Chapter 5
Maxwell's Equations

Abstract To describe the electromagnetic behavior of an eddy-current probe coil, and with the goal of becoming equipped to interpret the measured impedance of an eddy-current coil, this chapter describes expressions of Maxwell's equations in full and under the quasi-static regime that is of direct relevance to eddy-current NDE. From Maxwell's equations, equations governing the electromagnetic fields can be expressed in various ways. Interface conditions on the electromagnetic field is also provided. The interface conditions are needed, along with the governing equations, to set up a bounded system of equations that can be solved for the fields generated by eddy-current probe coils. Similarly, suitable governing equations and appropriate interface conditions provide the mathematical framework by which the influence of a test-piece—with or without a defect—on the probe impedance can be described.

5.1 Introduction

James Clerk Maxwell (1831–1879, Fig. 5.1) was a Scottish mathematician and physicist [1]. Early in life, he showed signs of mathematical talent, contributing an original research article to the Royal Society of Edinburgh at the age of only fifteen years. When he was reluctantly appointed as professor of experimental physics at the University of Cambridge (UK) later in life, however, he was not a great success as a lecturer. His lectures were too difficult for most students to understand and typically attracted an audience of only three or four. In research, he was brilliant. Maxwell made significant contributions to the understanding and theoretical descriptions of several important physical phenomena, including the kinetic theory of gases. His crowning achievement was in the field of electromagnetics, in which he expressed in mathematical form Faraday's speculations on the existence and effects of magnetic lines of force (Faraday had very little mathematical knowledge, remember). Maxwell gathered a few relatively simple equations that described the various known phenomena of electricity and magnetism, and coupled them together. He revealed that electricity and magnetism could not exist separately from one another but, if one was found, then the other existed as well. The field of *electromagnetism* was born.

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Fig. 5.1 James Clerk Maxwell, Scottish mathematician and physicist, 1831–1879 [2]

Maxwell showed that the oscillation of an electric charge produced an electromagnetic field that radiated outwards from its source at constant speed. This constant speed turned out to be the speed of light so Maxwell suggested that light itself was electromagnetic radiation! In addition, since charges could oscillate at any frequency, it seemed to Maxwell that an entire spectrum of electromagnetic radiation should exist, of which visible light constituted only a small part. As time went on, the existence of various other parts of the electromagnetic spectrum has been verified. All of today's wireless technology, and myriad other practical devices and theoretical endeavors, are founded on the work of Maxwell.

In the context of EC NDE, Maxwell's equations can be used to describe mathematically the interactions of a probe field with a test-piece, even to the point of being able to predict the change in coil impedance due to various types of defect in a structure. This mathematics is the only way to accurately determine these kinds of interactions. It is exciting and profound that mathematical physics can be used to calculate quantities of practical interest, that have a real impact on society in the context of inspections of aircraft, vehicles, bridges, nuclear power plants, and other structures whose integrity is critical to human and environmental safety.

5.2 Faraday's Law

Faraday's law is the first of Maxwell's equations that we shall examine. Faraday discovered that the induced electromotance in a closed circuit is equal to the time rate of change of the magnetic flux linkage in the circuit:

$$V = -\frac{d\lambda}{dt} = -N\frac{d\Psi}{dt} \tag{5.1}$$

where $\lambda = N\Psi$ is the flux linkage, N is the number of turns in the circuit, and Ψ is the magnetic flux through each turn. The negative sign is due to Lenz's Law, and shows that the induced voltage acts in such a way as to oppose the flux change producing it. This means that the direction of current flow in the circuit is such that the magnetic field produced by the induced current opposes the change in the original magnetic field. One consequence of Lenz's Law in EC NDE is that the direction of circulation of the induced eddy currents is opposite to that of the current flowing in the inducing coil.

From (5.1) a point form of Faraday's Law can be derived. First, express the right-hand-side of (5.1) in terms of the magnetic induction field $\mathbf{B}$, by replacing Ψ with the surface integral given in (4.16). For $N = 1$,

$$V = -\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{S}, \tag{5.2}$$

where S is an open surface bounded by path C that describes the circuit of interest, such as the circular path made by a loop of wire. The path is closed but permits a discontinuity in $\mathbf{E}$ integrated around that path. The discontinuity is mathematically necessary, to represent the practical incorporation of a voltage source into a drive circuit, or to allow for the measurement of a potential drop induced in a pick up circuit. Imagine a closed loop of wire whose ends are connected via a twisted pair to the termini of a variable voltage source, or to a voltmeter. The loop is closed, but a voltage drop exists across the ends of the wire. Expressing the voltage in terms of the line integral of $\mathbf{E}$, just mentioned, gives

$$\oint_C \mathbf{E}\cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{S} \tag{5.3}$$

Noting that it is the magnetic induction field $\mathbf{B}$ that is varying with time, rather than the loop area, we obtain

$$\oint_C \mathbf{E}\cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{S}. \tag{5.4}$$

Next apply Stokes' Theorem, Sect. 10.3.6, to the left-hand-side of (5.4). Stokes' Theorem requires that $\mathbf{E}$ has continuous derivatives on S but this condition does not exclude the possibility that its line integral ($V = \int \mathbf{E}\cdot d\mathbf{l}$) is discontinuous—a point whose necessity was just described. The application of Stokes' Theorem yields

$$\int_S \nabla\times\mathbf{E}\cdot d\mathbf{S} = -\int_S \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{S}. \tag{5.5}$$

For these two integrals to be equal, their integrands must be equal and, consequently,

$$\nabla\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \tag{5.6}$$

Expression (5.6) is Faraday's Law in point form, and is also one of Maxwell's equations. This relation is at the heart of EC NDE, describing the fact that a time-varying field of magnetic induction, produced by an eddy- current coil, produces an electric field (and hence induces eddy currents) in a metal test-piece nearby.

5.3 Maxwell–Ampère Law

The second of Maxwell's equations that we shall examine is the Maxwell–Ampère Law. This relation was born out of a deficiency that Maxwell perceived in Ampère's circuital theorem. Beginning with the latter, Ampère's circuital theorem states that the line integral of the *magnetostatic* field $\mathbf{H}$ around a closed path is equal to the net current I enclosed by the path,

$$\oint \mathbf{H} \cdot d\mathbf{l} = I. \tag{5.7}$$

To obtain Ampère's circuital theorem in point form, apply Stokes' Theorem to the left-hand-side of (5.7) and relate I to the current density as in (2.2). Then

$$\int_S \nabla \times \mathbf{H} \cdot d\mathbf{S} = \int_S \mathbf{J} \cdot d\mathbf{S} \tag{5.8}$$

and hence

$$\nabla \times \mathbf{H} = \mathbf{J}. \tag{5.9}$$

This relation is known as Ampère's law and tells us that the magnetostatic field $\mathbf{H}$ is not conservative, but that $\mathbf{J}$ is its *source*. (The curl, $\nabla\times$, of a conservative field is identically zero.)

Maxwell recognized that Ampère's Law is incomplete for time-varying fields, because it violates the requirement that current be continuous. For more detail, see [3]. Adding displacement current density $\mathbf{J}_d = \partial\mathbf{D}/\partial t$ to the conduction current density $\mathbf{J}$ already present in (5.9) gives

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}, \tag{5.10}$$

which is Maxwell's equation (based on Ampère's circuital theorem) for a time-varying field. Displacement current

$$I_d = \int_S \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{S}$$

is the "current" that flows in a dielectric (between the plates of a capacitor, for example).

The main relevance of the Maxwell–Ampère law to EC NDE is that it enables description of the magnetic fields associated with (i) the electric current flowing in an eddy-current coil and (ii) the eddy currents flowing in a test-piece.

5.3.1 *Quasi-static Regime*

Despite the fact that eddy currents must be induced by a *time-varying* field associated with the current flowing in an eddy-current drive coil, it turns out that the displacement current can be neglected in most EC NDE analyses. The following argument shows why this is the case.

The total current density in any conductor is the sum of the conduction current density and the displacement current density, given by

$$\mathbf{J} + \mathbf{J}_d = \left(\sigma + \epsilon \frac{\partial}{\partial t}\right) \mathbf{E}$$

where Ohm's Law (2.15) and constitutive relation (2.33) have been used to form the above expression. In phasor form, for sinusoidal time variation of the fields, the above sum can be expressed as $(\sigma + j\omega\epsilon)\mathbf{E}$. For typical metals tested by EC NDE, σ is on the order of 10 MS/m and the maximum frequency employed is around 10 MHz. Therefore,

$$|\mathbf{J}| \sim 10^7 |\mathbf{E}|$$

and

$$|\mathbf{J}_d| \sim 2\pi \times 10^7 \times 8.85 \times 10^{-12} |\mathbf{E}| \sim 10^{-3} |\mathbf{E}|$$

This means that $|\mathbf{J}| \gg |\mathbf{J}_d|$ and displacement current can be neglected even for the highest frequencies most commonly employed in EC NDE, to a very good approximation. Under these circumstances, EC NDE operates in a *quasi-static* regime in which

$$\nabla \times \mathbf{H} \approx \mathbf{J}. \tag{5.11}$$

This relation is obviously equivalent to Ampère's Law, although we should keep in mind that strictly Ampère's Law applies only to magnetostatic fields. The field produced by an EC coil is necessarily time-varying in order for eddy currents to be induced in a test-piece at all and the term "*quasi*-static" is used to remind us of this fact.

As a related point of interest, radio-frequency EC technology that operates up to 100 MHz has been developed in recent years for inspection of lower conductivity materials such as carbon–fiber-based composites [4] and ceramic–matrix composites. As inspection frequency increases and conductivity of the test-piece decreases it is clear from the above discussion that the quasi-static approximation becomes less accurate and, for very low conductivity materials (insulators) it is the displacement current that dominates [5].

5.4 Gauss' Law

The third of Maxwell's equations that we shall consider is Gauss' Law. Gauss' Law states that the total electric flux Ψ_e through any closed surface is equal to the total charge enclosed by that surface, Q_{enc}.

$$\Psi_e = \oint_S \mathbf{D} \cdot d\mathbf{S} = Q_{enc} \tag{5.12}$$

Further,

$$Q = \int_V \rho_v dV$$

where ρ_v is volume charge density (electric charge per unit volume at a point) measured in Coulombs per meter cubed (C/m^3) and V is the volume enclosed by surface S. Hence,

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_V \rho_v dV \tag{5.13}$$

and, applying the divergence theorem (also of Gauss, (10.48)) to the left-hand side of (5.13) yields

$$\int_V \nabla \cdot \mathbf{D}\, dV = \int_V \rho_v dV \tag{5.14}$$

from which the point form of Gauss' Law is obtained:

$$\nabla \cdot \mathbf{D} = \rho_v. \tag{5.15}$$

In other words, the strength of divergence of **D** from a point is determined by the electric charge per unit volume at that point. Relation (5.15) allows **D**, or **E** by (2.33), to be determined easily for many symmetric distributions of charge, but note that the relation always holds irrespective of the particular shape of the charge distribution.

5.5 Gauss' Law for Magnetic Fields

The fourth and final equation of Maxwell that we shall consider is Gauss' Law for magnetic fields; the counterpart of the equation examined in the previous section (Sect. 5.4) for electric fields. Unlike for **D**, the magnetic induction field **B** has no sources or sinks. This is a consequence of the fact that the fundamental source of the magnetic field is charge in motion, which always gives rise to a magnetic dipole rather than individual "magnetic charges". To derive Gauss' Law for magnetic fields, we can follow a development similar to that in the previous section, but now there is no "charge" enclosed in surface S and

$$\Psi = \oint_S \mathbf{B} \cdot d\mathbf{S} = \int_V \nabla \cdot \mathbf{B}\, dV = 0. \tag{5.16}$$

At a point,

$$\nabla \cdot \mathbf{B} = 0 \tag{5.17}$$

and we see that $\mathbf{B}$ is solenoidal (divergenceless). This means that the lines of $\mathbf{B}$ are always closed loops.

5.5.1 Magnetic Vector Potential

The form of Gauss' Law as written in (5.17) invites the definition of the magnetic vector potential $\mathbf{A}$ such that

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{5.18}$$

since, according to identity (10.44), the divergence of the curl of any vector is zero. One reason for defining the magnetic vector potential is that it is easier, for some configurations, to solve a problem in terms of the magnetic vector potential than in terms of the magnetic induction field itself. In this text, $\mathbf{A}$ is employed in the derivation of the analytical expression for $\mathbf{B}$ at all points in space due to a current loop in free space, Sect. 6.3.2.

5.6 Interface Conditions on the Electromagnetic Field

In later sections of this text, we shall see how Maxwell's equations may be manipulated to provide governing equations for the electromagnetic field in the vicinity of an eddy-current coil. When an eddy-current coil is brought near to a metal test-piece, the electromagnetic field due to the coil penetrates the conductor, and the field exists in more than one material (air and metal) at the same time. The conductor surface is a *boundary* or *interface* between the two dissimilar media, and the electromagnetic field obeys certain conditions there. These conditions are known as *boundary conditions* or *interface conditions*. In order to solve the governing equations and obtain a mathematical description of the electromagnetic field in a region of space occupied by more than one medium, we need to know the interface conditions that the fields must obey. The conditions on the four field quantities $\mathbf{E}$, $\mathbf{D}$, $\mathbf{H}$ and $\mathbf{B}$, and on the

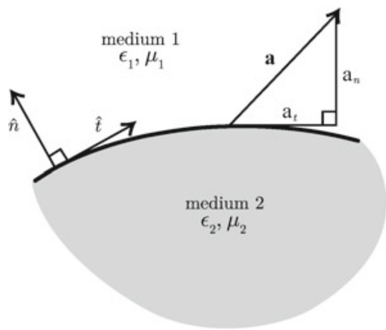

Fig. 5.2 A vector field **a** at the interface between two media with different dielectric and magnetic properties. The unit vectors normal ($\hat{n}$) and tangential ($\hat{t}$) to the interface are shown, along with the normal and tangential components of **a**

current density **J**, are presented in this section but not derived. For a derivation, see [3] or any standard undergraduate textbook on electricity and magnetism.

Consider a vector field that is oriented arbitrarily with respect to a boundary between two media, as shown in Fig. 5.2.

Electric Field It can be shown that *the tangential component of* **E** is continuous at the boundary. This means that the tangential component of **E** does not change as the boundary is traversed. Expressing this mathematically,

$$(\mathbf{E}_1 - \mathbf{E}_2) \times \hat{n} = 0 \qquad \text{or} \qquad E_{1t} - E_{2t} = 0. \tag{5.19}$$

In (5.19), the subscripts 1 and 2 refer to the two media and the subscripts n and t refer to the normal and tangential components of the vector at the interface, respectively.

Electric Displacement If ρ_s is the surface density of free charge placed deliberately on the interface, then it can be shown that *the jump* (discontinuity) *in the normal component of* **D** *at the boundary is equal to the free surface charge density on the boundary*:

$$(\mathbf{D}_1 - \mathbf{D}_2) \cdot \hat{n} = \rho_s \qquad \text{or} \qquad D_{1n} - D_{2n} = \rho_s. \tag{5.20}$$

Magnetic Field If $\mathbf{K} = K\hat{t}$ is a surface current measured in A/m that flows on the boundary, then *the jump in the tangential component of* **H** *at the boundary is equal to the surface current*:

$$(\mathbf{H}_1 - \mathbf{H}_2) \times \hat{n} = \mathbf{K} \qquad \text{or} \qquad H_{1t} - H_{2t} = K. \tag{5.21}$$

Magnetic Induction *The normal component of* **B** *is always continuous across a boundary*:

$$(\mathbf{B}_1 - \mathbf{B}_2) \cdot \hat{n} = 0 \qquad \text{or} \qquad B_{1n} - B_{2n} = 0. \tag{5.22}$$

Current Density From (5.19) with Ohm's law (2.15) the following boundary condition on the tangential component of the current density can be obtained:

Table 5.1 Interface conditions on the electromagnetic field and current density. The density of free surface charge at the boundary is represented by ρ_s (C/m^2). The surface current density at the boundary is K (A/m). In most cases of relevance to EC NDE, ρ_s and K are zero

Vector field	Tangential component	Normal component
Electric field, **E**	$E_{1t} - E_{2t} = 0$	$\epsilon_1 E_{1n} - \epsilon_2 E_{2n} = \rho_s$
Electric displacement, **D**	$D_{1t}/\epsilon_1 - D_{2t}/\epsilon_2 = 0$	$D_{1n} - D_{2n} = \rho_s$
Magnetic field, **H**	$H_{1t} - H_{2t} = K$	$\mu_1 H_{1n} - \mu_2 H_{2n} = 0$
Magnetic induction, **B**	$B_{1t}/\mu_1 - B_{2t}/\mu_2 = K$	$B_{1n} - B_{2n} = 0$
Current density, **J**	$J_{1t}/\sigma_1 - J_{2t}/\sigma_2 = 0$	$J_{1n} - J_{2n} = 0$

$$J_{1t}/\sigma_1 - J_{2t}/\sigma_2 = 0. \tag{5.23}$$

The normal component of the current density is continuous at the boundary:

$$J_{1n} - J_{2n} = 0. \tag{5.24}$$

Looking at continuity conditions (5.20) and (5.21), a parallel with Maxwell's equations (5.15) and (5.9) can be seen. In volumetric space, the volume charge density ρ_v gives rise to divergence in **D** as expressed in (5.9), whereas surface charge density ρ_s at an interface gives rise to a jump in D_n across the interface, (5.20). The behavior stems from the fact that ρ_v and ρ_s are *sources* of **D**. Similarly, the existence of a surface current **K** at a boundary gives rise to a jump in H_t across the boundary as expressed in (5.21), whereas the volume current density **J** gives rise to a circulating magnetic field, (5.9). Currents **J** and **K** are sources of **H**.

The constitutive relations (2.25) and (2.33) can be used with the above interface conditions (5.19)–(5.22) to obtain a full set of conditions on both the normal and tangential components of all four vector fields. These are given, along with conditions on the normal and tangential components of the current density **J**, in Table 5.1. Commonly, no surface charge density exists at a boundary and $\rho_s = 0$ in the above relations. Then, D_n is continuous at the boundary. Similarly, there is often no surface current at a boundary and $K = 0$. Then, H_t is continuous at the boundary.

Example: Interface conditions on the electric field The electric field just outside a cylindrical rod whose axis lies along the z-axis is given by $\mathbf{E}_2 = 60\hat{z}$ V/m. The conductivity of the rod is 46 MS/m. Find the current density in region 1, just inside the rod.

Solution: In a cylindrical coordinate system, the tangential component of the electric field at the rod surface is $\mathbf{E}_t = E_\phi\hat{\phi} + E_z\hat{z}$. Hence, $E_{1z} = E_{2z}$ and $J_z = \sigma E_{1z} = \sigma E_{2z} = 46 \times 10^6 \times 60 = 2.76 \times 10^9$ A/m^2.

Example: Interface conditions on the magnetic field Two extensive homogeneous isotropic ferrites (which can support no surface currents) meet on the plane $z = 0$.

For $z > 0$, $\mu_{r1} = 100$ and for $z < 0$, $\mu_{r2} = 20$. A uniform magnetic field $\mathbf{H}_1 = 5\hat{x} - 2\hat{y} + 3\hat{z}$ A/m exists for $z \geq 0$. Find $\mathbf{H}_2$ for $z \leq 0$.

Solution: $\mathbf{H}_2$ is also uniform. Considering first the component of the field normal to the boundary, in the z-direction, $\mu_1 H_{1z} = \mu_2 H_{2z}$ which means that

$$H_{2z} = \frac{\mu_{r1}}{\mu_{r2}} H_{1z} = \frac{100}{20} \times 3 = 15 \text{ A/m}.$$

Tangential to the boundary, in the absence of free surface currents, $H_{1t} = H_{2t}$. Hence

$$\mathbf{H}_{1t} = 5\hat{x} - 2\hat{y} = \mathbf{H}_{2t}.$$

Putting the tangential and normal components of the field together gives the solution

$$\mathbf{H}_2 = 5\hat{x} - 2\hat{y} + 15\hat{z} \text{ A/m}.$$

5.7 Summary

In this chapter, the equations of Maxwell have been described and their relation to EC NDE has been discussed. For ease of reference, Maxwell's equations in both differential and integral form are collected together in Table 5.2. It should be noted that, in most cases of relevance to EC NDE, the quasi-static approximation can be assumed, which means that $|\mathbf{J}| \gg |\partial\mathbf{D}/\partial t|$ and $\nabla \times \mathbf{H} \approx \mathbf{J}$. In conductors, it is also generally the case that $\rho_v = 0$. In this way, two of the four Maxwell's equations are simplified in their application to EC NDE.

The boundary conditions that govern the behavior of the electromagnetic fields and the current density at the interface between two media have also been described in this chapter and are summarized in Table 5.1. Again, some simplification is generally possible in problems of relevance to EC NDE. In particular, it is usually the case that ρ_s and K are zero in the treatment of EC boundary-value problems.

The stage is now set for proceeding to develop governing equations and boundary conditions which can be solved to compute quantities of relevance to EC NDE. This is the task to which we turn in the next chapter.

Table 5.2 Maxwell's equations in differential and integral form. ρ_v is volume density of free charge (C/m^3). $\mathbf{J}$ is current density (A/m^2). In most cases of relevance to EC NDE, $\nabla \times \mathbf{H} \approx \mathbf{J}$ and $\rho_v = 0$

Law	Differential form	Integral form
Faraday's law	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{S}$
Maxwell–Ampère law	$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$	$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right) \cdot d\mathbf{S}$
Gauss' law	$\nabla \cdot \mathbf{D} = \rho_v$	$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_V \rho_v dV$
Gauss' law for magnetic fields	$\nabla \cdot \mathbf{B} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$

5.8 Exercises

1. A long, cylindrical conductor has radius a and carries current I uniformly distributed over its cross section. Use Ampère's law to show that, *inside* the conductor, $H_\phi = I\rho/(2\pi a^2)$, $\rho < a$, where ρ and ϕ are coordinates of a cylindrical system whose axis coincides with the axis of the conductor.
2. Beginning with Maxwell's equations, derive

   (a) the magnetic field in air due to an infinitesimally thin, long, straight wire carrying current $\mathcal{I}$ in the z-direction

   $$\mathbf{H} = \hat{\phi}\frac{\mathcal{I}}{2\pi\rho}, \qquad \rho > 0,$$

   where ρ and ϕ are coordinates of the cylindrical system and

   (b) the electric field in air due to the same infinitesimally thin wire

   $$\mathbf{E} = \hat{z}\,\frac{\mathcal{I}}{2\pi} j\omega\mu_0 \ln\rho, \qquad \rho > 0,$$

   previously given in (2.17) and (2.14), respectively.
3. Explain what is meant by the *quasi-static regime* in the context of EC NDE. At frequencies used in EC NDE, for what class of materials is this (a) a good approximation and (b) a poor approximation?
4. If a **B**-field is specified everywhere by $B_x = ky$, $B_y = -kx$, $B_z = 0$, k being constant, find an expression for the current density **J** which would give rise to it.
5. The electric field just outside a cylindrical rod whose axis lies along the z-axis is given by $\mathbf{E} = 15\hat{z}$ V/m. The conductivity of the rod is 43 MS/m. Find the current density just inside the rod.
6. Two extensive homogeneous isotropic ferrites (which can support no surface currents) meet on the plane $z = 0$. For $z > 0$, $\mu_{r1} = 50$ in medium 1 and for $z < 0$, $\mu_{r2} = 5$ in medium 2. A uniform magnetic field $\mathbf{H}_1 = 3\hat{x} - 4\hat{y} + 2\hat{z}$ A/m exists for $z \geq 0$. Find $\mathbf{H}_2$ for $z \leq 0$.

References

1. Asimov, I.: Asimov's Biographical Encyclopedia of Science and Technology. Pan Books, London (1975)
2. Public domain image. http://en.wikipedia.org/wiki/Michael_Faraday. Accessed 2 July 2014
3. Sadiku, M.N.O.: Elements of Electromagnetics, 4th edn. Oxford University Press, New York (2007)
4. Heuer, H.: Review on quality assurance along the CFRP value chain - Non-destructive testing of fabrics, preforms and CFRP by HF radio wave techniques. Compos. B-Eng. **77**, 494–501 (2015)
5. Gäbler, S., Heuer, H., Heinrich, G.: Measuring and imaging permittivity of insulators using high-frequency eddy-current devices. IEEE Trans. Instrum. Meas. **64**, 2227–2238 (2015)