Chapter 6
Single Resource Revenue Management with Dependent Demands

6.1 Introduction

Revenue managers struggled for decades with the problem of finding optimal control mechanisms for fare class structures with dependent demands. In this context, a resource, such as seats on a plane, can be offered at different fares with potentially different restrictions and ancillary services, and the demand for those fares is interdependent. The question is what subset of the fares (or assortment of products) to offer for sale at any given time. Practitioners often use the term open, or open for sale, for a fare that is part of the offered assortment, and the term closed for fares that are not part of the offered assortment. For many years, practitioners preferred to model time implicitly by seeking extensions of Littlewood's rule and EMSR type heuristics to the case of dependent demands. Finding the right way to extend Littlewood's rule proved to be more difficult than anticipated. An alternative approach, favored by academics and gaining traction in industry, is to model time explicitly. In this chapter, we will explore both formulations but most of our attention is devoted to the more tractable model where time is treated explicitly.

In Sect. 6.2, we give a dynamic programming formulation for the revenue management problem with a single resource with dependent demands. In Sect. 6.2.2, we use a linear program to give an upper bound on the optimal total expected revenue and extract a bid-price heuristic from the linear program. In Sect. 6.2.3, we discuss a model where fares cannot be made available once they are closed. In Sect. 6.3, we focus on models that handle time implicitly. As we will see, these models are complicated by the fact that changing the protection level also changes the number of potential customers for higher fare classes. Nevertheless, we develop a heuristic that performs reasonably well.

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6.2 Explicit Time Models

In this section, we consider models where time is considered explicitly. Modeling time explicitly allows for time-varying arrival rates and time-varying discrete choice models. Customers arrive according to a time heterogeneous Poisson process with intensity $\{\lambda_t : 0 \leq t \leq T\}$, where T is the length of the sales horizon, and t represents the time-to-go. Thus, time T is the beginning of the sales horizon and time 0 is the end. The total expected number of customers that arrive during the last t units of time is $\Lambda_t := \int_0^t \lambda_s ds$.

The set of products is $N := \{1, \dots, n\}$. We obtain a revenue of p_j from the sale of product j. There is a single resource with limited capacity. The sale of each product consumes one unit of the resource. A consumer arriving at time-to-go t selects from the offered assortment based on a discrete choice model, say $\{\pi_{tj}(\cdot) : j \in N\}$. More precisely, if we offer the subset S of products, then the customer arriving at time t selects product $j \in S$ with probability $\pi_{tj}(S)$. Let $V(t, x)$ denote the maximum total expected revenue that can be attained over the last t units of the sale horizon with x units of capacity. Assume that at time-to-go t, we offer set $S \subseteq N$ and keep this set of fares open for δt units of time. If $\lambda_t \delta t \ll 1$, then the probability that a customer arrives and requests product $j \in S$ is $\lambda_t \pi_{tj}(S)\delta t + o(\delta t)$, so

$$
\begin{aligned}
V(t,x) =& \max_{S\subseteq N} \Bigg\{ \sum_{j\in S} \lambda_t\, \delta t\, \pi_{tj}(S)\, [p_j + V(t-\delta t, x-1)] \\
& \qquad + \Bigg(1 - \lambda_t\, \delta t \sum_{k\in S} \pi_{tj}(S)\Bigg) V(t-\delta t, x) \Bigg\} + o(\delta t) \\
=& V(t-\delta t, x) + \lambda_t\, \delta t\, \max_{S\subseteq N} \sum_{j\in S} (p_j - \Delta V(t-\delta t, x))\, \pi_{tj}(S) + o(\delta t) \\
=& V(t-\delta t, x) + \lambda_t\, \delta t\, \max_{S\subseteq N} R_t(S, \Delta V(t-\delta t, x)) + o(\delta t) \qquad (6.1)
\end{aligned}
$$

for $x \geq 1$, where $R_t(S, z) := \sum_{j\in S}(p_j - z)\, \pi_{tj}(S)$ and $\Delta V(t-\delta t, x) := V(t - \delta t, x) - V(t - \delta t, x - 1)$ for $x \geq 1$.

We can subtract $V(t - \delta t, x)$ from both sides of Eq. (6.1), divide by δt and take limits as $\delta t \downarrow 0$ to obtain the Hamilton–Jacobi–Bellman (HJB) equation

$$
\frac{\partial V(t,x)}{\partial t} = \lambda_t\, \mathcal{R}_t(\Delta V(t,x)) \qquad (6.2)
$$

with boundary conditions that $V(t, 0) = 0$ and $V(0, x) = 0$, where $\mathcal{R}_t(z) := \max_{S\subseteq N} R_t(S, z)$.

The value function $V(t, x)$ is often computed approximately by solving a discrete-time dynamic program based on (6.1). This involves rescaling time and the arrival rates, using $\delta t = 1$, and dropping the $o(\delta t)$ term. Time can be rescaled by a positive real number, say a, such that $T \leftarrow aT$ is an integer by setting $\lambda_t \leftarrow \frac{1}{a}\lambda_{t/a}$, $\pi_{tj}(S) \leftarrow \pi_{t/a,j}(S)$. The resulting dynamic program is given by

$$V(t, x) = V(t-1, x) + \lambda_t \mathcal{R}_t(\Delta V(t-1, x)), \tag{6.3}$$

with the same boundary conditions.

The generic optimization problem in formulations (6.2) and (6.3) is of the form $\mathcal{R}_t(z) := \max_{S \subseteq N} R_t(S, z)$, where $z \in \Re_+$ is a non-negative scalar representing the marginal value of capacity. Since there are 2^n subsets $S \subseteq N$, solving the assortment optimization problem could require the evaluation of the objective function for an exponential number of subsets. Moreover, the problem has to be solved for different values of $z = \Delta V(t, x)$ as the marginal value of capacity changes with the state of the system (t, x).

As discussed at the end of chapter on assortment optimization, for any choice model there is a collection $\mathcal{E} = \{E_j : j \in K\}$, $K = \{0, 1, \ldots, k\}$ of efficient sets that can be ordered so that $\Pi_j := \sum_{i \in E_j} \pi_i(E_j)$ is increasing in $j \in K$, and $E_0 = \emptyset$. Letting $R_j := R(E_j) := \sum_{k \in E_j} p_k \pi_k(E_j)$ we can define the slopes $u_j := (R_j - R_{j-1})/(\Pi_j - \Pi_{j-1})$ for $j = 1, \ldots, k$. Then $u_0 > u_1 > \ldots > u_k > u_{k+1} = 0$, where for convenience we set $u_0 = \infty$ and $u_{k+1} = 0$. Then, the efficient set E_j is optimal to offer for all $z \in [u_{j+1}, u_j]$ for $j = 0, 1, \ldots, k$. At boundary points u_j, both E_j and E_{j-1} are optimal. This implies that the index that maximizes $R_j - z\Pi_j$ over $j \in K$ is given by $a(z) := \max\{j : u_j > z\}$.

If we apply the idea of efficient sets in the context of (6.2) where different choice models may apply at different values of t, we let $z = \Delta V(t, x)$, and $\{u_{tj} : j \in K_t\}$ be the slopes corresponding to the efficient sets in $\mathcal{E}_t = \{E_{tj} : j \in K_t\}$. In this case,

$$a(t, x) := \max\{j \in K_t : u_{tj} > \Delta V(t, x)\}$$

is the index of the efficient set in $\mathcal{E}_t$ that maximizes $R_{tj} - \Delta V(t, x)\Pi_{tj}$.[1] Consequently, it is optimal to offer assortment $E_{t,a(t,x)}$ at state (t, x), corresponding to efficient set E_{tj} with index $j = a(t, x)$. For formulation (6.3), the definition of $a(t, x)$ is the same except that we use $\Delta V(t-1, x)$ instead of $\Delta V(t, x)$ on the right side. The following result is valid for both formulations (6.2) and (6.3).

Theorem 6.1 *The index $a(t, x)$ is decreasing in t and increasing in x over every time interval where the choice model is time invariant.*

Sometimes it is convenient to refer to action j as shorthand for offering efficient set E_{tj}. Thus, it is optimal to offer action $a(t, x)$ at state (t, x). As t increases, $\Delta V(t, x)$ increases and the optimal solution shifts to efficient sets with a smaller

[1] Strictly speaking we should say an index, but the index is unique except at boundary points.

probability of a sale. In contrast, as x increases, $\Delta V(t, x)$ decreases, and the optimal solution shifts to efficient sets with larger sale probability. In general, we cannot say that we close lower fares when t is large (or open lower fares when x is large) because the efficient sets need not be nested-by-fare. For choice models for which the efficient sets enjoy the nested-by-fare property, we can talk of opening and closing fares as the state dynamics change with the understanding that if a fare is open, then all higher fares will be open at the same time.

6.2.1 Formulation as an Independent Demand Model

Consider formulation (6.2) for the dependent demand model. Is it possible to transform this into an independent demand model? The answer is yes, provided that the efficient sets have been properly identified. The transformation into an independent demand model requires creating artificial products that have artificial, but independent demands. Given λ_t, and $(\Pi_{tj}, R_{tj}),\ j \in K_t$ for the dependent demand model, the transformation is obtained by setting $\tilde{\lambda}_{tj} := \lambda_t[\Pi_{tj} - \Pi_{t,j-1}]$ and $\tilde{p}_{tj} := u_{tj}$ for $j \in K_t$. The set of $\tilde{p}_{tj}$ are known as transformed fares, and are equal to the slopes between efficient fares.

Then, the independent demand formulation

$$\frac{\partial V(t,x)}{\partial t} = \sum_{j \in K_t} \tilde{\lambda}_{tj}[\tilde{p}_{tj} - \Delta V(t,x)]^+ \tag{6.4}$$

generates the correct value function. The proof of equivalence for the formulation (6.4) is left as an exercise.

The transformation is part of folklore and has appeared in many papers. The fact that the transformation works for the dynamic program has led some practitioners to conclude that the transformed data can be used as an input to Littlewood's rule or to heuristics such as the EMSR, as the transformed demands are independent. There are two problems with this approach. First, the transformation is often used without first identifying the efficient sets. More serious, perhaps, is the fact that Littlewood's rule and its variants require the low-before-high demand arrival pattern. This is tantamount to assuming that Poisson demands with parameters $\tilde{\lambda}_{tj}$ will arrive low-before-high, but these are artificial demands from customers willing to buy under action j but not under action $j - 1$. When capacity is allocated to this marginal customer, we cannot prevent some degree of demand cannibalization from customers willing to buy under action $j - 1$ into some of the fares in action j. We will return to this issue in Sect. 6.3.

6.2.2 Upper Bound and Bid-Price Heuristic

We will now present an upper bound on the value functions (6.2) for the case where the choice models are time invariant and later explain how to deal with the time variant case. The upper bound is based on approximate dynamic programming with affine value function approximations.

It is well known that a dynamic program can be solved as a mathematical program by making the value function at each state a decision variable. This leads to the formulation $V(T, c) = \min F(T, c)$ subject to the constraints $\partial F(t, x)/\partial t \geq \lambda_t[R_j - \Delta F(t, x)\Pi_j] \quad \forall(t, x)$ for all $j \in K$, where the decision variables are the class of non-negative functions $F(t, x)$ that are differential in x with boundary conditions $F(t, 0) = F(0, x) = 0$ for all $t \in [0, T]$ and all $x \in \{0, 1, \ldots, c\}$. At optimality $F(t, x) = V(t, x)$ for all $t \in [0, T]$ and all $x \in \{0, 1, \ldots, c\}$.

While this formulation is daunting, it becomes easier once we restrict the functions to be of the affine form

$$\tilde{F}(t, x) = \int_0^t \beta_s(x)ds + xz_t \quad \text{and} \quad z_t \geq 0.$$

We will further restrict ourselves to the invariant case: $\beta_s(x) = \beta_s$ for $x > 0$, $\beta_s(0) = 0$, $z_t = z$ for $t > 0$ and $z_0 = 0$. With this restriction, the partial derivative and marginal value of capacity have simple forms and the boundary conditions are satisfied. More precisely,

$$\partial \tilde{F}(t, x)/\partial t = \beta_t \quad \text{and} \quad \Delta \tilde{F}(t, x) = z \quad \forall t > 0, x > 0,$$

with $\tilde{F}(t, 0) = \tilde{F}(0, t) = 0$.

This reduces the program to $\tilde{V}(T, c) = \min \tilde{F}(T, c) = \min \int_0^T \beta_t dt + cz$, subject to $\beta_t \geq \lambda_t[R_j - z\Pi_j] \, \forall j \in K$. Since we have restricted the set of functions $F(t, x)$ to be affine we obtain an upper bound $\bar{V}(T, c) \geq V(T, c)$.

Since this is a minimization problem, the optimal choice for β_t is $\beta_t = \lambda_t \max_{j \in K}[R_j - z\Pi_j] = \lambda_t \mathcal{R}(z)$, where $\mathcal{R}(z) := \max_{j \in K}[R_j - z\Pi_j]$ is a decreasing, convex, non-negative and piecewise linear function of z. Consequently, the overall problem reduces to

$$\bar{V}(T, c) = \min_{z \geq 0} \left[\int_0^T \lambda_t \mathcal{R}(z)dt + cz \right] = \min_{z \geq 0} [\Lambda \mathcal{R}(z) + cz], \tag{6.5}$$

where $\Lambda := \int_0^T \lambda_t dt$ is the aggregate arrival rate over the sales horizon $[0, T]$. Notice that $\Lambda \mathcal{R}(z) + cz$ is convex in z.

We next link the upper bound to the function $Q(\rho)$ that was used in the previous chapter to define efficient sets. We reproduce the definition of $Q(\rho)$ here for convenience. Let $\Pi(S) := \sum_{i \in S} \pi_i(S)$ be the probability of a sale when assortment $S \subset N$ is offered, and let

$$
\begin{aligned}
Q(\rho) := \max & \sum_{S \subseteq N} R(S)t(S) \\
\text{subject to} \quad & \sum_{S \subseteq N} \Pi(S)t(S) \leq \rho \\
& \sum_{S \subseteq N} t(S) = 1 \\
& t(S) \geq 0 \quad \forall S \subseteq N,
\end{aligned}
$$

denote the maximum expected revenue from selecting a convex combination of assortments whose sale rate is bounded by the scalar $\rho \geq 0$. We are now ready to link $\bar{V}(T, c)$ and $Q(\rho)$.

Proposition 6.2

$$\bar{V}(T, c) = \Lambda Q(c/\Lambda).$$

Having established the upper bound, we now turn to finding an optimal solution to problem (6.5), which we will denote by $z(T, c)$. We will show that $z(T, c)$ is one of the slopes $u_j := (R_j - R_{j-1})/(\Pi_j - \Pi_{j-1})$ between consecutive efficient sets $\mathcal{E} = \{E_j, j \in K\}$, $K = \{0, 1, \ldots, k\}$. Let $\rho := c/\Lambda$ and define

$$a(T, c) := \min\{j \leq k : \Pi_j > \rho\},$$

and set $a(T, c) := k + 1$ if $\rho \geq \Pi_k$.

If $a(T, c) = k + 1$, then the marginal value of capacity is $z(T, c) = u_{k+1} := 0$, and it is optimal to offer the efficient set E_k. If $a(T, c) = j \leq k$, then $\Pi_{j-1} \leq \rho < \Pi_j$, and the marginal value of capacity is $z(T, c) = u_{a(T,c)} = u_j$, with the primal solution offering a convex combination of E_{j-1} and E_j, where the weight on set E_j positive unless $\rho = \Pi_{j-1}$ in which case it is optimal to offer set E_{j-1} all the time. In summary, $z(T, c) = u_{a(T,c)}$. If $z(T, c) = u_{k+1} = 0$, it is optimal to offer set $A(T, c) = E_k$. Otherwise, it is optimal to offer a convex combination of sets $E_{a(T,c)-1}$ and $E_{a(T,c)}$.

We now define a bid-price heuristic that offers set $E_j = k$ if $a(T, c) = k + 1$ and offers set $E_{a(T,c)}$ otherwise. This heuristic offers the efficient set with the highest sales probability that is part of the optimal solution to the primal problem. We can express this heuristic more succinctly by offering at state (T, c) the set

$$A(T, c) = E_{\min(a(T,c),k)}$$

while capacity is positive, and switching to $E_0 = \emptyset$ when capacity is exhausted. When $z(T, c) > 0$, we have $\Lambda \Pi_{a(T,c)} \geq c$, so the bid-price heuristic is likely to exhaust capacity before the end of the horizon. An obvious refinement is to compute $a(t_j, x_j)$ at reading dates $T = t_1 > t_2 > \ldots > t_J > t_{J+1} = 0$ and to offer the set

$$E_{\min(a(t_j,x_j),k)} \quad \forall\, x_j > 0,$$

Table 6.1 Efficient sets in Example 6.3

Index	Efficient set	Π_i	R_i	u_i
0	$\emptyset$	0	0	
1	{1}	0.50	500.00	1000
2	{1, 2}	$0.6\bar{6}$	$533.3\bar{3}$	200

Table 6.2 Upper bound and optimal actions in Example 6.3

c	ρ	$\bar{V}(T, c)$	$Z(T, c)$	t_1	t_2	Sales E_1	Sales E_2	Fare 1 sales	Fare 2 sales
12	0.30	12,000	1000	0.6	0.0	12	0	12	0
16	0.40	16,000	1000	0.8	0.0	16	0	16	0
20	0.50	20,000	1000	1.0	0.0	20	0	20	0
22	0.55	20,400	200	0.7	0.3	14	8	18	4
24	0.60	20,800	200	0.4	0.6	8	16	16	8
26	0.65	21,200	200	0.1	0.9	2	24	14	12
28	0.70	21,333	0	0.0	1.0	0	26.6	13.3	13.3
32	0.80	21,333	0	0.0	1.0	0	26.6	13.3	13.3

over time interval $(t_{j+1}, t_j]$, where $\rho_j := x_j/\Lambda_{t_j}$, and

$$a(t_j, x_j) := \min\{i \in K : \Pi_i > \rho_j\},$$

and $a(t_j, x_j) := k + 1$ if $\rho_j > \Pi_k$. This refinement helps curb sales at marginal fares.

Example 6.3 Suppose that $p_1 = 1000$, $p_2 = 600$, and a BAM with $v_0 = v_1 = v_2 = e^1$. Table 6.1 shows the efficient sets, together with the sale and revenue rates, and the slopes between efficient sets. We will assume that the aggregate arrival rate over the sales horizon $[0, T] = [0, 1]$ is $\lambda = 40$, so the expected number of customers to arrive over $[0, T]$ is $\Lambda = 40$. Table 6.2 provides the upper bound $\bar{V}(T, c)$ for different values of c. The table also provides $z(T, c)$ and the solution to the problem $Q(\rho)$ in terms of the proportion of time sets t_1 and t_2 that the efficient sets $E_1 = \{1\}$ and $E_2 = \{1, 2\}$ are offered. Notice that sales under action E_1 first increase and then decrease as c increases. While this may not be intuitive, the logical explanation is that when we have sufficient capacity we exclusively use E_2 because this is the efficient set that maximizes the revenue rate (since $R_2 > R_1$). When $\rho = c/\Lambda < \Pi_2$, we have insufficient capacity to sustain sales at E_2 and that is why we have $t_1 > 0$ for $c \leq 26 < \Lambda\Pi_2$.

If the discrete choice model is time varying, then we have $\mathcal{R}_t(z) = \max_{j \in K_t}[R_{tj} - z\Pi_{tj}]$, resulting in

$$\bar{V}(T, c) = \min_{z \geq 0} \left[\int_0^T \lambda_t \mathcal{R}_t(z) dt + cz \right],$$

where the objective function is also convex in z. For this model, it is also possible to find a bid-price heuristic but it is important to update the dual variable at least as frequently as the changes in the underlying choice model.

6.2.3 Monotone Fares

Formulations (6.2) and (6.3) implicitly assume that the capacity provider can offer any subset of fares at any state (t, x). This flexibility works well if customers are not strategic. Otherwise, customers may anticipate the possibility of lower fares and postpone their purchases in the hope of being offered lower fares at a later time. If customers act strategically, the capacity provider may counter by imposing restrictions that do not allow lower fares to reopen once they are closed. Actions to limit strategic customer behavior are commonly employed by revenue management practitioners, although competitive pressures sometimes force them to deviate from this goal.

Let $V_S(t, x)$ be the optimal expected revenue when the state is (t, x), and we are restricted to use only assortments that are subsets of S. The system starts at state (T, c) and $S = N$. If a strict subset U of S is used then all fares in $U' := \{j \in N : j \notin U\}$ are permanently closed and cannot be offered at a later state regardless of the evolution of sales. To obtain a discrete-time counterpart to (6.3), let

$$W_U(t, x) := V_U(t-1, x) + \lambda_t[R_t(U) - \pi_t(U)\Delta V_U(t-1, x)].$$

Then the dynamic program is given by

$$V_S(t, x) := \max_{U \subseteq S} W_U(t, x) \tag{6.6}$$

with boundary conditions $V_S(t, 0) = V_S(0, x) = 0$ for all $t \geq 0$, $x \geq 1$ and $S \subseteq N$. The goal of this formulation is to find $V_N(T, c)$ and the corresponding optimal control policy.

Notice that formally the state of the system has been expanded to (S, t, x) where S is the last offered set and (t, x) are, as usual, the time-to-go and the remaining inventory. Formulation (6.3) requires optimizing over all subsets $S \subseteq N$, while formulation (6.6) requires an optimization over all subsets $U \subseteq S$ for any given $S \subseteq N$. Obviously the complexity of these formulations is large if the number of fares is more than a handful. Airlines typically have over twenty different fares so the number of possible subsets gets large very quickly. Fortunately, in many cases, we do not need to do the optimization over all possible subsets. As we have seen, the optimization can be reduced to the set of efficient fares. For the p-GAM, we know that the collection of efficient sets is contained by the nested-by-fare sets $\{E_0, E_1, \ldots, E_n\}$ where $E_0 = \emptyset$ and $E_j := \{1, \ldots, j\}$ for $j = 1, \ldots, n$. For the p-GAM, and any other model for which the nested-by-fare property holds, the state

Table 6.3 Value functions for dynamic allocation policies in Example 6.5

c	$\rho = c/\Lambda$	$V_3(T, c)$	$V(T, c)$	$\bar{V}(T, c)$
4	0.16	3769	3871	4000
6	0.24	5356	5534	6000
8	0.32	6897	7013	7477
10	0.40	8259	8335	8950
12	0.48	9304	9382	10,423
14	0.56	9976	10,111	10,846
16	0.64	10,418	10,583	11,146
18	0.72	10,803	10,908	11,447
20	0.80	11,099	11,154	11,504
22	0.88	11,296	11,322	11,504
24	0.96	11,409	11,420	11,504
26	1.04	11,466	11,470	11,504
28	1.12	11,490	11,492	11,504
30	1.20	11,498	11,500	11,504
32	1.27	11,502	11,503	11,504

of the system reduces to (j, t, x) where E_j is the last offered set at (t, x). For such models, the formulation (6.6) reduces to

$$V_j(t, x) = \max_{k \le j} W_k(t, x) \tag{6.7}$$

where $V_j(t, x) := V_{E_j}(t, x)$ and

$$W_k(t, x) = V_k(t-1, x) + \lambda_t[R_{kt} - \Pi_{kt} \Delta V_k(t-1, x)],$$

$R_{kt} := \sum_{l \in S_k} p_l \pi_{lt}(E_k)$ and $\Pi_{kt} := \sum_{l \in S_k} \pi_{lt}(E_k)$. Let

$$a_j(t, x) := \max\{k \le j : W_k(t, x) = V_j(t, x)\}. \tag{6.8}$$

Theorem 6.4 *At state (j, t, x), it is optimal to offer efficient set*

$$E_{a_j(t,x)} := \{1, \ldots, a_j(t, x)\},$$

where $a_j(t, x)$ given by (6.8) is decreasing in t and increasing in x over time intervals where the choice model is time invariant.

The proof of this result follows the sample path arguments of the corresponding proof in the independent demand chapter. Clearly $V_1(t, x) \le V_2(t, x) \le V_n(t, x) \le V(t, x) \le \bar{V}(t, x)$.

Example 6.5 Table 6.3 presents the value functions $V(T, c)$ that results from solving the dynamic program (6.3), the upper bound $\bar{V}(T, c) = \Lambda Q(c/\Lambda)$, as well as the performance $V_3(T, c)$ corresponding to the dynamic program (6.7). All of these quantities are computed for the MNL model with fares $p_1 = \$1000$, $p_2 = \$800$, $p_3 = \$500$ with price sensitivity $\beta_p = -0.0035$, schedule quality $s_i = 200$ for $i = 1, 2, 3$ with quality sensitivity $\beta_s = 0.005$, and an outside alternative with $p_0 = \$1100$ and schedule quality $s_0 = 500$, Gumbel parameter $\phi = 1$, arrival rate $\lambda = 25$ and $T = 1$. Recall that for the MNL model, the attractiveness $v_i = e^{\phi\mu_i}$ where μ_i is the mean utility. In this case $\mu_i = \beta_p p_i + \beta_s s_i$. The computations were done with time rescaled by a factor $a = 10{,}000$. Not surprisingly $V_3(T, c) \leq V(T, c)$ as $V_3(T, c)$ constrains fares to remain closed once they are closed for the first time. However, the difference in revenues is relatively small except for small values of c.

6.3 Implicit Time Models

We now turn to models where the notion of time is implicit. The effort is mostly directed to finding extensions of Littlewood's rule to the case of dependent demands. We will assume that we are working with a choice model with efficient sets that are nested: $E_0 \subseteq E_1 \ldots \subseteq E_k$, even if they are not nested-by-fare. We continue using the notation $\Pi_j := \sum_{k \in E_j} \pi_k(E_j)$ and $R_j := \sum_{k \in E_j} p_k \pi_k(E_j)$, so the slopes $u_j := (R_j - R_{j-1})/(\Pi_j - \Pi_{j-1})$, $j = 1, \ldots, k$ are positive and decreasing. We will denote by $q_j := R_j/\Pi_j$ the average fare, conditioned on a sale, under efficient set E_j (action j) for $j \geq 1$ and define $q_0 = 0$.

Suppose that the total number of potential customers over the selling horizon is a random variable D. For example, D can be Poisson with parameter Λ. Let D_j be the total demand if only set E_j is offered. Then D_j is conditionally binomial with parameters D and Π_j, so if D is Poisson with parameter Λ, then D_j is Poisson with parameter $\Lambda \Pi_j$.

We will present an exact solution for the two-fare class problem and a heuristic for the multi-fare case. The solution to the two-fare class problem is, in effect, an extension of Littlewood's rule for discrete choice models. The heuristic for the multi-fare problem applies the two-fare result to each pair of consecutive actions, say j and $j + 1$, and selects the best j.

6.3.1 Two Fare Classes

For the two-fare case, while capacity is available, provider will offer either action 2 (associated with efficient set $E_2 = \{1, 2\}$) or action 1 (associated with efficient set $E_1 = \{1\}$). If the provider runs out of inventory, he offers action 0, corresponding to $E_0 = \emptyset$. Action 2 is optimal for ample capacity, while action 1 is optimal when capacity is scarce. Our task is to find an optimal number, say $y(c) \in \{0, \ldots, c\}$ of units to protect for sale under action 1.

To find $y(c)$, we start with an arbitrary protection level $y \in \{0, \ldots, c\}$. The expected revenue under action 2 is $q_2\, \mathbb{E}[\min(D_2, c-y)]$ where q_2 is the average fare per unit sold under action 2. Of the $(D_2 - c + y)^+$ customers denied bookings, a fraction $\beta := \Pi_1/\Pi_2$ will be willing to purchase under action 1. Thus, the demand under action 1 will be a conditionally binomial random variable, say $U(y)$, with a random number $(D_2 - c + y)^+$ of trials and success probability β. The expected revenue that results from allowing up to $c - y$ bookings under action 2 is given by

$$W_2(y, c) := q_1\, \mathbb{E}[\min(U(y), \max(y, c - D_2))] + q_2\, \mathbb{E}[\min(D_2, c - y)],$$

where the first term corresponds to the revenue under action 1. Conditioning the first term on the event $D_2 > c - y$, allows us to write

$$W_2(y, c) = q_1\, \mathbb{E}[\min(U(y), y)|D_2 > c-y)]\, \mathbb{P}(D_2 > c-y) + q_2\, \mathbb{E}[\min(D_2, c-y)].$$

The reader may be tempted to follow the marginal analysis idea presented in Chap. 1 for the independent demand case. In the independent demand case, the marginal value of protecting one more unit of capacity is realized only if the marginal unit is sold. The counterpart here would be $\mathbb{P}(U(y) \geq y|D_2 > c - y)$, and a naive application of marginal analysis would protect the y-th unit whenever $q_1\, \mathbb{P}(U(y) \geq y|D_2 > c - y) > q_2$.

However, with dependent demands, protecting one more unit of capacity *also* increases the potential demand under action 1 by one unit. This is because an additional customer is denied capacity under action 2 (when $D_2 > c - y$) and this customer may end up buying a unit of capacity under action 1 even when not all the y units are sold. Ignoring this can lead to very different results in terms of protection levels. The correct analysis is to acknowledge that an extra unit of capacity is sold to the marginal customer with probability $\beta\, \mathbb{P}(U(y-1) < y-1|D_2 > c-y)$. This suggests protecting the y-th unit whenever

$$q_1\, [\mathbb{P}(U(y) \geq y|D_2 > c - y) + \beta\, \mathbb{P}(U(y-1) < y - 1|D_2 > c - y)] > q_2.$$

To simplify the left-hand side, notice that conditioning on the decision of the marginal customer results in

$$\begin{aligned}\mathbb{P}(U(y) \geq y|D_2 > c - y) = \beta\, \mathbb{P}(U(y-1) \geq y - 1|D_2 > c - y)\\ + (1-\beta)\, \mathbb{P}(U(y-1) \geq y|D_2 > c - y).\end{aligned}$$

Combining terms leads to protecting the y-th unit whenever

$$q_1\, [\beta + (1-\beta)\, \mathbb{P}(U(y-1) \geq y|D_2 > c - y)] > q_2.$$

Let

$$r := u_2/q_1 = \frac{q_2 - \beta q_1}{(1-\beta)q_1}, \tag{6.9}$$

denote the critical fare ratio. In industry, the ratio r given by (6.9) is known as fare adjusted ratio, in contrast to the unadjusted ratio q_2/q_1 that results when $\beta = 0$.

The arguments above suggest that the optimal protection level can be obtained by selecting the largest $y \in \{1, \ldots, c\}$ such that $\mathbb{P}(U(y-1) \geq y | D_2 > c - y) > r$ provided that $\mathbb{P}(U(0) \geq 1 | D_2 \geq c) > r$ and to set $y = 0$ otherwise.

To summarize, an optimal protection level can be obtained by setting $y(c) = 0$ if $\mathbb{P}(U(0) \geq 1 | D_2 > c) \leq r$; otherwise setting

$$y(c) = \max\{y \in \{1, \ldots, c\} : \mathbb{P}(U(y-1) \geq y | D_2 > c - y) > r\}. \tag{6.10}$$

One important observation is that for dependent demands the optimal protection level $y(c)$ is first increasing and then decreasing in c. The reason is that for low capacity it is optimal to protect all the inventory for sale under action 1. However, for high capacity, it is optimal to allocate all the capacity to action 2. The intuition is that action 2 has a higher revenue rate, so with high capacity we give up trying to sell under action 1. This is clearly seen in Table 6.2 of Example 6.3. Heuristic solutions that propose protection levels of the form $\min(y^h, c)$, which are based on independent demand logic, are bound to do poorly when c is close to $\Lambda \Pi_2$.

One can derive Littlewood's rule for discrete choice models (6.10) formally by analyzing $\Delta W_2(y, c) := W_2(y, c) - W_2(y-1, c)$, the marginal value of protecting the y-th unit of capacity for sale under action 1.

Proposition 6.6

$$\Delta W_2(y, c) = [q_1(\beta + (1-\beta)\,\mathbb{P}(U(y-1) \geq y | D_2 > c - y) - q_2]\,\mathbb{P}(D_2 > c - y). \tag{6.11}$$

Moreover, the expression in brackets is decreasing in $y \in \{1, \ldots, c\}$.

Consequently, $\Delta W_2(y, c)$ has at most one sign change. If it does, then it must be from positive to negative. $W_2(y, c)$ is then maximized by the largest integer $y \in \{1, \ldots, c\}$, say $y(c)$, such that $\Delta W_2(y, c)$ is positive, and by $y(c) = 0$ if $\Delta W_2(1, c) < 0$. This confirms Littlewood's rule for discrete choice models (6.10).

6.3.2 *Heuristic Protection Levels*

While the computation of $y(c)$ and $V_2(c) = W_2(y(c), c)$ is not numerically difficult, the conditional probabilities involved may be difficult to understand conceptually. Moreover, the formulas do not provide intuition and do not generalize easily to multiple fares. In this section, we develop a simple heuristic to find near-optimal protection levels that provides some of the intuition that is lacking in the computation of optimal protection levels $y(c)$. In addition, the heuristic can easily be extended to multiple fares.

The heuristic consists of approximating the conditional binomial random variable $U(y-1)$ with parameters $(D_2 - c + y - 1)^+$ and β by its conditional expectation, namely by $(\text{Bin}(D_2, \beta) - \beta(c+1-y))^+$. Since $\text{Bin}(D_2, \beta)$ is just D_1, the approximation yields $(D_1 - \beta(c+1-y))^+$. We expect this approximation to be reasonable if $\mathbb{E}[D_1] \geq \beta(c+1-y)$. This is equivalent to the condition

$$c < y^p + \mathbb{E}[D_2 - D_1] = y^p + \Lambda\pi_2(1-\beta),$$

where we have $y^p = \max\{y \in \mathcal{N} : \mathbb{P}(D_1 \geq y) > r)\}$. In this case, $\mathbb{P}(U(y-1) \geq y|D_2 > c-y)$ can be approximated by the expression $\mathbb{P}(D_1 \geq (1-\beta)y+\beta(c+1))$. We think of $y^p = (1-\beta)y + \beta(c+1)$ as a pseudo-protection level that will be modified to obtain a heuristic protection level when the approximation is reasonable, e.g., when $c < y^p + \mathbb{E}[D_2 - D_1]$, by setting

$$y^h(c) = \max\left\{y \in \mathcal{N} : y \leq \frac{y^p - \beta(c+1)}{(1-\beta)}\right\} \wedge c.$$

If $c > y^p + \mathbb{E}[D_2 - D_1]$, we set $y^h(c) = 0$. Thus, the heuristic will stop protecting capacity for action 1 when c is sufficiently large! This makes sense since action 2 maximizes the expected revenue per customer and this is optimal when capacity is sufficiently abundant.

Notice that the heuristic involves three modifications to Littlewood's rule for independent demands. First, instead of using the first choice demand for fare 1, when both fares are open, we use the stochastically larger demand D_1 for fare 1, when it is the only open fare. Second, instead of using the ratio of the fares p_2/p_1 we use the modified fare ratio $r = u_2/q_1$ based on sell-up adjusted fare values. From this we obtain a pseudo-protection level y^p that is then modified to obtain $y^h(c)$. Finally, we keep $y^h(c)$ if capacity is scarce, e.g., if $c < y^p + E[D_2 - D_1]$ and set $y^h(c) = 0$ otherwise. In summary, the heuristic involves a different distribution, a fare adjustment, and a modification to the pseudo-protection level. The following example illustrates the performance of the heuristic.

Example 6.7 Suppose that $p_1 = 1000$, $p_2 = 600$, and a BAM with $v_0 = v_1 = v_2 = e^1$ and that $\Lambda = 40$ as in Example 6.3. We report the optimal protection level $y(c)$, the heuristic protection level $y^h(c)$, the upper bound $\bar{V}(c)$, the optimal expected revenue $V(c)$ of the uni-directional formulation (6.6), the performance $V_2(c)$ of $y(c)$ and the performance $V_2^h(c) = W_2(y^h(c), c)$ of $y^h(c)$, and the percentage gap between $(V_2(c) - V_2^h(c))/V_2(c)$ in Table 6.4. Notice that the performance of the static heuristic, $V_2^h(c)$, is almost as good as the performance $V_2(c)$ of the optimal policy under formulation (6.6).

Table 6.4 Performance of the heuristic for two-fare problem in Example 6.7

c	$y(c)$	$y^h(c)$	$\bar{V}(c)$	$V(c)$	$V_2(c)$	$V_2^h(c)$	Gap (%)
12	12	12	12,000	11,961	11,960	11,960	0.00
16	16	16	16,000	15,610	15,593	15,593	0.00
20	20	20	20,000	18,324	18,223	18,223	0.00
24	21	24	20,800	19,848	19,526	19,512	0.07
28	9	12	21,333	20,668	20,414	20,391	0.11
32	4	0	21,333	21,116	21,036	20,982	0.26
36	3	0	21,333	21,283	21,267	21,258	0.05
40	2	0	21,333	21,325	21,333	21,322	0.01

6.3.3 Theft Versus Standard Nesting and Arrival Patterns

The types of inventory controls used in the airline's reservation system along with the demand order of arrival are additional factors that must be considered in revenue management optimization. If $y(c) < c$, we allow up to $c - y(c)$ bookings under action 2 with *all* sales counting against the booking limit $c - y(c)$. In essence, the booking limit is imposed on action 2 (rather than on fare 2). This is known as theft nesting. Implementing theft nesting controls may be tricky if a capacity provider needs to exert controls through the use of standard nesting, i.e., when booking limits are only imposed on the lowest open fare. This modification may be required either because the system is built on the philosophy of standard nesting or because users are accustomed to thinking of imposing booking limits on the lowest open fare. Here we explore how one can adapt protection levels and booking limits for the dependent demand model to situations where controls must be exerted through standard nesting.

A fraction of sales under action 2 corresponds to sales under fare p_2. This fraction is given by $\omega := \pi_2(E_2)/\Pi_2$. So if booking controls need to be exerted directly on the sales at fare p_2, we can set booking limit $\omega(c - y(c))$ on sales at fare p_2. This is equivalent to using the larger protection level

$$\hat{y}(c) := (1 - \omega)c + \omega y(c) \tag{6.12}$$

for sales at fare 1. This modification makes implementation easier for systems designed for standard nesting controls, and it performs very well under a variety of demand arrival patterns.

It is possible to combine demand choice models with fare arrival patterns by sorting customers through their first choice demand and then assuming a low-before-high demand arrival pattern. For the two-fare case, the first choice demands for fare 1 and fare 2 are Poisson random variables with rates $\Lambda\pi_1(E_2)$ and $\Lambda\pi_2(E_2)$. Assume now that customers whose first choice demand is for fare 2 arrive first, perhaps because of purchasing restrictions associated with this fare. customers whose first choice is fare 2 will purchase this fare if available. They will consider upgrading to fare 1 if fare 2 is not available. One may wonder what kind of control is effective to deal with this arrival pattern. It turns out that setting protection level $\hat{y}(c)$ given by (6.12) for fare 1, with standard nesting, is optimal for this arrival pattern and is very robust to other (mixed) arrival patterns.

6.3.4 Multiple Fare Classes

For multiple fare classes, finding optimal protection levels can be very complex. However, if we limit our search to the best two consecutive efficient sets we can easily adapt the results from the two-fare class to deal with the multiple-fare class problem. For any $j \in \{1, \ldots, n-1\}$, consider the problem of allocating capacity between action j (corresponding to efficient set E_j) and action $j+1$ (corresponding to efficient set E_{j+1}) where action $j+1$ is offered first. In particular, suppose we want to protect $y \leq c$ units of capacity for action j against action $j+1$. We will then sell $\min(D_{j+1}, c-y)$ units under action $j+1$ at an average fare q_{j+1}. We will then move to action j with $\max(y, c-D_{j+1})$ units of capacity and residual demand $U_j(y)$, where $U_j(y)$ is conditionally binomial with parameters $(D_{j+1}-c+y)^+$ and $\beta_j := \Pi_j/\Pi_{j+1}$. Assuming we do not restrict sales under action j, the expected revenue under actions $j+1$ and j will be given by

$$W_{j+1}(y,c) := q_j \, \mathbb{E} \min(U_j(y), \max(y, c-D_{j+1})) + q_{j+1} \, \mathbb{E} \min(D_{j+1}, c-y). \qquad (6.13)$$

Notice that under action j we will either run out of capacity or will run out of customers. Indeed, if $U_j(y) \geq y$ then we run out of capacity, and if $U_j(y) < y$ then we run out of customers. Let $W_{j+1}(c) := \max_{y \leq c} W_{j+1}(y,c)$ and set $W_1(c) := q_1 \, \mathbb{E} \min(D_1, c)$. Clearly,

$$V_n(c) \geq \max_{1 \leq j \leq n} W_j(c), \qquad (6.14)$$

so a simple heuristic is to compute $W_j(c)$ for each $j \in \{1, \ldots, n\}$ and select j to maximize $W_j(c)$. To find an optimal protection level for E_j against E_{j+1}, we need to compute $\Delta W_{j+1}(y,c) = W_{j+1}(y,c) - W_{j+1}(y-1,c)$. For this we can repeat the analysis of the two-fare case to show that an optimal protection level for action E_j against action E_{j+1} is given by $y_j(c) = 0$ if $\Delta W_{j+1}(1,c) < 0$ and by

$$y_j(c) = \max\{y \in \{1, \ldots, c\} : \mathbb{P}(U_j(y-1) \geq y | D_{j+1} > c-y) > r_j\}, \qquad (6.15)$$

where

$$r_j := \frac{u_{j+1}}{q_j} = \frac{q_{j+1} - \beta_j q_j}{(1-\beta_j)q_j}.$$

Alternatively, we can use the heuristic described in the two-fare section to approximate $U_j(y-1)$ by $D_j - \beta_j(c+1-y)$ and use this in turn to approximate the conditional probability in (6.15) by $\mathbb{P}(D_j \geq y + \beta(c-y+1))$. This involves finding the pseudo-protection level

$$y_j^p = \max\{y \in \mathcal{N} : \mathbb{P}(D_j \geq y) > r_j\}.$$

If $c < y_j^p + d_{j+1}$, then

$$y_j^h(c) = \max\left\{y \in \mathcal{N}_+ : y \leq \frac{y_j^p - \beta_j(c+1)}{1-\beta_j}\right\} \wedge c, \tag{6.16}$$

and set $y^h(c) = 0$ if $c \geq y_j^p + d_{j+1}$.

We will let $V_n^h(c)$ be the expected revenues resulting from applying the protection levels.

Example 6.8 Consider now a three fare example with fares $p_1 = 1000$, $p_2 = 800$, $p_3 = 500$, schedule quality $s_i = 200$, $i = 1, 2, 3$, $\beta_p = -0.0035$, $\beta_s = 0.005$, $\phi = 1$. Then $v_1 = 0.082$, $v_2 = 0.165$, $v_3 = 0.472$. Assume that the outside alternative is a product with price $p_0 = 1100$ and schedule quality $s_0 = 500$ and that the expected number of potential customers is Poisson with parameter $\Lambda = 25$. Table 6.5 reports the protection levels $y_j(c)$ and $y_j^h(c)$ as well as $V_3(c)$ and $V_3^h(c)$ for $c \in \{4, 6, \ldots, 26, 28\}$. As shown in table, the heuristic performs very well with a maximum gap of 0.14% relative to $V_3(c)$ which was computed through exhaustive search. It is also instructive to see that $V_3^h(c)$ is not far from $V_3(c, T)$, as reported in Table 6.3, for the dynamic model. In fact, the average gap is less than 0.5% while the largest gap is 1.0% for $c = 18$.

Example 6.8 suggests that the heuristic for the static model works almost as well as the optimal dynamic program $V_n(T, c)$ for the case where efficient sets are nested-by-fare and fares cannot be opened once they are closed for the first time. Thus, the multi-fare heuristic described in this section works well to prevent strategic customers from gaming the system provided that the efficient fares are nested-by-fare as they are in a number of important applications. While the heuristic for the static model gives up a bit in terms of performance relative to the dynamic model, it has several advantages. First, the static model does not need the overall demand to be Poisson. Second, the static model does not need as much detail in terms of the arrival rates. These advantages are part of the reason why people in industry have a preference for static models, even though dynamic models are easier to understand, easier to solve to optimality, and just as easy to implement.

Table 6.5 Performance of Heuristic for three-fare problem in Example 6.8

c	ρ	$y_1(c)$	$y_2(c)$	$y_1^h(c)$	$y_2^h(c)$	$V_3(c)$	$V_3^h(c)$	Gap (%)
4	0.16	4	4	4	4	3769	3769	0.00
6	0.24	3	6	3	6	5310	5310	0.00
8	0.32	1	8	1	8	6845	6845	0.00
10	0.40	0	10	0	10	8217	8217	0.00
12	0.48	0	12	0	12	9288	9288	0.00
14	0.56	0	14	0	14	9971	9971	0.00
16	0.64	0	13	0	14	10,357	10,354	0.02
18	0.72	0	9	0	10	10,700	10,694	0.05
20	0.80	0	5	0	6	11,019	11,019	0.00
22	0.88	0	4	0	2	11,254	11,238	0.14
24	0.96	0	3	0	0	11,391	11,388	0.03
26	1.04	0	2	0	0	11,458	11,450	0.08
28	1.12	0	2	0	0	11,488	11,485	0.03

6.4 End of Chapter Problems

1. For the MNL model, let $\Pi_j = \sum_{k \in S_j} \pi_k(S_j)$ and $R_j = \sum_{k \in S_j} p_k \pi_k(S_j)$ for $j = 0, \ldots, n$. Consider the dynamic program

$$V(t, x) = V(t-1, x) + \lambda_t \max_{j \in K} \left[R_j - \Pi_j \Delta V(t-1, x) \right],$$

with boundary condition $V(t, 0) = V(0, x) = 0$ for $t \geq 0$ and $x \in \mathcal{N}$, where $M = \{0, 1, \ldots, n\}$. Let

$$u_j = \frac{R_j - R_{j-1}}{\Pi_j - \Pi_{j-1}}.$$

(a) Show that $u_j = (p_j - r_{j-1})/(1 - \pi_{j-1})$ and in particular that $u_1 = p_1$.
(b) Show that it is optimal to offer set $S_{a(t,x)}$ at state (t, x) where

$$a(t, x) = \max\{j : u_j \geq \Delta V(t, x)\}.$$

Hint: You may want to use the following two facts: 1) $\Delta V(t, x) \leq p_1$ and 2) u_j is decreasing in j for the MNL model.

2. Code the following dynamic program:

$$V(t, x) = V(t-1, x) + \lambda_t \max_{j} \left[R_j - \Pi_j \Delta V(t-1, x) \right] \tag{6.17}$$

with boundary condition $V(t, 0) = V(0, x) = 0$ for $t \geq 0$ and $x \in \mathcal{N}$.

Run the code for a flight with 3 fares $p_1 = 1150$, $p_2 = 950$, $p_3 = 650$, quality attributes $q_1 = 1000$, $q_2 = 850$, $q_3 = 750$, price sensitivity $\beta_p = -1$ and quality sensitivity $\beta_q = 1.25$. Suppose that the utility of fare i is $U_i = \mu_i + \epsilon_i$ where $\mu_i = \beta_p p_i + \beta_q q_i$, $i = 1, 2, 3$ and the ϵ_is are independent Gumbel random variables with parameter $\phi = 0.01$. Assume $\lambda_t = \lambda = 0.01$, $T = 10{,}000$. Find $V(T, c)$ for $c \in \{35, 40, 55, 60, 65, 70, 75, 80, 85, 90\}$.

3. Prove that the transformation that leads to the independent demand formulation (6.4) provides the correct value function.
4. Show that an alternative formulation is given by

$$\frac{\partial V(t,x)}{\partial t} = \max_{j \in M_t} \lambda_t \Pi_{tj}[q_{tj} - \Delta V(t,x)]$$

where $q_{tj} = R_{tj}/\Pi_{tj}$, and for convenience we define $q_{0t} = 0$. We can think of $\lambda_t \Pi_{tj}$ as the demand rate associated with average fare q_{tj}, which reduces the dynamic revenue management model with dependent demands to a dynamic pricing model with a finite price menu.
5. Consider a two-fare problem with dependent demands governed by a BAM with parameters $v_0 = 1$, $v_1 = 1.1$, $v_2 = 1.2$. Suppose that the fares are $p_1 = 1000$ and $p_2 = 720$ and that the total number of potential customers is Poisson with parameter $\Lambda = 55$.

(a) Determine the sale rate Π_i and the revenue rate R_i per arriving customer under action $i = 1, 2$, where $E_1 = \{1\}$ and $E_2 = \{1, 2\}$.
(b) For capacity values $c \in \{16, 17, \ldots, 35\}$ solve the linear problem

$$\begin{aligned} \Lambda R(c/\Lambda) \;=\; \max \quad & \Lambda[R_1 t_1 + R_2 t_2] \\ \text{s.t.} \quad & \Lambda[\Pi_1 t_1 + \Pi_2 t_2] \le c \\ & t_1 + t_2 + t_0 = 1 \\ & t_i \ge 0, \quad \forall i = 0, 1, 2, \end{aligned}$$

and determine the expected number of units $\Lambda \Pi_i t_i$ sold under action $i = 1, 2$.
(c) From your answer to part b, determine the optimal number of units sold for each fare $i = 1, 2$ for each value of $c \in \{16, \ldots, 35\}$. What happens to optimal number of sales for each fare $1 = 1, 2$ as c increases?
(d) Find the largest integer, say y^p, such that $P(D_1 \ge y) > r$ where D_1 is Poisson with parameter $\Lambda_1 = \Lambda \Pi_1$, $r = u_2/q_1$, $u_2 = (R_2 - R_1)/(\Pi_2 - \Pi_1)$ and $q_1 = R_1/\Pi_1 = p_1$.
(e) Let $\Lambda_2 = \Lambda \Pi_2$ and $\beta = \Lambda_1/\Lambda_2$. For each $c \in \{16, 11, \ldots, 35\}$, check if $c < y^p + \Lambda(\Pi_2 - \Pi_1)$ and if so, let

$$y^h(c) = \max\left\{ y \in \mathcal{N} : y \le \frac{y^p - \beta(c+1)}{1 - \beta} \right\} \wedge c,$$

and set $y^h(c) = 0$ otherwise.

(f) For each $c \in \{16, 11, \ldots, 36\}$, use simulation to compute the expected revenue using protection level $y^h(c)$ for action 1 against action 2. Compare the expected revenues to the upper bound $\Lambda R(c/\Lambda)$. For what value of c do you find the largest gap?

6.5 Bibliographical Remarks

Formulation in (6.3) and Theorem 6.1 are due to Talluri and van Ryzin (2004a). The formulation in that paper reduces to the one in Lee and Hersh (1993) when demands are independent. The fare and demand transformations that map λ_t and (Π_{tj}, R_{tj}), $j \in K_t$ into $(\hat{p}_{tj}, \hat{\lambda}_{tj})$, $j \in K_t$ as discussed in Sect. 6.2.1 appeared first in Kincaid and Darling (1963), as documented by Walczak et al. (2010). Fiig et al. (2010) and Walczak et al. (2010) proposed feeding the transformed data into a static model and using the EMSR-b heuristic to compute protection levels. Sierag et al. (2015) and Ge and Pan (2010) extend the work of Talluri and van Ryzin (2004a) to incorporate cancellations and overbooking into a single-resource revenue management problem.

The protection level formula in (6.10) is due to Gallego et al. (2009a). This formula is a reinterpretation of the main result in Brumelle et al. (1990). Efforts to transform the problem into an independent demand model include Belobaba and Weatherford (1996), and more recently by Fiig et al. (2010) and Walczak et al. (2010). Gallego et al. (2009b) show that setting protection level $\hat{y}(c)$ given by (6.12) with standard nesting is optimal and quite robust to other arrival patterns.

Cooper et al. (2006) and Cooper and Li (2012) develop models to study the consequences of specifying a simple customer behavior for choosing among the fare classes, when, in fact, the customer behavior is more complicated.

Appendix

Proof of Proposition 6.2 We can linearize (6.5) by introducing a new variable, say y, such that $y \geq R_j - z\Pi_j$ for all $j \in K$ and $z \geq 0$, which results in the linear program:

$$\begin{aligned} \bar{V}(T, c) = & \min_{z \geq 0} [\Lambda y + cz], \\ \text{subject to} \quad & \Lambda y + \Lambda \Pi_j z \geq \Lambda R_j \quad j \in K \\ & z \geq 0, \end{aligned}$$

where for convenience we have multiplied the constraints $y + \Pi_j z \geq R_j$, $j \in K$ by $\Lambda > 0$. The dual of this problem is given by

$$\bar{V}(T,c) = \Lambda \max \sum_{j \in K} R_j t_j$$

$$\text{subject to} \qquad \Lambda \sum_{j \in K} \Pi_j t_j \le c$$

$$\sum_{j \in K} t_j = 1$$

$$t_j \ge 0 \quad \forall j \in K.$$

This linear program decides the proportion of time, $t_j \in [0, 1]$, that each efficient set E_j is offered to maximize the revenue subject to the capacity constraint. Dividing the constraint by Λ and defining $\rho = c/\Lambda$ we see that $\bar{V}(T,c)/\Lambda = Q(\rho)$, or equivalently $\bar{V}(T,c) = \Lambda Q(c/\Lambda)$. □