8

Poisson Processes and Ruin Theory

We give in this chapter the main results on Poisson processes, which are basic examples of jump processes. Despite their elementary properties they are building blocks of jump process theory. We present various generalizations such as inhomogeneous Poisson processes and compound Poisson processes. These processes are not used to model financial prices, due to the simple character of their jumps and are in practice mixed with Brownian motion, as we shall present in ↣ Chapter 10. However, they represent the main model in insurance theory. We end this chapter with two sections about point processes and marked point processes.

The reader can refer to Çinlar [188], Cocozza-Thivent [190], Karlin and Taylor [515] and the last chapter in Shreve [795] for the study of standard Poisson processes, to Brémaud [124] for general Poisson processes, and to Jacod and Shiryaev [471], Kallenberg [504], Kingman [523], Last and Brandt [565], Neveu [669], Prigent [725] and Protter [727] for point processes, and to Mikosch [651, 652] for applications.

8.1 Counting Processes and Stochastic Integrals

A **counting process** is a process which increases in unit steps at isolated times and is constant between these times. It can be constructed as follows. Let $(T_n, n \geq 0)$ be a sequence of random variables defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that

$$T_0 = 0, \quad T_n < T_{n+1} \text{ for } T_n < \infty \,.$$

This sequence models the times when jumps occur. We define the family of random variables, for $t \geq 0$,

$$N_t = \begin{cases} n \text{ if } t \in [T_n, T_{n+1}[\\ +\infty \text{ otherwise,} \end{cases}$$

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or, equivalently,

$$N_t = \sum_{n\geq 1} \mathbb{1}_{\{T_n \leq t\}} = \sum_{n\geq 0} n\mathbb{1}_{\{T_n \leq t < T_{n+1}\}},\ N_0 = 0\,.$$

This counting process $(N_t, t \geq 0)$, associated with the sequence $(T_n, n \geq 0)$, is increasing and right-continuous. We denote by N_{t-} the left-limit of N_s when $s \to t, s < t$ and by $\Delta N_s = N_s - N_{s-}$ the jump process of N. The **explosion time** is the r.v. $T = \sup_n T_n$. In what follows, we reduce our attention to the case $T = \infty$.

Let $\mathbf{F}$ be a given filtration. A counting process is $\mathbf{F}$-adapted if and only if the random variables $(T_n, n \geq 1)$ are $\mathbf{F}$-stopping times. In that case, for any n, the set $\{N_t \leq n\} = \{T_{n+1} > t\}$ belongs to $\mathcal{F}_t$.

The natural filtration of N denoted by $\mathbf{F}^N$ where $\mathcal{F}_t^N = \sigma(N_s, s \leq t)$ is the smallest filtration $\mathbf{F}^N$ which satisfies the usual hypotheses and such that N is $\mathbf{F}^N$-adapted.

The **stochastic integral** $\int_0^t C_s dN_s$ is defined pathwise as a Stieltjes integral for every bounded measurable process (not necessarily $\mathbf{F}^N$-adapted) $(C_t, t \geq 0)$ by

$$(C\star N)_t := \int_0^t C_s dN_s = \int_{]0,t]} C_s dN_s := \sum_{n=1}^{\infty} C_{T_n} \mathbb{1}_{\{T_n \leq t\}}\,.$$

We emphasize that the integral $\int_0^t C_s dN_s$ is here an integral over the time interval $]0, t]$, where the upper limit t is included and the lower limit 0 excluded. This integral is finite since there is a finite number of jumps during the time interval $]0, t]$. We shall also write

$$\int_0^t C_s dN_s = \sum_{s\leq t} C_s \Delta N_s$$

where the right-hand side contains only a finite number of non-zero terms. The integral $\int_0^\infty C_s dN_s$ is defined as $\int_0^\infty C_s dN_s = \sum_{n=1}^\infty C_{T_n}$, when the right-hand side converges.

We shall also use the differential notation $d(C\star N)_t := C_t dN_t$.

We can associate a **random measure** to any counting process as follows. For any Borel set $\Lambda \subset \mathbb{R}^+$, for any ω, set

$$\mu(\omega, \Lambda) = \#\{n \geq 1\, : T_n(\omega) \in \Lambda\}\,.$$

For any ω, the map $\Lambda \to \mu(\omega, \Lambda)$ defines a positive measure on $\mathbb{R}^+$. One can note that $\mu(\omega, dt) = \sum_n \delta_{T_n(\omega)}(dt)$.

The random variable N_t can be written as

$$N_t(\omega) = \mu(\omega,]0, t]) = \int_{]0,t]} \mu(\omega, ds)$$

and the Stieltjes (or stochastic) integral as $\int_0^t C_s dN_s = \int_0^t C_s \mu(ds)$.

8.2 Standard Poisson Process

8.2.1 Definition and First Properties

The **standard Poisson process** is a counting process such that the random variables $(T_{n+1} - T_n, \, n \geq 0)$ are independent and identically distributed with exponential law of parameter λ with $\lambda > 0$. Hence, the explosion time is infinite and

$$\mathbb{P}(N_t = n) = e^{-\lambda t} \frac{(\lambda t)^n}{n!}.$$

The standard Poisson process can be redefined as follows (see e.g., Çinlar [188]): it is a counting process without explosion (i.e., $T = \infty$) such that
- for every $s, t \geq 0$ the r.v. $N_{t+s} - N_t$ is independent of $\mathcal{F}_t^N$,
- for every s, t, the r.v. $N_{t+s} - N_t$ has the same law as N_s.

or, in an equivalent way, a counting process without explosion whose increments are independent and stationary.

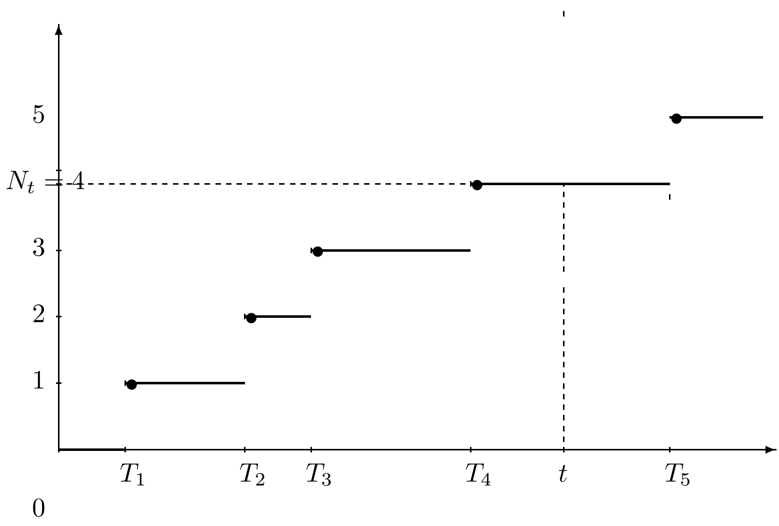

Fig. 8.1 Poisson process

Definition 8.2.1.1 *Let* $\mathbf{F}$ *be a given filtration and* λ *a positive constant. The process* N *is an* $\mathbf{F}$*-Poisson process with intensity* λ *if* N *is an* $\mathbf{F}$*-adapted process, such that for all positive numbers* (t,s)*, the r.v.* $N_{t+s} - N_t$ *is independent of* $\mathcal{F}_t$ *and follows the Poisson law with parameter* λs*.*

The random measure μ associated with a Poisson process is such that $\mu(\Lambda)$ is almost surely finite for any bounded set Λ (the number of jumps in any finite interval of time is almost surely finite), and $\mathbb{E}(\mu(\Lambda)) = \lambda|\Lambda|$ where $|\Lambda|$ is the Lebesgue measure of the set Λ.

We now recall some properties of Poisson processes.

• The time T_n when the n^{th}-jump of N occurs is the sum of n independent exponential r.v's, hence it has a Gamma law with parameters (n, λ):

$$\mathbb{P}(T_n \in dt) = \frac{(\lambda t)^{n-1}}{(n-1)!}\lambda e^{-\lambda t}\, \mathbb{1}_{\{t>0\}} dt,$$

and its Laplace transform, for $\mu > -\lambda$, is given by

$$\mathbb{E}(e^{-\mu T_n}) = \left(\frac{\lambda}{\lambda+\mu}\right)^n .$$

• From the properties of the Poisson distribution, it follows that for every $t > 0$,

$$\mathbb{E}(N_t) = \lambda t, \quad \text{Var}\,(N_t) = \lambda t$$

and for every $x > 0$, $t \geq 0, u, \alpha \in \mathbb{R}$

$$\mathbb{E}(x^{N_t}) = e^{\lambda t(x-1)}\,;\ \mathbb{E}(e^{iuN_t}) = e^{\lambda t(e^{iu}-1)}\,;\ \mathbb{E}(e^{\alpha N_t}) = e^{\lambda t(e^{\alpha}-1)} . \tag{8.2.1}$$

• Conditionally on $(N_t = n)$, the law of $(T_1, T_2, \ldots, T_n)$ is a multinomial distribution on $[0, t]$.

• Let, for t fixed and $i \geq 1$, $T_i^{(t)} := T_{N_t+i} - t$ where T_{N_t+i} is the time of the i-th jump which occurs after t. The sequence of times $(T_i^{(t)}, i \geq 1)$ has the same law as $(T_i, i \geq 1)$. This property is called the lack of memory of the Poisson process.

Exercise 8.2.1.2 Let N be a Poisson process. Prove that $N_t t^{-1} \to \lambda$ a.s. when t goes to infinity. ◁

Exercise 8.2.1.3 Let N be a Poisson process and T_n its n-th jump time. Prove that

$$\mathbb{P}(T_n \geq s|\mathcal{F}_t) = \mathbb{1}_{s \leq T_n \leq t} + \mathbb{1}_{t<T_n} \int_{s-t}^{\infty} \frac{\lambda(\lambda u)^{n-1-N_t}}{(n-1-N_t)!} e^{-\lambda u}\, \mathbb{1}_{\{u \geq 0\}}\, du\,.$$

◁

8.2.2 Martingale Properties

From the independence of the increments of the Poisson process, we derive the following martingale properties:

Proposition 8.2.2.1 *Let N be an* **F**-*Poisson process. For each $\alpha \in \mathbb{R}$, for each bounded Borel function h, the following processes are* **F**-*martingales:*

$$\begin{aligned}
&\text{(i)} && M_t := N_t - \lambda t, \\
&\text{(ii)} && M_t^2 - \lambda t = (N_t - \lambda t)^2 - \lambda t, && (8.2.2)\\
&\text{(iii)} && \exp(\alpha N_t - \lambda t(e^\alpha - 1)), && (8.2.3)\\
&\text{(iv)} && \exp\left(\int_0^t h(s)dN_s - \lambda \int_0^t (e^{h(s)} - 1)ds\right), \\
&\text{(v)} && \int_0^t h(s)dM_s\,, \\
&\text{(vi)} && \left(\int_0^t h(s)dM_s\right)^2 - \lambda \int_0^t h^2(s)ds\,.
\end{aligned}$$

PROOF: Let $s < t$. From the independence of the increments of the Poisson process, we obtain:

(i) $\mathbb{E}(M_t - M_s|\mathcal{F}_s) = \mathbb{E}(N_t - N_s) - \lambda(t-s) = 0$, hence M is a martingale.

(ii) The martingale property of M and the independence of the increments of the Poisson process imply

$$\begin{aligned}
\mathbb{E}(M_t^2 - M_s^2|\mathcal{F}_s) &= \mathbb{E}[(M_t - M_s)^2|\mathcal{F}_s] = \mathbb{E}[(N_t - N_s - \lambda(t-s))^2|\mathcal{F}_s] \\
&= \mathbb{E}[(N_t - N_s)^2] - \lambda^2(t-s)^2 \\
&= \mathbb{E}[N_{t-s}^2] - \lambda^2(t-s)^2 = \text{Var} N_{t-s}\,,
\end{aligned}$$

hence,

$$\mathbb{E}(M_t^2 - M_s^2|\mathcal{F}_s) = \lambda(t-s)\,,$$

and the process $(M_t^2 - \lambda t, t \geq 0)$ is a martingale.

(iii) From the form of the Laplace transform of N_t given in (8.2.1) and the independence of the increments, $\mathbb{E}[\exp[\alpha(N_t - N_s) - \lambda(t-s)(e^\alpha - 1)]\,|\mathcal{F}_s] = 1$, hence the martingale property of the process in (iii).

Assertions (iv-v-vi) can be proved first for elementary functions h of the form $h = \sum_i a_i \mathbb{1}_{]t_i, t_{i+1}]}$ and by then passing to the limit for general bounded Borel functions h. □

Exercise 8.2.2.2 Prove that, for any $\beta > -1$, any bounded Borel function h, and any bounded Borel function φ valued in $]-1, \infty[$, the processes

$$\exp[\ln(1+\beta)N_t - \lambda\beta t] = (1+\beta)^{N_t} e^{-\lambda\beta t},$$

$$\exp\left(\int_0^t h(s)dN_s + \lambda\int_0^t (1-e^{h(s)})ds\right)$$
$$= \exp\left(\int_0^t h(s)dM_s + \lambda\int_0^t (1+h(s)-e^{h(s)})ds\right),$$

$$\exp\left(\int_0^t \ln(1+\varphi(s))dN_s - \lambda\int_0^t \varphi(s)ds\right)$$
$$= \exp\left(\int_0^t \ln(1+\varphi(s))dM_s + \lambda\int_0^t (\ln(1+\varphi(s))-\varphi(s))ds\right),$$

are martingales.
`Hint:` These formulae are "avatars" of those of Proposition 8.2.2.1. ◁

Exercise 8.2.2.3 Prove (without using the following Proposition!) that the process $(\int_0^t N_{s-}dM_s, t \geq 0)$ is a martingale, and that the process $\int_0^t N_s dM_s$ is not a martingale. ◁

Definition 8.2.2.4 *The martingale $(M_t = N_t - \lambda t, t \geq 0)$ is called the* ***compensated process*** *of N, and λ the* ***intensity*** *of the process N.*

Remarks 8.2.2.5 (a) Note that the process M is a discontinuous martingale with bounded variation.

(b) We give an example of a martingale which is not square integrable. Let $X_t = \int_0^t \frac{1}{\sqrt{s}} dM_s$. The process X is a martingale, however, it is not square integrable.

The previous Proposition 8.2.2.1 can be generalized to predictable integrands:

Proposition 8.2.2.6 *Let N be an $\mathbf{F}$-Poisson process and let H be an $\mathbf{F}$-predictable bounded process. Then the following processes are martingales:*

$$\left.\begin{array}{l}
\text{(i)}\quad (H\star M)_t = \int_0^t H_s dM_s = \int_0^t H_s dN_s - \lambda\int_0^t H_s ds \\
\text{(ii)}\quad (H\star M)_t^2 - \lambda\int_0^t H_s^2 ds \\
\text{(iii)}\ \exp\left(\int_0^t H_s dN_s + \lambda\int_0^t (1-e^{H_s})ds\right) =: \mathcal{E}(H\star M)_t \\
\qquad\qquad = 1 + \int_0^t \mathcal{E}(H\star M)_{s-} H_s dM_s
\end{array}\right\} \quad (8.2.4)$$

Proof: One establishes (8.2.4) for predictable processes $(H_t, t \geq 0)$ of the form $H_t = K_S \mathbb{1}_{]S,T]}(t)$ where S and T are two stopping times and K_S is $\mathcal{F}_S$-measurable. In that case,

$$\int_0^t H_s dM_s = K_S(M_{T\wedge t} - M_{S\wedge t})$$

and the martingale property follows. Then, one passes to the limit. The same procedure can be applied to prove that the two processes (ii) and (iii) of (8.2.4) are martingales. □

We have used in (iii) the notation $\mathcal{E}(H \star M)_t$ for the Doléans-Dade exponential of the martingale $\int H_s dM_s$.

Comments 8.2.2.7 (a) If H satisfies $\mathbb{E}(\int_0^t |H_s| ds) < \infty$, the process in (i) is still a martingale.

(b) The results of Exercise 8.2.2.3 are now quite clear: in general, the martingale property (8.2.4) does not extend from predictable to adapted processes H. Indeed, from the definition of the stochastic integral w.r.t. N, and the fact that for every fixed s, $N_s - N_{s-} = 0, \mathbb{P}\, a.s.$,

$$\int_0^t (N_s - N_{s-}) dM_s = \int_0^t (N_s - N_{s-}) dN_s - \lambda \int_0^t (N_s - N_{s-}) ds$$
$$= N_t - \lambda \int_0^t (N_s - N_{s-}) ds = N_t \,.$$

Hence, the left-hand side, where one integrates the adapted (unpredictable) process $N_s - N_{s-}$ with respect to the martingale M, is not a martingale. Equivalently, the process

$$\int_0^t N_s dM_s = \int_0^t N_{s-} dM_s + N_t \,,$$

is not a martingale.

(c) Property (i) of Proposition 8.2.2.6 enables us to prove that the jump times $(T_i, i \geq 1)$ are not predictable. Indeed, if T_1 were a predictable stopping time, then the process $(\mathbb{1}_{\{t<T_1\}}, t \geq 0)$ would be predictable, however $\int_0^t \mathbb{1}_{\{s<T_1\}} dM_s = -\lambda(t \wedge T_1)$ is not a martingale. More generally, assume that T_i is predictable. Then, $(\int_0^t \mathbb{1}_{[T_i]}(s) dM_s, t \geq 0)$ would be a martingale and

$$\mathbb{E}\left(\int_0^t \mathbb{1}_{[T_i]}(s) dN_s\right) = \mathbb{E}\left(\mathbb{1}_{T_i \leq t}\left(N_{T_i} - N_{T_i-}\right)\right) = \mathbb{P}(T_i \leq t)$$

would be equal to $\mathbb{E}\left(\int_0^t \mathbb{1}_{[T_i]}(s) \lambda ds\right) = 0$, which is absurd.

Remark 8.2.2.8 Note that (i) and (ii) of Proposition 8.2.2.1 imply that the process $(M_t^2 - N_t; t \geq 0)$ is a martingale. Hence, there exist (at least) two increasing processes A such that $(M_t^2 - A_t, t \geq 0)$ is a martingale. The increasing process $(\lambda t, t \geq 0)$ is the predictable quadratic variation of M (denoted $\langle M \rangle$), whereas the increasing process $(N_t, t \geq 0)$ is the optional quadratic variation of M (denoted $[M]$). For any $\mu \in [0, 1]$, the process $(\mu N_t + (1-\mu)\lambda t; t \geq 0)$ is increasing and the process

$$M_t^2 - (\mu N_t + (1-\mu)\lambda t) = M_t^2 - \lambda t - \mu(N_t - \lambda t)$$

is a martingale. (See $\rightarrowtail$ Section 9.2 for the definition of quadratic variation if needed.)

8.2.3 Infinitesimal Generator

Proposition 8.2.3.1 *The Poisson process is a process with independent and stationary increments, and hence is a Markov process; its infinitesimal generator $\mathcal{L}$ is given by*

$$\mathcal{L}(f)(x) = \lambda[f(x+1) - f(x)]\,,$$

where f is a bounded Borel function.

PROOF: The Markov property follows from

$$\mathbb{E}(f(N_t)|\mathcal{F}_s^N) = \mathbb{E}(f(N_t - N_s + N_s)|\mathcal{F}_s^N) = F(t-s, N_s)$$

where $F(u,x) = \mathbb{E}(f(x+N_u))$ and $t \geq s$. We recall the definition of the infinitesimal generator:

$$\mathcal{L}(f)(x) = \lim_{t\to 0} \frac{1}{t}(\mathbb{E}(f(x+N_t)) - f(x))\,.$$

Hence, from $\mathbb{E}(f(x+N_t)) = \sum_{n=0}^{\infty} f(x+n)\mathbb{P}(N_t = n)$, we obtain

$$\frac{1}{t}(\mathbb{E}(f(x+N_t)) - f(x)) = e^{-\lambda t}\sum_{n=0}^{\infty} \frac{f(n+x)-f(x)}{t}\frac{(\lambda t)^n}{n!}\,.$$

From

$$e^{-\lambda t}\sum_{n\geq 2}\frac{(\lambda t)^n}{n!} \leq \frac{\lambda^2 t^2}{2}$$

the limit of $\frac{1}{t}(\mathbb{E}(f(x+N_t)) - f(x))$ when t goes to 0 is equal to the limit of $e^{-\lambda t}\frac{\lambda t}{t}(f(x+1) - f(x))$, that is to $\lambda(f(x+1) - f(x))$. □

Therefore, for any bounded Borel function f, the process

$$C_t^f = f(N_t) - f(0) - \int_0^t \mathcal{L}(f)(N_s)ds$$

is a martingale (see Proposition 1.1.14.2). Using that

$$f(N_t) - f(0) = \int_0^t (f(N_{s-}+1) - f(N_{s-}))\,dN_s\,, \tag{8.2.5}$$

the martingale $(C_t^f, t \geq 0)$ can be written as a stochastic integral with respect to the compensated martingale $(M_t = N_t - \lambda t, t \geq 0)$ as

$$C_t^f = \int_0^t [f(N_{s-}+1) - f(N_{s-})]dM_s\,.$$

Comment 8.2.3.2 Processes with independent and stationary increments are called Lévy processes, the reader may refer to $\rightarrowtail$ Chapter 11 for a more extended study.

Exercise 8.2.3.3 Extend formula (8.2.5) to functions f defined on $\mathbb{R}^+ \times \mathbb{N}$ that are C^1 with respect to the first variable, and prove that if β is a constant with $\beta > -1$ and $L_t = \exp(\log(1+\beta)N_t - \lambda\beta t)$, then $dL_t = L_{t-}\beta dM_t$.

More generally, let $L_t = (1+a)^{N_t}e^{-\lambda a t}$ for $a \in \mathbb{R}$. Prove that L satisfies $dL_t = L_{t-}adM_t$, i.e.,

$$L_t = 1 + \int_0^t L_{s-}\, adM_s = 1 + a\int_0^t L_{s-}\, dN_s - \lambda a \int_0^t L_{s-}\, ds\,.$$

Note that, for $a < -1$, L_t takes values in $\mathbb{R}$. The process L is the Doléans-Dade exponential of the martingale aM. ◁

Exercise 8.2.3.4 Let $T > 0$ be fixed and let $\varphi : [0, T] \to \mathbb{R}$ be a bounded Borel function and N a Poisson process. Prove that there exist a predictable process h and a constant c such that

$$\exp\left(\int_0^T \varphi(s)dN_s\right) = c + \int_0^T h_s dN_s\,.$$

`Hint:` Set $Z_t = \int_0^t \varphi(s)dN_s$. Then,

$$de^{Z_t} = \left(e^{Z_{t-}+\varphi(t)} - e^{Z_{t-}}\right)dN_t\,.$$

The reader may be interested to compare this simple result with the predictable representation theorem in Subsection 8.3.5. ◁

8.2.4 Change of Probability Measure: An Example

If N is a Poisson process with constant intensity λ, then, from Exercises 8.2.2.2 and 8.2.3.3, for $\beta > -1$, the process L defined by

$$L_t = (1+\beta)^{N_t}e^{-\lambda\beta t}$$

is a strictly positive martingale with expectation equal to 1. Let $\mathbb{Q}$ be the probability defined via $\mathbb{Q}|_{\mathcal{F}_t} = L_t\mathbb{P}|_{\mathcal{F}_t}$. From

$$\mathbb{E}_{\mathbb{Q}}(x^{N_t}) = \mathbb{E}_{\mathbb{P}}(L_t x^{N_t}) = e^{-\lambda\beta t}\mathbb{E}_{\mathbb{P}}([(1+\beta)x]^{N_t}) = \exp((1+\beta)\lambda t(x-1))$$

we deduce that the r.v. N_t follows the Poisson law with parameter $(1+\beta)\lambda t$ under $\mathbb{Q}$. Let $t_1 < \cdots < t_i < t_{i+1} < \cdots < t_n$ and let $(x_i, i \le n)$ be a sequence of positive real numbers. The equalities

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}\left(\prod_{i=1}^n x_i^{N_{t_{i+1}}-N_{t_i}}\right) &= \mathbb{E}_{\mathbb{P}}\left(e^{-\lambda\beta t}\prod_{i=1}^n((1+\beta)x_i)^{N_{t_{i+1}}-N_{t_i}}\right)\\ &= e^{-\lambda\beta t}\prod_{i=1}^n e^{-\lambda(t_{i+1}-t_i)}\, e^{\lambda(t_{i+1}-t_i)(1+\beta)x_i}\\ &= \prod_{i=1}^n e^{(1+\beta)\lambda(t_{i+1}-t_i)(x_i-1)}\end{aligned}$$

establish that, under $\mathbb{Q}$, $N_{t_{i+1}} - N_{t_i} \overset{\text{law}}{=} N_{t_{i+1}-t_i}$ is a Poisson r.v. with parameter $(1+\beta)\lambda(t_{i+1}-t_i)$ and that N has independent increments. Therefore, the process N is a $\mathbb{Q}$-Poisson process with intensity equal to $(1+\beta)\lambda$. Let us state this result as a proposition:

Proposition 8.2.4.1 *Let Π^λ be the probability on the canonical space which makes the coordinate process a Poisson process with intensity λ. Then, the following absolute continuity relationship holds:*

$$\boxed{\Pi^{(1+\beta)\lambda}|_{\mathcal{F}_t} = \left((1+\beta)^{N_t}e^{-\lambda\beta t}\right)\ \Pi^\lambda|_{\mathcal{F}_t}\,.}$$

Comment 8.2.4.2 One should note the analogy between the change of intensity of Poisson processes and the change of drift of a BM under a change of probability. However, let us point out a major difference. If $\mathbb{Q}$ is equivalent to $\mathbb{P}$, we know that if B is a $\mathbb{P}$-BM and $\widehat{B}$ is the martingale part of B under $\mathbb{Q}$, then $B_t^2 - t$ is a $\mathbb{P}$-martingale and $\widehat{B}_t^2 - t$ is a $\mathbb{Q}$-martingale (in other words the brackets are the same, i.e., $\langle B\rangle = \langle \widehat{B}\rangle$). If $\mathbb{Q}$ is equivalent to $\mathbb{P}$, and $M_t = N_t - \lambda t$ the compensated martingale associated with a Poisson process, the process $M_t^2 - \lambda t$ is a $\mathbb{P}$-martingale and the $\mathbb{P}$-(predictable) bracket of M is λt. We have proved above that the $\mathbb{Q}$-(predictable) bracket of $\widehat{M_t} = N_t - (1+\beta)\lambda t$ is $(1+\beta)\lambda t$. Hence, the predictable bracket is no longer the same under a change of probability. See ↣ Section 9.4 for a general Girsanov theorem and ↣ Subsection 11.3.1 for the case of Lévy processes.

8.2.5 Hitting Times

Let $x > 0$ and $T_x = \inf\{t, N_t \geq x\}$. Then, for $n-1 < x \leq n$, the hitting time $T_x = \inf\{t, N_t \geq n\} = \inf\{t, N_t = n\}$ is equal to the time of the n^{th}-jump of N, and hence has a Gamma (n, λ) law.

Exercise 8.2.5.1 Let $X_t = N_t + ct$. Compute $\mathbb{P}(\inf_{s\leq t} X_s \leq a)$. One should distinguish the cases $c > 0$ and $c < 0$. ◁

8.3 Inhomogeneous Poisson Processes

8.3.1 Definition

Instead of considering a constant intensity λ as before, now $(\lambda(t), t \geq 0)$ is an $\mathbb{R}^+$-valued Borel function satisfying $\int_0^t \lambda(u)du < \infty, \forall t$ and $\int_0^\infty \lambda(u)du = \infty$. An **inhomogeneous Poisson process** N with intensity λ is a counting process with independent increments which satisfies, for $t > s$,

$$\mathbb{P}(N_t - N_s = n) = e^{-\Lambda(s,t)} \frac{(\Lambda(s,t))^n}{n!} \tag{8.3.1}$$

where $\Lambda(s,t) = \Lambda(t) - \Lambda(s) = \int_s^t \lambda(u)du$, and $\Lambda(t) = \int_0^t \lambda(u)du$.

If $(T_n, n \geq 1)$ is the sequence of successive jump times associated with N, the law of T_n is:

$$\mathbb{P}(T_n \leq t) = \frac{1}{(n-1)!} \int_0^t \exp(-\Lambda(s)) \, (\Lambda(s))^{n-1} \, d\Lambda(s) \, .$$

It can easily be shown that an inhomogeneous Poisson process with deterministic intensity is an inhomogeneous Markov process. Moreover, since N_t has a Poisson law with parameter $\Lambda(t)$, one has $\mathbb{E}(N_t) = \Lambda(t), \mathrm{Var}(N_t) = \Lambda(t)$. For any real numbers u and α, for any $t \geq 0$,

$$\begin{aligned} \mathbb{E}(e^{iuN_t}) &= \exp((e^{iu} - 1)\Lambda(t)), \\ \mathbb{E}(e^{\alpha N_t}) &= \exp((e^{\alpha} - 1)\Lambda(t)) \, . \end{aligned}$$

An inhomogeneous Poisson process can be constructed as a deterministic time changed Poisson process, i.e., if $\widehat{N}$ is a Poisson process with constant intensity equal to 1, then $N_t = \widehat{N}_{\Lambda(t)}$ is an inhomogeneous Poisson process with intensity Λ.

We emphasize that we shall use the term Poisson process only when dealing with the standard Poisson process, i.e., when $\Lambda(t) = \lambda t$.

8.3.2 Martingale Properties

The martingale properties of a standard Poisson process can be extended to an inhomogeneous Poisson process:

Proposition 8.3.2.1 *Let N be an inhomogeneous Poisson process with deterministic intensity λ and $\mathbf{F}^N$ its natural filtration. The process*

$$M_t = N_t - \int_0^t \lambda(s)ds, \, t \geq 0$$

is an $\mathbf{F}^N$*-martingale. The increasing function* $\Lambda(t) := \int_0^t \lambda(s)ds$ *is called the (deterministic)* **compensator** *of* N*.*

Let ϕ *be an* $\mathbf{F}^N$*-predictable process such that* $\mathbb{E}(\int_0^t |\phi_s|\lambda(s)ds) < \infty$ *for every* t*. Then, the process* $(\int_0^t \phi_s dM_s, t \geq 0)$ *is an* $\mathbf{F}^N$*-martingale. In particular,*

$$\mathbb{E}\left(\int_0^t \phi_s \, dN_s\right) = \mathbb{E}\left(\int_0^t \phi_s \lambda(s) ds\right). \tag{8.3.2}$$

As in the constant intensity case, for any bounded $\mathbf{F}^N$-predictable process H, the following processes are martingales:

$$\begin{aligned}
&(i) && (H{\star}M)_t = \int_0^t H_s dM_s = \int_0^t H_s dN_s - \int_0^t \lambda(s) H_s ds\,,\\
&(ii) && (H{\star}M)_t^2 - \int_0^t \lambda(s) H_s^2 ds\,,\\
&(iii) && \exp\left(\int_0^t H_s dN_s - \int_0^t \lambda(s)(e^{H_s} - 1) ds\right).
\end{aligned}$$

8.3.3 Watanabe's Characterization of Inhomogeneous Poisson Processes

The study of inhomogeneous Poisson processes can be generalized to the case where the intensity is not absolutely continuous with respect to the Lebesgue measure. In this case, Λ is an increasing, right-continuous, deterministic function with value zero at time zero, and it satisfies $\Lambda(\infty) = \infty$. If N is a counting process with independent increments and if (8.3.1) holds, the process $(N_t - \Lambda(t), t \geq 0)$ is a martingale and for any bounded predictable process ϕ, the equality $\mathbb{E}(\int_0^t \phi_s \, dN_s) = \mathbb{E}(\int_0^t \phi_s d\Lambda(s))$ is satisfied for any t. This result admits a converse.

Proposition 8.3.3.1 (Watanabe's Characterization.) *Let* N *be a counting process and* Λ *an increasing, continuous function with value zero at time zero. Let us assume that the process* $(M_t := N_t - \Lambda(t),\ t \geq 0)$ *is a martingale. Then* N *is an inhomogeneous Poisson process with compensator* Λ*. It is a Poisson process if* $\Lambda(t) = \lambda t$*.*

PROOF: Let $s < t$ and $\theta > 0$.

$$\begin{aligned}
e^{\theta N_t} - e^{\theta N_s} &= \sum_{s<u\leq t} e^{\theta N_u} - e^{\theta N_{u-}}\\
&= \sum_{s<u\leq t} e^{\theta N_{u-}} (e^\theta - 1) \Delta N_u = (e^\theta - 1) \int_{]s,t]} e^{\theta N_{u-}} dN_u\\
&= (e^\theta - 1) \left(\int_{]s,t]} e^{\theta N_{u-}} dM_u + \int_{]s,t]} e^{\theta N_u} d\Lambda(u)\right).
\end{aligned}$$

By relying on the fact that the first integral is a martingale,

$$\begin{aligned}\mathbb{E}(e^{\theta N_t} - e^{\theta N_s}|\mathcal{F}_s) &= (e^\theta - 1)\mathbb{E}\left(\int_{]s,t]} e^{\theta N_u} d\Lambda(u)|\mathcal{F}_s\right) \\ &= (e^\theta - 1)\int_{]s,t]} \mathbb{E}\,(e^{\theta N_u}|\mathcal{F}_s) d\Lambda(u)\,.\end{aligned}$$

Let s be fixed and define $\phi(t) = \mathbb{E}(e^{\theta N_t}|\mathcal{F}_s)$. Then, for $t > s$,

$$\phi(t) = \phi(s) + (e^\theta - 1)\int_s^t \phi(u) d\Lambda(u)\,.$$

Solving this equation leads to

$$\phi(t) = e^{\theta N_s} \exp\left[(e^\theta - 1)\int_s^t d\Lambda(u)\right]\,.$$

This shows that the process N has independent increments and that, for $s < t$, the r.v. $N_t - N_s$ has a Poisson law with parameter $\Lambda(t) - \Lambda(s)$. □

8.3.4 Stochastic Calculus

In this section, M is the compensated martingale of an inhomogeneous Poisson process N with deterministic intensity $(\lambda(s), s \geq 0)$. From now on, we restrict our attention to integrals of predictable processes, even if the stochastic integral is defined in a more general setting.

Integration by Parts Formula

Let x and y be two predictable processes and define two processes X and Y as

$$X_t = x + \int_0^t x_s dN_s, \quad Y_t = y + \int_0^t y_s dN_s\,.$$

The jumps of X (resp. of Y) occur at the same times as the jumps of N and $\Delta X_s = x_s \Delta N_s, \Delta Y_s = y_s \Delta N_s$. The processes X and Y are of finite variation and are constant between two jumps. Then, it is easy to check that

$$X_t Y_t = xy + \sum_{s\leq t} \Delta(XY)_s = xy + \sum_{s\leq t} X_{s-}\Delta Y_s + \sum_{s\leq t} Y_{s-}\Delta X_s + \sum_{s\leq t} \Delta X_s\,\Delta Y_s$$

We shall write this equality as

$$\boxed{X_t Y_t = xy + \int_0^t Y_{s-} dX_s + \int_0^t X_{s-} dY_s + [X,Y]_t}$$

where (note that $(\Delta N_t)^2 = \Delta N_t$)

$$[X,Y]_t := \sum_{s\le t} \Delta X_s \, \Delta Y_s = \sum_{s\le t} x_s y_s \Delta N_s = \int_0^t x_s y_s dN_s \, .$$

More generally (a general discussion is proposed in $\rightarrowtail$ Chapter 9 and 10), if

$$\begin{aligned} dX_t &= h_t dt + x_t dN_t, \; X_0 = x \\ dY_t &= \widetilde{h}_t dt + y_t dN_t, \; Y_0 = y \, , \end{aligned}$$

one still gets

$$X_t Y_t = xy + \int_0^t Y_{s-} dX_s + \int_0^t X_{s-} dY_s + [X,Y]_t$$

where

$$[X,Y]_t = \int_0^t x_s y_s dN_s \, .$$

In particular, if $dX_t = x_t dM_t$ and $dY_t = y_t dM_t$, the process $X_t Y_t - [X,Y]_t$ is a local martingale.

Itô's Formula

For Poisson processes, Itô's formula is obvious as we now explain. We shall give an extension of this formula for more general processes in the following Chapter 9.

Let N be a Poisson process and f a bounded Borel function. The trivial equality

$$f(N_t) = f(N_0) + \sum_{0<s\le t} f(N_s) - f(N_{s^-}) \tag{8.3.3}$$

is the main step in obtaining Itô's formula for a Poisson process.

We can write the right-hand side of (8.3.3) as a stochastic integral:

$$\begin{aligned} \sum_{0<s\le t} f(N_s) - f(N_{s^-}) &= \sum_{0<s\le t} [f(N_{s^-}+1) - f(N_{s^-})] \, \Delta N_s \\ &= \int_0^t [f(N_{s^-}+1) - f(N_{s^-})] \, dN_s \, , \end{aligned}$$

hence, the canonical decomposition of the semi-martingale $f(N)$ as the sum of a martingale and an absolutely continuous adapted process is

$$f(N_t) = f(N_0) + \int_0^t [f(N_{s^-}+1) - f(N_{s^-})] dM_s + \int_0^t [f(N_{s^-}+1) - f(N_{s^-})] \lambda ds \, .$$

It is straightforward to generalize this result. Let

$$X_t = x + \int_0^t x_s dN_s = x + \sum_{T_n \le t} x_{T_n} ,$$

with x a predictable process. The process $(X_t, t \ge 0)$ has at time T_n, a jump of size $(\Delta X)_{T_n} = x_{T_n}$, and is constant between two consecutive jumps. The obvious identity

$$F(X_t) = F(X_0) + \sum_{s \le t} F(X_s) - F(X_{s-}) ,$$

holds for any bounded function F. The number of jumps before t is a.s. finite, and the sum is well defined. This formula can be written in an equivalent form:

$$\begin{aligned} F(X_t) - F(X_0) &= \sum_{s \le t} (F(X_s) - F(X_{s-})) \Delta N_s \\ &= \int_0^t (F(X_s) - F(X_{s-})) dN_s = \int_0^t (F(X_{s-} + x_s) - F(X_{s-})) dN_s \end{aligned}$$

where the integral on the right-hand side is a Stieltjes integral. More generally again, we have the following result

Proposition 8.3.4.1 *Let h be an adapted process, x a predictable process and*

$$dX_t = h_t dt + x_t dM_t = (h_t - x_t \lambda(t))dt + x_t dN_t$$

where N is an inhomogeneous Poisson process. Let $F \in C^{1,1}(\mathbb{R}^+ \times \mathbb{R})$. Then

$$\begin{aligned} F(t, X_t) = &\int_0^t [F(s, X_{s-} + x_s) - F(s, X_{s-})] dM_s \\ &+ \int_0^t (\partial_t F(s, X_s) + \partial_x F(s, X_s) h_s) \, ds \\ &+ \int_0^t (F(s, X_{s-} + x_s) - F(s, X_{s-}) - \partial_x F(s, X_{s-}) x_s) \, \lambda(s) ds \, . \end{aligned} \tag{8.3.4}$$

PROOF: Indeed, between two jumps of the process N, $dX_t = (h_t - \lambda(t)x_t)dt$, and for $T_n < s < t < T_{n+1}$,

$$F(t, X_t) = F(s, X_s) + \int_s^t \partial_t F(u, X_u) du + \int_s^t \partial_x F(u, X_u)(h_u - x_u \lambda(u)) du \, .$$

At jump times T_n, one has $F(T_n, X_{T_n}) = F(T_n, X_{T_n-}) + \Delta F(\cdot, X)_{T_n}$. Hence,

$$\begin{aligned} F(t, X_t) = F(0, X_0) &+ \int_0^t \partial_t F(s, X_s) ds + \int_0^t \partial_x F(s, X_s)(h_s - x_s \lambda(s)) \, ds \\ &+ \sum_{s \le t} (F(s, X_s) - F(s, X_{s-})) \, . \end{aligned}$$

This formula can be written as

$$\begin{aligned}F(t,X_t) - F(0,X_0) &= \int_0^t \partial_t F(s,X_s)ds + \int_0^t \partial_x F(s,X_s)(h_s - x_s\lambda(s))ds \\ &\quad + \int_0^t [F(s,X_s) - F(s,X_{s-})]dN_s \\ &= \int_0^t \partial_t F(s,X_s)ds + \int_0^t \partial_x F(s,X_{s-})dX_s \\ &\quad + \int_0^t [F(s,X_s) - F(s,X_{s-}) - \partial_x F(s,X_{s-})x_s]dN_s \\ &= \int_0^t \partial_t F(s,X_s)ds + \int_0^t \partial_x F(s,X_{s-})dX_s \\ &\quad + \int_0^t [F(s,X_{s-}+x_s) - F(s,X_{s-}) - \partial_x F(s,X_{s-})x_s]dN_s\,.\end{aligned}$$

One can also write

$$\begin{aligned}F(t,X_t) = F(0,X_0) &+ \int_0^t \partial_t F(s,X_s)ds + \int_0^t \partial_x F(s,X_{s-})dX_s \\ &+ \sum_{s\le t} [F(s,X_s) - F(s,X_{s-}) - \partial_x F(s,X_{s-})x_s \Delta N_s]\ .\end{aligned}$$

which is easy to memorize. The first three terms on the right-hand side are obtained from "ordinary" calculus, the fourth term takes into account the jumps of the left-hand side and of the stochastic integral on the right-hand side.

Remarks 8.3.4.2 (a) In the "ds" integrals, we can write X_{s-} or X_s, since, for any bounded Borel function f,

$$\int_0^t f(X_{s-})ds = \int_0^t f(X_s)ds\,.$$

Note that since dN_s a.s. $N_s = N_{s-} + 1$, one has

$$\int_0^t f(N_{s-})dN_s = \int_0^t f(N_s - 1)dN_s\,.$$

However, we systematically use the form $\int_0^t f(N_{s-})dN_s$, even though the integral $\int_0^t f(N_s - 1)dN_s$ has a meaning. The reason is that

$$\int_0^t f(N_{s-})dM_s = \int_0^t f(N_{s-})dN_s - \lambda \int_0^t f(N_s)ds$$

is a martingale, whereas $\int_0^t f(N_s - 1)dM_s$ is not.

(b) We have named Itô's formula a formula allowing us to write the process $F(t, X_t)$ as a sum of stochastic integrals, as in equation (8.3.5). In fact, the aim of Itô's formula is to give, under some suitable conditions on F, the canonical decomposition of the semi-martingale $F(t, X_t)$.

Exercise 8.3.4.3 Let N be a Poisson process with intensity λ. Prove that, if $S_t = S_0 e^{\mu t + \sigma N_t}$, then

$$dS_t = S_{t^-}(\mu dt + (e^\sigma - 1)dN_t)$$

and that S is a martingale iff $\mu = -\lambda(e^\sigma - 1)$. Prove that, for $a + 1 > 0$, the process $(L_t = \exp(N_t \ln(1+a) - \lambda a t), t \geq 0)$ is a martingale and that, if $\mathbb{Q}|_{\mathcal{F}_t} = L_t \mathbb{P}|_{\mathcal{F}_t}$, the process N is a $\mathbb{Q}$-Poisson process with intensity $\lambda(1+a)$. Note the progression made from Exercise 8.2.3.3. ◁

Exercise 8.3.4.4 The aim of this exercise is to prove that the linear equation $dZ_t = Z_{t^-}\mu dM_t$, $Z_0 = 1$ with $\mu > -1$ has a unique solution. Assume that Z^1 and Z^2 are two solutions. W.l.g., we can assume that Z^2 is strictly positive. Prove that Z^1/Z^2 satisfies an ordinary differential equation with unique solution equal to 1. ◁

8.3.5 Predictable Representation Property

Proposition 8.3.5.1 *Let $\mathbf{F}^N$ be the natural filtration of the standard Poisson process N and let $H \in L^2(\mathcal{F}^N_\infty)$ be a square integrable random variable. Then, there exists a unique $\mathbf{F}^N$-predictable process $(h_t, t \geq 0)$ such that*

$$H = \mathbb{E}(H) + \int_0^\infty h_s dM_s$$

and $\mathbb{E}(\int_0^\infty h_s^2 ds) < \infty$.

PROOF: The family of exponential random variables

$$Y = \exp\left(\int_0^\infty \varphi(s) dN_s - \lambda \int_0^\infty (e^{\varphi(s)} - 1) ds\right),$$

where φ is a bounded deterministic function with compact support, is total in $L^2(\mathcal{F}^N_\infty)$. Any Y in this family can be written as a stochastic integral with respect to dM. Indeed, from Exercise 8.2.2.2 the process

$$Y_t = \exp\left(\int_0^t \varphi(s) dN_s - \lambda \int_0^t (e^{\varphi(s)} - 1) ds\right) = \mathbb{E}(Y|\mathcal{F}^N_t)$$

is a martingale, and is the solution of

$$dY_t = Y_{t-}(e^{\varphi(t)} - 1)dM_t ,$$

so that,

$$Y = 1 + \int_0^\infty Y_{s-}(e^{\varphi(s)} - 1)dM_s .$$

Hence, with the notation of the statement, $h_s = Y_{s-}(e^{\varphi(s)} - 1)$. For more general random variables, the result follows by passing to the limit, owing to the isometry formula

$$\mathbb{E}\left(\int_0^\infty h_s dM_s\right)^2 = \lambda \mathbb{E}\left(\int_0^\infty h_s^2 ds\right) .$$

□

Comment 8.3.5.2 This result goes back to Brémaud and Jacod [125], Chou and Meyer [180], Davis [219].

8.3.6 Multidimensional Poisson Processes

Definition 8.3.6.1 *A process $(N^1, \ldots, N^d)$ is a d-dimensional* **F**-*Poisson process if each component N^j is a right-continuous adapted process such that $N_0^j = 0$ and if there exist positive constants λ_j such that for every $t \geq s \geq 0$ and every integer n_j*

$$\mathbb{P}\left[\bigcap_{j=1}^d (N_t^j - N_s^j = n_j)|\mathcal{F}_s\right] = \prod_{j=1}^d e^{-\lambda_j(t-s)} \frac{(\lambda_j(t-s))^{n_j}}{n_j!} .$$

Note that the processes $(N^j, j = 1, \ldots, d)$ are independent; more generally, for any s, the processes $\Big((N_{s+t}^j - N_s^j, j = 1, \ldots, d), t \geq 0\Big)$ are independent and also independent of $\mathcal{F}_s$.

Proposition 8.3.6.2 *An* **F**-*adapted process N is a d-dimensional* **F**-*Poisson process if and only if:*

(i) *each N^j is an* **F**-*Poisson process,*
(ii) *no two N^j's jump simultaneously $\mathbb{P}$ a.s..*

PROOF: We give the proof for $d = 2$.

(a) We assume (i) and (ii). For any pair (f, g) of bounded Borel functions, the process

$$X_t = \exp\left(\int_0^t f(s)dN_s^1 + \int_0^t g(s)dN_s^2\right)$$

satisfies

$$X_t = 1 + \sum_{0<s\leq t} \Delta X_s = 1 + \sum_{0<s\leq t} X_{s-}\left[\exp(f(s)\Delta N_s^1 + g(s)\Delta N_s^2) - 1\right].$$

From condition (ii)

$$X_t = 1 + \sum_{0<s\le t} X_{s-}\left[(e^{f(s)}-1)\Delta N^1_s + (e^{g(s)}-1)\Delta N^2_s\right],$$

hence, from the martingale property of the compensated process $N^i_t - \lambda_i t$:

$$\begin{aligned}\mathbb{E}(X_t) &= 1 + \mathbb{E}\left[\int_0^t X_{s-}\left((e^{f(s)}-1)\lambda_1 + (e^{g(s)}-1)\lambda_2\right)ds\right]\\ &= 1 + \int_0^t \mathbb{E}[X_s]\left((e^{f(s)}-1)\lambda_1 + (e^{g(s)}-1)\lambda_2\right)ds\,.\end{aligned}$$

Therefore, solving this equation, we find

$$\begin{aligned}\mathbb{E}(X_t) &= \exp\left(\int_0^t (e^{f(s)}-1)\lambda_1 ds\right)\exp\left(\int_0^t (e^{g(s)}-1)\lambda_2 ds\right)\\ &= \mathbb{E}\left[\exp\left(\int_0^t f(s)dN^1_s\right)\right]\mathbb{E}\left[\exp\left(\int_0^t g(s)dN^2_s\right)\right].\end{aligned}$$

The result follows.

(b) Conversely, if N is a d-dimensional Poisson process, then (i) and (ii) hold. □

Comment 8.3.6.3 Another proof follows from the predictable representation theorem valid for M^1 and M^2 individually. Let $H^i \in L^2(\mathcal{F}^i_\infty)$ for $i = 1, 2$. From $H^i = \mathbb{E}(H^i) + \int_0^\infty h^i_s dM^i_s$ and the integration by parts formula, we deduce that $\mathbb{E}(H^1H^2) = \mathbb{E}(H^1)\mathbb{E}(H^2)$ if and only if $[M^i, M^j] = 0$.

In order to construct correlated Poisson processes, one can proceed as follows. Let $(N^i, i = 1, 2, 3)$ be independent Poisson processes. Then the processes $\hat{N} = N^1 + N^2$ and $\tilde{N} = N^1 + N^3$ are correlated Poisson processes.

Exercise 8.3.6.4 Let $(N^i, i = 1, 2)$ be two independent Poisson processes. Prove that $N = N^1 + N^2$ is a Poisson process. Compute the compensator of N. Let $\tau^i = \inf\{t : N^i_t = 1\}$ and $\tau = \inf\{t : N_t = 1\}$. Compute $\mathbb{P}(\tau = \tau^1)$. ◁

8.4 Stochastic Intensity Processes

8.4.1 Doubly Stochastic Poisson Processes

Let $\mathbf{F}$ be a given filtration, where $\mathcal{F}_0$ is not the trivial σ-algebra; let N be a counting process which is $\mathbf{F}$-adapted and let λ be a positive process such that for any t, λ_t is $\mathcal{F}_0$-measurable and $\int_0^t \lambda_s ds < \infty, \mathbb{P}$ a.s.. Let $\Lambda(s,t) = \int_s^t \lambda_u du$. If

$$\mathbb{E}(e^{i\alpha(N_t - N_s)}|\mathcal{F}_s) = \exp\left((e^{i\alpha}-1)\Lambda(s,t)\right)$$

for any $t > s$ and any α, then N is called a **doubly stochastic Poisson process**. In that case,

$$\mathbb{P}(N_t - N_s = k|\mathcal{F}_s) = \exp(-\Lambda(s,t))\frac{(\Lambda(s,t))^n}{n!}$$

and the process $(N_t - \int_0^t \lambda_u du, t \geq 0)$ is an $\mathbf{F}$-martingale.

Comment 8.4.1.1 Doubly stochastic intensity processes are used in finance to model the intensity of default process (see Schönbucher [765]).

8.4.2 Inhomogeneous Poisson Processes with Stochastic Intensity

Definition 8.4.2.1 *Let $\mathbf{F}$ be a given filtration, N an $\mathbf{F}$-adapted counting process, and $(\lambda_t, t \geq 0)$ a positive $\mathbf{F}$-progressively measurable process such that for every t, $\Lambda_t := \int_0^t \lambda_s ds < \infty$, $\mathbb{P}$ a.s..*

*The process N is an **inhomogeneous Poisson process with stochastic intensity** λ if for every positive $\mathbf{F}$-predictable process $(\phi_t, t \geq 0)$ the following equality is satisfied:*

$$\mathbb{E}\left(\int_0^\infty \phi_s \, dN_s\right) = \mathbb{E}\left(\int_0^\infty \phi_s \lambda_s ds\right) .$$

Therefore $(M_t = N_t - \Lambda_t,\ t \geq 0)$ is an $\mathbf{F}$-local martingale and an $\mathbf{F}$-martingale if for every t, $\mathbb{E}(\Lambda_t) < \infty$.

Proposition 8.4.2.2 *Let N be an inhomogeneous Poisson process with stochastic intensity λ. Then, for any $\mathbf{F}$-predictable process ϕ such that $\forall t$, $\mathbb{E}(\int_0^t |\phi_s|\lambda_s ds) < \infty$, the process $(\int_0^t \phi_s dM_s,\ t \geq 0)$ is an $\mathbf{F}$-martingale.*

The intensity depends in an important manner of the reference filtration. For example, the $\mathbf{F}^N$-intensity of N is $\mathbb{E}(\lambda_s|\mathcal{F}_s^N)$, i.e.,

$$N_t - \int_0^t \mathbb{E}(\lambda_s|\mathcal{F}_s^N)ds$$

is an $\mathbf{F}^N$-martingale. This is a particular case of the general filtering formula given in Proposition 5.10.3.1.

An inhomogeneous Poisson process N with stochastic intensity λ_t can be viewed as a time change of a standard Poisson process $\tilde{N}$, i.e., $N_t = \tilde{N}_{\Lambda_t}$.

8.4.3 Itô's Formula

The formula obtained in Subsection 8.3.4 can be generalized to inhomogeneous Poisson processes with stochastic intensities.

8.4.4 Exponential Martingales

We now extend Exercise 8.2.3.3 to more general Doléans-Dade exponentials:

Proposition 8.4.4.1 *Let N be an inhomogeneous Poisson process with stochastic intensity $(\lambda_t,\, t \geq 0)$, and $(\mu_t,\, t \geq 0)$ a predictable process such that, for any t, $\int_0^t |\mu_s|\lambda_s\, ds < \infty$. Let $(T_n, n \geq 1)$ be the sequence of jump times of N. Then, the process L, the solution of*

$$dL_t = L_{t-}\mu_t dM_t, \quad L_0 = 1\,, \tag{8.4.1}$$

is a local martingale defined by

$$L_t = \begin{cases} \exp(-\int_0^t \mu_s\lambda_s\, ds) & \textit{if}\; t < T_1 \\ \prod_{n,T_n \leq t}(1+\mu_{T_n}) \exp(-\int_0^t \mu_s\lambda_s\, ds) & \textit{if}\; t \geq T_1\,. \end{cases} \tag{8.4.2}$$

Moreover, if μ is such that $\mu_s > -1\, a.s. \forall s$, then

$$L_t = \exp\left[-\int_0^t \mu_s\lambda_s ds + \int_0^t \ln(1+\mu_s)\, dN_s\right]\,.$$

Later, we shall simply write the equalities (8.4.2) as

$$L_t = \prod_{n,T_n \leq t} (1+\mu_{T_n}) \exp\left(-\int_0^t \mu_s\lambda_s\, ds\right)$$

with the understanding that $\prod_\emptyset = 1$.

Proof: From general results on SDE, the linear equation (8.4.1) admits a unique solution (see also Exercise 8.3.4.4). Between two consecutive jumps, the solution of the equation (8.4.1) satisfies

$$dL_t = -L_{t-}\mu_t\lambda_t dt$$

therefore, for $t \in [T_n, T_{n+1}[$, we obtain

$$L_t = L_{T_n} \exp\left(-\int_{T_n}^t \mu_s\lambda_s ds\right)\,.$$

The jumps of L occur at the same times as the jumps of N and the size of the jumps is $\Delta L_t = L_{t-}\mu_t \Delta N_t$, therefore $L_{T_n} = L_{T_n-}(1+\mu_{T_n})$. By backward recurrence on n, we get (8.4.2). □

The local martingale L is denoted by $\mathcal{E}(\mu\star M)$ and called the **Doléans-Dade exponential** of the process $\mu\star M$. The process L can also be written

$$L_t = \prod_{0<s\leq t} (1+\mu_s \Delta N_s\,) \exp\left[-\int_0^t \mu_s\, \lambda_s\, ds\right]\,. \tag{8.4.3}$$

Moreover, if for every t, $\mu_t > -1$, then L is a positive local martingale, therefore it is a supermartingale and

$$\begin{aligned} L_t &= \exp\left[-\int_0^t \mu_s\lambda_s ds + \sum_{s\le t}\ln(1+\mu_s)\varDelta N_s\right] \\ &= \exp\left[-\int_0^t \mu_s\lambda_s ds + \int_0^t \ln(1+\mu_s)\, dN_s\right] \\ &= \exp\left[\int_0^t [\ln(1+\mu_s)-\mu_s]\,\lambda_s\, ds + \int_0^t \ln(1+\mu_s)\, dM_s\right] . \end{aligned}$$

The process L is a martingale if $\forall t$, $\mathbb{E}(L_t) = 1$. This is the case if μ is bounded. We shall see a more general criterion in ↣ Subsection 9.4.3.

If μ is not greater than -1, then the process L defined in (8.4.2) is still a local martingale which satisfies $dL_t = L_{t^-}\mu_t dM_t$. However it may be negative.

Example 8.4.4.2 A useful example is the case where $\mu \equiv -1$. In this case, we obtain that $\mathbb{1}_{\{t<T_1\}} \exp\left(\int_0^t \lambda_s ds\right)$ is a local martingale. Note that we have obtained similar results in Chapter 7 for processes with a single jump.

8.4.5 Change of Probability Measure

We establish now a particular case of the general Girsanov theorem (see ↣ Section 9.4 for a general case).

Proposition 8.4.5.1 *Let μ be a predictable process such that $\mu > -1$ and $\int_0^t \lambda_s|\mu_s|ds < \infty$ a.s.. Let L be the positive exponential local martingale solution of $dL_t = L_{t^-}\mu_t dM_t$. Assume that L is a martingale and let $\mathbb{Q}$ be the probability measure (locally equivalent to $\mathbb{P}$) defined on $\mathcal{F}_t$ by $\mathbb{Q}|_{\mathcal{F}_t} = L_t\,\mathbb{P}|_{\mathcal{F}_t}$. Then, under $\mathbb{Q}$, the process M^μ defined as*

$$M_t^\mu := M_t - \int_0^t \mu_s\lambda_s ds = N_t - \int_0^t (\mu_s+1)\lambda_s\, ds\ , t\ge 0$$

is a local martingale.

Proof: From the integration by parts formula, we get

$$\begin{aligned} d(M^\mu L)_t &= M_{t^-}^\mu dL_t + L_{t^-} dM_t^\mu + d[L, M^\mu]_t \\ &= M_{t^-}^\mu dL_t + L_{t^-} dM_t^\mu + L_{t^-}\mu_t dN_t \\ &= M_{t^-}^\mu dL_t + L_{t^-} dM_t + L_{t^-}\mu_t dM_t = (M_{t^-}^\mu \mu_t + 1 + \mu_t)L_{t^-} dM_t\,, \end{aligned}$$

hence, the process $M^\mu L$ is a $\mathbb{P}$-local martingale and M^μ is a $\mathbb{Q}$-local martingale. If μ and λ are deterministic, the process N is a $\mathbb{Q}$-inhomogeneous Poisson process with deterministic intensity $(\mu(t)+1)\lambda(t)$. □

Comment 8.4.5.2 We have seen that a Poisson process with stochastic intensity can be viewed as a time-changed of a standard Poisson process. Here, we interpret a Poisson process with stochastic intensity as a Poisson process with constant intensity under a change of probability. Indeed, a Poisson process with intensity 1 under $\mathbb{P}$ is a Poisson process with stochastic intensity $(\lambda_t, t \geq 0)$ under $\mathbb{Q}^\lambda$, where $\mathbb{Q}^\lambda|_{\mathcal{F}_t} = L_t^\lambda \, \mathbb{P}|_{\mathcal{F}_t}$ and where $dL_t^\lambda = L_{t^-}^\lambda (\lambda_t - 1) dM_t$.

8.4.6 An Elementary Model of Prices Involving Jumps

Suppose that S is a stochastic process with dynamics given by

$$dS_t = S_{t^-}(b(t)dt + \phi(t)dM_t), \tag{8.4.4}$$

where M is the compensated martingale associated with an inhomogeneous Poisson process N with strictly positive deterministic intensity λ and where b, ϕ are deterministic continuous functions. We assume that $\phi > -1$ so that the process S remains strictly positive. The solution of (8.4.4) is

$$\begin{aligned} S_t &= S_0 \exp\left[-\int_0^t \phi(s)\lambda(s)ds + \int_0^t b(s)ds\right] \prod_{s \leq t}(1 + \phi(s)\Delta N_s) \\ &= S_0 \exp\left[\int_0^t b(s)ds\right] \exp\left[\int_0^t \ln(1+\phi(s))dN_s - \int_0^t \phi(s)\lambda(s)ds\right]. \end{aligned}$$

Hence $S_t \exp\left(-\int_0^t b(s)ds\right)$ is a strictly positive local martingale.

We assume now that S is the dynamics of the price of a financial asset under the historical probability measure. We denote by r the deterministic interest rate and by $R_t = \exp(-\int_0^t r(s)ds)$ the discount factor. It is important to give a necessary and sufficient condition under which the financial market with the asset S and the riskless asset is complete and arbitrage free when ϕ does not vanish. Therefore, our aim is to give conditions such that there exists a probability measure $\mathbb{Q}$, equivalent to $\mathbb{P}$, under which the discounted process SR is a local martingale.

Any $\mathbf{F}^M$-martingale admits a representation as a stochastic integral with respect to M. Hence, any strictly positive $\mathbf{F}^M$-martingale L can be written as $dL_t = L_{t^-}\mu_t dM_t$ where μ is an $\mathbf{F}^M$-predictable process such that $\mu > -1$ and, if $L_0 = 1$, the martingale L can be used as a Radon-Nikodým density. We are looking for conditions on μ such that the process RS is a $\mathbb{Q}$-local martingale where $d\mathbb{Q}|_{\mathcal{F}_t} = L_t d\mathbb{P}|_{\mathcal{F}_t}$; or equivalently, the process $(Y_t = R_t S_t L_t, t \geq 0)$ is a $\mathbb{P}$-local martingale. Integration by parts yields

$$\begin{aligned} dY_t &\overset{\text{mart}}{=} Y_{t^-}\left((b(t) - r(t))dt + \phi(t)\mu_t d[M]_t\right) \\ &\overset{\text{mart}}{=} Y_{t^-}\left(b(t) - r(t) + \phi(t)\mu_t\lambda(t)\right) dt\,. \end{aligned}$$

Hence, Y is a $\mathbb{P}$-local martingale if and only if $\mu_t = -\dfrac{b(t) - r(t)}{\phi(t)\lambda(t)}$.

Assume that $\mu > -1$ and define $\mathbb{Q}|_{\mathcal{F}_t} = L_t \mathbb{P}|_{\mathcal{F}_t}$. The process N is an inhomogeneous $\mathbb{Q}$-Poisson process with intensity $((\mu(s)+1)\lambda(s), s \geq 0)$ and

$$dS_t = S_{t_-}(r(t)dt + \phi(t)dM_t^{\mu})$$

where $(M^{\mu}(t) = N_t - \int_0^t (\mu(s)+1)\lambda(s)\,ds\,,\, t \geq 0)$ is the compensated $\mathbb{Q}$-martingale. Hence, the discounted price SR is a $\mathbb{Q}$-local martingale. In this setting, $\mathbb{Q}$ is the unique equivalent martingale measure.

The condition $\mu > -1$ is needed in order to obtain at least one e.m.m. and, from the fundamental theorem of asset pricing, to deduce the absence of arbitrage property.

If μ fails to be greater than -1, there does not exist an e.m.m. and there are arbitrages in the market. We now make explicit an arbitrage opportunity in the particular case when the coefficients are constant with $\phi > 0$ and $\dfrac{b-r}{\phi\lambda} > 1$, hence $\mu < -1$. The inequality

$$S_t = S_0 \exp[(b-\phi\lambda)t] \prod_{s \leq t}(1+\phi\Delta N_s) > S_0 e^{rt} \prod_{s \leq t}(1+\phi\Delta N_s) > S_0 e^{rt}$$

proves that an agent who borrows S_0 and invests in a long position in the underlying has an arbitrage opportunity, since his terminal wealth at time T $S_T - S_0 e^{rT}$ is strictly positive with probability one. Note that, in this example, the process $(S_t e^{-rt}, t \geq 0)$ is increasing.

Comment 8.4.6.1 We have required that ϕ and b are continuous functions in order to avoid integrability conditions. Obviously, we can generalize, to some extent, to the case of Borel functions. Note that, since we have assumed that $\phi(t)$ does not vanish, there is the equality of σ-fields

$$\sigma(S_s, s \leq t) = \sigma(N_s, s \leq t) = \sigma(M_s, s \leq t)\,.$$

8.5 Poisson Bridges

Let N be a Poisson process with constant intensity λ, $\mathcal{F}_t^N = \sigma(N_s, s \leq t)$ its natural filtration and $T > 0$ a fixed time. Let $\mathcal{G}_t = \sigma(N_s, s \leq t; N_T)$ be the natural filtration of N enlarged with the terminal value N_T of the process N.

8.5.1 Definition of the Poisson Bridge

Proposition 8.5.1.1 *The process*

$$\eta_t = N_t - \int_0^t \frac{N_T - N_s}{T-s} ds,\ t \leq T$$

is a $\mathbf{G}$*-martingale with predictable bracket*

$$\Lambda_t = \int_0^t \frac{N_T - N_s}{T-s} ds\,.$$

PROOF: For $0 < s < t < T$,

$$\mathbb{E}(N_t - N_s|\mathcal{G}_s) = \mathbb{E}(N_t - N_s|N_T - N_s) = \frac{t-s}{T-s}(N_T - N_s)$$

where the last equality follows from the fact that, if X and Y are independent with Poisson laws with parameters μ and ν respectively, then

$$\mathbb{P}(X = k|X + Y = n) = \frac{n!}{k!(n-k)!}\alpha^k(1-\alpha)^{n-k}$$

where $\alpha = \dfrac{\mu}{\mu+\nu}$. Hence,

$$\begin{aligned}\mathbb{E}\left(\int_s^t du \frac{N_T - N_u}{T-u}|\mathcal{G}_s\right) &= \int_s^t \frac{du}{T-u}\left(N_T - N_s - \mathbb{E}(N_u - N_s|\mathcal{G}_s)\right) \\ &= \int_s^t \frac{du}{T-u}\left(N_T - N_s - \frac{u-s}{T-s}(N_T - N_s)\right) \\ &= \int_s^t \frac{du}{T-s}(N_T - N_s) = \frac{t-s}{T-s}(N_T - N_s)\,.\end{aligned}$$

Therefore,

$$\mathbb{E}\left(N_t - N_s - \int_s^t \frac{N_T - N_u}{T-u}du|\mathcal{G}_s\right) = \frac{t-s}{T-s}(N_T - N_s) - \frac{t-s}{T-s}(N_T - N_s) = 0$$

and the result follows.

Therefore, η is a compensated **G**-Poisson process, time-changed by $\int_0^t \frac{N_T - N_s}{T-s}ds$, i.e., $\eta_t = \widetilde{M}(\int_0^t \frac{N_T - N_s}{T-s}ds)$ where $(\widetilde{M}(t), t \geq 0)$ is a compensated Poisson process. □

Comment 8.5.1.2 Poisson bridges are studied in Jeulin and Yor [496]. This kind of enlargement of filtration is used for modelling insider trading in Elliott and Jeanblanc [314], Grorud and Pontier [410] and Kohatsu-Higa and Øksendal [534].

8.5.2 Harness Property

The previous result may be extended in terms of the harness property.

Definition 8.5.2.1 *A process X fulfills the* **harness property** *if*

$$\mathbb{E}\left(\frac{X_t - X_s}{t-s}\,\middle|\, \mathcal{F}_{s_0],[T}\right) = \frac{X_T - X_{s_0}}{T - s_0}$$

for $s_0 \leq s < t \leq T$ where $\mathcal{F}_{s_0],[T} = \sigma(X_u, u \leq s_0, u \geq T)$.

A process with the harness property satisfies

$$\mathbb{E}\left(X_t \Big| \mathcal{F}_{s],[T}\right) = \frac{T-t}{T-s} X_s + \frac{t-s}{T-s} X_T \,,$$

and conversely.

Proposition 8.5.2.2 *If X satisfies the harness property, then, for any fixed T,*

$$M_t^T = X_t - \int_0^t du \frac{X_T - X_u}{T-u}, \; t < T$$

is an $\mathcal{F}_{t],[T}$-martingale and conversely.

PROOF: If X satisfies the harness property, it is easy to check that M^T is an $\mathcal{F}_{t],[T}$-martingale. Conversely, assume that M^T is an $\mathcal{F}_{t],[T}$-martingale. Let us prove that the harness property holds, i.e.,

$$\mathbb{E}\left(\frac{X_t - X_s}{t-s} \Big| \mathcal{F}_{s],[T}\right) = \frac{X_T - X_s}{T-s} \,.$$

From the hypothesis

$$\begin{aligned} \mathbb{E}(X_t - X_s \big| \mathcal{F}_{s],[T}) &= \int_s^t du \, \mathbb{E}\left(\frac{X_T - X_u}{T-u} \Big| \mathcal{F}_{s],[T}\right) \\ &= (X_T - X_s) \int_s^t \frac{du}{T-u} - \int_s^t \frac{du}{T-u} \mathbb{E}(X_u - X_s \big| \mathcal{F}_{s],[T}) \,. \end{aligned}$$

Therefore, for fixed s, T, the process $\varphi(u) = \mathbb{E}(X_u - X_s | \mathcal{F}_{s],[T})$ defined for $u \geq s$, satisfies

$$\varphi(t) = (X_T - X_s) \int_s^t \frac{du}{T-u} - \int_s^t \frac{du}{T-u} \varphi(u) \,.$$

It follows that φ is a solution of the ODE

$$\varphi'(t) = \frac{X_T - X_s}{T-t} - \varphi(t) \frac{1}{T-t}$$

with initial condition $\varphi(s) = 0$. This ODE has a unique solution given by $\varphi(t) = (t-s) \frac{X_T - X_s}{T-s}$. □

Comment 8.5.2.3 See Exercise 6.19 in Chaumont and Yor [161] for other properties. See also Jacod and Protter [470] and Exercise 12.3 in Yor [868]. We shall prove in ↣ Subsection 11.2.7 that any integrable Lévy process enjoys the harness property (see also Mansuy and Yor [621]). This property is used in Corcuera et al. [194] for studying insider trading.

8.6 Compound Poisson Processes

8.6.1 Definition and Properties

Definition 8.6.1.1 *Let $\lambda > 0$ and let F be a cumulative distribution function on $\mathbb{R}$. A (λ, F)-***compound Poisson process** *is a process $X = (X_t, t \geq 0)$ of the form*

$$X_t = \sum_{k=1}^{N_t} Y_k, \ X_0 = 0$$

where N is a Poisson process with intensity λ and the $(Y_k, k \geq 1)$ are i.i.d. random variables with law $F(y) = \mathbb{P}(Y_1 \leq y)$, independent of N (we use the convention that $\sum_{k=1}^{0} Y_k = 0$). We assume that $\mathbb{P}(Y_1 = 0) = 0$.

The process X differs from a Poisson process since the sizes of the jumps are random variables. We denote by $F(dy)$ the measure associated with F and by F^{*n} its n-th convolution, i.e.,

$$F^{*n}(y) = \mathbb{P}\left(\sum_{k=1}^{n} Y_k \leq y\right).$$

We use the convention $F^{*0}(y) = \mathbb{P}(0 \leq y) = \mathbb{1}_{[0,\infty[}(y)$.

Proposition 8.6.1.2 *A (λ, F)-compound Poisson process has stationary and independent increments (i.e., it is a Lévy process $\rightarrowtail$ Chapter 11); the cumulative distribution function of the r.v. X_t is*

$$\mathbb{P}(X_t \leq x) = e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} F^{*n}(x).$$

PROOF: Since the (Y_k) are i.i.d., one gets

$$\mathbb{E}\left(\exp(i\lambda \sum_{k=1}^{n} Y_k + i\mu \sum_{k=n+1}^{m} Y_k)\right) = (\mathbb{E}[\exp(i\lambda Y_1)]\,)^n \, (\mathbb{E}[\exp(i\mu Y_1)]\,)^{m-n}.$$

Then, setting $\psi(\lambda, n) = (\mathbb{E}[\exp(i\lambda Y_1)]\,)^n$, the independence and stationarity of the increments $(X_t - X_s)$ and X_s with $t > s$ follows from

$$\begin{aligned}\mathbb{E}(\exp(i\lambda X_s + i\mu(X_t - X_s))) &= \mathbb{E}(\psi(\lambda, N_s)\,\psi(\mu, N_t - N_s)) \\ &= \mathbb{E}(\psi(\lambda, N_s))\,\mathbb{E}(\psi(\mu, N_{t-s})).\end{aligned}$$

The independence of a finite sequence of increments follows by induction.

From the independence of N and the random variables $(Y_k, k \geq 1)$ and using the Poisson law of N_t, we get

$$\mathbb{P}(X_t \le x) = \sum_{n=0}^{\infty} \mathbb{P}\left(N_t = n, \sum_{k=1}^{n} Y_k \le x\right)$$
$$= \sum_{n=0}^{\infty} \mathbb{P}(N_t = n)\mathbb{P}\left(\sum_{k=1}^{n} Y_k \le x\right) = e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} F^{*n}(x)\,.$$

□

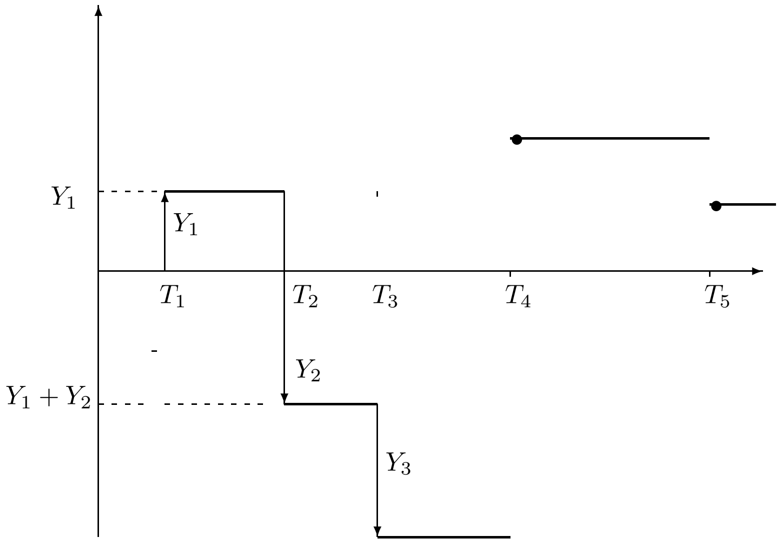

Fig. 8.2 Compound Poisson process

8.6.2 Integration Formula

If $Z_t = Z_0 + bt + X_t$ with X a (λ, F)-compound Poisson process, and if f is a C^1 function, the following obvious formula gives a representation of $f(Z_t)$ as a sum of integrals:

$$f(Z_t) = f(Z_0) + \int_0^t bf'(Z_s)ds + \sum_{s \le t} f(Z_s) - f(Z_{s^-})$$
$$= f(Z_0) + \int_0^t bf'(Z_s)ds + \sum_{s \le t} (f(Z_s) - f(Z_{s^-}))\Delta N_s$$
$$= f(Z_0) + \int_0^t bf'(Z_s)ds + \int_0^t (f(Z_s) - f(Z_{s^-}))\, dN_s\,.$$

It is possible to write this formula as

$$f(Z_t) = f(Z_0)+\int_0^t (bf'(Z_s)+(f(Z_s)-f(Z_{s-})\lambda)ds+\int_0^t (f(Z_s) - f(Z_{s-}))\, dM_s$$

however this equality does not give immediately the canonical decomposition of the semi-martingale $f(Z_t)$. Indeed, the reader can notice that the process $\int_0^t (f(Z_s) - f(Z_{s-}))\, dM_s$ is not a martingale. See $\rightarrowtail$ Subsection 8.6.4 for the decomposition of this semi-martingale.

Exercise 8.6.2.1 Prove that the infinitesimal generator of Z is given, for C^1 functions f such that f and f' are bounded, by

$$\mathcal{L}f(x) = bf'(x) + \lambda \int_{-\infty}^{\infty} (f(x+y) - f(x))\, F(dy)\,.$$

◁

8.6.3 Martingales

Proposition 8.6.3.1 *Let X be a (λ, F)-compound Poisson process such that $\mathbb{E}(|Y_1|) < \infty$. Then, the process $(Z_t = X_t - t\lambda\mathbb{E}(Y_1), t \geq 0)$ is a martingale and in particular, $\mathbb{E}(X_t) = \lambda t\mathbb{E}(Y_1) = \lambda t \int_{-\infty}^{\infty} yF(dy)$.*

If $\mathbb{E}(Y_1^2) < \infty$, the process $(Z_t^2 - t\lambda\mathbb{E}(Y_1^2), t \geq 0)$ is a martingale and $Var\,(X_t) = \lambda t\mathbb{E}(Y_1^2)$.

Proof: The martingale property of $(X_t - \mathbb{E}(X_t), t \geq 0)$ follows from the independence and stationarity of the increments of the process X. We leave the details to the reader. It remains to compute the expectation of the r.v. X_t as follows:

$$\begin{aligned}\mathbb{E}(X_t) &= \sum_{n=1}^{\infty} \mathbb{E}\left(\sum_{k=1}^{n} Y_k \mathbb{1}_{\{N_t=n\}}\right) = \sum_{n=1}^{\infty} n\mathbb{E}(Y_1)\mathbb{P}(N_t = n)\\ &= \mathbb{E}(Y_1)\sum_{n=1}^{\infty} n\mathbb{P}(N_t = n) = \lambda t\mathbb{E}(Y_1)\,.\end{aligned}$$

The proof of the second property can be done by the same method; however, it is more convenient to use the Laplace transform of X (See below, Proposition 8.6.3.4). □

Proposition 8.6.3.2 *Let $X_t = \sum_{i=1}^{N_t} Y_i$ be a (λ, F)-compound Poisson process, where the random variables Y_i are square integrable.*

Then $Z_t^2 - \sum_{i=1}^{N_t} Y_i^2$ is a martingale.

Proof: It suffices to write

$$Z_t^2 - \sum_{i=1}^{N_t} Y_i^2 = Z_t^2 - \lambda t \mathbb{E}(Y_1^2) - \left(\sum_{i=1}^{N_t} Y_i^2 - \lambda t \mathbb{E}(Y_1^2) \right) .$$

We have proved that $Z_t^2 - \lambda t \mathbb{E}(Y_1^2)$ is a martingale. Now, since $\sum_{i=1}^{N_t} Y_i^2$ is a compound Poisson process, $\sum_{i=1}^{N_t} Y_i^2 - \lambda t \mathbb{E}(Y_1^2)$ is a martingale. □

The process $A_t = \sum_{i=1}^{N_t} Y_i^2$ is an increasing process such that $X_t^2 - A_t$ is a martingale. Hence, as for a Poisson process, we have two (in fact an infinity of) increasing processes C_t such that $X_t^2 - C_t$ is a martingale. The particular process $C_t = t\lambda \mathbb{E}(Y_1^2)$ is predictable, whereas the process $A_t = \sum_{i=1}^{N_t} Y_i^2$ satisfies $\Delta A_t = (\Delta X_t)^2$. The predictable process $t\lambda \mathbb{E}(Y_1^2)$ is the predictable quadratic variation and is denoted $\langle X \rangle_t$, the process $\sum_{i=1}^{N_t} Y_i^2$ is the optional quadratic variation of X and is denoted $[X]_t$.

Proposition 8.6.3.3 *Let $X_t = \sum_{k=1}^{N_t} Y_k$ be a (λ, F)-compound Poisson process.*

(a) *Let $dS_t = S_{t-}(\mu dt + dX_t)$ (that is S is the Doléans-Dade exponential martingale $\mathcal{E}(U)$ of the process $U_t = \mu t + X_t$). Then,*

$$S_t = S_0 e^{\mu t} \prod_{k=1}^{N_t} (1 + Y_k) .$$

In particular, if $1 + Y_1 > 0, \mathbb{P}.a.s.$, then

$$S_t = S_0 \exp \left(\mu t + \sum_{k=1}^{N_t} \ln(1 + Y_k) \right) = S_0 e^{\mu t + X_t^*} = S_0 e^{U_t^*} .$$

Here, X^* is the (λ, F^*)-compound Poisson process $X_t^* = \sum_{k=1}^{N_t} Y_k^*$, where $Y_k^* = \ln(1 + Y_k)$ (hence $F^*(y) = F(e^y - 1)$) and

$$U_t^* = U_t + \sum_{s \le t} (\ln(1 + \Delta X_s) - \Delta X_s) = U_t + \sum_{k=1}^{N_t} (\ln(1 + Y_k) - Y_k) .$$

The process $(S_t e^{-rt}, t \ge 0)$ is a local martingale if and only if $\mu + \lambda \mathbb{E}(Y_1) = r$.

(b) *The process*

$$S_t = x \exp(bt + X_t) = x e^{V_t} \tag{8.6.1}$$

is a solution of

$$dS_t = S_{t-} dV_t^* ,\ S_0 = x$$

(i.e., $S_t = x\, \mathcal{E}(V^*)_t$) where

$$V_t^* = V_t + \sum_{s \le t} (e^{\Delta X_s} - 1 - \Delta X_s) = bt + \sum_{s \le t} (e^{\Delta X_s} - 1) .$$

The process S is a martingale if and only if

$$\lambda \int_{-\infty}^{\infty} (1 - e^y) F(dy) = b .$$

PROOF: The solution of

$$dS_t = S_{t^-} (\mu dt + dX_t), \; S_0 = x$$

is

$$S_t = x\mathcal{E}(U)_t = xe^{\mu t} \prod_{k=1}^{N_t} (1 + Y_k) = xe^{\mu t} e^{\sum_{k=1}^{N_t} \ln(1+Y_k)} = e^{\mu t + \sum_{k=1}^{N_t} Y_k^*}$$

where $Y_k^* = \ln(1 + Y_k)$. From

$$\mu t + \sum_{k=1}^{N_t} Y_k^* = \mu t + X_t + \sum_{k=1}^{N_t} Y_k^* - X_t = U_t + \sum_{s \le t} (\ln(1 + \Delta X_s) - \Delta X_s) ,$$

we obtain $S_t = xe^{U_t^*}$. Then,

$$\begin{aligned} d(e^{-rt} S_t) &= e^{-rt} S_{t^-} ((-r + \mu + \lambda \mathbb{E}(Y_1)) dt + dX_t - \lambda \mathbb{E}(Y_1) dt) \\ &= e^{-rt} S_{t^-} ((-r + \mu + \lambda \mathbb{E}(Y_1)) dt + dZ_t) , \end{aligned}$$

where $Z_t = X_t - \lambda \mathbb{E}(Y_1) t$ is a martingale. It follows that $e^{-rt} S_t$ is a local martingale if and only if $-r + \mu + \lambda \mathbb{E}(Y_1) = 0$.

The second assertion is the same as the first one, with a different choice of parametrization. Let

$$S_t = xe^{bt + X_t} = xe^{bt} \exp \left(\sum_1^{N_t} Y_k \right) = xe^{bt} \prod_{k=1}^{N_t} (1 + Y_k^*)$$

where $1 + Y_k^* = e^{Y_k}$. Hence, from part a), $dS_t = S_{t^-}(bdt + dV_t^*)$ where $V_t^* = \sum_{k=1}^{N_t} Y_k^*$. It remains to note that

$$bt + V_t^* = V_t + V_t^* - X_t = V_t + \sum_{s \le t} (e^{\Delta X_s} - 1 - \Delta X_s) .$$

□

We now denote by ν the positive measure $\nu(dy) = \lambda F(dy)$. Using this notation, a (λ, F)-compound Poisson process will be called a **ν-compound Poisson** process. This notation, which is not standard, will make the various formulae more concise and will be of constant use in ↣ Chapter 11 when dealing with Lévy 's processes which are a generalization of compound Poisson processes. Conversely, to any positive finite measure ν on $\mathbb{R}$, we can associate a cumulative distribution function by setting $\lambda = \nu(\mathbb{R})$ and $F(dy) = \nu(dy)/\lambda$ and construct a ν-compound Poisson process.

Proposition 8.6.3.4 *If X is a ν-compound Poisson process, let*

$$\mathcal{J}(\nu) = \left\{ \alpha : \int_{-\infty}^{\infty} e^{\alpha x} \nu(dx) < \infty \right\} .$$

The Laplace transform of the r.v. X_t is

$$\mathbb{E}(e^{\alpha X_t}) = \exp\left(-t \int_{-\infty}^{\infty} (1 - e^{\alpha x}) \nu(dx) \right) \mathit{for}\, \alpha \in \mathcal{J}(\nu).$$

The process

$$Z_t^{(\alpha)} = \exp\left(\alpha X_t + t \int_{-\infty}^{\infty} (1 - e^{\alpha x}) \nu(dx) \right)$$

is a martingale.

The characteristic function of the r.v. X_t is

$$\mathbb{E}(e^{iuX_t}) = \exp\left(-t \int_{-\infty}^{\infty} (1 - e^{iux}) \nu(dx) \right) .$$

PROOF: From the independence between the random variables $(Y_k, k \geq 1)$ and the process N,

$$\mathbb{E}(e^{\alpha X_t}) = \mathbb{E}\left(\exp\left(\alpha \sum_{k=1}^{N_t} Y_k \right) \right) = \mathbb{E}(\Phi(N_t))$$

where $\Phi(n) = \mathbb{E}\left(\exp\left(\alpha \sum_{k=1}^{n} Y_k \right) \right) = [\Psi_Y(\alpha)]^n$, with $\Psi_Y(\alpha) = \mathbb{E}\left(\exp(\alpha Y_1)\right)$. Now, $\mathbb{E}(\Phi(N_t)) = \sum_n [\Psi_Y(\alpha)]^n e^{-\lambda t} \frac{\lambda^n t^n}{n!} = \exp\left(-\lambda t(1 - \Psi_Y(\alpha)\right)$. The martingale property follows from the independence and stationarity of the increments of X. □

Taking the derivative w.r.t. α of $Z^{(\alpha)}$ and evaluating it at $\alpha = 0$, we obtain that the process Z of Proposition 8.6.3.1 is a martingale, and using the second derivative of $Z^{(\alpha)}$ evaluated at $\alpha = 0$, one obtains that $Z_t^2 - \lambda t \mathbb{E}(Y_1^2)$ is a martingale.

Proposition 8.6.3.5 *Let X be a ν-compound Poisson process, and f a bounded Borel function. Then, the process*

$$\exp\left(\sum_{k=1}^{N_t} f(Y_k) + t\int_{-\infty}^{\infty}(1-e^{f(x)})\nu(dx)\right)$$

is a martingale. In particular

$$\mathbb{E}\left(\exp\left(\sum_{k=1}^{N_t} f(Y_k)\right)\right) = \exp\left(-t\int_{-\infty}^{\infty}(1-e^{f(x)})\nu(dx)\right).$$

PROOF: The proof is left as an exercise. □

For any bounded Borel function f, we denote by $\nu(f) = \int_{-\infty}^{\infty} f(x)\nu(dx)$ the product $\lambda\mathbb{E}(f(Y_1))$. Then, one has the following proposition:

Proposition 8.6.3.6 (i) *Let X be a ν-compound Poisson process and f a bounded Borel function. The process*

$$M_t^f = \sum_{s\le t} f(\Delta X_s)\mathbb{1}_{\{\Delta X_s\neq 0\}} - t\nu(f)$$

is a martingale.

(ii) *Conversely, suppose that X is a pure jump process and that there exists a finite positive measure σ such that*

$$\sum_{s\le t} f(\Delta X_s)\mathbb{1}_{\{\Delta X_s\neq 0\}} - t\sigma(f)$$

is a martingale for any bounded Borel function f, then X is a σ-compound Poisson process.

PROOF: (i) From the definition of M^f,

$$\begin{aligned}\mathbb{E}(M_t^f) &= \sum_n \mathbb{E}(f(Y_n))\mathbb{P}(T_n < t) - t\nu(f) = \mathbb{E}(f(Y_1))\sum_n \mathbb{P}(T_n < t) - t\nu(f)\\ &= \mathbb{E}(f(Y_1))\mathbb{E}(N_t) - t\nu(f) = 0\,.\end{aligned}$$

The proof of the proposition is now standard and results from the computation of conditional expectations which leads to, for $s > 0$

$$\mathbb{E}(M_{t+s}^f - M_t^f|\mathcal{F}_t) = \mathbb{E}\left(\sum_{t<u\le t+s} f(\Delta X_u)\mathbb{1}_{\{\Delta X_u\neq 0\}} - s\nu(f)|\mathcal{F}_t\right) = 0\,.$$

Another proof relies on the fact that the process

$$\sum_{s\le t} f(\Delta X_s)\mathbb{1}_{\{\Delta X_s\neq 0\}} = \sum_{k=1}^{N_t} f(Y_k) = \sum_{k=1}^{N_t} Z_k$$

is a compound Poisson process, hence

$$\sum_{k=1}^{N_t} f(Y_k) - t\lambda\mathbb{E}(Z_1) = \sum_{k=1}^{N_t} f(Y_k) - t\lambda\mathbb{E}(f(Y_1))$$

is a martingale.

(ii) For the converse, we write

$$\begin{aligned} e^{iuX_t} &= 1 + \sum_{s\le t} e^{iuX_{s-}}(e^{iu\Delta X_s} - 1) \\ &= 1 + \int_0^t e^{iuX_{s-}} dM_s^f + \sigma(f)\int_0^t e^{iuX_s} ds \end{aligned}$$

where $f(x) = e^{iux} - 1$. Hence,

$$\mathbb{E}(e^{iuX_{t+s}}|\mathcal{F}_t) = e^{iuX_t} + \sigma(f)\int_0^s dr\, \mathbb{E}(e^{iuX_{t+r}}|\mathcal{F}_t)\,.$$

Setting $\varphi(s) = \mathbb{E}(e^{iuX_{t+s}}|\mathcal{F}_t)$, one gets $\varphi(s) = \varphi(0) + \sigma(f)\int_0^s \varphi(r)dr$, hence

$$\mathbb{E}(e^{iuX_{t+s}}|\mathcal{F}_t) = e^{iuX_t}\exp\left(s\int_{\mathbb{R}} \sigma(dx)(e^{iux} - 1)\right)\,.$$

The remainder of the proof is standard and left to the reader. □

Introducing the random measure $\mu = \sum_n \delta_{T_n, Y_n}$ on $\mathbb{R}^+ \times \mathbb{R}$ and denoting by $(f * \mu)_t$ the integral[1]

$$\int_0^t \int_{\mathbb{R}} f(x)\mu(\omega; ds, dx) = \sum_{k=1}^{N_t} f(Y_k)\,,$$

we obtain that

$$M_t^f = (f * \mu)_t - t\nu(f) = \int_0^t \int_{\mathbb{R}} f(x)(\mu(\omega; ds, dx) - ds\,\nu(dx))$$

is a martingale. (We shall generalize this fact when studying marked point processes in ↣ Section 8.8 and Lévy processes in ↣ Chapter 11.)

Example 8.6.3.7 Let $U_s = \alpha s + \sigma W_s$ where W is a standard Brownian motion and let N be a Poisson process with intensity 1, independent of W. Define the process Z as $Z_t = U_{N_t}$ (that is a time change of the drifted Brownian motion U). Conditionally on $N_1 = n$, the r.v. Z_1 has a $\mathcal{N}(\alpha n, \sigma^2 n)$ law. The process Z is a compound Poisson process

$$(Z_t, t \ge 0) \stackrel{\text{law}}{=} \left(\sum_{k=1}^{N_t} Y_k,\, t \ge 0\right) \text{ where } Y_k \stackrel{\text{law}}{=} \mathcal{N}(\alpha, \sigma^2)\,.$$

[1] Later, in Chapter 11, we shall often use $\mathbf{N}(ds, dx)$ instead of $\mu(ds, dx)$

Example 8.6.3.8 Let $X^{(i)}, i = 1, 2$ be two compound Poisson processes

$$X_t^{(i)} = \sum_{k=1}^{N_t^{(i)}} Y_k^{(i)}$$

where $Y^{(i)}, N^{(i)}, i = 1, 2$ are independent and $Y_1^{(i)}$ is a reflected normal r.v. (i.e., with density $f(y)\mathbb{1}_{\{y>0\}}$ where $f(y) = \frac{\sqrt{2}}{\sigma\sqrt{\pi}} e^{-y^2/(2\sigma^2)}$). The characteristic function of the r.v. $X_t^{(1)} - X_t^{(2)}$ is

$$\Psi(u) = e^{-2\lambda t} e^{\lambda t(\Phi(u)+\Phi(-u))}$$

with $\Phi(u) = \mathbb{E}(e^{iuY_1})$. From

$$\begin{aligned}\Phi(u) + \Phi(-u) = \mathbb{E}(e^{iuY_1} + e^{-iuY_1}) &= \int_0^\infty e^{iuy} f(y)dy + \int_0^\infty e^{-iuy} f(y)dy \\ &= \int_{-\infty}^\infty e^{iuy} f(y)dy = 2e^{-\sigma^2 u^2/2}\end{aligned}$$

we obtain

$$\Psi(u) = \exp(2\lambda t(e^{-\sigma^2 u^2/2} - 1)) \, .$$

This is the characteristic function of $\sigma W(N_t^{(1)} + N_t^{(2)})$ where W is a BM, evaluated at time $N_t^{(1)} + N_t^{(2)}$.

Exercise 8.6.3.9 Let X be a (λ, F)-compound Poisson process. Compute $\mathbb{E}(e^{iuX_t})$ in the following two cases:

(a) Merton's case [643]: The law F is a Gaussian law, with mean c and variance δ,

(b) Kou's case [540] (double exponential model): The law F of Y_1 is

$$F(dx) = \left(p\theta_1 e^{-\theta_1 x}\mathbb{1}_{\{x>0\}} + (1-p)\theta_2 e^{\theta_2 x}\mathbb{1}_{\{x<0\}}\right) dx \, ,$$

where $p \in]0,1[$ and $\theta_i, i = 1, 2$ are positive numbers. See $\rightarrowtail$ Example 10.4.4.5 for a generalization and an answer. ◁

Exercise 8.6.3.10 (Example of a Compound Poisson Process.) (See Sato [761], p. 21.) Let X be a ν-compound Poisson process with ν a probability measure of the form $\nu(dx) = p\delta_1(dx) + q\delta_{-1}(dx)$ where $\delta_a(dx)$ denotes the Dirac measure at a and where $q = 1 - p, p \in]0,1[$. Prove that

$$\mathbb{P}(X_t = k) = e^{-t} \left(\frac{p}{q}\right)^{k/2} I_k(2(pq)^{1/2}t)$$

where I_k is the Bessel function (see $\rightarrowtail$ A.5.2). ◁

Exercise 8.6.3.11 (Extension of Compound Poisson Process.) (See Sato [761], p. 143) Let X be a ν-compound Poisson process and h a bounded function. The sequence (T_k) is the sequence of jumps of the Poisson process N. Let $Z_t = \sum_{k=1}^{N_t} h(T_k, Y_k)$. Prove that

$$\mathbb{E}(e^{iuZ_t}) = \exp\left(\int_0^t ds \int (e^{iuh(s,y)} - 1)\nu(dy)\right) .$$

The process Z has independent non-homogeneous increments; it is called an additive process. ◁

8.6.4 Itô's Formula

Let X be a ν-compound Poisson process, and $Z_t = Z_0 + bt + X_t$. Then, Itô's formula

$$\begin{aligned} f(Z_t) - f(Z_0) &= b\int_0^t f'(Z_s)ds + \sum_{k,\, T_k \leq t} f(Z_{T_k}) - f(Z_{T_k-}) \\ &= b\int_0^t f'(Z_s)ds + \int_0^t \int_{\mathbb{R}} [f(Z_{s-} + y) - f(Z_{s-})]\, \mu(ds, dy) \end{aligned}$$

(where $\mu = \sum_{n=1}^{\infty} \delta_{T_n, Y_n}$) can be written as

$$f(Z_t) - f(Z_0) = \int_0^t ds\, (\mathcal{L}f)(Z_s) + M(f)_t$$

where $\mathcal{L}f(x) = bf'(x) + \int_{\mathbb{R}} (f(x+y) - f(x))\, \nu(dy)$ is the infinitesimal generator of Z and

$$M(f)_t = \int_0^t \int_{\mathbb{R}} [f(Z_{s-} + y) - f(Z_{s-})]\, (\mu(ds, dy) - ds\, \nu(dy))$$

is a local martingale.

8.6.5 Hitting Times

Let $Z_t = ct - \sum_{k=1}^{N_t} Y_k$ be a (λ, F)-compound Poisson process with a drift term $c > 0$ and $T(x) = \inf\{t : x + Z_t \leq 0\}$ where $x > 0$. The random variables Y can be interpreted as losses for insurance companies. The process Z is called the Cramer-Lundberg risk process.

► If $c = 0$ and if the support of the cumulative distribution function F is included in $[0, \infty[$, then the process Z is decreasing and

$$\{T(x) \geq t\} = \{Z_t + x \geq 0\} = \left\{x \geq \sum_{k=1}^{N_t} Y_k\right\} ,$$

hence,

$$\mathbb{P}(T(x) \geq t) = \mathbb{P}\left(x \geq \sum_{k=1}^{N_t} Y_k\right) = \sum_n \mathbb{P}(N_t = n) F^{*n}(x)\,.$$

For a cumulative distribution function F with support in $\mathbb{R}$,

$$\mathbb{P}(T(x) \geq t) = \sum_n \mathbb{P}(N_t = n)\mathbb{P}(Y_1 \leq x, Y_1 + Y_2 \leq x, \dots, Y_1 + \cdots + Y_n \leq x)\,.$$

▶ Assume now that $c \neq 0$, that the support of F is included in $[0, \infty[$ and that, for every u, $\mathbb{E}(e^{uY_1}) < \infty$. Setting $\psi(u) = cu + \int_0^\infty (e^{uy} - 1)\nu(dy)$, the process $(\exp(uZ_t - t\psi(u)), t \geq 0)$ is a martingale (Corollary 8.6.3.3). Since the process Z has no negative jumps, the level cannot be crossed with a jump and therefore $Z_{T(x)} = -x$. From Doob's optional sampling theorem, $\mathbb{E}(e^{uZ_{t\wedge T(x)} - (t\wedge T(x))\psi(u)}) = 1$ and when t goes to infinity, one obtains

$$\mathbb{E}(e^{-ux - T(x)\psi(u)} \mathbb{1}_{\{T(x)<\infty\}}) = 1\,.$$

Hence one gets the Laplace transform of $T(x)$

$$\mathbb{E}(e^{-\theta T(x)} \mathbb{1}_{\{T(x)<\infty\}}) = e^{x\psi^\sharp(\theta)}\,,$$

where $\psi^\sharp$ is the negative inverse of ψ (i.e., $\psi^\sharp(\theta)$ is the solution y of $\psi(y) = \theta$ for $\theta > 0$ which satisfies $y < 0$).

Example 8.6.5.1 (One-sided Exponential Law.)
If $F(dy) = \kappa e^{-\kappa y} \mathbb{1}_{\{y>0\}} dy$, one obtains $\psi(u) = cu - \frac{\lambda u}{\kappa + u}$, hence inverting ψ,

$$\mathbb{E}(e^{-\theta T(x)} \mathbb{1}_{\{T(x)<\infty\}}) = e^{x\psi^\sharp(\theta)}\,,$$

with

$$\psi^\sharp(\theta) = \frac{\lambda + \theta - \kappa c - \sqrt{(\lambda + \theta - \kappa c)^2 + 4\theta\kappa c}}{2c}\,.$$

Exercise 8.6.5.2 Let $X_t = \sum_{i=1}^{N_t} Y_i$ and $X_t^* = \sum_{i=1}^{N_t^*} Y_i^*$ be two compound Poisson processes, where N, N^* are independent Poisson processes with respective intensities λ and λ^*. We assume that the four random objects N, N^*, Y, Y^* are independent and that the law of Y_1 (resp. the law of Y_1^*) has support in $[0, \infty[$. Prove that $e^{-\rho(X_t - X_t^*)}$ is a martingale for ρ a root of $\lambda\mathbb{E}(e^{-\rho Y_1} - 1) + \lambda^*\mathbb{E}(e^{\rho Y_1^*} - 1) = 0$. ◁

8.6.6 Change of Probability Measure

Two questions may be asked:

(a) Starting from a ν-compound Poisson process X under $\mathbb{P}$, find some changes of measures $\mathbb{Q} \ll \mathbb{P}$ such that, under $\mathbb{Q}$, X is still a compound Poisson process.

(b) Given two compound Poisson processes, when are their distributions locally equivalent?

We treat point (a) and leave (b) to the reader (see $\rightarrowtail$ Exercise 8.6.6.3).

Let X be a ν-compound Poisson process, $\widetilde{\nu}$ a positive finite measure on $\mathbb{R}$ absolutely continuous w.r.t. ν, and $\widetilde{\lambda} = \widetilde{\nu}(\mathbb{R}) > 0$. Let

$$L_t = \exp\left(t(\lambda - \widetilde{\lambda}) + \sum_{s \le t} \ln\left(\frac{d\widetilde{\nu}}{d\nu}\right)(\Delta X_s)\right). \tag{8.6.2}$$

Proposition 8.6.3.4 proves that, if $\sum_{k=1}^{N_t} Z_k$ is a compound Poisson process, then

$$\exp\left(\sum_{k=1}^{N_t} Z_k + t\lambda\mathbb{E}(1 - e^{Z_1})\right)$$

is a martingale. It follows that

$$\exp\left(\sum_{k=1}^{N_t} f(Y_k) + t\int_{-\infty}^{\infty}(1 - e^{f(x)})\nu(dx)\right) \tag{8.6.3}$$

is a martingale, hence for $f = \ln\left(\frac{d\widetilde{\nu}}{d\nu}\right)$, the process L is a martingale. Set $\mathbb{Q}|_{\mathcal{F}_t} = L_t\mathbb{P}|_{\mathcal{F}_t}$.

Proposition 8.6.6.1 *Let X be a ν-compound Poisson process under $\mathbb{P}$. Define $d\mathbb{Q}|_{\mathcal{F}_t} = L_t d\mathbb{P}|_{\mathcal{F}_t}$ where L is given in (8.6.2). Then, the process X is a $\widetilde{\nu}$-compound Poisson process under $\mathbb{Q}$.*

PROOF: First we find the law of the random variable $X_t = \sum_{k=1}^{N_t} Y_k$ under $\mathbb{Q}$. Let $f = \ln\left(\frac{d\widetilde{\nu}}{d\nu}\right)$. Then

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}(e^{iuX_t}) &= \mathbb{E}_{\mathbb{P}}\left(e^{iuX_t}\exp\left(t(\lambda - \widehat{\lambda}) + \sum_{1}^{N_t} f(Y_k)\right)\right) \\
&= \sum_{n=0}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!}e^{t(\lambda - \widehat{\lambda})}\left(\mathbb{E}_{\mathbb{P}}(e^{iuY_1 + f(Y_1)})\right)^n \\
&= \sum_{n=0}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!}e^{t(\lambda - \widehat{\lambda})}\left(\mathbb{E}_{\mathbb{P}}\left(\frac{d\widehat{\nu}}{d\nu}(Y_1)\, e^{iuY_1}\right)\right)^n \\
&= \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!}e^{-t\widehat{\lambda}}\left(\frac{1}{\lambda}\int e^{iuy}d\widehat{\nu}(y)\right)^n = \exp\left(t\int(e^{iuy} - 1)\, d\widehat{\nu}(y)\right).
\end{aligned}$$

It remains to check that X has independent and stationary increments under $\mathbb{Q}$. Using Proposition 8.6.3.5, one gets, for $t > s$,

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}(e^{iu(X_t-X_s)}|\mathcal{F}_s) &= \frac{1}{L_s}\mathbb{E}_{\mathbb{P}}(L_t e^{iu(X_t-X_s)}|\mathcal{F}_s) \\ &= \exp\left((t-s)\int(e^{iux}-1)\widetilde{\nu}(dx)\right) .\end{aligned}$$

□

In that case, the change of measure changes the intensity (equivalently, the law of N_t) and the law of the jumps, but the independence of the Y^i is preserved and N remains a Poisson process. It is possible to change the measure using more general Radon-Nikodým densities, so that the process X does not remain a compound Poisson process.

Exercise 8.6.6.2 Prove that the process L defined in (8.6.3) satisfies

$$dL_t = L_{t-}\left(\int_{\mathbb{R}}(e^{f(y)}-1)(\mu(dt,dy)-\nu(dy)dt)\right) .$$

◁

Exercise 8.6.6.3 Prove that two compound Poisson processes with measures ν and $\widetilde{\nu}$ are locally absolutely continuous, only if ν and $\widetilde{\nu}$ are equivalent.
`Hint:` Use $\mathbb{E}\left(\left(\sum_{s\le t} f(\Delta X_s)\right)\right) = t\nu(f)$. ◁

8.6.7 Price Process

We consider, as in Mordecki [658], the stochastic differential equation

$$dS_t = (\alpha S_{t^-} + \beta)\,dt + (\gamma S_{t^-} + \delta)dX_t \tag{8.6.4}$$

where X is a ν-compound Poisson process.

Proposition 8.6.7.1 *The solution of* (8.6.4) *is a Markov process with infinitesimal generator*

$$\mathcal{L}(f)(x) = (\alpha x+\beta)f'(x) + \int_{-\infty}^{+\infty}[f(x+\gamma xy+\delta y)-f(x)]\,\nu(dy)\,,$$

for suitable f (in particular for $f \in C^1$ with compact support).

Proof: We use Stieltjes integration to write, path by path,

$$f(S_t) - f(x) = \int_0^t f'(S_{s-})(\alpha S_{s-}+\beta)\,ds + \sum_{0\le s\le t}\Delta(f(S_s))\,.$$

Hence,

$$\mathbb{E}(f(S_t)) - f(x) = \mathbb{E}\left(\int_0^t f'(S_s)(\alpha S_s + \beta)ds\right) + \mathbb{E}\left(\sum_{0\leq s\leq t} \Delta(f(S_s))\right).$$

From

$$\begin{aligned}\mathbb{E}\left(\sum_{0\leq s\leq t} \Delta(f(S_s))\right) &= \mathbb{E}\left(\sum_{0\leq s\leq t} f(S_{s-} + \Delta S_s) - f(S_{s-})\right)\\ &= \mathbb{E}\left(\int_0^t \int_{\mathbb{R}} d\nu(y)\,[f(S_{s-} + (\gamma S_{s-} + \delta)y) - f(S_{s-})]\right),\end{aligned}$$

we obtain the infinitesimal generator. □

Proposition 8.6.7.2 *The process* $(e^{-rt}S_t, t \geq 0)$ *where* S *is a solution of (8.6.4) is a local martingale if and only if*

$$\alpha + \gamma \int_{\mathbb{R}} y\nu(dy) = r, \quad \beta + \delta \int_{\mathbb{R}} y\nu(dy) = 0\,.$$

PROOF: Left as an exercise. □

Let $\widetilde{\nu}$ be a positive finite measure which is absolutely continuous with respect to ν and

$$L_t = \exp\left((\lambda - \widetilde{\lambda}) + \sum_{s\leq t} \ln\left(\frac{d\widetilde{\nu}}{d\nu}\right)(\Delta X_s)\right).$$

Let $\mathbb{Q}|_{\mathcal{F}_t} = L_t \mathbb{P}|_{\mathcal{F}_t}$. Under $\mathbb{Q}$,

$$dS_t = (\alpha S_{t^-} + \beta)\,dt + (\gamma S_{t^-} + \delta)dX_t$$

where X is a $\widetilde{\nu}$-compound Poisson process. The process $(S_t e^{-rt}, t \geq 0)$ is a $\mathbb{Q}$-martingale if and only if

$$\alpha + \gamma \int_{\mathbb{R}} y\widetilde{\nu}(dy) = r, \quad \beta + \delta \int_{\mathbb{R}} y\widetilde{\nu}(dy) = 0\,.$$

Hence, there is an infinite number of e.m.m's: one can change the intensity of the Poisson process, or the law of the jumps, while preserving the compound process setting. Of course, one can also change the probability so as to break the independence assumptions.

8.6.8 Martingale Representation Theorem

The martingale representation theorem will be presented in the following Section 8.8 on marked point processes.

8.6.9 Option Pricing

The valuation of perpetual American options will be presented in ↣ Subsection 11.9.1 in the chapter on Lévy processes, using tools related to Lévy processes. The reader can refer to the papers of Gerber and Shiu [388, 389] and Gerber and Landry [386] for a direct approach. The case of double-barrier options is presented in Sepp [781] for double exponential jump diffusions, the case of lookback options is studied in Nahum [664]. Asian options are studied in Bellamy [69].

8.7 Ruin Process

We present briefly some basic facts about the problem of ruin, where compound Poisson processes play an essential rôle.

8.7.1 Ruin Probability

In the **Cramer-Lundberg model** the surplus process of an insurance company is $x + Z_t$, with $Z_t = ct - X_t$, where $X_t = \sum_{k=1}^{N_t} Y_k$ is a compound Poisson process. Here, c is assumed to be positive, the Y_k are $\mathbb{R}^+$-valued and we denote by F the cumulative distribution function of Y_1. Let $T(x)$ be the first time when the surplus process falls below 0:

$$T(x) = \inf\{t > 0 \,:\, x + Z_t \leq 0\} .$$

The probability of ruin is $\Phi(x) = \mathbb{P}(T(x) < \infty)$. Note that $\Phi(x) = 1$ for $x < 0$.

Lemma 8.7.1.1 *If $\infty > \mathbb{E}(Y_1) \geq \frac{c}{\lambda}$, then for every x, ruin occurs with probability 1.*

Proof: Denoting by T_k the jump times of the process N, and setting

$$S_n = \sum_1^n [Y_k - c(T_k - T_{k-1})] ,$$

the probability of ruin is

$$\Phi(x) = \mathbb{P}(\inf_n(-S_n) < -x) = \mathbb{P}(\sup_n S_n > x) .$$

The strong law of large numbers implies

$$\lim_{n\to\infty} \frac{1}{n} S_n = \lim_{n\to\infty} \frac{1}{n} \sum_1^n [Y_k - c(T_k - T_{k-1})] = \mathbb{E}(Y_1) - \frac{c}{\lambda} .$$

□

8.7.2 Integral Equation

Let $\Psi(x) = 1 - \Phi(x) = \mathbb{P}(T(x) = \infty)$ where $T(x) = \inf\{t > 0 : x + Z_t \leq 0\}$. Obviously $\Psi(x) = 0$ for $x < 0$. From the Markov property, for $x \geq 0$

$$\Psi(x) = \mathbb{E}(\Psi(x + cT_1 - Y_1))$$

where T_1 is the first jump time of the Poisson process N. Thus

$$\Psi(x) = \int_0^\infty dt \lambda e^{-\lambda t} \mathbb{E}(\Psi(x + ct - Y_1))\,.$$

With the change of variable $y = x + ct$ we get

$$\Psi(x) = e^{\lambda x/c} \frac{\lambda}{c} \int_x^\infty dy e^{-\lambda y/c} \mathbb{E}(\Psi(y - Y_1))\,.$$

Differentiating w.r.t. x, we obtain

$$\begin{aligned} c\Psi'(x) &= \lambda\Psi(x) - \lambda\mathbb{E}(\Psi(x - Y_1)) = \lambda\Psi(x) - \lambda \int_0^\infty \Psi(x - y) dF(y) \\ &= \lambda\Psi(x) - \lambda \int_0^x \Psi(x - y) dF(y)\,. \end{aligned}$$

In the case where the Y_k's are exponential with parameter μ,

$$c\Psi'(x) = \lambda\Psi(x) - \lambda \int_0^x \Psi(x - y) \mu e^{-\mu y} dy\,.$$

Differentiating w.r.t. x and using the integration by parts formula leads to

$$c\Psi''(x) = (\lambda - c\mu)\Psi'(x)\,.$$

► For $\beta = \frac{1}{c}(\lambda - c\mu) < 0$, the solution of this differential equation is

$$\Psi(x) = c_1 \int_x^\infty e^{\beta t} dt + c_2$$

where c_1 and c_2 are two constants such that $\Psi(\infty) = 1$ and $\lambda\Psi(0) = c\Psi'(0)$. Therefore $c_2 = 1$, $c_1 = \frac{\lambda}{c} \frac{\lambda - \mu c}{c\mu} < 0$ and $\Psi(x) = 1 - \frac{\lambda}{c\mu} e^{\beta x}$. It follows that $\mathbb{P}(T(x) < \infty) = \frac{\lambda}{c\mu} e^{\beta x}$.
► If $\beta > 0$, then $\Psi(x) = 0$. Note that the condition $\beta > 0$ is equivalent to $\mathbb{E}(Y_1) \geq \frac{c}{\lambda}$.

8.7.3 An Example

Let $Z_t = ct - X_t$ where $X_t = \sum_{k=1}^{N_t} Y_k$ is a compound Poisson process. We denote by F the cumulative distribution function of Y_1 and we assume that

$F(0) = 0$, i.e., that the random variable Y_1 is $\mathbb{R}^+$-valued. Yuen et al. [871] assume that the insurer is allowed to invest in a portfolio, with stochastic return $R_t = rt + \sigma W_t + X^*_t$ where W is a Brownian motion and $X^*_t = \sum_{k=1}^{N^*_t} Y^*_k$ is a compound Poisson process. We assume that $(Y_k, Y^*_k, k \geq 1, N, N^*, W)$ are independent. We denote by F^* the cumulative distribution function of Y^*_1.

The risk process S associated with this model is defined as the solution $S_t(x)$ of the stochastic differential equation

$$S_t = x + Z_t + \int_0^t S_{s-} dR_s \,, \tag{8.7.1}$$

i.e.,

$$S_t(x) = U_t \left(x + \int_0^t U^{-1}_{s-} dZ_s \right)$$

where $U_t = e^{rt} \mathcal{E}(\sigma W)_t \prod_{k=1}^{N^*_t} (1 + Y^*_k)$. Note that the process S jumps at the time when the processes N or N^* jump and that

$$\Delta S_t = \Delta Z_t + S_{t-} \Delta R_t \,.$$

Let $T(x) = \inf\{t : S_t(x) < 0\}$ and $\Psi(x) = \mathbb{P}(T(x) = \infty) = \mathbb{P}(\inf_t S_t(x) \geq 0)$, the survival probability.

Proposition 8.7.3.1 *For $x \geq 0$, the function Ψ is the solution of the implicit equation*

$$\Psi(x) = \int_0^\infty \int_0^\infty \frac{\gamma}{2y^{2+\alpha+a}} p^\alpha_u(1, y) (D(y, u) + D^*(y, u)) \, dy du$$

where

$$\begin{aligned}
p^\alpha_u(z, y) &= \left(\frac{y}{z}\right)^\alpha \frac{y}{u} e^{-(z^2+y^2)/(2u)} I_\alpha \left(\frac{zy}{u}\right), \\
D^*(y, u) &= \frac{\lambda^*}{\lambda + \lambda^*} \int_{-1}^\infty \Psi((1 + z) y^{-2} (x + 4c\sigma^{-2} u)) \, dF^*(z), \\
D(y, u) &= \frac{\lambda}{\lambda + \lambda^*} \int_0^{y^{-2}(x+4c\sigma^{-2}u)} \Psi(y^{-2}(x + 4c\sigma^{-2} u) - z) \, dF(z),
\end{aligned}$$

$$a = \sigma^{-2}(2r - \sigma^2), \ \gamma = \frac{8(\lambda + \lambda^*)}{\sigma^2}, \ \alpha = (a^2 + \gamma^2)^{1/2} \,.$$

PROOF: Let τ (resp. τ^*) be the first time when the process N (resp. N^*) jumps, $T = \tau \wedge \tau^*$ and $m = \inf_{t \geq 0} S_t$. Note that, from the independence between N and N^*, we have $\mathbb{P}(\tau = \tau^*) = 0$. On the set $\{t < T\}$, one has $S_t = e^{rt} \mathcal{E}(\sigma W)_t \left(x + c \int_0^t e^{-rs} [\mathcal{E}(\sigma W)_s]^{-1} ds \right)$. We denote by V the process

$V_t = e^{rt}\mathcal{E}(\sigma W)_t(x + c\int_0^t e^{-rs}[\mathcal{E}(\sigma W)_s]^{-1}ds)$. The optional stopping theorem applied to the bounded martingale

$$M_t = \mathbb{E}(\mathbb{1}_{m\geq 0}|\mathcal{F}_t)$$

and the strong Markov property lead to

$$\Psi(x) = \mathbb{P}(m \geq 0) = M_0 = \mathbb{E}(M_T) = \mathbb{E}(\Psi(S_T))\,.$$

Hence,

$$\begin{aligned}
\Psi(x) &= \mathbb{E}(\Psi(S_\tau)\mathbb{1}_{\tau<\tau^*}) + \mathbb{E}(\Psi(S_{\tau^*})\mathbb{1}_{\tau^*<\tau})\\
&= \mathbb{E}(\Psi(V_\tau - Y_1)\mathbb{1}_{\tau<\tau^*}) + \mathbb{E}(\Psi(V_{\tau^*}(1+Y_1^*))\mathbb{1}_{\tau^*<\tau})\\
&= \int_0^\infty dt\lambda e^{-\lambda t}\mathbb{E}(\Psi(V_t - Y_1))\,\mathbb{P}(t<\tau^*)\\
&\qquad + \int_0^\infty dt\lambda^* e^{-\lambda^* t}\mathbb{E}(\Psi(V_t(1+Y_1^*)))\,\mathbb{P}(t<\tau)\\
&= \int_0^\infty e^{-(\lambda+\lambda^*)t}\left(\lambda\mathbb{E}[\Psi(V_t - Y_1)] + \lambda^*\mathbb{E}[\Psi(V_t(1+Y_1^*))]\right)dt\,.
\end{aligned}$$

Employing the change of variable $t = 4\sigma^{-2}s$,

$$\Psi(x) = \frac{4}{\sigma^2}\int_0^\infty e^{-4\sigma^{-2}(\lambda+\lambda^*)s}\left(\lambda\Upsilon(s) + \lambda^*\Upsilon^*(s)\right)ds\,,$$

where

$$\Upsilon(s) = \mathbb{E}[\Psi(X_s - Y_1)], \Upsilon^*(s) = \mathbb{E}[\Psi(Z_s(1+Y_1^*))]$$

and

$$X_s = e^{2(as+B_s)}\left(x + \frac{4c}{\sigma^2}\int_0^s e^{-2(at+B_t)}dt\right),\ B_s = \frac{\sigma}{2}W_{4\sigma^{-2}s}\,,$$

where $a = \frac{2r}{\sigma^2} - 1$. Hence, using the symmetry of BM,

$$\Upsilon(s) = \mathbb{E}\left[\Psi\left(e^{-2(B_s-as)}\left(x + \frac{4c}{\sigma^2}\int_0^s e^{2(B_t-at)}dt\right) - Y_1\right)\right].$$

Therefore

$$\begin{aligned}
&\frac{4}{\sigma^2}\int_0^\infty e^{-4\sigma^{-2}(\lambda+\lambda^*)s}\lambda\Upsilon(s)ds\\
&= \frac{\lambda}{\lambda+\lambda^*}\mathbb{E}\left[\Psi\left(e^{-2(B_\Theta - a\Theta)}\left(x + \frac{4c}{\sigma^2}\int_0^\Theta e^{2(B_t-at)}dt\right) - Y_1\right)\right]
\end{aligned}$$

where Θ is an exponential random variable, independent of B, with parameter $4(\lambda+\lambda^*)\sigma^{-2}$. The law of the pair

$$\left(e^{-2(B_\Theta - a\Theta)}, \int_0^\Theta e^{2(B_t - at)} dt\right)$$

was presented in Corollary 6.6.2.2. It follows that

$$\frac{4}{\sigma^2}\int_0^\infty e^{-4\sigma^{-2}(\lambda+\lambda^*)s}\lambda\Upsilon(s)ds = \int_0^\infty \int_0^\infty \frac{\gamma}{2y^{2+\alpha+a}} p_u^\alpha(1,y) D(y,u) dy du\,.$$

The study of the second term can be carried out by the same method. □

Comment 8.7.3.2 See the main papers of Klüppelberg [526], Paulsen [700], Paulsen and Gjessing [701], Yuen et al [871], the books of Asmussen [23], Embrechts et al. [322], Mikosch [650] and Mel'nikov [639] and the thesis of Loisel [602]. Many applications to ruin theory can be found in Gerber and his co-authors, e.g., in [387].

8.8 Marked Point Processes

We now generalize compound Poisson processes, introducing briefly a class of processes which are no longer Lévy processes: we introduce a spatial dimension for the size of jumps which are no longer i.i.d. random variables; moreover, the time intervals between two consecutive jumps are no longer independent. Let $(E, \mathcal{E})$ be a measurable space and $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space.

8.8.1 Random Measure

Definition 8.8.1.1 *A* ***random measure*** ϑ *on the space* $\mathbb{R}^+ \times E$ *is a family of positive measures* $(\vartheta(\omega; dt, dx); \omega \in \Omega)$ *defined on* $\mathbb{R}^+ \times E$ *such that, for* $[0,t] \times A \in \mathcal{B} \otimes \mathcal{E}$*, the map* $\omega \to \vartheta(\omega; [0,t], A)$ *is* $\mathcal{F}$*-measurable, and satisfying* $\vartheta(\omega; \{0\} \times E) = 0$.

8.8.2 Definition

Let (Z_n) be a sequence of random variables taking values in the measurable space $(E, \mathcal{E})$, and (T_n) an increasing sequence of positive random variables, with - to avoid explosion - $\lim_n T_n = +\infty$. We define the **marked point process** $\mathbf{N} = \{(T_n, Z_n)\}$ by: for each Borel set $A \subset E$,

$$N_t(A) = \sum_n \mathbb{1}_{\{T_n \le t\}} \mathbb{1}_{\{Z_n \in A\}}\,.$$

We associate with $\mathbf{N}$ a random measure μ by

$$\mu(\cdot; [0,t], A) = N_t(A)\,.$$

The natural filtration of $\mathbf{N}$ is

$$\mathcal{F}_t^{\mathbf{N}} = \sigma(N_s(A), s \le t, A \in \mathcal{E}) .$$

Let H be a map

$$(t, \omega, z) \in (\mathbb{R}^+, \Omega, E) \to H(t, \omega, z) \in \mathbb{R} .$$

The map H is predictable if it is $\mathcal{P} \otimes \mathcal{E}$ measurable. The random counting measure $\mu(\omega; ds, dz)$ acts on the set of predictable processes H as

$$\begin{aligned}(H \star \mu)_t &= \int_{]0,t]} \int_E H(s,z) \mu(ds, dz) = \sum_n H(T_n, Z_n) \mathbb{1}_{\{T_n \le t\}} \\ &= \sum_{n=1}^{N_t(E)} H(T_n, Z_n) ,\end{aligned}$$

where we have dropped ω in the notation.

Definition 8.8.2.1 *The* ***compensator*** *of μ is the (up to a null set) unique random measure ν such that, for every predictable process H,*

(i) the process $H \star \nu$ is predictable,
(ii) if moreover, the process $|H| \star \mu$ is increasing and locally integrable, the process $(H \star \mu - H \star \nu)$ is a local martingale.

The existence of a compensator is established in Brémaud and Jacod [125], Jacod and Shiryaev [471] and Kallenberg [505].

We now assume that $E = \mathbb{R}^d$. The compensator admits an explicit representation: let $G_n(dt, dz)$ be a regular version of the conditional distribution of (T_{n+1}, Z_{n+1}) with respect to $\mathcal{F}_{T_n}^N = \sigma\{((T_1, Z_1), \dots (T_n, Z_n)\}$. Then,

$$\nu(dt, dz) = \sum_n \mathbb{1}_{\{T_n < t \le T_{n+1}\}} \frac{G_n(dt, dz)}{G_n([t, \infty[\times \mathbb{R}^d)} . \tag{8.8.1}$$

A proof can be found in Prigent [725], Chapter 1 Proposition 1.1.30.

Comment 8.8.2.2 See Brémaud and Jacod [125], Brémaud [124], Prigent [725], Jacod [467] and Jacod and Shiryaev [471] for more details on marked point processes.

Warning 8.8.2.3 The notation in various papers in the literature can be very different from the above: authors may use $\mathbf{N}$ or N for various quantities.

8.8.3 An Integration Formula

Let $dX_t = \beta_t dt + \int_E \gamma(t,z)\mu(dt,dz)$, where β and γ are predictable and let F be a $C^{1,1}$ function. Then

$$\begin{aligned} dF(t,X_t) &= \partial_t F dt + \beta_t \, \partial_x F dt \\ &\quad + \int_E \left(F(t, X_{t-} + \gamma(t,z)) - F(t, X_{t-})\right) \mu(dt,dz) \end{aligned}$$

or, in an integrated form

$$\begin{aligned} F(t,X_t) &= F(0,X_0) + \int_0^t \partial_t F(s,X_s) ds + \int_0^t \beta_s \, \partial_x F(s,X_s) ds \\ &\quad + \sum_{n=1}^{N_t(E)} \left[F(T_n, X_{T_n}) - F(T_n, X_{T_n^-})\right]. \end{aligned}$$

8.8.4 Marked Point Processes with Intensity and Associated Martingales

In what follows, we assume that, for every $A \in \mathcal{E}$, the process $N_t(A)$ admits the $\mathbf{F}$-predictable intensity $\lambda_t(A)$, i.e., there exists a predictable process $(\lambda_t(A), t \geq 0)$ such that

$$N_t(A) - \int_0^t \lambda_s(A) ds$$

is a martingale. (The most common form of intensity is $\lambda_t(A) = \alpha_t m_t(A)$ where α_t is a positive predictable process and m_t a deterministic probability measure on $(E, \mathcal{E})$. In that case, $\nu(dt,dz) = \alpha_t m_t(A) dt$. We shall say that the marked point process admits $(\alpha_t, m_t(dz))$ as $\mathbb{P}$-local characteristics.)

If $X_t := \sum_{n=1}^{N_t(E)} H(T_n, Z_n)$ where H is an $\mathbf{F}$-predictable process that satisfies

$$\mathbb{E}\left(\int_{]0,t]} \int_E |H(s,z)| \lambda_s(dz) ds\right) < \infty$$

the process

$$X_t - \int_0^t \int_E H(s,z)\lambda_s(dz) ds = \int_{]0,t]} \int_E H(s,z) \left[\mu(ds,dz) - \lambda_s(dz) ds\right]$$

is a martingale and in particular

$$\mathbb{E}\left(\int_{]0,t]} \int_E H(s,z)\mu(ds,dz)\right) = \mathbb{E}\left(\int_{]0,t]} \int_E H(s,z)\lambda_s(dz) ds\right).$$

8.8.5 Girsanov's Theorem

Let μ be the random measure of a marked point process with intensity of the form $\lambda_t(A) = \alpha_t m_t(A)$ where m_t is, as above, a deterministic probability measure on $(E, \mathcal{E})$. Let $(\psi_t, h(t,z))$ be two predictable positive processes such that

$$\int_0^t \psi_s \alpha_s ds < \infty, \ \int_E h(t,z) m_t(dz) = 1\,.$$

Let L be the local martingale solution of

$$dL_t = L_{t^-} \int_E (\psi_t h_t(z) - 1)(\mu(dt,dz) - \alpha_t m_t(dz)dt)\,.$$

If $\mathbb{E}(L_t) = 1$, setting $\mathbb{Q}|_{\mathcal{F}_t} = L_t \mathbb{P}|_{\mathcal{F}_t}$, the marked point process has the $\mathbb{Q}$-local characteristics $(\psi_t \alpha_t, h(t,z) m_t(dz))$.

Exercise 8.8.5.1 Prove Proposition 8.6.6.1 using the above result. ◁

8.8.6 Predictable Representation Theorem

Let $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ be a probability space where $\mathbf{F}$ is the filtration generated by the marked point process $\mathbf{N}$. Then, any $(\mathbb{P}, \mathbf{F})$-square integrable martingale M admits the representation

$$M_t = M_0 + \int_0^t \int_E H(s,x)(\mu(ds,dx) - \lambda_s(dx)ds)$$

where H is a predictable process such that

$$\mathbb{E}\left(\int_0^t \int_E |H(s,x)|^2 \lambda_s(dx)ds\right) < \infty\,.$$

See Brémaud [124] for a proof. More generally

Proposition 8.8.6.1 *Let W be a Brownian motion M, $\mathbf{N}$ a marked point process and $\mathcal{F}_t = \sigma(W_s, \mathcal{F}_s^{\mathbf{N}}; s \le t)$ completed.*

Let $\widetilde{\mu}(ds,dz) = \mu(ds,dx) - \lambda_s(dx)ds$. Then, any $(\mathbb{P}, \mathbf{F})$-local martingale has the representation

$$M_t = M_0 + \int_0^t \varphi_s dW_s + \int_0^t \int_E H(s,z) \widetilde{\mu}(ds,dz) \qquad (8.8.2)$$

where φ is a predictable process such that $\int_0^t \varphi_s^2 ds < \infty$ and H is a predictable process such that $\int_0^t \int_E |H(s,x)| \lambda_s(dx)ds < \infty$. If M is a square integrable martingale, each term on the right-hand side of the representation (8.8.2) is square integrable, and

$$\mathbb{E}\left(\left(\int_0^t \varphi_s dW_s\right)^2\right) = \mathbb{E}\left(\int_0^t \varphi_s^2 ds\right)$$

$$\mathbb{E}\left(\left(\int_0^t \int_E H(s,z)\widetilde{\mu}(ds,dz)\right)^2\right) = \mathbb{E}\left(\int_0^t \int_E H^2(s,z)\lambda_s(dz)ds\right)$$

PROOF: We refer to Kunita and Watanabe [550], Kunita [549], and to Chapter III in the book of Jacod and Shiryaev [471]. □

Comment 8.8.6.2 Björk et al. [103] and Prigent [725, 724] gave the first applications to finance of Marked point processes, which are now studied by many authors, especially in a BSDE framework.

Exercise 8.8.6.3 Check that the process

$$S_t = \exp\left(\int_0^t \left(\beta_s - \frac{1}{2}\sigma_s^2\right) ds + \int_0^t \sigma_s dW_s\right) \prod_{n, T_n \le t} (1 + \gamma(T_n, Z_n))$$

is a solution of

$$dS_t = S_{t^-}\left(\beta_t dt + \sigma_t dW_t + \int_E \gamma(t,y)\mu(dt,dy)\right),$$

where μ is the random measure associated with the marked point process $\mathbf{N} = \{(T_n, Z_n)\}$. ◁

8.9 Poisson Point Processes

We end this chapter with a brief section on Poisson point processes, which are of major importance in the study of Brownian excursions.

8.9.1 Poisson Measures

Let $(E, \mathcal{E})$ be a measurable space. A random measure μ on $(E, \mathcal{E})$ is a Poisson measure with intensity ν, where ν is a σ-finite measure on $(E, \mathcal{E})$, if

(i) for every set $B \in \mathcal{E}$ with $\nu(B) < \infty$, $\mu(B)$ follows a Poisson distribution with parameter $\nu(B)$, and
(ii) for disjoint sets $B_i, i \le n$, the variables $\mu(B_i), i \le n$ are independent.

Example 8.9.1.1 Let π be a probability measure, $(Y_k, k \in \mathbb{N})$ i.i.d. random variables with law π and N a Poisson variable with mean m, independent of the Y_k's. The random measure $\sum_{k=1}^{N} \delta_{Y_k}$ is a Poisson measure with intensity $\nu = m\pi$. Here, δ_y is the Dirac measure at point y.

8.9.2 Point Processes

Let $(E, \mathcal{E})$ be a measurable space and δ an additional point. We introduce $E_\delta = E \cup \delta, \mathcal{E}_\delta = \sigma(\mathcal{E}, \{\delta\})$.

Definition 8.9.2.1 *Let* $\mathbf{e}$ *be a stochastic process defined on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*, taking values in* $(E_\delta, \mathcal{E}_\delta)$*. The process* $\mathbf{e}$ *is a point process if:*

(i) *the map* $(t, \omega) \to \mathbf{e}_t(\omega)$ *is* $\mathcal{B}(]0, \infty[) \otimes \mathcal{F}$*-measurable,*
(ii) *the set* $D_\omega = \{t : \mathbf{e}_t(\omega) \neq \delta\}$ *is a.s. countable.*

For every measurable set B of $]0, \infty[\times E$, we set

$$N^B(\omega) := \sum_{s \geq 0} \mathbb{1}_B(s, \mathbf{e}_s(\omega)) .$$

In particular, if $B =]0, t] \times \Gamma$, we write

$$N_t^\Gamma = N^B = \text{Card}\{s \leq t : \mathbf{e}(s) \in \Gamma\} .$$

Let the space $(\Omega, \mathbb{P})$ be endowed with a filtration $\mathbf{F}$. A point process is $\mathbf{F}$-adapted if, for any $\Gamma \in \mathcal{E}$, the process N^Γ is $\mathbf{F}$-adapted. For any $\Gamma \in \mathcal{E}_\delta$, we define a point process $\mathbf{e}^\Gamma$ by

$$\begin{aligned} \mathbf{e}_t^\Gamma(\omega) &= \mathbf{e}_t(\omega) \quad \text{if } \mathbf{e}_t(\omega) \in \Gamma \\ \mathbf{e}_t^\Gamma(\omega) &= \delta \quad \text{otherwise} \end{aligned}$$

Definition 8.9.2.2 *A point process* $\mathbf{e}$ *is discrete if* $N_t^E < \infty$ *a.s. for every* t*. It is said to be* σ*-discrete if there is a sequence* E_n *of sets with* $E = \cup E_n$ *such that each* $\mathbf{e}^{E_n}$ *is discrete.*

8.9.3 Poisson Point Processes

Definition 8.9.3.1 *An* $\mathbf{F}$*-Poisson point process* $\mathbf{e}$ *is a* σ*-discrete point process such that:*

(i) *the process* $\mathbf{e}$ *is* $\mathbf{F}$*-adapted,*
(ii) *for any* s *and* t *and any* $\Gamma \in \mathcal{E}$*,* $N_{s+t}^\Gamma - N_t^\Gamma$ *is independent from* $\mathcal{F}_t$ *and distributed as* N_s^Γ*.*

In particular, for any disjoint family $(\Gamma_i, i = 1, \ldots, d)$, the d-dimensional process $(N_t^{\Gamma_i}, i = 1, \cdots, d)$ is a Poisson process. Moreover, if N^Γ is finite almost surely, then $\mathbb{E}(N_t^\Gamma) < \infty$ and the quantity $\frac{1}{t}\mathbb{E}(N_t^\Gamma)$ does not depend on t.

Definition 8.9.3.2 *The* σ*-finite measure on* $\mathcal{E}$ *defined by*

$$\mathbf{n}(\Gamma) = \frac{1}{t}\mathbb{E}(N_t^\Gamma)$$

is called the characteristic measure of $\mathbf{e}$*.*

If $\mathbf{n}(\Gamma) < \infty$, the process $N_t^\Gamma - t\mathbf{n}(\Gamma)$ is an $\mathbf{F}$-martingale.

Proposition 8.9.3.3 (Compensation Formula.) *Let H be a predictable positive process (i.e., measurable with respect to $\mathcal{P} \otimes \mathcal{E}_\delta$) vanishing at δ. Then*

$$\mathbb{E}\left[\sum_{s \geq 0} H(s, \omega, \mathbf{e}_s(\omega))\right] = \mathbb{E}\left[\int_0^\infty ds \int_E H(s, \omega, u)\mathbf{n}(du)\right].$$

If, for any t, $\mathbb{E}\left[\int_0^t ds \int_E H(s, \omega, u)\mathbf{n}(du)\right] < \infty$, the compensated process

$$\sum_{s \leq t} H(s, \omega, \mathbf{e}_s(\omega)) - \int_0^t ds \int_E H(s, \omega, u)\mathbf{n}(du)$$

is a martingale.

PROOF: By the Monotone Class Theorem, it is enough to prove this formula for $H(s, \omega, u) = K(s, \omega)\mathbb{1}_\Gamma(u)$. In that case, $N_t^\Gamma - t\mathbf{n}(\Gamma)$ is an $\mathbf{F}$-martingale. □

Proposition 8.9.3.4 (Exponential Formula.) *If f is a $\mathcal{B} \otimes \mathcal{E}$-measurable function such that $\int_0^t ds \int_E |f(s, u)|\mathbf{n}(du) < \infty$ for every t, then,*

$$\mathbb{E}\left[\exp\left(i \sum_{0 < s \leq t} f(s, \mathbf{e}_s)\right)\right] = \exp\left(\int_0^t ds \int_E (e^{if(s,u)} - 1)\mathbf{n}(du)\right).$$

Moreover, if $f \geq 0$,

$$\mathbb{E}\left[\exp\left(-\sum_{0 < s \leq t} f(s, \mathbf{e}_s)\right)\right] = \exp\left(-\int_0^t ds \int_E (1 - e^{-f(s,u)})\mathbf{n}(du)\right).$$

8.9.4 The Itô Measure of Brownian Excursions

Let $(B_t, t \geq 0)$ be a Brownian motion and (τ_s) be the inverse of the local time (L_t) at level 0. The set $\cup_{s \geq 0}]\tau_{s-}(\omega), \tau_s(\omega)[$ is (almost surely) equal to the complement of the zeros set $\{u : B_u(\omega) = 0\}$. The excursion process $(\mathbf{e}_s, s \geq 0)$ is defined by

$$\mathbf{e}_s(\omega)(t) = \mathbb{1}_{\{t \leq \tau_s(\omega) - \tau_{s-}(\omega)\}} B_{(\tau_{s-(\omega)} + t)(\omega)}, t \geq 0.$$

This is a path-valued process $\mathbf{e} : \mathbb{R}^+ \times \Omega \to \Omega_*$, where

$$\Omega_* = \{\varepsilon : \mathbb{R}^+ \to \mathbb{R} : \exists V(\varepsilon) < \infty, \text{ with } \varepsilon(V(\varepsilon) + t) = 0, \forall t \geq 0$$
$$\varepsilon(u) \neq 0, \forall\, 0 < u < V(\varepsilon), \, \varepsilon(0) = 0, \, \varepsilon \text{ is continuous}\}.$$

Hence, $V(\varepsilon)$ is the lifetime of ε.

The starting point of Itô's excursion theory is that the excursion process is a Poisson Point Process; its **characteristic measure n**, called Itô's measure, evaluated on the set Γ, i.e., $\mathbf{n}(\Gamma)$, is the intensity of the Poisson process

$$N_t^\Gamma := \sum_{s\le t} \mathbb{1}_{\mathbf{e}_s \in \Gamma} .$$

The quantity $\mathbf{n}(\Gamma)$ is the positive real γ such that $N_t^\Gamma - t\gamma$ is an $(\mathcal{F}_{\tau_t})$-martingale.

Here, are some very useful descriptions of $\mathbf{n}$:

- **Itô**: Conditionally on $V = v$, the process

$$(|\epsilon_u|, u \le v)$$

is a BES^3 bridge of length v. The law of the lifetime V under $\mathbf{n}$ is

$$\mathbf{n}_V(dv) = \frac{dv}{\sqrt{2\pi v^3}} .$$

Thanks to the symmetry of Brownian motion, a full description of $\mathbf{n}$ is

$$\mathbf{n}(d\epsilon) = \int_0^\infty \mathbf{n}_V(dv)\, \frac{1}{2}(\Pi_+^v + \Pi_-^v)\,(d\epsilon)$$

where Π_+^v (resp. Π_-^v) is the law of the standard Bessel Bridge (resp. the law of its negative) with dimension 3 and length v.

- **Williams**: Let $M(\epsilon) = \sup_{u\le v} |\epsilon_u|$. Then, conditionally on $M = m$, the two processes $(\epsilon_u, u \le T_m)$ and $(\epsilon_{V-u}, u \le V - T_m)$ are two independent BES^3 processes considered up to their first hitting time of m, and

$$\mathbf{n}_M(dm) = \mathbf{n}(M(\epsilon) \in dm) = \frac{dm}{m^2} .$$

We leave to the reader the task of writing a disintegration formula for $\mathbf{n}$ with respect to $\mathbf{n}_M$.

Comment 8.9.4.1 See Jeanblanc et al. [483] for applications to decomposition of Brownian paths and Feynman-Kac formula. At the moment, there are very few applications to finance of excursion theory. One can cite Gauthier [376] for a study of Parisian options, and Chesney et al. [173] for Asian-Parisian options.