Chapter 1
Single Resource Revenue Management with Independent Demands

1.1 Introduction

In this chapter, we consider the single resource, independent demand revenue management problem with multiple fare classes. This problem arises in the airline industry where different fares for the same cabin are designed to cater to different market segments. As an example, a low fare may have advance purchase and length of stay restrictions and exclude ancillary services such as advance seat selection, luggage handling, and priority boarding. This low fare may target price-conscious consumers who travel for leisure on restricted budgets. On the other hand, a high fare designed for business consumers may be unrestricted, include ancillary services and be designed to be frequently available for late bookings. If requests for the low fare arrive first, the airline risks selling all of its capacity before seeing requests for the high fare. A key decision in revenue management is how much capacity to reserve for higher fare classes, or equivalently how much capacity to make available for lower fare classes. Throughout the chapter, we will refer to airline applications, but the reader should keep in mind that the models apply more generally.

We assume that the set of fare classes is given. This set of fare classes corresponds to a menu of prices, restrictions, and ancillary services. Demands for the different fare classes are assumed to be independent random variables. In particular, we assume that if a consumer finds that her preferred fare class is unavailable, then she will leave the system without purchasing anything. This assumption holds approximately when the difference in fares is large enough that demands for the different fare classes are decoupled or when consumers can find alternative sources of capacity for their preferred fare class, perhaps on a different flight or with a different carrier. In some cases, the demand for a fare class that is closed may be recaptured by other open fare classes. Demand recapture makes the independent assumption untenable. We will address this issue in a separate chapter on dependent demands based on discrete choice models.

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We will also assume that the capacity available for sale is fixed and cannot be modified or replenished during the selling horizon. Later in the chapter, we will discuss how to optimally select the initial capacity in situations where it can be purchased at an increasing convex cost. Unsold capacity is assumed to have a zero salvage value. This assumption is without loss of generality as any problem with positive salvage value can be transformed into an equivalent problem with zero salvage value.

The objective is to maximize the total expected revenue from all fare classes by dynamically choosing the fare classes to offer for sale during the selling horizon. In practice this is done by selectively denying requests for lower fare classes with the hope of using capacity for requests from higher fare classes. Consequently, the revenue curve that monitors the accumulation of the total revenue during the selling horizon may start low, but later catch up and hopefully exceed the revenue curve corresponding to a policy that accepts all requests.

In Sect. 1.2, we study the two fare class revenue management problem, where the capacity provider has to decide how much capacity to make available to the low-fare class before seeing the demand for the high fare class. In this formulation time is treated implicitly by assuming that requests for different fare classes arrive *sequentially*. The solution is given by Littlewood's rule. In Sect. 1.3, we present a dynamic programming formulation for multiple fare classes under the assumption that the requests for different fare classes arrive sequentially in a *low-before-high* order. Structural results such as concavity of the value function are also presented. While arrival order can be relaxed, the sequential arrival assumption cannot. In Sect. 1.4, we study the problem of setting initial capacity and its connections to the newsvendor problem. In Sect. 1.5, we present commonly used heuristics for the multiple fare class revenue management problem. In Sect. 1.6, we provide bounds on the optimal total expected revenue. In Sect. 1.7, we introduce a model that allows for non-sequential arrival patterns by modeling demands as independent Poisson processes with time-dependent arrival rates. In Sect. 1.8, we study models that do not allow fare classes to reopen once they are closed. This restriction may be helpful to cope with strategic consumers as it deters them from waiting for lower fares. In Sect. 1.9, we extend the analysis to the case of compound Poisson demands, where arriving consumers may request more than one unit of capacity. In Sect. 1.10, we conclude the chapter by comparing the performance of the multi-fare revenue problem under the sequential, low-before-high fare arrival pattern, to the performance of the more flexible model that allows for compound Poisson demands. Not surprisingly, the latter model outperforms the former when the sequential fare arrival pattern fails to hold.

1.2 Two Fare Classes

Consider an airline flight with c units of capacity. The capacity can be sold either at full fare for a price of p_1 or at a discounted fare for a price of $p_2 < p_1$. The discounted fare typically has advanced purchasing and usage restrictions. As an example, a round trip discounted fare may need to be purchased three weeks in advance and may require a Saturday night stay. We assume that all booked consumers will actually travel. This assumption avoids the need to overbook the capacity. We discuss how to deal with cancellations before the departure day and no-shows on the departure day in a separate chapter on overbooking models.

We assume a low-before-high fare class arrival order, which implies that demand for the discounted fare, say D_2, books before the demand for the full fare, say D_1. This arrival pattern holds approximately in practice and it is encouraged by the advance purchase restrictions imposed on the lower fare. Notice that low-before-high is a worst case arrival pattern in terms of revenues. Indeed, if the full fare consumers arrived first, then we would accept them up to the available capacity and use the residual capacity, if any, to satisfy demand from the discounted fare consumers. When the arrival pattern is low-before-high, it is critical to impose booking limits on the low-fare consumers, as otherwise the low-fare consumers may exhaust capacity and force the provider to deny capacity to consumers willing to pay higher fares.

Let c be the capacity on the flight. Suppose that we protect $y \in \{0, 1, \ldots, c\}$ units of capacity for the full fare before observing the actual demand for the discounted fare. This leaves $c - y$ units of capacity available to satisfy the demand for the discounted fare. We refer to $c - y$ as the booking limit for the discounted fare. Consequently, sales at the discounted fare are given by $\min\{c - y, D_2\}$. The remaining capacity is equal to $c - \min\{c - y, D_2\} = \max\{y, c - D_2\}$, so sales at the full fare are given by $\min\{\max\{y, c - D_2\}, D_1\}$. The total expected revenue from the two fare classes is therefore

$$W(y, c) := p_2\, \mathbb{E}\{\min\{c - y, D_2\}\} + p_1\, \mathbb{E}\{\min\{\max\{y, c - D_2\}, D_1\}\}. \tag{1.1}$$

The objective is to find the protection level y that maximizes the expected revenue $W(y, c)$. The extreme strategies $y = 0$ and $y = c$ correspond, respectively, to the case where no capacity is protected, and to the case where all of the capacity is protected for the full fare consumers. We will later discuss when these extreme strategies are optimal. In most cases, however, an intermediate strategy is optimal.

The fare ratio $r := p_2/p_1$ plays an important role in determining the optimal protection level. If the ratio is close to zero, then the full fare is substantially larger than the discounted fare and we would be inclined to protect more capacity for the full fare demand. If the ratio is close to one, then the discounted fare is close to the full fare and we would be inclined to accept more requests for the discounted fare since we can get almost the same revenue per unit of capacity. The distribution of the full fare demand is also important in determining the optimal protection level. If $\mathbb{P}\{D_1 \geq c\}$ is very large, then the full fare demand exceeds the available capacity with high probability, so it makes sense to protect the entire capacity for the full fare consumers as it is likely that the provider can sell all of the capacity at the full fare. However, if $\mathbb{P}\{D_1 \geq c\}$ is very small, then it is unlikely that all the capacity can be sold at the full fare, so fewer units should be protected for the full fare consumers. As we demonstrate shortly, the demand for the discounted fare has no influence on the optimal protection level.

We can use marginal analysis to study the tradeoff between accepting and rejecting a request for the discounted fare when we have y units of capacity available. If we accept this request, then we obtain a revenue of p_2 for the marginal unit. If we reject the request and close down the discount fare, then we will sell the y-th unit at fare p_1 only if the full fare demand D_1 is at least as large as y. Thus, it is intuitively optimal to reject the request for the discounted fare when $p_1 \, \mathbb{P}\{D_1 \geq y\} > p_2$. This argument suggests that an optimal protection level y_1^* is given by

$$y_1^* = \max\{y \in \mathbb{N}_+ : \mathbb{P}\{D_1 \geq y\} > r\}, \tag{1.2}$$

where $\mathbb{N}_+ = \{0, 1, \ldots\}$ is the set of non-negative integers. The formula for the optimal protection level in (1.2) is known as Littlewood's rule. Later we will show that (1.2) is a maximizer of (1.1).

Example 1.1 Suppose that D_1 is a Poisson random variable with mean 80, the full fare is $p_1 = \$100$ and the discounted fare is $p_2 = \$60$, so $r = 60/100 = 0.6$. To compute the optimal protection level y_1^*, we are interested in the cumulative tail distribution $\mathbb{P}\{D_1 \geq y\} = 1 - \mathbb{P}\{D_1 \leq y-1\}$. Since most statistical software packages return the value of the cumulative distribution $\mathbb{P}\{D_1 \leq y - 1\}$, rather than the value of the tail distribution $\mathbb{P}\{D_1 \geq y\}$, we see that y_1^* should satisfy $\mathbb{P}\{D_1 \leq y_1^* - 1\} < 1 - r \leq \mathbb{P}\{D_1 \leq y_1^*\}$. Since $\mathbb{P}\{D_1 \leq 77\} < 0.4 \leq \mathbb{P}\{D_1 \leq 78\}$, we conclude that $y_1^* = 78$. Consequently, it is optimal to protect 78 seats for the full fare consumers. If $c = 200$, then the booking limit for the discounted fare consumers is $c - y_1^* = 122$, which indicates that it is optimal to allow at most 122 bookings for the discounted fare class. If $c \leq y_1^*$, then all of the capacity should be protected for the full fare consumers, resulting in a booking limit of zero for the discounted fare class.

Remark 1.2 The following remarks are immediately derived from Littlewood's rule.

- The optimal protection level y_1^* is independent of the distribution of the discounted fare demand D_2.
- If $\mathbb{P}\{D_1 \geq y_1^* + 1\} = r$, then $y_1^* + 1$ is also optimal protection level, so both y_1^* and $y_1^* + 1$ result in the same total expected revenue. Protecting the $y_1^* + 1$ units of capacity increases the variance of the revenue, but it reduces the probability of rejecting requests from the full fare consumers.

From Littlewood's rule in (1.2), we see that the extreme strategy $y_1^* = 0$ is optimal when $\mathbb{P}\{D_1 \geq 1\} \leq r$ and $y_1^* = c$ is optimal when $\mathbb{P}\{D_1 \geq c\} > r$.

1.2.1 Continuous Demand Distributions

Although the demand in revenue management models is inherently a discrete quantity, it can be easier to model the demand with a continuous random variable. If the demand D_1 from fare class 1 is a continuous random variable with cumulative distribution function $F_1(y) = \mathbb{P}\{D_1 \leq y\}$, then the optimal protection level is

$$y_1^* = F_1^{-1}(1 - r),$$

where $F_1^{-1}(\cdot)$ denotes the inverse of $F_1(\cdot)$. In particular, if D_1 is a normal random variable with mean μ_1 and standard deviation σ_1, then we have

$$y_1^* = \mu_1 + \sigma_1 \, \Phi^{-1}(1 - r),$$

where $\Phi(\cdot)$ is the cumulative distribution function of the standard normal random variable. This formula can be used to understand how the protection level changes as a function of the demand parameters. Notice that if $r < 1/2$, then $\Phi^{-1}(r) < 0$, so $y_1^* < \mu_1$ and y_1^* decreases with σ_1. Similarly, if $r > 1/2$, then $\Phi^{-1}(r) > 0$, so $y_1^* > \mu_1$ and y_1^* increases with σ_1. If $r = 1/2$, then $\Phi^{-1}(r) = 0$ so that $y_1^* = \mu_1$ and it does not depend on σ_1.

Example 1.3 Suppose that D_1 is a normal random variable with mean 80 and standard deviation 9, the full fare is $p_1 = \$100$ and the discount fare is $p_2 = \$60$. Thus, we have $y_1^* = 80 + 3 \times \Phi^{-1}(1 - 0.6) = 77.72$. Since $r < 1/2$, we have $y_1^* < 80$. Notice that the solution is quite close to that of Example 1.1. This is because a Poisson random variable with mean 80 can be well approximated by a normal random variable with mean 80 and standard deviation $9 \approx \sqrt{80}$.

1.2.2 *Quality of Service, Salvage Values, and Callable Products*

As we will now see, Littlewood's rule can result in poor service to consumers who prefer the high fare when the fare ratio is high. The probability of denying service to at least one high fare consumer is known as the full fare spill rate. Since at least one consumer is denied service when $D_1 > \max\{y_1^*, c - D_2\}$, we have

$$\mathbb{P}\{D_1 > \max\{y_1^*, c - D_2\}\} \leq \mathbb{P}\{D_1 > y_1^*\} \leq r < \mathbb{P}\{D_1 \geq y_1^*\},$$

where the last two inequalities follow from (1.2). We call $\mathbb{P}\{D_1 > y_1^*\}$ the maximal spill rate. Notice that if the inequality $y_1^* \geq c - D_2$ holds with high probability, as it typically does in practice when the discount fare demand D_2 is large relative to c, then the spill rate approaches the maximal spill rate which is, by design, close to the ratio r.

High spill rates may lead to the loss of full fare consumers to competition. To see this, imagine two airlines, each offering a discounted fare and a full fare in the same market, where the fare ratio r is large and there is high demand for the discounted fare. In this situation, the maximal spill rate is high, indicating that with high probability at least one fare class 1 consumer will be denied capacity. Suppose Airline A uses Littlewood's rule with spill rates close to r, which implies that Airline A turns down fare class 1 consumers with high probability. Airline B can protect more seats for the full fare consumers than recommended by Littlewood's rule. By doing so, Airline B sacrifices revenue in the short run but can attract some of the fare class 1 consumers spilled by Airline A. Over time, Airline A may see a decrease in class fare 1 demand as a secular change and protect even fewer seats for fare class 1 consumers. Meanwhile, Airline B will see an increase in class fare 1 consumers. At this point, Airline B can switch to the optimal protection level recommended by Littlewood's rule, deriving larger revenues in the long run. In essence, Airline B has strategically traded discounted fare consumers for full fare consumers with Airline A.

One way to cope with high spill rates and their adverse strategic consequences is to impose a penalty cost of ρ for each unit of full fare demand that is not served. This penalty is supposed to measure the ill will incurred when service is denied to a full fare consumer. Imposing a penalty ρ for each unit of full fare demand that is not served can be shown to be equivalent to increasing the fare of fare class 1 to $p_1 + \rho$ in the corresponding optimization problem. This leads to a modification of Littlewood's rule, resulting in optimal protection level given by

$$y_1^* = \max\left\{y \in \mathbb{N}_+ : \mathbb{P}\{D_1 \geq y\} > \frac{p_2}{p_1 + \rho}\right\}. \tag{1.3}$$

Since $p_2/(p_1+\rho) < p_2/p_1$, we get lower maximal spill rates by imposing a penalty for each unit of fare class 1 demand that is not served. Obviously, this adjustment in maximal spill rates comes at the expense of having larger protection levels and lower total expected revenues. This is, in essence, a sacrifice in expected revenues to keep a stream of consumers from defecting to competitors.

To improve the spill rate without sacrificing sales at the discount fare, the airline can modify the discount fare offering by adding a restriction that allows the airline to recall or buy back capacity when needed. This approach leads to revenue management with callable products. Callable products can be sold either by giving consumers an upfront discount or by giving them a compensation if and when capacity is recalled. When managed correctly, callable products lead to better capacity utilization, provide better service to full fare consumers, and induce demand from consumers who are attracted either to the upfront discount or to the compensation when the capacity is recalled. Callable products are common in the secondary market for event tickets as they provide a hedge against uncertainty in the supply in the sense that the liability for failing to deliver capacity is limited to the monetary compensation tied to the callable product.

We can also account for salvage values for the capacity that is not sold at the end of the selling horizon. Suppose there is a salvage value $s < p_2$ on excess capacity after the arrival of the full fare demand. This salvage value can be interpreted as the revenue from standby tickets or last minute travel deals. A salvage value of s for each unit of unsold capacity is equivalent to decreasing the fare of the fare classes 1 and 2, respectively, to $p_1 - s$ and $p_2 - s$. Therefore, using Littlewood's rule, the optimal protection level is given by

$$y_1^* = \max\left\{y \in \mathbb{N}_+ : \mathbb{P}\{D_1 \geq y\} > \frac{p_2 - s}{p_1 - s}\right\}. \tag{1.4}$$

1.3 Multiple Fare Classes

In this section, we present an exact solution to the multiple fare class problem using dynamic programming. The analysis is somewhat technical and readers may prefer to first focus on the dynamic programming formulation in (1.6) and the main results in Proposition 1.5, Theorem 1.6, and Corollary 1.7 before going over the details of the analysis.

We assume that the capacity provider has c units of perishable capacity to be allocated among n fare classes, where the fares are indexed so that $p_n < \ldots < p_1$. Lower fares typically have severe time-of-purchase and traveling restrictions. Given the time-of-purchase restriction, it is natural to assume, as we do, a low-before-high fare arrival order, with fare class n arriving first and fare class 1 arriving last. We use $N = \{1, \ldots, n\}$ to denote the set of fare classes. Let D_j denote the random demand for fare class j. We assume that the demand random variables $D_1, \ldots, D_n$ are independent of each other with finite means $\mu_j := \mathbb{E}\{D_j\} < \infty$ for all $j \in N$.

Let $V_j(x)$ denote the optimal total expected revenue from fare classes $j, j-1 \ldots, 1$ given x units of remaining capacity just before facing the demand for fare class j. To write a dynamic program, we first review the sequence of events in stage j (just before the arrival of demand for fare class j):

- Given x units of remaining capacity select protection level $y \in \{0, \ldots, x\}$ for fare classes $j-1, j-2, \ldots, 1$ and make $x-y$ units of capacity available for sale to fare class j.
- Observe demand for fare class j. The capacity sold to fare class j is given by $\min\{x-y, D_j\}$, and the revenue generated is $p_j \min\{x-y, D_j\}$.
- The remaining capacity before facing demand for fare class $j-1$ is given by $x - \min\{x-y, D_j\} = \max\{y, x-D_j\}$.

Let $W_j(y, x)$ be the optimal expected revenue from fare classes $j, j-1, \ldots, 1$ assuming that we protect $y \leq x$ units of capacity for the fare classes $j-1, j-2, \ldots, 1$. In this case, the functions $\{W_j(\cdot,\cdot) : j = n, \ldots, 1\}$ and $\{V_j(\cdot) : j = n, \ldots, 1\}$ satisfy the relationship

$$W_j(y, x) = p_j\, \mathbb{E}\{\min\{x-y, D_j\}\} + \mathbb{E}\{V_{j-1}(\max\{y, x-D_j\})\}. \tag{1.5}$$

In the expression above, $p_j\, \mathbb{E}\{\min\{x-y, D_j\}\}$ is the expected revenue obtained from fare class j given that we have x units of remaining capacity when facing the demand for fare class j and we protect y units of capacity for fare classes $j-1, j-2, \ldots 1$. On the other hand, $V_{j-1}(\max\{y, x-D_j\})$ is the optimal expected revenue obtained from fare classes $j-1, j-2, \ldots, 1$, given that we have $\max\{y, x-D_j\}$ units of remaining capacity before facing the demand from fare class $j-1$. Given x units of remaining capacity, we maximize $W_j(y, x)$ over $y \in \{0, \ldots, x\}$ resulting in the dynamic program:

$$\begin{aligned} V_j(x) &= \max_{y \in \{0,\ldots,x\}} W_j(y, x) \\ &= \max_{y \in \{0,\ldots,x\}} \Big\{ p_j \mathbb{E}\{\min\{x-y, D_j\}\} + \mathbb{E}\{V_{j-1}(\max\{y, x-D_j\})\} \Big\}. \end{aligned} \tag{1.6}$$

By convention, we have $V_0(x) = 0$, since we do not collect any revenue when there are no fare classes left to arrive. By definition, $V_n(c)$ is the optimal total expected revenue for the multiple fare class problem with n fare classes and an initial capacity of c units. Assuming that computing each one of the expectations in (1.6) takes constant time, we can solve the dynamic program in $O(c^2)$ operations.

1.3.1 Structure of the Optimal Policy

For any function f with integer domain, let $\Delta f(x) = f(x) - f(x-1)$.

The following lemma is important in establishing structural results.

Lemma 1.4 *Let $g(x) := \mathbb{E}\{G(\min\{X, x\})\}$, where X is an integer valued random variable with $\mathbb{E}X < \infty$, and $G(\cdot)$ is a function over the integers. Then,*

$$\Delta g(x) = \Delta G(x)\, \mathbb{P}\{X \geq x\}.$$

Similarly, let $h(x) := \mathbb{E}\{H(\max\{X, x\})\}$, where X is an integer valued random variable with $\mathbb{E}X < \infty$, and $H(\cdot)$ is a function over the integers. Then,

$$\Delta h(x) = \Delta H(x)\, \mathbb{P}\{X < x\}.$$

The next two results provide the key results of this section.

Proposition 1.5 *The value functions computed through (1.6) satisfy the following properties.*

- *$\Delta V_j(x)$ is decreasing in $x \in \{1, \ldots, c\}$.*
- *$\Delta V_j(x)$ is increasing in $j \in \{1, \ldots, n\}$.*

Since $\Delta V_j(x)$ is decreasing in x, the value function $V_j(\cdot)$ is concave. The following theorem speaks to the monotonicity of the protection levels.

Theorem 1.6 *For all $j = n, \ldots, 1$, the function $W_j(y, x)$ is unimodal in y and the maximizer of $W_j(y, x)$ over $y \in \{0, \ldots, x\}$ is given by $\min\{y^*_{j-1}, x\}$, where*

$$y^*_{j-1} = \max\{y \in \mathbb{N}_+ : \Delta V_{j-1}(y) > p_j\}. \tag{1.7}$$

Moreover, $y^*_{n-1} \geq y^*_{n-2} \geq \ldots \geq y^*_1 \geq y^*_0 = 0$. Thus, optimal protection levels are monotone in the number of stages left until the end of the selling horizon.

For the case of $n = 2$, we get a formal proof of Littlewood's rule.

Corollary 1.7

$$y^*_1 = \max\{y \in \mathbb{N}_+ : p_1 \mathbb{P}\{D_1 \geq y\} > p_2\}.$$

Likewise, it is possible to show (1.3) and (1.4) by appropriately modifying $V_0(x)$.

Table 1.1 Optimal total expected revenues $V_j(c)$ for the values of capacity $c \in \{50, 100, 150, 200, 250, 300, 350\}$ and $j = 1, 2, 3, 4, 5$

c	Load factor	$V_1(c)$	$V_2(c)$	$V_3(c)$	$V_4(c)$	$V_5(c)$
50	560%	1500	3427	3427	3427	3427
100	280%	1500	3900	5441	5441	5441
150	187%	1500	3900	5900	7189	7189
200	140%	1500	3900	5900	7825	8159
250	112%	1500	3900	5900	7825	8909
300	93%	1500	3900	5900	7825	9564
350	80%	1500	3900	5900	7825	9625

Remark 1.8 The following remarks apply for the structure of the optimal policy:

- Let x_j be the remaining capacity just before facing the demand for fare class j. Then capacity is allocated to fare class j only if $x_j > y^*_{j-1}$ with at most $[x_j - y^*_{j-1}]^+$ bookings allowed.
- The protection level y^*_{j-1} is independent of the distribution of the demand from fare classes $n, n-1, \ldots, j$.
- The policy is implemented as follows: At stage n, we start with $x_n = c$ units of inventory and we protect $y_{n-1}(x_n) = \min\{y^*_{n-1}, x_n\}$ units of capacity for fares $n-1, n-2, \ldots, 1$. Therefore, we allow up to $[x_n - y^*_{n-1}]^+$ units of capacity to be sold to fare class n. We sell $\min\{[x_n - y^*_{n-1}]^+, D_n\}$ units of capacity to fare class n and we have a remaining capacity of $x_{n-1} = x_n - \min\{[x_n - y^*_{n-1}]^+, D_n\}$ at stage $n-1$. We protect $y_{n-2}(x_{n-1}) = \min\{y^*_{n-2}, x_{n-1}\}$ units of capacity for fares $n-2, n-1, \ldots, 1$. Therefore, we allow up to $[x_{n-1} - y^*_{n-2}]^+$ units of capacity to be sold to fare class $n-1$. We continue until we reach stage 1 with x_1 units of capacity, allowing $(x_1 - y_0)^+ = (x_1 - 0)^+ = x_1$ to be sold to fare class 1.

Example 1.9 Suppose that there are five fare classes. The demand for all fare classes is a Poisson random variable. The fares and the expected demand for the five fare classes are given by $(p_5, p_4, p_3, p_2, p_1) = (15, 35, 40, 60, 100)$ and $(\mathbb{E}\{D_5\}, \mathbb{E}\{D_4\}, \mathbb{E}\{D_3\}, \mathbb{E}\{D_2\}, \mathbb{E}\{D_1\}) = (120, 55, 50, 40, 15)$. For this problem instance, the optimal protection levels are $y^*_4 = 169$, $y^*_3 = 101$, $y^*_2 = 54$, and $y^*_1 = 14$. In Table 1.1, we show the total expected revenue $V_j(c)$ obtained from fare classes $j, j-1, \ldots, 1$ when we have c units of remaining capacity at the beginning of stage j, as well as the corresponding load factors $\sum_{j=1}^{5} \mathbb{E}\{D_j\}/c = 280/c$.

The effect of restricting capacity for low fares is apparent in the pattern of total expected revenues across fare classes. For example, the total expected revenues $V_2(50)$, $V_3(50)$, $V_4(50)$, and $V_5(50)$ are all equal to \$3427 since fare classes 5, 4, and 3 are rationed when $c = 50$. On the other hand, $V_1(350)$ through $V_5(350)$ vary from \$1500 to \$9625 since there is enough capacity to accommodate all or nearly all demand from the five fare classes. In Fig. 1.1, we show the marginal value of capacity $\Delta V_j(x)$ when we have x units of remaining capacity at the beginning of stage j. The marginal value of capacity increases as we have fewer units of capacity and as we have more stages left until the end of the selling horizon.

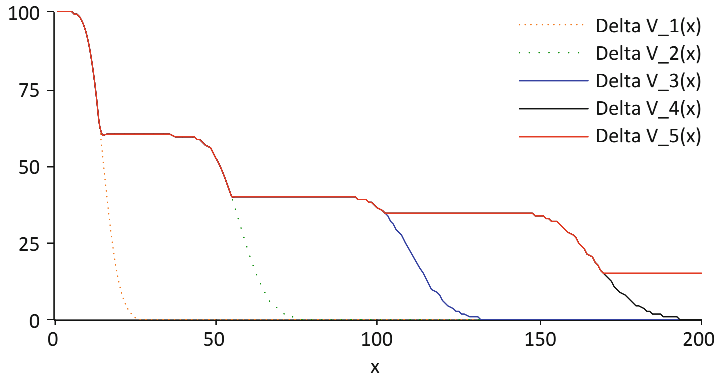

Fig. 1.1 Marginal value of capacity $\Delta V_j(x)$ as a function of x for $j = 1, 2, 3, 4, 5$

Although computing the optimal policy is not numerically onerous, some computations can be streamlined to obtain a more efficient implementation. It can be shown that

$$\Delta V_j(x) = p_j \mathbb{P}\{D_j \geq x - y^*_{j-1}\} + \sum_{k=0}^{x-y^*_{j-1}-1} \Delta V_{j-1}(x-k) \mathbb{P}\{D_j = k\} \tag{1.8}$$

for all $x > y^*_{j-1}$, and that $\Delta V_j(x) = \Delta V_{j-1}(x)$ if $x \leq y^*_{j-1}$. The proof of this result is left as an exercise.

1.3.2 Nonmonotone Fares

It is possible to relax the low-before-high assumption while retaining the assumption that requests for the different fare classes arrive sequentially. Proposition 1.5 holds as stated. The optimal protection level y^*_{j-1} is computed as stated in Theorem 1.6, but optimal protection levels are not necessarily monotone. Clearly $y^*_{j-1} = 0$ whenever $p_j > \max\{p_{j-1}, p_{j-2}, \ldots p_1,\}$ since it is optimal to accept all requests for fare class j up to capacity since there is no point in protecting capacity to sell it later at a lower fare! As an example, suppose that $p_3 < p_2$ and $p_2 > p_1$. Since no fare classes are arriving after fare class 1, it is optimal to serve the demand from fare class 1 as much as possible, which implies that $V_1(x) = p_1 \mathbb{E}\{\min\{x, D_1\}\}$. Using Lemma 1.4, we obtain $\Delta V_1(x) = p_1 \mathbb{P}\{D_1 \geq x\} < p_2$, where the inequality uses the fact that $p_2 > p_1$. In this case, (1.7) implies that $y^*_1 = 0$. Since $y^*_1 = 0$, we see that $V_2(x) = p_2 \mathbb{E} \min\{x, D_2\} + \mathbb{E}\{V_1([x - D_1]^+)\} = p_2 \mathbb{E} \min\{x, D_2\} + p_1 \mathbb{E}\{\min\{D_1, [x - D_2]^+\}\}$. One can check that

$$\Delta V_2(x) = p_2 \mathbb{P}\{D_2 \geq x\} + p_1 \mathbb{P}\{D_2 < x \leq D[1,2]\} \geq p_2 \mathbb{P}\{D_2 \geq x\},$$

where we use $D[i, j] := \sum_{k=i}^{j} D_k$. The optimal amount of capacity to protect for fare classes 2 and 1 is given by $y_2^* = \max\{y \in \mathbb{N}_+ : \Delta V_2(y) > p_3\}$. On the other hand, if there were no fare classes arriving after fare class 2, then by Littlewood's rule, the optimal amount of capacity to protect for fare classes 2 would be $\max\{y \in \mathbb{N}_+ : \mathbb{P}\{D_2 \geq y\} > p_3/p_2\}$. Since $\Delta V_2(x) > p_2 \mathbb{P}\{D_2 \geq x\}$, the capacity protected for fare classes 2 and 1 is larger than it would be when there was no demand for fare class 1.

1.4 The Generalized Newsvendor Problem

Consider the problem of selecting c to maximize $\Pi_n(c) := V_n(c) - K(c)$, where $V_n(c)$ is the solution to the multi-fare revenue management problem and $K(c)$ is the cost of procuring c units of capacity. This problem is relevant in revenue management when capacity decisions are made. We will assume that $K(c)$ is increasing convex. Two plausible models are the linear model: $K(c) = k\,c$ for some unit cost k, or the fixed cost model with a capacity limit: $K(0) = 0$, $K(c) = k$ for all $0 < c \leq \bar{c}$ and $K(c) = \infty$ for $c > \bar{c}$.

The smallest maximizer of $\Pi_n(c)$, say c^*, is characterized in the following proposition.

Proposition 1.10 *The smallest optimal procurement quantity is given by*

$$c^* = \max\{c \in \mathbb{N}_+ \ : \ \Delta V_n(c) > \Delta K(c)\}.$$

For the linear model, we can write c^* as a function of k, yielding

$$c(k) = \max\{c \in \mathbb{N}_+ \ : \ \Delta V_n(c) > k\}.$$

The order quantities at the price points are the corresponding protection levels of the corresponding revenue management problem, so

$$c(p_{j+1}) = y_j \quad \forall \ j \in \{1, \ldots, n-1\},$$

with $c(k)$ increasing in k.

For the fixed cost model, $c^* = \bar{c}$ if $V_n(\bar{c}) > k$ and $c^* = 0$ otherwise.

Figure 1.2 depicts $c(k)$ for the data of Example 1.9, with $c(60) = 14$, $c(40) = 54$, $c(35) = 101$, and $c(15) = 169$.

In a retail setting, the low-before-high arrival pattern is unlikely to hold, and a more useful model is to seek c to maximize $\Pi_n(c) = V_n(c) - k\,c$, where

$$V_n(c) := \sum_{j=1}^{n} p_j\, \mathbb{E}\{\min\{D_j, (c - D[1, j-1])^+\}\}$$

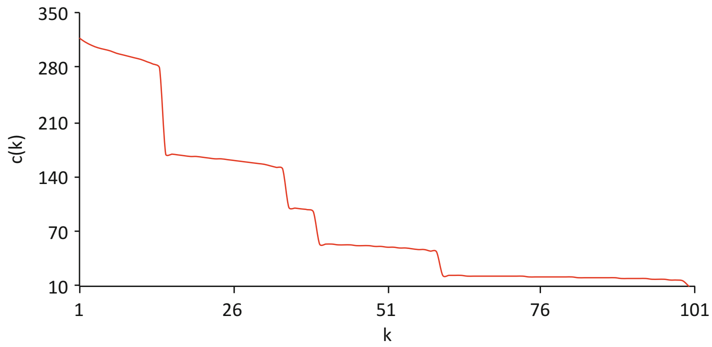

Fig. 1.2 Optimal capacity as a function of cost for Example 1.9

is the expected revenue corresponding to the high-before-low arrival pattern: $D_1, D_2, \ldots, D_n$ at prices $p_1 > p_2 > \ldots > p_n$. The marginal value of capacity can be written as

$$\Delta V_n(c) = \sum_{j=1}^{n} (p_j - p_{j+1})\, \mathbb{P}\{D[1, j] > c\},$$

where for convenience we define $p_{n+1} = 0$. This leads directly to the following result.

Corollary 1.11 *In a retail setting, with a high-before-low arrival pattern, an optimal order quantity for the linear cost model $K(c) = k\,c$ is given by*

$$c(k) = \max\{c \in \mathbb{N}_+ : \sum_{j=1}^{n} (p_j - p_{j+1})\, \mathbb{P}\{D[1, j] > c\} > k\}.$$

The classical newsvendor problem corresponds to the case $n = 2$ with $p = p_1 > p_2 = s$, where s is the salvage value and demand at the salvage value is infinite (an implicit assumption that is seldom discussed in the context of the newsvendor model). The model presented here is more general, even if $n = 2$, as it allows for random demand at the salvage value. In fact, it allows for random demand at all discounted prices $p_2, \ldots, p_n$.

Integrating dynamic pricing into the newsvendor problem allows for multiple price discounts, and this results in higher profits as more sales occur at price points above the salvage value.

Example 1.12 Consider a newsvendor problem with four prices and independent normally distributed demands as given in Table 1.2. At $k = 30$, the optimal order quantity is $c = 121$, resulting in an expected profit of \$4,101.05. Suppose instead, that the retailer used only price $p_1 = 100$ and salvage value $s = 20$ at which all residual capacity can be sold. In this case, the retailer will order $c = 30$ units and an expected profit of \$1,666.67.

Table 1.2 Problem parameters for Example 1.12

j	p_j	μ_j	σ_j
1	100	125	5
2	70	36	6
3	50	50	10
4	25	100	20

Further research is needed in terms of selecting a multi-price schedule $p_1 > p_2 > \ldots > p_n$ to maximize expected profits taking into account strategic behavior as consumers may want to wait for a lower price at the cost of deriving lower utility from the product and at the risk of being rationed.

1.5 Heuristics for Multiple Fare Classes

Several heuristics for the multiple fare class problem were developed in the 1980s. These heuristics are essentially extensions of Littlewood's rule. The most important heuristics are known as EMSR-a and EMSR-b, where EMSR stands for expected marginal seat revenue. For a while, some of these heuristics were even thought to be optimal by their proponents, until comparisons with optimal policies based on dynamic programming were carried out in the 1990s. By that time heuristics were already part of implemented systems and industry practitioners were reluctant to replace them with the solutions provided by dynamic programming. There are several reasons for this reluctance. People feel more comfortable with something they understand. Also, the performance gap between the heuristics and the optimal policies tends to be small. Finally, there is a feeling among some users that the heuristics may be more robust to demand estimation errors.

EMSR-a is based on the idea of adding protection levels produced by applying Littlewood's rule to each pair of fare classes. Suppose that we are at stage j and we need to decide how much capacity to protect for fare classes $j-1, j-2, \ldots, 1$. We can use Littlewood's rule to decide how much capacity to protect for fare class k demand against fare class j for all $k = j-1, \ldots, 1$. More precisely, we compute y^*_{kj} as

$$y^*_{kj} = \max\left\{y \in \mathbb{N}_+ : \mathbb{P}\{D_k \geq y\} > \frac{p_j}{p_k}\right\}.$$

so that y^*_{kj} is the amount of capacity that we protect for fare class k when serving the demand for fare class j. In this case, EMSR-a heuristic protects

$$y^a_{j-1} := \sum_{k=1}^{j-1} y^*_{kj}$$

units of capacity for fare classes $j-1, \ldots, 1$ when serving the demand for fare class j. In particular, if the demand D_k for fare class k is a normal random variable with mean μ_k and standard deviation σ_k, then $y^*_{jk} = \mu_k + \sigma_k \Phi^{-1}(1 - p_j/p_k)$, so

$$y^a_{j-1} = \mu[1, j-1] + \sum_{k=1}^{j-1} \sigma_k \Phi^{-1}(1 - p_j/p_k),$$

we use $\mu[i, j] := \sum_{k=i}^{j} \mu_k$. Notice that EMSR-a heuristic involves $j - 1$ calls to Littlewood's rule to compute the protection level y^a_{j-1}.

EMSR-b heuristic is based on a single call to Littlewood's rule to compute each protection level. Suppose that we are at stage j and we need to decide how much capacity to protect for fare classes $j-1, j-2, \ldots, 1$. At this point in the problem, we assume that there are two fare classes, one fare class corresponding to fare class j and another fare class corresponding to the aggregation of fare classes $j-1, j-2, \ldots, 1$. The demand from fare class j is given by D_j, and the demand from the fare class that corresponds to the aggregation of fare classes $j-1, j-2, \ldots, 1$ is $D[1, j-1]$. The fare associated with fare class j is p_j. To compute the fare associated with the fare class that corresponds to the aggregation of fare classes $j-1, j-2, \ldots, 1$, we use a weighted sum of the fares of the aggregated fare classes and set the fare associated with the aggregated fare class as

$$\bar{p}_{j-1} = \sum_{k=1}^{j-1} p_k \frac{\mu_k}{\mu[1, j-1]}.$$

When serving the demand for fare class j, to decide how many units of capacity to protect for fare classes $j-1, j-2, \ldots, 1$, we compute the protection level in a two fare class problem, where the demand random variables for the two fare classes are D_j and $D[1, j-1]$, whereas the fares for the two fare classes are p_j and $\bar{p}_{j-1}$. Thus, when serving the demand for fare class j, EMSR-b heuristic protects

$$y^b_{j-1} = \max\left\{ y \in \mathbb{N}_+ : \mathbb{P}\{D[1, j-1] \geq y\} > \frac{p_j}{\bar{p}_{j-1}} \right\}$$

units of capacity for fare classes $j-1, j-2, \ldots, 1$. We note that using EMSR-b heuristic requires the distribution of $D[1, j-1] := \sum_{k=1}^{j-1} D_k$. Computing the distribution of $D[1, j-1]$ requires a convolution, but in some cases, such as the case where the demand has normal or Poisson distribution, the distribution of $D[1, j-1]$ can be easily obtained, since sums of independent normal or Poisson random variables are, respectively, also normal or Poisson random variables. In the special case where the demand for fare class j is a normal random variable with mean μ_j and standard deviation σ_j, we obtain

$$y^b_{j-1} = \mu[1, j-1] + \sigma[1, j-1]\, \Phi^{-1}(1 - p_j/\bar{p}_{j-1}),$$

where $\sigma[1, j-1]$ is the standard deviation of the demand $D[1, j-1]$ from aggregated fare class. More specifically, we have $\sigma[1, j-1] = \sqrt{\sum_{k=1}^{j-1} \sigma_k^2}$.

Once we compute protection levels either by using EMSR-a or EMSR-b heuristic, we use these protection levels to make capacity allocation decisions in the same way we use the optimal protection levels that are computed through the dynamic programming formulation of the problem. In particular, using $y_{n-1}^h, y_{n-2}^h, \ldots, y_1^h$ to denote the protection level computed by any heuristic, when serving the demand for fare class j, we protect y_{j-1}^h units of capacity for fare classes $j-1, j-2, \ldots, 1$. Thus, if we have x units of remaining capacity and $x \leq y_{j-1}^h$, then we do not make any capacity available for fare class j, so we do not serve any demand from fare class j and the remaining capacity at the next stage is still x. If $x > y_{j-1}^h$, then we make $x - y_{j-1}^h$ unit of capacity available for fare class j, so sales to fare class j equal $\min\{x - y_{j-1}^h, D_j\}$ and the remaining capacity that we have at the next stage is $x - \min\{x - y_{j-1}^h, D_j\} = \max\{y_{j-1}^h, x - D_j\}$. Let $V_j^h(x)$ be the total expected revenue obtained from the fare classes $j, j-1, \ldots, 1$ with x units of remaining capacity at stage j using heuristic protection levels $y_{n-1}^h, y_{n-2}^h, \ldots, y_1^h$, we can compute $\{V_j^h(\cdot) : j = n, \ldots, 1\}$ by using the recursion:

$$V_j^h(x) = \begin{cases} V_{j-1}^h(x) & \text{if } x \leq y_{j-1}^* \\ p_j\, \mathbb{E}\{\min\{x - y_{j-1}^*, D_j\}\} & \\ \qquad\qquad + \mathbb{E}\{V_{j-1}^h(\max\{y_{j-1}^*, x - D_j\})\} & \text{if } x > y_{j-1}^* \end{cases}$$

with the boundary condition that $V_0^h(x) = 0$.

Alternatively, the values $V_j^h(x)$ can be estimated using Monte Carlo simulation. Using $D_n^k, D_{n-1}^k, \ldots, D_1^k$ to denote the k-th sample, we can simulate the decisions made by using the protection levels $y_{n-1}^h, y_{n-2}^h, \ldots, y_1^h$. We start with a capacity of $x_n^k = c$ at stage n. Given that we have x_j^k units of remaining capacity at stage n, if $x_j^k \leq y_{j-1}^h$, then we do not make any capacity available for fare class j, so $s_j^k = 0$. If $x_j^k > y_{j-1}^h$, then we make $x_j^k - y_{j-1}^h$ units of capacity available for fare class j, so $s_j^k = \min\{x_j^k - y_{j-1}^h, D_j^k\}$. The remaining capacity at stage $j+1$ is $x_{j+1}^k = x_j^k - s_k^k$. For the k-th sample, the total revenue is $\sum_{j=1}^n p_j\, s_j^k$. Averaging the total revenue over many demand samples provides an estimate of the total expected revenue obtained by a set of protection levels.

Example 1.13 We have applied the EMSR-a and EMSR-b heuristics to the problem instance in Example 1.9. Recall that the optimal protection levels for this problem instance are $y_4^* = 169$, $y_3^* = 101$, $y_2^* = 54$, and $y_1^* = 14$. The protection levels provided by EMSR-a heuristic are $y_4^a = 171$, $y_3^a = 97$, $y_2^a = 53$, and $y_1^a = 14$. The protection levels provided by EMSR-b heuristic are $y_4^b = 166$, $y_3^b = 102$, $y_2^b = 54$, and $y_1^b = 14$. In Table 1.3, we show the total expected revenues obtained by the two heuristics and the optimal total expected revenue for different values of initial

Table 1.3 Performance of EMSR-a and EMSR-b heuristics for Example 1.13

c	Load factor	$V_5^a(c)$	$V_5^b(c)$	$V_5(c)$
50	560%	3427	3427	3427
100	280%	5432	5441	5441
150	187%	7181	7189	7189
200	140%	8157	8151	8159
250	112%	8907	8901	8909
300	93%	9564	9563	9564
350	80%	9625	9625	9625

Table 1.4 Problem parameters for Example 1.14

j	p_j	μ_j	σ_j	y_j^*	y_j^a	y_j^b
1	1050	17.3	5.8	16.7	16.7	16.7
2	567	45.1	15.0	42.5	38.7	50.9
3	534	39.6	13.2	72.3	55.7	83.2
4	520	34.0	11.3			

capacity. In particular, $V_5^a(c)$ and $V_5^b(c)$ correspond to the total expected revenues obtained by EMSR-a and EMSR-b heuristics with c units of initial capacity, whereas $V_5(c)$ corresponds to the optimal total expected revenue.

As seen in Table 1.3, the performance of both heuristics is close to optimal. We recall that this problem instance involves Poisson demand random variables and a low-before-high fare arrival order. The heuristics continue to perform well if the demand random variables are compound Poisson and the aggregate demand is approximated by a gamma random variable. However, the model constructed in this chapter makes strong assumptions about the arrival order of the fares and the heuristics may not perform well when the arrival order assumption does not hold.

Two more examples are presented below.

Example 1.14 There are four fare classes. The demand for each fare class j is normally distributed with mean μ_j and standard deviation σ_j. Table 1.4 shows the problem parameters and the protection levels computed by EMSR-a and EMSR-b heuristics, as well as the optimal protection levels. There are considerable discrepancies between the protection levels computed by the different approaches. The discrepancies are less severe in the later stages, since we essentially deal with a problem with a small number of fare classes in the later stages.

The total expected revenues obtained by the optimal policy, estimated through a simulation study with 500,000 replications, are shown in Table 1.5. Capacity is varied from 80 to 160 to create load factors in the range 1.7–0.85. The percent suboptimality of the two heuristics is also reported. For this problem instance, EMSR-a performs slightly better than EMSR-b, but both perform quite well, despite the discrepancies in the protection levels.

Example 1.15 This problem instance is similar to the one in Example 1.14. The only difference is in the fares of fare classes 2 and 3. The problem parameters and the protection levels are shown in Table 1.6. The total expected revenues

Table 1.5 Performance of EMRS-a and EMSR-b heuristics for Example 1.14

c	Load factor	$V_n(c)$	EMSR-a % Sub	EMSR-b % Sub
80	1.70	49,642	0.33%	0.43%
90	1.51	54,855	0.24%	0.52%
100	1.36	60,015	0.13%	0.44%
110	1.24	65,076	0.06%	0.34%
120	1.13	69,801	0.02%	0.21%
130	1.05	73,926	0.01%	0.10%
140	0.97	77,252	0.00%	0.04%
150	0.91	79,617	0.00%	0.01%
160	0.85	81,100	0.00%	0.00%

Table 1.6 Problem parameters and protection levels for Example 1.15

j	p_j	μ_j	σ_j	y_j^*	y_j^a	y_j^b
1	1050	17.3	5.8	9.7	9.8	9.8
2	950	45.1	15.0	54.0	50.4	53.3
3	699	39.6	13.2	98.2	91.6	96.8
4	520	34.0	11.3			

Table 1.7 Performance of EMRS-a and EMSR-b heuristics for Example 1.15

c	Load factor	$V_n(c)$	EMSR-a % Sub	EMSR-b % Sub
80	1.70	67,505	0.10%	0.00%
90	1.51	74,003	0.06%	0.00%
100	1.36	79,615	0.40%	0.02%
110	1.24	84,817	0.35%	0.02%
120	1.13	89,963	0.27%	0.01%
130	1.05	94,860	0.15%	0.01%
140	0.97	99,164	0.06%	0.01%
150	0.91	102,418	0.01%	0.00%
160	0.85	104,390	0.00%	0.00%

obtained by the optimal policy, estimated through a simulation study with 500,000 replications, are shown in Table 1.7, as well as the percent suboptimality gaps of the two heuristics. For this problem instance, both heuristics continue to perform well and EMSR-b has a slight edge.

1.6 Bounds on Optimal Expected Revenue

In this section, we show how to quickly compute lower and upper bounds on $V_n(c)$. It is natural to ask why we need bounds on $V_n(c)$ when we can compute this quantity exactly in $O(c^2)$ operations. Although we can compute $V_n(c)$ for the single resource, multiple fare problem, we will later encounter problems where exact computations are either not possible or very time consuming. In such cases, having bounds on the optimal total expected revenue becomes useful. The techniques that we develop in this section form a stepping stone for computing bounds on the optimal total expected revenue for more complicated revenue management problems.

To obtain an upper bound on $V_n(c)$, consider the perfect foresight problem where the demand vector $D = (D_n, \ldots, D_1)$ is known in advance. Having access to the demand for all fare classes in advance allows us to optimally allocate the capacity to the different fare classes by solving the knapsack type problem

$$\bar{V}(c \mid D) := \max \left\{ \sum_{j=1}^{n} p_j\, x_j : \sum_{j=1}^{n} x_j \le c,\ 0 \le x_j \le D_j\ \forall\, j = 1, \ldots, n \right\}. \tag{1.9}$$

For each realization D of the demand random variables, advance knowledge results in a total revenue that is at least as large as the total revenue collected by the optimal policy with sequential arrivals and unknown demands. As a result, $\mathbb{E}\{\bar{V}(c \mid D)\} \ge V_n(c)$. For convenience, we will denote this upper bound as $V_n^U(c) := \mathbb{E}\{\bar{V}(c \mid D)\}$.

The solution to problem (1.9) can be written in explicit form as it is optimal to serve the demand from fare class 1 as much as possible before serving the demand from fare class 2. Therefore, the optimal value of the decision variable x_j is given by $\min\{D_j, (c - D[1, j-1])^+\}$. In this expression, we note that $(c - D[1, j-1])^+$ is the remaining capacity after we satisfy the demand from fare classes $1, 2, \ldots, j-1$ as much as possible. Therefore, the optimal objective value of problem (1.9) is given by

$$\bar{V}(c \mid D) = \sum_{j=1}^{n} p_j\ \min\{D_j, (c - D[1, j-1])^+\}.$$

Taking expectations, results in $V_n^U(c)$ as

$$\begin{aligned} V_n^U(c) &= \sum_{j=1}^{n} p_j\, \mathbb{E}\{\min\{D_j, (c - D[1, j-1])^+\}\} \\ &= \sum_{j=1}^{n} p_j\, (\mathbb{E}\{\min\{D[1, j], c\}\} - \mathbb{E}\{\min\{D[1, j-1], c\}\}) \\ &= \sum_{j=1}^{n} (p_j - p_{j+1})\, \mathbb{E}\{\min\{D[1, j], c\}\}, \\ &= \sum_{j=1}^{n} (p_j - p_{j+1}) \sum_{k=1}^{c} \mathbb{P}\{D[1, j] \ge k\} \end{aligned} \tag{1.10}$$

where we define $p_{n+1} \equiv 0$. The second equality follows from the fact that $\min\{D_j, (c - D[1, j-1])^+\} = \min\{D[1, j], c\} - \min\{D[1, j-1], c\}$. Computing this upper bound requires the evaluation of $\mathbb{E}\{\min\{D[1, j], c\}\}$. If $D[1, j]$ is a non-negative integer random variable, then $\mathbb{E}\{\min\{D[1, j], c\}\} = \sum_{k=1}^{c} \mathbb{P}\{D[1, j] \geq k\}$ and this justifies the last equality.

To make the upper bound more tractable, notice that $\bar{V}(c \mid D)$ is concave in D, so by Jensen's inequality $V_n^U(c) = \mathbb{E}\{\bar{V}(c \mid D)\} \leq \bar{V}(c \mid \mathbb{E}\{D\}) := \bar{V}_n(c)$. Letting $\mu_j := \mathbb{E}\{D_j\}$, we see that

$$\bar{V}_n(c) = \max \left\{ \sum_{j=1}^{n} p_j\, x_j \ : \ \sum_{j=1}^{n} x_j \leq c, \quad 0 \leq x_j \leq \mu_j \ \ \forall\, j = 1, \ldots, n \right\} \tag{1.11}$$

is the solution to a linear program (1.11) known as the fluid model or the deterministic capacity allocation problem. It is a knapsack problem. Similar to problem (1.9), the optimal value of the decision variable x_j is given by $\min\{\mu_j, (c - \mu[1, j-1])^+\}$. In this case, the optimal objective value of problem (1.11) is

$$\bar{V}_n(c) = \sum_{j=1}^{n} (p_j - p_{j+1}) \min\{\mu[1, j], c\},$$

where $\mu[1, j] := \sum_{i=1}^{j} \mu_i$.

A lower bound on the optimal total expected revenue can be obtained by following a policy that uses zero protection levels. In this case, since the demand from fare class n arrives before the demand from fare class $n-1$, we serve the demand from fare class n as much as possible before servicing the demand from fare class $n-1$. Thus, the sales for fare class j are given by $\min\{D_j, (c - D[j+1, n])^+\}$, where $c - D[j+1, n])^+$ captures the remaining capacity after we satisfy the demand from fare classes $n, n-1, \ldots, j+1$. Therefore, the total expected revenue obtained by the policy that uses zero protection levels is $V^L(c) = \sum_{j=1}^{n} p_j\, \mathbb{E}\{\min\{D_j, (c - D[j+1, n])^+\}\}$. Similar to our approach in (1.10), we can simplify this expression as

$$\begin{aligned} V_n^L(c) &= \sum_{j=1}^{n} p_j\, \mathbb{E}\{\min\{D_k, (c - D[j+1, n])^+\}\} \\ &= \sum_{j=1}^{n} p_j (\mathbb{E} \min\{D[j, n], c\} - \mathbb{E} \min\{D[j+1, n], c\}) \\ &= \sum_{j=1}^{n} (p_j - p_{j-1})\, \mathbb{E}\{\min\{D[j, n], c\}\}, \end{aligned} \tag{1.12}$$

where we define $p_0 = 0$. Notice that all of the terms in the sum are negative except for the term with $j = 1$. By the preceding discussion, it follows that $V_n^L(c) \leq V_n(c)$, since the total expected revenue $V_n^L(c)$ is computed under the possible suboptimal protection levels, which are all equal to zero. The above arguments justify the following proposition.

Proposition 1.16 *For the multiple fare class problem, we have*

$$V_n^L(c) \leq V_n(c) \leq V_n^U(c) \leq \bar{V}_n(c).$$

1.6.1 Revenue Opportunity Model

The bounds presented here can help with the so-called revenue opportunity model. The revenue opportunity is the spread between the ex-post optimal revenue using the estimated uncensored demand, and the revenue that results from not applying booking controls. Demand uncensoring refers to a statistical technique that attempts to estimate actual demand from the observed sales, which may be constrained by booking limits. The ex-post optimal revenue is a hindsight optimization and is equivalent to our perfect foresight model, resulting in revenue $\bar{V}(c|D)$, where D is the uncensored demand. On the other hand, the revenue that results from not applying booking controls is given by $V_n^L(c|D)$, which corresponds to the expression in (1.12) before taking the expectation. So for a given realization of demand, a measure of the revenue opportunity is defined as is $\bar{V}(c|D) - V_n^L(c|D)$. The achieved revenue opportunity is the difference between the actual revenue from applying optimal or heuristic controls and the lower bound. The ratio of the achieved revenue opportunity to the revenue opportunity is often called the percentage achieved revenue opportunity. The revenue opportunity $\bar{V}(c|D) - V_n^L(c|D)$ is sometimes approximated by its expectation $V_n^U(c) - V_n^L(c)$. Tables 1.8 and 1.9 show there is a significant revenue opportunity, particularly for $c \leq 140$. Thus, one use for the revenue opportunity model is to identify situations where revenue management has the most potential so that more effort can be put where it is most needed. The revenue opportunity model has also been used to show the benefits of using network-based controls versus using leg-based controls in networks.

Example 1.17 Tables 1.8 and 1.9 report $V_n^L(c)$, $V_n(c)$, $V_n^U(c)$, and $\bar{V}_n(c)$ for the data of Examples 1.14 and 1.15, respectively. Notice that $V_n^U(c)$ represents a significant improvement over the better known bound $\bar{V}_n(c)$, particularly for intermediate values of capacity. The spread $V_n^U(c) - V_n^L(c)$ between the lower and upper bound is a gauge of the potential improvements in revenues from using an optimal or heuristic admission control policy. If capacity is scarce relative to the potential demand, then the relative gap is large, and the potential for applying revenue management solutions is also relatively large. This is because significant improvements in revenues can be obtained from rationing capacity to lower fares. As capacity increases, the relative gap decreases indicating that less can be gained by rationing capacity. At very high levels of capacity, it is optimal to accept all requests, so there is nothing to be gained from the use of an optimal admission control policy.

Table 1.8 Optimal expected revenue and bounds for Example 1.14

c	$V_n^L(c)$	$V_n(c)$	$V_n^U(c)$	$\bar{V}_n(c)$
80	42,728	49,642	53,039	53,315
90	48,493	54,855	58,293	58,475
100	54,415	60,015	63,366	63,815
110	60,393	65,076	68,126	69,043
120	66,180	69,801	72,380	74,243
130	71,398	73,926	75,923	79,443
140	75,662	77,252	78,618	82,563
150	78,751	79,617	80,456	82,563
160	80,704	81,100	81,564	82,563

Table 1.9 Optimal expected revenue and bounds for Example 1.15

c	$V_n^L(c)$	$V_n(c)$	$V_n^U(c)$	$\bar{V}_n(c)$
80	52,462	67,505	72,717	73,312
90	61,215	74,003	79,458	80,302
100	70,136	79,615	85,621	87,292
110	78,803	84,817	91,122	92,850
120	86,728	89,963	95,819	98,050
130	93,446	94,869	99,588	103,250
140	98,630	99,164	102,379	106,370
150	102,209	102,418	104,251	106,370
160	104,385	104,390	105,368	106,370

1.7 General Fare Arrival Patterns with Poisson Demands

So far we have suppressed the time dimension. The order of the arrivals has provided us with stages that are a proxy for time. In this section, we consider models where time is considered explicitly. There are advantages of including time as part of the model as this allows for a more precise formulation of the consumer arrival process. For example, we can relax the low-before-high fare arrival assumption and allow for interleaved arrivals for different fare classes. On the other hand, the advantage of flexibility comes at the cost of estimating arrival rates for each of the fare classes over the sales horizon. If arrival rates are not estimated accurately, then adding the time dimension may hurt rather than help performance. In addition, formulations where time is handled explicitly usually assume that the demand for each fare class follows a Poisson process, whereas our earlier models based on sequential fare arrivals do not have this restriction. Here, we will provide formulations for both the Poisson and the compound Poisson cases. The compound Poisson model is flexible enough to fit any mean and variance of demand for each fare class. In Sect. 1.10, we compare the performance of the formulation with sequential fare arrivals to the formulation that allows for compound Poisson demands. Not surprisingly, optimal policies designed for arbitrary fare arrival patterns are superior.

1.7.1 Model

The length of the selling horizon is T, and time t will represent the time left until the end of the selling horizon. As before, there are n fare classes indexed by $\{1, \dots, n\}$. We assume that consumers will leave the system when their preferred class is not available. Consumers requesting fare class j may be rejected because the fare is intentionally not made available in the hope of selling the capacity at a higher fare. There may also be time-of-purchase restrictions on some fares. We use M_t to denote the set of valid fares at time-to-go t. Typically $M_t = \{1, \dots, n\}$ for large t, but low fares are dropped from M_t as time-of-purchase restrictions become binding. Consumers requesting fare class j arrive according to a Poisson process with arrival rate function $\{\lambda_{jt} : 0 \le t \le T\}$. The number of consumers that arrive during the last t units of time and request fare class j, say N_{jt}, is Poisson with parameter $\Lambda_{jt} := \int_0^t \lambda_{js}\, 1(j \in M_s)\, ds$, where $1(j \in M_s) = 1$ if $j \in M_s$ and 0 otherwise. We will use the shorthand notation $\Lambda_j := \Lambda_{jT}$ to denote the total expected demand for fare class $j \in \{1, \dots, n\}$. We assume, without loss of generality, that $p_1 > p_2 > \dots > p_n$.

Let $V(t, x)$ denote the maximum expected revenue that can be attained over the last t units of time from x units of capacity. We will develop both discrete and continuous time dynamic programs to compute $V(t, x)$. We now argue that the probability that there is exactly one request for fare class j over the interval $(t - \delta t, t]$ is $\lambda_{jt}\delta t + o(\delta t)$. Let $N_j(t - \delta t, t]$ denote the number of requests for fare class j over the interval $(t - \delta t, t]$. This is a Poisson random variable with mean $\int_{t-\delta t}^t \lambda_{js}\, ds = \lambda_{jt}\, \delta t + o(\delta t)$. Then,

$$\mathbb{P}\{N_j(t - \delta t, t] = 1\} = \lambda_{jt}\, \delta t\ \exp(-\lambda_{jt}\, \delta t) + o(\delta t) = \lambda_{jt}\, \delta t + o(\delta t),$$

while the probability that there are no requests for the other fare classes over the same interval is

$$\begin{aligned}\mathbb{P}\{N_k(t - \delta t, t] = 0, \forall k \neq j\} &= \exp\Big(-\sum_{k \neq j} \lambda_{kt}\delta t\Big) + o(\delta t) \\ &= 1 - \sum_{k \neq j} \lambda_{kt}\delta t + o(\delta t).\end{aligned}$$

Multiplying the two terms above and rearranging, we obtain $\lambda_{jt}\delta t + o(\delta t)$, as claimed.

Let $\Delta V(t, x) = V(t, x) - V(t, x - 1)$ for $x \ge 1$ and $t \ge 0$ and consider time steps of size $\delta t \ll 1$. Notice that

$$
\begin{aligned}
V(t,x) &= \sum_{j \in M_t} \lambda_{jt}\, \delta t \, \max\{p_j + V(t-\delta t, x-1), V(t-\delta t, x)\} \\
&\quad + \Big(1 - \sum_{j \in M_t} \lambda_{jt}\, \delta t\Big) V(t-\delta t, x) + o(\delta t) \\
&= V(t-\delta t, x) + \delta t \sum_{j \in M_t} \lambda_{jt} [p_j - \Delta V(t-\delta t, x)]^+ + o(\delta t) \qquad (1.13)
\end{aligned}
$$

with boundary conditions $V(t,0) = 0$ and $V(0,x) = 0$ for all $x \geq 0$. In the first equality, the first term on the right-hand side corresponds to the arrival of one request for fare class j, so a decision must be made between accepting the request earning $p_j + V(t-\delta t, x-1)$ or rejecting it and earning $V(t-\delta t, x)$, since we move to the next time period with the capacity x in the latter case. The second term on the right-hand side of the first equality corresponds to the case where no request arrives in the interval $(t-\delta t, t]$, resulting in expected revenue $V(t-\delta t, x)$. The second equality follows by arranging the terms. Subtracting $V(t-\delta t, x)$ from both sides of the equality in (1.13), dividing by δt and taking the limit as $\delta t \downarrow 0$, we obtain the Hamilton–Jacobi–Bellman (HJB) equation

$$
\frac{\partial V(t,x)}{\partial t} = \mathcal{R}_t(\Delta V(t,x)). \qquad (1.14)
$$

where

$$
\mathcal{R}_t(z) := \sum_{j \in M_t} \lambda_{jt}\, [p_j - z]^+
$$

is a decreasing convex function of z. The boundary conditions are as before $V(t,0) = V(0,x) = 0$. The equation tells us that the rate at which $V(t,x)$ grows with t is the sum of the arrival rates times the positive part of the fares net of the marginal value of capacity $\Delta V(t,x)$ at state (t,x). We can think of the right-hand side as the profit rate when the marginal cost is set equal to marginal value $\Delta V(t,x) = V(t,x) - V(t,x-1)$ of the xth unit of capacity.

1.7.2 Optimal Policy and Structural Results

Notice that fare j is accepted at state (t,x) if and only if $p_j \geq \Delta V(t,x)$. Thus, if fare j is accepted, then all fares $k \leq j$ are accepted since $p_k \geq p_j \geq \Delta V(t,x)$. This suggest that we find the index for the lowest acceptable fare by letting, for each time t and capacity x,

$$
a(t,x) := \max\{j : p_j \geq \Delta V(t,x)\}.
$$

In this case, if we are at time t with capacity x, then it is optimal to accept all fares in the active set

$$
A(t,x) := \{j \in M_t : j \leq a(t,x)\},
$$

and to reject all fares in the complement $R(t,x) := \{j \in \{1,\ldots,n\} : j \notin A(t,x)\}$. Note that the active set $A(t,x)$ essentially defines an admission control policy. The following theorem provides some structural results about the value function $V(t,x)$, its increments $\Delta V(t,x)$, and the admission control policy $A(t,x)$.

Theorem 1.18 *The value function $V(t,x)$ is increasing in t and in x. The increment of the value function $\Delta V(t,x)$ is decreasing in x and increasing in t. Moreover, $a(t,x)$ and $A(t,x)$ are increasing in x.*

If the arrival rates and the set of valid fares are stationary so that $\lambda_{jt} = \lambda_j > 0$ for all j and $M_t = M = \{1,\ldots,n\}$ for all t, then $a(t,x)$ and $A(t,x)$ are decreasing in t and $V(t,x)$ is strictly increasing and concave in t.

Notice that we can also express the optimal policy in terms of dynamic protection levels $y_j(t)$, $j = 1,\ldots,n-1$, $0 \le t \le T$, which are given by

$$y_j(t) := \max\{x : a(t,x) = j\},$$

Thus, if $x \le y_j(t)$, then fares $k > j$ should be closed. This observation follows because $a(t,x)$ is increasing in x.

1.7.3 Discrete-Time Formulation

The value function in (1.14) can be accurately computed by solving and pasting the HJB equation over a discrete mesh. Alternatively, $V(t,x)$ can be approximately computed by using a discrete time dynamic programming formulation. A discrete time dynamic programming formulation emerges from (1.13) by rescaling time, setting $\delta t = 1$, and dropping the $o(\delta t)$ term. This can be done by selecting $k > 1$, so that kT is an integer, and setting $\lambda_{jt} \leftarrow \frac{1}{k}\lambda_{j,t/k}$, for $t \in [0, kT]$. The scale factor k should be sufficiently large so that after scaling, we have $\sum_{j\in M_t} \lambda_{jt} \ll 1$, e.g., $\sum_{j\in M_t} \lambda_{jt} \le 0.01$ for all $t \in [0,T]$, with $T \leftarrow kT$. The resulting discrete time dynamic program is given by

$$\begin{aligned} V(t,x) = \sum_{j\in M_t} \lambda_{jt} \ \max\{p_j + V(t-1,x-1), V(t-1,x)\} \\ + \Big(1 - \sum_{j\in M_t} \lambda_{jt}\Big) V(t-1,x) \\ = V(t-1,x) + \sum_{j\in M_t} \lambda_{jt}[p_j - \Delta V(t-1,x)]^+ \\ = V(t-1,x) + \mathcal{R}_t(\Delta V(t-1,x)), \end{aligned} \tag{1.15}$$

with the same boundary conditions. Computing $V(t,x)$ via (1.15) is quite easy and fairly accurate if time is scaled appropriately. For each time period t and capacity x,

Table 1.10 Optimal total expected revenues in Example 1.19

c	50	100	150	200	250	300	350
$V(T, c)$	3553.6	5654.9	7410.1	8390.6	9139.3	9609.6	9625.0

the complexity is order $O(n)$, so the overall computational complexity to compute $V(t, x)$ for all t and x is $O(ncT)$.

From the dynamic program in (1.15), it is optimal to accept a request for fare class j when we have $p_j \geq \Delta V(t-1, x)$, or equivalently, when $p_j + V(t-1, x-1) \geq V(t-1, x)$. The latter condition compares the immediate revenue from fare class j plus the value of being in a state with one less unit of capacity at the next period with the value of not accepting the request and being in a state with the same capacity at the next period. Letting $a(t, x) := \max\{j : p_j \geq \Delta V(t-1, x)\}$, if we are at time period t with capacity x, then it is optimal to accept all fares in the active set

$$A(t, x) := \{j \in M_t : j \leq a(t, x)\},$$

and to reject all fares in the complement $R(t, x) := \{j \in \{1, \ldots, n\} : j \notin A(t, x)\}$. All of the structural results of Theorem 1.18 continue to hold for the discrete time model.

Example 1.19 Consider Example 1.9 with five fare classes with fares $p_1 = \$100$, $p_2 = \$60$, $p_3 = \$40$, $p_4 = \$35$, and $p_5 = \$15$. We also assume that the arrival rates are uniform over the horizon $[0, T]$, i.e., $\lambda_j = \Lambda_j / T$, and independent Poisson demands with means $\Lambda_1 = 15$, $\Lambda_2 = 40$, $\Lambda_3 = 50$, $\Lambda_4 = 55$, and $\Lambda_5 = 120$ and $T = 1$. The scaling factor was selected so that $\sum_{i=1}^{5} \Lambda_i / k < 0.01$ resulting in $T \leftarrow kT = 2800$. In Table 1.10, we present the expected revenues $V(T, c)$ for $c \in \{50, 100, 150, 200, 250, 300, 350\}$.

1.8 Monotonic Fare Offerings

The dynamic programs in (1.14) and (1.15) implicitly assume that fares can be opened and closed at any time. To see how a closed fare may reopen, suppose that $a(t, x) = j$ so set $A(t, x) = \{k \in M_t : k \leq j\}$ is offered at state (t, x), but an absence of sales may trigger fare $j + 1$ to open as $a(s, x)$ increases and as the time-to-go s decreases. This can lead to the emergence of strategic consumers or third parties that specialize in exploiting inter-temporal fare arbitrage opportunities, where one waits for a lower fare class to be available. To avoid such strategic behavior, the capacity provider may commit to a policy of never opening fares once they are closed. Handling monotonic fares requires modifying the dynamic programming formulation into something akin to the dynamic program where time was handled implicitly through prefixed arrival order of the fare classes. In particular, let $V_j(t, x)$

be the maximum expected revenue from state (t, x) that can be obtained by offering any set $S_{it} = \{k \in M_t, k \leq i\}$ with $i \leq j$, so that we do not open any fares cheaper than fare class j. Let $W_k(t, x)$ be the expected revenue from accepting fares S_{kt} at state (t, x) and then following an optimal policy. More precisely,

$$\begin{aligned} W_k(t,x) &= \sum_{i \in S_{kt}} \lambda_{it}[p_i + V_k(t-1, x-1)] + (1 - \sum_{i \in S_{kt}} \lambda_{it}) V_k(t-1, x) \\ &= V_k(t-1,x) + \lambda_t \left[r_{kt} - \pi_{kt} \Delta V_k(t-1,x)\right] \\ &= V_k(t-1,x) + \lambda_t \, \pi_{kt} \left[q_{kt} - \Delta V_k(t-1,x)\right], \end{aligned}$$

where $\Delta V_k(t,x) = V_k(t,x) - V_k(t,x-1)$, and the quantities λ_t, π_{kt}, and q_{kt} are as defined as $\lambda_t := \sum_{j \in M_t} \lambda_{jt}$, $\pi_{jt} := \sum_{k \in S_{jt}} \lambda_{kt}/\lambda_t$, and $r_{jt} := \sum_{k \in S_{jt}} p_k \lambda_{kt}/\lambda_t$. Then, $V_j(t,x)$ satisfies

$$V_j(t,x) = \max_{k \leq j} W_k(t,x) \tag{1.16}$$

with the boundary conditions $V_j(t, 0) = V_j(0, x) = 0$ for all $t \geq 0$, $x \in \mathbb{N}_+$ and $j = 1, \ldots, n$. It follows immediately that $V_j(t, x)$ is monotone increasing in j. The complexity to compute each $V_j(t, x)$ is $O(1)$, so the complexity to compute $V_j(t, x)$ for all $j = 1, \ldots, n$, $x = 1, \ldots, c$ is $O(nc)$. Since there are T time periods, the overall complexity is $O(ncT)$. While computing $V_j(t, x)$ numerically is fairly simple, it is satisfying to know more about the structure of optimal policies as this gives both managerial insights and can simplify computations. The proof of the structural results are intricate and subtle, but they parallel the results for the dynamic programs in (1.14) and (1.15).

Lemma 1.20 *The value functions $\{V_j(t, x) : j = 1, \ldots, n,\ x = 1, \ldots, c,\ t = 1, \ldots, T\}$ computed through the dynamic program in (1.16) satisfy the following properties.*

- *$\Delta V_j(t, x)$ is decreasing in $x \in \{1, \ldots, c\}$, so the marginal value of capacity is diminishing.*
- *$\Delta V_j(t, x)$ is increasing in $j \in \{1, \ldots, n\}$, so the marginal value of capacity increases when we have more flexibility in terms of opening and closing fare classes.*
- *$\Delta V_j(t, x)$ is increasing in t, so the marginal value of capacity increases as the time-to-go increases.*

Let

$$a_j(t, x) := \max\{k \leq j : W_k(t, x) = V_j(t, x)\}.$$

In words, $a_j(t, x)$ is the index of the lowest open fare that is optimal to post if we are at time t with a capacity of x and we are allowed to use any fares in S_{jt}. Also, let

$$A_j(t, x) := \{k \in M_t : k \leq a_j(t, x)\}.$$

Then, it follows that $A_j(t, x)$ is the optimal set of fares to open at state (j, t, x). Clearly $V_i(t, x) = V_j(t, x)$ for all $i \in \{a_j(t, x), \ldots, j\}$. The following lemma asserts that $a_j(t, x)$ is increasing in x and in j and gives conditions for $a_j(t, x)$ to be decreasing in t.

Lemma 1.21 *The index $a_j(t, x)$ is increasing in x and in j. Furthermore, $a_j(t, x)$ is decreasing in t if the arrival rates λ_{jt} are time invariant and $M_t = M$ for all t. Moreover, $a_j(t, x) = k < j$ implies $a_i(t, x) = k$ for all $i \geq k$.*

It is possible to think of the policy in terms of protection levels, as well as in terms of stopping sets. Indeed, let $T_j := \{(t, x) : V_j(t, x) = V_{j-1}(t, x)\}$. We can think of T_j as the stopping set for fare j as it is optimal to close down fare j upon entering set T_j. For each t, let $y_j(t) := \max\{x \in \mathbb{N}_+ : (t, x) \in T_{j+1}\}$. We can think of $y_j(t)$ as the protection level for fares in S_j against higher fares. The following theorem is the counterpart to Theorem 1.6.

Theorem 1.22 *The optimal policy computed through the dynamic program in (1.16) satisfies the following properties.*

- *$A_j(t, x)$ is increasing in x and j. Furthermore, $A_j(t, x)$ is decreasing in t if the problem parameters is time invariant.*
- *$T_1 \subseteq T_2 \subseteq \ldots \subseteq T_n$.*
- *$y_j(t)$ is increasing in t and in j.*
- *If $x \leq y_j(t)$ then $V_i(t, x) = V_j(t, x)$ for all $i > j$.*

The policy is implemented as follows. The starting state is (n, T, c) as we can use any of the fares $\{1, \ldots, n\}$, we have T units of time-to-go and c is the initial inventory. At any state (j, t, x), we post fares $A_j(t, x) = \{1, \ldots, a_j(t, x)\}$. If a unit is sold during period t, then the state is updated to $(a_j(t, x - 1), t - 1, x - 1)$ since all fares in the set $A_j(t, x)$ are allowed, the time-to-go is $t - 1$ and the inventory is $x - 1$. If no sales occur during period t, the state is updated to $(a_j(t, x), t - 1, x)$. The process continues until either $t = 0$ or $x = 0$.

Example 1.23 Considering the same data in Example 1.19, in Table 1.11, we give the expected revenues $V_j(T, c)$, $j = 1, \ldots, 5$ and $V(T, c)$ for $c \in \{50, 100, 150, 200, 250\}$. The first row is $V_5(T, c)$ from Example 1.19. Notice that $V(T, c) \geq V_j(T, c)$, since the optimal total expected revenue $V(T, c)$ is computed under the assumption that a closed fare class can be opened again. The difference in optimal total expected revenues $V(T, c) - V_5(T, c)$ due to the flexibility of opening and closing fares may be significant for some small values of c. For example, the difference is 1.7% for $c = 50$. However, the difference is small for larger values of c, and attempting to go for the extra revenue by opening an already closed fare may invite strategic consumers to wait for lower fares or for third parties to arbitrage the system by pre-selling capacity and then optimizing the time-of-purchase to exploit predictable price dynamics.

We close this section with a remark on mark-up and mark-down policies. Let us go back to the broader pricing interpretation coupled with the monotonic fare

Table 1.11 Optimal total expected revenues with monotonic fare offerings in Example 1.23

c	50	100	150	200	250	300	350
$V(T,c)$	3553.6	5654.9	7410.1	8390.6	9139.3	9609.6	9625.0
$V_5(T,c)$	3494.5	5572.9	7364.6	8262.8	9072.3	9607.2	9625.0
$V_4(T,c)$	3494.5	5572.9	7364.6	7824.9	7825.0	7825.0	7825.0
$V_3(T,c)$	3494.5	5572.9	5900.0	5900.0	5900.0	5900.0	5900.0
$V_2(T,c)$	3494.5	3900.0	3900.0	3900.0	3900.0	3900.0	3900.0
$V_1(T,c)$	1500.0	1500.0	1500.0	1500.0	1500.0	1500.0	1500.0

formulation in (1.16). In many applications the price menu $p_{jt}, j = 1, \ldots, n$ is time invariant, but the associated sales rates $\pi_{jt}, j = 1, \ldots, n$ are time varying. In addition, we will assume that there is a price p_{0t} such that $\pi_{0t} = 0$ for all t. This technicality helps with the formulation as a means of turning off demand when the system runs out of inventory. Recalling that we focus on monotonic fare offerings, the case $p_{1t} \geq p_{2t} \geq \ldots \geq p_{nt}$ and $\pi_{1t} \leq \pi_{2t} \leq \ldots \leq \pi_{nt}$ is known as the mark-up problem, while the case $p_{1t} \leq p_{2t} \leq \ldots \leq p_{nt}$ and $\pi_{1t} \geq \pi_{2t} \geq \ldots \geq \pi_{nt}$ is known as the mark-down problem. The former model is relevant in revenue management while the second is relevant in retailing.

For the revenue management formulation, the problem can be viewed as determining when to mark-up, i.e. switch from action j to $j-1$. The optimal mark-up times are random as they depend on the evolution of sales under the optimal policy. Suppose that the current state is (j, t, x), so the last action was j, the time-to-go is t and the inventory is x. We want to determine whether we should continue using action j or switch to action $j - 1$. We know that if $x > y_{j-1}(t)$, then we should keep action j and if $x \leq y_{j-1}(t)$ then we should close fare class j. Let $T_j := \{(t, x) : x \leq y_{j-1}(t)\}$, so it is optimal to stop action j upon first entering set T_j. Notice that a mark-up occurs when the current inventory falls below a curve, so low inventories trigger mark-ups, and mark-ups are triggered by sales. The retailing formulation also has a threshold structure, but this time a mark-down is triggered by inventories that are high relative to a curve, so the optimal timing of a mark-down is triggered by the absence of sales.

1.9 Compound Poisson Demands

The formulations of the dynamic programs in (1.14) and (1.15) implicitly assume that each request is for a single unit of capacity. Instead, suppose that each arrival is for a random number of units. More specifically, suppose that size of a request for fare class j is a random variable Z_j, and that the probability mass function $P_j(z) := \mathbb{P}\{Z_j = z\}$ is known for fare class each j. As before, we assume independent demands for the different fare classes. We seek to generalize the dynamic programs in (1.14) and (1.15) so that at each state (t, x) we can decide whether or not to accept a fare request for a certain fare class and of a certain size. If we have a request for fare class j and for size z, then the expected revenue from accepting the request is $z p_j + V(t-1, x-z)$ and the expected revenue from rejecting the request is $V(t-1, x)$. Let $\Delta_z V(t,x) = V(t,x) - V(t, x-z) = \Delta V(t,x) + \Delta V(t, x-1) + \ldots + \Delta V(t, x-z+1)$ for all $z \leq x$. We set $\Delta_z V(t,x) = \infty$ if $z > x$. With this notation, the difference between accepting and rejecting a request for z units at state (t, x) is given by $z p_j - \Delta_z V(t,x)$, and it is optimal to accept the request whenever this quantity is non-negative. Notice that any request for $z > x$ is rejected as capacity is insufficient. (A different model is needed if a fraction of the consumers are willing to take partial orders.) The dynamic program in (1.14) with compound Poisson demands is given by

$$\frac{\partial V(t,x)}{\partial t} = \sum_{j \in M_t} \lambda_{jt} \sum_{z=1}^{x} P_j(z)[z p_j - \Delta_z V(t,x)]^+, \tag{1.17}$$

with boundary conditions $V(t,0) = V(0,x) = 0$. The optimal policy is to accept a request of size $z \leq x$ for fare class j, if $z p_j \geq \Delta_z V(t,x)$ and to reject all requests of size $z > x$. For $z \leq x$, define

$$j(z|t,x) := \arg\max \left\{ j : p_j \geq \frac{\Delta_z V(t,x)}{z} \right\},$$

so if we are at time-to-go t with remaining capacity of x, then it is optimal to accept requests of size z for all fares in the set

$$A(z|t,x) := \{ j \in M_t : j \leq j(z|t,x) \}.$$

The discrete time dynamic program in (1.15) with compound Poisson demands is given by

$$V(t,x) = V(t-1,x) + \sum_{j \in M_t} \lambda_{jt} \sum_{z=1}^{x} P_j(z)\,[z p_j - \Delta_z V(t-1,x)]^+, \tag{1.18}$$

with the same boundary conditions, and the optimal controls are of the same form except that $\Delta V(t-1,x)$ is used in defining $j(z|t,x)$.

For compound Poisson demands, we can no longer claim that the marginal value of capacity $\Delta V(t,x)$ is decreasing in x, although it is still true that $\Delta V(t,x)$ is increasing in t. To see why $\Delta V(t,x)$ is not monotone in x, consider a problem where the majority of the requests are for two units and requests are seldom for one unit. Then the marginal value of capacity for even values of x may be larger than the marginal value of capacity for odd values of x. Consequently, some of the structure may be lost. For example, it may be optimal to accept a request of a single unit of capacity when x is odd, but not if x is even. However, even if some of the structure is lost, the computations involved to solve the dynamic program in (1.18)

Table 1.12 Value function $V(T, c)$ in Example 1.24 with compound Poisson demand

c	50	100	150	200	250	300
$V(T, c)$	3837	6463	8451	10,241	11,724	12,559

Table 1.13 The first differences $\Delta V(208, x)$ in Example 1.24 with compound Poisson demand

x	1	2	3	4	5	6
$\Delta V(208, x)$	70.05	66.48	59.66	60.14	54.62	50.41

are straightforward as long as the distribution of Z_j is known. Airlines, for example, have a very good idea of the distribution of Z_j for different fare classes.

Example 1.24 Consider the same data in Example 1.19 with fares $p_1 = \$100$, $p_2 = \$60$, $p_3 = \$40$, $p_4 = \$35$, and $p_5 = \$15$ and independent Poisson requests with means $\Lambda_1 = 15$, $\Lambda_2 = 40$, $\Lambda_3 = 50$, $\Lambda_4 = 55$, $\Lambda_5 = 120$ over the horizon $[0, 1]$. Now, we will assume that the distribution of the demand sizes is given by $\mathbb{P}\{Z_j = 1\} = 0.65$, $\mathbb{P}\{Z_j = 2\} = 0.25$, $\mathbb{P}\{Z_j = 3\} = 0.05$, and $\mathbb{P}\{Z_j = 4\} = 0.05$ for all fare classes $j = 1, \ldots, 5$. Notice that $\mathbb{E}[Z_j] = 1.5$ and $\mathbb{E}[Z_j^2] = 2.90$. We will assume that $M_t = \{1, \ldots, n\}$ for all $t \in [0, T]$. Our computations are based on the dynamic program in (1.18) with a rescaled time horizon $T \leftarrow kT = 2800$, and rescaled arrival rates $\lambda_j \leftarrow \lambda_j/k$ for all $j = 1, \ldots, n$. Table 1.12 provides $V(T, c)$ for $c \in \{50, 100, 150, 200, 250, 300, 350\}$. Table 1.13 provides $\Delta V(t, x)$ for $t = 207$ in the rescaled horizon for $x \in \{1, \ldots, 6\}$ to illustrate the behavior of the policy. The reader can verify that at state $(t, x) = (208, 3)$, it is optimal to accept a request for one unit at fare p_2, and to reject the request if the request is for two units. If we have one more unit of inventory, so the state is $(t, x) = (208, 4)$ then it is optimal to reject a request for one unit at fare p_2, and to accept the request if it is for two units. The reason for this behavior is that the value of $\Delta V(t, x)$ is not monotone decreasing at $x = 4$.

1.10 Sequential vs. Mixed Arrival Formulations

In this section, we compare the performance of sequential policies obtained using the dynamic program in (1.6) with the performance of formulation (1.18) that allows for mixed arrivals and compound Poisson demands. Allowing for arbitrary arrivals provides more flexibility so it should not be surprising that policies based on a more flexible model would do better. Computational studies should concentrate on measuring the gap between the two. The gap is fairly small when the arrival rates are sequential and the low-before-high assumption holds, and more generally when the arrival rates follow a prescribed order that is consistent with the computations of the protection levels.

Table 1.14 Sub-optimality of EMSR-b with standard nesting vs optimal dynamic policy for Example 1.25

c	50	100	150	200	250	300
$V^s(T,c)$	3653	6177	8187	9942	11,511	12,266
$V(T,c)$	3837	6463	8451	10,241	11,724	12,559
% Sub	4.8%	4.4%	3.1%	2.9%	1.8%	2.3%

Comparing sequential and dynamic policies when the fare arrival rates do not follow a specific pattern is more difficult because revenues depend heavily on how the protection levels from the sequential policy are implemented. Two possible implementations are possible. Under theft nesting a request of size z for fare class j is accepted if $x - z \geq y_{j-1}$, where x is the current inventory and y_{j-1} is the protection level for fares $\{1, \ldots, j-1\}$. This method is called theft nesting because the remaining inventory x at time-to-go t deducts all previous bookings regardless of fare class. In contrast, standard nesting is implemented by accepting a size z request for fare j if $x - z \geq (y_{j-1} - b[1, j-1])^+$, where $b[1, j-1]$ are the observed bookings of fares $[1, j-1]$ at state (t, x). In practice, standard nesting works much better than theft nesting when the fare arrival pattern is not low-to-high. This makes sense because standard nesting does not insist on protecting y_{j-1} units for fares $\{1, \ldots, j-1\}$ even though we have already booked $b[1, j-1]$ units of these fares. Consequently, we use standard nesting in comparing sequential policies versus dynamic policies to give sequential policies a fighting chance.

Example 1.25 Consider the data of Example 1.24. Let $V(T, c)$ be the value function at the beginning of the selling horizon with initial capacities computed through the dynamic program in (1.18). Thus, $V(T, c)$ is the optimal total expected revenue under the compound Poisson arrivals. Let $V^s(T, c)$ be the total expected revenue collected by the sequential EMSR-b policy under standard nesting. In Table 1.14, we compare $V(T, c)$ with $V^s(T, c)$. Part of the gap between $V^s(T, c)$ and $V(T, c)$ can be reduced by frequently recomputing the booking limits applying the EMSR-b heuristic during the sales horizon. However, this is not enough to overcome the disadvantage of the EMSR-b heuristic when applied to mixed arrival patterns.

We end by noticing that it is possible to show that the upper bound $V_n^U(c)$ for $V_n(c)$, developed in Sect. 1.6 for the model with fixed arrival order for fares, is still a valid upper bound for $V(T, c)$ computed under arbitrary arrival pattern.

1.11 End of Chapter Problems

1. A coffee shop gets a daily allocation of 100 bagels. The bagels can be either sold individually at $1.00 each or can be used later in the day for sandwiches. Each bagel sold as a sandwich provides a revenue of $1.50 independent of the other ingredients.

Table 1.15 Fare classes, fares and demand distributions

Class	Fare	Demand distribution
1	\$600	Poisson(25)
2	\$475	Poisson(30)
3	\$265	Poisson(29)
4	\$130	Poisson(30)

(a) Suppose that demand for bagel sandwiches is estimated to be Poisson with parameter 80. How many bagels would you reserve for sandwiches?
(b) Compare the expected revenue of the solution of part (a) to the expected revenue of the heuristic that does not reserve capacity for sandwiches assuming that the demand for individual bagels is Poisson with parameter 150?
(c) Answer Part (a) if the demand for bagel sandwiches is normal with mean 100 and standard deviation 20.

2. Suppose capacity is 120 seats and there are four fares. The demand distributions for the different fares are given in Table 1.15.

   Determine the optimal protection levels. [Hints: The sum of independent Poisson random variables is Poisson with the obvious choice of parameter to make the means match. If D is Poisson with parameter λ, then $\mathbb{P}\{D = k+1\} = \mathbb{P}\{D = k\}\lambda/(k+1)$ for any non-negative integer k.
3. Consider a parking lot in a community near Manhattan. The parking lot has 100 parking spaces. The parking lot attracts both commuters and daily parkers. The parking lot manager knows that he can fill the lot with commuters at a monthly fee of \$180 each. The parking lot manager has conducted a study and has found that the expected monthly revenue from x parking spaces dedicated to daily parkers is approximated well by the quadratic function $R(x) = 300x - 1.5x^2$ over the range $x \in \{0, 1, \ldots, 100\}$. Note: Assume for the purpose of the analysis that parking slots rented to commuters cannot be used for daily parkers even if some commuters do not always use their slots.

   (a) What would the expected monthly revenue of the parking lot be if all the capacity is allocated to commuters?
   (b) What would the expected monthly revenue of the parking lot be if all the capacity is allocated to daily parkers?
   (c) How many units should the parking manager allocate to daily parkers and how many to commuters?
   (d) What is the expected revenue under the optimal allocation policy?

4. A fashion retailer has decided to remove a certain item of clothing from the racks in 1 week to make room for a new item. There are currently 80 units of the item and the current sale price is \$150 per unit. Consider the following three strategies assuming that any units remaining at the end of the week can be sold to a jobber at \$30 per unit.

Table 1.16 Fare classes, fares and demand distributions

Class	Fare	Demand distribution
1	\$500	Poisson(45)
2	\$380	Poisson(55)
3	\$215	Poisson(50)
4	\$180	Poisson(100)

(a) Keep the current price. Find the expected revenue under this strategy under the assumption that demand at the current price is Poisson with parameter 50.

(b) Lower the price to \$90 per unit. Find the expected revenue under this strategy under the assumption that demand at \$90 is Poisson with parameter 120.

(c) Keep the price at \$150 but e-mail a 40% discount coupon for the item to a population of price sensitive consumers that would not buy the item at \$150. The coupon is valid only for the first day and does not affect the demand for the item at \$150. Compute the expected revenue under this strategy assuming that you can control the number of coupons e-mailed so that demand from the coupon population is Poisson with parameter x for values of x in the set $\{0, 5, 10, 15, 20, 25, 30, 35\}$. In your calculations assume that demand from coupon holders arrives before demand from consumers willing to pay the full price. Assume also that you cannot deny capacity to a coupon holder as long as capacity is available (so capacity cannot be protected for consumers willing to pay the full price). What value of x would you select? You can assume, as in parts (a) and (b) that any leftover units are sold to the jobber at \$30 per unit.

5. Prove that Eq. (1.8) holds.
6. Suppose we have a capacity of 220 seats and four fare classes. The fares and demand distribution for each fare class are given in Table 1.16. In all cases, except where noted, we will assume a low-to-high fare class arrival pattern.

(a) Determine the optimal protection levels using dynamic programming.

(b) Determine the protection levels under the EMSR-a heuristic

(c) Determine the protection levels under the EMSR-b heuristic

(d) Use simulation or the exact method to estimate the expected sales for each fare class and the total expected revenues for the policies determined in parts (a)–(c).

(e) Find the expected revenue under a policy that does not protect inventory for higher fare classes assuming the arrival pattern is low-to-high.

(f) Find the expected revenue of the policy in part (e) if the fare class arrival pattern is high-to-low.

(g) Solve the linear programming described in class to obtain an upper bound on the expected revenue of the optimal policy.

Table 1.17 Dynamic booking control with booking limits and protection levels

	Booking limits				Protection levels					
	1	2	3	4	1	2	3	4	Request	Action
1	50	45	37	22	5	13	28	50	4 seats in Class 4	
2									5 seats in Class 3	
3									7 seats in Class 2	
4									5 seats in Class 4	
5									5 seats in Class 1	
6									5 seats in Class 4	
7									6 seats in Class 3	
8									3 seats in Class 2	
9									1 seats in Class 1	
10									2 seats in Class 3	
11										

7. Consider a flight with a capacity of 50 seats and four fare classes. Suppose that we implement *nested protection levels* starting with $(y_1, y_2, y_3, y_4) = (5, 13, 28, 50)$. Table 1.17 shows a series of booking requests. For this problem, each request must be accepted on all-or-none basis, i.e. given a request of m units, we can only sell m units or none at all. Determine whether each request would be accepted, and update the booking limits and protection levels accordingly.
8. Suppose that you are the capacity provider for a popular event. The face value of the tickets is \$100 per seat, and the venue can hold 350 individuals. The \$100 tickets go on sale a month before the event. Assume demand for \$100 tickets is at least 350. You estimate that demand from people willing to pay \$300 for a ticket the day of the event can be modeled as a negative binomial with parameters $r = 36$ and $p = 1/4$ (mean 144 and variance 432). More precisely, the probability mass function of demand for \$300 tickets is $\mathbb{P}\{D_1 = k\} = \binom{k-1}{35}(1/4)^{36}(3/4)^{k-36}$ for integer values of $k \geq 36$.
   (a) How many tickets should you reserve for sale at \$300?
   (b) Evaluate the expected revenue of the strategy of part (a) and determine the average number of unsold seats under the strategy of part (a).
   (c) Suppose now that you sell the \$100 tickets with a callable option that allows you to buy them back for \$130 if needed (you can assume consumers are willing to accept this deal). Suppose that you exercise the option of buying back \$100 tickets at \$130 when demand for \$300 tickets exceeds the number you reserved for them in part (a). Use simulation to evaluate the expected revenue of this strategy and determine the average number of unsold seats. You can continue to assume that demand for \$100 tickets exceeds the capacity of the venue for the purpose of your calculations.

Table 1.18 Fare classes, fares and demand distributions

j	p_j	$\mathbb{E}[D_j]$
1	\$75	8
2	\$100	21
3	\$75	31
4	\$60	20

(d) Consider now a refinement of the strategy in part (c) where you can fine tune the number of tickets that you reserve for sale at \$300. How many tickets would you reserve? Compute the expected profit under the new strategy and also the expected number of unsold seats.

9. Find the optimal protection levels for the data in Table 1.18 and compute the optimal expected revenues $V_1(c)$, $V_2(c)$, $V_3(c)$, and $V_4(c)$ for $c \in \{50, 55, 60, 65, 70, 75, 80\}$ assuming Poisson demands.
10. Modify Problem 9 so that $p_1 = \$125$ and compute optimal protection levels and the value function $V_4(c)$ for $c \in \{50, 55, 60, 65, 70, 75, 80\}$.
11. Compute the upper bound $V^H(c)$ and $\bar{V}(c)$ and the lower bound $V^L(c)$ and the spread $V^H(c) - V^L(c)$ for Problem 9 for $c \in \{50, 55, 60, 65, 70, 75, 80\}$.
12. Use the discrete time dynamic programs to compute $V(T, c)$ and $V_j(T, c)$, $j = 1, 2, 3, 4$ for the data of Problem 9, for the values of the capacity $c \in \{50, 55, 60, 65, 70, 75, 80\}$ for the following arrival rate models:

(a) Uniform arrival rates, e.g. $\lambda_{tj} = \Lambda_j = E[D_j]$ for $0 \le t \le T = 1$. Be sure to rescale time so that $T = a$ is an integer large enough so that $\sum_{j=1}^{3} \mathbb{E}[D_j]/a \le 0.01$. What accounts for the difference between $V(T, c)$ and $V_4(T, c)$? What accounts for the difference between $V_4(T, c)$ and $V_4(c)$?
(b) Low-to-high arrival rates: Dividing the selling horizon $[0, T] = [0, 1]$ into 4 sub-intervals $[t_{j-1}, t_j]$, $j = 1, \ldots, 4$ with $t_j = j/4$, and set $\lambda_{jt} = 4\Lambda_j$ over $t \in [t_{j-1}, t_j]$ and $\lambda_{jt} = 0$ otherwise. Again, be sure to rescale the system so that $T = a$ is an integer large enough so that $\max_j \max_t \lambda_{jt}/a \le 0.01$. What accounts for the difference between $V(T, c)$ and $V_4(T, c)$? What accounts for the difference between $V_4(T, c)$ and $V_4(c)$?

13. Show that the upper bound (1.11) holds for the model presented in Sect. 1.7 with $\mu_i = \Lambda_i$ for all $i \in N$. Find the dual for the formulation and show that you can reduce this to a single dimensional convex problem in the dual of the capacity constraint.

1.12 Bibliographic Remarks

Talluri and van Ryzin (2004b), Phillips (2005) and Ozer and Phillips (2012) are comprehensive reference books on revenue management and pricing. Weatherford and Bolidy (1992), McGill and van Ryzin (1999), van Ryzin and Talluri (2003) and van Ryzin and Talluri (2005) give reviews of the literature on the subject. For the two-fare class model, a formula for the optimal protection level as a function of $\mathbb{P}\{D_1 \geq y\}$ and r was proposed by Littlewood (1972). His arguments were not formal; however, they were later justified by Bathia and Prakesh (1973) and Richter (1982). Our discussion of quality of service and salvage values borrows from Brumelle et al. (1990). Gallego et al. (2008a) discuss how callable products can improve the quality of service. The two-fare class model has connections to the newsvendor problem. A coverage of the newsvendor problem can be found in Zipkin (2000). Cachon and Kok (2007a) study the newsvendor problem when the salvage value of the product depends on how much inventory is left over. Gallego and Moon (1993) and Perakis and Roels (2008) focus on the newsvendor problem when the demand distribution is not known fully. Boyaci and Ozer (2010) consider a problem of information acquisition for capacity planning. Levi et al. (2015) give bounds for sample average approximation solution to the newsvendor problem, which extend to the two-fare class model. Hu et al. (2016a) consider a newsvendor problem where the customers choose between products.

Wollmer (1992) uses dynamic programming to obtain the optimal policy for the multi-fare problem with discrete demands and fixed arrival rate for the fare classes. Curry (1990) derives optimality conditions when demands are assumed to follow a continuous distribution. Brumelle and McGill (1993) allow for either discrete or continuous demand distributions and make a connection with the theory of optimal stopping. The reader is referred to Robinson (1995) for the case where the arrival pattern of the fare classes is not necessary low-to-high. The papers by van Ryzin and McGill (2000) and Kunnumkal and Topaloglu (2009) give an algorithm for computing the optimal protection levels only by using samples of the random demand, instead of using the demand distributions. Ball and Queyranne (2009) and Ma et al. (2018) provide a competitive analysis for single-leg revenue management problems.

Credit for the EMSR heuristics is sometimes given to the American Airlines team working on revenue management problems shortly after deregulation. The first published account of these heuristics appears in Simpson (1985), Belobaba (1987) and Belobaba (1989). Ratliff (2005) reports that EMSR-b usually provides improved performance on real world problems, especially ones involving nested inventory controls. Diwan (2010) numerically compares various approaches for single-leg revenue management problems.

Examples 1.14 and 1.15 are from Wollmer (1992). The reader is referred to Chandler and Ja (2007) and Temath et al. (2010) for further information on the uses of the revenue opportunity model. Lee and Hersh (1993) first proposed a model that is equivalent to our discrete-time formulation with arbitrary arrival patterns. Brumelle and Walczak (2003) present more general results in this vein.

Weatherford et al. (1993) consider problems with diversion possibilities between the different fare classes. Revenue management problems have clear connections to dynamic packing problems studied in Kleywegt and Papastavrou (1998). Belobaba and Farkas (1999) focus on the interactions between revenue management decisions and estimating the spill rate between different classes. Zhao and Zheng (2001) consider the case with monotonic fare offerings but with only two fare classes. The sample path based proof technique that can be found in their paper becomes useful in numerous revenue management settings. In particular, this approach can be used to show the results related to monotonic fare offerings in this chapter. Mark-up and the mark-down problems can be studied from the point of view of stopping times. We refer the reader to Feng and Gallego (1995, 2000), and Feng and Xiao (2000) for mark-up and mark-down problems.

Gupta and Cooper (2005) and Cooper and Gupta (2006) provide comparisons between revenues in different systems with different demand distributions.

Appendix

Proof of Lemma 1.4 Taking expectations yields $g(x) = G(x)\mathbb{P}\{X \geq x\} + \sum_{j \leq x-1} G(j)\,\mathbb{P}\{X = j\}$ and $g(x-1) = G(x-1)\,\mathbb{P}\{X \geq x\} + \sum_{j \leq x-1} G(j)\,\mathbb{P}\{X = j\}$. Taking the difference yields $\Delta g(x) = \Delta G(x)\mathbb{P}\{X \geq x\}$. Similarly, taking expectations, we have $h(x) = H(x)\,\mathbb{P}\{X < x\} + \sum_{j \geq x} H(j)\,\mathbb{P}\{X = j\}$. Similarly, we also have $h(x-1) = H(x-1)\,\mathbb{P}\{X < x\} + \sum_{j \geq x} H(j)\,\mathbb{P}\{X = j\}$. Taking the difference, we see that $\Delta h(x) = \Delta H(x)\,\mathbb{P}\{X < x\}$. □

Proof of Proposition 1.5 We will prove the result by induction on j. The result holds for $j = 1$ since $\Delta V_1(y) = p_1\,\mathbb{P}\{D_1 \geq y\}$ is decreasing in y, and clearly $\Delta V_1(y) = p_1\,\mathbb{P}\{D_1 \geq y\} \geq \Delta V_0(y) = 0$. Assume that the result is true for V_{j-1}. It follows from the dynamic programming equation that

$$V_j(x) = \max_{y \leq x} \left\{ W_j(y, x) \right\},$$

where, for any $y \leq x$,

$$W_j(y, x) = \mathbb{E}\{p_j \min\{D_j, x - y\}\} + \mathbb{E}\{V_{j-1}(\max\{x - D_j, y\})\}.$$

Directly using the definition of $W_j(y, x)$, for $y \in \{1, \ldots, x\}$, we can show that

$$\Delta W_j(y, x) = W_j(y, x) - W_j(y-1, x) = [\Delta V_{j-1}(y) - p_j]\,\mathbb{P}\{D_j > x - y\}.$$

Since $\Delta V_{j-1}(y)$ is decreasing in y by the inductive hypothesis, we see that $W_j(y, x) \geq W_j(y-1, x)$ if $\Delta V_{j-1}(y) > p_j$ and $W_j(y, x) \leq W_j(y-1, x)$ if $\Delta V_{j-1}(y) \leq p_j$. Consider the expression

$$y_{j-1} = \max\{y \in \mathbb{N}_+ : \Delta V_{j-1}(y) > p_j\},$$

where the definition of $\Delta V_j(y)$ is extended to $y = 0$ for all j by setting $\Delta V_j(0) = p_1$. If $x \geq y_{j-1}$, then

$$V_j(x) = \max_{y \leq x} W_j(y, x) = W_j(y_{j-1}, x).$$

On the other hand, if $x < y_{j-1}$, then

$$V_j(x) = \max_{y \leq x} W_j(y, x) = W_j(x, x).$$

In summary, we have

$$\begin{aligned} V_j(x) &= W_j(\min(x, y_{j-1}), x) \\ &= \begin{cases} V_{j-1}(x), & \text{if } x \leq y_{j-1} \\ \mathbb{E}\{p_j \min\{D_j, x - y_{j-1}\}\} & \\ \qquad + \mathbb{E}\{V_{j-1}(\max\{x - D_j, y_{j-1}\})\} & \text{if } x > y_{j-1}. \end{cases} \end{aligned}$$

Using the expression above, computing $\Delta V_j(x) = V_j(x) - V_j(x-1)$ for $x \in \mathbb{N}_+$ results in

$$\Delta V_j(x) = \begin{cases} \Delta V_{j-1}(x), & \text{if } x \leq y_{j-1} \\ \mathbb{E}\{\min\{p_j, \Delta V_{j-1}(x - D_j)\}\} & \text{if } x > y_{j-1}. \end{cases}$$

We will now use this result to show that $\Delta V_j(x)$ is itself decreasing in x. Since $\Delta V_j(x) = \Delta V_{j-1}(x)$ for $x \leq y_{j-1}$ and $\Delta V_{j-1}(x)$ is decreasing in x, we only need to worry about the case $x > y_{j-1}$. However, in this case, we have

$$\Delta V_j(x) = E \min(p_j, \Delta V_{j-1}(x - D_j))$$

is decreasing in x, since $\Delta V_{j-1}(x)$ is itself decreasing in x. Lastly, at y_{j-1}, using the expression for the first difference $\Delta V_j(x)$ above,

$$\begin{aligned} \Delta V_j(y_{j-1}) = \Delta V_{j-1}(y_{j-1}) &> p_j \\ &\geq \mathbb{E}\{\min\{p_j, \Delta V_{j-1}(x - D_j)\}\} = \Delta V_j(y_{j-1} + 1), \end{aligned}$$

showing that $\Delta V_{j-1}(x)$ is decreasing at $x = y_{j-1}$ as well.

Now, we show that $\Delta V_j(x) \geq \Delta V_{j-1}(x)$. For $x > y_{j-1}$, we have

$$\min\{p_j, \Delta V_{j-1}(x - D_j)\} \geq \min\{p_j, \Delta V_{j-1}(x)\} = \Delta V_{j-1}(x),$$

where the inequality follows since $\Delta V_{j-1}(x)$ is decreasing in x, and the equality holds since $x > y_{j-1}$. Taking expectations we see that $\Delta V_j(x) \geq \Delta V_{j-1}(x)$ on $x > y_{j-1}$. Lastly, note that $\Delta V_j(x) = \Delta V_{j-1}(x)$ on $x \leq y_{j-1}$. □

Proof of Theorem 1.6 By Lemma 1.4, we have

$$\begin{aligned}\mathbb{E}\{\min\{x-y, D_j\}\} - \mathbb{E}\{\min\{x-(y-1), D_j\}\} \\ &= -\mathbb{E}\{\min\{x-y+1, D_j\}\} + \mathbb{E}\{\min\{x-y, D_j\}\} \\ &= -\mathbb{P}\{D_j \geq x-y+1) = -\mathbb{P}\{D_j > x-y\}.\end{aligned}$$

Similarly, $\mathbb{E}\{V_{j-1}(\max\{y, x-D_j\})\} - \mathbb{E}\{V_{j-1}(\max\{y-1, x-D_j\})\} = \Delta V_{j-1}(y) \times \mathbb{P}\{x-D_j < y\} = \Delta V_{j-1}(y)\, \mathbb{P}\{D_j > x-y\}$. This implies that

$$\Delta W_j(y, x) = W_j(y, x) - W_j(y-1, x) = (\Delta V_{j-1}(y) - p_j)\, \mathbb{P}\{D_j > x-y\},$$

thus, the sign of $\Delta W_j(y, x)$ is dictated by the sign of $\Delta V_{j-1}(y) - p_j$.

We now show that $W_j(y, x)$ is a unimodal in y. Letting y^*_{j-1} be as defined in (1.7), for all $y > y^*_{j-1}$, we have $\Delta V_{j-1}(y) \leq p_j$. Therefore, $\Delta W_j(y, x) \leq 0$ for all $y > y^*_{j-1}$. Similarly, we have $\Delta V_{j-1}(y^*_{j-1}) > p_j$, but since $\Delta V_j(x)$ is decreasing in x by the first part of Proposition 1.5, it follows that $\Delta V_{j-1}(y) > p_j$ for all $y \leq y^*_{j-1}$. Therefore, $\Delta W_j(y, x) \geq 0$ for all $y \leq y^*_{j-1}$. Having $\Delta W_j(y, x) \geq 0$ for all $y \leq y^*_{j-1}$ and $\Delta W_j(y, x) \leq 0$ for all $y > y^*_{j-1}$ implies that $W_j(y, x)$ is unimodal in y and its maximizer occurs at y^*_{j-1}. So, the maximizer of $W_j(y, x)$ over $y \in \{0, \dots, x\}$ occurs at $\min\{y^*_{j-1}, x\}$. Third, we show that the optimal protection levels are monotone in the fare classes. By the definition of y^*_{j-1}, we have $\Delta V_{j-1}(y^*_{j-1}) > p_j$, and since $\Delta V_j(x) \geq \Delta V_{j-1}(x)$ by the second part of Proposition 1.5, we obtain $\Delta V_j(y^*_{j-1}) \geq \Delta V_{j-1}(y^*_{j-1}) > p_j > p_{j+1}$, which implies that $\Delta V_j(y^*_{j-1}) > p_{j+1}$. In this case, since y^*_j is given by $\max\{y \in \mathbb{N}_+ : \Delta V_j(y) > p_{j+1}\}$, it must be the case that $y^*_j \geq y^*_{j-1}$. □

Proof of Corollary 1.7 Let $G(x) = p_1 x$, then $V_1(x) = g(x) = \mathbb{E}\{G(\min\{D_1, x\})\}$, so $\Delta V_1(x) = \Delta g(x) = p_1\, \mathbb{P}\{D_1 \geq x\}$. Then, by Theorem 1.6,

$$y_1 = \max\{y \in \mathbb{N}_+ : p_1\, \mathbb{P}\{D_1 \geq x\} > p_2\}$$

which coincides with Littlewood's rule. □

Proof of Proposition 1.10 Since $\Pi_n(c, k)$ is the difference of a concave and a linear function, $\Pi_n(c, k)$ is itself concave. The marginal value of adding the c-th unit of capacity is $\Delta V_n(c) - k$ so the c-th unit increases profits as long as $\Delta V_n(c) > k$. Therefore, the smallest optimal capacity is given by $c(k)$. (Notice that $c(k) + 1$ may be also optimal if $\Delta V_n(c(k) + 1) = k$.) Note that $c(k)$ is decreasing in k since $\Delta V_n(c)$ is decreasing in c. Suppose that $k = p_{j+1}$. To establish $c(p_{j+1}) = y_j$, it is enough to show that $\Delta V_n(y_j) > p_{j+1} \geq \Delta V_n(y_j + 1)$. By definition, $y_j = \max\{y \in \mathbb{N}_+ : \Delta V_j(y) > p_{j+1}\}$, so that we have $\Delta V_j(y_j) > p_{j+1} \geq \Delta V_j(y_j + 1)$. Since it is optimal to protect up to y_j units of capacity for sale at fares $j, j-1, \dots, 1$, it follows that $V_n(c) = V_j(c)$ for all $c \leq y_j$, and consequently $\Delta V_n(y_j) =$

$\Delta V_j(y_j) > p_{j+1}$. Now $\Delta V_n(y_j + 1)$ can be written as a convex combination of p_{j+1} and $\Delta V_j(y_j+1) \leq p_{j+1}$ which implies that $\Delta V_n(y_j+1) \leq p_{j+1}$, as desired. □

Proof of Theorem 1.18 Since $\mathcal{R}_t(z) \geq 0$, it follows from (1.14) that $V(t,x)$ is increasing in t, with strict inequality as long as there is a fare $k \in M_t$ such that $p_k > \Delta V(t,x)$ and $\lambda_{kt} > 0$. To show that $V(t,x+1) \geq V(t,x)$ consider a sample path argument where the system with $x+1$ units of inventory uses the optimal policy for the system with x units of inventory until either the system with x units runs out of stock or time runs out. If the system with x units of inventory runs out at time s, then the system with $x+1$ units of inventory can still collect $V(s,1) \geq 0$. On the other hand, if time runs out the two systems collect the same revenue. Consequently, the system with $x+1$ units of inventory makes at least as much revenue resulting in $V(t,x+1) \geq V(t,x)$.

Clearly $\Delta V(t,1) \leq \Delta V(t,0) = \infty$. Assume as the inductive hypothesis that $\Delta V(t,y)$ is decreasing in $y \leq x$ for all $t \geq 0$. We want to show that $\Delta V(t,x+1) \leq \Delta V(t,x)$, or equivalently that

$$V(t,x+1) + V(t,x-1) \leq V(x) + V(x). \tag{1.19}$$

We will use a sample path argument to establish inequality (1.19). Consider four systems, one with $x+1$ units of inventory, one with $x-1$ units of inventory, and two with x units of inventory. Assume that we follow the optimal policy for the system with $x+1$ and for the system with $x-1$ that are on the left-hand side of inequality (1.19). For the two systems on the right, we use the sub-optimal policies designed for $x+1$ and $x-1$ units of inventory, respectively. We follow these policies until one of the following events occurs: time runs out, the difference in inventories for the systems on the left drops to 1, or the inventory of the system with $x-1$ units drops to zero. After that time we follow optimal policies for all four systems. To establish inequality (1.19), we will show that the revenues obtained for the systems in the right are at least as large as for the systems on the left, even though sub-optimal policies are used for the systems in the right. This is obviously true if we run out of time since the realized revenues of the two systems on the right are exactly equal to the realized revenues from the two systems on the left. Assume now that at time $s \in (0,t)$, the difference in inventories on the two systems on the left-hand side drops to 1, so that the states are $(s, y+1)$ and (s,y) for some $y < x$. This means that system on the left with $x+1$ units of inventory had $x-y$ units of sale and the system with $x-1$ units of inventory had $x-1-y$ units of sale. This implies that the system on the right that was following the policy designed for $x+1$ reaches state (s,y), while the system that was using the policy designed for $x-1$ reaches state $(s,y+1)$. Clearly, the additional optimal expected revenues over $[0,s]$ for each pair of systems is $V(s,y+1) + V(s,y) = V(s,y) + V(s,y+1)$, showing that the system on the right gets as much revenue as the system on the left even if sub-optimal polices are used for part of the horizon. Finally, if the inventory of the system with $x-1$ units of inventory drops to 0 at some time $s \in [0,t)$, so that state of the systems on the left are, respectively, (s, y) and $(s, 0)$ for some y, such that $1 < y \leq x$, while the systems on the left are $(s, y-1)$ and $(s, 1)$. From the inductive hypothesis, we know that $\Delta V(s, y) \leq \Delta V(s, 1)$ for all $y \leq x$ and all $s \leq t$. Consequently,

$$V(s, y) = V(s, y) + V(s, 0) \leq V(s, y-1) + V(s, 1),$$

and once again the pair of systems on the right result in at least as much revenue even though sub-optimal policies are used for part of the sales horizon.

We now show that $\Delta V(t, x)$ is increasing in t. This is equivalent to

$$\frac{\partial V(t, x)}{\partial t} = \mathcal{R}_t(\Delta V(t, x)) \geq \mathcal{R}_t(\Delta V(t, x-1)) = \frac{\partial V(t, x-1)}{\partial t},$$

but this is true on account of $\mathcal{R}_t(z)$ being decreasing in z and $\Delta V(t, x)$ being decreasing in x.

Notice that the set $a(t, x)$ is increasing in x since $\Delta V(t, x)$ is decreasing in x. Consequently, the set $A(t, x)$ is also increasing in x.

We now show that $V(t, x)$ is strictly increasing in t when $d_t(p) = d(p)$ is time invariant. This is because

$$\frac{\partial V(t, x)}{\partial t} = r(\Delta V(t, x)) \geq r(\Delta V(t, 1)) = r(V(t, 1)) > 0,$$

where the first inequality follows because r is decreasing and $V(t, 1) = \Delta V(t, 1) \geq \Delta V(t, x)$ for all $x \geq 1$. The strict inequality follows because $V(t, 1)$ must be below p_1 as otherwise if $V(t, 1) = p_1$, then the single unit of inventory must be priced at p_1 over the horizon $[0, t]$ and must sell with probability one over that interval. However, there is a positive probability equal to $e^{-\lambda_1 t}$ that the unit does not sell, so $V(t, 1) < p_1$, implying that $r(V(t, 1)) > 0$, so $V(t, x)$ is strictly increasing in t.

To show that $V(t, x)$ is concave, notice that since $r(z)$ is almost everywhere differentiable, then

$$\frac{\partial^2 V(t, x)}{\partial t^2} = r'(\Delta V(t, x)) \frac{\partial \Delta V(t, x)}{\partial t} \leq 0,$$

follows since $r'(z) \leq 0$, on account of $r(z)$ being decreasing in z, and from the fact that $\Delta V(t, x)$ is increasing in t. The fact that $r(z)$ is not differentiable at points $p_j \in M$ does not change the argument because we can take the right derivative of r and things work well given that $\Delta V(t, x)$ is increasing in t. □

Proof of Lemma 1.20 We will first show part that $\Delta V_j(t, x)$ is decreasing in x which is equivalent to showing that $2V_j(t, x) \geq V_j(t, x+1) + V_j(t, x-1)]$ for all $x \geq 1$. Let A be an optimal admission control rule starting from state $(t, x+1)$ and let B be an optimal admission control rule starting from $(t, x-1)$. These admission control rules are mappings from the state space to subsets $S_k = \{1, \ldots, k\}, k =$

$0, 1, \ldots, j$ where $S_0 = \emptyset$ is the optimal control whenever a system runs out of inventory. Consider four systems: two starting from state (t, x), using control rules A' and B', respectively, and one each starting from $(t, x+1)$ and $(t, x-1)$, using control rule A and B, respectively. Our goal is to specify heuristic control rules A' and B' that together make the expected revenues of the two systems starting with (t, x) at least as large as the expected revenues from the systems starting at $(t, x+1)$ and $(t, x-1)$. This will imply that $2V_j(t, x) \geq V_j(t, x+1) + V_j(t, x-1)$.

We will use the control rules $A' = A \cap B$ and $B' = A \cup B$ until the first time, if ever, the remaining inventory of the system (t, x) controlled by A' is equal to the remaining inventory of the system $(t, x+1)$ controlled by A. This will happen the first time, if ever, there is a sale under A and not under A', i.e. a sale under A but not under B. Let t' be the first time this happens, if it happens before the end of the horizon, and set $t' = 0$ otherwise. If $t' > 0$ then we apply policy $A' = A$ and $B' = B$ over $s \in [0, t')$. We claim that the expected revenue from the two systems starting with (t, x) is the same as the expected revenue from the other two systems. This is because the sales and revenues up to, but before t', are the same in the two systems. At t' sales occur only for the system (t, x) controlled by B' and the system $(t, x+1)$ controlled by A, and the revenues from the two sales are identical. After the sales at t', the inventory of the system (t, x) controlled by A' becomes identical to the inventory of the system $(t, x+1)$ controlled by A while the inventory of the system (t, x) controlled by B' becomes identical to the inventory of the system $(t, x-1)$ controlled by B. Since the policy switches to $A' = A$ and $B' = B$, then sales and revenues are the same over $[0, t')$. If $t' = 0$, then the sales of the two systems are the same during the entire horizon.

It remains to verify that inventories don't become negative. Prior to time t', the systems remain balance in the sense that system (t, x) governed by A' always has one unit of inventory less than system $(t, x+1)$ governed by A and system (t, x) governed by B' has one more unit of inventory than system $(t, x-1)$ governed by B. Thus the only two systems that could potential run out of inventory before t' are A' and B.

Since sales under $A' = A \cap B$ are more restricted than sales under B, the inventory of system (t, x) governed by A' will always be at least one unit since at most $x - 1$ units of sale are allowed under B. Therefore the only way the system can run out of inventory is if system $(t, x-1)$ runs out of inventory under B before t'. However, in this case, sales would stop under systems A' and B, while sales will continue under $B' = A$ and A so revenues will continue to be the same until the first sale under A at which point we reached t'. This shows that even if the system $(t, x-1)$ runs out of inventory under B the two systems continue to have the same revenues over the entire horizon. Consequently $2\Delta V_j(t, x) \geq V_j(t, x+1) + V_j(t, x-1)$ for all $x \geq 1$.

To show that $\Delta V_j(t, x)$ is increasing in j, it is enough to show that

$$V_j(t, x) + V_{j-1}(t, x-1) \geq V_j(t, x-1) + V_{j-1}(t, x).$$

To do this, we again use a sample path argument. Let A be an optimal admission control rule for the system $(j, t, x-1)$ and B be an admission control rule for the system $(j-1, t, x)$. Let A' and B' be heuristic admission rules applied, respectively, to the systems (j, t, x) and $(j-1, t, x-1)$. Our goal is to exhibit heuristics A' and B' such that when applied to the systems (j, t, x) and $(j-1, t, x-1)$ they generate as much revenue as the applying A to $(j, t, x-1)$ and B to $(j-1, t, x)$. This will imply that $V_j(t, x) + V_{j-1}(t, x-1) \geq V_j(t, x-1) + V_{j-1}(t, x)$.

Let $A' = A \cup B$ and $B' = A \cap B$ and let t' be the first time there is a sale under $A \cup B$ without a corresponding sale in A, so there is a sale under B but not under A. If $t' = 0$, then the revenues of the sets of two systems are equal. If $t' > 0$ switch at that point to the policy $A' = A$ and $B' = B$. Then sales and revenues under both sets of two systems are equal up to t'. At t' there are sales for the system (j, t, x) and $(j-1, t, x-1)$ that generate the same revenues. Moreover, the inventories of the two sets of two systems have the same inventories immediately after the sale at t'. Since the policy then switches to $A' = A$ and $B' = B$ then sales and revenues are the same for the two set of systems over $s \in [0, t')$. The only system in danger to run out of inventory is system (j, t, x) under $A' = A \cup B$, but that system has the same number of sales as the system $(j, t, x-1)$ under A up to t'. Therefore the system (j, t, x) has at least one unit of inventory up to t'.

To show that $\Delta V_j(t, x)$ is increasing in t it is enough to show that

$$V_j(t, x) + V_j(t-1, x-1) \geq V_j(t, x-1) + V_j(t-1, x).$$

To do this we again use a sample path argument. Let A be an optimal admission control rule for the system $(t, x-1)$ and B be an optimal admission control rule for the system $(t-1, x)$ Let A' and B' be heuristic admission rules applied, respectively, to the systems (t, x) and $(t-1, x-1)$. Our goal is to exhibit heuristics A' and B' such that when applied to the systems (t, x) and $(t-1, x-1)$ they generate as much revenue as the applying A to $(t, x-1)$ and B to $(t-1, x)$. This will imply that $V_j(t, x) + V_j(t-1, x-1) \geq V_j(t, x-1) + V_j(t-1, x)$. Let $A' = A \cup B$ and $B' = A \cap B$ and let t' be the first time there is a sale under A' without a corresponding sale in A, so there is a sale under B but not under A. If $t' = 0$ then the revenues of the sets of two systems are equal. If $t' > 0$ switch at that point to the policy $A' = A$ and $B' = B$. Then sales and revenues under both sets of two systems are equal up to t'. At t', there are sales for the system (t, x) and $(t-1, x)$ that generate the same revenues. Moreover, the inventories of the two sets of two systems have the same inventories immediately after the sale at t'. Since the policy then switches to $A' = A$ and $B' = B$, then sales and revenues are the same for the two set of systems over $s \in [0, t')$. The only system in danger to run out of inventory is system $(t-1, x-1)$ under $B' = A \cup B$, but that system has the same number of sales as the system $(t-1, x)$ under B up to t'. Therefore, the system $(t-1, x-1)$ has at least one unit of inventory up to t'. □

Proof of Lemma 1.21 We will first show that $a_j(t, x)$ can also be characterized as $a_j(t, x) = \max\{k \leq j : p_k \geq \Delta V_k(t-1, x)\}$. The result will then follow from

Lemma 1.20. First notice that if $a_j(t,x) = k < j$ then $V_i(t,x) = V_k(t,x)$ for all $i \in \{k,\ldots,j\}$. Moreover, $a_j(t,x) = k < j$ implies that $W_k(t,x) > W_{k+1}(t,x)$. Consequently, $0 > W_{k+1}(t,x) - W_k(t,x) = (p_{k+1} - \Delta V_{k+1}(t-1,x))\lambda_{k+1}$, so $p_{k+1} < \Delta V_{k+1}(t-1,x)$. Conversely, if $p_k \geq \Delta V_k(t-1,x)$ then $W_k(t,x) - W_{k-1}(t,x) \geq (p_k - \Delta V_k(t-1,x))\lambda_k \geq 0$ so $W_k(t,x) \geq W_{k-1}(t,x)$. With the new characterization, we now turn to the monotonicity of $a_j(t,x) = \max\{k \leq j : p_k \geq \Delta V_k(t-1,x)\}$. The monotonicity with respect to j is obvious because it expands the set over which we are maximizing. To see the monotonicity with respect to t, notice that $\Delta V_k(t,x) \geq \Delta V_k(t-1,x)$ so k is excluded from the set whenever $\Delta V_k(t-1,x) \leq p_k < \Delta V_k(t,x)$. To see the monotonicity with respect to x, notice that $\Delta V_k(t-1,x+1) \leq \Delta V_k(t,x) \leq p_k$ implies that k contributes positively at state $(t-1,x+1)$ whenever it contributes at $(t-1,x)$. □

Proof of Theorem 1.22 The properties of $A_j(t,x)$ follow from the properties of $a_j(t,x)$ established in Lemma 1.21. Note that $T_j = \{(t,x) : a_j(t,x) < j\}$. From Lemma 1.21, $a_j(t,x) < j$ implies that $a_i(t,x) < i$ for all $i > j$, so $T_j \subseteq T_i$ for all $i > j$. This implies that $y_j(t)$ is increasing in j for any $t \geq 0$. If $t' > t$, then $a_{j+1}(t', y_j(t)) \leq a_{j+1}(t, y_j(t)) < j+1$, so $y_j(t') \geq y_j(t)$. Note that $y_j(t) \leq y_i(t)$ for all $i > j$, then $x \leq y_j(t)$ implies $V_{i+1}(t,x) = V_i(t,x)$ for all $i \geq j$ and therefore $V_i(t,x) = V_j(t,x)$ for all $i > j$. □