14

Vertex Colourings

Contents

14.1 Chromatic Number

Recall that a *k-vertex-colouring*, or simply a *k-colouring*, of a graph $G = (V, E)$ is a mapping $c : V \to S$, where S is a set of k *colours*; thus, a k-colouring is an assignment of k colours to the vertices of G. Usually, the set S of colours is taken to be $\{1, 2, \ldots, k\}$. A colouring c is *proper* if no two adjacent vertices are assigned the same colour. Only loopless graphs admit proper colourings.

Alternatively, a k-colouring may be viewed as a partition $\{V_1, V_2, \ldots, V_k\}$ of V, where V_i denotes the (possibly empty) set of vertices assigned colour i. The sets

V_i are called the *colour classes* of the colouring. A proper k-colouring is then a k-colouring in which each colour class is a stable set. In this chapter, we are only concerned with proper colourings. It is convenient, therefore, to refer to a proper colouring as a 'colouring' and to a proper k-colouring as a 'k-colouring'.

A graph is *k-colourable* if it has a k-colouring. Thus a graph is 1-colourable if and only if it is empty, and 2-colourable if and only if it is bipartite. Clearly, a loopless graph is k-colourable if and only if its underlying simple graph is k-colourable. Therefore, in discussing vertex colourings, we restrict our attention to simple graphs.

The minimum k for which a graph G is k-colourable is called its *chromatic number*, and denoted $\chi(G)$. If $\chi(G) = k$, the graph G is said to be *k-chromatic*. The triangle, and indeed all odd cycles, are easily seen to be 3-colourable. On the other hand, they are not 2-colourable because they are not bipartite. They therefore have chromatic number three: they are 3-chromatic. A 4-chromatic graph known as the *Hajós graph* is shown in Figure 14.1. The complete graph K_n has chromatic number n because no two vertices can receive the same colour. More generally, every graph G satisfies the inequality

$$\chi \geq \frac{n}{\alpha} \tag{14.1}$$

because each colour class is a stable set, and therefore has at most α vertices.

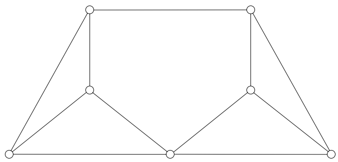

Fig. 14.1. The Hajós graph: a 4-chromatic graph

Colouring problems arise naturally in many practical situations where it is required to partition a set of objects into groups in such a way that the members of each group are mutually compatible according to some criterion. We give two examples of such problems. Others will no doubt occur to the reader.

Example 14.1 EXAMINATION SCHEDULING
The students at a certain university have annual examinations in all the courses they take. Naturally, examinations in different courses cannot be held concurrently if the courses have students in common. How can all the examinations be organized in as few parallel sessions as possible? To find such a schedule, consider the graph G whose vertex set is the set of all courses, two courses being joined by an edge if they give rise to a conflict. Clearly, stable sets of G correspond to conflict-free groups of courses. Thus the required minimum number of parallel sessions is the chromatic number of G.

Example 14.2 CHEMICAL STORAGE
A company manufactures n chemicals $C_1, C_2, \ldots, C_n$. Certain pairs of these chemicals are incompatible and would cause explosions if brought into contact with each other. As a precautionary measure, the company wishes to divide its warehouse into compartments, and store incompatible chemicals in different compartments. What is the least number of compartments into which the warehouse should be partitioned? We obtain a graph G on the vertex set $\{v_1, v_2, \ldots, v_n\}$ by joining two vertices v_i and v_j if and only if the chemicals C_i and C_j are incompatible. It is easy to see that the least number of compartments into which the warehouse should be partitioned is equal to the chromatic number of G.

If H is a subgraph of G and G is k-colourable, then so is H. Thus $\chi(G) \geq \chi(H)$. In particular, if G contains a copy of the complete graph K_r, then $\chi(G) \geq r$. Therefore, for any graph G,

$$\chi \geq \omega \tag{14.2}$$

The odd cycles of length five or more, for which $\omega = 2$ and $\chi = 3$, show that this bound for the chromatic number is not sharp. More surprisingly, as we show in Section 14.3, there exist graphs with arbitrarily high girth and chromatic number.

A GREEDY COLOURING HEURISTIC

Because a graph is 2-colourable if and only if it is bipartite, there is a polynomial-time algorithm (for instance, using breadth-first search) for deciding whether a given graph is 2-colourable. In sharp contrast, the problem of 3-colourability is already $\mathcal{NP}$-complete. It follows that the problem of finding the chromatic number of a graph is $\mathcal{NP}$-hard. In practical situations, one must therefore be content with efficient heuristic procedures which perform reasonably well. The most natural approach is to colour the vertices in a greedy fashion, as follows.

Heuristic 14.3 THE GREEDY COLOURING HEURISTIC

INPUT: a graph G
OUTPUT: a colouring of G

1. *Arrange the vertices of G in a linear order: $v_1, v_2, \ldots, v_n$.*
2. *Colour the vertices one by one in this order, assigning to v_i the smallest positive integer not yet assigned to one of its already-coloured neighbours.*

It should be stressed that the number of colours used by this greedy colouring heuristic depends very much on the particular ordering chosen for the vertices. For example, if $K_{n,n}$ is a complete bipartite graph with parts $X := \{x_1, x_2, \ldots, x_n\}$ and $Y := \{y_1, y_2, \ldots, y_n\}$, then the bipartite graph $G[X, Y]$ obtained from this graph by deleting the perfect matching $\{x_i y_i : 1 \leq i \leq n\}$ would require n colours if the vertices were listed in the order $x_1, y_1, x_2, y_2, \ldots, x_n, y_n$. On the other hand, only two colours would be needed if the vertices were presented in the order $\{x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n\}$; indeed there is always an ordering which yields an optimal colouring (Exercise 14.1.9). The problem is that it is hard to know in advance which orderings will produce optimal colourings.

Nevertheless, the number of colours used by the greedy heuristic is never greater than $\Delta + 1$, regardless of the order in which the vertices are presented. When a vertex v is about to be coloured, the number of its neighbours already coloured is clearly no greater than its degree $d(v)$, and this is no greater than the maximum degree, Δ. Thus one of the colours $1, 2, \ldots, \Delta + 1$ will certainly be available for v. We conclude that, for any graph G,

$$\chi \leq \Delta + 1 \tag{14.3}$$

In other words, every k-chromatic graph has a vertex of degree at least $k - 1$. In fact, every k-chromatic graph has at least k vertices of degree at least $k - 1$ (Exercise 14.1.3b).

The bound (14.3) on the chromatic number gives essentially no information on how many vertices of each colour there are in a $(\Delta + 1)$-colouring. A far-reaching strengthening of inequality (14.3) was obtained by Hajnal and Szemerédi (1970), who showed that every graph G admits a balanced $(\Delta + 1)$-colouring, that is, one in which the numbers of vertices of each colour differ by at most one. A shorter proof of this theorem was found by Kierstead and Kostochka (2006).

Brooks' Theorem

Although the bound (14.3) on the chromatic number is best possible, being attained by odd cycles and complete graphs, Brooks (1941) showed that these are the only connected graphs for which equality holds.

Our proof of Brooks' Theorem is similar in spirit to one given by Lovász (1975b), but makes essential use of DFS-trees. In particular, we appeal to a result of Chartrand and Kronk (1968), who showed that cycles, complete graphs, and complete bipartite graphs whose parts are of equal size are the only graphs with the property that every DFS-tree is a Hamilton path rooted at one of its ends (see Exercise 6.1.10).

Theorem 14.4 Brooks' Theorem
If G is a connected graph, and is neither an odd cycle nor a complete graph, then $\chi \leq \Delta$.

Proof Suppose first that G is not regular. Let x be a vertex of degree δ and let T be a search tree of G rooted at x. We colour the vertices with the colours $1, 2, \ldots, \Delta$ according to the greedy heuristic, selecting at each step a leaf of the subtree of T induced by the vertices not yet coloured, assigning to it the smallest available colour, and ending with the root x of T. When a vertex v different from x is about to be coloured, it is adjacent in T to at least one uncoloured vertex, and so is adjacent in G to at most $d(v) - 1 \leq \Delta - 1$ coloured vertices. It is therefore assigned one of the colours $1, 2, \ldots, \Delta$. Finally, when x is coloured, it, too, is assigned one of the colours $1, 2, \ldots, \Delta$, because $d(x) = \delta \le \Delta - 1$. The greedy heuristic therefore produces a Δ-colouring of G.

Suppose now that G is regular. If G has a cut vertex x, then $G = G_1 \cup G_2$, where G_1 and G_2 are connected and $G_1 \cap G_2 = \{x\}$. Because the degree of x in G_i is less than $\Delta(G)$, neither subgraph G_i is regular, so $\chi(G_i) \le \Delta(G_i) = \Delta(G)$, $i = 1, 2$, and $\chi(G) = \max\{\chi(G_1), \chi(G_2)\} \le \Delta(G)$ (Exercise 14.1.2). We may assume, therefore, that G is 2-connected.

If every depth-first search tree of G is a Hamilton path rooted at one of its ends, then G is a cycle, a complete graph, or a complete bipartite graph $K_{n,n}$ (Exercise 6.1.10). Since, by hypothesis, G is neither an odd cycle nor a complete graph, $\chi(G) = 2 \le \Delta(G)$.

Suppose, then, that T is a depth-first search tree of G, but not a path. Let x be a vertex of T with at least two children, y and z. Because G is 2-connected, both $G - y$ and $G - z$ are connected. Thus y and z are either leaves of T or have proper descendants which are joined to ancestors of x. It follows that $G' := G - \{y, z\}$ is connected. Consider a search tree T' with root x in G'. By colouring y and z with colour 1, and then the vertices of T' by the greedy heuristic as above, ending with the root x, we obtain a Δ-colouring of G. □

Colourings of Digraphs

A (proper) *vertex colouring* of a digraph D is simply a vertex colouring of its underlying graph G, and its *chromatic number* $\chi(D)$ is defined to be the chromatic number $\chi(G)$ of G. Why, then, consider colourings of digraphs? It turns out that the chromatic number of a digraph provides interesting information about its subdigraphs. The following theorem of Gallai (1968a) and Roy (1967) tells us that digraphs with high chromatic number always have long directed paths. It can be viewed as a common generalization of a theorem about chains in posets (see Exercise 2.1.23) and Rédei's Theorem on directed Hamilton paths in tournaments (Theorem 2.3).

Theorem 14.5 The Gallai–Roy Theorem
Every digraph D contains a directed path with χ vertices.

Proof Let k be the number of vertices in a longest directed path of D. Consider a maximal acyclic subdigraph D' of D. Because D' is a subdigraph of D, each directed path in D' has at most k vertices. We k-colour D by assigning to vertex v the colour $c(v)$, where $c(v)$ is the number of vertices of a longest directed path in D' starting at v. Let us show that this colouring is proper.

Consider any arc (u, v) of D. If (u, v) is an arc of D', let vPw be a longest directed v-path in D'. Then $u \notin V(P)$, otherwise $vPuv$ would be a directed cycle in D'. Thus $uvPw$ is a directed u-path in D', implying that $c(u) > c(v)$.

If (u, v) is not an arc of D', then $D' + (u, v)$ contains a directed cycle, because the subdigraph D' is maximally acyclic, so D' contains a directed (v, u)-path P. Let Q be a longest directed u-path in D'. Because D' is acyclic, $V(P) \cap V(Q) = \{u\}$.

Thus PQ is a directed v-path in D', implying that $c(v) > c(u)$. In both cases, $c(u) \neq c(v)$. □

Exercises

14.1.1 Chvátal Graph
The *Chvátal graph*, shown in Figure 14.2, is a 4-regular graph of girth four on twelve vertices. Show that this graph is 4-chromatic. (V. Chvátal)

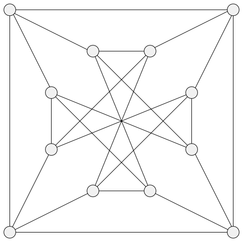

Fig. 14.2. The Chvátal graph: a 4-chromatic 4-regular graph of girth four

$\star$**14.1.2** Show that $\chi(G) = \max\{\chi(B) : B \text{ a block of } G\}$.

$\star$**14.1.3**

a) In a k-colouring of a k-chromatic graph, show that there is a vertex of each colour which is adjacent to vertices of every other colour.
b) Deduce that every k-chromatic graph has at least k vertices of degree at least $k-1$.

14.1.4 Show that $\chi(G) \leq k_1k_2$ if and only if $G = G_1 + G_2$, where $\chi(G_i) \leq k_i$, $i = 1, 2$. (S.A. Burr)

14.1.5 k-Degenerate Graph
A graph is k-*degenerate* if it can be reduced to K_1 by repeatedly deleting vertices of degree at most k.

a) Show that a graph is k-degenerate if and only if every subgraph has a vertex of degree at most k.
b) Characterize the 1-degenerate graphs.
c) Show that every k-degenerate graph is $(k+1)$-colourable.
d) Using Exercise 14.1.4, deduce that the union of a k-degenerate graph and an ℓ-degenerate graph is $(k+1)(\ell+1)$-colourable.

14.1.6 Establish the following bounds on the chromatic number of the Kneser graph $KG_{m,n}$.

$$\frac{n}{m} \leq \chi(KG_{m,n}) \leq n - 2m + 2$$

(Lovász (1978) proved the conjecture of Kneser (1955) that the upper bound is sharp; see, also, Bárány (1978) and Greene (2002).)

14.1.7 Show that, for any graph G, $\chi \geq n^2/(n^2 - 2m)$.

14.1.8 Let G be a graph in which any two odd cycles intersect. Show that:

a) $\chi \leq 5$,
b) if $\chi = 5$, then G contains a copy of K_5.

14.1.9 Given any graph G, show that there is an ordering of its vertices such that the greedy heuristic, applied to that ordering, yields a colouring with χ colours.

14.1.10 Let G have degree sequence $(d_1, d_2, \ldots, d_n)$, where $d_1 \geq d_2 \geq \cdots \geq d_n$.

a) Using a greedy heuristic, show that $\chi \leq \max\{\min\{d_i + 1, i\} : 1 \leq i \leq n\}$.
b) Deduce that $\chi \leq \lceil (2m)^{1/2} \rceil$. (D.J.A. Welsh and M.B. Powell)

14.1.11

a) Show that $\chi(G)\chi(\overline{G}) \geq n$.
b) Using Exercise 14.1.10, deduce that $2\sqrt{n} \leq \chi(G) + \chi(\overline{G}) \leq n + 1$.
(E.A. Nordhaus and J.W. Gaddum)

14.1.12 Let k be a positive integer, and let G be a graph which contains no cycle of length $1 \pmod{k}$. Show that G is k-colourable. (Zs. Tuza)

14.1.13 Catlin Graph
The *composition* $G[H]$ was defined in Exercise 12.3.9.

a) Show that $\chi(G[H]) \leq \chi(G)\chi(H)$, for any two graphs G and H.
b) The graph $C_5[K_3]$ shown in Figure 14.3 is known as the *Catlin graph*. Show that $\chi(C_5[K_3]) < \chi(C_5)\chi(K_3)$. (P. Catlin)

14.1.14 Let G be the graph $C_5[K_n]$.

a) Show that $\chi = \lceil \frac{5n}{2} \rceil$.
b) Deduce that $\chi = \lceil (\omega + \Delta + 1)/2 \rceil$. (A. Kostochka)

$\star$**14.1.15**

a) Show that every graph G has an orientation each of whose induced subdigraphs has a kernel.
b) Consider any such orientation D. Show that G is $(\Delta^+(D) + 1)$-colourable.
c) Deduce inequality (14.3).

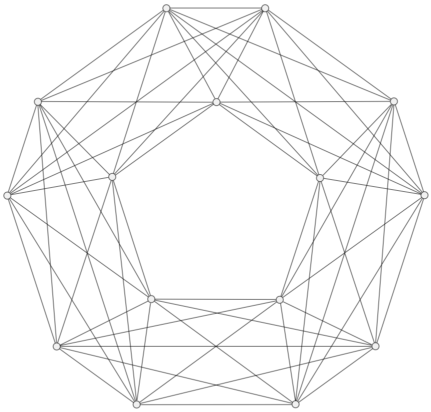

Fig. 14.3. The Catlin graph $C_5[K_3]$

14.1.16 The Erdős–Szekeres Theorem

a) Let D be a digraph with $\chi \geq kl + 1$, and let f be a real-valued function defined on V. Show that D contains either a directed path $(u_0, u_1, \ldots, u_k)$ with $f(u_0) \leq f(u_1) \leq \cdots \leq f(u_k)$ or a directed path $(v_0, v_1, \ldots, v_l)$ with $f(v_0) > f(v_1) > \cdots > f(v_l)$. (V. Chvátal and J. Komlós)
b) Deduce that any sequence of $kl+1$ distinct integers contains either an increasing subsequence of $k+1$ terms or a decreasing sequence of $l+1$ terms. (P. Erdős and G. Szekeres)

14.1.17 Let G be an undirected graph. Show that

$$\chi(G) = \min \{\lambda(D) : D \text{ an orientation of } G\}$$

where $\lambda(D)$ denotes the number of vertices in a longest directed path of D.

14.1.18 Weak Product
The *weak product* of graphs G and H is the graph $G \times H$ with vertex set $V(G) \times V(H)$ and edge set $\{((u, u'), (v, v')) : (u, v) \in E(G), (u', v') \in E(H)\}$. Show that, for any two graphs G and H, $\chi(G \times H) \leq \min\{\chi(G), \chi(H)\}$. (S. Hedetniemi)

14.1.19 Chromatic Number of a Hypergraph
The *chromatic number* $\chi(H)$ of a hypergraph $H := (V, \mathcal{F})$ is the least number of colours needed to colour its vertices so that no edge of cardinality more than one is monochromatic. (This is one of several ways of defining the chromatic number of a hypergraph; it is often referred to as the *weak chromatic number*.) Determine the chromatic number of:

a) the Fano hypergraph (Figure 1.15a),
b) the Desargues hypergraph (Figure 1.15b).

14.1.20

a) Show that the Hajós graph (Figure 14.1) is a unit-distance graph.
(P. O'Donnell has shown that there exists a 4-chromatic unit-distance graph of arbitrary girth.)
b) Let G be a unit-distance graph. Show that $\chi \leq 7$ by considering a plane hexagonal lattice and finding a suitable 7-face colouring of it.

14.1.21 Show that:

a) if $\chi(G) = 2k$, then G has a bipartite subgraph with at least $mk/(2k-1)$ edges,
b) if $\chi(G) = 2k+1$, then G has a bipartite subgraph with at least $m(k+1)/(2k+1)$ edges. (L.D. Andersen, D. Grant and N. Linial)

14.1.22 Let $G := (V, E)$ be a graph, and let $f(G)$ be the number of proper k-colourings of G. By applying the inequality of Exercise 13.2.3, show that

$$k^n\left(1 - \frac{m}{k}\right) \leq f(G) \leq k^n\left(1 - \frac{m}{k+m-1}\right)$$

14.1.23 Let G be a 5-regular graph on $4k$ vertices, the union of a Hamilton cycle C and k disjoint copies $G_1, G_2, \ldots, G_k$ of K_4. Let F and F' be the two 1-factors of G contained in C, and let F_i be a 1-factor of G_i, $1 \leq i \leq k$. By combining a 2-vertex colouring of $F \cup_i F_i$ with a 2-vertex colouring of $F' \cup_i F'_i$, where F'_i is an appropriately chosen 1-factor of G_i, $1 \leq i \leq k$, deduce that $\chi(G) = 4$.
(N. Alon)

14.1.24 Let G be a 3-chromatic graph on n vertices. Show how to find, in polynomial time, a proper colouring of G using no more than $3\sqrt{n}$ colours.
(A. Wigderson)
(Blum and Karger (1997) have described a polynomial-time algorithm for colouring a 3-chromatic graph on n vertices using $O(n^{3/14})$ colours.)

14.1.25 Let G be a simple connected claw-free graph with $\alpha \geq 3$.

a) Show that $\Delta \leq 4(\omega - 1)$ by induction on n, proceeding as follows.
   - ▷ If G is separable, apply induction.
   - ▷ If G is 2-connected, let x be a vertex of degree Δ and set $X := N(x) \cup \{x\}$. Show that $\alpha(G[X]) = 2$. Deduce that $Y := V \setminus X \neq \emptyset$.
   - ▷ If $\alpha(G - v) \geq 3$ for some $v \in Y$, apply induction.
   - ▷ If $\alpha(G - v) = 2$ for all $v \in Y$, show that Y consists either of a single vertex or of two nonadjacent vertices.
   - ▷ Show that, in the former case, $N(x)$ is the union of four cliques, and in the latter case, the union of two cliques.
   - ▷ Conclude.

b) Deduce that $\chi \leq 4(\omega - 1)$. (M. Chudnovsky and P.D. Seymour)

(Chudnovsky and Seymour have in fact shown that $\chi \leq 2\omega$.)

14.1.26

a) Show that every digraph D contains a spanning branching forest F in which the sets of vertices at each level are stable sets of D (the vertices at level zero being the roots of the components of F).
b) Deduce the Gallai–Roy Theorem (14.5).
c) A (k, l)-*path* is an oriented path of length $k + l$ obtained by identifying the terminal vertices of a directed path of length k and a directed path of length l. Let D be a digraph and let k and l be positive integers such that $k + l = \chi$. Deduce from (a) that D contains either a $(k, l-1)$-path or a $(k-1, l)$-path.

(A. El-Sahili and M. Kouider)

14.1.27 Let k be a positive integer. Show that every infinite k-chromatic graph contains a finite k-chromatic subgraph. (N.G. de Bruijn and P. Erdős)

14.2 Critical Graphs

When dealing with colourings, it is helpful to study the properties of a special class of graphs called colour-critical graphs. We say that a graph G is *colour-critical* if $\chi(H) < \chi(G)$ for every proper subgraph H of G. Such graphs were first investigated by Dirac (1951). Here, for simplicity, we abbreviate the term 'colour-critical' to 'critical'. A k-*critical* graph is one that is k-chromatic and critical. Note that a minimal k-chromatic subgraph of a k-chromatic graph is k-critical, so every k-chromatic graph has a k-critical subgraph. The *Grötzsch graph*, a 4-critical graph discovered independently by Grötzsch (1958/1959) and, independently by Mycielski (1955), is shown in Figure 14.4 (see Exercise 14.3.1).

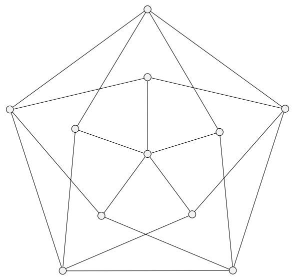

Fig. 14.4. The Grötzsch graph: a 4-critical graph

Theorem 14.6 *If G is k-critical, then $\delta \geq k - 1$.*

Proof By contradiction. Let G be a k-critical graph with $\delta < k-1$, and let v be a vertex of degree δ in G. Because G is k-critical, $G-v$ is $(k-1)$-colourable. Let $\{V_1, V_2, \ldots, V_{k-1}\}$ be a $(k-1)$-colouring of $G-v$. The vertex v is adjacent to $\delta < k-1$ vertices. It therefore must be nonadjacent in G to every vertex in some V_j. But then $\{V_1, V_2, \ldots, V_j \cup \{v\}, \ldots, V_{k-1}\}$ is a $(k-1)$-colouring of G, a contradiction. Thus $\delta \geq k-1$. □

Theorem 14.6 implies that every k-chromatic graph has at least k vertices of degree at least $k-1$, as noted already in Section 14.1.

Let S be a vertex cut of a connected graph G, and let the components of $G-S$ have vertex sets $V_1, V_2, \ldots, V_t$. Recall that the subgraphs $G_i := G[V_i \cup S]$ are the *S-components* of G. We say that colourings of $G_1, G_2, \ldots, G_t$ *agree* on S if, for every $v \in S$, vertex v is assigned the same colour in each of the colourings.

Theorem 14.7 *No critical graph has a clique cut.*

Proof By contradiction. Let G be a k-critical graph. Suppose that G has a clique cut S. Denote the S-components of G by $G_1, G_2, \ldots, G_t$. Because G is k-critical, each G_i is $(k-1)$-colourable. Furthermore, because S is a clique, the vertices of S receive distinct colours in any $(k-1)$-colouring of G_i. It follows that there are $(k-1)$-colourings of $G_1, G_2, \ldots, G_t$ which agree on S. These colourings may be combined to yield a $(k-1)$-colouring of G, a contradiction. □

Corollary 14.8 *Every critical graph is nonseparable.* □

By Theorem 14.7, if a k-critical graph has a 2-vertex cut $\{u, v\}$, then u and v cannot be adjacent. We say that a $\{u, v\}$-component G_i of G is of *type 1* if every $(k-1)$-colouring of G_i assigns the same colour to u and v, and of *type 2* if every $(k-1)$-colouring of G_i assigns distinct colours to u and v. Figure 14.5 depicts the $\{u, v\}$-components of the Hajós graph with respect to a 2-vertex cut $\{u, v\}$. Observe that there are just two $\{u, v\}$-components, one of each type. Dirac (1953) showed that this is always so in critical graphs.

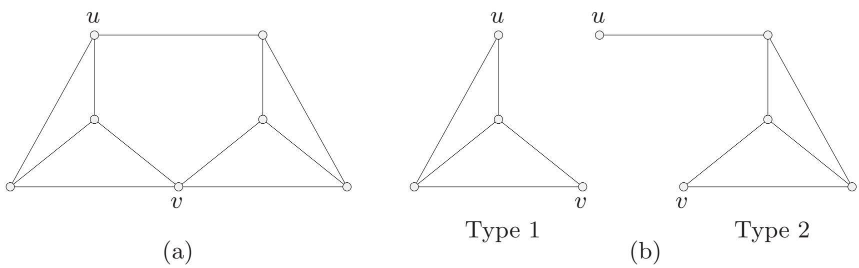

Fig. 14.5. (a) A 2-vertex cut $\{u, v\}$ of the Hajós graph, (b) its two $\{u, v\}$-components

Theorem 14.9 *Let G be a k-critical graph with a 2-vertex cut $\{u, v\}$, and let e be a new edge joining u and v. Then:*

1. *$G = G_1 \cup G_2$, where G_i is a $\{u, v\}$-component of G of type i, $i = 1, 2$,*
2. *both $H_1 := G_1 + e$ and $H_2 := G_2 / \{u, v\}$ are k-critical.*

Proof
1. Because G is critical, each $\{u, v\}$-component of G is $(k-1)$-colourable. Now there cannot exist $(k-1)$-colourings of these $\{u, v\}$-components all of which agree on $\{u, v\}$, as such colourings would together yield a $(k-1)$-colouring of G. Therefore there are two $\{u, v\}$-components G_1 and G_2 such that no $(k-1)$-colouring of G_1 agrees with any $(k-1)$-colouring of G_2. Clearly one, say G_1, must be of type 1, and the other, G_2, of type 2. Because G_1 and G_2 are of different types, the subgraph $G_1 \cup G_2$ of G is not $(k-1)$-colourable. The graph G being critical, we deduce that $G = G_1 \cup G_2$.
2. Because G_1 is of type 1, H_1 is k-chromatic. We prove that H_1 is critical by showing that, for every edge f of H_1, the subgraph $H_1 \setminus f$ is $(k-1)$-colourable. This is clearly so if $f = e$, since in this case $H_1 \setminus e = G_1$. Let f be some other edge of H_1. In any $(k-1)$-colouring of $G \setminus f$, the vertices u and v receive different colours, because G_2 is a subgraph of $G \setminus f$. The restriction of such a colouring to the vertices of G_1 is a $(k-1)$-colouring of $H_1 \setminus f$. Thus H_1 is k-critical. An analogous argument shows that H_2 is k-critical. □

Exercises

14.2.1 Show that $\chi(G) \leq 1 + \max\{\delta(F) : F \subseteq G\}$.

14.2.2 Show that the only 1-critical graph is K_1, the only 2-critical graph is K_2, and the only 3-critical graphs are the odd cycles of length three or more.

14.2.3 Show that the Chvátal graph (Figure 14.2) is 4-critical.

14.2.4 Let G be the 4-regular graph derived from the cartesian product of a triangle $x_1x_2x_3x_1$ and a path $y_1y_2y_3y_4y_5$ by identifying the vertices (x_1, y_1) and (x_1, y_5), (x_2, y_1) and (x_3, y_5), and (x_3, y_1) and (x_2, y_5). Show that G is 4-critical.

(T. Gallai)

14.2.5 Let $G = \mathrm{CG}(\mathbb{Z}_n, S)$ be a circulant, where $n \equiv 1 \pmod 3$, $|S| = k$, $1 \in S$, and $i \equiv 2 \pmod 3$ for all $i \in S$, $i \neq 1$. Show that G is a 4-critical k-regular k-connected graph. (L.S. Melnikov)

14.2.6 Uniquely Colourable Graph
A k-chromatic graph G is *uniquely k-colourable*, or simply *uniquely colourable*, if any two k-colourings of G induce the same partition of V.

a) Determine the uniquely 2-colourable graphs.

b) Generalize Theorem 14.7 by showing that no vertex cut of a critical graph induces a uniquely colourable subgraph.

14.2.7

a) Show that if u and v are two vertices of a critical graph G, then $N(u) \not\subseteq N(v)$.
b) Deduce that no k-critical graph has exactly $k+1$ vertices.

14.2.8 Show that:

a) $\chi(G_1 \vee G_2) = \chi(G_1) + \chi(G_2)$,
b) $G_1 \vee G_2$ is critical if and only if both G_1 and G_2 are critical.

14.2.9 Hajós Join
Let G_1 and G_2 be disjoint graphs, and let $e_1 := u_1v_1$ and $e_2 := u_2v_2$ be edges of G_1 and G_2, respectively. The graph obtained from G_1 and G_2 by identifying u_1 and u_2, deleting e_1 and e_2, and adding a new edge v_1v_2 is called a *Hajós join* of G_1 and G_2. Show that the Hajós join of two graphs is k-critical if and only if both graphs are k-critical. (G. Hajós)

14.2.10 For $n = 4$ and all $n \geq 6$, construct a 4-critical graph on n vertices.

14.2.11 Schrijver Graph
Let $S := \{1, 2, \ldots, n\}$. The *Schrijver graph* $SG_{m,n}$ is the subgraph of the Kneser graph $KG_{m,n}$ induced by the m-subsets of S which contain no two consecutive elements in the cyclic order $(1, 2, \ldots, n, 1)$.

a) Draw the Schrijver graph $SG_{3,8}$.
b) Show that this graph is 4-chromatic, whereas every vertex-deleted subgraph of it is 3-chromatic.

(Schrijver (1978) has shown that $SG_{m,n}$ is $(n-2m+2)$-chromatic, and that every vertex-deleted subgraph of it is $(n-2m+1)$-chromatic.)

14.2.12

a) Let G be a k-critical graph with a 2-vertex cut $\{u, v\}$. Show that $d(u)+d(v) \geq 3k-5$.
b) Deduce Brooks' Theorem (14.4) for graphs with 2-vertex cuts.

14.2.13 Show that Brooks' Theorem (14.4) is equivalent to the following statement: if G is k-critical ($k \geq 4$) and not complete, then $2m \geq (k-1)n + 1$.
(Dirac (1957) sharpened this bound to $2m \geq (k-1)n + (k-3)$.)

14.2.14 A hypergraph H is *k-critical* if $\chi(H) = k$, but $\chi(H') < k$ for every proper subhypergraph H' of H. Show that:

a) the only 2-critical hypergraph is K_2,
b) the Fano hypergraph (depicted in Figure 1.15a) is 3-critical.

14.2.15 Let $H := (V, \mathcal{F})$ be a 3-critical hypergraph, where $V := \{v_1, v_2, \ldots, v_n\}$ and $\mathcal{F} := \{F_1, F_2, \ldots, F_m\}$, and let $\mathbf{M}$ be the incidence matrix of H.

a) Suppose that the rows of $\mathbf{M}$ are linearly dependent, so that there are real numbers λ_i, $1 \le i \le n$, not all zero, such that $\sum\{\lambda_i : v_i \in F_j\} = 0$, $1 \le j \le m$. Set $Z := \{i : \lambda_i = 0\}$, $P := \{i : \lambda_i > 0\}$, and $N := \{i : \lambda_i < 0\}$. Show that:
   i) $H' := H[Z]$ has a 2-colouring $\{R, B\}$,
   ii) H has the 2-colouring $\{R \cup P, B \cup N\}$.
b) Deduce that the rows of $\mathbf{M}$ are linearly independent.
c) Conclude that $|\mathcal{F}| \ge |V|$. (P.D. Seymour)

14.2.16 Let G be a k-chromatic graph which has a colouring in which each colour is assigned to at least two vertices. Show that G has a k-colouring with this property. (T. Gallai)

14.2.17

a) By appealing to Theorem 2.5, show that a bipartite graph with average degree $2k$ or more contains a path of length $2k + 1$. (A. Gyarfás and J. Lehel)
b) Using Exercise 14.1.21, deduce that every digraph D contains an antidirected path of length at least $\chi/4$.

14.2.18 An *antidirected cycle* in a digraph is a cycle of even length whose edges alternate in direction.

a) Find a tournament on five vertices which contains no antidirected cycle.
b) Show that every 8-chromatic digraph contains an antidirected cycle. (D. Grant, F. Jaeger, and C. Payan)

14.3 Girth and Chromatic Number

As we noted in the previous section, a graph which contains a large clique necessarily has a high chromatic number. On the other hand, and somewhat surprisingly, there exist triangle-free graphs with arbitrarily high chromatic number. Recursive constructions of such graphs were first described by (Blanche) Descartes (see Ungar and Descartes (1954) and Exercise 14.3.3). Later, Erdős (1961a) applied the probabilistic method to demonstrate the existence of graphs with arbitrarily high girth and chromatic number.

Theorem 14.10 *For each positive integer k, there exists a graph with girth at least k and chromatic number at least k.*

Proof Consider $G \in \mathcal{G}_{n,p}$, and set $t := \lceil 2p^{-1} \log n \rceil$. By Theorem 13.6, almost surely $\alpha(G) \le t$. Let X be the number of cycles of G of length less than k. By linearity of expectation (13.4),

$$E(X) = \sum_{i=3}^{k-1} \frac{(n)_i}{2i} p^i < \sum_{i=0}^{k-1} (np)^i = \frac{(np)^k - 1}{np - 1}$$

where $(n)_i$ denotes the falling factorial $n(n-1)\cdots(n-i+1)$. Markov's Inequality (13.4) now yields:

$$P(X > n/2) < \frac{E(X)}{n/2} < \frac{2((np)^k - 1)}{n(np - 1)}$$

Therefore, if $p := n^{-(k-1)/k}$,

$$P(X > n/2) < \frac{2(n-1)}{n(n^{1/k} - 1)} \to 0 \quad \text{as } n \to \infty$$

in other words, G almost surely has no more than $n/2$ cycles of length less than k.

It follows that, for n sufficiently large, there exists a graph G on n vertices with stability number at most t and no more than $n/2$ cycles of length less than k. By deleting one vertex of G from each cycle of length less than k, we obtain a graph G' on at least $n/2$ vertices with girth at least k and stability number at most t. By inequality (14.1),

$$\chi(G') \geq \frac{v(G')}{\alpha(G')} \geq \frac{n}{2t} \sim \frac{n^{1/k}}{8 \log n}$$

It suffices, now, to choose n large enough to guarantee that $\chi(G') \geq k$. □

Mycielski's Construction

Note that the above proof is nonconstructive: it merely asserts the *existence* of graphs with arbitrarily high girth and chromatic number. Recursive *constructions* of such graphs were given by Lovász (1968a) and also by Nešetřil and Rödl (1979). We describe here a simpler construction of triangle-free k-chromatic graphs, due to Mycielski (1955).

Theorem 14.11 *For any positive integer k, there exists a triangle-free k-chromatic graph.*

Proof For $k = 1$ and $k = 2$, the graphs K_1 and K_2 have the required property. We proceed by induction on k. Suppose that we have already constructed a triangle-free graph G_k with chromatic number $k \geq 2$. Let the vertices of G_k be $v_1, v_2, \ldots, v_n$. Form the graph G_{k+1} from G_k as follows: add $n+1$ new vertices $u_1, u_2, \ldots, u_n, v$, and then, for $1 \leq i \leq n$, join u_i to the neighbours of v_i in G_k, and also to v. For example, if $G_2 := K_2$, then G_3 is the 5-cycle and G_4 the Grötzsch graph (see Figure 14.6).

The graph G_{k+1} certainly has no triangles. For, because $u_1, u_2, \ldots, u_n$ is a stable set in G_{k+1}, no triangle can contain more than one u_i; and if $u_i v_j v_k u_i$

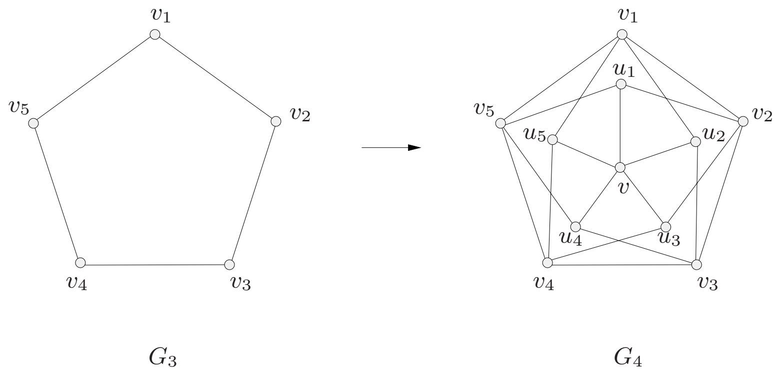

Fig. 14.6. Mycielski's construction

were a triangle in G_{k+1}, then $v_iv_jv_kv_i$ would be a triangle in G_k, contrary to our assumption.

We now show that G_{k+1} is $(k+1)$-chromatic. Note, first, that G_{k+1} is $(k+1)$-colourable, because any k-colouring of G_k can be extended to a $(k+1)$-colouring of G_{k+1} by assigning the colour of v_i to u_i, $1 \leq i \leq n$, and then assigning a new colour to v. Therefore, it remains to show that G_{k+1} is not k-colourable.

Suppose that G_{k+1} has a k-colouring. This colouring, when restricted to $\{v_1, v_2, \dots, v_n\}$, is a k-colouring of the k-chromatic graph G_k. By Exercise 14.1.3, for each colour j, there exists a vertex v_i of colour j which is adjacent in G_k to vertices of every other colour. Because u_i has precisely the same neighbours in G_k as v_i, the vertex u_i must also have colour j. Therefore, each of the k colours appears on at least one of the vertices u_i. But no colour is now available for the vertex v, a contradiction. We infer that G_{k+1} is indeed $(k+1)$-chromatic, and the theorem follows by induction. □

Other examples of triangle-free graphs with arbitrarily high chromatic number are the shift graphs (see Exercise 14.3.2).

Exercises

14.3.1 Let $G_2 := K_2$, and let G_k be the graph obtained from G_{k-1} by Mycielski's construction, $k \geq 3$. Show that G_k is a k-critical graph on $3 \cdot 2^{k-2} - 1$ vertices.

14.3.2 Shift Graph
The *shift graph* SG_n is the graph whose vertex set is the set of 2-subsets of $\{1, 2, \dots, n\}$, there being an edge joining two pairs $\{i, j\}$ and $\{k, l\}$, where $i < j$ and $k < l$, if and only if $j = k$. Show that SG_n is a triangle-free graph of chromatic number $\lceil \log_2 n \rceil$. (P. Erdős and A. Hajnal)

14.3.3 Let G be a k-chromatic graph on n vertices with girth at least six, where $k \geq 2$. Form a new graph H as follows.

▷ Take $\binom{kn}{n}$ disjoint copies of G and a set S of kn new vertices, and set up a one-to-one correspondence between the copies of G and the n-element subsets of S.
▷ For each copy of G, pair up its vertices with the members of the corresponding n-element subset of S and join each pair by an edge.

Show that H has chromatic number at least $k+1$ and girth at least six.
(B. Descartes)

14.4 Perfect Graphs

Inequality (14.2), which states that $\chi \geq \omega$, leads one to ask which graphs G satisfy it with equality. One soon realizes, however, that this question as it stands is not particularly interesting, because if H is any k-colourable graph and G is the disjoint union of H and K_k, then $\chi(G) = \omega(G) = k$. Berge (1963) noted that such artificial examples may be avoided by insisting that inequality (14.2) hold not only for G but also for all of its induced subgraphs. He called such graphs G 'perfect', and observed that the graphs satisfying this property include many basic families of graphs, such as bipartite graphs, line graphs of bipartite graphs, chordal graphs, and comparability graphs. He also noted that well-known min–max theorems concerning these seemingly disparate families of graphs simply amount to saying that they are perfect.

A graph G is *perfect* if $\chi(H) = \omega(H)$ for every induced subgraph H of G; otherwise, it is *imperfect*. An imperfect graph is *minimally imperfect* if each of its proper induced subgraphs is perfect. The triangular prism and the octahedron are examples of perfect graphs (Exercise 14.4.1), whereas the odd cycles of length five or more, as well as their complements, are minimally imperfect (Exercise 14.4.2). The cycle C_7 and its complement $\overline{C_7}$ are shown in Figure 14.7.

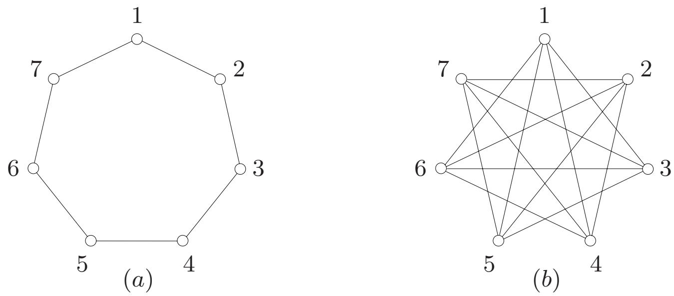

Fig. 14.7. The minimally imperfect graphs (a) C_7, and (b) $\overline{C_7}$

Being 2-colourable, bipartite graphs are clearly perfect. The fact that their line graphs are perfect is a consequence of a theorem concerning edge colourings of bipartite graphs (see Exercise 17.1.17). By Theorem 9.20, every chordal graph has a simplicial decomposition, and this property can be used to show that chordal graphs are perfect (Exercise 14.4.3). Comparability graphs are perfect too. That this is so may be deduced from a basic property of partially ordered sets (see Exercise 14.4.4).

The Perfect Graph Theorem

Berge (1963) observed that all the perfect graphs in the above classes also have perfect complements. For example, the König–Rado Theorem (8.30) implies that the complement of a bipartite graph is perfect, and Dilworth's Theorem (19.5) implies that the complement of a comparability graph is perfect. Based on this empirical evidence, Berge (1963) conjectured that a graph is perfect if and only if its complement is perfect. This conjecture was verified by Lovász (1972b), resulting in what is now known as the *Perfect Graph Theorem.*

Theorem 14.12 The Perfect Graph Theorem
A graph is perfect if and only if its complement is perfect. □

Shortly thereafter, A. Hajnal (see Lovász (1972a)) proposed the following beautiful characterization of perfect graphs. This, too, was confirmed by Lovász (1972a).

Theorem 14.13 *A graph G is perfect if and only if every induced subgraph H of G satisfies the inequality*

$$v(H) \leq \alpha(H)\omega(H)$$

Observe that the above inequality is invariant under complementation, because $v(\overline{H}) = v(H)$, $\alpha(\overline{H}) = \omega(H)$, and $\omega(\overline{H}) = \alpha(H)$. Theorem 14.13 thus implies the Perfect Graph Theorem (14.12).

The proof that we present of Theorem 14.13 is due to Gasparian (1996). It relies on an elementary rank argument (the proof technique of Linear Independence discussed in Section 2.4). We need the following property of minimally imperfect graphs.

Proposition 14.14 *Let S be a stable set in a minimally imperfect graph G. Then $\omega(G - S) = \omega(G)$.*

Proof We have the following string of inequalities (Exercise 14.4.5).

$$\omega(G - S) \leq \omega(G) \leq \chi(G) - 1 \leq \chi(G - S) = \omega(G - S)$$

Because the left and right members are the same, equality holds throughout. In particular, $\omega(G - S) = \omega(G)$. □

We can now establish a result on the structure of minimally imperfect graphs. This plays a key role in the proof of Theorem 14.13.

Lemma 14.15 *Let G be a minimally imperfect graph with stability number α and clique number ω. Then G contains $\alpha\omega + 1$ stable sets $S_0, S_1, \ldots, S_{\alpha\omega}$ and $\alpha\omega + 1$ cliques $C_0, C_1, \ldots, C_{\alpha\omega}$ such that:*

- ▷ *each vertex of G belongs to precisely α of the stable sets S_i,*
- ▷ *each clique C_i has ω vertices,*
- ▷ *$C_i \cap S_i = \emptyset$, for $0 \le i \le \alpha\omega$,*
- ▷ *$|C_i \cap S_j| = 1$, for $0 \le i < j \le \alpha\omega$.*

Proof Let S_0 be a stable set of α vertices of G, and let $v \in S_0$. The graph $G - v$ is perfect because G is minimally imperfect. Thus $\chi(G - v) = \omega(G - v) \le \omega(G)$. This means that for any $v \in S_0$, the set $V \setminus \{v\}$ can be partitioned into a family $\mathcal{S}_v$ of ω stable sets. Denoting $\{\cup\mathcal{S}_v : v \in S_0\}$ by $\{S_1, S_2, \ldots, S_{\alpha\omega}\}$, it can be seen that $\{S_0, S_1, \ldots, S_{\alpha\omega}\}$ is a family of $\alpha\omega + 1$ stable sets of G satisfying the first property above.

By Proposition 14.14, $\omega(G - S_i) = \omega(G)$, $0 \le i \le \alpha\omega$. Therefore there exists a maximum clique C_i of G that is disjoint from S_i. Because each of the ω vertices in C_i lies in α of the stable sets S_j, $0 \le i \le \alpha\omega$, and because no two vertices of C_i can belong to a common stable set, $|C_i \cap S_j| = 1$, for $0 \le i < j \le \alpha\omega$. □

Let us illustrate Lemma 14.15 by taking G to be the minimally imperfect graph $\overline{C_7}$, labelled as shown in Figure 14.7b. Here $\alpha = 2$ and $\omega = 3$. Applying the procedure described in the proof of the lemma, we obtain the following seven stable sets and seven cliques.

$$\begin{array}{lllllll} S_0 = 12, & S_1 = 23, & S_2 = 45, & S_3 = 67, & S_4 = 34, & S_5 = 56, & S_6 = 17 \\ C_0 = 357, & C_1 = 146, & C_2 = 136, & C_3 = 135, & C_4 = 257, & C_5 = 247, & C_6 = 246 \end{array}$$

(where we write 12 for the set $\{1, 2\}$, and so on.) The incidence matrices **S** and **C** of these families are shown in Figure 14.8.

S :

	S_0	S_1	S_2	S_3	S_4	S_5	S_6
1	1	0	0	0	0	0	1
2	1	1	0	0	0	0	0
3	0	1	0	0	1	0	0
4	0	0	1	0	1	0	0
5	0	0	1	0	0	1	0
6	0	0	0	1	0	1	0
7	0	0	0	1	0	0	1

C :

	C_0	C_1	C_2	C_3	C_4	C_5	C_6
1	0	1	1	1	0	0	0
2	0	0	0	0	1	1	1
3	1	0	1	1	0	0	0
4	0	1	0	0	0	1	1
5	1	0	0	1	1	0	0
6	0	1	1	0	0	0	1
7	1	0	0	0	1	1	0

Fig. 14.8. Incidence matrices of families of stable sets and cliques of $\overline{C_7}$

We are now ready to prove Theorem 14.13.

Proof Suppose that G is perfect, and let H be an induced subgraph of G. Because

G is perfect, H is $\omega(H)$-colourable, implying that $v(H) \leq \alpha(H)\omega(H)$. We prove the converse by showing that if G is minimally imperfect, then $v(G) \geq \alpha(G)\omega(G)+1$.

Consider the families $\{S_i : 0 \leq i \leq \alpha\omega\}$ and $\{C_i : 0 \leq i \leq \alpha\omega\}$ of stable sets and cliques described in Lemma 14.15. Let $\mathbf{S}$ and $\mathbf{C}$ be the $n \times (\alpha\omega+1)$ incidence matrices of these families. It follows from Lemma 14.15 that $\mathbf{S}^t\mathbf{C} = \mathbf{J} - \mathbf{I}$, where $\mathbf{J}$ is the square matrix of order $\alpha\omega + 1$ all of whose entries are 1 and $\mathbf{I}$ is the identity matrix of order $\alpha\omega + 1$. Now $\mathbf{J} - \mathbf{I}$ is a nonsingular matrix (with inverse $(1/\alpha\omega)\mathbf{J} - \mathbf{I}$). Its rank is thus equal to its order, $\alpha\omega + 1$. Hence both $\mathbf{S}$ and $\mathbf{C}$ are also of rank $\alpha\omega + 1$. But these matrices have n rows, so $n \geq \alpha\omega + 1$. □

Two consequences of the Perfect Graph Theorem are (Exercise 14.4.6):

Corollary 14.16 *A graph G is perfect if and only if, for any induced subgraph H of G, the maximum number of vertices in a stable set of H is equal to the minimum number of cliques required to cover all the vertices of H.* □

Corollary 14.17 *The Shannon capacity of a perfect graph G is equal to its stability number: $\Theta(G) = \alpha(G)$.* □

Corollary 14.17 prompts the problem of determining the Shannon capacities of the minimally imperfect graphs. Of these, only $\Theta(C_5)$ is known (see Exercise 12.1.14). It would be interesting to determine $\Theta(C_7)$.

The Strong Perfect Graph Theorem

If a graph is perfect, then so are all of its induced subgraphs. This means that one can characterize perfect graphs by describing all minimally imperfect graphs. We have remarked that the odd cycles of length five or more are minimally imperfect, as are their complements. Berge (1963) proposed the conjecture that these are the only minimally imperfect graphs; equivalently, that a graph is perfect if and only if it contains no odd cycle of length at least five, or its complement, as an induced subgraph. He named this conjecture, whose truth would imply the Perfect Graph Theorem, the *Strong Perfect Graph Conjecture*. Some forty years later, it was proved by Chudnovsky et al. (2006).

Theorem 14.18 The Strong Perfect Graph Theorem
A graph is perfect if and only if it contains no odd cycle of length at least five, or its complement, as an induced subgraph. □

This theorem was a major achievement, as much effort had been expended over the years on attempts to settle the Strong Perfect Graph Conjecture. Furthermore, a polynomial-time recognition algorithm for perfect graphs was developed shortly thereafter by Chudnovsky et al. (2005).

Perfect graphs play an important role in combinatorial optimization and polyhedral combinatorics. Schrijver (2003) dedicates three chapters of his scholarly treatise to this widely studied area. The survey article by Chudnovsky et al. (2003) includes an excellent account of some of the recent developments in the subject. The original motivations for the study of perfect graphs, and its early history, are described by Berge (1996, 1997).

Exercises

14.4.1 Show that the triangular prism and the octahedron are perfect graphs.

14.4.2 For each $k \geq 2$, show that both C_{2k+1} and $\overline{C_{2k+1}}$ are minimally imperfect graphs.

14.4.3

a) Let G be a chordal graph and $(X_1, X_2, \ldots, X_k)$ a simplicial decomposition of G. Show that $\chi = \max\{|X_i| : 1 \leq i \leq k\}$.
b) Deduce that every chordal graph is perfect.

14.4.4 Using the result stated in Exercise 2.1.23, show that every comparability graph is perfect.

$\star$**14.4.5** Verify the three inequalities in the proof of Proposition 14.14.

14.4.6 Prove Corollaries 14.16 and 14.17.

14.4.7 Without appealing to the Strong Perfect Graph Theorem, show that every minimally imperfect graph G satisfies the relation $n = \alpha\omega + 1$.

14.4.8 Deduce from Theorem 14.13 that the problem of recognizing perfect graphs belongs to co-$\mathcal{NP}$. (K. Cameron; V. Chvátal)

14.5 List Colourings

In most practical colouring problems, there are restrictions on the colours that may be assigned to certain vertices. For example, in the chemical storage problem of Example 14.2, radioactive substances might require special storage facilities. Thus in the corresponding graph there is a list of colours (appropriate storage compartments) associated with each vertex (chemical). In an admissible colouring (assignment of compartments to chemicals), the colour of a vertex must be chosen from its list. This leads to the notion of list colouring.

Let G be a graph and let L be a function which assigns to each vertex v of G a set $L(v)$ of positive integers, called the *list* of v. A colouring $c : V \to \mathbb{N}$ such that $c(v) \in L(v)$ for all $v \in V$ is called a *list colouring* of G with respect to L, or an L-*colouring*, and we say that G is L-*colourable*. Observe that if $L(v) = \{1, 2, \ldots, k\}$ for all $v \in V$, an L-colouring is simply a k-colouring. For instance, if G is a bipartite graph and $L(v) = \{1, 2\}$ for all vertices v, then G has the L-colouring which assigns colour 1 to all vertices in one part and colour 2 to all vertices in the other part. Observe, also, that assigning a list of length one to a vertex amounts to precolouring the vertex with that colour.

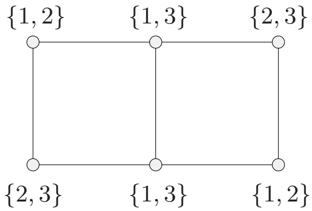

Fig. 14.9. A bipartite graph whose list chromatic number is three

List Chromatic Number

At first glance, one might believe that a k-chromatic graph in which each list $L(v)$ is of length at least k necessarily has an L-colouring. However, this is not so. It can be checked that the bipartite graph shown in Figure 14.9 has no list colouring with respect to the indicated lists. On the other hand, if arbitrary lists of length three are assigned to the vertices of this graph, it will have a compatible list colouring (Exercise 14.5.1).

A graph G is said to be *k-list-colourable* if it has a list colouring whenever all the lists have length k. Every graph G is clearly n-list-colourable. The smallest value of k for which G is k-list-colourable is called the *list chromatic number* of G, denoted $\chi_L(G)$. For example, the list chromatic number of the graph shown in Figure 14.9 is equal to three, whereas its chromatic number is two. (More generally, there exist 2-chromatic graphs whose list chromatic number is arbitrarily large, see Exercise 14.5.5.)

Bounds on the list chromatic numbers of certain graphs can be found by means of kernels. This might seem odd at first, because the kernel (introduced in Section 12.1) is a notion concerning directed graphs, whereas the list chromatic number is one concerning undirected graphs. The following theorem (a strengthening of Exercise 14.1.15) provides a link between kernels and list colourings.

Theorem 14.19 *Let G be a graph, and let D be an orientation of G each of whose induced subdigraphs has a kernel. For $v \in V$, let $L(v)$ be an arbitrary list of at least $d^+_D(v) + 1$ colours. Then G admits an L-colouring.*

Proof By induction on n, the statement being trivial for $n = 1$. Let V_1 be the set of vertices of D whose lists include colour 1. (We may assume that $V_1 \neq \emptyset$ by renaming colours if necessary.) By assumption, $D[V_1]$ has a kernel S_1. Colour the vertices of S_1 with colour 1, and set $G' := G - S_1$, $D' := D - S_1$ and $L'(v) := L(v) \setminus \{1\}$, $v \in V(D')$. For any vertex v of D' whose list did not contain colour 1,

$$|L'(v)| = |L(v)| \geq d^+_D(v) + 1 \geq d^+_{D'}(v) + 1$$

and for any vertex v of D' whose list did contain colour 1,

$$|L'(v)| = |L(v)| - 1 \geq d^+_D(v) \geq d^+_{D'}(v) + 1$$

The last inequality holds because, in D, the vertex v dominates some vertex of the kernel S_1, so its outdegree in D' is smaller than in D. By induction, G' has an L'-colouring. When combined with the colouring of S_1, this yields an L-colouring of G. □

As a simple illustration of Theorem 14.19, consider the case where D is an acyclic orientation of G. Because every acyclic digraph has a kernel (Exercise 12.1.10b), D satisfies the hypothesis of the theorem. Clearly, $d^+_D(v) \leq \Delta^+(D) \leq \Delta(G)$. Theorem 14.19 therefore tells us that G has a list colouring whenever each list is comprised of $\Delta + 1$ colours.

A similar approach can be applied to list colourings of interval graphs. Woodall (2001) showed that every interval graph G has an acyclic orientation D with $\Delta^+ \leq \omega - 1$ (Exercise 14.5.10). Appealing to Theorem 14.19 yields the following result.

Corollary 14.20 *Every interval graph G has list chromatic number ω.* □

Exercises

$\star$**14.5.1** Show that the list chromatic number of the graph shown in Figure 14.9 is equal to three.

14.5.2

a) Show that $\chi_L(K_{3,3}) = 3$.
b) Using the Fano plane, obtain an assignment of lists to the vertices of $K_{7,7}$ which shows that $\chi_L(K_{7,7}) > 3$.

14.5.3 Generalize Brooks' Theorem (14.4) by proving that if G is a connected graph, and is neither an odd cycle nor a complete graph, then G is Δ-list-colourable. (P. Erdős, A.L. Rubin, and H. Taylor; V.G. Vizing)

14.5.4 Show that $K_{m,n}$ is k-list-colourable for all $k \geq \min\{m, n\} + 1$.

$\star$**14.5.5** Show that $\chi_L(K_{n,n^n}) = n + 1$. (N. Alon and M. Tarsi)

14.5.6 By choosing as lists the edges of the non-2-colourable hypergraph whose existence was established in Exercise 13.2.15, show that $\chi_L(K_{n,n}) \geq c_n \log_2 n$, where $c_n \sim 1$.

14.5.7 Let S be a set of cardinality $2k - 1$, where $k \geq 1$. Consider the complete bipartite graph $K_{n,n}$, where $n = \binom{2k-1}{k}$, in which the lists attached to the vertices in each part are the k-subsets of S. Show that $K_{n,n}$ has no list colouring with this assignment of lists. (P. Erdős, A.L. Rubin, and H. Taylor)

14.5.8 A *theta graph* $\mathrm{TG}_{k,l,m}$ is a graph obtained by joining two vertices by three internally disjoint paths of lengths k, l, and m. Show that:

a) $TG_{2,2,2k}$ is 2-list-colourable for all $k \geq 1$,
b) a connected simple graph is 2-list-colourable if and only if the subgraph obtained by recursively deleting vertices of degree one is an isolated vertex, an even cycle, or a theta graph $TG_{2,2,2k}$, where $k \geq 1$.

(P. Erdős, A.L. Rubin, and H. Taylor)

14.5.9 Let $G = (V, E)$ be a simple graph. For $v \in V$, let $L(v)$ be a list of k or more colours. Suppose that, for each vertex v and each colour in $L(v)$, no more than $k/2e$ neighbours of v have that same colour in their lists (where e is the base of natural logarithms). By applying the Local Lemma (Theorem 13.12), show that G has a list colouring with respect to L. (B.A. Reed)

$\star$**14.5.10** Let G be an interval graph.

a) Show that G has an acyclic orientation D with $\Delta^+ = \omega - 1$.
b) Deduce that $\chi_L = \chi = \omega$. (D.R. Woodall)

14.6 The Adjacency Polynomial

We have already seen how linear algebraic techniques can be used to prove results in graph theory, for instance by means of rank arguments (see the inset in Chapter 2) or by studying the eigenvalues of the adjacency matrix of the graph (see the inset in Chapter 3). In this section, we develop yet another algebraic tool, this time related to polynomials, and apply it to obtain results on list colouring. To this end, we define a natural polynomial associated with a graph, indeed so natural that it is often referred to as the *graph polynomial*.

Let G be a graph with vertex set $V := \{v_1, v_2, \ldots, v_n\}$. Set $\mathbf{x} := (x_1, x_2, \ldots, x_n)$. The *adjacency polynomial* of G is the multivariate polynomial

$$A(G, \mathbf{x}) := \prod_{i<j} \{(x_i - x_j) : v_i v_j \in E\}$$

Upon expanding $A(G, \mathbf{x})$ we obtain 2^m monomials (some of which might cancel out). Each of these monomials is obtained by selecting exactly one variable from every factor $x_i - x_j$, and thus corresponds to an orientation of G: we orient the edge $v_i v_j$ of G in such a way that the vertex corresponding to the chosen variable is designated to be the tail of the resulting arc.

For example, if G is the graph shown in Figure 14.10, its adjacency polynomial is given by

$$A(G, \mathbf{x}) = (x_1 - x_2)(x_1 - x_3)(x_1 - x_4)(x_2 - x_3)(x_3 - x_4) \tag{14.4}$$

There are $2^5 = 32$ terms in the expansion of this expression before cancellation, whereas after cancellation only 24 terms remain:

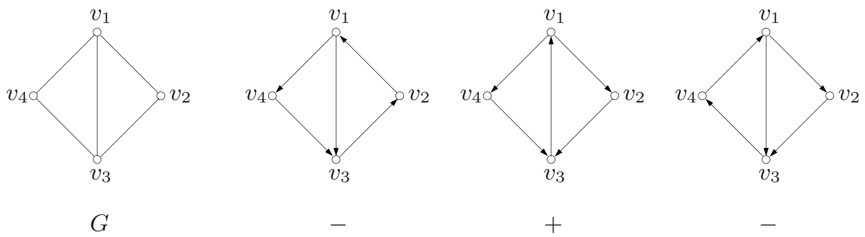

Fig. 14.10. A labelled graph G and the three orientations corresponding to the term $x_1^2x_2x_3x_4$ of its adjacency polynomial

$$\begin{aligned}A(G,\mathbf{x}) = \quad & x_1^3x_2x_3 - x_1^3x_2x_4 - x_1^3x_3^2 + x_1^3x_3x_4 - x_1^2x_2^2x_3 + x_1^2x_2^2x_4 \\ & - x_1^2x_2x_3x_4 + x_1^2x_2x_4^2 + x_1^2x_3^3 - x_1^2x_3x_4^2 + x_1x_2^2x_3^2 - x_1x_2^2x_4^2 \\ & - x_1x_2x_3^3 + x_1x_2x_3^2x_4 - x_1x_3^3x_4 + x_1x_3^2x_4^2 - x_2^2x_3^2x_4 + x_2^2x_3x_4^2 \\ & + x_2x_3^3x_4 - x_2x_3^2x_4^2 + x_2x_3x_4^3 - x_2x_4^4 - x_3^2x_4^3 + x_3x_4^4\end{aligned}$$

The graph G has the three orientations with outdegree sequence $(2,1,1,1)$ shown in Figure 14.10. These orientations are precisely the ones which correspond to the monomial $x_1^2x_2x_3x_4$. Observe that the coefficient of this term in A(G,$\mathbf{x}$) is -1. This is because two of the three terms in the expansion of the product (14.4) have a negative sign, whereas the remaining one has a positive sign.

As a second example, consider the complete graph K_n. We have

$$A(K_n,\mathbf{x}) = \prod_{1\le i<j\le n}(x_i - x_j) = \begin{vmatrix} x_1^{n-1} & x_2^{n-1} & \dots & x_n^{n-1} \\ x_1^{n-2} & x_2^{n-2} & \dots & x_n^{n-2} \\ \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \dots & \cdot \\ x_1 & x_2 & \dots & x_n \\ 1 & 1 & \dots & 1 \end{vmatrix}$$

The number of monomials in the expansion of this *Vandermonde determinant* is $n!$ (Exercise 14.6.1) which is much smaller (due to cancellation of terms) than $2^{\binom{n}{2}}$, the number of monomials in the expansion of the adjacency polynomial.

In order to express the adjacency polynomial of a graph in terms of its orientations, we need a little notation. In the expansion of $A(G,\mathbf{x})$, each monomial occurs with a given sign. We associate this same sign with the corresponding orientation D of G by defining

$$\sigma(D) := \prod\{\sigma(e) : a \in A(D)\}$$

where

$$\sigma(a) := \begin{cases} +1 \text{ if } a = (v_i, v_j) \text{ with } i < j \\ -1 \text{ if } a = (v_i, v_j) \text{ with } i > j \end{cases}$$

For example, the three orientations of the graph G in Figure 14.10 have the signs indicated.

Now let $\mathbf{d} := (d_1, d_2, \ldots, d_n)$ be a sequence of nonnegative integers whose sum is m. We define the *weight* of $\mathbf{d}$ by

$$w(\mathbf{d}) := \sum \sigma(D)$$

where the sum is taken over all orientations D of G whose outdegree sequence is $\mathbf{d}$. Setting

$$\mathbf{x}^{\mathbf{d}} := \prod_{i=1}^{n} x_i^{d_i}$$

we can now express the adjacency polynomial as:

$$A(G, \mathbf{x}) = \sum_{\mathbf{d}} w(\mathbf{d})\mathbf{x}^{\mathbf{d}}$$

In order to understand the relevance of the latter expression to list colourings, we need an algebraic tool developed by Alon (1999) and known as the *Combinatorial Nullstellensatz*, by analogy with a celebrated theorem of D. Hilbert.

PROOF TECHNIQUE: THE COMBINATORIAL NULLSTELLENSATZ

The Combinatorial Nullstellensatz is based on the following proposition, a generalization to n variables of the fact that a polynomial of degree d in one variable has at most d distinct roots.

Proposition 14.21 *Let f be a nonzero polynomial over a field F in the variables $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, of degree d_i in x_i, $1 \le i \le n$. Let L_i be a set of $d_i + 1$ elements of F, $1 \le i \le n$. Then there exists $\mathbf{t} \in L_1 \times L_2 \times \cdots \times L_n$ such that $f(\mathbf{t}) \neq 0$.*

Proof As noted above, the case $n = 1$ simply expresses the fact that a polynomial of degree d in one variable has at most d distinct roots. We proceed by induction on n, where $n \ge 2$.

We first express f as a polynomial in x_n whose coefficients f_j are polynomials in the variables $x_1, x_2, \ldots, x_{n-1}$:

$$f = \sum_{j=0}^{d_n} f_j x_n^j$$

Because f is nonzero by hypothesis, f_j is nonzero for some j, $0 \le j \le d_n$. By induction, there exist $t_i \in L_i$, $1 \le i \le n-1$, such that $f_j(t_1, t_2, \ldots, t_{n-1}) \neq 0$. Therefore the polynomial $\sum_{j=0}^{d_n} f_j(t_1, t_2, \ldots, t_{n-1})\, x_n^j$ is nonzero. Applying the case $n = 1$ to this polynomial, we deduce that $f(t_1, t_2, \ldots, t_n) \neq 0$ for some $t_n \in L_n$. $\square$

THE COMBINATORIAL NULLSTELLENSATZ (CONTINUED)

Theorem 14.22 THE COMBINATORIAL NULLSTELLENSATZ
Let f be a polynomial over a field F in the variables $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. Suppose that the total degree of f is $\sum_{1=1}^{n} d_i$ and that the coefficient in f of $\prod_{i=1}^{n} x_i^{d_i}$ is nonzero. Let L_i be a set of $d_i + 1$ elements of F, $1 \leq i \leq n$. Then there exists $\mathbf{t} \in L_1 \times L_2 \times \cdots \times L_n$ such that $f(\mathbf{t}) \neq 0$.

Proof For $1 \leq i \leq n$, set

$$f_i := \prod_{t \in L_i} (x_i - t)$$

Then f_i is a polynomial of degree $|L_i| = d_i + 1$, with leading term $x_i^{d_i+1}$, so we may write $f_i = g_i + x_i^{d_i+1}$, where g_i is a polynomial in x_i of degree at most d_i. By repeatedly substituting $-g_i$ for $x_i^{d_i+1}$ in the polynomial f, we obtain a new polynomial in which the degree of x_i does not exceed d_i. Performing this substitution operation for all i, $1 \leq i \leq n$, results in a polynomial g of degree at most d_i in x_i, $1 \leq i \leq n$.
Moreover, because $f_i(t) = 0$ for all $t \in L_i$, we have $t^{d_i+1} = -g_i(t)$ for all $t \in L_i$, $1 \leq i \leq n$. It follows that

$$g(\mathbf{t}) = f(\mathbf{t}) \quad \text{for all } \mathbf{t} \in L_1 \times L_2 \times \cdots \times L_n$$

Observe that every monomial of g is of total degree strictly less than $\sum_{1=1}^{n} d_i$, apart from the monomial $\prod_{i=1}^{n} x_i^{d_i}$, which is unchanged. Thus g is nonzero. By Proposition 14.21, applied to g, there exists $\mathbf{t} \in L_1 \times L_2 \times \cdots \times L_n$ such that $g(\mathbf{t}) \neq 0$. This implies that $f(\mathbf{t}) \neq 0$. $\square$

Corollary 14.23 *Let G be a graph, and let D be an orientation of G without directed odd cycles. Then G is $(\mathbf{d}+\mathbf{1})$-list-colourable, where $\mathbf{d}$ is the outdegree sequence of D.*

Proof Every orientation of G with outdegree sequence $\mathbf{d}$ has the same sign as D (Exercise 14.6.2a). Therefore $w(\mathbf{d}) \neq 0$. The result follows on applying Theorem 14.22 with $f(\mathbf{x}) = A(G, \mathbf{x})$. $\square$

Corollary 14.24 *If G has an odd number of orientations D with outdegree sequence $\mathbf{d}$, then G is $(\mathbf{d}+\mathbf{1})$-list-colourable.*

Proof In this case $w(\mathbf{d})$ is also odd, thus nonzero. $\square$

Further applications of the Combinatorial Nullstellensatz are given in the exercises which follow.

Exercises

14.6.1 Show that the number of monomials in the expansion of the Vandermonde determinant of order n is $n!$

14.6.2

a) Let G be a graph, and let D be an orientation of G with outdegree sequence $\mathbf{d}$.
   i) If D' is an orientation of G with outdegree sequence $\mathbf{d}$, show that $\sigma(D') = \sigma(D)$ if and only if $|A(D) \setminus A(D')|$ is even.
   ii) Deduce that if D has no directed odd cycles, then all orientations of G with outdegree sequence $\mathbf{d}$ have the same sign.
b) For a graph G, denote by $G(\mathbf{d})$ the graph whose vertices are the orientations of G with outdegree sequence $\mathbf{d}$, two such orientations D and D' being adjacent in $G(\mathbf{d})$ if and only if $A(D) \setminus A(D')$ is the arc set of a directed cycle. Denote by $B(\mathbf{d})$ the spanning subgraph of $G(\mathbf{d})$ whose edges correspond to directed odd cycles. Show that:
   i) $G(\mathbf{d})$ is connected,
   ii) $B(\mathbf{d})$ is bipartite.

14.6.3 Let T be a transitive tournament on n vertices with outdegree sequence $\mathbf{d} := (d_1, d_2, \ldots, d_n)$, where $d_1 \leq d_2 \leq \cdots \leq d_n$.

a) Express the number of directed triangles of T in terms of n and d.
b) Deduce that if $G = K_n$ and $\mathbf{d} \neq (0, 1, 2, \ldots, n-1)$, then the bipartite graph $B(\mathbf{d})$ (defined in Exercise 14.6.2) has parts of equal size.
c) Deduce that $(0, 1, 2, \ldots, n-1)$ is the only sequence $\mathbf{d}$ such that $w(\mathbf{d}) \neq 0$.

14.6.4 Let $G(x, y)$ be a graph, where $N(x) \setminus \{y\} = N(y) \setminus \{x\}$, and let D be an orientation of G with $d^+(x) = d^+(y)$. Show that $w(\mathbf{d}) = 0$, where $\mathbf{d}$ is the outdegree sequence of D. (S. Ceroi)

14.6.5 The Fleischner–Stiebitz Theorem
Let G be a 4-regular graph on $3k$ vertices, the union of a cycle of length $3k$ and k pairwise disjoint triangles.

a) Show that the number of eulerian orientations of G with a given sign is even.
b) Fleischner and Stiebitz (1992) have shown (by induction on n) that the total number of eulerian orientations of G is congruent to $2 \pmod 4$. Deduce that G is 3-list-colourable and thus 3-colourable.
   (H. Fleischner and M. Stiebitz)
   (Sachs (1993) has shown that the number of 3-colourings of G is odd.)

14.6.6

a) For a graph G, as in Exercise 21.4.5, define $d^*(G) := \max\{d(F) : F \subseteq G\}$, the maximum of the average degrees of the subgraphs of G. Show that every bipartite graph G is $(\lceil d^*/2 \rceil + 1)$-list-colourable.

b) Deduce that every planar bipartite graph is 3-list-colourable.
c) Find a planar bipartite graph whose list chromatic number is three.

(N. Alon and M. Tarsi)

14.6.7 The Cauchy–Davenport Theorem
Let A and B be nonempty subsets of $\mathbb{Z}_p$, where p is a prime. Define the *sum* $A+B$ of A and B by $A+B := \{a+b : a \in A, b \in B\}$.

a) If $|A|+|B| > p$, show that $A+B = \mathbb{Z}_p$.
b) Suppose that $|A|+|B| \leq p$ and also that $|A+B| \leq |A|+|B|-2$. Let C be a set of $|A|+|B|-2$ elements of $\mathbb{Z}_p$ that contains $A+B$. Consider the polynomial $f(x,y) := \prod_{c \in C}(x+y-c)$. Show that:
   i) $f(a,b) = 0$ for all $a \in A$ and all $b \in B$,
   ii) the coefficient of $x^{|A|-1}y^{|B|-1}$ in $f(x,y)$ is nonzero.
c) By applying the Combinatorial Nullstellensatz, deduce the *Cauchy–Davenport Theorem:* if A and B are nonempty subsets of $\mathbb{Z}_p$, where p is a prime, then either $A+B = \mathbb{Z}_p$ or $|A+B| \geq |A|+|B|-1$.

(N. Alon, M.B. Nathanson, and I.Z. Rusza)

14.6.8 Let $G = (V,E)$ be a loopless graph with average degree greater than $2p-2$ and maximum degree at most $2p-1$, where p is a prime. Show that G has a p-regular subgraph by proceeding as follows.

Consider the polynomial f over $\mathbb{Z}_p$ in the variables $\mathbf{x} = (x_e : e \in E)$ defined by

$$f(\mathbf{x}) := \prod_{v \in V}\left(1 - \left(\sum_{e \in E} m_{ve}x_e\right)^{p-1}\right) - \prod_{e \in E}(1 - x_e)$$

a) Show that:
   i) the degree of f is $e(G)$,
   ii) the coefficient of $\prod_{e \in E} x_e$ in f is nonzero.
b) Deduce from the Combinatorial Nullstellensatz that $f(\mathbf{c}) \neq 0$ for some vector $\mathbf{c} = (c_e : e \in E) \in \{0,1\}^E$.
c) Show that $\mathbf{c} \neq \mathbf{0}$ and $\mathbf{Mc} = \mathbf{0}$.
d) By considering the spanning subgraph of G with edge set $\{e \in E : c_e = 1\}$, deduce that G has a p-regular subgraph.
e) Deduce, in particular, that every 4-regular loopless graph with one additional link contains a 3-regular subgraph.

(N. Alon, S. Friedland, and G. Kalai)

(Tashkinov (1984) proved that every 4-regular simple graph contains a 3-regular subgraph.)

14.7 The Chromatic Polynomial

We have seen how the adjacency polyomial provides insight into the complex topic of graph colouring. Here, we discuss another polynomial related to graph colouring, the chromatic polynomial. In this final section, we permit loops and parallel edges.

In the study of colourings, some insight can be gained by considering not only the existence of k-colourings but the number of such colourings; this approach was developed by Birkhoff (1912/13) as a possible means of attacking the Four-Colour Conjecture.

We denote the number of distinct k-colourings $c : V \to \{1, 2, \ldots, k\}$ of a graph G by $C(G,k)$. Thus $C(G,k) > 0$ if and only if G is k-colourable. In particular, if G has a loop then $C(G,k) = 0$. Two colourings are to be regarded as distinct if some vertex is assigned different colours in the two colourings; in other words, if $\{V_1, V_2, \ldots, V_k\}$ and $\{V'_1, V'_2, \ldots, V'_k\}$ are two k-colourings, then $\{V_1, V_2, \ldots, V_k\} = \{V'_1, V'_2, \ldots, V'_k\}$ if and only if $V_i = V'_i$ for $1 \le i \le k$. A triangle, for example, has six distinct 3-colourings.

If G is empty, then each vertex can be independently assigned any one of the k available colours, so $C(G,k) = k^n$. On the other hand, if G is complete, then there are k choices of colour for the first vertex, $k-1$ choices for the second, $k-2$ for the third, and so on. Thus, in this case, $C(G,k) = k(k-1)\cdots(k-n+1)$.

There is a simple recursion formula for $C(G,k)$, namely:

$$C(G,k) = C(G \setminus e, k) - C(G \,/\, e, k) \tag{14.5}$$

where e is any link of G. Formula (14.5) bears a close resemblance to the recursion formula for $t(G)$, the number of spanning trees of G (Proposition 4.9). We leave its proof as an exercise (14.7.1). The formula gives rise to the following theorem.

Theorem 14.25 *For any loopless graph G, there exists a polynomial $P(G,x)$ such that $P(G,k) = C(G,k)$ for all nonnegative integers k. Moreover, if G is simple and e is any edge of G, then $P(G,x)$ satisfies the recursion formula:*

$$P(G,x) = P(G \setminus e, x) - P(G \,/\, e, x) \tag{14.6}$$

The polynomial $P(G,x)$ is of degree n, with integer coefficients which alternate in sign, leading term x^n, and constant term zero.

Proof By induction on m. If $m = 0$, then $C(G,k) = k^n$, and the polynomial $P(G,x) = x^n$ satisfies the conditions of the theorem trivially.

Suppose that the theorem holds for all graphs with fewer than m edges, where $m \ge 1$, and let G be a loopless graph with m edges. If G is not simple, define $P(G,x) := P(H,x)$, where H is the underlying simple graph of G. By induction, H satisfies the conditions of the theorem, so G does also. If G is simple, let e be an edge of G. Both $G \setminus e$ and $G \,/\, e$ have $m-1$ edges and are loopless. By induction, there exist polynomials $P(G \setminus e, x)$ and $P(G \,/\, e, x)$ such that, for all nonnegative integers k,

$$P(G \setminus e, k) = C(G \setminus e, k) \quad \text{and} \quad P(G / e, k) = C(G / e, k) \tag{14.7}$$

Furthermore, there are nonnegative integers $a_1, a_2, \ldots, a_{n-1}$ and $b_1, b_2, \ldots, b_{n-1}$ such that:

$$P(G \setminus e, x) = \sum_{i=1}^{n-1} (-1)^{n-i} a_i x^i + x^n \quad \text{and} \quad P(G / e, x) = \sum_{i=1}^{n-1} (-1)^{n-i-1} b_i x^i \tag{14.8}$$

Define $P(G, x) := P(G \setminus e, x) - P(G / e, x)$, so that the desired recursion (14.6) holds. Applying (14.6), (14.7), and (14.5), we have:

$$P(G, k) = P(G \setminus e, k) - P(G / e, k) = C(G \setminus e, k) - C(G / e, k) = C(G, k)$$

and applying (14.6) and (14.8) yields

$$P(G, x) = P(G \setminus e, x) - P(G / e, x) = \sum_{i=1}^{n-1} (-1)^{n-i} (a_i + b_i) x^i + x^n$$

Thus $P(G, x)$ satisfies the stated conditions. □

The polynomial $P(G, x)$ is called the *chromatic polynomial* of G. Formula (14.6) provides a means of calculating chromatic polynomials recursively. It can be used in either of two ways:

i) by repeatedly applying the recursion $P(G, x) = P(G \setminus e, x) - P(G / e, x)$, thereby expressing $P(G, x)$ as an integer linear combination of chromatic polynomials of empty graphs,
ii) by repeatedly applying the recursion $P(G \setminus e, x) = P(G, x) + P(G / e, x)$, thereby expressing $P(G, x)$ as an integer linear combination of chromatic polynomials of complete graphs.

Method (i) is more suited to graphs with few edges, whereas (ii) can be applied more efficiently to graphs with many edges (see Exercise 14.7.2).

The calculation of chromatic polynomials can sometimes be facilitated by the use of a number of formulae relating the chromatic polynomial of a graph to the chromatic polynomials of certain subgraphs (see Exercises 14.7.6a, 14.7.7, and 14.7.8). However, no polynomial-time algorithm is known for finding the chromatic polynomial of a graph. (Such an algorithm would clearly provide a polynomial-time algorithm for computing the chromatic number.)

Although many properties of chromatic polynomials have been found, no one has yet discovered which polynomials are chromatic. It has been conjectured by Read (1968) that the sequence of coefficients in any chromatic polynomial must first rise in absolute value and then fall; in other words, that no coefficient may be flanked by two coefficients having greater absolute value. But even if true, this property together with the properties listed in Theorem 14.25 would not be enough to characterize chromatic polynomials. For example, the polynomial $x^4 - 3x^3 + 3x^2$

satisfies all of these properties but is not the chromatic polynomial of any graph (Exercise 14.7.3b).

By definition, the value of the chromatic polynomial $P(G, x)$ at a positive integer k is the number of k-colourings of G. Surprisingly, evaluations of the polynomial at certain other special values of x also have interesting interpretations. For example, it was shown by Stanley (1973) that $(-1)^n P(G, -1)$ is the number of acyclic orientations of G (Exercise 14.7.11).

Roots of chromatic polynomials, or *chromatic roots*, exhibit a rather curious behaviour. Using the recursion (14.6), one can show that 0 is the only real chromatic root less than 1 (Exercise 14.7.9); note that 0 is a chromatic root of every graph and 1 is a chromatic root of every nonempty loopless graph. Jackson (1993b) extended these observations by proving that no chromatic polynomial can have a root in the interval $(1, 32/27]$. Furthermore, Thomassen (1997c) showed that the only real intervals that are free of chromatic roots are $(-\infty, 0)$, $(0, 1)$, and $(1, 32/27]$. Thomassen (2000) also established an unexpected link between chromatic roots and Hamilton paths.

In the context of plane triangulations, the values of $P(G, x)$ at the *Beraha numbers* $B_k := 2 + 2\cos(2\pi/k)$, $k \geq 1$, are remarkably small, suggesting that the polynomial might have roots close to these numbers (see Tutte (1970)).

For a survey of this intriguing topic, we refer the reader to Read and Tutte (1988).

Exercises

$\star$**14.7.1** Prove the recursion formula (14.5).

14.7.2

a) Calculate the chromatic polynomial of the 3-star $K_{1,3}$ by using the recursion $P(G, x) = P(G \backslash e, x) - P(G / e, x)$ to express it as an integer linear combination of chromatic polynomials of empty graphs.
b) Calculate the chromatic polynomial of the 4-cycle C_4 by using the recursion $P(G \backslash e, x) = P(G, x) + P(G / e, x)$ to express it as an integer linear combination of chromatic polynomials of complete graphs.

14.7.3

a) Show that if G is simple, then the coefficient of x^{n-1} in $P(G, x)$ is $-m$.
b) Deduce that no graph has chromatic polynomial $x^4 - 3x^3 + 3x^2$.

14.7.4 Show that:

a) if G is a tree, then $P(G, x) = x(x-1)^{n-1}$,
b) if G is connected and $P(G, x) = x(x-1)^{n-1}$, then G is a tree.

14.7.5 Show that if G is a cycle of length n, then $P(G, x) = (x-1)^n + (-1)^n(x-1)$.

14.7.6

a) Show that $P(G \vee K_1, x) = xP(G, x-1)$.
b) Using (a) and Exercise 14.7.5, show that if G is a wheel with n spokes, then $P(G, x) = x(x-2)^n + (-1)^n x(x-2)$.

14.7.7

a) Show that if G and H are disjoint, then $P(G \cup H, x) = P(G, x)P(H, x)$.
b) Deduce that the chromatic polynomial of a graph is equal to the product of the chromatic polynomials of its components.

14.7.8 If $G \cap H$ is complete, show that $P(G \cup H, x)P(G \cap H, x) = P(G, x)P(H, x)$.

14.7.9 Show that zero is the only real root of $P(G, x)$ smaller than one.

14.7.10 Show that no real root of $P(G, x)$ can exceed n. (L. Lovász)

$\star$**14.7.11** Show that the number of acyclic orientations of a graph G is equal to $(-1)^n P(G, -1)$. (R.P. Stanley)

$\star$**14.7.12** Let G be a graph. For a subset S of E, denote by $c(S)$ the number of components of the spanning subgraph of G with edge set S. Show that $P(G, x) = \sum_{S \subseteq E} (-1)^{|S|} x^{c(S)}$. (H. Whitney)

14.8 Related Reading

Fractional Colourings

A vertex colouring $\{V_1, V_2, \ldots, V_k\}$ of a graph $G = (V, E)$ can be viewed as expressing the incidence vector $\mathbf{1} := (1, 1, \ldots, 1)$ of V as the sum of the incidence vectors of the stable sets $V_1, V_2, \ldots, V_k$. This suggests the following relaxation of the notion of vertex colouring.

A *fractional colouring* of a graph $G = (V, E)$ is an expression of $\mathbf{1}$ as a nonnegative rational linear combination of incidence vectors of stable sets of G. The least sum of the coefficients in such an expression is called the *fractional chromatic number* of G, denoted $\chi^*(G)$. Thus

$$\chi^* := \min\left\{\sum \lambda_S : \sum \lambda_S f_S = \mathbf{1}\right\}$$

where the sums are taken over all stable sets S of G. The fractional chromatic number is clearly a lower bound on the chromatic number. However, it is still $\mathcal{NP}$-hard to compute this parameter.

By applying linear programming duality and using the fact that the stable sets of a graph G are the cliques of its complement $\overline{G}$, it can be shown that

$\chi^*(G) = \alpha^{**}(\overline{G})$. Thus $\chi^*(G)$ is an upper bound for the Shannon capacity of $\overline{G}$ (see (12.2)).

The fractional chromatic number is linked to list colourings in a simple way. A graph G is (k,l)-*list-colourable* if, from arbitrary lists $L(v)$ of k colours, sets $C(v)$ of l colours can be chosen so that $C(u) \cap C(v) = \emptyset$ whenever $uv \in E$. It was shown by Alon et al. (1997) that $\chi^* = \inf\{k/l : G \text{ is } (k,l)\text{-list-colourable}\}$.

Further properties of the fractional chromatic number can be found in Scheinerman and Ullman (1997) and Schrijver (2003).

Homomorphisms and Circular Colourings

A *homomorphism* of a graph G into another graph H is a mapping $f : V(G) \to V(H)$ such that $f(u)f(v) \in E(H)$ for all $uv \in E(G)$. When H is the complete graph K_k, a homomorphism from G into H is simply a k-colouring of G. Thus the concept of a homomorphism may be regarded as a generalization of the notion of vertex colouring studied in this chapter. Many intriguing unsolved problems arise when one considers homomorphisms of graphs into graphs which are not necessarily complete (see Hell and Nešetřil (2004)). One particularly interesting instance is described below.

Let k and d be two positive integers such that $k \geq 2d$. A (k,d)-*colouring* of a graph G is a function $f : V \to \{1, 2, \ldots, k\}$ such that $d \leq |f(u) - f(v)| \leq k - d$ for all $uv \in E$. Thus a $(k,1)$-colouring of a graph is simply a proper k-colouring, and a (k,d)-colouring is a homomorphism from the graph into $\overline{C_k^{d-1}}$, the complement of the $(d-1)$st power of a k-cycle. Vince (1988) (see also Bondy and Hell (1990)) showed that, for any graph G, $\min\{k/d\colon G \text{ has a } (k,d)\text{-colouring}\}$ exists. This minimum, denoted by $\chi_c(G)$, is known as the *circular chromatic number* of G. (The name of this parameter derives from an alternative definition, due to X. Zhu, in which the vertices are associated with arcs of a circle, adjacent vertices corresponding to disjoint arcs.) One can easily show that $\chi(G) - 1 < \chi_c(G) \leq \chi(G)$, so $\chi(G) = \lceil \chi_c(G) \rceil$. However, there are graphs whose chromatic numbers are the same but whose circular chromatic numbers are different. For example, $\chi_c(K_3) = 3$ whereas $\chi_c(C_5) = 5/2$. One challenging unsolved problem in this area is to characterize the graphs for which these two parameters are equal. This question remains unsolved even for planar graphs. The comprehensive survey by Zhu (2001) contains many other intriguing problems.