3

Hitting Times: A Mix of Mathematics and Finance

In this chapter, a Brownian motion $(W_t, t \geq 0)$ starting from 0 is given on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and $\mathbf{F} = (\mathcal{F}_t, t \geq 0)$ is its natural filtration. As before, the function $\mathcal{N}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du$ is the cumulative function of a standard Gaussian law $\mathcal{N}(0,1)$. We establish well known results on first hitting times of levels for BM, BM with drift and geometric Brownian motion, and we study barrier and lookback options. However, we emphasize that the main results on barrier option valuation are obtained below without any knowledge of hitting time laws but using only the strong Markov property. In the last part of the chapter, we present applications to the structural approach of default risk and real options theory and we give a short presentation of American options.

For a continuous path process X, we denote by $T_a(X)$ (or, if there is no ambiguity, T_a) the first hitting time of the level a for the process X defined as

$$T_a(X) = \inf\{t \geq 0 \,:\, X_t = a\}\,.$$

The first time when X is above (resp. below) the level a is

$$T_a^+ = \inf\{t \geq 0 \,:\, X_t \geq a\}, \quad \text{resp.} \quad T_a^- = \inf\{t \geq 0 \,:\, X_t \leq a\}\,.$$

For $X_0 = x$ and $a > x$, we have $T_a^+ = T_a$, and $T_a^- = 0$ whereas for $a < x$, we have $T_a^- = T_a$, and $T_a^+ = 0$. In what follows, we shall write hitting time for first hitting time. We denote by M_t^X (resp. m_t^X) the running maximum (resp. minimum)

$$M_t^X = \sup_{s \leq t} X_s, \; m_t^X = \inf_{s \leq t} X_s\,.$$

In case X is a BM, we shall frequently omit the superscript and denote by M_t the running maximum of the BM. In this chapter, no martingale will be denoted M_t!

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3.1 Hitting Times and the Law of the Maximum for Brownian Motion

We first study the law of the pair of random variables (W_t, M_t) where M is the maximum process of the Brownian motion W, i.e., $M_t := \sup_{s \le t} W_s$. In a similar way, we define the minimum process m as $m_t := \inf_{s \le t} W_s$. Let us remark that the process M is an increasing process, with positive values, and that $M \overset{\text{law}}{=} (-m)$. Then, we deduce the law of the hitting time of a given level by the Brownian motion.

3.1.1 The Law of the Pair of Random Variables (W_t, M_t)

Let us prove the reflection principle.

Proposition 3.1.1.1 (Reflection principle.) *For $y \ge 0$, $x \le y$, one has:*

$$\boxed{\mathbb{P}(W_t \le x\,, M_t \ge y) = \mathbb{P}(W_t \ge 2y - x)\,.} \tag{3.1.1}$$

PROOF: Let $T_y^+ = \inf\{t : W_t \ge y\}$ be the first time that the BM W is greater than y. This is an $\mathbf{F}$-stopping time and $\{T_y^+ \le t\} = \{M_t \ge y\}$ for $y \ge 0$. Furthermore, for $y \ge 0$ and by relying on the continuity of Brownian motion paths, $T_y^+ = T_y$ and $W_{T_y} = y$. Therefore

$$\mathbb{P}(W_t \le x\,, M_t \ge y) = \mathbb{P}(W_t \le x\,, T_y \le t) = \mathbb{P}(W_t - W_{T_y} \le x - y\,, T_y \le t)\,.$$

For the sake of simplicity, we denote $\mathbb{E}_{\mathbb{P}}(\mathbb{1}_A | T_y) = \mathbb{P}(A | T_y)$. By relying on the strong Markov property, we obtain

$$\begin{aligned}\mathbb{P}(W_t - W_{T_y} \le x - y\,, T_y \le t) &= \mathbb{E}(\mathbb{1}_{\{T_y \le t\}}\, \mathbb{P}(W_t - W_{T_y} \le x - y\,|T_y)) \\ &= \mathbb{E}(\mathbb{1}_{\{T_y \le t\}}\, \Phi(T_y))\end{aligned}$$

with $\Phi(u) = \mathbb{P}(\widetilde{W}_{t-u} \le x - y)$ where $(\widetilde{W}_u := W_{T_y+u} - W_{T_y}, u \ge 0)$ is a Brownian motion independent of $(W_t, t \le T_y)$. The process $\widetilde{W}$ has the same law as $-\widetilde{W}$. Therefore $\Phi(u) = \mathbb{P}(\widetilde{W}_{t-u} \ge y - x)$ and by proceeding backward

$$\begin{aligned}\mathbb{E}(\mathbb{1}_{\{T_y \le t\}}\, \Phi(T_y)) &= \mathbb{E}[\mathbb{1}_{\{T_y \le t\}} \mathbb{P}(W_t - W_{T_y} \ge y - x\,|T_y)] \\ &= \mathbb{P}(W_t \ge 2y - x\,, T_y \le t)\,.\end{aligned}$$

Hence,

$$\mathbb{P}(W_t \le x, M_t \ge y) = \mathbb{P}(W_t \ge 2y - x, M_t \ge y)\,. \tag{3.1.2}$$

The right-hand side of (3.1.2) is equal to $\mathbb{P}(W_t \ge 2y - x)$ since, from $x \le y$ we have $2y - x \ge y$ which implies that, on the set $\{W_t \ge 2y - x\}$, one has

$M_t \geq y$ (i.e., the hitting time T_y is smaller than t). □

From the symmetry of the normal law, it follows that

$$\mathbb{P}(W_t \leq x, M_t \geq y) = \mathbb{P}(W_t \geq 2y - x) = \mathcal{N}\left(\frac{x-2y}{\sqrt{t}}\right).$$

We now give the joint law of the pair of r.v's (W_t, M_t) for fixed t.

Theorem 3.1.1.2 *Let W be a BM starting from 0 and $M_t = \sup_{s \leq t} W_s$. Then,*

$$\textit{for} \quad y \geq 0, x \leq y, \quad \mathbb{P}(W_t \leq x, M_t \leq y) = \mathcal{N}\left(\frac{x}{\sqrt{t}}\right) - \mathcal{N}\left(\frac{x-2y}{\sqrt{t}}\right) \tag{3.1.3}$$

$$\begin{aligned} \textit{for} \quad y \geq 0, x \geq y, \quad \mathbb{P}(W_t \leq x, M_t \leq y) &= \mathbb{P}(M_t \leq y) \\ &= \mathcal{N}\left(\frac{y}{\sqrt{t}}\right) - \mathcal{N}\left(\frac{-y}{\sqrt{t}}\right), \end{aligned} \tag{3.1.4}$$

$$\textit{for} \quad y \leq 0, \quad \mathbb{P}(W_t \leq x, M_t \leq y) = 0.$$

The distribution of the pair of r.v's (W_t, M_t) is

$$\boxed{\mathbb{P}(W_t \in dx, M_t \in dy) = \mathbb{1}_{\{y \geq 0\}} \mathbb{1}_{\{x \leq y\}} \frac{2(2y-x)}{\sqrt{2\pi t^3}} \exp\left(-\frac{(2y-x)^2}{2t}\right) dx\, dy} \tag{3.1.5}$$

Proof: From the reflection principle it follows that, for $y \geq 0, x \leq y$,

$$\begin{aligned} \mathbb{P}(W_t \leq x, M_t \leq y) &= \mathbb{P}(W_t \leq x) - \mathbb{P}(W_t \leq x, M_t \geq y) \\ &= \mathbb{P}(W_t \leq x) - \mathbb{P}(W_t \geq 2y - x), \end{aligned}$$

hence the equality (3.1.3) is obtained.

For $0 \leq y \leq x$, since $M_t \geq W_t$ we get:

$$\mathbb{P}(W_t \leq x, M_t \leq y) = \mathbb{P}(W_t \leq y, M_t \leq y) = \mathbb{P}(M_t \leq y).$$

Furthermore, by setting $x = y$ in (3.1.3)

$$\mathbb{P}(W_t \leq y, M_t \leq y) = \mathcal{N}\left(\frac{y}{\sqrt{t}}\right) - \mathcal{N}\left(\frac{-y}{\sqrt{t}}\right),$$

hence the equality (3.1.5) is obtained. Finally, for $y \leq 0$,

$$\mathbb{P}(W_t \leq x, M_t \leq y) = 0$$

since $M_t \geq M_0 = 0$. □

Note that we have also proved that the process B defined for $y > 0$ as

$$B_t = W_t \mathbb{1}_{\{t < T_y\}} + (2y - W_t) \mathbb{1}_{\{T_y \leq t\}}$$

is a Brownian motion.

Comment 3.1.1.3 It is remarkable that Bachelier [39, 40] obtained the reflection principle for Brownian motion, extending the result of Désiré André for random walks. See Taqqu [819] for a presentation of Bachelier's work.

Remark 3.1.1.4 Let $T_0 = \inf\{t > 0 : W_t = 0\}$. Then $\mathbb{P}(T_0 = 0) = 1$.

Exercise 3.1.1.5 We have proved that

$$\mathbb{P}(W_t \in dx, M_t \in dy) = \mathbb{1}_{\{y \geq 0\}} \mathbb{1}_{\{x \leq y\}} \frac{1}{\sqrt{t}} g\left(\frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}}\right) dx\, dy$$

where

$$g(x, y) = \frac{2(2y - x)}{\sqrt{2\pi}} \exp\left(-\frac{(2y - x)^2}{2}\right).$$

Prove that $(M_t, W_t, t \geq 0)$ is a Markov process and give its semi-group in terms of g. ◁

3.1.2 Hitting Times Process

Proposition 3.1.2.1 *Let W be a Brownian motion and, for any $y > 0$, define $T_y = \inf\{t : W_t = y\}$. The increasing process $(T_y, y \geq 0)$ has independent and stationary increments. It enjoys the scaling property*

$$(T_{\lambda y}, y \geq 0) \stackrel{\text{law}}{=} (\lambda^2 T_y, y \geq 0).$$

Proof: The increasing property follows from the continuity of paths of the Brownian motion. For $z > y$,

$$T_z - T_y = \inf\{t \geq 0 : W_{T_y + t} - W_{T_y} = z - y\}.$$

Hence, the independence and the stationarity properties follow from the strong Markov property. From the scaling property of BM, for $\lambda > 0$,

$$T_{\lambda y} = \inf\left\{t : \frac{1}{\lambda} W_t = y\right\} \stackrel{\text{law}}{=} \lambda^2 \inf\{t : \widehat{W}_t = y\}$$

where $\widehat{W}$ is the BM defined by $\widehat{W}_t = \frac{1}{\lambda} W_{\lambda^2 t}$. □

The process $(T_y, y \geq 0)$ is a particular stable subordinator (with index 1/2) (see ↣ Section 11.6). Note that this process is not continuous but admits a right-continuous left-limited version. The non-continuity property may seem surprising at first, but can easily be understood by looking at the following case. Let W be a BM and $T_1 = \inf\{t : W_t = 1\}$. Define two random times g and θ as

$$g = \sup\{t \le T_1 : W_t = 0\},\ \theta = \inf\left\{t \le g : W_t = \sup_{s\le g} W_s\right\}$$

and $\Sigma = W_\theta$. Obviously

$$\theta = T_\Sigma < g < T_{\Sigma+} := \inf\{t : W_t > \Sigma\}\,.$$

See Karatzas and Shreve [513] Chapter 6, Theorem 2.1. for more comments and ↣ Example 11.2.3.5 for a different explanation.

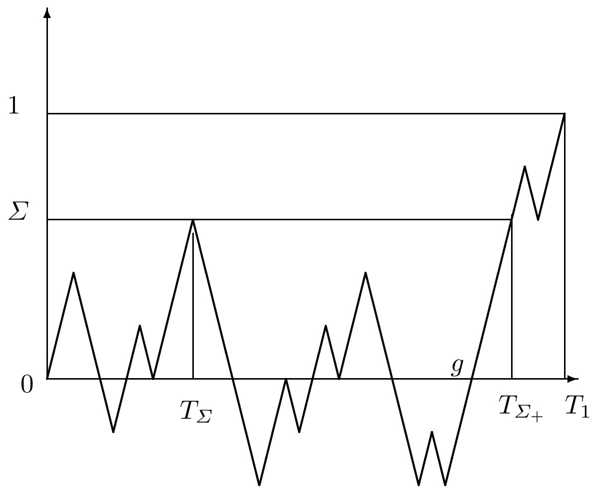

Fig. 3.1 Non continuity of T_y

3.1.3 Law of the Maximum of a Brownian Motion over $[0, t]$

Proposition 3.1.3.1 *For fixed t, the random variable M_t has the same law as $|W_t|$.*

PROOF: This follows from the equality (3.1.4) which states that

$$\mathbb{P}(M_t \le y) = \mathbb{P}(W_t \le y) - \mathbb{P}(W_t \le -y)\,.$$

□

Comments 3.1.3.2 (a) Obviously, the process M does not have the same law as the process $|W|$. Indeed, the process M is an increasing process, whereas this is not the case for the process $|W|$. Nevertheless, there are some further equalities in law, e.g., $M-W \stackrel{\text{law}}{=} |W|$, this identity in law taking place between processes (see Lévy's equivalence Theorem 4.1.7.2 in ↣ Subsection 4.1.7).

(b) Seshadri's result states that the two random variables $M_t(M_t - W_t)$ and W_t are independent and that $M_t(M_t - W_t)$ has an exponential law (see Yor [867, 869]).

Exercise 3.1.3.3 Prove that as a consequence of the reflection principle (formula (3.1.1)), for any fixed t:
(i) $2M_t - W_t$ is distributed as $\|B_t^{(3)}\|$ where $B^{(3)}$ is a 3-dimensional BM, starting from 0,
(ii) given $2M_t - W_t = r$, both M_t and $M_t - W_t$ are uniformly distributed on $[0, r]$.
This result is a small part of Pitman's theorem (see ↣ Comments 4.1.7.3 and ↣ Section 5.7). ◁

3.1.4 Laws of Hitting Times

For $x > 0$, the law of the hitting time T_x of the level x is now easily deduced from

$$\begin{aligned}\mathbb{P}(T_x \leq t) &= \mathbb{P}(x \leq M_t) = \mathbb{P}(x \leq |W_t|) \\ &= \mathbb{P}(x \leq |G|\sqrt{t}) = \mathbb{P}\left(\frac{x^2}{G^2} \leq t\right),\end{aligned} \tag{3.1.6}$$

where, as usual, G stands for a Gaussian random variable, with zero expectation and unit variance. Hence,

$$\boxed{T_x \stackrel{\text{law}}{=} \frac{x^2}{G^2}} \tag{3.1.7}$$

and the density of the r.v. T_x is given by:

$$\mathbb{P}(T_x \in dt) = \frac{x}{\sqrt{2\pi t^3}} \exp\left(-\frac{x^2}{2t}\right) \mathbb{1}_{\{t \geq 0\}}\, dt\,.$$

For $x < 0$, we have, using the symmetry of the law of BM

$$T_x = \inf\{t : W_t = x\} = \inf\{t : -W_t = -x\} \stackrel{\text{law}}{=} T_{-x}$$

and it follows that, for any $x \neq 0$,

$$\boxed{\mathbb{P}(T_x \in dt) = \frac{|x|}{\sqrt{2\pi t^3}} \exp\left(-\frac{x^2}{2t}\right) \mathbb{1}_{\{t\geq 0\}}\, dt\,.} \tag{3.1.8}$$

In particular, for $x \neq 0$, $\mathbb{P}(T_x < \infty) = 1$ and $\mathbb{E}(T_x) = \infty$. More precisely, $\mathbb{E}((T_x)^\alpha) < \infty$ if and only if $\alpha < 1/2$, which is immediate from (3.1.7).

Remark 3.1.4.1 Note that, for $x > 0$, from the explicit form of the density of T_x given in (3.1.8), we have

$$t\mathbb{P}(T_x \in dt) = x\mathbb{P}(W_t \in dx)\,.$$

This relation, known as Kendall's identity (see Borovkov and Burq [110]) will be generalized in ↣ Subsection 11.5.3.

Exercise 3.1.4.2 Prove that, for $0 \leq a < b$,

$$\mathbb{P}(W_s \neq 0, \forall t \in [a,b]) = \frac{2}{\pi} \arcsin\sqrt{\frac{a}{b}}\,.$$

`Hint:` From elementary properties of Brownian motion, we have

$$\begin{aligned}\mathbb{P}(W_s \neq 0, \forall s \in [a,b]) &= \mathbb{P}(\forall s \in [a,b],\, W_s - W_a \neq -W_a)\\ &= \mathbb{P}(\forall s \in [a,b],\, W_s - W_a \neq W_a) = \mathbb{P}(\widehat{T}_{W_a} > b-a)\,,\end{aligned}$$

where $\widehat{T}$ is associated with the BM $(\widehat{W}_t = W_{t+a} - W_a, t \geq 0)$. Using the scaling property, we compute the right-hand side of this equality

$$\begin{aligned}\mathbb{P}(W_s \neq 0, \forall s \in [a,b]) &= \mathbb{P}(aW_1^2 \widehat{T}_1 > b-a) = \mathbb{P}\left(\frac{G^2}{\widehat{G}^2} > \frac{b}{a} - 1\right)\\ &= \mathbb{P}\left(\frac{1}{1+C^2} < \frac{a}{b}\right) = \frac{2}{\pi}\arcsin\left(\sqrt{\frac{a}{b}}\right)\,,\end{aligned}$$

where G and $\widehat{G}$ are two independent standard Gaussian variables and C a standard Cauchy variable (see ↣ A.4.2 for the required properties of Gaussian variables). ◁

Exercise 3.1.4.3 Prove that $\sigma(M_s - W_s, s \leq t) = \sigma(W_s, s \leq t)$.
`Hint:` This equality follows from $\int_0^t \mathbb{1}_{\{M_s - W_s = 0\}} d(M_s - W_s) = M_t$. Use the fact that dM_s is carried by $\{s : M_s = B_s\}$. ◁

Exercise 3.1.4.4 The right-hand side of formula (3.1.5) reads, on the set $y \geq 0, y - x \geq 0$,

$$\frac{\mathbb{P}(T_{y-x} \in dt)}{dt} dxdy = \frac{2y-x}{t} p_t(2y-x)dxdy$$

Check simply that this probability has total mass equal to 1! ◁

3.1.5 Law of the Infimum

The law of the infimum of a Brownian motion may be obtained by relying on the same procedure as the one used for the maximum. It can also be deduced by observing that

$$m_t := \inf_{s\le t} W_s = -\sup_{s\le t}(-W_s) = -\sup_{s\le t}(B_s)$$

where $B = -W$ is a Brownian motion. Hence

$$\begin{aligned}
&\text{for} \quad y\le 0, x\ge y \qquad && \mathbb{P}(W_t\ge x, m_t\ge y) = \mathcal{N}\left(\frac{-x}{\sqrt{t}}\right) - \mathcal{N}\left(\frac{2y-x}{\sqrt{t}}\right),\\
&\text{for} \quad y\le 0,\, x\le y \qquad && \mathbb{P}(W_t\ge x, m_t\ge y) = \mathcal{N}\left(\frac{-y}{\sqrt{t}}\right) - \mathcal{N}\left(\frac{y}{\sqrt{t}}\right)\\
& && \qquad\qquad = 1 - 2\mathcal{N}\left(\frac{y}{\sqrt{t}}\right),\\
&\text{for} \quad y\ge 0 \qquad && \mathbb{P}(W_t\ge x, m_t\ge y) = 0\,.
\end{aligned}$$

In particular, for $y \le 0$, the second equality reduces to

$$\mathbb{P}(m_t\ge y) = \mathcal{N}\left(\frac{-y}{\sqrt{t}}\right) - \mathcal{N}\left(\frac{y}{\sqrt{t}}\right).$$

If the Brownian motion W starts from z at time 0 and if T_0 is the first hitting time of 0, i.e., $T_0 = \inf\{t : W_t = 0\}$, then, for $z > 0, x > 0$, we obtain

$$\mathbb{P}_z(W_t\in dx, T_0\ge t) = \mathbb{P}_0(W_t+z\in dx, T_{-z}\ge t) = \mathbb{P}_0(W_t+z\in dx, m_t\ge -z)\,.$$

The right-hand side of this equality can be obtained by differentiating w.r.t. x the following equality, valid for $x \ge 0, z \ge 0$ (hence $x - z \ge -z, -z \le 0$)

$$\mathbb{P}(W_t\ge x-z, m_t\ge -z) = \mathcal{N}\left(-\frac{x-z}{\sqrt{t}}\right) - \mathcal{N}\left(-\frac{x+z}{\sqrt{t}}\right).$$

Thus, we obtain, using the notation (1.4.2)

$$\boxed{\begin{aligned}
\mathbb{P}_z(W_t\in dx, T_0\ge t) &= \frac{\mathbb{1}_{\{x\ge 0\}}}{\sqrt{2\pi t}}\left[\exp\left(-\frac{(z-x)^2}{2t}\right) - \exp\left(-\frac{(z+x)^2}{2t}\right)\right]dx\,,\\
&= \mathbb{1}_{\{x\ge 0\}}(p_t(z-x) - p_t(z+x))dx\,.
\end{aligned}} \tag{3.1.9}$$

3.1.6 Laplace Transforms of Hitting Times

The law of first hitting time of a level y is characterized by its Laplace transforms, which is given in the next proposition.

Proposition 3.1.6.1 *Let T_y be the first hitting time of $y \in \mathbb{R}$ for a standard Brownian motion. Then, for $\lambda > 0$*

$$\mathbb{E}\left[\exp\left(-\frac{\lambda^2}{2}T_y\right)\right] = \exp(-|y|\lambda)\,.$$

PROOF: Recall that, for any $\lambda \in \mathbb{R}$, the process $(\exp(\lambda W_t - \frac{1}{2}\lambda^2 t),\, t \geq 0)$ is a martingale. Now, for $y \geq 0$, $\lambda \geq 0$ the martingale

$$(\exp(\lambda W_{t\wedge T_y} - \frac{1}{2}\lambda^2(t \wedge T_y)),\ t \geq 0)$$

is bounded by $e^{\lambda y}$, hence it is u.i.. Using $\mathbb{P}(T_y < \infty) = 1$, Doob's optional sampling theorem yields

$$\mathbb{E}\left[\exp\left(\lambda W_{T_y} - \frac{1}{2}\lambda^2 T_y\right)\right] = 1\,.$$

Since $W_{T_y} = y$, we obtain the Laplace transform of T_y. The case where $y < 0$ follows since $W \stackrel{\text{law}}{=} -W$. □

Warning 3.1.6.2 In order to apply Doob's optional sampling theorem, we had to check carefully that the martingale $\exp(\lambda W_{t\wedge T_y} - \frac{1}{2}\lambda^2(t \wedge T_y))$ is uniformly integrable. In the case $\lambda > 0$ and $y < 0$, a wrong use of this theorem would lead to the equality between 1 and

$$\mathbb{E}[\exp(\lambda W_{T_y} - \frac{1}{2}\lambda^2 T_y)] = e^{\lambda y}\mathbb{E}\left[\exp\left(-\frac{1}{2}\lambda^2 T_y\right)\right],$$

that is the two quantities $\mathbb{E}[\exp(-\frac{1}{2}\lambda^2 T_y)]$ and $\exp(-y\lambda)$ would be the same. This is obviously false since the quantity $\mathbb{E}[\exp(-\frac{1}{2}\lambda^2 T_y)]$ is smaller than 1 whereas $\exp(-y\lambda)$ is strictly greater than 1.

Remark 3.1.6.3 From the equality (3.1.8) and Proposition 3.1.6.1, we check that for $\lambda > 0$

$$\exp(-|y|\lambda) = \int_0^\infty dt\,\frac{|y|}{\sqrt{2\pi t^3}}\,\exp\left(-\frac{y^2}{2t}\right)\exp\left(-\frac{\lambda^2 t}{2}\right). \tag{3.1.10}$$

This equality may be directly obtained, in the case $y > 0$, by checking that the function

$$H(\mu) = \int_0^\infty dt\,\frac{1}{\sqrt{t^3}}\,e^{-\mu t}\exp\left(-\frac{1}{t}\right)$$

satisfies $\mu H'' + \frac{1}{2}H' - H = 0$. A change of function $G(\sqrt{\mu}) = H(\mu)$ leads to $\frac{1}{4}G'' - G = 0$, and the form of H follows. Let us remark that, for $y > 0$, one can write the equality (3.1.10) in the form

$$1 = \int_0^\infty dt \, \frac{y}{\sqrt{2\pi t^3}} \exp\left(-\frac{1}{2}\left(\frac{y}{\sqrt{t}} - \lambda\sqrt{t}\right)^2\right). \tag{3.1.11}$$

Note that the quantity

$$\frac{y}{\sqrt{2\pi t^3}} \exp\left(-\frac{1}{2}\left(\frac{y}{\sqrt{t}} - \lambda\sqrt{t}\right)^2\right)$$

in the right-hand member is the density of the hitting time of the level y by a drifted Brownian motion (see ↣ formula (3.2.3)). Another proof relies on the knowledge of the resolvent of the Brownian motion: the result can be obtained via a differentiation w.r.t. y of the equality obtained in Exercise 1.4.1.7

$$\int_0^\infty e^{-\lambda^2 t/2} p_t(0,y)dt = \int_0^\infty e^{-\lambda^2 t/2} \frac{1}{\sqrt{2\pi t}} e^{-\frac{y^2}{2t}} dt = \frac{1}{\lambda} e^{-|y|\lambda}$$

Comment 3.1.6.4 We refer the reader to Lévy's equivalence ↣ Theorem 4.1.7.2 which allows translation of all preceding results to the running maximum involving results on the Brownian motion local time.

Exercise 3.1.6.5 Let $T_a^* = \inf\{t \geq 0 : |W_t| = a\}$. Using the fact that the process $(e^{-\lambda^2 t/2} \cosh(\lambda W_t), t \geq 0)$ is a martingale, prove that

$$\mathbb{E}(\exp(-\lambda^2 T_a^*/2)) = [\cosh(a\lambda)]^{-1}.$$

See ↣ Subsection 3.5.1 for the density of T_a^*. ◁

Exercise 3.1.6.6 Let $\tau = \inf\{t : M_t - W_t > a\}$. Prove that M_τ follows the exponential law with parameter a^{-1}.
`Hint:` The exponential law stems from

$$\mathbb{P}(M_\tau > x + y | M_\tau > y) = \mathbb{P}(\tau > T_{x+y} | \tau > T_y) = \mathbb{P}(M_\tau > x).$$

The value of the mean of M_τ is obtained by passing to the limit in the equality $\mathbb{E}(M_{\tau\wedge n}) = \mathbb{E}(M_{\tau\wedge n} - W_{\tau\wedge n})$. ◁

Exercise 3.1.6.7 Let W be a Brownian motion, $\mathbf{F}$ its natural filtration and $M_t = \sup_{s\leq t} W_s$. Prove that, for $t < 1$,

$$\mathbb{E}(f(M_1)|\mathcal{F}_t) = F(1-t, W_t, M_t)$$

with

$$F(s,a,b) = \sqrt{\frac{2}{\pi s}} \left(f(b) \int_0^{b-a} e^{-u^2/(2s)} du + \int_b^\infty f(u) \exp\left(-\frac{(u-a)^2}{2s}\right) du \right).$$

`Hint:` Note that

$$\sup_{s\leq 1} W_s = \sup_{s\leq t} W_s \vee \sup_{t\leq s\leq 1} W_s = \sup_{s\leq t} W_s \vee (\widehat{M}_{1-t} + W_t)$$

where $\widehat{M}_s = \sup_{u\leq s} \widehat{W}_u$ for $\widehat{W}_u = W_{u+t} - W_t$.

Another method consists in an application of $\rightarrowtail$ Theorem 4.1.7.8. Apply Doob's Theorem to the martingale $h(M_t)(M_t - W_t) + \int_{M_t}^{\infty} du\, h(u)$. $\triangleleft$

Exercise 3.1.6.8 Let a and σ be continuous deterministic functions, B a BM and X the solution of $dX_t = a(t)X_t dt + \sigma(t)dB_t$, $X_0 = x$.

Let $T_0 = \inf\{t \geq 0,\, X_t \leq 0\}$. Prove that, for $x > 0, y > 0$,

$$\mathbb{P}(X_t \geq y, T_0 \leq t) = \mathbb{P}(X_t \leq -y)\,.$$

`Hint:` Use the fact that $X_t e^{-A_t} = W^{(x)}_{\alpha(t)}$ where $A_t = \int_0^t a(s)ds$ and $W^{(x)}$ is a Brownian motion starting from x. Here α denotes the increasing function $\alpha(t) = \int_0^t e^{-2A(s)}\sigma^2(s)ds$. Then, use the reflection principle to obtain $\mathbb{P}(W^{(x)}_u \geq z, T_0 \leq u) = \mathbb{P}(W^{(x)}_u \leq -z)$. We refer the reader to $\rightarrowtail$ Theorem 4.1.7.2 which allows computations relative to the maximum M to be couched in terms of Brownian local time.

$\triangleleft$

Exercise 3.1.6.9 Let f be a (bounded) function. Prove that

$$\lim_{t\to\infty} \sqrt{t}\,\mathbb{E}(f(M_t)|\mathcal{F}_s) = c(f(M_s)(M_s - W_s) + F(M_s))$$

where c is a constant and $F(x) = \int_x^{\infty} du f(u)$.

`Hint:` Write $M_t = M_s \vee (W_s + \widehat{M}_{t-s})$ where $\widehat{M}$ is the supremum of a Brownian motion $\widehat{W}$, independent of $W_u, u \leq s$. $\triangleleft$

3.2 Hitting Times for a Drifted Brownian Motion

We now study the first hitting times for the process $X_t = \nu t + W_t$, where W is a Brownian motion and ν a constant. Let $M_t^X = \sup(X_s, s \leq t)$, $m_t^X = \inf(X_s, s \leq t)$ and $T_y(X) = \inf\{t \geq 0 \,|\, X_t = y\}$. We recall that $\mathbf{W}^{(\nu)}$ denotes the law of the Brownian motion with drift ν, i.e., $\mathbf{W}^{(\nu)}(X_t \in A)$ is the probability that a Brownian motion with drift ν belongs to A at time t.

3.2.1 Joint Laws of (M^X, X) and (m^X, X) at Time t

Proposition 3.2.1.1 *For $y \geq 0, y \geq x$*

$$\mathbf{W}^{(\nu)}(X_t \leq x, M_t^X \leq y) = \mathcal{N}\left(\frac{x - \nu t}{\sqrt{t}}\right) - e^{2\nu y}\mathcal{N}\left(\frac{x - 2y - \nu t}{\sqrt{t}}\right)$$

and for $y \leq 0, y \leq x$

$$\mathbf{W}^{(\nu)}(X_t \geq x, m_t^X \geq y) = \mathcal{N}\left(\frac{-x+\nu t}{\sqrt{t}}\right) - e^{2\nu y}\mathcal{N}\left(\frac{-x+2y+\nu t}{\sqrt{t}}\right).$$

PROOF: From Cameron-Martin's theorem (see Proposition 1.7.5.2)

$$\mathbf{W}^{(\nu)}(X_t \leq x, M_t^X \geq y) = \mathbb{E}\left[\exp\left(\nu W_t - \frac{\nu^2}{2}t\right) \mathbb{1}_{\{W_t \leq x, M_t^W \geq y\}}\right].$$

From the reflection principle (3.1.2) for $y \geq 0, x \leq y$, it holds that

$$\mathbb{P}(W_t \leq x, M_t^W \geq y) = \mathbb{P}(x \geq 2y - W_t, M_t^W \geq y),$$

hence, on the set $y \geq 0, x \leq y$, one has

$$\mathbb{P}(W_t \in dx, M_t^W \in dy) = \mathbb{P}(2y - W_t \in dx, M_t^W \in dy).$$

It follows that

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(\nu W_t - \frac{\nu^2}{2}t\right) \mathbb{1}_{\{W_t \leq x, M_t^W \geq y\}}\right] \\
&= \mathbb{E}\left[\exp\left(\nu(2y - W_t) - \frac{\nu^2}{2}t\right) \mathbb{1}_{\{2y - W_t \leq x, M_t^W \geq y\}}\right] \\
&= e^{2\nu y}\mathbb{E}\left[\exp\left(-\nu W_t - \frac{\nu^2}{2}t\right) \mathbb{1}_{\{W_t \geq 2y - x\}}\right].
\end{aligned}$$

Applying Cameron-Martin's theorem again we obtain

$$\mathbb{E}\left[\exp\left(-\nu W_t - \frac{\nu^2}{2}t\right) \mathbb{1}_{\{W_t \geq 2y - x\}}\right] = \mathbf{W}^{(-\nu)}(X_t \geq 2y - x).$$

It follows that for $y \geq 0, y \geq x$,

$$\begin{aligned}
\mathbf{W}^{(\nu)}(X_t \leq x, M_t^X \geq y) &= e^{2\nu y}\mathbb{P}(W_t \geq 2y - x + \nu t) \\
&= e^{2\nu y}\mathcal{N}\left(\frac{-2y + x - \nu t}{\sqrt{t}}\right).
\end{aligned}$$

Therefore, for $y \geq 0$ and $y \geq x$,

$$\begin{aligned}
\mathbf{W}^{(\nu)}(X_t \leq x, M_t^X \leq y) &= \mathbf{W}^{(\nu)}(X_t \leq x) - \mathbf{W}^{(\nu)}(X_t \leq x, M_t^X \geq y) \\
&= \mathcal{N}\left(\frac{x - \nu t}{\sqrt{t}}\right) - e^{2\nu y}\mathcal{N}\left(\frac{x - 2y - \nu t}{\sqrt{t}}\right),
\end{aligned}$$

and for $y \leq 0, y \leq x$,

$$\begin{aligned}\mathbf{W}^{(\nu)}(X_t \geq x, m_t^X \leq y) &= \mathbb{P}(W_t + \nu t \geq x, \inf_{s \leq t}(W_s + \nu s) \leq y) \\ &= \mathbb{P}(-W_t - \nu t \leq -x, \sup_{s \leq t}(-W_s - \nu s) \geq -y) \\ &= \mathbb{P}(W_t - \nu t \leq -x, \sup_{s \leq t}(W_s - \nu s) \geq -y) \\ &= e^{2\nu y} \mathcal{N}\left(\frac{2y - x + \nu t}{\sqrt{t}}\right). \end{aligned} \quad (3.2.1)$$

The result of the proposition follows. □

Corollary 3.2.1.2 *Let $X_t = \nu t + W_t$ and $M_t^X = \sup_{s \leq t} X_s$. The joint density of the pair X_t, M_t^X is*

$$\mathbf{W}^{(\nu)}(X_t \in dx, M_t^X \in dy) = \mathbb{1}_{x<y}\mathbb{1}_{0<y}\frac{2(2y-x)}{\sqrt{2\pi t^3}} e^{\nu x - \frac{1}{2}\nu^2 t - \frac{1}{2t}(2y-x)^2} dx dy$$

Exercise 3.2.1.3 Prove that for $y \geq 0$ and $y \geq x$

$$\mathbf{W}^{(\nu)}(X_t \leq x, M_t^X \geq y) = e^{2\nu y}\mathbb{P}(W_t + \nu t \leq x - 2y)$$

and that for $y \leq 0$ and $y \leq x$

$$\mathbf{W}^{(\nu)}(X_t \geq x, m_t^X \leq y) = e^{2\nu y}\mathbb{P}(W_t + \nu t \geq x - 2y).$$

◁

3.2.2 Laws of Maximum, Minimum, and Hitting Times

The laws of the maximum and the minimum of a drifted Brownian motion are deduced from the obvious equalities

$$\mathbf{W}^{(\nu)}(M_t^X \leq y) = \mathbf{W}^{(\nu)}(X_t \leq y, M_t^X \leq y)$$

and $\mathbf{W}^{(\nu)}(m_t^X \geq y) = \mathbf{W}^{(\nu)}(X_t \geq y, m_t^X \geq y)$. The right-hand sides of these equalities are computed from Proposition 3.2.1.1. In a closed form, we obtain

$$\begin{aligned}
\mathbf{W}^{(\nu)}(M_t^X \leq y) &= \mathcal{N}\left(\frac{y - \nu t}{\sqrt{t}}\right) - e^{2\nu y}\mathcal{N}\left(\frac{-y - \nu t}{\sqrt{t}}\right), \quad y \geq 0 \\
\mathbf{W}^{(\nu)}(M_t^X \geq y) &= \mathcal{N}\left(\frac{-y + \nu t}{\sqrt{t}}\right) + e^{2\nu y}\mathcal{N}\left(\frac{-y - \nu t}{\sqrt{t}}\right), \quad y \geq 0 \\
\mathbf{W}^{(\nu)}(m_t^X \geq y) &= \mathcal{N}\left(\frac{-y + \nu t}{\sqrt{t}}\right) - e^{2\nu y}\mathcal{N}\left(\frac{y + \nu t}{\sqrt{t}}\right), \quad y \leq 0 \\
\mathbf{W}^{(\nu)}(m_t^X \leq y) &= \mathcal{N}\left(\frac{y - \nu t}{\sqrt{t}}\right) + e^{2\nu y}\mathcal{N}\left(\frac{y + \nu t}{\sqrt{t}}\right), \quad y \leq 0.
\end{aligned}$$

For $y > 0$, from the equality $\mathbf{W}^{(\nu)}(T_y(X) \geq t) = \mathbf{W}^{(\nu)}(M_t^X \leq y)$, we deduce that the law of the random variable $T_y(X)$ is

$$\mathbf{W}^{(\nu)}(T_y(X) \in dt) = \frac{y}{\sqrt{2\pi t^3}} e^{\nu y} \exp\left(-\frac{1}{2}\left(\frac{y^2}{t} + \nu^2 t\right)\right) \mathbb{1}_{\{t\geq 0\}} dt \qquad (3.2.2)$$

or, in a more pleasant form

$$\boxed{\mathbf{W}^{(\nu)}(T_y(X) \in dt) = \frac{y}{\sqrt{2\pi t^3}} \exp\left(-\frac{1}{2t}(y - \nu t)^2\right) \mathbb{1}_{\{t\geq 0\}}\, dt\,.} \qquad (3.2.3)$$

This law is the inverse Gaussian law with parameters (y, ν). (See $\rightarrowtail$ Appendix A.4.4.)

Note that, for $\nu < 0$ and $y > 0$, when $t \to \infty$ in $\mathbf{W}^{(\nu)}(T_y \geq t)$, we obtain $\mathbf{W}^{(\nu)}(T_y = \infty) = 1 - e^{2\nu y}$. In this case, the density of T_y under $\mathbf{W}^{(\nu)}$ is defective. For $\nu > 0$ and $y > 0$, we obtain $\mathbf{W}^{(\nu)}(T_y = \infty) = 1$, which can also be obtained from (3.1.11). See also Exercise 1.2.3.10.

Let us point out the simple (Cameron-Martin) absolute continuity relationship between the Brownian motion with drift ν and the Brownian motion with drift $-\nu$: from both formulae

$$\begin{cases} \mathbf{W}^{(\nu)}|_{\mathcal{F}_t} = \exp\left(\nu X_t - \frac{1}{2}\nu^2 t\right) \mathbf{W}|_{\mathcal{F}_t} \\ \mathbf{W}^{(-\nu)}|_{\mathcal{F}_t} = \exp\left(-\nu X_t - \frac{1}{2}\nu^2 t\right) \mathbf{W}|_{\mathcal{F}_t} \end{cases} \qquad (3.2.4)$$

we deduce

$$\mathbf{W}^{(\nu)}|_{\mathcal{F}_t} = \exp(2\nu X_t)\mathbf{W}^{(-\nu)}|_{\mathcal{F}_t}\,. \qquad (3.2.5)$$

(See $\rightarrowtail$ Exercise 3.6.6.4 for an application of this relation.) In particular, we obtain again, using Proposition 1.7.1.4,

$$\mathbf{W}^{(\nu)}(T_y < \infty) = e^{2\nu y}, \text{ for } \nu y < 0\,.$$

Exercise 3.2.2.1 Let $X_t = W_t + \nu t$ and $m_t^X = \inf_{s\leq t} X_s$. Prove that, for $y < 0, y < x$

$$\mathbb{P}(m_t^X \leq y | X_t = x) = \exp\left(-\frac{2y(y-x)}{t}\right).$$

`Hint:` Note that, from Cameron-Martin's theorem, the left-hand side does not depend on ν. $\triangleleft$

3.2.3 Laplace Transforms

From Cameron-Martin's relationship (3.2.4),

$$\mathbf{W}^{(\nu)}\left(\exp\left(-\frac{\lambda^2}{2}T_y(X)\right)\right) = \mathbb{E}\left(\exp\left(\nu W_{T_y} - \frac{\nu^2 + \lambda^2}{2}T_y(W)\right)\right),$$

where $\mathbf{W}^{(\nu)}(\cdot)$ is the expectation under $\mathbf{W}^{(\nu)}$. From Proposition 3.1.6.1, the right-hand side equals

$$e^{\nu y}\mathbb{E}\left[\exp\left(-\frac{1}{2}(\nu^2+\lambda^2)T_y(W)\right)\right] = e^{\nu y}\exp\left(-|y|\sqrt{\nu^2+\lambda^2}\right).$$

Therefore

$$\mathbf{W}^{(\nu)}\left(\exp\left[-\frac{\lambda^2}{2}T_y(X)\right]\right) = e^{\nu y}\exp\left(-|y|\sqrt{\nu^2+\lambda^2}\right). \tag{3.2.6}$$

In particular, letting λ go to 0 in (3.2.6), in the case $\nu y < 0$

$$\mathbf{W}^{(\nu)}(T_y < \infty) = e^{2\nu y},$$

which proves again that the probability that a Brownian motion with strictly positive drift hits a negative level is not equal to 1. In the case $\nu y \geq 0$, obviously $\mathbf{W}^{(\nu)}(T_y < \infty) = 1$. This is explained by the fact that $(W_t + \nu t)/t$ goes to ν when t goes to infinity, hence the drift drives the process to infinity. In the case $\nu y > 0$, taking the derivative (w.r.t. $\lambda^2/2$) of (3.2.6) for $\lambda = 0$, we obtain $\mathbf{W}^{(\nu)}(T_y(X)) = y/\nu$. When $\nu y < 0$, the expectation of the stopping time is equal to infinity.

3.2.4 Computation of $\mathbf{W}^{(\nu)}(\mathbb{1}_{\{T_y(X)<t\}}\, e^{-\lambda T_y(X)})$

We present the computation of $\mathbf{W}^{(\nu)}[\mathbb{1}_{\{T_y(X)<t\}}\exp(-\lambda T_y(X))]$. This will be useful for finance purposes, for example while studying Boost options in ↣ Subsection 3.9.2 and last passage times (↣ Subsections 4.3.9 and 5.6.4). Obviously, the computation could be done using the density of T_y, however this direct method is rather tedious.

For any γ, Cameron-Martin's theorem leads to

$$\begin{aligned}&\mathbf{W}^{(\nu)}(e^{-\lambda T_y(X)}\mathbb{1}_{\{T_y(X)<t\}})\\ &= \mathbf{W}^{(\gamma)}\left(e^{-\lambda T_y(X)}\exp\left[(\nu-\gamma)X_{T_y} - \frac{\nu^2-\gamma^2}{2}T_y\right]\mathbb{1}_{\{T_y(X)<t\}}\right).\end{aligned}$$

Choosing γ such that $2\lambda = \gamma^2 - \nu^2$, we obtain

$$\mathbf{W}^{(\nu)}(e^{-\lambda T_y(X)}\mathbb{1}_{\{T_y(X)<t\}}) = \exp[(\nu-\gamma)y]\,\mathbf{W}^{(\gamma)}(T_y(X) < t).$$

Hence, using the results on the law of the hitting time established in Subsection 3.2.2 for $y > 0$,

$$\mathbf{W}^{(\nu)}(e^{-\lambda T_y}\mathbb{1}_{\{T_y<t\}}) = e^{(\nu-\gamma)y}\mathcal{N}\left(\frac{\gamma t - y}{\sqrt{t}}\right) + e^{(\nu+\gamma)y}\mathcal{N}\left(\frac{-\gamma t - y}{\sqrt{t}}\right)$$

and, for $y < 0$

$$\mathbf{W}^{(\nu)}(e^{-\lambda T_y}\mathbb{1}_{\{T_y<t\}}) = e^{(\nu-\gamma)y}\mathcal{N}\left(\frac{-\gamma t+y}{\sqrt{t}}\right) + e^{(\nu+\gamma)y}\mathcal{N}\left(\frac{\gamma t+y}{\sqrt{t}}\right).$$

Setting

$$H(a,y,t) := e^{-ay}\mathcal{N}\left(\frac{at-y}{\sqrt{t}}\right) + e^{ay}\mathcal{N}\left(\frac{-at-y}{\sqrt{t}}\right), \tag{3.2.7}$$

we get

$$\begin{aligned}\mathbf{W}^{(\nu)}(e^{-\lambda T_y}\mathbb{1}_{\{T_y<t\}}) &= e^{\nu y}H(\gamma,|y|,t)\\ &= e^{\nu y}H(\sqrt{2\lambda+\nu^2},|y|,t).\end{aligned}$$

In particular, for $\nu = 0$,

$$\mathbb{E}(e^{-\lambda T_y(W)}\mathbb{1}_{\{T_y(W)<t\}}) = H(\sqrt{2\lambda},|y|,t).$$

3.2.5 Normal Inverse Gaussian Law

Let (W,B) be a two-dimensional Brownian motion, $X_t = x+\nu t+W_t$, and $T_y^{(\mu)} = \inf\{t : \mu t + B_t = y\}$. Then, the density of $X_{T_y^{(\mu)}}$ is the **Normal Inverse Gaussian law** $NIG(\alpha,\nu,x,y)$ where $\alpha = \sqrt{\nu^2+\mu^2}$. (If needed, see $\rightarrowtail$ Appendix A.4.5 for the expression of the density.) This can be checked from

$$\mathbb{P}(X_{T_y^{(\mu)}} \in A) = \int_0^\infty \mathbb{P}(X_u \in A)\mathbb{P}(T_y^{(\mu)} \in du)$$

and the integral representation of the Bessel function K_ν.

Another method of finding the law of $X_{T_y^{(\mu)}}$ is to compute its characteristic function as follows:

$$\begin{aligned}\mathbb{E}\left(\exp(i\zeta(x+\nu T_y^{(\mu)}+W_{T_y^{(\mu)}}))\right) &= \mathbb{E}\left(\exp(i\zeta(x+\nu T_y^{(\mu)}) - \frac{\zeta^2}{2}T_y^{(\mu)})\right)\\ &= \exp(i\zeta x)\mathbb{E}\left(\exp\left[(i\zeta\nu - \frac{\zeta^2}{2})T_y^{(\mu)}\right]\right)\end{aligned}$$

$$\begin{aligned}&= \exp(i\zeta x)e^{\mu y}\mathbb{E}\left(\exp\left[-\frac{1}{2}(\zeta^2+\mu^2-2i\zeta\nu)T_y^{(0)}\right]\right)\\ &= \exp(i\zeta x)e^{\mu y}e^{-y\sqrt{(\zeta-i\nu)^2+\mu^2+\nu^2}}.\end{aligned}$$

Comment 3.2.5.1 See Barndorff-Nielsen [51], Eberlein [289] and Barndorff-Nielsen et al. [53] for applications of these laws in finance.

3.3 Hitting Times for Geometric Brownian Motion

Let us assume that

$$dS_t = S_t(\mu dt + \sigma dW_t) \, , S_0 = x > 0 \tag{3.3.1}$$

with $\sigma > 0$, i.e.,

$$S_t = x \exp\left((\mu - \sigma^2/2)t + \sigma W_t\right) = xe^{\sigma X_t} \, ,$$

where $X_t = \nu t + W_t$, $\nu = (\mu - \sigma^2/2)\,\sigma^{-1}$. We denote by $T_a(S)$ the first hitting time of a by the process S and $T_\alpha(X)$ the first hitting time of α by the process X. From

$$T_a(S) = \inf\{t \geq 0 \, : \, S_t = a\} = \inf\{t \geq 0 \, : \, X_t = \frac{1}{\sigma}\ln(a/x)\}$$

we obtain $T_a(S) = T_\alpha(X)$ where

$$\alpha = \frac{1}{\sigma}\ln(a/x) \, .$$

When another level b is considered for the geometric Brownian motion S, we shall denote

$$\beta = \frac{1}{\sigma}\ln(b/x) \, .$$

Using the previous results, we give below the law of the hitting time, as well as the law of the maximum M_t^S (resp. minimum m_t^S) of S over the time interval $[0, t]$.

3.3.1 Laws of the Pairs (M_t^S, S_t) and (m_t^S, S_t)

We deduce, from the results obtained for drifted Brownian motion in Proposition 3.2.1.1, that for $a > b, a > x$

$$\begin{aligned}\mathbb{P}_x(S_t \leq b, M_t^S \leq a) &= \mathbf{W}^{(\nu)}(X_t \leq \beta, M_t^X \leq \alpha) \\ &= \mathcal{N}\left(\frac{\beta - \nu t}{\sqrt{t}}\right) - e^{2\nu\alpha}\mathcal{N}\left(\frac{\beta - 2\alpha - \nu t}{\sqrt{t}}\right)\end{aligned}$$

whereas, for $b > a, a < x$

$$\begin{aligned}\mathbb{P}_x(S_t \geq b, m_t^S \geq a) &= \mathbf{W}^{(\nu)}(X_t \geq \beta, m_t^X \geq \alpha) \\ &= \mathcal{N}\left(\frac{-\beta + \nu t}{\sqrt{t}}\right) - e^{2\nu\alpha}\mathcal{N}\left(\frac{-\beta + 2\alpha + \nu t}{\sqrt{t}}\right) .\end{aligned}$$

Proposition 3.3.1.1 *Let $S_t = xe^{\mu t + \sigma W_t}$ and $M_t^S = \sup_{s \leq t} S_s$. The joint density of the pair S_t, M_t^S is*

$$\mathbb{P}(S_t \in dz, M_t^S \in dy)$$
$$= \frac{2}{\sigma^3\sqrt{2\pi t^3}} \frac{\ln(y^2/(xz))}{zy} \exp\left(-\frac{\ln^2(y^2/(xz))}{2\sigma^2 t} + \frac{\rho}{\sigma}\ln(z/x) - \frac{\rho^2 t}{2}\right) dzdy$$

where $\rho = \mu/\sigma + \sigma/2$.

It follows that, for $a > x$ (or $\alpha > 0$)

$$\begin{aligned}\mathbb{P}_x(T_a(S) < t) &= \mathbf{W}^{(\nu)}(T_\alpha(X) < t) \\ &= \mathcal{N}\left(\frac{-\alpha + \nu t}{\sqrt{t}}\right) + e^{2\nu\alpha}\mathcal{N}\left(\frac{-\nu t - \alpha}{\sqrt{t}}\right) \end{aligned} \tag{3.3.2}$$

and, for $a < x$ (or $\alpha < 0$)

$$\mathbb{P}_x(T_a(S) < t) = \mathcal{N}\left(\frac{\alpha - \nu t}{\sqrt{t}}\right) + e^{2\nu\alpha}\mathcal{N}\left(\frac{\nu t + \alpha}{\sqrt{t}}\right) . \tag{3.3.3}$$

The density of the hitting time $T_a(S)$ is obtained by differentiation, or more directly, from (3.2.3) and the equality $T_a(S) = T_\alpha(X)$:

$$\mathbb{P}_x(T_a(S) \in dt) = \frac{dt}{\sqrt{2\pi t^3}} \alpha \exp\left(-\frac{1}{2t}(\alpha - \nu t)^2\right) \mathbb{1}_{\{t \geq 0\}} . \tag{3.3.4}$$

Exercise 3.3.1.2 Prove that, for $a > S_0$, and $t \leq T$

$$\mathbb{P}(T_a(S) > T|\mathcal{F}_t) = \mathbb{1}_{\{\max_{s\leq t} S_s < a\}} \left(\mathcal{N}(d_1) - \left(\frac{a}{S_t}\right)^{2(r-\delta-\sigma^2/2)\sigma^{-2}} \mathcal{N}(d_2)\right)$$

with

$$\begin{aligned} d_1 &= \frac{1}{\sigma\sqrt{T-t}}\left(\ln\left(\frac{a}{S_t}\right) - \left(r - \delta - \frac{\sigma^2}{2}\right)\right) \\ d_2 &= \frac{1}{\sigma\sqrt{T-t}}\left(\ln\left(\frac{S_t}{a}\right) - \left(r - \delta - \frac{\sigma^2}{2}\right)\right) . \end{aligned}$$

◁

3.3.2 Laplace Transforms

From the equality $T_a(S) = T_\alpha(X)$,

$$\mathbb{E}_x\left(\exp\left[-\frac{\lambda^2}{2}T_a(S)\right]\right) = \mathbf{W}^{(\nu)}\left(\exp\left[-\frac{\lambda^2}{2}T_\alpha(X)\right]\right) .$$

Therefore, from (3.2.6)

$$\mathbb{E}_x\left(\exp\left[-\frac{\lambda^2}{2}T_a(S)\right]\right) = \exp\left(\nu\alpha - |\alpha|\sqrt{\nu^2 + \lambda^2}\right) . \tag{3.3.5}$$

3.3.3 Computation of $\mathbb{E}(e^{-\lambda T_a(S)} \mathbb{1}_{\{T_a(S)<t\}})$

For $a > x$ (or $\alpha > 0$) we obtain, using the results of Subsection 3.2.4 about drifted Brownian motion, and choosing γ such that $2\lambda = \gamma^2 - \nu^2$,

$$\mathbb{E}_x(e^{-\lambda T_a(S)} \mathbb{1}_{\{T_a(S)<t\}}) = e^{(\nu-\gamma)\alpha} \mathcal{N}\left(\frac{\gamma t - \alpha}{\sqrt{t}}\right) + e^{(\gamma+\nu)\alpha} \mathcal{N}\left(\frac{-\gamma t - \alpha}{\sqrt{t}}\right).$$

In the case $\lambda = \mu$, $2\lambda + \nu^2 = (\mu\sigma^{-1} + \sigma/2)^2$, we choose $\gamma = -(\mu\sigma^{-1} + \sigma/2)$ so that $\gamma + \nu = -\sigma, \nu - \gamma = 2\mu/\sigma$. Then, for $a > x$

$$\mathbb{E}_x(e^{-\mu T_a(S)} \mathbb{1}_{\{T_a(S)<t\}}) = e^{2\mu\alpha/\sigma} \mathcal{N}\left(\frac{\gamma t - \alpha}{\sqrt{t}}\right) + e^{-\alpha\sigma} \mathcal{N}\left(\frac{-\gamma t - \alpha}{\sqrt{t}}\right).$$

In the case where $a < x$, we obtain

$$\mathbb{E}_x(e^{-\mu T_a(S)} \mathbb{1}_{\{T_a(S)<t\}}) = e^{2\mu\alpha/\sigma} \mathcal{N}\left(\frac{\alpha - \gamma t}{\sqrt{t}}\right) + e^{-\alpha\sigma} \mathcal{N}\left(\frac{\gamma t + \alpha}{\sqrt{t}}\right).$$

3.4 Hitting Times in Other Cases

3.4.1 Ornstein-Uhlenbeck Processes

Proposition 3.4.1.1 *Let $(X_t, t \geq 0)$ be an OU process defined as*

$$dX_t = -kX_t\, dt + dW_t, \quad X_0 = x,$$

and $T_0 = \inf\{t \geq 0 : X_t = 0\}$. For any $x > 0$, the density function of T_0 equals

$$f(t) = \frac{x}{\sqrt{2\pi}} \exp\left(\frac{kx^2}{2}\right) \exp\left(\frac{k}{2}(t - x^2 \coth(kt))\right) \left(\frac{k}{\sinh(kt)}\right)^{3/2}.$$

PROOF: We present here the proof of Alili et al. [10]. As proved in Corollary 2.6.1.2, the OU process can be written $X_t = e^{-kt}(x + \int_0^t e^{ks} dW_s)$. Hence

$$\begin{aligned} T_0 &= \inf\{t \geq 0 : X_t = 0\} = \inf\{t : x + \int_0^t e^{ks} dW_s = 0\} \\ &= \inf\{t : \widehat{W}_{A(t)} = -x\} \end{aligned}$$

where we have written the martingale $\int_0^t e^{ks} dW_s$ as a Brownian motion $\widehat{W}$, time changed by $A(t) = \int_0^t e^{2ks} ds$ (see $\rightarrowtail$ Section 5.1 for comments). It follows that $A(T_0) = T_{-x}(\widehat{W})$, hence

$$\begin{aligned} \mathbb{P}_x(T_0 \in dt) &= A'(t) \mathbb{P}_0(T_{-x}(\widehat{W}) \in dA(t)) \\ &= e^{2kt} \exp\left(-\frac{x^2}{2A(t)}\right) \frac{|x|}{\sqrt{2\pi A^3(t)}}\, dt. \end{aligned}$$

Some easy computations, based on $A(t) = \frac{\sinh(kt)}{k} e^{kt}$ lead to the result. □

Comments 3.4.1.2 (a) We shall present a different proof in $\rightarrowtail$ Subsection 6.5.2. See also $\rightarrowtail$ Subsection 5.3.7.

(b) Ricciardi and Sato [732] obtained, for $x > a$, that the density of the hitting time of a is

$$-ke^{k(x^2-a^2)/2}\sum_{n=1}^{\infty}\frac{D_{\nu_{n,a}}(x\sqrt{2k})}{D'_{\nu_{n,a}}(a\sqrt{2k})}e^{-k\nu_{n,a}t}$$

where $0 < \nu_{1,a} < \cdots < \nu_{n,a} < \cdots$ are the zeros of $\nu \to D_\nu(-a)$. Here D_ν is the parabolic cylinder function with index ν (see $\rightarrowtail$ Appendix A.5.4). The expression $D'_{\nu_{n,a}}$ denotes the derivative of $D_\nu(a)$ with respect to ν, evaluated at point $\nu = \nu_{n,a}$. Note that the formula in Leblanc et al. [573] for the law of the hitting time of a is only valid for $a = 0$. See also the discussion in Subsection 3.4.1.

(c) See other related results in Borodin and Salminen [109], Alili et al. [10], Göing-Jaeschke and Yor [398, 397], Novikov [679, 678], Patie [697], Pitman and Yor [719], Salminen [752], Salminen et al. [755] and Shepp [786].

Exercise 3.4.1.3 Prove that the Ricciardi and Sato result given in Comments 3.4.1.2 (b) allows us to express the density of

$$\tau := \inf\{t : x + W_t = \sqrt{1+2kt}\}\,.$$

`Hint:` The hitting time of a for an OU process is

$$\inf\{t \,:\, e^{-kt}(x + \widehat{W}_{A(t)}) = a\} = \inf\{u \,:\, x + \widehat{W}_u = ae^{kA^{-1}(u)}\}\,.$$

$\lhd$

3.4.2 Deterministic Volatility and Nonconstant Barrier

Valuing barrier options has some interest in two different frameworks:

(i) in a Black and Scholes model with deterministic volatility and a constant barrier
(ii) in a Black and Scholes model with a barrier which is a deterministic function of time.

As we discuss now, these two problems are linked. Let us study the case where the process S is a geometric BM with deterministic volatility $\sigma(t)$:

$$dS_t = S_t(rdt + \sigma(t)dW_t), \quad S_0 = x\,,$$

and let $T_a(S)$ be the first hitting time of a constant barrier a:

$$T_a(S) = \inf\{t \,:\, S_t = a\} = \inf\left\{t \,:\, rt - \frac{1}{2}\int_0^t \sigma^2(s)ds + \int_0^t \sigma(s)dW_s = \alpha\right\}\,,$$

where $\alpha = \ln(a/x)$. The process $U_t = \int_0^t \sigma(s)dW_s$ is a Gaussian martingale and can be written as $Z_{A(t)}$ where Z is a BM and $A(t) = \int_0^t \sigma^2(s)ds$ (see $\rightarrowtail$ Section 5.1 for a general presentation of time change). Let C be the inverse of the function A. Then,

$$T_a(S) = \inf\{t : rt - \frac{1}{2}A(t) + Z_{A(t)} = \alpha\} = \inf\left\{C(u) : rC(u) - \frac{1}{2}u + Z_u = \alpha\right\}$$

hence, the computation of the law of $T_a(S)$ reduces to the study of the hitting time of the non-constant boundary $\alpha - rC(u)$ by the drifted Brownian motion $(Z_u - \frac{1}{2}u, u \geq 0)$. This is a difficult and as yet unsolved problem (see references and comments below).

Comments 3.4.2.1 Deterministic Barriers and Brownian Motion. Groeneboom [409] studies the case

$$T = \inf\{t : x + W_t = \alpha t^2\} = \inf\{t : X_t = -x\}$$

where $X_t = W_t - \alpha t^2$. He shows that the densities of the first passage times for the process X can be written as functionals of a Bessel process of dimension 3, by means of the Cameron-Martin formula. For any $x > 0$ and $\alpha < 0$,

$$\mathbb{P}_x(T \in dt) = 2(\alpha c)^2 \sum_{n=0}^{\infty} \exp\left(\lambda_n/c - \frac{2}{3}\alpha^2 t^3\right) \frac{\mathrm{Ai}(\lambda_n - 2\alpha c x)}{\mathrm{Ai}'(\lambda_n)},$$

where λ_n are the zeros on the negative half-line of the Airy function Ai, the unique bounded solution of $u'' - xu = 0$, $u(0) = 1$, and $c = (1/2\alpha^2)^{1/3}$. (See $\rightarrowtail$ Appendix A.5.5 for a closed form.) This last expression was obtained by Salminen [753].

Breiman [122] studies the case of a square root boundary when the stopping time T is $T = \inf\{t : W_t = \sqrt{\alpha + \beta t}\}$ and relates this study to that of the first hitting times of an OU process.

The hitting time of a nonlinear boundary by a Brownian motion is studied in a general framework in Alili's thesis [6], Alili and Patie [9], Daniels [210], Durbin [285], Ferebee [344], Hobson et al. [443], Jennen and Lerche [491, 492], Kahalé [503], Lerche [581], Park and Paranjape [695], Park and Schuurmann [696], Patie's thesis [697], Peskir and Shiryaev [708], Robbins and Siegmund [734], Salminen [753] and Siegmund and Yuh [798].

Deterministic Barriers and Diffusion Processes. We shall study hitting times for Bessel processes in $\rightarrowtail$ Chapter 6 and for diffusions in Subsection 5.3.6. See Borodin and Salminen [109], Delong [245], Kent [519] or Pitman and Yor [715] for more results on first hitting time distributions for diffusions. See also Barndorff-Nielsen et al. [52], Kent [520, 521], Ricciardi et al. [732, 731], and Yamazato [854]. We shall present in $\rightarrowtail$ Subsection 5.4.3 a method based on the Fokker-Planck equation in the case of general diffusions.

3.5 Hitting Time of a Two-sided Barrier for BM and GBM

3.5.1 Brownian Case

For $a < 0 < b$ let T_a, T_b be the two hitting times of a and b, where

$$T_y = \inf\{t \geq 0 \,:\, W_t = y\}\,,$$

and let $T^* = T_a \wedge T_b$ be the exit time from the interval $[a, b]$. As before M_t denotes the maximum of the Brownian motion over the interval $[0, t]$ and m_t the minimum.

Proposition 3.5.1.1 *Let W be a BM starting from x and let $T^* = T_a \wedge T_b$. Then, for any a, b, x with $a < x < b$*

$$\mathbb{P}_x(T^* = T_a) = \mathbb{P}_x(T_a < T_b) = \frac{b-x}{b-a}$$

and $\mathbb{E}_x(T^*) = (x-a)(b-x)$.

PROOF: We apply Doob's optional sampling theorem to the bounded martingale $(W_{t\wedge T_a \wedge T_b}, t \geq 0)$, so that

$$x = \mathbb{E}_x(W_{T_a \wedge T_b}) = a\mathbb{P}_x(T_a < T_b) + b\mathbb{P}_x(T_b < T_a)\,,$$

and using the obvious equality

$$\mathbb{P}_x(T_a < T_b) + \mathbb{P}_x(T_b < T_a) = 1\,,$$

one gets $\mathbb{P}_x(T_a < T_b) = \dfrac{b-x}{b-a}$.

The process $\{W^2_{t\wedge T_a \wedge T_b} - (t \wedge T_a \wedge T_b), t \geq 0\}$ is a bounded martingale, hence applying Doob's optional sampling theorem again, we get

$$x^2 = \mathbb{E}_x(W^2_{t\wedge T_a \wedge T_b}) - \mathbb{E}_x(t \wedge T_a \wedge T_b)\,.$$

Passing to the limit when t goes to infinity, we obtain

$$x^2 = a^2\mathbb{P}_x(T_a < T_b) + b^2\mathbb{P}_x(T_b < T_a) - \mathbb{E}_x(T_a \wedge T_b)\,,$$

hence $\mathbb{E}_x(T_a \wedge T_b) = x(b+a) - ab - x^2 = (x-a)(b-x)$. □

Comment 3.5.1.2 The formula established in Proposition 3.5.1.1 will be very useful in giving a definition for the scale function of a diffusion (see Subsection 5.3.2).

Proposition 3.5.1.3 *Let W be a BM starting from 0, and let $a < 0 < b$. The Laplace transform of $T^* = T_a \wedge T_b$ is*

$$\mathbb{E}_0 \left[\exp\left(-\frac{\lambda^2}{2}T^*\right)\right] = \frac{\cosh[\lambda(a+b)/2]}{\cosh[\lambda(b-a)/2]} .$$

The joint law of (M_t, m_t, W_t) is given by

$$\mathbb{P}_0(a \le m_t < M_t \le b, W_t \in E) = \int_E \varphi(t,y)\,dy \tag{3.5.1}$$

where, for $y \in [a,b]$,

$$\begin{aligned}\varphi(t,y) &= \mathbb{P}_0(W_t \in dy\,,\, T^* > t)\,/dy \\ &= \sum_{n=-\infty}^{\infty} p_t(y + 2n(b-a)) - p_t(2b - y + 2n(b-a))\end{aligned} \tag{3.5.2}$$

and p_t is the Brownian density

$$p_t(y) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{y^2}{2t}\right) .$$

PROOF: We only give the proof of the form of the Laplace transform. We refer the reader to formula 5.7 in Chapter X of Feller [343], and Freedman [357], for the form of the joint law. The Laplace transform of T^* is obtained by Doob's optional sampling theorem. Indeed, the martingale

$$\exp\left(\lambda\left(W_{t\wedge T^*} - \frac{a+b}{2}\right) - \frac{\lambda^2\,(t \wedge T^*)}{2}\right)$$

is bounded and T^* is finite, hence

$$\begin{aligned}\exp\left[-\lambda\left(\frac{a+b}{2}\right)\right] &= \mathbb{E}\left[\exp\left(\lambda\left(W_{T^*} - \frac{a+b}{2}\right) - \frac{\lambda^2\,T^*}{2}\right)\right] \\ &= \exp\left(\lambda\frac{b-a}{2}\right)\,\mathbb{E}\left[\exp\left(-\frac{\lambda^2\,T^*}{2}\right)\mathbb{1}_{\{T^*=T_b\}}\right] \\ &\quad + \exp\left(\lambda\frac{a-b}{2}\right)\,\mathbb{E}\left[\exp\left(-\frac{\lambda^2\,T^*}{2}\right)\mathbb{1}_{\{T^*=T_a\}}\right]\end{aligned}$$

and using $-W$ leads to

$$\begin{aligned}\exp\left[-\lambda\left(\frac{a+b}{2}\right)\right] &= \mathbb{E}\left[\exp\left(\lambda\left(-W_{T^*} - \frac{a+b}{2}\right) - \frac{\lambda^2\,T^*}{2}\right)\right] \\ &= \exp\left(\lambda\frac{-3b-a}{2}\right)\,\mathbb{E}\left[\exp\left(-\frac{\lambda^2\,T^*}{2}\right)\mathbb{1}_{\{T^*=T_b\}}\right] \\ &\quad + \exp\left(\lambda\frac{-b-3a}{2}\right)\,\mathbb{E}\left[\exp\left(-\frac{\lambda^2\,T^*}{2}\right)\mathbb{1}_{\{T^*=T_a\}}\right] .\end{aligned}$$

By solving a linear system of two equations, the following result is obtained:

$$\begin{cases} \mathbb{E}\left[\exp\left(-\frac{\lambda^2 T^*}{2}\right)\mathbb{1}_{\{T^*=T_b\}}\right] = \frac{\sinh(-\lambda a)}{\sinh(\lambda(b-a))} \\ \mathbb{E}\left[\exp\left(-\frac{\lambda^2 T^*}{2}\right)\mathbb{1}_{\{T^*=T_a\}}\right] = \frac{\sinh(\lambda b)}{\sinh(\lambda(b-a))} \end{cases} . \tag{3.5.3}$$

The proposition is finally derived from

$$\mathbb{E}\left[e^{-\lambda^2 T^*/2}\right] = \mathbb{E}\left[e^{-\lambda^2 T^*/2)}\mathbb{1}_{\{T^*=T_b\}}\right] + \mathbb{E}\left[e^{-\lambda^2 T^*/2}\mathbb{1}_{\{T^*=T_a\}}\right] .$$

□

By inverting this Laplace transform using series expansions, written in terms of $e^{-\lambda c}$ (for various c) which is the Laplace transform in $\lambda^2/2$ of T_c, the density of the exit time T^* of $[a,b]$ for a BM starting from $x \in [a,b]$ follows: for $y \in [a,b]$,

$$\mathbb{P}_x(B_t \in dy, T^* > t) = dy \sum_{n\in\mathbb{Z}} p_t(y-x+2n(b-a)) - p_t(2b-y-x+2n(b-a))$$

and the density of T^* is

$$\mathbb{P}_x(T^* \in dt) = (\mathrm{ss}_t(b-x, b-a) + \mathrm{ss}_t(x-a, b-a))\, dt$$

where, using the notation of Borodin and Salminen [109],

$$\mathrm{ss}_t(u,v) = \frac{1}{\sqrt{2\pi t^3}} \sum_{k=-\infty}^{\infty} (v-u+2kv)e^{-(v-u+2kv)^2/2t} . \tag{3.5.4}$$

In particular,

$$\mathbb{P}_x(T^* \in dt, B_{T^*} = a) = \mathrm{ss}_t(x-a, b-a)dt .$$

In the case $-a = b$ and $x = 0$, we get the formula obtained in Exercise 3.1.6.5 for $T_b^* = \inf\{t : |B_t| = b\}$:

$$\mathbb{E}_0\left[\exp\left(-\frac{\lambda^2}{2}T_b^*\right)\right] = (\cosh(b\lambda))^{-1}$$

and inverting the Laplace transform leads to the density

$$\mathbb{P}_0(T_b^* \in dt) = \frac{1}{b^2} \sum_{n=-\infty}^{\infty} \left(n+\frac{1}{2}\right) e^{-(1/2)(n+1/2)^2\pi^2 t/b^2} dt .$$

□

Comments 3.5.1.4 (a) Let $M_1^* = \sup_{s\leq 1} |B_s|$ where B is a d-dimensional Brownian motion. As a consequence of Brownian scaling, $M_1^* \stackrel{\text{law}}{=} (T_1^*)^{-1/2}$ where $T_1^* = \inf\{t : |B_t| = 1\}$. In [774], Schürger computes the moments of the random variable M_1^* using the formula established in Exercise 1.1.12.4. See also Biane and Yor [86] and Pitman and Yor [720].

(b) Proposition 3.5.1.1 can be generalized to diffusions by using the corresponding scale functions. See $\rightarrowtail$ Subsection 5.3.2.

(c) The law of the hitting time of a two-sided barrier was studied in Bachelier [40], Borodin and Salminen [109], Cox and Miller [204], Freedman [357], Geman and Yor [384], Harrison [420], Karatzas and Shreve [513], Kunitomo and Ikeda [551], Knight [528], Itô and McKean [465] (Chapter I) and Linetsky [593]. See also Biane, Pitman and Yor [85].

(d) Another approach, following Freedman [357] and Knight [528] is given in [RY], Chap. III, Exercise 3.15.

(e) The law of T^* and generalizations can be obtained using spider-martingales (see Yor [868], p. 107).

3.5.2 Drifted Brownian Motion

Let $X_t = \nu t + W_t$ be a drifted Brownian motion and $T^*(X) = T_a(X) \wedge T_b(X)$ with $a < 0 < b$. From Cameron-Martin's theorem, writing T^* for $T^*(X)$,

$$\begin{aligned}\mathbf{W}^{(\nu)}\left(\exp\left(-\frac{\lambda^2}{2}T^*\right)\right) &= \mathbb{E}\left(\exp\left(\nu W_{T^*} - \frac{\nu^2}{2}T^*\right)\exp\left(-\frac{\lambda^2}{2}T^*\right)\right) \\ &= \mathbb{E}(\mathbb{1}_{\{T^*=T_a\}}\, e^{\nu W_{T^*}-(\nu^2+\lambda^2)T^*/2}) + \mathbb{E}(\mathbb{1}_{\{T^*=T_b\}}\, e^{\nu W_{T^*}-(\nu^2+\lambda^2)T^*/2}) \\ &= e^{\nu a}\mathbb{E}(\mathbb{1}_{\{T^*=T_a\}}\, e^{-(\nu^2+\lambda^2)T^*/2}) + e^{\nu b}\mathbb{E}(\mathbb{1}_{\{T^*=T_b\}}\, e^{-(\nu^2+\lambda^2)T^*/2}).\end{aligned}$$

From the result (3.5.3) obtained in the case of a standard BM, it follows that

$$\mathbf{W}^{(\nu)}\left(\exp\left(-\frac{\lambda^2}{2}T^*\right)\right) = \exp(\nu a)\frac{\sinh(\mu b)}{\sinh(\mu(b-a))} + \exp(\nu b)\frac{\sinh(-\mu a)}{\sinh(\mu(b-a))}$$

where $\mu^2 = \nu^2 + \lambda^2$. Inverting the Laplace transform,

$$\begin{aligned}&\mathbb{P}_x(T^* \in dt) \\ &\quad = e^{-\nu^2 t/2}\left(e^{\nu(a-x)}\mathrm{ss}_t(b-x, b-a) + e^{\nu(b-x)}\mathrm{ss}_t(x-a, b-a)\right)dt,\end{aligned}$$

where the function ss is defined in (3.5.4). In the particular case $a = -b$, the Laplace transform is

$$\mathbf{W}^{(\nu)}\left(\exp\left(-\frac{\lambda^2}{2}T^*\right)\right) = \frac{\cosh(\nu b)}{\cosh(b\sqrt{\nu^2+\lambda^2})}.$$

The formula (3.5.1) can also be extended to drifted Brownian motion thanks to the Cameron-Martin relationship.

3.6 Barrier Options

In this section, we study the price of barrier options in the case where the underlying asset S follows the Garman-Kohlhagen risk-neutral dynamics

$$dS_t = S_t((r-\delta)dt + \sigma dW_t)\,, \tag{3.6.1}$$

where r is the risk-free interest rate, δ the dividend yield generated by the asset and W a BM. If needed, we shall denote by $(S_t^x, t \geq 0)$ the solution of (3.6.1) with initial condition x. In a closed form,

$$S_t^x = xe^{(r-\delta)t}e^{\sigma W_t - \sigma^2 t/2}\,.$$

We follow closely El Karoui [297] and El Karoui and Jeanblanc [300]. In a first step, we recall some properties of standard Call and Put options. We also recall that an option is **out-of-the-money** (resp. in-the-money) if its intrinsic value $(S_t - K)^+$ is equal to 0 (resp. strictly positive).

3.6.1 Put-Call Symmetry

In the particular case where $r = \delta = 0$, Garman and Kohlhagen's formulae (2.7.4) for the time-t price of a European call C_E^* and a put option P_E^* with strike price K and maturity T on the underlying asset S reduce to

$$C_E^*\,(x,K,T-t) = x\mathcal{N}\left[d_1\left(\frac{x}{K}, T-t\right)\right] - K\mathcal{N}\left[d_2\left(\frac{x}{K}, T-t\right)\right] \tag{3.6.2}$$

$$P_E^*\,(x,K,T-t) = K\mathcal{N}\left[d_1\left(\frac{K}{x}, T-t\right)\right] - x\mathcal{N}\left[d_2\left(\frac{K}{x}, T-t\right)\right] \tag{3.6.3}$$

The functions d_i are defined on $\mathbb{R}^+ \times [0,T]$ as:

$$\begin{aligned} d_1(y,u) &:= \frac{1}{\sqrt{\sigma^2\,u}}\ln(y) + \frac{1}{2}\sqrt{\sigma^2\,u} \\ d_2(y,u) &:= d_1(y,u) - \sqrt{\sigma^2\,u}\,, \end{aligned} \tag{3.6.4}$$

and x is the value of the underlying at time t. Note that these formulae do not depend on the sign of σ and $d_1(y,u) = -d_2(1/y,\,u)$.

In the general case, the time-t prices of a European call C_E and a put option P_E with strike price K and maturity T on the underlying currency S are

$$\begin{aligned} C_E\,(x,K;r,\delta;T-t) &= C_E^*(xe^{-\delta(T-t)}, Ke^{-r(T-t)}, T-t) \\ P_E\,(x,K;r,\delta;T-t) &= P_E^*(xe^{-\delta(T-t)}, Ke^{-r(T-t)}, T-t) \end{aligned}$$

or, in closed form

$$C_E\ (x,K;r,\delta;T-t) = xe^{-\delta(T-t)}\mathcal{N}\left[d_1\left(\frac{xe^{-\delta(T-t)}}{Ke^{-r(T-t)}}, T-t\right)\right] - Ke^{-r(T-t)}\mathcal{N}\left[d_2\left(\frac{xe^{-\delta(T-t)}}{Ke^{-r(T-t)}}, T-t\right)\right] \quad (3.6.5)$$

$$P_E\ (x,K;r,\delta;T-t) = Ke^{-r(T-t)}\mathcal{N}\left[d_1\left(\frac{Ke^{-r(T-t)}}{xe^{-\delta(T-t)}}, T-t\right)\right] - xe^{-\delta(T-t)}\mathcal{N}\left[d_2\left(\frac{Ke^{-r(T-t)}}{xe^{-\delta(T-t)}}, T-t\right)\right]. \quad (3.6.6)$$

Notation: The quantity $C_E^*(\alpha,\beta;u)$ depends on three arguments: the first one, α, is the value of the underlying, the second one β is the value of the strike, and the third one, u, is the time to maturity. For example, $C_E^*\ (K,x;T-t)$ is the time-t value of a call on an underlying with time-t value equal to K and strike x. We shall use the same kind of convention for the function $C_E(x,K;r,\delta;u)$ which depends on 5 arguments.

As usual, $\mathcal{N}$ represents the cumulative distribution function of a standard Gaussian variable.

If σ is a deterministic function of time, $d_i(y,T-t)$ has to be changed into $d_i(y,T,t)$, where

$$\begin{aligned} d_1(y;T,t) &= \frac{1}{\Sigma_{t,T}}\ln(y) + \frac{1}{2}\Sigma_{t,T} \\ d_2(y;T,t) &= d_1(y;T,t) - \Sigma_{t,T} \end{aligned} \quad (3.6.7)$$

with $\Sigma_{t,T}^2 = \int_t^T \sigma^2(s)ds$.

Note that, from the definition and the fact that the geometric Brownian motion (solution of (3.6.1)) satisfies $S_t^{\lambda x} = \lambda S_t^x$, the call (resp. the put) is a homogeneous function of degree 1 with respect to the first two arguments, the spot and the strike:

$$\begin{aligned} \lambda C_E\ (x,K;r,\delta;T-t) &= C_E\ (\lambda x,\lambda K;r,\delta;T-t) \\ \lambda P_E\ (x,K;r,\delta;T-t) &= P_E\ (\lambda x,\lambda K;r,\delta;T-t)\,. \end{aligned} \quad (3.6.8)$$

This can also be checked from the formula (3.6.5). The **Deltas**, i.e., the first derivatives of the option price with respect to the underlying, are given by

$$\text{DeltaC}(x,K;r,\delta;T-t) = e^{-\delta(T-t)}\mathcal{N}\left[d_1\left(\frac{xe^{-\delta(T-t)}}{Ke^{-r(T-t)}}, T-t\right)\right]$$

$$\text{DeltaP}(x,K;r,\delta;T-t) = -e^{-\delta(T-t)}\mathcal{N}\left[d_2\left(\frac{Ke^{-r(T-t)}}{xe^{-\delta(T-t)}}, T-t\right)\right].$$

The Deltas are homogeneous of degree 0 in the first two arguments, the spot and the strike:

$$\begin{aligned} \text{DeltaC}\ (x,K;r,\delta;T-t) &= \text{DeltaC}\ (\lambda x,\lambda K;r,\delta;T-t), \\ \text{DeltaP}\ (x,K;r,\delta;T-t) &= \text{DeltaP}\ (\lambda x,\lambda K;r,\delta;T-t)\,. \end{aligned} \tag{3.6.9}$$

Using the explicit formulae (3.6.5, 3.6.6), the following result is obtained.

Proposition 3.6.1.1 *The put-call symmetry is given by the following expressions*

$$\begin{aligned} C^*_E(K,x;T-t) &= P^*_E(x,K;T-t) \\ P_E(x,K;r,\delta;T-t) &= C_E(K,x;\delta,r;T-t)\,. \end{aligned}$$

PROOF: The formula is straightforward from the expressions (3.6.2, 3.6.3) of C^*_E and P^*_E. Hence, the general case for C_E and P_E follows. This formula is in fact obvious when dealing with exchange rates: the seller of US dollars is the buyer of Euros. From the homogeneity property, this can also be written

$$P_E(x,K;r,\delta;T-t) = xKC_E(1/x,1/K;\delta,r;T-t)\,. \qquad \square$$

Remark 3.6.1.2 A different proof of the put-call symmetry which does not use the closed form formulae (3.6.2, 3.6.3) relies on Cameron-Martin's formula and a change of numéraire. Indeed

$$C_E(x,K,r,\delta,T) = \mathbb{E}_{\mathbb{Q}}(e^{-rT}(S_T-K)^+) = \mathbb{E}_{\mathbb{Q}}(e^{-rT}(S_T/x)(x-KxS_T^{-1})^+)\,.$$

The process $Z_t = e^{-(r-\delta)t}S_t/x$ is a strictly positive martingale with expectation 1. Set $\widehat{\mathbb{Q}}|_{\mathcal{F}_t} = Z_t\mathbb{Q}|_{\mathcal{F}_t}$. Under $\widehat{\mathbb{Q}}$, the process $Y_t = xK(S_t)^{-1}$ follows dynamics $dY_t = Y_t((\delta-r)dt-\sigma dB_t)$ where B is a $\widehat{\mathbb{Q}}$-Brownian motion, and $Y_0 = K$. Hence,

$$C_E(x,K,r,\delta,T) = \mathbb{E}_{\mathbb{Q}}(e^{-\delta T}Z_T(x-Y_T)^+) = \widehat{\mathbb{E}}(e^{-\delta T}(x-Y_T)^+)\,,$$

and the right-hand side represents the price of a put option on the underlying Y, when δ is the interest rate, r the dividend, K the initial value of the underlying asset, $-\sigma$ the volatility and x the strike. It remains to note that the value of a put option is the same for σ and $-\sigma$.

Comments 3.6.1.3 (a) This symmetry relation extends to American options (see Carr and Chesney [147], McDonald and Schroder [633] and Detemple [251]). See ↣ Subsections 10.4.2 and 11.7.3 for an extension to mixed diffusion processes and Lévy processes.

(b) The homogeneity property does not extend to more general dynamics.

Exercise 3.6.1.4 Prove that

$$\begin{aligned} C_E(x,K;r,\delta;T-t) &= P^*_E(Ke^{-\mu(T-t)},xe^{\mu(T-t)};T-t) \\ &= e^{-\mu(T-t)}P^*_E(K,xe^{2\mu(T-t)};T-t)\,, \end{aligned}$$

where $\mu = r-\delta$ is called the cost of carry. ◁

3.6.2 Binary Options and Δ's

Among the exotic options traded on the market, binary options are the simplest ones. Their valuation is straightforward, but hedging is more difficult. Indeed, the hedging ratio is discontinuous in the neighborhood of the strike price.

A **binary call** (in short BinC) (resp. binary put, BinP) is an option that generates one monetary unit if the underlying value is higher (resp. lower) than the strike, and 0 otherwise. In other words, the payoff is $\mathbb{1}_{\{S_T \geq K\}}$ (resp. $\mathbb{1}_{\{S_T \leq K\}}$). Binary options are also called digital options.

Since $\frac{1}{h}\left((x-k)^+ - (x-(k+h))^+\right) \to \mathbb{1}_{\{x \geq k\}}$ as $h \to 0$, the value of a binary call is the limit, as $h \to 0$ of the call-spread

$$\frac{1}{h}[C(x,K,T) - C(x,K+h,T)]\,,$$

i.e., is equal to the negative of the derivative of the call with respect to the strike. Along the same lines, a binary put is the derivative of the put with respect to the strike.

By differentiating the formula obtained in Exercise 3.6.1.4 with respect to the variable K, we obtain the following formula:

Proposition 3.6.2.1 *In the Garman-Kohlhagen framework, with carrying cost $\mu = r - \delta$ the following results are obtained:*

$$\begin{aligned}\mathrm{BinC}(x,K;r,\delta;T-t) &= -e^{-\mu(T-t)}\mathrm{DeltaP}_E^*(K, xe^{2\mu(T-t)}; T-t)\\ &= e^{-r(T-t)}\mathcal{N}\left[d_2\left(\frac{xe^{\mu(T-t)}}{K}, T-t\right)\right] \qquad (3.6.10)\end{aligned}$$

$$\begin{aligned}\mathrm{BinP}(x,K;r,\delta;T-t) &= e^{-\mu(T-t)}\mathrm{DeltaC}_E^*(K, xe^{2\mu(T-t)}; T-t)\\ &= e^{-r(T-t)}\,\mathcal{N}\left[d_1\left(\frac{K}{xe^{\mu(T-t)}}, T-t\right)\right]\,, \qquad (3.6.11)\end{aligned}$$

where d_1, d_2 are defined in (3.6.4).

Exercise 3.6.2.2 Prove that

$$\begin{cases}\mathrm{DeltaC}\ (x,K;r,\delta) = \dfrac{1}{x}\left[C_E\ (x,K;r,\delta) + K\mathrm{BinC}\ (x,K;r,\delta)\right]\\[2ex] \mathrm{DeltaP}\ (x,K;r,\delta) = \dfrac{1}{x}\left[P_E\ (x,K;r,\delta) - K\mathrm{BinP}\ (x,K;r,\delta)\right]\end{cases} \qquad (3.6.12)$$

where the quantities are evaluated at time $T-t$. ◁

Comments 3.6.2.3 The price of a BinC can also be computed via a PDE approach, by solving

$$\begin{cases} \partial_t u + \frac{1}{2}\sigma^2 x^2 \partial_{xx} u + \mu\, \partial_x u = ru \\ u(x,T) = \mathbb{1}_{\{K<x\}} \, . \end{cases} \tag{3.6.13}$$

See Ingersoll [459] and Rubinstein and Reiner [747] for a discussion on binary options. Navatte and Quittard-Pinon [667] have studied binary options in a stochastic interest case (one factor Gaussian model); their results are extended to a Lévy model in Eberlein and Kluge [291].

3.6.3 Barrier Options: General Characteristics

Practitioners give the name *barrier options* to options with a payoff that depends on whether or not the underlying value has reached a given level (the barrier) before maturity. They are particular types of path-dependent options, because the final payoff depends on the asset price trajectory and they are classified into two categories:

- *Knock-out options:* The option ceases to exist at the first passage time of the underlying value at the barrier.
- *Knock-in options:* The option is activated as soon as the barrier is reached.

Let us consider for instance:

- A DOC (**down-and-out call**) with strike K, barrier L and maturity T is the option to buy the underlying at price K (at maturity T) if the underlying value never falls below the (low) barrier L before time T. The value of a DOC is therefore null for $S_0 < L$ and, for $S_0 \geq L$,

  $$\mathrm{DOC}(S_0, K, L, T) := \mathbb{E}_{\mathbb{Q}}(e^{-rT}(S_T - K)^+ \mathbb{1}_{\{T<T_L\}})$$

  where:

  $$T_L := \inf\{t \,|\, S_t \leq L\} = \inf\{t \,|\, S_t = L\} \, .$$

  In what follows, we consider DOC options only in the case $S_0 \geq L$.
- An UOC (**up-and-out call**) has the same characteristics but the (high) barrier H is above the initial underlying value, $S_0 \leq H$. Its price is

  $$\mathrm{UOC}(S_0, K, H, T) := \mathbb{E}_{\mathbb{Q}}(e^{-rT}(S_T - K)^+ \mathbb{1}_{\{T<T_H\}})$$

  where $T_H := \inf\{t \,|\, S_t \geq H\} = \inf\{t \,|\, S_t = H\}$.
- A DIC (**down-and-in call**) is activated if the underlying value falls below the barrier L before time T. Its price is, for $S_0 > L$,

  $$\mathrm{DIC}(S_0, K, L, T) := \mathbb{E}_{\mathbb{Q}}(e^{-rT}(S_T - K)^+ \mathbb{1}_{\{T>T_L\}}) \, .$$

- An UIC (**up-and-in call**) is activated as soon as the underlying value hits the barrier H from below. Its price is

$$\mathrm{UIC}(S_0, K, H, T) := \mathbb{E}_{\mathbb{Q}}(e^{-rT}(S_T - K)^+ \mathbb{1}_{\{T > T_H\}}) .$$

The same definitions apply to puts, binary options and bonds. For example

- A DIP is a **down-and-in put**.
- A **binary down-and-in call** (BinDIC) is a binary call, activated only if the underlying value falls below the barrier, before maturity. The payoff is $\mathbb{1}_{\{S_T > K\}} \mathbb{1}_{\{T_L < T\}}$.
- A DIB (**down-and-in bond**) is a product which generates one monetary unit at maturity if the barrier L has been reached beforehand by the underlying. Its value is $\mathbb{E}_{\mathbb{Q}}(e^{-rT}\mathbb{1}_{\{T_L < T\}}) = e^{-rT}\mathbb{Q}(T_L < T)$.

Barrier options are often used on currency markets. Their prices are smaller than the corresponding standard European prices. This provides an advantage for the marketing of these products. However, they are more difficult to hedge.

Depending on the "at the barrier" intrinsic value, these exotic options can be classified further :

- A barrier option that is out of the money when the barrier L is reached is called a **regular** option. As an example, note that the time-t intrinsic value $(x - K)^+ \mathbb{1}_{\{T_L \le t\}}$ of a DIC such that $K \ge L$ is equal to 0 for $x = L$.
- A barrier option which is in the money when the barrier is reached is called a **reverse** option.
- Some barrier options generate a *rebate* received in cash when the barrier L is reached. The value of the rebate corresponds to the payoff of a binary option. In particular, the rebate is often chosen in such a way that the payoff continuity is kept at the barrier, e.g., if the payoff is $f(S_T)$ at time T, the rebate is $f(L)$.

Let us remark that by relying on the absence of arbitrage opportunities, to being long on one in-option and on one out-option is equivalent to be long on a plain-vanilla option. Therefore, we restrict our attention to in-options.

Comments 3.6.3.1 (a) Barrier options are studied in a discrete time setting in Wilmott et al. [847], Chesney et al. [176], Musiela and Rutkowski [661], Zhang [872], Pliska [721] and Wilmott [846].

(b) In continuous time, the main papers are Andersen et al. [16], Rubinstein and Reiner [746], Bowie and Carr [116], Rich [733], Heynen and Kat [434], Douady [262], Carr and Chou [148], Baldi et al. [42], Linetsky [593] and Suchanecki [814]. Broadie et al. [130] present some correction terms between discrete and continuous time barrier options.

(c) Roberts and Shortland [735] study a case where the underlying has time dependent coefficients. The books of Kat [516], Musiela and Rutkowski [661], Zhang [872] and Wilmott [846] contain more information. Taleb [818] studies hedging strategies.

3.6.4 Valuation and Hedging of a Regular Down-and-In Call Option When the Underlying is a Martingale

In this section, we suppose that the barrier option is written on an underlying S *without carrying costs* – hence a martingale – (i.e., $\mu = 0$ or $r = \delta$) with dynamics having *deterministic* volatility:

$$dS_t = S_t \sigma(t) dW_t\,.$$

Furthermore, when there is no ambiguity, the instantaneous time t, the maturity time T and the volatility will not appear as arguments in the formulae. The value of the underlying at time t is denoted by x.

Let L be the barrier. We denote by $\mathrm{DIC}^M(x, K, L)$ the DIC option price and by $C_E^M(x, K)$ (resp. $P_E^M(x, K)$) the standard European call (resp. put) price (where the first variable is the underlying and the second variable is the strike), when the underlying is a martingale (hence the superscript M). Relying on the assumption that the carrying cost is zero, the symmetry formula established in Proposition 3.6.1.1 is

$$C_E^M(x, K) = P_E^M(K, x)\,. \tag{3.6.14}$$

In particular $\partial_K C_E^M(x, K) = \mathrm{Delta} P_E^M(K, x)$.

We now follow closely Carr et al. [149]. We recall that for a regular DIC option, the barrier L is lower than the strike ($K \geq L$).

Proposition 3.6.4.1 *Consider a regular DIC option on an underlying without carrying costs.*

(a) *Its price is given by:*

$$\textit{for} \quad x \leq L, \quad \mathrm{DIC}^M(x, K, L) = C_E^M(x, K)\,, \tag{3.6.15}$$

$$\textit{for} \quad x \geq L, \quad \mathrm{DIC}^M(x, K, L) = \frac{K}{L} P_E^M\left(x, \frac{L^2}{K}\right) = C_E^M\left(L, K\frac{x}{L}\right)\,, \tag{3.6.16}$$

(b) *The static hedging consists of:*

(i) *a long call for $x \leq L$,*

(ii) *for $x \geq L$, a long position of K/L puts of strike L^2/K.*

PROOF: We shall give a proof "without mathematics."

▶ If the value x of the underlying (at date t) is smaller than the barrier L, the option is already activated, therefore it is a plain vanilla option and the equality (3.6.15) is satisfied.

▶ If the value of the underlying (at date t) is higher than the barrier, we proceed as follows. Let t be fixed and denote by

$$T_L = \inf\{s \geq t\,; S_s \leq L\} \tag{3.6.17}$$

the first passage time after t of the underlying value below the barrier.

In order to price the option at time t, by relying on the absence of arbitrage opportunities, we compute the option value at date T_L, and we denote by V this value. In a second step, we compute the value at time t of the claim V, to be received at time T_L.

At the barrier, the level of the underlying is known and only the remaining maturity $T - T_L$ is unknown. The DIC^M option is equivalent to a call of maturity $T - T_L$ on an underlying with value L, i.e., $C_E^M(L, K, T - T_L)$. The underlying is a martingale, and the volatility is deterministic. Therefore, the underlying dynamics with starting time T_L and starting point L is, conditionally with respect to the past before T_L, log-normally distributed. The symmetry formula (3.6.14) and the homogeneity of the put price yield

$$C_E^M(L, K, T - T_L) = \frac{K}{L} P_E^M\left(L, \frac{L^2}{K}, T - T_L\right) . \tag{3.6.18}$$

Now, since the underlying is equal to the barrier, the down-and-in option values are equal to standard option values, in particular, for any strike k, one has $\text{DIP}^M(L, k, L) = P_E^M(L, k)$. Therefore, formula (3.6.18) implies that the option $\text{DIC}^M(x, K, L)$ is equivalent to K/L options $\text{DIP}^M\left(x, L^2K^{-1}, L\right)$.

At maturity, the terminal payoff of the DIP is strictly positive only if the underlying value is below L^2K^{-1}. Since $L \le K$, the quantity L^2K^{-1} is smaller than L. Hence, if the DIP is in the money at maturity, the barrier L was reached with probability 1; therefore, the barrier is no longer relevant in pricing the option. The $\text{DIP}^M\left(x, L^2K^{-1}, L\right)$ barrier option is thus equal to the plain vanilla $P_E^M\left(x, L^2K^{-1}\right)$ for $L \le K$ and the result is obtained.

In order to conclude, the symmetry formula is applied again. □

Corollary 3.6.4.2 *In an explicit form,*

$$\text{for } x \le L,\ \text{DIC}^M = e^{-r(T-t)}\left\{x\mathcal{N}\left(d_1\left(\frac{x}{K}, T-t\right)\right) - K\mathcal{N}\left(d_2\left(\frac{x}{K}, T-t\right)\right)\right\},$$

$$\begin{aligned}\text{for } x \ge L,\ \text{DIC}^M = e^{-r(T-t)}&\left\{L\mathcal{N}\left(d_1\left(\frac{L^2}{Kx}, T-t\right)\right)\right.\\ &\left. -\frac{Kx}{L}\mathcal{N}\left(d_2\left(\frac{L^2}{Kx}, T-t\right)\right)\right\},\end{aligned}$$

where d_1, d_2 are defined in (3.6.4).

Proposition 3.6.4.3 *The price of a regular up-and-in put ($H \ge K$) on an underlying without carrying cost is given by:*

(i) *for $x \ge L$,* $\text{UIP}^M(x, K, H) = P_E^M(x, K)$*,*

(ii) *for $x \le H$,* $\text{UIP}^M(x, K, H) = \dfrac{K}{H} C_E^M\left(x, \dfrac{H^2}{K}\right)$.

Proposition 3.6.4.4 *Let $x \geq L$. The regular binary option* BinDIC^M *satisfies*

(i)

$$\text{BinDIC}^M(x,K,L) = \frac{x}{L}\,\text{BinC}^M\left(L, \frac{Kx}{L}\right), \tag{3.6.19}$$

(ii)

$$\begin{aligned}\text{DeltaDIC}^M(x,K,L) &= -\frac{K}{L}\text{BinC}^M\left(L, \frac{Kx}{L}\right)\\ &= -\frac{Ke^{-rT}}{L}\mathcal{N}\left(d_2\left(\frac{L^2}{xK}\,,T\right)\right). \end{aligned}\tag{3.6.20}$$

The price of the DIB *option is given by*

$$\text{DIB}^M(x,L) = e^{-rT}\left[\frac{x}{L}\mathcal{N}\left(d_2(L/x\,,T)\right) + \mathcal{N}\left(d_1(L/x\,,T)\right)\right]. \tag{3.6.21}$$

The binary put value is obtained by proceeding along the same lines.

PROOF: By definition, $\text{BinDIC}^M(x,K,L) = -\partial_K \text{DIC}^M(x,K,L)$. We differentiate the first and third terms of (3.6.16) with respect to K. We get

$$\text{BinDIC}^M(x,K,L) = -\frac{x}{L}\partial_K C_E^M\left(L, \frac{Kx}{L}\right) = -\frac{x}{L}\text{DeltaP}^{\text{M}}\left(\frac{Kx}{L}, L\right)$$

where we have used the symmetry formula for the second equality. It remains to apply Proposition 3.6.2.1 to obtain (i). By differentiating the two sides of the first and second terms of equality (3.6.16) w.r.t. x, one gets

$$\text{DeltaDIC}^{\text{M}}(x,K,L) = \frac{K}{L}\text{DeltaP}^{\text{M}}\left(x, \frac{L^2}{K}\right) = \frac{K}{L}\text{DeltaP}^{\text{M}}\left(\frac{Kx}{L}, L\right)$$

where we have used the homogeneity property of degree 0 for the last equality, hence (ii) is obtained using Proposition 3.6.2.1 again. The payoff of the DIB option is equal to 1 if the barrier is reached before time T, and, using

$$\{T_L \leq T\} = \{T_L \leq T,\, S_T > L\} \cup \{S_T \leq L\},$$

we obtain

$$\begin{aligned}\text{DIB}^M(x,L) &= \text{BinDIC}^M(x,L,L) + \text{BinP}^M(x,L)\\ &= \frac{x}{L}\text{BinC}^M(L,x) + \text{BinP}^M(x,L)\\ &= e^{-rT}\left\{\frac{x}{L}\mathcal{N}\left(d_2\left(\frac{L}{x}\,,T\right)\right) + \mathcal{N}\left(d_1\left(\frac{L}{x}\,,T\right)\right)\right\}.\end{aligned}$$

One can check that the value of the DIB is smaller than 1. □

Hedge of a Regular Down-and-In Call Option

In this section, we do not write the time argument in $d_i(x,T)$. A static hedge for a DIC regular option consists in holding K/L puts as long as the underlying value remains above the barrier, and a standard call after the barrier is crossed. At the barrier, the put-call symmetry implies the continuity of the price. This is not the case for the hedge ratio, which admits a right limit given from (3.6.20) by

$$\varDelta_+ \mathrm{DIC}^M(L,K,L) = -\frac{Ke^{-rT}}{L}\mathcal{N}\left(d_2(LK^{-1})\right)$$

whereas, from (3.6.15) the left limit is

$$\varDelta_- \mathrm{DIC}^M(L,K,L) = \mathrm{DeltaC}^M(L,K) = e^{-rT}\mathcal{N}\left(d_1(LK^{-1})\right) .$$

Hence, the Delta is not continuous at the barrier and admits a negative jump equal to minus the discounted probability that the underlying with starting point K reaches the barrier before T: indeed from (3.6.21)

$$\begin{aligned}[\varDelta_+ - \varDelta_-]\mathrm{DIC}^M(L,K,L) &= -\frac{Ke^{-rT}}{L}\mathcal{N}\left(d_2(LK^{-1})\right) - e^{-rT}\mathcal{N}\left(d_1(LK^{-1})\right)\\ &= -\mathrm{DIB}^M(K,L)\,.\end{aligned}$$

The absolute value of the jump is smaller than 1.

3.6.5 Mathematical Results Deduced from the Previous Approach

In this section, we do not write the time argument T in $d_i(x,T)$. We consider a *martingale* $(S_t, t \geq 0)$ with *deterministic* volatility $\sigma = (\sigma(t), t \geq 0)$ which represents the price of an asset without carrying costs under the risk neutral probability $\mathbb{Q}$, that is

$$S_t = x \exp\left(\int_0^t \sigma(s)dW_s - \frac{1}{2}\int_0^t \sigma^2(s)ds\right) . \qquad (3.6.22)$$

A Result on Change of Probability

In a first step, we translate the symmetry formula in terms of a change of probability: equality (3.6.14) reads for any K,

$$\mathbb{E}_{\mathbb{Q}}((S_T - K)^+) = \mathbb{E}_{\mathbb{Q}}((x - KS_T/x)^+)\,.$$

We note that if X and Y are positive random variables with density, satisfying $\mathbb{E}((X-K)^+) = \mathbb{E}((Y-K)^+)$ for any $K \geq 0$, then $X \overset{\text{law}}{=} Y$. Therefore, from

$$\mathbb{E}_{\mathbb{Q}}((S_T - K)^+) = \mathbb{E}_{\mathbb{Q}}((x - KS_T/x)^+) = \mathbb{E}_{\mathbb{Q}}\left(\frac{S_T}{x}\left(\frac{x^2}{S_T} - K\right)^+\right)$$

it follows that the law of S_T under $\mathbb{Q}$ is equal to the law of x^2/S_T under $\widehat{\mathbb{Q}}$, where $\widehat{\mathbb{Q}}|_{\mathcal{F}_T} = \frac{S_T}{x}\mathbb{Q}|_{\mathcal{F}_T}$.

One can also obtain the same result using Cameron-Martin's relationship.

Exercise 3.6.5.1 Let X be a integrable random variable with density φ such that $\mathbb{E}(f(X)) = \mathbb{E}(Xf(1/X))$ for any bounded function f.

Prove that $\varphi(x) = \frac{1}{x^2}\varphi(\frac{1}{x})$. Check that the density of $X = e^{B_T - T/2}$ satisfies this equality.

`Hint:` Consider $\xi(s) := \mathbb{E}(X^s)$ for $s \in \mathbb{C}$, which satisfies $\xi(s) = \xi(1-s)$. ◁

Joint Law of (m_T^S, S_T)

Here, we assume that $x \geq L$. Let us introduce the first passage time below the barrier:

$$T_L = \inf\{t : S_t \leq L\}$$

where we set $\inf(\emptyset) = +\infty$ and note that

$$\{T_L \leq T\} = \left\{\inf_{0\leq t\leq T} S_t \leq L\} = \{m_T^S \leq L\right\},$$

where $m_t^S = \inf_{s\leq t} S_s$. The prices at time 0 for barrier and binary options are given as:

$$\begin{aligned}
\mathrm{DIC}^M(x,K,L) &= e^{-rT}\mathbb{E}_Q[\mathbb{1}_{\{T_L\leq T\}}(S_T - K)^+],\\
\mathrm{BinDIC}^M(x,K,L) &= e^{-rT}\mathbb{Q}[\{T_L \leq T\} \cap \{S_T \geq K\}]\\
&= e^{-rT}\mathbb{Q}[\{m_T^S \leq L\} \cap \{S_T \geq K\}].
\end{aligned}$$

Proposition 3.6.5.2 *Let $(S_t, t \geq 0)$ be a martingale with the following dynamics*

$$dS_t = S_t\sigma(t)dW_t\,,\ S_0 = x$$

where W is a $\mathbb{Q}$-Brownian motion, with initial value x with $x \geq L$.

For any $K \geq L$, the law of the pair (m_T^S, S_T) is given by

$$\mathbb{Q}(m_T^S \leq L,\, S_T \geq K) = \frac{x}{L}\mathbb{Q}\left(S_T \geq \frac{Kx^2}{L^2}\right) = \frac{x}{L}\mathcal{N}\left(d_2\left(\frac{L^2}{Kx}\right)\right)$$

and the law of the minimum $m_T^S = \inf_{t\leq T} S_t$:

$$\mathbb{Q}(m_T^S \leq L) = \frac{x}{L}\mathcal{N}\left(d_2(Lx^{-1})\right) + \mathcal{N}\left(d_1(Lx^{-1})\right),$$

where d_1, d_2 are given by (3.6.7).

PROOF: Formula (3.6.19) leads to

$$\mathbb{Q}(m_T^S \leq L, S_T \geq K) = \frac{x}{L}\mathbb{Q}\left(L\frac{S_T}{x} \geq \frac{K}{L}x\right) = \frac{x}{L}\mathbb{Q}\left(S_T \geq \frac{Kx^2}{L^2}\right).$$

The law of the minimum follows, taking $K = L$. □

The equality

$$\mathbb{Q}(m_T^S \leq L,\, S_T \geq K) = \frac{x}{L}\mathbb{Q}\left(S_T \geq \frac{Kx^2}{L^2}\right)$$

corresponds to the reflection principle obtained for Brownian motion. Indeed, writing, for $x = 1$, $S_t = e^{\sigma X_t}$ where $X_t = W_t - \nu t$ and $\nu = \sigma/2$ and taking the logarithm, when σ is constant, one obtains the formula given in Exercise 3.2.1.3 for the drifted Brownian motion:

$$\mathbb{P}(W_T - \nu T \geq \alpha, \inf_{0\leq t\leq T}(W_t - \nu t) \leq \beta) = e^{2\nu\beta}\mathbb{P}(W_T - \nu T \geq \alpha - 2\beta)\,.$$

By considering current prices, we shall obtain the conditional distribution (with respect to the information at time t) of the underlying value at time T and its minimum on the time interval (t, T). Let $S_t = y$ and let $m_t^S = \inf_{s\leq t} S_s = m$ (with $m \leq y$) be the minimum over the time interval $[0, t]$. In the case $m \leq L$, the barrier has been reached during the time interval $[0, t]$, whereas the barrier has not been reached when $m > L$. In the second case, the two events $(\inf_{0\leq u\leq T} S_u \leq L)$ and $(\inf_{t\leq u\leq T} S_u \leq L)$ are identical.

The equality (3.6.19) concerning barrier options

$$\text{BinDIC}^M(S_t, K, L, T-t) = \frac{S_t}{L}\text{BinC}^M\left(L, \frac{KS_t}{L}, T-t\right)$$

can be written, on the set $\{T_L \geq t\}$, as follows:

$$\mathbb{Q}(\{T_L \leq T\} \cap \{S_T \geq K\}|\mathcal{F}_t) = \mathbb{Q}(\{\inf_{t\leq u\leq T} S_u \leq L\} \cap \{S_T \geq K\}|\mathcal{F}_t)$$
$$= \frac{S_t}{L}\mathbb{Q}\left(S_T\frac{L}{S_t} \geq \frac{KS_t}{L}|\mathcal{F}_t\right) = \frac{S_t}{L}\mathcal{N}\left(d_2\left(\frac{L^2}{KS_t}, T-t\right)\right)\,. \qquad (3.6.23)$$

The equality (3.6.23) gives the conditional distribution function of the pair $(m^S[t, T], S_T)$ where $m^S[t, T] = \min_{t\leq s\leq T} S_s$, on the set $\{T_L \geq t\}$, as a differentiable function.

Hence, the conditional law of the pair $(m^S[t, T], S_T)$ with respect to $\mathcal{F}_t$ admits a density $f(h, k)$ on the set $0 < h < k$ which can be computed from the density p of a log-normal random variable with expectation 1 and with variance $\Sigma_{t,T}^2 = \int_t^T \sigma^2(s)ds$,

$$p(y) = \frac{1}{y\Sigma_{t,T}\sqrt{2\pi}}\exp\left(-\frac{1}{2\Sigma_{t,T}^2}\left(\ln(y) - \frac{1}{2}\Sigma_{t,T}^2\right)^2\right)\,. \qquad (3.6.24)$$

Indeed,

$$\mathbb{Q}(m^S[t,T] \le L, S_T \ge K | S_t = x) = \frac{x}{L}\mathcal{N}\left(d_2\left(\frac{L^2}{Kx}, T-t\right)\right)$$
$$= \frac{x}{L}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_2(L^2/(Kx),,T-t)} e^{-u^2/2}du = \frac{x}{L}\int_{Kx/L^2}^{+\infty} p(y)dy\,.$$

Hence, we obtain the following proposition:

Proposition 3.6.5.3 *Let $dS_t = \sigma(t)S_t dB_t$. The conditional density f of the pair $(\inf_{t\le u\le T} S_u, S_T)$ is given, on the set $\{0 < h < k\}$, by*

$$\mathbb{Q}(\inf_{t\le u\le T} S_u \in dh, S_T \in dk|\, S_t = x)$$
$$= \left(-\frac{3x^2}{h^4}p(kxh^{-2}) - \frac{2kx^3}{h^6}p'(kxh^{-2})\right)\, dh\, dk\,.$$

where p is defined in (3.6.24).

Comment 3.6.5.4 In the case $dS_t = \sigma(t)S_t dB_t$, the law of $(S_T, \sup_{s\le T} S_s)$ can also be obtained from results on BM. Indeed,

$$S_s = S_0 \exp\left(\int_0^s \sigma(u)dB_u - \frac{1}{2}\int_0^s \sigma^2(u)du\right)$$

can be written using a change of time as

$$S_t = S_0 \exp\left(B_{\Sigma_t} - \frac{1}{2}\Sigma_t\right)$$

where B is a BM and $\Sigma(t) = \int_0^t \sigma^2(u)du$. The law of $(S_T, \sup_{s\le T} S_s)$ is deduced from the law of $(B_u, \sup_{s\le u} B_s)$ where $u = \Sigma_T$.

3.6.6 Valuation and Hedging of Regular Down-and-In Call Options: The General Case

Valuation

We shall keep the same notation for options. However under the risk neutral probability, the dynamics of the underlying are now:

$$dS_t = S_t\left((r-\delta)dt + \sigma dW_t\right), S_0 = x\,. \qquad (3.6.25)$$

A standard method exploiting the martingale framework consists of studying the associated forward price $S_t^F = S_t e^{(r-\delta)(T-t)}$. This is a martingale under the risk-neutral forward probability measure. In this case, it is necessary to discount the barrier.

We can avoid this problem by noticing that any log-normally distributed asset is the power of a martingale asset. In what follows, we shall denote by DIC^S the price of a DIC option on the underlying S with dynamics given by equation (3.6.25).

Lemma 3.6.6.1 *Let S be an underlying whose dynamics are given by* (3.6.25) *under the risk-neutral probability $\mathbb{Q}$. Then, setting*

$$\gamma = 1 - \frac{2(r-\delta)}{\sigma^2}, \tag{3.6.26}$$

(i) *the process $S^\gamma = (S_t^\gamma, t \geq 0)$ is a martingale with dynamics*

$$dS_t^\gamma = S_t^\gamma \, \widehat{\sigma} \, dW_t$$

where $\widehat{\sigma} = \gamma\sigma$.

(ii) *for any positive Borel function f*

$$\mathbb{E}_{\mathbb{Q}}(f(S_T)) = \mathbb{E}_{\mathbb{Q}}\left(\left(\frac{S_T}{x}\right)^\gamma f\left(\frac{x^2}{S_T}\right)\right).$$

Proof: The proof of (i) is obvious. The proof of (ii) was the subject of Exercise 1.7.3.7 (see also Exercise 3.6.5.1). □

The important fact is that the process $S_t^\gamma = \exp(\widehat{\sigma} W_t - \frac{1}{2}\widehat{\sigma}^2 t)$ is a martingale, hence we can apply the results of Subsection 3.6.4.

The valuation and the instantaneous replication of the BinDIC^S on an underlying S with dynamics (3.6.25), and more generally of a DIC option, are possible by relying on Lemma 3.6.6.1.

Theorem 3.6.6.2 *The price of a regular down-and-in binary option on an underlying with dynamics (3.6.25) is, for $x \geq L$,*

$$\text{BinDIC}^S(x,K,L) = \left(\frac{x}{L}\right)^\gamma \text{BinC}^S\left(L, \frac{Kx}{L}\right). \tag{3.6.27}$$

The price of a regular DIC option is, for $x \geq L$,

$$\text{DIC}^S(x,K,L) = \left(\frac{x}{L}\right)^{\gamma-1} C_E^S\left(L, \frac{Kx}{L}\right). \tag{3.6.28}$$

Proof: In the first part of the proof, we assume that γ is positive, so that the underlying with carrying cost is an increasing function of the underlying martingale.

It is therefore straightforward to value the binary options:

$$\begin{aligned} \text{BinC}^S(x,K;\sigma) &= \text{BinC}^M(x^\gamma, K^\gamma; \widehat{\sigma}) \\ \text{BinDIC}^S(x,K,L;\sigma) &= \text{BinDIC}^M(x^\gamma, K^\gamma, L^\gamma; \widehat{\sigma}), \end{aligned}$$

where we indicate (when it seems important) the value of the volatility, which is σ for S and $\widehat{\sigma}$ for S^γ. The right-hand sides of the last two equations are known from equation (3.6.19):

$$\begin{aligned}\text{BinDIC}^M(x^\gamma, K^\gamma, L^\gamma; \widehat{\sigma}) &= \left(\frac{x}{L}\right)^\gamma \text{BinC}^M\left(L^\gamma, \left(\frac{Kx}{L}\right)^\gamma; \widehat{\sigma}\right) \\ &= \left(\frac{x}{L}\right)^\gamma \text{BinC}^S\left(L, \frac{Kx}{L}; \sigma\right). \end{aligned} \tag{3.6.29}$$

Hence, we obtain the equality (3.6.27). Note that, from formulae (3.6.10) and (3.6.9) (we drop the dependence w.r.t. σ)

$$\begin{aligned}\text{BinC}^S\left(L, \frac{Kx}{L}\right) &= -e^{-\mu T}\text{DeltaP}^S\left(\frac{Kx}{L}, Le^{2\mu T}\right) \\ &= -e^{-\mu T}\text{DeltaP}^S\left(x, \frac{(Le^{\mu T})^2}{K}\right).\end{aligned}$$

By taking the integral of this option's value between K and $+\infty$, the price DIC^S is obtained

$$\begin{aligned}\text{DIC}^S(x,K,L) &= \int_K^\infty \text{BinDIC}^S(x,k,L)dk = \left(\frac{x}{L}\right)^\gamma \int_K^\infty \text{BinC}^S\left(L, k\frac{x}{L}\right) dk \\ &= \left(\frac{x}{L}\right)^{\gamma-1} C_E^S\left(L, \frac{Kx}{L}\right).\end{aligned}$$

By relying on the put-call symmetry relationship of Proposition 3.6.1.1, and on the homogeneity property (3.6.8), the equality

$$\text{DIC}^S(x,L,K) = \left(\frac{x}{L}\right)^{\gamma-1} \frac{K}{L} P_E^S\left(x, \frac{L^2}{K}\right)$$

is obtained.

When γ is negative, a DIC binary option on the underlying becomes a UIP binary option on an underlying which is a martingale. In particular,

$$\text{BinDIC}^S(x,K,L;\sigma) = \text{BinUIP}^M\left(x^\gamma, K^\gamma, L^\gamma; \widehat{\sigma}\right),$$

and

$$\text{BinP}^M\left(L^\gamma, \left(\frac{Kx}{L}\right)^\gamma; \widehat{\sigma}\right) = \text{BinC}^S\left(L, \frac{Kx}{L}; \sigma\right)$$

because the payoffs of the two options are the same. From Proposition 3.6.4.3 corresponding to UIP options, we obtain

$$\begin{aligned}\text{BinUIP}^M(x^\gamma, K^\gamma, H^\gamma \widehat{\sigma}) &= \left(\frac{x}{H}\right)^\gamma \text{BinP}^M\left(H^\gamma, \left(\frac{Kx}{H}\right)^\gamma; \widehat{\sigma}\right) \\ &= \left(\frac{x}{H}\right)^\gamma \text{BinC}^S\left(H, \frac{Kx}{H}; \sigma\right).\end{aligned}$$

□

Remark 3.6.6.3 Let us remark that, when $\mu = 0$ (i.e., $\gamma = 1$) the equality (3.6.28) is formula (3.6.16). The presence of carrying costs induces us to consider a forward boundary, already introduced by Carr and Chou [148], in order to give two-sided bounds for the option's price. Indeed, if μ is positive and $(x/L)^{\gamma-1} \leq 1$, the right-hand side gives Carr's upper bound, while if μ is negative, the lower bound is obtained.

Therefore, the smaller $\frac{2\mu}{\sigma^2}$, the more accurate is Carr's approximation. This is also the case when x is close to L, because at the boundary, the two formulae are the same.

Hedging of the Regular Down-and-In Call Option in the General Case

As for the case of a regular DIC option without carrying costs, the Delta is discontinuous at the boundary. By relying on the above developments and on equation (3.6.29), the following equation is obtained

$$\begin{aligned}\Delta_+ \mathrm{DIC}^S(L,K,L) &= \frac{\gamma - 1}{L}\, C_E^S(L,K) - \frac{K}{L}\mathrm{BinC}^S(L,K)\\ &= \frac{\gamma}{L}\, C_E^S(L,K) - \mathrm{DeltaC}^S(L,K)\,.\end{aligned}$$

Thus,

$$(\Delta_+ - \Delta_-)\mathrm{DIC}^S(L,K,L) = \frac{\gamma}{L} C_E^S(L,K) - 2\,\mathrm{DeltaC}^S(L,K)\,.$$

However, the absolute value of this quantity is not always smaller than 1, as it was in the case without carrying costs. Therefore, depending on the level of the carrying costs, the discontinuity can be either positive or negative.

Exercise 3.6.6.4 Recover (ii) with the help of formula (3.2.4) which expresses a simple absolute continuity relationship between Brownian motions with opposite drifts ◁

Exercise 3.6.6.5 A power put option (see Exercise 2.3.1.5) is an option with payoff $S_T^\alpha (K - S_T)^+$, its price is denoted $\mathrm{PowP}^\alpha(x,K)$. Prove that there exists γ such that

$$\mathrm{DIC}^S(x,K,L) = \frac{1}{L^\gamma}\mathrm{PowP}^{\gamma-1}(Kx, L^2)\,.$$

`Hint:` From (ii) in Lemma 3.6.6.1, $\mathrm{DIC}^S(x,K,L) = \frac{1}{L^\gamma}\mathbb{E}(S_T^\gamma(\frac{L^2}{S_T} - K)^+)$. ◁

3.6.7 Valuation and Hedging of Reverse Barrier Options

Valuation of the Down-and-In Bond

The payoff of a down-and-in bond (DIB) is one monetary unit at maturity, if the barrier is reached before maturity. It is straightforward to obtain these prices by relying on BinDIC(x, L, L) prices and on a standard binary put. Indeed, the payoff of the BinDIC option is one monetary unit if the underlying value is greater than L and if the barrier is hit. The payoff of the standard binary put is also 1 if the underlying value is below the barrier at maturity. Being long on these two options generates a payoff of 1 if the barrier was reached before maturity. Hence,

$$\text{for } x \geq L, \text{DIB}(x, L) = \text{BinP}(x, L) + \text{BinDIC}(x, L, L)$$
$$\text{for } x \leq L, \text{DIB}(x, L) = B(0, T)\,.$$

By relying on equations (3.6.10, 3.6.11, 3.6.28) and on Black and Scholes' formula, we obtain, for $x \geq L$,

$$\begin{aligned}\text{DIB}(x, L) &= \text{BinP}^S(x, L) + \left(\frac{x}{L}\right)^\gamma \text{BinC}^S(L, x) \\ &= e^{-rT}\left[\mathcal{N}\left(d_1\left(\frac{L}{xe^{\mu T}}\right)\right) + \frac{x^\gamma}{L^\gamma}\mathcal{N}\left(d_2\left(\frac{Le^{\mu T}}{x}\right)\right)\right]. \end{aligned} \quad (3.6.30)$$

Example 3.6.7.1 Prove the following relationships:

$$\begin{aligned} & DIC^S(x, L, L) + L\,\text{BinDIC}^S(x, L, L) \\ &= \left(\frac{x}{L}\right)^{\gamma-1} e^{-\mu T}\left[P_E^S(x, Le^{2\mu T}) - L\frac{x}{L}\text{Delta}P_E^S(x, Le^{2\mu T})\right] \\ &= \left(\frac{x}{L}\right)^{\gamma-1} e^{\mu T} L\,\text{BinP}^S(x, Le^{2\mu T})\,, \end{aligned}$$

$$\begin{aligned}\text{DIB}(x, L) &= \text{BinP}^S(x, L) + \left(\frac{x}{L}\right)^{\gamma-1} e^{\mu T}\text{BinP}^S(x, Le^{2\mu T}) \\ &\quad - \frac{1}{L}\text{DIC}^S(x, L, L)\,. \end{aligned} \quad (3.6.31)$$

`Hint:` Use formulae (3.6.12) and (3.6.28).

Valuation of a Reverse DIC, Case $K < L$

Let us study the reverse DIC option, with strike smaller than the barrier, that is $K \leq L$. Depending on the value of the underlying with respect to the barrier at maturity, the payoff of such an option can be decomposed. Let us consider the case where $x \geq L$.

- The option with a payoff $(S_T - K)^+$ if the underlying value is higher than L at maturity and if the barrier was reached can be hedged with a DIC(x, L, L) with payoff $(S_T - L)$ at maturity if the barrier was reached and by $(L - K)$ BinDIC(x, L, L) options, with a payoff $L - K$ if the barrier was reached.
- The option with a payoff $(S_T - K)^+$ if the underlying value is between K and L at maturity (which means that the barrier was reached) can be hedged by the following portfolio:

$$-P_E(x,L) + P_E(x,K) + (L-K)\mathrm{DIB}(x,L)\,.$$

Indeed the corresponding payoff is

$$\begin{aligned}(S_T-K)^+\mathbb{1}_{\{K\le S_T\le L\}} &= (S_T-L-K+L)\mathbb{1}_{\{K\le S_T\le L\}}\\ &= (S_T-L)\mathbb{1}_{\{K\le S_T\le L\}} + (L-K)\mathbb{1}_{\{K\le S_T\le L\}}\\ &= (S_T-L)\mathbb{1}_{\{S_T\le L\}} - (S_T-L)\mathbb{1}_{\{S_T\le K\}}\\ &\qquad + (L-K)\mathbb{1}_{\{S_T\le L\}} - (L-K)\mathbb{1}_{\{S_T\le K\}}\\ &= -(L-S_T)^+ + (K-S_T)^+ + (L-K)\mathbb{1}_{\{S_T\le L\}}\,.\end{aligned}$$

This very general formula is a simple consequence of the no arbitrage principle and can be obtained without specific assumptions concerning the underlying dynamics, unlike the DIB valuation formula.

The hedging of such an option requires plain vanilla options, regular DIC options with the barrier equal to the strike, and $\mathrm{DIB}(x,L)$ options, and is not straightforward. The difficulty corresponds to the hedging of the standard binary option.

In the particular case of a deterministic volatility, by relying on (3.6.31),

$$\begin{aligned}\mathrm{DIC}_{rev}(x,K,L) = \left(\frac{K}{L}-1\right)&\mathrm{DIC}(x,L,L) - P_E(x,L) + P_E(x,K)\\ &+ (L-K)\mathrm{BinP}(x,L)\\ &+ (L-K)\left(\frac{x}{L}\right)^{\gamma-1} e^{\mu T}\mathrm{BinP}(x,Le^{2\mu T})\,.\end{aligned}$$

3.6.8 The Emerging Calls Method

Another way to understand barrier options is the study of the first passage time of the underlying at the barrier, and of the prices of the calls at this first passage time. This corresponds to integration of the calls with respect to the hitting time distribution.

Let us assume that the initial underlying value x is higher than the barrier, i.e., $x > L$. We denote, as usual,

$$T_L = \inf\{t \,:\, S_t \le L\}$$

the hitting time of the barrier L.

The term $e^{rT}\mathrm{DIB}(x,L,T)$ is equal to the probability that the underlying reaches the barrier before maturity T. Hence, its derivative, i.e., the quantity $f_L(x,t) = \partial_T[e^{rT}\mathrm{DIB}(x,L,T)]_{T=t}$ is the density $\mathbb{Q}(T_L \in dt)/dt$, and the following decomposition of the barrier option is obtained:

$$\mathrm{DIC}(x,K,L,T) = \int_0^T C_E(L,K,T-\tau)e^{-r\tau} f_L(x,\tau)d\tau\,. \tag{3.6.32}$$

The density f_L is obtained by differentiating $e^{rT}\text{DIB}$ with respect to T in (3.6.30). Hence

$$f_L(x,t) = \frac{h}{\sqrt{2\pi t^3}} \exp(-\frac{1}{2t}(h-\nu t)^2)\,,$$

where

$$h = \frac{1}{\sigma}\ln\left(\frac{x}{L}\right),\ \nu = \frac{\mu}{\sigma} - \frac{\sigma}{2}\,.$$

(See Subsection 3.2.2 for a different proof.)

3.6.9 Closed Form Expressions

Here, we give the previous results in a closed form.

▶ For $K \le L$,

$$\begin{aligned}\text{DIC}^S(L,K) = S_0\Big(\mathcal{N}(z_1) - \mathcal{N}(z_2) + \left(\frac{L}{x}\right)^{\frac{2r}{\sigma^2}+1}\mathcal{N}(z_3)\Big)\\ - Ke^{-rT}\Big(\mathcal{N}(z_4) - \mathcal{N}(z_5) + \left(\frac{L}{x}\right)^{\frac{2r}{\sigma^2}-1}\mathcal{N}(z_6)\Big)\end{aligned}$$

where

$$\begin{aligned}
z_1 &= \frac{1}{\sigma\sqrt{T}}\left(\left(r+\frac{1}{2}\sigma^2\right)T + \ln\left(\frac{x}{K}\right)\right), & z_4 &= z_1 - \sigma\sqrt{T}\\
z_2 &= \frac{1}{\sigma\sqrt{T}}\left(\left(r+\frac{1}{2}\sigma^2\right)T + \ln\left(\frac{x}{L}\right)\right), & z_5 &= z_2 - \sigma\sqrt{T}\\
z_3 &= \frac{1}{\sigma\sqrt{T}}\left(\left(r+\frac{1}{2}\sigma^2\right)T - \ln\left(\frac{x}{L}\right)\right), & z_6 &= z_3 - \sigma\sqrt{T}\,.
\end{aligned}$$

▶ In the case $K \ge L$, we find that

$$\text{DIC}^S(L,K) = x\left(\frac{L}{x}\right)^{\frac{2r}{\sigma^2}+1}\mathcal{N}(z_7) - Ke^{-rT}\left(\frac{L}{x}\right)^{\frac{2r}{\sigma^2}-1}\mathcal{N}(z_8)$$

where

$$\begin{aligned}
z_7 &= \frac{1}{\sigma\sqrt{T}}\left(\ln(L^2/xK) + \left(r+\frac{1}{2}\sigma^2\right)T\right)\\
z_8 &= z_7 - \sigma\sqrt{T}\,.
\end{aligned}$$

3.7 Lookback Options

A **lookback** option on the minimum is an option to buy at maturity T the underlying S at a price equal to K times the minimum value m_T^S of the underlying during the maturity period (here, $m_T^S = \min_{0\le u\le T} S_u$). The terminal payoff is $(S_T - Km_T^S)^+$. We assume in this section that the dynamics of the underlying asset value under the risk-adjusted probability is given in a Garman-Kohlhagen model by equation (3.6.25).

3.7.1 Using Binary Options

The BinDIC^S price formula can be used in order to value and hedge options on a minimum. Let $\mathrm{MinC}^S(x, K)$ be the price of the lookback option. The terminal payoff can be written

$$(S_T - Km_T^S)^+ = \int_0^{+\infty} \mathbb{1}_{\{S_T \ge k \ge Km_T^S\}}\, dk\,.$$

The expectation of this quantity can be expressed in terms of barrier options:

$$\begin{aligned}\mathrm{MinC}^S(x,K) = e^{-rT}\mathbb{E}_{\mathbb{Q}}((S_T - Km_T^S)^+) &= \int_0^{+\infty} \mathrm{BinDIC}^S\left(x, k, \frac{k}{K}\right) dk \\ &= \int_0^{xK} \mathrm{BinDIC}^S\left(x, k, \frac{k}{K}\right) dk + \int_{xK}^{\infty} \mathrm{BinDIC}^S\left(x, k, \frac{k}{K}\right) dk \\ &= I_1 + I_2\,.\end{aligned}$$

In the second integral I_2, since $x < k/K$, the BinDIC is activated at time 0 and $\mathrm{BinDIC}^S\big(x, k, \frac{k}{K}\big) = \mathrm{BinC}^S(x, k)$, hence

$$I_2 = e^{-rT}\int_{xK}^{\infty} \mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{S_T\ge k\}})dk = e^{-rT}\mathbb{E}_{\mathbb{Q}}((S_T - xK)^+) = C_E^S(x, xK)\,.$$

The first term I_1 is more difficult to compute than I_2. From Theorem 3.6.6.2, we obtain, for $k < Kx$,

$$\mathrm{BinDIC}^S\left(x, k, \frac{k}{K}\right) = \left(\frac{xK}{k}\right)^{\gamma} \mathrm{BinC}^S\left(\frac{k}{K}, xK\right),$$

where γ is the real number such that $(S_t^\gamma, t \ge 0)$ is a martingale, i.e., $S_t = xM_t^{1/\gamma}$ where M is a martingale with initial value 1. From the identity $\mathrm{BinC}^S(x,K) = e^{-rT}\mathbb{Q}(xM_T^{1/\gamma} > K)$, we get:

$$\begin{aligned}\int_0^{xK} \mathrm{BinDIC}^S\left(x, k, \frac{k}{K}\right) dk &= \int_0^{xK} \left(\frac{xK}{k}\right)^{\gamma} \mathrm{BinC}^S\left(\frac{k}{K}, xK\right) dk \\ &= e^{-rT}\mathbb{E}_{\mathbb{Q}}\left(\int_0^{xK} \left(\frac{xK}{k}\right)^{\gamma} \mathbb{1}_{\{kM_T^{1/\gamma} > xK^2\}} dk\right)\end{aligned}$$

$$= e^{-rT}(xK)^{\gamma}\mathbb{E}_{\mathbb{Q}}\left(\int_0^{\infty} k^{-\gamma}\mathbb{1}_{\{xK>k>xK^2M_T^{-1/\gamma}\}}dk\right).$$

For $\gamma \neq 1$, the integral can be computed as follows:

$$\begin{aligned}
&\int_0^{xK} \text{BinDIC}^S\left(x,k,\frac{k}{K}\right)dk \\
&= e^{-rT}\frac{(xK)^{\gamma}}{1-\gamma}\mathbb{E}_{\mathbb{Q}}\left[\left((xK)^{1-\gamma}-(xK^2M_T^{-1/\gamma})^{1-\gamma}\right)^+\right] \\
&= e^{-rT}\frac{xK}{1-\gamma}\mathbb{E}_{\mathbb{Q}}\left[\left(1-K^{1-\gamma}M_T^{-(1-\gamma)/\gamma}\right)^+\right] \\
&= e^{-rT}\frac{xK}{1-\gamma}\mathbb{E}_{\mathbb{Q}}\left[\left(1-\frac{K^{1-\gamma}S_T^{\gamma-1}}{x^{\gamma-1}}\right)^+\right].
\end{aligned}$$

Using Itô's formula and recalling that $1-\gamma = \frac{2\mu}{\sigma^2}$, we have

$$d(S_t^{\gamma-1}) = S_t^{\gamma-1}\left(\mu dt - \frac{2\mu}{\sigma}dW_t\right)$$

hence the following formula is derived

$$\text{MinC}^S(x,K) = x\left[C_E^S(1,K;\sigma)+\frac{K\sigma^2}{2\mu}P_E^S\left(K^{1-\gamma},1;\frac{2\mu}{\sigma}\right)\right]$$

where $C_E^S(x,K;\sigma)$ (resp. $P_E^S(x,K;\sigma)$) is the call (resp. put) value on an underlying with carrying cost μ and volatility σ with strike K. The price at date t is $\text{MinC}^S(S_t, Km_t^S; T-t)$ where $m_t^S = \min_{s\le t} S_s$.

For $\gamma = 1$ we obtain

$$\text{MinC}^S(x,K) = C_E^S(x,xK) + xK\mathbb{E}_{\mathbb{Q}}\left[\left(\ln\frac{S_T}{xK}\right)^+\right].$$

Let $C_{\ln}^S(x,K)$ be the price of an option with payoff $(\ln(S_T/x)-\ln K)^+$, then

$$\text{MinC}^S(x,K) = C_E^S(x,xK) + xKC_{\ln}^S(x,xK).$$

3.7.2 Traditional Approach

The payoff for a **standard lookback** call option is $S_T - m_T^S$. Let us remark that the quantity $S_T - m_T^S$ is positive. The price of such an option is

$$\text{MinC}^S(x,1;T) = e^{-rT}\mathbb{E}_{\mathbb{Q}}(S_T - m_T^S)$$

whereas $\text{MinC}^S(x,1;T-t)$, the price at time t, is given by

$$\mathrm{MinC}^S(x,1;T-t) = e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}(S_T - m_T^S|\mathcal{F}_t)\,.$$

We now forget the superscript S in order to simplify the notation. The relation $m_T = m_t \wedge m_{t,T}$, with $m_{t,T} = \inf\{S_u, u \in [t,T]\}$ leads to

$$e^{-rt}\mathrm{MinC}^S(x,1;T-t) = e^{-rT}\mathbb{E}_{\mathbb{Q}}(S_T|\mathcal{F}_t) - e^{-rT}\mathbb{E}_{\mathbb{Q}}(m_t \wedge m_{t,T}|\mathcal{F}_t)\,.$$

Using the $\mathbb{Q}$-martingale property of the process $(e^{-\mu t}S_t, t \geq 0)$, the first term is $e^{-rt}e^{-\delta(T-t)}S_t$. As far as the second term is concerned, the expectation is decomposed as follows:

$$\mathbb{E}_{\mathbb{Q}}(m_t \wedge m_{t,T}|\mathcal{F}_t) = \mathbb{E}_{\mathbb{Q}}(m_t \mathbb{1}_{\{m_t < m_{t,T}\}}|\mathcal{F}_t) + \mathbb{E}_{\mathbb{Q}}(m_{t,T}\mathbb{1}_{\{m_{t,T} < m_t\}}|\mathcal{F}_t)\,.$$

Using measurability and independence arguments, we obtain

$$\mathbb{E}_{\mathbb{Q}}(m_t \mathbb{1}_{\{m_t < m_{t,T}\}}|\mathcal{F}_t) = m_t\, \Phi(T-t, m_t, S_t)$$

where $\Phi(u,m,x) = \mathbb{Q}(m < x m_u^Y)$, with $Y \overset{\text{law}}{=} (S/S_0)$. An explicit expression for Φ is obtained from the results concerning the law of the minimum of the drifted Brownian motion or by relying on barrier options results:

$$\Phi(u,m,x) = \mathcal{N}(d - \sigma\sqrt{u}) - \left(\frac{x}{m}\right)^{1-2\mu/\sigma^2} \mathcal{N}\left(-d + \frac{2\mu}{\sigma}\sqrt{u}\right)$$

where

$$d = d_1\left(\frac{xe^{ru}}{m}\right) = \frac{\ln\left(\frac{x}{m}\right) + (\mu + \sigma^2/2)u}{\sigma\sqrt{u}}\,.$$

The quantity

$$\mathbb{E}_{\mathbb{Q}}(m_{t,T}\mathbb{1}_{\{m_{t,T} < m_t\}}|\mathcal{F}_t)$$

can be written $\Psi(T-t, m_t, S_t)$ with $\Psi(u,m,x) = \mathbb{E}_{\mathbb{Q}}(x m_u \mathbb{1}_{\{x m_u < m\}})$ which can be computed from the law of m_u. The following proposition (obtained also in the previous section, setting $K = 1$) is derived:

Proposition 3.7.2.1 *The lookback option price is*

$$\mathrm{Min}^S(S_t,1;T-t) = S_t e^{-\delta(T-t)}\mathcal{N}(d_t) - e^{-r(T-t)} m_t \mathcal{N}\left(d_t - \sigma\sqrt{T-t}\right)$$
$$+ e^{-r(T-t)}\frac{S_t\sigma^2}{2\mu}\left[\left(\frac{m_t}{S_t}\right)^{\frac{2\mu}{\sigma^2}} \mathcal{N}\left(-d_t + \frac{2\mu\sqrt{T-t}}{\sigma}\right) - e^{r(T-t)}\mathcal{N}(-d_t)\right]$$

with $d_t = \dfrac{1}{\sigma\sqrt{T-t}}\ln\left(\dfrac{S_t}{m_t} + \left(\mu + \dfrac{1}{2}\sigma^2\right)(T-t)\right)$ *and* $m_t = \inf_{s \leq t} S_s$.

Comment 3.7.2.2 Other results on lookback options are presented in Conze and Viswanathan [193] and He et al. [426]. A PDE approach for European options whose terminal payoff involves path-dependent lookback variables is presented in Xu and Kwok [853]. See also Elliott and Kopp [317] p. 182–183 for the case $\delta = 0$ and Musiela and Rutkowski [661] p. 214–218 and Shreve [795], p. 314–320.

3.8 Double-barrier Options

The payoff of a double-barrier option is $(S_T - K)^+$ if the underlying asset has remained in the range $[L, H]$ for all times between 0 and maturity, otherwise, the payoff is null. Its price is

$$C_{db}(x, K, L, H, T) := \mathbb{E}_{\mathbb{Q}}(e^{-rT}(S_T - K)^+ \mathbb{1}_{\{T^* > T\}})$$

where $T^* := T_H(S) \wedge T_L(S)$. We give the computation of

$$\mathbb{E}_{\mathbb{Q}}(e^{-rT}(S_T - K)^+ \mathbb{1}_{\{T^* < T\}}) = \mathbb{E}_{\mathbb{Q}}(e^{-rT}(S_T - K)^+) - C_{db}(x, K, L, H, T),$$

in the case where the risk-neutral dynamics of S are

$$dS_t = S_t(rdt + \sigma dW_t);$$

the price of the double barrier will follow. With a change of probability the quantity $\mathbb{E}_{\mathbb{Q}}(e^{-rT}(S_T - K)^+ \mathbb{1}_{\{T^* < T\}})$ can be written as

$$e^{-(r+\frac{1}{2}\nu^2)T}\mathbb{E}_{\mathbb{Q}}((xe^{\sigma B_T} - K)^+ e^{\nu B_T} \mathbb{1}_{\{T^* < T\}}),$$

where B is a generic BM. The explicit computation can be performed using the law of the pair (B_T, T^*) which may be obtained from the two-sided series (3.5.2).

Another approach is to proceed as in Geman and Yor [384] where the Laplace transform Φ of $\varphi(t) = \mathbb{E}_{\mathbb{Q}}[e^{\nu B_t}(xe^{\sigma B_t} - K)^+ \mathbb{1}_{\{T^* < t\}}]$ is computed. From Markov's property

$$\begin{aligned}\Phi(\lambda) &= \int_0^\infty \exp\left(-\frac{\lambda^2 t}{2}\right) \varphi(t)\, dt = \mathbb{E}_{\mathbb{Q}}\left(\int_{T^*}^\infty \exp\left(-\frac{\lambda^2 t}{2}\right) \psi(B_t)\, dt\right) \\ &= \mathbb{E}_{\mathbb{Q}}\left(\exp\left(-\frac{\lambda^2 T^*}{2}\right) \int_0^\infty \exp\left(-\frac{\lambda^2 t}{2}\right) \psi(\widetilde{B}_t + B_{T^*})\, dt\right)\end{aligned}$$

where $\psi(y) = e^{\nu y}(xe^{\sigma y} - K)^+$ and $\widetilde{B} = (\widetilde{B}_t = B_{t+T^*} - B_{T^*}; t \geq 0)$ is a Brownian motion independent of $(B_s, s \leq T^*)$. The computation of the expectation can be simplified by splitting the expression into two parts depending on the stopping time values:

$$\Phi(\lambda) = \Psi(h)\mathbb{E}_{\mathbb{Q}}\left[e^{-\lambda^2 T^*/2}\mathbb{1}_{\{T^* = T_h\}}\right] + \Psi(\ell)\mathbb{E}_{\mathbb{Q}}\left[e^{-\lambda^2 T^*/2}\mathbb{1}_{\{T^* = T_\ell\}}\right],$$

where $h = \ln(H/x)\sigma^{-1}, \ell = \ln(L/x)\sigma^{-1}$ and, from Exercise 1.4.1.7

$$\Psi(z) = \mathbb{E}\int_0^\infty e^{-\lambda^2 t/2}\psi(\widetilde{B}_t + z)dt = \frac{1}{\lambda}\int_{-\infty}^\infty e^{-\lambda|z-y|}\psi(y)dy.$$

We have obtained an explicit form for

$$\mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\frac{\lambda^2 T^*}{2}\right)\mathbb{1}_{\{T^*=T_h\}}\right]$$

in the proof of Proposition 3.5.1.3; we now present the computation of $\Psi(x)$.
▶ Let $K \in [L, H]$ and let $k = \ln(K/x)\sigma^{-1}, \ell = \ln(L/x)\sigma^{-1}$. For values of λ such that $\nu + \sigma - \lambda < 0$, and by relying on the resolvent:

$$\begin{aligned}\Psi(h) &= g(h,\lambda)[Kg(h,\nu-\lambda) - xg(h,\sigma+\nu-\lambda)] \\ &\quad + g(-h,\lambda)\big[x(g(h,\sigma+\nu+\lambda) - g(k,\sigma+\nu+\lambda)) \\ &\qquad - K(g(h,\nu+\lambda) - g(k,\nu+\lambda))\big]\end{aligned}$$

with $g(u,\alpha) = \frac{1}{u}e^{u\alpha}$ and

$$\Psi(\ell) = \frac{e^{\lambda\ell}}{\lambda}[Kg(k,\nu-\lambda) - xg(k,\sigma+\nu-\lambda)]\,.$$

▶ For $K < L$, and $z = \ell$ or $z = k$, we find

$$\begin{aligned}\Psi(z) &= g(h,\lambda)\big(Kg(h,\nu-\lambda) - xg(z,\sigma+\nu-\lambda)\big) \\ &\quad + g(-h,\lambda)\Big(x\left(g(h,\sigma+m+\lambda) - g(z,\sigma+\nu+\lambda)\right) \\ &\qquad - K\left(g(h,\nu+\lambda) - g(z,\nu+\lambda)\right)\Big)\,.\end{aligned}$$

The Laplace transform must now be inverted.

The main papers concerning double-barrier options are those of Kunitomo and Ikeda [551], Geman and Yor [384], Goldman et al. [399], Pelsser [704], Hui et al. [600], Schröder [768] and Davydov and Linetsky [226].

3.9 Other Options

We give a few examples of other traded options. We assume as previously that

$$dS_t = S_t((r-\delta)dt + \sigma dW_t),\; S_0 = x$$

under the risk-neutral probability $\mathbb{Q}$ and we denote by $T_a = T_a(S)$ the first time when level a is reached by the process S.

3.9.1 Options Involving a Hitting Time

Digital Options

The *asset-or-nothing* options depend on an exercise price K. The terminal payoff is equal to the value of the underlying, if it is in the money at maturity and 0 otherwise, i.e., $S_T\mathbb{1}_{\{S_T \geq K\}}$. The strike price plays the rôle of a barrier.

The value of such an option is $e^{-rT}\mathbb{E}_{\mathbb{Q}}(S_T \mathbb{1}_{\{S_T \geq K\}})$ and is straightforward to evaluate. Indeed, this is the first term in the Black and Scholes formula (2.3.3).

These options can also have an up-and-in feature which depends on a barrier. The price is $e^{-rT}\mathbb{E}_{\mathbb{Q}}(S_T \mathbb{1}_{\{S_T \geq K\}} \mathbb{1}_{\{T_L > T\}})$. They are used for hedging barrier options.

Barrier Forward-start or Early-ending Options

In this case, the barrier is activated at time T', with $T' < T$ where T is the maturity. In the case of an up-and-out forward-start call option, the payoff is $(S_T - K)^+ \mathbb{1}_{\{T_H^{T'} \geq T\}}$ with $T_H^{T'} = \inf\{u \geq T' \,:\, S_u \geq H\}$. For *early-ending* options, the barrier is active only until T'.

3.9.2 Boost Options

The BOOST (Banking On Overall Stability) options were introduced in the market by Société Générale in 1994. They are characterized by two levels, a and b, with $a \leq b$. When the boundary of a given range $[a, b]$ is reached for the first time, the BOOST option terminates, and its owner receives a payoff equal to a daily amount multiplied by the number of days during which the underlying asset remained in the range before the first exit. A BOOST option is, most of the time, a strictly decreasing function of the volatility; therefore it enables its owner to bet on a decrease in the volatility.

One-level

The one-level BOOST pays, at maturity, an amount equal to the time that the underlying asset remains continuously above a level a. Therefore, its price is

$$\mathbb{E}_{\mathbb{Q}}[e^{-rT}(T \wedge T_a)] = e^{-rT} T \, \mathbb{Q}(T < T_a) + e^{-rT}\mathbb{E}_{\mathbb{Q}}(T_a \mathbb{1}_{\{T_a < T\}}) \,.$$

Assume that $a < x$ and let us introduce, as in Subsection 3.2.4,

$$\Psi(\lambda) := \mathbb{E}(e^{-\lambda T_a(S)} \mathbb{1}_{\{T_a(S) < T\}}) = e^{(\nu-\gamma)\alpha} \mathcal{N}\left(\frac{\alpha - \gamma T}{\sqrt{T}}\right) + e^{(\nu+\gamma)\alpha} \mathcal{N}\left(\frac{\alpha + \gamma T}{\sqrt{T}}\right)$$

with $\nu = (r - \delta)(\sigma)^{-1} - \sigma/2$, $\gamma^2 = 2\lambda + \nu^2$, $\alpha = \sigma^{-1} \ln(a/x)$.

Then $\mathbb{E}(T_a \mathbb{1}_{\{T_a < T\}}) = -\Psi'(0)$, i.e.,

$$\begin{aligned} \mathbb{E}(T_a \mathbb{1}_{\{T_a < T\}}) = &-\frac{\alpha}{\nu}\left[\mathcal{N}\left(-\frac{\nu T + \alpha}{\sqrt{T}}\right) - e^{2\nu\alpha} \mathcal{N}\left(\frac{\nu T + \alpha}{\sqrt{T}}\right)\right] \\ &+ \frac{\sqrt{T}}{\nu\sqrt{2\pi}}\left[\exp\left[-\frac{1}{2T}(\nu T - \alpha)^2\right] - e^{2\nu\alpha} \exp\left[-\frac{1}{2T}(\nu T + \alpha)^2\right]\right] \end{aligned}$$

and

$$\mathbb{Q}(T < T_a) = 1 - \Psi(0) = \mathcal{N}\left(\frac{-\alpha + \nu T}{\sqrt{T}}\right) - e^{2\nu\alpha}\mathcal{N}\left(\frac{\alpha + \nu T}{\sqrt{T}}\right).$$

Corridor

The BOOST option value $B_{cor}(S_0, T)$ is given by the expected discounted payoff,

$$B_{cor}(S_0, T) := \mathbb{E}_{\mathbb{Q}}(e^{-rT^*}T^*\mathbb{1}_{\{T^*<T\}} + e^{-rT}T\mathbb{1}_{\{T^*\geq T\}}) \tag{3.9.1}$$

with

$$T^* = T_a(S) \wedge T_b(S).$$

We suppose that $a < S_0 < b$. The valuation problem reduces to the knowledge of the law of T^*.

Let us consider a **perpetual corridor BOOST with payment at hit**. Its price is given by (3.9.1) with $T = \infty$, i.e.,

$$B_{cor}(S_0, \infty) := \mathbb{E}_{\mathbb{Q}}(e^{-rT^*}T^*).$$

The problem reduces to the computation of $\Psi(\lambda) = \mathbb{E}_{\mathbb{Q}}(\exp(-\frac{1}{2}\lambda^2 T^*))$. Indeed, the computation of $\mathbb{E}_{\mathbb{Q}}(e^{-rT^*}T^*)$ will follow after differentiation with respect to λ: $\mathbb{E}_{\mathbb{Q}}(e^{-rT^*}T^*) = -\dfrac{\Psi'(\sqrt{2r})}{\sqrt{2r}}$. Let us remark that

$$T^* = \inf\{t | X_t \leq \alpha \text{ or } X_t \geq \beta\} := T^*(X)$$

where $X_t = \nu t + B_t = (\frac{r-\delta}{\sigma} - \frac{\sigma}{2})t + B_t$. Using the results obtained in Subsection 3.5.2, we get, in the case $\frac{a}{x} = \frac{x}{b}$,

$$\mathbb{E}_{\mathbb{Q}}(e^{-rT^*}) = \frac{b}{x}\frac{x^\theta + b^\theta}{x^{\theta-2} + b^{\theta-2}}$$

with

$$\theta = -\frac{2\nu}{\sigma} = -\frac{2(r-\delta)}{\sigma^2} + 1.$$

It follows that

$$\mathbb{E}(T^*e^{-rT^*}) = \frac{2b\,(bx)^{\theta-2}}{x\sigma^2\,[x^{\theta-2} + b^{\theta-2}]^2}(x^2 - b^2)\,\ln\frac{x}{b}.$$

Comments 3.9.2.1 (a) Many other examples are presented in Haug [425], Kat [516], Pechtl [703], and Zhang [872].

(b) Crucial hedging problems are not considered here: we refer to Bhansali [84] and Taleb [818].

(c) BOOST options have been studied by Douady [263] and Leblanc [572]. Path-dependent options with payoff of the form

$$\left(\int_0^T \mathbb{1}_{\{S_s \geq a\}} ds - K\right)^+ \mathbb{1}_{\{M_T^S \leq b\}}$$

are studied in Fujita et al. [367].

3.9.3 Exponential Down Barrier Option

We apply the results given in Subsections 3.3.1 and 3.2.2 to obtain the price of an option with a deterministic exponential barrier. As usual, we work in the Black and Scholes model where the dynamics of the underlying stock value in the risk-neutral economy are:

$$dS_t = S_t(rdt + \sigma dB_t),\ S_0 = x$$

where the risk-free rate r and the volatility σ are constant and where B is a Brownian motion under the risk-neutral probability $\mathbb{Q}$. The barrier $b(t)$ is a deterministic function of time

$$b(t) = z\exp(\eta t),$$

where $z < x$, $\eta > 0$ and $ze^{\eta T} < K$. The first hitting time of the barrier is the time τ

$$\tau = \inf\{t \geq 0,\ S_t \leq b(t)\} = \inf\{t \geq 0,\ \widehat{S}_t \leq z\}$$

where $\widehat{S}_t = S_t e^{-\eta t}$. The dynamics of $\widehat{S}$ are:

$$d\widehat{S}_t = \widehat{S}_t((r-\eta)dt + \sigma dB_t),\ \widehat{S}_0 = x\,.$$

We assume that the payoff $(K - S_\tau)^+ = (K - S_\tau)$ is paid at hit, i.e., at time τ in the case $\tau < T$ and that, if $T \leq \tau$, the payoff is $(K - S_T)^+$, paid at T, where K is the strike price. Therefore, the value of this down-paid at hit option with exponential barrier is given by:

$$\begin{aligned}
&P^{\eta,z}_{\text{expbar}}(S_0, T) \\
&= \mathbb{E}_{\mathbb{Q}}((K - S_\tau)^+ e^{-r\tau}\mathbb{1}_{\{\tau<T\}}) + e^{-rT}\mathbb{E}_{\mathbb{Q}}((K - S_T)^+\mathbb{1}_{\{\tau\geq T\}}) \\
&= \mathbb{E}_{\mathbb{Q}}((K - S_\tau)e^{-r\tau}\mathbb{1}_{\{\tau<T\}}) + e^{(\eta-r)T}\mathbb{E}_{\mathbb{Q}}((e^{-\eta T}K - \widehat{S}_T)^+\mathbb{1}_{\{\tau\geq T\}}) \\
&= \int_0^T (K - b(t))e^{-rt}\mathbb{Q}(\tau \in dt) \\
&\qquad + e^{(\eta-r)T}\int_z^{Ke^{-\eta T}} (Ke^{-\eta T} - y)\,\mathbb{Q}(\widehat{S}_T \in dy, \widehat{m}_T > z)
\end{aligned}$$

where $\widehat{m}_T$ is the minimum

$$\widehat{m}_T = \inf_{u\in[0,T]} \widehat{S}_u \,.$$

By relying on the dynamics of the process $\widehat{S}$ and on Subsections 3.2.2 and 3.3.1 the two densities are known: setting

$$\alpha = \frac{\ln(z/x)}{\sigma}\,, \qquad \text{and} \quad \nu = \frac{r-\eta}{\sigma} - \frac{\sigma}{2}\,,$$

we obtain

$$\mathbb{Q}(\tau \in dt) = |\alpha| \frac{1}{\sqrt{2\pi t^3}} \exp\left(-\frac{1}{2t}(\alpha - \nu t)^2\right) dt$$

and, for $y > z, x > z$,

$$\mathbb{Q}(\widehat{S}_T \in dy, m_T > z) = -\frac{d}{dy}\mathbb{Q}(\widehat{S}_T \geq y, m_T > z)\,.$$

Hence, setting $\beta(y) = \frac{\ln(y/x)}{\sigma}$

$$\mathbb{Q}(\widehat{S}_T \in dy, m_T > z)/dy =$$

$$\frac{1}{\sigma y\sqrt{2\pi T}} \left(\exp\left(-\frac{(-\beta(y)+\nu T)^2}{2T}\right) - e^{2\nu\alpha}\exp\left(-\frac{(-\beta(y)+2\alpha+\nu T)^2}{2T}\right)\right)$$

The value of the option follows.

Comments 3.9.3.1 By assuming that the exercise boundary of the American put (see $\rightarrowtail$ Section 3.11) written on a non-dividend-paying stock is an exponential function of time to expiration, Omberg [686] obtains an approximation of the put price P_A. The author makes the assumption that the exercise boundary for an American put can be approximated by

$$b_{p,z,\eta}(t) = z^* \exp(\eta^* t),\, t \in]0,T]$$

where the two unknowns $z^* = b_p(0)$ and η^* are positive and constant. Each function of this form corresponds to a possible exercise policy which is defined as follows: to exercise the put as soon as the underlying process S reaches $b_{p,z,\eta}$ before maturity, that is to say at time τ if $\tau < T$, or at maturity if the put is in the money and if $\tau \geqslant T$. In this context, the put option value is given by means of the previous computation:

$$P_A(S_0,T) = \sup_{z,\eta} P^{\eta,z}_{\text{expbar}}(S_0,T)$$

and z^*, η^* are the values of (z,η) which maximise this expression. By simplifying further the option value, Omberg [686] obtains a weighted sum of cumulative functions of the standard Gaussian law.

It is worthwhile mentioning that the above approximation is in reality a lower bound for the put value, since an exponential exercise boundary is in general suboptimal. Indeed, for example, at maturity, it is known that the exercise boundary is a non-differentiable function of time (the slope is infinite). As shown in equation (3.11.7), the approximation of the exercise boundary near to maturity is different from an exponential function of time. However, as shown by Omberg, the level of accuracy obtained with this approximation formula is high.

3.10 A Structural Approach to Default Risk

Credit risk, or default risk, concerns the case where a promised payoff is not delivered if some event (the default) happens before the delivery date. The default occurs at time τ where τ is a random variable.

In the structural approach, a default event is specified in terms of the evolution of the firm's assets. Given the value of the assets of the firm, the aim is to deduce the value of corporate debt.

3.10.1 Merton's Model

In this approach – pioneered by Merton [642] – the default occurs if the assets of the firm are insufficient to meet payments on debt *at maturity.* The firm is financed by the issue of bonds, and the face value L of the bonds must be paid at time T. At time T, the bondholders will receive $\min(V_T, L)$ where L is the debt value and V_T the value of the firm. Thus, writing

$$\min(V_T, L) = L - (L - V_T)^+$$

we are essentially dealing with an option pricing problem. Merton assumes that the risk-neutral dynamics of the value of the firm are

$$dV_t = V_t(rdt + \sigma dB_t),\ V_0 = v > L\,,$$

where r is the (constant) risk-free interest rate, and σ is the constant volatility. In that context, the contingent claim pricing methodology can be used: the market where $(V_t, t \geq 0)$ is a tradeable asset is complete and arbitrage free, the equivalent martingale measure is the historical one, hence the value of the corporate bonds at time t is

$$\mathbb{E}(e^{-r(T-t)} \min(V_T, L)|\mathcal{F}_t) = Le^{-r(T-t)} - P_E(t, V_t, L)$$

where $P_E(t, x, L)$ is the value at time t of a put option on the underlying V with strike L and maturity T.

We denote by $P(t,T) = e^{-r(T-t)}$ the value of a default-free zero-coupon and by $D(t,T)$ the value of the defaultable zero-coupon of maturity T, with payment $L = 1$, i.e.,

$$D(t,T) = e^{-r(T-t)}\mathbb{E}(\mathbb{1}_{\{V_T>1\}} + V_T\mathbb{1}_{\{V_T<1\}}|\mathcal{F}_t)\,.$$

Then, from the valuation formula for the European put option

$$D(t,T) = V_t \mathcal{N}(-d_1(V_t, T-t)) + P(t,T)\mathcal{N}(d_2(V_t, T-t)),$$

where

$$d_1(V_t, T-t) = \frac{\log(V_t) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2(V_t, T-t) = \frac{\log(V_t) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}.$$

We denote by

$$Y(t,T) = -\frac{\ln P(t,T)}{T-t},$$

and

$$Y_d(t,T) = -\frac{\ln D(t,T)}{T-t},$$

the yield to maturity. The spread on corporate debt, i.e.,

$$S(t,T) = Y_d(t,T) - Y(t,T)$$

is

$$S(t,T) = -\frac{1}{T-t}\ln\left(\mathcal{N}(d_2(V_t,t)) + \frac{V_t}{P(t,T)}\mathcal{N}(-d_1(V_t,t))\right).$$

We can specify the probability of default given the information at date t: if the dynamics of the firm are

$$dV_t = V_t(\mu dt + \sigma dB_t)$$

under the historical probability,

$$\mathbb{P}(V_T \leq L|\mathcal{F}_t) = \mathcal{N}(-d_t)$$

where now

$$d_t = \frac{1}{\sigma\sqrt{T-t}}\left(\ln(V_t/L) + (\mu - \sigma^2/2)(T-t)\right)$$

is the so-called *distance-to-default.*

Comment 3.10.1.1 Computation in the case where L is not assumed to be equal to 1 can be found, e.g., in Bielecki and Rutkowski [99]. If the default barrier is an exponential function, computations can be done using the previous subsection. Results are given in Bielecki and Rutkowski [99].

3.10.2 First Passage Time Models

Merton's model does not allow for a premature default; Black and Cox [104] extend Merton's model to the case where safety covenants provide the firm's bondholders with the right to force the firm into bankruptcy and obtain the ownership of the assets. They postulate that as soon as the firm's asset cross a lower threshold, the bondholders take over the firm. The safety convenant takes the form of an exponential. In this subsection, the model is simplified. We assume that the firm defaults when its value falls below a pre-specified level, i.e.,

$$\tau = T_L(V) = \inf\{t \,:\, V_t \leq L\},$$

where $V_0 \geq L$. In this case, the default time τ is a stopping time in the asset's filtration. The valuation of a defaultable claim X reduces to the problem of pricing the claim $X\mathbb{1}_{\{T<\tau\}}$. The valuation of the defaultable claim within the structural approach is a standard problem which needs the knowledge of the law of the pair (τ, X).

Let us assume that

$$dV_t = V_t((r-\delta)dt + \sigma dB_t)\,,$$

where δ stands for the dividend yield. The value of a defaultable T-maturity bond with face value 1 and $L \leq 1$ is $D(t,T) = P(t,T)\mathbb{E}(\mathbb{1}_{\{T<\tau\}}|\mathcal{F}_t)$, i.e., using the results on hitting time of a barrier for a geometric BM (see Exercise 3.3.1.2):

$$D(t,T) = P(t,T)\left(\mathcal{N}(b_1(V_t, T-t)) - \left[\frac{L}{V_t}\right]^{2\nu\sigma^{-2}} \mathcal{N}(b_2(V_t, T-t))\right)$$

where

$$\begin{aligned} b_1(x, T-t) &= \frac{1}{\sigma\sqrt{T-t}}\left(\ln(x/L) + \nu(T-t)\right) \\ b_2(x, T-t) &= \frac{1}{\sigma\sqrt{T-t}}\left(\ln(L/x) + \nu(T-t)\right)\,. \end{aligned}$$

Here, $\nu = r - \delta - \sigma^2/2$.

We now assume that a rebate β is paid at default time when it occurs before maturity. Assume that $\theta := \nu^2 + 2\sigma^2(r-\delta) > 0$. Then prior to the company's default (that is on the set $\{\tau > t\}$) the price of a defaultable bond equals

$$\begin{aligned} D(t,T) &= P(t,T)\big(\mathcal{N}\big(b_1(V_t, T-t)\big) - Z_t^{2\nu\sigma^{-2}}\mathcal{N}\big(b_2(V_t, T-t)\big)\big) \\ &\quad + \beta V_t\big(Z_t^{\theta\sigma^{-2}+1+\zeta}\mathcal{N}\big(b_3(V_t, T-t)\big) + Z_t^{\theta\sigma^{-2}+1-\zeta}\mathcal{N}\big(b_4(V_t, T-t)\big)\big), \end{aligned}$$

where $Z_t = L/V_t$, $\zeta = \sigma^{-2}\sqrt{\theta}$ and

$$b_3(V_t, T-t) = \frac{\ln(L/V_t) + \zeta\sigma^2(T-t)}{\sigma\sqrt{T-t}},$$
$$b_4(V_t, T-t) = \frac{\ln(L/V_t) - \zeta\sigma^2(T-t)}{\sigma\sqrt{T-t}}.$$

The general formulae (for L different from 1 and with an exponential barrier) can be obtained using results given in Subsection 3.9.3. See also Bielecki et al. [91].

Extensions: Zhou's Model

Zhou [877] studies the case where the dynamics of the firm's value is

$$dV_t = V_{t-}\left((\mu - \lambda\, c)dt + \sigma dW_t + dX_t\right)$$

where W is a Brownian motion, X a compound Poisson process with the jumps distributed as Y_1 where $\ln Y_1$ follows a Gaussian law with mean a and variance b^2, and $c = \exp(a + b^2/2)$. This choice of parameters implies that $V_t e^{\mu t}$ is a martingale (see ↣ Subsection 8.6.3). In the first part, Zhou studies Merton's problem in that setting. In the second part, he gives an approximation for the law of the first passage if the default time is $\tau = \inf\{t : V_t \leq L\}$.

Comment 3.10.2.1 Credit risk is presented in a more detailed form in Bielecki and Rutkowski [99] and Schönbucher [765] . The reader can also refer to the survey paper of Bielecki et al. [91]. See also ↣ Chapter 7.

3.11 American Options

An American option gives its owner the right to exercise at any time τ between the initial time and maturity (see Samuelson [757][1]). We refer to Elliott and Kopp [316] for a general presentation of American options and to Carr et al. [154] for a decomposition of prices. McKean [635] was the first to exhibit the relation between the evaluation problem and a free boundary problem.

[1] We reproduce the following comments, from Jarrow and Protter [480]. This is the paper that first coined the terms European and American options. According to a private communication with R.C. Merton, prior to writing the paper, P. Samuelson went to Wall Street to discuss options with industry professionals. His Wall Street contact explained that there were two types of options available, one more complex - that could be exercised any time prior to maturity, and one more simple - that could be exercised only at the maturity date, and that only the more sophisticated European mind (as opposed to the American mind) could understand the former. In response, when Samuelson wrote the paper, he used these as prefixes and reversed the ordering.

Let us consider a currency (resp. a stock) and let us assume that its dynamics under the risk-neutral probability $\mathbb{Q}$, are given by the Garman-Kohlhagen model:

$$dS_t = S_t((r-\delta)dt + \sigma dW_t)$$

where $(W_t, t \geq 0)$ is a $\mathbb{Q}$-Brownian motion, r and δ are the domestic and foreign risk-free interest rates (resp. the risk-free interest rate and the dividend rate) and σ is the currency volatility. These parameters are constant, σ is strictly positive and at least one of the positive parameters r and δ is strictly positive. We denote by $C_A(S_t, T-t)$ (resp. $P_A(S_t, T-t)$) the time-t price of an American call (resp. put) of maturity T and strike price K.

3.11.1 American Stock Options

Let us recall some well known facts on American options. The value of an **American call option** (resp. **put**) of maturity T and strike K, is

$$C_A(S_0, T) = \sup_{\tau \in \mathcal{T}(T)} \mathbb{E}_{\mathbb{Q}}(e^{-r\tau}(S_\tau - K)^+),$$

(resp. $\sup_{\tau \in \mathcal{T}(T)} \mathbb{E}_{\mathbb{Q}}(e^{-r\tau}(K - S_\tau)^+)$) where $\mathcal{T}(T)$ is the set of stopping times τ with values in $[0, T]$. Obviously, the value of an American call is greater than the value of a European call with same maturity and strike.

Lemma 3.11.1.1 *The value of an American call is equal to the value of a European call if the stock does not pay dividends before maturity ($\delta = 0$).*

PROOF: Indeed, from the convexity of $x \to (x - Ke^{-rT})^+$, the martingale property of the process $(e^{-rt}S_t, t \geq 0)$, and Jensen's inequality, the process $((e^{-rt}S_t - Ke^{-rT})^+, t \geq 0)$ is a $\mathbb{Q}$-submartingale. Hence, for any stopping time τ bounded by T,

$$\mathbb{E}_{\mathbb{Q}}((e^{-r\tau}S_\tau - Ke^{-rT})^+) \leq \mathbb{E}_{\mathbb{Q}}(e^{-rT}(S_T - K)^+).$$

The inequality

$$\mathbb{E}_{\mathbb{Q}}(e^{-r\tau}(S_\tau - K)^+) \leq \mathbb{E}_{\mathbb{Q}}((e^{-r\tau}S_\tau - Ke^{-rT})^+)$$

leads to $\sup_\tau \mathbb{E}_{\mathbb{Q}}(e^{-r\tau}(S_\tau - K)^+) \leq \mathbb{E}_{\mathbb{Q}}(e^{-rT}(S_T - K)^+)$ and the result follows (the reverse inequality is obvious). □

In the particular case of infinite maturity, an American option is called perpetual. The value of a perpetual American call $C_A(x, \infty)$ is x. Indeed, for any t,

$$x - e^{-rt}K \leq \mathbb{E}_{\mathbb{Q}}(e^{-rt}(S_t - K)^+) \leq C_A(x, \infty) \leq x$$

and the result follows when t goes to infinity. The limit of the value of a European call maturity T, when T goes to infinity is also equal to x, as can be seen from the Black-Scholes formula (see Theorem 2.3.2.1).

Exercise 3.11.1.2 The payoff of a capitalized-strike American put option is $(Ke^{rt} - S_t)^+$ if exercised at time t. Prove that the price of this option is the price of a European put, with strike $e^{rT}K$. ◁

3.11.2 American Currency Options

The exercise boundaries are defined as follows. For an American currency call (resp. put) of maturity T and for a given time $t, t \in [0,T]$,

$$\begin{cases} b_c(T-t) = \inf \{x \geq 0 \, : \, x - K = C_A(x, T-t)\} \, , \\ b_p(T-t) = \sup \{x \geq 0 \, : \, K - x = P_A(x, T-t)\} \, . \end{cases} \tag{3.11.1}$$

The **exercise boundary** for the American call (resp. put) gives for each time t before maturity the critical level at which the American option should be exercised. In the continuation region, i.e., when the underlying asset value is below (resp. above) the exercise boundary, the time value of the American call is strictly positive. In the stopping region, i.e., when the underlying asset value is above (resp. below) the exercise boundary, the time value is equal to zero and therefore it is worthwhile to exercise the option. As we recalled, for a non-dividend paying stock, it is never optimal to exercise the American call option before maturity. The exercise boundary for the call is therefore infinite before maturity. However, for currencies, it could be optimal to exercise the American call option strictly before maturity, in order to invest at the foreign interest rate instead of the domestic one. Hence, the exercise boundary given by the equation (3.11.1) is finite when $\delta > 0$.

By relying upon the proof of Proposition 2.7.1.1 for European options, the PDE that the option price satisfies in the continuation region, is obtained and is the same as in the European case:

$$\frac{\sigma^2}{2} x^2 \frac{\partial^2 C_A}{\partial x^2}(x,u) + (r-\delta) x \frac{\partial C_A}{\partial x}(x,u) - rC_A(x,u) - \frac{\partial C_A}{\partial u}(x,u) = 0 \, . \tag{3.11.2}$$

Proposition 3.11.2.1 *The American currency call price satisfies the following decomposition:*

$$\begin{aligned} C_A(S_t, T-t) = C_E(S_t, T-t) + \delta S_t \int_t^T e^{-\delta(s-t)} \mathcal{N}(d_1(S_t, b_c(T-s), s-t)) ds \\ - rK \int_t^T e^{-r(s-t)} \mathcal{N}(d_2(S_t, b_c(T-s), s-t)) ds \end{aligned} \tag{3.11.3}$$

with

$$\begin{aligned} d_1(x,y,u) &= \frac{\ln(x/y) + (r - \delta + \sigma^2/2)u}{\sigma\sqrt{u}} \, , \\ d_2(x,y,u) &= d_1(x,y,u) - \sigma\sqrt{u} \, . \end{aligned}$$

PROOF: Apply Itô's lemma to the process S and the function

$$\widetilde{C}(x,s) = e^{-r(s-t)} C_A(x, T-s)$$

on the interval $[t, T]$. Then,

$$e^{-r(T-t)} C_A(S_T, 0) = C_A(S_t, T-t) + \int_t^T \mathcal{A}\widetilde{C}(S_s, s) ds + \sigma \int_t^T S_s \frac{\partial \widetilde{C}}{\partial x}(S_s, s) dW_s \,, \tag{3.11.4}$$

where $\mathcal{A}$ is defined by:

$$\mathcal{A} = \frac{\sigma^2}{2} x^2 \frac{\partial^2}{\partial x^2} + (r-\delta) x \frac{\partial}{\partial x} + \frac{\partial}{\partial s} \,.$$

Now, in the continuation region the American call price satisfies the PDE given in equation (3.11.2) and therefore $\mathcal{A}\tilde{C}(S_s, s)$ is equal to zero. In the stopping region the American call is equal to its intrinsic value, and therefore, for $x > b_c(s)$:

$$\mathcal{A}\tilde{C}(x,s) = (r-\delta)x + r(K-x) = (rK - \delta x) \mathbb{1}_{\{x > b_c(s)\}} \,. \tag{3.11.5}$$

The last integral on the right-hand side of equation (3.11.4) is a martingale. By applying the expectation operator to this equation and by relying on the equality (3.11.5), we obtain

$$\begin{aligned} C_A(S_t, T-t) &= e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}}((S_T - K)^+ | \mathcal{F}_t) \\ &\quad - \int_t^T e^{-r(s-t)} \mathbb{E}_{\mathbb{Q}}((rK - \delta S_s) \mathbb{1}_{\{S_s > b_c(T-s)\}} | \mathcal{F}_t) ds \,. \end{aligned}$$

From Subsection 2.7.1, where the Garman and Kohlhagen model was derived, the decomposition given by equation (3.11.3) is obtained. □

Along the same lines, a decomposition for the put price can be derived .

Proposition 3.11.2.2 *The American currency put price satisfies the following decomposition:*

$$\begin{aligned} P_A(S_t, T-t) &= P_E(S_t, T-t) \\ &\quad + rK \int_t^T e^{-r(s-t)} \mathcal{N}(-d_2(S_s, b_p(T-s), s-t)) ds \\ &\quad - \delta S_t \int_t^T e^{-\delta(s-t)} \mathcal{N}(-d_1(S_s, b_p(T-s), s-t)) ds \,, \end{aligned} \tag{3.11.6}$$

with d_i given in Proposition 3.11.2.1 and b_p the exercise boundary for the put defined in (3.11.1).

By relying on Barles et al. [44] for non-dividend paying stock options ($\delta = 0$), an approximation of the American put exercise boundary near expiration T can be given:

$$b_p(T-t) \approx K(1 - \sigma\sqrt{(T-t)\,|\ln(T-t)|}) \tag{3.11.7}$$

for $t < T$.

By substituting the results given by (3.11.7) into equation (3.11.6), an approximation of the American put price is obtained, for small maturities.

3.11.3 Perpetual American Currency Options

PDE Approach

When the option's maturity tends to infinity, the following ODE is obtained:

$$\frac{\sigma^2}{2}x^2 C_A''(x) + (r-\delta)xC_A'(x) - rC_A(x) = 0 \tag{3.11.8}$$

where now the following notation is used:

$$C_A(x) = C_A(x, +\infty)\,.$$

We denote by L^* the limit when T goes to infinity of the monotonic function b_c (see (3.11.1)). As seen later, L^* is finite if $\delta > 0$.

The general solution of the equation (3.11.8) is of the form $a_1 x^{\gamma_1} + a_2 x^{\gamma_2}$ where γ_1 and γ_2 are the two roots of the polynomial

$$\frac{\sigma^2}{2}\gamma^2 + \left(r - \delta - \frac{\sigma^2}{2}\right)\gamma - r \tag{3.11.9}$$

which admits a positive and a negative root. The call price being an increasing function of the exchange rate, only the positive root

$$\gamma_1 = \frac{-\nu + \sqrt{\nu^2 + 2r}}{\sigma} \tag{3.11.10}$$

will be retained, and $C_A(x) = a_1 x^{\gamma_1}$. It can be observed that $\gamma_1 > 1$. Here ν is defined (as in Section 3.3) by:

$$\nu = \frac{1}{\sigma}\left(r - \delta - \frac{\sigma^2}{2}\right). \tag{3.11.11}$$

(Note that if $\delta = 0$, then $\gamma_1 = 1$.) Now, the parameter a_1 and the boundary L^* are obtained from the boundary conditions:

$$C_A(L^*) = a_1(L^*)^{\gamma_1} = L^* - K,\ C_A'(L^*) = a_1\gamma_1(L^*)^{\gamma_1 - 1} = 1$$

i.e., the option price and its derivative are continuous with respect to the underlying asset value at the exercise boundary. The continuity of the derivative at the boundary is assumed (this last property is the **smooth-fit** principle or smooth-pasting condition). It is not obvious that this property holds, see Elliott and Kopp [316] p.203. Therefore

$$a_1 = \frac{L^* - K}{(L^*)^{\gamma_1}}, \quad L^* = \frac{\gamma_1}{\gamma_1 - 1} K \geq K \,. \tag{3.11.12}$$

It follows that, in the **continuation region** (for $x < L^*$), the perpetual American call price is given by:

$$C_A(x) = (L^* - K) \left(\frac{x}{L^*} \right)^{\gamma_1} .$$

By relying on equation (3.11.12)

$$C_A(x) = \frac{K}{\gamma_1 - 1} e^{-\gamma_1 \ln\left(\frac{\gamma_1 K}{\gamma_1 - 1}\right)} x^{\gamma_1} \,. \tag{3.11.13}$$

In the stopping region, (for $x \geq L^*$): $C_A(x) = x - K$.

Martingale Approach

In order to derive the price of an American call, the martingale approach can also be used. In this framework the option's value is given by

$$C_A(S_t) = \sup_\tau \mathbb{E}_\mathbb{Q}((S_\tau - K)e^{-r(\tau - t)} | \mathcal{F}_t) \,,$$

where τ runs over all stopping times greater than t.

Let $t = 0$ and assume that the boundary is constant. By continuity of the Brownian motion if S_0 is in the continuation region (i.e., S_0 is smaller than the boundary):

$$C_A(S_0) = \sup_L \left[(L - K)\mathbb{E}_\mathbb{Q}(e^{-rT_L}) \right] \tag{3.11.14}$$

where T_L is the first passage time of the underlying asset value out of the continuation region:

$$T_L = \inf \{ t \geq 0 \ / \ S_t \geq L \} \ .$$

(See Elliott and Kopp, p. 196 [316] for a proof that it is possible to restrict attention to that family of stopping times.) The optimal value L^* is obtained by equating the derivative of $(L - K)\mathbb{E}_\mathbb{Q}(e^{-rT_L})$ with respect to L to zero, hence

$$L^* = \frac{-\mathbb{E}_\mathbb{Q}(e^{-rT_{L^*}})}{[\partial \mathbb{E}_\mathbb{Q}(e^{-rT_L})/\partial L]_{L=L^*}} + K \,. \tag{3.11.15}$$

Therefore,

$$C_A(S_0) = \frac{-\left(\mathbb{E}_\mathbb{Q}(e^{-rT_{L^*}}) \right)^2}{[\partial \mathbb{E}_\mathbb{Q}(e^{-rT_L})/\partial L]_{L=L^*}} \,. \tag{3.11.16}$$

Using equation (3.3.5), the Laplace transform of the hitting time T_L is

$$\mathbb{E}_{\mathbb{Q}}(e^{-rT_L}) = e^{-(-\nu+\sqrt{\nu^2+2r})\frac{1}{\sigma}\ln(L/S_0)} = e^{-\gamma_1 \ln(L/S_0)} \tag{3.11.17}$$

where the parameter γ_1 is defined in (3.11.10). We can thus derive the value of the exercise boundary from (3.11.15) which can be written $L^* = \frac{L^*}{\gamma_1} + K$. We get $L^* = \frac{\gamma_1}{\gamma_1 - 1}K \geq K$ and by relying on equation (3.11.16), the solution given by (3.11.13) is obtained.

The same procedure allows us to derive the put price as

$$P_A(S_0) = (K - L_*)\left(\frac{S_0}{L_*}\right)^{\gamma_2}. \tag{3.11.18}$$

and the exercise boundary for the perpetual American put is constant and given by $L_* = \gamma_2 K/(\gamma_2 - 1)$, where γ_2 is the negative root of (3.11.9). Let us remark that the put-call symmetry for American options (see Detemple [251]) can also be used:

$$P_A(S_0, K, r, \delta) = C_A(K, S_0, \delta, r) \tag{3.11.19}$$

where option prices are now indexed by four arguments. This symmetry comes basically from the fact that the right to sell a foreign currency corresponds to the right to buy the domestic one, and can be proved from a change of numéraire. Let us check that formulae (3.11.18) and (3.11.19) agree. The put-call symmetry formula (3.11.19) implies that

$$P_A(S_0) = (\ell - S_0)\left(\frac{K}{\ell}\right)^{\gamma}$$

where γ is the positive root of

$$\frac{\sigma^2}{2}\gamma^2 + \left(\delta - r - \frac{\sigma^2}{2}\right)\gamma - \delta = 0$$

and $\ell = \frac{\gamma}{\gamma - 1}S_0$. Note that $\gamma > 1$ and $1 - \gamma$ satisfies

$$\frac{\sigma^2}{2}(1-\gamma)^2 + \left(r - \delta - \frac{\sigma^2}{2}\right)(1-\gamma) - r = 0$$

hence $1 - \gamma = \gamma_2$, the negative root of (3.11.9). Now,

$$P_A(S_0) = (S_0)^{1-\gamma}K^{\gamma}(\gamma - 1)^{\gamma - 1}\left(\frac{1}{\gamma}\right)^{\gamma},$$

and the relation $\gamma_2 = 1 - \gamma$ yields

$$P_A(S_0) = (S_0)^{\gamma_2}K^{1-\gamma_2}\left(\frac{1}{-\gamma_2}\right)^{\gamma_2}(1-\gamma_2)^{\gamma_2 - 1},$$

which is (3.11.18).

By relying on the symmetrical relationship between American put and call boundaries (see Carr and Chesney [147] , Detemple [251]) the perpetual American put exercise boundary can also be obtained when T tends to infinity:

$$b_c(K, r, \delta, T - t) b_p(K, \delta, r, T - t) = K^2$$

where the exercise boundary is indexed by four arguments.

3.12 Real Options

Real options represent an important and relatively new trend in Finance and often involve the use of hitting times. Therefore, this topic will be briefly introduced in this chapter. In many circumstances, the standard NPV (Net Present Value) approach could generate wrong answers to important questions: "What are the relevant investments and when should the decision to invest be made?". This standard investment choice method consists of computing the NPV, i.e., the expected sum of the discounted difference between earnings and costs. Depending on the sign of the NPV, the criterion recommends acceptance (if it is positive) or rejection (otherwise) of the investment project. This approach is very simple and does not always model the complexity of the investment choice problem. First of all, this method presupposes that the earning and cost expectations can be estimated in a reliable way. Thus, the uncertainty inherent to many investment projects is not taken into account in an appropriate way. Secondly, this method is very sensitive to the level of the discount rate and the estimation of the this parameter is not always straightforward.

Finally, it is a static approach for a dynamical problem. Implicitly the question is: "Should the investment be undertaken now, or never?" It neglects the opportunity (one may use also the term *option*) to wait, in order to obtain more information, and to make the decision to invest or not to invest in an optimal way. In many circumstances, the timing aspects are not trivial and require specific treatment. By relying on the concept of a financial option, and more specifically on the concept of an American option (an optimal stopping theory), the investment choice problem can be tackled in a more appropriate way.

3.12.1 Optimal Entry with Stochastic Investment Costs

Mc Donald and Siegel's model [634], which corresponds to one of the seminal articles in the field of real options, is now briefly presented. As shown in their paper, some real option problems can be more complex than usual option pricing ones. They consider a firm with the following investment opportunity: at any time t, the firm can pay K_t to install the investment project which generates a sum of expected discounted future net cash-flows denoted V_t. The investment is irreversible. In their model, costs are stochastic and the maturity is infinite. It corresponds, therefore, to an extension of the perpetual American option pricing model with a stochastic strike price. See also Bellalah [68], Dixit and Pindyck [254] and Trigeorgis [820].

Let us assume that, under the historical probability $\mathbb{P}$, the dynamics of V (resp. K), the project-expected sum of discounted positive (resp. negative) instantaneous cash-flows (resp. costs) generated by the project- are given by:

$$\begin{cases} dV_t = V_t\,(\alpha_1 dt + \sigma_1 dW_t) \\ dK_t = K_t(\alpha_2 dt + \sigma_2 dB_t)\,. \end{cases}$$

The two trends α_1, α_2, the two volatilities σ_1 and σ_2, the correlation coefficient ρ of the two $\mathbb{P}$-Brownian motions W and B, and the discount rate r, are supposed to be constant. We also assume that $r > \alpha_i, i = 1, 2$.

If the investment date is t, the payoff of the real option is $(V_t - K_t)^+$. At time 0, the investment opportunity value is therefore given by

$$\begin{aligned} C_{RO}(V_0, K_0) &:= \sup_{\tau \in \mathcal{T}} \mathbb{E}_{\mathbb{P}}(e^{-r\tau}(V_\tau - K_\tau)^+) \\ &= \sup_{\tau \in \mathcal{T}} \mathbb{E}_{\mathbb{P}} \left(e^{-r\tau} K_\tau \left(\frac{V_\tau}{K_\tau} - 1 \right)^+ \right) \end{aligned}$$

where $\mathcal{T}$ is the set of stopping times, i.e, the set of possible investment dates.

Now, using that $K_t = K_0 e^{\alpha_2 t} e^{\sigma_2 B_t - \frac{1}{2}\sigma_2^2 t}$, the same kind of change of probability measure (change of numéraire) as in Subsection 2.7.2 leads to

$$C_{RO}(V_0, K_0) = K_0 \sup_{\tau \in \mathcal{T}} \mathbb{E}_{\mathbb{Q}} \left(e^{-(r-\alpha_2)\tau} \left(\frac{V_\tau}{K_\tau} - 1 \right)^+ \right) .$$

Here the probability measure $\mathbb{Q}$ is defined by its Radon-Nikodým derivative with respect to $\mathbb{P}$ on the σ-algebra $\mathcal{F}_t = \sigma(W_s, B_s, s \le t)$ by

$$\mathbb{Q}|_{\mathcal{F}_t} = \exp\left(-\frac{\sigma_2^2}{2} t + \sigma_2 B_t \right) \mathbb{P}|_{\mathcal{F}_t} .$$

The valuation of the investment opportunity then corresponds to that of a perpetual American option. As in Subsection 2.7.2, the dynamics of $X = V/K$ are obtained

$$dX_t/X_t = (\alpha_1 - \alpha_2)dt + \Sigma d\widehat{W}_t .$$

Here

$$\Sigma = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}$$

and $(\widehat{W}_t, t \ge 0)$ is a $\mathbb{Q}$-Brownian motion. Therefore, from the results obtained in Subsection 3.11.3 in the case of perpetual American option

$$C_{RO}(V_0, K_0) = K_0(L^* - 1)\left(\frac{V_0/K_0}{L^*}\right)^{\epsilon} \tag{3.12.1}$$

with

$$L^* = \frac{\epsilon}{\epsilon - 1}\,, \tag{3.12.2}$$

and

$$\epsilon = \sqrt{\left(\frac{\alpha_1 - \alpha_2}{\Sigma^2} - \frac{1}{2}\right)^2 + \frac{2(r - \alpha_2)}{\Sigma^2}} - \left(\frac{\alpha_1 - \alpha_2}{\Sigma^2} - \frac{1}{2}\right). \tag{3.12.3}$$

Let us now assume that spanning holds, that is, in this context, that there exist two assets perfectly correlated with V and K and with the same standard deviation as V and K. We can then rely on risk neutrality, and discounting at the risk-free rate.

Let us denote by α_1^* and α_2^* respectively the expected returns of assets 1 and 2 perfectly correlated respectively with V and K. Let us define δ_1 and δ_2 by

$$\delta_1 = \alpha_1^* - \alpha_1, \quad \delta_2 = \alpha_2^* - \alpha_2$$

These parameters play the rôle of the dividend yields in the exchange option context (see Section 2.7.2), and are constant in this framework (see Gibson and Schwartz [391] for stochastic convenience yields). The quantity δ_1 is an opportunity cost of delaying the investment and keeping the option to invest alive and δ_2 is an opportunity cost saved by differing installation. The trends $r - \delta_1$ (i.e., α_1 minus the risk premium associated with V which is equal to $\alpha_1^* - r$) and $r - \delta_2$ (equal to $\alpha_2 - (\alpha_2^* - r)$) should now be used instead of the trends α_1 and α_2, respectively. In this setting, r is the risk-free rate. Thus, equations (3.12.1) and (3.12.2) still give the solution, but with

$$\epsilon = \sqrt{\left(\frac{\delta_2 - \delta_1}{\Sigma^2} - \frac{1}{2}\right)^2 + \frac{2\delta_2}{\Sigma^2}} - \left(\frac{\delta_2 - \delta_1}{\Sigma^2} - \frac{1}{2}\right) \tag{3.12.4}$$

instead of equation (3.12.3). In the neo-classical framework it is optimal to invest if expected discounted earnings are higher than expected discounted costs, i.e., if X_t is higher than 1. When the risk is appropriately taken into account, the optimal time to invest is the first passage time of the process $(X_t, t \geq 0)$ for a level L^* strictly greater than 1, as shown in equation (3.12.2).

As seen above, in the real option framework usually different stochastic processes are involved (see also, for example, Loubergé et al. [604]). Results obtained by Hu and Øksendal [447] and Villeneuve [829], who consider the American option valuation with several underlyings, can therefore be very useful.

3.12.2 Optimal Entry in the Presence of Competition

If instead of a monopolistic situation, competition is introduced, by relying on Lambrecht and Perraudin [561], the value of the investment opportunity can be derived. Let us assume that the discounted sum K_t of instantaneous cost is now constant.

Two firms are involved. Only the first one behaves strategically. Both are potentially willing to invest a sum K in the same investment project. They consider only this investment project. The decision to invest is supposed to be irreversible and can be made at any time. Hence the real option is a perpetual American option. The investors are risk-neutral. Let us denote by r the constant interest rate. In this risk-neutral economy, the dynamics of S, the instantaneous cash-flows generated by the investment project, are given by

$$dS_t = S_t(\alpha dt + \sigma dW_t)\,.$$

Let us define V as the expected sum of positive instantaneous cash-flows S. The processes V and S have the same dynamics. Indeed, for $r > \alpha$:

$$\begin{aligned} V_t &= \mathbb{E}\left(\int_t^\infty e^{-r(u-t)} S_u du | \mathcal{F}_t\right) = e^{rt}\int_t^\infty e^{-(r-\alpha)u}\mathbb{E}(e^{-\alpha u} S_u|\mathcal{F}_t)du \\ &= e^{rt}\int_t^\infty e^{-(r-\alpha)u} e^{-\alpha t} S_t du = \frac{S_t}{r-\alpha}\,. \end{aligned}$$

In this model, the authors assume that firm 1 (resp. 2) completely loses the option to invest if firm 2 (resp. 1) invests first, and therefore considers the investment decision of a firm threatened by preemption.

Firm 1 behaves strategically in an incomplete information setting. This firm conjectures that firm 2 will invest when the underlying value reaches some level L_2^* and that L_2^* is an independent draw from a distribution G. The authors assume that G has a continuously differentiable density $g = G'$ with support in the interval $[L_2^D, L_2^U]$. The uncertainty in the investment level of the competitor comes from the fact that this level depends on competitor's investment costs which are not known with certainty and therefore only conjectured.

The structure of learning implied by the model is the following. Since firm 2 invests only when the underlying S hits for the first time the threshold L_2^*, firm 1 learns about firm 2 only when the underlying reaches a new supremum. Indeed, in this case, there are two possibilities. Firm 2 can either invest and firm 1 learns that the trigger level is the current S_t, but it is too late to invest for firm 1, or wait and firm 1 learns that L_2^* lies in a smaller interval than it has previously known, i.e., in $[M_t, L_2^U]$, where M_t is the supremum at time t: $M_t = \sup_{0\le u\le t} S_u$.

In this context, firm 1 behaves strategically, in that it looks for the optimal exercise level L_1^*, i.e., the trigger value which maximizes the conditional expectation of the discounted realized payoff. Indeed, the value C_S to firm 1, the strategic firm, is therefore

$$C_S(S_t, M_t) = \sup_L \left(\frac{L}{r-\alpha} - K\right) \mathbb{E}\left(e^{-r(T_L-t)} \mathbb{1}_{\{L_2^* > L\}} | \mathcal{F}_t \vee (L_2^* > M_t)\right)$$

where the stopping time T_L is the first passage time of the process S for level L after time t:

$$T_L = \inf\{u \geqslant t,\ S_u \geqslant L\}\,.$$

The payoff is realized only if the competitor is preempted, i.e., if $L_2^* > L$. If $M_t > L_2^D$, the value to the firm depends not only on the instantaneous value S_t of the underlying, but also on M_t which represents the knowledge accumulated by firm 1 about firm 2: the fact that up until time t, firm 1 was not preempted by firm 2, i.e., $L_2^* > M_t > L_2^D$. If $M_t \leq L_2^D$, the knowledge of M_t does not represent any worthwhile information and therefore

$$C_S(S_t, M_t) = \sup_L \left(\frac{L}{r-\alpha} - K\right) \mathbb{E}(e^{-r(T_L-t)} \mathbb{1}_{\{L_2^* > L\}} | \mathcal{F}_t), \text{ if } M_t \leq L_2^D\,.$$

From now on, let us assume that $M_t > L_2^D$. Hence, by independence between the r.v. L_2^* and the stopping time $T_L = \inf\{t \geqslant 0 \,:\, S_t \geqslant L\}$

$$C_S(S_t, M_t) = \sup_L (C_{NS}(S_t, L)\mathbb{P}(L_2^* > L \mid L_2^* > M_t))\,,$$

where the value of the non strategic firm $C_{NS}(S_t, L)$ is obtained by relying on equation (3.11.17):

$$C_{NS}(S_t, L) = \left(\frac{L}{r-\alpha} - K\right)\left(\frac{S_t}{L}\right)^{\gamma}\,,$$

and from equations (3.11.10–3.11.11) $\gamma = \frac{-\nu+\sqrt{2r+\nu^2}}{\sigma} > 0$ and $\nu = \frac{\alpha-\sigma^2/2}{\sigma}$. Now, in the specific case where the lower boundary L_2^D is higher than the optimal trigger value in the monopolistic case, the solution is known:

$$C_S(S_t, M_t) = C_{NS}\left(S_t, \frac{\gamma}{\gamma-1}(r-\alpha)K\right), \text{ if } L_2^D \geqslant \frac{\gamma}{\gamma-1}(r-\alpha)K\,.$$

Indeed, in this case the presence of the competition does not induce any change in the strategy of firm 1. It cannot be preempted, because the production costs of firm 2 are too high.

In the general case, when $L_2^D < \frac{\gamma}{\gamma-1}(r-\alpha)K$ and $(r-\alpha)K < L_2^U$ (otherwise the competitor will always preempt), knowing that potential candidates for L_1^* are higher than M_t:

$$C_S(S_t, M_t) = \sup_L \left(\frac{L}{r-\alpha} - K\right)\left(\frac{S_t}{L}\right)^{\gamma} \frac{\mathbb{P}(L_2^* > L)}{\mathbb{P}(L_2^* > M_t)}$$

i.e.,

$$C_S(S_t, M_t) = \sup_L \left(\left(\frac{L}{r-\alpha} - K \right) \left(\frac{S_t}{L} \right)^{\gamma} \frac{1 - G(L)}{1 - G(M_t)} \right) .$$

This optimization problem implies the following result. L_1^* is the solution of the equation

$$x = \frac{\gamma + h(x)}{\gamma - 1 + h(x)} (r - \alpha) K$$

with

$$h(x) = \frac{x g(x)}{1 - G(x)} .$$

The function: $y \to \frac{\gamma + y}{\gamma - 1 + y}$ is decreasing, hence the trigger level is smaller in presence of competition than in the monopolistic case:

$$L_1^* < \frac{\gamma}{\gamma - 1} (r - \alpha) K .$$

Indeed, the threat of preemption generates incentives to invest earlier than in the monopolist case.

The value to firm 1 is

$$C_S(S_t, M_t) = \left(\frac{L_1^*}{r - \alpha} - K \right) \left(\frac{S_t}{L_1^*} \right)^{\gamma} \frac{1 - G(L_1^*)}{1 - G(M_t)} .$$

Let us now consider a specific case. If L_2^* is uniformly distributed on the interval $[L_2^D, L_2^U]$, then:

$$\begin{aligned} C_S(S_t, M_t) &= \sup_L \left[\left(\frac{L}{r-\alpha} - K \right) \left(\frac{S_t}{L} \right)^{\gamma} \frac{(L_2^U - L)/(L_2^U - L_2^D)}{(L_2^U - M_t)/(L_2^U - L_2^D)} \right] \\ &= \sup_L \left[\left(\frac{L}{r-\alpha} - K \right) \left(\frac{S_t}{L} \right)^{\gamma} \frac{L_2^U - L}{L_2^U - M_t} \right] . \end{aligned}$$

In this case

$$h(x) = \frac{x/(L_2^U - L_2^D)}{(L_2^U - x)/(L_2^U - L_2^D)} = \frac{x}{L_2^U - x}$$

and L_1^* satisfies

$$x = \frac{\gamma + \frac{x}{L_2^U - x}}{\gamma - 1 + \frac{x}{L_2^U - x}} (r - \alpha) K$$

i.e.,

$$(\gamma - 2) x^2 + (1 - \gamma)(L_2^U + (r - \alpha) K) x + \gamma (r - \alpha) K L_2^U = 0 .$$

Hence, for $\gamma \neq 2$

$$L_1^* = \frac{(\gamma - 1)(L_2^U + (r - \alpha) K) + \sqrt{\Delta}}{2(\gamma - 2)}$$

with

$$\begin{aligned}\Delta &= (1-\gamma)^2(L_2^U + (r-\alpha)K)^2 - 4(\gamma-2)\gamma(r-\alpha)KL_2^U \\ &= (L_2^U - (r-\alpha)K)^2\gamma^2 - 2(L_2^U - (r-\alpha)K)^2\gamma + (L_2^U + (r-\alpha)K)^2 .\end{aligned}$$

It is straightforward to show that this discriminant is positive for any γ and therefore that L_1^* is well defined. For $\gamma = 2$, $L_1^* = \frac{2(r-\alpha)KL_2^U}{L_2^U+(r-\alpha)K}$.

3.12.3 Optimal Entry and Optimal Exit

Let us now modify the model of Lambrecht and Perraudin [561] as follows. There is no competition; the decision to invest is no longer irreversible; however, the decision to *disinvest* is irreversible and can be made at any time after the decision to invest has been taken. There are entry costs K_i and exit costs K_d.

Therefore, there are two embedded perpetual American options in such a model: First an American call that corresponds to the investment decision and a put that corresponds to the disinvestment decision.

The value to the firm VF, at initial time is therefore

$$VF(S_0) = \sup_{L_i, L_d} \left(\phi(L_i)\mathbb{E}(e^{-rT_{L_i}}) + \psi(L_d)\mathbb{E}(e^{-rT_{L_d}})\right)$$

where

$$\begin{aligned}\phi(\ell) &= \frac{\ell}{r-\alpha} - K - K_i \\ \psi(\ell) &= K - \frac{\ell}{r-\alpha} - K_d\end{aligned}$$

and where the stopping times T_{L_i} and T_{L_d} correspond respectively to the first passage time of the process S at level L_i (investment) and to the first passage time of the process S at level L_d, after T_{L_i} (disinvestment):

$$\begin{aligned}T_{L_i} &= \inf\{t \geq 0,\ S_t \geq L_i\} \\ T_{L_d} &= \inf\{t \geq T_{L_i},\ S_t \leq L_d\} .\end{aligned}$$

Indeed, the right to disinvest gives an additional value to the firm. In case of a decline of the underlying process S, for example at level L_d, by paying K_d, the firm has the right to avoid the expected discounted losses at this level: $\frac{L_d}{r-\alpha} - K$.

Hence, from Markov's property:

$$VF(S_0) = \sup_{L_i, L_d} \mathbb{E}(e^{-rT_{L_i}}) \left(\phi(L_i) + \psi(L_d)\mathbb{E}(e^{-r(T_{L_d} - T_{L_i})})\right) .$$

From Subsection 3.11.3, one gets

$$VF(S_0) = \sup_{L_i, L_d} \left(\frac{S_0}{L_i}\right)^{\gamma_1} \left[\phi(L_i) + \psi(L_d)\left(\frac{L_i}{L_d}\right)^{\gamma_2}\right]$$

with

$$\gamma_1 = \frac{-\nu + \sqrt{2r + \nu^2}}{\sigma} \geq 0,\ \gamma_2 = \frac{-\nu - \sqrt{2r + \nu^2}}{\sigma} \leq 0$$

and again

$$\nu = \frac{\alpha - \sigma^2/2}{\sigma}\,.$$

This optimization problem yields

$$L_d^* = \frac{\gamma_2}{\gamma_2 - 1}(r - \alpha)(K - K_d) < (r - \alpha)(K - K_d)$$

which corresponds to the standard exercise boundary of the perpetual put (see Subsection 3.11.3). It is a decreasing function of the exit cost K_d. Indeed, if this cost increases, there is less incentive to disinvest. The quantity L_i^* is a solution of

$$x = \frac{\gamma_1}{\gamma_1 - 1}(r - \alpha)(K + K_i) - \frac{\gamma_1 - \gamma_2}{\gamma_1 - 1}\left(\frac{x}{L_d^*}\right)^{\gamma_2}((r - \alpha)(K - K_d) - L_d^*)$$

hence,

$$L_i^* \leq \frac{\gamma_1}{\gamma_1 - 1}(r - \alpha)(K + K_i)$$

i.e., the possibility to disinvest gives to the firm incentives to invest earlier than in the irreversible investment case.

The value to the firm is therefore

$$VF(S_0) = \left(\frac{S_0}{L_i^*}\right)^{\gamma_1} \left[\phi(L_i^*) + \psi(L_d^*)\left(\frac{L_i^*}{L_d^*}\right)^{\gamma_2}\right]\,.$$

3.12.4 Optimal Exit and Optimal Entry in the Presence of Competition

Let us now assume that the firm has already invested and is in a monopolistic situation. It has the opportunity to disinvest. The decision to disinvest is not irreversible. However, even if the firm has the option to invest again after the decision to quit has been made, the monopolistic situation will be over: the firm will face competition. In this case, the firm will be threatened by preemption and the Lambrecht and Perraudin [561] setting will be used. There are exit costs K_d and entry costs K_i. Let us use the previous notation.

By relying on the last subsections the value $VF(S_t)$ to the firm is

$$V_t - K + \sup_{L_d, L_i}\left[\psi(L_d)\mathbb{E}(e^{-r(T_{L_d} - t)}|\mathcal{F}_t) + \phi(L_i)\mathbb{E}(e^{-r(T_{L_i} - t)}\mathbb{1}_{L_2^* > L_i}|\mathcal{F}_t)\right]$$

i.e., setting $t = 0$,

$$VF(S_0) = V_0 - K + \sup_{L_d, L_i} \left(\frac{S_0}{L_d} \right)^{\gamma_2} \left[\psi(L_d) + \phi(L_i) \left(\frac{L_d}{L_i} \right)^{\gamma_1} (1 - G(L_i)) \right] .$$

Indeed, if firm 1 cannot disinvest, its value is $V_0 - K$; however if it has the opportunity to disinvest, it adds value to the firm. Furthermore, if firm 1 decides to disinvest, as long as it is not preempted by the competition, it has the opportunity to invest again. This explains the last term on the right-hand side: the maximization of the discounted payoff generated by a perpetual American put and by a perpetual American call times the probability of avoiding preemption.

Let us remark that the value to the firm does not depend on the supremum M_t of the underlying. As long as it is active, firm 1 does not accumulate any knowledge about firm 2. The supremum M_t no longer represents the knowledge accumulated by firm 1 about firm 2. Even if $M_t > L_2^D$, it does not mean that: $L_2^* \geqslant M_t > L_2^D$. While firm 1 does not disinvest, the knowledge of M_t does not represent any worthwhile information because firm 2 cannot invest.

This optimization problem generates the following result. L_i^* is the solution of the equation:

$$x = \frac{\gamma_1 + h(x)}{\gamma_1 - 1 + h(x)} (r - \alpha) K$$

with

$$h(x) = \frac{x g(x)}{1 - G(x)} ,$$

and L_d^* is the solution z of the equation

$$z = \frac{\gamma_2}{\gamma_2 - 1} (r - \alpha)(K - K_d) + \frac{\gamma_1 - \gamma_2}{1 - \gamma_2} \left(\frac{z}{L_i^*} \right)^{\gamma_1} (L_i^* - (r - \alpha)(K + K_i))(1 - G(L_i^*)) .$$

The value to the firm is therefore:

$$VF(S_0) = V_0 - K + \left(\frac{S_0}{L_d^*} \right)^{\gamma_2} \left[\psi(L_d^*) + \phi(L_i^*) \left(\frac{L_d^*}{L_i^*} \right)^{\gamma_1} (1 - G(L_i^*)) \right] .$$

A good reference concerning optimal investment and disinvestment decisions, with or without lags, is Gauthier [376].

3.12.5 Optimal Entry and Exit Decisions

Let us keep the notation of the preceding subsections and still assume risk neutrality. Furthermore, let us assume now that there is no competition. Hence, we can restrict the discussion to only one firm. If at the initial time the firm has not yet invested, it has the possibility of investing at a cost K_i at any time and of disinvesting later at a cost K_d. The number of investment and disinvestment dates is not bounded. After each investment date the option to disinvest is activated and after each disinvestment date, the option to invest is activated.

Therefore, depending on the last decision of the firm before time t (to invest or to disinvest), there are two possible states for the firm: active or inactive.

In this context, the following theorem gives the values to the firm in these states.

Theorem 3.12.5.1 *Assume that in the risk-neutral economy, the dynamics of S, the instantaneous cash-flows generated by the investment project, are given by:*

$$dS_t = S_t(\alpha dt + \sigma dW_t)\,.$$

Assume further that the discounted sum of instantaneous investment cost K is constant and that $\alpha < r$ where r is the risk-free interest rate.

If the firm is inactive, i.e., if its last decision was to disinvest, its value is

$$VF_d(S_t) = \frac{1}{\gamma_1 - \gamma_2}\left(\frac{S_t}{L_i^*}\right)^{\gamma_1}\left(\frac{L_i^*}{r-\alpha} - \gamma_2 \phi(L_i^*)\right)\,.$$

If the firm is active, i.e., if its last decision was to invest, its value is

$$VF_i(S_t) = \frac{1}{\gamma_1 - \gamma_2}\left(\frac{S_t}{L_d^*}\right)^{\gamma_2}\left(\frac{L_d^*}{r-\alpha} + \gamma_1 \psi(L_d^*)\right) + \frac{S_t}{r-\alpha} - K\,.$$

Here, the optimal entry and exit thresholds, L_i^* and L_d^* are solutions of the following set of equations with unknowns (x, y)

$$\begin{aligned}
&\frac{1-(y/x)^{\gamma_1-\gamma_2}}{\gamma_1-\gamma_2}\left(\gamma_1 K_i - \gamma_2\left(\frac{x}{r-\alpha} - K\right) + \frac{x}{r-\alpha}\right)\\
&\qquad = \psi(y)\left(\frac{x}{y}\right)^{\gamma_2} - K_i\left(\frac{y}{x}\right)^{\gamma_1-\gamma_2} + \frac{x}{r-\alpha} - K\\
&\frac{1-(y/x)^{\gamma_1-\gamma_2}}{\gamma_1-\gamma_2}\left(\frac{y}{r-\alpha} + \gamma_2\psi(y)\right)\\
&\qquad = \phi(x)\left(\frac{y}{x}\right)^{\gamma_1} + \psi(y)\left(\frac{y}{x}\right)^{\gamma_1-\gamma_2}
\end{aligned}$$

with

$$\gamma_1 = \frac{-\nu + \sqrt{2r+\nu^2}}{\sigma} \geq 0,\;\; \gamma_2 = \frac{-\nu - \sqrt{2r+\nu^2}}{\sigma} \leq 0$$

and

$$\nu = \frac{\alpha - \sigma^2/2}{\sigma}\,.$$

In the specific case where $K_i = K_d = 0$, the optimal thresholds are

$$L_i^* = L_d^* = (r-\alpha)K\,.$$

PROOF: In the inactive state, the value of the firm is

$$VF_d(S_t) = \sup_{L_i} \mathbb{E}\left(e^{-r(T_{L_i}-t)}(VF_i(S_{T_{L_i}}) - K_i)|\mathcal{F}_t\right)$$

where T_{L_i} is the first passage time of the process S, after time t, for the possible investment boundary L_i

$$T_{L_i} = \inf\{u \geq t,\ S_u \geq L_i\}$$

i.e., by continuity of the underlying process S:

$$VF_d(S_t) = \sup_{L_i} \mathbb{E}\left(e^{-r(T_{L_i}-t)}(VF_i(L_i) - K_i)|\mathcal{F}_t\right).$$

Along the same lines:

$$VF_i(S_t) = \sup_{L_d} \mathbb{E}\left(e^{-r(T_{L_d}-t)}\left(VF_d(L_d) + \psi(L_d)\right)|\mathcal{F}_t\right) + \frac{S_t}{r-\alpha} - K$$

where T_{L_d} is the first passage time of the process S, after time t, for the possible disinvestment boundary L_d

$$T_{L_d} = \inf\{u \geq t,\ S_u \leq L_d\}.$$

Indeed, at a given time t, without exit options, the value to the active firm would be $\frac{S_t}{r-\alpha} - K$. However, by paying K_d, it has the option to disinvest for example at level L_d. At this level, the value to the firm is $VF_d(L_d)$ plus the value of the option to quit $K - \frac{L_d}{r-\alpha}$ (the put option corresponding to the avoided losses minus the cost K_d).

Therefore

$$VF_d(S_t) = \sup_{L_i} f_d(L_i) \tag{3.12.5}$$

where the function f_d is defined by

$$f_d(x) = \left(\frac{S_t}{x}\right)^{\gamma_1}(VF_i(x) - K_i) \tag{3.12.6}$$

where

$$VF_i(S_t) = \sup_{L_d} f_i(L_d) \tag{3.12.7}$$

and

$$f_i(x) = \left(\frac{S_t}{x}\right)^{\gamma_2}(VF_d(x) + \psi(L_d)) + \frac{S_t}{r-\alpha} - K. \tag{3.12.8}$$

Let us denote by $L_i^*$and L_d^* the optimal trigger values, i.e., the values which maximize the functions f_d and f_i. An inactive (resp. active) firm will find it optimal to remain in this state as long as the underlying value S remains below

L_i^* (resp. above L_d^*) and will invest (resp. disinvest) as soon as S reaches L_i^* (resp. L_d^*).

By setting S_t equal to L_d^* in equation (3.12.5) and to L_i^* in equation (3.12.7), the following equations are obtained:

$$VF_d(L_d^*) = \left(\frac{L_d^*}{L_i^*}\right)^{\gamma_1} (VF_i(L_i^*) - K_i)\,,$$
$$VF_i(L_i^*) = \left(\frac{L_i^*}{L_d^*}\right)^{\gamma_2} (VF_d(L_d^*) + \psi(L_d^*)) + \frac{L_i^*}{r-\alpha} - K\,.$$

The two unknowns $VF_d(L_d^*)$ and $VF_i(L_i^*)$ satisfy:

$$\left(1 - \left(\frac{L_d^*}{L_i^*}\right)^{\gamma_1-\gamma_2}\right) VF_d(L_d^*) = \phi(L_i^*)\left(\frac{L_d^*}{L_i^*}\right)^{\gamma_1} + \psi(L_d^*)\left(\frac{L_d^*}{L_i^*}\right)^{\gamma_1-\gamma_2} \tag{3.12.9}$$

$$\left(1 - \left(\frac{L_d^*}{L_i^*}\right)^{\gamma_1-\gamma_2}\right) VF_i(L_i^*) = \psi(L_d^*)\left(\frac{L_i^*}{L_d^*}\right)^{\gamma_2} - K_i\left(\frac{L_d^*}{L_i^*}\right)^{\gamma_1-\gamma_2} + \frac{L_i^*}{r-\alpha} - K\,. \tag{3.12.10}$$

Let us now derive the thresholds L_d^* and L_i^* required in order to obtain the value to the firm. From equation (3.12.8)

$$\frac{\partial f_i}{\partial x}(L_d) = \left(\frac{S_t}{L_d}\right)^{\gamma_2}\left(-\frac{\gamma_2}{L_d}(VF_d(L_d) + \psi(L_d)) + \frac{dVF_d}{dx}(L_d) - \frac{1}{r-\alpha}\right)$$

and from equation (3.12.6)

$$\frac{\partial f_d}{\partial x}(L_i) = \left(\frac{S_t}{L_i}\right)^{\gamma_1}\left(-\frac{\gamma_1}{L_i}(VF_i(L_i) - K_i) + \frac{dVF_i}{dx}(L_i)\right)\,.$$

Therefore the equation $\frac{\partial f_i}{\partial x}(L_d) = 0$ is equivalent to

$$\frac{\gamma_2}{L_d^*}(VF_d(L_d^*) + \psi(L_d^*)) = \frac{dVF_d}{dx}(L_d^*) - \frac{1}{r-\alpha}$$

or, from equations (3.12.5) and (3.12.6):

$$\frac{\gamma_2}{L_d^*}(VF_d(L_d^*) + \psi(L_d^*)) = \frac{\gamma_1}{L_d^*} VF_d(L_d^*) - \frac{1}{r-\alpha}$$

i.e.,

$$VF_d(L_d^*) = \frac{1}{\gamma_1 - \gamma_2}\left(\frac{L_d^*}{r-\alpha} + \gamma_2\psi(L_d^*)\right)\,. \tag{3.12.11}$$

Moreover, the equation

$$\frac{\partial f_d}{\partial x}(L_i) = 0$$

is equivalent to

$$\frac{\gamma_1}{L_i}(VF_i(L_i^*) - K_i) = \frac{dVF_i}{dx}(L_i^*)$$

i.e, by relying on equations (3.12.7) and (3.12.8)

$$\frac{\gamma_1}{L_i}(VF_i(L_i^*) - K_i) = \frac{\gamma_2}{L_i}\left(VF_i(L_i^*) - \left(\frac{L_i^*}{r-\alpha} - K\right)\right) + \frac{1}{r-\alpha}$$

i.e.,

$$VF_i(L_i^*) = \frac{1}{\gamma_1 - \gamma_2}\left(\gamma_1 K_i - \gamma_2\left(\frac{L_i^*}{r-\alpha} - K\right) + \frac{L_i^*}{r-\alpha}\right) . \qquad (3.12.12)$$

Therefore, by substituting $VF_d(L_d^*)$ and $VF_i(L_i^*)$, obtained in (3.12.11) and (3.12.12) respectively in equations (3.12.7) and (3.12.5), the values to the firm in the active and inactive states are derived.

Finally, by substituting in (3.12.9) the value of $VF_d(L_d^*)$ obtained in (3.12.11) and in (3.12.10) the value of $VF_i(L_i^*)$ obtained in (3.12.12), a set of two equations is derived. This set admits $L_i^*$and L_d^* as solutions.

In the specific case where $K_i = K_d = 0$, from (3.12.9) and (3.12.10) the investment and abandonment thresholds satisfy $L_i^* = L_d^*$. However we know that the investment threshold is higher than the investment cost and that the abandonment threshold is smaller $L_i^* \geq (r-\alpha)K \geq L_d^*$. Thus

$$L_i^* = L_d^* = (r-\alpha)K ,$$

and the theorem is proved.

By relying on a differential equation approach, Dixit [253] (and also Dixit and Pyndick [254]) solve the same problem (see also Brennan and Schwartz [127] for the evaluation of mining projects). The value-matching and smooth pasting conditions at investment and abandonment thresholds generate a set of four equations, which in our notation is

$$\begin{aligned}
VF_i(L_d^*) - VF_d(L_d^*) &= -K_d \\
VF_i(L_i^*) - VF_d(L_i^*) &= K_i \\
\frac{dVF_i}{dx}(L_d^*) - \frac{dVF_d}{dx}(L_d^*) &= 0 \\
\frac{dVF_i}{dx}(L_i^*) - \frac{dVF_d}{dx}(L_i^*) &= 0 .
\end{aligned}$$

In the probabilistic approach developed in this subsection, the first two equations correspond respectively to (3.12.7) for $S_t = L_d^*$ and to (3.12.5) for $S_t = L_i^*$.

The last two equations are obtained from the set of equations (3.12.5) to (3.12.8).