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Continuous-Path Random Processes: Mathematical Prerequisites

Historically, in mathematical finance, continuous-time processes have been considered from the very beginning, e.g., Bachelier [39, 41] deals with Brownian motion, which has continuous paths. This may justify making our starting point in this book to deal with continuous-path random processes, for which, in this first chapter, we recall some well-known facts. We try to give all the definitions and to quote all the important facts for further use. In particular, we state, without proofs, results on stochastic calculus, change of probability and stochastic differential equations.

For proofs, the reader can refer to the books of Revuz and Yor [730], denoted hereafter [RY], Chung and Williams [186], Ikeda and Watanabe [456], Karatzas and Shreve [513], Lamberton and Lapeyre [559], Rogers and Williams [741, 742] and Williams, R. [845]. See also the reviews of Varadhan [826], Watanabe [836] and Rao [729]. The books of Øksendal [684] and Wong and Hajek [850] cover a large part of stochastic calculus.

1.1 Some Definitions

1.1.1 Measurability

Given a space Ω, a **σ-algebra** on Ω is a class $\mathcal{F}$ of subsets of Ω, such that $\mathcal{F}$ is closed under complements and countable intersection (hence under countable union) and $\emptyset \in \mathcal{F}$ (hence, $\Omega \in \mathcal{F}$). For a given class $\mathcal{C}$ of subsets of Ω, we denote by $\sigma(\mathcal{C})$ the smallest σ-algebra which contains $\mathcal{C}$ (i.e., the intersection of all the σ-algebras containing $\mathcal{G}$).

A **measurable space** $(\Omega, \mathcal{F})$ is a space Ω endowed with a σ-algebra $\mathcal{F}$. **A measurable map** X from $(\Omega, \mathcal{F})$ to another measurable space $(E, \mathcal{E})$ is a map from Ω to E such that, for any $B \in \mathcal{E}$, the set

$$X^{-1}(B) := \{\omega \in \Omega : X(\omega) \in B\}$$

belongs to $\mathcal{F}$.

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A real-valued random variable (r.v.) on $(\Omega, \mathcal{F})$ is a measurable map from $(\Omega, \mathcal{F})$ to $(\mathbb{R}, \mathcal{B})$ where $\mathcal{B}$ is the Borel σ-algebra, i.e., the smallest σ-algebra that contains the intervals.

Let X be a real-valued random variable on a measurable space $(\Omega, \mathcal{F})$. The σ-algebra generated by X, denoted $\sigma(X)$, is $\sigma(X) := \{X^{-1}(B)\,; B \in \mathcal{B}\}$. Doob's theorem asserts that any $\sigma(X)$-measurable real-valued r.v. can be written as $h(X)$ where h is a **Borel function**, i.e., a measurable map from $(\mathbb{R}, \mathcal{B})$ to $(\mathbb{R}, \mathcal{B})$ (a function such that $h^{-1}(B) := \{x \in \mathbb{R} : h(x) \in B\} \in \mathcal{B}$ for any $B \in \mathcal{B}$). The set of bounded Borel functions on a measurable space $(E, \mathcal{E})$ (i.e., the measurable maps from $(E, \mathcal{E})$ to $(\mathbb{R}, \mathcal{B})$) will be denoted by $b(\mathcal{E})$. If $\mathcal{H}$ is a σ-algebra on Ω, *we shall make the slight abuse of notation* by writing $X \in \mathcal{H}$ for: X is an $\mathcal{H}$-measurable r.v. and $X \in b\mathcal{H}$ for: X is a bounded r.v. in $\mathcal{H}$.

Let $(X_i, i \in I)$ be a set of random variables. There exists a unique r.v. with values in $\bar{\mathbb{R}}$, denoted $\mathrm{esssup}_i X_i$ (**essential supremum** of the family $(X_i; i \in I)$) such that, for any r.v. Y,

$$X_i \leq Y a.s. \, \forall i \in I \Longleftrightarrow \mathrm{esssup}_i X_i \leq Y\,.$$

If the family is countable, $\mathrm{esssup}_i X_i = \sup_i X_i$. In the case where the set I is not countable, the map $\sup_i X_i$ (pointwise supremum) may not be a random variable.

1.1.2 Monotone Class Theorem

We will frequently use the monotone class theorem which we state without proof (see Dellacherie and Meyer [242], Chapter 1). We give two different versions of that theorem, one dealing with sets, the other with functions.

Theorem 1.1.2.1 *Let $\mathcal{C}$ be a collection of subsets of Ω such that*

- $\Omega \in \mathcal{C}$,
- *if* $A, B \in \mathcal{C}$ *and* $A \subset B$*, then* $B \backslash A = B \cap A^c \in \mathcal{C}$,
- *if* A_n *is an increasing sequence of elements of* $\mathcal{C}$*, then* $\cup_n A_n \in \mathcal{C}$.

Then, if $\mathcal{F} \subset \mathcal{C}$ *where* $\mathcal{F}$ *is closed under finite intersections, then* $\sigma(\mathcal{F}) \subset \mathcal{C}$.

Theorem 1.1.2.2 *Let $\mathcal{V}$ be a vector space of bounded real-valued functions on Ω such that*

- *the constant functions are in* $\mathcal{V}$,
- *if* h_n *is an increasing sequence of positive elements of* $\mathcal{V}$ *such that* $h = \sup h_n$ *is bounded, then* $h \in \mathcal{V}$.

If $\mathcal{G}$ *is a subset of* $\mathcal{V}$ *which is stable under pointwise multiplication, then* $\mathcal{V}$ *contains all the bounded* $\sigma(\mathcal{G})$*-measurable functions.*

1.1.3 Probability Measures

A **probability measure** $\mathbb{P}$ on a measurable space $(\Omega, \mathcal{F})$ is a map from $\mathcal{F}$ to $[0,1]$ such that:

- $\mathbb{P}(\Omega) = 1$,
- $\mathbb{P}(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$ for any countable family of disjoint sets $A_i \in \mathcal{F}$, i.e., such that $A_i \cap A_j = \emptyset$ for $i \neq j$.

Note that, for $A \in \mathcal{F}$, $\mathbb{P}(A) = 1 - \mathbb{P}(A^c)$ where A^c is the complement set of A, hence $P(\emptyset) = 0$.

We shall often write, for J a countable set, $\mathbb{P}(A_j, j \in J)$ for $\mathbb{P}(\cap_{j \in J} A_j)$.

Warning 1.1.3.1 The property $\mathbb{P}(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$ does not extend to a non-countable family.

A measurable space $(\Omega, \mathcal{F})$ endowed with a probability measure $\mathbb{P}$ is called a **probability space**.

The "elementary" negligible sets are the sets $A \in \mathcal{F}$ such that $\mathbb{P}(A) = 0$. Sets $\Gamma \subset \Gamma'$ with $\Gamma' \in \mathcal{F}$ and $\mathbb{P}(\Gamma') = 0$ are said to be $(\mathbb{P}, \mathcal{F})$**-negligible**.

If $(\Omega, \mathcal{F})$ is a measurable space and $\mathbb{P}$ a probability measure on $\mathcal{F}$, the **completion** of $\mathcal{F}$ with respect to $\mathbb{P}$ is the σ-algebra of subsets A of Ω such that there exist A_1 and A_2 in $\mathcal{F}$ with $A_1 \subset A \subset A_2$ and $\mathbb{P}(A_1) = \mathbb{P}(A_2)$ (or, equivalently, $\mathbb{P}(A_2 \cap A_1^c) = 0$). In particular, the completion of $\mathcal{F}$ contains all the $\mathbb{P}$-negligible sets.

1.1.4 Filtration

A **filtration** $\mathbf{F} = (\mathcal{F}_t, t \geq 0)$ is a family of σ-algebras $\mathcal{F}_t$ on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$, which is increasing, i.e., such that $\mathcal{F}_s \subset \mathcal{F}_t$ for $s < t$ (that is: if $A \in \mathcal{F}_s$, then $A \in \mathcal{F}_t$ for $s < t$). We note $\mathcal{F}_\infty = \vee_{t \in \mathbb{R}} \mathcal{F}_t$.

It is generally assumed that the filtration satisfies the so-called "**usual hypotheses,**" that is,

(i) the filtration is right-continuous, i.e., $\mathcal{F}_t = \cap_{u>t} \mathcal{F}_u$,

(ii) the σ-algebra $\mathcal{F}_0$ contains the $(\mathbb{P}, \mathcal{F})$-negligible sets of $\mathcal{F}_\infty$.

Usually, (but not always) the σ-algebra $\mathcal{F}_0$ is the trivial σ-algebra, up to completion.

A probability space endowed with a filtration which satisfies the usual hypotheses is called a **filtered probability space**.

We shall say that a filtration $\mathbf{G}$ is larger than $\mathbf{F}$, and write $\mathbf{F} \subset \mathbf{G}$, if $\mathcal{F}_t \subset \mathcal{G}_t$, $\forall t$.

Comment 1.1.4.1 It is important that the *usual hypotheses* are satisfied in order to be able to apply general results on stochastic processes, especially when studying processes with jumps.

1.1.5 Law of a Random Variable, Expectation

The law of a real-valued r.v. X defined on the space $(\Omega, \mathcal{F}, \mathbb{P})$ is the probability measure $\mathbb{P}_X$ on $(\mathbb{R}, \mathcal{B})$ defined by

$$\forall A \in \mathcal{B},\ \mathbb{P}_X(A) = \mathbb{P}(X \in A)\,.$$

It is the image on $(\mathbb{R}, \mathcal{B})$ of $\mathbb{P}$ by the map $\omega \to X(\omega)$. This definition extends to an $\mathbb{R}^n$-valued random variable, and, more generally, to an E-valued random variable (a measurable map from $(\Omega, \mathcal{F})$ to $(E, \mathcal{E})$). If X and Y have the same law, we shall write $X \stackrel{\text{law}}{=} Y$.

The **cumulative distribution function** of a real valued r.v. X is the right-continuous function F defined as $F(x) = \mathbb{P}(X \leq x)$.

The expectation of a positive random variable Z is defined as

$$\mathbb{E}(Z) = \int Z d\mathbb{P} = \int_{\mathbb{R}^+} x\, d\mathbb{P}_Z(x)\,,$$

and, if $\mathbb{E}(|X|) < \infty$, then $\mathbb{E}(X) = \mathbb{E}(X^+) - \mathbb{E}(X^-)$. In case of ambiguity, we shall denote by $\mathbb{E}_{\mathbb{P}}$ the expectation with respect to the probability measure $\mathbb{P}$. The r.v. X is said to be $\mathbb{P}$-integrable (or integrable if there is no ambiguity) if $\mathbb{E}(|X|) < \infty$.

There are a few important transforms T of probabilities (on $\mathbb{R}$, say) which characterize a given probability μ, i.e., such that the map $\mu \to T(\mu)$ is one-to-one.

- The **Fourier transform** $F_\mu(t) = \int_{\mathbb{R}} e^{itx} \mu(dx)$ (where $t \in \mathbb{R}$).
- The **Laplace transform** $L_\mu(\lambda) = \int_{\mathbb{R}} e^{-\lambda x} \mu(dx)$ defined on the interval $\{\lambda \in \mathbb{R} : \mathbb{E}(e^{-\lambda X}) < \infty\}$. Note that the Laplace transform is well defined on $\mathbb{R}^+$ if X is positive. We shall also use, when it is defined, the Laplace transform $\mathbb{E}(e^{\lambda X})$, $\lambda \in \mathbb{R}$.

1.1.6 Independence

A family of random variables $(X_i, i \in I)$, defined on the space $(\Omega, \mathcal{F}, \mathbb{P})$, is said to be **independent** if, for any n distinct indices $(i_1, i_2, \ldots, i_n)$ with $i_k \in I$ and for any $(A_1, \ldots, A_n)$ where $A_k \in \mathcal{B}$,

$$\mathbb{P}\left(\cap_{k=1}^n (X_{i_k} \in A_k)\right) = \prod_{k=1}^n \mathbb{P}(X_{i_k} \in A_k)\,.$$

A classical application of the monotone class theorem is that, if the r.vs $(X_i, i \in I)$ are independent, then, with the same notation as above, for any bounded Borel functions f_k,

$$\mathbb{E}\left(\prod_{k=1}^{n} f_k(X_{i_k})\right) = \prod_{k=1}^{n} \mathbb{E}\left(f_k(X_{i_k})\right) .$$

The converse holds true as well. In particular, two random variables X and Y are independent if and only if, for any pair of bounded Borel functions f and g, $\mathbb{E}(f(X)g(Y)) = \mathbb{E}(f(X))\mathbb{E}(g(Y))$. For the independence property to hold true, it suffices that this equality is satisfied for "enough" functions, for example:

- for f, g of the form $f = \mathbb{1}_{]-\infty,a]}, g = \mathbb{1}_{]-\infty,b]}$ for every pair of real numbers (a, b), i.e.,
$$\mathbb{P}(X \leq a,\, Y \leq b) = \mathbb{P}(X \leq a)\,\mathbb{P}(Y \leq b)\,,$$
- for f, g of the form $f(x) = e^{i\lambda x}, g(x) = e^{i\mu x}$ for every pair of real numbers (λ, μ), i.e.,
$$\mathbb{E}(e^{i(\lambda X+\mu Y)}) = \mathbb{E}(e^{i\lambda X})\,\mathbb{E}(e^{i\mu Y})\,.$$
- in the case where X and Y are positive random variables, for f, g of the form $f(x) = e^{-\lambda x}, g(x) = e^{-\mu x}$ for every pair of positive real numbers (λ, μ), i.e.,
$$\mathbb{E}(e^{-\lambda X} e^{-\mu Y}) = \mathbb{E}(e^{-\lambda X})\mathbb{E}(e^{-\mu Y})\,.$$

It is important to note that if X and Y are independent r.vs, then for any bounded Borel function Φ defined on $\mathbb{R}^2$, $\mathbb{E}(\Phi(X, Y)) = \mathbb{E}(\varphi(X))$ where $\varphi(x) = \mathbb{E}(\Phi(x, Y))$. This result can be seen as a consequence of the monotone class theorem, or as an application of Fubini's theorem.

1.1.7 Equivalent Probabilities and Radon-Nikodým Densities

Let $\mathbb{P}$ and $\mathbb{Q}$ be two probabilities defined on the same measurable space $(\Omega, \mathcal{F})$. The probability $\mathbb{Q}$ is said to be **absolutely continuous** with respect to $\mathbb{P}$, (denoted $\mathbb{Q} \ll \mathbb{P}$) if $\mathbb{P}(A) = 0$ implies $\mathbb{Q}(A) = 0$, for any $A \in \mathcal{F}$. In that case, there exists a positive, $\mathcal{F}$-measurable random variable L, called the **Radon-Nikodým density** of $\mathbb{Q}$ with respect to $\mathbb{P}$, such that

$$\forall A \in \mathcal{F}, \mathbb{Q}(A) = \mathbb{E}_{\mathbb{P}}(L\mathbb{1}_A)\,.$$

This random variable L satisfies $\mathbb{E}_{\mathbb{P}}(L) = 1$ and for any $\mathbb{Q}$-integrable random variable X, $\mathbb{E}_{\mathbb{Q}}(X) = \mathbb{E}_{\mathbb{P}}(XL)$. The notation $\frac{d\mathbb{Q}}{d\mathbb{P}} = L$ (or $\mathbb{Q}|_{\mathcal{F}} = L\,\mathbb{P}|_{\mathcal{F}}$) is in common use, in particular in the chain of equalities

$$\mathbb{E}_{\mathbb{Q}}(X) = \int X d\mathbb{Q} = \int X \frac{d\mathbb{Q}}{d\mathbb{P}}\, d\mathbb{P} = \int XLd\mathbb{P} = \mathbb{E}_{\mathbb{P}}(XL)\,.$$

The probabilities $\mathbb{P}$ and $\mathbb{Q}$ are said to be **equivalent**, (this will be denoted $\mathbb{P} \sim \mathbb{Q}$), if they have the same negligible sets, i.e., if for any $A \in \mathcal{F}$,

$$\mathbb{Q}(A) = 0 \Leftrightarrow \mathbb{P}(A) = 0\,,$$

or equivalently, if $\mathbb{Q} \ll \mathbb{P}$ and $\mathbb{P} \ll \mathbb{Q}$. In that case, there exists a strictly positive, $\mathcal{F}$-measurable random variable L, such that $\mathbb{Q}(A) = \mathbb{E}_{\mathbb{P}}(L\mathbb{1}_A)$. Note that $\frac{d\mathbb{P}}{d\mathbb{Q}} = L^{-1}$ and $\mathbb{P}(A) = \mathbb{E}_{\mathbb{Q}}(L^{-1}\mathbb{1}_A)$.

Conversely, if L is a strictly positive $\mathcal{F}$-measurable r.v., with expectation 1 under $\mathbb{P}$, then $\mathbb{Q} = L \cdot \mathbb{P}$ defines a probability measure on $\mathcal{F}$, equivalent to $\mathbb{P}$. From the definition of equivalence, if a property holds almost surely (a.s.) with respect to $\mathbb{P}$, it also holds a.s. for any probability $\mathbb{Q}$ equivalent to $\mathbb{P}$. Two probabilities $\mathbb{P}$ and $\mathbb{Q}$ on the same filtered probability space $(\Omega, \mathbf{F})$ are said to be locally equivalent[1] if they have the same negligible sets on $\mathcal{F}_t$, for every $t \geq 0$, i.e., if $\mathbb{Q}|_{\mathcal{F}_t} \sim \mathbb{P}|_{\mathcal{F}_t}$. In that case, there exists a strictly positive $\mathbf{F}$-adapted process $(L_t, t \geq 0)$ such that $\mathbb{Q}|_{\mathcal{F}_t} = L_t \mathbb{P}|_{\mathcal{F}_t}$. (See ↣ Subsection 1.7.1 for more information.) Furthermore, if τ is a stopping time (see ↣ Subsection 1.2.3), then

$$\mathbb{Q}|_{\mathcal{F}_\tau \cap \{\tau < \infty\}} = L_\tau \cdot \mathbb{P}|_{\mathcal{F}_\tau \cap \{\tau < \infty\}} .$$

This will be important when dealing with Girsanov's theorem and explosion times (See ↣ Proposition 1.7.5.3).

Warning 1.1.7.1 If $\mathbb{P} \sim \mathbb{Q}$ and X is a $\mathbb{P}$-integrable random variable, it is not necessarily $\mathbb{Q}$-integrable.

1.1.8 Construction of Simple Probability Spaces

In order to construct a random variable with a given law, say a Gaussian law, the canonical approach is to take $\Omega = \mathbb{R}$, $X : \Omega \to \mathbb{R}; X(\omega) = \omega$ the identity map and $\mathbb{P}$ the law on $\Omega = \mathbb{R}$ with the Gaussian density with respect to the Lebesgue measure, i.e.,

$$\mathbb{P}(d\omega) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\omega^2}{2}\right) d\omega$$

(recall that here ω is a real number). Then the cumulative distribution function of the random variable X is

$$F_X(x) = \mathbb{P}(X \leq x) = \int_\Omega \mathbb{1}_{\{\omega \leq x\}} \mathbb{P}(d\omega) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\omega^2}{2}\right) d\omega .$$

Hence, the map X is a Gaussian random variable. The construction of a real valued r.v. with any given law can be carried out using the same idea; for example, if one needs to construct a random variable with an exponential law, then, similarly, one may choose $\Omega = \mathbb{R}$ and the density $e^{-\omega}\mathbb{1}_{\{\omega \geq 0\}}$.

For two independent variables, we choose $\Omega = \Omega_1 \times \Omega_2$ where $\Omega_i, i = 1, 2$ are two copies of $\mathbb{R}$. On each Ω_i, one constructs a random variable as above,

[1] This commonly used terminology often refers to a sequence (T_n) of stopping times, with $T_n \uparrow \infty$ a.s.; here, it is preferable to restrict ourselves to the deterministic case $T_n = n$.

and defines $\mathbb{P} = \mathbb{P}_1 \otimes \mathbb{P}_2$ where the product probability $\mathbb{P}_1 \otimes \mathbb{P}_2$ is first defined on the sets $A_1 \times A_2$ for $A_i \in \mathcal{B}$, the Borel σ-field of $\mathbb{R}$, as

$$(\mathbb{P}_1 \otimes \mathbb{P}_2)(A_1 \times A_2) = \mathbb{P}_1(A_1)\mathbb{P}_2(A_2)\,,$$

and then extended to $\mathcal{B} \times \mathcal{B}$.

1.1.9 Conditional Expectation

Let X be an integrable random variable on the space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{H}$ a σ-algebra contained in $\mathcal{F}$, i.e., $\mathcal{H} \subseteq \mathcal{F}$. The **conditional expectation** of X given $\mathcal{H}$ is the almost surely unique $\mathcal{H}$-measurable random variable Z such that, for any bounded $\mathcal{H}$-measurable random variable Y,

$$\mathbb{E}(ZY) = \mathbb{E}(XY)\,.$$

The conditional expectation is denoted $\mathbb{E}(X|\mathcal{H})$ and the following properties hold (see, for example Breiman [123], Williams [842, 843]):

- If X is $\mathcal{H}$-measurable, $\mathbb{E}(X|\mathcal{H}) = X$, a.s.
- $\mathbb{E}(\mathbb{E}(X|\mathcal{H})) = \mathbb{E}(X)$.
- If $X \geq 0$, then $\mathbb{E}(X|\mathcal{H}) \geq 0$ a.s.
- Linearity: If Y is an integrable random variable and $a, b \in \mathbb{R}$,

$$\mathbb{E}(aX + bY|\mathcal{H}) = a\mathbb{E}(X|\mathcal{H}) + b\mathbb{E}(Y|\mathcal{H}), \quad \text{a.s.}$$

- If $\mathcal{G}$ is another σ-algebra and $\mathcal{G} \subseteq \mathcal{H}$, then

$$\mathbb{E}(\mathbb{E}(X|\mathcal{G})|\mathcal{H}) = \mathbb{E}(\mathbb{E}(X|\mathcal{H})|\mathcal{G}) = \mathbb{E}(X|\mathcal{G}), \quad \text{a.s.}$$

- If Y is $\mathcal{H}$-measurable and XY is integrable, $\mathbb{E}(XY|\mathcal{H}) = Y\mathbb{E}(X|\mathcal{H})$ a.s.
- Jensen's inequality: If f is a convex function such that $f(X)$ is integrable,

$$\mathbb{E}(f(X)|\mathcal{H}) \geq f(\mathbb{E}(X|\mathcal{H})), \quad \text{a.s.}$$

In the particular case where $\mathcal{H}$ is the σ-algebra generated by a r.v. Y, then $\mathbb{E}(X|\sigma(Y))$, which is usually denoted by $\mathbb{E}(X|Y)$, is $\sigma(Y)$-measurable, hence there exists a Borel function φ such that $\mathbb{E}(X|Y) = \varphi(Y)$. The function φ is uniquely defined up to a $\mathbb{P}_Y$-negligible set. The notation $\mathbb{E}(X|Y = y)$ is often used for $\varphi(y)$.

If X is an $\mathbb{R}^p$-valued random variable, and Y an $\mathbb{R}^n$-valued random variable, there exists a family of measures (conditional laws) $\mu(dx, y)$ such that, for any bounded Borel function h

$$\mathbb{E}(h(X)|Y = y) = \int h(x)\mu(dx, y)\,.$$

If (X, Y) are independent random variables, and h is a bounded Borel function, then $\mathbb{E}(h(X, Y)|Y) = \Psi(Y)$, where $\Psi(y) = \mathbb{E}(h(X, y))$, i.e., the conditional law of X given $Y = y$ does not depend on y.

Note that, if X is square integrable, then $\mathbb{E}(X|\mathcal{H})$ may be defined as the projection of X on the space $L^2(\Omega,\mathcal{H})$ of $\mathcal{H}$-measurable square integrable random variables. The conditional variance of a square integrable random variable X is

$$\mathrm{var}(X|\mathcal{H}) = \mathbb{E}(X^2|\mathcal{H}) - (\mathbb{E}(X|\mathcal{H}))^2 \, .$$

Definition 1.1.9.1 *Two σ-algebras $\mathcal{G}_1$ and $\mathcal{G}_2$ are said to be conditionally independent with respect to the σ-algebra $\mathcal{H}$ if $\mathbb{E}(G_1G_2|\mathcal{H}) = \mathbb{E}(G_1|\mathcal{H})\mathbb{E}(G_2|\mathcal{H})$ for any bounded random variables $G_i \in \mathcal{G}_i$. Two random variables X and Y are **conditionally independent** with respect to the σ-algebra $\mathcal{H}$ if $\sigma(X)$ and $\sigma(Y)$ are conditionally independent with respect to $\mathcal{H}$.*

This may be extended obviously to any finite family of r.v's. Two infinite families of random variables are conditionally independent if any finite subfamilies are conditionally independent.

1.1.10 Stochastic Processes

Definition 1.1.10.1 *A continuous time process X on $(\Omega,\mathcal{F},\mathbb{P})$ is a family of random variables $(X_t, t \geq 0)$, such that the map $(\omega,t) \to X_t(\omega)$ is $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}^+)$ measurable.*

We emphasize that when speaking of processes, we always mean a measurable process.

A process X is **continuous** if, for almost all ω, the map $t \to X_t(\omega)$ is continuous. The process is continuous on the right with limits on the left (in short **càdlàg** following the French acronym[2] if, for almost all ω, the map $t \to X_t(\omega)$ is càdlàg.

Definition 1.1.10.2 *A process X is **increasing** if $X_0 = 0$, X is right-continuous, and $X_s \leq X_t$, a.s. for $s \leq t$.*

Definition 1.1.10.3 *Let $(\Omega,\mathcal{F},\mathbf{F},\mathbb{P})$ be a filtered probability space. The process X is $\mathbf{F}$-adapted if for any $t \geq 0$, the random variable X_t is $\mathcal{F}_t$-measurable.*

The **natural filtration $\mathbf{F}^X$** of a stochastic process X is the smallest filtration $\mathbf{F}$ which satisfies the usual hypotheses and such that X is $\mathbf{F}$-adapted. We shall write in short (with an abuse of notation) $\mathcal{F}_t^X = \sigma(X_s, s \leq t)$.

[2] In French, continuous on the right is **c**ontinu **à** **d**roite, and with limits on the left is admettant des **l**imites **à** **g**auche. We shall also use càd for continuous on the right. The use of this acronym comes from P-A. Meyer.

Let $\mathbf{G} = (\mathcal{G}_t, t \geq 0)$ be another filtration on Ω. If $\mathbf{G}$ is larger than $\mathbf{F}$, and if X is an $\mathbf{F}$-adapted process, it is also $\mathbf{G}$-adapted.

Definition 1.1.10.4 *A real-valued process X is* ***progressively measurable*** *with respect to a given filtration $\mathbf{F} = (\mathcal{F}_t, t \geq 0)$, if, for every t, the map $(\omega, s) \to X_s(\omega)$ from $\Omega \times [0,t]$ into $\mathbb{R}$ is $\mathcal{F}_t \times \mathcal{B}([0,t])$-measurable.*

Any càd (or càg) $\mathbf{F}$-adapted process is progressively measurable. An $\mathbf{F}$-progressively measurable process is $\mathbf{F}$-adapted. If X is progressively measurable, then

$$\mathbb{E}\left(\int_0^\infty X_t dt\right) = \int_0^\infty \mathbb{E}(X_t)\, dt,$$

where the existence of one of these expressions implies the existence of the other.

Definition 1.1.10.5 *Two processes $(X_t, t \geq 0)$ and $(Y_t, t \geq 0)$ have the same law if, for any n and any $(t_1, t_2, \dots, t_n)$*

$$(X_{t_1}, X_{t_2}, \dots, X_{t_n}) \stackrel{\text{law}}{=} (Y_{t_1}, Y_{t_2}, \dots, Y_{t_n}).$$

We shall write in short $X \stackrel{\text{law}}{=} Y$, or $X \stackrel{\text{law}}{=} \mu$ for a given probability law μ (on the canonical space).

The process X is a modification of Y if, for any t, $\mathbb{P}(X_t = Y_t) = 1$. The process X is **indistinguishable from** (or a **version** of) Y if $\{\omega : X_t(\omega) = Y_t(\omega), \forall t\}$ is a measurable set and $\mathbb{P}(X_t = Y_t, \forall t) = 1$. If X and Y are modifications of each other and are a.s. continuous, they are indistinguishable.

Let us state without proof a sufficient condition for the existence of a continuous version of a stochastic process.

Theorem 1.1.10.6 (Kolmogorov.) *If a collection $(X_t, t \geq 0)$ of random variables satisfies*

$$\mathbb{E}(|X_t - X_s|^p) \leq C|t-s|^{1+\epsilon}$$

for some $C > 0$, $p > 0$ and $\epsilon > 0$, then this collection admits a modification $(\widetilde{X}_t, t \geq 0)$ which is a.s. continuous, i.e., out of a negligible set, the map $t \to \widetilde{X}_t(\omega)$ is continuous.

Proof: See, e.g., Ikeda and Watanabe [456], p. 20. □

Throughout the book, we shall see many applications of this theorem, in particular, for the existence of a.s. continuous Brownian paths (see ↣ Section 1.4).

Definition 1.1.10.7 *A process X has*
*- **independent increments** if for any pair $(s,t) \in \mathbb{R}_+^2$, the random variable $X_{t+s} - X_s$ is independent of $\mathcal{F}_s^X$,*
*- **stationary increments** if for any pair $(s,t) \in \mathbb{R}_+^2$,*

$$X_{t+s} - X_s \stackrel{\text{law}}{=} X_t\,.$$

A process is stationary if

$$\forall \text{ fixed } s > 0,\ (X_{t+s} - X_s, t \geq 0) \stackrel{\text{law}}{=} (X_t, t \geq 0)\,.$$

Definition 1.1.10.8 *A càd process A is of **finite variation** on $[0,t]$ if*

$$V_A(t,\omega) := \sup \sum_{i=1}^{n} |A_{t_i}(\omega) - A_{t_{i-1}}(\omega)| = \int_0^t |dA_s(\omega)|$$

is a.s. finite, where the supremum is taken over all finite partitions (t_i) of $[0,t]$.

A càd process A is of finite variation if it is of finite variation on any compact $[0,t]$. A càd finite variation process is the difference between two increasing processes. A càd finite variation process A is said to be integrable if $\mathbb{E}(\int_0^\infty |dA_s|) < \infty$. In the definition of finite variation processes, we do not restrict attention to adapted processes. Note that finite variation càd processes are càdlàg.

Exercise 1.1.10.9 One might naively think that a collection $(X_t, t \in \mathbb{R}^+)$ of independent r.v's may be chosen "measurably," i.e., with the map

$$(\mathbb{R}^+ \times \Omega, \mathcal{B}_{\mathbb{R}^+} \times \mathcal{F}) \to (\mathbb{R}, \mathcal{B}_{\mathbb{R}})\ : (t,\omega) \to X_t(\omega)$$

being measurable, so that X is a "true" process. Prove that if the X_t's are centered and $\sup_t \mathbb{E}(X_t^2) < \infty$, then no measurable choice can be constructed, except $X = 0$.
`Hint:` $\mathbb{E}(\int_0^t X_s ds)^2 = \int_0^t \int_0^t \mathbb{E}(X_s X_u) ds\, du$ would be equal to 0, hence X would be null. $\triangleleft$

1.1.11 Convergence

A sequence of processes Z^n converges in $L^2(\Omega \times [0,T])$ to a process Z if $\mathbb{E} \int_0^T |Z_s^n - Z_s|^2 ds$ converges to 0.

A sequence of processes Z^n converges uniformly on compacts in probability (ucp) to a process Z if, for any t, $\sup_{0 \leq s \leq t} |Z_s^n - Z_s|$ converges to 0 in probability.

1.1.12 Laplace Transform

The Laplace transform $\mathbb{E}(e^{\lambda X})$ of a r.v. X is well defined for $\lambda \geq 0$ when X is a negative r.v. (here, we use a slightly unorthodox definition of Laplace transform, with $\lambda \geq 0$). In some cases, the Laplace transform can be defined for every $\lambda \in \mathbb{R}$, as in the following important case, where we denote by $\mathcal{N}(\mu, \sigma^2)$ a Gaussian law with mean μ and variance σ^2:

Proposition 1.1.12.1 Laplace transform of a Gaussian variable. *The law of the random variable X is $\mathcal{N}(\mu, \sigma^2)$ if and only if, for any $\lambda \in \mathbb{R}$,*

$$\boxed{\mathbb{E}(\exp(\lambda X)) = \exp\left(\mu\lambda + \frac{1}{2}\lambda^2\sigma^2\right).}$$

This property extends to any $\lambda \in \mathbb{C}$, and to Gaussian random vectors: X is a d-dimensional Gaussian vector with mean μ and covariance matrix Σ if and only if for any $\lambda \in \mathbb{R}^d$,

$$\mathbb{E}(\exp(\lambda^* X)) = \exp\left(\lambda^*\mu + \frac{1}{2}\lambda^*\Sigma\lambda\right),$$

where the star stands for the transposition operator. If the matrix Σ is invertible, the random vector X admits the density

$$(2\pi)^{-d/2}(\det \Sigma)^{-1/2} \exp\left(-\frac{1}{2}(x-\mu)^*\Sigma^{-1}(x-\mu)\right).$$

Comment 1.1.12.2 Let $(X_t, t \geq 0)$ be a (measurable) process, $\lambda > 0$ and f a positive Borel function. Then, if Θ is a random variable, independent of X, with exponential law ($\mathbb{P}(\Theta \in dt) = \lambda e^{-\lambda t}\mathbb{1}_{\{t>0\}}dt$), one has

$$\mathbb{E}(f(X_\Theta)) = \lambda\mathbb{E}\left(\int_0^\infty e^{-\lambda t}f(X_t)dt\right) = \lambda\int_0^\infty e^{-\lambda t}\mathbb{E}\left(f(X_t)\right)dt\,.$$

Hence, if the process X is continuous, the value of $\mathbb{E}(f(X_\Theta))$ (for all λ and all bounded Borel functions f) characterizes the law of X_t, for any t, i.e., the law of the marginals of the process X. The law of the process assumed to be positive, may be characterized by $\mathbb{E}(\exp[-\int \mu(dt)X_t])$ for all positive measures μ on $(\mathbb{R}^+, \mathcal{B})$.

Exercise 1.1.12.3 Laplace Transforms for the Square of Gaussian Law. Let $X \overset{\text{law}}{=} \mathcal{N}(m, \sigma^2)$ and $\lambda > 0$. Prove that

$$\boxed{\mathbb{E}(e^{-\lambda X^2}) = \frac{1}{\sqrt{1+2\lambda\sigma^2}}\exp\left(-\frac{m^2\lambda}{1+2\lambda\sigma^2}\right)}$$

and more generally that

$$\mathbb{E}(\exp\{-\lambda X^2 + \mu X\}) = \frac{\widehat{\sigma}}{\sigma} \exp\left(\frac{\widehat{\sigma}^2}{2}\left(\mu + \frac{m}{\sigma^2}\right)^2 - \frac{m^2}{2\sigma^2}\right),$$

with $\widehat{\sigma}^2 = \dfrac{\sigma^2}{1+2\lambda\sigma^2}$. ◁

Exercise 1.1.12.4 Moments and Laplace Transform. If X is a positive random variable, prove that its negative moments are given by, for $r > 0$:

$$\boxed{\text{(a)} \qquad \mathbb{E}(X^{-r}) = \frac{1}{\Gamma(r)} \int_0^\infty t^{r-1} \mathbb{E}(e^{-tX}) dt}$$

where Γ is the Gamma function (see ↣ Subsection A.5.1 if needed) and its positive moments are, for $0 < r < 1$

$$\boxed{\text{(b)} \qquad \mathbb{E}(X^r) = \frac{r}{\Gamma(1-r)} \int_0^\infty \frac{1 - \mathbb{E}(e^{-tX})}{t^{r+1}} dt}$$

and for $n < r < n+1$, if $\phi(t) = \mathbb{E}(e^{-tX})$ belongs to C^n

$$\boxed{\text{(c)} \qquad \mathbb{E}(X^r) = \frac{r-n}{\Gamma(n+1-r)} \int_0^\infty (-1)^n \frac{\phi^{(n)}(0) - \phi^{(n)}(t)}{t^{r+1-n}} dt\,.}$$

`Hint:` For example, for (b), use Fubini's theorem and the fact that, for $0 < r < 1$,

$$s^r \Gamma(1-r) = r \int_0^\infty \frac{1 - e^{-st}}{t^{r+1}} dt\,.$$

For $r = n$, one has $\mathbb{E}(X^n) = (-1)^n \phi^{(n)}(0)$. See Schürger [774] for more results and applications. ◁

Exercise 1.1.12.5 Chi-squared Law. A noncentral chi-squared law $\chi^2(\delta, \alpha)$ with δ degrees of freedom and noncentrality parameter α has the density

$$\begin{aligned} f(x;\delta,\alpha) &= 2^{-\delta/2} \exp\left(-\frac{1}{2}(\alpha + x)\right) x^{\frac{\delta}{2}-1} \sum_{n=0}^\infty \left(\frac{\alpha}{4}\right)^n \frac{x^n}{n!\Gamma(n+\delta/2)} \mathbb{1}_{\{x>0\}} \\ &= \frac{e^{-\alpha/2}}{2\alpha^{\nu/2}} e^{-x/2} x^{\nu/2} I_\nu(\sqrt{x\alpha}) \mathbb{1}_{\{x>0\}}\,, \end{aligned}$$

where I_ν is the usual modified Bessel function (see ↣ Subsection A.5.2). Its cumulative distribution function is denoted $\chi^2(\delta, \alpha; \cdot)$.

Let $X_i, i = 1, \ldots, n$ be independent random variables with $X_i \stackrel{\text{law}}{=} \mathcal{N}(m_i, 1)$. Check that $\sum_{i=1}^n X_i^2$ is a noncentral chi-squared variable with n degrees of freedom, and noncentrality parameter $\sum_{i=1}^n m_i^2$. $\triangleleft$

1.1.13 Gaussian Processes

A real-valued process $(X_t, t \geq 0)$ is a **Gaussian process** if any finite linear combination $\sum_{i=1}^n a_i X_{t_i}$ is a Gaussian variable. In particular, for each $t \geq 0$, the random variable X_t is a Gaussian variable. The law of a Gaussian process is characterized by its mean function $\varphi(t) = \mathbb{E}(X_t)$ and its covariance function $c(t, s) = \mathbb{E}(X_t X_s) - \varphi(t)\varphi(s)$ which satisfies

$$\sum_{i,j} \lambda_i \bar{\lambda}_j \, c(t_i, t_j) \geq 0, \, \forall \lambda \in \mathbb{C}^n \, .$$

Note that this property holds for every square integrable process, but that, conversely a Gaussian process may always be associated with a pair (φ, c) satisfying the previous conditions. See Janson [479] for many results on Gaussian processes.

1.1.14 Markov Processes

The $\mathbb{R}^d$-valued process X is said to be a **Markov process** if for any t, the past $\mathcal{F}_t^X = \sigma(X_s, s \leq t)$ and the future $\sigma(X_{t+u}, u \geq 0)$ are conditionally independent with respect to X_t, i.e., for any t, for any bounded random variable $Y \in \sigma(X_u, u \geq t)$:

$$\mathbb{E}(Y|\mathcal{F}_t^X) = \mathbb{E}(Y|X_t) \, .$$

This is equivalent to: for any bounded Borel function f, for any times $t > s \geq 0$

$$\mathbb{E}(f(X_t)|\mathcal{F}_s^X) = \mathbb{E}(f(X_t)|X_s) \, .$$

A **transition probability** is a family $(P_{s,t}, 0 \leq s < t)$ of probabilities such that the Chapman-Kolmogorov equation holds:

$$P_{s,t}(x, A) = \int P_{s,u}(x, dy) P_{u,t}(y, A) = \mathbb{P}(X_t \in A | X_s = x) \, .$$

A Markov process with transition probability $P_{s,t}$ satisfies

$$\mathbb{E}(f(X_t)|X_s) = P_{s,t} f(X_s) = \int f(y) P_{s,t}(X_s, dy) \, ,$$

for any $t > s \geq 0$, for every bounded Borel function f. If $P_{s,t}$ depends only on the difference $t - s$, the Markov process is said to be a **time-homogeneous Markov process** and we simply write P_t for $P_{0,t}$. Results for homogeneous Markov processes can be formally extended to inhomogeneous Markov processes by adding a time dimension to the space, i.e., by considering the process $((X_t, t), t \geq 0)$. For a time-homogeneous Markov process

$$\mathbb{P}_x(X_{t_1} \in A_1, \ldots, X_{t_n} \in A_n) = \int_{A_1} P_{t_1}(x, dx_1) \cdots \int_{A_n} P_{t_n - t_{n-1}}(x_{n-1}, dx_n)\,,$$

where $\mathbb{P}_x$ means that $X_0 = x$.

The (strong) **infinitesimal generator** of a time-homogeneous Markov process is the operator $\mathcal{L}$ defined as

$$\mathcal{L}(f)(x) = \lim_{t \to 0} \frac{\mathbb{E}_x(f(X_t)) - f(x)}{t}\,,$$

where $\mathbb{E}_x$ denotes the expectation for the process starting from x at time 0. The domain of the generator is the set $\mathcal{D}(\mathcal{L})$ of bounded Borel functions f such that this limit exists in the norm $\|f\| = \sup |f(x)|$.

Let X be a time-homogeneous Markov process. The associated **semigroup** $P_t f(x) = \mathbb{E}_x(f(X_t))$ satisfies

$$\frac{d}{dt}(P_t f) = P_t \mathcal{L} f = \mathcal{L} P_t f,\ f \in \mathcal{D}(\mathcal{L})\,. \tag{1.1.1}$$

(See, for example, Kallenberg [505] or [RY], Chapter VII.)

A Markov process is said to be **conservative** if $P_t(x, \mathbb{R}^d) = 1$ for all t and $x \in \mathbb{R}^d$. A nonconservative process can be made conservative by adding an extra state ∂ (called the cemetery state) to $\mathbb{R}^d$. The conservative transition function P_t^∂ is defined by

$$\begin{aligned}
&P_t^\partial(x, A) := P_t(x, A), \quad x \in \mathbb{R}^d, A \in \mathcal{B}\,,\\
&P_t^\partial(x, \partial) := 1 - P_t(x, \mathbb{R}^d), \quad x \in \mathbb{R}^d,\\
&P_t^\partial(\partial, A) := \delta_{\{\partial\}}(A), \quad A \in \mathbb{R}^d \cup \partial\,.
\end{aligned}$$

Definition 1.1.14.1 *The **lifetime** of (the conservative process) X is the $\mathbf{F}^X$-stopping time*

$$\zeta(\omega) := \inf\{t \geq 0 \,:\, X_t(\omega) = \partial\}\,.$$

See $\rightarrowtail$ Section 1.2.3 for the definition of stopping time.

Proposition 1.1.14.2 *Let X be a time-homogeneous Markov process with infinitesimal generator $\mathcal{L}$. Then, for any function f in the domain $\mathcal{D}(\mathcal{L})$ of the generator*

$$M_t^f := f(X_t) - f(X_0) - \int_0^t \mathcal{L} f(X_s) ds$$

is a martingale with respect to $\mathbb{P}_x, \forall x$. Moreover, if τ is a bounded stopping time

$$\mathbb{E}_x(f(X_\tau)) = f(x) + \mathbb{E}_x\left(\int_0^\tau \mathcal{L} f(X_s) ds\right)\,.$$

PROOF: See ↣ Section 1.2 for the definition of martingale. From

$$M_{t+s}^f - M_s^f = f(X_{t+s}) - f(X_s) - \int_s^{t+s} \mathcal{L}f(X_u)du$$

and the Markov property, one deduces

$$\mathbb{E}_x(M_{t+s}^f - M_s^f | \mathcal{F}_s) = \mathbb{E}_{X_s}(M_t^f)\,. \tag{1.1.2}$$

From (1.1.1),

$$\frac{d}{dt}\mathbb{E}_x[f(X_t)] = \mathbb{E}_x[\mathcal{L}f(X_t)],\ f \in \mathcal{D}(\mathcal{L})$$

hence, by integration

$$\mathbb{E}_x[f(X_t)] = f(x) + \int_0^t ds\, \mathbb{E}_x[\mathcal{L}f(X_s)]\,.$$

It follows that, for any x, $\mathbb{E}_x(M_t^f)$ equals 0, hence $\mathbb{E}_{X_s}(M_t^f) = 0$ and from (1.1.2), that M^f is a martingale. □

The family $(U_\alpha, \alpha > 0)$ of kernels defined by

$$U_\alpha f(x) = \int_0^\infty e^{-\alpha t}\mathbb{E}_x[f(X_t)]dt$$

is called the **resolvent** of the Markov process.(See also ↣ Subsection 5.3.6.)

The **strong Markov property** holds if for any finite stopping time T and any $t \geq 0$, (see ↣ Subsection 1.2.3 for the definition of a stopping time) and for any bounded Borel function f,

$$\mathbb{E}(f(X_{T+t})|\mathcal{F}_T^X) = \mathbb{E}(f(X_{T+t})|X_T)\,.$$

It follows that, for any pair of finite stopping times T and S, and any bounded Borel function f

$$\mathbb{1}_{\{S>T\}}\mathbb{E}(f(X_S)|\mathcal{F}_T^X) = \mathbb{1}_{\{S>T\}}\mathbb{E}(f(X_S)|X_T)\,.$$

Proposition 1.1.14.3 *Let X be a strong Markov process with continuous paths and b a continuous function. Define the first passage time of X over b as*

$$T_b = \inf\{t > 0 | X_t \geq b(t)\}\,.$$

Then, for $x \leq b(0)$ and $y > b(t)$

$$\mathbb{P}_x(X_t \in dy) = \int_0^t \mathbb{P}(X_t \in dy | X_s = b(s))F(ds)$$

where F is the law of T_b.

Sketch of the Proof: Let $B \subset [b(t), \infty[$.

$$\begin{aligned}\mathbb{P}_x(X_t \in B) = \mathbb{P}_x(X_t \in B, T_b \le t) &= \mathbb{E}_x(\mathbb{1}_{\{T_b \le t\}} \mathbb{E}_x(\mathbb{1}_{\{X_t \in B\}} | T_b)) \\ &= \int_0^t \mathbb{E}_x(\mathbb{1}_{\{X_t \in B\}} | T_b = s) \mathbb{P}_x(T_b \in ds) \\ &= \int_0^t \mathbb{P}(X_t \in B | X_s = b(s)) \mathbb{P}_x(T_b \in ds) \,.\end{aligned}$$

For a complete proof, see Peskir [707]. □

Definition 1.1.14.4 *Let X be a Markov process. A Borel set A is said to be* ***polar*** *if*

$$\mathbb{P}_x(T_A < \infty) = 0, \quad \textit{for every } x \in \mathbb{R}^d$$

where $T_A = \inf\{t > 0 \,:\, X_t \in A\}$.

This notion will be used (see ↣ Proposition 1.4.2.1) to study some particular cases.

Comment 1.1.14.5 See Blumenthal and Getoor [107], Chung [184], Dellacherie et al. [241], Dynkin [288], Ethier and Kurtz [336], Itô [462], Meyer [648], Rogers and Williams [741], Sharpe [785] and Stroock and Varadhan [812], for further results on Markov processes. Proposition 1.1.14.3 was obtained in Fortet [355] (see Peskir [707] for applications of this result to Brownian motion). Further examples of deterministic barriers will be given in ↣ Chapter 3.

Exercise 1.1.14.6 Let W be a Brownian motion (see ↣ Section 1.4 if needed), x, ν, σ real numbers, $X_t = x \exp(\nu t + \sigma W_t)$ and $M_t^X = \sup_{s \le t} X_s$. Prove that the process $(Y_t = M_t^X / X_t, t \ge 0)$ is a Markov process. This fact (proved by Lévy) is used in particular in Shepp and Shiryaev [787] for the valuation of Russian options and in Guo and Shepp [412] for perpetual lookback American options. ◁

1.1.15 Uniform Integrability

A family of random variables $(X_i, i \in I)$, is **uniformly integrable** (u.i.) if $\sup_{i \in I} \int_{|X_i| \ge a} |X_i| d\mathbb{P}$ goes to 0 when a goes to infinity.

If $|X_i| \le Y$ where Y is integrable, then $(X_i, i \in I)$ is u.i., but the converse does not hold.

Let $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ be a filtered probability space and X an $\mathcal{F}_\infty$-measurable integrable random variable. The family $(\mathbb{E}(X|\mathcal{F}_t), t \ge 0)$ is u.i.. More generally, if $(\Omega, \mathcal{F}, \mathbb{P})$ is a given probability space and X an integrable r.v., the family $\{\mathbb{E}(X|\mathcal{G}), \mathcal{G} \subseteq \mathcal{F}\}$ is u.i.

Very often, one uses the fact that if $(X_i, i \in I)$ is bounded in L^2, i.e., $\sup_i \mathbb{E}(X_i^2) < \infty$ then, it is a u.i. family.

Among the main uses of uniform integrability, the following is the most important: if $(X_n, n \geq 1)$ is u.i. and $X_n \xrightarrow{P} X$, then $X_n \xrightarrow{L^1} X$.

1.2 Martingales

Although our aim in this chapter is to discuss continuous path processes, there would be no advantage in this section of limiting ourselves to the scope of continuous martingales. We shall restrict our attention to continuous martingales in $\rightarrowtail$ Section 1.3.

1.2.1 Definition and Main Properties

Definition 1.2.1.1 *An* $\mathbf{F}$*-adapted process* $X = (X_t, t \geq 0)$*, is an* $\mathbf{F}$*-martingale (resp. sub-martingale, resp. super-martingale) if*

- $\mathbb{E}(|X_t|) < \infty$*, for every* $t \geq 0$*,*
- $\mathbb{E}(X_t|\mathcal{F}_s) = X_s$ *(resp.* $\mathbb{E}(X_t|\mathcal{F}_s) \geq X_s$*, resp.* $\mathbb{E}(X_t|\mathcal{F}_s) \leq X_s$*) a.s. for every pair* (s, t) *such that* $s < t$*.*

Roughly speaking, an $\mathbf{F}$-martingale is a process which is $\mathbf{F}$-conditionally constant, and a super-martingale is conditionally decreasing. Hence, one can ask the question: is a super-martingale the sum of a martingale and a decreasing process? Under some weak assumptions, the answer is positive (see the Doob-Meyer theorem quoted below as Theorem 1.2.1.6).

Example 1.2.1.2 The basic example of a martingale is the process X defined as $X_t := \mathbb{E}(X_\infty|\mathcal{F}_t)$, where X_∞ is a given $\mathcal{F}_\infty$-measurable integrable r.v.. In fact, X is a uniformly integrable martingale if and only if $X_t := \mathbb{E}(X_\infty|\mathcal{F}_t)$, for some $X_\infty \in L^1(\mathcal{F}_\infty)$.

Sometimes, we shall deal with processes indexed by $[0, T]$, which may be considered by a simple transformation as the above processes. If the filtration $\mathbf{F}$ is right-continuous, it is possible to show that any martingale has a càdlàg version.

If M is an $\mathbf{F}$-martingale and $\mathbf{H} \subseteq \mathbf{F}$, then $\mathbb{E}(M_t|\mathcal{H}_t)$ is an $\mathbf{H}$-martingale. In particular, if M is an $\mathbf{F}$-martingale, then it is an $\mathbf{F}^M$-martingale. A process is said to be a martingale if it is a martingale with respect to its natural filtration.

From now on, any martingale (super-martingale, sub-martingale) will be taken to be right-continuous with left-hand limits.

Warning 1.2.1.3 If M is an $\mathbf{F}$-martingale and $\mathbf{F} \subset \mathbf{G}$, it is *not* true in general that M is a $\mathbf{G}$-martingale (see $\rightarrowtail$ Section 5.9 on enlargement of filtrations for a discussion on that specific case).

Example 1.2.1.4 If X is a process with independent increments such that the r.v. X_t is integrable for any t, the process $(X_t - \mathbb{E}(X_t), t \geq 0)$ is a martingale. Sometimes, these processes are called self-similar processes (see $\rightarrowtail$ Chapter 11 for the particular case of Lévy processes).

Definition 1.2.1.5 *A process X is of the class (D), if the family of random variables $(X_\tau, \tau$ finite stopping time) is u.i..*

Theorem 1.2.1.6 (Doob-Meyer Decomposition Theorem) *The process $(X_t; t \geq 0)$ is a sub-martingale (resp. a super-martingale) of class (D) if and only if $X_t = M_t + A_t$ (resp. $X_t = M_t - A_t$) where M is a uniformly integrable martingale and A is an increasing predictable[3] process with $\mathbb{E}(A_\infty) < \infty$.*

PROOF: See Dellacherie and Meyer [244] Chapter VII, 12 or Protter [727] Chapter III. □

If M is a martingale such that $\sup_t \mathbb{E}(|M_t|) < \infty$ (i.e., M is L^1 bounded), there exists an integrable random variable M_∞ such that M_t converges almost surely to M_∞ when t goes to infinity (see [RY], Chapter I, Theorem 2.10). This holds, in particular, if M is uniformly integrable and in that case $M_t \rightarrow_{L^1} M_\infty$ and $M_t = \mathbb{E}(M_\infty|\mathcal{F}_t)$. However, an L^1-bounded martingale is not necessarily uniformly integrable as the following example shows:

Example 1.2.1.7 The martingale $M_t = \exp\left(\lambda W_t - \frac{\lambda^2}{2}t\right)$ where W is a Brownian motion (see $\rightarrowtail$ Section 1.4) is L^1 bounded (indeed $\forall t, \mathbb{E}(M_t) = 1$). From $\lim_{t\to\infty} \frac{W_t}{t} = 0$, a.s., we get that

$$\lim_{t\to\infty} M_t = \lim_{t\to\infty} \exp\left(t\left(\lambda\frac{W_t}{t} - \frac{\lambda^2}{2}\right)\right) = \lim_{t\to\infty} \exp\left(-t\frac{\lambda^2}{2}\right) = 0\,,$$

hence this martingale is not u.i. on $[0, \infty[$ (if it were, it would imply that M_t is null!).

Exercise 1.2.1.8 Let M be an $\mathbf{F}$-martingale and Z an adapted (bounded) continuous process. Prove that, for $0 < s < t$,

$$\mathbb{E}\left(M_t \int_s^t Z_u du|\mathcal{F}_s\right) = \mathbb{E}\left(\int_s^t M_u Z_u du|\mathcal{F}_s\right). \qquad \triangleleft$$

Exercise 1.2.1.9 Consider the interval $[0, 1]$ endowed with Lebesgue measure λ on the Borel σ-algebra $\mathcal{B}$. Define $\mathcal{F}_t = \sigma\{A : A \subset [0, t], A \in \mathcal{B}\}$. Let f be an integrable function defined on $[0, 1]$, considered as a random variable.

[3] See Subsection 1.2.3 for the definition of predictable processes. In the particular case where X is continuous, then A is continuous.

Prove that

$$\mathbb{E}(f|\mathcal{F}_t)(u) = f(u)\mathbb{1}_{\{u\leq t\}} + \mathbb{1}_{\{u>t\}}\frac{1}{1-t}\int_t^1 dx f(x)\,. \qquad \triangleleft$$

Exercise 1.2.1.10 Give another proof that $\lim_{t\to\infty} M_t = 0$ in the above Example 1.2.1.7 by using $T_{-a} = \inf\{t : W_t = -a\}$. $\triangleleft$

1.2.2 Spaces of Martingales

We denote by $\mathbf{H}^2$ (resp. $\mathbf{H}^2[0,T]$) the subset of **square integrable** martingales (resp. defined on [0,T]), i.e., martingales such that $\sup_t \mathbb{E}(M_t^2) < \infty$ (resp. $\sup_{t\leq T}\mathbb{E}(M_t^2) < \infty$). From Jensen's inequality, if M is a square integrable martingale, M^2 is a sub-martingale. It follows that the martingale M is square integrable on $[0,T]$ if and only if $\mathbb{E}(M_T^2) < \infty$.

If $M \in \mathbf{H}^2$, the process M is u.i. and $M_t = \mathbb{E}(M_\infty|\mathcal{F}_t)$. From Fatou's lemma, the random variable M_∞ is square integrable and

$$\mathbb{E}(M_\infty^2) = \lim_{t\to\infty}\mathbb{E}(M_t^2) = \sup_t \mathbb{E}(M_t^2)\,.$$

From $M_t^2 \leq \mathbb{E}(M_\infty^2|\mathcal{F}_t)$, it follows that $(M_t^2, t\geq 0)$ is uniformly integrable.

Doob's inequality states that, if $M \in \mathbf{H}^2$, then $\mathbb{E}(\sup_t M_t^2) \leq 4\mathbb{E}(M_\infty^2)$. Hence, $\mathbb{E}(\sup_t M_t^2) < \infty$ is equivalent to $\sup_t \mathbb{E}(M_t^2) < \infty$. More generally, if M is a martingale or a positive sub-martingale, and $p > 1$,

$$\|\sup_{t\leq T}|M_t|\|_p \leq \frac{p}{p-1}\sup_{t\leq T}\|M_t\|_p\,. \tag{1.2.1}$$

Obviously, the Brownian motion (see $\rightarrowtail$ Section 1.4) does not belong to $\mathbf{H}^2$, however, it belongs to $\mathbf{H}^2([0,T])$ for any T.

We denote by $\mathbf{H}^1$ the set of martingales M such that $\mathbb{E}(\sup_t|M_t|) < \infty$. More generally, the space of martingales such that $M^* = \sup_t |M_t|$ is in L^p is denoted by $\mathbf{H}^p$. For $p > 1$, one has the equivalence

$$M^* \in L^p \Leftrightarrow M_\infty \in L^p\,.$$

Thus the space $\mathbf{H}^p$ for $p > 1$ may be identified with $L^p(\mathcal{F}_\infty)$. Note that $\sup_t \mathbb{E}(|M_t|) \leq \mathbb{E}(\sup_t |M_t|)$, hence any element of $\mathbf{H}^1$ is L^1 bounded, but the converse if not true (see Azéma et al. [36]).

1.2.3 Stopping Times

Definitions

An $\mathbb{R}^+ \cup \{+\infty\}$-valued random variable τ is a **stopping time** with respect to a given filtration $\mathbf{F}$ (in short, an $\mathbf{F}$-stopping time), if $\{\tau \leq t\} \in \mathcal{F}_t$, $\forall t \geq 0$.

If the filtration $\mathbf{F}$ is right-continuous, it is equivalent to demand that $\{\tau < t\}$ belongs to $\mathcal{F}_t$ for every t, or that the left-continuous process $\mathbb{1}_{]0,\tau]\}}(t)$ is an $\mathbf{F}$-adapted process). If $\mathbf{F} \subset \mathbf{G}$, any $\mathbf{F}$-stopping time is a $\mathbf{G}$-stopping time.

If τ is an $\mathbf{F}$-stopping time, the σ-algebra of events prior to τ, $\mathcal{F}_\tau$ is defined as:

$$\mathcal{F}_\tau = \{A \in \mathcal{F}_\infty \,:\, A \cap \{\tau \leq t\} \in \mathcal{F}_t, \, \forall t\}.$$

If X is $\mathbf{F}$-progressively measurable and τ a $\mathbf{F}$-stopping time, then the r.v. X_τ is $\mathcal{F}_\tau$-measurable on the set $\{\tau < \infty\}$.

The σ-algebra $\mathcal{F}_{\tau-}$ is the smallest σ-algebra which contains $\mathcal{F}_0$ and all the sets of the form $A \cap \{t < \tau\}, t > 0$ for $A \in \mathcal{F}_t$.

Definition 1.2.3.1 *A stopping time τ is* ***predictable*** *if there exists an increasing sequence (τ_n) of stopping times such that almost surely*

(i) $\lim_n \tau_n = \tau$,

(ii) $\tau_n < \tau$ *for every n on the set $\{\tau > 0\}$.*

A stopping time τ is ***totally inaccessible*** *if $\mathbb{P}(\tau = \vartheta < \infty) = 0$ for any predictable stopping time ϑ (or, equivalently, if for any increasing sequence of stopping times $(\tau_n, n \geq 0)$, $\mathbb{P}(\{\lim \tau_n = \tau\} \cap A) = 0$ where $A = \cap_n \{\tau_n < \tau\}$).*

If X is an $\mathbf{F}$-adapted process and τ a stopping time, the ($\mathbf{F}$-adapted) process X^τ where $X^\tau_t := X_{t\wedge\tau}$ is called the **process** X **stopped** at τ.

Example 1.2.3.2 If τ is a **random time**, (i.e., a positive random variable), the smallest filtration with respect to which τ is a stopping time is the filtration generated by the process $D_t = \mathbb{1}_{\{\tau \leq t\}}$. The completed σ-algebra $\mathcal{D}_t$ is generated by the sets $\{\tau \leq s\}, s \leq t$ or, equivalently, by the random variable $\tau \wedge t$. This kind of times will be of great importance in $\rightarrowtail$ Chapter 7 to model default risk events.

Example 1.2.3.3 If X is a continuous process, and a a real number, the first time T_a^+ (resp. T_a^-) when X is greater (resp. smaller) than a, is an $\mathbf{F}^X$-stopping time

$$T_a^+ = \inf\{t \,:\, X_t \geq a\}, \quad \text{resp. } T_a^- = \inf\{t \,:\, X_t \leq a\}\,.$$

From the continuity of the process X, if the process starts below a (i.e., if $X_0 < a$), one has $T_a^+ = T_a$ where $T_a = \inf\{t \,:\, X_t = a\}$, and $X_{T_a} = a$ (resp. if $X_0 > a$, $T_a^- = T_a$). Note that if $X_0 \geq a$, then $T_a^+ = 0$, and $T_a > 0$.

More generally, if X is a continuous $\mathbb{R}^d$-valued processes, its **first entrance** time into a closed set F, i.e., $T_F = \inf\{t \,:\, X_t \in F\}$, is a stopping time (see [RY], Chapter I, Proposition 4.6.). If a real-valued process is progressive with respect to a standard filtration, the first entrance time of a Borel set is a stopping time.

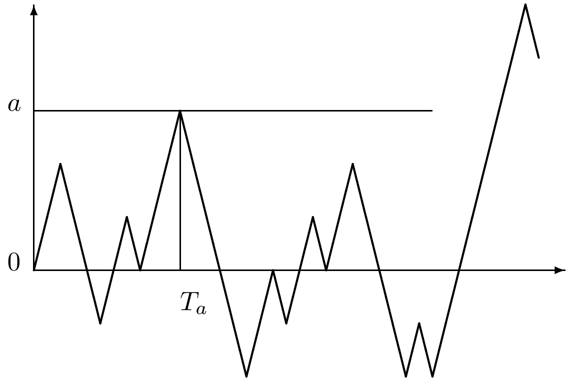

Fig. 1.1 First hitting time of a level a

Optional and Predictable Process

If τ and ϑ are two stopping times, the **stochastic interval** $]\!]\vartheta, \tau]\!]$ is the set $\{(t, \omega) : \vartheta(\omega) < t \leq \tau(\omega)\}$.

The **optional σ-algebra** $\mathcal{O}$ is the σ-algebra generated on $\mathcal{F} \times \mathcal{B}$ by the stochastic intervals $[\![\tau, \infty[\![$ where τ is an $\mathbf{F}$-stopping time.

The **predictable σ-algebra** $\mathcal{P}$ is the σ-algebra generated on $\mathcal{F} \times \mathcal{B}$ by the stochastic intervals $]\!]\vartheta, \tau]\!]$ where ϑ and τ are two $\mathbf{F}$-stopping times such that $\vartheta \leq \tau$.

A process X is said to be **F-predictable** (resp. $\mathbf{F}$-optional) if the map $(\omega, t) \to X_t(\omega)$ is $\mathcal{P}$-measurable (resp. $\mathcal{O}$-measurable).

Example 1.2.3.4 An adapted càg process is predictable.

Martingales and Stopping Times

If M is an $\mathbf{F}$-martingale and τ an $\mathbf{F}$-stopping time, the stopped process M^τ is an $\mathbf{F}$-martingale.

Theorem 1.2.3.5 (Doob's Optional Sampling Theorem.) *If M is a uniformly integrable martingale (e.g., bounded) and ϑ, τ are two stopping times with $\vartheta \leq \tau$, then*

$$M_\vartheta = \mathbb{E}(M_\tau | \mathcal{F}_\vartheta) = \mathbb{E}(M_\infty | \mathcal{F}_\vartheta), \ a.s.$$

If M is a positive super-martingale and ϑ, τ a pair of stopping times with $\vartheta \leq \tau$, then

$$\mathbb{E}(M_\tau | \mathcal{F}_\vartheta) \leq M_\vartheta \, .$$

Warning 1.2.3.6 This theorem often serves as a basic tool to determine quantities defined up to a first hitting time and laws of hitting times. However, in many cases, the u.i. hypothesis has to be checked carefully. For example, if W is a Brownian motion, (see the definition in $\rightarrowtail$ Section 1.4), and T_a the first hitting time of a, then $\mathbb{E}(W_{T_a}) = a$, while a blind application of Doob's theorem would lead to equality between $\mathbb{E}(W_{T_a})$ and $W_0 = 0$. The process $(W_{t \wedge T_a}, t \geq 0)$ is not uniformly integrable, but $(W_{t \wedge T_a}, t \leq t_0)$ is, and obviously so is $(W_{t \wedge T_{-c} \wedge T_a}, t \geq 0)$ (here, $-c < 0 < a$).

The following proposition is an easy converse to Doob's optional sampling theorem:

Proposition 1.2.3.7 *If M is an adapted integrable process, and if for any two-valued stopping time τ, $\mathbb{E}(M_\tau) = \mathbb{E}(M_0)$, then M is a martingale.*

PROOF: Let $s < t$ and $\Gamma_s \in \mathcal{F}_s$. The random time

$$\tau = \begin{cases} s & \text{on } \Gamma_s^c \\ t & \text{on } \Gamma_s \end{cases}$$

is a stopping time, hence $\mathbb{E}(M_t \mathbb{1}_{\Gamma_s}) = \mathbb{E}(M_s \mathbb{1}_{\Gamma_s})$ and the result follows. □

The adapted integrable process M is a martingale if and only if the following property is satisfied ([RY], Chapter II, Sect. 3): if ϑ, τ are two bounded stopping times with $\vartheta \leq \tau$, then

$$M_\vartheta = \mathbb{E}(M_\tau | \mathcal{F}_\vartheta), \text{ a.s.}$$

Comments 1.2.3.8 (a) Knight and Maisonneuve [530] proved that a random time τ is an **F**-stopping time if and only if, for any bounded **F**-martingale M, $\mathbb{E}(M_\infty | \mathcal{F}_\tau) = M_\tau$. Here, $\mathcal{F}_\tau$ is the σ-algebra generated by the random variables Z_τ, where Z is any **F**-optional process. (See Dellacherie et al. [241], page 141, for more information.)
(b) Note that there exist some random times τ which are not stopping times, but nonetheless satisfy $\mathbb{E}(M_0) = \mathbb{E}(M_\tau)$ for any bounded **F**-martingale (see Williams [844]). Such times are called pseudo-stopping times. (See $\rightarrowtail$ Subsection 5.9.4 and Comments 7.5.1.3.)

Definition 1.2.3.9 *A continuous uniformly integrable martingale M belongs to BMO space if there exists a constant m such that*

$$\mathbb{E}(\langle M \rangle_\infty - \langle M \rangle_\tau | \mathcal{F}_\tau) \leq m$$

for any stopping time τ.

See $\rightarrowtail$ Subsection 1.3.1 for the definition of the bracket $\langle . \rangle$. It can be proved (see, e.g., Dellacherie and Meyer [244], Chapter VII,) that the space BMO is the dual of $\mathbf{H}^1$.

See Kazamaki [517] and Doléans-Dade and Meyer [257] for a study of Bounded Mean Oscillation (BMO) martingales.

Exercise 1.2.3.10 A Useful Lemma: Doob's Maximal Identity.
(1) Let M be a positive continuous martingale such that $M_0 = x$.
 (i) Prove that if $\lim_{t\to\infty} M_t = 0$, then

$$\mathbb{P}(\sup M_t > a) = \left(\frac{x}{a}\right) \wedge 1 \tag{1.2.2}$$

and $\sup M_t \overset{\text{law}}{=} \dfrac{x}{U}$ where U is a random variable with a uniform law on $[0,1]$. (See [RY], Chapter 2, Exercise 3.12.)
 (ii) Conversely, if $\sup M_t \overset{\text{law}}{=} \dfrac{x}{U}$, show that $M_\infty = 0$.
(2) Application: Find the law of $\sup_t(B_t - \mu t)$ for $\mu > 0$. (Use Example 1.2.1.7). For $T_a^{(-\mu)} = \inf\{t : B_t - \mu t \geq a\}$, compute $\mathbb{P}(T_a^{(-\mu)} < \infty)$.
`Hint:` Apply Doob's optional sampling theorem to $T_y \wedge t$ and prove, passing to the limit when t goes to infinity, that

$$a = \mathbb{E}(M_{T_y}) = y\mathbb{P}(T_y < \infty) = y\mathbb{P}(\sup M_t \geq y)\,. \qquad \lhd$$

1.2.4 Local Martingales

Definition 1.2.4.1 *An adapted, right-continuous process M is an* $\mathbf{F}$-***local martingale*** *if there exists a sequence of stopping times (τ_n) such that:*

- *The sequence τ_n is increasing and $\lim_n \tau_n = \infty$, a.s.*
- *For every n, the stopped process $M^{\tau_n}\mathbb{1}_{\{\tau_n>0\}}$ is an $\mathbf{F}$-martingale.*

A sequence of stopping times such that the two previous conditions hold is called a localizing or reducing sequence. If M is a local martingale, it is always possible to choose the localizing sequence $(\tau_n, n \geq 1)$ such that each martingale $M^{\tau_n}\mathbb{1}_{\{\tau_n>0\}}$ is uniformly integrable.

Let us give some criteria that ensure that a local martingale is a martingale:

- Thanks to Fatou's lemma, a positive local martingale M is a supermartingale. Furthermore, it is a martingale if (and only if!) its expectation is constant ($\forall t$, $\mathbb{E}(M_t) = \mathbb{E}(M_0)$).
- A local martingale is a uniformly integrable martingale if and only if it is of the class (D) (see Definition 1.2.1.5).
- A local martingale is a martingale if and only if it is of the class (DL), that is, if for every $a > 0$ the family of random variables $(X_\tau, \tau \in \mathcal{T}_a)$ is uniformly integrable, where $\mathcal{T}_a$ is the set of stopping times smaller than a.
- If a local martingale M is in $\mathbf{H}^1$, i.e., if $\mathbb{E}(\sup_t |M_t|) < \infty$, then M is a uniformly integrable martingale (however, not every uniformly integrable martingale is in $\mathbf{H}^1$).

Later, we shall give explicit examples of local martingales which are not martingales. They are called strict local martingales (see, e.g., $\rightarrowtail$ Example 6.1.2.6 and $\rightarrowtail$ Subsection 6.4.1). Note that there exist strict local martingales with constant expectation (see $\rightarrowtail$ Exercise 6.1.5.6).

Doob-Meyer decomposition can be extended to general sub-martingales:

Proposition 1.2.4.2 *A process X is a sub-martingale (resp. a super-martingale) if and only if $X_t = M_t + A_t$ (resp. $X_t = M_t - A_t$) where M is a local martingale and A an increasing predictable process.*

We also use the following definitions:
A local martingale M is locally square integrable if there exists a localizing sequence of stopping times (τ_n) such that $M^{\tau_n} \mathbb{1}_{\{\tau_n > 0\}}$ is a square integrable martingale.
An increasing process A is locally integrable if there exists a localizing sequence of stopping times such that A^{τ_n} is integrable.
By similar localization, we may define locally bounded martingales, local super-martingales, and locally finite variation processes.

Let us state without proof (see [RY]) the following important result.

Proposition 1.2.4.3 *A continuous local martingale of locally finite variation is a constant.*

Warning 1.2.4.4 If X is a positive local super-martingale, then it is a super-martingale. If X is a positive local sub-martingale, it is not necessarily a sub-martingale (e.g., a positive strict local martingale is a positive local sub-martingale and a super-martingale).

Note that a locally integrable increasing process A does not necessarily satisfy $\mathbb{E}(A_t) < \infty$ for any t. As an example, if $A_t = \int_0^t ds/R_s^2$ where R is a 2-dimensional Bessel process (see $\rightarrowtail$ Chapter 6) then A is locally integrable, however $\mathbb{E}(A_t) = \infty$, since, for any $s > 0$, $\mathbb{E}(1/R_s^2) = \infty$.

Comment 1.2.4.5 One can also define a continuous quasi-martingale as a continuous process X such that

$$\sup \sum_{i=1}^{p(n)} \mathbb{E}|\mathbb{E}(X_{t_{i+1}^n} - X_{t_i^n} | \mathcal{F}_{t_i^n})| < \infty$$

where the supremum is taken over the sequences $0 < t_i^n < t_{i+1}^n < T$. Super-martingales (sub-martingales) are quasi-martingales. In that case, the above condition reads

$$\mathbb{E}(|X_T - X_0|) < \infty \,.$$

1.3 Continuous Semi-martingales

A d-dimensional **continuous semi-martingale** is an $\mathbb{R}^d$-valued process X such that each component X^i admits a decomposition as $X^i = M^i + A^i$ where M^i is a continuous local martingale with $M_0^i = 0$ and A^i is a continuous adapted process with locally finite variation. This decomposition is unique (see [RY]), and we shall say in short that M is the martingale part of the continuous semi-martingale X. This uniqueness property, which is not shared by general semi-martingales motivated us to restrict our study of semi-martingales at first to the continuous ones. Later ($\rightarrowtail$ Chapter 9) we shall consider general semi-martingales.

1.3.1 Brackets of Continuous Local Martingales

If M is a continuous local martingale, there exists a unique continuous increasing process $\langle M \rangle$, called the bracket (or predictable quadratic variation) of M such that $(M_t^2 - \langle M \rangle_t, \, t \geq 0)$ is a continuous local martingale (see [RY] Chap IV, Theorem 1.3 for the existence).

The process $\langle M \rangle$ is equal to the limit in probability of the quadratic variation $\sum_i (M_{t_{i+1}^n} - M_{t_i^n})^2$, where $0 = t_0^n < t_1^n < \cdots < t_{p(n)}^n = t$, when $\sup_{0 \leq i \leq p(n)-1} (t_{i+1}^n - t_i^n)$ goes to zero (see [RY], Chapter IV, Section 1). [4] Note that the limit of $\sum_i (M_{t_{i+1}^n} - M_{t_i^n})^2$ depends neither on the filtration nor on the probability measure on the space $(\Omega, \mathcal{F})$ (assuming that M remains a semi-martingale with respect to this filtration or to this probability) and the process $\langle M \rangle$ is $\mathbf{F}^M$-adapted.

Example 1.3.1.1 If W is a Brownian motion (defined in $\rightarrowtail$ Section 1.4),

$$\langle W \rangle_t = \lim \sum_{i=0}^{p(n)-1} (W_{t_{i+1}^n} - W_{t_i^n})^2 = t \, .$$

Here, the limit is in the L^2 sense (hence, in the probability sense). If $\sum_n \sup_i (t_{i+1}^n - t_i^n) < \infty$, the convergence holds also in the a.s. sense (see Kallenberg [505]). This is in particular the case for a dyadic sequence, where $t_i^n = \frac{i}{2^n} t$.

Definition 1.3.1.2 *If M and N are two continuous local martingales, the unique continuous process $(\langle M, N \rangle_t, \, t \geq 0)$ with locally finite variation such that $MN - \langle M, N \rangle$ is a continuous local martingale is called the* ***predictable bracket*** *(or the predictable covariation process) of M and N.*

[4] This is why the term *quadratic variation* is often used instead of bracket.

Let us remark that $\langle M \rangle = \langle M, M \rangle$ and

$$\langle M, N \rangle = \frac{1}{2} [\langle M + N \rangle - \langle M \rangle - \langle N \rangle] = \frac{1}{4} [\langle M + N \rangle - \langle M - N \rangle] \,.$$

These last identities are known as the polarization equalities.

In particular, if the bracket $\langle X, Y \rangle$ of two martingales X and Y is equal to zero, the product XY is a local martingale and X and Y are said to be **orthogonal**. Note that this is the case if X and Y are independent.

We present now some useful results, related to the predictable bracket. For the proofs, we refer to [RY], Chapter IV.

- A continuous local martingale M converges a.s. as t goes to infinity on the set $\{\langle M \rangle_\infty < \infty\}$.
- The **Kunita-Watanabe inequality** states that

  $$|\langle M, N \rangle| \le \langle M \rangle^{1/2} \langle N \rangle^{1/2} \,.$$

  More generally, for h, k positive measurable processes

  $$\int_0^t h_s\, k_s |d\langle M, N \rangle_s| \le \left(\int_0^t h_s^2 d\langle M \rangle_s \right)^{1/2} \left(\int_0^t k_s^2 d\langle N \rangle_s \right)^{1/2} \,.$$

- The **Burkholder-Davis-Gundy** (BDG) inequalities state that for $0 \le p < \infty$, there exist two universal constants c_p and C_p such that if M is a continuous local martingale,

  $$c_p\, \mathbb{E}[(\sup_t |M_t|)^p] \le \mathbb{E}(\langle M \rangle_\infty^{p/2}) \le C_p\, \mathbb{E}[(\sup_t |M_t|)^p] \,.$$

  (See Lenglart et al. [576] for a complete study.) It follows that, if a continuous local martingale M satisfies $\mathbb{E}(\langle M \rangle_\infty^{1/2}) < \infty$, then M is a martingale. Indeed, $\mathbb{E}(\sup_t |M_t|) < \infty$ (i.e., $M \in \mathbf{H}^1$) and, by dominated convergence, the martingale property follows.

We now introduce some spaces of processes, which will be useful for stochastic integration.

Definition 1.3.1.3 *For* $\mathbf{F}$ *a given filtration and* $M \in \mathbf{H}^{c,2}$*, the space of square integrable continuous* $\mathbf{F}$*-martingales, we denote by* $L^2(M, \mathbf{F})$ *the Hilbert space of equivalence classes of elements of* $\mathcal{L}^2(M)$*, the space of* $\mathbf{F}$*-progressively measurable processes* K *such that*

$$\mathbb{E}[\int_0^\infty K_s^2 d\langle M \rangle_s] < \infty \,.$$

We shall sometimes write only $L^2(M)$ when there is no ambiguity. If M is a continuous local martingale, we call $L^2_{loc}(M)$ the space of progressively measurable processes K such that there exists a sequence of stopping times (τ_n) increasing to infinity for which

$$\text{for every } n, \quad \mathbb{E}\left(\int_0^{\tau_n} K_s^2 d\langle M\rangle_s\right) < \infty \,.$$

The space $L^2_{loc}(M)$ consists of all progressively measurable processes K such that

$$\text{for every } t, \quad \int_0^t K_s^2 d\langle M\rangle_s < \infty \, a.s..$$

A continuous local martingale belongs to $\mathbf{H}^{c,2}$ (and is a martingale) if and only if $M_0 \in L^2$ and $\mathbb{E}(\langle M\rangle_\infty) < \infty$.

1.3.2 Brackets of Continuous Semi-martingales

Definition 1.3.2.1 *The bracket (or the predictable quadratic covariation) $\langle X, Y\rangle$ of two continuous semi-martingales X and Y is defined as the bracket of their local martingale parts M^X and M^Y.*

The bracket $\langle X, Y\rangle := \langle M^X, M^Y\rangle$ is also the limit in probability of the quadratic covariation of X and Y, i.e.,

$$\sum_{i=0}^{p(n)-1} (X_{t^n_{i+1}} - X_{t^n_i})(Y_{t^n_{i+1}} - Y_{t^n_i}) \tag{1.3.1}$$

for $0 = t^n_0 \leq t^n_1 \leq \cdots \leq t_{p(n)} = t$ when $\sup_{0\leq i\leq p(n)-1}(t^n_{i+1} - t^n_i)$ goes to 0. Indeed, the bounded variation parts A^X and A^Y do not contribute to the limit of the expression (1.3.1).

If τ is a stopping time, and X a semi-martingale, the stopped process X^τ is a semi-martingale and if Y is another semi-martingale, the bracket of the τ-stopped semi-martingales is the τ-stopped bracket:

$$\langle X^\tau, Y\rangle = \langle X^\tau, Y^\tau\rangle = \langle X, Y\rangle^\tau \,.$$

Remark 1.3.2.2 Let M be a continuous martingale of the form

$$M_t = \int_0^t \varphi_s dW_s$$

where φ is a continuous adapted process (such that $\int_0^t \varphi_s^2 ds < \infty$) and W a Brownian motion (see $\rightarrowtail$ Sections 1.4 and 1.5.1 for definitions). The quadratic variation $\langle M\rangle$ is the process

$$\langle M\rangle_t = \int_0^t \varphi_s^2 ds = \mathbb{P} - \lim \sum_{i=1}^{p(n)} (M_{t^n_{i+1}} - M^n_{t_i})^2 \,,$$

hence, $\mathcal{F}^M_t$ contains $\sigma(\varphi_s^2, s \leq t)$.

Exercise 1.3.2.3 Let M be a Gaussian martingale with bracket $\langle M \rangle$. Prove that the process $\langle M \rangle$ is deterministic.
`Hint:` The Gaussian property implies that, for $t > s$, the r.v. $M_t - M_s$ is independent of $\mathcal{F}_s^M$, hence

$$\mathbb{E}((M_t - M_s)^2 | \mathcal{F}_s^M) = \mathbb{E}((M_t - M_s)^2) = A(t) - A(s)$$

with $A(t) = \mathbb{E}(M_t^2)$ which is deterministic. $\triangleleft$

1.4 Brownian Motion

1.4.1 One-dimensional Brownian Motion

Let X be an $\mathbb{R}$-valued *continuous* process starting from 0 and $\mathbf{F}^X$ its natural filtration.

Definition 1.4.1.1 *The continuous process X is said to be a Brownian motion, (in short, a BM), if one of the following equivalent properties is satisfied:*

(i) *The process X has stationary and independent increments, and for any $t > 0$, the r.v. X_t follows the $\mathcal{N}(0, t)$ law.*
(ii) *The process X is a Gaussian process, with mean value equal to 0 and covariance $t \wedge s$.*
(iii) *The processes $(X_t, t \geq 0)$ and $(X_t^2 - t, t \geq 0)$ are $\mathbf{F}^X$-local martingales.*
(iii′) *The process X is an $\mathbf{F}^X$-local martingale with bracket t.*
(iv) *For every real number λ, the process $\left(\exp\left(\lambda X_t - \frac{\lambda^2}{2} t\right), t \geq 0\right)$ is an $\mathbf{F}^X$-local martingale.*
(v) *For every real number λ, the process $\left(\exp\left(i\lambda X_t + \frac{\lambda^2}{2} t\right), t \geq 0\right)$ is an $\mathbf{F}^X$-local martingale.*

To establish the existence of Brownian motion, one starts with the canonical space $\Omega = C(\mathbb{R}^+, \mathbb{R})$ of continuous functions. The canonical process $X_t : \omega \to \omega(t)$ (ω is now a generic continuous function) is defined on Ω. There exists a unique probability measure on this space Ω such that the law of X satisfies the above properties. This probability measure is called Wiener measure and is often denoted by $\mathbf{W}$ in deference to Wiener (1923) who proved its existence. We refer to [RY] Chapter I, for the proofs.

It can be proved, as a consequence of Kolmogorov's continuity criterion 1.1.10.6 that a process (not assumed to be continuous) which satisfies (i) or (ii) admits in fact a continuous modification. There exist discontinuous processes that satisfy (iii) (e.g., the martingale associated with a Poisson process, see $\rightarrowtail$ Chapter 8).

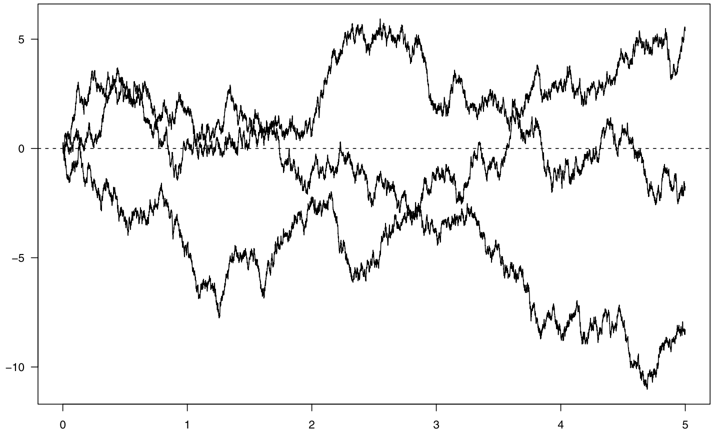

Fig. 1.2 Simulation of Brownian paths

Extending Definition 1.4.1.1, a continuous process X is said to be a BM with respect to a filtration $\mathbf{F}$ larger than $\mathbf{F}^X$ if for any (t, s), the random variable $X_{t+s} - X_t$ is independent of $\mathcal{F}_t$ and is $\mathcal{N}(0, s)$ distributed.

The **transition probability** of the Brownian motion starting from x (i.e., such that $\mathbb{P}_x(W_0 = x) = 1$) is $p_t(x, y)$ defined as

$$p_t(x, y)dy = \mathbb{P}_x(W_t \in dy) = \mathbb{P}_0(x + W_t \in dy)$$

and

$$\boxed{p_t(x, y) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{1}{2t}(x - y)^2\right) .} \tag{1.4.1}$$

We shall also use the notation $p_t(x)$ for $p_t(0, x) = p_t(x, 0)$, hence

$$p_t(x, y) = p_t(x - y) .$$

We shall prove in $\rightarrowtail$ Exercise 1.5.3.3 Lévy's characterization of Brownian motion, which is a generalization of (iii) above.

Theorem 1.4.1.2 (Lévy's Characterization of Brownian Motion.) *The process X is an $\mathbf{F}$-Brownian motion if and only if the processes $(X_t, t \geq 0)$ and $(X_t^2 - t, t \geq 0)$ are continuous $\mathbf{F}$-local martingales.*

In this case, the processes are $\mathbf{F}^X$-local martingales, and in fact $\mathbf{F}^X$-martingales. If X is a Brownian motion, the local martingales in (iv) and (v) Definition 1.4.1.1 are martingales. See also [RY], Chapter IV, Theorem 3.6.

An important fact is that in a Brownian filtration, i.e., in a filtration generated by a BM, every stopping time is predictable ([RY], Chapter IV, Corollary 5.7) which is equivalent to the property that all martingales are continuous.

Comment 1.4.1.3 In order to prove property (a), it must be established that $\lim_{t\to 0} tW_{1/t} = 0$, which follows from $(W_t, t > 0) \stackrel{\text{law}}{=} (tW_{1/t}, t > 0)$.

Definition 1.4.1.4 *A process $X_t = \mu t + \sigma B_t$ where B is a Brownian motion is called a drifted Brownian motion, with drift μ.*

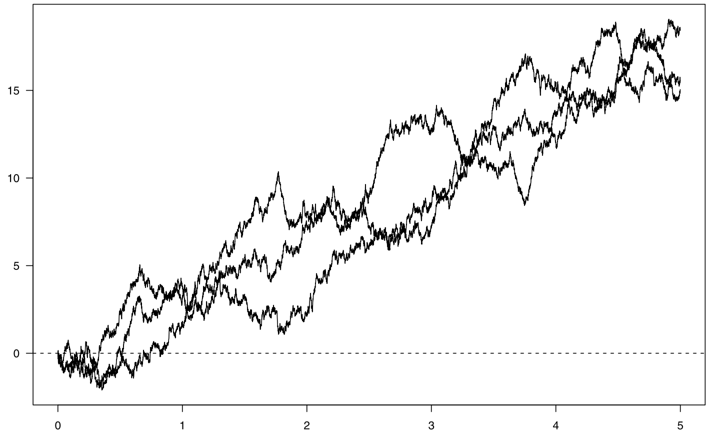

Fig. 1.3 Simulation of drifted Brownian paths $X_t = 3(t + B_t)$

Example 1.4.1.5 Let W be a Brownian motion. Then,

(a) The processes $(-W_t, t \geq 0)$ and $(tW_{1/t}, t \geq 0)$ are BMs. The second result is called the **time inversion** property of the BM.
(b) For any $c \in \mathbb{R}^+$, the process $(\frac{1}{c}W_{c^2 t}, t \geq 0)$ is a BM (scaling property).
(c) The process $B_t = \int_0^t \operatorname{sgn}(W_s)dW_s$ is a Brownian motion with respect to $\mathbf{F}^W$ (and to $\mathbf{F}^B$): indeed the processes B and $(B_t^2 - t, t \geq 0)$ are $\mathbf{F}^W$-martingales. (See $\rightarrowtail$ 1.5.1 for the definition of the stochastic integral and the proofs of the martingale properties). It can be proved that the natural filtration of B is strictly smaller than the filtration of W (see $\rightarrowtail$ Section 5.8).

(d) The process $\widehat{B}_t = W_t - \int_0^t W_s \frac{ds}{s}$ is a Brownian motion with respect to $\mathbf{F}^B$) (but not w.r.t. $\mathbf{F}^W$): indeed, the process $\widehat{B}$ is a Gaussian process and an easy computation establishes that its mean is 0 and its covariance is $s \wedge t$. It can be noted that the process $\widehat{B}$ is not an $\mathbf{F}^W$-martingale and that its natural filtration is strictly smaller than the filtration of W (see $\rightarrowtail$ Section 5.8).

Comment 1.4.1.6 A Brownian filtration is large enough to contain a strictly smaller Brownian filtration (see Examples 1.4.1.5, (c) and (d)). On the other hand, if the processes $W^{(i)}, i = 1, 2$ are independent real-valued Brownian motions, it is not possible to find a real-valued Brownian motion B such that $\sigma(B_s, s \le t) = \sigma(W_s^{(1)}, W_s^{(2)}, s \le t)$. This will be proved using the predictable representation theorem. (See $\rightarrowtail$ Subsection 1.6.1.)

Exercise 1.4.1.7 Prove that, for $\lambda > 0$, one has

$$\boxed{\int_0^\infty e^{-\lambda t} p_t(x, y) dt = \frac{1}{\sqrt{2\lambda}} e^{-|x-y|\sqrt{2\lambda}} .}$$

Prove that if f is a bounded Borel function, and $\lambda > 0$,

$$\mathbb{E}_x(\int_0^\infty e^{-\lambda^2 t/2} f(W_t) dt) = \frac{1}{\lambda} \int_{-\infty}^\infty e^{-\lambda|y-x|} f(y) dy . \qquad \triangleleft$$

Exercise 1.4.1.8 Prove that (v) of Definition 1.4.1.1 characterizes a BM, i.e., if the process $(Z_t = \exp(i\lambda X_t + \frac{\lambda^2}{2} t), t \ge 0)$ is a $\mathbf{F}^X$-local martingale for any λ, then X is a BM.
`Hint:` Establish that Z is a martingale, then prove that, for $t > s$,

$$\forall A \in \mathcal{F}_s,\ \mathbb{E}[\mathbb{1}_A \exp(i\lambda(X_t - X_s))] = \mathbb{P}(A) \exp\left(-\tfrac{1}{2}\lambda^2 (t-s)\right) . \qquad \triangleleft$$

Exercise 1.4.1.9 Prove that, for any $\lambda \in \mathbb{C}$, $(e^{-\lambda^2 t/2} \cosh(\lambda W_t), t \ge 0)$ is a martingale. $\triangleleft$

Exercise 1.4.1.10 Let W be a BM and φ be an adapted process.
(a) Prove that $\int_0^t \varphi_s dW_s$ is a BM if and only if $|\varphi_s| = 1,\ ds$ a.s.
(b) Assume now that φ is deterministic. Prove that $W_t - \int_0^t ds\, \varphi_s\, W_s$ is a BM if and only if $\varphi \equiv 0$ or $\varphi \equiv \frac{1}{s},\ ds$ a.s..
`Hint:` The function φ satisfies, for $t > s$,

$$\mathbb{E}\left((W_t - \int_0^t du\, \varphi_u\, W_u)\, (W_s - \int_0^s du\, \varphi_u\, W_u) \right) = s$$

if and only if $s\varphi_s = \varphi_s \int_0^s du\, u\, \varphi_u$. $\triangleleft$

1.4.2 d-dimensional Brownian Motion

A continuous process $X = (X^1, \dots, X^d)$, taking values in $\mathbb{R}^d$ is a d-dimensional Brownian motion if one of the following equivalent properties is satisfied:

- all its components X^i are independent Brownian motions.
- The processes X^i and $(X^i_t X^j_t - \delta_{i,j} t, t \geq 0)$, where $\delta_{i,j}$ is the Kronecker symbol ($\delta_{i,j} = 1$ if $i = j$ and $\delta_{i,j} = 0$ otherwise) are continuous local $\mathbf{F}^X$-martingales.
- For any $\lambda \in \mathbb{R}^d$, the process $\left(\exp\left(i\lambda \centerdot X_t + \frac{\|\lambda\|^2}{2} t\right), t \geq 0\right)$ is a continuous $\mathbf{F}^X$-local martingale, where the notation $\lambda \centerdot x$ indicates the Euclidian scalar product between λ and x.

Proposition 1.4.2.1 *Let B be a $\mathbb{R}^d$-valued Brownian motion, and T_x the first hitting time of x, defined as $T_x = \inf\{t > 0 : B_t = x\}$.*

- *If $d = 1$, $\mathbb{P}(T_x < \infty) = 1$, for every $x \in \mathbb{R}$,*
- *If $d \geq 2$, $\mathbb{P}(T_x < \infty) = 0$, for every $x \in \mathbb{R}^d$, i.e., the one-point sets are polar.*
- *If $d \leq 2$, the BM is recurrent, i.e., almost surely, the set $\{t : B_t \in A\}$ is unbounded for all open subsets $A \in \mathbb{R}^d$.*
- *If $d \geq 3$, the BM is transient, more precisely, $\lim_{t\to\infty} |B_t| = +\infty$ almost surely.*

Proof: We refer to [RY], Chapter V, Section 2. □

1.4.3 Correlated Brownian Motions

If W^1 and W^2 are two independent BMs and ρ a constant satisfying $|\rho| \leq 1$, the process

$$W^3 = \varrho W^1 + \sqrt{1 - \varrho^2}\, W^2$$

is a BM, and $\langle W^1, W^3 \rangle_t = \varrho t$. This leads to the following definition.

Definition 1.4.3.1 *Two $\mathbf{F}$-Brownian motions B and W are said to be $\mathbf{F}$-correlated with correlation ρ if $\langle B, W \rangle_t = \rho t$.*

Proposition 1.4.3.2 *The components of the 2-dimensional correlated BM (B, W) are independent if and only if $\rho = 0$.*

Proof: If the Brownian motions are independent, their product is a martingale, hence $\rho = 0$. Note that this can also be proved using the integration by parts formula (see $\rightarrowtail$ Subsection 1.5.2).

If the bracket is null, then the product BW is a martingale, and it follows that for $t > s$,

$$\mathbb{E}(B_s W_t) = \mathbb{E}(B_s \mathbb{E}(W_t|\mathcal{F}_s)) = \mathbb{E}(B_s W_s) = 0\,.$$

Therefore, the Gaussian processes W and B are uncorrelated, hence they are independent. $\square$

If B and W are correlated BMs, the process $(B_t W_t - \rho t, t \geq 0)$ is a martingale and $\mathbb{E}(B_t W_t) = \rho t$. From the Cauchy-Schwarz inequality, it follows that $|\rho| \leq 1$. In the case $|\rho| < 1$, the process X defined by the equation

$$W_t = \rho B_t + \sqrt{1-\rho^2} X_t$$

is a Brownian motion independent of B. Indeed, it is a continuous martingale, and it is easy to check that its bracket is t. Moreover $\langle X, B \rangle = 0$.

Note that, for any pair $(a, b) \in \mathbb{R}^2$ the process $Z_t = aB_t + bW_t$ is, up to a multiplicative factor, a Brownian motion. Indeed, setting $c = \sqrt{a^2 + b^2 + 2ab\rho}$ the two processes $\left(\widetilde{Z}_t := \frac{1}{c} Z_t, t \geq 0\right)$ and $(\widetilde{Z}_t^2 - t, t \geq 0)$ are continuous martingales, hence $\widetilde{Z}$ is a Brownian motion.

Proposition 1.4.3.3 *Let $B_t = \Gamma W_t$ where W is a d-dimensional Brownian motion and $\Gamma = (\gamma_{i,j})$ is a $d \times d$ matrix with $\sum_{j=1}^d \gamma_{i,j}^2 = 1$. The process B is a vector of correlated Brownian motions, with correlation matrix $\rho = \Gamma \Gamma^*$.*

Exercise 1.4.3.4 Prove Proposition 1.4.3.3. $\lhd$

Exercise 1.4.3.5 Let B be a Brownian motion and let $\widehat{B}_t = B_t - \int_0^t ds \frac{B_s}{s}$. Prove that for every t, the r.v's B_t and $\widehat{B}_t$ are not correlated, hence are independent. However, clearly, the two Brownian motions B and $\widehat{B}$ are not independent. There is no contradiction with our previous discussion, as $\widehat{B}$ is not an $\mathbf{F}^B$-Brownian motion. $\lhd$

Remark 1.4.3.6 It is possible to construct two Brownian motions W and B such that the pair (W, B) is not a Gaussian process. For example, let W be a Brownian motion and set $B_t = \int_0^t \operatorname{sgn}(W_s) dW_s$ where the stochastic integral is defined in $\rightarrowtail$ Subsection 1.5.1. The pair (W, B) is not Gaussian, since $aW_t + B_t = \int_0^t (a + \operatorname{sgn}(W_s)) dW_s$ is not a Gaussian process. Indeed, its bracket is not deterministic, whereas the bracket of a Gaussian martingale is deterministic (see Exercise 1.3.2.3). Note that $\langle B, W \rangle_t = \int_0^t \operatorname{sgn}(W_s) ds$, hence the bracket is not of the form as in Definition 1.4.3.1. Nonetheless, there is some "correlation" between these two Brownian motions.

1.5 Stochastic Calculus

Let $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ be a filtered probability space. We recall very briefly the definition of a stochastic integral with respect to a square integrable martingale. We refer the reader to [RY] for details.

1.5.1 Stochastic Integration

An elementary **F**-predictable process is a process K which can be written

$$K_t := K_0 \mathbb{1}_{\{0\}}(t) + \sum_i K_i \mathbb{1}_{]T_i,T_{i+1}]}(t)\,,$$

with

$$0 = T_0 < T_1 < \cdots < T_i < \cdots \text{ and } \lim_i T_i = +\infty\,.$$

Here, the T_i's are **F**-stopping times and the r.v's K_i are $\mathcal{F}_{T_i}$-measurable and uniformly bounded, i.e., there exists a constant C such that $\forall i$, $|K_i| \le C$ a.s..

Let M be a continuous local martingale.

▶ For any elementary predictable process K, the stochastic integral $\int_0^t K_s dM_s$ is defined path-by-path as

$$\int_0^t K_s dM_s := \sum_{i=0}^{\infty} K_i (M_{t\wedge T_{i+1}} - M_{t\wedge T_i})\,.$$

▶ The stochastic integral $\int_0^t K_s dM_s$ can be defined for any continuous process $K \in L^2(M)$ as follows. For any $p \in \mathbb{N}$, one defines the sequence of stopping times

$$\begin{aligned}
T_0 &:= 0\\
T_1^p &:= \inf\left\{t \,:\, |K_t - K_0| > \frac{1}{p}\right\}\\
T_n^p &:= \inf\left\{t > T_{n-1}^p \,:\, |K_t - K_{T_{n-1}^p}| > \frac{1}{p}\right\}\,.
\end{aligned}$$

Set $K_s^{(p)} = \sum_i K_{T_i^p} \mathbb{1}_{]T_i^p, T_{i+1}^p]}(s)$. The sequence $\int_0^t K_s^{(p)} dM_s$ converges in L^2 to a continuous local martingale denoted by $(K\star M)_t := \int_0^t K_s dM_s$.

▶ Then, by density arguments, one can define the stochastic integral for any process $K \in L^2(M)$, and by localization for $K \in L^2_{loc}(M)$.

If $M \in \mathbf{H}^{c,2}$, there is an isometry between $L^2(M)$ and the space of stochastic integrals, i.e.,

$$\mathbb{E}\left(\int_0^t K_s^2\, d\langle M\rangle_s\right) = \mathbb{E}\left(\int_0^t K_s dM_s\right)^2\,.$$

(See [RY], Chapter IV for details.)

Let M and N belong to $\mathbf{H}^{c,2}$ and $\phi \in L^2(M), \psi \in L^2(N)$. For the martingales X and Y, where $X_t = (\phi \star M)_t$ and $Y_t = (\psi \star N)_t$, we have $\langle X \rangle_t = \int_0^t \phi_s^2 d\langle M \rangle_s$ and $\langle X, Y \rangle_t = \int_0^t \psi_s \phi_s d\langle M, N \rangle_s$. In particular, for any fixed T, the process $(X_t, t \leq T)$ is a square integrable martingale.

If X is a semi-martingale, the integral of a predictable process K, where $K \in L^2_{loc}(M) \cap L^1_{loc}(|dA|)$ with respect to X is defined to be

$$\int_0^t K_s dX_s = \int_0^t K_s dM_s + \int_0^t K_s dA_s$$

where $\int_0^t K_s dA_s$ is defined path-by-path as a Stieltjes integral (we have required that $\int_0^t |K_s(\omega)| \, |dA_s(\omega)| < \infty$).

For a Brownian motion, we obtain in particular the following proposition:

Proposition 1.5.1.1 *Let W be a Brownian motion, τ a stopping time and θ an adapted process such that $\mathbb{E}\left(\int_0^\tau \theta_s^2 ds\right) < \infty$. Then $\mathbb{E}\left(\int_0^\tau \theta_s dW_s\right) = 0$ and $\mathbb{E}\left(\int_0^\tau \theta_s dW_s\right)^2 = \mathbb{E}\left(\int_0^\tau \theta_s^2 ds\right)$.*

PROOF: We apply the previous results with $\widetilde{\theta} = \theta \mathbb{1}_{\{]0,\tau]\}}$.

Comment 1.5.1.2 In the previous proposition, the integrability condition $\mathbb{E}\left(\int_0^\tau \theta_s^2 ds\right) < \infty$ is important (the case where $\tau = \inf\{t : W_t = a\}$ and $\theta = 1$ is an example where the condition does not hold).

In general, there is the inequality

$$\mathbb{E}\left(\int_0^\tau K_s dM_s\right)^2 \leq \mathbb{E}\left(\int_0^\tau K_s^2 \, d\langle M \rangle_s\right) \tag{1.5.1}$$

and it may happen that

$$\mathbb{E}\left(\int_0^\tau K_s^2 \, d\langle M \rangle_s\right) = \infty, \quad \text{and } \mathbb{E}\left(\int_0^\tau K_s dM_s\right)^2 < \infty \, .$$

This is the case if $K_t = 1/R_t^2$ for $t \geq 1$ and $K_t = 0$ for $t < 1$ where R is a Bessel process of dimension 3 and M the driving Brownian motion for R (see $\rightarrowtail$ Section 6.1).

Comment 1.5.1.3 In the case where K is continuous, the stochastic integral $\int K_s dM_s$ is the limit of the "Riemann sums" $\sum_i K_{u_i}(M_{t_{i+1}} - M_{t_i})$ where $u_i \in [t_i, t_{i+1}[$. But these sums do not converge pathwise because the paths of M are a.s. not of bounded variation. This is why we use L^2 convergence. It can be proved that the Riemann sums converge uniformly on compacts in probability to the stochastic integral.

Exercise 1.5.1.4 Let b and θ be continuous deterministic functions. Prove that the process $Y_t = \int_0^t b(u)du + \int_0^t \theta(u)dW_u$ is a Gaussian process, with mean $\mathbb{E}(Y_t) = \int_0^t b(u)du$ and covariance $\int_0^{s \wedge t} \theta^2(u)du$. $\triangleleft$

Exercise 1.5.1.5 Prove that, if W is a Brownian motion, from the definition of the stochastic integral as an L^2 limit, $\int_0^t W_s dW_s = \frac{1}{2}(W_t^2 - t)$. $\triangleleft$

1.5.2 Integration by Parts

The integration by parts formula follows directly from the definition of the bracket. If (X, Y) are two continuous semi-martingales, then

$$d(XY) = XdY + YdX + d\langle X, Y\rangle$$

or, in an integrated form

$$X_tY_t = X_0Y_0 + \int_0^t X_s dY_s + \int_0^t Y_s dX_s + \langle X, Y\rangle_t\,.$$

Definition 1.5.2.1 *Two square integrable continuous martingales are orthogonal if their product is a martingale.*

Exercise 1.5.2.2 If two martingales are independent, they are orthogonal. Check that the converse does not hold.
`Hint:` Let B and W be two independent Brownian motions. The martingales W and M where $M_t = \int_0^t W_s dB_s$ are orthogonal and not independent. Indeed, the martingales W and M satisfy $\langle W, M\rangle = 0$. However, the bracket of M, that is $\langle M\rangle_t = \int_0^t W_s^2 ds$ is $\mathbf{F}^W$-adapted. One can also note that

$$\mathbb{E}\left(\exp\left(i\lambda \int_0^t W_s dB_s\right) |\mathcal{F}_\infty^W\right) = \exp\left(-\frac{\lambda^2}{2}\int_0^t W_s^2 ds\right)\,,$$

and the right-hand side is not a constant as it would be if the independence property held. The martingales M and N where $N_t = \int_0^t B_s dW_s$ (or M and $\widetilde{N}_t := \int_0^t W_s dW_s$) are also orthogonal and not independent. $\triangleleft$

Exercise 1.5.2.3 Prove that the two martingales N and $\widetilde{N}$, defined in Exercise 1.5.2.2 are not orthogonal although as r.v's, for fixed t, N_t and $\widetilde{N}_t$ are orthogonal in L^2. $\triangleleft$

1.5.3 Itô's Formula: The Fundamental Formula of Stochastic Calculus

The vector space of semi-martingales is invariant under "smooth" transformations, as established by Itô (see [RY] Chapter IV, for a proof):

Theorem 1.5.3.1 (Itô's formula.) *Let F belong to $C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^d, \mathbb{R})$ and let $X = M + A$ be a continuous d-dimensional semi-martingale. Then the process $(F(t, X_t),\, t \geq 0)$ is a continuous semi-martingale and*

$$\boxed{\begin{aligned} F(t,X_t) = F(0,X_0) &+ \int_0^t \frac{\partial F}{\partial t}(s,X_s)ds + \sum_{i=1}^d \int_0^t \frac{\partial F}{\partial x_i}(s,X_s)dX_s^i \\ &+ \quad \frac{1}{2}\sum_{i,j}\int_0^t \frac{\partial^2 F}{\partial x_j\,\partial x_i}(s,X_s)d\langle X^i,X^j\rangle_s\,. \end{aligned}}$$

Hence, the bounded variation part of $F(t,X_t)$ is

$$\int_0^t \frac{\partial F}{\partial t}(s,X_s)ds + \sum_{i=1}^d \int_0^t \frac{\partial F}{\partial x_i}(s,X_s)dA_s^i \\ + \frac{1}{2}\sum_{i,j}\int_0^t \frac{\partial^2 F}{\partial x_j \partial x_i}(s,X_s)d\langle X^i,X^j\rangle_s\,. \tag{1.5.2}$$

An important consequence is the following: in the one-dimensional case, if X is a martingale ($X = M$) and $d\langle M\rangle_t = h(t)dt$ with h deterministic (i.e., X is a Gaussian martingale), and if F is a $C^{1,2}$ function such that $\partial_t F + h(t)\frac{1}{2}\partial_{xx}F = 0$, then the process $F(t,X_t)$ is a local martingale. A similar result holds in the d-dimensional case.

Note that the application of Itô's formula does not depend on whether or not the processes (A_t^i) or $\langle M^i,M^j\rangle_t$ are absolutely continuous with respect to Lebesgue measure. In particular, if $F \in C^{1,1,2}(\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^d, \mathbb{R})$ and V is a continuous bounded variation process, then

$$dF(t,V_t,X_t) = \frac{\partial F}{\partial t}(t,V_t,X_t)dt + \frac{\partial F}{\partial v}(t,V_t,X_t)dV_t + \sum_i \frac{\partial F}{\partial x_i}(t,V_t,X_t)dX_t^i \\ + \frac{1}{2}\sum_{i,j}\frac{\partial^2 F}{\partial x_j\,\partial x_i}(t,V_t,X_t)d\langle X^i,X^j\rangle_t\,.$$

We now present an extension of Itô's formula, which is useful in the study of stochastic flows and in some cases in finance, when dealing with factor models (see Douady and Jeanblanc [264]) or with credit derivatives dynamics in a multi-default setting (see Bielecki et al. [96]).

Theorem 1.5.3.2 (Itô-Kunita-Ventzel's formula.) *Let $F_t(x)$ be a family of stochastic processes, continuous in $(t,x) \in (\mathbb{R}^+ \times \mathbb{R}^d)$ a.s. satisfying:*

(i) *For each $t > 0$, $x \to F_t(x)$ is C^2 from $\mathbb{R}^d$ to $\mathbb{R}$.*
(ii) *For each x, $(F_t(x), t \geq 0)$ is a continuous semi-martingale*

$$dF_t(x) = \sum_{j=1}^n f_t^j(x)dM_t^j$$

where M^j are continuous semi-martingales, and $f^j(x)$ are stochastic processes continuous in (t,x), such that $\forall s>0$, $x \to f^j_s(x)$ are C^1 maps, and $\forall x$, $f^j(x)$ are adapted processes.

Let $X=(X^1,\dots,X^d)$ be a continuous semi-martingale. Then

$$F_t(X_t) = F_0(X_0) + \sum_{j=1}^n \int_0^t f^j_s(X_s)dM^j_s + \sum_{i=1}^d \int_0^t \frac{\partial F_s}{\partial x_i}(X_s)dX^i_s + \sum_{i=1}^d \sum_{j=1}^n \int_0^t \frac{\partial f_s}{\partial x_i}(X_s)d\langle M^j, X^i\rangle_s + \frac{1}{2}\sum_{i,k=1}^d \int_0^t \frac{\partial^2 F_s}{\partial x_i \partial x_k} d\langle X^k, X^i\rangle_s \,.$$

PROOF: We refer to Kunita [546] and Ventzel [828]. □

Exercise 1.5.3.3 Prove Theorem 1.4.1.2, i.e., if X is continuous, X_t and $X_t^2 - t$ are martingales, then X is a BM.
`Hint:` Apply Itô's formula to the complex valued martingale $\exp(i\lambda X_t + \frac{1}{2}\lambda^2 t)$ and use Exercise 1.4.1.8. ◁

Exercise 1.5.3.4 Let $f \in C^{1,2}([0,T]\times\mathbb{R}^d, \mathbb{R})$. We write $\partial_x f(t,x)$ for the row vector $\left[\frac{\partial f}{\partial x_i}(t,x)\right]_{i=1,\dots,d}$; $\partial_{xx} f(t,x)$ for the matrix $\left[\frac{\partial^2 f}{\partial x_i\,\partial x_j}(t,x)\right]_{i,j}$, and $\partial_t f(t,x)$ for $\frac{\partial f}{\partial t}(t,x)$. Let $B=(B^1,\dots,B^n)$ be an n-dimensional Brownian motion and $Y_t = f(t,X_t)$, where X_t satisfies $dX^i_t = \mu^i_t dt + \sum_{j=1}^n \eta^{i,j}_t dB^j_t$. Prove that

$$dY_t = \left\{\partial_t f(t,X_t) + \partial_x f(t,X_t)\mu_t + \frac{1}{2}\left[\eta_t \partial_{xx} f(t,X_t)\eta_t^T\right]\right\}dt + \partial_x f(t,X_t)\eta_t\, dB_t \,.$$

◁

Exercise 1.5.3.5 Let B be a d-dimensional Brownian motion, with $d \geq 2$ and β defined as

$$d\beta_t = \frac{1}{\|B_t\|} B_t \cdot dB_t = \frac{1}{\|B_t\|}\sum_{i=1}^d B^i_t dB^i_t, \quad \beta_0 = 0.$$

Prove that β is a Brownian motion. This will be the starting point of the study of Bessel processes (see ⤚ Chapter 6). ◁

Exercise 1.5.3.6 Let $dX_t = b_t dt + dB_t$ where B is a Brownian motion and b a given bounded $\mathbf{F}^B$-adapted process. Let

$$L_t = \exp\left(-\int_0^t b_s dB_s - \frac{1}{2}\int_0^t b_s^2 ds\right) .$$

Show that L and LX are local martingales. (This will be used while dealing with Girsanov's theorem in ⤚ Section 1.7.) ◁

Exercise 1.5.3.7 Let X and Y be continuous semi-martingales. The Stratonovich integral of X w.r.t. Y may be defined as

$$\int_0^t X_s \circ dY_s = \int_0^t X_s dY_s + \frac{1}{2}\langle X, Y\rangle_t \,.$$

Prove that

$$\int_0^t X_s \circ dY_s = (ucp) \lim_{n\to\infty} \sum_{i=0}^{p(n)-1} \left(\frac{X_{t_i^n} + X_{t_{i+1}^n}}{2}\right)(Y_{t_{i+1}^n} - Y_{t_i^n})\,,$$

where $0 = t_0 < t_1^n < \cdots < t_{p(n)}^n = t$ is a subdivision of $[0, T]$ such that $\sup_i(t_{i+1}^n - t_i^n)$ goes to 0 when n goes to infinity. Prove that for $f \in C^3$, we have

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) \circ dX_s \,.$$

For a Brownian motion, the Stratonovich integral may also be approximated as

$$\int_0^t \varphi(B_s) \circ dB_s = \lim_{n\to\infty} \sum_{i=0}^{p(n)-1} \varphi(B_{(t_i+t_{i+1})/2})(B_{t_{i+1}} - B_{t_i})\,,$$

where the limit is in probability; however, such an approximation does not hold in general for continuous semi-martingales (see Yor [859]). See Stroock [811], page 226, for a discussion on the C^3 assumption on f in the integral form of $f(X_t)$. The Stratonovich integral can be extended to general semi-martingales (not necessarily continuous): see Protter [727], Chapter 5. $\triangleleft$

Exercise 1.5.3.8 Let B be a BM and $M_t^B := \sup_{s\le t} B_s$. Let $f(t, x, y)$ be a $C^{1,2,1}(\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+)$ function such that

$$\frac{1}{2} f_{xx} + f_t = 0$$
$$f_x(t, 0, y) + f_y(t, 0, y) = 0 \,.$$

Prove that $f(t, M_t^B - B_t, M_t^B)$ is a local martingale. In particular, for $h \in C^1$

$$h(M_t^B) - h'(M_t^B)(M_t^B - B_t)$$

is a local martingale. See Carraro et al. [157] and El Karoui and Meziou [304] for application to finance. $\triangleleft$

Exercise 1.5.3.9 (Kennedy Martingales.) Let $B_t^{(\mu)} := B_t + \mu t$ be a BM with drift μ and $M^{(\mu)}$ its running maximum, i.e., $M_t^{(\mu)} = \sup_{s\le t} B_s^{(\mu)}$. Let $R_t = M_t^{(\mu)} - B_t^{(\mu)}$ and $T_a = T_a(R) = \inf\{t : R_t \ge a\}$.

1. Set $\mu = 0$. Prove that, for any (α, β) the process

$$e^{-\alpha M_t - \frac{1}{2}\beta^2 t}\left(\cosh(\beta(M_t - B_t)) + \frac{\alpha}{\beta}\sinh(\beta(M_t - B_t))\right)$$

   is a martingale. Deduce that

$$\mathbb{E}\left(\exp\left(-\alpha M_{T_a} - \frac{1}{2}\beta^2 T_a\right)\right) = \beta\left(\beta\cosh\beta a + \alpha\sinh\beta a\right)^{-1} := \varphi(\alpha, \beta; a)\,.$$

2. For any μ, prove that

$$\mathbb{E}\left(\exp\left(-\alpha M^{(\mu)}_{T_a} - \frac{1}{2}\beta^2 T_a\right)\right) = e^{-\mu a}\varphi(\alpha_\mu, \beta_\mu; a)$$

   where $\alpha_\mu = \alpha - \mu, \beta_\mu = \sqrt{\beta^2 + \mu^2}$.

$\triangleleft$

Exercise 1.5.3.10 Let $(B^{(\mu)}_t, t \geq 0)$ be a Brownian motion with drift μ, and let b, c be real numbers. Define
$X_t = \exp(-cB^{(\mu)}_t)\left(x + \int_0^t \exp(bB^{(\mu)}_s)ds\right)$. Prove that

$$X_t = x - c\int_0^t X_s dB^{(\mu)}_s + \frac{c^2}{2}\int_0^t X_s ds + \int_0^t e^{(b-c)B^{(\mu)}_s} ds\,.$$

In particular, for $b = c$, X is a diffusion (see $\rightarrowtail$ Section 5.3) with infinitesimal generator

$$\frac{c^2}{2}x^2\partial_{xx} + \left[\left(\frac{c^2}{2} - c\mu\right)x + 1\right]\partial_x\,.$$

(See Donati-Martin et al. [258].) $\triangleleft$

Exercise 1.5.3.11 Let $B^{(\mu)}$ be as defined in Exercice 1.5.3.9 and let $M^{(\mu)}$ be its running maximum. Prove that, for $t < T$,

$$\mathbb{E}(M^{(\mu)}_T | \mathcal{F}_t) = M^{(\mu)}_t + \int_{M^{(\mu)}_t - B^{(\mu)}_t}^{\infty} G(T - t, u)\,du$$

where $G(T - t, u) = \mathbb{P}(M^{(\mu)}_{T-t} > u)$. $\triangleleft$

Exercise 1.5.3.12 Let $M_t = \int_0^t (X_s dY_s - Y_s dX_s)$ where X and Y are two real-valued independent Brownian motions. Prove that

$$M_t = \int_0^t \sqrt{X_s^2 + Y_s^2}\,dB_s$$

where B is a BM. Prove that

$$X_t^2 + Y_t^2 = 2\int_0^t (X_u dY_u + Y_u dX_u) + 2t$$
$$= 2\int_0^t \sqrt{X_u^2 + Y_u^2}\, d\beta_u + 2t$$

where β is a Brownian motion, with $d\langle B, \beta\rangle_t = 0$. ◁

1.5.4 Stochastic Differential Equations

We start with a general result ([RY], Chapter IX). Let $\mathcal{W} = C(\mathbb{R}^+, \mathbb{R}^d)$ be the space of continuous functions from $\mathbb{R}^+$ into $\mathbb{R}^d$, $w(s)$ the coordinate mappings and $\mathcal{B}_t = \sigma(w(s), s \le t)$. A function f defined on $\mathbb{R}^+ \times \mathcal{W}$ is said to be **predictable** if it is predictable as a process defined on $\mathcal{W}$ with respect to the filtration $(\mathcal{B}_t)$. If X is a continuous process defined on a probability space $(\Omega, \mathbf{F}, \mathbb{P})$, we write $f(t, X_\centerdot)$ for the value of f at time t on the path $t \to X_t(\omega)$. We emphasize that we write $X_\centerdot$ because $f(t, X_\centerdot)$ may depend on the path of X up to time t.

Definition 1.5.4.1 *Let g and f be two predictable functions on $\mathcal{W}$ taking values in the sets of $d \times n$ matrices and n-dimensional vectors, respectively. A solution of the stochastic differential equation* $\mathbf{e}$*(f,g) is a pair (X, B) of adapted processes on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration* $\mathbf{F}$ *such that:*

- *The n-dimensional process B is a standard* $\mathbf{F}$*-Brownian motion.*
- *For $i = 1, \dots, d$ and for any $t \in \mathbb{R}^+$*

$$X_t^i = X_0^i + \int_0^t f_i(s, X_\centerdot)ds + \sum_{j=0}^n \int_0^t g_{i,j}(s, X_\centerdot)dB_s^j\,. \qquad \mathbf{e}(f,g)$$

We shall also write this equation as

$$dX_t^i = f_i(t, X_\centerdot)dt + \sum_{j=0}^n g_{i,j}(t, X_\centerdot)dB_t^j\,.$$

Definition 1.5.4.2 *(1) There is* ***pathwise uniqueness*** *for* $\mathbf{e}$*(f,g) if whenever two pairs (X, B) and $(\widehat{X}, \widehat{B})$ are solutions defined on the same probability space with $B = \widehat{B}$ and $X_0 = \widehat{X}_0$, then X and $\widehat{X}$ are indistinguishable.*

(2) There is ***uniqueness in law*** *for* $\mathbf{e}$*(f,g) if whenever (X, B) and $(\widehat{X}, \widehat{B})$ are two pairs of solutions possibly defined on different probability spaces with $X_0 \overset{\text{law}}{=} \widehat{X}_0$, then $X \overset{\text{law}}{=} \widehat{X}$.*

(3) A solution (X, B) is said to be ***strong*** *if X is adapted to the filtration* $\mathbf{F}^B$*. A general solution is often called a* ***weak*** *solution, and if not strong, a* ***strictly weak*** *solution.*

Theorem 1.5.4.3 *Assume that f and g satisfy the Lipschitz condition, for a constant $K > 0$, which does not depend on t,*

$$\|f(t,w) - f(t,w')\| + \|g(t,w) - g(t,w')\| \leq K \sup_{s \leq t} \|w(s) - w'(s)\| .$$

Then, $\mathbf{e}$*(f,g) admits a unique strong solution, up to indistinguishability.*

See [RY], Chapter IX for a proof. The following theorem, due to Yamada and Watanabe (see also [RY] Chapter IX, Theorem 1.7) establishes a hierarchy between different uniqueness properties.

Theorem 1.5.4.4 *If pathwise uniqueness holds for* $\mathbf{e}$*(f,g), then uniqueness in law holds and the solution is strong.*

Example 1.5.4.5 Pathwise uniqueness is strictly stronger than uniqueness in law. For example, in the one-dimensional case, let $\sigma(x) = \text{sgn}(x)$, with $\text{sgn}(0) = -1$. Any solution (X, B) of $\mathbf{e}(0, \sigma)$ (meaning that $g(t, X_\centerdot) = \sigma(X_t)$) starting from 0 is a standard BM, thus uniqueness in law holds. On the other hand, if β is a BM, and $B_t = \int_0^t \text{sgn}(\beta_s) d\beta_s$, then (β, B) and $(-\beta, B)$ are two solutions of $\mathbf{e}(0, \sigma)$ (indeed, $dB_t = \sigma(\beta_t) d\beta_t$ is equivalent to $d\beta_t = \sigma(\beta_t) dB_t$), and pathwise uniqueness does not hold. If (X, B) is any solution of $\mathbf{e}(0, \sigma)$, then $B_t = \int_0^t \text{sgn}(X_s) dX_s$, and $\mathbf{F}^B = \mathbf{F}^{|X|}$ which establishes that any solution is strictly weak (see ↣ Comments 4.1.7.9 and ↣ Subsection 5.8.2 for the study of the filtrations).

A simple case is the following:

Theorem 1.5.4.6 *Let $b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : [0, T] \times \mathbb{R}^d \to \mathbb{R}^{d \times n}$ be Borel functions satisfying*

$$\|b(t,x)\| + \|\sigma(t,x)\| \leq C(1 + \|x\|) \, , x \in \mathbb{R}^d, t \in [0, T],$$
$$\|b(t,x) - b(t,y)\| + \|\sigma(t,x) - \sigma(t,y)\| \leq C\|x - y\| \, , x, y \in \mathbb{R}^d, t \in [0, T]$$

and let X_0 be a square integrable r.v. independent of the n-dimensional Brownian motion B. Then, the stochastic differential equation (SDE)

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, \; t \leq T, \; X_0 = x$$

has a unique continuous strong solution, up to indistinguishability. Moreover, this process is a strong (inhomogeneous) Markov process.

Sketch of the Proof: The proof relies on Picard's iteration procedure.
▶ In a first step, one considers the mapping $Z \to K(Z)$ where

$$(K(Z))_t = x + \int_0^t b(s, Z_s)ds + \int_0^t \sigma(s, Z_s)dB_s \, ,$$

and one defines a sequence $(X^n)_{n=0}^\infty$ of processes by setting $X^0 = x$, and $X^n = K(X^{n-1})$. Then, one proves that

$$\mathbb{E}\left(\sup_{s\le t}(X_s^n - X_s^{n-1})^2\right) \le kc^n \frac{t^n}{n!}$$

where k, c are constants. This proves the existence of a solution.
▶ In a second step, one establishes the uniqueness by use of Gronwall's lemma. See [RY], Chapter IV for details. □

The solution depends continuously on the initial value.

Example 1.5.4.7 Geometric Brownian Motion. If B is a Brownian motion and μ, σ are two real numbers, the solution S of

$$dS_t = S_t(\mu dt + \sigma dB_t)$$

is called a geometric Brownian motion with parameters μ and σ. The process S will often be written in this book as

$$S_t = S_0 \exp(\mu t + \sigma B_t - \sigma^2 t/2) = S_0 \exp(\sigma X_t) \tag{1.5.3}$$

where

$$X_t = \nu t + B_t,\ \nu = \frac{\mu}{\sigma} - \frac{\sigma}{2}\,. \tag{1.5.4}$$

The process $(S_t e^{-\mu t}, t \ge 0)$ is a martingale. The Markov property of S may be seen from the equality

$$S_t = S_s \exp(\mu(t-s) + \sigma(B_t - B_s) - \sigma^2(t-s)/2),\ t > s\,.$$

Let s be fixed. The process $Y_u = \exp(\mu u + \sigma \widehat{B}_u - \sigma^2 u/2), u \ge 0)$ where $\widehat{B}_u = B_{s+u} - B_s$ is independent of $\mathcal{F}_s^S$ and has the same law as S_u/S_0. Moreover, the decomposition $S_t = S_s Y_{t-s}$, for $t > s$ where Y is independent of $\mathcal{F}_s^S$ and has the same law as S/S_0 will be of frequent use.

Example 1.5.4.8 Affine Coefficients: Method of Variation of Constants. The solution of

$$dX_t = (a(t)X_t + b(t))dt + (c(t)X_t + f(t))dB_t,\ X_0 = x$$

where a, b, c, f are (bounded) Borel functions is $X = YZ$ where Y is the solution of

$$dY_t = Y_t[a(t)dt + c(t)dB_t],\ Y_0 = 1$$

and

$$Z_t = x + \int_0^t Y_s^{-1}[b(s) - c(s)f(s)]ds + \int_0^t Y_s^{-1} f(s) dB_s\,.$$

Note that one can write Y in a closed form as

$$Y_t = \exp\left(\int_0^t a(s)ds + \int_0^t c(s)dB_s - \frac{1}{2}\int_0^t c^2(s)ds\right)$$

Remark 1.5.4.9 Under Lipschitz conditions on the coefficients, the solution of

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t,\ t \le T,\ X_0 = x \in \mathbb{R}$$

is a homogeneous Markov process. More generally, under the conditions of Theorem 1.5.4.6, the solution of

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t,\ t \le T,\ X_0 = x \in \mathbb{R}$$

is an inhomogeneous Markov process. The pair (X_t, t) is a homogeneous Markov process.

Definition 1.5.4.10 (Explosion Time.) *Suppose that X is a solution of an SDE with locally Lipschitz coefficients. Then, a localisation argument allows to define unambiguously, for every n, $(X_t, t \le \tau_n)$, when τ_n is the first exit time from $[-n, n]$. Let $\tau = \sup \tau_n$. When $\tau < \infty$, we say that X explodes at time τ.*

If the functions $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^n$ are continuous, the SDE

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t \tag{1.5.5}$$

admits a weak solution up to its explosion time.

Under the regularity assumptions

$$\|\sigma(x) - \sigma(y)\|^2 \le C|x-y|^2\, r(|x-y|^2),\ \text{ for } |x-y| < 1$$
$$|b(x) - b(y)| \le C|x-y|\, r(|x-y|^2),\ \text{ for } |x-y| < 1\,,$$

where $r :]0,1[\to \mathbb{R}^+$ is a C^1 function satisfying

(i) $\lim_{x\to 0} r(x) = +\infty$,

(ii) $\lim_{x\to 0} \dfrac{xr'(x)}{r(x)} = 0$,

(iii) $\displaystyle\int_0^a \frac{ds}{sr(s)} = +\infty$, for any $a > 0$,

Fang and Zhang [340, 341] have established the pathwise uniqueness of the solution of the equation (1.5.5).

If, for $|x| \ge 1$,

$$\|\sigma(x)\|^2 \le C\,(|x|^2\, \rho(|x|^2) + 1)$$
$$|b(x)| \le C\,(|x|\, \rho(|x|^2) + 1)$$

for a function ρ of class C^1 satisfying

(i) $\lim_{x\to\infty} \rho(x) = +\infty\,,$

(ii) $\lim_{x\to\infty} \dfrac{x\rho'(x)}{\rho(x)} = 0\,,$

(iii) $\displaystyle\int_1^\infty \frac{ds}{s\rho(s)+1} = +\infty$,

then, the solution of the equation (1.5.5) does not explode.

1.5.5 Stochastic Differential Equations: The One-dimensional Case

In the case of dimension one, the following result requires less regularity for the existence of a solution of the equation

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s\,. \tag{1.5.6}$$

Theorem 1.5.5.1 *Suppose* $\varphi\ :]0,\infty[\to]0,\infty[$ *is a Borel function such that* $\int_{0+} da/\varphi(a) = +\infty$.
Under any of the following conditions:

(i) *the Borel function* b *is bounded, the function* σ *does not depend on the time variable and satisfies*

$$|\sigma(x) - \sigma(y)|^2 \leq \varphi(|x - y|)$$

and $|\sigma| \geq \epsilon > 0$ *,*

(ii) $|\sigma(s,x) - \sigma(s,y)|^2 \leq \varphi(|x - y|)$ *and* b *is Lipschitz continuous,*

(iii) *the function* σ *does not depend on the time variable and satisfies*

$$|\sigma(x) - \sigma(y)|^2 \leq |f(x) - f(y)|$$

where f *is a bounded increasing function,* $\sigma \geq \epsilon > 0$ *and* b *is bounded,*

the equation (1.5.6) admits a unique solution which is strong, and the solution X *is a Markov process.*

See [RY], Chapter IV, Section 3 for a proof. Let us remark that condition (iii) on σ holds in particular if σ is bounded below and has bounded variation: indeed

$$|\sigma(x) - \sigma(y)|^2 \leq V|\sigma(x) - \sigma(y)| \leq V|f(x) - f(y)|$$

with $V = \int |d\sigma|$ and $f(x) = \int_{-\infty}^x |d\sigma(y)|$.

The existence of a solution for $\sigma(x) = \sqrt{|x|}$ and more generally for the case $\sigma(x) = |x|^\alpha$ with $\alpha \geq 1/2$ can be proved using $\varphi(a) = ca$. For $\alpha \in [0, 1/2[$, pathwise uniqueness does not hold, see Girsanov [394], McKean [637], Jacod and Yor [472].

This criterion does not extend to higher dimensions. As an example, let Z be a complex valued Brownian motion. It satisfies

$$Z_t^2 = 2\int_0^t Z_s dZ_s = 2\int_0^t |Z_s| d\gamma_s$$

where $\gamma_t = \displaystyle\int_0^t \frac{Z_s dZ_s}{|Z_s|}$ is a $\mathbb{C}$-valued Brownian motion (see also $\rightarrowtail$ Subsection 5.1.3). Now, the equation $\zeta_t = 2\int_0^t \sqrt{|\zeta_s|}d\gamma_s$ where γ is a Brownian motion admits at least two solutions: the constant process 0 and the process Z.

Comment 1.5.5.2 The proof of (iii) was given in the homogeneous case, using time change and Cameron-Martin's theorem, by Nakao [666] and was improved by LeGall [566]. Other interesting results are proved in Barlow and Perkins [49], Barlow [46], Brossard [132] and Le Gall [566].

The reader will find in $\rightarrowtail$ Subsection 5.5.2 other results about existence and uniqueness of stochastic differential equations.

It is useful (and sometimes unavoidable!) to allow solutions to explode. We introduce an absorbing state δ so that the processes are $\mathbb{R}^d \cup \delta$-valued. Let τ be the explosion time (see Definition 1.5.4.10) and set $X_t = \delta$ for $t > \tau$.

Proposition 1.5.5.3 *Equation* $\mathbf{e}(f, g)$ *has no exploding solution if*

$$\sup_{s \le t} |f(s, x_{\bullet})| + \sup_{s \le t} |g(s, x_{\bullet})| \le c(1 + \sup_{s \le t} |x_{\bullet}|) .$$

PROOF: See Kallenberg [505] and Stroock and Varadhan [812]. □

Example 1.5.5.4 Zvonkin's Argument. The equation

$$dX_t = dB_t + b(X_t)dt$$

where b is a bounded Borel function has a solution. Indeed, assume that there is a solution and let $Y_t = h(X_t)$ where h satisfies $\frac{1}{2}h''(x) + b(x)h'(x) = 0$ (so h is of the form

$$h(x) = C \int_0^x dy \, \exp(-2\widehat{b}(y)) + D$$

where $\widehat{b}$ is an antiderivative of b, hence h is strictly monotone). Then

$$Y_t = h(x) + \int_0^t h'(h^{-1}(Y_s))dB_s .$$

Since $h' \circ h^{-1}$ is Lipschitz, Y exists, hence X exists. The law of X is

$$\mathbb{P}_x^{(b)}|_{\mathcal{F}_t} = \exp\left(\int_0^t b(X_s)dX_s - \frac{1}{2}\int_0^t b^2(X_s)ds\right) \mathbf{W}_x|_{\mathcal{F}_t} .$$

In a series of papers, Engelbert and Schmidt [331, 332, 333] prove results concerning existence and uniqueness of solutions of

$$X_t = x + \int_0^t \sigma(X_s)dB_s$$

that we recall now (see Cherny and Engelbert [168], Karatzas and Shreve [513] p. 332, or Kallenberg [505]). Let

$$N_\sigma = \{x \in \mathbb{R} : \sigma(x) = 0\}$$
$$I_\sigma = \{x \in \mathbb{R} : \int_{-a}^{a} \sigma^{-2}(x+y)dy = +\infty, \forall a > 0\}.$$

The condition $I_\sigma \subset N_\sigma$ is necessary and sufficient for the existence of a solution for arbitrary initial value, and $N_\sigma \subset I_\sigma$ is sufficient for uniqueness in law of solutions. These results are generalized to the case of SDE with drift by Rutkowski [751].

Example 1.5.5.5 The equation

$$dX_t = \frac{1}{2}X_t dt + \sqrt{1+X_t^2}\,dB_t, \quad X_0 = 0$$

admits the unique solution $X_t = \sinh(B_t)$. Indeed, it suffices to note that, setting $\varphi(x) = \sinh(x)$, one has $d\varphi(B_t) = b(X_t)dt + \sigma(X_t)dW_t$ where

$$\sigma(x) = \varphi'(\varphi^{-1}(x)) = \sqrt{1+x^2},\ b(x) = \frac{1}{2}\varphi''(\varphi^{-1}(x)) = \frac{x}{2}. \tag{1.5.7}$$

More generally, if φ is a strictly increasing, C^2 function, which satisfies $\varphi(-\infty) = -\infty, \varphi(\infty) = \infty$, the process $Z_t = \varphi(B_t)$ is a solution of

$$Z_t = Z_0 + \int_0^t \varphi' \circ \varphi^{-1}(Z_s)dB_s + \frac{1}{2}\int_0^t \varphi'' \circ \varphi^{-1}(Z_s)ds.$$

One can characterize more explicitly SDEs of this form. Indeed, we can check that

$$dZ_t = b(Z_t)dt + \sigma(Z_t)dB_t$$

where

$$b(z) = \frac{1}{2}\sigma(z)\sigma'(z). \tag{1.5.8}$$

Example 1.5.5.6 Tsirel'son's Example. Let us give Tsirel'son's example [822] of an equation with diffusion coefficient equal to one, for which there is no strong solution, as an SDE of the form $dX_t = f(t, X_\centerdot)dt + dB_t$. Introduce the bounded function T on path space as follows: let $(t_i, i \in -\mathbb{N})$ be a sequence of positive reals which decrease to 0 as i decreases to $-\infty$. Let

$$T(s, X_\centerdot) = \sum_{k \in -\mathbb{N}^*} \left[\!\left[\frac{X_{t_k} - X_{t_{k-1}}}{t_k - t_{k-1}} \right]\!\right] \mathbb{1}_{]t_k, t_{k+1}]}(s).$$

Here, $[[x]]$ is the fractional part of x. Then, the equation $e(T, 1)$ has no strong solution because, for each fixed k, $\left[\!\left[\frac{X_{t_k} - X_{t_{k-1}}}{t_k - t_{k-1}} \right]\!\right]$ is independent of B, and uniformly distributed on $[0, 1]$. Thus Zvonkin's result does not extend to the case where the coefficients depend on the past of the process. See Le Gall and Yor [568] for further examples.

Example 1.5.5.7 Some stochastic differential equations of the form

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t$$

can be reduced to an SDE with affine coefficients (see Example 1.5.4.8) of the form

$$dY_t = (a(t)Y_t + b(t))dt + (c(t)Y_t + f(t))dW_t\,,$$

by a change of variable $Y_t = U(t, X_t)$. Many examples are provided in Kloeden and Platen [524]. For example, the SDE

$$dX_t = -\frac{1}{2}\exp(-2X_t)dt + \exp(-X_t)dW_t$$

can be transformed (with $U(x) = e^x$) to $dY_t = dW_t$. Hence, the solution is $X_t = \ln(W_t + e^{X_0})$ up to the explosion time $\inf\{t : W_t + e^{X_0} = 0\}$.

Flows of SDE

Here, we present some results on the important topic of the stochastic flow associated with the initial condition.

Proposition 1.5.5.8 *Let*

$$X_t^x = x + \int_0^t b(s, X_s^x)ds + \int_0^t \sigma(s, X_s^x)dW_s$$

and assume that the functions b and σ are globally Lipschitz and have locally Lipschitz first partial derivatives. Then, the explosion time is equal to ∞. Furthermore, the solution is continuously differentiable w.r.t. the initial value, and the process $Y_t = \partial_x X_t$ satisfies

$$Y_t = 1 + \int_0^t Y_s\,\partial_x b(s, X_s^x)ds + \int_0^t Y_s \partial_x \sigma(s, X_s^x)dW_s\,.$$

We refer to Kunita [547, 548] or Protter, Chapter V [727] for a proof.

SDE with Coefficients Depending of a Parameter

We assume that $b(t, x, a)$ and $\sigma(t, x, a)$, defined on $\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}$, are C^2 with respect to the two last variables x, a, with bounded derivatives of first and second order.

Let

$$X_t = x + \int_0^t b(s, X_s, a)ds + \int_0^t \sigma(s, X_s, a)dW_s$$

and $Z_t = \partial_a X_t$. Then,

$$\begin{aligned} Z_t = & \int_0^t \left(\partial_a b(s, X_s, a) + Z_s \partial_x b(s, X_s, a)\right) ds \\ & + \int_0^t \left(\partial_a \sigma(s, X_s, a) + Z_s \partial_x \sigma(s, X_s, a)\right) dW_s\,. \end{aligned}$$

See Métivier [645].

Comparison Theorem

We conclude this paragraph with a comparison theorem.

Theorem 1.5.5.9 (Comparison Theorem.) *Let*

$$dX_i(t) = b_i(t, X_i(t))dt + \sigma(t, X_i(t))dW_t\,,\ i = 1, 2$$

where $b_i, i = 1, 2$ are bounded Borel functions and at least one of them is Lipschitz and σ satisfies (ii) or (iii) of Theorem 1.5.5.1. Suppose also that $X_1(0) \geq X_2(0)$ and $b_1(x) \geq b_2(x)$. Then $X_1(t) \geq X_2(t)\,, \forall t, a.s.$

PROOF: See [RY], Chapter IX, Section 3. □

Exercise 1.5.5.10 Consider the equation $dX_t = \mathbb{1}_{\{X_t \geq 0\}} dB_t$. Prove (in a direct way) that this equation has no solution starting from 0. Prove that the equation $dX_t = \mathbb{1}_{\{X_t > 0\}} dB_t$ has a solution.
`Hint:` For the first part, one can consider a smooth function f vanishing on $\mathbb{R}^+$. From Itô's formula, it follows that X remains positive, and the contradiction is obtained from the remark that X is a martingale. ◁

Comment 1.5.5.11 Doss and Süssmann Method. Let σ be a C^2-function with bounded derivatives of the first two orders, and let b be Lipschitz continuous. Let h be the solution of the ODE

$$\frac{\partial h}{\partial t}(x, t) = \sigma(h(x, t)),\ h(x, 0) = x\,.$$

Let X be a continuous semi-martingale which vanishes at time 0 and let D be the solution of the ODE

$$\frac{dD_t}{dt} = b(h(D_t, X_t(\omega)))\,\exp\left\{-\int_0^{X_t(\omega)} \sigma'(h(D_s, s))ds\right\},\ D_0 = y\,.$$

Then, $Y_t = h(D_t, X_t)$ is the unique solution of

$$Y_t = y + \int_0^t \sigma(Y_s) \circ dX_s + \int_0^t b(Y_s)ds$$

where $\circ$ stands for the Stratonovich integral (see Exercise 1.5.3.7). See Doss [261] and Süssmann [815].

1.5.6 Partial Differential Equations

We now give an important relation between two problems: to compute the (conditional) expectation of a function of the terminal value of the solution of an SDE and to solve a second-order PDE with boundary conditions.

Proposition 1.5.6.1 *Let $\mathcal{A}$ be the second-order operator defined on $C^{1,2}$ functions by*

$$\mathcal{A}(\varphi)(t,x) = \frac{\partial \varphi}{\partial t}(t,x) + b(t,x)\frac{\partial \varphi}{\partial x}(t,x) + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 \varphi}{\partial x^2}(t,x)\,.$$

Let X be the diffusion (see $\rightarrowtail$ Section 5.3)

$$dX_t = b(t,X_t)dt + \sigma(t,X_t)dW_t\,.$$

We assume that this equation admits a unique solution. Then, for $f \in C_b(\mathbb{R})$ the bounded solution to the Cauchy problem

$$\mathcal{A}\varphi = 0,\ \varphi(T,x) = f(x)\,, \tag{1.5.9}$$

is given by

$$\varphi(t,x) = \mathbb{E}(f(X_T)|X_t = x)\,.$$

Conversely, if $\varphi(t,x) = \mathbb{E}(f(X_T)|X_t = x)$ is $C^{1,2}$, then it solves (1.5.9).

PROOF: From the Markov property of X, the process

$$\varphi(t,X_t) = \mathbb{E}(f(X_T)|X_t) = \mathbb{E}(f(X_T)|\mathcal{F}_t)\,,$$

is a martingale. Hence, its bounded variation part is equal to 0. From (1.5.2), assuming that $\varphi \in C^{1,2}$,

$$\partial_t \varphi + b(t,x)\partial_x \varphi + \frac{1}{2}\sigma^2(t,x)\partial_{xx}\varphi = 0\,.$$

The smoothness of φ is established from general results on diffusions under suitable conditions on b and σ (see Kallenberg [505], Theorem 17-6 and Durrett [286]). □

Exercise 1.5.6.2 Let $dX_t = rX_t dt + \sigma(X_t)dW_t$, Ψ a bounded continuous function and $\psi(t,x) = \mathbb{E}(e^{-r(T-t)}\Psi(X_T)|X_t = x)$. Assuming that ψ is $C^{1,2}$, prove that

$$\partial_t \psi + rx\partial_x \psi + \frac{1}{2}\sigma^2(x)\partial_{xx}\psi = r\psi,\ \psi(T,x) = \Psi(x)\,. \qquad \triangleleft$$

1.5.7 Doléans-Dade Exponential

Let M be a continuous local martingale. For any $\lambda \in \mathbb{R}$, the process

$$\mathcal{E}(\lambda M)_t := \exp\left(\lambda M_t - \frac{\lambda^2}{2}\langle M \rangle_t\right)$$

is a positive local martingale (hence, a super-martingale), called the **Doléans-Dade exponential** of λM (or, sometimes, the stochastic exponential of λM). It is a martingale if and only if $\forall t,\ \mathbb{E}(\mathcal{E}(\lambda M)_t) = 1$.

If $\lambda \in L^2(M)$, the process $\mathcal{E}(\lambda M)$ is the unique solution of the stochastic differential equation

$$dY_t = Y_t\, \lambda_t dM_t,\ Y_0 = 1\,.$$

This definition admits an extension to semi-martingales as follows. If X is a continuous semi-martingale vanishing at 0, the **Doléans-Dade exponential** of X is the unique solution of the equation

$$Z_t = 1 + \int_0^t Z_s dX_s\,.$$

It is given by

$$\mathcal{E}(X)_t := \exp\left(X_t - \frac{1}{2}\langle X\rangle_t\right)\,.$$

Let us remark that in general $\mathcal{E}(\lambda M)\,\mathcal{E}(\mu M)$ is not equal to $\mathcal{E}((\lambda+\mu)M)$. In fact, the general formula

$$\mathcal{E}(X)_t\,\mathcal{E}(Y)_t = \mathcal{E}(X + Y + \langle X, Y\rangle)_t \tag{1.5.10}$$

leads to

$$\mathcal{E}(\lambda M)_t \mathcal{E}(\mu M)_t = \mathcal{E}((\lambda+\mu)M + \lambda\mu\langle M\rangle)_t\,,$$

hence, the product of the exponential local martingales $\mathcal{E}(M)\mathcal{E}(N)$ is a local martingale if and only if the local martingales M and N are orthogonal.

Example 1.5.7.1 For later use (see ↣ Proposition 2.6.4.1) we present the following computation. Let f and g be two continuous functions and W a Brownian motion starting from x at time 0. The process

$$Z_t = \exp\left(\int_0^t [f(s)W_s + g(s)]dW_s - \frac{1}{2}\int_0^t [f(s)W_s + g(s)]^2 ds\right)$$

is a local martingale. Using ↣ Proposition 1.7.6.4, it can be proved that it is a martingale, therefore its expectation is equal to 1. It follows that

$$\mathbb{E}\left(\exp\left[\int_0^t [f(s)W_s + g(s)]dW_s - \frac{1}{2}\int_0^t [f^2(s)W_s^2 + 2W_s f(s)g(s)]ds\right]\right)$$
$$= \exp\left(\frac{1}{2}\int_0^t g^2(s)ds\right)\,.$$

If moreover f and g are C^1, integration by parts yields

$$\int_0^t g(s)dW_s = g(t)W_t - g(0)W_0 - \int_0^t g'(s)W_s ds$$
$$\int_0^t f(s)W_s dW_s = \frac{1}{2}\left(W_t^2 f(t) - W_0^2 f(0) - \int_0^t f(s)ds - \int_0^t f'(s)W_s^2 ds\right)\,,$$

therefore,

$$\mathbb{E}\Bigg[\exp\Bigg(g(t)W_t + \frac{1}{2}f(t)W_t^2 \\ - \frac{1}{2}\int_0^t \left([f^2(s)+f'(s)]W_s^2 + 2W_s(f(s)g(s)+g'(s))\right) ds\Bigg)\Bigg]$$

$$= \exp\left(g(0)W_0 + \frac{1}{2}\left(f(0)W_0^2 + \int_0^t f(s)ds + \int_0^t g^2(s)ds\right)\right).$$

Exercise 1.5.7.2 Check formula (1.5.10), by showing, e.g., that both sides satisfy the same linear SDE. ◁

Exercise 1.5.7.3 Let H and Z be continuous semi-martingales. Check that the solution of the equation $X_t = H_t + \int_0^t X_s dZ_s$, is

$$X_t = \mathcal{E}(Z)_t \left(H_0 + \int_0^t \frac{1}{\mathcal{E}(Z)_s}(dH_s - d\langle H, Z\rangle_s)\right).$$

See Protter [727], Chapter V, Section 9, for the case where H, Z are general semi-martingales. ◁

Exercise 1.5.7.4 Prove that if θ is a bounded function, then the process $(\mathcal{E}(\theta\star W)_t, t \leq T)$ is a u.i. martingale.
`Hint:`

$$\exp\left(\int_0^t \theta_s dW_s - \frac{1}{2}\int_0^t \theta_s^2 ds\right) \leq \exp\left(\sup_{t\leq T}\int_0^t \theta_s dW_s\right) = \exp \widehat{\beta}_{\int_0^T \theta_s^2 ds}$$

with $\widehat{\beta}_t = \sup_{u\leq t}\beta_u$ where β is a BM. ◁

Exercise 1.5.7.5 Multiplicative Decomposition of Positive Sub-martingales. Let $X = M + A$ be the Doob-Meyer decomposition of a strictly positive continuous sub-martingale. Let Y be the solution of

$$dY_t = Y_t \frac{1}{X_t} dM_t, Y_0 = X_0$$

and let Z be the solution of $dZ_t = -Z_t \frac{1}{X_t} dA_t, Z_0 = 1$. Prove that $U = Y/Z$ satisfies $dU_t = U_t \frac{1}{X_t} dX_t$ and deduce that $U = X$.
`Hint:` Use that the solution of $dU_t = U_t \frac{1}{X_t} dX_t$ is unique. See Meyer and Yoeurp [649] and Meyer [647] for a generalization to discontinuous sub-martingales. Note that this decomposition states that a strictly positive continuous sub-martingale is the product of a martingale and an increasing process. ◁

1.6 Predictable Representation Property

1.6.1 Brownian Motion Case

Let W be a real-valued Brownian motion and $\mathbf{F}^W$ its natural filtration. We recall that the space $L^2(W)$ was presented in Definition 1.3.1.3.

Theorem 1.6.1.1 *Let $(M_t, t \geq 0)$ be a square integrable $\mathbf{F}^W$-martingale (i.e., $\sup_t \mathbb{E}(M_t^2) < \infty$). There exists a constant μ and a unique predictable process m in $L^2(W)$ such that*

$$\forall t, \quad M_t = \mu + \int_0^t m_s \, dW_s \, .$$

If M is an $\mathbf{F}^W$-local martingale, there exists a unique predictable process m in $L^2_{loc}(W)$ such that

$$\forall t, \quad M_t = \mu + \int_0^t m_s \, dW_s \, .$$

PROOF: The first step is to prove that for any square integrable $\mathcal{F}^W_\infty$-measurable random variable F, there exists a unique predictable process H such that

$$F = \mathbb{E}(F) + \int_0^\infty H_s \, dW_s \, , \tag{1.6.1}$$

and $\mathbb{E}[\int_0^\infty H_s^2 ds] < \infty$. Indeed, the space of random variables F of the form (1.6.1) is closed in L^2. Moreover, it contains any random variable of the form

$$F = \exp\left(\int_0^\infty f(s)dW_s - \frac{1}{2}\int_0^\infty f(s)^2 ds\right)$$

with $f = \sum_i \lambda_i \mathbb{1}_{]t_{i-1},t_i]}$, $\lambda_i \in \mathbb{R}^d$, and this space is total in L^2. Then density arguments complete the proof. See [RY], Chapter V, for details. □

Example 1.6.1.2 A special case of Theorem 1.6.1.1 is when $M_t = f(t, W_t)$ where f is a smooth function (hence, f is space-time harmonic, i.e., it satisfies $\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2} = 0$). In that case, Itô's formula leads to $m_s = \partial_x f(s, W_s)$.

This theorem holds in the multidimensional Brownian setting. Let W be a n-dimensional BM and M be a square integrable $\mathbf{F}^W$-martingale. There exists a constant μ and a unique n-dimensional predictable process m in $L^2(W)$ such that

$$\forall t, \quad M_t = \mu + \sum_{i=1}^n \int_0^t m_s^i \, dW_s^i \, .$$

Corollary 1.6.1.3 *Every $\mathbf{F}^W$-local martingale admits a continuous version.*

As a consequence, every optional process in a Brownian filtration is predictable.
From now on, we shall abuse language and say that every $\mathbf{F}^W$-local martingale is continuous.

Corollary 1.6.1.4 *Let W be a $\mathbf{G}$-Brownian motion with natural filtration $\mathbf{F}$. Then, for every square integrable $\mathbf{G}$-adapted process φ,*

$$\mathbb{E}\left(\int_0^t \varphi_s dW_s | \mathcal{F}_t\right) = \int_0^t \mathbb{E}(\varphi_s|\mathcal{F}_s) dW_s \, ,$$

where $\mathbb{E}(\varphi_s|\mathcal{F}_s)$ denotes the predictable version of the conditional expectation.

PROOF: Since the r.v. $\int_0^t \mathbb{E}(\varphi_s|\mathcal{F}_s)dW_s$ is $\mathcal{F}_t$-measurable, it suffices to check that, for any bounded r.v. $F_t \in \mathcal{F}_t$

$$\mathbb{E}\left(F_t \int_0^t \varphi_s dW_s\right) = \mathbb{E}\left(F_t \int_0^t \mathbb{E}(\varphi_s|\mathcal{F}_s)dW_s\right).$$

The predictable representation theorem implies that $F_t = \mathbb{E}(F_t) + \int_0^t f_s dW_s$, for some $\mathbf{F}$-predictable process $f \in L^2(W)$, hence

$$\begin{aligned} \mathbb{E}\left(F_t \int_0^t \varphi_s dW_s\right) &= \mathbb{E}\left(\int_0^t f_s \varphi_s ds\right) = \int_0^t \mathbb{E}(f_s \varphi_s) ds \\ &= \int_0^t \mathbb{E}(f_s \mathbb{E}(\varphi_s|\mathcal{F}_s)) ds = \mathbb{E}\left(\int_0^t f_s \mathbb{E}(\varphi_s|\mathcal{F}_s) ds\right) \\ &= \mathbb{E}\left(\left\{\mathbb{E}(F_t) + \int_0^t f_s dW_s\right\} \int_0^t \mathbb{E}(\varphi_s|\mathcal{F}_s) dW_s\right) \, , \end{aligned}$$

which ends the proof. □

Example 1.6.1.5 If $F = \int_0^\infty ds\, h(s, W_s)$ where $\int_0^\infty ds\, \mathbb{E}(|h(s, W_s)|) < \infty$, then from the Markov property, $M_t = \mathbb{E}(F|\mathcal{F}_t) = \int_0^t ds\, h(s, W_s) + \varphi(t, W_t)$, for some function φ. Assuming that φ is smooth, the martingale property of M and Itô's formula lead to

$$h(t, W_t) + \partial_t \varphi(t, W_t) + \frac{1}{2} \partial_{xx} \varphi(t, W_t) = 0$$

and $M_t = \varphi(0,0) + \int_0^t \partial_x \varphi(s, W_s) dW_s$. See the papers of Graversen et al. [405] and Shiryaev and Yor [793] for some examples of functionals of the Brownian motion which are explicitly written as stochastic integrals.

Proposition 1.6.1.6 *Let $M_t = \mathbb{E}(f(W_T)|\mathcal{F}_t)$, for $t \le T$ where f is a C_b^1 function. Then,*

$$M_t = \mathbb{E}(f(W_T)) + \int_0^t \mathbb{E}(f'(W_T)|\mathcal{F}_s) dW_s = \mathbb{E}(f(W_T)) + \int_0^t P_{T-s}(f')(W_s) dW_s \, .$$

PROOF: From the independence and stationarity of the increments of the Brownian motion,

$$\mathbb{E}(f(W_T)|\mathcal{F}_t) = \psi(t, W_t)$$

where $\psi(t,x) = \mathbb{E}(f(x + W_{T-t}))$. Itô's formula and the martingale property of $\psi(t, W_t)$ lead to

$$\partial_x \psi(t,x) = \mathbb{E}(f'(x + W_{T-t})) = \mathbb{E}(f'(W_T)|W_t = x)\,.$$

□

Comment 1.6.1.7 In a more general setting, one can use Malliavin's derivative. For T fixed, and $h \in L^2([0,T])$, we define $W(h) = \int_0^T h(s)dW_s$. Let $F = f(W(h_1), \ldots, W(h_n))$ where f is a smooth function. The derivative of F is defined as the process $(D_t F, t \leq T)$ by

$$D_t F = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(W(h_1), \ldots, W(h_n)) h_i(t)\,.$$

The Clark-Ocone representation formula states that for random variables which satisfy some suitable integrability conditions,

$$F = \mathbb{E}(F) + \int_0^T \mathbb{E}(D_t F|\mathcal{F}_t) dW_t\,.$$

We refer the reader to the books of Nualart [681] for a study of Malliavin calculus and of Malliavin and Thalmaier [616] for applications in finance. See also the issue [560] of *Mathematical Finance* devoted to applications to finance of Malliavin calculus.

Exercise 1.6.1.8 Let $W = (W^1, \ldots, W^d)$ be a d-dimensional BM. Is the space of martingales $\sum_{i=1}^{d} \int_0^t H_i(W^i_\cdot)_s dW^i_s$ dense in the space of square integrable martingales?
`Hint:` The answer is negative. Look for $Y \in L^2(\mathcal{W}_\infty)$ such that Y is orthogonal to all these variables. ◁

1.6.2 Towards a General Definition of the Predictable Representation Property

Besides the Predictable Representation Property (PRP) of Brownian motion, let us recall the Kunita-Watanabe orthogonal decomposition of a martingale M with respect to another one X:

Lemma 1.6.2.1 ***(Kunita-Watanabe Decomposition.)*** *Let X be a given continuous local $\mathbf{F}$-martingale. Then, every continuous $\mathbf{F}$-local martingale M vanishing at 0 may be uniquely written*

$$M = H \star X + N \tag{1.6.2}$$

where H is predictable and N is a local martingale orthogonal to X.

Referring to the Brownian motion case (previous subsection), one may wonder for which local martingales X it is true that every N in (1.6.2) is a constant. This leads us to the following definition.

Definition 1.6.2.2 *A continuous local martingale X enjoys the* ***predictable representation property*** *(PRP) if for any $\mathbf{F}^X$-local martingale $(M_t, t \geq 0)$, there is a constant m and an $\mathbf{F}^X$-predictable process $(m_s, s \geq 0)$ such that*

$$M_t = m + \int_0^t m_s dX_s,\ t \geq 0.$$

Exercise 1.6.2.3 Prove that $(m_s, s \geq 0)$ is unique in $L^2_{loc}(X)$. ◁

More generally, a continuous $\mathbf{F}$-local martingale X enjoys the $\mathbf{F}$-predictable representation property if any $\mathbf{F}$-adapted martingale M can be written as $M_t = m + \int_0^t m_s\, dX_s$, with $\int_0^t m_s^2 d\langle X\rangle_s < \infty$. We do not require in that last definition that $\mathbf{F}$ is the natural filtration of X. (See an important example in ↣ Subsection 1.7.7.)

We now look for a characterization of martingales that enjoy the PRP. Given a continuous $\mathbf{F}$-adapted process Y, we denote by $\mathcal{M}(Y)$ the subset of probability measures $\mathbb{Q}$ on $(\Omega, \mathbf{F})$, for which the process Y is a $(\mathbb{Q}, \mathbf{F})$-local martingale. This set is convex. A probability measure $\mathbb{P}$ is called extremal in $\mathcal{M}(Y)$ if whenever $\mathbb{P} = \lambda\mathbb{P}_1 + (1-\lambda)\mathbb{P}_2$ with $\lambda \in]0, 1[$ and $\mathbb{P}_1, \mathbb{P}_2 \in \mathcal{M}(Y)$, then $\mathbb{P} = \mathbb{P}_1 = \mathbb{P}_2$.

Note that if $\mathbb{P} = \lambda\mathbb{P}_1 + (1-\lambda)\mathbb{P}_2$, then $\mathbb{P}_1$ and $\mathbb{P}_2$ are absolutely continuous with respect to $\mathbb{P}$. However, the $\mathbb{P}_i$'s are not necessarily equivalent. The following theorem relates the PRP for Y under $\mathbb{P} \in \mathcal{M}(Y)$ and the extremal points of $\mathcal{M}(Y)$.

Theorem 1.6.2.4 *The process Y enjoys the PRP with respect to $\mathbf{F}^Y$ and $\mathbb{P}$ if and only if $\mathbb{P}$ is an extremal point of $\mathcal{M}(Y)$.*

PROOF: See Jacod [468], Yor [861] and Jacod and Yor [472]. □

Comments 1.6.2.5 (a) The PRP is essential in finance and is deeply linked with Delta hedging and completeness of the market. If the price process enjoys the PRP under an equivalent probability measure, the market is complete. It is worthwhile noting that the key process is the price process itself, rather than the processes that may drive the price process. See ↣ Subsection 2.3.6 for more details.

(b) We compare Theorems 1.6.1.1 and 1.6.2.4. It turns out that the Wiener measure is an extremal point in $\mathcal{M}$, the set of martingale laws on $C(\mathbb{R}^+, \mathbb{R})$ where $Y_t(\omega) = \omega(t)$. This extremality property follows from Lévy's characterization of Brownian motion.

(c) Let us give an example of a martingale which does not enjoy the PRP. Let $M_t = \int_0^t e^{aB_s - a^2 s/2} d\beta_s = \int_0^t \mathcal{E}(aB)_s d\beta_s$, where B, β are two independent one-dimensional Brownian motions. We note that $d\langle M\rangle_t = (\mathcal{E}(aB)_t)^2 dt$, so that $(\mathcal{E}_t := \mathcal{E}(aB)_t, t \geq 0)$ is $\mathbf{F}^M$-adapted and hence is an $\mathbf{F}^M$-martingale. Since $\mathcal{E}_t = 1 + a\int_0^t \mathcal{E}_s dB_s$, the martingale $\mathcal{E}$ cannot be obtained as a stochastic integral w.r.t. β or equivalently w.r.t. M. In fact, every $\mathbf{F}^M$-martingale can be written as the sum of a stochastic integral with respect to M (or equivalently to β) and a stochastic integral with respect to B.

(d) It is often asked what is the minimal number of orthogonal martingales needed to obtain a representation formula in a given filtration. We refer the reader to Davis and Varaiya [224] who defined the notion of multiplicity of a filtration. See also Davis and Obłój [223] and Barlow et al. [50].

Example 1.6.2.6 We give some examples of martingales that enjoy the PRP.

(a) Let W be a BM and $\mathbf{F}$ its natural filtration. Set $X_t = x + \int_0^t x_s \, dW_s$ where $(x_s, s \geq 0)$ is continuous and does not vanish. Then X enjoys the PRP.

(b) A continuous martingale is a time-changed Brownian motion. Let X be a martingale, then $X_t = \beta_{\langle X\rangle_t}$ where β is a Brownian motion. If $\langle X\rangle$ is measurable with respect to β, then X is said to be pure, and $\mathbb{P}_X$ is extremal. However, the converse does not hold. See Yor [862].

Exercise 1.6.2.7 Let $\mathcal{M}_{\mathbb{P}}(X) = \{\mathbb{Q} \ll \mathbb{P} : X \text{ is a } \mathbb{Q}\text{-martingale}\}$. For any convex set $\mathcal{K}$, we denote by $\mathrm{ext}(\mathcal{K})$ the set of extremal points of $\mathcal{K}$. Prove that

$$\mathrm{ext}(\mathcal{M}_{\mathbb{P}}(X)) = \mathrm{ext}(\mathcal{M}(X)) \cap \mathcal{M}_{\mathbb{P}}(X)\,.$$

An open question is: does the equality

$$\mathrm{ext}(\mathcal{M}_{\mathbb{P}}^{eq}(X)) = \mathrm{ext}\mathcal{M}(X) \cap \mathcal{M}_{\mathbb{P}}^{eq}(X)$$

where $\mathcal{M}_{\mathbb{P}}^{eq}(X) = \{\mathbb{Q} \sim \mathbb{P} : X \text{ is a } \mathbb{Q}\text{-martingale}\}$, hold? ◁

Exercise 1.6.2.8 We present an example where the representation of a bounded r.v. considered as the terminal variable of a martingale can be explicitly computed. Let B be a Brownian motion and $T_a = \inf\{t \geq 0 : B_t = a\}$ where $a > 0$.

1. Using the Doléans-Dade exponential of λB, prove that, for $\lambda > 0$

$$\mathbb{E}(e^{-\lambda^2 T_a/2} | \mathcal{F}_t) = e^{-\lambda a} + \lambda \int_0^{T_a \wedge t} e^{-\lambda(a - B_u) - \lambda^2 u/2} dB_u \qquad (1.6.3)$$

and that

$$e^{-\lambda^2 T_a/2} = e^{-\lambda a} + \lambda \int_0^{T_a} e^{-\lambda(a - B_u) - \lambda^2 u/2} dB_u\,.$$

Check that $\mathbb{E}(\int_0^{T_a} (e^{-\lambda(a-B_u) - \lambda^2 u/2})^2 du) < \infty$. Prove that (1.6.3) is not true for $\lambda < 0$, i.e., that, in the case $\mu := -\lambda > 0$ the quantities

$\mathbb{E}(e^{-\mu^2 T_a/2}|\mathcal{F}_t)$ and $e^{\mu a} - \mu \int_0^{T_a \wedge t} e^{\mu(a-B_u)-\mu^2 u/2}\, dB_u$ are not equal. Prove that, nonetheless,

$$e^{-\lambda^2 T_a/2} = e^{\lambda a} - \lambda \int_0^{T_a} e^{\lambda(a-B_u)-\lambda^2 u/2} dB_u$$

but $\mathbb{E}(\int_0^{T_a} (e^{\lambda(a-B_u)-\lambda^2 u/2})^2\, du) = \infty$. Deduce, from the previous results, that

$$\sinh(\lambda a) = \lambda \int_0^{T_a} e^{-\lambda^2 u/2} \cosh((a-B_u)\lambda)\, dB_u\,.$$

2. By differentiating the Laplace transform of T_a, and using the fact that φ satisfies the Kolmogorov equation $\partial_t \varphi(t,x) = \frac{1}{2}\partial_{xx}\varphi(t,x)$, (see $\rightarrowtail$ Subsection 5.4.1), prove that

$$\lambda e^{-\lambda c} = 2 \int_0^{\infty} e^{-\lambda^2 t/2} \partial_t \varphi(t,c)\, dt$$

where $\varphi(t,x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/(2t)}$.

3. Prove that, for any bounded Borel function f

$$\mathbb{E}(f(T_a)|\mathcal{F}_t) = \mathbb{E}(f(T_a)) + 2\int_0^{T_a \wedge t} dB_s \int_0^{\infty} f(u+s) \frac{\partial}{\partial u}\varphi(u, B_s - a) du\,.$$

4. Deduce that, for fixed T,

$$\mathbb{1}_{\{T_a < T\}} = \mathbb{P}(T_a < T) + 2\int_0^{T_a \wedge T} \varphi(T-s, B_s - a)\, dB_s\,.$$

See Shiryaev and Yor [793], Graversen et al. [405] for other examples. ◁

1.6.3 Dudley's Theorem

In the previous exercise, we were careful to check the integrability of the stochastic integrals. This may be contrasted with Dudley's result [269], which states that every $\mathcal{F}_T^W$-random variable can be represented as an Itô stochastic integral $\int_0^T \theta_s dW_s$ where θ is predictable and satisfies $\int_0^T \theta_s^2 ds < \infty$, a.s. where W is a Brownian motion. In fact, this result has no relation with the predictable representation property, as shown by Émery et al. [330]. Indeed, the authors proved that, in a filtration where any martingale is continuous, if τ is a stopping time and X is an $\mathcal{F}_\tau$-measurable random variable, there exists a local martingale M, null at 0, such that $M_\tau = X$.

Comment 1.6.3.1 In mathematical finance, Dudley's result is related to arbitrage opportunities (see $\rightarrowtail$ Chapter 2 for the definition of financial terms if needed). Let us study the simple case where $dS_t = S_t \sigma dW_t$, $S_0 = x > 0$

is the price of the risky asset and where the interest rate is null. Consider a process θ such that $\int_0^T \theta_s^2 ds < \infty$, a.s.. and $\int_0^T \theta_s dW_s = 1$ (the existence is a consequence of Dudley's theorem). Had we chosen $\pi_s = \theta_s/(S_s\sigma)$ as the risky part of a self-financing strategy with a zero initial wealth, then we would obtain an arbitrage opportunity. However, the wealth X associated with this strategy, i.e., $X_t = \int_0^t \theta_s dW_s$ is not bounded below (otherwise, X would be a super-martingale with initial value equal to 0, hence $\mathbb{E}(X_T) \leq 0$). These strategies are linked with the well-known doubling strategy of coin tossing (see Harrison and Pliska [422]).

1.6.4 Backward Stochastic Differential Equations

In deterministic case studies, it is easy to solve an ODE with a terminal condition just by time reversal. In a stochastic setting, if one insists that the solution is adapted w.r.t. a given filtration, it is not possible in general to use time reversal.

A probability space $(\Omega, \mathcal{F}, \mathbb{P})$, an n-dimensional Brownian motion W and its natural filtration $\mathbf{F}$, an $\mathcal{F}_T$-measurable square integrable random variable ζ and a family of $\mathbf{F}$-adapted, $\mathbb{R}^d$-valued processes $f(t, \centerdot, x, y), x, y \in \mathbb{R}^d \times \mathbb{R}^{d\times n}$ are given (we shall, as usual, forget the dependence in ω and write only $f(t, x, y)$). The problem we now consider is to solve a stochastic differential equation where the terminal condition ζ as well as the form of the drift term f (called the generator) are given, however, the diffusion term is left unspecified.

The **Backward Stochastic Differential Equation** (BSDE) (f, ζ) has the form

$$\begin{aligned} -dX_t &= f(t, X_t, Y_t)\, dt - Y_t \centerdot dW_t \\ X_T &= \zeta\,. \end{aligned}$$

Here, we have used the usual convention of signs which is in force while studying BSDEs. The solution of a BSDE is *a pair (X, Y) of adapted processes* which satisfy

$$X_t = \zeta + \int_t^T f(s, X_s, Y_s)\, ds - \int_t^T Y_s \centerdot dW_s\,, \tag{1.6.4}$$

where X is $\mathbb{R}^d$-valued and Y is $d \times n$-matrix valued.

We emphasize that the diffusion coefficient Y is a part of the solution, as it is clear from the obvious case when f is null: in that case, we are looking for a martingale with given terminal value. Hence, the quantity Y is the predictable process arising in the representation of the martingale X in terms of the Brownian motion.

Example 1.6.4.1 Let us study the easy case where f is a deterministic function of time (or a given process such that $\int_0^T f_s ds$ is square integrable) and

$d = n = 1$. If there exists a solution to $X_t = \zeta + \int_t^T f(s)\,ds - \int_t^T Y_s\,dW_s$, then the $\mathbf{F}$-adapted process $X_t + \int_0^t f(s)\,ds$ is equal to $\zeta + \int_0^T f(s)\,ds - \int_t^T Y_s\,dW_s$. Taking conditional expectation w.r.t. $\mathcal{F}_t$ of the two sides, and assuming that Y is square integrable, we get

$$X_t + \int_0^t f(s)\,ds = \mathbb{E}(\zeta + \int_0^T f(s)\,ds|\mathcal{F}_t) \tag{1.6.5}$$

therefore, the process $X_t + \int_0^t f(s)\,ds$ is an $\mathbf{F}$-martingale with terminal value $\zeta + \int_0^T f(s)\,ds$. (A more direct proof is to write $dX_t + f(t)dt = Y_t dW_t$.) The predictable representation theorem asserts that there exists an adapted square integrable process Y such that $X_t + \int_0^t f(s)\,ds = X_0 + \int_0^t Y_s dW_s$ and the pair (X, Y) is the solution of the BSDE. The process X can be written in terms of the generator f and the terminal condition as $X_t = \mathbb{E}(\zeta + \int_t^T f(s)ds|\mathcal{F}_t)$. In particular, if $\zeta^1 \geq \zeta^2$ and $f_1 \geq f_2$, and if X^i is the solution of (f_i, ζ^i) for $i = 1, 2$, then, for $t \in [0, T]$, $X_t^1 \geq X_t^2$.

Definition 1.6.4.2 *Let $L^2([0,T] \times \Omega; \mathbb{R}^d)$ be the set of $\mathbb{R}^d$-valued square integrable $\mathbf{F}$-progressively measurable processes, i.e., processes Z such that*

$$\mathbb{E}\left[\int_0^T \|Z_s\|^2 ds\right] < \infty\,.$$

Theorem 1.6.4.3 *Let us assume that for any $(x, y) \in \mathbb{R}^n \times \mathbb{R}^{d\times n}$, the process $f(\,\bullet\,, x, y)$ is progressively measurable, with $f(\,\bullet\,, 0, 0) \in L^2([0,T] \times \Omega; \mathbb{R}^d)$ and that the function $f(t, \bullet, \bullet)$ is uniformly Lipschitz, i.e., there exists a constant K such that*

$$\|f(t, x_1, y_1) - f(t, x_2, y_2)\| \leq K[\,\|x_1 - x_2\| + \|y_1 - y_2\|\,], \quad \forall t, \mathbb{P}, a.s.$$

Then there exists a unique pair (X, Y) of adapted processes belonging to $L^2([0,T] \times \Omega; \mathbb{R}^n) \times L^2([0,T] \times \Omega, \mathbb{R}^{d\times n})$ which satisfies (1.6.4).

Sketch of the Proof: Example (1.6.4.1) provides the proof when f does not depend on (x, y). The general case is established using Picard's iteration: let Φ be the map $\Phi(x, y) = (X, Y)$ where (x, y) is a pair of adapted processes and (X, Y) is the solution of

$$-dX_t = f(t, x_t, y_t)\,dt - Y_t \,\bullet\, dW_t,\ X_T = \zeta\,.$$

The map Φ is proved to be a contraction.
The uniqueness is proved by introducing the norm $\|\Phi\|_\beta^2 = \mathbb{E}(\int_0^T e^{\beta s}|\phi_s|ds)$ and giving a priori estimates of the norm $\|Y_1 - Y_2\|_\beta$ for two solutions of the BSDE. See Pardoux and Peng [694] and El Karoui et al. [309] for details. □

An important result is the following comparison theorem for BSDE

Theorem 1.6.4.4 *Let $f^i, i = 1, 2$ be two real-valued processes satisfying the previous hypotheses and $f^1(t, x, y) \leq f^2(t, x, y)$. Let ζ^i be two $\mathcal{F}_T$-measurable, square integrable real-valued random variables such that $\zeta^1 \leq \zeta^2$ a.s.. Let (X^i, Y^i) be the solution of*

$$-dX_t^i = f^i(t, X_t^i, Y_t^i)\, dt - Y_t^i \centerdot dW_t\, , X_T^i = \zeta.$$

Then $X_t^1 \leq X_t^2, \forall t \leq T$.

Linear Case. Let us consider the particular case of a linear generator $f : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ defined as $f(t, x, y) = a_t x + b_t \centerdot y + c_t$ where a, b, c are bounded adapted processes. We define the adjoint process Γ as the solution of the SDE

$$\begin{cases} d\Gamma_t = \Gamma_t[a_t dt + b_t \centerdot dW_t] \\ \Gamma_0 \;\; = 1 \end{cases} . \tag{1.6.6}$$

Theorem 1.6.4.5 *Let $\zeta \in \mathcal{F}_T$, square integrable. The solution of the linear BSDE*

$$-dX_t = (a_t X_t + b_t \centerdot Y_t + c_t)dt - Y_t \centerdot dW_t,\ X_T = \zeta$$

is given by

$$X_t = (\Gamma_t)^{-1}\mathbb{E}\left(\Gamma_T \zeta + \int_t^T \Gamma_s c_s ds | \mathcal{F}_t\right) .$$

Proof: If (X, Y) is a solution of

$$-dX_t = (a_t X_t + b_t \centerdot Y_t + c_t)dt - Y_t \centerdot dW_t$$

with the terminal condition $X_T = \zeta$, then

$$-d\widehat{X}_t = \widehat{c}_t dt - Y_t \centerdot (dW_t - b_t dt),\ \widehat{X}_T = \zeta \exp\left(\int_0^T a_s ds\right)$$

where $\widehat{X}_t = X_t \exp\big(\int_0^t a_s ds\big)$ and $\widehat{c}_t = c_t \exp\big(\int_0^t a_s ds\big)$. We use Girsanov's theorem (see $\rightarrowtail$ Section 1.7) to eliminate the term $Y \centerdot b$. Let $\mathbb{Q}|_{\mathcal{F}_t} = L_t\, \mathbb{P}|_{\mathcal{F}_t}$ where $dL_t = L_t b_t \centerdot dW_t$. Then,

$$-d\widehat{X}_t = \widehat{c}_t dt - Y_t \centerdot d\widetilde{W}_t$$

where $\widetilde{W}$ is a $\mathbb{Q}$-Brownian motion and the process $\widehat{X}_t + \int_0^t \widehat{c}_s ds$ is a $\mathbb{Q}$-martingale with terminal value $\zeta + \int_0^T \widehat{c}_s ds$. Hence, $\widehat{X}_t = \mathbb{E}_{\mathbb{Q}}(\zeta + \int_t^T \widehat{c}_s ds | \mathcal{F}_t)$. The result follows by application of Exercise 1.2.1.8. □

Backward stochastic differential equations are of frequent use in finance. Suppose, for example, that an agent would like to obtain a terminal wealth

X_T while consuming at a given rate c (an adapted positive process). The financial market consists of d securities

$$dS_t^i = S_t^i(b_i(t)dt + \sum_{j=1}^{d} \sigma_{i,j}(t)dW_t^{(j)})$$

and a riskless bond with interest rate denoted by r. We assume that the market is complete and arbitrage free (see $\rightarrowtail$ Chapter 2 if needed). The wealth associated with a portfolio $(\pi_i, i = 0, \dots, d)$ is the sum of the wealth invested in each asset, i.e., $X_t = \pi_0(t)S_t^0 + \sum_{i=1}^d \pi_i(t)S_t^i$. The self-financing condition for a portfolio with a given consumption c, i.e.,

$$dX_t = \pi_0(t)dS_t^0 + \sum_{i=1}^{d} \pi_i(t)dS_t^i - c_t dt$$

allows us to write

$$dX_t = X_t r dt + \pi_t \centerdot (b_t - r\mathbf{1})dt - c_t dt + \pi_t \centerdot \sigma_t dW_t\,,$$

where $\mathbf{1}$ is the d-dimensional vector with all components equal to 1. Therefore, the pair (wealth process, portfolio) is obtained via the solution of the BSDE

$$dX_t = f(t, X_t, Y_t)dt + Y_t \centerdot dW_t,\ X_T \text{ given}$$

with $f(t, \cdot, x, y) = rx + y \centerdot \sigma_t^{-1}(b_t - r\mathbf{1}) - c_t$ and the portfolio $(\pi_i, i = 1, \dots, d)$ is given by $\pi_t = Y_t \centerdot \sigma_t^{-1}$. This is a particular case of a linear BSDE. Then, the process Γ introduced in (1.6.6) satisfies

$$d\Gamma_t = \Gamma_t(rdt + \sigma_t^{-1}(b_t - r\mathbf{1})dW_t),\ \Gamma_0 = 1$$

and Γ_t is the product of the discounted factor e^{-rt} and the strictly positive martingale L, which satisfies

$$dL_t = L_t\sigma_t^{-1}(b_t - r\mathbf{1})dW_t, L_0 = 1\,,$$

i.e., $\Gamma_t = e^{-rt}L_t$. If $\mathbb{Q}$ is defined as $\mathbb{Q}|_{\mathcal{F}_t} = L_t\mathbb{P}|_{\mathcal{F}_t}$, denoting $R_t = e^{-rt}$, the process $R_tX_t + \int_0^t c_sR_s ds$ is a local martingale under the e.m.m. $\mathbb{Q}$ (see $\rightarrowtail$ Chapter 2 if needed). Therefore,

$$\Gamma_t X_t = \mathbb{E}_{\mathbb{P}}\left(X_T\Gamma_T + \int_t^T c_s\Gamma_s ds|\mathcal{F}_t\right)\,.$$

In particular, the value of wealth at time t needed to hedge a positive terminal wealth X_T and a positive consumption is always positive. Moreover, from the comparison theorem, if $X_T^1 \le X_T^2$ and $c^1 \le c^2$, then $X_t^1 \le X_t^2$. This can be explained using the arbitrage principle. If a contingent claim ζ_1 is greater than a contingent claim ζ_2, and if there is no consumption, then the initial wealth is the price of ζ_1 and is greater than the price of ζ_2.

Exercise 1.6.4.6 Quadratic BSDE: an example. This exercise provides an example where there exists a solution although the Lipschitz condition is not satisfied.
Let a and b be two constants and ζ a bounded $\mathcal{F}_T$-measurable r.v.. Prove that the solution of $-dX_t = (aY_t^2 + bY_t)dt - Y_t dW_t, X_T = \zeta$ is

$$X_t = \frac{1}{2a}\left(\frac{1}{2}b^2(t-T) - bW_t + \ln \mathbb{E}\left(e^{bW_T + 2a\zeta}|\mathcal{F}_t\right)\right).$$

`Hint:` First, prove that the solution of the BSDE

$$-dX_t = aY_t^2 dt - Y_t dW_t,\ X_T = \zeta$$

is $X_t = \frac{1}{2a}\ln \mathbb{E}(e^{2a\zeta}|\mathcal{F}_t)$. Then, using Girsanov's theorem, the solution of

$$-dX_t = (aY_t^2 + bY_t)dt - Y_t dW_t, X_T = \zeta$$

is given by

$$X_t = \frac{1}{2a}\ln \widehat{\mathbb{E}}(e^{2a\zeta}|\mathcal{F}_t)$$

where $\widehat{\mathbb{Q}}_{|\mathcal{F}_t} = e^{bW_t - \frac{1}{2}b^2 t}\mathbb{P}_{|\mathcal{F}_t}$. Therefore,

$$\begin{aligned} X_t &= \frac{1}{2a}\ln\left(\mathbb{E}(e^{bW_T - \frac{1}{2}b^2 T}e^{2a\zeta}|\mathcal{F}_t)e^{-bW_t + \frac{1}{2}b^2 t}\right) \\ &= \frac{1}{2a}\left(\ln \mathbb{E}\left(e^{bW_T - \frac{1}{2}b^2 T}e^{2a\zeta}|\mathcal{F}_t\right) - bW_t + \frac{1}{2}b^2 t\right). \quad \triangleleft \end{aligned}$$

Comments 1.6.4.7 (a) The main references on this subject are the collective book [303], the book of Ma and Yong [607], the El Karoui and Quenez lecture in [308], El Karoui et al. [309] and Buckdhan's lecture in [134]. See also the seminal papers of Lepeltier and San Martin [578, 579] where general existence theorems for continuous generators with linear growth are established.

(b) In El Karoui and Rouge [310], the indifference price is characterized as a solution of a BSDE with a quadratic generator.

(c) BSDEs are used to solve control problems in Bielecki et al. [98], Hamadène [419], Hu and Zhou [448] and Mania and Tevzadze [619].

(d) Backward stochastic differential equations are also studied in the case where the driving martingale is a process with jumps. The reader can refer to Barles et al. [43], Royer [744], Nualart and Schoutens [683] and Rong [743].

(e) Reflected BSDE are studied by El Karoui and Quenez [308] in order to give the price of an American option, without using the notion of a Snell envelope.

(f) One of the main applications of BSDE is the notion of non-linear expectation (or G-expectation), and the link between this notion and risk measures (see Peng [705, 706]).

1.7 Change of Probability and Girsanov's Theorem

1.7.1 Change of Probability

We start with a general filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ where, as usual $\mathcal{F}_0$ is trivial.

Proposition 1.7.1.1 *Let $\mathbb{P}$ and $\mathbb{Q}$ be two equivalent probabilities on $(\Omega, \mathcal{F}_T)$. Then, there exists a strictly positive $(\mathbb{P}, \mathbf{F})$-martingale $(L_t, t \leq T)$, such that $\mathbb{Q}|_{\mathcal{F}_t} = L_t \, \mathbb{P}|_{\mathcal{F}_t}$, that is $\mathbb{E}_{\mathbb{Q}}(X) = \mathbb{E}_{\mathbb{P}}(L_t X)$ for any $\mathcal{F}_t$-measurable positive random variable X with $t \leq T$. Moreover, $L_0 = 1$ and $\mathbb{E}_{\mathbb{P}}(L_t) = 1$, $\forall t \leq T$.*

PROOF: If $\mathbb{P}$ and $\mathbb{Q}$ are equivalent on $(\Omega, \mathcal{F}_T)$, from the Radon-Nikodým theorem there exists a strictly positive $\mathcal{F}_T$-measurable random variable L_T such that $\mathbb{Q} = L_T \mathbb{P}$ on $\mathcal{F}_T$. From the definition of $\mathbb{Q}$, the expectation under $\mathbb{Q}$ of any $\mathcal{F}_T$-measurable $\mathbb{Q}$-integrable r.v. X is defined as $\mathbb{E}_{\mathbb{Q}}(X) = \mathbb{E}_{\mathbb{P}}(L_T X)$. In particular, $\mathbb{E}_{\mathbb{P}}(L_T) = 1$.

The process $L = (L_t = \mathbb{E}_{\mathbb{P}}(L_T | \mathcal{F}_t), t \leq T)$ is a $(\mathbb{P}, \mathbf{F})$-martingale and is the Radon-Nikodým density of $\mathbb{Q}$ with respect to $\mathbb{P}$ on $\mathcal{F}_t$. Indeed, if X is $\mathcal{F}_t$-measurable (hence $\mathcal{F}_T$-measurable) and $\mathbb{Q}$-integrable

$$\mathbb{E}_{\mathbb{Q}}(X) = \mathbb{E}_{\mathbb{P}}(L_T X) = \mathbb{E}_{\mathbb{P}}[\mathbb{E}_{\mathbb{P}}(X L_T | \mathcal{F}_t)] = \mathbb{E}_{\mathbb{P}}[X \mathbb{E}_{\mathbb{P}}(L_T | \mathcal{F}_t)] = \mathbb{E}_{\mathbb{P}}(X L_t).$$

□

Note that $\mathbb{P}|_{\mathcal{F}_T} = (L_T)^{-1} \mathbb{Q}|_{\mathcal{F}_T}$ so that, for any positive r.v. $Y \in \mathcal{F}_T$, $\mathbb{E}_{\mathbb{P}}(Y) = \mathbb{E}_{\mathbb{Q}}(L_T^{-1} Y)$ and L^{-1} is a $\mathbb{Q}$-martingale.

We shall speak of the law of a random variable (or of a process) under $\mathbb{P}$ or under $\mathbb{Q}$ to make precise the choice of the probability measure on the space Ω. From the equivalence between the measures, a property which holds $\mathbb{P}$-a.s. holds also $\mathbb{Q}$-a.s. However, a $\mathbb{P}$-integrable random variable is not necessarily $\mathbb{Q}$-integrable.

Definition 1.7.1.2 *A probability $\mathbb{Q}$ on a filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ is said to be locally equivalent to $\mathbb{P}$ if there exists a strictly positive $\mathbf{F}$-martingale L such that $\mathbb{Q}|_{\mathcal{F}_t} = L_t \, \mathbb{P}|_{\mathcal{F}_t}$, $\forall t$. The martingale L is called the Radon-Nikodým density of $\mathbb{Q}$ w.r.t. $\mathbb{P}$.*

Warning 1.7.1.3 This definition, which is standard in mathematical finance, is different from the more general one used by the Strasbourg school, where locally refers to a sequence of **F**-stopping times, increasing to infinity.

Proposition 1.7.1.4 *Let $\mathbb{P}$ and $\mathbb{Q}$ be locally equivalent, with Radon-Nikodým density L. Then, for any stopping time τ,*

$$\mathbb{Q}|_{\mathcal{F}_\tau \cap (\tau < \infty)} = L_\tau \, \mathbb{P}|_{\mathcal{F}_\tau \cap (\tau < \infty)} \, .$$

Proof: Let $A \in \mathcal{F}_\tau$. Then,

$$\mathbb{Q}(\mathbb{1}_A \mathbb{1}_{\{\tau \le t\}}) = \mathbb{E}_\mathbb{P}(L_t \mathbb{1}_A \mathbb{1}_{\{\tau \le t\}}) = \mathbb{E}_\mathbb{P}(L_\tau \mathbb{1}_A \mathbb{1}_{\{\tau \le t\}})\,.$$

The result follows by letting $t \to \infty$. □

Proposition 1.7.1.4 may be quite useful to shift computations under $\mathbb{Q}$ into computations under $\mathbb{P}$ when L_τ has a simple expression. See ↣ Subsection 3.2.3 and ↣ Exercice 4.3.5.7.

Proposition 1.7.1.5 (Bayes Formula.) *Suppose that $\mathbb{Q}$ and $\mathbb{P}$ are equivalent on $\mathcal{F}_T$ with Radon-Nikodým density L. Let X be a $\mathbb{Q}$-integrable $\mathcal{F}_T$-measurable random variable, then, for $t < T$*

$$\mathbb{E}_\mathbb{Q}(X|\mathcal{F}_t) = \frac{\mathbb{E}_\mathbb{P}(L_T X|\mathcal{F}_t)}{L_t}\,.$$

Proof: The proof follows immediately from the definition of conditional expectation. To check that the $\mathcal{F}_t$-measurable r.v. $Z = \dfrac{\mathbb{E}_\mathbb{P}(L_T X|\mathcal{F}_t)}{L_t}$ is the $\mathbb{Q}$-conditional expectation of X, we prove that $\mathbb{E}_\mathbb{Q}(F_t X) = \mathbb{E}_\mathbb{Q}(F_t Z_t)$ for any bounded $\mathcal{F}_t$-measurable random variable F_t. This follows from the equalities

$$\begin{aligned}\mathbb{E}_\mathbb{Q}(F_t X) &= \mathbb{E}_\mathbb{P}(L_T F_t X) = \mathbb{E}_\mathbb{P}(F_t \mathbb{E}_\mathbb{P}(X L_T|\mathcal{F}_t)) \\ &= \mathbb{E}_\mathbb{Q}(F_t L_t^{-1} \mathbb{E}_\mathbb{P}(X L_T|\mathcal{F}_t)) = \mathbb{E}_\mathbb{Q}(F_t Z)\,.\end{aligned}$$

□

Proposition 1.7.1.6 *Let $\mathbb{P}$ and $\mathbb{Q}$ be two locally equivalent probability measures with Radon-Nikodým density L. A process M is a $\mathbb{Q}$-martingale if and only if the process LM is a $\mathbb{P}$-martingale. By localization, this result remains true for local martingales.*

Proof: Let M be a $\mathbb{Q}$-martingale. From the Bayes formula, we obtain, for $s \le t$,

$$M_s = \mathbb{E}_\mathbb{Q}(M_t|\mathcal{F}_s) = \frac{\mathbb{E}_\mathbb{P}(L_t M_t|\mathcal{F}_s)}{L_s}$$

and the result follows. The converse part is now obvious. □

Exercise 1.7.1.7 Let $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ be a filtered probability space and denote by $(L_t, t \ge 0)$ the Radon-Nikodým density of $\mathbb{Q}$ with respect to $\mathbb{P}$. Then, if $\widetilde{\mathbf{F}}$ is a subfiltration of $\mathbf{F}$, prove that $\mathbb{Q}|_{\widetilde{\mathcal{F}}_t} = \widetilde{L}_t \mathbb{P}|_{\widetilde{\mathcal{F}}_t}$, where $\widetilde{L}_t = \mathbb{E}_\mathbb{P}(L_t|\widetilde{\mathcal{F}}_t)$. ◁

Exercise 1.7.1.8 Give conditions on the function h so that the measure $\mathbb{Q}$ defined on $\mathcal{F}_T$ as $\mathbb{Q} = h(W_T)\mathbb{P}$ is a probability equivalent to $\mathbb{P}$. Prove that, for $t < T$ $\mathbb{Q}|_{\mathcal{F}_t} = L_t \mathbb{P}|_{\mathcal{F}_t}$ where

$$L_t = \int_{-\infty}^{\infty} dy\, h(y) \frac{e^{-(y-W_t)^2/(2(T-t))}}{\sqrt{2\pi(T-t)}} .$$

Prove that

$$L_t = 1 + \int_0^t dW_s \int_{-\infty}^{\infty} dy \frac{h(y)e^{-(y-W_s)^2/(2(T-s))}}{\sqrt{2\pi(T-s)}} \frac{y-W_s}{T-s} .$$

For $h \in C^1$ with compact support, prove that

$$L_t = 1 + \int_0^t dW_s \int_{-\infty}^{\infty} dy \frac{h'(y)e^{-(y-W_s)^2/(2(T-s))}}{\sqrt{2\pi(T-s)}} .$$

See Baudoin [60] for applications. $\triangleleft$

Exercise 1.7.1.9 (1) Let f a Borel function satisfying $0 < \int_0^\infty f^2(u)du < \infty$. Compute, for any t, $\mathbb{P}\left(\int_0^\infty f(s)dW_s > 0 \,|\mathcal{F}_t\right) =: Z_t^f$. Prove that, as a consequence $Z_t^f > 0$ a.s., but $\mathbb{P}(Z_\infty^f = 0) = 1/2$.

(2) Prove that there exist pairs $(\mathbb{Q}, \mathbb{P})$ of probabilities that are locally equivalent, but $\mathbb{Q}$ is not equivalent to $\mathbb{P}$ on $\mathcal{F}_\infty$. $\triangleleft$

1.7.2 Decomposition of $\mathbb{P}$-Martingales as $\mathbb{Q}$-semi-martingales

Theorem 1.7.2.1 *Let $\mathbb{P}$ and $\mathbb{Q}$ be locally equivalent, with Radon-Nikodým density L. We assume that the process L is continuous.*

If M is a continuous $\mathbb{P}$-local martingale, then the process $\widetilde{M}$ defined by

$$d\widetilde{M} = dM - \frac{1}{L} d\langle M, L\rangle$$

is a continuous $\mathbb{Q}$-local martingale. If N is another continuous $\mathbb{P}$-local martingale,

$$\langle M, N\rangle = \langle \widetilde{M}, \widetilde{N}\rangle = \langle M, \widetilde{N}\rangle .$$

Proof: From Proposition 1.7.1.6, it is enough to check that $\widetilde{M}L$ is a $\mathbb{P}$-local martingale, which follows easily from Itô's calculus. $\square$

Corollary 1.7.2.2 *Under the hypotheses of Theorem 1.7.2.1, we may write the process L as a Doléans-Dade martingale: $L_t = \mathcal{E}(\zeta)_t$, where ζ is an $\mathbf{F}$-local martingale. The process $\widetilde{M} = M - \langle M, \zeta\rangle$ is a $\mathbb{Q}$-local martingale.*

1.7.3 Girsanov's Theorem: The One-dimensional Brownian Motion Case

If the filtration $\mathbf{F}$ is generated by a Brownian motion W, and $\mathbb{P}$ and $\mathbb{Q}$ are locally equivalent, with Radon-Nikodým density L, the martingale L admits a representation of the form $dL_t = \psi_t dW_t$. Since L is strictly positive, this equality takes the form $dL_t = -\theta_t L_t dW_t$, where $\theta = -\psi/L$. (The minus sign will be convenient for further use in finance (see $\rightarrowtail$ Subsection 2.2.2), to obtain the usual risk premium). It follows that

$$L_t = \exp\left(-\int_0^t \theta_s dW_s - \frac{1}{2}\int_0^t \theta_s^2 ds\right) = \mathcal{E}(\zeta)_t\,,$$

where $\zeta_t = -\int_0^t \theta_s dW_s$.

Proposition 1.7.3.1 (Girsanov's Theorem) *Let W be a $(\mathbb{P}, \mathbf{F})$-Brownian motion and let θ be an $\mathbf{F}$-adapted process such that the solution of the SDE*

$$dL_t = -L_t \theta_t dW_t,\ L_0 = 1$$

is a martingale. We set $\mathbb{Q}|_{\mathcal{F}_t} = L_t\,\mathbb{P}|_{\mathcal{F}_t}$. Then the process W admits a $\mathbb{Q}$-semi-martingale decomposition $\widetilde{W}$ as $W_t = \widetilde{W}_t - \int_0^t \theta_s ds$ where $\widetilde{W}$ is a $\mathbb{Q}$-Brownian motion.

PROOF: From $dL_t = -L_t\theta_t dW_t$, using Girsanov's theorem 1.7.2.1, we obtain that the decomposition of W under $\mathbb{Q}$ is $\widetilde{W}_t - \int_0^t \theta_s ds$. The process W is a $\mathbb{Q}$-semi-martingale and its martingale part $\widetilde{W}$ is a BM. This last fact follows from Lévy's theorem, since the bracket of W does not depend on the (equivalent) probability. □

Warning 1.7.3.2 Using a real-valued, or complex-valued martingale density L, with respect to Wiener measure, induces a real-valued or complex-valued measure on path space. The extension of the Girsanov theorem in this framework is tricky; see Dellacherie et al. [241], paragraph (39), page 349, as well as Ruiz de Chavez [748] and Begdhdadi-Sakrani [66].

When the coefficient θ is deterministic, we shall refer to this result as **Cameron-Martin's** theorem due to the origin of this formula [137], which was extended by Maruyama [626], Girsanov [393], and later by Van Schuppen and Wong [825].

Example 1.7.3.3 Let S be a geometric Brownian motion

$$dS_t = S_t(\mu dt + \sigma dW_t)\,.$$

Here, W is a Brownian motion under a probability $\mathbb{P}$. Let $\theta = (\mu - r)/\sigma$ and $dL_t = -\theta L_t dW_t$. Then, $B_t = W_t + \theta t$ is a Brownian motion under $\mathbb{Q}$, where $\mathbb{Q}|_{\mathcal{F}_t} = L_t\,\mathbb{P}|_{\mathcal{F}_t}$ and

$$dS_t = S_t(rdt + \sigma dB_t)\,.$$

Comment 1.7.3.4 In the previous example, the equality

$$S_t(\mu dt + \sigma dW_t) = S_t(rdt + \sigma dB_t)$$

holds under both $\mathbb{P}$ and $\mathbb{Q}$. The rôle of the probabilities $\mathbb{P}$ and $\mathbb{Q}$ makes precise the dynamics of the driving process W (or B). Therefore, the equation can be computed in an "algebraic" way, by setting $dB_t = dW_t + \theta dt$. This leads to

$$\mu dt + \sigma dW_t = rdt + \sigma[dW_t + \theta dt] = rdt + \sigma dB_t\,.$$

The explicit computation of S can be made with W or B

$$\begin{aligned} S_t &= S_0 \exp\left(\mu t + \sigma W_t - \frac{1}{2}\sigma^2 t\right) \\ &= S_0 \exp\left(rt + \sigma B_t - \frac{1}{2}\sigma^2 t\right)\,. \end{aligned}$$

As a consequence, the importance of the probability appears when we compute the expectations

$$\mathbb{E}_{\mathbb{P}}(S_t) = S_0 e^{\mu t},\ \mathbb{E}_{\mathbb{Q}}(S_t) = S_0 e^{rt}\,,$$

with the help of the above formulae. Note that $(S_t e^{-\mu t}, t \geq 0)$ is a $\mathbb{P}$-martingale and that $(S_t e^{-rt}, t \geq 0)$ is a $\mathbb{Q}$-martingale.

Example 1.7.3.5 Let

$$dX_t = a\,dt + 2\sqrt{X_t}dW_t \tag{1.7.1}$$

where we choose $a \geq 0$ so that there exists a positive solution $X_t \geq 0$. (See $\rightarrowtail$ Chapter 6 for more information.) Let F be a C^1 function. The continuity of F implies that the local martingale

$$L_t = \exp\left(\int_0^t F(s)\sqrt{X_s}dW_s - \frac{1}{2}\int_0^t F^2(s)X_s ds\right)$$

is in fact a martingale, therefore $\mathbb{E}(L_t) = 1$. From the definition of X and the integration by parts formula,

$$\begin{aligned} \int_0^t F(s)\sqrt{X_s}dW_s &= \frac{1}{2}\int_0^t F(s)(dX_s - ads) \qquad (1.7.2) \\ &= \frac{1}{2}\left(F(t)X_t - F(0)X_0 - \int_0^t F'(s)X_s ds - a\int_0^t F(s)ds\right)\,. \end{aligned}$$

Therefore, one obtains the general formula

$$\begin{aligned} &\mathbb{E}\left[\exp\left(\frac{1}{2}\left\{F(t)X_t - \int_0^t [F'(s) + F^2(s)]X_s ds\right\}\right)\right] \\ &= \exp\left(\frac{1}{2}\left[F(0)X_0 + a\int_0^t F(s)ds\right]\right)\,. \end{aligned}$$

In the particular case $F(s) = -k/2$, setting

$$\mathbb{Q}|_{\mathcal{F}_t} = L_t \, \mathbb{P}|_{\mathcal{F}_t} \, ,$$

we obtain

$$dX_t = k(\theta - X_t)dt + 2\sqrt{X_t}dB_t = (a - kX_t)dt + 2\sqrt{X_t}dB_t \tag{1.7.3}$$

where B is a $\mathbb{Q}$-Brownian motion. Hence, if $\mathbb{Q}^a$ is the law of the process (1.7.1) and ${}^k\mathbb{Q}^a$ the law of the process defined in (1.7.3) with $a = k\theta$, we get from (1.7.2) the absolute continuity relationship

$${}^k\mathbb{Q}^a|_{\mathcal{F}_t} = \exp\left(\frac{k}{4}(at - X_t + x) - \frac{k^2}{8}\int_0^t X_s ds\right)\mathbb{Q}^a|_{\mathcal{F}_t} \, .$$

See Donati-Martin et al. [258] for more information.

Exercise 1.7.3.6 See Exercise 1.7.1.8 for the notation. Prove that B defined by

$$dB_t = dW_t - \frac{\displaystyle\int_{-\infty}^{\infty} dy \, h'(y) e^{-(y-W_t)^2/(2(T-t))}}{\displaystyle\int_{-\infty}^{\infty} dy \, h(y) e^{-(y-W_t)^2/(2(T-t))}} dt$$

is a $\mathbb{Q}$-Brownian motion. See Baudoin [61] for an application to finance. ◁

Exercise 1.7.3.7 (1) Let $dS_t = S_t \sigma dW_t, S_0 = x$. Prove that for any bounded function f,

$$\mathbb{E}(f(S_T)) = \mathbb{E}\left(\frac{S_T}{x} f\left(\frac{x^2}{S_T}\right)\right) .$$

(2) Prove that, if $dS_t = S_t(\mu dt + \sigma dW_t)$, there exists γ such that S^γ is a martingale. Prove that for any bounded function f,

$$\mathbb{E}(f(S_T)) = \mathbb{E}\left(\left(\frac{S_T}{x}\right)^\gamma f\left(\frac{x^2}{S_T}\right)\right) .$$

Prove that, for bounded function f,

$$\mathbb{E}(S_T^\alpha f(S_T)) = x^\alpha e^{\mu(\alpha)T}\mathbb{E}\left(f(e^{\alpha\sigma^2 T} S_T)\right) ,$$

where $\mu(\alpha) = \alpha(\mu + \frac{1}{2}\sigma^2(\alpha - 1))$. See ↣ Lemma 3.6.6.1 for application to finance. ◁

Exercise 1.7.3.8 Let W be a $\mathbb{P}$-Brownian motion, and $B_t = W_t + \nu t$ be a $\mathbb{Q}$-Brownian motion, under a suitable change of probability. Check that, in the case $\nu > 0$, the process e^{W_t} tends towards 0 under $\mathbb{Q}$ when t goes to infinity, whereas this is not the case under $\mathbb{P}$. ◁

Comment 1.7.3.9 The relation obtained in question (1) in Exercise 1.7.3.7 can be written as

$$\mathbb{E}(\varphi(W_T - \sigma T/2)) = \mathbb{E}(e^{-\sigma(W_T + \sigma T/2)}\varphi(W_T + \sigma T/2))$$

which is an "h-process" relationship between a Brownian motion with drift $\sigma/2$ and a Brownian motion with drift $-\sigma/2$.

Exercise 1.7.3.10 Examples of a martingale with respect to two different probabilities:
Let W be a $\mathbb{P}$-BM, and set $d\mathbb{Q}|_{\mathcal{F}_t} = L_t d\mathbb{P}|_{\mathcal{F}_t}$ where $L_t = \exp(\lambda W_t - \frac{1}{2}\lambda^2 t)$. Prove that the process X, where

$$X_t = W_t - \int_0^t \frac{W_s}{s} ds$$

is a Brownian motion with respect to its natural filtration under both $\mathbb{P}$ and $\mathbb{Q}$.
`Hint:` (a) Under $\mathbb{P}$, for any t, $(X_u, u \leq t)$ is independent of W_t and is a Brownian motion.

(b) Replacing W_u by $(W_u + \lambda u)$ in the definition of X does not change the value of X. (See Atlan et al. [26].) See also $\rightarrowtail$ Example 5.8.2.3. $\triangleleft$

1.7.4 Multidimensional Case

Let W be an n-dimensional Brownian motion and θ an n-dimensional adapted process such that $\int_0^t ||\theta_s||^2 ds < \infty, a.s.$. Define the local martingale L as the solution of $dL_t = L_t \theta_t \centerdot dW_t = L_t(\sum_{i=1}^n \theta_t^i dW_t^i)$, so that

$$L_t = \exp\left(\int_0^t \theta_s \centerdot dW_s - \frac{1}{2}\int_0^t ||\theta_s||^2 ds\right) .$$

If L is a martingale, the n-dimensional process $(\widetilde{W}_t = W_t - \int_0^t \theta_s ds, t \geq 0)$ is a $\mathbb{Q}$-martingale, where $\mathbb{Q}$ is defined by $\mathbb{Q}|_{\mathcal{F}_t} = L_t \, \mathbb{P}|_{\mathcal{F}_t}$. Then, $\widetilde{W}$ is an n-dimensional Brownian motion (and in particular its components are independent).

If W is a Brownian motion with correlation matrix Λ, then, since the brackets do not depend on the probability, under $\mathbb{Q}$, the process

$$\widetilde{W}_t = W_t - \int_0^t \theta_s \centerdot \Lambda \, ds$$

is a correlated Brownian motion with the same correlation matrix Λ.

1.7.5 Absolute Continuity

In this section, we describe Girsanov's transformation in terms of absolute continuity. We start with elementary remarks. In what follows, $\mathbf{W}_x$ denotes the Wiener measure such that $\mathbf{W}_x(X_0 = x) = 1$ and $\mathbf{W}$ stands for $\mathbf{W}_0$. The notation $\mathbf{W}^{(\nu)}$ for the law of a BM with drift ν on the canonical space will be used:

$$\mathbf{W}^{(\nu)}[F(X_t, t \leq T)] = \mathbb{E}[F(\nu t + W_t, t \leq T)]\,.$$

On the left-hand side the process X is the canonical process, whose law is that of a Brownian motion with drift ν, on the right-hand side, W stands for a standard Brownian motion.

The right-hand side could be written as $\mathbf{W}^{(0)}[F(\nu t + X_t, t \leq T)]$. We also use the notation $\mathbf{W}^{(f)}$ for the law of the solution of $dX_t = f(X_t)dt + dW_t$.

Comment 1.7.5.1 Throughout our book, $(X_t, t \geq 0)$ may denote a particular stochastic process, often defined in terms of BM, or $(X_t, t \geq 0)$ may be the canonical process on $C(\mathbb{R}^+, \mathbb{R}^d)$. Each time, the context should not bring any ambiguity.

Proposition 1.7.5.2 (Cameron-Martin's Theorem.)
The Cameron-Martin theorem reads:

$$\mathbf{W}^{(\nu)}[F(X_t, t \leq T)] = \mathbf{W}^{(0)}[e^{\nu X_T - \nu^2 T/2} F(X_t, t \leq T)]\,.$$

More generally:

Proposition 1.7.5.3 (Girsanov's Theorem.) *Assume that the solution of $dX_t = f(X_t)dt + dW_t$ does not explode. Then, Girsanov's theorem reads: for any T,*

$$\mathbf{W}^{(f)}[F(X_t, t \leq T)]$$
$$= \mathbf{W}^{(0)}\left[\exp\left(\int_0^T f(X_s)dX_s - \frac{1}{2}\int_0^T f^2(X_s)ds\right) F(X_t, t \leq T)\right].$$

This result admits a useful extension to stopping times (in particular to explosion times):

Proposition 1.7.5.4 *Let ζ be the explosion time of the solution of the SDE $dX_t = f(X_t)dt + dW_t$. Then, for any stopping time $\tau \leq \zeta$,*

$$\mathbf{W}^{(f)}[F(X_t, t \leq \tau)]$$
$$= \mathbf{W}^{(0)}\left[\exp\left(\int_0^\tau f(X_s)dX_s - \frac{1}{2}\int_0^\tau f^2(X_s)ds\right) F(X_t, t \leq \tau)\right].$$

Example 1.7.5.5 From Cameron-Martin's theorem applied to the particular random variable $F(X_t, t \le \tau) = h(e^{\sigma X_\tau})$, we deduce

$$\begin{aligned}\mathbf{W}^{(\nu)}(h(e^{\sigma X_\tau})) = \mathbb{E}(h(e^{\sigma(W_\tau+\nu\tau)})) &= \mathbf{W}^{(0)}(e^{-\nu^2\tau/2+\nu X_\tau} h(e^{\sigma X_\tau})) \\ &= \mathbb{E}(e^{-\nu^2\tau/2} e^{\nu W_\tau} h(e^{\sigma W_\tau}))\,.\end{aligned}$$

Example 1.7.5.6 If $T_a(S)$ is the first hitting time of a for the geometric Brownian motion $S = xe^{\sigma X}$ defined in (1.5.3), with $a > x$ and $\sigma > 0$, and $T_\alpha(X)$ is the first hitting time of $\alpha = \frac{1}{\sigma}\ln(a/x)$ for the drifted Brownian motion X defined in (1.5.4), then

$$\begin{aligned}\mathbb{E}(F(S_t, t \le T_a(S))) &= \mathbf{W}^{(\nu)}\left[F(xe^{\sigma X_t}, t \le T_\alpha(X))\right] \\ &= \mathbf{W}^{(0)}\left[e^{\nu\alpha - \frac{\nu^2}{2}T_\alpha(X)} F(xe^{\sigma X_t},\, t \le T_\alpha(X))\right] \\ &= \mathbb{E}\left(e^{\nu\alpha - \frac{\nu^2}{2}T_\alpha(W)} F(xe^{\sigma W_t},\, t \le T_\alpha(W))\right)\,. \end{aligned} \tag{1.7.4}$$

Exercise 1.7.5.7 Let W be a standard Brownian motion, $a > 1$, and τ the stopping time $\tau = \inf\{t : e^{W_t - t/2} > a\}$. Prove that, $\forall \lambda \ge 1/2$,

$$\mathbb{E}\left(\mathbb{1}_{\{\tau<\infty\}} \exp(\lambda W_\tau - \frac{1}{2}\lambda^2\,\tau\right) = 1\,.$$

`Hint:`

$$\mathbb{E}\left(\mathbb{1}_{\tau<\infty} \exp\left(\lambda W_\tau - \frac{1}{2}\lambda^2\,\tau\right)\right) = \mathbf{W}^{(\lambda)}(\tau < \infty)\,.$$

The process $(W_t - \frac{1}{2}t, t \ge 0)$ is, under $\mathbf{W}^{(\lambda)}$, a BM with drift $\lambda - \frac{1}{2}$. ◁

Exercise 1.7.5.8 Let W be a $\mathbb{P}$-Brownian motion and $d\mathbb{Q}|_{\mathcal{F}_t} = e^{W_t - t/2} d\mathbb{P}|_{\mathcal{F}_t}$. Let $\tau = \inf\{t : W_t = -m\}$ for $m > 0$. Compute $\mathbb{P}(\tau < \infty)$ and $\mathbb{Q}(\tau < \infty)$.
`Hint:` $\mathbb{P}(\tau < \infty) = 1$, and using results on hitting times of BM (see ↣ Proposition 3.1.6.1) $\mathbb{Q}(\tau < \infty) = e^{-m}\mathbb{E}_{\mathbb{P}}(e^{-\tau/2}) = e^{-2m}$. ◁

1.7.6 Condition for Martingale Property of Exponential Local Martingales

As noted previously, if $\mathbb{Q}$ is a probability measure equivalent to $\mathbb{P}$, then the Radon-Nikodým density is a martingale: A strict local martingale cannot be a density between two probabilities.

In many cases we have to solve a problem of the following form: let W be a Brownian motion and

$$X_t^{\Phi} := W_t - \int_0^t ds\,\Phi_s \tag{1.7.5}$$

where Φ is an $\mathbf{F}^W$-predictable process such that $\int_0^1 ds\,|\Phi_s| < \infty$; find a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$, such that $(X_t^{\Phi}, t \le 1)$ is a $(\mathbb{Q}, \mathbf{F})$-martingale.

Suppose that $\mathbb{Q}$ exists. Then $\mathbb{Q}|_{\mathcal{F}_t} = L_t \mathbb{P}|_{\mathcal{F}_t}$ and $d\langle L, W\rangle_t = \Phi_t L_t dt$. Hence $L_t = 1 + \int_0^t L_s \Phi_s dW_s$ and $\int_0^t ds\, \Phi_s^2 < \infty, a.s.$. It remains to check that the local martingale L is a martingale. The positive local martingale L is a supermartingale and is a martingale when $\mathbb{E}(L_t) = 1$. We give below criteria which can be more widely applied. A first condition is due to Novikov.

Proposition 1.7.6.1 (Novikov's Condition.) *If the continuous martingale ζ satisfies:*

$$\mathbb{E}\left(\exp\left(\frac{1}{2}\langle\zeta\rangle_\infty\right)\right) < \infty \tag{1.7.6}$$

then ζ belongs to $\mathbf{H}^p$ for every $p \in [1, \infty[$ and $L = \mathcal{E}(\zeta)$ is a uniformly integrable martingale.

PROOF: See [RY], Chapter VIII, Proposition 1.15. □

The constant 1/2 in (1.7.6) is the best possible (see Kazamaki [517], Chapter 1, Example 1.5).

In the case where $\zeta_t = \int_0^t \theta_s dW_s$, Novikov's condition reads

$$\mathbb{E}\left(\exp\left(\frac{1}{2}\int_0^\infty \theta_s^2 ds\right)\right) < \infty\,.$$

Obviously, if we restrict our attention to the time interval $[0, T]$, Novikov's condition

$$\mathbb{E}\left(\exp\left(\frac{1}{2}\int_0^T \theta_s^2 ds\right)\right) < \infty \tag{1.7.7}$$

implies that $(L_t; 0 \le t \le T)$ is a martingale where $dL_t = \theta_t L_t dW_t$. Note that, Novikov's condition (1.7.7) is satisfied whenever θ is bounded.

It should be noted that if the local martingale $\mathcal{E}(\zeta)$ is uniformly integrable, i.e., if the family of r.v. $(\mathcal{E}(\zeta)_t, t \ge 0)$ is u.i., it is not necessarily a martingale (see Kazamaki [517], Chapter 1, Example 1.1. for a counter-example). If the local martingale $\mathcal{E}(\zeta)$ belongs to class D, i.e., if the family of r.v. $(\mathcal{E}(\zeta)_\tau, \tau$ stopping time) is u.i., then $\mathcal{E}(\zeta)$ is a martingale. The process $\mathcal{E}(\zeta)$ can be a martingale which is not uniformly integrable: take $\zeta = B$ where B is a Brownian motion.

Let us give two theorems (see Kazamaki [517]).

Theorem 1.7.6.2 (Kazamaki's Criterion.) *If ζ is a continuous local martingale such that the process $\exp(\frac{1}{2}\zeta)$ is a uniformly integrable submartingale, then the process $L = \mathcal{E}(\zeta)$ is a uniformly integrable martingale.*

Theorem 1.7.6.3 (BMO Criterion.) *Let ζ be a continuous martingale in BMO, then the process $L = \mathcal{E}(\zeta)$ is a uniformly integrable martingale.*

These conditions are often difficult to check and the following proposition is a useful tool. In a Markovian case, an easy condition is the following:

Proposition 1.7.6.4 (Non-explosion Criteria.) *Let $\zeta_t = \int_0^t b(s, W_s)dW_s$ where b satisfies*

$$\begin{cases} |b(s,x) - b(s,y)| \leq C|x-y| \,, \\ \sup_{s\leq t} |b(s,0)| \quad \leq C \,. \end{cases}$$

Then, the process $Z_t = \exp(\zeta_t - \frac{1}{2}\langle\zeta\rangle_t)\,; t \geq 0$ is a martingale. More generally, Z is a martingale as soon as the stochastic equation

$$dX_t = b(t, X_t)dt + dW_t,\; X_0 = 0$$

has a unique solution in law, without explosion.

PROOF: If the stochastic differential equation $X_t = W_t + \int_0^t b(s, X_s)ds$ has a unique solution, its law is locally equivalent to the Wiener measure (here, locally refers to the existence of a localizing sequence of stopping times). Let $T_n = \inf\{t : |X_t| = n\}$. We define an equivalent probability measure $\mathbf{W}^b$ via:

$$\mathbf{W}^b|_{\mathcal{F}_{t\wedge T_n}} = \exp\left[\int_0^{t\wedge T_n} b(s, X_s)dX_s - \frac{1}{2}\int_0^{t\wedge T_n} b^2(s, X_s)ds\right] \mathbf{W}|_{\mathcal{F}_{t\wedge T_n}} \,.$$

Then, for any $F_t \in \mathcal{F}_t$

$$\mathbf{W}^b(F_t \mathbb{1}_{\{t<T_n\}}) = \mathbf{W}\left(F_t \mathbb{1}_{\{t<T_n\}} \exp\left[\int_0^t b(s, X_s)dX_s - \frac{1}{2}\int_0^t b^2(s, X_s)ds\right]\right)$$

Letting n go to infinity, and using the fact that $T_n \to \infty$ both under $\mathbf{W}^b$ and $\mathbf{W}$, we obtain:

$$\mathbf{W}^b|_{\mathcal{F}_t} = \exp\left[\int_0^t b(s, X_s)dX_s - \frac{1}{2}\int_0^t b^2(s, X_s)ds\right] \mathbf{W}|_{\mathcal{F}_t} \,,$$

hence, the process

$$\exp\left(\int_0^t b(s, X_s)dX_s - \frac{1}{2}\int_0^t b^2(s, X_s)ds\right),\; t \geq 0$$

is a martingale. □

In the particular case $b(x) = \lambda x$ of the OU process, we deduce that the process

$$\exp\left(\lambda\frac{B_t^2 - t}{2} - \frac{\lambda^2}{2}\int_0^t ds B_s^2\right), \quad t \geq 0$$

is a martingale, for any $\lambda \in \mathbb{R}$.

Example 1.7.6.5 If $dX_t = dB_t + f(X_t)dt$, where $f : \mathbb{R} \to \mathbb{R}$ is a C^1-function, the Feller criterion (see McKean [636] or $\rightarrowtail$ Proposition 5.3.3.4) gives a sufficient condition for no explosion. Note that if $\mathbf{W}_x^{(f)}$ is the law of the solution, and τ the explosion time, then

$$\begin{aligned}\mathbf{W}_x^{(f)}|_{\mathcal{F}_t \cap \{t<\tau\}} &= \exp\left(\int_0^t f(X_s)dX_s - \frac{1}{2}\int_0^t f^2(X_s)ds\right)\mathbf{W}_x|_{\mathcal{F}_t}\\ &= \exp\left(F(X_t) - F(x) - \frac{1}{2}\int_0^t (f^2 + f')(X_s)ds\right)\mathbf{W}_x|_{\mathcal{F}_t}\end{aligned}$$

where F is an antiderivative of f. If $f(x) = |x|^\gamma$ with $\gamma > 1$, then there is explosion. In the case $f(x) = cx^{2n}$, one gets

$$\begin{aligned}\mathbb{P}^{(c,n)}(\tau > t) &= \mathbb{E}\left(\exp\left(c\int_0^t B_s^{2n} dB_s - \frac{c^2}{2}\int_0^t B_s^{4n} ds\right)\right)\\ &= \mathbb{E}\left(\exp\left(ct^{n+1/2}\int_0^1 B_s^{2n} dB_s - \frac{c^2}{2}t^{2n+1}\int_0^1 B_s^{4n} ds\right)\right),\end{aligned}$$

which gives an implicit description of the law of τ in terms of the joint law of $\left(\int_0^1 B_s^{2n} dB_s, \int_0^1 B_s^{4n} ds\right)$.

Example 1.7.6.6 Let us give one example of a local martingale which is not a martingale (we say that the local martingale is a strict local martingale). Let α be a positive real number and

$$dX_t = X_t Y_t^\alpha \sigma dB_t; \quad dY_t = Y_t a\, dB_t\,.$$

Using the fact that the process Z defined by $dZ_t = Z_t a dW_t + Z_t^{\alpha+1}\mu dt$ with $\mu > 0$ has a finite explosion time, Sin [800] proves that the process X is a strict local martingale.

Comment 1.7.6.7 There is an extensive literature on uniformly integrable exponential martingales. Let us mention Cherny and Shiryaev [169], Choulli et al. [181], Kazamaki [517] and Lépingle and Mémin [580].

1.7.7 Predictable Representation Property under a Change of Probability

Let $\mathbf{F}$ be the filtration of a Brownian motion W and θ an $\mathbf{F}$-adapted process such that the local martingale $L_t := \exp(\int_0^t \theta_s dW_s - \frac{1}{2}\int_0^t \theta_s^2\, ds)$ is a martingale. Let $\mathbb{Q}$ be the probability law, equivalent to $\mathbb{P}$ on $\mathcal{F}_t$ for any t, defined as $\mathbb{Q}|_{\mathcal{F}_t} = L_t\, \mathbb{P}|_{\mathcal{F}_t}$. Girsanov's theorem implies that $\widetilde{W}_t := W_t - \int_0^t \theta_s ds$ is an $(\mathbf{F}, \mathbb{Q})$-Brownian motion. Since, obviously, the process $\widetilde{W}$ is $\mathbf{F}$-adapted, the inclusion $\widetilde{\mathcal{F}}_t = \sigma(\widetilde{W}_s, s \le t) \subseteq \mathcal{F}_t$ holds. If θ is deterministic, then both filtrations are equal, but this is not the case in general (see Tsirel'son's example 1.5.5.6 or [822]). However, the representation theorem (see Section 1.6.1) extends to this framework.

Proposition 1.7.7.1 *Let W be a Brownian motion under $\mathbb{P}$, $\mathbf{F}$ its natural filtration, and $\mathbb{Q}$ a probability measure locally equivalent to $\mathbb{P}$. Let $\widetilde{W}$ be the martingale part of the $\mathbb{Q}$-semimartingale W. If M is a $(\mathbf{F}, \mathbb{Q})$-local martingale, there exists an $\mathbf{F}$-predictable process H such that*

$$\forall t, \quad M_t = M_0 + \int_0^t H_s \, d\widetilde{W}_s \,.$$

PROOF: It is enough to write the predictable representation of the $\mathbb{P}$-martingale ML as $M_t L_t = M_0 + \int_0^t \psi_s dW_s$. From Itô's formula and the obvious relation $M = (ML)L^{-1}$, the process M can be written as a stochastic integral w.r.t. $\widetilde{W}$. □

We have here an example of a "weakly Brownian filtration." We shall give other examples in ↣ Chapter 5.

Exercise 1.7.7.2 Prove the result recalled in Comment 1.4.1.6.
`Hint:` If $W_T^{(i)}$ could be written as $\int_0^T \phi_s^{(i)} dB_s$ for $i = 1, 2$, the properties of $\phi^{(i)}$ would lead to a contradiction. ◁

1.7.8 An Example of Invariance of BM under Change of Measure

Let $\mathbb{P}$ and $\mathbb{Q}$ be two equivalent probabilities on $(\Omega, \mathcal{F})$ and X a r.v. (or a process). We present a simple condition under which the law of X is the same under $\mathbb{P}$ and $\mathbb{Q}$, as well as an example (see also ↣ Example 1.7.3.10).

Proposition 1.7.8.1 *Let X be a real-valued $\mathbf{F}$-Brownian motion under $\mathbb{P}$ and L be the Radon-Nikodým density of $\mathbb{Q}$ w.r.t. $\mathbb{P}$. Then X is a $\mathbb{Q}$-Brownian motion if and only if X and L are $(\mathbf{F}, \mathbb{P})$-orthogonal martingales*

PROOF: Note that

$$\widetilde{X}_t = X_t - \int_0^t \frac{d\langle X, L\rangle_s}{L_s}$$

is a $(\mathbf{F}, \mathbb{Q})$- Brownian motion. □

This result admits an extension to the multidimensional case: Let W be an n-dimensional Brownian motion and $X_t = x + \int_0^t x_s \centerdot dW_s$ where $(x_t, t \geq 0)$ is an n-dimensional predictable process. The process X is a BM if and only if $|x_t|^2 = 1, ds \times d\mathbb{P} a.s.$ Let L be a Radon-Nikodým density. The process L admits the representation $L_t = 1 + \int_0^t \ell_s \centerdot dW_s$. The process X is a $(\mathbf{F}, \mathbb{Q})$-Brownian motion if and only if $x_t \centerdot \ell_t = 0$, $dt \times d\mathbb{P}$ a.s.

Example 1.7.8.2 If $W = (X, Y)$ is a 2-dimensional Brownian motion starting from (a, b), the pair (x, ℓ) where $x_t = W_t / |W_t|$ (stopped at the first time $|W|$ vanishes) and $\ell_t = (Y_t, -X_t)$ satisfies the previous condition.