4

Complements on Brownian Motion

In the first part of this chapter, we present the definition of local time and the associated Tanaka formulae, first for Brownian motion, then for more general *continuous* semi-martingales. In the second part, we give definitions and basic properties of Brownian bridges and Brownian meander. This is motivated by the fact that, in order to study complex derivative instruments, such as passport options or Parisian options, some knowledge of local times, bridges and excursions with respect to BM in particular and more generally for diffusions, is useful. We give some applications to exotic options, in particular to Parisian options.

The main mathematical references on these topics are Chung and Williams [186], Kallenberg [505], Karatzas and Shreve [513], [RY], Rogers and Williams [742] and Yor [864, 867, 868].

4.1 Local Time

4.1.1 A Stochastic Fubini Theorem

Let X be a semi-martingale on a filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$, μ a bounded measure on $\mathbb{R}$, and H, defined on $\mathbb{R}^+ \times \Omega \times \mathbb{R}$, a $\mathcal{P} \otimes \mathcal{B}$ bounded measurable map, where $\mathcal{P}$ is the $\mathbf{F}$-predictable σ-algebra. Then

$$\int_0^t dX_s \left(\int \mu(da) H(s, \omega, a) \right) = \int \mu(da) \left(\int_0^t dX_s \, H(s, \omega, a) \right) .$$

More precisely, both sides are well defined and are equal.

This result can be proven for $H(s, \omega, a) = h(s, \omega)\varphi(a)$, then for a general H as above by applying the MCT. We leave the details to the reader.

4.1.2 Occupation Time Formula

Theorem 4.1.2.1 (Occupation Time Formula.) *Let B be a one-dimensional Brownian motion. There exists a family of increasing processes, the*

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local times of B, $(L_t^x, t \geq 0;\, x \in \mathbb{R})$*, which may be taken jointly continuous in* (x,t)*, such that, for every Borel bounded function* f

$$\boxed{\int_0^t f(B_s)\,ds = \int_{-\infty}^{+\infty} L_t^x f(x)\,dx\,.} \qquad (4.1.1)$$

In particular, for every t *and for every Borel set* A*, the Brownian occupation time of* A *between* 0 *and* t *satisfies*

$$\nu(t,A) := \int_0^t \mathbb{1}_{\{B_s \in A\}}\,ds = \int_{-\infty}^{\infty} \mathbb{1}_A(x)\,L_t^x\,dx\,. \qquad (4.1.2)$$

PROOF: To prove Theorem 4.1.2.1, we consider the left-hand side of the equality (4.1.1) as "originating" from the second order correction term in Itô's formula. Here are the details.

Let us assume that f is a continuous function with compact support. Let

$$F(x) := \int_{-\infty}^{x} dz \int_{-\infty}^{z} dy f(y) = \int_{-\infty}^{\infty} (x-y)^+ f(y)dy\,.$$

Consequently, F is C^2 and $F'(x) = \int_{-\infty}^{x} f(y)\,dy = \int_{-\infty}^{\infty} f(y)\,\mathbb{1}_{\{x>y\}}\,dy$. Itô's formula applied to F and the stochastic Fubini theorem yield

$$\int_{-\infty}^{\infty} (B_t - y)^+ f(y)dy = \int_{-\infty}^{\infty} (B_0 - y)^+ f(y)dy + \int_{-\infty}^{\infty} dy f(y) \int_0^t \mathbb{1}_{\{B_s > y\}} dB_s$$
$$+ \frac{1}{2}\int_0^t f(B_s)ds\,.$$

Therefore

$$\frac{1}{2}\int_0^t f(B_s)ds = \int_{-\infty}^{\infty} dy f(y) \left((B_t - y)^+ - (B_0 - y)^+ - \int_0^t \mathbb{1}_{\{B_s > y\}} dB_s \right) \qquad (4.1.3)$$

and formula (4.1.1) is obtained by setting

$$\frac{1}{2} L_t^y = (B_t - y)^+ - (B_0 - y)^+ - \int_0^t \mathbb{1}_{\{B_s > y\}} dB_s\,. \qquad (4.1.4)$$

Furthermore, it may be proven from (4.1.4), with the help of Kolmogorov's continuity criterion (see Theorem 1.1.10.6), that L_t^y may be chosen jointly continuous with respect to the two variables y and t (see [RY], Chapter VI for a detailed proof). □

Had we started from $G'(x) = -\int_x^{\infty} f(y)\,dy = -\int_{-\infty}^{\infty} f(y)\,\mathbb{1}_{\{x<y\}}\,dy$, we would have obtained the following occupation time formula

$$\int_0^t f(B_s)\,ds = \int_{-\infty}^{\infty} \widetilde{L}_t^y\, f(y)dy\,, \tag{4.1.5}$$

with

$$\frac{1}{2}\widetilde{L}_t^y = (B_t - y)^- - (B_0 - y)^- + \int_0^t \mathbb{1}_{\{B_s<y\}} dB_s\,.$$

Therefore,

$$(B_t - y)^- = (B_0 - y)^- - \int_0^t \mathbb{1}_{\{B_s<y\}} dB_s + \frac{1}{2}\widetilde{L}_t^y\,.$$

Note that $L_t^y - \widetilde{L}_t^y = B_t - B_0 - \int_0^t \mathbb{1}_{\{B_s\neq y\}} dB_s = 2\int_0^t dB_s \mathbb{1}_{\{B_s=y\}}$, hence

$$(B_t - y)^- = (B_0 - y)^- - \int_0^t \mathbb{1}_{\{B_s\leq y\}} dB_s + \frac{1}{2}L_t^y\,.$$

Furthermore, the integral $\displaystyle\int_0^t dB_s \mathbb{1}_{\{B_s=y\}}$ is equal to 0, because its second order moment is equal to 0; indeed:

$$\mathbb{E}\left(\int_0^t dB_s \mathbb{1}_{\{B_s=y\}}\right)^2 = \int_0^t \mathbb{P}(B_s = y)ds = 0\,.$$

Hence, $L^y = \widetilde{L}^y$.

Comments 4.1.2.2 (a) In the occupation time formula (4.1.1), the time t may be replaced by any random time τ.

(b) The concept and several constructions (different from the above) of local time in the case of Brownian motion are due to Lévy [585].

(c) Existence of local times for Markov processes whose points are regular for themselves is developed in Blumenthal and Getoor [107]. Occupation densities for general stochastic processes are discussed in Geman and Horowitz [377]. Local times for diffusions are presented in $\rightarrowtail$ Section 5.5 and in Borodin and Salminen [109].

(d) Continuity results for Brownian local times are due to Trotter [821], and many results can be found in the collective book [37].

4.1.3 An Approximation of Local Time

The quantity L_t^x is called **the local time** of the Brownian motion at level x between 0 and t. From (4.1.1), we obtain the equality

$$L_t^x = \lim_{\epsilon\to 0} \frac{1}{2\epsilon} \int_0^t \mathbb{1}_{[x-\epsilon,x+\epsilon]}(B_s)\,ds\,,$$

where the limit holds a.s.. It can also be shown that it holds in L^2. This approximation shows in particular that $(L_t^x, t \geq 0)$, the local time at level x, is an increasing process. An important property (see [RY], Chapter VI) is that, for fixed x, the support of the random measure dL_t^x is precisely the set $\{t \geq 0 : B_t = x\}$. In other words, for $x = 0$, say, the local time (at level 0) increases only on the set of zeros of the Brownian motion B. In particular

$$\int_0^t f(B_s) dL_s^0 = f(0) L_t^0 \,.$$

Exercise 4.1.3.1 Let H be a measurable map defined on $\mathbb{R}^+ \times \Omega \times \mathbb{R}$. Prove that, for any random time τ,

$$\int_0^\tau H(s,\omega,B_s) ds = \int_{-\infty}^{\infty} dx \int_0^\tau H(s,\omega,x)\, d_s L_s^x \,,$$

where the notation $d_s L_s^x$ makes precise that x is fixed and the measure $d_s L_s^x$ is on $\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+)$. ◁

4.1.4 Local Times for Semi-martingales

The same approach can be applied to continuous semi-martingales X (see ↣ Subsection 4.1.8). In this case, the two quantities L and $\tilde{L}$ obtained from equations (4.1.4) and (4.1.5) where B is changed to X can be different, and the continuity property does not necessarily hold. There are also different definitions of local time, the reader is referred to ↣ Section 5.5.

4.1.5 Tanaka's Formula

Tanaka's formulae are variants of Itô's formula for the absolute value and the positive and negative parts of a BM.

Proposition 4.1.5.1 (Tanaka's Formulae.) *Let B be a Brownian motion and L_t^x its local time at level x between 0 and t. For every t,*

$$(B_t - x)^+ = (B_0 - x)^+ + \int_0^t \mathbb{1}_{\{B_s > x\}}\, dB_s + \frac{1}{2} L_t^x \tag{4.1.6}$$

$$(B_t - x)^- = (B_0 - x)^- - \int_0^t \mathbb{1}_{\{B_s \leq x\}}\, dB_s + \frac{1}{2} L_t^x \tag{4.1.7}$$

$$|B_t - x| = |B_0 - x| + \int_0^t \text{sgn}\,(B_s - x)\, dB_s + L_t^x \tag{4.1.8}$$

where $\text{sgn}(x) = 1$ *if* $x > 0$ *and* $\text{sgn}(x) = -1$ *if* $x \le 0$.

PROOF: The first two formulae follow directly from the definition. The last equality is obtained by summing term by term the two previous ones. □

Comment 4.1.5.2 If Itô's formula could be applied to $|B|$, without taking care of the discontinuity at 0 of the derivative of $|x|$, then arguing that BM spends Lebesgue measure zero time in a given state, one would obtain the equality of $|B_t|$ and $\int_0^t \text{sgn}\,(B_s)\, dB_s$. This is obviously absurd, since $|B_t|$ is positive and $\int_0^t \text{sgn}\,(B_s)\, dB_s$ is a centered variable. Indeed, the process $(\int_0^t \text{sgn}\,(B_s)\, dB_s,\, t \ge 0)$ is a Brownian motion, see Example 1.4.1.5. Therefore, the local time spent at level 0 by the original Brownian motion B is quite meaningful, in that Tanaka's formulae are expressions of the Doob-Meyer decomposition of the sub-martingales $(B_t - x)^+$ and $|B_t - x|$ where $\frac{1}{2} L_t^x$ and L_t^x are the corresponding increasing processes.

More generally, Tanaka's formulae may be extended to develop $f(B_t)$ as a semi-martingale when f is locally the difference of two convex functions:

$$f(B_t) = f(B_0) + \int_0^t (D_- f)(B_s)\, dB_s + \frac{1}{2} \int_{\mathbb{R}} L_t^a f''(da) \tag{4.1.9}$$

where $D_- f$ is the left derivative of f and f'' is the second derivative in the distribution sense, meaning

$$\int f''(da) g(a) = \int f(a) g''(a) da$$

for any twice differentiable function g with compact support.

Note that if f is a C^1 function, and is also C^2 on $\mathbb{R} \setminus \{a_1, \ldots, a_n\}$, for a finite number of points $(a_i, i = 1, \ldots, n)$,

$$f(B_t) = f(B_0) + \int_0^t f'(B_s) dB_s + \frac{1}{2} \int_0^t g(B_s) ds$$

where $g(x)dx$ is the second derivative of f in the distribution sense. In that case, there is no local time apparent in the formula.

More generally, if f is locally a difference of two convex functions, which is C^2 on $\mathbb{R} \setminus \{a_1, \ldots, a_n\}$, then

$$f(B_t) = f(B_0) + \int_0^t f'(B_s) dB_s + \frac{1}{2} \int_0^t g(B_s) ds + \frac{1}{2} \sum_{i=1}^n L_t^{a_i} (f'(a_i^+) - f(a_i^-))\,.$$

Warning 4.1.5.3 Some authors (e.g., Karatzas and Shreve [513]) choose a different normalization of local times starting from the occupation formula. Hence in their version of Tanaka's formulae, a coefficient other than 1/2 appears. These different conventions should be considered with care as they may be a source of errors. On the other hand, the most common choice is the coefficient 1/2, which allows the extension of Itô's formula as in (4.1.9).

Comment 4.1.5.4 If B is a Brownian motion, a necessary and sufficient condition for $f(B)$ to be a semi-martingale is that f is locally a difference of two convex functions. In [179], Chitashvili and Mania describe the functions $f(t,x)$ such that $(f(t,B_t), t \geq 0)$ is a semi-martingale. See also Chitashvili and Mania [178], Çinlar et al. [189], Föllmer et al. [349], Kunita [546] and Wang [834] for different generalizations of Itô's formula.

Exercise 4.1.5.5 Scaling Properties of the Local Time. Prove that for any $\lambda > 0$,

$$(L^x_{\lambda^2 t};\, x,t \geq 0) \stackrel{\text{law}}{=} (\lambda L^{x/\lambda}_t;\, x,t \geq 0)\,.$$

In particular, the following equality in law holds true

$$(L^0_{\lambda^2 t}, t \geq 0) \stackrel{\text{law}}{=} (\lambda L^0_t, t \geq 0)\,.$$

◁

Exercise 4.1.5.6 Let $\tau_\ell = \inf\{t > 0 \,:\, L^0_t > \ell\}$. Prove that

$$\mathbb{P}(\forall \ell \geq 0, B_{\tau_\ell} = B_{\tau_{\ell-}} = 0) = 1\,.$$

◁

Exercise 4.1.5.7 Let $dS_t = S_t(r(t)dt + \sigma dW_t)$ where r is a deterministic function and let h be a convex function satisfying $xh'(x) - h(x) \geq 0$. Prove that $\exp(-\int_0^t r(s)ds)\, h(S_t) = R_t h(S_t)$ is a local sub-martingale.
`Hint:` Apply the Itô-Tanaka formula to obtain that

$$\begin{aligned} R(t)h(S_t) = h(x) &+ \int_0^t R(u)r(u)(S_u h'(S_u) - h(S_u))du \\ &+ \frac{1}{2}\int h''(da)\int_0^t R(s)d_sL^a_s + \text{ loc. mart.}. \end{aligned}$$

◁

4.1.6 The Balayage Formula

We now give some other applications of the MCT to stochastic integration, thus obtaining another kind of extension of Itô's formula.

Proposition 4.1.6.1 (Balayage Formula.) *Let Y be a continuous semi-martingale and define*

$$g_t = \sup\{s \leq t : Y_s = 0\},$$

with the convention $\sup\{\emptyset\} = 0$. *Then*

$$h_{g_t} Y_t = h_0 Y_0 + \int_0^t h_{g_s} dY_s$$

for every predictable, locally bounded process h.

PROOF: By the MCT, it is enough to show this formula for processes of the form $h_u = \mathbf{1}_{[0,\tau]}(u)$, where τ is a stopping time. In this case,

$$h_{g_t} = \mathbb{1}_{\{g_t \le \tau\}} = \mathbb{1}_{\{t \le d_\tau\}} \quad \text{where} \quad d_\tau = \inf\{s \ge \tau : Y_s = 0\}.$$

Hence,

$$h_{g_t} Y_t = \mathbb{1}_{\{t \le d_\tau\}} Y_t = Y_{t \wedge d_\tau} = Y_0 + \int_0^t \mathbb{1}_{\{s \le d_\tau\}} dY_s = h_0 Y_0 + \int_0^t h_{g_s} dY_s.$$

□

Let $Y_t = B_t$, then from the balayage formula we obtain that

$$h_{g_t} B_t = \int_0^t h_{g_s} dB_s$$

is a local martingale with increasing process $\int_0^t h_{g_s}^2 ds$.

Exercise 4.1.6.2 Let $\varphi : \mathbb{R}^+ \to \mathbb{R}$ be a locally bounded real-valued function, and L the local time of the Brownian motion at level 0. Prove that $(\varphi(L_t) B_t, t \ge 0)$ is a Brownian motion time changed by $\int_0^t \varphi^2(L_s) ds$.
`Hint:` Note that for $h_s = \varphi(L_s)$, one has $h_s = h_{g_s}$, then use the balayage formula. Note also that one could prove the result first for $\varphi \in C^1$ and then pass to the limit. ◁

4.1.7 Skorokhod's Reflection Lemma

The following real variable lemma will allow us in particular to view local times as supremum processes.

Lemma 4.1.7.1 *Let* y *be a continuous function. There is a unique pair of functions* (z, k) *such that*

(i) $k(0) = 0$, *k is an increasing continuous function*
(ii) $z(t) = -y(t) + k(t) \ge 0$
(iii) $\int_0^t \mathbb{1}_{\{z(s)>0\}} dk(s) = 0$,

This pair is given by

$$k^*(t) = \sup_{0<s\le t} (y(s)) \vee 0, \ z^*(t) = -y(t) + k^*(t).$$

PROOF: The pair $k^*(t) = \sup_{0<s\leq t}(y(s)) \vee 0, z^*(t) = -y(t) + k^*(t)$ satisfies the required properties. Let us prove that the solution is unique. Let (z_1, k_1) and (z_2, k_2) be two pairs of solutions. Then, since $z_1 - z_2$ has bounded variation, from the integration by parts formula,

$$0 \leq (z_1 - z_2)^2(t) = 2\int_0^t (z_1(s) - z_2(s))\, d(k_1(s) - k_2(s))\,.$$

From (iii), the right-hand side of the above equality is equal to

$$-2\int_0^t z_2(s)\, dk_1(s) - 2\int_0^t z_1(s)\, dk_2(s)$$

which is negative. Hence, $z_1 = z_2$. □

Note that, if y is increasing, then $z = 0$. We now give an important consequence of the Skorokhod lemma:

Theorem 4.1.7.2 (Lévy's Equivalence Theorem.) *Let B be a Brownian motion starting at 0, L its local time at level 0 and $M_t = \sup_{s\leq t} B_s$. The two-dimensional processes $(|B|, L)$ and $(M - B, M)$ have the same law, i.e.,*

$$(|B_t|, L_t\,;\, t \geq 0) \stackrel{\text{law}}{=} (M_t - B_t, M_t\,;\, t \geq 0)\,.$$

PROOF: Tanaka's formula implies that

$$|B_t| = \int_0^t \text{sgn}(B_s)dB_s + L_t^0$$

where L^0, the local time of B, is an increasing process. Therefore, $(|B|, L^0)$ is a solution of Skorokhod's lemma associated with the Brownian motion $\beta_t = -\int_0^t \text{sgn}(B_s)dB_s$. Hence, $L_t^0 = \sup_{s\leq t} \beta_s$. By denoting $M_t = \sup_{s\leq t} B_s$, we obtain the decompositions

$$|B_t| = -\beta_t + L_t^0$$
$$M_t - B_t = (-B_t) + M_t\,.$$

The pair $(M - B, M)$ is a solution to Skorokhod's lemma associated with the Brownian motion B, because M increases only on the set $M - B = 0$. Hence, the processes $(|B|, L)$ and $(M - B, M)$ have the same law. □

Comments 4.1.7.3 (a) We have proved, in Proposition 3.1.3.1 that, for any fixed t, $M_t \stackrel{\text{law}}{=} |B_t|$. Here, we obtain that the processes $M - B$ and $|B|$ have the same law. In particular, for fixed t, $M_t - B_t \stackrel{\text{law}}{=} |B_t|$. We also have $M_t \stackrel{\text{law}}{=} L_t$.

(b) As a consequence of Skorokhod's lemma, if $\beta_t = \int_0^t \operatorname{sgn}(B_s)dB_s$, then it is easily shown that $\sigma(\beta_s, s \le t) = \sigma(|B_s|, s \le t)$. See ↣ Subsection 5.8.2 for comments. This may be contrasted with the equality obtained in 3.1.4.3:

$$\sigma(M_s - B_s, s \le t) = \sigma(B_s, s \le t)\,.$$

(c) There are various identities in law involving the BM and its maximum process. From Lévy's theorem, one obtains that

$$(|B_t| + L_t;\, t \ge 0) \stackrel{\text{law}}{=} (2M_t - B_t;\, t \ge 0)\,.$$

From ↣ Exercise 4.1.7.12, we obtain that, for every t, $2M_t - B_t \stackrel{\text{law}}{=} R_t$ where R is a BES^3 process (see ↣ Chapter 6 if needed). Pitman [712] has extended this result at the level of processes, proving that

$$(2M_t - B_t, M_t;\, t \ge 0) \stackrel{\text{law}}{=} (R_t, J_t;\, t \ge 0)$$

where R is a BES^3 process and $J_t = \inf_{s \ge t} R_s$ (see ↣ Section 5.7). Hence, it also holds that

$$(|B_t| + L_t, L_t;\, t \ge 0) \stackrel{\text{law}}{=} (R_t, J_t;\, t \ge 0)\,.$$

We now present further consequences of Lévy's theorem:

Example 4.1.7.4 Let $(\tau_\ell, \ell \ge 0)$ be the inverse of the local time $(L_t^0, t \ge 0)$ defined as $\tau_\ell = \inf\{t : L_t^0 > \ell\}$, and let T_x be the first hitting time of x. Then $(T_x, x \ge 0) \stackrel{\text{law}}{=} (\tau_x, x \ge 0)$. Indeed, from Lévy's equivalence Theorem 4.1.7.2 $(M_t, t \ge 0) \stackrel{\text{law}}{=} (L_t, t \ge 0)$. Hence the same equality holds for the inverse processes. As a consequence, we note that

$$(L_t^x,\, t \ge 0) \stackrel{\text{law}}{=} \left((L_t^0 - |x|)^+,\, t \ge 0\right)\,.$$

Indeed, on the one hand

$$(L_t^x,\, t \ge 0) = (L^x_{T_x + (t - T_x)^+},\, t \ge 0) \stackrel{\text{law}}{=} (L^0_{(t-T_x)^+},\, t \ge 0)$$

where L^0 and T_x are independent. On the other hand

$$\left((L^0_{\tau_\ell + (t-\tau_\ell)^+} - \ell)^+,\, t \ge 0\right) \stackrel{\text{law}}{=} (L^0_{(t-\widehat{\tau}_\ell)^+},\, t \ge 0)$$

where $\widehat{\tau}_\ell$ is independent of $(L_t^0,\, t \ge 0)$. To conclude, we use $\widehat{\tau}_\ell \stackrel{\text{law}}{=} T_\ell$, and take $\ell = |x|$.

Example 4.1.7.5 For fixed t, let θ_t be the first time at which the Brownian motion reaches its maximum over the time interval $[0,t]$:

$$\theta_t := \inf\{s \le t \,|\, B_s = M_t\} = \inf\{s \le t \,|\, B_s = \sup_{u\le t} B_u\}\,.$$

If $t = 1$, we obtain

$$\begin{aligned}(\theta_1 \le u) &= \left\{\sup_{u\le s\le 1} B_s \le \sup_{s\le u} B_s\right\}\\ &= \left\{\sup_{u\le s\le 1} (B_s - B_u) + B_u \le \sup_{s\le u} B_s\right\} = \left\{\sup_{0\le v\le 1-u} \widehat{B}_v + B_u \le M_u\right\},\end{aligned}$$

where $\widehat{B}$ is a BM independent of $(B_s, s \le u)$. Setting $\widehat{M}_u = \sup_{s\le u} \widehat{B}_s$, we get from Lévy's Theorem 4.1.7.2 and Proposition 3.1.3.1

$$\begin{aligned}\mathbb{P}(\theta_1 \le u) &= \mathbb{P}(\widehat{M}_{1-u} \le M_u - B_u) = \mathbb{P}(|\widehat{B}_{1-u}| \le |B_u|)\\ &= \mathbb{P}(\sqrt{1-u}|\widehat{B}_1| \le \sqrt{u}|B_1|) = \mathbb{P}\left(\frac{|B_1|}{|\widehat{B}_1|} \ge \frac{\sqrt{1-u}}{\sqrt{u}}\right)\\ &= \mathbb{P}\left(C^2 \ge \frac{1-u}{u}\right) = \mathbb{P}\left(u \ge \frac{1}{1+C^2}\right)\end{aligned}$$

where C follows the standard Cauchy law (see ↣ Appendix A.4.2). Hence, for $u \le 1$,

$$\mathbb{P}(\theta_1 \le u) = \frac{2}{\pi}\text{arc sin}\sqrt{u}\,.$$

Finally, by scaling, for $s \le t$,

$$\mathbb{P}(\theta_t \le s) = \frac{2}{\pi}\text{arc sin}\sqrt{\frac{s}{t}}\,,$$

therefore, θ_t is Arcsine distributed on $[0,t]$. Note the non-trivial identity in law $\theta_t \stackrel{\text{law}}{=} A_t^+$ where $A_t^+ = \int_0^t \mathbb{1}_{\{B_s>0\}} ds$ (see Subsection 2.5.2). As a direct application of Lévy's equivalence theorem, we obtain

$$\theta_t \stackrel{\text{law}}{=} g_t = \sup\{s \le t \,:\, B_s = 0\}\,.$$

Proceeding along the same lines, we obtain the equality

$$\mathbb{P}(M_t \in dx, \theta_t \in du) = \frac{x}{\pi u\sqrt{u(t-u)}} \exp\left(-\frac{x^2}{2u}\right) \mathbb{1}_{\{0\le x, 0\le u\le t\}}\, du\, dx \tag{4.1.10}$$

and from the previous equalities and using the Markov property

$$\begin{aligned}\mathbb{P}(\theta_1 \le u|\mathcal{F}_u) &= \mathbb{P}(\sup_{u\le s\le 1} (B_s - B_u) + B_u \le \sup_{s\le u} B_s\,|\mathcal{F}_u)\\ &= \mathbb{P}(\widehat{M}_{1-u} \le M_u - B_u|\mathcal{F}_u) = \Psi(1-u, M_u - B_u)\,.\end{aligned}$$

Here,

$$\Psi(u,x) = \mathbb{P}(\widehat{M}_u \le x) = \mathbb{P}(|B_u| \le x) = \frac{2}{\sqrt{2\pi}} \int_0^{x/\sqrt{u}} \exp\left(-\frac{y^2}{2}\right) dy\,.$$

Note that, for $x > 0$, the density of M_t at x can also be obtained from the equality (4.1.10). Hence, we have the equality

$$\int_0^t du \frac{x}{\pi\sqrt{u^3(t-u)}} \exp\left(-\frac{x^2}{2u}\right) = \sqrt{\frac{2}{\pi t}} e^{-x^2/(2t)}\,. \tag{4.1.11}$$

We also deduce from Lévy's theorem that the right-hand side of (4.1.10) is equal to $\mathbb{P}(L_t \in dx, g_t \in du)$.

Example 4.1.7.6 From Lévy's identity, it is straightforward to obtain that $\mathbb{P}(L^a_\infty = \infty) = 1$.

Example 4.1.7.7 As discussed in Pitman [713], the law of the pair (L^x_1, B_1) may be obtained from Lévy's identity: for $y > 0$,

$$\mathbb{P}(L^x_1 \in dy, B_1 \in db) = \frac{|x| + y + |b-x|}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(|x| + y + |b-x|)^2\right) dydb\,.$$

Proposition 4.1.7.8 *Let φ be a C^1 function. Then, the process*

$$\boxed{\varphi(M_t) - (M_t - B_t)\varphi'(M_t)}$$

is a local martingale.

PROOF: As a first step we assume that φ is C^2. Then, from integration by parts and using the fact that M is increasing

$$(M_t - B_t)\varphi'(M_t) = \int_0^t \varphi'(M_s)\, d(M_s - B_s) + \int_0^t (M_s - B_s)\varphi''(M_s) dM_s\,.$$

Now, we note that $\int_0^t (M_s - B_s)\varphi''(M_s)dM_s = 0$, since dM_s is carried by $\{s : M_s = B_s\}$, and that $\int_0^t \varphi'(M_s)dM_s = \varphi(M_t) - \varphi(0)$. The result follows. The general case is obtained using the MCT. □

Comment 4.1.7.9 As we mentioned in Example 1.5.4.5, any solution of Tanaka's SDE $X_t = X_0 + \int_0^t \mathrm{sgn}(X_s)dB_s$ is a Brownian motion. We can check that there are indeed weak solutions to this equation: start with a Brownian motion X and construct the BM $B_t = \int_0^t \mathrm{sgn}(X_s)dX_s$. This Brownian motion is equal to $|X| - L$, so B is adapted to the filtration generated by $|X|$ which is strictly smaller than the filtration generated by X. Hence, the equation $X_t = X_0 + \int_0^t \mathrm{sgn}(X_s)dB_s$ has no strong solution. Moreover, one can find infinitely many solutions, e.g., $\epsilon_{g_t} X_t$, where ϵ is a ± 1-valued predictable process, and $g_t = \sup\{s \le t : X_s = 0\}$.

Exercise 4.1.7.10 Prove Proposition 4.1.7.8 as a consequence of the balayage formula applied to $Y_t = M_t - B_t$. ◁

Exercise 4.1.7.11 Using the balayage formula, extend the result of Proposition 4.1.7.8 when φ' is replaced by a bounded Borel function. ◁

Exercise 4.1.7.12 Prove, using Theorem 3.1.1.2, that the joint law of the pair $(|B_t|, L_t^0)$ is

$$\mathbb{P}(|B_t| \in dx,\, L_t^0 \in d\ell) = \mathbb{1}_{\{x\geq 0\}}\mathbb{1}_{\{\ell\geq 0\}}\frac{2(x+\ell)}{\sqrt{2\pi t^3}}\exp\left(-\frac{(x+\ell)^2}{2t}\right)dx\,d\ell\,.$$

◁

Exercise 4.1.7.13 Let φ be in C_b^1. Prove that $(\varphi(L_t^0) - |B_t|\varphi'(L_t^0), t \geq 0)$ is a martingale. Let $T_a^* = \inf\{t \geq 0 : |B_t| = a\}$. Prove that $L_{T_a^*}^0$ follows the exponential law with parameter $1/a$.
`Hint:` Use Proposition 4.1.7.8 together with Lévy's Theorem. Then, compute the Laplace transform of $L_{T_a^*}^0$ by means of the optional stopping theorem. The second part may also be obtained as a particular case of ↣ Azéma's lemma 5.2.2.5. ◁

Exercise 4.1.7.14 Let y be a continuous positive function vanishing at 0: $y(0) = 0$. Prove that there exists a unique pair of functions (z, k) such that

(i) $k(0) = 0$, where k is an increasing continuous function
(ii) $z(t) + k(t) = y(t), z(t) \geq 0$
(iii) $\int_0^t \mathbb{1}_{\{z(s)>0\}}dk(s) = 0$
(iv) $\forall t, \exists d(t) \geq t,\ z(d(t)) = 0$

`Hint:` $k^*(t) = \inf_{s\geq t}(y(s))$. ◁

Exercise 4.1.7.15 Let S be a price process, assumed to be a continuous local martingale, and φ a C^1 concave, increasing function. Denote by S^* the running maximum of S. Prove that the process $X_t = \varphi(S_t^*) + \varphi'(S_t^*)(S_t - S_t^*)$ is the value of the self-financing strategy with a risky investment given by $S_t\varphi'(S_t^*)$, which satisfies the floor constraint $X_t \geq \varphi(S_t)$.
`Hint:` Using an extension of Proposition 4.1.7.8, X is a local martingale. It is easy to check that $X_t = X_0 + \int_0^t \varphi'(S_s^*)dS_s$. For an intensive study of this process in finance, see El Karoui and Meziou [305]. The equality $X_t \geq \varphi(S_t)$ follows from concavity of φ. ◁

4.1.8 Local Time of a Semi-martingale

As mentioned above, local times can also be defined in greater generality for semi-martingales. The same approach as the one used in Subsection 4.1.2 leads to the following:

Theorem 4.1.8.1 (Occupation Time Formula.) *Let X be a continuous semi-martingale. There exists a family of increasing processes (**Tanaka-Meyer local times**) $(L_t^x(X), t \geq 0 \,;\, x \in \mathbb{R})$ such that for every bounded measurable function φ*

$$\int_0^t \varphi(X_s)\, d\langle X\rangle_s = \int_{-\infty}^{+\infty} L_t^x(X)\varphi(x)\, dx\,. \tag{4.1.12}$$

There is a version of L_t^x which is jointly continuous in t and right-continuous with left limits in x. (If X is a continuous martingale, its local time may be chosen jointly continuous.) In the sequel, we always choose this version. This local time satisfies

$$L_t^x(X) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_0^t \mathbb{1}_{[x,x+\epsilon[}(X_s) d\langle X\rangle_s\,.$$

If Z is a continuous local martingale,

$$L_t^x(Z) = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_0^t \mathbb{1}_{]x-\epsilon,x+\epsilon[}(Z_s) d\langle Z\rangle_s\,.$$

The same result holds with any random time in place of t.

For a continuous semi-martingale $X = Z + A$,

$$L_t^x(X) - L_t^{x-}(X) = 2\int_0^t \mathbb{1}_{\{X_s=x\}} dX_s = 2\int_0^t \mathbb{1}_{\{X_s=x\}} dA_s\,. \tag{4.1.13}$$

In particular,

$$L_t^0(|X|) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_0^t \mathbb{1}_{]-\epsilon,\epsilon[}(X_s) d\langle X\rangle_s = L_t^0(X) + L_t^{0-}(X),$$

hence

$$L_t^0(|X|) = 2L_t^0(X) - 2\int_0^t \mathbb{1}_{\{X_s=0\}} dA_s\,.$$

Example 4.1.8.2 A Non-Continuous Local Time. Let Z be a continuous martingale and X be the semi-martingale

$$X_t = aZ_t^+ - bZ_t^- = \int_0^t dZ_s (a\mathbb{1}_{\{Z_s>0\}} + b\mathbb{1}_{\{Z_s<0\}}) + \frac{a-b}{2} L_t^0(Z)\,.$$

Then, it follows from (4.1.13) that $L_t^0(X) - L_t^{0-}(X) = (a-b)L_t^0(Z)$. In particular, for the reflected BM, i.e., for X when $Z_t = B_t, a = 1, b = -1$, we get $L^0(|B|) - L^{0-}(|B|) = 2L^0(B)$. Note that $L^{0-}(|B|) = 0$, hence $L^0(|B|) = 2L^0(B)$.

Example 4.1.8.3 Let $Y_t = |B_t|$. Tanaka's formula gives:

$$|B_t| = \int_0^t \text{sgn}(B_s) dB_s + L_t$$

where $(L_t, t \geq 0)$ denotes the local time of $(B_t; t \geq 0)$ at $y = 0$. By an application of the balayage formula, we obtain

$$h_{g_t} |B_t| = \int_0^t h_{g_s} \text{sgn}(B_s) dB_s + \int_0^t h_s dL_s$$

having used the fact that $L_{g_s} = L_s$. Consequently, replacing, if necessary, h by $|h|$, we see that the process $\int_0^t |h_s| dL_s$ is the local time at 0 of $(h_{g_t} B_t, t \geq 0)$.

Tanaka-Meyer Formulae

As before we set $\text{sgn}(x) = 1$ for $x > 0$ and $\text{sgn}(x) = -1$ for $x \leq 0$. Let X be a continuous semi-martingale. For every (t, x),

$$|X_t - x| = |X_0 - x| + \int_0^t \text{sgn}\,(X_s - x)\, dX_s + L_t^x(X)\,, \tag{4.1.14}$$

$$(X_t - x)^+ = (X_0 - x)^+ + \int_0^t \mathbb{1}_{\{X_s > x\}}\, dX_s + \frac{1}{2} L_t^x(X)\,, \tag{4.1.15}$$

$$(X_t - x)^- = (X_0 - x)^- - \int_0^t \mathbb{1}_{\{X_s \leq x\}}\, dX_s + \frac{1}{2} L_t^x(X)\,. \tag{4.1.16}$$

In particular, $|X - x|, (X - x)^+$ and $(X - x)^-$ are semi-martingales.

Proposition 4.1.8.4 (Lévy's Equivalence Theorem for Drifted Brownian Motion.) *Let $B^{(\nu)}$ be a BM with drift ν, i.e., $B_t^{(\nu)} = B_t + \nu t$, and $M_t^{(\nu)} = \sup_{s \leq t} B_s^{(\nu)}$. Then*

$$(M_t^{(\nu)} - B_t^{(\nu)}, M_t^{(\nu)}\,; t \geq 0) \stackrel{\text{law}}{=} (|X_t^{(\nu)}|, L_t(X^{(\nu)})\,; t \geq 0) \tag{4.1.17}$$

where $X^{(\nu)}$ is the (unique) strong solution of

$$dX_t = dB_t - \nu\, \text{sgn}(X_t)\, dt, X_0 = 0\,.$$

Proof: Let $X^{(\nu)}$ be the strong solution of

$$dX_t = dB_t - \nu \operatorname{sgn}(X_t)\, dt, X_0 = 0$$

(see Theorem 1.5.5.1 for the existence of X) and apply Tanaka's formula. Then,

$$|X_t^{(\nu)}| = \int_0^t \operatorname{sgn}(X_s^{(\nu)}) \left(dB_s - \nu \operatorname{sgn}(X_s^{(\nu)})\, ds\right) + L_t^0(X^{(\nu)})$$

where $L^0(X^{(\nu)})$ is the Tanaka-Meyer local time of $X^{(\nu)}$ at level 0. Hence, setting $\beta_t = \int_0^t \operatorname{sgn} X_s^{(\nu)}\, dB_s$,

$$|X_t^{(\nu)}| = (\beta_t - \nu t) + L_t^0(X^{(\nu)})$$

and the result follows from Skorokhod's lemma. □

Comments 4.1.8.5 (a) Note that the processes $|B^{(\nu)}|$ and $M^{(\nu)} - B^{(\nu)}$ do not have the same law (hence the right-hand side of (4.1.17) cannot be replaced by $(|B_t^{(\nu)}|, L_t(B^{(\nu)}), t \geq 0))$. Indeed, for $\nu > 0$, $B_t^{(\nu)}$ goes to infinity as t goes to ∞, whereas $M_t^{(\nu)} - B_t^{(\nu)}$ vanishes for some arbitrarily large values of t. Pitman and Rogers [714] extended the result of Pitman [712] and proved that

$$(|B_t^{(\nu)}| + L_t^{(\nu)},\ t \geq 0) \stackrel{\text{law}}{=} (2M_t^{(\nu)} - B_t^{(\nu)},\ t \geq 0)\,.$$

(b) The equality in law of Proposition 4.1.8.4 admits an extension to the case $dB_t^{(a)} = a_t(B_t^{(a)})dt + dB_t$ and $X^{(a)}$ the unique weak solution of

$$dX_t^{(a)} = dB_t - a_t(X_t^{(a)}) \operatorname{sgn}(X_t^{(a)})dt, X_0^{(a)} = 0$$

where $a_t(x)$ is a bounded predictable family. The equality

$$(M^{(a)} - B^{(a)}, M^{(a)}) \stackrel{\text{law}}{=} (|X^{(a)}|, L(X^{(a)}))$$

is proved in Shiryaev and Cherny [792].

We discuss here the Itô-Tanaka formula for strict local continuous martingales, as it is given in Madan and Yor [614].

Theorem 4.1.8.6 *Let S be a positive continuous strict local martingale, τ an $\mathbf{F}^S$-stopping time, a.s. finite, and K a positive real number. Then*

$$\mathbb{E}((S_\tau - K)^+) = (S_0 - K)^+ + \frac{1}{2}\mathbb{E}(L_\tau^K) - \mathbb{E}(S_0 - S_\tau)$$

where L^K is the local time of S at level K.

PROOF: We prove that

$$M_t = \frac{1}{2}L_t^K - (S_t - K)^+ + S_t = \frac{1}{2}L_t^K + (S_t \wedge K)$$

is a uniformly integrable martingale. In a first step, from Tanaka's formula, M is a (positive) local martingale, hence a super-martingale and $\mathbb{E}(L_t^K) \leq 2S_0$. Since L is an increasing process, it follows that $\mathbb{E}(L_\infty^K) \leq 2S_0$ and the process M is a uniformly integrable martingale. We then apply the optimal stopping theorem at time τ. □

Comment 4.1.8.7 It is important to see that, if the discounted price process is a martingale under the e.m.m., then the put-call parity holds: indeed, taking expectation of discounted values of $(S_T - K)^+ = S_T - K + (K - S_T)^+$ leads to $C(x,T) = x - Ke^{-rT} + P(x,T)$. This is no more the case if discounted prices are strict local martingales. See Madan and Yor [614], Cox and Hobson [203], Pal and Protter [692].

4.1.9 Generalized Itô-Tanaka Formula

Theorem 4.1.9.1 *Let X be a continuous semi-martingale, f a convex function, $D_- f$ its left derivative and $f''(dx)$ its second derivative in the distribution sense. Then,*

$$f(X_t) = f(X_0) + \int_0^t D_- f(X_s)dX_s + \frac{1}{2}\int_{\mathbb{R}} L_t^x(X)f''(dx)$$

holds.

Corollary 4.1.9.2 *Let X be a continuous semi-martingale, f a C^1 function and assume that there exists a measurable function h, integrable on any finite interval $[-a,a]$ such that $f'(y) - f'(x) = \int_x^y h(z)dz$. Then, Itô's formula*

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)dX_s + \frac{1}{2}\int_0^t h(X_s)d\langle X\rangle_s$$

holds.

PROOF: In this case, f is locally the difference of two convex functions and $f''(dx) = h(x)dx$. Indeed, for every $\varphi \in C_b^\infty$,

$$\langle f'', \varphi\rangle \;=\; -\langle f', \varphi'\rangle \;= -\int dx f'(x)\varphi'(x) = \int dz h(z)\varphi(z)\,.$$

□

In particular, if f is a C^1 function, which is C^2 on $\mathbb{R}\setminus\{a_1, \ldots, a_n\}$, for a finite number of points (a_i), then

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)dX_s + \frac{1}{2}\int_0^t g(X_s)d\langle X^c\rangle_s \,.$$

Here $\mu(dx) = g(x)dx$ is the second derivative of f in the distribution sense and X^c the continuous martingale part of X (see ↣ Subsection 9.3.3).

Exercise 4.1.9.3 Let X be a semi-martingale such that $d\langle X\rangle_t = \sigma^2(t, X_t)dt$. Assuming that the law of the r.v. X_t admits a density $\varphi(t,x)$, prove that, under some regularity assumptions,

$$\mathbb{E}(d_t L_t^x) = \varphi(t,x)\sigma^2(t,x)dt \,. \qquad \triangleleft$$

4.2 Applications

4.2.1 Dupire's Formula

In a general stochastic volatility model, with

$$dS_t = S_t\left(\alpha(t,S_t)dt + \sigma_t dB_t\right),$$

it follows that $\langle S\rangle_t = \int_0^t S_u^2\sigma_u^2 du$, therefore

$$\sigma_u^2 = \frac{d}{du}\left(\int_0^u \frac{d\langle S\rangle_s}{S_s^2}\right)$$

is $\mathbf{F}^S$-adapted. However, despite the fact that this process (the square of the volatility) is, from a mathematical point of view, adapted to the filtration of prices, it is not directly observed on the markets, due to the lack of information on prices. See ↣ Section 6.7 for some examples of stochastic volatility models. In that general setting, the volatility is a functional of prices.

Under the main assumption that the volatility is a function of time and of the current value of the underlying asset, i.e., that the underlying process follows the dynamics

$$dS_t = S_t\left(\alpha(t,S_t)dt + \sigma(t,S_t)dB_t\right)\,,$$

Dupire [283, 284] and Derman and Kani [250] give a relation between the volatility and the price of European calls. The function $\sigma^2(t,x)$, called the **local volatility**, is a crucial parameter for pricing and hedging derivatives.

We recall that the **implied volatility** is the value of σ such that the price of a call is equal to the value obtained by applying the Black and Scholes formula. The interested reader can also refer to Berestycki et al. [73] where a link between local volatility and implied volatility is given. The authors also propose a calibration procedure to reconstruct a local volatility.

Proposition 4.2.1.1 (Dupire Formula.) *Assume that the European call prices $C(K,T) = \mathbb{E}(e^{-rT}(S_T - K)^+)$ for any maturity T and any strike K*

are known. If, under the risk-neutral probability, the stock price dynamics are given by

$$dS_t = S_t \left(r dt + \sigma(t, S_t) dW_t \right) \tag{4.2.1}$$

where σ *is a deterministic function, then*

$$\boxed{\frac{1}{2} K^2 \sigma^2(T, K) = \frac{\partial_T C(K,T) + rK \partial_K C(K,T)}{\partial^2_{KK} C(K,T)}}$$

where ∂_T *(resp.* ∂_K*) is the partial derivative operator with respect to the maturity (resp. the strike).*

PROOF: (a) We note that, differentiating with respect to K the equality $e^{-rT}\mathbb{E}((S_T - K)^+) = C(K,T)$, we obtain

$$\partial_K C(K,T) = -e^{-rT} \mathbb{P}(S_T > K)$$

and that, assuming the existence of a density $\varphi(T, x)$ of S_T,

$$\varphi(T, K) = e^{rT} \partial_{KK} C(K,T) \,.$$

(b) We now follow Leblanc [572] who uses the local time technology, whereas the original proof of Dupire (see ↣ Subsection 5.4.2) does not. Tanaka's formula applied to the semi-martingale S gives

$$(S_T - K)^+ = (S_0 - K)^+ + \int_0^T \mathbb{1}_{\{S_s > K\}} dS_s + \frac{1}{2} \int_0^T dL^K_s(S) \,.$$

Therefore, using integration by parts

$$e^{-rT}(S_T - K)^+ = (S_0 - K)^+ - r \int_0^T e^{-rs} (S_s - K)^+ ds$$
$$+ \int_0^T e^{-rs} \mathbb{1}_{\{S_s > K\}} dS_s + \frac{1}{2} \int_0^T e^{-rs} dL^K_s(S) \,.$$

Taking expectations, for every pair (K, T),

$$\begin{aligned} C(K,T) &= \mathbb{E}(e^{-rT}(S_T - K)^+) \\ &= (S_0 - K)^+ + \mathbb{E}\left(\int_0^T e^{-rs} r S_s \mathbb{1}_{\{S_s > K\}} ds \right) \\ &\quad - r\mathbb{E}\left(\int_0^T e^{-rs} (S_s - K) \mathbb{1}_{\{S_s > K\}} ds \right) + \frac{1}{2} \mathbb{E}\left(\int_0^T e^{-rs} dL^K_s(S) \right) . \end{aligned}$$

From the definition of the local time,

$$\mathbb{E}\left(\int_0^T e^{-rs} dL_s^K(S)\right) = \int_0^T e^{-rs}\varphi(s,K)K^2\sigma^2(s,K)ds$$

where $\varphi(s,\cdot)$ is the density of the r.v. S_s (see Exercise 4.1.9.3). Therefore,

$$C(K,T) = (S_0 - K)^+ + rK\int_0^T e^{-rs}\mathbb{P}(S_s > K)\, ds \\ + \frac{1}{2}\int_0^T e^{-rs}\varphi(s,K)K^2\sigma^2(s,K)ds\,.$$

Then, by differentiating w.r.t. T, one obtains

$$\partial_T C(K,T) = rKe^{-rT}\mathbb{P}(S_T > K) + \frac{1}{2}e^{-rT}\varphi(T,K)K^2\sigma^2(T,K)\,. \quad (4.2.2)$$

(c) We now use the result found in (a) to write (4.2.2) as

$$\partial_T C(K,T) = -rK\partial_K C(K,T) + \frac{1}{2}\sigma^2(T,K)K^2\partial_{KK}C(K,T)$$

which is the required result. □

Comments 4.2.1.2 (a) Atlan [25] presents examples of stochastic volatility models where a local volatility can be computed.

(b) Dupire result is deeply linked with Gyöngy's theorem [414] which studies processes with given marginals. See also Brunich [133] and Hirsch and Yor [438, 439].

4.2.2 Stop-Loss Strategy

This strategy is also said to be the "all or nothing" strategy. A strategic allocation (a reference portfolio) with value V_t is given in the market. The investor would like to build a strategy, based on V, such that the value of the investment is greater than a benchmark, equal to $KP(t,T)$ where K is a constant and $P(t,T)$ is the price at time t of a zero-coupon with maturity T. We assume, w.l.g., that the initial value of V is greater than $KP(0,T)$. The stop-loss strategy relies upon the following argument: the investor takes a long position in the strategic allocation.

The first time when $V_t \leq KP(t,T)$ the investor invests his total wealth of the portfolio to buy K zero-coupon bonds. When the situation is reversed, the orders are inverted and all the wealth is invested in the strategic allocation. Hence, at maturity, the wealth is $\max(V_T, K)$. See Andreasen et al. [18], Carr and Jarrow [153] and Sondermann [804] for comments.

The well-known drawback of this method is that it cannot be applied in practice when the price of the risky asset fluctuates around the floor

$G_t = KP(t,T)$, because of transaction costs. Moreover, even in the case of constant interest rate, the strategy is not self-financing. Indeed, the value of this strategy is greater than $KP(t,T)$. If such a strategy were self-financing, and if there were a stopping time τ such that its value equalled$KP(\tau,T)$, then it would remain equal to $KP(t,T)$ after time τ, and this is obviously not the case. (See Lakner [558] for details.) It may also be noted that the discounted process $e^{-rt}\max(V_t, KP(t,T))$ is not a martingale under the risk-neutral probability measure (and the process $\max(V_t, KP(t,T))$ is not the value of a self-financing strategy). More precisely,

$$e^{-rt}\max(V_t, KP(t,T)) \overset{\text{mart}}{=} L_t$$

where L is the local time of $(V_t e^{-rt}, t \geq 0)$ at the level Ke^{-rT}.

Sometimes, practitioners introduce a corridor around the floor and change the strategy only when the asset price is outside this corridor. More precisely, the value of the portfolio is

$$V_t \mathbb{1}_{\{t<T_1\}} + (K-\epsilon)\mathbb{1}_{\{T_1 \leq t < T_2\}} + V_t \mathbb{1}_{\{T_2 \leq t < T_3\}} + \dots$$

where

$$\begin{aligned} &T_1 = \inf\{t : V_t \leq K - \epsilon\}, T_2 = \inf\{t : t > T_1, V_t \geq K + \epsilon\}, \\ &T_3 = \inf\{t : t > T_2, V_t \leq K - \epsilon\} \dots \quad . \end{aligned}$$

The terminal value of the portfolio when the width of the corridor tends to 0 can be shown to converge a.s. to $\max(V_T, K) - L_T^K$, where L_T^K represents the local time of $(V_t, t \in [0,T])$ at level K.

4.2.3 Knock-out BOOST

Let (a,b) be a pair of positive real numbers with $b < a$. The **knock-out BOOST** studied in Leblanc [572] is an option which pays, at maturity, the time that the underlying asset has remained above a level b, until the first time the asset reaches the level a. We assume that the underlying follows a geometric Brownian motion, i.e., $S_t = xe^{\sigma X_t}$ where X is a BM with drift ν. In symbols, the value of this knock-out BOOST option is

$$\text{KOB}(a,b;T) = E_{\mathbb{Q}}\left(e^{-rT}\int_0^{T\wedge T_a} \mathbb{1}_{(S_s>b)}\,ds\right).$$

Let α be the level relative to X, i.e., $\alpha = \dfrac{1}{\sigma}\ln\dfrac{a}{x}$. From the occupation time formula (4.1.1) and the fact that $L^y_{T\wedge T_\alpha}(X) = 0$ for $y > \alpha$, we obtain that, for every function f

$$\mathbf{W}^{(\nu)}\left(\int_0^{T\wedge T_\alpha} f(X_s)\,ds\right) = \int_{-\infty}^{\alpha} f(y)\mathbf{W}^{(\nu)}[L^y_{T_\alpha\wedge T}]dy\,,$$

where, as in the previous chapter $\mathbf{W}^{(\nu)}$ is the law of a drifted Brownian motion (see Section 3.2). Hence, if $\beta = \frac{1}{\sigma}\ln\frac{b}{x}$,

$$\mathrm{KOB}(a,b;T) = \mathbf{W}^{(\nu)}\left(e^{-rT}\int_0^{T\wedge T_\alpha} \mathbb{1}_{\{X_s>\beta\}}\,ds\right) = e^{-rT}\int_\beta^\alpha \Psi_{\alpha,\nu}(y)\,dy$$

where $\Psi_{\alpha,\nu}(y) = \mathbf{W}^{(\nu)}(L^y_{T_\alpha\wedge T})$.

The computation of $\Psi_{\alpha,\nu}$ can be performed using Tanaka's formula. Indeed, for $y<\alpha$, using the occupation time formula,

$$\begin{aligned}\frac{1}{2}\Psi_{\alpha,\nu}(y) &= \mathbf{W}^{(\nu)}[(X_{T_\alpha\wedge T}-y)^+] - (-y)^+ - \nu\mathbf{W}^{(\nu)}\left(\int_0^{T_\alpha\wedge T}\mathbb{1}_{\{X_s>y\}}ds\right)\\ &= \mathbf{W}^{(\nu)}[(X_{T_\alpha\wedge T}-y)^+] - (-y)^+ - \nu\int_y^\alpha \Psi_{\alpha,\nu}(z)dz\\ &= (\alpha-y)^+\mathbf{W}^{(\nu)}(T_\alpha<T) + \mathbf{W}^{(\nu)}\left[(X_T-y)^+\mathbb{1}_{\{T_\alpha>T\}}\right]\\ &\qquad - (-y)^+ - \nu\int_y^\alpha\Psi_{\alpha,\nu}(z)dz\,. \end{aligned} \tag{4.2.3}$$

Let us compute explicitly the expectation of the local time in the case $T=\infty$ and $\alpha\nu>0$. The formula (4.2.3) reads

$$\begin{aligned}&\frac{1}{2}\Psi_{\alpha,\nu}(y) = (\alpha-y)^+ - (-y)^+ - \nu\int_y^\alpha\Psi_{\alpha,\nu}(z)dz,\\ &\Psi_{\alpha,\nu}(\alpha) = 0\,.\end{aligned}$$

This gives

$$\Psi_{\alpha,\nu}(y) = \begin{cases}\dfrac{1}{\nu}(1-\exp(2\nu(y-\alpha))) & \text{for } 0\le y\le\alpha\\[2ex] \dfrac{1}{\nu}(1-\exp(-2\nu\alpha))\,\exp(2\nu y) & \text{for } y\le 0.\end{cases}$$

In the general case, differentiating (4.2.3) with respect to y gives for $y\le\alpha$

$$\begin{aligned}\frac{1}{2}\Psi'_{\alpha,\nu}(y) &= -\mathbf{W}^{(\nu)}(T_\alpha<T) - \mathbf{W}^{(\nu)}(T_\alpha>T, X_T>y) + \mathbb{1}_{\{y<0\}} + \nu\Psi_{\alpha,\nu}(y)\\ &= -1 + \mathbf{W}^{(\nu)}(T_\alpha>T, X_T<y) + \mathbb{1}_{\{y<0\}} + \nu\Psi_{\alpha,\nu}(y)\\ &= -1 + \mathcal{N}\left(\frac{y-\nu T}{\sqrt{T}}\right) - e^{2\nu\alpha}\mathcal{N}\left(\frac{y-2\alpha-\nu t}{\sqrt{T}}\right) + \mathbb{1}_{\{y<0\}} + \nu\Psi_{\alpha,\nu}(y)\,.\end{aligned}$$

It follows that $\Psi_{\alpha,\nu}(y) =$

$$2e^{2\nu y}\int_y^\alpha e^{-2\nu x}\left(-1+\mathcal{N}\left(\frac{x-\nu T}{\sqrt{T}}\right) - e^{2\nu\alpha}\mathcal{N}\left(\frac{x-2\alpha-\nu t}{\sqrt{T}}\right) + \mathbb{1}_{\{x<0\}}\right)dx\,.$$

4.2.4 Passport Options

An interesting application of local time is the study of passport options. We do not present this problem here, mainly because this is related to optimization problems which are beyond the scope of this book. See Delbaen and Yor [239], Henderson [430], Henderson and Hobson [431], Shreve and Večeř [797].

4.3 Bridges, Excursions, and Meanders

Given a process $(X_t, t \geq 0)$, we shall denote by $X^{[a,b]}$, for a pair of random times $0 < a < b$, the scaled process

$$X_t^{[a,b]} = \frac{1}{\sqrt{b-a}} X_{a+t(b-a)},\ 0 \leq t \leq 1. \tag{4.3.1}$$

In what follows, B is a BM starting from 0 with natural filtration $\mathbf{F}$.

4.3.1 Brownian Motion Zeros

Let $\mathcal{Z}(\omega)$ be the random set

$$\mathcal{Z} = \{t \geq 0 \, : \, B_t = 0\} \, .$$

The complementary set $\mathcal{Z}^c$ is open and is therefore a countable union of maximal open intervals. The set $\mathcal{Z}$ does not have isolated points and has zero Lebesgue measure, as a consequence of the occupation density formula (4.1.2) where $A = \{0\}$.

Exercise 4.3.1.1 Let $(\tau_\ell, \ell \geq 0)$ be the inverse of the local time at level 0, defined in Example 4.1.7.4. Prove that, if $u \in \mathcal{Z}$, then $u = \tau_s$ or $u = \tau_{s-}$ for some s.
`Hint:` if $u \in \mathcal{Z}$, either $L_{u+\epsilon} - L_u > 0$ for every ϵ, and $u = \tau_s$ for $s = L_u$, or L is constant and $u = \tau_{s-}$ for $s = L_u$. ◁

4.3.2 Excursions

Let t be a fixed time and let $g_t = \sup\{s \leq t \, : B_s = 0\}$ be the **last passage time** at level 0 before time t and $d_t = \inf\{s \geq t \, : B_s = 0\}$ the **first passage time** at level zero after time t. The **Brownian excursion** which straddles t is the path

$$(B_{g_t+u}\, ; 0 \leq u \leq d_t - g_t) \, .$$

The normalized excursion is taken to be the process $(B_u^{[g_t,d_t]}, 0 \leq u \leq 1)$, or sometimes it is defined as its absolute value.
It is worth noting that g_t is not an $\mathbf{F}$-stopping time, whereas d_t is an $\mathbf{F}$-stopping time.

Let us remark that, for u in the interval (g_t, d_t), the sign of B_u remains constant.

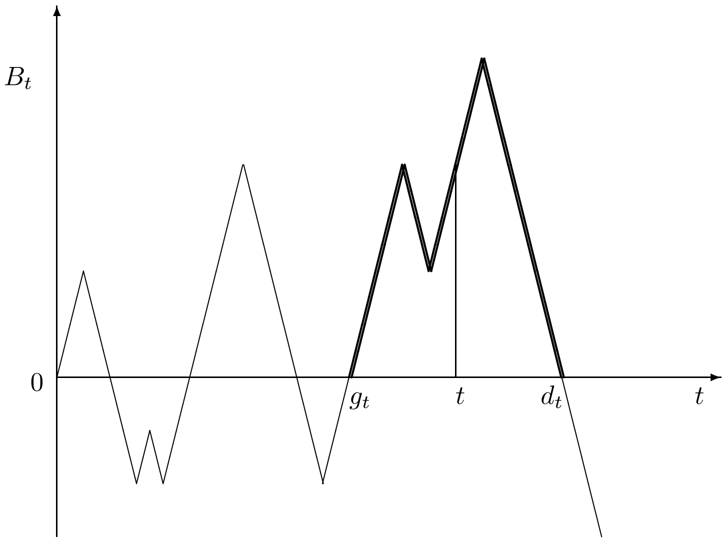

Fig. 4.1 Excursion of a Brownian motion straddling t

4.3.3 Laws of T_x, d_t and g_t

We study here the laws of the random variables T_x, d_t and g_t.

Proposition 4.3.3.1 *Let $T_x = \inf\{t : B_t = x\}$ and $M_t = \sup_{s\leq t} B_s$. Then:*

$$T_x \stackrel{\text{law}}{=} x^2 T_1 \stackrel{\text{law}}{=} \left(\frac{x}{M_1}\right)^2 \stackrel{\text{law}}{=} \left(\frac{x}{B_1}\right)^2 .$$

PROOF: By scaling $T_x \stackrel{\text{law}}{=} x^2 T_1$ and $M_t \stackrel{\text{law}}{=} \sqrt{t} M_1$. Furthermore,

$$\mathbb{P}(T_1 \geq u) = \mathbb{P}(M_u \leq 1) = \mathbb{P}(\sqrt{u} M_1 \leq 1) = \mathbb{P}\left(\left(\frac{1}{M_1}\right)^2 \geq u\right)$$

which implies the remaining equalities, using that $B_1^2 \stackrel{\text{law}}{=} M_1^2$ (see Proposition 3.1.3.1). □

Proposition 4.3.3.2 (i) *The law of d_u is that of $u(1+C^2)$ where C is a Cauchy random variable with density $\frac{1}{\pi}\frac{1}{1+x^2}$.*

(ii) *The variable g_t is Arcsine distributed:*

$$\mathbb{P}(g_t \in ds) = \frac{1}{\pi} \frac{1}{\sqrt{s(t-s)}} \mathbb{1}_{\{s \le t\}} \, ds \,.$$

PROOF: By definition, $d_u = u + \inf\{v \,|\, B_{u+v} - B_u = -B_u\}$. The process $\widehat{B} = (\widehat{B}_t = B_{t+u} - B_u, t \ge 0)$ is a Brownian motion independent of B_u. Let $\widehat{T}_a$ be the first hitting time of a associated with this process $\widehat{B}$. By using results of the previous proposition and the scaling property of Brownian motion, we obtain

$$d_u \stackrel{\text{law}}{=} u + \widehat{T}_{-B_u} \stackrel{\text{law}}{=} u + B_u^2 \widehat{T}_1 \stackrel{\text{law}}{=} u + u B_1^2 \widehat{T}_1 \stackrel{\text{law}}{=} u \left(1 + \frac{B_1^2}{\widehat{B}_1^2} \right)$$

and therefore from the explicit computation of the law of $B_1^2 / \widehat{B}_1^2$ (see ↣ Appendix A.4.2)

$$d_u \stackrel{\text{law}}{=} u(1 + C^2)\,, \; C \text{ with density } \frac{1}{\pi} \frac{1}{1+x^2} \,.$$

From $\{g_t < u\} = \{t < d_u\}$ we deduce, for all t and u,

$$\frac{d_u}{u} \stackrel{\text{law}}{=} \frac{t}{g_t} \stackrel{\text{law}}{=} 1 + C^2 \,;$$

consequently, g_t is Arcsine distributed. □

These results can be extended to the last time before 1 when a Brownian motion reaches level a.

Proposition 4.3.3.3 *Let* $g_1^a = \sup\{t \le 1 \,:\, B_t = a\}$, *where* $\sup(\emptyset) = 1$. *The law of* g_1^a *is*

$$\mathbb{P}(g_1^a \in dt) = \exp\left(-\frac{a^2}{2t} \right) \frac{dt}{\pi \sqrt{t(1-t)}} \mathbb{1}_{\{0<t<1\}} \,, \qquad (4.3.2)$$

$$\mathbb{P}(g_1^a = 1) = \mathbb{P}(|G| \le a)$$

where G *is a standard Gaussian random variable. The r.v.*

$$d_1^a = \inf\{u \ge 1 \,:\, B_u = a\}$$

has the same law as $1 + \dfrac{(a-G)^2}{\widetilde{G}^2}$ *where* G *and* $\widetilde{G}$ *are independent standard Gaussian random variables.*

PROOF: From the equality, with $t < 1$,

$$\{g_1^a \le t\} = \{T_a \le t\} \cap \{\widehat{g}^0_{1-T_a} \le t - T_a\}$$

where $\widehat{g}^0$ is relative to the Brownian motion $(\widehat{B}_u = B_{u+T_a} - B_{T_a}, u \ge 0)$, i.e., $\widehat{g}^0_t = \sup\{s \le t \,:\, \widehat{B}_s = 0\}$, one obtains

$$\mathbb{P}(g_1^a \le t) = \int_0^t \mathbb{P}(T_a \in du)\mathbb{P}(\widehat{g}_{1-u}^0 \le t - u\} \,.$$

The laws of T_a and $\widehat{g}_{1-u}^0$ are known, and some easy computation leads to

$$\mathbb{P}(g_1^a \le t) = \frac{a}{\pi\sqrt{2\pi}} \int_0^t \frac{dv}{\sqrt{1-v}} \int_0^v du \frac{e^{-a^2/(2u)}}{\sqrt{u^3(v-u)}} \,.$$

It remains to recall that, from (4.1.11) the second integral on the right-hand side is known.

Note that the right-hand side of (4.3.2) is a sub-probability, and that the missing mass is

$$\mathbb{P}(g_1^a = 1) = \mathbb{P}(T_a \ge 1) = \mathbb{P}(|G| \le a) \,,$$

where G is the standard Gaussian variable.

Let $d_t^a(B) = \inf\{u \ge t \,:\, B_u = a\}$. We obtain

$$\begin{aligned} d_t^a(B) &= t + \inf\{u \ge 0 \,:\, B_{u+t} - B_t = a - B_t\} \\ &= t + \widehat{T}_{a-B_t} \overset{\text{law}}{=} t + \frac{(a-B_t)^2}{G^2} \,. \end{aligned} \tag{4.3.3}$$

Here, $\widehat{T}_b = \inf\{u \ge 0 : \widehat{B}_u = b\}$, where $\widehat{B}$ is a Brownian motion independent of $\mathcal{F}_t$, and G is a standard Gaussian variable, independent of B. □

Comments 4.3.3.4 (a) Formula (4.3.2) plays an important rôle in the discussion of quantiles of Brownian motion in Yor [866] (formula (3.b) therein).

(b) We recall that we already saw the occurrence of the Arcsine law in Subsection 2.5.2 and Example 4.1.7.5.

Exercise 4.3.3.5 The aim of this exercise is to provide an explanation of the fact, obtained in Proposition 4.3.3.3, that

$$\mathbb{P}(|G| \le 1) + \int_0^1 \mathbb{P}(g_1^a \in dt) = 1 \,.$$

From the equality $G^2 \overset{\text{law}}{=} 2\mathbf{e}g_1$ where $\mathbf{e}$ is exponentially distributed with parameter 1 and G is a standard Gaussian variable (see ↣ Appendix A.4.2), prove that $\mathbb{P}(|G| > a) = \mathbb{E}(e^{-a^2/(2g_1)})$ and conclude. ◁

Exercise 4.3.3.6 Let

$$\begin{aligned} g_a^{(\nu)} &= \sup\{t \,:\, B_t + \nu t = a\} \\ T_a^{(\nu)} &= \inf\{t \,:\, B_t + \nu t = a\} \end{aligned}$$

Prove that

$$(T_a^{(\nu)}, g_a^{(\nu)}) \overset{\text{law}}{=} \left(\frac{1}{g_\nu^{(a)}}, \frac{1}{T_\nu^{(a)}} \right) .$$

See Bentata and Yor [72] for related results. ◁

4.3.4 Laws of (B_t, g_t, d_t)

We now study the laws of the pairs of r.v's (B_t, d_t) and (B_t, g_t) for fixed t.

Proposition 4.3.4.1 *The joint laws of the pairs (B_t, d_t) and (B_t, g_t) are given by:*

$$\mathbb{P}(B_t \in dx, d_t \in ds) = \mathbb{1}_{\{s \geq t\}} \frac{|x|}{2\pi\sqrt{t(s-t)^3}} \exp\left(-\frac{sx^2}{2t(s-t)}\right) dx\, ds\,, \quad (4.3.4)$$

$$\mathbb{P}(B_t \in dx, g_t \in ds) = \mathbb{1}_{\{s \leq t\}} \frac{|x|}{2\pi\sqrt{s(t-s)^3}} \exp\left(-\frac{x^2}{2(t-s)}\right) dx\, ds\,. \quad (4.3.5)$$

PROOF: We begin with the law of (B_t, d_t). From the Markov property we derive

$$\begin{aligned}\mathbb{P}(B_t \in dx, d_t \in ds) &= \mathbb{P}(B_t \in dx)\mathbb{P}(d_t \in ds | B_t = x)\\ &= \mathbb{P}(B_t \in dx)\mathbb{P}_x(T_0 \in ds - t)\\ &= \mathbb{P}(B_t \in dx)\mathbb{P}_0(T_x \in ds - t)\,,\end{aligned}$$

and the two expressions on the right-hand side of the latter equation are well known.

For the second law, we use time inversion for the pair (B, g). Let us define $\{\hat{B}_t = tB_{1/t}, t > 0\}$ a standard Brownian motion and let $\widehat{g}$ be related to $\widehat{B}$ via $\widehat{g}_u = \sup\{s < u : \widehat{B}_s = 0\}$. We begin with an identity in law between d_t and $g_{1/t}$:

$$\begin{aligned} d_t &= \inf\{s \geq t : B_s = 0\} = \inf\{s^{-1} \geq t : B_{1/s} = 0\}\\ &= \inf\{s^{-1} \geq t : sB_{1/s} = 0\} = \inf\{s^{-1} \geq t : \widehat{B}_s = 0\}\\ &= 1/\sup\left\{u \leq \frac{1}{t} : \widehat{B}_u = 0\right\} = \frac{1}{\widehat{g}_{1/t}}\,.\end{aligned}$$

Therefore, since $B_t = t\hat{B}_{1/t}$, we have

$$\mathbb{P}(B_t \leq x, g_t \leq s) = \mathbb{P}\left(\widehat{B}_{1/t} \leq \frac{x}{t}, \widehat{d}_{1/t} \geq \frac{1}{s}\right).$$

Denoting by $f_t(x,s)$ the density of the pair $(\widehat{B}_t, \widehat{d}_t)$, and using the first part of the proof:

$$\frac{1}{dsdx}\mathbb{P}(B_t \in dx, g_t \in ds) = \frac{\partial^2}{\partial x \partial s}\mathbb{P}\left(\widehat{B}_{1/t} \leq \frac{x}{t}, \widehat{d}_{1/t} \geq \frac{1}{s}\right) = \frac{1}{ts^2} f_{1/t}\left(\frac{x}{t}, \frac{1}{s}\right).$$

The result follows from this. □

Comment 4.3.4.2 The reader will find in Chung [183] another proof of (4.3.5) based on the following remark, which uses the law of the pair (B_t, m_t^B) established in Subsection 3.1.5:

$$\begin{aligned}\mathbb{P}(g_t \leq s, B_s \in dx, B_t \in dy) &= \mathbb{P}(B_s \in dx, B_u \neq 0, \forall u \in [s,t], B_t \in dy) \\ &= \mathbb{P}(B_s \in dx)\, \mathbb{P}_x(B_{t-s} \in dy,\, T_0 > t-s) \\ &= \mathbb{P}(B_s \in dx)\, \mathbb{P}_0(B_{t-s} + x \in dy,\, m_{t-s}^B > -x)\end{aligned}$$

$$= \frac{e^{-x^2/(2s)}}{\sqrt{2\pi s}} \frac{1}{\sqrt{2\pi(t-s)}} \left(\exp\left(-\frac{(x-y)^2}{2(t-s)}\right) - \exp\left(-\frac{(x+y)^2}{2(t-s)}\right)\right) dx\, dy\,.$$

By integrating with respect to dx, and differentiating with respect to s, the result is obtained.

Exercise 4.3.4.3 Let $t > 0$ be fixed and $\theta_t = \inf\{s \leq t \,|\, M_t = B_s\}$ where $M_t = \sup_{s \leq t} B_s$. Prove that

$$(M_t, \theta_t) \stackrel{\text{law}}{=} (|B_t|, g_t) \stackrel{\text{law}}{=} (L_t, g_t)\,.$$

`Hint:` Use the equalities (4.1.10) and (4.3.4) and Lévy's theorem. ◁

4.3.5 Brownian Bridge

The **Brownian bridge** $(b_t, 0 \leq t \leq 1)$ is defined as the conditioned process $(B_t, t \leq 1 | B_1 = 0)$. Note that $B_t = (B_t - tB_1) + tB_1$ where, from the Gaussian property, the process $(B_t - tB_1, t \leq 1)$ and the random variable B_1 are independent. Hence $(b_t, 0 \leq t \leq 1) \stackrel{\text{law}}{=} (B_t - tB_1,\, 0 \leq t \leq 1)$. The Brownian bridge process is a Gaussian process, with zero mean and covariance function $s(1-t), s \leq t$. Moreover, it satisfies $b_0 = b_1 = 0$.

Each of the Gaussian processes X, Y and Z where

$$X_t = (1-t)\int_0^t \frac{dB_s}{1-s}\,;\, 0 \leq t \leq 1$$

$$Z_t = tB_{(1/t)-1}\,;\, 0 \leq t \leq 1$$

$$Y_t = (1-t)B\left(\frac{t}{1-t}\right)\,;\, 0 \leq t \leq 1$$

has the same properties, and is a Brownian bridge. Note that the apparent difficulty in defining the above processes at time 0 or 1 may be resolved by extending it continuously to $[0, 1]$.

Since $(W_{1-t} - W_1,\, t \leq 1) \stackrel{\text{law}}{=} (W_t,\, t \leq 1)$, the Brownian bridge is invariant under time reversal.

We can represent the Brownian bridge between 0 and y during the time interval $[0,1]$ as

$$(B_t - tB_1 + ty;\ t \le 1)$$

and we denote by $\mathbf{W}^{(1)}_{0\to y}$ its law on the canonical space. More generally, $\mathbf{W}^{(T)}_{x\to y}$ denotes the law of the Brownian bridge between x and y during the time interval $[0,T]$, which may be expressed as

$$\left(x + B_t - \frac{t}{T}B_T + \frac{t}{T}(y-x);\ t \le T\right),$$

where $(B_t; t \le T)$ is a standard BM starting from 0.

Theorem 4.3.5.1 *For every t, $\mathbf{W}^{(t)}_{x\to y}$ is equivalent to $\mathbf{W}_x$ on $\mathcal{F}_s$ for $s < t$.*

PROOF: Let us consider a more general case: suppose $((X_t;\ t \ge 0), (\mathcal{F}_t), \mathbb{P}_x)$ is a real valued Markov process with semigroup

$$P_t(x,dy) = p_t(x,y)dy,$$

and F_s is a non-negative $\mathcal{F}_s$-measurable functional. Then, for $s \le t$, and any function f

$$\mathbb{E}_x[F_s f(X_t)] = \mathbb{E}_x[F_s\, P_{t-s}f(X_s)]\,.$$

On the one hand

$$\begin{aligned}\mathbb{E}_x[F_s\, P_{t-s}f(X_s)] &= \mathbb{E}_x[F_s \int f(y)\, p_{t-s}(X_s,y)\, dy]\\ &= \int f(y)\mathbb{E}_x[F_s p_{t-s}(X_s,y)]\, dy\,.\end{aligned}$$

On the other hand

$$\mathbb{E}_x[F_s f(X_t)] = \mathbb{E}_x[\mathbb{E}_x[F_s|X_t] f(X_t)] = \int dy f(y) p_t(x,y) \mathbb{E}^{(t)}_{x\to y}(F_s)\,,$$

where $\mathbb{P}^{(t)}_{x\to y}$ is the probability measure associated with the bridge (for a general definition of Markov bridges, see Fitzsimmons et al. [346]) between x and y during the time interval $[0,t]$. Therefore,

$$\mathbb{E}^{(t)}_{x\to y}(F_s) = \frac{\mathbb{E}_x[F_s p_{t-s}(X_s,y)]}{p_t(x,y)}\,.$$

Thus

$$\boxed{\mathbb{P}^{(t)}_{x\to y}|_{\mathcal{F}_s} = \frac{p_{t-s}(X_s,y)}{p_t(x,y)}\mathbb{P}_x|_{\mathcal{F}_s}\,.} \tag{4.3.6}$$

□

Sometimes, we shall denote X under $\mathbb{P}^{(t)}_{x\to y}$ by $(X^{(t)}_{x\to y}(s), s \le t)$.

If X is an n-dimensional Brownian motion and $x = y = 0$ we have, for $s < t$,

$$\boxed{\mathbf{W}^{(t)}_{0\to 0}|_{\mathcal{F}_s} = \left(\frac{t}{t-s}\right)^{n/2} \exp\left(\frac{-|X_s|^2}{2(t-s)}\right) \mathbf{W}_0|_{\mathcal{F}_s}\,.} \tag{4.3.7}$$

As a consequence of (4.3.7), identifying the density as the exponential martingale $\mathcal{E}(Z)$, where $Z_s = -\int_0^s \frac{X_u}{t-u}\,dX_u$, we obtain the canonical decomposition of the standard Brownian bridge (under $\mathbf{W}^{(t)}_{0\to 0}$) as:

$$X_s = B_s - \int_0^s du\,\frac{X_u}{t-u}, \quad s < t, \tag{4.3.8}$$

where $(B_s, s \le t)$ is a Brownian motion under $\mathbf{W}^{(t)}_{0\to 0}$. (This decomposition may be related to the harness property in $\rightarrowtail$ Definition 8.5.2.1.)

Therefore, we obtain that the standard Brownian bridge b is a solution of the following stochastic equation

$$\begin{cases} db_t = -\dfrac{b_t}{1-t}\,dt + dB_t\,;\ 0 \le t < 1 \\ \\ b_0 = 0\,. \end{cases}$$

Proposition 4.3.5.2 *Let $X_t = \mu t + \sigma B_t$ where B is a BM, and for fixed T, $(X^{(T)}_{0\to y}(t), t \le T)$ is the associated bridge. Then, the law of the bridge does not depend on μ, and in particular*

$$\mathbb{P}(X^{(T)}_{0\to y}(t) \in dx) = \frac{dx}{\sigma\sqrt{2\pi t}}\sqrt{\frac{T}{T-t}}\exp\left(-\frac{1}{2\sigma^2}\left(\frac{x^2}{t} + \frac{(y-x)^2}{T-t} - \frac{y^2}{T}\right)\right) \tag{4.3.9}$$

PROOF: The fact that the law does not depend on μ can be viewed as a consequence of Girsanov's theorem. The form of the density is straightforward from the computation of the joint density of (X_t, X_T), or from (4.3.6). □

Proposition 4.3.5.3 *Let $B^{(t)}_{x\to z}$ be a Brownian bridge, starting from x at time 0 and ending at z at time t, and $M^{br}_t = \sup_{0\le s\le t} B^{(t)}_{x\to z}(s)$. Then, for any $m > z \vee x$,*

$$\mathbb{P}^{(t)}_{x\to z}(M^{br}_t \le m) = 1 - \exp\left(-\frac{(z+x-2m)^2}{2t} + \frac{(z-x)^2}{2t}\right).$$

In particular, let b *be a standard Brownian bridge* $(x = z = 0, t = 1)$*. Then,*

$$\sup_{0\le s\le 1} b_s \stackrel{\text{law}}{=} \frac{1}{2}R\,,$$

where R *is Rayleigh distributed with density* $x\exp\left(-\frac{1}{2}x^2\right)\mathbb{1}_{\{x\ge 0\}}$*. If* $\ell^a_1(b)$ *denotes the local time of* b *at level* a *at time 1, then for every* a

$$\ell^a_1(b) \stackrel{\text{law}}{=} (R - 2|a|)^+\,. \tag{4.3.10}$$

PROOF: Let B be a standard Brownian motion and $M^B_t = \sup_{0\le s\le t} B_s$. Then, for every $y > 0$ and $x \le y$, equality (3.1.3) reads

$$\mathbb{P}(B_t \in dx\,, M^B_t \le y) = \frac{dx}{\sqrt{2\pi t}}\exp\left(-\frac{x^2}{2t}\right) - \frac{dx}{\sqrt{2\pi t}}\exp\left(-\frac{(2y-x)^2}{2t}\right),$$

hence,

$$\begin{aligned}\mathbb{P}(M^B_t \le y|B_t = x) &= \frac{\mathbb{P}(B_t \in dx\,, M^B_t \le y)}{\mathbb{P}(B_t \in dx)} = 1 - \exp\Big(-\frac{(2y-x)^2}{2t} + \frac{x^2}{2t}\Big)\\ &= 1 - \exp\left(-\frac{2y^2 - 2xy}{t}\right).\end{aligned}$$

More generally,

$$\mathbb{P}(\sup_{s\le t} B_s + x \le y|B_t + x = z) = \mathbb{P}(M^B_t \le y - x|B_t = z - x)$$

hence

$$\mathbb{P}_x(\sup_{0\le s\le t} B^{(t)}_{x\to z}(s) \le y) = 1 - \exp\left(-\frac{(z+x-2y)^2}{2t} + \frac{(z-x)^2}{2t}\right).$$

The result on local time follows by conditioning w.r.t. B_1 the equality obtained in Example 4.1.7.7. □

Theorem 4.3.5.4 *Let* B *be a Brownian motion. For every* t*, the process* $B^{[0,g_t]}$ *defined by:*

$$B^{[0,g_t]} = \left(\tfrac{1}{\sqrt{g_t}}B_{ug_t}, u \le 1\right) \tag{4.3.11}$$

is a Brownian bridge $B^{(1)}_{0\to 0}$ *independent of the* σ*-algebra* $\sigma\{g_t, B_{g_t+u}, u \ge 0\}$*.*

PROOF: By scaling, it suffices to prove the result for $t = 1$. Let $\widehat{B}_t = tB_{1/t}$. As in the proof of Proposition 4.3.4.1, $\widehat{d}_1 = \frac{1}{g_1}$. Then,

$$\frac{1}{\sqrt{g_1}} B(ug_1) = u\sqrt{g_1}\, \widehat{B}\left(\frac{1}{ug_1}\right) = \frac{u}{\sqrt{\widehat{d}_1}} \left[\widehat{B}\left(\frac{1}{g_1} + \frac{1}{g_1}(\frac{1}{u} - 1)\right) - \widehat{B}\left(\frac{1}{g_1}\right)\right]$$
$$= \frac{u}{\sqrt{\widehat{d}_1}} \left[\widehat{B}\left(\widehat{d}_1 + \widehat{d}_1\left(\frac{1}{u} - 1\right)\right) - \widehat{B}(\widehat{d}_1)\right].$$

Knowing that $(\widehat{B}_{\widehat{d}_1+s} - \widehat{B}_{\widehat{d}_1}; s \geq 0)$ is a Brownian motion independent of $\mathcal{F}_{\widehat{d}_1}$ and that $\widehat{B}_{\widehat{d}_1} = 0$, the process $\widetilde{B}_u = \frac{1}{\sqrt{\widehat{d}_1}} \widehat{B}_{\widehat{d}_1 + \widehat{d}_1 u}$ is also a Brownian motion independent of $\mathcal{F}_{\widehat{d}_1}$. Therefore $t\widetilde{B}_{(\frac{1}{t}-1)}$ is a Brownian bridge independent of $\mathcal{F}_{\widehat{d}_1}$ and the result is proved. □

Example 4.3.5.5 Let B be a real-valued Brownian motion under $\mathbb{P}$ and

$$X_t = B_t - \int_0^t \frac{B_s}{s} ds\,.$$

This process X is an $\mathbf{F}^*$-Brownian motion where $\mathbf{F}^*$ is the filtration generated by the bridges, i.e.,

$$\mathcal{F}_t^* = \sigma\left\{B_u - \frac{u}{t} B_t,\, u \leq t\right\}.$$

Let $L_t = \exp(\lambda B_t - \frac{\lambda^2 t}{2})$ and $\mathbb{Q}_{|\mathcal{F}_t} = L_t \mathbb{P}_{|\mathcal{F}_t}$. Then $\mathbb{Q}_{|\mathcal{F}_t^*} = \mathbb{P}_{|\mathcal{F}_t^*}$.

Comments 4.3.5.6 (a) It can be proved that $|B|^{[g_1,d_1]}$ has the same law as a BES^3 bridge and is independent of

$$\sigma(B_u, u \leq g_1) \vee \sigma(B_u, u \geq d_1) \vee \sigma(\text{sgn}(B_1)).$$

(b) For a study of Bridges in a general Markov setting, see Fitzsimmons et al. [346].

(c) Application to fast simulation of Brownian bridge in finance can be found in Pagès [691], Metwally and Atiya [646]. We shall study Brownian bridges again when dealing with enlargements of filtrations, in ⤚ Subsection 5.9.2.

Exercise 4.3.5.7 Let $T_a = \inf\{t : |X_t| = a\}$. Give the law of T_a under $\mathbf{W}_{0\to 0}^{(t)}$.

`Hint:` : $\mathbf{W}_{0\to 0}^{(t)}\left(f(T_a \mathbb{1}_{\{T_a<t\}})\right) = \mathbf{W}\left(f(T_a)\mathbb{1}_{\{T_a<t\}} \frac{t}{(t-T_a)^{n/2}} e^{-\frac{a^2}{2(t-T_a)}}\right).$ ◁

4.3.6 Slow Brownian Filtrations

If ζ is a random time, i.e., a random variable such that $\zeta > 0$ a.s., we define the σ-field $\mathcal{F}_\zeta^-$ of the past up to ζ as the σ-algebra generated by the variables h_ζ, where h is a generic predictable process.

Likewise, we may define $\mathcal{F}_\zeta^+$ as the σ-algebra generated by the variables h_ζ, where h is a generic $\mathbf{F}$-progressively measurable process.

In particular, we consider, as in Dellacherie et al. [241] the σ-algebras $\mathcal{F}_{g_t}^-$ and $\mathcal{F}_{g_t}^+$. The following properties are satisfied:

- Both $(\mathcal{F}_{g_t}^-, t \geq 0)$ and $(\mathcal{F}_{g_t}^+, t \geq 0)$ are increasing and are called the slow Brownian filtrations, $(\mathcal{F}_{g_t}^-, t \geq 0)$ being the strict slow Brownian filtration and $(\mathcal{F}_{g_t}^+, t \geq 0)$ the wide slow Brownian filtration.
- For fixed t, there is the double identity

$$\mathcal{F}_{g_t}^+ = \cap_{\epsilon>0} \mathcal{F}_{g_t+\epsilon}^- = \mathcal{F}_{g_t}^- \vee \sigma(\text{sgn} B_t) .$$

This shows that $\mathcal{F}_{g_t}^+$ is the σ-algebra of the immediate future after g_t and the second identity provides the independent complement $\sigma(\text{sgn} B_t)$ which needs to be added to $\mathcal{F}_{g_t}^-$ to capture $\mathcal{F}_{g_t}^+$. See Barlow et al. [50].

4.3.7 Meanders

Definition 4.3.7.1 *The Brownian meander of length 1 is the process defined by:*

$$m_u := \frac{1}{\sqrt{1-g_1}} |B_{g_1+u(1-g_1)}|; \ (u \leq 1) .$$

We begin with a very useful result:

Proposition 4.3.7.2 *The law of m_1 is the Rayleigh law whose density is*

$$x \exp(-x^2/2) \, \mathbb{1}_{\{x \geq 0\}} .$$

Consequently, $m_1 \stackrel{\text{law}}{=} \sqrt{2\mathbf{e}}$ holds.

Proof: From (4.3.5),

$$\mathbb{P}(B_1 \in dx, g_1 \in ds) = \mathbb{1}_{\{s \leq 1\}} \frac{|x| \, dx \, ds}{2\pi \sqrt{s(1-s)^3}} \exp\left(-\frac{x^2}{2(1-s)}\right) .$$

We deduce, for $x > 0$,

$$\begin{aligned}
\mathbb{P}(m_1 \in dx) &= \int_{s=0}^1 \mathbb{P}(m_1 \in dx, g_1 \in ds) = \int_{s=0}^1 \mathbb{P}(\frac{|B_1|}{\sqrt{1-s}} \in dx, g_1 \in ds) \\
&= dx \, \mathbb{1}_{\{x \geq 0\}} \int_0^1 ds \frac{2x(1-s)}{2\pi \sqrt{s(1-s)^3}} \exp\left(-\frac{x^2(1-s)}{2(1-s)}\right) \\
&= 2x \, dx \, \mathbb{1}_{\{x \geq 0\}} \exp\left(-x^2/2\right) \int_0^1 ds \frac{1}{2\pi\sqrt{s(1-s)}} \\
&= x e^{-x^2/2} \mathbb{1}_{\{x \geq 0\}} dx ,
\end{aligned}$$

where we have used the fact that $\int_0^1 ds \frac{1}{\pi\sqrt{s(1-s)}} = 1$, from the property of the arcsin density. □

We continue with a more global discussion of meanders in connection with the slow Brownian filtrations. For any given t, by scaling, the law of the process

$$m_u^{(t)} = \frac{1}{\sqrt{t-g_t}} |B_{g_t+u(t-g_t)}| \, , u \le 1$$

does not depend on t. Furthermore, this process is independent of $\mathcal{F}_{g_t}^+$ and in particular of g_t and $\text{sgn}(B_t)$. All these properties extend also to the case when t is replaced by τ, any $\mathcal{F}_{g_t}^-$-stopping time.

Note that, from $|B_1| = \sqrt{1-g_1} m_1$ where m_1 and $\sqrt{1-g_1}$ are independent, we obtain from the particular case of the beta-gamma algebra (see ↣ Appendix A.4.2) $G^2 \overset{\text{law}}{=} 2\mathbf{e}g_1$ where $\mathbf{e}$ is exponentially distributed with parameter 1, G is a standard Gaussian variable, and g_1 and $\mathbf{e}$ are independent.

Comment 4.3.7.3 For more properties of the Brownian meander, see Biane and Yor [87] and Bertoin and Pitman [82].

4.3.8 The Azéma Martingale

We now introduce the Azéma martingale which is an $(\mathcal{F}_{g_t}^+)$-martingale and enjoys many remarkable properties.

Proposition 4.3.8.1 *Let B be a Brownian motion. The process*

$$\mu_t = (\text{sgn} B_t)\, \sqrt{t-g_t}, \; t \ge 0$$

is an $(\mathcal{F}_{g_t}^+)$-martingale. Let

$$\Psi(z) = \int_0^\infty x \exp\left(zx - \frac{x^2}{2}\right) dx = 1 + z\sqrt{2\pi}\, \mathcal{N}(z) e^{z^2/2} \, . \tag{4.3.12}$$

The process

$$\exp\left(-\frac{\lambda^2}{2} t\right) \Psi(\lambda \mu_t), \; t \ge 0$$

is an $(\mathcal{F}_{g_t}^+)$-martingale.

PROOF: Following Azéma and Yor [38] closely, we project the $\mathbf{F}$-martingale B on $\mathcal{F}_{g_t}^+$. From the independence property of the meander and $\mathcal{F}_{g_t}^+$, we obtain

$$\mathbb{E}(B_t | \mathcal{F}_{g_t}^+) = \mathbb{E}(m_1^{(t)}\, \mu_t | \mathcal{F}_{g_t}^+) = \mu_t \mathbb{E}(m_1^{(t)}) = \sqrt{\frac{\pi}{2}}\, \mu_t \, . \tag{4.3.13}$$

Hence, $(\mu_t, t \ge 0)$ is an $(\mathcal{F}_{g_t}^+)$-martingale. In a second step, we project the $\mathbf{F}$-martingale $\exp(\lambda B_t - \frac{1}{2}\lambda^2 t)$ on the filtration $(\mathcal{F}_{g_t}^+)$:

$$\mathbb{E}(\exp(\lambda B_t - \frac{\lambda^2}{2}t)|\mathcal{F}_{g_t}^+) = \mathbb{E}\left(\exp(\lambda\, m_1^{(t)}\, \mu_t - \frac{\lambda^2}{2}t)|\mathcal{F}_{g_t}^+\right)$$

and, from the independence property of the meander and $\mathcal{F}_{g_t}^+$, we get

$$\mathbb{E}\left(\exp\left(\lambda B_t - \frac{\lambda^2}{2}t\right)|\mathcal{F}_{g_t}^+\right) = \exp\left(-\frac{\lambda^2}{2}t\right)\Psi(\lambda\mu_t)\,, \tag{4.3.14}$$

where Ψ is defined in (4.3.12) as

$$\Psi(z) = \mathbb{E}(\exp(zm_1\,)) = \int_0^\infty x\exp\left(zx - \frac{x^2}{2}\right)dx\,.$$

Obviously, the process in (4.3.14) is a $(\mathcal{F}_{g_t}^+)$-martingale. □

Comment 4.3.8.2 Some authors (e.g. Protter [726]) define the Azéma martingale as $\sqrt{\frac{\pi}{2}}\mu_t$, which is precisely the projection of the BM on the wide slow filtration, hence in further computations as in the next exercise, different multiplicative factors appear.

Note that the Azéma martingale is not continuous.

Exercise 4.3.8.3 Prove that the projection on the σ-algebra $\mathcal{F}_{g_t}^+$ of the **F**-martingale $(B_t^2 - t, t \geq 0)$ is $2(t - g_t) - t$, hence the process

$$\mu_t^2 - (t/2) = (t/2) - g_t$$

is an $(\mathcal{F}_{g_t}^+)$-martingale. ◁

4.3.9 Drifted Brownian Motion

We now study how our previous results are modified when working with a BM with drift. More precisely, we consider $X_t = x + \mu\, t + \sigma\, B_t$ with $\sigma > 0$. In order to simplify the proofs, we write $g^a(X)$ for $g_1^a(X) = \sup\{t \leq 1 \,:\, X_t = a\}$. The law of $g^a(X)$ may be obtained as follows

$$\begin{aligned} g^a(X) &= \sup\{t \leq 1 \,:\, \mu t + \sigma B_t = a - x\} \\ &= \sup\{t \leq 1 \,:\, \nu t + B_t = \alpha\}\,, \end{aligned}$$

where $\nu = \mu/\sigma$ and $\alpha = (a - x)/\sigma$. From Girsanov's theorem, we deduce

$$\mathbb{P}(g^a(X) \leq t) = \mathbb{E}\Big(\mathbb{1}_{\{g^\alpha \leq t\}}\exp\left(\nu B_1 - \frac{\nu^2}{2}\right)\Big)\,, \tag{4.3.15}$$

where

$$g^\alpha = g_1^\alpha(B) = \sup\{t \leq 1 \,:\, B_t = \alpha\}.$$

Then, using that $|B_1| = m_1\sqrt{1-g_1}$ where m_1 is the value at time 1 of the Brownian meander,

$$\mathbb{P}(g^a(X) \le t) = \exp\Big(-\frac{\nu^2}{2}\Big)\,\mathbb{E}\Big(\mathbb{1}_{\{g^\alpha < t\}} \exp\big(\nu\epsilon m_1\sqrt{1-g^\alpha}\big)\Big) \qquad (4.3.16)$$

where ϵ is a Bernoulli random variable; furthermore, the random variables g^α, ϵ, and m_1 are mutually independent. Therefore, since m_1 follows the Rayleigh law

$$\mathbb{P}(m_1 \in dy) = y\exp\left(-\frac{y^2}{2}\right)\mathbb{1}_{\{y\ge 0\}}\,dy\,,$$

we obtain

$$\begin{aligned}\mathbb{P}(g^a(X) \le t) &= \exp\Big(-\frac{\nu^2}{2}\Big)\int_0^t \frac{1}{\pi\sqrt{u(1-u)}}\exp\Big(-\frac{(a-x)^2}{2u\sigma^2}\Big)\Upsilon(\nu,u)\,du\\ &:= \Psi(x,a,t), \qquad (4.3.17)\end{aligned}$$

where

$$\begin{aligned}\Upsilon(\nu,u) &= \mathbb{E}(\exp(\nu\epsilon m_1\sqrt{1-u}))\\ &= \frac{1}{2}\Big(\int_0^\infty e^{\nu y\sqrt{1-u}}ye^{-y^2/2}\,dy + \int_0^\infty e^{-\nu y\sqrt{1-u}}ye^{-y^2/2}\,dy\Big),\end{aligned}$$

that is,

$$\Upsilon(\nu,u) = \int_0^\infty \cosh(\nu y\sqrt{1-u})\,ye^{-y^2/2}\,dy.$$

Lemma 4.3.9.1 *Let $X_t = \nu t + B_t$. We have, for $t < 1$*

$$\mathbb{P}(g^a(X) > t|\mathcal{F}_t) = \mathbb{1}_{\{T_a(X)\le t\}}e^{\nu(\alpha - X_t)}H(\nu, |\alpha - X_t|, 1-t)\,,$$

where, for $y > 0$

$$H(\nu,y,s) = e^{-\nu y}\mathcal{N}\left(\frac{\nu s - y}{\sqrt{s}}\right) + e^{\nu y}\mathcal{N}\left(\frac{-\nu s - y}{\sqrt{s}}\right).$$

Proof: From the absolute continuity relationship, we obtain, for $t < 1$

$$\mathbf{W}^{(\nu)}(g^a(X) \le t|\mathcal{F}_t) = \zeta_t^{-1}\,\mathbf{W}^{(0)}(\zeta_1\mathbb{1}_{\{g^a(X)\le t\}}|\mathcal{F}_t),$$

where

$$\zeta_t = \exp\big(\nu X_t - \frac{t\nu^2}{2}\big)\,. \qquad (4.3.18)$$

Therefore, from the equality

$$\{g^a(X) \le t\} = \{T_a(X) \le t\} \cap \{d_t^a(X) > 1\}$$

we obtain

$$\mathbf{W}^{(0)}(\zeta_1 \mathbb{1}_{\{g^a \le t\}} | \mathcal{F}_t)$$
$$= \exp\left(\nu X_t - \nu^2/2\right) \mathbb{1}_{\{T_a(X) \le t\}} \mathbf{W}^{(0)}\left(\exp[\nu(X_1 - X_t)] \mathbb{1}_{\{d_t^a(X) > 1\}} | \mathcal{F}_t\right).$$

Using the independence properties of Brownian motion and equality (4.3.3), we get

$$\begin{aligned} &\mathbf{W}^{(0)}\left(\exp[\nu(X_1 - X_t)] \mathbb{1}_{\{d_t^a(X) > 1\}} | \mathcal{F}_t\right) \\ &= \mathbf{W}^{(0)}\left(\exp[\nu Z_{1-t}] \mathbb{1}_{\{T_{a-X_t}(Z) > 1-t\}} | \mathcal{F}_t\right) \\ &= \Theta(a - X_t, 1 - t) \end{aligned}$$

where $Z_t = X_1 - X_t \stackrel{\text{law}}{=} X_{1-t}$ is independent of $\mathcal{F}_t$ under $\mathbf{W}^{(0)}$ and

$$\Theta(x, s) := \mathbf{W}^{(0)}\left(e^{\nu X_s} \mathbb{1}_{\{T_x \ge s\}}\right) = e^{s\nu^2/2} - \mathbf{W}^{(0)}\left(e^{\nu X_s} \mathbb{1}_{\{T_x < s\}}\right).$$

By conditioning with respect to $\mathcal{F}_{T_x}$, we obtain (see Subsection 3.2.4 for the computation of H)

$$\begin{aligned} &\mathbf{W}^{(0)}\left(e^{\nu X_s} \mathbb{1}_{\{T_x < s\}}\right) \\ &= e^{\nu x} \mathbf{W}^{(0)}\left(\mathbb{1}_{\{T_x < s\}} e^{\frac{\nu^2}{2}(s - T_x)} \mathbf{W}^{(0)}\left(e^{\nu(X_s - X_{T_x}) - \frac{\nu^2}{2}(s - T_x)} | \mathcal{F}_{T_x}\right)\right) \\ &= e^{\nu x} \mathbf{W}^{(0)}\left(\mathbb{1}_{\{T_x < s\}} e^{\frac{\nu^2}{2}(s - T_x)}\right) = e^{\nu x + s\nu^2/2} H(\nu, |x|, s)\,. \end{aligned}$$

Therefore,

$$\Theta(a - X_t, 1 - t) = e^{(1-t)\nu^2/2}(1 - e^{\nu(a - X_t)} H(\nu, |a - X_t|, 1 - t))$$

and

$$\begin{aligned} \mathbf{W}^{(\nu)}(g^a(X) \le t | \mathcal{F}_t) &= \mathbb{1}_{\{T_a(X) \le t\}} \exp\left(\frac{(t-1)\nu^2}{2}\right) \Theta(a - X_t, 1 - t) \\ &= \mathbb{1}_{\{T_a(X) \le t\}} \left(1 - e^{\nu(a - X_t)} H(\nu, |a - X_t|, 1 - t)\right). \end{aligned}$$

□

4.4 Parisian Options

In this section, our aim is to price an exotic option which we describe below, in a Black and Scholes framework: the underlying asset satisfies the stochastic differential equation

$$dS_t = S_t((r - \delta)\, dt + \sigma\, dW_t) \tag{4.4.1}$$

where W is a Brownian motion under the risk-neutral probability $\mathbb{Q}$, and w.l.g. $\sigma > 0$. In a closed form,

$$S_t = S_0 e^{\sigma X_t}$$

where $X_t = W_t + \nu t$ and $\nu = \frac{r-\delta}{\sigma} - \frac{\sigma}{2}$. The owner of an **up-and-out Parisian** option (UOPa) loses its value if the stock price reaches a level H (H is for **H**igh) and remains constantly above this level for a time interval longer than D (the delay). A **down-and-in** Parisian option (DIPa) is activated if the stock price falls below a **L**ow level L and remains constantly below this level for a time interval longer than D. For a delay equal to zero, the Parisian option reduces to a standard barrier option. When the delay is extended beyond maturity, the UOPa option reduces to a standard European option. In the intermediate case, the option presents its "Parisian" feature and becomes a flexible financial tool which has some interesting properties: for instance, for some values of the parameters, when the underlying asset price is close to the out-barrier or when the size of the delay is small, its value is a decreasing function of the volatility. Therefore, it allows traders to bet in a simple manner on a decrease of volatility. Last but not least, as far as down-and-out barrier options are concerned, an influential agent in the market who has written such options and sees the price approaching the barrier may try to push the price further down, even momentarily and the cost of doing so may be smaller than the option payoff. In the case of Parisian options, this would be more difficult and expensive.

Parisian options, or more precisely Parisian times (the time when the option is activated or deactivated) are useful for modelling bankruptcy time; we note that following Chapter 11 of the United States Bankruptcy Code concerning reorganization of a business allows the firm to wait a certain time before being declared in bankruptcy.

For a generic continuous process Y and a given $t > 0$, we introduce $g_t^b(Y)$, the last time before t at which the process Y was at level b, i.e.,

$$g_t^b(Y) = \sup\{s \le t : Y_s = b\}.$$

For an UOPa option we need to consider the first time at which the underlying asset S is above H for a period greater than D, i.e.,

$$\begin{aligned} G_D^{+,H}(S) &= \inf\{t > 0 : (t - g_t^H(S))\mathbb{1}_{\{S_t > H\}} \ge D\} \\ &= \inf\{t > 0 : (t - g_t^h(X))\mathbb{1}_{\{X_t > h\}} \ge D\} = G_D^{+,h}(X) \end{aligned}$$

where $h = \ln(H/S_0)/\sigma$. If this stopping time occurs before the maturity then the UOPa option is worthless. The price of an UOPa call option is

$$\begin{aligned} \text{UOPa}(S_0, H, D; T) &= E_{\mathbb{Q}}\left(e^{-rT}(S_T - K)^+ \mathbb{1}_{\{G_D^{+,H}(S) > T\}}\right) \\ &= E_{\mathbb{Q}}\left(e^{-rT}(S_0 e^{\sigma X_T} - K)^+ \mathbb{1}_{\{G_D^{+,h}(X) > T\}}\right) \end{aligned}$$

or, using a change of probability (see Example 1.7.5.5)

$$\text{UOPa}(S_0, H, D; T) = e^{-(r+\nu^2/2)T}\mathbb{E}\left(e^{\nu W_T}(S_0 e^{\sigma W_T} - K)^+ \mathbb{1}_{\{G_D^{+,h}(W)>T\}}\right),$$

where W is a Brownian motion. The sum of the prices of an up-and-in (UIPa) and an UOPa option with the same strike and delay is obviously the price of a plain-vanilla European call.

In the same way, the value of a DIPa option with level L is defined using

$$G_D^{-,L}(S) = \inf\{t > 0 : (t - g_t^L(S))\mathbb{1}_{\{S_t<L\}} \geq D\}$$

which equals, in terms of X,

$$G_D^{-,\ell}(X) = \inf\{t > 0 : (t - g_t^\ell(X))\mathbb{1}_{\{X_t<\ell\}} \geq D\}$$

with $\ell = \dfrac{1}{\sigma}\ln(L/S_0)$. Then, the value of a DIPa option is equal to

$$\begin{aligned}\text{DIPa}(S_0, L, D; T) &= E_{\mathbb{Q}}\left(e^{-rT}(S_T - K)^+ \mathbb{1}_{\{G_D^{-,L}(S)<T\}}\right)\\ &= e^{-(r+\nu^2/2)T}\mathbb{E}\left(e^{\nu W_T}(S_0 e^{\sigma W_T} - K)^+ \mathbb{1}_{\{G_D^{-,\ell}(W)<T\}}\right)\\ &:= e^{-(r+\nu^2/2)T}\,{}^\star\text{DIPa}(S_0, L, D; T),\end{aligned}$$

where in this section, we define the general "star" transformation of a function f as

$${}^\star f(t) = e^{(r+\nu^2/2)t} f(t).$$

In the case $S_0 > L$, the computation of ${}^\star\text{DIPa}(S_0, L, D; T)$ can be reduced to the case $L = S_0$, i.e., $\ell = 0$. Indeed, for the option to be activated, the level L has to be reached by the process S (or equivalently, the level ℓ has to be reached by the process W) before the maturity T. Therefore, introducing $T_\ell = \inf\{t : W_t = \ell\}$, we obtain

$$\begin{aligned}&{}^\star\text{DIPa}(S_0, L, D; T) = \mathbb{E}(e^{\nu W_T}(S_0 e^{\sigma W_T} - K)^+ \mathbb{1}_{\{G_D^{-,\ell}(W)<T\}})\\ &= \mathbb{E}\left(e^{\nu(W_T - W_{T_\ell} + \ell)}(S_0 e^{\sigma(W_T - W_{T_\ell}+\ell)} - K)^+ \mathbb{1}_{\{G_D^{-,\ell}(W)<T\}}\right)\\ &= e^{\nu\ell}\mathbb{E}\left(e^{\nu Z_{T-T_\ell}}(S_0 e^{\sigma(\ell + Z_{T-T_\ell})} - K)^+ \mathbb{1}_{\{G_D^{-,0}(Z)<T-T_\ell\}}\right)\end{aligned}$$

where $Z_t = W_{t+T_\ell} - W_{T_\ell}$ is a BM independent of T_ℓ. Let us now introduce F_ℓ, the cumulative distribution function of T_ℓ.

$$\begin{aligned}&{}^\star\text{DIPa}(S_0, L, D; T)\\ &= e^{\nu\ell}\int_0^T dF_\ell(u)\mathbb{E}(e^{\nu Z_{T-u}}(S_0 e^{\sigma(Z_{T-u}+\ell)} - K)^+ \mathbb{1}_{\{G_D^{-,0}(Z)<T-u\}})\\ &= e^{\nu\ell}\int_0^T dF_\ell(u)\,{}^\star\text{DIPa}(S_0, S_0, D; T-u).\end{aligned}$$

We have used the fact that the computation of the law of the Parisian time below a level ℓ for a Brownian motion starting at level ℓ reduces to the law of the Parisian time below level 0 for a standard Brownian motion (starting from 0). Nevertheless, in the next subsection, we shall present a different approach.

4.4.1 The Law of $(G_D^{-,\ell}(W)\,,\,W_{G_D^{-,\ell}})$

In a first step, we compute the law of the pair (Parisian time, Brownian motion at the Parisian time) for a level $\ell = 0$.

Proposition 4.4.1.1 *Let W be a Brownian motion and $G_D^- := G_D^{-,0}(W)$. The random variables G_D^- and $W_{G_D^-}$ are independent and*

$$\mathbb{P}(W_{G_D^-} \in dx) = \frac{-x}{D} \exp\left(-\frac{x^2}{2D}\right) \mathbb{1}_{\{x<0\}}\, dx, \qquad (4.4.2)$$

$$\mathbb{E}\left(\exp\left(-\frac{\lambda^2}{2} G_D^-\right)\right) = \frac{1}{\Psi(\lambda\sqrt{D}\,)} \qquad (4.4.3)$$

where $\Psi(z) = \int_0^\infty x \exp\left(zx - \frac{x^2}{2}\right) dx = 1 + z\sqrt{2\pi}\mathcal{N}(z)e^{z^2/2}$.

PROOF: We have defined in Subsection 4.3.6 the wide slow Brownian filtration $(\mathcal{F}^+_{g_t}, t \geq 0)$. The r.v. G_D^- is an $(\mathcal{F}^+_{g_t}, t \geq 0)$- hence an $(\mathcal{F}_t, t \geq 0)$- stopping time. From results on meanders recalled in Subsection 4.3.7, the process

$$\left(\frac{1}{\sqrt{D}} \left|W_{g_{G_D^-}} + uD\right| , u \leq 1\right)$$

is a Brownian meander independent of $\mathcal{F}^+_{g_{G_D^-}}$, since $G_D^- = g_{G_D^-} + D$, the r.v. $\frac{1}{\sqrt{D}} W_{G_D^-}$ is distributed as $-m_1$, hence

$$\mathbb{P}(W_{G_D^-} \in dx) = \frac{-x}{D} \exp\left(-\frac{x^2}{2D}\right) \mathbb{1}_{\{x<0\}}\, dx,$$

and the variables G_D^- and $W_{G_D^-}$ are independent. From Proposition 4.3.8.1, the process

$$\Psi(-\lambda\mu_{t\wedge G_D^-}) \exp\left(-\frac{\lambda^2}{2}(t \wedge G_D^-)\right) ,\ t \geq 0\,,$$

(where μ denotes the Azéma martingale) is a $\mathcal{F}^+_{g_t}$-local martingale. Since, for $\lambda > 0$, $0 < -\lambda\mu_{t\wedge G_D^-} < \lambda D$, this process is bounded. Hence, using the optional sampling theorem at G_D^-, we obtain

$$\mathbb{E}\left(\Psi(-\lambda\mu_{G_D^-}\right) \exp(\left(-\frac{\lambda^2}{2}\, G_D^-\right) = \Psi(0) = 1$$

and the left-hand side equals $\Psi(\lambda\sqrt{D}\,)\,\mathbb{E}(\exp(-\frac{\lambda^2}{2}\,G_D^-))$. The formula

$$\mathbb{E}\left(\exp\left(-\frac{\lambda^2}{2}\,G_D^-\right)\right) = \frac{1}{\Psi(\lambda\sqrt{D}\,)}$$

follows. □

From the above proposition, we can easily deduce the law of the pair $(G_D^{-,\ell}, W_{G_D^{-,\ell}})$ in the case $\ell < 0$, as we now show.

Corollary 4.4.1.2 *Let $\ell < 0$. The random variables $G_D^{-,\ell}$ and $W_{G_D^{-,\ell}}$ are independent and their laws are given by*

$$\mathbb{P}(W_{G_D^{-,\ell}} \in dx) = \frac{dx}{D}\,\mathbb{1}_{\{x<\ell\}}\,(\ell - x)\exp\left(-\frac{(x-\ell)^2}{2D}\right) \quad (4.4.4)$$

$$\mathbb{E}\left(\exp\left(-\frac{\lambda^2}{2}G_D^{-,\ell}\right)\right) = \frac{\exp(\ell\lambda)}{\Psi(\lambda\sqrt{D})}\,. \quad (4.4.5)$$

PROOF: This study may be reduced to the previous one, with the help of the stopping time $T_\ell = T_\ell(W)$. Since

$$G_D^{-,\ell} = T_\ell + \widehat{G}_D^-$$

where

$$\widehat{G}_D^- = \inf\{t \geq 0 \,:\, \mathbb{1}_{\{\widehat{W}_t \leq 0\}}(t - g_t^0(\widehat{W})) \geq D\}$$

with $\widehat{W}_t = W_{T_\ell + t} - W_{T_\ell}$, it follows, from the independence between T_ℓ and $\widehat{G}_D^-$, that

$$\mathbb{E}\left(\exp\left(-\frac{\lambda^2}{2}G_D^{-,\ell}\right)\right) = \mathbb{E}\left(\exp\left(-\frac{\lambda^2}{2}T_\ell\right)\right)\,\mathbb{E}\left(\exp\left(-\frac{\lambda^2}{2}\widehat{G}_D^-\right)\right)\,.$$

The Laplace transform of the hitting time T_ℓ is known (see Proposition 3.1.6.1) and $\widehat{G}_D^- \stackrel{\text{law}}{=} G_D^-$; hence, by application of equality (4.4.3)

$$\mathbb{E}\left(\exp\left(-\frac{\lambda^2}{2}G_D^{-,\ell}\right)\right) = \frac{\exp(\ell\lambda)}{\Psi(\lambda\sqrt{D})}\,.$$

We obtain finally from (4.4.2) that

$$\begin{aligned}\mathbb{P}(W_{G_D^{-,\ell}} \in dx) &= \mathbb{P}(\widehat{W}_{\widehat{G}_D^-} - \ell \in dx - \ell) \\ &= \frac{dx}{D}\,\mathbb{1}_{\{x<\ell\}}\,(\ell - x)\exp\left(-\frac{(x-\ell)^2}{2D}\right)\,.\end{aligned}$$

Note in particular that, since $\Psi(0) = 1$, $\mathbb{P}(G_D^{-,\ell} < \infty) = 1$. □

Proposition 4.4.1.3 *In the case $\ell > 0$, the random variables $G_D^{-,\ell}$ and $W_{G_D^{-,\ell}}$ are independent. Their laws are characterized by*

$$\mathbb{E}(\exp(-\lambda G_D^{-,\ell})) = e^{-\lambda D}\,(1 - F_\ell(D)) + \frac{1}{\Psi(\sqrt{2\lambda D})} H(\sqrt{2\lambda}, \ell, D)\,,$$

where F_ℓ is the cumulative distribution function of T_ℓ and the function H is defined in (3.2.7), *and*

$$\mathbb{P}(W_{G_D^{-,\ell}} \in dx) = \mathbb{1}_{\{x \leq \ell\}} dx \left[e^{-(x-\ell)^2/(2D)} \mathbb{P}(T_\ell < D) \frac{\ell - x}{D} \right.$$
$$\left. + \frac{1}{\sqrt{2\pi D}} \left(e^{-x^2/(2D)} - e^{-(x-2\ell)^2/(2D)} \right) \right] .$$

Proof: In the case $\ell > 0$, the first excursion below ℓ begins at $t = 0$. We now use the obvious equality

$$\mathbb{E}(\exp(-\lambda G_D^{-,\ell})) = \mathbb{E}(\mathbb{1}_{\{T_\ell < D\}} \exp(-\lambda G_D^{-,\ell})) + \mathbb{E}(\mathbb{1}_{\{T_\ell > D\}} \exp(-\lambda G_D^{-,\ell}))\,.$$

On the set $\{T_\ell > D\}$, we have $G_D^{-,\ell} = D$. Therefore,

$$\begin{aligned}\mathbb{E}(\mathbb{1}_{\{T_\ell > D\}} \exp(-\lambda G_D^{-,\ell})) &= \exp(-\lambda D)\mathbb{P}(T_\ell > D) \\ &= \exp(-\lambda D)\,(1 - F_\ell(D))\,.\end{aligned}$$

Here, F_ℓ is the cumulative distribution function of T_ℓ (see formula 3.1.6). On the set $\{T_\ell < D\}$, we write, as in the proof of the previous corollary, $G_D^{-,\ell} = T_\ell + \widehat{G}_D^-$. Hence, on $(T_\ell < D)$, we have:

$$\mathbb{E}\left(\exp(-\lambda G_D^{-,\ell}) \,|\, \mathcal{F}_{T_\ell}\right) = \exp(-\lambda T_\ell)\,\mathbb{E}\left(\exp(-\lambda \widehat{G}_D^-)\right) .$$

Therefore, $\mathbb{E}\big(\mathbb{1}_{\{T_\ell < D\}} \exp(-\lambda G_D^{-,\ell})\big) = \dfrac{1}{\Psi(\sqrt{2\lambda D})}\mathbb{E}(\mathbb{1}_{\{T_\ell < D\}} \exp(-\lambda T_\ell))$. The quantity $\mathbb{E}(\mathbb{1}_{\{T_\ell < D\}} \exp(-\lambda T_\ell))$ has been computed in Subsection 3.2.4, and is equal to $H(\sqrt{2\lambda}, \ell, D)$ (see formula (3.2.7)).

It follows that

$$\mathbb{E}(\exp(-\lambda G_D^{-,\ell})) = e^{-\lambda D}\,(1 - F_\ell(D)) + \frac{1}{\Psi(\sqrt{2\lambda D})} H(\sqrt{2\lambda}, \ell, D)\,.$$

The law of $W_{G_D^{-,\ell}}$ can easily be deduced from the following three equalities:

$$W_{G_D^{-,\ell}} = (\ell + \widehat{W}_{\widehat{G}_D^-})\mathbb{1}_{\{T_\ell < D\}} + W_D \mathbb{1}_{\{T_\ell > D\}}$$

$$\mathbb{P}(\ell + \widehat{W}_{\widehat{G}_D^-} \in dx, T_\ell < D) = \mathbb{P}(T_\ell < D)\mathbb{1}_{\{x \leq \ell\}}(\ell - x)\,\exp\left(-\frac{(x-\ell)^2}{2D}\right)\frac{dx}{D}$$

$$\mathbb{P}(W_D \in dx, T_\ell > D) = \frac{dx}{\sqrt{2\pi D}} \left(\exp\left(-\frac{x^2}{2D}\right) - \exp\left(-\frac{(x-2\ell)^2}{2D}\right)\right) \mathbb{1}_{\{x \le \ell\}} .$$

□

Comment 4.4.1.4 The independence property of a stopping time τ and the position of the Brownian motion at that time B_τ is a fairly rare phenomenon for Brownian stopping times; it is satisfied for $\tau = G_D^{-,\ell}$. It can be proved, for example that, if T is a bounded stopping time such that T and W_T are independent, then T is a constant. A more general study of stopping times which enjoy this independence property can be found in De Meyer et al. [246, 247]. See also the following exercise.

Exercise 4.4.1.5 Let $T_a^* = \inf\{t : |W_t| = a\}$. Prove that the r.v's T_a^* and $W_{T_a^*}$ are independent and show that $W_{T_a^*}$ is symmetric with values $\pm a$. See Section 3.5. ◁

4.4.2 Valuation of a Down-and-In Parisian Option

We have seen that the price of a down-and-in Parisian option is given by

$$\mathrm{DIPa}(S_0, L, D; T) = e^{-(r+\nu^2/2)T}\, {}^\star\mathrm{DIPa}(S_0, L, D; T)$$

where

$${}^\star\mathrm{DIPa}(S_0, L, D; T) = \mathbb{E}\left(\mathbb{1}_{\{G_D^{-,\ell} \le T\}} \mathbb{E}\left(e^{\nu W_T}(S_0 e^{\sigma W_T} - K)^+ | \mathcal{F}_{G_D^{-,\ell}}\right)\right) .$$

From the strong Markov property

$${}^\star\mathrm{DIPa}(S_0, L, D; T) = \mathbb{E}(\mathbb{1}_{\{G_D^{-,\ell} \le T\}} \mathcal{P}_{T - G_D^{-,\ell}}(\psi)(W_{G_D^{-,\ell}}))$$

with

$$\begin{cases} \psi(y) = e^{\nu y}(S_0 e^{\sigma y} - K)^+, \\ \mathcal{P}_t f(z) = \dfrac{1}{\sqrt{2\pi t}} \displaystyle\int_{-\infty}^{\infty} f(y) \exp\left(-\frac{(y-z)^2}{2t}\right) dy \, . \end{cases}$$

Denote by φ the density of $W_{G_D^{-,\ell}}$ and recall that $G_D^{-,\ell}$ and $W_{G_D^{-,\ell}}$ are independent. Then,

$$\begin{aligned} {}^\star\mathrm{DIPa}(S_0, L, D; T) &= \textstyle\int_{-\infty}^{\infty} \varphi(dz)\, \mathbb{E}(\mathbb{1}_{\{G_D^{-,\ell} \le T\}} \mathcal{P}_{T - G_D^{-,\ell}}(\psi)(z)) \\ &= \textstyle\int_{-\infty}^{\infty} dy\, \psi(y)\, h_\ell(T, y) \end{aligned} \tag{4.4.6}$$

where the function h_ℓ is defined by

$$h_\ell(t, y) = \int_{-\infty}^{\infty} \varphi(dz)\, \gamma(t, y - z) \tag{4.4.7}$$

with

$$\gamma(t,x) = \mathbb{E}\left(\frac{\mathbb{1}_{\{G_D^{-,\ell}\le t\}}}{\sqrt{2\pi(t-G_D^{-,\ell})}}\exp\left(-\frac{x^2}{2(t-G_D^{-,\ell})}\right)\right).$$

Then, replacing ψ by its value, we obtain

$$^\star\text{DIPa}(S_0,L;D,T) = \int_k^\infty dy\, e^{\nu y}(S_0 e^{\sigma y}-K)h_\ell(T,y) \tag{4.4.8}$$

where $k = \frac{1}{\sigma}\ln\frac{K}{S_0}$. The computation of this quantity relies on the knowledge of h_ℓ; however this function h_ℓ is only known through its time Laplace transform $\widehat{h}_\ell$ which is given in the following two theorems.

Theorem 4.4.2.1 *In the case $S_0 > L$ (i.e., $\ell < 0$) the function $t \to h_\ell(t,y)$ is characterized by its Laplace transform: for $\lambda > 0$,*

$$\widehat{h}_\ell(\lambda,y) = \frac{e^{\ell\sqrt{2\lambda}}}{D\sqrt{2\lambda}\,\Psi(\sqrt{2\lambda D})}\int_0^\infty dz\, z\exp\left(-\frac{z^2}{2D}-|y+z-\ell|\sqrt{2\lambda}\right)$$

where $\Psi(z)$ is defined in (4.3.12). If $y > \ell$, then

$$\widehat{h}_\ell(\lambda,y) = \frac{\Psi(-\sqrt{2\lambda D})}{\Psi(\sqrt{2\lambda D})}\,\frac{e^{(2\ell-y)\sqrt{2\lambda}}}{\sqrt{2\lambda}}.$$

Proof: In the case $S_0 > L$, from (4.4.4), the density φ of $W_{G_D^{-,\ell}}$ is

$$\varphi(x) = \mathbb{P}(W_{G_D^{-,\ell}}\in dx)/dx = \frac{1}{D}(\ell-x)\,\exp\left(-\frac{(x-\ell)^2}{2D}\right)\mathbb{1}_{\{x\le\ell\}}.$$

The function h_ℓ is defined in terms of φ and γ. Thus, the knowledge of the Laplace transform of γ will lead to the knowledge of $\widehat{h}_\ell$.

For $\lambda > 0$, we obtain, with an obvious change of variable,

$$\begin{aligned}\int_0^\infty dt\, e^{-\lambda t}\gamma(t,x) &= \mathbb{E}\left[\int_{G_D^{-,\ell}}^\infty dt\,\frac{e^{-\lambda t}}{\sqrt{2\pi(t-G_D^{-,\ell})}}\exp\left(-\frac{x^2}{2(t-G_D^{-,\ell})}\right)\right]\\ &= \mathbb{E}(e^{-\lambda G_D^{-,\ell}})\int_0^\infty dt\,\exp\left(-\frac{x^2}{2t}\right)\frac{e^{-\lambda t}}{\sqrt{2\pi t}}.\end{aligned} \tag{4.4.9}$$

The integral on the right of (4.4.9) is the resolvent kernel of Brownian motion and is equal to $\frac{1}{\sqrt{2\lambda}}e^{-|x|\sqrt{2\lambda}}$. By substituting this result in (4.4.9) and using the Laplace transform of $G_D^{-,\ell}$ given in (4.4.5), we obtain:

$$\int_0^\infty dt\, e^{-\lambda t}\gamma(t,x) = \frac{e^{-(|x|-\ell)\sqrt{2\lambda}}}{\sqrt{2\lambda}\,\Psi(\sqrt{2\lambda D})}\,. \tag{4.4.10}$$

Therefore, from the definition (4.4.7) of h_ℓ, its Laplace transform, $\widehat{h}_\ell(\lambda,y)$ is given by

$$\begin{aligned}\int_0^\infty dt\, e^{-\lambda t}h_\ell(t,y) &= \int_{-\infty}^{\ell}\frac{dz}{D}\,(\ell-z)e^{-\frac{(\ell-z)^2}{2D}}\int_0^\infty dt\, e^{-\lambda t}\gamma(t,y-z)\\ &= \int_0^\infty \frac{du}{D}\,ue^{-\frac{u^2}{2D}}\int_0^\infty dt\, e^{-\lambda t}\gamma(t,y+u-\ell)\end{aligned}$$

$$= \frac{e^{\ell\sqrt{2\lambda}}}{D\sqrt{2\lambda}\,\Psi(\sqrt{2\lambda D})}\int_0^\infty du\, u\,\exp\left(-\frac{u^2}{2D}-|y+u-\ell|\sqrt{2\lambda}\right).$$

The corresponding integral

$$K_{\lambda,D}(a) := \frac{1}{D}\int_0^\infty du\, u\,\exp\left(-\frac{u^2}{2D}-|u+a|\sqrt{2\lambda}\right)$$

can be easily evaluated as follows.

▶ If $a>0$, using the change of variables $u=z\sqrt{D}$, we obtain

$$K_{\lambda,D}(a) = \exp(-a\sqrt{2\lambda})\Psi(-\sqrt{2\lambda D})$$

and this leads to the formula for $y>\ell$.

▶ If $a<0$, a similar method leads to

$$K_{\lambda,D}(a) = e^{a\sqrt{2\lambda}} + 2\sqrt{\pi\lambda D}e^{\lambda D}$$
$$\times\left(e^{a\sqrt{2\lambda}}\left[\mathcal{N}\left(\frac{-a}{\sqrt{D}}-\sqrt{2\lambda D}\right)-\mathcal{N}\left(-\sqrt{2\lambda D}\right)\right]-e^{-a\sqrt{2\lambda}}\mathcal{N}\left(\frac{a}{\sqrt{D}}-\sqrt{2\lambda D}\right)\right).$$

As a partial check, note that if $D=0$, the Parisian option is a standard barrier option. The previous computation simplifies and we obtain

$$\widehat{h}_\ell(\lambda,y) = \frac{e^{\ell\sqrt{2\lambda}}}{\sqrt{2\lambda}}e^{-|\ell-y|\sqrt{2\lambda}}\,.$$

It is easy to invert $\widehat{h}_\ell$ and we are back to the formula (3.6.28) for the price of a DIC option obtained in Theorem 3.6.6.2 . □

Remark 4.4.2.2 The quantity $\Psi(-\sqrt{2\lambda D})$ is a Laplace transform, as well as the quantity $\dfrac{e^{(2\ell-y)\sqrt{2\lambda}}}{\sqrt{2\lambda}}$. Therefore, in order to invert $\widehat{h}_\ell$ in the case $y>\ell$, it suffices to invert the Laplace transform $\dfrac{1}{\Psi(\sqrt{2\lambda D})}$. This is not easy: see Schröder [770] for some computation.

Theorem 4.4.2.3 *In the case $S_0 < L$ (i.e., $\ell > 0$), the function $h_\ell(t,y)$ is characterized by its Laplace transform, for $\lambda > 0$,*

$$\begin{aligned}\widehat{h}_\ell(\lambda, y) &= \widehat{g}(t,y) \\ &+ \frac{1}{D\sqrt{2\lambda}\,\Psi(\sqrt{2\lambda D})} H(\sqrt{2\lambda}, \ell, D) \int_0^\infty dz\, z \exp\left(-\frac{z^2}{2D} - |y-\ell+z|\sqrt{2\lambda}\right).\end{aligned}$$

where g is defined in the following equality (4.4.12), and H is defined in (3.2.7).

Proof: In the case $\ell > 0$, the Laplace transform of $h_\ell(.,y)$ is more complicated. Denoting again by φ the law of $W_{G_D^{-,\ell}}$, we obtain

$$\int_0^\infty dt\, e^{-\lambda t} h_\ell(t,y) = \mathbb{E}\left(\int_{-\infty}^\infty \varphi(dz)\, e^{-\lambda G_D^{-,\ell}} \frac{1}{\sqrt{2\lambda}} \exp(-|y-z|\sqrt{2\lambda})\right).$$

Using the previous results, and the cumulative distribution function F_ℓ of T_ℓ,

$$\begin{aligned}&\int_0^\infty dt\, e^{-\lambda t} h_\ell(t,y) = \qquad (4.4.11)\\ &\frac{1}{D\sqrt{2\lambda}\,\Psi(\sqrt{2\lambda D})} \int_0^\infty dz\, z \exp\left(-\frac{z^2}{2D} - |y-\ell+z|\sqrt{2\lambda}\right) \int_0^D F_\ell(dx)\, e^{-\lambda x} \\ &+ \frac{e^{-\lambda D}}{2\sqrt{\lambda \pi D}} \int_{-\infty}^\ell dz \left(\exp\left(-\frac{z^2}{2D}\right) - \exp - \frac{(z-2\ell)^2}{2D}\right) e^{-|y-z|\sqrt{2\lambda}}.\end{aligned}$$

We know from Remark 3.1.6.3 that $\frac{1}{\sqrt{2\lambda}} \exp(-|a|\sqrt{2\lambda})$ is the Laplace transform of $\frac{1}{\sqrt{2\pi t}} \exp(-\frac{a^2}{2t})$. Hence, the second term on the right-hand side of (4.4.11) is the time Laplace transform of $g(\cdot, y)$ where

$$g(t,y) = \frac{\mathbb{1}_{\{t>D\}}}{2\pi\sqrt{D(t-D)}} \int_{-\infty}^\ell e^{\frac{(y-z)^2}{2(t-D)}} \left(e^{-\frac{z^2}{2D}} - e^{-\frac{(z-2\ell)^2}{2D}}\right) dz\,. \qquad (4.4.12)$$

We have not be able go further in the inversion of the Laplace transform.

A particular case: If $y > \ell$, the first term on the right-hand side of (4.4.11) is equal to

$$\frac{\Psi(-\sqrt{2\lambda D})}{\Psi(\sqrt{2\lambda D})} \frac{e^{-(y-\ell)\sqrt{2\lambda}}}{\sqrt{2\lambda}} \int_0^D F_\ell(dx) e^{-\lambda x}.$$

This term is the product of four Laplace transforms; however, the inverse transform of $\frac{1}{\Psi(\sqrt{2\lambda D})}$ is not identified. □

Comment 4.4.2.4 Parisian options are studied in Avellaneda and Wu [30], Chesney et al. [175], Cornwall et al. [196], Dassios [213], Gauthier [376] and Haber et al. [415]. Numerical analysis is carried out in Bernard et al. [76], Costabile [198], Labart and Lelong [556] and Schröder [770]. An approximation with an implied barrier is done in Anderluh and Van der Weide [14]. Double-sided Parisian options are presented in Anderluh and Van der Weide [15], Dassios and Wu [215, 216, 217] and Labart and Lelong [557]. The "Parisian" time models a default time in Çetin et al. [158] and in Chen and Suchanecki [162, 163]. Cumulative Parisian options are developed in Detemple [252], Hugonnier [451] and Moraux [657]. Their Parisian name is due to their birth place as well as to the meanders of the Seine River which lead many tourists to excursions around Paris.

Exercise 4.4.2.5 We have just introduced Parisian down-and-in options with a call feature, denoted here C_{DIPa}. One can also define Parisian up-and-in options P_{UIPa} with a put feature, i.e., with payoff $(K - S_T)^+ \mathbb{1}_{\{G_D^{+,L} < T\}}$. Prove the symmetry formula

$$C_{\text{DIPa}}(S_0, K, L; r, \delta; D, T) = K S_0 P_{\text{UIPa}}(S_0^{-1}, K^{-1}, L^{-1}, \delta, r; D, T)\,.$$

◁

4.4.3 PDE Approach

In Haber et al. [415] and in Wilmott [846], the following PDE approach to valuation of Parisian option is presented, in the case $\delta = 0$. The value at time t of a down-and-out Parisian option is a function of the three variables $t, S_t, t - g_t$, i.e., DOPa $= \Phi(T-t, S_t, t-g_t)$ and the discounted price process $e^{-rt}\Phi(T-t, S_t, t-g_t)$ is a $\mathbb{Q}$-martingale. Using the fact that $(g_t, t \geq 0)$ is an increasing process, Itô's calculus gives

$$d[e^{-rt}\Phi(t, S_t, t-g_t)] = e^{-rt}\Big[-r\Phi dt + (\partial_t \Phi)\, dt + (\partial_x \Phi)\, dS_t + (\partial_u \Phi)\,(dt - dg_t) + \frac{1}{2}\sigma^2 S_t^2 (\partial_{xx}\Phi)\, dt\Big]$$

between two jumps of g_t. (Here, u is the third variable of the function Φ). Therefore, the dt terms must sum to 0 giving

$$\begin{cases} -r\Phi + \partial_t\Phi + xr\partial_x\Phi + \partial_u\Phi + \dfrac{1}{2}\sigma^2 x^2 \partial_{xx}\Phi = 0, & \text{for } u < D \\ \partial_u \Phi(t, x, 0) = 0\,. \end{cases}$$

with the boundary conditions

$$\begin{cases} \Phi(t,x,u) = \Phi(t,x,0), & \text{for } x \geq L \\ \Phi(t,x,u) = 0, & \text{for } u \geq D,\ x < L\,. \end{cases}$$

4.4.4 American Parisian Options

American Parisian options are also considered. Grau [403] combined Monte Carlo simulations and PDE solvers (see also Grau and Kallsen [404]) in order to price European and American Parisian options. The PDE approaches developed by Haber et al. [415] and Wilmott [846] can also be used in order to value these options. In the same setting, where the risk-neutral dynamics of the underlying are given by (4.4.1), Chesney and Gauthier [172] developed a probabilistic approach for the pricing of American Parisian options. They derived the following result for currency options:

Proposition 4.4.4.1 *The price of an American Parisian down-and-out call (*ADOPa*) can be decomposed as follows:*

$$\begin{aligned}
&\mathrm{ADOPa}\left(S_0, L, D, T\right) = \mathrm{DOPa}\left(S_0, L, D, T\right) \\
&+ \delta S_0 \int_0^T e^{-\alpha u} \mathbb{E}\left[\mathbb{1}_{\{W_u \geq \bar{b}(u)\}} \mathbb{1}_{\{u < G_D^{-,\ell}(W)\}} \exp\left(\left(\nu + \sigma\right) W_u\right)\right] du \\
&- rK \int_0^T e^{-\alpha u} \mathbb{E}\left[\mathbb{1}_{\{W_u \geq \bar{b}(u)\}} \mathbb{1}_{\{u < G_D^{-,\ell}(W)\}} \exp\left(\nu W_u\right)\right] du
\end{aligned}$$

where

$$\begin{aligned}
&\alpha = r + \frac{\nu^2}{2},\ \nu = \frac{1}{\sigma}\left(r - \delta - \frac{\sigma^2}{2}\right),\ \bar{b}(u) = \frac{1}{\sigma} \ln\left(\frac{b_c(T-u)}{S_0}\right), \\
&\ell = \frac{1}{\sigma} \ln\left(\frac{L}{S_0}\right) \leq 0
\end{aligned}$$

and where $\{b_c(T-u), u \in [0,T]\}$ *is the exercise boundary (see Section* 3.11 *for the general definition). Here, the process* W *is a Brownian motion.*

This decomposition can also be written as follows:

$$\begin{aligned}
\mathrm{ADOPa}\left(S_0, L, D, T\right) &= \mathrm{DOPa}\left(S_0, L, D, T\right) + \delta \int_0^T \mathrm{DOPa}(S_0, b_c(T-u), u) du \\
&+ \delta \int_0^T \left(b_c(T-u) - \frac{r}{\delta} K\right) \mathrm{BinDOCPa}\left(S_0, b_c(T-u), u\right)\, du
\end{aligned}$$

where $\mathrm{DOPa}(S_0, b_c(T-u), u)$ is the price of the European Parisian down-and-out call option with maturity u, strike price $b_c(T-u)$, barrier L and delay D, $\mathrm{BinDOCPa}\left(S_0, b_c(T-u), u\right)$ is the price of a Parisian binary call (see Subsections 3.6.2 and 3.6.3 for the definitions of binary calls and binary barrier options) which generates at maturity a pay-off of one monetary unit if the underlying value is higher than the strike price and if the first instant –when the underlying price spends consecutively more than D units of time under the level L_1 – is greater than the maturity u. Otherwise, the payoff is equal to zero.

Denote by ADOPa (S_0, L, D) the price of a perpetual American Parisian option. The following proposition is obtained:

Proposition 4.4.4.2 *The price of a perpetual American Parisian down-and-out call is given by:*

$$\begin{aligned}\mathrm{ADOPa}\,(S_0, L, D) &= \delta \int_0^{+\infty} \mathrm{DOPa}(S_0, L_c, u)\, du \\ &\quad + \delta \int_0^{+\infty} \left(L_c - \frac{r}{\delta} K\right) \mathrm{BinDOCPa}\,(S_0, L_c, u)\; du\end{aligned}$$

or

$$\mathrm{ADOPa}(S_0, L, D) = \left(1 - \frac{\Psi(-\kappa\sqrt{D})}{\Psi(\kappa\sqrt{D})} e^{2\ell\kappa}\right) \frac{1}{\sigma\kappa} \left(\frac{S_0}{L_c}\right)^{\gamma} \left(\frac{\delta L_c}{\gamma - 1} - \frac{r}{\gamma} K\right)$$

with $\kappa = \sqrt{2r + \nu^2}$, $\gamma = \frac{-\nu + \sqrt{2r+\nu^2}}{\sigma}$ *and where the exercise boundary* L_c *is defined implicitly by:*

$$L_c - K = \left(1 - \frac{\Psi(-\kappa\sqrt{D})}{\Psi(\kappa\sqrt{D})} \left(\frac{L}{L_c}\right)^{2\frac{\kappa}{\sigma}}\right) \frac{1}{\sigma\kappa} \left(\frac{\delta L_c}{\gamma - 1} - \frac{r}{\gamma} K\right)$$

where the function Ψ *is defined in equation* (4.3.12).

Solutions when the excursion has already started and for the "in" barrier case are also derived. The latter case is easier to analyze. Indeed, in this setting, the option holder cannot do or decide anything before the option is activated; once the option is activated then it does not have a barrier anymore, but is just a plain vanilla American call. The exercise frontier for an American Parisian "in" barrier option is therefore the exercise frontier of the corresponding plain vanilla option, starting at the activation time.