.
d) Deduce that $h(D') \equiv h(D) \pmod 2$.
e) Conclude that every tournament has an odd number of directed Hamilton paths. (L. Rédei)

2.2.23

a) Let S be a set of n points in the plane, the distance between any two of which is at most one. Show that there are at most n pairs of points of S at distance exactly one. (P. Erdős)
b) For each $n \geq 3$, describe such a set S for which the number of pairs of points at distance exactly one is n.

2.2.24 Let G be a simple graph on n vertices and m edges, with minimum degree δ and maximum degree Δ.

a) Show that there is a simple Δ-regular graph H which contains G as an induced subgraph.
b) Let H be such a graph, with $v(H) = n + r$. Show that:
   i) $r \geq \Delta - \delta$,
   ii) $r\Delta \equiv n\Delta \pmod 2$,
   iii) $r\Delta \geq n\Delta - 2m \geq r\Delta - r(r - 1)$.

(Erdős and Kelly (1967) showed that if r is the smallest positive integer which satisfies the above three conditions, then there does indeed exist a simple Δ-regular graph H on $n + r$ vertices which contains G as an induced subgraph.)

2.2.25 Let G be a simple graph on n vertices, where $n \geq 4$, and let k be an integer, $2 \leq k \leq n - 2$. Suppose that all induced subgraphs of G on k vertices have the same number of edges. Show that G is either empty or complete.

2.3 Modifying Graphs

We have already discussed some simple ways of modifying a graph, namely deleting or adding vertices or edges. Here, we describe several other local operations on graphs. Although they do not give rise to subgraphs or supergraphs, it is natural and convenient to introduce them here.

Vertex Identification and Edge Contraction

To *identify* nonadjacent vertices x and y of a graph G is to replace these vertices by a single vertex incident to all the edges which were incident in G to either x or y. We denote the resulting graph by $G / \{x, y\}$ (see Figure 2.5a). To *contract* an edge e of a graph G is to delete the edge and then (if the edge is a link) identify its ends. The resulting graph is denoted by G / e (see Figure 2.5b).

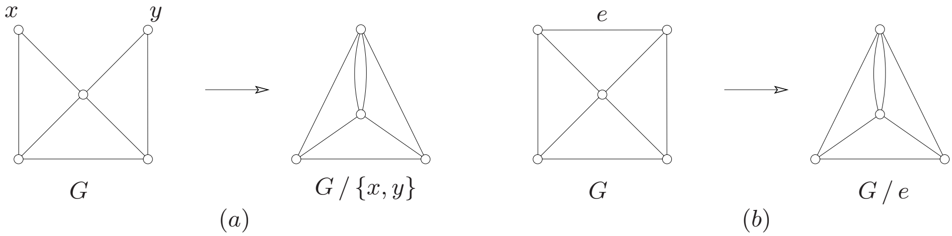


Fig. 2.5. (a) Identifying two vertices, and (b) contracting an edge

Vertex Splitting and Edge Subdivision

The inverse operation to edge contraction is vertex splitting. To *split* a vertex v is to replace v by two adjacent vertices, v' and v'', and to replace each edge incident to v by an edge incident to either v' or v'' (but not both, unless it is a loop at v), the other end of the edge remaining unchanged (see Figure 2.6a). Note that a vertex of positive degree can be split in several ways, so the resulting graph is not unique in general.

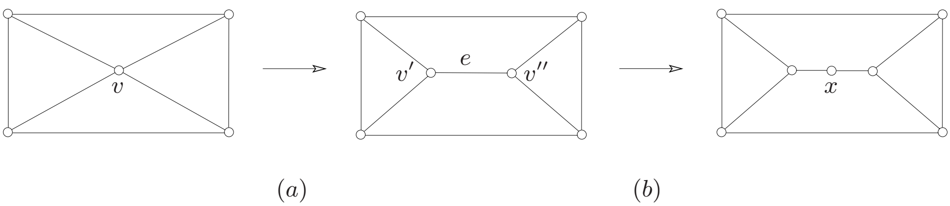


Fig. 2.6. (a) Splitting a vertex, and (b) subdividing an edge

A special case of vertex splitting occurs when exactly one link, or exactly one end of a loop, is assigned to either v' or v''. The resulting graph can then be viewed as having been obtained by subdividing an edge of the original graph, where to *subdivide* an edge e is to delete e, add a new vertex x, and join x to the ends of e (when e is a link, this amounts to replacing e by a path of length two, as in Figure 2.6b).

Exercises

2.3.1

a) Show that $c(G\,/\,e) = c(G)$ for any edge e of a graph G.
b) Let G be an acyclic graph, and let $e \in E$.
 i) Show that $G\,/\,e$ is acyclic.
 ii) Deduce that $m = n - c$.

2.4 Decompositions and Coverings

Decompositions

A *decomposition* of a graph G is a family $\mathcal{F}$ of edge-disjoint subgraphs of G such that

$$\cup_{F\in\mathcal{F}} E(F) = E(G) \tag{2.2}$$

If the family $\mathcal{F}$ consists entirely of paths or entirely of cycles, we call $\mathcal{F}$ a *path decomposition* or *cycle decomposition* of G.

Every loopless graph has a trivial path decomposition, into paths of length one. On the other hand, not every graph has a cycle decomposition. Observe that if a graph has a cycle decomposition $\mathcal{C}$, the degree of each vertex is twice the number of cycles of $\mathcal{C}$ to which it belongs, so is even. A graph in which each vertex has even degree is called an *even graph*. Thus, a graph which admits a cycle decomposition is necessarily even. Conversely, as was shown by Veblen (1912/13), every even graph admits a cycle decomposition.

Theorem 2.7 Veblen's Theorem
A graph admits a cycle decomposition if and only if it is even.

Proof We have already shown that the condition of evenness is necessary. We establish the converse by induction on $e(G)$.

Suppose that G is even. If G is empty, then $E(G)$ is decomposed by the empty family of cycles. If not, consider the subgraph F of G induced by its vertices of positive degree. Because G is even, F also is even, so every vertex of F has degree two or more. By Theorem 2.1, F contains a cycle C. The subgraph $G' := G \setminus E(C)$ is even, and has fewer edges than G. By induction, G' has a cycle decomposition $\mathcal{C}'$. Therefore G has the cycle decomposition $\mathcal{C} := \mathcal{C}' \cup \{C\}$. $\square$

There is a corresponding version of Veblen's Theorem for digraphs (see Exercise 2.4.2).

Proof Technique: Linear Independence

Algebraic techniques can occasionally be used to solve problems where combinatorial methods fail. Arguments involving the ranks of appropriately chosen matrices are particularly effective. Here, we illustrate this technique by giving a simple proof, due to Tverberg (1982), of a theorem of Graham and Pollak (1971) on decompositions of complete graphs into complete bipartite graphs. There are many ways in which a complete graph can be decomposed into complete bipartite graphs. For example, K_4 may be decomposed into six copies of K_2, into three copies of $K_{1,2}$, into the stars $K_{1,1}$, $K_{1,2}$, and $K_{1,3}$, or into $K_{2,2}$ and two copies of K_2. What Graham and Pollak showed is that, no matter how K_n is decomposed into complete bipartite graphs, there must be at least $n-1$ of them in the decomposition. Observe that this bound can always be achieved, for instance by decomposing K_n into the stars $K_{1,k}$, $1 \le k \le n-1$.

Theorem 2.8 *Let $\mathcal{F} := \{F_1, F_2, \ldots, F_k\}$ be a decomposition of K_n into complete bipartite graphs. Then $k \ge n-1$.*

Proof Let $V := V(K_n)$ and let F_i have bipartition (X_i, Y_i), $1 \le i \le k$. Consider the following system of $k+1$ homogeneous linear equations in the variables x_v, $v \in V$:

$$\sum_{v \in V} x_v = 0, \qquad \sum_{v \in X_i} x_v = 0, \quad 1 \le i \le k$$

Suppose that $k < n-1$. Then this system, consisting of fewer than n equations in n variables, has a solution $x_v = c_v$, $v \in V$, with $c_v \ne 0$ for at least one $v \in V$. Thus

$$\sum_{v \in V} c_v = 0 \text{ and } \sum_{v \in X_i} c_v = 0, \quad 1 \le i \le k$$

Because $\mathcal{F}$ is a decomposition of K_n,

$$\sum_{vw \in E} c_v c_w = \sum_{i=1}^{k} \left(\sum_{v \in X_i} c_v \right) \left(\sum_{w \in Y_i} c_w \right)$$

Therefore

$$0 = \left(\sum_{v \in V} c_v \right)^2 = \sum_{v \in V} c_v^2 + 2 \sum_{i=1}^{k} \left(\sum_{v \in X_i} c_v \right) \left(\sum_{w \in Y_i} c_w \right) = \sum_{v \in V} c_v^2 > 0$$

a contradiction. We conclude that $k \ge n-1$. □

Further proofs based on linear independence arguments are outlined in Exercises 2.4.9 and 14.2.15.

Coverings

We now define the related concept of a covering. A *covering* or *cover* of a graph G is a family $\mathcal{F}$ of subgraphs of G, not necessarily edge-disjoint, satisfying (2.2). A covering is *uniform* if it covers each edge of G the same number of times; when this number is k, the covering is called a k-*cover*. A 1-cover is thus simply a decomposition. A 2-cover is usually called a *double cover*. If the family $\mathcal{F}$ consists entirely of paths or entirely of cycles, the covering is referred to as a *path covering* or *cycle covering*. Every graph which admits a cycle covering also admits a uniform cycle covering (Exercise 3.5.7).

The notions of decomposition and covering crop up frequently in the study of graphs. In Section 3.5, we discuss a famous unsolved problem concerning cycle coverings, the Cycle Double Cover Conjecture. The concept of covering is also useful in the study of another celebrated unsolved problem, the Reconstruction Conjecture (see Section 2.7, in particular Exercise 2.7.11).

Exercises

2.4.1 Let e be an edge of an even graph G. Show that $G\,/\,e$ is even.

$\star$**2.4.2** Even Directed Graph
A digraph D is *even* if $d^-(v) = d^+(v)$ for each vertex $v \in V$. Prove the following directed version of Veblen's Theorem (2.7): *A directed graph admits a decomposition into directed cycles if and only if it is even.*

2.4.3 Find a decomposition of K_{13} into three copies of the circulant $\mathrm{CG}(\mathbb{Z}_{13}, \{1, -1, 5, -5\})$.

2.4.4

a) Show that K_n can be decomposed into copies of K_p only if $n-1$ is divisible by $p-1$ and $n(n-1)$ is divisible by $p(p-1)$. For which integers n do these two conditions hold when p is a prime?
b) For k a prime power, describe a decomposition of K_{k^2+k+1} into copies of K_{k+1}, based on a finite projective plane of order k.

2.4.5 Let n be a positive integer.

a) Describe a decomposition of K_{2n+1} into Hamilton cycles.
b) Deduce that K_{2n} admits a decomposition into Hamilton paths.

$\star$**2.4.6** Consider the graph obtained from the Petersen graph by replacing each of the five edges in a 1-factor by two parallel edges, as shown in Figure 2.7. Show that every cycle decomposition of this 4-regular graph includes a 2-cycle.

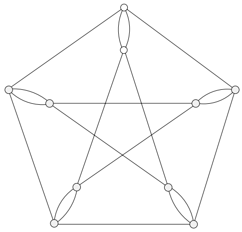


Fig. 2.7. The Petersen graph with a doubled 1-factor

2.4.7 Let G be a connected graph with an even number of edges.

a) Show that G can be oriented so that the outdegree of each vertex is even.
b) Deduce that G admits a decomposition into paths of length two.

2.4.8 Show that every loopless digraph admits a decomposition into two acyclic digraphs.

2.4.9 Give an alternative proof of the de Bruijn–Erdős Theorem (see Exercise 1.3.15b) by proceeding as follows. Let $\mathbf{M}$ be the incidence matrix of a geometric configuration $(P, \mathcal{L})$ which has at least two lines and in which any two points lie on exactly one line.

a) Show that the columns of $\mathbf{M}$ span $\mathbb{R}^n$, where $n := |P|$.
b) Deduce that $\mathbf{M}$ has rank n.
c) Conclude that $|\mathcal{L}| \geq |P|$.

2.5 Edge Cuts and Bonds

Edge Cuts

Let X and Y be sets of vertices (not necessarily disjoint) of a graph $G = (V, E)$. We denote by $E[X, Y]$ the set of edges of G with one end in X and the other end in Y, and by $e(X, Y)$ their number. If $Y = X$, we simply write $E(X)$ and $e(X)$ for $E[X, X]$ and $e(X, X)$, respectively. When $Y = V \setminus X$, the set $E[X, Y]$ is called the *edge cut* of G associated with X, or the *coboundary* of X, and is denoted by $\partial(X)$; note that $\partial(X) = \partial(Y)$ in this case, and that $\partial(V) = \emptyset$. In this notation, a graph $G = (V, E)$ is bipartite if $\partial(X) = E$ for some subset X of V, and is connected if $\partial(X) \neq \emptyset$ for every nonempty proper subset X of V. The edge cuts of a graph are illustrated in Figure 2.8.

An edge cut $\partial(v)$ associated with a single vertex v is a *trivial* edge cut; this is simply the set of all links incident with v. If there are no loops incident with v, it follows that $|\partial(v)| = d(v)$. Accordingly, in the case of loopless graphs, we refer to $|\partial(X)|$ as the *degree* of X and denote it by $d(X)$.

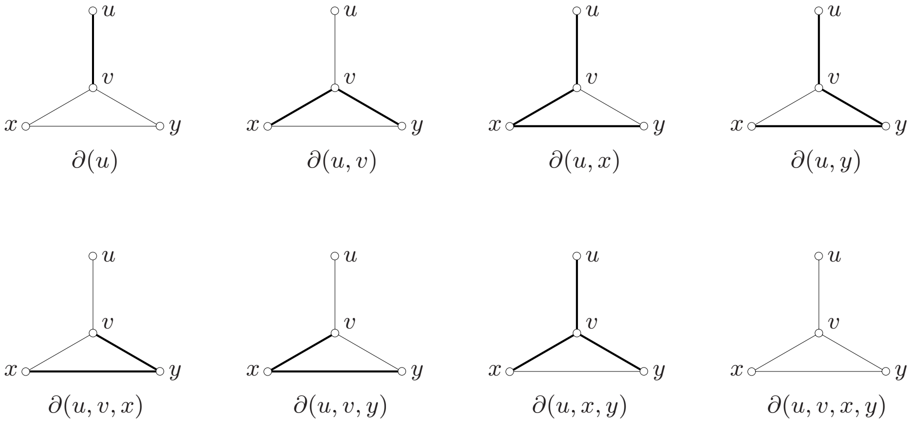


Fig. 2.8. The edge cuts of a graph

The following theorem is a natural generalization of Theorem 1.1, the latter theorem being simply the case where $X = V$. Its proof is based on the technique of counting in two ways, and is left as an exercise (2.5.1a).

Theorem 2.9 *For any graph G and any subset X of V,*

$$|\partial(X)| = \sum_{v \in X} d(v) - 2e(X) \qquad \square$$

Veblen's Theorem (2.7) characterizes even graphs in terms of cycles. Even graphs may also be characterized in terms of edge cuts, as follows.

Theorem 2.10 *A graph G is even if and only if $|\partial(X)|$ is even for every subset X of V.*

Proof Suppose that $|\partial(X)|$ is even for every subset X of V. Then, in particular, $|\partial(v)|$ is even for every vertex v. But, as noted above, $\partial(v)$ is just the set of all links incident with v. Because loops contribute two to the degree, it follows that all degrees are even. Conversely, if G is even, then Theorem 2.9 implies that all edge cuts are of even cardinality. $\square$

The operation of symmetric difference of spanning subgraphs was introduced in Section 2.1. The following propositions show how edge cuts behave with respect to symmetric difference.

Proposition 2.11 *Let G be a graph, and let X and Y be subsets of V. Then*

$$\partial(X) \triangle \partial(Y) = \partial(X \triangle Y)$$

Proof Consider the Venn diagram, shown in Figure 2.9, of the partition of V

$$(X \cap Y, \quad X \setminus Y, \quad Y \setminus X, \quad \overline{X} \cap \overline{Y})$$

determined by the partitions $(X, \overline{X})$ and $(Y, \overline{Y})$, where $\overline{X} := V \setminus X$ and $\overline{Y} := V \setminus Y$. The edges of $\partial(X)$, $\partial(Y)$, and $\partial(X \triangle Y)$ between these four subsets of V are indicated schematically in Figure 2.10. It can be seen that $\partial(X) \triangle \partial(Y) = \partial(X \triangle Y)$. $\square$

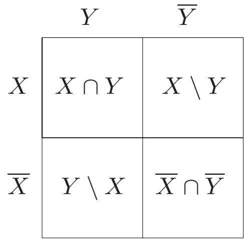


Fig. 2.9. Partition of V determined by the partitions $(X, \overline{X})$ and $(Y, \overline{Y})$

Corollary 2.12 *The symmetric difference of two edge cuts is an edge cut.* $\square$

We leave the proof of the second proposition to the reader (Exercise 2.5.1b).

Proposition 2.13 *Let F_1 and F_2 be spanning subgraphs of a graph G, and let X be a subset of V. Then*

$$\partial_{F_1 \triangle F_2}(X) = \partial_{F_1}(X) \triangle \partial_{F_2}(X) \qquad \square$$

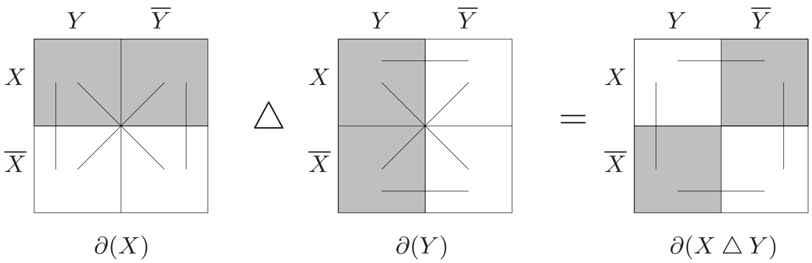


Fig. 2.10. The symmetric difference of two cuts

Bonds

A *bond* of a graph is a minimal nonempty edge cut, that is, a nonempty edge cut none of whose nonempty proper subsets is an edge cut. The bonds of the graph whose edge cuts are depicted in Figure 2.8 are shown in Figure 2.11.

The following two theorems illuminate the relationship between edge cuts and bonds. The first can be deduced from Proposition 2.11 (Exercise 2.5.1c). The second provides a convenient way to check when an edge cut is in fact a bond.

Theorem 2.14 *A set of edges of a graph is an edge cut if and only if it is a disjoint union of bonds.* □

Theorem 2.15 *In a connected graph G, a nonempty edge cut $\partial(X)$ is a bond if and only if both $G[X]$ and $G[V \setminus X]$ are connected.*

Proof Suppose, first, that $\partial(X)$ is a bond, and let Y be a nonempty proper subset of X. Because G is connected, both $\partial(Y)$ and $\partial(X \setminus Y)$ are nonempty. It follows that $E[Y, X \setminus Y]$ is nonempty, for otherwise $\partial(Y)$ would be a nonempty proper subset of $\partial(X)$, contradicting the supposition that $\partial(X)$ is a bond. We conclude that $G[X]$ is connected. Likewise, $G[V \setminus X]$ is connected.

Conversely, suppose that $\partial(X)$ is not a bond. Then there is a nonempty proper subset Y of V such that $X \cap Y \neq \emptyset$ and $\partial(Y) \subset \partial(X)$. But this implies (see Figure 2.10) that $E[X \cap Y, X \setminus Y] = E[Y \setminus X, \overline{X} \cap \overline{Y}] = \emptyset$. Thus $G[X]$ is not connected if $X \setminus Y \neq \emptyset$. On the other hand, if $X \setminus Y = \emptyset$, then $\emptyset \subset Y \setminus X \subset V \setminus X$, and $G[V \setminus X]$ is not connected. □

Cuts in Directed Graphs

If X and Y are sets of vertices (not necessarily disjoint) of a digraph $D = (V, A)$, we denote the set of arcs of D whose tails lie in X and whose heads lie in Y by $A(X, Y)$, and their number by $a(X, Y)$. This set of arcs is denoted by $A(X)$ when $Y = X$, and their number by $a(X)$. When $Y = V \setminus X$, the set $A(X, Y)$ is called the *outcut* of D associated with X, and denoted by $\partial^+(X)$. Analogously, the set $A(Y, X)$ is called the *incut* of D associated with X, and denoted by $\partial^-(X)$. Observe that $\partial^+(X) = \partial^-(V \setminus X)$. Note, also, that $\partial(X) = \partial^+(X) \cup \partial^-(X)$. In

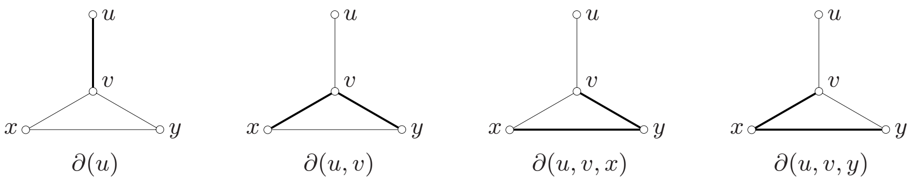


Fig. 2.11. The bonds of a graph

the case of loopless digraphs, we refer to $|\partial^+(X)|$ and $|\partial^-(X)|$ as the *outdegree* and *indegree* of X, and denote these quantities by $d^+(X)$ and $d^-(X)$, respectively.

A digraph D is called *strongly connected* or *strong* if $\partial^+(X) \neq \emptyset$ for every nonempty proper subset X of V (and thus $\partial^-(X) \neq \emptyset$ for every nonempty proper subset X of V, too).

Exercises

$\star$**2.5.1**

a) Prove Theorem 2.9.
b) Prove Proposition 2.13.
c) Deduce Theorem 2.14 from Proposition 2.11.

$\star$**2.5.2** Let D be a digraph, and let X be a subset of V.

a) Show that $|\partial^+(X)| = \sum_{v \in X} d^+(v) - a(X)$.
b) Suppose that D is even. Using the Principle of Directional Duality, deduce that $|\partial^+(X)| = |\partial^-(X)|$.
c) Deduce from (b) that every connected even digraph is strongly connected.

2.5.3 Let G be a graph, and let X and Y be subsets of V. Show that $\partial(X \cup Y) \triangle \partial(X \cap Y) = \partial(X \triangle Y)$.

$\star$**2.5.4** Let G be a loopless graph, and let X and Y be subsets of V.

a) Show that:

$$d(X) + d(Y) = d(X \cup Y) + d(X \cap Y) + 2e(X \setminus Y, Y \setminus X)$$

b) Deduce the following *submodular inequality* for degrees of sets of vertices.

$$d(X) + d(Y) \geq d(X \cup Y) + d(X \cap Y)$$

c) State and prove a directed analogue of this submodular inequality.

$\star$**2.5.5** An *odd graph* is one in which each vertex is of odd degree. Show that a graph G is odd if and only if $|\partial(X)| \equiv |X| \pmod 2$ for every subset X of V.

$\star$**2.5.6** Show that each arc of a strong digraph is contained in a directed cycle.

2.5.7 Directed Bond
A *directed bond* of a digraph is a bond $\partial(X)$ such that $\partial^-(X) = \emptyset$ (in other words, $\partial(X)$ is the outcut $\partial^+(X)$).

a) Show that an arc of a digraph is contained either in a directed cycle, or in a directed bond, but not both. (G.J. Minty)
b) Deduce that:
   i) a digraph is acyclic if and only if every bond is a directed bond,

ii) a digraph is strong if and only if no bond is a directed bond.

$\star$**2.5.8** FEEDBACK ARC SET
A *feedback arc set* of a digraph D is a set S of arcs such that $D \setminus S$ is acyclic. Let S be a minimal feedback arc set of a digraph D. Show that there is a linear ordering of the vertices of D such that the arcs of S are precisely those arcs whose heads precede their tails in the ordering.

2.5.9 Let (D, w) be a weighted oriented graph. For $v \in V$, set $w^+(v) := \sum\{w(a) : a \in \partial^+(v)\}$. Suppose that $w^+(v) \geq 1$ for all $v \in V \setminus \{y\}$, where $y \in V$. Show that D contains a directed path of weight at least one, by proceeding as follows.

a) Consider an arc $(x, y) \in \partial^-(y)$ of maximum weight. Contract this arc to a vertex y', delete all arcs with tail y', and replace each pair $\{a, a'\}$ of multiple arcs (with head y') by a single arc of weight $w(a)+w(a')$, all other arcs keeping their original weights. Denote the resulting weighted digraph by (D', w'). Show that if D' contains a directed path of weight at least one, then so does D.
b) Deduce, by induction on V, that D contains a directed path of weight at least one. (B. BOLLOBÁS AND A.D. SCOTT)

2.6 Even Subgraphs

By an *even subgraph* of a graph G we understand a *spanning* even subgraph of G, or frequently just the edge set of such a subgraph. Observe that the first two subgraphs in Figure 2.4 are both even, as is their symmetric difference. Indeed, it is an easy consequence of Proposition 2.13 that the symmetric difference of even subgraphs is always even.

Corollary 2.16 *The symmetric difference of two even subgraphs is an even subgraph.*

Proof Let F_1 and F_2 be even subgraphs of a graph G, and let X be a subset of V. By Proposition 2.13,

$$\partial_{F_1 \triangle F_2}(X) = \partial_{F_1}(X) \triangle \partial_{F_2}(X)$$

By Theorem 2.10, $\partial_{F_1}(X)$ and $\partial_{F_2}(X)$ are both of even cardinality, so their symmetric difference is too. Appealing again to Theorem 2.10, we deduce that $F_1 \triangle F_2$ is even. $\square$

As we show in Chapters 4 and 21, the even subgraphs of a graph play an important structural role. When discussing even subgraphs (and only in this context), by a *cycle* we mean the edge set of a cycle. By the same token, we use the term *disjoint cycles* to mean edge-disjoint cycles. With this convention, the cycles of a graph are its minimal nonempty even subgraphs, and Theorem 2.7 may be restated as follows.

Theorem 2.17 *A set of edges of a graph is an even subgraph if and only if it is a disjoint union of cycles.* □

The Cycle and Bond Spaces

Even subgraphs and edge cuts are related in the following manner.

Proposition 2.18 *In any graph, every even subgraph meets every edge cut in an even number of edges.*

Proof We first show that every cycle meets every edge cut in an even number of edges. Let C be a cycle and $\partial(X)$ an edge cut. Each vertex of C is either in X or in $V \setminus X$. As C is traversed, the number of times it crosses from X to $V \setminus X$ must be the same as the number of times it crosses from $V \setminus X$ to X. Thus $|E(C) \cap \partial(X)|$ is even.

By Theorem 2.17, every even subgraph is a disjoint union of cycles. It follows that every even subgraph meets every edge cut in an even number of edges. □

We denote the set of all subsets of the edge set E of a graph G by $\mathcal{E}(G)$. This set forms a vector space of dimension m over $GF(2)$ under the operation of symmetric difference. We call $\mathcal{E}(G)$ the *edge space* of G. With each subset X of E, we may associate its *incidence vector* $\mathbf{f}_X$, where $f_X(e) = 1$ if $e \in X$ and $f_X(e) = 0$ if $e \notin X$. The function which maps X to $\mathbf{f}_X$ for all $X \subseteq E$ is an isomorphism from $\mathcal{E}$ to $(GF(2))^E$ (Exercise 2.6.2).

By Corollary 2.16, the set of all even subgraphs of a graph G forms a subspace $\mathcal{C}(G)$ of the edge space of G. We call this subspace the *cycle space* of G, because it is generated by the cycles of G. Likewise, by Corollary 2.12, the set of all edge cuts of G forms a subspace $\mathcal{B}(\mathcal{G})$ of $\mathcal{E}(G)$, called the *bond space* (Exercise 2.6.4a,b). Proposition 2.18 implies that these two subspaces are orthogonal. They are, in fact, orthogonal complements (Exercise 2.6.4c).

In Chapter 20, we extend the above concepts to arbitrary fields, in particular to the field of real numbers.

Exercises

2.6.1 Show that:

a) a graph G is even if and only if E is an even subgraph of G,
b) a graph G is bipartite if and only if E is an edge cut of G.

$\star$**2.6.2** Show that the edge space $\mathcal{E}(G)$ is a vector space over $GF(2)$ with respect to the operation of symmetric difference, and that it is isomorphic to $(GF(2))^E$.

2.6.3

a) Draw all the elements of the cycle and bond spaces of the wheel W_4.

b) How many elements are there in each of these two vector spaces?

$\star$**2.6.4** Show that:

a) the cycles of a graph generate its cycle space,
b) the bonds of a graph generate its bond space,
c) the bond space of a graph G is the row space of its incidence matrix $\mathbf{M}$ over $GF(2)$, and the cycle space of G is its orthogonal complement.

2.6.5 How many elements are there in the cycle and bond spaces of a graph G?

2.6.6 Show that every graph G has an edge cut $[X, Y]$ such that $G[X]$ and $G[Y]$ are even.

2.7 Graph Reconstruction

Two graphs G and H on the same vertex set V are called *hypomorphic* if, for all $v \in V$, their vertex-deleted subgraphs $G - v$ and $H - v$ are isomorphic. Does this imply that G and H are themselves isomorphic? Not necessarily: the graphs $2K_1$ and K_2, though not isomorphic, are clearly hypomorphic. However, these two graphs are the only known nonisomorphic pair of hypomorphic simple graphs, and it was conjectured in 1941 by Kelly (1942) (see also Ulam (1960)) that there are no other such pairs. This conjecture was reformulated by Harary (1964) in the more intuitive language of reconstruction. A *reconstruction* of a graph G is any

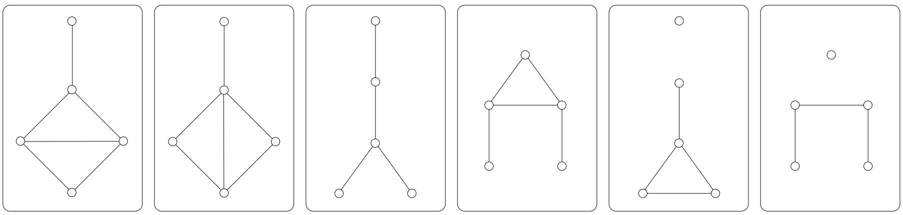


Fig. 2.12. The deck of a graph on six vertices

graph that is hypomorphic to G. We say that a graph G is *reconstructible* if every reconstruction of G is isomorphic to G, in other words, if G can be 'reconstructed' up to isomorphism from its vertex-deleted subgraphs. Informally, one may think of the (unlabelled) vertex-deleted subgraphs as being presented on cards, one per card. The problem of reconstructing a graph is then that of determining the graph from its *deck* of cards. The reader is invited to reconstruct the graph whose deck of six cards is shown in Figure 2.12.

THE RECONSTRUCTION CONJECTURE

Conjecture 2.19 *Every simple graph on at least three vertices is reconstructible.*

The Reconstruction Conjecture has been verified by computer for all graphs on up to ten vertices by McKay (1977). In discussing it, we implicitly assume that our graphs have at least three vertices.

One approach to the Reconstruction Conjecture is to show that it holds for various classes of graphs. A class of graphs is *reconstructible* if every member of the class is reconstructible. For instance, regular graphs are easily shown to be reconstructible (Exercise 2.7.5). One can also prove that disconnected graphs are reconstructible (Exercise 2.7.11). Another approach is to prove that specific parameters are reconstructible. We call a graphical parameter *reconstructible* if the parameter takes the same value on all reconstructions of G. A fundamental result of this type was obtained by Kelly (1957). For graphs F and G, we adopt the notation of Lauri and Scapellato (2003) and use $\binom{G}{F}$ to denote the number of copies of F in G. For instance, if $F = K_2$, then $\binom{G}{F} = e(G)$, if $F = G$, then $\binom{G}{F} = 1$, and if $v(F) > v(G)$, then $\binom{G}{F} = 0$.

Lemma 2.20 KELLY'S LEMMA
For any two graphs F and G such that $v(F) < v(G)$, the parameter $\binom{G}{F}$ is reconstructible.

Proof Each copy of F in G occurs in exactly $v(G) - v(F)$ of the vertex-deleted subgraphs $G - v$ (namely, whenever the vertex v is not present in the copy). Therefore

$$\binom{G}{F} = \frac{1}{v(G) - v(F)} \sum_{v \in V} \binom{G-v}{F}$$

Since the right-hand side of this identity is reconstructible, so too is the left-hand side. $\square$

Corollary 2.21 *For any two graphs F and G such that $v(F) < v(G)$, the number of subgraphs of G that are isomorphic to F and include a given vertex v is reconstructible.*

Proof This number is $\binom{G}{F} - \binom{G-v}{F}$, which is reconstructible by Kelly's Lemma. $\square$

Corollary 2.22 *The size and the degree sequence are reconstructible parameters.*

Proof Take $F = K_2$ in Kelly's Lemma and Corollary 2.21, respectively. □

An edge analogue of the Reconstruction Conjecture was proposed by Harary (1964). A graph is *edge-reconstructible* if it can be reconstructed up to isomorphism from its edge-deleted subgraphs.

THE EDGE RECONSTRUCTION CONJECTURE

Conjecture 2.23 *Every simple graph on at least four edges is edge-reconstructible.*

Note that the bound on the number of edges is needed on account of certain small counterexamples (see Exercise 2.7.2). The notions of *edge reconstructibility* of classes of graphs and of graph parameters are defined in an analogous manner to those of reconstructibility, and there is an edge version of Kelly's Lemma, whose proof we leave as an exercise (Exercise 2.7.13a).

Lemma 2.24 KELLY'S LEMMA: EDGE VERSION
For any two graphs F and G such that $e(F) < e(G)$, the parameter $\binom{G}{F}$ is edge reconstructible. □

Because edge-deleted subgraphs are much closer to the original graph than are vertex-deleted subgraphs, it is intuitively clear (but not totally straightforward to prove) that the Edge Reconstruction Conjecture is no harder than the Reconstruction Conjecture (Exercise 2.7.14). Indeed, a number of approaches have been developed which are effective for edge reconstruction, but not for vertex reconstruction. We describe below one of these approaches, Möbius Inversion.

PROOF TECHNIQUE: MÖBIUS INVERSION

We discussed earlier the proof technique of counting in two ways. Here, we present a more subtle counting technique, that of *Möbius Inversion*. This is a generalization of the *Inclusion-Exclusion Formula*, a formula which expresses the cardinality of the union of a family of sets $\{A_i : i \in T\}$ in terms of the cardinalities of intersections of these sets:

$$|\cup_{i \in T} A_i| = \sum_{\emptyset \subset X \subseteq T} (-1)^{|X|-1} |\cap_{i \in X} A_i| \tag{2.3}$$

the case of two sets being the formula $|A \cup B| = |A| + |B| - |A \cap B|$.

Möbius Inversion (continued)

Theorem 2.25 The Möbius Inversion Formula
Let $f : 2^T \to \mathbb{R}$ be a real-valued function defined on the subsets of a finite set T. Define the function $g : 2^T \to \mathbb{R}$ by

$$g(S) := \sum_{S \subseteq X \subseteq T} f(X) \tag{2.4}$$

Then, for all $S \subseteq T$,

$$f(S) = \sum_{S \subseteq X \subseteq T} (-1)^{|X|-|S|} g(X) \tag{2.5}$$

Remark. Observe that (2.4) is a linear transformation of the vector space of real-valued functions defined on 2^T. The Möbius Inversion Formula (2.5) simply specifies the inverse of this transformation.

Proof By the Binomial Theorem,

$$\sum_{S \subseteq X \subseteq Y} (-1)^{|X|-|S|} = \sum_{|S| \le |X| \le |Y|} \binom{|Y|-|S|}{|X|-|S|} (-1)^{|X|-|S|} = (1-1)^{|Y|-|S|}$$

which is equal to 0 if $S \subset Y$, and to 1 if $S = Y$. Therefore,

$$\begin{aligned} f(S) &= \sum_{S \subseteq Y \subseteq T} f(Y) \sum_{S \subseteq X \subseteq Y} (-1)^{|X|-|S|} \\ &= \sum_{S \subseteq X \subseteq T} (-1)^{|X|-|S|} \sum_{X \subseteq Y \subseteq T} f(Y) = \sum_{S \subseteq X \subseteq T} (-1)^{|X|-|S|} g(X) \qquad \square \end{aligned}$$

We now show how the Möbius Inversion Formula can be applied to the problem of edge reconstruction. This highly effective approach was introduced by Lovász (1972c) and refined successively by Müller (1977) and Nash-Williams (1978).

The idea is to count the mappings between two simple graphs G and H on the same vertex set V according to the intersection of the image of G with H. Each such mapping is determined by a permutation σ of V, which one extends to $G = (V, E)$ by setting $\sigma(G) := (V, \sigma(E))$, where $\sigma(E) := \{\sigma(u)\sigma(v) : uv \in E\}$. For each spanning subgraph F of G, we consider the permutations of G which map the edges of F onto edges of H and the remaining edges of G onto edges of $\overline{H}$. We denote their number by $|G \to H|_F$, that is:

$$|G \to H|_F := |\{\sigma \in S_n : \ \sigma(G) \cap H = \sigma(F)\}|$$

Möbius Inversion (continued)

In particular, if $F = G$, then $|G \to H|_F$ is simply the number of embeddings of G in H, which we denote for brevity by $|G \to H|$, and if F is empty, $|G \to H|_F$ is the number of embeddings of G in the complement of H; that is, $|G \to \overline{H}|$. These concepts are illustrated in Figure 2.13 for all spanning subgraphs F of G when $G = K_1 + K_{1,2}$ and $H = 2K_2$. Observe that, for any subgraph F of G,

$$\sum_{F \subseteq X \subseteq G} |G \to H|_X = |F \to H| \tag{2.6}$$

and that

$$|F \to H| = \mathrm{aut}(F) \binom{H}{F} \tag{2.7}$$

where $\mathrm{aut}(F)$ denotes the number of automorphisms of F, because the subgraph F of G can be mapped onto each copy of F in H in $\mathrm{aut}(F)$ distinct ways.

Lemma 2.26 Nash-Williams' Lemma
Let G be a graph, F a spanning subgraph of G, and H an edge reconstruction of G that is not isomorphic to G. Then

$$|G \to G|_F - |G \to H|_F = (-1)^{e(G)-e(F)} \mathrm{aut}(G) \tag{2.8}$$

Proof By (2.6) and (2.7),

$$\sum_{F \subseteq X \subseteq G} |G \to H|_X = \mathrm{aut}(F) \binom{H}{F}$$

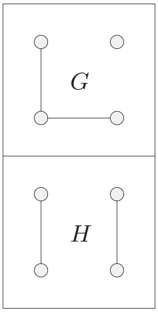
We invert this identity by applying the Möbius Inversion Formula (identifying each spanning subgraph of G with its edge set), to obtain:

$$|G \to H|_F = \sum_{F \subseteq X \subseteq G} (-1)^{e(X)-e(F)} \mathrm{aut}(X) \binom{H}{X}$$

Therefore,

$$|G \to G|_F - |G \to H|_F = \sum_{F \subseteq X \subseteq G} (-1)^{e(X)-e(F)} \mathrm{aut}(X) \left(\binom{G}{X} - \binom{H}{X} \right)$$

Because H is an edge reconstruction of G, we have $\binom{G}{X} = \binom{H}{X}$ for every proper spanning subgraph X of G, by the edge version of Kelly's Lemma (2.24). Finally, $\binom{G}{G} = 1$, whereas $\binom{H}{G} = 0$ since $e(H) = e(G)$ and $H \ncong G$. $\square$



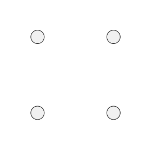
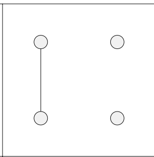
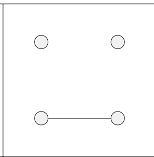
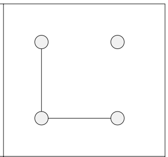
F				
$\lvert G \to G\rvert_F$	10	6	6	2
$\lvert G \to H\rvert_F$	8	8	8	0

Fig. 2.13. Counting mappings

MÖBIUS INVERSION (CONTINUED)

Theorem 2.27 *A graph G is edge reconstructible if there exists a spanning subgraph F of G such that either of the following two conditions holds.*

(i) $|G \to H|_F$ *takes the same value for all edge reconstructions H of G,*
(ii) $|F \to G| < 2^{e(G)-e(F)-1}\mathrm{aut}(G)$.

Proof Let H be an edge reconstruction of G. If condition (i) holds, the left-hand side of (2.8) is zero whereas the right-hand side is nonzero. The inequality of condition (ii) is equivalent, by (2.6), to the inequality

$$\sum_{F \subseteq X \subseteq G} |G \to G|_X < 2^{e(G)-e(F)-1}\mathrm{aut}(G)$$

But this implies that $|G \to G|_X < \mathrm{aut}(G)$ for some spanning subgraph X of G such that $e(G) - e(X)$ is even, and identity (2.8) is again violated (with $F := X$). Thus, in both cases, Nash-Williams' Lemma implies that H is isomorphic to G. □

Choosing F as the empty graph in Theorem 2.27 yields two sufficient conditions for the edge reconstructibility of a graph in terms of its edge density, due to Lovász (1972) and Müller (1977), respectively (Exercise 2.7.8).

Corollary 2.28 *A graph G is edge reconstructible if either $m > \frac{1}{2}\binom{n}{2}$ or $2^{m-1} > n!$* □

Two other applications of the Möbius Inversion Formula to graph theory are given in Exercises 2.7.17 and 14.7.12. For further examples, see Whitney (1932b). Theorem 2.25 was extended by Rota (1964) to the more general context of partially ordered sets.

It is natural to formulate corresponding conjectures for digraphs (see Harary (1964)). Tools such as Kelly's Lemma apply to digraphs as well, and one might be led to believe that the story is much the same here as for undirected graphs. Most surprisingly, this is not so. Several infinite families of nonreconstructible digraphs, and even nonreconstructible tournaments, were constructed by Stockmeyer (1981) (see Exercise 2.7.18). One such pair is shown in Figure 2.14. We leave its verification to the reader (Exercise 2.7.9).

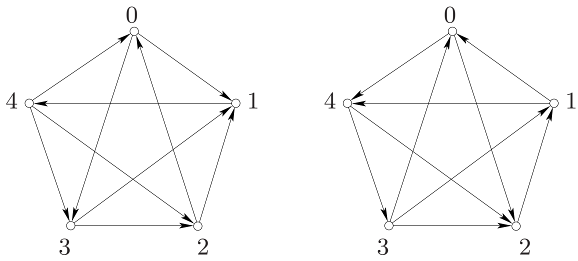


Fig. 2.14. A pair of nonreconstructible tournaments

We remark that there also exist infinite families of nonreconstructible hypergraphs (see Exercise 2.7.10 and Kocay (1987)) and nonreconstructible infinite graphs (see Exercise 4.2.10). Further information on graph reconstruction can be found in the survey articles by Babai (1995), Bondy (1991), and Ellingham (1988), and in the book by Lauri and Scapellato (2003).

Exercises

2.7.1 Find two nonisomorphic graphs on six vertices whose decks both include the first five cards displayed in Figure 2.12. (P.K. Stockmeyer)

2.7.2 Find a pair of simple graphs on two edges, and also a pair of simple graphs on three edges, which are not edge reconstructible.

2.7.3 Two dissimilar vertices u and v of a graph G are called *pseudosimilar* if the vertex-deleted subgraphs $G - u$ and $G - v$ are isomorphic.

a) Find a pair of pseudosimilar vertices in the graph of Figure 2.15.
b) Construct a tree with a pair of pseudosimilar vertices.

(F. Harary and E.M. Palmer)

2.7.4 A class $\mathcal{G}$ of graphs is *recognizable* if, for each graph $G \in \mathcal{G}$, every reconstruction of G also belongs to $\mathcal{G}$. The class $\mathcal{G}$ is *weakly reconstructible* if, for each graph $G \in \mathcal{G}$, every reconstruction of G that belongs to $\mathcal{G}$ is isomorphic to G. Show that a class of graphs is reconstructible if and only if it is both recognizable and weakly reconstructible.

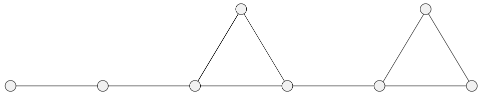


Fig. 2.15. A graph containing a pair of pseudosimilar vertices (Exercise 2.7.3)

2.7.5

a) Show that regular graphs are both recognizable and weakly reconstructible.
b) Deduce that this class of graphs is reconstructible.

2.7.6

a) Let G be a connected graph on at least two vertices, and let P be a maximal path in G, starting at x and ending at y. Show that $G - x$ and $G - y$ are connected.
b) Deduce that a graph on at least three vertices is connected if and only if at least two vertex-deleted subgraphs are connected.
c) Conclude that the class of disconnected graphs is recognizable.

2.7.7 Verify identity (2.6) for the graphs G and H of Figure 2.13, and for all spanning subgraphs F of G.

$\star$**2.7.8** Deduce Corollary 2.28 from Theorem 2.27.

2.7.9 Show that the two tournaments displayed in Figure 2.14 form a pair of nonreconstructible tournaments. (P.K. Stockmeyer)

2.7.10 Consider the hypergraphs G and H with vertex set $V := \{1, 2, 3, 4, 5\}$ and respective edge sets

$$\mathcal{F}(G) := \{123, 125, 135, 234, 345\} \quad \text{and} \quad \mathcal{F}(H) := \{123, 135, 145, 234, 235\}$$

Show that (G, H) is a nonreconstructible pair.

2.7.11 Let G be a graph, and let $\mathcal{F} := (F_1, F_2, \ldots, F_k)$ be a sequence of graphs (not necessarily distinct). A *covering* of G by $\mathcal{F}$ is a sequence $(G_1, G_2, \ldots, G_k)$ of subgraphs of G such that $G_i \cong F_i$, $1 \le i \le k$, and $\cup_{i=1}^k G_i = G$. We denote the number of coverings of G by $\mathcal{F}$ by $c(\mathcal{F}, G)$. For example, if $\mathcal{F} := (K_2, K_{1,2})$, the coverings of G by $\mathcal{F}$ for each graph G such that $c(\mathcal{F}, G) > 0$ are as indicated in Figure 2.16 (where the edge of K_2 is shown as a dotted line).

a) Show that, for any graph G and any sequence $\mathcal{F} := (F_1, F_2, \ldots, F_k)$ of graphs such that $v(F_i) < v(G)$, $1 \le i \le k$, the parameter

$$\sum_X c(\mathcal{F}, X) \binom{G}{X}$$

is reconstructible, where the sum extends over all unlabelled graphs X such that $v(X) = v(G)$. (W.L. Kocay)

b) Applying Exercise 2.7.11a to all families $\mathcal{F} := (F_1, F_2, \ldots, F_k)$ such that $\sum_{i=1}^{k} v(F_i) = v(G)$, deduce that the class of disconnected graphs is weakly reconstructible.

c) Applying Exercise 2.7.6c, conclude that this class is reconstructible.

(P.J. Kelly)

2.7.12 Let G and H be two graphs on the same vertex set V, where $|V| \geq 4$. Suppose that $G - \{x, y\} \cong H - \{x, y\}$ for all $x, y \in V$. Show that $G \cong H$.

⋆2.7.13

a) Prove the edge version of Kelly's Lemma (Lemma 2.24).
b) Using the edge version of Kelly's Lemma, show that the number of isolated vertices is edge reconstructible.
c) Deduce that the Edge Reconstruction Conjecture is valid for all graphs provided that it is valid for all graphs without isolated vertices.

2.7.14

a) By applying Exercise 2.7.11a, show that the (vertex) deck of any graph without isolated vertices is edge reconstructible.
b) Deduce from Exercise 2.7.13c that the Edge Reconstruction Conjecture is true if the Reconstruction Conjecture is true.

(D.L. Greenwell)

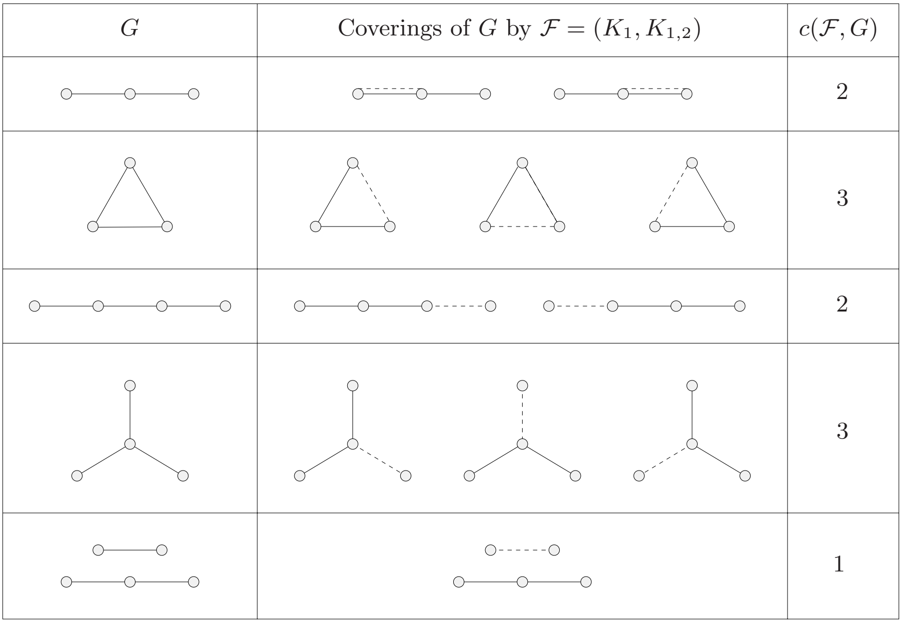


Fig. 2.16. Covering a graph by a sequence of graphs (Exercise 2.7.11)

2.7.15 Let $\{A_i : i \in T\}$ be a family of sets. For $S \subseteq T$, define $f(S) := |(\cap_{i\in S} A_i) \setminus (\cup_{i\in T\setminus S} A_i)|$ and $g(S) := |\cap_{i\in S} A_i|$, where, by convention, $\cap_{i\in\emptyset} A_i = \cup_{i\in T} A_i$.

a) Show that $g(S) = \sum_{S\subseteq X\subseteq T} f(X)$.
b) Deduce from the Möbius Inversion Formula (2.5) that

$$\sum_{\emptyset\subseteq X\subseteq T} (-1)^{|X|} |\cap_{i\in X} A_i| = 0.$$

c) Show that this identity is equivalent to the Inclusion–Exclusion Formula (2.3).

2.7.16 Use the Binomial Theorem to establish the Inclusion-Exclusion Formula (2.3) directly, without appealing to Möbius Inversion.

2.7.17 Consider the lower-triangular matrix $\mathbf{A}_n$ whose rows and columns are indexed by the isomorphism types of the graphs on n vertices, listed in increasing order of size, and whose (X, Y) entry is $\binom{X}{Y}$.

a) Compute $\mathbf{A}_3$ and $\mathbf{A}_4$.
b) For $k \in \mathbb{Z}$, show that the (X, Y) entry of $(\mathbf{A}_n)^k$ is $k^{e(X)-e(Y)}\binom{X}{Y}$.

(X. Buchwalder)

2.7.18 Consider the Stockmeyer tournament ST_n, defined in Exercise 1.5.11.

a) Show that each vertex-deleted subgraph of ST_n is self-converse.
b) Denote by $odd(ST_n)$ and $even(ST_n)$ the subtournaments of ST_n induced by its odd and even vertices, respectively. For $n \geq 1$, show that $odd(ST_n) \cong ST_{n-1} \cong even(ST_n)$.
c) Deduce, by induction on n, that $ST_n - k \cong ST_n - (2^n - k + 1)$ for all $k \in V(ST_n)$.

(W. Kocay)

d) Consider the following two tournaments obtained from ST_n by adding a new vertex 0. In one of these tournaments, 0 dominates the odd vertices and is dominated by the even vertices; in the other, 0 dominates the even vertices and is dominated by the odd vertices. Show that these two tournaments on $2^n + 1$ vertices form a pair of nonreconstructible digraphs.

(P.K. Stockmeyer)

2.7.19 To *switch* a vertex of a simple graph is to exchange its sets of neighbours and non-neighbours. The graph so obtained is called a *switching* of the graph. The collection of switchings of a graph G is called the (switching) *deck* of G. A graph is *switching-reconstructible* if every graph with the same deck as G is isomorphic to G.

a) Find four pairs of graphs on four vertices which are not switching-reconstruct -ible.
b) Let G be a graph with n odd. Consider the collection $\mathcal{G}$ consisting of the n^2 graphs in the decks of the graphs which comprise the deck of G.
   i) Show that G is the only graph which occurs an odd number of times in $\mathcal{G}$.
   ii) Deduce that G is switching-reconstructible.
c) Let G be a graph with $n \equiv 2 \pmod 4$. Show that G is switching-reconstructible.

(R.P. Stanley; N. Alon)

2.8 Related Reading

Path and Cycle Decompositions

Veblen's Theorem (2.7) tells us that every even graph can be decomposed into cycles, but it says nothing about the number of cycles in the decomposition. One may ask how many or how few cycles there can be in a cycle decomposition of a given even graph. These questions are not too hard to answer in special cases, such as when the graph is complete (see Exercises 2.4.4 and 2.4.5a). Some forty years ago, G. Hajós conjectured that *every simple even graph on n vertices admits a decomposition into at most* $(n-1)/2$ *cycles* (see Lovász (1968b)). Surprisingly little progress has been made on this simply stated problem. An analogous conjecture on path decompositions was proposed by T. Gallai at about the same time (see Lovász (1968b)), namely that *every simple connected graph on n vertices admits a decomposition into at most* $(n+1)/2$ *paths*. This bound is sharp if all the degrees are odd, because in any path decomposition each vertex must be an end of at least one path. Lovász (1968b) established the truth of Gallai's conjecture in this case (see also Donald (1980)).

Legitimate Decks

In the Reconstruction Conjecture (2.19), the deck of vertex-deleted subgraphs of a graph is supplied, the goal being to determine the graph. A natural problem, arguably even more fundamental, is to characterize such decks. A family $\mathcal{G} := \{G_1, G_2, \ldots, G_n\}$ of n graphs, each of order $n-1$, is called a *legitimate deck* if there is at least one graph G with vertex set $\{v_1, v_2, \ldots, v_n\}$ such that $G_i \cong G - v_i$, $1 \le i \le n$. The *Legitimate Deck Problem* asks for a characterization of legitimate decks. This problem was raised by Harary (1964). It was shown by Harary et al. (1982) and Mansfield (1982) that the problem of recognizing whether a deck is legitimate is as hard (in a sense to be discussed in Chapter 8) as that of deciding whether two graphs are isomorphic.

The various counting arguments deployed to attack the Reconstruction Conjecture provide natural necessary conditions for legitimacy. For instance, the proof of Kelly's Lemma (2.20) tells us that if $\mathcal{G}$ is the deck of a graph G, then $\binom{G}{F} = \sum_{i=1}^{n} \binom{G_i}{F}/(n - v(F))$ for every graph F on fewer than n vertices. Because the left-hand side is an integer, $\sum_{i=1}^{n} \binom{G_i}{F}$ must be a multiple of $n - v(F)$. It is not hard to come up with an illegitimate deck which passes this test. Indeed, next to nothing is known on the Legitimate Deck Problem. A more general problem would be to characterize, for a fixed integer k, the vectors $(\binom{G}{F} : v(F) = k)$, where G ranges over all graphs on n vertices. Although trivial for $k = 2$, the problem is unsolved already for $k = 3$ and appears to be very hard. Even determining the minimum number of triangles in a graph on n vertices with a specified number of edges is a major challenge (see Razborov (2006), where a complex asymptotic formula, derived by highly nontrivial methods, is given).

Ultrahomogeneous Graphs

A simple graph is said to be *k-ultrahomogeneous* if any isomorphism between two of its isomorphic induced subgraphs on k or fewer vertices can be extended to an automorphism of the entire graph. It follows directly from the definition that every graph is 0-ultrahomogeneous, that 1-ultrahomogeneous graphs are the same as vertex-transitive graphs, and that complements of k-ultrahomogeneous graphs are k-ultrahomogeneous.

Cameron (1980) showed that any graph which is 5-ultrahomogeneous is k-ultrahomogeneous for all k. Thus it is of interest to classify the k-ultrahomogeneous graphs for $1 \leq k \leq 5$. The 5-ultrahomogeneous graphs were completely described by Gardiner (1976). They are the self-complementary graphs C_5 and $L(K_{3,3})$, and the Turán graphs $T_{k,rk}$, for all $k \geq 1$ and $r \geq 1$, as well as their complements. These graphs all have rather simple structures. There is, however, a remarkable 4-ultrahomogeneous graph. It arises from a very special geometric configuration, discovered by Schläfli (1858), consisting of twenty-seven lines on a cubic surface, and is known as the *Schläfli graph*. Here is a description due to Chudnovsky and Seymour (2005).

The vertex set of the graph is $\mathbb{Z}_3^3$, two distinct vertices (a, b, c) and (a', b', c') being joined by an edge if $a' = a$ and either $b' = b$ or $c' = c$, or if $a' = a + 1$ and $b' \neq c$. This construction results in a 16-regular graph on twenty-seven vertices. The subgraph induced by the sixteen neighbours of a vertex of the Schläfli graph is isomorphic to the complement of the *Clebsch graph*, shown in Figure 12.9. In turn, the subgraph induced by the neighbour set of a vertex of the complement of the Clebsch graph is isomorphic to the complement of the Petersen graph. Thus, one may conclude that the Clebsch graph is 3-ultrahomogeneous and that the Petersen graph is 2-ultrahomogeneous. By employing the classification theorem for finite simple groups, Buczak (1980) showed that the the Schläfli graph and its complement are the only two graphs which are 4-ultrahomogeneous without being 5-ultrahomogeneous.

The notion of ultrahomogeneity may be extended to infinite graphs. The countable random graph G described in Exercise 13.2.18 has the property that if F and F' are isomorphic induced subgraphs of G, then any isomorphism between F and F' can be extended to an automorphism of G. Further information about ultrahomogeneous graphs may be found in Cameron (1983) and Devillers (2002).