Chapter 11
Competitive Assortment and Price Optimization

11.1 Introduction

In the models that we studied thus far, we considered the decisions made by a single firm. The implicit assumption in our development was that the other firms do not react to the decisions of each other. Naturally, this is almost never the case. When a firm decreases its prices, fearing loss of customers, its competitors may also decrease its prices. Both online and brick-and-mortar retail stores consider the assortments offered by the other stores when making planning their assortments. There is vast literature on modeling competition. Nevertheless, despite the fact that competition is the rule rather than an exception and there is vast literature on modeling competition, the development of operational models that can drive real-time decision making under competition is in its infancy. In most operational models, it is often the case that the competition is ignored or modeled rather simplistically. Perhaps, the most important reason for this is that explicitly modeling competition often times results in intractable models. Thus, for the sake of computational tractability, the reactions of the other firms are ignored. Furthermore, the data that drive the operational models are often collected in a competitive environment, and one usually naively hopes that building a noncompetitive model driven by data collected in a competitive environment will take care of the competition itself, but of course, this hope is not based on any scientific evidence.

Competition is a critical area for improvement for operational revenue management models, and we are starting to see more and more models in the literature that explicitly try to incorporate competition. In this chapter, we give a glimpse of two models. In Sect. 11.2, multiple firms compete in an environment where they choose the assortments they offer to their customers. The model here is static in the sense that there is no time dimension. In Sect. 11.3, multiple firms compete in their pricing decisions, there is limited inventory and the sales take place over time.

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11.2 Competitive Assortment Optimization

In this section, we consider a competitive assortment optimization problem between two firms, when the customers choose among the products offered by the firms according to the multinomial logit (MNL) model.

11.2.1 Problem Formulation

Consider two firms each of which has access to different sets of products. Among the set of products that a firm has access to, the firm chooses a subset, or an assortment, of products to offer to the customers. Considering all the products offered by *both* firms, a customer chooses among the products according to the MNL model. The goal of each firm is to choose an assortment of products to offer to maximize the expected revenue that it obtains from a customer. We index the firms by $\{1, -1\}$. For $i \in \{1, -1\}$, we use N_i to denote the set of products that firm i has access to. In other words, firm i offers an assortment within the set of products N_i. The set of all products is given by $N = N_1 \cup N_{-1}$. Let $v_j > 0$ be the attraction value of product $j \in N$, and v_0 be the attraction value of the no-purchase option. Let $V(S) := \sum_{j \in S} v_j$ denote the total attraction value of the products in set S. If the two firms offer subsets (S_1, S_{-1}) with $S_1 \subseteq N_1$ and $S_{-1} \subseteq N_{-1}$, then a customer chooses product $j \in S_1 \cup S_{-1}$ with probability

$$\pi_j(S_1, S_{-1}) := \frac{v_j}{v_0 + V(S_1) + V(S_{-1})}.$$

For $i \in \{1, -1\}$, we use $\mathcal{F}_i$ to denote the set of feasible assortments that can be offered by firm i. For example, each firm may be constrained by the number of products that they can display to their customers. Alternatively, each product may occupy a certain amount of space and the total space consumption of the products offered by a firm may have to be below a certain space limit. The revenue associated with product $j \in N$ is $p_j > 0$. Given that the two firms offer the assortments of products $(S_1, S_{-1}) \in \mathcal{F}_1 \times \mathcal{F}_{-1}$, the expected revenue that firm i obtains from a customer is

$$R_i(S_i, S_{-i}) := \sum_{j \in S_i} p_j \, \pi_j(S_i, S_{-i}) = \frac{\sum_{j \in S_i} p_j \, v_j}{v_0 + V(S_i) + V(S_{-i})}. \tag{11.1}$$

Therefore, if firm $-i$ offers the subset S_{-i} of products, then firm i maximizes its expected revenue by solving the problem

$$\max_{S_i \in \mathcal{F}_i} R_i(S_i, S_{-i}). \tag{11.2}$$

An optimal solution to the problem above is a best response of firm i to the assortment S_{-i} offered by firm $-i$. We say that the assortments $(S_1^*, S_{-1}^*) \in \mathcal{F}_1 \times \mathcal{F}_{-1}$ are a Nash equilibrium, if S_i^* is a best response to S_{-i}^* for all $i \in \{1, -1\}$. In the rest of our discussion, we show that a Nash equilibrium for competitive assortment optimization exists. We characterize a Pareto-dominating equilibrium in the sense that the expected revenue for each firm in the Pareto-dominating equilibrium is at least as large as its corresponding expected revenue in any other equilibria. Lastly, we compare the assortments in a Nash equilibrium with those chosen by a central planner to maximize the total expected revenue obtained by the two firms.

11.2.2 Existence of Equilibrium

Let $z_i^*(S_{-i})$ denote the optimal objective value of problem (11.2). In other words, $z_i^*(S_{-i})$ is the best expected revenue that firm i can achieve when firm $-i$ offers the assortment S_{-i}. Noting the expected revenue expression in (11.1), we have

$$z_i^*(S_{-i}) \geq \frac{\sum_{j \in S_i} p_j\, v_j}{v_0 + V(S_i) + V(S_{-i})} \qquad \forall\, S_i \in \mathcal{F}_i,$$

and the inequality above holds as equality at an optimal solution to problem (11.2). Since $V(S_i) = \sum_{j \in S_i} v_j$, this inequality is equivalent to

$$[v_0 + V(S_{-i})]\, z_i^*(S_{-i}) \geq \sum_{j \in S_i} (p_j - z_i^*(S_{-i}))\, v_j \qquad \forall\, S_i \in \mathcal{F}_i,$$

with equality holding at an optimal solution to (11.2), so

$$[v_0 + V(S_{-i})]\, z_i^*(S_{-i}) = \max_{S_i \in \mathcal{F}_i} \left\{ \sum_{j \in S_i} (p_j - z_i^*(S_{-i}))\, v_j \right\}.$$

Therefore, an optimal solution to problem (11.2) can be obtained by solving the problem:

$$\max_{S_i \in \mathcal{F}_i} \left\{ \sum_{j \in S_i} (p_j - z_i^*(S_{-i}))\, v_j \right\}. \tag{11.3}$$

Throughout, we assume that if problem (11.2) or (11.3) has multiple optimal solutions, then we choose a solution S_i that has the largest total attraction value $V(S_i)$. Note that problem (11.3) is not immediately useful to solve problem (11.2) because solving problem (11.3) requires knowing $z_i^*(S_{-i})$ and we do not know $z_i^*(S_{-i})$ before solving problem (11.2)! Nevertheless, we will use problem (11.3) to show the existence of Nash equilibria and to characterize the properties of such equilibria.

Consider two assortments $\hat{S}_{-i}$ and $\tilde{S}_{-i}$ that could be offered by firm $-i$. Let $\hat{S}_i$ be a best response of firm i to the assortment $\hat{S}_{-i}$ and $\tilde{S}_i$ be a best response of firm i to the assortment $\tilde{S}_{-i}$. In the next lemma, we present a key monotonicity result.

Lemma 11.1 *If $V(\hat{S}_{-i}) \leq V(\tilde{S}_{-i})$, then $V(\hat{S}_i) \leq V(\tilde{S}_i)$.*

The lemma above establishes a monotonicity property for the best response of each firm, where if firm $-i$ offers an assortment with a larger total attraction value, then firm i, in its best response, also offers an assortment with a larger total attraction value. By using this lemma, we will be able to show that a tatonnement process converges to a Nash equilibrium. In the process, we will also establish the existence of Nash equilibria. To describe the tatonnement process, we define the sequence of assortments $\{(\hat{S}_1^t, \hat{S}_{-1}^t) : t = 0, 1, \ldots\}$ offered by the two firms as follows. We start with $\hat{S}_1^0 = \varnothing$ and $\hat{S}_{-1}^0 = \varnothing$. Using $(\hat{S}_1^t, \hat{S}_{-1}^t)$, we compute $(\hat{S}_1^{t+1}, \hat{S}_{-1}^{t+1})$ as

$$\hat{S}_1^{t+1} \in \arg\max_{S_1 \in \mathcal{F}_1} R_1(S_1, \hat{S}_{-1}^t) \qquad \text{and} \qquad \hat{S}_{-1}^{t+1} \in \arg\max_{S_{-1} \in \mathcal{F}_{-1}} R_{-1}(S_{-1}, \hat{S}_1^{t+1}).$$

Thus, $\hat{S}_1^{t+1}$ is a best response of firm 1 to the assortment $\hat{S}_{-1}^t$ offered by firm -1, whereas $\hat{S}_{-1}^{t+1}$ is a best response of firm -1 to the assortment $\hat{S}_1^{t+1}$ offered by firm 1. In the next theorem, we use this tatonnement process to show that there exists a Nash equilibrium. In the proof, using Lemma 11.1, we argue that the sequence of the total attraction values in the assortments generated by the tatonnement process converges, in which case, we are able to construct a Nash equilibrium by using the limit of this sequence.

Theorem 11.2 *There exists a Nash equilibrium.*

In the proof of Theorem 11.2, we use Lemma 11.1 to argue that the sequence of assortments $\{(\hat{S}_1^t, \hat{S}_{-1}^t) : t = 0, 1, \ldots\}$ generated in the tatonnement process satisfies $V(\hat{S}_i^{t+1}) \geq V(\hat{S}_i^t)$ for all $i \in \{1, -1\}$. Thus, there exists an iteration counter $t_0 \geq 0$ in the tatonnement process such that $V(\hat{S}_1^{t_0}) = V(\hat{S}_1^{t_0+1}) = V(\hat{S}_1^{t_0+2}) = \ldots$ and $V(\hat{S}_{-1}^{t_0}) = V(\hat{S}_{-1}^{t_0+1}) = V(\hat{S}_{-1}^{t_0+2}) = \ldots$. In this case, we are able to show that $(\hat{S}_1^{t_0+1}, \hat{S}_{-1}^{t_0})$ is a Nash equilibrium. We refer to $(\hat{S}_1^{t_0+1}, \hat{S}_{-1}^{t_0})$ as a Nash equilibrium generated by the tatonnement process. In the tatonnement process, we started with the assortments $\hat{S}_1^0 = \hat{S}_{-1}^0 = \varnothing$, but the choice of $\hat{S}_1^0$ is irrelevant because we compute $\hat{S}_1^1$ as a best response to $\hat{S}_{-1}^0$ and we compute $\hat{S}_{-1}^1$ as a best response to $\hat{S}_1^1$. Thus, $\hat{S}_1^0$ does not play any role in the tatonnement process. Also, by using the same argument in the proof of Theorem 11.2, we can show that the tatonnement process would yield a Nash equilibrium even if we choose $\hat{S}_1^0$ and $\hat{S}_{-1}^0$ arbitrarily, but as we show in the next section, an equilibrium that we reach by choosing $\hat{S}_1^0 = \hat{S}_{-1}^0 = \varnothing$ Pareto dominates any equilibria. Therefore, when we say a Nash equilibrium generated by the tatonnement process, we will mean the one obtained by starting with $\hat{S}_1^0 = \hat{S}_{-1}^0 = \varnothing$.

11.2.3 Properties of Equilibrium

There can be multiple Nash equilibria in general, but it turns out that a Nash equilibrium generated by the tatonnement process will always Pareto dominate the others. In other words, the expected revenue of each firm in a Nash equilibrium generated by the tatonnement process is at least as large as its corresponding expected revenue in any other Nash equilibrium. We show this result in the next theorem.

Theorem 11.3 *A Nash equilibrium generated by the tatonnement process is Pareto dominant.*

The key to the result above is to show that the total attraction value of each assortment in a Nash equilibrium generated by the tatonnement process is no larger than its corresponding total attraction value in another Nash equilibria. Next, we compare the assortments offered in the absence of competition and the assortments offered by a central planner with the assortments offered in a Nash equilibrium.

In the absence of competition, firm i finds an assortment to offer by solving the problem $\max_{S_i \in \mathcal{F}_i} R_i(S_i, \varnothing)$. Let $(S_1^{\text{NC}}, S_{-1}^{\text{NC}})$ be the assortments offered by the two firms in the absence of competition, where the superscript NC stands for no competition. Also, if there were a central planner that chooses the assortments offered by the two firms to maximize the total expected revenue obtained by the two firms, then she would solve the problem

$$\max_{(S_1, S_{-1}) \in \mathcal{F}_1 \times \mathcal{F}_{-1}} \Big\{ R_1(S_1, S_{-1}) + R_{-1}(S_{-1}, S_1) \Big\}. \tag{11.4}$$

Let $(S_1^{\text{CP}}, S_{-1}^{\text{CP}})$ be the assortments offered by the central planner, where the superscript CP stands for central planner. In the next theorem, we show that the total attraction value of the products offered by each firm in any equilibrium is at least as large as the total attraction value of the products offered by the corresponding firm in the absence of competition. Furthermore, the total attraction value of the products offered by each firm in any equilibrium is also at least as large as the total attraction value of the products offered by the corresponding firm under the solution of the central planner. These results indicate that competition has the tendency to increase the total attraction values of the products offered by each firm. In other words, to deal with competition, the firms enlarge their assortments by offering assortments with larger total attraction values.

Theorem 11.4 *Let (S_1^*, S_{-1}^*) be any Nash equilibrium, $(S_1^{\text{NC}}, S_{-1}^{\text{NC}})$ be the assortments offered by the two firms in the absence of competition, and $(S_1^{\text{CP}}, S_{-1}^{\text{CP}})$ be the assortments offered by the central planner. Then, $V(S_i^{\text{NC}}) \le V(S_i^*)$ and $V(S_i^{\text{CP}}) \le V(S_i^*)$ for all $i \in \{1, -1\}$.*

Note that the result above holds for any Nash equilibrium.

11.3 Dynamic Pricing Under Competition

In this section, we consider dynamic pricing in an oligopolistic market with a mix of substitutable and complementary perishable products. Each firm has a fixed initial stock of items and competes in setting prices to sell them over a finite sales horizon. Customers sequentially arrive at the market, make a choice that includes the no-purchase alternative, and then leave the system. Assuming deterministic customer arrival rates, we show that any equilibrium strategy has a simple structure involving a finite set of time-invariant shadow prices measuring capacity externalities that firms exert on each other. This simple structure sheds light on dynamic revenue management problems under competition and demand uncertainty. Indeed, it turns out that the equilibrium solutions from the deterministic game provide precommitted and contingent heuristic policies that are asymptotic equilibria for the stochastic game when demand and supply are sufficiently large.

11.3.1 Problem Formulation

We consider a market of m competing firms selling differentiated perishable products over a finite horizon $[0, T]$. At time $t = 0$, each firm i has an initial inventory of c_i units of a unique product. We count the time forwards and use t for the elapsed time and $s = T - t$ for the remaining time. Let $p(t)$ be the vector of prices at time t, and let $d(t, p(t))$ be the vector of product demands at time t at prices $p(t)$, and let $r_i(t, p) = p_i d_i(t, p)$ be the revenue rate for firm i at time t when the price vector is $p = (p_i, p_{-i})$, where p_i is the priced offered by firm i and p_{-i} is the vector of prices from firms other than firm i. We make the following assumptions.

1. (a) The demand for firm i, $d_i(t, p)$ is continuously differentiable in p for all i and all t.
   (b) The aggregate demand $\int_0^T d_i(t, p(t))dt$ for firm i is pseudo-convex in its price path $p_i(t),\ t \in [0, T]$.
2. (a) The aggregate revenue $\int_0^T r_i(t, p(t))dt$ for firm i is pseudo-concave in its price path $p_i(t),\ t \in [0, T]$.
   (b) There exist a function $R_i(t)$ such that $r_i(t, p) \leq R_i(t)$ and $\int_0^T R_i(t)dt < \infty$.
3. (a) There exist a choke price $p_i(t, p_{-i})$ such that

$$\lim_{p_i \to p_i(t, p_{-i})} d_i(t, p) = 0 \quad \text{and} \quad \lim_{p_i \to p_i(t, p_{-i})} r_i(t, p) = 0.$$

   Moreover, the choke price is always an available option for each firm.
   (b) Other than the choke price, firm i chooses prices from a compact and convex subset $\mathcal{P}_i(t, p_{-i})$ of $\{p_i \in \Re_+ : d_i(t, p_i, p_{-i}) \geq 0\}$.

   (c) The salvage value of the products at the end of the horizon is zero, and all other costs are sunk.

4. All firms have perfect knowledge about the inventory levels of other firms at any time.

As examples of possible demand functions for the firms, consider the MNL demand function

$$d_i(t, p) = \lambda(t) \frac{\beta_i(t) \exp(-\alpha_i(t) p_i)}{a_0(t) + \sum_j \beta_j(t) \exp(-\alpha_j(t) p_j)},$$

where $\lambda(t), a_0(t), \alpha_i(t), \beta_i(t) > 0$ for all i and t. As a second example, consider now the linear demand function

$$d_i(t, p) = a_i(t) - b_i(t) p_i + \sum_{j \neq i} c_{ij}(t)\, p_j,$$

where $a_i(t), b_i(t) > 0$ for all i. These linear demand functions can arise from a representative consumer maximizing a quadratic utility function and can accommodate substitute and complementary products depending on whether $c_{ij}(t)$ is positive or negative. It can be shown that Assumptions 1 and 2 are satisfied both by the MNL model and the linear demand model.

Assumption 3(a) ensures that a firm immediately exits the market on a stockout. In this case, customers who originally prefer the stockout firm will spill over to the remaining firms that still have positive inventory. The spillover is endogenized from the demand model according to customers' preferences and product substitutability. Moreover, in view of Assumption 3(b), firms can use the choke price before it runs out of stock. The compactness assumption of $\mathcal{P}_i(t, p_{-i})$ fails to hold for some models, like the MNL. Fortunately, for the MNL model there are ways around that avoid compactness.

Assumption 3(c) is without loss of generality. Assumption 4 is standard in game theory and it is realistic in an airline setting as major airlines offer a feature of previewing seat availability from their websites.

Let $x(t) \in [0, c]$ be the joint inventory at time t. A joint open-loop strategy $p(t)$ depends only on time t and the initial inventory $x(0) = c$. In contrast, a feedback strategy $p(t, x(t))$ depends on t and the current inventory $x(t)$. The set of all open-loop strategies is denoted by $\mathcal{P}_0$, and the set of all feedback strategies is denoted by $\mathcal{P}_F$.

Let $D[0, T]$ denote the set of all right-continuous real-valued functions with left limits defined on $[0, T]$, where the left discontinuities allow for price jumps after a sale. Given a price control path $p \in D[0, T]^m$, we denote the total profit for firm i by

$$J_i[p] = \int_0^T r_i(t, p_i(t)) dt.$$

Moreover, under $p \in D[0, T]^m$, the inventory of product i evolves according to

$$\dot{x}_i(t) = -d_i(t, p(t)) \quad 0 \leq t \leq T$$

starting from $x_i(0) = c_i$.

The objective of each firm is to maximize its own total revenue over the sales horizon subject to *all* capacity constraints over the entire sales horizon. Thus, firm's i problem is

$$\max_{p_i(t), t \in [0,T]} \int_0^T r_i(t, p(t))dt$$

subject to

$$x_j(t) = c_j - \int_0^t d_j(v, p(v))dv \geq 0 \quad \forall t \in [0, T], \; j = 1, \ldots, m.$$

Firms simultaneously solve their own revenue maximization problems subject to a joint set of constraints, giving rise to a game with coupled strategy constraints for all firms. These are known as generalized Nash games with coupled constraints. If some pricing policy results in negative inventory at some time, then it will be eliminated from the joint feasible strategy space. In other words, all firms face a joint set of constraints, $x(t) \geq 0$ for all $t \in [0, T]$ in selecting feasible strategies.

A generalized open-loop Nash equilibrium (OLNE) is an open-loop control path $p \in \mathcal{P}_0$ such that $p \in D[0, T]^m$, and $p_i(t), 0 \leq t \in T$ solves firm i's problem for all i. Likewise, a feedback loop Nash Equilibrium (FNE) is a feedback control path $p \in \mathcal{P}_F$ such that $p \in D[0, T]^m$, and $p_i(t, x(t))$ solves firm i's problem for all i.

In a nonzero-sum differential game, open-loop and feedback strategies are generally different. However, re-solving the OLNE with the current time and inventory level continuously over time results in an FNE, which generates the same price path and inventory trajectory as those of the OLNE with the same initial time and inventory level. Because of this relationship, we call an OLNE, an equilibrium strategy.

11.3.2 Equilibrium Results

The following theorem gives important existence results.

Theorem 11.5 *If the choke price $p_i^\infty(t, p_{-i})$ is in the convex and compact set $\mathcal{P}_i(t, p_{-i})$ for each i, then an equilibrium exists. For the MNL model, an equilibrium exists where firms never use the choke price.*

The first part of Theorem 11.5 applies to the linear demand model, but it does not apply to the MNL model, because for the MNL model, the sets $\mathcal{P}_i(t, p_{-i})$ are not compact. Nevertheless, the second part guarantees the existence of an equilibrium that does not involve the choke price for any of the firms.

Necessary Conditions

From Pontryagin's maximum principle for constrained set space, the following are necessary conditions for an OLNE.

Theorem 11.6 *If an open-loop pricing policy p^* is an OLNE, then there exists a non-negative, $m \times m$ matrix of non-negative shadow prices μ_{ij}, such that for any t such that $x_i^*(t) > 0$, the open-loop policy maximizes*

$$r_i(t, p_i, p_{-i}^*(t)) - \sum_j \mu_{ij} d_j(t, p_i, p_{-i}^*(t)).$$

Moreover, $\mu_{ij} x_j^*(T) = 0$ for all j. Let $E_i = \{t \in [0, T] : x_i^*(t) = 0\}$, and if E_i is non-empty, define $t_i = \inf E_i$. For all $t \in [t_i, T]$, firm i uses the choke price $p^\infty(t, p_{-i}^*(t))$. There exists a decreasing shadow price process $\mu_{ij}(t) \in [0, \mu_{ij}]$ for all j and $t \in [t_i, T]$ such that the choke price $p_i^\infty(t)$ maximizes

$$r_i(t, p_i, p_{-i}^*(t)) - \sum_j \mu_{ij}(t) d_j(t, p_i, p_{-i}^*(t)).$$

As a result, the OLNE has a simple structure. First, there exist an $m \times m$ matrix of finite, non-negative, time invariant, shadow prices μ_{ij}. At time t, let $S(t) = \{i : x_i^*(t) > 0\}$ be the set of firms with positive inventories. Then every firm in $S(t)$ simultaneously solves the problem

$$\max_{p_i} \left\{ r_i(t, p_i, p_{-i}) - \sum_{j \in S(t)} \mu_{ij} d_j(t, p_i, p_{-i}) \right\},$$

where $p_i \in \mathcal{P}_i(t, p_{-i}) \cup p_i^\infty(t, p_{-i})$ and all firms $i \notin S(t)$ use their choke price. Notice that the set of allowable prices for firms in $S(t)$ include the choke price, thus a firm may use its choke price even if it has positive inventory.

We now illustrate how capacity externalities influence the equilibrium pricing. Fix an arbitrary time t. If firms i and j offer substitutable products, then firm j's scarce capacity exerts an externality on firm i by pushing up firm i's price: Since firm j has limited capacity, it has a tendency to increase its own price due to the self-inflicted capacity externality. Because of the substitutability between products from firms i and j, the price competition between the two firms will be alleviated so that firm i can also post a higher price. On the other hand, if firms i and j offer complementary products, then firm j's scarce capacity exerts an externality on firm i by pushing down firm i's price: While firm j has a tendency to increase its own price, due to the complementarity between products from firms i and j, firm i has to undercut its price to compensate for the price increase of firm j. By a similar reasoning, on stockout, a product's market exit by posting choke prices will be a boon for its substitutable products and a bane for its complementary products.

An important special case is that of time invariant demands $d(t, p) = d(p)$. In this case, the price trajectories and the available products in the market remain constant before the first stockout event, between any two consecutive stockout events, and after the final stockout event until the end of the sales horizon.

Sufficient Conditions
Consider now a bounded rational OLNE where the matrix of shadow prices is diagonal, so $\mu_{ii} \geq 0$ and $\mu_{ij} = 0$ for all $i \neq j$. Such a bounded rational equilibrium may arise if firms only care about their own capacity constraint. This bounded rational equilibrium may also arise if firms do not have inventory information for their competitors, and equilibrium outcomes emerge from repeated best responses. Moreover, it can also arise when firms proceed under the assumption that the competitors have sufficiently large capacities as if they would never stock out.

Theorem 11.7 *If $r_i(t, p)$ is concave in p_i and $d_i(t, p)$ is convex in p_j for all i, j, t, then the necessary conditions are also sufficient. Moreover, if $\int_0^T [r_i(t, p(t)) - \mu(t) d_i(t, p(t))]dt$ is pseudo-concave in $p_i(t), 0 \leq t \leq T$ for all $\mu(t) \geq 0$, $0 \leq t \leq T$, then the necessary conditions together with $\mu_{ij}(t) = 0$ for all $i \neq j$ and t are also sufficient for a bounded rational OLNE.*

The first part of Theorem 11.7 applies to the linear demand model, but fails for the MNL model. On the other hand, second set of sufficient conditions apply to the MNL model.

11.3.3 Comparative Statics

If all products are substitutable such that the price competition is (log-)supermodular, then a decrease in the initial capacity level of any firm leads to higher equilibrium prices at any time for all firms in a bounded rational OLNE. Consider now a duopoly selling complementary products. If the price competition is (log-)submodular, then a decrease in the initial capacity level of one firm leads to higher equilibrium prices at any time for the firm itself and lower equilibrium prices at any time for the other firm in a bounded rational OLNE.

Uniqueness
A normalized OLNE has a matrix of constant shadow prices where all the rows are the same, so $\mu_{ij} = \mu_j$ independent of i for all i, j. In essence, all firms use the same set of shadow prices for a firm's capacity constraint in their best-response problems. Suppose that $d_i(t, p)$ is twice continuously differentiable in p for all i and t. If $d_i(t, p)$ is convex in p_j for all i, j, t, and

$$\frac{\partial^2 r_i(t, p)}{\partial p_i^2} + \sum_{j \neq i} \left| \frac{\partial^2 r_i(t, p)}{\partial p_i \partial p_j} \right| < 0$$

for all i, t, then there exist a unique normalized OLNE.

Unfortunately, this result does not apply to the MNL model as the demand function is not convex in p. For this reason, we present alternative conditions for uniqueness that apply to the MNL model. Assume that $d_i(t, p)$ is twice continuously differentiable in p for all i and t. If $\partial d_i(t, p)/\partial p_i < 0$ for all i, t, and the Jacobian and Hessian matrix of the demand function $d(t, p)$ with respect to p are negative semidefinitive for all $p \in \mathcal{P}$, then there exist at most one bounded rational OLNE for any vector of diagonal shadow prices. Moreover, there exists a unique bounded rational OLNE for some vector of diagonal shadow prices. It is possible to verify that the MNL model satisfies these latter set of conditions.

Coupled with the existence results, we know that there exists at least one bounded rational OLNE, and that there exists at least one vector of diagonal shadow prices such that its corresponding bounded rational OLNE is unique.

Applications

In dynamic Bertrand–Edgeworth models, firms may avoid head-to-head competition and take turns acting as monopolists. This can also happen in our model depending on the inter-temporal demand structure; however, it is possible to show that this can never happen under the MNL model. On the other hand, examples exist where firms run out of stock before the end of the sales horizon in equilibrium even if demands are stationary. However, among all such equilibria, the one using the whole sales horizon Pareto dominates all others and it is the unique bounded rational OLNE.

11.3.4 Asymptotic Optimality for the Stochastic Case

We extend the differential game to account for demand uncertainty by considering its stochastic-game counterpart in continuous time. We show that the solutions suggested by the differential game capture the essence and provide a good approximation to the stochastic game. Given a feasible pricing policy u, we denote the revenue for firm i as $G_i(u)$. A policy u^* is a Markovian equilibrium if $G(u_i, u^*_{-i}) \leq G(u^*_i, u^*_{-i})$ for all i. An equilibrium can be found, in theory, by simultaneously solving the corresponding Hamilton–Jacobi–Bellman equations for all the firms. It can be shown that using an affine functional approximation for the value functions of all the firms coincides with the differential game we have studied earlier.

Using k as an index, we consider a sequence of problems with demand rate $d^k(t, p) = kd(t, p)$ and capacity $c^k = kc$. Let $\bar{G}^k_i(u) = G^k(u)/k$ be the revenue form firm i. In a stochastic game, a feasible policy u^* is called an asymptotic Nash equilibrium in the limiting regime of the sequence of scaled stochastic games, if for any $\epsilon > 0$ and all i, there exists an l such that for all $k > l$, $\bar{G}^l_i(u_i, u^*_{-i}) \leq \bar{G}^l(u^*)+\epsilon$ for all feasible policies (u_i, u^*_{-i}).

Theorem 11.8 *Any OLNE heuristic corresponding to an OLNE of the differential game is an asymptotic Nash equilibrium in the limiting regime of the sequence of scaled stochastic games.*

Under the stochastic regime, firms may prefer to use a re-solving feedback strategy that updates prices continuously based on the state of the system as a potentially better heuristic. The following theorem tells us that this feedback policy is also asymptotically optimal.

Theorem 11.9 *The re-solving feedback heuristic is an asymptotic Nash equilibrium in the limiting regime of the sequence of scaled stochastic games.*

The last two results are of practical value to capacity providers that can be assured that heuristics based on the differential game have good asymptotic properties.

11.4 End of Chapter Problems

1. Consider a version of the Markov chain (MC) choice model that we studied in the chapters on choice modeling and assortment optimization, but we make pricing decisions, instead of assortment offer decisions. With probability λ_i, a consumer arriving into the system is primarily interested in product i and checks its price p_i. The consumer makes a purchase with $e^{-\alpha_i p_i}$ and leaves the system generating a revenue equal to p_i. Otherwise, the consumer rejects product i, and transitions to product j with probability ρ_{ij} and checks its price p_j. Assume that $\sum_{j \in N} \rho_{ij} < 1$ for all $i \in N$ so that a customer visiting product i and not purchasing this product transitions to the no-purchase option and leaves the system with probability $1 - \sum_{j \in N} \rho_{ij}$. The customer transitions among the products until she makes a purchase decision or she decides to leave without a purchase.
   (a) Given that we charge the prices $p = \{p_i : i \in N\}$, write a system of equations that we can solve to compute the purchase probability of each product.
   (b) Let g_i be the optimal expected revenue from a customer currently visiting product i. Write a dynamic program that can be used to compute $\{g_i : i \in N\}$.
2. We continue with the MC choice model setup discussed in the previous problem. Assume that the set of products N are partitioned into the sets N^1 and N^{-1}. There are two firms, firm 1 and firm -1. Firm i owns and sets the prices of products in the set N^i. The customers make a choice over the whole set of products N according to the MC choice model. If a customer purchases a product owned by firm i, then firm i generates a revenue. Each firm is interested in maximizing its own expected revenue.
   (a) Assume that firm -1 charges the prices $\{p_i^{-1} : i \in N^{-1}\}$ for the products that it owns. Let g_i^1 be the optimal expected revenue of firm 1 from a customer that is currently visiting product i. Write a dynamic program that can be used to compute $\{g_i^1 : i \in N^1 \cup N^{-1}\}$. Note that g_i^1 is nonzero for $i \in N^{-1}$.

In particular, firm 1 can generate nonzero revenue from a customer visiting a product owned by firm -1 because this customer may decide not to purchase the product and subsequently transition to a product owned by firm 1.

(b) What is the optimal price that firm 1 should charge for product $i \in N^1$ as a function of $\{g_j^1 : j \in N^1 \cup N^{-1}\}$?

3. In the second part of the previous problem, we derived how to compute the best response of firm 1 to the prices $\{p_i^{-1} : i \in N^{-1}\}$ charged by firm -1. Let $\hat{p}^1 = \{\hat{p}_i^1 : i \in N^1\}$ be the best response of firm 1 to the prices $\hat{p}^{-1} = \{\hat{p}_i^{-1} : i \in N^{-1}\}$ charged by firm -1. Let $\tilde{p}^1 = \{\tilde{p}_i^1 : i \in N^1\}$ be the best response of firm 1 to the prices $\tilde{p}^{-1} = \{\tilde{p}_i^{-1} : i \in N^{-1}\}$ charged by firm -1.

   (a) Show that if $\hat{p}_i^{-1} \geq \tilde{p}_i^{-1}$ for all $i \in N^{-1}$, then $\hat{p}_i^1 \geq \tilde{p}_i^1$ for all $i \in N^1$.
   (b) Using the previous part, show that there exists a Nash equilibrium for the two firms.

4. Consider a symmetric duopoly market with firms 1 and 2 each selling one product by setting prices p_1 and p_2, respectively. The two products are substitutable. The demand system that governs the market has a linear form:

$$d_1(p_1, p_2) = a - p_1 + \gamma p_2,$$
$$d_2(p_1, p_2) = a - p_2 + \gamma p_1,$$

   where $a > 0$ and $\gamma \in [0, 1)$. Both firms have the identical marginal cost of z to procure, produce, and distribute their products. Note that though the two firms are symmetric, nothing prevents them from adopting asymmetric decisions.

   (a) As γ increases, how does the total sales $d_1(p_1, p_2) + d_2(p_1, p_2)$ change for a given price vector (p_1, p_2)? Is there any issue with this monotonicity property?
   (b) If both firms simultaneously set *prices* to maximize their profit, what is the price equilibrium?
   (c) If both firms simultaneously make decisions on the *sales quantity* to maximize their profit, what is the market equilibrium outcome in terms of prices?
   (d) Compare the equilibrium outcomes in parts (b) and (c). Provide an intuitive explanation why the comparison you observe hold.
   (e) Consider a two-stage sequential game in which firms maximize their profit. In the first stage, both firms simultaneously decide on the *capacity* of their production and distribution. As a result, how many each can sell will be capped by the capacity level determined by themselves. In the second stage, given the capacity level they build in the first stage, both firms simultaneously set *prices* to maximize their profit. For this two-stage game, what is the price equilibrium in the second stage given the equilibrium capacity level set in the first stage?

(f) Compare the equilibrium outcomes in parts (c) and (e). Provide an intuitive explanation why the comparison you observe hold.
(g) On top of the two-stage sequential game of part (e), suppose in the second stage, firms can produce, distribute, and sell more than the capacity level each sets in the first stage. The downside is the additional quantity produced, distributed, and sold beyond the capacity level incurs an *additional* cost z' per unit beyond z. What is the equilibrium market outcome you expect to see from this modified two-stage sequential game?

11.5 Bibliographical Remarks

The competitive assortment optimization model that we discuss in this chapter is based on Besbes and Saure (2016). The authors extend the results that we discuss in this chapter to the case where there are common products that can be offered by both firms. If a customer chooses such a common product, then she makes the purchase decision from either of the firms with equal probabilities. Also, the authors analyze the setting where the firms choose the assortment of products to offer, as well as the prices of the products in the offered assortment. The competitive pricing model presented in this chapter is due to Gallego and Hu (2014). We refer the reader to that paper for the details of the analysis. Related models also appear in Federgruen and Hu (2015) and Federgruen and Hu (2018).

Hopp and Xu (2008) study a dynamic price and assortment optimization problem under competition. The authors adopt a fluid approximation framework, where the demand takes on its expected value. They establish the existence of a Nash equilibrium and provide conditions for uniqueness. Gallego et al. (2006a) study competitive pricing problems under the MNL model. Chen and Chen (2017) incorporate the network effects into the problem in a duopoly setting. Anderson and de Palma (1992) and Li and Huh (2011) study competitive pricing problems under the nested logit (NL) model. In the first paper, the authors assume that all products have the same price sensitivity, whereas in the second paper, the authors assume that the products in the same nest have the same price sensitivity. Gallego and Wang (2014) extend these results to the case where the products can have arbitrary price sensitivities. Cachon and Kok (2007b) use the NL model to analyze the decisions made by category managers, who focus on the expected revenue obtained from a customer purchasing a product in their own category. The authors characterize the potential revenue loss and provide remedies to attain expected revenues close to those that can be obtained by a central planner. Kok and Xu (2011) study the structural properties of the best-response dynamics when the customers choose according to the NL model. Cooper et al. (2015) develop a model to understand the consequences of ignoring the competition while estimating the customer demand. Feng and Hu (2017) study a competitive product investment model to understand the customer herding behavior.

There is a large and still growing body of literature on competitive models in network revenue management. There is work focusing on dynamic pricing models with competition; see Perakis and Sood (2006), Gallego et al. (2006b), Xu and Hopp (2006), Kachani et al. (2007), Levin et al. (2009), Adida and Perakis (2010a), Martinez-de-Albeniz and Talluri (2011), Caro and Martinez-de-Albeniz (2012) and Kirshner et al. (2018). There is also work on competitive assortment models; see Heese and Martinez-de-Albeniz (2018). Lastly, there is work on studying price or quantity competition in static problems that include either a single or two time periods; see Farahat and Perakis (2009), Nalca et al. (2010, 2013), Farahat and Perakis (2010), Martinez-de-Albeniz and Roels (2011), Afeche et al. (2014), Wang and Hu (2014), Cho and Tang (2014), Bazhanov et al. (2015), Nazerzadeh and Perakis (2016), Aviv et al. (2017, 2018) and Cachon and Feldman (2017).

Appendix

Proof of Lemma 11.1 If $V(\hat{S}_{-i}) = V(\tilde{S}_{-i})$, then since the objective function of problem (11.2) depends on S_{-i} only through $V(S_{-i})$, a best response of firm i to the assortment $\hat{S}_{-i}$ is also a best response of firm i to the assortment $\tilde{S}_{-i}$. In this case, the result follows immediately. Assume without loss of generality that $V(\hat{S}_{-i}) < V(\tilde{S}_{-i})$, and assume for a contradiction that $V(\hat{S}_i) > V(\tilde{S}_i)$. From (11.1), for any $S_i \in \mathcal{F}_i$ and $S_i \neq \varnothing$, $R_i(S_i, \hat{S}_{-i}) > R_i(S_i, \tilde{S}_{-i})$. Consequently,

$$z_i^*(\hat{S}_{-i}) = \max_{S_i \in \mathcal{F}_i} R_i(S_i, \hat{S}_{-i}) > \max_{S_i \in \mathcal{F}_i} R_i(S_i, \tilde{S}_{-i}) = z_i^*(\tilde{S}_{-i}),$$

so that $z_i^*(\hat{S}_{-i}) > z_i^*(\tilde{S}_{-i})$, where we implicitly assume that there exists a nonempty feasible solution to the two maximization problems above; otherwise, $\hat{S}_i = \tilde{S}_i = \varnothing$ and the result trivially holds. On the other hand, by the discussion before the lemma, a best response of firm i to the assortment S_{-i} is given by an optimal solution to problem (11.3). Therefore, $\hat{S}_i$ is an optimal solution to problem (11.3) after replacing $z_i^*(S_{-i})$ with $z_i^*(\hat{S}_{-i})$. In other words, $\hat{S}_i$ is an optimal solution to the problem

$$\max_{S_i \in \mathcal{F}_i} \left\{ \sum_{j \in S_i} (p_j - z_i^*(\hat{S}_{-i}))\, v_j \right\}.$$

Since $\tilde{S}_i$ is a feasible but not necessarily an optimal solution to the problem above, it follows that

$$\sum_{j \in \hat{S}_i} (p_j - z_i^*(\hat{S}_{-i}))\, v_j \geq \sum_{j \in \tilde{S}_i} (p_j - z_i^*(\hat{S}_{-i}))\, v_j.$$

Interchanging the roles of $\hat{S}_{-i}$ and $\tilde{S}_{-i}$ and following the same argument, we see that

$$\sum_{j\in\tilde{S}_i}(p_j - z_i^*(\tilde{S}_{-i}))\, v_j \geq \sum_{j\in\hat{S}_i}(p_j - z_i^*(\tilde{S}_{-i}))\, v_j.$$

Adding the two inequalities yields, it follows that $z_i^*(\hat{S}_{-i})\,[V(\tilde{S}_i) - V(\hat{S}_i)] \geq z_i^*(\tilde{S}_{-i})\,[V(\tilde{S}_i) - V(\hat{S}_i)]$. The last inequality contradicts the fact that $z_i^*(\hat{S}_{-i}) > z_i^*(\tilde{S}_{-i})$ and $V(\hat{S}_i) > V(\tilde{S}_i)$. □

Proof of Theorem 11.2 We use induction over the iterations to show that $V(\hat{S}_1^t) \geq V(\hat{S}_1^{t-1})$ and $V(\hat{S}_{-1}^t) \geq V(\hat{S}_{-1}^{t-1})$ for all $t = 1, 2, \ldots$. The result trivially holds for $t = 1$, since we have $\hat{S}_1^0 = \hat{S}_{-1}^0 = \varnothing$ so that $V(\hat{S}_1^0) = V(\hat{S}_{-1}^0) = 0$. Assuming that $V(\hat{S}_1^t) \geq V(\hat{S}_1^{t-1})$ and $V(\hat{S}_{-1}^t) \geq V(\hat{S}_{-1}^{t-1})$, we proceed to showing that $V(\hat{S}_1^{t+1}) \geq V(\hat{S}_1^t)$ and $V(\hat{S}_{-1}^{t+1}) \geq V(\hat{S}_{-1}^t)$. By definition $\hat{S}_1^t$ is a best response of firm 1 to the assortment $\hat{S}_{-1}^{t-1}$, whereas $\hat{S}_1^{t+1}$ is a best response of firm 1 to the assortment $\hat{S}_{-1}^t$. Since $V(\hat{S}_{-1}^{t-1}) \leq V(\hat{S}_{-1}^t)$ by the induction hypothesis, by Lemma 11.1, it follows that $V(\hat{S}_1^t) \leq V(\hat{S}_1^{t+1})$. Similarly, by definition, $\hat{S}_{-1}^t$ is a best response of firm -1 to the assortment $\hat{S}_1^t$, whereas $\hat{S}_{-1}^{t+1}$ is a best response of firm -1 to the assortment $\hat{S}_1^{t+1}$. We just established that $V(\hat{S}_1^t) \leq V(\hat{S}_1^{t+1})$, so by Lemma 11.1, we obtain $V(\hat{S}_{-1}^t) \leq V(\hat{S}_{-1}^{t+1})$. This discussion completes the induction argument, so that we have $V(\hat{S}_1^t) \geq V(\hat{S}_1^{t-1})$ and $V(\hat{S}_{-1}^t) \geq V(\hat{S}_{-1}^{t-1})$ for all $t = 1, 2, \ldots$. Since the number of possible assortments is finite and the sequences $\{V(\hat{S}_1^t) : t = 0, 1, \ldots\}$ and $\{V(\hat{S}_{-1}^t) : t = 0, 1, \ldots\}$ are increasing, these sequences converge. Therefore, there exists $t_0 \geq 0$ such that $V(\hat{S}_1^{t_0}) = V(\hat{S}_1^{t_0+1}) = V(\hat{S}_1^{t_0+2}) = \ldots$ and $V(\hat{S}_{-1}^{t_0}) = V(\hat{S}_{-1}^{t_0+1}) = V(\hat{S}_{-1}^{t_0+2}) = \ldots$.

We claim that $(\hat{S}_1^{t_0+1}, \hat{S}_{-1}^{t_0})$ is a Nash equilibrium. By definition of the tatonnement process, $\hat{S}_1^{t_0+1}$ is a best response of firm 1 to the assortment $\hat{S}_{-1}^{t_0}$ offered by firm -1. It only remains to argue that $\hat{S}_{-1}^{t_0}$ is a best response of firm -1 to the assortment $\hat{S}_1^{t_0+1}$ offered by firm 1. By the definition of the tatonnement process, note that $\hat{S}_{-1}^{t_0}$ is a best response of firm -1 to the assortment $\hat{S}_1^{t_0}$. The best response of firm -1 to the assortment $\hat{S}_1^{t_0}$ is computed by solving the problem $\max_{S_{-1}\in\mathcal{F}_{-1}} R_{-1}(S_{-1}, \hat{S}_1^{t_0})$. By (11.1), $R_{-1}(S_{-1}, \hat{S}_1^{t_0})$ depends on $\hat{S}_1^{t_0}$ only through $V(\hat{S}_1^{t_0})$. Since $V(\hat{S}_1^{t_0}) = V(\hat{S}_1^{t_0+1})$, it follows that $\hat{S}_{-1}^{t_0}$ is also a best response of firm -1 to the assortment $\hat{S}_1^{t_0+1}$, establishing the claim. □

Proof of Theorem 11.3 Let (S_1^*, S_{-1}^*) be any Nash equilibrium. Assume that the sequence of assortments $\{(\hat{S}_1^t, \hat{S}_{-1}^t) : t = 0, 1, \ldots\}$ is generated by the tatonnement process. We use induction over the iterations of the tatonnement process to show that $V(\hat{S}_1^t) \leq V(S_1^*)$ and $V(\hat{S}_{-1}^t) \leq V(S_{-1}^*)$ for all $t = 0, 1, \ldots$. The result trivially holds for $t = 0$, since we have $\hat{S}_1^0 = \hat{S}_{-1}^0 = \varnothing$. Assuming that $V(\hat{S}_1^t) \leq V(S_1^*)$ and $V(\hat{S}_{-1}^t) \leq V(S_{-1}^*)$, we proceed to showing that $V(\hat{S}_1^{t+1}) \leq V(S_1^*)$ and $V(\hat{S}_{-1}^{t+1}) \leq V(S_{-1}^*)$. By definition of the tatonnement process, $\hat{S}_1^{t+1}$ is a best response of firm 1 to the assortment $\hat{S}_{-1}^t$. Also, by the definition of a Nash equilibrium, S_1^* is a best response of firm 1 to the assortment S_{-1}^*. In this case, since we have $V(\hat{S}_{-1}^t) \leq V(S_{-1}^*)$, by Lemma 11.1, we obtain $V(\hat{S}_1^{t+1}) \leq V(S_1^*)$. Similarly, by the definition of the tatonnement process, $\hat{S}_{-1}^{t+1}$ is a best response of firm -1 to the assortment $\hat{S}_1^{t+1}$. By the definition of a Nash equilibrium, S_{-1}^* is a best response of firm -1 to the assortment S_1^*. Since we have just shown that $V(\hat{S}_1^{t+1}) \leq V(S_1^*)$, by Lemma 11.1, we obtain $V(\hat{S}_{-1}^{t+1}) \leq V(S_{-1}^*)$, completing the induction argument.

By the preceding discussion, the sequence $\{(\hat{S}_1^t, \hat{S}_{-1}^t) : t = 0, 1, \ldots\}$ of assortments generated by the tatonnement process satisfies $V(\hat{S}_1^t) \leq V(S_1^*)$ and $V(\hat{S}_{-1}^t) \leq V(S_{-1}^*)$ for all $t = 0, 1, \ldots$. In this case, letting $(\hat{S}_1^{t_0+1}, \hat{S}_{-1}^{t_0})$ be a Nash equilibrium generated by the tatonnement process, we have $V(\hat{S}_1^{t_0+1}) \leq V(S_1^*)$ and $V(\hat{S}_{-1}^{t_0}) \leq V(S_{-1}^*)$. By (11.1), $R_i(S_i, S_{-i})$ is decreasing in $V(S_{-i})$. Therefore, since $V(\hat{S}_{-1}^{t_0}) \leq V(S_{-1}^*)$, we get $R_1(S_1, \hat{S}_{-1}^{t_0}) \geq R_1(S_1, S_{-1}^*)$ for all $S_1 \in \mathcal{F}_1$. Also, since $(\hat{S}_1^{t_0+1}, \hat{S}_{-1}^{t_0})$ and (S_1^*, S_{-1}^*) are Nash equilibria, we have $\hat{S}_1^{t_0+1} \in \arg\max_{S_1 \in \mathcal{F}_1} R_1(S_1, \hat{S}_{-1}^{t_0})$ and $S_1^* \in \arg\max_{S_1 \in \mathcal{F}_1} R_1(S_1, S_{-1}^*)$, because the assortments offered by firm 1 must be a best response to the assortments offered by firm -1 in any Nash equilibrium. In this case, we obtain

$$R_1(\hat{S}_1^{t_0+1}, \hat{S}_{-1}^{t_0}) = \max_{S_1 \in \mathcal{F}_1} R_1(S_1, \hat{S}_{-1}^{t_0}) \geq \max_{S_1 \in \mathcal{F}_1} R_1(S_1, S_{-1}^*) = R_1(S_1^*, S_{-1}^*),$$

where the inequality uses the fact that $R_1(S_1, \hat{S}_{-1}^{t_0}) \geq R_1(S_1, S_{-1}^*)$ for all $S_1 \in \mathcal{F}_1$. The chain of inequalities above shows that the expected revenue of firm 1 in the equilibrium $(\hat{S}_1^{t_0+1}, \hat{S}_{-1}^{t_0})$ is at least as large as its expected revenue in the equilibrium (S_1^*, S_{-1}^*). We can use a similar argument to show that the same statement holds for firm -1 as well. □

Proof of Theorem 11.4 By its definition, S_i^{NC} is a best response of firm i to the empty assortment. Also, by the definition of a Nash equilibrium, S_i^* is a best response of firm i to the assortment S_{-i}^*. Since $V(\varnothing) = 0 \leq V(S_{-i}^*)$, by Lemma 11.1, we obtain $V(S_i^{\text{NC}}) \leq V_i(S_i^*)$, establishing the first inequality in the theorem. To show the second inequality in the theorem, let z^* be the optimal objective value of problem (11.4) and $z_i^*(S_{-i}^*)$ be the optimal objective value of problem (11.2) after replacing S_{-i} with S_{-i}^*. Because (S_1^*, S_{-1}^*) is a feasible but not necessarily an optimal solution to problem (11.4), we have

$$z^* = R_1(S_1^{\rm CP}, S_{-1}^{\rm CP}) + R_{-1}(S_{-1}^{\rm CP}, S_1^{\rm CP})$$
$$\geq R_1(S_1^*, S_{-1}^*) + R_{-1}(S_{-1}^*, S_1^*) = z_1^*(S_{-1}^*) + z_{-1}^*(S_1^*),$$

where the last equality follows from the fact that S_i^* is a best response of firm i to the assortment S_{-i}^*. In problem (11.2), each firm can trivially obtain a strictly positive expected revenue by offering any product. Therefore, $z_1^*(S_{-1}^*) > 0$ and $z_{-1}^*(S_1^*) > 0$, in which case, the chain of inequalities above implies that $z^* > z_1^*(S_{-1}^*)$ and $z^* > z_{-1}^*(S_1^*)$.

Note that in problem (11.4), if we fix the assortment S_{-1} at its optimal value $S_{-1}^{\rm CP}$ and optimize only over the assortment S_1, then setting $S_1 = S_1^{\rm CP}$ would still yield an optimal solution. Therefore, $S_1^{\rm CP}$ is an optimal solution to the problem

$$\max_{S_1 \in \mathcal{F}_1} \left\{ R_1(S_1, S_{-1}^{\rm CP}) + R_{-1}(S_{-1}^{\rm CP}, S_1) \right\} = \max_{S_1 \in \mathcal{F}_1} \frac{\sum_{j \in S_1} p_j\, v_j + \sum_{j \in S_{-1}^{\rm CP}} p_j\, v_j}{v_0 + V(S_1) + V(S_{-1}^{\rm CP})}$$

yielding the optimal objective value z^*. In this case, we have

$$z^* \geq \frac{\sum_{j \in S_1} p_j\, v_j + \sum_{j \in S_{-1}^{\rm CP}} p_j\, v_j}{v_0 + V(S_1) + V(S_{-1}^{\rm CP})} \qquad \forall\, S_1 \in \mathcal{F}_1,$$

and the inequality above holds as equality at the optimal solution $S_1^{\rm CP}$. Following the same sequence of steps that we used to obtain problem (11.3), it follows that

$$[v_0 + V(S_{-1}^{\rm CP})]\, z^* \geq \sum_{j \in S_1} (p_j - z^*)\, v_j + \sum_{j \in S_{-1}^{\rm CP}} p_j\, v_j \qquad \forall\, S_1 \in \mathcal{F}_1,$$

and the inequality holds as equality at the optimal solution $S_1^{\rm CP}$. Therefore, $S_1^{\rm CP}$ is an optimal solution to the problem

$$\max_{S_1 \in \mathcal{F}_1} \left\{ \sum_{j \in S_1} (p_j - z^*)\, v_j \right\},$$

but since S_1^* is a feasible but not necessarily an optimal solution to the problem above, we obtain

$$\sum_{j \in S_1^{\rm CP}} (p_j - z^*)\, v_j \geq \sum_{j \in S_1^*} (p_j - z^*)\, v_j .$$

Also, since S_1^* is a best response of firm 1 to the assortment S_{-1}^*, S_1^* is an optimal solution to problem (11.3) with $i = 1$ and $S_{-i} = S_{-i}^*$. However, since $S_1^{\rm CP}$ is a feasible but not necessarily an optimal solution to this problem, we get

$$\sum_{j \in S_1^*} (p_j - z_1^*(S_{-1}^*))\, v_j \geq \sum_{j \in S_1^{\mathrm{CP}}} (p_j - z_1^*(S_{-1}^*))\, v_j.$$

Adding the last two inequalities, we obtain

$$z^* \,[V(S_1^*) - V(S_1^{\mathrm{CP}})] \geq z_1^*(S_{-1}^*)\,[V(S_1^*) - V(S_1^{\mathrm{CP}})].$$

If $V(S_1^*) < V(S_1^{\mathrm{CP}})$, then the last inequality implies that $z^* \leq z_1^*(S_{-1}^*)$, which contradicts the fact that $z^* > z_1^*(S_{-1}^*)$. Therefore, we must have $V(S_1^*) \geq V(S_1^{\mathrm{CP}})$. A symmetric argument also shows that $V(S_{-1}^*) \geq V(S_{-1}^{\mathrm{CP}})$, establishing the second inequality in the theorem. □