5

Complements on Continuous Path Processes

In this chapter, we present the important notion of time change, which will be crucial when studying applications to finance in a Lévy process setting. We then introduce the operation of dual predictable projection, which will be an important tool when working with the reduced form approach in the default risk framework (of course, it has many other applications as will appear clearly in subsequent chapters). We present important facts about general homogeneous diffusions, in particular concerning their Green functions, scale functions and speed measures. These three quantities are of great interest when valuing options in a general setting. We study applications related to last passage times. A section is devoted to enlargements of filtrations, an important subject when dealing with insider trading.

The books of Borodin and Salminen [109], Itô [462], Itô and McKean [465], Karlin and Taylor [515], Karatzas and Shreve [513], Kallenberg [505], Knight [528], Øksendal [684], [RY] and Rogers and Williams [741, 742] are highly recommended. See also the review of Varadhan [826].

An excellent reference for the study of first hitting times of a fixed level for a diffusion is the book of Borodin and Salminen [109] where many results can be found. The general theory of stochastic processes is presented in Dellacherie [240], Dellacherie and Meyer [242, 244] and Dellacherie, Meyer and Maisonneuve [241]. Some results about the general theory of processes can also be found in $\rightarrowtail$ Chapter 9.

5.1 Time Changes

5.1.1 Inverse of an Increasing Process

In this paragraph, we deal with processes on a probability space but do not make any reference to a given filtration. Let us recall that by definition (see Subsection 1.1.10) an increasing process is equal to 0 at time 0; it is right

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continuous and of course increasing. Let A be an increasing process and let C be the **right inverse** of A, that is the increasing process defined by:

$$C_u = \inf\{t : A_t > u\} \tag{5.1.1}$$

where $\inf\{\emptyset\} = \infty$. We shall use C_u or $C(u)$ for the value at time u of the process C. The process C is right-continuous and satisfies

$$C_{u-} = \inf\{t : A_t \geq u\}$$

and $\{C_u > t\} = \{A_t < u\}$. We also have $A_{C_s} \geq s$ and $A_t = \inf\{u : C_u > t\}$. (See [RY], Chapter 0, section 4 for details.) Moreover, if A is continuous and strictly increasing, C is continuous and $C(A_t) = t$.

Proposition 5.1.1.1 *Time changing in integrals can be effected as follows: if f is a positive Borel function*

$$\int_{[0,\infty[} f(s)\, dA_s = \int_0^\infty f(C_u)\, \mathbb{1}_{\{C_u<\infty\}} du\,.$$

PROOF: For $f = \mathbb{1}_{[0,v]}$, the formula reads

$$A_v = \int_0^\infty \mathbb{1}_{\{C_u \leq v\}} du$$

and is a consequence of the definition of C. The general formula follows from the monotone class theorem. □

5.1.2 Time Changes and Stopping Times

In this section, $\mathbf{F}$ is a right-continuous filtration, and A is a right-continuous adapted increasing process with right inverse C. From the identity

$$\{C_u \leq t\} = \{A_t \geq u\}\,,$$

we see that $(C_u, u \geq 0)$ is a family of $\mathbf{F}$-stopping times. This leads us to define a **time change** C as a family $(C_u, u \geq 0)$ of stopping times such that the map $u \to C_u$ is a.s. increasing and right continuous. We denote by $\mathbf{F}_C$ the filtration $\mathbf{F}_C = (\mathcal{F}_{C_t}, t \geq 0)$. For every t the r.v. A_t is an $\mathbf{F}_C$-stopping time (indeed $\{A_t < u\} = \{C_u > t\}$).

Example 5.1.2.1 We have studied a very special case of time change while dealing with Ornstein-Uhlenbeck processes in Section 2.6. These processes are obtained from a Brownian motion by means of a deterministic time change.

Example 5.1.2.2 Let W be a Brownian motion and let

$$T_t = \inf\{s \geq 0 : W_s > t\} = \inf\left\{s \geq 0 \,:\, \max_{u \leq s} W_u > t\right\}$$

be the right-continuous inverse of $M_t = \max_{u \leq t} W_u$. The process $(T_t, t \geq 0)$ is increasing, and right-continuous (see Subsection 3.1.2). See $\rightarrowtail$ Section 11.8 for applications.

Exercise 5.1.2.3 Let (B, W) be a two-dimensional Brownian motion and define

$$T_t = \inf\{s \geq 0 : W_s > t\}\,.$$

Prove that $(Y_t = B_{T_t}, t \geq 0)$ is a Cauchy process, i.e., a process with independent and stationary increments, such that Y_t has a Cauchy law with characteristic function $\exp(-t|u|)$.
`Hint:` $\mathbb{E}(e^{iuB_{T_t}}) = \int e^{-\frac{1}{2}u^2 T_t(\omega)}\mathbb{P}(d\omega) = \mathbb{E}(e^{-\frac{1}{2}u^2 T_t}) = e^{-t|u|}$. ◁

5.1.3 Brownian Motion and Time Changes

Proposition 5.1.3.1 (Dubins-Schwarz's Theorem.) *A continuous martingale M such that*

$$\langle M \rangle_\infty = \infty$$

is a time-changed Brownian motion. In other words, there exists a Brownian motion W such that $M_t = W_{\langle M \rangle_t}$.

Sketch of the Proof: Let $A = \langle M \rangle$ and define the process W as $W_u = M_{C_u}$ where C is the inverse of A. One can then show that W is a continuous local martingale, with bracket $\langle W \rangle_u = \langle M \rangle_{C_u} = u$. Therefore, W is a Brownian motion, and replacing u by A_t in $W_u = M_{C_u}$, one obtains $M_t = W_{A_t}$. □

Comments 5.1.3.2 (a) This theorem was proved in Dubins and Schwarz [268]. It admits a partial extension due to Knight [527] to the multidimensional case: if M is a d-dimensional martingale such that $\langle M^i, M^j \rangle = 0, i \neq j$ and $\langle M^i \rangle_\infty = \infty, \forall i$, then the process $W = (M^i_{C_i(t)}, i \leq d, t \geq 0)$ is a d-dimensional Brownian motion w.r.t. its natural filtration, where the process C_i is the inverse of $\langle M^i \rangle$. See, e.g., Rogers and Williams [741]. The assumption $\langle M \rangle_\infty = \infty$ can be relaxed (See [RY], Chapter V, Theorem 1.10).

(b) Let us mention another two-dimensional extension of Dubins and Schwarz's theorem for complex valued local martingales which generalize complex Brownian motion. Getoor and Sharpe [390] introduced the notion of a continuous conformal local martingale as a process $Z = X + iY$, valued in $\mathbb{C}$, the complex plane, where X and Y are real valued continuous local martingales and Z^2 is a local martingale. A prototype is the complex-valued

Brownian motion. If Z is a continuous conformal local martingale, then, from $Z_t^2 = X_t^2 - Y_t^2 + 2iX_tY_t$, we deduce that $\langle X \rangle_t = \langle Y \rangle_t$ and $\langle X, Y \rangle_t = 0$. Hence, applying Knight's result to the two-dimensional local martingale (X, Y), there exists a complex-valued Brownian motion B such that $Z = B_{\langle X \rangle}$. In fact, in this case, B can be shown to be a Brownian motion w.r.t. $(\mathcal{F}_{\alpha_u}, u \geq 0)$, where $\alpha_u = \inf\{t : \langle X \rangle_t > u\}$. If $(Z_t, t \geq 0)$ denotes now a $\mathbb{C}$-valued Brownian motion, and $f : \mathbb{C} \to \mathbb{C}$ is holomorphic, then $(f(Z_t), t \geq 0)$ is a conformal martingale. The $\mathbb{C}$-extension of the Dubins-Schwarz-Knight theorem may then be written as:

$$f(Z_t) = \widehat{Z}_{\int_0^t |f'(Z_u)|^2 du}, t \geq 0 \tag{5.1.2}$$

where f' is the $\mathbb{C}$-derivative of f, and $\widehat{Z}$ denotes another $\mathbb{C}$-valued Brownian motion. This is an extremely powerful result due to Lévy, which expresses the conformal invariance of $\mathbb{C}$-valued Brownian motion. It is easily shown, as a consequence, using the exponential function that, if $Z_0 = a$, then $(Z_t, t \geq 0)$ shall never visit $b \neq a$ (of course, almost surely). As a consequence, (5.1.2) may be extended to any meromorphic function from $\mathbb{C}$ to itself, when $P(Z_0 \in S) = 0$ with S the set of singular points of f.

(c) See Jacod [468], Chapter 10 for a detailed study of time changes, and El Karoui and Weidenfeld [311] and Le Jan [569].

Exercise 5.1.3.3 Let f be a non-constant holomorphic function on $\mathbb{C}$ and $Z = X + iY$ a complex Brownian motion. Prove that there exists another complex Brownian motion B such that $f(Z_t) = f(Z_0) + B(\int_0^t |f'(Z_s)|^2 d\langle X \rangle_s)$ (see [RY], Chapter 5). As an example, $\exp(Z_t) = 1 + B_{\int_0^t ds \, \exp(2X_s)}$. ◁

We now come back to a study of real-valued continuous local martingales.

Lemma 5.1.3.4 *Let M be a continuous local martingale with $\langle M \rangle_\infty = \infty$, W the Brownian motion such that $M_t = W_{\langle M \rangle_t}$ and C the right-inverse of $\langle M \rangle$. If H is an adapted process such that for any t,*

$$\int_0^t H_s^2 d\langle M \rangle_s \left(= \int_0^{\langle M \rangle_t} H_{C_u}^2 \, du \right) < \infty ,$$

then

$$\int_0^t H_s dM_s = \int_0^{\langle M \rangle_t} H_{C_u} dW_u , \qquad \int_0^{C_t} H_s dM_s = \int_0^t H_{C_u} dW_u .$$

Lemma 5.1.3.5 *Let $X_t = \int_0^{C_t} H_s dW_s$, where C is a time change with respect to $\mathbf{F}$, differentiable with respect to time. Assume that $C'_t \neq 0$ for any t. Then,*

$$dX_t = H_{C_t} \sqrt{C'_t} \, dB_t ,$$

where B is an $\mathbf{F}_C$-Brownian motion.

PROOF: From the previous lemma

$$\int_0^{C_t} H_s dW_s = \int_0^t H_{C_u} dW_{C_u} ,$$

hence, $dX_t = H_{C_t} dW_{C_t}$. The process $(W_{C_u}, u \geq 0)$ is a local martingale with bracket C_u. The process

$$B_t = \int_0^t \frac{1}{\sqrt{C'_u}} dW_{C_u}$$

is a Brownian motion. □

Remarks 5.1.3.6 (a) Up to an enlargement of probability space, one can generalize the previous lemma to the case where the condition $C'_t \neq 0$ does not hold, but where we keep the assumption that C is differentiable. (The proof is left to the reader.)

(b) A time-changed local martingale is not necessarily a local martingale with respect to the time-changed filtration. As seen in Example 5.1.2.2, if T_t is the first hitting time of the level t for the Brownian motion B, the process $t \to T_t$ is increasing and is a time change. However, $B_{T_t} = t$ is not a local martingale. This illustrates, although very roughly, Monroe's theorem (see Remark 5.1.3.6) which states that any semi-martingale (even discontinuous) is a time changed Brownian motion. [655, 656]

However, if X is a continuous $\mathbf{F}$-local martingale and C a continuous time change, then $(X_{C_t})_{t\geq 0}$ is a continuous $\mathbf{F}_C$-local martingale. (See [RY], Chapter V, Section 1).

Comments 5.1.3.7 (a) Changes of time are extensively used for finance purposes in the papers of Geman, Madan and Yor [379, 385, 380, 381].

(b) The "pli cacheté" of Doeblin [255] may have been one of the first papers studying time changes.

(c) Further extensions to Markov processes are found in Volkonski [831]. See also McKean's paper [637] for other aspects of this major idea and important applications to Bessel processes in ↣ Chapter 6.

Exercise 5.1.3.8 Let Y be the solution of

$$dY_t = (cY_t + kY_t^2)dt + \sqrt{Y_t} dW_t$$

Prove that $Y_t = Z(\int_0^t Y_s ds)$ where $dZ(u) = (c + kZ(u))du + d\widehat{W}_u$. ◁

Exercise 5.1.3.9 Let Z be a complex BM $Z_t = X_t + iY_t$. Consider the two martingales $|Z_t|^2 - 2t$ and $\int_0^t (X_s dY_s - Y_s dX_s)$. Prove that

$$\frac{1}{2}\left(|Z_t|^2 - 2t\right) + i \int_0^t (X_s dY_s - Y_s dX_s)$$

is a conformal martingale which can be represented as $\widehat{Z}_u = \beta_u + i\gamma_u$ time-changed by $\int_0^t |Z_s|^2 ds$ with β and γ two independent BM's. Prove that $\sigma(\beta_u, u \geq 0) = \sigma(|Z_t|, t \geq 0)$, hence γ and $|Z|$ are independent. $\triangleleft$

5.2 Dual Predictable Projections

In this section, after recalling some basic facts about optional and predictable projections, we introduce the concept of a dual predictable projection[1], which leads to the fundamental notion of predictable compensators. We recommend the survey paper of Nikeghbali [674].

Recall that a process is said to be **optional** if it is measurable with respect to the σ-algebra on $\mathbb{R}^+ \times \Omega$ generated by càdlàg $\mathbf{F}$-adapted processes, considered as mappings on $\mathbb{R}^+ \times \Omega$, whereas a **predictable** process is measurable with respect to the σ-algebra on $\mathbb{R}^+ \times \Omega$ generated by càg $\mathbf{F}$-adapted processes (see ↣ Subsection 9.1.3 for comments).

5.2.1 Definitions

Let X be a bounded (or positive) process, and $\mathbf{F}$ a given filtration. The **optional projection** of X is the unique optional process ${}^{(o)}X$ which satisfies: for any $\mathbf{F}$-stopping time τ

$$\mathbb{E}(X_\tau \mathbb{1}_{\{\tau<\infty\}}) = \mathbb{E}({}^{(o)}X_\tau \mathbb{1}_{\{\tau<\infty\}}) . \tag{5.2.1}$$

For any $\mathbf{F}$-stopping time τ, let $\Gamma \in \mathcal{F}_\tau$ and apply the equality (5.2.1) to the stopping time $\tau_\Gamma = \tau \mathbb{1}_\Gamma + \infty \mathbb{1}_{\Gamma^c}$. We get the re-inforced identity:

$$\mathbb{E}(X_\tau \mathbb{1}_{\{\tau<\infty\}} | \mathcal{F}_\tau) = {}^{(o)}X_\tau \mathbb{1}_{\{\tau<\infty\}} .$$

In particular, if A is an increasing process, then, for $s \leq t$:

$$\mathbb{E}({}^{(o)}A_t - {}^{(o)}A_s | \mathcal{F}_s) = \mathbb{E}(A_t - A_s | \mathcal{F}_s) \geq 0 . \tag{5.2.2}$$

Note that, for any t, $\mathbb{E}(X_t|\mathcal{F}_t) = {}^{(o)}X_t$. However, $\mathbb{E}(X_t|\mathcal{F}_t)$ is defined almost surely for any t; thus uncountably many null sets are involved, hence, a priori, $\mathbb{E}(X_t|\mathcal{F}_t)$ is not a well-defined process whereas ${}^{(o)}X$ takes care of this difficulty.

Likewise, the **predictable projection** of X is the unique predictable process ${}^{(p)}X$ such that for any $\mathbf{F}$-predictable stopping time τ

$$\mathbb{E}(X_\tau \mathbb{1}_{\{\tau<\infty\}}) = \mathbb{E}({}^{(p)}X_\tau \mathbb{1}_{\{\tau<\infty\}}) . \tag{5.2.3}$$

As above, this identity reinforces as

$$\mathbb{E}(X_\tau \mathbb{1}_{\{\tau<\infty\}} | \mathcal{F}_{\tau-}) = {}^{(p)}X_\tau \mathbb{1}_{\{\tau<\infty\}} ,$$

for any $\mathbf{F}$-predictable stopping time τ (see Subsection 1.2.3 for the definition of $\mathcal{F}_{\tau-}$).

[1] See Dellacherie [240] for the notion of dual optional projection.

Example 5.2.1.1 Let τ and ϑ be two stopping times such that $\vartheta \leq \tau$ and Z a bounded r.v.. Let $X = Z\mathbb{1}_{]\!]\vartheta,\tau]\!]}$. Then, ${}^{(o)}X = U\mathbb{1}_{]\!]\vartheta,\tau]\!]}$, ${}^{(p)}X = V\mathbb{1}_{]\!]\vartheta,\tau]\!]}$ where U (resp. V) is the right-continuous (resp. left-continuous) version of the martingale $(\mathbb{E}(Z|\mathcal{F}_t), t \geq 0)$.

Let τ and ϑ be two stopping times such that $\vartheta \leq \tau$ and X a positive process. If A is an increasing optional process, then, since $\mathbb{1}_{]\!]\vartheta,\tau]\!]}(t)$ is predictable

$$\mathbb{E}\left(\int_\vartheta^\tau X_t dA_t\right) = \mathbb{E}\left(\int_\vartheta^\tau {}^{(o)}X_t dA_t\right).$$

If A is an increasing predictable process, then

$$\mathbb{E}\left(\int_\vartheta^\tau X_t dA_t\right) = \mathbb{E}\left(\int_\vartheta^\tau {}^{(p)}X_t dA_t\right).$$

The notion of interest in this section is that of **dual predictable projection**, which we define as follows:

Proposition 5.2.1.2 *Let $(A_t,\, t \geq 0)$ be an integrable increasing process (not necessarily $\mathbf{F}$-adapted). There exists a unique $\mathbf{F}$-predictable increasing process $(A_t^{(p)}, t \geq 0)$, called the dual predictable projection of A such that*

$$\mathbb{E}\left(\int_0^\infty H_s dA_s\right) = \mathbb{E}\left(\int_0^\infty H_s dA_s^{(p)}\right)$$

for any positive $\mathbf{F}$-predictable process H.

In the particular case where $A_t = \int_0^t a_s ds$, one has

$$A_t^{(p)} = \int_0^t {}^{(p)}a_s ds \tag{5.2.4}$$

PROOF: See Dellacherie [240], Chapter V, Dellacherie and Meyer [244], Chapter 6 paragraph (73), page 148, or Protter [727] Chapter 3, Section 5. □

This definition extends to the difference between two integrable (for simplicity) increasing processes. The terminology "dual predictable projection" refers to the fact that it is the random measure $d_t A_t(\omega)$ which is relevant when performing that operation. If X is bounded and A has integrable variation (not necessarily adapted), then

$$\mathbb{E}((X\star A^{(p)})_\infty) = \mathbb{E}(({}^{(p)}X\star A)_\infty).$$

This is equivalent to: for $s < t$,

$$\mathbb{E}(A_t - A_s|\mathcal{F}_s) = \mathbb{E}(A_t^{(p)} - A_s^{(p)}|\mathcal{F}_s). \tag{5.2.5}$$

If A is adapted (not necessarily predictable), then $(A_t - A_t^{(p)}, t \geq 0)$ is a martingale. In that case, $A_t^{(p)}$ is also called the predictable compensator of A.

More generally, from Proposition 5.2.1.2 and (5.2.5), the process ${}^{(o)}A - A^{(p)}$ is a martingale.

Proposition 5.2.1.3 *If A is increasing, the process ${}^{(o)}A$ is a sub-martingale and $A^{(p)}$ is the predictable increasing process in the Doob-Meyer decomposition of the sub-martingale ${}^{(o)}A$.*

Example 5.2.1.4 Let W be a Brownian motion, $M_t = \sup_{s \leq t} W_s$ its running maximum, and $R_t = 2M_t - W_t$. Then, from ↣ Pitman's Theorem 5.7.2.1 and its Corollary 5.7.2.2, for any positive Borel function f,

$$\mathbb{E}(f(M_t)|\mathcal{F}_t^R) = \int_0^1 dx f(R_t x)\,,$$

hence, $\mathbb{E}(2M_t|\mathcal{F}_t^R) = R_t$ and the predictable projection of $2M_t$ is R_t. On the other hand, from Pitman's theorem

$$R_t = \beta_t + \int_0^t \frac{ds}{R_s}\,,$$

where β is a Brownian motion, therefore, the dual predictable projection of $2M_t$ on $\mathcal{F}_t^R$ is $\int_0^t \frac{ds}{R_s}$. Note that the difference between these two projections is the (Brownian) martingale β.

In a general setting, the predictable projection of an increasing process A is a sub-martingale whereas the dual predictable projection is an increasing process. The predictable projection and the dual predictable projection of an increasing process A are equal if and only if $({}^{(p)}A_t, t \geq 0)$ is increasing.

It will also be convenient to introduce the following terminology:

Definition 5.2.1.5 *If ϑ is a random time, we call the* ***predictable compensator*** *associated with ϑ the dual predictable projection A^ϑ of the increasing process $\mathbb{1}_{\{\vartheta \leq t\}}$. This dual predictable projection A^ϑ satisfies*

$$\mathbb{E}(k_\vartheta) = \mathbb{E}\left(\int_0^\infty k_s dA_s^\vartheta\right) \tag{5.2.6}$$

for any positive, predictable process k.

5.2.2 Examples

In the sequel, we present examples of computation of dual predictable projections. We end up with Azéma's lemma, providing the law of the predictable compensator associated with the last passage at 0 of a BM before T, evaluated at a (stopping) time T. See also Knight [529] and ↣ Sections 5.6 and 7.4.

Example 5.2.2.1 Let $(B_s)_{s\geq 0}$ be an $\mathbf{F}-$ Brownian motion starting from 0 and $B^{(\nu)}_s = B_s + \nu s$. Let $\mathbf{G}^{(\nu)}$ be the filtration generated by the process $(|B^{(\nu)}_s|, s \geq 0)$ (which coincides with the one generated by $(B^{(\nu)}_s)^2$). We now compute the decomposition of the semi-martingale $(B^{(\nu)})^2$ in the filtration $\mathbf{G}^{(\nu)}$ and the dual predictable projection (with respect to $\mathbf{G}^{(\nu)}$) of the finite variation process $\int_0^t B^{(\nu)}_s ds$.

Itô's lemma provides us with the decomposition of the process $(B^{(\nu)})^2$ in the filtration $\mathbf{F}$:

$$(B^{(\nu)}_t)^2 = 2\int_0^t B^{(\nu)}_s dB_s + 2\nu \int_0^t B^{(\nu)}_s ds + t\,. \tag{5.2.7}$$

To obtain the decomposition in the filtration $\mathbf{G}^{(\nu)}$ we remark that, on the canonical space, denoting as usual by X the canonical process,

$$\mathbf{W}^{(0)}(e^{\nu X_s}|\mathcal{F}^{|X|}_s) = \cosh(\nu X_s)$$

which leads to the equality:

$$\mathbf{W}^{(\nu)}(X_s|\mathcal{F}^{|X|}_s) = \frac{\mathbf{W}^{(0)}(X_s e^{\nu X_s}|\mathcal{F}^{|X|}_s)}{\mathbf{W}^{(0)}(e^{\nu X_s}|\mathcal{F}^{|X|}_s)} = X_s \tanh(\nu X_s) \equiv \psi(\nu X_s)/\nu\,,$$

where $\psi(x) = x\tanh(x)$. We now come back to equality (5.2.7). Due to (5.2.4), we have just shown that:

$$\text{The dual predictable projection of } 2\nu\int_0^t B^{(\nu)}_s ds \text{ is } 2\int_0^t ds\, \psi(\nu B^{(\nu)}_s)\,. \tag{5.2.8}$$

As a consequence,

$$(B^{(\nu)}_t)^2 - 2\int_0^t ds\,\psi(\nu B^{(\nu)}_s) - t$$

is a $\mathbf{G}^{(\nu)}$-martingale with increasing process $4\int_0^t (B^{(\nu)}_s)^2 ds$. Hence, there exists a $\mathbf{G}^{(\nu)}$-Brownian motion β such that

$$(B_t + \nu t)^2 = 2\int_0^t |B_s + \nu s| d\beta_s + 2\int_0^t ds\,\psi(\nu(B_s + \nu s)) + t\,. \tag{5.2.9}$$

Exercise 5.2.2.2 Prove that, more generally than (5.2.8), the dual predictable projection of $\int_0^t f(B^{(\nu)}_s)ds$ is $\int_0^t \mathbb{E}(f(B^{(\nu)}_s)|\mathcal{G}^{(\nu)}_s)ds$ and that

$$\mathbb{E}(f(B^{(\nu)}_s)|\mathcal{G}^{(\nu)}_s) = \frac{f(B^{(\nu)}_s)e^{\nu B^{(\nu)}_s} + f(-B^{(\nu)}_s)e^{-\nu B^{(\nu)}_s}}{2\cosh(\nu B^{(\nu)}_s)}\,.$$

◁

Exercise 5.2.2.3 Prove that if $(\alpha_s, s \geq 0)$ is an increasing predictable process and X a positive measurable process, then

$$\left(\int_0^{\cdot} X_s d\alpha_s\right)_t^{(p)} = \int_0^t {}^{(p)}X_s d\alpha_s \,.$$

In particular

$$\left(\int_0^{\cdot} X_s ds\right)_t^{(p)} = \int_0^t {}^{(p)}X_s ds \,.$$

◁

Example 5.2.2.4 Let B be a Brownian motion and $Y_t = |B_t|$. Tanaka's formula gives

$$|B_t| = \int_0^t \mathrm{sgn}(B_s) dB_s + L_t$$

where L denotes the local time of $(B_t; t \geq 0)$ at level 0. By an application of the balayage formula (see Subsection 4.1.6), we obtain (we recall that g_t denotes the last passage time at level 0 before t)

$$h_{g_t}|B_t| = \int_0^t h_{g_s} \mathrm{sgn}(B_s) dB_s + \int_0^t h_s dL_s$$

where we have used the fact that $L_{g_s} = L_s$. Consequently, replacing, if necessary, h by $|h|$, we see that the process $\int_0^t |h_s| dL_s$ is the local time at 0 of $(h_{g_t} B_t, t \geq 0)$. Let now τ be a stopping time such that $(B_{t\wedge\tau}; t \geq 0)$ is uniformly integrable, and satisfies $\mathbb{P}(B_\tau = 0) = 0$. Then, it follows from the balayage formula that, for every predictable and bounded process h

$$\mathbb{E}\left(h_{g_\tau}|B_\tau|\right) = \mathbb{E}\left(\int_0^\tau h_s dL_s\right). \tag{5.2.10}$$

As an example, consider $\tau = T_a^* = \inf\{t : |B_t| = a\}$; we have

$$\mathbb{E}\left(h_{g_{T_a^*}}\right) = \frac{1}{a}\mathbb{E}\left(\int_0^{T_a^*} h_s dL_s\right),$$

whence we conclude that the predictable compensator $(A_t^\vartheta; t \geq 0)$ associated with $\vartheta := g_{T_a^*}$ is given by

$$A_t^\vartheta = \frac{1}{a} L_{t\wedge T_a^*}.$$

In the general case, applying (5.2.10) to the variable $\xi_{g_\tau} = \mathbb{E}\left(|B_\tau| | \mathcal{F}_{g_\tau}\right)$, where $(\xi_u; u \geq 0)$ is a predictable process (note that $\mathbb{P}(\xi_{g_\tau} = 0) = 0$, as a consequence of $\mathbb{P}(B_\tau = 0) = 0$) we obtain

$$\mathbb{E}\left(k_{g_\tau}\right) = \mathbb{E}\left(\int_0^\tau \frac{k_s}{\xi_s} dL_s\right) \tag{5.2.11}$$

from which we deduce that the predictable compensator associated with g_τ is

$$A_t = \int_0^{t\wedge\tau} \frac{dL_s}{\xi_s}. \tag{5.2.12}$$

In general, finding ξ may necessitate some work, but in some cases, e.g., $\tau = \inf\{t : |B_t| = \alpha_t\}$, for a continuous adapted process (α_t) such that $\alpha_t \equiv \alpha_{g_t}$, no extra computation is needed, since: $|B_\tau| = \alpha_{g_\tau}$ is $\mathcal{F}_{g_\tau}$ measurable, hence we can take: $\xi_u = \alpha_u$; finally $A_t = \int_0^{t\wedge\tau} \frac{dL_s}{\alpha_s}$.

We finish this subsection with the following interesting lemma which, in some generality, gives the law of A_τ.

Lemma 5.2.2.5 (Azéma.) *Let B be a BM and τ a stopping time such that $(B_{t\wedge\tau}; t \geq 0)$ is uniformly integrable, and satisfies $\mathbb{P}(B_\tau = 0) = 0$. Let A be the predictable compensator associated with g_τ. Then, A_τ is an exponential variable with mean 1.*

PROOF: Since, as a consequence of equality (5.2.12), $A_\tau = A_{g_\tau}$, we have for every $\lambda \geq 0$

$$\mathbb{E}\left(e^{-\lambda A_\tau}\right) = \mathbb{E}\left(\int_0^\tau e^{-\lambda A_s} dA_s\right),$$

as a consequence of (5.2.6) applied to $\vartheta = g_\tau$ and $k_t = \exp(-\lambda A_t)$. Thus, we obtain

$$\mathbb{E}\left(e^{-\lambda A_\tau}\right) = \mathbb{E}\left(\frac{1 - e^{-\lambda A_\tau}}{\lambda}\right),$$

or equivalently, $\mathbb{E}\left(e^{-\lambda A_\tau}\right) = \frac{1}{1+\lambda}$. The desired result follows immediately. □

Note that a corollary of this result provides the law of the local time of the BM at the time $T_a^* = \inf\{t : |B_t| = a\}$: $L_{T_a^*}$ is an exponential variable, with mean a.

Exercise 5.2.2.6 Let $\mathbf{F} \subset \mathbf{G}$ and let $G_t - \int_0^t \gamma_s ds$ be a $\mathbf{G}$-martingale. Recalling that ${}^{(o)}X$ is the $\mathbf{F}$-optional projection of a process X, prove that ${}^{(o)}G_t - \int_0^t {}^{(o)}\gamma_s ds$ is an $\mathbf{F}$-martingale. ◁

5.3 Diffusions

In this section, we present the main facts on linear diffusions, following closely the presentation of Chapter 2 in Borodin and Salminen [109]. We refer to

Durrett [287], Itô and McKean [465], Linetsky [595] and Rogers and Williams [742] for other studies of general diffusions.

A linear diffusion is a strong Markov process with continuous paths taking values on an interval I with left-end point $\ell \geq -\infty$ and right-end point $r \leq \infty$. We denote by ζ the life time of X (see Definition 1.1.14.1). We assume in what follows (unless otherwise stated) that all the diffusions we consider are **regular**, i.e., they satisfy $\mathbb{P}_x(T_y < \infty) > 0, \forall x, y \in I$ where $T_y = \inf\{t : X_t = y\}$.

5.3.1 (Time-homogeneous) Diffusions

In this book, we shall mainly consider diffusions which are Itô processes: let b and σ be two real-valued functions which are Lipschitz on the interval I, such that $\sigma(x) > 0$ for all x in the interval I. Then, there exists a unique solution to

$$X_t = x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s \,, \tag{5.3.1}$$

starting at point $x \in]\ell, r[$, up to the first exit time $T_{\ell,r} = T_\ell(X) \wedge T_r(X)$. In this case, X is a time-homogeneous diffusion.

In fact, the Lipschitz assumption is not quite necessary; see Theorem 1.5.5.1 for some finer assumptions on b and σ.

Solutions of

$$X_t = x + \int_0^t b(X_s, s)ds + \int_0^t \sigma(X_s, s)dW_s \,, \tag{5.3.2}$$

with time dependent coefficients b and σ are called time-inhomogeneous diffusions; for these processes, the following results do not apply.

From now on, we shall only consider diffusions of the type (5.3.1), and we drop the term "time-homogeneous." We mention furthermore that, in general studies of diffusions (see Borodin and Salminen [109]), a rôle is also played by a killing measure; however, since we shall not use this item in our presentation, we do not introduce it.

5.3.2 Scale Function and Speed Measure

Scale Function

Definition 5.3.2.1 *Let X be a diffusion on I and $T_y = \inf\{t \geq 0 : X_t = y\}$, for $y \in I$. A scale function is an increasing function from I to $\mathbb{R}$ such that, for $x \in [a, b]$*

$$\mathbb{P}_x(T_a < T_b) = \frac{s(x) - s(b)}{s(a) - s(b)} \,. \tag{5.3.3}$$

Obviously, if s^* is a scale function, then so is $\alpha s^* + \beta$ for any (α, β), with $\alpha > 0$ and any scale function can be written as $\alpha s^* + \beta$.

Proposition 5.3.2.2 *The process* $(s(X_t), 0 \leq t < T_{\ell,r})$ *is a local martingale. The scale function satisfies*

$$\frac{1}{2}\sigma^2(x)s''(x) + b(x)s'(x) = 0\,.$$

PROOF: For any finite stopping time $\tau < T_{\ell,r}$, the equality

$$\mathbb{E}_x\left(\frac{s(X_\tau) - s(b)}{s(a) - s(b)}\right) = \frac{s(x) - s(b)}{s(a) - s(b)}$$

follows from the Markov property. □

In the case of diffusions of the form (5.3.1), a (differentiable) scale function is

$$s(x) = \int_c^x \exp\left(-2\int_c^u b(v)/\sigma^2(v)\,dv\right) du \tag{5.3.4}$$

for some choice of $c \in]\ell, r[$. The increasing process of $s(X)$ being

$$A_t = \int_0^t (s'\sigma)^2(X_u)du,$$

(by an application of Itô's formula), the local martingale $(s(X_t), t < T_{\ell,r})$ can be written as a time changed Brownian motion: $s(X_t) = \beta_{A_t}$.

In the case of constant coefficients with $b < 0$ (resp. $b > 0$) and $\sigma \neq 0$, the diffusion is defined on $\mathbb{R}$, $T_{\ell,r} = \infty$, and we may choose $s(x) = \exp\left(-2bx/\sigma^2\right)$, (resp. $s(x) = -\exp\left(-2bx/\sigma^2\right)$) so that s is a strictly increasing function and $s(-\infty) = 0, s(\infty) = \infty$ (resp. $s(-\infty) = -\infty, s(\infty) = 0$).

A diffusion is said to be in **natural scale** if $s(x) = x$. In this case, if $I = \mathbb{R}$, the diffusion $(X_t, t \geq 0)$ is a local martingale.

Speed Measure

The **speed measure m** is defined as the measure such that the infinitesimal generator of X can be written as

$$\mathcal{A}f(x) = \frac{d}{d\mathbf{m}}\frac{d}{ds}f(x)$$

where

$$\frac{d}{ds}f(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{s(x+h) - s(x)},$$

and

$$\frac{d}{d\mathbf{m}}g(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{\mathbf{m}(x, x+h)}.$$

In the case of diffusions of the form (5.3.1), the speed measure is absolutely continuous with respect to Lebesgue measure, i.e., $\mathbf{m}(dx) = m(x)dx$, hence

$$\begin{aligned}\mathcal{A}f(x) &= \frac{d}{d\mathbf{m}}\frac{d}{ds}f(x) = \frac{1}{m(x)}\frac{d}{dx}\left(\frac{1}{s'}\frac{d}{dx}f\right) \\ &= \frac{1}{m(x)\,s'(x)}f''(x) - \frac{s''(x)}{m(x)\,(s')^2(x)}f'(x) \\ &= \frac{1}{m(x)\,s'(x)}f''(x) + \frac{2\,b(x)}{m(x)\,s'(x)\,\sigma^2(x)}f'(x)\end{aligned}$$

where the last equality comes from formula (5.3.4). Since in this case the infinitesimal generator has the form

$$\mathcal{A}f(x) = \frac{1}{2}\sigma^2(x)f''(x) + b(x)f'(x),$$

the density of the speed measure is

$$\boxed{m(x) = \frac{2}{\sigma^2(x)s'(x)}\,.} \tag{5.3.5}$$

The density of the speed measure satisfies

$$\frac{1}{2}\left(\sigma^2(x)m(x)\right)'' + (s(x)b(x)))'(x) = 0\,.$$

It is important to consider the local martingale $s(X_t)$ only strictly before the hitting time of the boundary. The reader should keep in mind the example of the reflected Brownian motion, which is not a martingale, although $s(x) = x$ (see $\rightarrowtail$ Proposition 6.1.2.4).

If X is a diffusion with scale function s, we have seen that $s(X_t) = \beta_{A_t}$, where β is a Brownian motion. In terms of speed measure, the increasing process A is the inverse of

$$C_u = \frac{1}{2}\int_0^u m(\beta_s)ds = \frac{1}{2}\int \mathbf{m}(dz)L_u^z(\beta)\,.$$

Remark 5.3.2.3 Beware that some authors define the speed measure with a factor 1/2, that is $\mathcal{A}f(x) = \frac{1}{2}\frac{d}{d\mathbf{m}}\frac{d}{ds}f(x)$. Our convention, without this factor 1/2 is the same as Borodin and Salminen [109].

Exercise 5.3.2.4 Prove that, if $s(X)$ is a martingale, then, equality (5.3.3) holds. $\lhd$

5.3.3 Boundary Points

Definition 5.3.3.1 *The boundary points are classified as follows:*

- *The left-hand point ℓ is an* ***exit boundary*** *if, for any $x \in]\ell, r[$,*

$$\int_\ell^x \mathbf{m}(]y,x[)s'(y)dy < \infty$$

and an ***entrance boundary*** *if, for any $x \in]\ell, r[$,*

$$\int_\ell^x \mathbf{m}(]\ell,y[)s'(y)dy < \infty\,.$$

- *The right-hand point r is an* ***exit boundary*** *if, for any $x \in]\ell, r[$,*

$$\int_x^r \mathbf{m}(]x,y[)s'(y)dy < \infty$$

and an ***entrance boundary*** *if, for any $x \in]\ell, r[$,*

$$\int_x^r \mathbf{m}(]y,r[)s'(y)dy < \infty\,.$$

- *A boundary point which is both entrance and exit is called* ***non-singular****.*
- *A boundary point that is neither entrance nor exit is called* ***natural****.*

A diffusion reaches its non-singular boundaries with positive probability, and it is possible to start a diffusion from a non-singular boundary.

An example where 0 is an entrance boundary is given by the BES^3 process (see $\rightarrowtail$ Chapter 6), or more generally by a BES^δ with $\delta \geq 2$. We recall that a BES^δ process with $\delta \geq 2$ does not return to 0 after it has left this point.

Definition 5.3.3.2 *Let X be a diffusion. The point ℓ is said to be* ***instantaneously reflecting*** *if $\mathbf{m}(\{\ell\}) = 0$.*

For the reflected BM $|B|$, the point 0 is instantaneously reflecting and the Lebesgue measure of the set $\{t : |B_t| = 0\}$ is zero.

Example 5.3.3.3 We present, following Borodin and Salminen [109], the computation of the scale function and speed measure for some important diffusion processes:

- **Drifted Brownian motion.**
  Suppose $X_t = B_t + \nu t$. A scale function for X is $s(x) = \exp(-2\nu x)$ for $\nu < 0$, and $s(x) = -\exp(-2\nu x)$ for $\nu > 0$. The density of the speed measure is $m(x) = 2e^{2\nu x}$. The lifetime is ∞.

- **Geometric Brownian motion.** Let $dS_t = S_t(\mu dt + \sigma dB_t)$. We have seen in Lemma 3.6.6.1 that $S_t^{1-\gamma}$ is a martingale for $\gamma = 2\mu/\sigma^2$. Hence
  - a scale function of S is $s(x) = -(x^{1-\gamma})/(1-\gamma)$ for $\gamma \neq 1$ and $\ln x$ for $\gamma = 1$,
  - the density of the speed measure is $m(x) = 2x^{\gamma-2}/\sigma^2$.

  The boundary points 0 and ∞ are natural.
  - If $\gamma > 1$, then $\lim_{t\to\infty} S_t(\omega) = \infty$, a.s.,
  - if $\gamma < 1$, then $\lim_{t\to\infty} S_t(\omega) = 0$, a.s.,
  - if $\gamma = 1$, then $\liminf_{t\to\infty} S_t(\omega) = 0, \limsup_{t\to\infty} S_t(\omega) = \infty$ a.s..

- **Reflected Brownian motion.**
  The process $X_t = |W_t|$ is a diffusion on $[0, \infty[$. The left-hand point 0 is a non-singular boundary point. The scale function is $s(x) = x$, the density of the speed measure is $m(x) = 2$.

- **Bessel processes.** A Bessel process (see ↣ Section 6.1) with dimension δ and index $\nu = \frac{\delta}{2} - 1$ is a diffusion on $]0, \infty[$, or on $[0, \infty[$ depending on the value of ν and the boundary conditions at 0.
  For all values of ν, the boundary point ∞ is natural. The boundary point 0 is
  - exit-non-entrance if $\nu \leq -1$
  - nonsingular if $-1 < \nu < 0$
  - entrance-not exit if $\nu \geq 0$.

  In the nonsingular case, the boundary condition at 0 is usually reflection or killing. A scale function for a $\text{BES}^{(\nu)}$ is $s(x) = x^{-2\nu}$ for $\nu < 0$, $s(x) = \ln x$ for $\nu = 0$ and $s(x) = -x^{-2\nu}$ for $\nu > 0$. It follows that a scale function for a $\text{BESQ}^{(\nu)}$ is
  - $s(x) = x^{-\nu}$ for $\nu < 0$,
  - $s(x) = \ln x$ for $\nu = 0$ and
  - $s(x) = -x^{-\nu}$ for $\nu > 0$.

  See ↣ Proposition 6.1.2.4 for more information.
  For $\nu > 0$, the density of the speed measure is $m(x) = \nu^{-1} x^{2\nu+1}$.

- **Affine equation.**
  Let
  $$dX_t = (\alpha X_t + 1)dt + \sqrt{2}\, X_t dW_t\,, X_0 = x\,.$$

The scale function derivative is $s'(x) = x^{-\alpha}e^{1/x}$ and the speed density function is $m(x) = x^{\alpha}e^{-1/x}$.

- **OU and Vasicek processes.**
Let r be a (k, σ) Ornstein-Uhlenbeck process. A scale function derivative is $s'(x) = \exp(kx^2/\sigma^2)$. If r is a $(k, \theta; \sigma)$ Vasicek process (see Section 2.6), $s'(x) = \exp k(x-\theta)^2/\sigma^2$.

The first application of the concept of speed measure is Feller's test for non-explosion (see Definition 1.5.4.10). We shall see in the sequel that speed measures are very useful tools.

Proposition 5.3.3.4 (Feller's Test for non-explosion.) *Let b, σ belong to $C^1(\mathbb{R})$, and let X be the solution of*

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t$$

with τ its explosion time. The process does not explode, i.e., $\mathbb{P}(\tau = \infty) = 1$ if and only if

$$\int_{-\infty}^{0} [s(x) - s(-\infty)]\, m(x)dx = \int_{0}^{\infty} [s(\infty) - s(x)]\, m(x)dx = \infty\,.$$

PROOF: see McKean [637], page 65. □

Comments 5.3.3.5 This proposition extends the case where the coefficients b and σ are only locally Lipschitz. Khasminskii [522] developed Feller's test for multidimensional diffusion processes (see McKean [637], page 103, Rogers and Williams [742], page 299). See Borodin and Salminen [109], Breiman [123], Freedman [357], Knight [528], Rogers and Williams [741] or [RY] for more information on speed measures.

Exercise 5.3.3.6 Let $dX_t = \theta dt + \sigma\sqrt{X_t}dW_t$, $X_0 > 0$, where $\theta > 0$ and, for $a < x < b$ let $\psi_{a,b}(x) = \mathbb{P}_x(T_b(X) < T_a(X))$. Prove that

$$\psi_{a,b}(x) = \frac{x^{1-\nu} - a^{1-\nu}}{b^{1-\nu} - a^{1-\nu}}$$

where $\nu = 2\theta/\sigma^2$. Prove also that if $\nu > 1$, then T_0 is infinite and that if $\nu < 1$, $\psi_{0,b}(x) = (x/b)^{1-\nu}$. Thus, the process $(1/X_t, t \geq 0)$ explodes in the case $\nu < 1$. ◁

5.3.4 Change of Time or Change of Space Variable

In a number of computations, it is of interest to time change a diffusion into BM by means of the scale function of the diffusion. It may also be of interest to relate diffusions of the form

$$X_t = x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s$$

to those for which $\sigma = 1$, that is $Y_t = y + \beta_t + \int_0^t du\, \mu(Y_u)$ where β is a Brownian motion. For this purpose, one may proceed by means of a change of time or change of space variable, as we now explain.

(a) Change of Time

Let $A_t = \int_0^t \sigma^2(X_s)ds$ and assume that $|\sigma| > 0$. Let $(C_u, u \geq 0)$ be the inverse of $(A_t, t \geq 0)$. Then

$$X_{C_u} = x + \beta_u + \int_0^u dC_h\, b(X_{C_h})$$

From $h = \int_0^{C_h} \sigma^2(X_s)ds$, we deduce $dC_h = \dfrac{dh}{\sigma^2(X_{C_h})}$, hence

$$Y_u := X_{C_u} = x + \beta_u + \int_0^u dh\, \frac{b}{\sigma^2}(Y_h)$$

where β is a Brownian motion.

(b) Change of Space Variable

Assume that $\varphi(x) = \displaystyle\int_0^x \frac{dy}{\sigma(y)}$ is well defined and that φ is of class C^2. From Itô's formula

$$\begin{aligned}\varphi(X_t) &= \varphi(x) + \int_0^t \varphi'(X_s)dX_s + \frac{1}{2}\int_0^t \varphi''(X_s)\sigma^2(X_s)ds \\ &= \varphi(x) + W_t + \int_0^t ds \left(\frac{b}{\sigma}(X_s) - \frac{1}{2}\sigma'(X_s)\right).\end{aligned}$$

Hence, setting $Z_t = \varphi(X_t)$, we get

$$Z_t = z + W_t + \int_0^t \widehat{b}(Z_s)ds$$

where $\widehat{b}(z) = \frac{b}{\sigma}(\varphi^{-1}(z)) - \frac{1}{2}\sigma'(\varphi^{-1}(z))$.

Comment 5.3.4.1 See Doeblin [255] for some interesting applications.

5.3.5 Recurrence

Definition 5.3.5.1 *A diffusion X with values in I is said to be* ***recurrent*** *if*

$$\mathbb{P}_x(T_y < \infty) = 1, \ \forall x, y \in I\,.$$

If not, the diffusion is said to be ***transient****.*

It can be proved that the homogeneous diffusion X given by (5.3.1) on $]\ell, r[$ is recurrent if and only if $s(\ell+) = -\infty$ and $s(r-) = \infty$. (See [RY], Chapter VII, Section 3, for a proof given as an exercise.)

Example 5.3.5.2 A one-dimensional Brownian motion is a recurrent process, a Bessel process (see ↣ Chapter 6) with index strictly greater than 0 is a transient process. For the (recurrent) one-dimensional Brownian motion, the times T_y are large, i.e., $\mathbb{E}_x(T_y^\alpha) < \infty,$ for $x \neq y$ if and only if $\alpha < 1/2$.

5.3.6 Resolvent Kernel and Green Function

Resolvent Kernel

The resolvent of a Markov process X is the family of operators $f \to R_\lambda f$

$$R_\lambda f(x) = \mathbb{E}_x\left(\int_0^\infty e^{-\lambda t} f(X_t)dt\right)\,.$$

The resolvent kernel of a diffusion is the density (with respect to Lebesgue measure) of the resolvent operator, i.e., the Laplace transform in t of the transition density $p_t(x, y)$:

$$R_\lambda(x, y) = \int_0^\infty e^{-\lambda t} p_t(x, y)dt\,, \tag{5.3.6}$$

where $\lambda > 0$ for a recurrent diffusion and $\lambda \geq 0$ for a transient diffusion. It satisfies

$$\frac{1}{2}\sigma^2(x)\frac{\partial^2 R_\lambda}{\partial x^2} + b(x)\frac{\partial R_\lambda}{\partial x} - \lambda R_\lambda = 0 \quad \text{for } x \neq y$$

and $R_\lambda(x, x) = 1$. The Sturm-Liouville O.D.E.

$$\frac{1}{2}\sigma^2(x)u''(x) + b(x)u'(x) - \lambda u(x) = 0 \tag{5.3.7}$$

admits two linearly independent continuous positive solutions (the basic solutions) $\Phi_{\lambda\uparrow}(x)$ and $\Phi_{\lambda\downarrow}(x)$, with $\Phi_{\lambda\uparrow}$ increasing and $\Phi_{\lambda\downarrow}$ decreasing, which are determined up to constant factors.

A straightforward application of Itô's formula establishes that $e^{-\lambda t}\Phi_{\lambda\uparrow}(X_t)$ and $e^{-\lambda t}\Phi_{\lambda\downarrow}(X_t)$ are local martingales, for $\lambda > 0$, hence, using carefully

Doob's optional stopping theorem, we obtain the Laplace transform of the first hitting times:

$$\mathbb{E}_x\left(e^{-\lambda T_y}\right) = \begin{cases} \Phi_{\lambda\uparrow}(x)/\Phi_{\lambda\uparrow}(y) & \text{if } x < y \\ \Phi_{\lambda\downarrow}(x)/\Phi_{\lambda\downarrow}(y) & \text{if } x > y \end{cases} . \tag{5.3.8}$$

Green Function

Let $p_t^{(m)}(x,y)$ be the transition probability function relative to the speed measure $m(y)dy$:

$$\mathbb{P}_x(X_t \in dy) = p_t^{(m)}(x,y)m(y)dy\,. \tag{5.3.9}$$

It is a known and remarkable result that $p_t^{(m)}(x,y) = p_t^{(m)}(y,x)$ (see Chung [185] and page 149 in Itô and McKean [465]).

The Green function is the density with respect to the speed measure of the resolvent operator: using $p_t^{(m)}(x,y)$, the transition probability function relative to the speed measure, there is the identity

$$G_\lambda(x,y) := \int_0^\infty e^{-\lambda t} p_t^{(m)}(x,y)dt = w_\lambda^{-1}\Phi_{\lambda\uparrow}(x \wedge y)\Phi_{\lambda\downarrow}(x \vee y)\,,$$

where the Wronskian

$$w_\lambda := \frac{\Phi'_{\lambda\uparrow}(y)\Phi_{\lambda\downarrow}(y) - \Phi_{\lambda\uparrow}(y)\Phi'_{\lambda\downarrow}(y)}{s'(y)} \tag{5.3.10}$$

depends only on λ and not on y. Obviously

$$m(y)G_\lambda(x,y) = R_\lambda(x,y)\,,$$

hence

$$R_\lambda(x,y) = w_\lambda^{-1}m(y)\Phi_{\lambda\uparrow}(x \wedge y)\Phi_{\lambda\downarrow}(x \vee y)\,. \tag{5.3.11}$$

A diffusion is transient if and only if $\lim_{\lambda\to 0} G_\lambda(x,y) < \infty$ for some $x, y \in I$ and hence for all $x, y \in I$.

Comment 5.3.6.1 See Borodin and Salminen [109] and Pitman and Yor [718, 719] for an extended study. Kent [520] proposes a methodology to invert this Laplace transform in certain cases as a series expansion. See Chung [185] and Chung and Zhao [187] for an extensive study of Green functions. Many authors call Green functions our resolvent.

5.3.7 Examples

Here, we present examples of computations of functions $\Phi_{\lambda\downarrow}$ and $\Phi_{\lambda\uparrow}$ for certain diffusions.

- **Brownian motion with drift μ: $X_t = \mu t + \sigma W_t$.** In this case, the basic solutions of
$$\frac{1}{2}\sigma^2 u'' + \mu u' = \lambda u$$
are
$$\Phi_{\lambda\uparrow}(x) = \exp\left[\frac{x}{\sigma^2}\left(-\mu + \sqrt{2\lambda\sigma^2 + \mu^2}\right)\right],$$
$$\Phi_{\lambda\downarrow}(x) = \exp\left[-\frac{x}{\sigma^2}\left(\mu + \sqrt{2\lambda\sigma^2 + \mu^2}\right)\right].$$

- **Geometric Brownian motion: $dX_t = X_t(\mu dt + \sigma dW_t)$.**
The basic solutions of
$$\frac{1}{2}\sigma^2 x^2 u'' + \mu x u' = \lambda u$$
are
$$\Phi_{\lambda\uparrow}(x) = x^{\frac{1}{\sigma^2}(-\mu + \frac{\sigma^2}{2} + \sqrt{2\lambda\sigma^2 + (\mu - \sigma^2/2)^2})},$$
$$\Phi_{\lambda\downarrow}(x) = x^{-\frac{1}{\sigma^2}(\mu - \frac{\sigma^2}{2} + \sqrt{2\lambda\sigma^2 + (\mu - \sigma^2/2)^2})}.$$

- **Bessel process with index ν.** Let $dX_t = dW_t + \left(\nu + \frac{1}{2}\right)\frac{1}{X_t}dt$. For $\nu > 0$, the basic solutions of
$$\frac{1}{2}u'' + \left(\nu + \frac{1}{2}\right)\frac{1}{x}u' = \lambda u$$
are
$$\Phi_{\lambda\uparrow}(x) = x^{-\nu} I_\nu(x\sqrt{2\lambda}),\ \Phi_{\lambda\downarrow}(x) = x^{-\nu} K_\nu(x\sqrt{2\lambda}),$$
where I_ν and K_ν are the classical Bessel functions with index ν (see ↣ Appendix A.5.2).

- **Affine Equation.**
Let
$$dX_t = (\alpha X_t + \beta)dt + \sqrt{2} X_t dW_t,$$
with $\beta \neq 0$. The basic solutions of
$$x^2 u'' + (\alpha x + \beta)u' = \lambda u$$
are

$$\Phi_{\lambda\uparrow}(x) = \left(\frac{\beta}{x}\right)^{(\nu+\mu)/2} M\left(\frac{\nu+\mu}{2}, 1+\mu, \frac{\beta}{x}\right),$$

$$\Phi_{\lambda\downarrow}(x) = \left(\frac{\beta}{x}\right)^{(\nu+\mu)/2} U\left(\frac{\nu+\mu}{2}, 1+\mu, \frac{\beta}{x}\right)$$

where M and U denote the Kummer functions (see $\rightarrowtail$ A.5.4 in the Appendix) and $\mu = \sqrt{\nu^2 + 4\lambda}$, $1+\nu = \alpha$.

- **Ornstein-Uhlenbeck and Vasicek Processes.** Let $k > 0$ and

$$dX_t = k(\theta - X_t)dt + \sigma dW_t\,, \tag{5.3.12}$$

a Vasicek process. The basic solutions of

$$\frac{1}{2}\sigma^2 u'' + k(\theta - x)u' = \lambda u$$

are

$$\Phi_{\lambda\uparrow}(x) = \exp\left(\frac{k\,(x-\theta)^2}{2\sigma^2}\right) D_{-\lambda/k}\left(-\frac{x-\theta}{\sigma}\sqrt{2k}\right),$$

$$\Phi_{\lambda\downarrow}(x) = \exp\left(\frac{k\,(x-\theta)^2}{2\sigma^2}\right) D_{-\lambda/k}\left(\frac{x-\theta}{\sigma}\sqrt{2k}\right).$$

Here, D_ν is the parabolic cylinder function with index ν (see $\rightarrowtail$ Appendix A.5.4).

Comment 5.3.7.1 For OU processes, i.e., in the case $\theta = 0$ in equation (5.3.12), Ricciardi and Sato [732] obtained, for $x > a$, that the density of the hitting time of a is

$$-ke^{k(x^2-a^2)/2} \sum_{n=1}^{\infty} \frac{D_{\nu_{n,a}}(x\sqrt{2k})}{D'_{\nu_{n,a}}(a\sqrt{2k})} e^{-k\nu_{n,a}t}$$

where $0 < \nu_{1,a} < \cdots < \nu_{n,a} < \cdots$ are the zeros of $\nu \to D_\nu(-a)$. The expression $D'_{\nu_{n,a}}$ denotes the derivative of $D_\nu(a)$ with respect to ν, evaluated at the point $\nu = \nu_{n,a}$. Note that the formula in Leblanc et al. [573] for the law of the hitting time of a is only valid for $a = 0, \theta = 0$. See also the discussion in Subsection 3.4.1.

Extended discussions on this topic are found in Alili et al. [10], Göing-Jaeschke and Yor [398, 397], Novikov [678], Patie [697] or Borodin and Salminen [109].

- **CEV Process.**
The **c**onstant **e**lasticity of **v**ariance process (See $\rightarrowtail$ Section 6.4) follows

$$dS_t = S_t(\mu dt + S_t^\beta dW_t)\,.$$

In the case $\beta < 0$, the basic solutions of

$$\frac{1}{2}x^{2\beta+2}u''(x) + \mu x u'(x) = \lambda u(x)$$

are

$$\Phi_{\lambda\uparrow}(x) = x^{\beta+1/2}e^{\epsilon x/2}M_{k,m}(x),\ \Phi_{\lambda\downarrow}(x) = x^{\beta+1/2}e^{\epsilon x/2}W_{k,m}(x)$$

where M and W are the Whittaker functions (see $\rightarrowtail$ Subsection A.5.7) and

$$\epsilon = \mathrm{sgn}(\mu\beta),\ m = -\frac{1}{4\beta},\ k = \epsilon\left(\frac{1}{2} + \frac{1}{4\beta}\right) - \frac{\lambda}{2|\mu\beta|}\,.$$

See Davydov and Linetsky [225].

Exercise 5.3.7.2 Prove that the process

$$X_t = \exp(aB_t + bt)\left(x + \int_0^t ds \exp(-aB_s - bs)\right)$$

satisfies

$$X_t = x + a\int_0^t X_u dB_u + \int_0^t \left(\left(\frac{a^2}{2} + b\right)X_u + 1\right)du\,.$$

(See Donati-Martin et al. [258] for further properties of this process, and application to Asian options.) More generally, consider the process

$$dY_t = (aY_t + b)dt + (cY_t + d)dW_t\,,$$

where $c \neq 0$. Prove that, if $X_t = cY_t + d$, then

$$dX_t = (\alpha X_t + \beta)dt + X_t dW_t$$

with $\alpha = a/c, \beta = b - da/c$. From $T_\alpha(Y^y) = T_{c\alpha+d}(X^{cx+d})$, deduce the Laplace transform of first hitting times for the process Y. ◁

5.4 Non-homogeneous Diffusions

5.4.1 Kolmogorov's Equations

Let

$$Lf(s,x) = b(s,x)\partial_x f(s,x) + \frac{1}{2}\sigma^2(s,x)\partial^2_{xx}f(s,x)\,.$$

A **fundamental solution** of

$$\partial_s f(s,x) + Lf(s,x) = 0 \tag{5.4.1}$$

is a positive function $p(x,s;y,t)$ defined for $0 \le s < t$, $x,y \in \mathbb{R}$, such that for any function $\varphi \in C_0(\mathbb{R})$ and any $t > 0$ the function

$$f(s,x) = \int_{\mathbb{R}} \varphi(y)p(s,x;t,y)dy$$

is bounded, is of class $C^{1,2}$, satisfies (5.4.1) and obeys $\lim_{s\uparrow t} f(s,x) = \varphi(x)$.

If b and σ are real valued bounded and continuous functions $\mathbb{R}^+ \times \mathbb{R}$ such that

(i) $\sigma^2(t,x) \ge c > 0$,
(ii) there exists $\alpha \in]0,1]$ such that for all (x,y), for all $s,t \ge 0$,

$$|b(t,x) - b(s,y)| + |\sigma^2(t,x) - \sigma^2(s,y)| \le K(|t-s|^\alpha + |x-y|^\alpha)\,,$$

then the equation

$$\partial_s f(s,x) + Lf(s,x) = 0$$

admits a strictly positive fundamental solution p. For fixed (y,t) the function $u(s,x) = p(s,x;t,y)$ is of class $C^{1,2}$ and satisfies the backward Kolmogorov equation that we present below. If in addition, the functions $\partial_x b(t,x)$, $\partial_x \sigma(t,x)$, $\partial_{xx}\sigma(t,x)$ are bounded and Hölder continuous, then for fixed (x,s) the function $v(t,y) = p(s,x;t,y)$ is of class $C^{1,2}$ and satisfies the forward Kolmogorov equation that we present below.

Note that a time-inhomogeneous diffusion process can be treated as a homogeneous process. Instead of X, consider the space-time diffusion process (t, X_t) on the enlarged state space $\mathbb{R}^+ \times \mathbb{R}^d$.

We give Kolmogorov's equations for the general case of inhomogeneous diffusions

$$dX_t = b(t,X_t)dt + \sigma(t,X_t)dW_t\,.$$

Proposition 5.4.1.1 *The transition probability density $p(s,x;t,y)$ defined for $s < t$ as $\mathbb{P}_{x,s}(X_t \in dy) = p(s,x;t,y)dy$ satisfies the two partial differential equations (recall δ_x is the Dirac measure at x):*

- *The* ***backward Kolmogorov equation:***

$$\begin{cases} \dfrac{\partial}{\partial s}p(s,x;t,y) + \dfrac{1}{2}\sigma^2(s,x)\dfrac{\partial^2}{\partial x^2}p(s,x;t,y) + b(s,x)\dfrac{\partial}{\partial x}p(s,x;t,y) = 0\,, \\ \lim_{s\to t} p(s,x;t,y)dy = \delta_x(dy)\,. \end{cases}$$

- *The* ***forward Kolmogorov equation***

$$\begin{cases} \dfrac{\partial}{\partial t}p(s,x;t,y) - \dfrac{1}{2}\dfrac{\partial^2}{\partial y^2}\big(p(s,x;t,y)\sigma^2(t,y)\big) + \dfrac{\partial}{\partial y}\big(p(s,x;t,y)b(t,y)\big) = 0\,, \\ \lim_{t\to s} p(s,x;t,y)dy = \delta_x(dy)\,. \end{cases}$$

Sketch of the Proof: The backward equation is really straightforward to derive. Let φ be a C^2 function with compact support. For any fixed t, the martingale $\mathbb{E}(\varphi(X_t)|\mathcal{F}_s)$ is equal to $f(s, X_s) = \int_{\mathbb{R}} \varphi(y)p(s, X_s; t, y)dy$ since X is a Markov process. An application of Itô's formula to $f(s, X_s)$ leads to its decomposition as a semi-martingale. Since it is in fact a true martingale its bounded variation term must be equal to zero. This result being true for every φ, it provides the backward equation.

The forward equation is in a certain sense the dual of the backward one. Recall that if φ is a C^2 function with compact support, then

$$\mathbb{E}_{s,x}(\varphi(X_t)) = \int_{\mathbb{R}} \varphi(y)p(s, x; t, y)dy\,.$$

From Itô's formula, for $t > s$

$$\varphi(X_t) = \varphi(X_s) + \int_s^t \varphi'(X_u)dX_u + \frac{1}{2}\int_s^t \varphi''(X_u)\sigma^2(u, X_u)du\,.$$

Hence, taking (conditional) expectations

$$\begin{aligned}\mathbb{E}_{s,x}(\varphi(X_t)) &= \varphi(x) + \int_s^t \mathbb{E}_{s,x}\left(\varphi'(X_u)b(u, X_u) + \frac{1}{2}\sigma^2(u, X_u)\varphi''(X_u)\right)du \\ &= \varphi(x) + \int_s^t du \int_{\mathbb{R}} \left(\varphi'(y)b(u, y) + \frac{1}{2}\sigma^2(u, y)\varphi''(y)\right)p(s, x; u, y)dy\,.\end{aligned}$$

From the integration by parts formula (in the sense of distributions if the coefficients are not smooth enough) and since φ and φ' vanish at ∞:

$$\begin{aligned}\int_{\mathbb{R}} \varphi(y)p(s, x; t, y)dy &= \varphi(x) - \int_s^t du \int_{\mathbb{R}} \varphi(y)\frac{\partial}{\partial y}\left(b(u, y)p(s, x; u, y)\right)dy \\ &\quad + \frac{1}{2}\int_s^t du \int_{\mathbb{R}} \varphi(y)\frac{\partial^2}{\partial y^2}\left(\sigma^2(u, y)p(s, x; u, y)\right)dy\,.\end{aligned}$$

Differentiating with respect to t, we obtain that

$$\frac{\partial}{\partial t}p(s, x; t, y) = -\frac{\partial}{\partial y}(b(t, y)p(s, x; t, y)) + \frac{1}{2}\frac{\partial^2}{\partial y^2}\left(\sigma^2(t, y)p(s, x; t, y)\right)\,.$$

□

Note that for homogeneous diffusions, the density

$$p(x; t, y) = \mathbb{P}_x(X_t \in dy)/dy$$

satisfies the backward Kolmogorov equation

$$\frac{1}{2}\sigma^2(x)\frac{\partial^2 p}{\partial x^2}(x; t, y) + b(x)\frac{\partial p}{\partial x}(x; t, y) = \frac{\partial}{\partial t}p(x; t, y)\,.$$

Comments 5.4.1.2 (a) The Kolmogorov equations are the topic studied by Doeblin [255] in his now celebrated " pli cacheté n^0 11668".

(b) We refer to Friedman [361] p.141 and 148, Karatzas and Shreve [513] p.328, Stroock and Varadhan [812] and Nagasawa [663] for the multidimensional case and for regularity assumptions for uniqueness of the solution to the backward Kolmogorov equation. See also Itô and McKean [465], p.149 and Stroock [810].

5.4.2 Application: Dupire's Formula

Under the assumption that the underlying asset follows

$$dS_t = S_t(rdt + \sigma(t, S_t)dW_t)$$

under the risk-neutral probability, Dupire [284, 283] established a formula relating the local volatility $\sigma(t, x)$ and the value $C(T, K)$ of a European Call where K is the strike and T the maturity, i.e.,

$$\frac{1}{2}K^2\sigma^2(T, K) = \frac{\partial_T C(T, K) + rK\partial_K C(T, K)}{\partial^2_{KK} C(T, K)} .$$

We have established this formula using a local-time methodology in Subsection 4.2.1; here we present the original proof of Dupire as an application of the Kolmogorov backward equation. Let $f(T, x)$ be the density of the random variable S_T, i.e.,

$$f(T, x)dx = \mathbb{P}(S_T \in dx) .$$

Then,

$$C(T, K) = e^{-rT}\int_0^\infty (x-K)^+ f(T, x)dx = e^{-rT}\int_K^\infty (x-K)f(T, x)dx$$

$$= e^{-rT}\int_K^\infty dx f(T, x)\int_K^x dy = e^{-rT}\int_K^\infty dy\int_y^\infty f(T, x)dx . \quad (5.4.2)$$

By differentiation with respect to K,

$$\frac{\partial C}{\partial K}(T, K) = -e^{-rT}\int_0^\infty \mathbb{1}_{\{x>K\}} f(T, x)dx = -e^{-rT}\int_K^\infty f(T, x)dx ,$$

hence, differentiating again

$$\frac{\partial^2 C}{\partial K^2}(T, K) = e^{-rT} f(T, K) \quad (5.4.3)$$

which allows us to obtain the law of the underlying asset from the prices of the European options. For notational convenience, we shall now write $C(t, x)$

instead of $C(T,K)$. From (5.4.3), $f(t,x) = e^{rt}\frac{\partial^2 C}{\partial x^2}(t,x)$, hence differentiating both sides of this equality w.r.t. t gives

$$\frac{\partial}{\partial t} f = re^{rt}\frac{\partial^2 C}{\partial x^2} + e^{rt}\frac{\partial^2}{\partial x^2}\frac{\partial}{\partial t} C\,.$$

The density f satisfies the forward Kolmogorov equation

$$\frac{\partial f}{\partial t}(t,x) - \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(x^2\sigma^2(t,x)f(t,x)\right) + \frac{\partial}{\partial x}(rxf(t,x)) = 0\,,$$

or

$$\frac{\partial f}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(x^2\sigma^2 f\right) - rf - rx\frac{\partial}{\partial x} f\,. \tag{5.4.4}$$

Replacing f and $\frac{\partial f}{\partial t}$ by their expressions in terms of C in (5.4.4), we obtain

$$re^{rt}\frac{\partial^2 C}{\partial x^2} + e^{rt}\frac{\partial^2}{\partial x^2}\frac{\partial}{\partial t}C = e^{rt}\frac{1}{2}\frac{\partial^2}{\partial x^2}\left(x^2\sigma^2\frac{\partial^2 C}{\partial x^2}\right) - re^{rt}\frac{\partial^2 C}{\partial x^2} - rxe^{rt}\frac{\partial}{\partial x}\frac{\partial^2 C}{\partial x^2}$$

and this equation can be simplified as follows

$$\begin{aligned}\frac{\partial^2}{\partial x^2}\frac{\partial}{\partial t}C &= \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(x^2\sigma^2\frac{\partial^2 C}{\partial x^2}\right) - 2r\frac{\partial^2 C}{\partial x^2} - rx\frac{\partial}{\partial x}\frac{\partial^2 C}{\partial x^2}\\ &= \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(x^2\sigma^2\frac{\partial^2 C}{\partial x^2}\right) - r\left(2\frac{\partial^2 C}{\partial x^2} + x\frac{\partial}{\partial x}\frac{\partial^2 C}{\partial x^2}\right)\\ &= \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(x^2\sigma^2\frac{\partial^2 C}{\partial x^2}\right) - r\,\frac{\partial^2}{\partial x^2}\left(x\frac{\partial C}{\partial x}\right)\,,\end{aligned}$$

hence,

$$\frac{\partial^2}{\partial x^2}\frac{\partial C}{\partial t} = \frac{\partial^2}{\partial x^2}\left(\frac{1}{2}x^2\sigma^2\frac{\partial^2 C}{\partial x^2} - rx\frac{\partial C}{\partial x}\right)\,.$$

Integrating twice with respect to x shows that there exist two functions α and β, depending only on t, such that

$$\frac{1}{2}x^2\sigma^2(t,x)\frac{\partial^2 C}{\partial x^2}(t,x) = rx\frac{\partial C}{\partial x}(t,x) + \frac{\partial C}{\partial t}(t,x) + \alpha(t)x + \beta(t)\,.$$

Assuming that the quantities

$$\begin{cases} x^2\sigma^2(t,x)\dfrac{\partial^2 C}{\partial x^2}(t,x) = e^{-rt}x^2\sigma^2(t,x)f(t,x)\\ x\dfrac{\partial C}{\partial x}(t,x) = -e^{-rt}x\displaystyle\int_x^\infty f(t,y)dy\\ \dfrac{\partial C}{\partial t}(t,x)\end{cases}$$

go to 0 as x goes to infinity, we obtain $\lim_{x\to\infty}\alpha(t)x + \beta(t) = 0, \forall t$, hence $\alpha(t) = \beta(t) = 0$ and

$$\frac{1}{2}x^2\sigma^2(t,x)\frac{\partial^2 C}{\partial x^2}(t,x) = rx\frac{\partial C}{\partial x}(t,x) + \frac{\partial C}{\partial t}(t,x)\,.$$

The value of $\sigma(t,x)$ in terms of the call prices follows. □

5.4.3 Fokker-Planck Equation

Proposition 5.4.3.1 *Let $dX_t = b(t,X_t)dt + \sigma(t,X_t)dB_t$, and assume that h is a deterministic function such that $X_0 > h(0)$, $\tau = \inf\{t \geq 0 : X_t \leq h(t)\}$ and*

$$g(t,x)dx = \mathbb{P}(X_t \in dx, \tau > t)\,.$$

The function $g(t,x)$ satisfies the Fokker-Planck equation

$$\frac{\partial}{\partial t}g(t,x) = -\frac{\partial}{\partial x}\big(b(t,x)g(t,x)\big) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\big(\sigma^2(t,x)g(t,x)\big)\,;\; x > h(t)$$

and the boundary conditions

$$\lim_{t\to 0} g(t,x)dx = \delta(x - X_0)$$
$$g(t,x)|_{x=h(t)} = 0\,.$$

PROOF: The proof follows that of the backward Kolmogorov equation.

▶ We first note that

$$\begin{aligned}
\mathbb{E}(\varphi(X_{t\wedge\tau})) &= \mathbb{E}(\varphi(X_t)\mathbb{1}_{\{t\leq\tau\}}) + \mathbb{E}(\varphi(X_\tau)\mathbb{1}_{\{\tau<t\}}) \\
&= \int_{\mathbb{R}} \varphi(x)g(t,x)dx + \mathbb{E}(\varphi(h(\tau))\mathbb{1}_{\{\tau<t\}}) \\
&= \int_{\mathbb{R}} \varphi(x)g(t,x)dx + \int_0^t \varphi(h(u))\mu(du)
\end{aligned}$$

where μ is the law of τ.

▶ If φ is a C^2 function with compact support,

$$\varphi(X_{t\wedge\tau}) = \varphi(X_{s\wedge\tau}) + \int_{s\wedge\tau}^{t\wedge\tau} \varphi'(X_u)dX_u + \frac{1}{2}\int_{s\wedge\tau}^{t\wedge\tau} \varphi''(X_u)\sigma^2(u,X_u)du\,,$$

hence,

$$\begin{aligned}
\mathbb{E}(\varphi(X_{t\wedge\tau})) &= \mathbb{E}(\varphi(X_{s\wedge\tau})) + \mathbb{E}\left(\int_s^t \mathbb{1}_{\{u<\tau\}}\varphi'(X_u)b(u,X_u)du\right) \\
&\qquad + \frac{1}{2}\mathbb{E}\left(\int_s^t \mathbb{1}_{\{u<\tau\}}\varphi''(X_u)\sigma^2(u,X_u)du\right) \\
&= \int \varphi(x)g(s,x)dx + \int_0^s \varphi(h(v))\mu(dv) \\
&\quad + \int_s^t du\int_{\mathbb{R}} dx\left(\varphi'(x)b(u,x) + \frac{1}{2}\varphi''(x)\sigma^2(u,x)\right)g(x,u)\,.
\end{aligned}$$

This identity holds for any function φ of class C^2, therefore, using integration by parts for the last integral, and differentiation with respect to t, we get the result. The law of τ is obtained by integration w.r.t. x. □

Using the Fokker-Planck equation, Iyengar [466], He et al. [426] and Zhou [876] established the following result.

Proposition 5.4.3.2 *Let $X^i_t = \alpha_i t + \sigma_i W^i_t$ where W^1, W^2 are two correlated Brownian motions, with correlation ρ, and let m^i_t be the running minimum of X^i. The probability density*

$$\mathbb{P}(X^1_t \in dx_1, X^2_t \in dx_2, m^1_t \in dm_1, m^2_t \in dm_2) = p(x_1, x_2, t; m_1, m_2) dx_1 dx_2 dm_1 dm_2$$

is given by

$$p(x_1, x_2, t;\, m_1, m_2) = \frac{e^{a_1 x_1 + a_2 x_2 + bt}}{\sigma_1 \sigma_2 \sqrt{1-\rho^2}}\, h(x_1, x_2, t; m_1, m_2) \tag{5.4.5}$$

with

$$h(x_1, x_2, t; m_1, m_2) = \frac{2}{\beta t} e^{-(r^2+r_0^2)/(2t)} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi\theta_0}{\beta}\right) \sin\left(\frac{n\pi\theta}{\beta}\right) I_{(n\pi)/\beta}\left(\frac{rr_0}{t}\right)$$

where I_ν is the modified Bessel function of index ν and

$$a_1 = \frac{\alpha_1\sigma_2 - \rho\alpha_2\sigma_1}{(1-\rho^2)\sigma_1^2\sigma_2}, \quad a_2 = \frac{\alpha_2\sigma_1 - \rho\alpha_1\sigma_2}{(1-\rho^2)\sigma_1\sigma_2^2}$$

$$b = -\alpha_1 a_1 - \alpha_2 a_2 + \frac{1}{2}\left(\sigma_1^2 a_1^2 + \sigma_2^2 a_2^2\right) + \rho\sigma_1\sigma_2 a_1 a_2$$

$$\beta = \tan^{-1}\left(-\frac{\sqrt{1-\rho^2}}{\rho}\right), \quad \textit{for}\, \rho < 0$$

$$= \pi - \tan^{-1}\left(\frac{\sqrt{1-\rho^2}}{\rho}\right), \quad \textit{for}\, \rho > 0$$

$$z_1 = \frac{1}{\sqrt{1-\rho^2}}\left[\left(\frac{x_1 - m_1}{\sigma_1}\right) - \rho\left(\frac{x_2 - m_2}{\sigma_2}\right)\right], \quad z_2 = \frac{x_2 - m_2}{\sigma_2}$$

$$z_{10} = \frac{1}{\sqrt{1-\rho^2}}\left[-\frac{m_1}{\sigma_1} + \rho\frac{m_2}{\sigma_2}\right], \quad z_{20} = -\frac{m_2}{\sigma_2}$$

$$r = \sqrt{z_1^2 + z_2^2}, \quad \tan\theta = \frac{z_2}{z_1}, \quad \theta \in [0, \beta]$$

$$r_0 = \sqrt{z_{10}^2 + z_{20}^2}, \quad \tan\theta_0 = \frac{z_{20}}{z_{10}}, \quad \theta_0 \in [0, \beta].$$

The joint law with the maximum M_i is

$$\begin{aligned}&\mathbb{P}(X_t^1 \in dx_1, X_t^2 \in dx_2, m_t^1 \geq m_1, M_t^2 \leq M_2)\\ &= p(x_1, -x_2, t; m_1, -M_2, \alpha_1, -\alpha_2, \sigma_1, \sigma_2, -\rho) dx_1 dx_2\end{aligned}$$

where $p(x_1, x_2, t; m_1, m_2; \alpha_1, \alpha_2, \sigma_1, \sigma_2, \rho)$ is the density given in (5.4.5).

Comments 5.4.3.3 (a) The knowledge of the multidimensional laws of such variables is important in the structural approach of credit risk. However, the complexity of the above formula makes it difficult to implement. Let us mention that the wrong formula given in Bielecki and Rutkowski [99] in the first edition has been corrected in the second printing. See also the recent paper of Patras [698] where a proof using probabilistic and geometric tools is given and Blanchet-Scalliet and Patras [106] for application to counterparty risk.

(b) Recently, Rogers and Shepp [739] have studied the correlation $c(\rho)$ of the maxima of correlated BMs. Denoting by $M_t^i = \sup_{s \leq t} W_s^i$ the running supremum of the BM W^i, they established that

$$c(\rho) = (\cos\alpha) \int_0^\infty du \frac{\cosh(\alpha u)}{\sinh(u\pi/2)} \tanh(u\gamma)$$

where α is given in terms of the correlation coefficient ρ between the BMs as $\alpha = \arcsin(\rho) \in [\pi/2, \pi/2]$ and $2\gamma = \alpha + \pi/2$.

The proof relies on three steps: the first one is to compute the joint law of (M_Θ^1, M_Θ^2) for Θ an exponential random variable with parameter λ, independent of (W^1, W^2). If

$$F(x_1, x_2) = \mathbb{P}(x_1 \leq M_\Theta^1, x_2 \leq M_\Theta^2),$$

then it is easy to check that

$$c(\rho) = \lambda \int_0^\infty \int_0^\infty f(x_1, x_2) dx_1 dx_2$$

In a second step, the authors note that, since $\mathbb{P}(M_\Theta^1 > x_i) = e^{-\sqrt{2\lambda}x_i}$, then

$$F(x_1, x_2) = e^{-\sqrt{2\lambda}x_1} + e^{-\sqrt{2\lambda}x_2} - \mathbb{P}(M_\Theta^1 < x_1, M_\Theta^2 < x_2).$$

They introduce $X_t^i = M_t^i - W_t^i$ and obtain

$$\mathbb{P}(M_\Theta^1 < x_1, M_\Theta^2 < x_2) = \mathbb{P}(\tau \leq \Theta | X_0^1 = x_1,\, X_0^2 = x_2)$$

where $\tau = \inf\{t : X_t^1 X_t^2 = 0\}$. The last step consists of the computation of

$$\widehat{F}(x_1, x_2) = \mathbb{P}(\tau \leq \Theta | X_0^1 = x_1,\, X_0^2 = x_2) = \mathbb{E}(e^{-\lambda\tau} | X_0^1 = x_1,\, X_0^2 = x_2)$$

which satisfies

$$2\lambda \widehat{f}(x_1, x_2) = (\partial_{x_1x_1}^2 + 2\rho \partial_{x_1} \partial_{x_2} + \partial_{x_1x_1}^2) \widehat{f}(x_1, x_2)$$

with the boundary condition $\widehat{f} = 1$ on the axes.

5.4.4 Valuation of Contingent Claims

Suppose $V_f(x,T)$ is the value of a contingent claim with payoff $f(S_T)$, i.e., $V_f(x,T) = \mathbb{E}_{\mathbb{Q}}(e^{-rT} f(S_T))$ where

$$dS_t = S_t((r-\kappa)dt + \sigma(S_t)dW_t),\ S_0 = x$$

under the risk-adjusted probability $\mathbb{Q}$. In terms of the transition probability of S relative to the Lebesgue measure, that is $\mathbb{Q}(S_T \in dy) = p_T(x,y)dy$ the value of the claim is:

$$V_f(x,T) = \mathbb{E}_{\mathbb{Q}}(e^{-rT} f(S_T)) = e^{-rT} \int_0^\infty f(y) p_T(x,y)dy\,.$$

Therefore, the quantity $e^{-rT} p_T(x,y)$ can be interpreted as the price of a security with the Dirac measure payoff δ_y. It is called the price of an Arrow-Debreu security or the pricing kernel. The Laplace transform of V_f with respect to the maturity is

$$\widehat{V}_f(x,\lambda) = \int_0^\infty e^{-\lambda T} V_f(x,T)dT\,.$$

This can be written as

$$\widehat{V}_f(x,\lambda) = \int_0^\infty dT e^{-\lambda T} e^{-rT} \int_0^\infty dy f(y) p_T(x,y) = \frac{1}{\lambda} \mathbb{E}_{\mathbb{Q}}(e^{-r\mathbf{e}} f(S_{\mathbf{e}}))$$

where $\mathbf{e}$ is an exponential random variable with parameter λ which is independent of $(S_t, t \geq 0)$; this is the so-called exponential weighing, or Canadization, an expression due to Carr [146], who uses this tool to price options. In terms of an Arrow-Debreu security, we obtain that

$$\widehat{V}_f(x,\lambda) = \int_0^\infty f(y) \widehat{A}(y,\lambda)dy\,.$$

Here, $\widehat{A}$ is the Laplace transform of the price of an Arrow-Debreu security,

$$\widehat{A}(y,\lambda) = \int_0^\infty e^{-\lambda t} e^{-rt} p_t(x,y)dt = R_{\lambda+r}(x,y)\,.$$

We have seen in (5.3.11) that the resolvent is given in terms of the fundamental solutions of the ODE (5.3.7), hence

$$\widehat{V}_f(x,\lambda) = \\ w_\nu^{-1} \left(\Phi_{\nu\downarrow}(x) \int_0^x m(y) f(y) \Phi_{\nu\uparrow}(y)dy + \Phi_{\nu\uparrow}(x) \int_x^\infty m(y) f(y) \Phi_{\nu\downarrow}(y)dy \right)$$

where $\nu = r + \lambda$.

5.5 Local Times for a Diffusion

5.5.1 Various Definitions of Local Times

We assume that $(X_t, t \geq 0)$ is a regular diffusion on $\mathbb{R}$, with a C^1 scale function s and speed measure $\mathbf{m}$. As discussed in Itô and McKean [465], Borodin and Salminen [109] and [RY], there exists a jointly continuous family of local times $\ell_t^x(X)$, sometimes called **Itô-McKean local times** or **diffusion local times**, defined by the following occupation density formula

$$\int_0^t du\, f(X_u) = \int_{\mathbb{R}} \mathbf{m}(dx)\, f(x)\ell_t^x(X) \tag{5.5.1}$$

for all positive Borel functions f.

The process $(Y_t = s(X_t), t \geq 0)$ is a local martingale, and, as such (see formula (4.1.16)), it admits a **Tanaka-Meyer local time** $(L_t^y(Y), t \geq 0)$ at level y, which is characterized by the property that

$$\left((Y_t - y)^+ - \frac{1}{2}L_t^y(Y)\,,\, t \geq 0\right)$$

is a local martingale.

Assuming that $\mathbf{m}(dx) = m(x)dx$, there exists an **occupation local time** λ_t^x which is defined via the occupation time formula

$$\int_0^t f(X_u)du = \int_{\mathbb{R}} dx f(x)\lambda_t^x(X)\,.$$

Lemma 5.5.1.1 *Let X be a diffusion, s a scale function and $Y = s(X)$. For all x and $t \geq 0$, one has*

$$L_t^x(X) = \frac{1}{s'(x)} L_t^{s(x)}(Y),\;\; L_t^{s(x)}(Y) = 2\ell_t^x(X)\,.$$

Hence, $\ell_t^x(X)$ is the Tanaka-Meyer diffusion local time of $s(X)$ at level $s(x)$. Assuming that the density m exists,

$$\lambda_t^x = m(x)\ell_t^x\,.$$

PROOF: Let $L^y(Y)$ be the Tanaka-Meyer local time of $Y = s(X)$.

$$\begin{aligned}\int_{\mathbb{R}} f(y)L_t^y(Y)dy &= \int_0^t f(Y_u)d\langle Y\rangle_u = \int_0^t f(s(X_u))(s'(X_u))^2 d\langle X\rangle_u \\ &= \int_{\mathbb{R}} f(s(x))\,(s'(x))^2 L_t^x(X)dx \\ &= \int_{\mathbb{R}} f(y)\,s'(s^{-1}(y))L_t^{s^{-1}(y)}(X)dy\,.\end{aligned}$$

Hence

$$L_t^y(Y) = s'(s^{-1}(y))L_t^{s^{-1}(y)}(X)$$

so that

$$L_t^{s(x)}(Y) = s'(x)L_t^x(X)\,. \tag{5.5.2}$$

From the definition of $L_t^x(X)$, and recalling that $m(x)\sigma^2(x) = \frac{2}{s'(x)}$ (see equality 5.3.5), one obtains, on the one hand

$$\int_0^t d\langle X\rangle_u f(X_u) = \int_{\mathbb{R}} f(x)L_t^x(X)dx\,.$$

On the other hand,

$$\begin{aligned}\int_0^t d\langle X\rangle_u f(X_u) &= \int_0^t \sigma^2(X_u)f(X_u)du \\ &= \int_{\mathbb{R}} m(x)\sigma^2(x)f(x)\ell_t^x(X)dx = \int_{\mathbb{R}} \frac{2}{s'(x)}f(x)\ell_t^x dx\end{aligned}$$

and it follows that (see formula (5.3.2))

$$L_t^x(X) = \frac{2}{s'(x)}\ell_t^x(X)\,,$$

hence, from (5.5.2), $L_t^{s(x)}(Y) = 2\ell_t^x(X)$. □

We recall that (see equality (5.3.9), there exists a density $p^{(m)}$ such that

$$\mathbb{E}_{x_0}(f(X_u)) = \int m(dx)p_u^{(m)}(x_0,x)f(x)\,.$$

Consequently

$$\mathbb{E}_{x_0}(\ell_t^x(X)) = \int_0^t du\, p_u^{(m)}(x_0,x)\,.$$

5.5.2 Some Diffusions Involving Local Time

Example 5.5.2.1 Skew Brownian Motion. The skew BM with parameter α is a process Y satisfying $Y_t = W_t + \alpha L_t^0(Y)$ where W is a Brownian motion, $L^0(Y)$ is the Tanaka-Meyer local time of the process Y at level 0, and $\alpha \leq 1/2$. Note that this process, which turns out to be a continuous strong Markov process, is not an Itô process. In order to prove the existence of the skew Brownian motion, we look for a function φ of the form $\beta y^+ - \gamma y^-$ for two constants β and γ such that $\varphi(Y_t)$ is a martingale, which solves an SDE. Using Tanaka's formula, we obtain

$$\begin{aligned}\varphi(Y_t) &= \beta\left(\int_0^t \mathbb{1}_{\{Y_s>0\}} dY_s + \frac{1}{2}L_t^0(Y)\right) - \gamma\left(-\int_0^t \mathbb{1}_{\{Y_s\le 0\}} dY_s + \frac{1}{2}L_t^0(Y)\right)\\ &= \beta\left(\int_0^t \mathbb{1}_{\{Y_s>0\}} dW_s + \frac{1}{2}L_t^0(Y)\right)\\ &\qquad\qquad - \gamma\left(-\int_0^t \mathbb{1}_{\{Y_s\le 0\}} dW_s - \alpha L_t^0(Y) + \frac{1}{2}L_t^0(Y)\right)\\ &= \int_0^t \left(\beta\mathbb{1}_{\{Y_s>0\}} + \gamma\mathbb{1}_{\{Y_s\le 0\}}\right) dW_s + \frac{1}{2}\left(\beta - \gamma + 2\alpha\gamma\right) L_t^0(Y)\,.\end{aligned}$$

Hence, for $\beta - \gamma + 2\alpha\gamma = 0$, $\beta > 0$ and $\gamma > 0$, the process $X_t = \varphi(Y_t)$ is a martingale solution of the stochastic differential equation

$$dX_t = (\beta\mathbb{1}_{X_t>0} + \gamma\mathbb{1}_{X_t\le 0})dW_t\,. \tag{5.5.3}$$

This SDE has no strong solution for β and γ strictly positive but has a unique strictly weak solution (see Barlow [47]).

The process Y is such that $|Y|$ is a reflecting Brownian motion. Indeed,

$$dY_t^2 = 2Y_t(dW_t + \alpha dL_t^0(Y)) + dt = 2Y_t dW_t + dt\,.$$

Walsh [833] proved that, conversely, the only continuous diffusions whose absolute values are reflected BM's are the skew BM's. It can be shown that for fixed $t > 0$, $Y_t \overset{\text{law}}{=} \epsilon|W_t|$ where W is a BM independent of the Bernoulli r.v. ϵ, $\mathbb{P}(\epsilon = 1) = p, \mathbb{P}(\epsilon = -1) = 1 - p$ where $p = \frac{1}{2(1-\alpha)}$.

The relation (4.1.13) between $L_t^0(Y)$ and $L_t^{0-}(Y)$ reads

$$L_t^0(Y) - L_t^{0-}(Y) = 2\int_0^t \mathbb{1}_{\{Y_s=0\}} dY_s\,.$$

The integral $\int_0^t \mathbb{1}_{\{Y_s=0\}} dW_s$ is null and $\int_0^t \mathbb{1}_{\{Y_s=0\}} dL_s^0(Y) = L_t^0(Y)$, hence

$$L_t^0(Y) - L_t^{0-}(Y) = 2\alpha L_t^0(Y)$$

that is $L_t^{0-}(Y) = L_t^0(Y)(1-2\alpha)$, which proves the nonexistence of a skew BM for $\alpha > 1/2$.

Comment 5.5.2.2 For several studies of skew Brownian motion, and more generally of processes Y satisfying

$$Y_t = \int_0^t \sigma(Y_s)dB_s + \int \nu(dy)L_t^y(Y)$$

we refer to Barlow [47], Harrison and Shepp [424], Ouknine [687], Le Gall [567], Lejay [575], Stroock and Yor [813] and Weinryb [838].

Example 5.5.2.3 Sticky Brownian Motion. Let $x > 0$. The solution of

$$X_t = x + \int_0^t \mathbb{1}_{\{X_s>0\}} dW_s + \theta \int_0^t \mathbb{1}_{\{X_s=0\}} ds \tag{5.5.4}$$

with $\theta > 0$ is called sticky Brownian motion with parameter θ. From Tanaka's formula,

$$X_t^- = -\theta \int_0^t \mathbb{1}_{\{X_s=0\}} ds + \frac{1}{2} L_t(X) .$$

The process $\theta \int_0^t \mathbb{1}_{\{X_s=0\}} ds$ is increasing, hence, from Skorokhod's lemma, $L_t(X) = 2\theta \int_0^t \mathbb{1}_{\{X_s=0\}} ds$ and $X_t^- = 0$. Hence, we may write the equation (5.5.4) as

$$X_t = x + \int_0^t \mathbb{1}_{\{X_s>0\}} dW_s + \frac{1}{2} L_t(X)$$

which enables us to write

$$X_t = \beta\left(\int_0^t \mathbb{1}_{\{X_s>0\}} ds\right)$$

where $(\beta(u), u \geq 0)$ is a reflecting BM starting from x. See Warren [835] for a thorough study of sticky Brownian motion.

Exercise 5.5.2.4 Let $\theta > 0$ and X be the sticky Brownian motion with $X_0 = 0$.

(1) Prove that $L_t^x(X) = 0$, for every $x < 0$; then, prove that $X_t \geq 0$, a.s.

(2) Let $A_t^+ = \int_0^t ds \, \mathbb{1}_{\{X_s>0\}}, A_t^0 = \int_0^t ds \, \mathbb{1}_{\{X_s=0\}}$, and define their inverses $\alpha_u^+ = \inf\{t : A_t^+ > u\}$ and $\alpha_u^0 = \inf\{t : A_t^0 > u\}$. Identify the law of $(X_{\alpha_u^+}, u \geq 0)$.

(3) Let G be a Gaussian variable, with unit variance and 0 expectation. Prove that, for any $u > 0$ and $t > 0$

$$\alpha_u^+ \stackrel{\text{law}}{=} u + \frac{1}{\theta}\sqrt{u}\,|G| \, ; \; A_t^+ \stackrel{\text{law}}{=} \left(\sqrt{t + \frac{G^2}{4\theta^2}} - \frac{|G|}{2\theta}\right)^2$$

deduce that

$$A_t^0 \stackrel{\text{law}}{=} \frac{|G|}{\theta}\sqrt{t + \frac{G^2}{4\theta^2}} - \frac{G^2}{2\theta^2}$$

and compute $\mathbb{E}(A_t^0)$.

`Hint:` The process $X_{\alpha_u^+} = W_u^+ + \theta A_{\alpha_u^+}^0$ where W_u^+ is a BM and $A_{\alpha_u^+}^0$ is an increasing process, constant on $\{u : X_{\alpha_u^+} > 0\}$, solves Skorokhod equation. Therefore it is a reflected BM. The obvious equality $t = A_t^+ + A_t^0$ leads to $\alpha_u^+ = u + A_{\alpha_u^+}^0$, and $a A_{\alpha_u^+}^0 \stackrel{\text{law}}{=} L_u^0$. ◁

5.6 Last Passage Times

We now present the study of the law (and the conditional law) of some last passage times for diffusion processes. In this section, W is a standard Brownian motion and its natural filtration is $\mathbf{F}$. These random times have been studied in Jeanblanc and Rutkowski [486] as theoretical examples of default times, in Imkeller [457] as examples of insider private information and, in a pure mathematical point of view, in Pitman and Yor [715] and Salminen [754].

5.6.1 Notation and Basic Results

If τ is a random time, then, it is easy to check that the process $\mathbb{P}(\tau > t|\mathcal{F}_t)$ is a super-martingale. Therefore, it admits a Doob-Meyer decomposition.

Lemma 5.6.1.1 *Let τ be a positive random time and*

$$\mathbb{P}(\tau > t|\mathcal{F}_t) = M_t - A_t$$

the Doob-Meyer decomposition of the super-martingale $Z_t = \mathbb{P}(\tau > t|\mathcal{F}_t)$. Then, for any predictable positive process H,

$$\mathbb{E}(H_\tau) = \mathbb{E}\left(\int_0^\infty dA_u H_u\right).$$

PROOF: For any process H of the form $H = \Lambda_s \mathbb{1}_{]s,t]}$ with $\Lambda_s \in b\mathcal{F}_s$, one has

$$\mathbb{E}(H_\tau) = \mathbb{E}(\Lambda_s \mathbb{1}_{]s,t]}(\tau)) = \mathbb{E}(\Lambda_s(A_t - A_s)).$$

The result follows from MCT. □

Comment 5.6.1.2 The reader will find in Nikeghbali and Yor [676] a multiplicative decomposition of the super-martingale Z as $Z_t = n_t D_t$ where D is a decreasing process and n a local martingale, and applications to enlargement of filtration.

We now show that, in a diffusion setup, A_t and M_t may be computed explicitly for some random times τ.

5.6.2 Last Passage Time of a Transient Diffusion

Proposition 5.6.2.1 *Let X be a transient homogeneous diffusion such that $X_t \to +\infty$ when $t \to \infty$, and s a scale function such that $s(+\infty) = 0$ (hence, $s(x) < 0$ for $x \in \mathbb{R}$) and $\Lambda_y = \sup\{t : X_t = y\}$ the last time that X hits y. Then,*

$$\mathbb{P}_x(\Lambda_y > t|\mathcal{F}_t) = \frac{s(X_t)}{s(y)} \wedge 1.$$

PROOF: We follow Pitman and Yor [715] and Yor [868], p.48, and use that under the hypotheses of the proposition, one can choose a scale function such that $s(x) < 0$ and $s(+\infty) = 0$ (see Sharpe [784]).

Observe that

$$\mathbb{P}_x\big(\Lambda_y > t|\mathcal{F}_t\big) = \mathbb{P}_x\Big(\inf_{u\geq t} X_u < y \,\Big|\, \mathcal{F}_t\Big) = \mathbb{P}_x\Big(\sup_{u\geq t}(-s(X_u)) > -s(y) \,\Big|\, \mathcal{F}_t\Big)$$
$$= \mathbb{P}_{X_t}\Big(\sup_{u\geq 0}(-s(X_u)) > -s(y)\Big) = \frac{s(X_t)}{s(y)} \wedge 1,$$

where we have used the Markov property of X, and the fact that if M is a continuous local martingale with $M_0 = 1$, $M_t \geq 0$, and $\lim_{t\to\infty} M_t = 0$, then

$$\sup_{t\geq 0} M_t \overset{\text{law}}{=} \frac{1}{U},$$

where U has a uniform law on $[0,1]$ (see Exercise 1.2.3.10). □

Lemma 5.6.2.2 *The* $\mathbf{F}^X$*-predictable compensator* A *associated with the random time* Λ_y *is the process* A *defined as* $A_t = -\frac{1}{2s(y)} L_t^{s(y)}(Y)$, *where* $L(Y)$ *is the local time process of the continuous martingale* $Y = s(X)$.

PROOF: From $x \wedge y = x - (x-y)^+$, Proposition 5.6.2.1 and Tanaka's formula, it follows that

$$\frac{s(X_t)}{s(y)} \wedge 1 = M_t + \frac{1}{2s(y)} L_t^{s(y)}(Y) = M_t + \frac{1}{s(y)} \ell_t^y(X)$$

where M is a martingale. The required result is then easily obtained. □

We deduce the law of the last passage time:

$$\mathbb{P}_x(\lambda_y > t) = \left(\frac{s(x)}{s(y)} \wedge 1\right) + \frac{1}{s(y)} \mathbb{E}_x(\ell_t^y(X))$$
$$= \left(\frac{s(x)}{s(y)} \wedge 1\right) + \frac{1}{s(y)} \int_0^t du\, p_u^{(m)}(x,y)\,.$$

Hence, for $x < y$

$$\mathbb{P}_x(\Lambda_y \in dt) = -\frac{dt}{s(y)} p_t^{(m)}(x,y) = -\frac{dt}{s(y)m(y)} p_t(x,y)$$
$$= -\frac{\sigma^2(y)s'(y)}{2s(y)} p_t(x,y) dt\,. \tag{5.6.1}$$

For $x > y$, we have to add a mass at point 0 equal to

$$1 - \left(\frac{s(x)}{s(y)} \wedge 1\right) = 1 - \frac{s(x)}{s(y)} = \mathbb{P}_x(T_y < \infty)\,.$$

Example 5.6.2.3 Last Passage Time for a Transient Bessel Process: For a Bessel process of dimension $\delta > 2$ and index ν (see $\rightarrowtail$ Chapter 6), starting from 0,

$$\mathbb{P}_0^\delta(\Lambda_a < t) = \mathbb{P}_0^\delta(\inf_{u\geq t} R_u > a) = \mathbb{P}_0^\delta(\sup_{u\geq t} R_u^{-2\nu} < a^{-2\nu})$$
$$= \mathbb{P}_0^\delta\left(\frac{R_t^{-2\nu}}{U} < a^{-2\nu}\right) = \mathbb{P}_0^\delta(a^{2\nu} < UR_t^{2\nu}) = \mathbb{P}_0^\delta\left(\frac{a^2}{R_1^2 U^{1/\nu}} < t\right).$$

Thus, the r.v. $\Lambda_a = \frac{a^2}{R_1^2 U^{1/\nu}}$ is distributed as $\frac{a^2}{2\gamma(\nu+1)\beta_{\nu,1}} \stackrel{\text{law}}{=} \frac{a^2}{2\gamma(\nu)}$ where $\gamma(\nu)$ is a gamma variable with parameter ν:

$$\mathbb{P}(\gamma(\nu) \in dt) = \mathbb{1}_{\{t\geq 0\}} \frac{t^{\nu-1}e^{-t}}{\Gamma(\nu)} dt.$$

Hence,

$$\mathbb{P}_0^\delta(\Lambda_a \in dt) = \mathbb{1}_{\{t\geq 0\}} \frac{1}{t\Gamma(\nu)} \left(\frac{a^2}{2t}\right)^\nu e^{-a^2/(2t)} dt. \tag{5.6.2}$$

We might also find this result directly from the general formula (5.6.1) and apply formula (6.2.3) for the expression of the density.

Proposition 5.6.2.4 *For H a positive predictable process*

$$\mathbb{E}_x(H_{\Lambda_y} | \Lambda_y = t) = \mathbb{E}_x(H_t | X_t = y)$$

and, for $y > x$,

$$\mathbb{E}_x(H_{\Lambda_y}) = \int_0^\infty \mathbb{E}_x(\Lambda_y \in dt)\, \mathbb{E}_x(H_t | X_t = y).$$

In the case $x > y$,

$$\mathbb{E}_x(H_{\Lambda_y}) = H_0 \left(1 - \frac{s(x)}{s(y)}\right) + \int_0^\infty \mathbb{E}_x(\Lambda_y \in dt)\, \mathbb{E}_x(H_t | X_t = y).$$

PROOF: We have shown in the previous Proposition 5.6.2.1 that

$$\mathbb{P}_x(\Lambda_y > t | \mathcal{F}_t) = \frac{s(X_t)}{s(y)} \wedge 1.$$

From Itô-Tanaka's formula

$$\frac{s(X_t)}{s(y)} \wedge 1 = \frac{s(x)}{s(y)} \wedge 1 + \int_0^t \mathbb{1}_{\{X_u > y\}}\, d\frac{s(X_u)}{s(y)} - \frac{1}{2} L_t^{s(y)}(s(X)).$$

It follows, using Lemma 5.6.1.1 that

$$\begin{aligned}\mathbb{E}_x(H_{\Lambda_x}) &= \frac{1}{2}\mathbb{E}_x\left(\int_0^\infty H_u\, d_u L_u^{s(y)}(s(X))\right)\\ &= \frac{1}{2}\mathbb{E}_x\left(\int_0^\infty \mathbb{E}_x(H_u|X_u=y)\, d_u L_u^{s(y)}(s(X))\right).\end{aligned}$$

Therefore, replacing H_u by $H_u g(u)$, we get

$$\mathbb{E}_x\left(H_{\Lambda_x} g(\Lambda_x)\right) = \frac{1}{2}\mathbb{E}_x\left(\int_0^\infty g(u)\,\mathbb{E}_x\left(H_u|X_u=y\right)\, d_u L_u^{s(y)}(s(X))\right). \quad (5.6.3)$$

Consequently, from (5.6.3), we obtain

$$\begin{aligned}\mathbb{P}_x\left(\Lambda_y \in du\right) &= \frac{1}{2} d_u \mathbb{E}_x\left(L_u^{s(y)}(s(X))\right)\\ \mathbb{E}_x\left(H_{\Lambda_y}|\Lambda_y=t\right) &= \mathbb{E}_x(H_t|X_t=y).\end{aligned}$$

□

Remark 5.6.2.5 In the literature, some studies of last passage times employ time inversion. See an example in the next Exercise 5.6.2.6.

Exercise 5.6.2.6 Let X be a drifted Brownian motion with positive drift ν and Λ_y^ν its last passage time at level y. Prove that

$$\mathbb{P}_x(\Lambda_y^{(\nu)} \in dt) = \frac{\nu}{\sqrt{2\pi t}} \exp\left(-\frac{1}{2t}(x-y+\nu t)^2\right) dt,$$

and

$$\mathbb{P}_x(\Lambda_y^{(\nu)} = 0) = \begin{cases} 1 - e^{-2\nu(x-y)}, & \text{for } x > y \\ 0 & \text{for } x < y. \end{cases}$$

Prove, using time inversion that, for $x = 0$,

$$\Lambda_y^{(\nu)} \overset{\text{law}}{=} \frac{1}{T_\nu^{(y)}}$$

where

$$T_a^{(b)} = \inf\{t : B_t + bt = a\}$$

See Madan et al. [611]. ◁

5.6.3 Last Passage Time Before Hitting a Level

Let $X_t = x + \sigma W_t$ where the initial value x is positive and σ is a positive constant. We consider, for $0 < a < x$ the last passage time at the level a before hitting the level 0, given as $g_{T_0}^a(X) = \sup\{t \le T_0 : X_t = a\}$, where

$$T_0 = T_0(X) = \inf\{t \ge 0 : X_t = 0\}.$$

(In a financial setting, T_0 can be interpreted as the time of bankruptcy.) Then, setting $\alpha = (a-x)/\sigma$, $T_{-x/\sigma}(W) = \inf\{t : W_t = -x/\sigma\}$ and $d_t^\alpha(W) = \inf\{s \geq t : W_s = \alpha\}$

$$\mathbb{P}_x\big(g^a_{T_0}(X) \leq t|\mathcal{F}_t\big) = \mathbb{P}\big(d_t^\alpha(W) > T_{-x/\sigma}(W)|\mathcal{F}_t\big)$$

on the set $\{t < T_{-x/\sigma}(W)\}$. It is easy to prove that

$$\mathbb{P}\big(d_t^\alpha(W) < T_{-x/\sigma}(W)|\mathcal{F}_t\big) = \Psi(W_{t\wedge T_{-x/\sigma}(W)}, \alpha, -x/\sigma),$$

where the function $\Psi(\cdot, a, b) : \mathbb{R} \to \mathbb{R}$ equals, for $a > b$,

$$\Psi(y, a, b) = \mathbb{P}_y(T_a(W) > T_b(W)) = \begin{cases} (a-y)/(a-b) & \text{for } b < y < a, \\ 1 & \text{for } a < y, \\ 0 & \text{for } y < b. \end{cases}$$

(See Proposition 3.5.1.1 for the computation of Ψ.) Consequently, on the set $\{T_0(X) > t\}$ we have

$$\mathbb{P}_x\big(g^a_{T_0}(X) \leq t|\mathcal{F}_t\big) = \frac{(\alpha - W_{t\wedge T_0})^+}{a/\sigma} = \frac{(\alpha - W_t)^+}{a/\sigma} = \frac{(a - X_t)^+}{a}. \qquad (5.6.4)$$

As a consequence, applying Tanaka's formula, we obtain the following result.

Lemma 5.6.3.1 *Let $X_t = x + \sigma W_t$, where $\sigma > 0$. The $\mathbf{F}$-predictable compensator associated with the random time $g^a_{T_0(X)}$ is the process A defined as $A_t = \frac{1}{2\alpha} L^\alpha_{t\wedge T_{-x/\sigma}(W)}(W)$, where $L^\alpha(W)$ is the local time of the Brownian Motion W at level $\alpha = (a-x)/\sigma$.*

5.6.4 Last Passage Time Before Maturity

In this subsection, we study the last passage time at level a of a diffusion process X before the fixed horizon (maturity) T. We start with the case where $X = W$ is a Brownian motion starting from 0 and where the level a is null:

$$g_T = \sup\{t \leq T : W_t = 0\}.$$

Lemma 5.6.4.1 *The $\mathbf{F}$-predictable compensator associated with the random time g_T equals*

$$A_t = \sqrt{\frac{2}{\pi}} \int_0^{t\wedge T} \frac{dL_s}{\sqrt{T-s}},$$

where L is the local time at level 0 of the Brownian motion W.

PROOF: It suffices to give the proof for $T = 1$, and we work with $t < 1$. Let G be a standard Gaussian variable. Then

$$\mathbb{P}\Big(\frac{a^2}{G^2} > 1 - t\Big) = \Phi\Big(\frac{|a|}{\sqrt{1-t}}\Big),$$

where $\Phi(x) = \sqrt{\frac{2}{\pi}} \int_0^x \exp(-\frac{u^2}{2})du$. For $t < 1$, the set $\{g_1 \leq t\}$ is equal to $\{d_t > 1\}$. It follows from (4.3.3) that

$$\mathbb{P}(g_1 \leq t | \mathcal{F}_t) = \Phi\left(\frac{|W_t|}{\sqrt{1-t}}\right) .$$

Then, the Itô-Tanaka formula combined with the identity

$$x\Phi'(x) + \Phi''(x) = 0$$

leads to

$$\begin{aligned}\mathbb{P}(g_1 \leq t | \mathcal{F}_t) &= \int_0^t \Phi'\left(\frac{|W_s|}{\sqrt{1-s}}\right) d\left(\frac{|W_s|}{\sqrt{1-s}}\right) + \frac{1}{2}\int_0^t \frac{ds}{1-s}\, \Phi''\left(\frac{|W_s|}{\sqrt{1-s}}\right) \\ &= \int_0^t \Phi'\left(\frac{|W_s|}{\sqrt{1-s}}\right) \frac{\text{sgn}(W_s)}{\sqrt{1-s}}\, dW_s + \int_0^t \frac{dL_s}{\sqrt{1-s}}\, \Phi'\left(\frac{|W_s|}{\sqrt{1-s}}\right) \\ &= \int_0^t \Phi'\left(\frac{|W_s|}{\sqrt{1-s}}\right) \frac{\text{sgn}(W_s)}{\sqrt{1-s}}\, dW_s + \sqrt{\frac{2}{\pi}} \int_0^t \frac{dL_s}{\sqrt{1-s}} .\end{aligned}$$

It follows that the **F**-predictable compensator associated with g_1 is

$$A_t = \sqrt{\frac{2}{\pi}} \int_0^t \frac{dL_s}{\sqrt{1-s}},\ (t < 1) .$$

□

These results can be extended to the last time before T when the Brownian motion reaches the level α, i.e., $g_T^\alpha = \sup\{t \leq T : W_t = \alpha\}$, where we set $\sup(\emptyset) = T$. The predictable compensator associated with g_T^α is

$$A_t = \sqrt{\frac{2}{\pi}} \int_0^{t\wedge T} \frac{dL_s^\alpha}{\sqrt{T-s}},$$

where L^α is the local time of W at level α.

We now study the case where $X_t = x + \mu\, t + \sigma\, W_t$, with constant coefficients μ and $\sigma > 0$. Let

$$\begin{aligned}g_1^a(X) &= \sup\{t \leq 1 : X_t = a\} \\ &= \sup\{t \leq 1 : \nu t + W_t = \alpha\}\end{aligned}$$

where $\nu = \mu/\sigma$ and $\alpha = (a-x)/\sigma$. From Lemma 4.3.9.1, setting

$$V_t = \alpha - \nu t - W_t = (a - X_t)/\sigma ,$$

we obtain

$$\mathbb{P}(g_1^a(X) \leq t | \mathcal{F}_t) = (1 - e^{\nu V_t} H(\nu, |V_t|, 1-t)) \mathbb{1}_{\{T_0(V) \leq t\}},$$

where

$$H(\nu, y, s) = e^{-\nu y} \mathcal{N}\left(\frac{\nu s - y}{\sqrt{s}}\right) + e^{\nu y} \mathcal{N}\left(\frac{-\nu s - y}{\sqrt{s}}\right).$$

Using Itô's lemma, we obtain the decomposition of $1 - e^{\nu V_t} H(\nu, |V_t|, 1-t)$ as a semi-martingale $M_t + C_t$.

We note that C increases only on the set $\{t : X_t = a\}$. Indeed, setting $g_1^a(X) = g$, for any predictable process H, one has

$$\mathbb{E}(H_g) = \mathbb{E}\left(\int_0^\infty dC_s H_s\right)$$

hence, since $X_g = a$,

$$0 = \mathbb{E}(\mathbb{1}_{X_g \neq a}) = \mathbb{E}\left(\int_0^\infty dC_s \mathbb{1}_{X_s \neq a}\right).$$

Therefore, $dC_t = \kappa_t dL_t^a(X)$ and, since L increases only at points such that $X_t = a$ (i.e., $V_t = 0$), one has

$$\kappa_t = H_x'(\nu, 0, 1-t).$$

The martingale part is given by $dM_t = m_t dW_t$ where

$$m_t = e^{\nu V_t} \left(\nu H(\nu, |V_t|, 1-t) - \mathrm{sgn}(V_t)\, H_x'(\nu, |V_t|, 1-t)\right).$$

Therefore, the predictable compensator associated with $g_1^a(X)$ is

$$\int_0^t \frac{H_x'(\nu, 0, 1-s)}{e^{\nu V_s} H(\nu, 0, 1-s)} dL_s^a.$$

Exercise 5.6.4.2 The aim of this exercise is to compute, for $t < T < 1$, the quantity $\mathbb{E}(h(W_T)\mathbb{1}_{\{T<g_1\}}|\mathcal{G}_t)$, which is the price of the claim $h(S_T)$ with barrier condition $\mathbb{1}_{\{T<g_1\}}$.

Prove that

$$\mathbb{E}(h(W_T)\mathbb{1}_{\{T<g_1\}}|\mathcal{F}_t) = \mathbb{E}(h(W_T)|\mathcal{F}_t) - \mathbb{E}\Big(h(W_T)\Phi\Big(\frac{|W_T|}{\sqrt{1-T}}\Big)\,\Big|\,\mathcal{F}_t\Big),$$

where

$$\Phi(x) = \sqrt{\frac{2}{\pi}} \int_0^x \exp\left(-\frac{u^2}{2}\right) du.$$

Define $k(w) = h(w)\Phi(|w|/\sqrt{1-T})$. Prove that $\mathbb{E}\big(k(W_T)\,\big|\,\mathcal{F}_t\big) = \widetilde{k}(t, W_t)$, where

$$\begin{aligned}\widetilde{k}(t,a) &= \mathbb{E}\Big(k(W_{T-t}+a)\Big)\\ &= \frac{1}{\sqrt{2\pi(T-t)}} \int_{\mathbb{R}} h(u)\Phi\Big(\frac{|u|}{\sqrt{1-T}}\Big) \exp\Big(-\frac{(u-a)^2}{2(T-t)}\Big) du.\end{aligned}$$

◁

5.6.5 Absolutely Continuous Compensator

From the preceding computations, the reader might think that the **F**-predictable compensator is always singular w.r.t. the Lebesgue measure. This is not the case, as we show now. We are indebted to Michel Émery for this example.

Let W be a Brownian motion and let $\tau = \sup\{t \le 1 : W_1 - 2W_t = 0\}$, that is the last time before 1 when the Brownian motion is equal to half of its terminal value at time 1. Then,

$$\{\tau \le t\} = \left\{\inf_{t\le s\le 1} 2W_s \ge W_1 \ge 0\right\} \cup \left\{\sup_{t\le s\le 1} 2W_s \le W_1 \le 0\right\}.$$

▶ The quantity

$$\mathbb{P}(\tau \le t, W_1 \ge 0|\mathcal{F}_t) = \mathbb{P}\left(\inf_{t\le s\le 1} 2W_s \ge W_1 \ge 0|\mathcal{F}_t\right)$$

can be evaluated using the equalities

$$\begin{aligned}\left\{\inf_{t\le s\le 1} W_s \ge \frac{W_1}{2} \ge 0\right\} &= \left\{\inf_{t\le s\le 1}(W_s - W_t) \ge \frac{W_1}{2} - W_t \ge -W_t\right\}\\ &= \left\{\inf_{0\le u\le 1-t}(\widetilde{W}_u) \ge \frac{\widetilde{W}_{1-t}}{2} - \frac{W_t}{2} \ge -W_t\right\},\end{aligned}$$

where $(\widetilde{W}_u = W_{t+u} - W_t, u \ge 0)$ is a Brownian motion independent of $\mathcal{F}_t$. It follows that

$$\mathbb{P}\left(\inf_{t\le s\le 1} W_s \ge \frac{W_1}{2} \ge 0|\mathcal{F}_t\right) = \Psi(1-t, W_t)\,,$$

where

$$\begin{aligned}\Psi(s,x) &= \mathbb{P}\left(\inf_{0\le u\le s} \widetilde{W}_u \ge \frac{\widetilde{W}_s}{2} - \frac{x}{2} \ge -x\right) = \mathbb{P}\left(2M_s - W_s \le \frac{x}{2},\, W_s \le \frac{x}{2}\right)\\ &= \mathbb{P}\left(2M_1 - W_1 \le \frac{x}{2\sqrt{s}},\, W_1 \le \frac{x}{2\sqrt{s}}\right).\end{aligned}$$

▶ The same kind of computation leads to

$$\mathbb{P}\left(\sup_{t\le s\le 1} 2W_s \le W_1 \le 0|\mathcal{F}_t\right) = \Psi(1-t, -W_t)\,.$$

▶ The quantity $\Psi(s,x)$ can now be computed from the joint law of the maximum and of the process at time 1; however, we prefer to use Pitman's theorem (see ↣ Section 5.7): let $\widetilde{U}$ be a r.v. uniformly distributed on $[-1,+1]$ independent of $R_1 := 2M_1 - W_1$, then

$$\mathbb{P}(2M_1 - W_1 \leq y, W_1 \leq y) = \mathbb{P}(R_1 \leq y, \widetilde{U} R_1 \leq y)$$
$$= \frac{1}{2}\int_{-1}^{1} \mathbb{P}(R_1 \leq y, uR_1 \leq y) du\,.$$

For $y > 0$,

$$\frac{1}{2}\int_{-1}^{1} \mathbb{P}(R_1 \leq y, uR_1 \leq y) du = \frac{1}{2}\int_{-1}^{1} \mathbb{P}(R_1 \leq y) du = \mathbb{P}(R_1 \leq y)\,.$$

For $y < 0$

$$\int_{-1}^{1} \mathbb{P}(R_1 \leq y, uR_1 \leq y) du = 0\,.$$

Therefore

$$\mathbb{P}(\tau \leq t | \mathcal{F}_t) = \Psi(1-t, W_t) + \Psi(1-t, -W_t) = \rho\left(\frac{|W_t|}{\sqrt{1-t}}\right)$$

where

$$\rho(y) = \mathbb{P}(R_1 \leq y) = \sqrt{\frac{2}{\pi}} \int_0^y x^2 e^{-x^2/2} dx\,.$$

Then $Z_t = \mathbb{P}(\tau > t | \mathcal{F}_t) = 1 - \rho(\frac{|W_t|}{\sqrt{1-t}})$. We can now apply Tanaka's formula to the function ρ. Noting that $\rho'(0) = 0$, the contribution to the Doob-Meyer decomposition of Z of the local time of W at level 0 is 0. Furthermore, the increasing process A of the Doob-Meyer decomposition of Z is given by

$$dA_t = \left(\frac{1}{2}\rho''\left(\frac{|W_t|}{\sqrt{1-t}}\right)\frac{1}{1-t} + \frac{1}{2}\rho'\left(\frac{|W_t|}{\sqrt{1-t}}\right)\frac{|W_t|}{\sqrt{(1-t)^3}}\right) dt$$
$$= \frac{1}{1-t}\,\frac{|W_t|}{\sqrt{1-t}}\,e^{-W_t^2/2(1-t)} dt\,.$$

We note that A may be obtained as the dual predictable projection on the Brownian filtration of the process $A_s^{(W_1)}, s \leq 1$, where $(A_s^{(x)}, s \leq 1)$ is the compensator of τ under the law of the Brownian bridge $\mathbb{P}_{0\to x}^{(1)}$.

5.6.6 Time When the Supremum is Reached

Let W be a Brownian motion, $M_t = \sup_{s \leq t} W_s$ and let τ be the time when the supremum on the interval $[0, 1]$ is reached, i.e.,

$$\tau = \inf\{t \leq 1 : W_t = M_1\} = \sup\{t \leq 1 : M_t - W_t = 0\}\,.$$

Let us denote by ζ the positive continuous semimartingale

$$\zeta_t = \frac{M_t - W_t}{\sqrt{1-t}}, \; t < 1.$$

Let $F_t = \mathbb{P}(\tau \leq t|\mathcal{F}_t)$. Since $F_t = \Phi(\zeta_t)$, (where $\Phi(x) = \sqrt{\frac{2}{\pi}} \int_0^x \exp(-\frac{u^2}{2})du$, see Example 4.1.7.5) using Itô's formula, we obtain the canonical decomposition of F as follows:

$$\begin{aligned} F_t &= \int_0^t \Phi'(\zeta_u)\, d\zeta_u + \frac{1}{2}\int_0^t \Phi''(\zeta_u)\, \frac{du}{1-u} \\ &\overset{(i)}{=} -\int_0^t \Phi'(\zeta_u) \frac{dW_u}{\sqrt{1-u}} + \sqrt{\frac{2}{\pi}} \int_0^t \frac{dM_u}{\sqrt{1-u}} \overset{(ii)}{=} U_t + \tilde{F}_t, \end{aligned}$$

where $U_t = -\int_0^t \Phi'(\zeta_u) \dfrac{dW_u}{\sqrt{1-u}}$ is a martingale and $\widetilde{F}$ a predictable increasing process. To obtain (i), we have used that $x\Phi' + \Phi'' = 0$; to obtain (ii), we have used that $\Phi'(0) = \sqrt{2/\pi}$ and also that the process M increases only on the set

$$\{u \in [0,t] : M_u = W_u\} = \{u \in [0,t] : \zeta_u = 0\}.$$

5.6.7 Last Passage Times for Particular Martingales

Proposition 5.6.7.1 *Let X be a continuous positive local martingale such that $X_0 = x$, and $\lim_{t\to\infty} X_t = 0$. Let $\Sigma_t = \sup_{s\leq t} X_s$ the (continuous) supremum process. We consider the last passage time of the process X at the level Σ_∞:*

$$\begin{aligned} g &= \sup\{t \geq 0 : \quad X_t = \Sigma_\infty\} \\ &= \sup\{t \geq 0 : \quad \Sigma_t - X_t = 0\}\,. \end{aligned} \tag{5.6.5}$$

Consider the supermartingale

$$Z_t = \mathbb{P}\left(g > t \mid \mathcal{F}_t\right).$$

Then:

(i) *the multiplicative decomposition of the supermartingale Z reads*

$$Z_t = \frac{X_t}{\Sigma_t},$$

(ii) *The Doob-Meyer (additive decomposition) of Z is:*

$$Z_t = m_t - \log\left(\Sigma_t\right), \tag{5.6.6}$$

where m is the $\mathbf{F}$-martingale

$$m_t = \mathbb{E}\left[\log \Sigma_\infty | \mathcal{F}_t\right].$$

PROOF: We recall the Doob's maximal identity 1.2.3.10. Applying (1.2.2) to the martingale $(Y_t := X_{T+t}, t \geq 0)$ for the filtration $\mathbf{F}^T := (\mathcal{F}_{t+T}, t \geq 0)$, where T is a $\mathbf{F}$-stopping time, we obtain that

$$\mathbb{P}\left(\Sigma^T > a|\mathcal{F}_T\right) = \left(\frac{X_T}{a}\right) \wedge 1, \tag{5.6.7}$$

where

$$\Sigma^T := \sup_{u \geq T} X_u.$$

Hence $\frac{X_T}{\Sigma^T}$ is a uniform random variable on $(0,1)$, independent of $\mathcal{F}_T$. The multiplicative decomposition of Z follows from

$$\mathbb{P}\left(g > t \mid \mathcal{F}_t\right) = \mathbb{P}\left(\sup_{u \geq t} X_u \geq \Sigma_t \mid \mathcal{F}_t\right) = \left(\frac{X_t}{\Sigma_t}\right) \wedge 1 = \frac{X_t}{\Sigma_t}$$

From the integration by parts formula applied to $\frac{X_t}{\Sigma_t}$, and using the fact that X, hence Σ are continuous, we obtain

$$dZ_t = \frac{dX_t}{\Sigma_t} - X_t \frac{d\Sigma_t}{(\Sigma_t)^2}$$

Since $d\Sigma_t$ charges only the set $\{t \,:\, X_t = \Sigma_t\}$, one has

$$dZ_t = \frac{dX_t}{\Sigma_t} - \frac{d\Sigma_t}{\Sigma_t} = \frac{dX_t}{\Sigma_t} - d(\ln \Sigma_t)$$

From the uniqueness of the Doob-Meyer decomposition, we obtain that the predictable increasing part of the submartingale Z is $\ln \Sigma_t$, hence

$$Z_t = m_t - \ln \Sigma_t$$

where m is a martingale. The process Z is of class (D), hence m is a uniformly integrable martingale. From $Z_\infty = 0$, one obtains that $m_t = \mathbb{E}(\ln \Sigma_\infty | \mathcal{F}_t)$. □

Remark 5.6.7.2 From the Doob-Meyer (additive) decomposition of Z, we have $1 - Z_t = (1 - m_t) + \ln \Sigma_t$. From Skorokhod's reflection lemma presented in Subsection 4.1.7 we deduce that

$$\ln \Sigma_t = \sup_{s \leq t} m_s - 1$$

We now study the Azéma supermartingale associated with the random time L, a last passage time or the end of a predictable set Γ, i.e.,

$$L(\omega) = \sup\{t \,:\, (t,\omega) \in \Gamma\}$$

(See ↣ Section 5.9.4 for properties of these times in an enlargement of filtration setting).

Proposition 5.6.7.3 *Let L be the end of a predictable set. Assume that all the $\mathbf{F}$-martingales are continuous and that L avoids the $\mathbf{F}$-stopping times. Then, there exists a continuous and nonnegative local martingale N, with $N_0 = 1$ and $\lim_{t\to\infty} N_t = 0$, such that:*

$$Z_t = \mathbb{P}(L > t \mid \mathcal{F}_t) = \frac{N_t}{\Sigma_t}$$

where $\Sigma_t = \sup_{s\le t} N_s$. The Doob-Meyer decomposition of Z is

$$Z_t = m_t - A_t$$

and the following relations hold

$$\begin{aligned} N_t &= \exp\left(\int_0^t \frac{dm_s}{Z_s} - \frac{1}{2}\int_0^t \frac{d\langle m\rangle_s}{Z_s^2}\right) \\ \Sigma_t &= \exp(A_t) \\ m_t &= 1 + \int_0^t \frac{dN_s}{\Sigma_s} = \mathbb{E}(\ln S_\infty | \mathcal{F}_t) \end{aligned}$$

PROOF: As recalled previously, the Doob-Meyer decomposition of Z reads $Z_t = m_t - A_t$ with m and A continuous, and dA_t is carried by $\{t : Z_t = 1\}$. Then, for $t < T_0 := \inf\{t : Z_t = 0\}$

$$-\ln Z_t = -\left(\int_0^t \frac{dm_s}{Z_s} - \frac{1}{2}\int_0^t \frac{d\langle m\rangle_s}{Z_s^2}\right) + A_t$$

From Skorokhod's reflection lemma (see Subsection 4.1.7) we deduce that

$$A_t = \sup_{u\le t}\left(\int_0^u \frac{dm_s}{Z_s} - \frac{1}{2}\int_0^u \frac{d\langle m\rangle_s}{Z_s^2}\right)$$

Introducing the local martingale N defined by

$$N_t = \exp\left(\int_0^t \frac{dm_s}{Z_s} - \frac{1}{2}\int_0^t \frac{d\langle m\rangle_s}{Z_s^2}\right),$$

it follows that

$$Z_t = \frac{N_t}{\Sigma_t}$$

and

$$\Sigma_t = \sup_{u\le t} N_u = \exp\left(\sup_{u\le t}\left(\int_0^u \frac{dm_s}{Z_s} - \frac{1}{2}\int_0^u \frac{d\langle m\rangle_s}{Z_s^2}\right)\right) = e^{A_t}$$

□

The three following exercises are from the work of Bentata and Yor [72].

Exercise 5.6.7.4 Let M be a positive martingale, such that $M_0 = 1$ and $\lim_{t\to\infty} M_t = 0$. Let $a \in [0,1[$ and define $G_a = \sup\{t : M_t = a\}$. Prove that

$$\mathbb{P}(G_a \le t|\mathcal{F}_t) = \left(1 - \frac{M_t}{a}\right)^+$$

Assume that, for every $t > 0$, the law of the r.v. M_t admits a density $(m_t(x), x \ge 0)$, and $(t,x) \to m_t(x)$ may be chosen continuous on $(0,\infty)^2$ and that $d\langle M\rangle_t = \sigma_t^2 dt$, and there exists a jointly continuous function $(t,x) \to \theta_t(x) = \mathbb{E}(\sigma_t^2 | M_t = x)$ on $(0,\infty)^2$. Prove that

$$\mathbb{P}(G_a \in dt) = \left(1 - \frac{M_0}{a}\right)\delta_0(dt) + \mathbb{1}_{\{t>0\}}\frac{1}{2a}\theta_t(a)m_t(a)dt$$

`Hint:` Use Tanaka's formula to prove that the result is equivalent to $d_t\mathbb{E}(L_t^a(M)) = dt\theta_t(a)m_t(a)$ where L is the Tanaka-Meyer local time (see Subsection 5.5.1). ◁

Exercise 5.6.7.5 Let B be a Brownian motion and

$$T_a^{(\nu)} = \inf\{t : B_t + \nu t = a\}$$
$$G_a^{(\nu)} = \sup\{t : B_t + \nu t = a\}$$

Prove that

$$(T_a^{(\nu)}, G_a^{(\nu)}) \stackrel{\text{law}}{=} \left(\frac{1}{G_\nu^{(a)}}, \frac{1}{T_\nu^{(a)}}\right)$$

Give the law of the pair $(T_a^{(\nu)}, G_a^{(\nu)})$. ◁

Exercise 5.6.7.6 Let X be a transient diffusion, such that

$$\mathbb{P}_x(T_0 < \infty) = 0, x > 0$$
$$\mathbb{P}_x(\lim_{t\to\infty} X_t = \infty) = 1, x > 0$$

and note s the scale function satisfying $s(0^+) = -\infty, s(\infty) = 0$. Prove that for all $x, t > 0$,

$$\mathbb{P}_x(G_y \in dt) = \frac{-1}{2s(y)} p_t^{(m)}(x,y)dt$$

where $p^{(m)}$ is the density transition w.r.t. the speed measure m. ◁

5.7 Pitman's Theorem about $(2M_t - W_t)$

5.7.1 Time Reversal of Brownian Motion

In our proof of Pitman's theorem, we shall need two results about time reversal of Brownian motion which are of interest by themselves:

Lemma 5.7.1.1 *Let W be a Brownian motion, L its local time at level 0 and $\tau_\ell = \inf\{t : L_t \geq \ell\}$. Then*

$$(W_u,\, u \leq \tau_\ell | \tau_\ell = t) \overset{\text{law}}{=} (W_u,\, u \leq t | L_t = \ell, W_t = 0)$$

As a consequence,

$$(W_{\tau_\ell - u},\, u \leq \tau_\ell) \overset{\text{law}}{=} (W_u,\, u \leq \tau_\ell)$$

Proof: Assuming the first property, we show how the second one is deduced. The scaling property allows us to restrict attention to the case $\ell = 1$. Since the law of the Brownian bridge is invariant under time reversal (see Section 4.3.5), we get that

$$(W_u, u \leq t | W_t = 0) \overset{\text{law}}{=} (W_{t-u}, u \leq t | W_t = 0)\,.$$

This identity implies

$$((W_u, u \leq t), L_t | W_t = 0) \overset{\text{law}}{=} ((W_{t-u}, u \leq t), L_t | W_t = 0)\,.$$

Therefore

$$(W_u, u \leq \tau_1 | \tau_1 = t) \overset{\text{law}}{=} (W_u, u \leq t | L_t = 1, W_t = 0)$$

$$\overset{\text{law}}{=} (W_{t-u}, u \leq t | L_t = 1, W_t = 0) \overset{\text{law}}{=} (W_{\tau_1 - u}, u \leq \tau_1 | \tau_1 = t)\,.$$

We conclude that

$$(W_{\tau_1 - u}; u \leq \tau_1)(W_u; u \leq \tau_1)\,.$$

□

The second result about time reversal is a particular case of a general result for Markov processes due to Nagasawa. We need some references to the Bessel process of dimension 3 (see ↣ Chapter 6).

Theorem 5.7.1.2 (Williams' Time Reversal Result.) *Let W be a BM, T_a the first hitting time of a by W and R a Bessel process of dimension 3 starting from 0, and Λ_a its last passage time at level a. Then*

$$(a - W_{T_a - t},\, t \leq T_a) \overset{\text{law}}{=} (R_t,\, t \leq \Lambda_a)\,.$$

Proof: We refer to [RY], Chapter VII. □

5.7.2 Pitman's Theorem

Here again, the Bessel process of dimension 3 (denoted as BES^3) plays an essential rôle (see ↣ Chapter 6).

Theorem 5.7.2.1 (Pitman's Theorem.) *Let W be a Brownian motion and $M_t = \sup_{s \le t} W_s$. The following identity in law holds*

$$(2M_t - W_t, M_t; t \ge 0) \stackrel{\text{law}}{=} (R_t, J_t; t \ge 0)$$

where $(R_t; t \ge 0)$ is a BES^3 *process starting from 0 and $J_t = \inf_{s \ge t} R_s$.*

PROOF: We note that it suffices to prove the identity in law between the first two components, i.e.,

$$(2M_t - W_t; t \ge 0) \stackrel{\text{law}}{=} (R_t; t \ge 0) . \tag{5.7.1}$$

Indeed, the equality (5.7.1) implies

$$(2M_t - W_t, \inf_{s \ge t}(2M_s - W_s); t \ge 0) \stackrel{\text{law}}{=} \left(R_t, \inf_{s \ge t} R_s; t \ge 0\right) .$$

We prove below that $M_t = \inf_{s \ge t}(2M_s - W_s)$. Hence, the equality

$$(2M_t - W_t, M_t; t \ge 0) \stackrel{\text{law}}{=} (R_t, J_t; t \ge 0) .$$

holds.

▶ We prove $M_t = \inf_{s \ge t}(2M_s - W_s)$ in two steps. First, note that for $s \ge t$, $2M_s - W_s \ge M_s \ge M_t$ hence $M_t \le \inf_{s \ge t}(2M_s - W_s)$.

In a second step, we introduce $\theta_t = \inf\{s \ge t : M_s = W_s\}$. Since the increasing process M increases only when $M = W$, it is obvious that $M_t = M_{\theta_t}$. From $M_{\theta_t} = 2M_{\theta_t} - W_{\theta_t} \ge \inf_{s \ge \theta_t}(2M_s - W_s)$ we deduce that $M_t = \inf_{s \ge \theta_t}(2M_s - W_s) \ge \inf_{s \ge t}(2M_s - W_s)$. Therefore, the equality $M_t = \inf_{s \ge t}(2M_s - W_s)$ holds.

▶ We now prove the desired result (5.7.1) with the help of Lévy's identity: the two statements

$$(2M_t - W_t; t \ge 0) \stackrel{\text{law}}{=} (R_t; t \ge 0)$$

and

$$(|W_t| + L_t; t \ge 0) \stackrel{\text{law}}{=} (R_t; t \ge 0) ,$$

are equivalent (we recall that L denotes the local time at 0 of W). Hence, we only need to prove that, for every ℓ,

$$(|W_t| + L_t; t \le \tau_\ell) \stackrel{\text{law}}{=} (R_t; t \le \Lambda_\ell) \tag{5.7.2}$$

where

$$\tau_\ell = \inf\{t : L_t \ge \ell\} \quad \text{and} \quad \Lambda_\ell = \sup\{t : R_t = \ell\}.$$

Accordingly, using Lemma 5.7.1.1, the equality (5.7.2) is equivalent to:

$$(|W_{\tau_\ell - t}| + (\ell - L_{\tau_\ell - t}); t \leq \tau_\ell) \stackrel{law}{=} (R_t; t \leq \Lambda_\ell)\,.$$

By Lévy's identity, this is equivalent to:

$$(\ell - W_{T_\ell - t}; t \leq T_\ell) \stackrel{\text{law}}{=} (R_t; t \leq \Lambda_\ell)$$

which is precisely Williams' time reversal theorem.

□

Corollary 5.7.2.2 *Let* $\widetilde{R}_t = 2M_t - W_t$, $\mathcal{R}_t = \sigma\{\widetilde{R}_s; s \leq t\}$, *and let* T *be an* $(\mathcal{R}_t)$ *stopping time. Then, conditionally on* $\mathcal{R}_T$, *the r.v.* M_T *(and, consequently, the r.v.* $M_T - W_T$*) is uniformly distributed on* $[0, \widetilde{R}_T]$. *Hence,* $\dfrac{M_T - W_T}{\widetilde{R}_T}$ *is uniform on* $[0,1]$ *and independent of* $\mathcal{R}_T$.

PROOF: Using Pitman's theorem, the statement of the corollary is equivalent to: if $(R^a_s; s \geq 0)$ is a BES^3_a process, $\inf_{s\geq 0} R^a_s$ is uniform on $[0, a]$, which follows from the useful lemma of Exercise 1.2.3.10.

Consequently for $x < y$

$$\mathbb{P}(M_u \leq x | \widetilde{R}_u = y) = \mathbb{P}(Uy \leq x) = x/y\,.$$

□

The property featured in the corollary entails an intertwining property between the semigroups of BM and BES^3 which is detailed in the following exercise.

Exercise 5.7.2.3 Denote by (P_t) and (Q_t) respectively the semigroups of the Brownian motion and of the BES^3. Prove that $Q_t \Lambda = \Lambda P_t$ where

$$\Lambda \,:\, f \to \Lambda f(r) = \frac{1}{2r} \int_{-r}^{+r} dx f(x)\,.$$

◁

Exercise 5.7.2.4 With the help of Corollary 5.7.2.2 and the Cameron-Martin formula, prove that the process $2M^{(\mu)}_t - W^{(\mu)}_t$, where $W^{(\mu)}_t = W_t + \mu t$, is a diffusion whose generator is $\frac{1}{2}\frac{d^2}{dx^2} + \mu \coth \mu x \,\frac{d}{dx}$. ◁

5.8 Filtrations

In the Black-Scholes model with constant coefficients, i.e.,

$$dS_t = S_t(\mu dt + \sigma dW_t),\; S_0 = x \tag{5.8.1}$$

where μ, σ and x are constants, the filtration $\mathbf{F}^S$ generated by the asset prices

$$\mathcal{F}_t^S := \sigma(S_s, s \le t)$$

is equal to the filtration $\mathbf{F}^W$ generated by W. Indeed, the solution of (5.8.1) is

$$S_t = x \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t \right) \tag{5.8.2}$$

which leads to

$$W_t = \frac{1}{\sigma}\left(\ln \frac{S_t}{S_0} - \left(\mu - \frac{\sigma^2}{2}\right) t \right) . \tag{5.8.3}$$

From (5.8.2), any function of S_t is a function of W_t, and $\mathcal{F}_t^S \subset \mathcal{F}_t^W$. From (5.8.3) the reverse inclusion holds.

This result remains valid for μ and σ deterministic functions, as long as $\sigma(t) > 0, \forall t$.

However, in general, the source of randomness is not so easy to identify; likewise models which are chosen to calibrate the data may involve more complicated filtrations. We present here a discussion of such set-ups. Our present aim is not to give a general framework but to study some particular cases.

5.8.1 Strong and Weak Brownian Filtrations

Amongst continuous-time processes, Brownian motion is undoubtedly the most studied process, and many characterizations of its law are known. It may thus seem a little strange that, deciding whether or not a filtration $\mathbf{F}$, on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$, is the natural filtration $\mathbf{F}^B$ of a Brownian motion $(B_t, t \ge 0)$ is a very difficult question and that, to date, no necessary and sufficient criterion has been found.

However, the following necessary condition can already discard a number of unsuitable "candidates," in a reasonably efficient manner: in order that $\mathbf{F}$ be a Brownian filtration, it is necessary that there exists an $\mathbf{F}$-Brownian motion β such that all $\mathbf{F}$-martingales may be written as $M_t = c + \int_0^t m_s d\beta_s$ for some $c \in \mathbb{R}$ and some predictable process m which satisfies $\int_0^t ds\, m_s^2 < \infty$. If needed, the reader may refer to $\rightarrowtail$ Section 9.5 for the general definition of the predictable representation property (PRP). This leads us to the following definition.

Definition 5.8.1.1 *A filtration $\mathbf{F}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\mathcal{F}_0$ is $\mathbb{P}$ a.s. trivial is said to be **weakly Brownian** if there exists an $\mathbf{F}$-Brownian motion β such that β has the predictable representation property with respect to $\mathbf{F}$.*

*A filtration $\mathbf{F}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\mathcal{F}_0$ is $\mathbb{P}$ a.s. trivial is said to be **strongly Brownian** if there exists an $\mathbf{F}$-BM β such that $\mathcal{F}_t = \mathcal{F}_t^\beta$.*

Implicitly, in the above definition, we assume that β is one-dimensional, but of course, a general discussion with d-dimensional Brownian motion can be developed.

Note that a strongly Brownian filtration is weakly Brownian since the Brownian motion enjoys the PRP. Since the mid-nineties, the study of weak Brownian filtrations has made quite some progress, starting with the proof by Tsirel'son [823] that the filtration of Walsh's Brownian motion as defined in Walsh [833] (see also Barlow and Yor [50]) taking values in $N \geq 3$ rays is weakly Brownian, but not strongly Brownian. See, in particular, the review paper of Émery [327] and notes and comments in Chapter V of [RY].

▶ We first show that weakly Brownian filtrations are left globally invariant under locally equivalent changes of probability. We start with a weakly Brownian filtration $\mathbf{F}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and we consider another probability $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ such that $\mathbb{Q}|_{\mathcal{F}_t} = L_t \mathbb{P}|_{\mathcal{F}_t}$.

Proposition 5.8.1.2 *If* $\mathbf{F}$ *is weakly Brownian under* $\mathbb{P}$ *and* $\mathbb{Q}$ *is locally equivalent to* $\mathbb{P}$*, then* $\mathbf{F}$ *is also weakly Brownian under* $\mathbb{Q}$*.*

PROOF: Let M be an $(\mathbf{F}, \mathbb{Q})$-local martingale, then ML is an $(\mathbf{F}, \mathbb{P})$-local martingale, hence $N_t := M_t L_t = c + \int_0^t n_s d\beta_s$ for some Brownian motion β defined on $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$, independently from M. Similarly, $dL_s = \ell_s d\beta_s$. Therefore, we have

$$\begin{aligned} M_t = \frac{N_t}{L_t} &= N_0 + \int_0^t \frac{dN_s}{L_s} - \int_0^t \frac{N_s dL_s}{L_s^2} + \int_0^t \frac{N_s d\langle L \rangle_s}{L_s^3} - \int_0^t \frac{d\langle N, L \rangle_s}{L_s^2} \\ &= c + \int_0^t \frac{n_s}{L_s} d\beta_s - \int_0^t \frac{N_s \ell_s}{L_s^2} d\beta_s + \int_0^t \frac{N_s \ell_s^2}{L_s^3} ds - \int_0^t \frac{n_s \ell_s}{L_s^2} ds \\ &= c + \int_0^t \left(\frac{n_s}{L_s} - \frac{N_s \ell_s}{L_s^2} \right) \left(d\beta_s - \frac{d\langle \beta, L \rangle_s}{L_s} \right) . \end{aligned}$$

Thus, $(\widetilde{\beta}_t := \beta_t - \int_0^t \frac{d\langle \beta, L \rangle_s}{L_s};\ t \geq 0)$, the Girsanov transform of the original Brownian motion β, allows the representation of all $(\mathbf{F}, \mathbb{Q})$-martingales. □

▶ We now show that weakly Brownian filtrations are left globally invariant by "nice" time changes. Again, we consider a weakly Brownian filtration $\mathbf{F}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $A_t = \int_0^t a_s ds$ where $a_s > 0$, $d\mathbb{P} \otimes ds$ a.s., be a strictly increasing, $\mathbf{F}$ adapted process, such that $A_\infty = \infty$, $\mathbb{P}$ a.s..

Proposition 5.8.1.3 *If* $\mathbf{F}$ *is weakly Brownian under* $\mathbb{P}$ *and* τ_u *is the right-inverse of the strictly increasing process* $A_t = \int_0^t a_s ds$*, then* $(\mathcal{F}_{\tau_u}, u \geq 0)$ *is also weakly Brownian under* $\mathbb{P}$*.*

PROOF: It suffices to be able to represent any $(\mathcal{F}_{\tau_u}, u \geq 0)$-square integrable martingale in terms of a given $(\mathcal{F}_{\tau_u}, u \geq 0)$-Brownian motion $\widetilde{\beta}$. Consider $\widetilde{M}$ a square integrable $(\mathcal{F}_{\tau_u}, u \geq 0)$-martingale. From our hypothesis, we know that $\widetilde{M}_\infty = c + \int_0^\infty m_s d\beta_s$, where β is an $\mathbf{F}$-Brownian motion and m is an $\mathbf{F}$-predictable process such that $\mathbb{E}\left(\int_0^\infty ds\, m_s^2\right) < \infty$. Thus, we may write

$$\widetilde{M}_\infty = c + \int_0^\infty \frac{m_s}{\sqrt{a_s}} \sqrt{a_s}\, d\beta_s\,. \tag{5.8.4}$$

It remains to define $\widetilde{\beta}$ the $(\mathcal{F}_{\tau_u}, u \geq 0)$-Brownian motion which satisfies $\int_0^t \sqrt{a_s} d\beta_s := \widetilde{\beta}_{A_t}$. Going back to (5.8.4), we obtain

$$\widetilde{M}_\infty = c + \int_0^\infty \frac{m_{\tau_u}}{\sqrt{a_{\tau_u}}}\, d\widetilde{\beta}_u\,.$$

□

These two properties do not extend to strongly Brownian filtrations. In particular, $\mathbf{F}$ may be strongly Brownian under $\mathbb{P}$ and only weakly Brownian under $\mathbb{Q}$ (see Dubins et al. [267], Barlow et al. [48]).

5.8.2 Some Examples

In what follows, we shall sometimes write Brownian filtration for strongly Brownian filtration.

Let $\mathbf{F}$ be a Brownian filtration, M an $\mathbf{F}$-martingale and $\mathbf{F}^M = (\mathcal{F}_t^M)$ the natural filtration of M.

(a) Reflected Brownian Motion. Let B be a Brownian motion and $\widetilde{B}_t = \int_0^t \mathrm{sgn}(B_s) dB_s$. The process $\widetilde{B}$ is a Brownian motion in the filtration $\mathbf{F}^{|B|}$. From $L_t = \sup_{s \leq t}(-\widetilde{B}_s)$, it follows that $\mathcal{F}_t^{\widetilde{B}} = \mathcal{F}_t^{|B|}$, hence, $\mathbf{F}^{|B|}$ is strongly Brownian and different from $\mathbf{F}$ since the r.v. $\mathrm{sgn}(B_t)$ is independent of $(|B_s|, s \leq t)$.

(b) Discontinuous Martingales Originating from a Brownian Setup. We give an example where there exists $\mathbf{F}^M$-discontinuous martingales. Let $M_t := \int_0^t \mathbb{1}_{\{B_s<0\}} dB_s$. Tanaka's formula leads to

$$B_t^- = -\int_0^t \mathbb{1}_{\{B_s<0\}} dB_s + \frac{1}{2} L_t\,.$$

The natural filtration of M, i.e., $\mathbf{F}^M$ is equal to the natural filtration of the process $(B_t^-, t \geq 0)$. The $\mathbf{F}^M$-martingale

$$\mathbb{E}\left(B_t^+ - \frac{1}{2} L_t | \mathcal{F}_t^M\right) = -\frac{1}{2} L_t + \mathbb{1}_{\{B_t>0\}} \sqrt{t - g_t}\, \mathbb{E}(m_1)\,,$$

(where we use here the notation of Section 4.3) is discontinuous, thus $\mathbf{F}^M$ is not even weakly Brownian. We refer to Williams [841] for a discussion.

(c) A Note about the PRP. Let $\mathbf{F}$ be a filtration and suppose that for a given $\mathbf{F}$-martingale M, any $\mathbf{F}$-martingale $(N_t, t \geq 0)$ vanishing at 0 can be written as $N_t = \int_0^t n_s dM_s$. This does not imply that $\sigma(M_s, s \leq t)$ equals $\mathcal{F}_t$ (in fact this is at the heart of the distinction between strongly and weakly Brownian filtrations). For example, let $\widetilde{B}_t = \int_0^t \mathrm{sgn}\,(B_s)\, dB_s$. As we have seen in the first example above, $\mathcal{F}_t^{\widetilde{B}} = \sigma(|B_s|, s \leq t)$ and is strictly smaller than $\mathbf{F}$. Nevertheless, any $\mathbf{F}$-martingale $(N_t, t \geq 0)$ with $N_0 = 0$ can be represented as

$$N_t = \int_0^t \nu_s dB_s = \int_0^t \nu_s \mathrm{sgn}\,(B_s)\, \mathrm{sgn}\,(B_s)\, dB_s = \int_0^t n_s d\widetilde{B}_s\,,$$

where $n_s = \nu_s\, \mathrm{sgn}\,(B_s)$.

(d) Another Example. Let $Y_t = \int_0^t B_s dW_s$ where W and B are independent Brownian motions. From

$$Y_t = \int_0^t |B_s| \mathrm{sgn}(B_s) dW_s = \int_0^t |B_s| d\widehat{W}_s$$

where $\widehat{W}_t = \int_0^t \mathrm{sgn}(B_s) dW_s$, it follows that

$$\mathcal{F}_t^Y = \sigma\{|B_s|, \widehat{W}_s, s \leq t\} = \sigma\{\widehat{B}_s, \widehat{W}_s, s \leq t\}\,,$$

where $\widehat{B}_t = \int_0^t \mathrm{sgn}(B_s) dB_s$ is a BM independent of $\widehat{W}$. Any $\mathbf{F}^Y$-martingale can be written as

$$y + \int_0^t \varphi_s d\widehat{B}_s + \int_0^t \psi_s d\widehat{W}_s\,,$$

for two $\mathbf{F}^Y$-predictable processes ψ and φ.

(e) Filtration Generated by a Stochastic Integral with Non-vanishing Integrator. Let $X_t = \int_0^t H_s dW_s$ where W is a $\mathbf{G}$-Brownian motion for some filtration $\mathbf{G}$, and H is a strictly positive continuous $\mathbf{G}$-adapted process (we do not require that $\mathbf{G}$ is the natural filtration of W). Then $\mathcal{F}_t^X = \sigma(H_s, W_s;\, s \leq t)$.

The case where the integrator may vanish is not so easy. Here are other examples.

(f) Tsirel'son's drift. Let

$$\mathbf{W}^{(T)}|_{\mathcal{F}_t} = \exp\left(\int_0^t T(s, X_\centerdot) dX_s - \frac{1}{2}\int_0^t T^2(s, X_\centerdot) ds\right) \mathbf{W}|_{\mathcal{F}_t}$$

where T is Tsirel'son drift (see Example 1.5.5.6). The process

$$X_t^{(T)} = X_t - \int_0^t T(s, X_{\bullet})ds$$

is a $\mathbf{W}^{(T)}$-Brownian motion whose filtration is strictly smaller than $\mathbf{F}$; however, $\mathbf{F}$ is the natural filtration of a $\mathbf{W}^{(T)}$-Brownian motion.

More generally, if

$$\mathbf{W}^b|_{\mathcal{F}_t} = \exp\left(\int_0^t b(s, X_{\bullet})dX_s - \frac{1}{2}\int_0^t b^2(s, X_{\bullet})ds\right) \mathbf{W}|_{\mathcal{F}_t} ,$$

the process

$$X_t^b = X_t - \int_0^t b(s, X.)ds$$

is a $\mathbf{W}^b, \mathbf{F}$-Brownian motion. Dubins et al. [267] established that there exist infinitely many b's such that $\mathbf{F}$ is not the natural filtration of a $\mathbf{W}^b$ Brownian motion, i.e., $\mathbf{F}$ is not strongly Brownian under $\mathbf{W}^b$. See also Emery and Schachermayer [329].

(g) Let W and B be two independent Brownian motions, and let $Z = BW$. From $B_tW_t = \int_0^t (B_s dW_s + W_s dB_s)$ one obtains that $B_t^2 + W_t^2$ is measurable w.r.t $\mathcal{F}_t^Z$. Hence, the random variables $\frac{1}{\sqrt{2}}|B_t + W_t|$ and $\frac{1}{\sqrt{2}}|B_t - W_t|$ are $\mathcal{F}_t^Z$-measurable. The processes $\beta_t^{(\pm)} = \frac{1}{\sqrt{2}}(B_t \pm W_t)$ are independent Brownian motions. The filtration $\mathbf{F}^Z$ is generated by two independent reflected BMs, hence from **a)** above, it is generated by two independent Brownian motions.

Exercise 5.8.2.1 Let B and W be two independent Brownian motions and $Y_t = aB_t + bW_t$. Prove that $\sigma(Y_s, s \le t) \subset \sigma(B_s, W_s, s \le t)$ and that the inclusion is strict.

Let N_1 and N_2 be two independent Poisson processes and $Y_t = aN_{1,t} + bN_{2,t}$, where $a \neq b$. Prove that $\sigma(Y_s, s \le t) = \sigma(N_{1,s}, N_{2,s}, s \le t)$. ◁

Exercise 5.8.2.2 Let B and W be two independent Brownian motions, a and b two strictly positive numbers with $a \neq b$ and $Y_t = aB_t^2 + bW_t^2$. Prove that $\sigma(Y_s, s \le t) = \sigma(B_s^2, W_s^2, s \le t)$.

Generalize this result to the case $Y_t = \sum_{i=1}^n a_i (B_t^i)^2$ where $a_i > 0$ and $a_i \neq a_j$ for $i \neq j$. Prove that the filtration of Y is that of an n-dimensional Brownian motion.

`Hint:` Compute the bracket of Y and iterate this procedure. ◁

Example 5.8.2.3 Example of a martingale with respect to two different probabilities:

Let $B = (B_1, B_2)$ be a two-dimensional BM, and $R_t^2 = B_1^2(t) + B_2^2(t)$. The process

$$L_t = \exp\left(\int_0^t (B_1(s)dB_1(s) + B_2(s)dB_2(s)) - \frac{1}{2}\int_0^t R_s^2 ds\right)$$

is a martingale. Let $\mathbb{Q}|_{\mathcal{F}_t} = L_t \mathbb{P}|_{\mathcal{F}_t}$. The process

$$X_t = \int_0^t (B_2(s)dB_1(s) - B_1(s)dB_2(s))$$

is a $\mathbb{P}$ (and a $\mathbb{Q}$) martingale. The process R^2 is a BESQ under $\mathbb{P}$ and a CIR under $\mathbb{Q}$ (see $\rightarrowtail$ Chapter 6). See also Example 1.7.3.10.

Comment 5.8.2.4 In [328], Emery and Schachermayer show that there exists an absolutely continuous strictly increasing time-change such that the time-changed filtration is no longer Brownian.

5.9 Enlargements of Filtrations

In general, if **G** is a filtration larger than **F**, it is not true that an **F**-martingale remains a martingale in the filtration **G** (an interesting example is Azéma's martingale μ (see Subsection 4.3.8): this discontinuous $\mathbf{F}^\mu$-martingale is not an $\mathcal{F}^B$-martingale, it is not even a $\mathcal{F}^B$-semi-martingale; see $\rightarrowtail$ Example 9.4.2.3).

In the seminal paper [461], Itô studies the definition of the integral of a non-adapted process of the form $f(B_1, B_s)$ for some function f, with respect to a Brownian motion B. From the end of the seventies, Barlow, Jeulin and Yor started a systematic study of the problem of enlargement of filtrations: namely which **F**-martingales M remain **G**-semi-martingales and if it is the case, what is the semi-martingale decomposition of M in **G**?

Up to now, four lecture notes volumes have been dedicated to this question: Jeulin [493], Jeulin and Yor [497], Yor [868] and Mansuy and Yor [622]. See also related chapters in the books of Protter [727] and Dellacherie, Maisonneuve and Meyer [241]. Some important papers are Brémaud and Yor [126], Barlow [45], Jacod [469, 468] and Jeulin and Yor [495].

These results are extensively used in finance to study two specific problems occurring in insider trading: existence of arbitrage using strategies adapted w.r.t. the large filtration, and change of prices dynamics, when an **F**-martingale is no longer a **G**-martingale.

We now study mathematically the two situations.

5.9.1 Immersion of Filtrations

Let **F** and **G** be two filtrations such that $\mathbf{F} \subset \mathbf{G}$. Our aim is to study some conditions which ensure that **F**-martingales are **G**-semi-martingales, and one can ask in a first step whether all **F**-martingales are **G**-martingales. This last property is equivalent to $\mathbb{E}(D|\mathcal{F}_t) = \mathbb{E}(D|\mathcal{G}_t)$, for any t and $D \in L^1(\mathcal{F}_\infty)$.

Let us study a simple example where $\mathbf{G} = \mathbf{F} \vee \sigma(D)$ where $D \in L^1(\mathcal{F}_\infty)$ and D is not $\mathcal{F}_0$-measurable. Obviously $\mathbb{E}(D|\mathcal{G}_t) = D$ is a **G**-martingale and $\mathbb{E}(D|\mathcal{F}_t)$ is a **F**-martingale. However $\mathbb{E}(D|\mathcal{G}_0) \neq \mathbb{E}(D|\mathcal{F}_0)$, and some **F**-martingales are not **G**-martingales.

The filtration **F** is said to be **immersed** in **G** if any square integrable **F**-martingale is a **G**-martingale (Tsirel'son [824], Émery [327]). This is also referred to as the $(\mathcal{H})$ hypothesis by Brémaud and Yor [126] which was defined as:

$(\mathcal{H})$ Every **F**-square integrable martingale is a **G**-square integrable martingale.

Proposition 5.9.1.1 *Hypothesis* $(\mathcal{H})$ *is equivalent to any of the following properties:*

$(\mathcal{H}1)$ $\forall\, t \geq 0$, *the* σ*-fields* $\mathcal{F}_\infty$ *and* $\mathcal{G}_t$ *are conditionally independent given* $\mathcal{F}_t$*.*
$(\mathcal{H}2)$ $\forall\, t \geq 0,\ \forall\, G_t \in L^1(\mathcal{G}_t)$, $\mathbb{E}(G_t|\mathcal{F}_\infty) = \mathbb{E}(G_t|\mathcal{F}_t)$.
$(\mathcal{H}3)$ $\forall\, t \geq 0,\ \forall\, F \in L^1(\mathcal{F}_\infty)$, $\mathbb{E}(F|\mathcal{G}_t) = \mathbb{E}(F|\mathcal{F}_t)$.

In particular, $(\mathcal{H})$ *holds if and only if every* **F***-local martingale is a* **G***-local martingale.*

PROOF:
▶ $(\mathcal{H}) \Rightarrow (\mathcal{H}1)$. Let $F \in L^2(\mathcal{F}_\infty)$ and assume that hypothesis $(\mathcal{H})$ is satisfied. This implies that the martingale $F_t = \mathbb{E}(F|\mathcal{F}_t)$ is a **G**-martingale such that $F_\infty = F$, hence $F_t = \mathbb{E}(F|\mathcal{G}_t)$. It follows that for any t and any $G_t \in L^2(\mathcal{G}_t)$:

$$\mathbb{E}(FG_t|\mathcal{F}_t) = \mathbb{E}(G_t\mathbb{E}(F|\mathcal{G}_t)|\mathcal{F}_t) = \mathbb{E}(G_t\mathbb{E}(F|\mathcal{F}_t)|\mathcal{F}_t) = \mathbb{E}(G_t|\mathcal{F}_t)\mathbb{E}(F|\mathcal{F}_t)$$

which is equivalent to $(\mathcal{H}1)$.
▶ $(\mathcal{H}1) \Rightarrow (\mathcal{H})$. Let $F \in L^2(\mathcal{F}_\infty)$ and $G_t \in L^2(\mathcal{G}_t)$. Under $(\mathcal{H}1)$,

$$\mathbb{E}(F\mathbb{E}(G_t|\mathcal{F}_t)) = \mathbb{E}(\mathbb{E}(F|\mathcal{F}_t)\mathbb{E}(G_t|\mathcal{F}_t)) \overset{\mathcal{H}1}{=} \mathbb{E}(\mathbb{E}(FG_t|\mathcal{F}_t)) = \mathbb{E}(FG_t)$$

which is $(\mathcal{H})$.
▶ $(\mathcal{H}2) \Rightarrow (\mathcal{H}3)$. Let $F \in L^2(\mathcal{F}_\infty)$ and $G_t \in L^2(\mathcal{G}_t)$. If $(\mathcal{H}2)$ holds, then it is easy to prove that, for $F \in L^2(\mathcal{F}_\infty)$,

$$\mathbb{E}(G_t\mathbb{E}(F|\mathcal{F}_t)) = \mathbb{E}(F\mathbb{E}(G_t|\mathcal{F}_t)) \overset{\mathcal{H}2}{=} \mathbb{E}(FG_t) = \mathbb{E}(G_t\mathbb{E}(F|\mathcal{G}_t)),$$

which implies $(\mathcal{H}3)$. The general case follows by approximation.
▶ Obviously $(\mathcal{H}3)$ implies $(\mathcal{H})$. □

In particular, under $(\mathcal{H})$, if W is an **F**-Brownian motion, then it is a **G**-martingale with bracket t, since such a bracket does not depend on the filtration. Hence, it is a **G**-Brownian motion.

A trivial (but useful) example for which $(\mathcal{H})$ is satisfied is $\mathbf{G} = \mathbf{F} \vee \mathbf{F}^1$ where $\mathbf{F}$ and $\mathbf{F}^1$ are two filtrations such that $\mathcal{F}_\infty$ is independent of $\mathcal{F}^1_\infty$.

We now present two propositions, in which setup the immersion property is preserved under change of probability.

Proposition 5.9.1.2 *Assume that the filtration* $\mathbf{F}$ *is immersed in* $\mathbf{G}$ *under* $\mathbb{P}$*, and let* $\mathbb{Q}|_{\mathcal{G}_t} = L_t \mathbb{P}|_{\mathcal{G}_t}$ *where* L *is assumed to be* $\mathbf{F}$*-adapted. Then,* $\mathbf{F}$ *is immersed in* $\mathbf{G}$ *under* $\mathbb{Q}$.

PROOF: Let N be a $(\mathbf{F}, \mathbb{Q})$-martingale, then $(N_t L_t, t \geq 0)$ is a $(\mathbf{F}, \mathbb{P})$-martingale, and since $\mathbf{F}$ is immersed in $\mathbf{G}$ under $\mathbb{P}$, $(N_t L_t, t \geq 0)$ is a $(\mathbf{G}, \mathbb{P})$-martingale which implies that N is a $(\mathbf{G}, \mathbb{Q})$-martingale. □

In the next proposition, we do not assume that the Radon-Nikodým density is $\mathbf{F}$-adapted.

Proposition 5.9.1.3 *Assume that* $\mathbf{F}$ *is immersed in* $\mathbf{G}$ *under* $\mathbb{P}$*, and define* $\mathbb{Q}|_{\mathcal{G}_t} = L_t \mathbb{P}|_{\mathcal{G}_t}$ *and* $\Lambda_t = \mathbb{E}(L_t|\mathcal{F}_t)$*. Assume that all* $\mathbf{F}$*-martingales are continuous and that the* $\mathbf{G}$*-martingale* L *is continuous. Then,* $\mathbf{F}$ *is immersed in* $\mathbf{G}$ *under* $\mathbb{Q}$ *if and only if the* $(\mathbf{G}, \mathbb{P})$*-local martingale*

$$\int_0^t \frac{dL_s}{L_s} - \int_0^t \frac{d\Lambda_s}{\Lambda_s} := \mathcal{L}(L)_t - \mathcal{L}(\Lambda)_t$$

is orthogonal to the set of all $(\mathbf{F}, \mathbb{P})$*-local martingales.*

PROOF: We prove that any $(\mathbf{F}, \mathbb{Q})$-martingale is a $(\mathbf{G}, \mathbb{Q})$-martingale. Every $(\mathbf{F}, \mathbb{Q})$-martingale M^Q may be written as

$$M_t^Q = M_t^P - \int_0^t \frac{d\langle M^P, \Lambda \rangle_s}{\Lambda_s}$$

where M^P is an $(\mathbf{F}, \mathbb{P})$-martingale. By hypothesis, M^P is a $(\mathbf{G}, \mathbb{P})$-martingale and, from Girsanov's theorem, $M_t^P = N_t^Q + \int_0^t \frac{d\langle M^P, L \rangle_s}{L_s}$ where N^Q is an $(\mathbf{F}, \mathbb{Q})$-martingale. It follows that

$$\begin{aligned} M_t^Q &= N_t^Q + \int_0^t \frac{d\langle M^P, L \rangle_s}{L_s} - \int_0^t \frac{d\langle M^P, \Lambda \rangle_s}{\Lambda_s} \\ &= N_t^Q + \int_0^t d\langle M^P, \mathcal{L}(L) - \mathcal{L}(\Lambda) \rangle_s \,. \end{aligned}$$

Thus M^Q is an $(\mathbf{G}, \mathbb{Q})$ martingale if and only if $\langle M^P, \mathcal{L}(L) - \mathcal{L}(\Lambda) \rangle_s = 0$. □

Exercise 5.9.1.4 Assume that hypothesis $(\mathcal{H})$ holds under $\mathbb{P}$. Let

$$\mathbb{Q}|_{\mathcal{G}_t} = L_t \mathbb{P}|_{\mathcal{G}_t}; \quad \mathbb{Q}|_{\mathcal{F}_t} = \widehat{L}_t \mathbb{P}|_{\mathcal{F}_t} \,.$$

Prove that hypothesis $(\mathcal{H})$ holds under $\mathbb{Q}$ if and only if:

$$\forall X \geq 0,\ X \in \mathcal{F}_\infty, \quad \frac{\mathbb{E}(XL_\infty|\mathcal{G}_t)}{L_t} = \frac{\mathbb{E}(X\widehat{L}_\infty|\mathcal{F}_t)}{\widehat{L}_t}$$

See Nikeghbali [674]. ◁

5.9.2 The Brownian Bridge as an Example of Initial Enlargement

Rather than studying ab initio the general problem of initial enlargement, we discuss an interesting example. Let us start with a BM $(B_t, t \geq 0)$ and its natural filtration $\mathbf{F}^B$. Define a new filtration as $\mathcal{G}_t = \mathcal{F}_t^B \vee \sigma(B_1)$. In this filtration, the process $(B_t, t \geq 0)$ is no longer a martingale. It is easy to be convinced of this by looking at the process $(\mathbb{E}(B_1|\mathcal{G}_t), t \leq 1)$: this process is identically equal to B_1, not to B_t, hence $(B_t, t \geq 0)$ is not a $\mathbf{G}$-martingale. However, $(B_t, t \geq 0)$ is a $\mathbf{G}$-semi-martingale, as follows from the next proposition

Proposition 5.9.2.1 *The decomposition of B in the filtration $\mathbf{G}$ is*

$$B_t = \beta_t + \int_0^{t\wedge 1} \frac{B_1 - B_s}{1-s} ds$$

where β is a $\mathbf{G}$-Brownian motion.

PROOF: We have seen, in (4.3.8), that the canonical decomposition of Brownian bridge under $\mathbf{W}^{(1)}_{0\to 0}$ is

$$X_t = \beta_t - \int_0^t ds\, \frac{X_s}{1-s}, \quad t \leq 1\,.$$

The same proof implies that the decomposition of B in the filtration $\mathbf{G}$ is

$$B_t = \beta_t + \int_0^{t\wedge 1} \frac{B_1 - B_s}{1-s} ds\,.$$

□

It follows that if M is an $\mathbf{F}$-local martingale such that $\int_0^1 \frac{1}{\sqrt{1-s}} d|\langle M, B\rangle|_s$ is finite, then

$$M_t = \widehat{M}_t + \int_0^{t\wedge 1} \frac{B_1 - B_s}{1-s} d\langle M, B\rangle_s$$

where $\widehat{M}$ is a $\mathbf{G}$-local martingale.

Comments 5.9.2.2 (a) As we shall see in ↣ Subsection 11.2.7, Proposition 5.9.2.1 can be extended to integrable Lévy processes: if X is a Lévy process which satisfies $\mathbb{E}(|X_t|) < \infty$ and $\mathbf{G} = \mathbf{F}^X \vee \sigma(X_1)$, the process

$$X_t - \int_0^{t\wedge 1} \frac{X_1 - X_s}{1-s}\, ds,$$

is a **G**-martingale.

(b) The singularity of $\frac{B_1 - B_t}{1-t}$ at $t = 1$, i.e., the fact that $\frac{B_1 - B_t}{1-t}$ is not square integrable between 0 and 1 prevents a Girsanov measure change transforming the $(\mathbb{P}, \mathbf{G})$ semi-martingale B into a $(\mathbb{Q}, \mathbf{G})$ martingale. Let

$$dS_t = S_t(\mu dt + \sigma dB_t)$$

and enlarge the filtration with S_1 (or equivalently, with B_1). In the enlarged filtration, setting $\zeta_t = \frac{B_1 - B_t}{1-t}$, the dynamics of S are

$$dS_t = S_t((\mu + \sigma\zeta_t)dt + \sigma d\beta_t)\,,$$

and there does not exist an e.m.m. such that the discounted price process $(e^{-rt}S_t, t \le 1)$ is a **G**-martingale. However, for any $\epsilon \in]0,1]$, there exists a uniformly integrable **G**-martingale L defined as

$$dL_t = \frac{\mu - r + \sigma\zeta_t}{\sigma} L_t d\beta_t,\ t \le 1 - \epsilon, \quad L_0 = 1\,,$$

such that, setting $d\mathbb{Q}|_{\mathcal{G}_t} = L_t d\mathbb{P}|_{\mathcal{G}_t}$, the process $(e^{-rt}S_t, t \le 1 - \epsilon)$ is a $(\mathbb{Q}, \mathbf{G})$-martingale.

This is the main point in the theory of insider trading where the knowledge of the terminal value of the underlying asset creates an arbitrage opportunity, which is effective at time 1.

5.9.3 Initial Enlargement: General Results

Let **F** be a Brownian filtration generated by B. We consider $\mathcal{F}_t^{(L)} = \mathcal{F}_t \vee \sigma(L)$ where L is a real-valued random variable. More precisely, in order to satisfy the usual hypotheses, redefine

$$\mathcal{F}_t^{(L)} = \cap_{\epsilon > 0} \left\{ \mathcal{F}_{t+\epsilon} \vee \sigma(L) \right\}\ .$$

We recall that there exists a family of regular conditional distributions $\lambda_t(\omega, dx)$ such that $\lambda_t(\cdot, A)$ is a version of $\mathbb{E}(\mathbb{1}_{\{L \in A\}} | \mathcal{F}_t)$ and for any ω, $\lambda_t(\omega, \cdot)$ is a probability on $\mathbb{R}$.

Proposition 5.9.3.1 (Jacod's Criterion.) *Suppose that, for each $t < T$, $\lambda_t(\omega, dx) \ll \nu(dx)$ where ν is the law of L. Then, every **F**-semi-martingale $(X_t, t < T)$ is also an $\mathcal{F}_t^{(L)}$-semi-martingale.*

*Moreover, if $\lambda_t(\omega, dx) = p_t(\omega, x)\nu(dx)$ and if X is an **F**-martingale, its decomposition in the filtration $\mathcal{F}_t^{(L)}$ is*

$$X_t = \widetilde{X}_t + \int_0^t \frac{d\langle p.(L), X\rangle_s}{p_s(L)}\,.$$

In a more general setting (see Yor [868]), for a bounded Borel function f, let $(\lambda_t(f), t \geq 0)$ be the continuous version of the martingale $(\mathbb{E}(f(L)|\mathcal{F}_t), t \geq 0)$. There exists a predictable kernel $\lambda_t(dx)$ such that

$$\lambda_t(f) = \int \lambda_t(dx) f(x)\,.$$

From the predictable representation property applied to the martingale $\mathbb{E}(f(L)|\mathcal{F}_t)$, there exists a predictable process $\widehat{\lambda}(f)$ such that

$$\lambda_t(f) = \mathbb{E}(f(L)) + \int_0^t \widehat{\lambda}_s(f) dB_s\,.$$

Proposition 5.9.3.2 *We assume that there exists a predictable kernel* $\widehat{\lambda}_t(dx)$ *such that*

$$dt \text{ a.s.,} \quad \widehat{\lambda}_t(f) = \int \widehat{\lambda}_t(dx) f(x)\,.$$

Assume furthermore that $dt \times d\mathbb{P}$ *a.s. the measure* $\widehat{\lambda}_t(dx)$ *is absolutely continuous with respect to* $\lambda_t(dx)$*:*

$$\widehat{\lambda}_t(dx) = \rho(t,x)\lambda_t(dx)\,.$$

Then, if X *is an* $\mathbf{F}$*-martingale, there exists a* $\mathbf{F}^{(L)}$*-martingale* $\widehat{X}$ *such that*

$$X_t = \widehat{X}_t + \int_0^t \rho(s,L) d\langle X, B\rangle_s\,.$$

Sketch of the proof: Let X be an $\mathbf{F}$-martingale, f a given bounded Borel function and $F_t = \mathbb{E}(f(L)|\mathcal{F}_t)$. From the hypothesis

$$\begin{aligned} F_t &= \mathbb{E}(f(L)) + \int_0^t \widehat{\lambda}_s(f) dB_s \\ &= \mathbb{E}(f(L)) + \int_0^t \left(\int \rho(s,x)\lambda_s(dx) f(x) \right) dB_s\,. \end{aligned}$$

Then, for $A_s \in \mathcal{F}_s$, $s < t$:

$$\begin{aligned} \mathbb{E}(\mathbb{1}_{A_s} f(L)(X_t - X_s)) &= \mathbb{E}(\mathbb{1}_{A_s}(F_t X_t - F_s X_s)) = \mathbb{E}(\mathbb{1}_{A_s}(\langle F, X\rangle_t - \langle F, X\rangle_s)) \\ &= \mathbb{E}\left(\mathbb{1}_{A_s} \int_s^t d\langle X, B\rangle_u \, \widehat{\lambda}_u(f) \right) \\ &= \mathbb{E}\left(\mathbb{1}_{A_s} \int_s^t d\langle X, B\rangle_u \int \lambda_u(dx) f(x)\rho(u,x) \right)\,. \end{aligned}$$

Therefore, $V_t = \int_0^t \rho(u,L)\, d\langle X, B\rangle_u$ satisfies

$$\mathbb{E}(\mathbb{1}_{A_s} f(L)(X_t - X_s)) = \mathbb{E}(\mathbb{1}_{A_s} f(L)(V_t - V_s))\,.$$

It follows that, for any $G_s \in \mathcal{F}_s^{(L)}$,

$$\mathbb{E}(\mathbb{1}_{G_s}(X_t - X_s)) = \mathbb{E}(\mathbb{1}_{G_s}(V_t - V_s)),$$

hence, $(X_t - V_t, t \geq 0)$ is an $\mathbf{F}^{(L)}$-martingale. □

Let us write the result of Proposition 5.9.3.2 in terms of Jacod's criterion. If $\lambda_t(dx) = p_t(x)\nu(dx)$, then

$$\lambda_t(f) = \int p_t(x) f(x) \nu(dx).$$

Hence,

$$d\langle \lambda.(f), B\rangle_t = \widehat{\lambda}_t(f)dt = \int dx f(x)\, d_t\langle p.(x), B\rangle_t$$

and

$$\widehat{\lambda}_t(dx) = d_t\langle p.(x), B\rangle_t = \frac{d_t\langle p.(x), B\rangle_t}{p_t(x)} p_t(x) dx$$

therefore,

$$\widehat{\lambda}_t(dx)dt = \frac{d_t\langle p.(x), B\rangle_t}{p_t(x)} \lambda_t(dx).$$

In the case where $\lambda_t(dx) = \Phi(t,x)dx$, with $\Phi > 0$, it is possible to find ψ such that

$$\Phi(t,x) = \Phi(0,x) \exp\left(\int_0^t \psi(s,x)dB_s - \frac{1}{2}\int_0^t \psi^2(s,x)ds\right)$$

and it follows that $\widehat{\lambda}_t(dx) = \psi(t,x)\lambda_t(dx)$. Then, if X is an $\mathbf{F}$-martingale of the form $X_t = x + \int_0^t x_s dB_s$, the process $(X_t - \int_0^t ds\, x_s\, \psi(s,L), t \geq 0)$ is an $\mathbf{F}^{(L)}$-martingale.

Example 5.9.3.3 We now give some examples taken from Mansuy and Yor [622] in a Brownian set-up for which we use the preceding. Here, B is a standard Brownian motion.

▶ **Enlargement with B_1.** We compare the results obtained in Subsection 5.9.2 and the method presented in Subsection 5.9.3. Let $L = B_1$. From the Markov property

$$\mathbb{E}(g(B_1)|\mathcal{F}_t) = \mathbb{E}(g(B_1 - B_t + B_t)|\mathcal{F}_t) = F_g(B_t, 1-t)$$

where $F_g(y, 1-t) = \int g(x) p_{1-t}(y,x)dx$ and $p_s(y,x) = \frac{1}{\sqrt{2\pi s}} \exp\left(-\frac{(x-y)^2}{2s}\right)$. It follows that $\lambda_t(dx) = \frac{1}{\sqrt{2\pi(1-t)}} \exp\left(-\frac{(x-B_t)^2}{2(1-t)}\right) dx$. Then

$$\lambda_t(dx) = p_t^x \mathbb{P}(B_1 \in dx)$$

with

$$p_t^x = \frac{1}{\sqrt{(1-t)}} \exp\left(-\frac{(x-B_t)^2}{2(1-t)} + \frac{x^2}{2}\right).$$

From Itô's formula,

$$dp_t^x = p_t^x \frac{x-B_t}{1-t} dB_t .$$

It follows that $d\langle p^x, B\rangle_t = p_t^x \frac{x-B_t}{1-t} dt$, hence

$$B_t = \widetilde{B}_t + \int_0^t \frac{x-B_s}{1-s} ds .$$

Note that, in the notation of Proposition 5.9.3.2, one has

$$\widehat{\lambda}_t(dx) = \frac{x-B_t}{1-t} \frac{1}{\sqrt{2\pi(1-t)}} \exp\left(-\frac{(x-B_t)^2}{2(1-t)}\right) dx .$$

▶ **Enlargement with** $M^B = \sup_{s\le 1} B_s$**.** From Exercise 3.1.6.7,

$$\mathbb{E}(f(M^B)|\mathcal{F}_t) = F(1-t, B_t, M_t^B)$$

where $M_t^B = \sup_{s\le t} B_s$ with

$$F(s,a,b) = \sqrt{\frac{2}{\pi s}} \left(f(b) \int_0^{b-a} e^{-u^2/(2s)} du + \int_b^\infty f(u) e^{-(u-a)^2/(2s)} du \right)$$

and

$$\lambda_t(dy) = \sqrt{\frac{2}{\pi(1-t)}} \left\{ \delta_y(M_t^B) \int_0^{M_t^B - B_t} \exp\left(-\frac{u^2}{2(1-t)}\right) du \right.$$
$$\left. + \mathbb{1}_{\{y > M_t^B\}} \exp\left(-\frac{(y-B_t)^2}{2(1-t)}\right) dy \right\} .$$

Hence, by differentiation w.r.t. $x(= B_t)$, i.e., more precisely, by applying Itô's formula

$$\widehat{\lambda}_t(dy) = \sqrt{\frac{2}{\pi(1-t)}} \left\{ \delta_y(M_t^B) \exp\left(-\frac{(M_t^B - B_t)^2}{2(1-t)}\right) \right.$$
$$\left. + \mathbb{1}_{\{y > M_t^B\}} \frac{y-B_t}{1-t} \exp\left(-\frac{(y-B_t)^2}{2(1-t)}\right) \right\} .$$

It follows that

$$\rho(t,x) = \mathbb{1}_{\{x > M_t^B\}} \frac{x - B_t}{1-t} + \mathbb{1}_{\{M_t^B = x\}} \frac{1}{\sqrt{1-t}} \frac{\Phi'}{\Phi} \left(\frac{x - B_t}{\sqrt{1-t}} \right)$$

with $\Phi(x) = \sqrt{\frac{2}{\pi}} \int_0^x e^{-\frac{u^2}{2}} du$.

More examples can be found in Jeulin [493] and Mansuy and Yor [622]. Matsumoto and Yor [629] consider the case where $L = \int_0^\infty ds \exp(2(B_s - \nu s))$. See also Baudoin [61].

Exercise 5.9.3.4 Assume that the hypotheses of Proposition 5.9.3.1 hold and that $1/p_\infty(\cdot, L)$ is integrable with expectation $1/c$. Prove that under the probability R defined as

$$dR|_{\mathcal{F}_\infty} = c/p_\infty(\cdot, L) d\mathbb{P}|_{\mathcal{F}_\infty}$$

the r.v. L is independent of $\mathcal{F}_\infty$. This fact plays an important rôle in Grorud and Pontier [411]. ◁

5.9.4 Progressive Enlargement

We now consider a different case of enlargement, more precisely the case where τ is a finite random time, i.e., a finite non-negative random variable, and we denote

$$\mathcal{F}_t^\tau = \cap_{\epsilon > 0} \left\{ \mathcal{F}_{t+\epsilon} \vee \sigma(\tau \wedge (t+\epsilon)) \right\} .$$

Proposition 5.9.4.1 *For any $\mathbf{F}^\tau$-predictable process H, there exists an $\mathbf{F}$-predictable process h such that $H_t \mathbb{1}_{\{t \le \tau\}} = h_t \mathbb{1}_{\{t \le \tau\}}$. Under the condition $\forall t, \mathbb{P}(\tau \le t | \mathcal{F}_t) < 1$, the process $(h_t, t \ge 0)$ is unique.*

Proof: We refer to Dellacherie [245] and Dellacherie et al. [241], page 186. The process h may be recovered as the ratio of the $\mathbf{F}$-predictable projections of $H_t \mathbb{1}_{\{t < \tau\}}$ and $\mathbb{1}_{\{t < \tau\}}$:

$$h_t = \frac{\mathbb{E}(H_t \mathbb{1}_{\{t < \tau\}} | \mathcal{F}_t)}{\mathbb{P}(t < \tau | \mathcal{F}_t)} .$$ □

Immersion Setting

Let us first investigate the case where the $(\mathcal{H})$ hypothesis holds.

Lemma 5.9.4.2 *In the progressive enlargement setting, $(\mathcal{H})$ holds between $\mathbf{F}$ and $\mathbf{F}^\tau$ if and only if one of the following equivalent conditions holds:*

$$\begin{array}{lll} \text{(i)} \ \forall (t,s),\ s \le t, & \mathbb{P}(\tau \le s | \mathcal{F}_\infty) = \mathbb{P}(\tau \le s | \mathcal{F}_t), \\ \text{(ii)} \ \forall t, & \mathbb{P}(\tau \le t | \mathcal{F}_\infty) = \mathbb{P}(\tau \le t | \mathcal{F}_t). \end{array} \qquad (5.9.1)$$

PROOF: If (ii) holds, then (i) holds too. If (i) holds, $\mathcal{F}_\infty$ and $\sigma(t \wedge \tau)$ are conditionally independent given $\mathcal{F}_t$. The property follows. This result can also be found in Dellacherie and Meyer [243]. □

Note that, if $(\mathcal{H})$ holds, then (ii) implies that the process $\mathbb{P}(\tau \leq t|\mathcal{F}_t)$ is decreasing.

Example: assume that $\mathbf{F} \subset \mathbf{G}$ where $(\mathcal{H})$ holds for $\mathbf{F}$ and $\mathbf{G}$. Let τ be a $\mathbf{G}$-stopping time. Then, $(\mathcal{H})$ holds for $\mathbf{F}$ and $\mathbf{F}^\tau$.

General Setting

We denote by Z^τ the $\mathbf{F}$-super-martingale $\mathbb{P}(\tau > t|\mathcal{F}_t)$, also called the Azéma supermartingale (introduced in [35]). We assume in what follows

(A) The random time τ avoids the $\mathbf{F}$-stopping times, i.e., $\mathbb{P}(\tau = \vartheta) = 0$ for any $\mathbf{F}$-stopping time ϑ.

Under **(A)**, the $\mathbf{F}$-dual predictable projection of the process $D_t := \mathbb{1}_{\tau \leq t}$, denoted A^τ, is continuous. Indeed, if ϑ is a jump time of A^τ, it is predictable, and

$$\mathbb{E}(A^\tau_\vartheta - A^\tau_{\vartheta-}) = \mathbb{E}(\mathbb{1}_{\tau=\vartheta}) = 0\,;$$

the continuity of A^τ follows.

Proposition 5.9.4.3 *The canonical decomposition of the semi-martingale Z^τ is*

$$Z^\tau_t = \mathbb{E}(A^\tau_\infty|\mathcal{F}_t) - A^\tau_t = \mu^\tau_t - A^\tau_t$$

where $\mu^\tau_t := \mathbb{E}(A^\tau_\infty|\mathcal{F}_t)$.

PROOF: From the definition of the dual predictable projection, for any predictable process H, one has

$$\mathbb{E}(H_\tau) = \mathbb{E}\left(\int_0^\infty H_u dA^\tau_u\right)\,.$$

Let t be fixed and $F_t \in \mathcal{F}_t$. Then, the process $H_u = F_t \mathbb{1}_{\{t<u\}}, u \geq 0$ is $\mathbf{F}$-predictable. Then

$$\mathbb{E}(F_t \mathbb{1}_{\{t<\tau\}}) = \mathbb{E}(F_t(A^\tau_\infty - A^\tau_t))\,.$$

It follows that $\mathbb{E}(A^\tau_\infty|\mathcal{F}_t) = Z^\tau_t + A^\tau_t$. □

Comment 5.9.4.4 It can be proved that the martingale

$$\mu^\tau_t := \mathbb{E}(A^\tau_\infty|\mathcal{F}_t) = A^\tau_t + Z^\tau_t$$

is BMO (see Definition 1.2.3.9).

It is proved in Yor [860] that if X is an $\mathbf{F}$-martingale then the processes $X_{t\wedge\tau}$ and $X_t(1-D_t)$ are $\mathbf{F}^\tau$ semi-martingales. Furthermore, the decompositions of the $\mathbf{F}$-martingales in the filtration $\mathbf{F}^\tau$ are known up to time τ (Jeulin and Yor [495]).

Proposition 5.9.4.5 *Every $\mathbf{F}$-martingale M stopped at time τ is an $\mathbf{F}^\tau$-semi-martingale with canonical decomposition*

$$M_{t\wedge\tau} = \widetilde{M}_t + \int_0^{t\wedge\tau} \frac{d\langle M, \mu^\tau\rangle_s}{Z^\tau_{s-}}\,,$$

where $\widetilde{M}$ is an $\mathbf{F}^\tau$-local martingale. The process

$$\mathbb{1}_{\{\tau\le t\}} - \int_0^{t\wedge\tau} \frac{1}{Z^\tau_{s-}} dA^\tau_s$$

is an $\mathbf{F}^\tau$-martingale.

PROOF: Let H be an $\mathbf{F}^\tau$-predictable process. There exists an $\mathbf{F}$-predictable process h such that $H_t\mathbb{1}_{\{t\le\tau\}} = h_t\mathbb{1}_{\{t\le\tau\}}$, hence, if M is an $\mathbf{F}$-martingale, for $s<t$,

$$\begin{aligned}
&\mathbb{E}(H_s(M_{t\wedge\tau} - M_{s\wedge\tau})) = \mathbb{E}(H_s\mathbb{1}_{\{s<\tau\}}(M_{t\wedge\tau} - M_{s\wedge\tau}))\\
&= \mathbb{E}(h_s\mathbb{1}_{\{s<\tau\}}(M_{t\wedge\tau} - M_{s\wedge\tau}))\\
&= \mathbb{E}\left(h_s(\mathbb{1}_{\{s<\tau\le t\}}(M_\tau - M_s) + \mathbb{1}_{\{t<\tau\}}(M_t - M_s))\right)
\end{aligned}$$

From the definition of Z,

$$\mathbb{E}\left(h_s\mathbb{1}_{\{s<\tau\le t\}}M_\tau\right) = -\mathbb{E}\left(h_s\int_s^t M_u dZ_u\right)$$

and, noting that

$$\int_s^t M_u dZ_u - M_sZ_s + Z_tM_t = \int_s^t Z_u dM_u + \langle M, Z\rangle_t - \langle M, Z\rangle_s$$

we get, from the martingale property of M

$$\begin{aligned}
&\mathbb{E}(H_s(M_{t\wedge\tau} - M_{s\wedge\tau})) = \mathbb{E}(h_s(\langle M, \mu^\tau\rangle_t - \langle M, \mu^\tau\rangle_s))\\
&= \mathbb{E}\left(h_s\int_s^t \frac{d\langle M, \mu^\tau\rangle_u}{Z^\tau_{u-}} Z^\tau_{u-}\right) = \mathbb{E}\left(h_s\int_s^t \frac{d\langle M, \mu^\tau\rangle_u}{Z^\tau_{u-}}\,\mathbb{E}(\mathbb{1}_{\{u<\tau\}}|\mathcal{F}_u)\right)\\
&= \mathbb{E}\left(h_s\int_s^t \frac{d\langle M, \mu^\tau\rangle_u}{Z^\tau_{u-}}\mathbb{1}_{\{u<\tau\}}\right) = \mathbb{E}\left(h_s\int_{s\wedge\tau}^{t\wedge\tau} \frac{d\langle M, \mu^\tau\rangle_u}{Z^\tau_{u-}}\right).
\end{aligned}$$

The result follows. □

Pseudo-stopping Times

As we have mentioned, if $(\mathcal{H})$ holds, the process $(Z_t^\tau, t \geq 0)$ is a decreasing process. The converse is not true. The decreasing property of Z^τ is closely related with the definition of pseudo-stopping times, a notion developed from D. Williams example (see Example 5.9.4.8 below).

Definition 5.9.4.6 *A random time τ is a pseudo-stopping time if, for any bounded $\mathbf{F}$-martingale M, $\mathbb{E}(M_\tau) = M_0$.*

Proposition 5.9.4.7 *The random time τ is a pseudo-stopping time if and only if one of the following equivalent properties holds:*

- *For any local $\mathbf{F}$-martingale m, the process $(m_{t\wedge\tau}, t \geq 0)$ is a local $\mathbf{F}^\tau$-martingale,*
- $A_\infty^\tau = 1$,
- $\mu_t^\tau = 1, \forall t \geq 0$,
- *The process Z^τ is a decreasing $\mathbf{F}$-predictable process.*

PROOF: We refer to Nikeghbali and Yor [675]. □

Example 5.9.4.8 The first example of a pseudo-stopping time was given by Williams [844]. Let B be a Brownian motion and define the stopping time $T_1 = \inf\{t : B_t = 1\}$ and the random time $\theta = \sup\{t < T_1 : B_t = 0\}$. Set

$$\tau = \sup\{s < \theta : B_s = M_s^B\}$$

where M_s^B is the running maximum of the Brownian motion. Then, τ is a pseudo-stopping time. Note that $\mathbb{E}(B_\tau)$ is not equal to 0; this illustrates the fact we cannot take any martingale in Definition 5.9.4.6. The martingale $(B_{t\wedge T_1}, t \geq 0)$ is neither bounded, nor uniformly integrable. In fact, since the maximum M_θ^B $(=B_\tau)$ is uniformly distributed on $[0,1]$, one has $\mathbb{E}(B_\tau) = 1/2$.

Honest Times

For a general random time τ, it is not true that $\mathbf{F}$-martingales are $\mathbf{F}^\tau$-semi-martingales. Here is an example: due to the separability of the Brownian filtration, there exists a bounded random variable τ such that $\mathcal{F}_\infty = \sigma(\tau)$. Hence, $\mathcal{F}_{\tau+t}^\tau = \mathcal{F}_\infty, \forall t$ so that the $\mathbf{F}^\tau$-martingales are constant after τ. Consequently, $\mathbf{F}$-martingales are not $\mathbf{F}^\tau$-semi-martingales.

On the other hand, there exists an interesting class of random times τ such that $\mathbf{F}$-martingales are $\mathbf{F}^\tau$-semi-martingales.

Definition 5.9.4.9 *A random time is honest if it is the end of a predictable set, i.e., $\tau(\omega) = \sup\{t : (t,\omega) \in \Gamma\}$, where Γ is an $\mathbf{F}$-predictable set.*

In particular, an honest time is $\mathcal{F}_\infty$-measurable. If X is a transient diffusion, the last passage time Λ_a (see Proposition 5.6.2.1) is honest. Jeulin [493] established that an $\mathcal{F}_\infty$-measurable random time is honest if and only if it is equal, on $\{\tau < t\}$, to an $\mathcal{F}_t$-measurable random variable.

A key point in the proof of the next Proposition 5.9.4.10 is the following description of $\mathbf{F}^\tau$-predictable processes: if τ, an $\mathcal{F}_\infty$-measurable random time, is honest, and if H is an $\mathbf{F}^\tau$-predictable process, then there exist two $\mathbf{F}$-predictable processes h and $\widetilde{h}$ such that

$$H_t = h_t \mathbb{1}_{\{\tau > t\}} + \widetilde{h}_t \mathbb{1}_{\{\tau \le t\}} \,.$$

(See Jeulin [493] for a proof.)

Proposition 5.9.4.10 *Let τ be honest. Then, if X is an $\mathbf{F}$-local martingale, there exists an $\mathbf{F}^\tau$-local martingale $\widetilde{X}$ such that*

$$X_t = \widetilde{X}_t + \int_0^{t\wedge\tau} \frac{d\langle X, \mu^\tau\rangle_s}{Z^\tau_{s-}} - \int_\tau^{\tau\vee t} \frac{d\langle X, \mu^\tau\rangle_s}{1 - Z^\tau_{s-}} \,.$$

PROOF: Let M be an $\mathbf{F}$-martingale which belongs to $\mathbf{H}^1$ and $G_s \in \mathcal{F}^\tau_s$. We define a $\mathbf{G}^\tau$ predictable process H as $H_u = \mathbb{1}_{G_s}\mathbb{1}_{]s,t]}(u)$. For $s < t$, one has, using the decomposition of $\mathbf{G}^\tau$ predictable processes:

$$\begin{aligned}\mathbb{E}(\mathbb{1}_{G_s}(M_t - M_s)) &= \mathbb{E}\left(\int_0^\infty H_u dM_u\right) \\ &= \mathbb{E}\left(\int_0^\tau h_u dM_u\right) + \mathbb{E}\left(\int_\tau^\infty \widetilde{h}_u dM_u\right) \,.\end{aligned}$$

Noting that $\int_0^t \widetilde{h}_u dM_u$ is a martingale yields $\mathbb{E}\left(\int_0^\infty \widetilde{h}_u dM_u\right) = 0$,

$$\begin{aligned}\mathbb{E}(\mathbb{1}_{G_s}(M_t - M_s)) &= \mathbb{E}\left(\int_0^\tau (h_u - \widetilde{h}_u) dM_u\right) \\ &= \mathbb{E}\left(\int_0^\infty dA^\tau_v \int_0^v (h_u - \widetilde{h}_u) dM_u\right) \,.\end{aligned}$$

By integration by parts, with $N_t = \int_0^t (h_u - \widetilde{h}_u) dM_u$, we get

$$\mathbb{E}(\mathbb{1}_{G_s}(M_t - M_s)) = \mathbb{E}(N_\infty A^\tau_\infty) = \mathbb{E}\left(\int_0^\infty (h_u - \widetilde{h}_u) d\langle M, \mu^\tau\rangle_u\right) \,.$$

Now, it remains to note that

$$
\begin{aligned}
&\mathbb{E}\left(\int_0^\infty H_u\left(\frac{d\langle M,\mu^\tau\rangle_u}{Z_{u-}}\mathbb{1}_{\{u\le\tau\}} - \frac{d\langle M,\mu^\tau\rangle_u}{1-Z_{u-}}\mathbb{1}_{\{u>\tau\}}\right)\right)\\
&= \mathbb{E}\left(\int_0^\infty\left(h_u\frac{d\langle M,\mu^\tau\rangle_u}{Z_{u-}}\mathbb{1}_{\{u\le\tau\}} - \widetilde{h}_u\frac{d\langle M,\mu^\tau\rangle_u}{1-Z_{u-}}\mathbb{1}_{\{u>\tau\}}\right)\right)\\
&= \mathbb{E}\left(\int_0^\infty\left(h_u d\langle M,\mu^\tau\rangle_u - \widetilde{h}_u d\langle M,\mu^\tau\rangle_u\right)\right)\\
&= \mathbb{E}\left(\int_0^\infty (h_u - \widetilde{h}_u) d\langle M,\mu^\tau\rangle_u\right)
\end{aligned}
$$

to conclude the result in the case $M \in \mathbf{H}^1$. The general result follows by localization. □

Example 5.9.4.11 Let W be a Brownian motion, and $\tau = g_1$, the last time when the BM reaches 0 before time 1, i.e., $\tau = \sup\{t \le 1 : W_t = 0\}$. Using the computation of Z^{g_1} in Subsection 5.6.4 and Proposition 5.9.4.10, we obtain the decomposition of the Brownian motion in the enlarged filtration

$$
\begin{aligned}
W_t = \widetilde{W}_t - \int_0^t \mathbb{1}_{[0,\tau]}(s)\,\frac{\Phi'}{1-\Phi}\left(\frac{|W_s|}{\sqrt{1-s}}\right)\frac{\operatorname{sgn}(W_s)}{\sqrt{1-s}}\,ds\\
+ \mathbb{1}_{\{\tau\le t\}}\operatorname{sgn}(W_1)\int_\tau^t \frac{\Phi'}{\Phi}\left(\frac{|W_s|}{\sqrt{1-s}}\right)ds
\end{aligned}
$$

where $\Phi(x) = \sqrt{\frac{2}{\pi}}\int_0^x \exp(-u^2/2)du$.

Comments 5.9.4.12 (a) The $(\mathcal{H})$ hypothesis was studied by Brémaud and Yor [126] and Mazziotto and Szpirglas [632], and in a financial setting by Kusuoka [552], Elliott et al. [315] and Jeanblanc and Rutkowski [486, 487].

(b) An incomplete list of authors concerned with enlargement of filtration in finance for insider trading is: Amendinger [12], Amendinger et al. [13], Baudoin [61], Corcuera et al. [194], Eyraud-Loisel [338], Florens and Fougère [347], Gasbarra et al. [374], Grorud and Pontier [410], Hillairet [436], Imkeller [457], Imkeller et al. [458], Karatzas and Pikovsky [512], Kohatsu-Higa [532, 533] and Kohatsu-Higa and Øksendal [534].

(c) Enlargement theory is also used to study asymmetric information, see e. g. Föllmer et al. [353] and progressive enlargement is an important tool for the study of default in the reduced form approach by Bielecki et al. [91, 92, 93], Elliott et al.[315] and Kusuoka [552] (see $\rightarrowtail$ Chapter 7).

(d) See also the papers of Ankirchner et al. [19] and Yoeurp [858].

(e) Note that the random time τ presented in Subsection 5.6.5 is not the end of a predictable set, hence, is not honest. However, **F**-martingales are semi-martingales in the progressive enlarged filtration: it suffices to note that **F**-martingales are semi-martingales in the filtration initially enlarged with W_1.

5.10 Filtering the Information

A priori, one might think somewhat naïvely that the drift term in the historical dynamics of the asset plays no rôle in contingent claims valuation. Nevertheless, working in the filtration generated by the asset shows the importance of this parameter. We present here some results, linked with filtering theory. However, we do not present the theory in detail, and the reader can refer to Lipster and Shiryaev [598] and Brémaud [124] for processes with jumps.

5.10.1 Independent Drift

Suppose that $dB_t^{(Y)} = Y dt + dB_t$, $B_0^{(Y)} = 0$ where Y is some r.v. independent of B and with law ν. The following proposition describes the distribution of $B^{(Y)}$.

Proposition 5.10.1.1 *The law of $B^{(Y)}$ is $\mathbf{W}^{h_\nu}$ defined as*

$$\mathbf{W}^{h_\nu}|_{\mathcal{F}_t} = h_\nu(X_t, t)\,\mathbf{W}|_{\mathcal{F}_t}\,.$$

Here, $h_\nu(x,t) = \int \nu(dy) \exp(yx - \frac{y^2}{2}t)$.

PROOF: Let F be a functional on $C([0,t],\mathbb{R})$. Using the independence between Y and B, and the Cameron-Martin theorem, we get

$$\begin{aligned}\mathbb{E}[F(B_s^{(Y)}, s \le t)] = \mathbb{E}[F(sY + B_s, s \le t)] &= \int \nu(dy)\mathbb{E}[F(sy + B_s, s \le t)] \\ &= \int \nu(dy)\mathbb{E}\left[F(B_s, s \le t)\,\exp\left(yB_t - \frac{y^2}{2}t\right)\right] \\ &= \mathbb{E}[F(B_s; s \le t) h_\nu(B_t, t)]\,.\end{aligned}$$

□

We now give the canonical decomposition of $B^{(Y)}$ in its own filtration. Let $\mathbf{W}^{h_\nu}|_{\mathcal{F}_t} = h_\nu(X_t, t)\,\mathbf{W}|_{\mathcal{F}_t} = L_t\,\mathbf{W}|_{\mathcal{F}_t}$. Therefore, the bracket $\langle X, L\rangle_t$ is equal to $\int_0^t \partial_x h_\nu(X_s, s)\,ds$, and, from Girsanov's theorem,

$$\beta_t = X_t - \int_0^t ds \frac{\partial_x h_\nu}{h_\nu}(X_s, s)$$

is a $\mathbf{W}^{h_\nu}$-martingale, more precisely a $\mathbf{W}^{h_\nu}$-Brownian motion and

$$X_t = \beta_t + \int_0^t ds \frac{\partial_x h_\nu}{h_\nu}(X_s, s)\,.$$

The canonical decomposition of $B^{(Y)}$ is

$$B_t^{(Y)} = \gamma_t + \int_0^t ds \, \frac{\partial_x h_\nu}{h_\nu}(B_s^{(Y)}, s) \, .$$

where γ is a BM with respect to the natural filtration of $B^{(Y)}$.

The next proposition describes the conditional law of Y, given $B^{(Y)}$.

Proposition 5.10.1.2 *If $f : \mathbb{R} \to \mathbb{R}^+$ is a Borel function, then*

$$\pi_t(f) := \mathbb{E}(f(Y)|B_s^{(Y)}, s \le t) = \frac{h_{(f\cdot\nu)}(B_t^{(Y)}, t)}{h_\nu(B_t^{(Y)}, t)}$$

where $h_{(f\cdot\nu)}(x,t) = \int \nu(dy) f(y) \exp(yx - \frac{y^2}{2}t)$ *and*

$$\pi_t(f) = 1 + \int_0^t \left(\partial_x \frac{h_{(f\cdot\nu)}}{h_\nu} \right) (B_s^{(Y)}, s) d\gamma_s \, .$$

Proof: On the one hand

$$\mathbb{E}(f(Y)F(B_s^{(Y)}, s \le t) = \mathbb{E}(F(B_s, s \le t) \, h_{(f\cdot\nu)}(B_t, t)) \, . \tag{5.10.1}$$

On the other hand, if

$$\Phi(B_s^{(Y)}, s \le t) = \mathbb{E}(f(Y)|B_s^{(Y)}, s \le t) \, ,$$

the left-hand side of (5.10.1) is equal to

$$\mathbb{E}\left(\Phi(B_s^{(Y)}, s \le t)F(B_s^{(Y)}, s \le t)\right) = \mathbb{E}\left(\Phi(B_s, s \le t)F(B_s, s \le t)h_\nu(B_t, t)\right) \, . \tag{5.10.2}$$

It follows that

$$\pi_t(f) = \Phi(B_s^{(Y)}, s \le t) = \frac{h_{(f\cdot\nu)}(B_t^{(Y)}, t)}{h_\nu(B_t^{(Y)}, t)} \, .$$

The expression of $\pi_t(f)$ as a stochastic integral follows directly from this expression of $\pi_t(f)$ (and the martingale property of $\pi_t(f)$). □

5.10.2 Other Examples of Canonical Decomposition

The above result can be generalized to the case where

$$dX_t = dW_t + (f(t)\tilde{W}_t + h(t)X_t)dt$$

where $\tilde{W}$ is independent of W. In that case, studied by Föllmer et al. [353], the canonical decomposition of X is

$$X_t = \beta_t + \int_0^t (f(u)k_u(X_v; v \le u) + h(u)X_u)\, du$$

where

$$k_u(X_s; s \le u) = \frac{1}{\Psi'(u)} \int_0^u \Psi(v)\,(f(v)dX_v - f(v)h(v)X_v dv)$$

with Ψ the fundamental solution of the Sturm-Liouville equation

$$\Psi''(t) = f^2(t)\Psi(t)$$

with boundary conditions $\Psi(0) = 0, \Psi'(0) = 1$.

5.10.3 Innovation Process

The following formula plays an important rôle in filtering theory and will be illustrated below.

Proposition 5.10.3.1 *Let $dX_t = Y_t dt + dW_t$, where W is an $\mathbf{F}$-Brownian motion and Y an $\mathbf{F}$-adapted process. Define $\widehat{Y}_t = \mathbb{E}(Y_t|\mathcal{F}_t^X)$, the optional projection of Y on $\mathbf{F}^X$. Then, the process*

$$Z_t := X_t - \int_0^t \widehat{Y}_s ds$$

is an $\mathbf{F}^X$-Brownian motion, called the innovation process.

PROOF: Note that, for $t > s$,

$$\begin{aligned}
\mathbb{E}(Z_t|\mathcal{F}_s^X) &= \mathbb{E}(X_t|\mathcal{F}_s^X) - \mathbb{E}\left(\int_0^t \widehat{Y}_u du|\mathcal{F}_s^X\right) \\
&= \mathbb{E}(W_t|\mathcal{F}_s^X) + \mathbb{E}\left(\int_0^t Y_u du|\mathcal{F}_s^X\right) - \int_0^s \widehat{Y}_u du - \mathbb{E}\left(\int_s^t \widehat{Y}_u du|\mathcal{F}_s^X\right).
\end{aligned}$$

From the inclusion $\mathcal{F}_t^X \subset \mathcal{F}_t$ and the fact that W is an $\mathbf{F}$-martingale, we obtain $\mathbb{E}(W_t|\mathcal{F}_s^X) = \mathbb{E}(W_s|\mathcal{F}_s^X)$. Therefore, by using

$$\int_s^t \mathbb{E}(Y_u|\mathcal{F}_s^X) du = \int_s^t \mathbb{E}(\widehat{Y}_u|\mathcal{F}_s^X) du$$

we obtain

$$\begin{aligned}
\mathbb{E}(Z_t|\mathcal{F}_s^X) &= \mathbb{E}(W_s|\mathcal{F}_s^X) + \mathbb{E}\left(\int_0^t Y_u du|\mathcal{F}_s^X\right) - \int_0^s \widehat{Y}_u du - \mathbb{E}\left(\int_s^t \widehat{Y}_u du|\mathcal{F}_s^X\right) \\
&= \mathbb{E}(X_s|\mathcal{F}_s^X) + \int_s^t \mathbb{E}(Y_u|\mathcal{F}_s^X) du - \int_0^s \widehat{Y}_u du - \mathbb{E}\left(\int_s^t \widehat{Y}_u du|\mathcal{F}_s^X\right) \\
&= X_s + \int_s^t \mathbb{E}(\widehat{Y}_u|\mathcal{F}_s^X) du - \int_0^s \widehat{Y}_u du - \mathbb{E}\left(\int_s^t \widehat{Y}_u du|\mathcal{F}_s^X\right) \\
&= X_s - \int_0^s \widehat{Y}_u du\,.
\end{aligned}$$

□

Proposition 5.10.3.1 is in fact a particular case of the more general result that follows, which is of interest if Z is not $\mathbf{F}$-adapted.

Proposition 5.10.3.2 *Let Z be a measurable process such that $\mathbb{E}(\int_0^t |Z_u|du)$ is finite for every t. Then, $\mathbb{E}(\int_0^t Z_u du|\mathcal{F}_t)$ is an $\mathbf{F}$-semi-martingale which decomposes as $M_t + \int_0^t du\, \mathbb{E}(Z_u|\mathcal{F}_u)$, where M is a martingale.*

PROOF: We leave the proof to the reader. □

Example 5.10.3.3 As an example, take $Z_u = B_1, \forall u$, with B a Brownian motion. Then

$$\mathbb{E}\left(\int_0^t duB_1|\mathcal{F}_t\right) = tB_t = M_t + \int_0^t duB_u\,.$$

Comment 5.10.3.4 The paper of Pham and Quenez [711] and the paper of Lefebvre et al. [574] study the problem of optimal consumption under partial observation, by means of filtering theory. See also Nakagawa [665] for an application to default risk.