Chapter 11
Systems Driven by an Infinite Dimensional Brownian Noise

In Chap. 8 we gave a representation for positive functionals of a Hilbert space valued Brownian motion. This chapter will apply the representation to study the large deviation properties of infinite dimensional small noise stochastic dynamical systems. In the application, the driving noise is given by a Brownian sheet, and so in this chapter we will present a sufficient condition analogous to Condition 9.1 (but there will be no Poisson noise in this chapter) that covers the setting of such noise processes (see Condition 11.15). Another formulation of an infinite dimensional Brownian motion that will be needed in Chap. 12 is as a sequence of independent Brownian motions regarded as a $\mathscr{C}([0,T]:\mathbb{R}^{\infty})$-valued random variable. We also present the analogous sufficient condition (Condition 11.12) for an LDP to hold for this type of driving noise.

To illustrate the approach we consider a class of reaction–diffusion stochastic partial differential equations (SPDE), for which well-posedness has been studied in [174]. Previous works that prove an LDP for this SPDE include [170, 235]. The proof of the Laplace principle proceeds by verification of Condition 11.15. Just as in Chap. 10, the key ingredients in the verification of this condition are the well-posedness and compactness for sequences of controlled versions of the original SPDE [Theorems 11.23, 11.24, and 11.25]. Also as in Chap. 10, the techniques and estimates used to prove such properties for the original (uncontrolled) stochastic model can be applied here as well, and indeed proofs for the controlled SPDEs proceed in very much the same way as those of their uncontrolled counterparts.

The chapter is organized as follows. In Sect. 11.1 we recall some common formulations of an infinite dimensional Brownian motion and relations between them. Starting from the variational representation for a Hilbert space valued Brownian motion from Chap. 8, we present analogous representations for these equivalent formulations of infinite dimensional Brownian motion. Then starting from the sufficient condition for Hilbert space valued Brownian motion given in Chap. 9, we state the corresponding sufficient conditions for a uniform Laplace principle to hold for these

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other formulations in Sect. 11.2. The illustration of how the conditions are verified is given in Sect. 11.3, which studies the large deviation properties for a family of stochastic reaction–diffusion equations.

11.1 Formulations of Infinite Dimensional Brownian Motion

An infinite dimensional Brownian motion arises in a natural fashion in the study of stochastic processes with a spatial parameter. We refer the reader to [69, 169, 243] for numerous examples in the physical sciences in which infinite dimensional Brownian motions are used to model the driving noise for stochastic dynamical systems. Depending on the application of interest, the infinite dimensional nature of the driving noise may be expressed in a variety of forms. Some examples include Hilbert space valued Brownian motion (as was considered in Chap. 8); cylindrical Brownian motion; an infinite sequence of iid standard (1-dimensional) Brownian motions; and space-time Brownian sheets. In what follows, we describe these various formulations and explain how they relate to each other. We will be concerned only with processes defined over a fixed time horizon, and thus fix $T \in (0, \infty)$, and all filtrations and stochastic processes will be defined over the horizon $[0, T]$. Reference to T will be omitted unless essential. Let $(\Omega, \mathscr{F}, P)$ be a probability space with a filtration $\{\mathscr{F}_t\}_{0 \le t \le T}$ satisfying the usual conditions. Let $(\mathscr{H}, \langle \cdot, \cdot \rangle)$ be a real separable Hilbert space. Let Λ be a symmetric strictly positive trace class operator on $\mathscr{H}$. Recall that an $\mathscr{H}$-valued continuous stochastic process $\{W(t)\}_{0 \le t \le T}$ defined on $(\Omega, \mathscr{F}, P)$ is called a Λ-Wiener process with respect to $\{\mathscr{F}_t\}$ if for every nonzero $h \in \mathscr{H}$, $\{\langle \Lambda h, h \rangle^{-1/2} \langle W(t), h \rangle\}$ is a one-dimensional standard $\{\mathscr{F}_t\}$-Wiener process.

Another formulation for an infinite dimensional Brownian motion, which will be used in Chap. 12 for the study of stochastic flows of diffeomorphisms, is as follows. Let $\{\beta_i\}_{i \in \mathbb{N}}$ be an infinite sequence of independent standard one-dimensional, $\{\mathscr{F}_t\}$-Brownian motions. We denote the product space of countably infinite copies of the real line by $\mathbb{R}^\infty$. Note that a sequence of independent standard Brownian motions $\{\beta_i\}_{i \in \mathbb{N}}$ can be regarded as a random variable with values in $\mathscr{C}([0, T] : \mathbb{R}^\infty)$, where $\mathbb{R}^\infty$ is equipped with the usual topology of coordinatewise convergence, which can be metrized using the distance

$$d(u, v) \doteq \sum_{k=1}^{\infty} \frac{|u_k - v_k| \wedge 1}{2^k}.$$

It is easily checked that with this metric, $\mathbb{R}^\infty$ is a Polish space. Thus $\beta = \{\beta_i\}_{i \in \mathbb{N}}$ is a random variable with values in the Polish space $\mathscr{C}([0, T] : \mathbb{R}^\infty)$, and can be regarded as another model of an infinite dimensional Brownian motion.

Let $\{e_i\}_{i \in \mathbb{N}}$ be a complete orthonormal system (CONS) for the Hilbert space $\mathscr{H}$ such that $\Lambda e_i = \lambda_i e_i$, where λ_i is the strictly positive ith eigenvalue of Λ, which corresponds to the eigenvector e_i. Since Λ is a trace class operator, $\sum_{i=1}^{\infty} \lambda_i < \infty$. Define

$$\beta_i(t) \doteq \frac{1}{\sqrt{\lambda_i}} \langle W(t), e_i \rangle, \ 0 \le t \le T, \ i \in \mathbb{N},$$

where W as before is a Λ-Wiener process with respect to $\{\mathscr{F}_t\}$. It is easy to check that $\{\beta_i\}$ is a sequence of independent standard $\{\mathscr{F}_t\}$-Brownian motions. Thus starting from a Λ-Wiener process, one can produce an infinite collection of independent standard Brownian motions in a straightforward manner. Conversely, given a collection of independent standard Brownian motions $\{\beta_i\}_{i\in\mathbb{N}}$ and $(\Lambda, \{e_i, \lambda_i\})$ as above, one can obtain a Λ-Wiener process W by setting

$$W(t) \doteq \sum_{i=1}^{\infty} \sqrt{\lambda_i} \beta_i(t) e_i, \ \ 0 \le t \le T. \tag{11.1}$$

The right side of (11.1) clearly converges in $\mathscr{L}^2(P)$ for each fixed t. Furthermore, one can check that the series also converges in $\mathscr{C}([0, T] : \mathscr{H})$ almost surely [69, Theorem 4.3]. These observations lead to the following result.

Proposition 11.1 *There exist measurable maps* $f : \mathscr{C}([0, T] : \mathbb{R}^\infty) \to \mathscr{C}([0, T] : \mathscr{H})$ *and* $g : \mathscr{C}([0, T] : \mathscr{H}) \to \mathscr{C}([0, T] : \mathbb{R}^\infty)$ *such that* $f(\beta) = W$ *and* $g(W) = \beta$ *a.s.*

Remark 11.2 Consider the Hilbert space $l_2 \doteq \{x = (x_1, x_2, \dots) : x_i \in \mathbb{R}$ and $\sum_{i=1}^{\infty} x_i^2 < \infty\}$ with the inner product $\langle x, y\rangle_0 \doteq \sum_{i=1}^{\infty} x_i y_i$. Let $\{\lambda_i\}_{i\in\mathbb{N}}$ be a sequence of strictly positive numbers such that $\sum_{i=1}^{\infty} \lambda_i < \infty$. Then the Hilbert space $\bar{l}_2 \doteq \{x = (x_1, x_2, \dots) : x_i \in \mathbb{R}$ and $\sum_{i=1}^{\infty} \lambda_i x_i^2 < \infty\}$ with the inner product $\langle x, y\rangle \doteq \sum_{i=1}^{\infty} \lambda_i x_i y_i$ contains l_2, and the embedding map $\iota : l_2 \to \bar{l}_2$, $\iota(x) = x$ is Hilbert–Schmidt. Furthermore, the infinite sequence of real Brownian motions β takes values in $\bar{l}_2$ almost surely and can be regarded as a $\bar{l}_2$-valued Λ-Wiener process, where Λ is defined by $\langle \Lambda x, y\rangle = \sum_{i=1}^{\infty} \lambda_i^2 x_i y_i$, $x, y \in \bar{l}_2$.

Equation (11.1) above can be interpreted as saying that the sequence $\{\lambda_i\}$ (or equivalently the trace class operator Λ) injects a "coloring" to a white noise such that the resulting process has greater regularity. In some models of interest, such coloring is obtained indirectly in terms of (state-dependent) diffusion coefficients. It is natural in such situations to consider the driving noise a "cylindrical Brownian motion" rather than a Hilbert space valued Brownian motion. Let $(\mathscr{H}, \langle\cdot,\cdot\rangle)$ as before be a real separable Hilbert space and fix a probability space and a filtration as above.

Definition 11.3 A family $\{B_t(h) = B(t, h) : t \in [0, T], h \in \mathscr{H}\}$ of real random variables is said to be an $\{\mathscr{F}_t\}$**-cylindrical Brownian motion** if the following hold.

(a) For every $h \in \mathscr{H}$ with $\|h\| = 1$, $\{B(t, h)\}$ is a standard $\mathscr{F}_t$-Wiener process.

(b) For every $t \in [0, T]$, $a_1, a_2 \in \mathbb{R}$ and $f_1, f_2 \in \mathscr{H}$,

$$B(t, a_1 f_1 + a_2 f_2) = a_1 B(t, f_1) + a_2 B(t, f_2) \ \text{ a.s.}$$

If $\{B_t(h)\}$ is a cylindrical Brownian motion and $\{e_i\}$ is a CONS in $\mathscr{H}$, setting $\beta_i(t) \doteq B(t, e_i)$, we see that $\{\beta_i\}$ is a sequence of independent standard one-dimensional $\{\mathscr{F}_t\}$-Brownian motions. Conversely, given a sequence $\{\beta_i\}_{i\in\mathbb{N}}$ of independent standard one-dimensional $\{\mathscr{F}_t\}$-Brownian motions,

$$B_t(h) \doteq \sum_{i=1}^{\infty} \beta_i(t)\langle e_i, h\rangle \tag{11.2}$$

defines a cylindrical Brownian motion on $\mathscr{H}$. For each $h \in \mathscr{H}$, the series in (11.2) converges in $\mathscr{L}^2(P)$ and a.s. in $\mathscr{C}([0, T] : \mathbb{R})$.

Proposition 11.4 *Let B be a cylindrical Brownian motion as in Definition 11.3 and let β be as constructed in the last paragraph. Then $\sigma\{B_s(h) : 0 \le s \le t, h \in \mathscr{H}\} = \sigma\{\beta(s) : 0 \le s \le t\}$ for all $t \in [0, T]$. In particular, if X is a $\sigma\{B(s, h) : 0 \le s \le T, h \in \mathscr{H}\}$-measurable random variable, then there exists a measurable map $g : \mathscr{C}([0, T] : \mathbb{R}^\infty) \to \mathbb{R}$ such that $g(\beta) = X$ a.s.*

In yet other stochastic dynamical systems, the driving noise is given as a space-time white noise process, also referred to as a Brownian sheet. In what follows, we introduce this stochastic process and describe its relationship with the formulations considered above. Let $(\Omega, \mathscr{F}, P, \{\mathscr{F}_t\})$ be as before and fix a bounded open subset $O \subset \mathbb{R}^d$. We follow standard usage and denote both cylindrical Brownian motions by B [more precisely by $B_t(h)$] and also Brownian sheets by B [in this case $B(t, x)$]. The intended use should be clear from context.

Definition 11.5 A family of real-valued Gaussian random variables

$$\{B(t, x),\ (t, x) \in [0, T] \times O\}$$

is called a **Brownian sheet** if the following hold.

(a) If $(t, x) \in [0, T] \times O$, then $E(B(t, x)) = 0$.

(b) If $0 \le s \le t \le T$, then $\{B(t, x) - B(s, x), x \in O\}$ is independent of $\mathscr{F}_s$.

(c) $\operatorname{Cov}(B(t, x), B(s, y)) = \lambda(A_{t,x} \cap A_{s,y})$, where λ is Lebesgue measure on $[0, T] \times O$ and

$$A_{t,x} \doteq \left\{(s, y) \in [0, T] \times O : 0 \le s \le t \text{ and } y_j \le x_j,\ j = 1, \ldots, d\right\}.$$

(d) The map $(t, u) \mapsto B(t, u)$ from $[0, T] \times O$ to $\mathbb{R}$ is uniformly continuous a.s.

Due to the uniform continuity property of part (d), $B = \{B(t, x),\ (t, x) \in [0, T] \times O\}$ can be regarded as a random variable with values in the Polish space $\mathscr{C}([0, T] \times \bar{O} : \mathbb{R})$, the space of continuous functions from $[0, T] \times \bar{O}$ to $\mathbb{R}$, equipped with the uniform topology.

To introduce stochastic integrals with respect to a Brownian sheet, we need the following definitions and notation, which are largely taken from [169].

Definition 11.6 (*Elementary and simple functions*) A function $f : O \times [0,T] \times \Omega \to \mathbb{R}$ is elementary if there exist $a,b \in [0,T]$, $a \le b$, a bounded $\{\mathscr{F}_a\}$-measurable random variable X, and $A \in \mathscr{B}(O)$ such that

$$f(x,s,\omega) = X(\omega)1_{(a,b]}(s)1_A(x).$$

A finite sum of elementary functions is referred to as a simple function. We denote by $\bar{\mathscr{S}}$ the class of all simple functions.

We now introduce the $\{\mathscr{F}_t\}$-predictable σ-field on $\Omega \times [0,T] \times O$. The definition is analogous to that of a predictable σ-field on $\Omega \times [0,T]$ introduced in Chap. 8 and is denoted by the same symbol.

Definition 11.7 (*Predictable σ-field*) The $\{\mathscr{F}_t\}$-predictable σ-field $\mathscr{PF}$ on $\Omega \times [0,T] \times O$ is the σ-field generated by $\bar{\mathscr{S}}$. A function $f : \Omega \times [0,T] \times O \to \mathbb{R}$ is called an $\{\mathscr{F}_t\}$-predictable process if it is $\mathscr{PF}$-measurable.

Remark 11.8 In Chap. 8 we considered a probability space supporting a Hilbert space valued Wiener process and defined the classes of integrands/controls $\mathscr{A}_b$, $\mathscr{A}$, $\bar{\mathscr{A}}_b$, and $\bar{\mathscr{A}}$. The first two are predictable with respect to the filtration generated by the Wiener process and either have a finite $\mathscr{L}^2$ norm a.s. ($\mathscr{A}$) or satisfy a uniform bound on this norm a.s. ($\mathscr{A}_b$), and the last two are analogous, save being $\{\mathscr{F}_t\}$-predictable (see the definitions given after Definition 8.2). In this chapter we will need the analogous processes for a number of alternative formulations of infinite dimensional Brownian motion. With some abuse of notation, we use the same symbols to denote the classes with the analogous predictability and boundedness properties for all these different formulations. The class intended in any circumstance will be clear from the context.

Thus analogous to the class of integrands $\bar{\mathscr{A}}$ introduced in Chap. 8, consider the class of all $\{\mathscr{F}_t\}$-predictable processes f such that $\int_{[0,T]\times O} f^2(s,x)ds\,dx < \infty$ a.s., and denote this class by $\bar{\mathscr{A}}$. Classes $\mathscr{A}_b$, $\mathscr{A}$, and $\bar{\mathscr{A}}_b$ are defined similarly. For all $f \in \bar{\mathscr{A}}$, the stochastic integral $M_t(f) \doteq \int_{[0,t]\times O} f(s,u)B(ds \times du)$, $t \in [0,T]$, is well defined as in Chap. 2 of [243]. Furthermore, for all $f \in \bar{\mathscr{A}}$, $\{M_t(f)\}_{0\le t\le T}$ is a continuous $\{\mathscr{F}_t\}$-local martingale. More properties of the stochastic integral can be found in Appendix D.2.4, and in much greater detail in [243].

Consider the Hilbert space $\mathscr{L}^2(O) \doteq \{f : O \to \mathbb{R} : \int_O f^2(x)dx < \infty\}$ equipped with the usual inner product. Let $\{\phi_i\}_{i\in\mathbb{N}}$ be a CONS in $\mathscr{L}^2(O)$. Then it is easy to verify that $\beta = \{\beta_i\}_{i\in\mathbb{N}}$ defined by $\beta_i(t) \doteq \int_{[0,t]\times O} \phi_i(x)B(ds \times dx)$, $i \in \mathbb{N}$, $t \in [0,T]$ is a sequence of independent standard real Brownian motions. Also, for $(t,x) \in [0,T] \times O$,

$$B(t,x) = \sum_{i=1}^{\infty} \beta_i(t) \int_O \phi_i(y)1_{(-\infty,x]}(y)dy \tag{11.3}$$

(where $(-\infty,x] = \{y : y_i \le x_i \text{ for all } i = 1,\ldots,d\}$), and the series in (11.3) converges in $\mathscr{L}^2(P)$ for each (t,x). From these considerations, it follows that

$$\sigma\{B(t,x),\ t\in[0,T],\ x\in O\}=\sigma\{\beta_i(t),\ i\in\mathbb{N},\ t\in[0,T]\}. \tag{11.4}$$

As a consequence of (11.4) and Lemma E.1 in the appendix, we have the following result.

Proposition 11.9 *There exists a measurable map $g:\mathscr{C}([0,T]:\mathbb{R}^\infty)\to\mathscr{C}([0,T]\times\bar{O}:\mathbb{R})$ such that $B=g(\beta)$ a.s., where β is as defined by $\beta_i(t)\doteq\int_{[0,t]\times O}\phi_i(x)B(ds\times dx)$.*

11.1.1 The Representations

In Chap. 8 we presented a variational representation for positive functionals of a Hilbert space valued Brownian motion. Using this representation and the results of Sect. 11.1, we can obtain analogous representations for other formulations of an infinite dimensional Brownian motion. Let $(\Omega,\mathscr{F},P,\{\mathscr{F}_t\})$ and $\beta=\{\beta_i\}$ be as in Sect. 11.1. Recall that β is a $\mathscr{C}([0,T]:\mathbb{R}^\infty)$-valued random variable.

Let $\mathscr{P}\mathscr{F}$ be the $\{\mathscr{F}_t\}$-predictable σ-field on $[0,T]\times\Omega$ as introduced in Definition 8.2. For a Hilbert space $\mathscr{H}_0$, let $\bar{\mathscr{A}}$ as in Chap. 8 be the collection of all $\mathscr{H}_0$-valued $\{\mathscr{F}_t\}$-predictable processes for which $\int_0^T\|\phi(s)\|_0^2ds<\infty$ a.s., where $\|\cdot\|_0$ is the norm in the Hilbert space $\mathscr{H}_0$. We also recall the classes $\mathscr{A}_b$, $\mathscr{A}$ and $\bar{\mathscr{A}}_b$ introduced in Chap. 8. Note that when $\mathscr{H}_0=l_2$, every $u\in\bar{\mathscr{A}}$ can be written as $u=\{u_i\}_{i\in\mathbb{N}}$, where for each i, u_i is a real-valued $\{\mathscr{F}_t\}$-predictable process and $\sum_{i=1}^\infty\int_0^T|u_i(s)|^2ds<\infty$ a.s. The following result is a consequence of Theorem 8.3, Proposition 11.1, and Remark 11.2.

Theorem 11.10 *Let G be a bounded measurable function mapping $\mathscr{C}([0,T]:\mathbb{R}^\infty)$ into $\mathbb{R}$. Then with $\mathscr{H}_0=l_2$, we have*

$$-\log Ee^{-G(\beta)}=\inf_{u=\{u_i\}\in\mathscr{R}}E\left[\frac{1}{2}\int_0^T\sum_{i=1}^\infty|u_i(s)|^2ds+G\left(\beta^u\right)\right],$$

where $\beta_i^{u_i}=\beta_i+\int_0^\cdot u_i(s)ds$, $i\in\mathbb{N}$, $\beta^u\doteq\{\beta_i^{u_i}\}_{i\in\mathbb{N}}$, and $\mathscr{R}$ can be $\bar{\mathscr{A}}$, $\bar{\mathscr{A}}_b$, $\mathscr{A}$, or $\mathscr{A}_b$.

Proof Taking $\mathscr{H}=\bar{l}_2$ introduced in Remark 11.2, it follows from Proposition 11.1 that there is a measurable map $g:\mathscr{C}([0,T]:\mathscr{H})\to\mathscr{C}([0,T]:\mathbb{R}^\infty)$ such that $\beta=g(W)$, where W is as defined in (11.1) with $\{\lambda_i\}$ as in Remark 11.2 and e_i as the vector with the ith coordinate $1/\sqrt{\lambda_i}$ and remaining coordinates 0. Note that the function g can be explicitly written as

$$[g(x)]_i(t)=\frac{1}{\sqrt{\lambda_i}}\langle x(t),e_i\rangle=x_i(t),\quad x\in\mathscr{C}([0,T]:\mathscr{H}),\ i\in\mathbb{N},\ t\in[0,T].$$

From Theorem 8.3, we then have

$$
\begin{aligned}
-\log Ee^{-G(\beta)} &= -\log Ee^{-G(g(W))} \\
&= \inf_{u=\{u_i\}\in\mathscr{R}} E\left[\frac{1}{2}\int_0^T \sum_{i=1}^{\infty} |u_i(s)|^2 ds + G\left(g(W^u)\right)\right],
\end{aligned}
$$

where $W^u(t) \doteq W(t) + \int_0^t u(s)ds$. The result now follows on observing that for all $u \in \mathscr{R}$, $g(W^u) = \beta^u$. ☐

We next note the representation theorem for a Brownian sheet that follows from Proposition 11.9, Theorem 11.10, and an application of Girsanov's theorem. In the statement below, $\bar{\mathscr{A}}$, $\bar{\mathscr{A}}_b$, $\mathscr{A}$, and $\mathscr{A}_b$ are as introduced below Definition 11.7.

Theorem 11.11 *Let $G : \mathscr{C}([0,T] \times \bar{O} : \mathbb{R}) \to \mathbb{R}$ be a bounded measurable map. Let B be a Brownian sheet as in Definition 11.5. Then*

$$
-\log Ee^{-G(B)} = \inf_{u\in\mathscr{R}} E\left[\frac{1}{2}\int_0^T \int_O u^2(s,r)drds + G(B^u)\right],
$$

where $B^u(t,x) = B(t,x) + \int_0^t \int_{(-\infty,x]\cap O} u(s,y)dyds$ and $\mathscr{R}$ can be $\bar{\mathscr{A}}$, $\bar{\mathscr{A}}_b$, $\mathscr{A}$, or $\mathscr{A}_b$.

Proof We consider only the case $\mathscr{R} = \mathscr{A}_b$, and note that all remaining cases can be treated similarly. Let g be as in Proposition 11.9. To apply the proposition, we need to refer to the analogous set of control processes used in Theorem 11.10, which we denote by $\mathscr{A}_b^\beta$. Then with β as defined above (11.3), we have

$$
\begin{aligned}
-\log Ee^{-G(B)} &= -\log Ee^{-G(g(\beta))} \\
&= \inf_{\hat{u}=\{\hat{u}_i\}\in\mathscr{A}_b^\beta} E\left[\frac{1}{2}\int_0^T \sum_{i=1}^{\infty} |\hat{u}_i(s)|^2 ds + G\left(g(\beta^{\hat{u}})\right)\right]. \qquad (11.5)
\end{aligned}
$$

Note that there is a one-to-one correspondence between elements of $\mathscr{A}_b$ and $\mathscr{A}_b^\beta$ given through the relations

$$
\begin{aligned}
u(t,x) &\doteq \sum_{i=1}^{\infty} \hat{u}_i(t)\phi_i(x),\ \ (t,x) \in [0,T] \times O \text{ for } \{\hat{u}_i\} \in \mathscr{A}_b^\beta, \\
\hat{u}_i(t) &\doteq \int_O u(t,x)\phi_i(x)dx,\ \ t \in [0,T] \text{ for } u \in \mathscr{A}_b.
\end{aligned}
$$

Furthermore,

$$
\int_0^T \sum_{i=1}^{\infty} |\hat{u}_i(s)|^2 ds = \int_0^T \int_O u^2(s,r)drds. \qquad (11.6)
$$

Finally, from Girsanov's theorem, with any u and $\hat{u}$ given by the above relations there is a measure Q that is mutually absolutely continuous with respect to P and is such that under Q, $(\beta^{\hat{u}}, B^u)$ have the same law as (β, B) under P. Thus $G(g(\beta^{\hat{u}})) = G(B^u)$ a.s., and the result now follows from (11.5) and (11.6). □

The analogous representation holds for cylindrical Brownian motion, with a similar proof. We omit the details.

11.2 General Sufficient Condition for an LDP

In this section we will present sufficient conditions for a uniform Laplace principle that are similar to those presented in Sect. 9.2.1 but with the driving noise a Brownian sheet or an infinite sequence of real Brownian motions (i.e., the $\mathbb{R}^\infty$-valued random variable β), rather than a Hilbert space valued Brownian motion. For simplicity, we do not include a Poisson noise here, although that setting can be covered in a similar manner.

Let, as in Sect. 11.1, $\beta = \{\beta_i\}$ be a sequence of independent standard real $\{\mathscr{F}_t\}$-Brownian motions on $(\Omega, \mathscr{F}, P, \{\mathscr{F}_t\})$. Recall that β is a $\mathscr{C}([0,T] : \mathbb{R}^\infty)$-valued random variable. For each $\varepsilon > 0$, let $\mathscr{G}^\varepsilon : \mathscr{Z} \times \mathscr{C}([0,T] : \mathbb{R}^\infty) \to \mathscr{E}$ be a measurable map, where $\mathscr{Z}$ and $\mathscr{E}$ are Polish spaces, and define

$$X^{\varepsilon,z} \doteq \mathscr{G}^\varepsilon(z, \sqrt{\varepsilon}\beta). \tag{11.7}$$

We now consider the Laplace principle for the family $\{X^{\varepsilon,z}\}$ and introduce the analogue of Condition 9.1 for this setting. In the assumption, S_M and $\bar{\mathscr{A}}_{b,M}$ (the deterministic controls with squared $\mathscr{L}^2$ norm bounded by M and $\{\mathscr{F}_t\}$-predictable processes that take values in S_M, respectively) are defined as in (8.1) and below (8.2), with $\mathscr{H}_0$ there replaced by the Hilbert space l_2.

Condition 11.12 *There exists a measurable map $\mathscr{G}^0 : \mathscr{Z} \times \mathscr{C}([0,T] : \mathbb{R}^\infty) \to \mathscr{E}$ such that the following hold.*

(a) *For every $M < \infty$ and compact set $K \subset \mathscr{Z}$, the set*

$$\Gamma_{M,K} \doteq \left\{ \mathscr{G}^0\left(z, \int_0^\cdot u(s)ds\right) : u \in S_M, z \in K \right\}$$

is a compact subset of $\mathscr{E}$.

(b) *Consider $M < \infty$ and families $\{u^\varepsilon\} \subset \bar{\mathscr{A}}_{b,M}$ and $\{z^\varepsilon\} \subset \mathscr{Z}$ such that u^ε converges in distribution (as S_M-valued random elements) to u and $z^\varepsilon \to z$ as $\varepsilon \to 0$. Then*

$$\mathscr{G}^\varepsilon\left(z^\varepsilon, \sqrt{\varepsilon}\beta + \int_0^\cdot u^\varepsilon(s)ds\right) \to \mathscr{G}^0\left(z, \int_0^\cdot u(s)ds\right),$$

as $\varepsilon \to 0$ in distribution.

The proof of the following uses a straightforward reduction to Theorem 9.2.

Theorem 11.13 *Let $X^{\varepsilon,z}$ be as in (11.7) and suppose that Condition 11.12 holds. For $z \in \mathscr{Z}$ and $\phi \in \mathscr{E}$ let*

$$I_z(\phi) \doteq \inf_{\left\{u\in\mathscr{L}^2([0,T]:l_2):\phi=\mathscr{G}^0\left(z,\int_0^\cdot u(s)ds\right)\right\}} \left[\frac{1}{2}\sum_{i=1}^{\infty}\int_0^T |u_i(s)|^2 ds\right]. \quad (11.8)$$

Suppose that for all $\phi \in \mathscr{E}$, $z \mapsto I_z(\phi)$ is a lower semicontinuous map from $\mathscr{Z}$ to $[0,\infty]$. Then for all $z \in \mathscr{E}_0$, $\phi \mapsto I_z(\phi)$ is a rate function on $\mathscr{E}$, and the family $\{I_z(\cdot),\ z \in \mathscr{Z}\}$ of rate functions has compact level sets on compacts. Furthermore, the family $\{X^{\varepsilon,z}\}$ satisfies the Laplace principle on $\mathscr{E}$ with rate function I_z, uniformly on compact subsets of $\mathscr{Z}$.

Proof From Remark 11.2 we can regard β as an $\mathscr{H}$-valued Λ-Wiener process, where $\mathscr{H} = \bar{l}_2$ and Λ is a trace class operator, as defined in Remark 11.2. Also, one can check that $\mathscr{H}_0 \doteq \Lambda^{1/2}\mathscr{H} = l_2$. Since the embedding map $\iota : \mathscr{C}([0,T] : \bar{l}_2) \to \mathscr{C}([0,T] : \mathbb{R}^\infty)$ is measurable (in fact continuous), $\hat{\mathscr{G}}^\varepsilon : \mathscr{Z} \times \mathscr{C}([0,T] : \bar{l}_2) \to \mathscr{E}$ defined by $\hat{\mathscr{G}}^\varepsilon(z,v) \doteq \mathscr{G}^\varepsilon(z,\iota(v))$, $(z,v) \in \mathscr{Z} \times \mathscr{C}([0,T] : \bar{l}_2)$ is a measurable map for every $\varepsilon \geq 0$. Note also that for $\varepsilon > 0$, $X^{\varepsilon,z} = \hat{\mathscr{G}}^\varepsilon(z,\sqrt{\varepsilon}\beta)$ a.s. Since Condition 11.12 holds, we have that parts (a) and (b) of Condition 9.1 are satisfied with $\mathscr{G}^\varepsilon$ there replaced by $\hat{\mathscr{G}}^\varepsilon$ for $\varepsilon \geq 0$ (note that there is no Poisson noise here) and with W replaced with β. Define $\hat{I}_z(\phi)$ by the right side of (9.4) but with $\mathscr{G}^0$ replaced by $\hat{\mathscr{G}}^0$, $S^{\hat{\mathscr{G}}}_{z,\phi} \doteq \left\{f \in \mathscr{L}^2([0,T] : \mathscr{H}_0) : \phi = \mathscr{G}^0(z, \int_0^\cdot f(s)ds)\right\}$, and $\bar{L}_T(q)$ replaced by $\frac{1}{2}\int_0^T \|f(s)\|_0^2\, ds$, so that

$$\hat{I}_z(\phi) = \inf_{f\in S^{\mathscr{G}}_{z,\phi}} \left[\frac{1}{2}\int_0^T \|f(s)\|_0^2\, ds\right].$$

Clearly $I_z(\phi) = \hat{I}_z(\phi)$ for all $(z,\phi) \in \mathscr{Z} \times \mathscr{E}$. The result is now an immediate consequence of Theorem 9.2. □

Remark 11.14 Since for $t \in (0,T)$, $\sum_{i=1}^\infty (\beta_i(t))^2 = \infty$ a.s., the $\mathbb{R}^\infty$-valued random variable $\beta(t)$ does not lie in the subset l_2 of $\mathbb{R}^\infty$. However, for any sequence $\{\lambda_i\}$ as in Remark 11.2, $\sum_{i=1}^\infty \lambda_i(\beta_i(t))^2 < \infty$ a.s., which shows that the support of $\beta(t)$ does lie in the larger Hilbert space $\bar{l}_2$. In fact, $t \mapsto \beta(t)$ is a.s. a continuous map from $[0,T]$ to $\bar{l}_2$, and it is easily checked that it defines a Λ-Wiener process with sample paths in $\bar{l}_2$. This identification of β with a Hilbert space valued Wiener process allows us to leverage Theorem 9.2 in establishing Theorem 11.13. Note that there are many different possible choices of sequences $\{\lambda_i\}$ (and corresponding Hilbert spaces $\bar{l}_2$) and any of them can be used to prove the theorem, which itself does not involve any specific Hilbert space.

To close this section, we consider the Laplace principle for functionals of a Brownian sheet. Let B be a Brownian sheet as in Definition 11.5. Let $\mathscr{G}^\varepsilon : \mathscr{Z} \times \mathscr{C}([0,T] \times \bar{O} : \mathbb{R}) \to \mathscr{E}$, $\varepsilon > 0$, be a family of measurable maps. Define $X^{\varepsilon,z} \doteq \mathscr{G}^\varepsilon(z,\sqrt{\varepsilon}B)$.

We now provide sufficient conditions for a Laplace principle to hold for the family $\{X^{\varepsilon,z}\}$.

Analogous to the classes defined in (8.1), we introduce for $N \in (0, \infty)$,

$$S_N \doteq \left\{ \phi \in \mathscr{L}^2([0,T] \times O) : \int_{[0,T]\times O} \phi^2(s,r)dsdr \le N \right\},$$

$$\bar{\mathscr{A}}_{b,N} \doteq \{u \in \bar{\mathscr{A}} : u(\omega) \in S_N, P\text{-a.s.}\}. \tag{11.9}$$

Once more, S_N is endowed with the weak topology on $\mathscr{L}^2([0,T] \times O)$, under which it is a compact metric space. For $u \in \mathscr{L}^2([0,T] \times O)$, define $\text{Int}(u) \in \mathscr{C}([0,T] \times O : \mathbb{R})$ by

$$\text{Int}(u)(t,x) \doteq \int_{[0,t]\times(O\cap(-\infty,x])} u(s,y)dsdy, \tag{11.10}$$

where as before, $(-\infty, x] \doteq \{y : y_i \le x_i \text{ for all } i = 1, \ldots, d\}$.

Condition 11.15 *There exists a measurable map $\mathscr{G}^0 : \mathscr{Z} \times \mathscr{C}([0,T] \times O : \mathbb{R}) \to \mathscr{E}$ such that the following hold.*

(a) *For every $M < \infty$ and compact set $K \subset \mathscr{Z}$, the set*

$$\Gamma_{M,K} \doteq \left\{\mathscr{G}^0(z, \text{Int}(u)) : u \in S_M,\ z \in K\right\}$$

is a compact subset of $\mathscr{E}$, where $\text{Int}(u)$ *is as in* (11.10).

(b) *Consider $M < \infty$ and families $\{u^\varepsilon\} \subset \bar{\mathscr{A}}_{b,M}$ and $\{z^\varepsilon\} \subset \mathscr{Z}$ such that u^ε converges in distribution (as S_M-valued random elements) to u and $z^\varepsilon \to z$ as $\varepsilon \to 0$. Then*

$$\mathscr{G}^\varepsilon\left(z^\varepsilon, \sqrt{\varepsilon}B + \text{Int}(u^\varepsilon)\right) \to \mathscr{G}^0\left(z, \text{Int}(u)\right)$$

in distribution as $\varepsilon \to 0$.

For $f \in \mathscr{E}$ and $z \in \mathscr{Z}$, define

$$I_z(f) = \inf_{\left\{u\in\mathscr{L}^2([0,T]\times O): f=\mathscr{G}^0(z,\text{Int}(u))\right\}} \left[\frac{1}{2}\int_{[0,T]\times O} u^2(s,r)drds\right]. \tag{11.11}$$

Theorem 11.16 *Let $\mathscr{G}^0 : \mathscr{Z} \times \mathscr{C}([0,T] \times O : \mathbb{R}) \to \mathscr{E}$ be a measurable map satisfying Condition 11.15. Suppose that for all $f \in \mathscr{E}$, $z \mapsto I_z(f)$ is a lower semicontinuous map from $\mathscr{Z}$ to $[0,\infty]$. Then for every $z \in \mathscr{Z}$, $I_z : \mathscr{E} \to [0,\infty]$, defined by (11.11), is a rate function on $\mathscr{E}$, and the family $\{I_z, z \in \mathscr{Z}\}$ of rate functions has compact level sets on compacts. Furthermore, the family $\{X^{z,\varepsilon}\}$ satisfies the Laplace principle on $\mathscr{E}$ with rate function I_z, uniformly for z in compact subsets of $\mathscr{Z}$.*

Proof Let $\{\phi_i\}_{i=1}^{\infty}$ be a CONS in $\mathscr{L}^2(O)$ and let

$$\beta_i(t) \doteq \int_{[0,t]\times O} \phi_i(x) B(ds \times dx), \quad t \in [0, T], \ i \in \mathbb{N}.$$

Then $\beta = \{\beta_i\}$ is a sequence of independent standard real Brownian motions, and it can be regarded as a $\mathscr{C}([0, T] : \mathbb{R}^{\infty})$-valued random variable. Furthermore, (11.3) is satisfied, and from Proposition 11.9, there is a measurable map $g : \mathscr{C}([0, T] : \mathbb{R}^{\infty}) \to \mathscr{C}([0, T] \times O : \mathbb{R})$ such that $g(\beta) = B$ a.s. For $\varepsilon > 0$, define $\hat{\mathscr{G}}^{\varepsilon} : \mathscr{Z} \times \mathscr{C}([0, T] : \mathbb{R}^{\infty}) \to \mathscr{E}$ by $\hat{\mathscr{G}}^{\varepsilon}(z, \sqrt{\varepsilon} v) \doteq \mathscr{G}^{\varepsilon}(z, \sqrt{\varepsilon} g(v))$, $(z, v) \in \mathscr{Z} \times \mathscr{C}([0, T] : \mathbb{R}^{\infty})$. Clearly, $\hat{\mathscr{G}}^{\varepsilon}$ is a measurable map and $\hat{\mathscr{G}}^{\varepsilon}(z, \sqrt{\varepsilon}\beta) = X^{\varepsilon,z}$ a.s. Next, note that

$$\left\{ v \in \mathscr{C}([0, T] : \mathbb{R}^{\infty}) : v(\cdot) = \int_0^{\cdot} \hat{u}(s) ds, \text{ for some } \hat{u} \in \mathscr{L}^2([0, T] : l_2) \right\}$$

is a measurable subset of $\mathscr{C}([0, T] : \mathbb{R}^{\infty})$. For $\hat{u} \in \mathscr{L}^2([0, T] : l_2)$, define $u_{\hat{u}} \in \mathscr{L}^2([0, T] \times O)$ by

$$u_{\hat{u}}(t, x) \doteq \sum_{i=1}^{\infty} \hat{u}_i(t) \phi_i(x), \ (t, x) \in [0, T] \times O.$$

Define $\hat{\mathscr{G}}^0 : \mathscr{Z} \times \mathscr{C}([0, T] : \mathbb{R}^{\infty}) \to \mathscr{E}$ by

$$\hat{\mathscr{G}}^0(z, v) \doteq \mathscr{G}^0(z, \mathrm{Int}(u_{\hat{u}})) \quad \text{if } v = \int_0^{\cdot} \hat{u}(s) ds \text{ and } \hat{u} \in \mathscr{L}^2([0, T] : l_2),$$

and set $\hat{\mathscr{G}}^0(z, v) \doteq 0$ for all other (z, v). Note that

$$\left\{ \hat{\mathscr{G}}^0 \left(z, \int_0^{\cdot} \hat{u}(s) ds \right) : \hat{u} \in S_M, z \in K \right\} = \left\{ \mathscr{G}^0 \left(z, \mathrm{Int}(u) \right) : u \in S_M, z \in K \right\},$$

where S_M on the left side is the one introduced above Condition 11.12, and S_M on the right side is the one introduced above (11.9). Since Condition 11.15 holds, we have that part (a) of Condition 11.12 holds with $\mathscr{G}^0$ there replaced by $\hat{\mathscr{G}}^0$. Next, an application of Girsanov's theorem (see the proof of Theorem 11.11) gives that for every $\hat{u}^{\varepsilon} \in \bar{\mathscr{A}}_{b,M}$ (where the latter class is as in Condition 11.12),

$$g\left(\beta + \frac{1}{\sqrt{\varepsilon}} \int_0^{\cdot} \hat{u}^{\varepsilon}(s) ds \right) = B + \frac{1}{\sqrt{\varepsilon}} \mathrm{Int}(u_{\hat{u}^{\varepsilon}}),$$

a.s. In particular, for every $M < \infty$ and families $\{\hat{u}^{\varepsilon}\} \subset \bar{\mathscr{A}}_{b,M}$ and $\{z^{\varepsilon}\} \subset \mathscr{Z}$ such that $\hat{u}^{\varepsilon}$ converges in distribution (as S_M-valued random elements) to $\hat{u}$ and $z^{\varepsilon} \to z$, we have

$$\begin{aligned}\lim_{\varepsilon\to 0}\hat{\mathscr{G}}^\varepsilon\left(z^\varepsilon, \sqrt{\varepsilon}\beta + \int_0^\cdot \hat{u}^\varepsilon(s)ds\right) &= \lim_{\varepsilon\to 0}\mathscr{G}^\varepsilon\left(z^\varepsilon, \sqrt{\varepsilon}B + \mathrm{Int}(u_{\hat{u}^\varepsilon})\right)\\ &= \mathscr{G}^0\left(z, \mathrm{Int}(u_{\hat{u}})\right)\\ &= \hat{\mathscr{G}}^0\left(z, \int_0^\cdot \hat{u}(s)ds\right).\end{aligned}$$

Thus w.p.1, part (b) of Condition 11.12 is satisfied with $\mathscr{G}^\varepsilon$ replaced by $\hat{\mathscr{G}}^\varepsilon$, $\varepsilon \geq 0$. The result now follows on noting that if $\hat{I}_z(f)$ is defined by the right side of (11.8) but with $\mathscr{G}^0$ there replaced by $\hat{\mathscr{G}}^0$, then $\hat{I}_z(f) = I_z(f)$ for all $(z, f) \in \mathscr{Z} \times \mathscr{E}$. □

11.3 Reaction–Diffusion SPDE

In this section we will use results from Sect. 11.2, and in particular Theorem 11.16, to study the small noise large deviation principle for a class of SPDE that was considered in [174]. The class includes, as a special case, the reaction–diffusion SPDEs considered in [235] (see Remark 11.22). The main result of the section is Theorem 11.21, which establishes the uniform Laplace principle for such SPDE.

As the discussion at the beginning of this Part III of the book indicates, this is but one of many possible applications of the abstract LDP (Theorem 9.2), though for this particular application we of course use the version appropriate for a Brownian sheet (Theorem 11.16). A main purpose of the presentation is to illustrate the claim that the essential issue in proving an LDP is a good qualitative theory for controlled versions of the original system under a law of large numbers scaling. Since we do not wish to prove this qualitative theory again, in this section we extensively apply results proved elsewhere, and in that sense, this section is not self-contained. This situation illustrates the fact that in any particular application of Theorem 9.2, one needs a thorough understanding of the qualitative properties of the infinite dimensional system under consideration.

11.3.1 The Large Deviation Theorem

Let $(\Omega, \mathscr{F}, P)$ be a probability space with a filtration $\{\mathscr{F}_t\}_{0\leq t\leq T}$ satisfying the usual conditions. Let $O \subset \mathbb{R}^d$ be a bounded open set and $\{B(t, x) : (t, x) \in \mathbb{R}_+ \times O\}$ a Brownian sheet on this filtered probability space. Consider the SPDE

$$dX(t,r) = [L(t)X(t,r) + R(t, r, X(t,r))]drdt + \sqrt{\varepsilon}A(t, r, X(t,r))\,B(dr \times dt) \tag{11.12}$$

with initial condition $X(0, r) = x(r)$. Here $\{L(t)\}_{0\leq t<\infty}$ is a family of linear, closed, densely defined operators on $\mathscr{C}(O)$ that generates a two-parameter strongly continuous semigroup (see [174, Sect. 1]) $\{U(t,s)\}_{0\le s\le t}$ on $\mathscr{C}(O)$, with kernel function $G(t,s,r,q)$, $0 \le s \le t, r, q \in O$. Thus for $f \in \mathscr{C}(O)$,

$$[U(t,s)f](r) = \int_O G(t,s,r,q)f(q)dq,\ r \in O,\ 0 \le s \le t \le T.$$

Also, A and R are measurable maps from $[0,T] \times O \times \mathbb{R}$ to $\mathbb{R}$ and $\varepsilon \in (0,\infty)$. By a solution of the SPDE (11.12), we mean the following.

Definition 11.17 A random field $X = \{X(t,r) : t \in [0,T], r \in O\}$ is called a mild solution of the stochastic partial differential equation (11.12) with initial condition ξ if $(t,r) \mapsto X(t,r)$ is continuous, $X(t,r)$ is $\{\mathscr{F}_t\}$-measurable for all $t \in [0,T]$ and $r \in O$, and if a.s. for all $t \in [0,T]$,

$$\begin{aligned} X(t,r) = &\int_O G(t,0,r,q)x(q)dq + \int_0^t \int_O G(t,s,r,q)R\left(s,q,X(s,q)\right)dqds \\ &+ \sqrt{\varepsilon}\int_0^t \int_O G(t,s,r,q)A\left(s,q,X(s,q)\right)B(dq\times ds). \end{aligned} \tag{11.13}$$

Implicit in Definition 11.17 is the requirement that the integrals in (11.13) be well defined. We will shortly introduce conditions on G, A, and R that ensure that for a continuous adapted random field X, all the integrals in (11.13) are meaningful. As a convention, we take $G(t,s,r,q)$ to be zero when $0 \le t < s \le T$, $r,q \in O$.

For $u \in \bar{\mathscr{A}}_{b,N}$ [which was defined in (11.9)], the controlled analogue of (11.13) is

$$\begin{aligned} Y(t,r) = &\int_O G(t,0,r,q)x(q)dq + \int_0^t \int_O G(t,s,r,q)R(s,q,Y(s,q))dqds \\ &+ \sqrt{\varepsilon}\int_0^t \int_O G(t,s,r,q)A(s,q,Y(s,q))B(dq\times ds) \\ &+ \int_0^t \int_O G(t,s,r,q)A(s,q,Y(s,q))u(s,q)dqds. \end{aligned} \tag{11.14}$$

The main work in proving an LDP for (11.13) is to prove qualitative properties (existence and uniqueness, tightness properties, and stability under perturbations) for solutions to (11.14). We begin by discussing the known qualitative theory for (11.13).

For $\alpha \in (0,\infty)$, let $\mathbb{B}_\alpha \doteq \{\psi \in \mathscr{C}(O) : \|\psi\|_\alpha < \infty\}$ be the Banach space with norm

$$\|\psi\|_\alpha \doteq \|\psi\|_0 + \sup_{r,q\in O, r\ne q} \frac{|\psi(r)-\psi(q)|}{\|r-q\|^\alpha}, \tag{11.15}$$

where $\|\psi\|_0 \doteq \sup_{r\in O}|\psi(r)|$. The Banach space $\mathbb{B}_\alpha([0,T]\times O)$ is defined as in (11.15) but with O replaced by $[0,T]\times O$, and for notational convenience we denote this space by $\mathbb{B}_\alpha^T$. For $\alpha = 0$, $\mathbb{B}_0^T$ is the space of all continuous maps from $[0,T]\times \bar{O}$ to $\mathbb{R}$ endowed with the sup-norm. The following will be a standing assumption for this section. In the assumption, $\bar{\alpha}$ is a fixed constant, and the large deviation principle will be proved in the topology of $\mathscr{C}([0,T]:\mathbb{B}_\alpha)$, for any fixed $\alpha \in (0,\bar{\alpha})$. Using the contraction principle, this large deviation principle provides large deviation asymptotics for the evaluation $X(t,r)$ for every fixed $(t,r) \in [0,T]\times O$, and for many other functionals as well, e.g., $\sup_{t\in[0,T]} \|X(t,\cdot)\|_\alpha$, $\sup_{(t,r)\in[0,T]\times O} |X(t,r)|$. Recall that $O \subset \mathbb{R}^d$. The following condition is taken from [170].

Condition 11.18 *The following two conditions hold.*

(a) *There exist constants $K(T) < \infty$ and $\gamma \in (d,\infty)$ such that*

(i) *for all $t, s \in [0,T]$, $r \in O$,*

$$\int_O |G(t,s,r,q)|dq \le K(T); \tag{11.16}$$

(ii) *for all $0 \le s < t \le T$ and $r, q \in O$,*

$$|G(t,s,r,q)| \le K(T)(t-s)^{-\frac{d}{\gamma}}; \tag{11.17}$$

(iii) *if $\bar{\alpha} \doteq \frac{\gamma-d}{2\gamma}$, then for all $\alpha \in (0,\bar{\alpha})$ and for all $0 \le s < t_1 \le t_2 \le T,\ r_1, r_2, q \in O$,*

$$\begin{aligned}&|G(t_1,s,r_1,q) - G(t_2,s,r_2,q)| \\ &\quad \le K(T)\left[(t_2-t_1)^{1-\frac{d}{\gamma}}(t_1-s)^{-1} + |r_1-r_2|^{2\alpha}(t_1-s)^{-\frac{d+2\alpha}{\gamma}}\right];\end{aligned} \tag{11.18}$$

(iv) *for all $z, y \in \mathbb{R}$, $r \in O$, and $0 \le t \le T$,*

$$|R(t,r,z) - R(t,r,y)| + |A(t,r,z) - A(t,r,y)| \le K(T)|z-y|$$

and

$$|R(t,r,z)| + |A(t,r,z)| \le K(T)(1+|z|). \tag{11.19}$$

(b) *For all $\alpha \in (0,\bar{\alpha})$ and $\xi \in \mathbb{B}_\alpha$, $\hat{\xi}(t) \doteq \int_O G(t,0,\cdot,q)\xi(q)dq$ belongs to $\mathbb{B}_\alpha$ and $\hat{\xi} \in \mathscr{C}([0,T]:\mathbb{B}_\alpha)$. The map $\xi \mapsto \hat{\xi}$ is a continuous map from $\mathbb{B}_\alpha$ to $\mathscr{C}([0,T]:\mathbb{B}_\alpha)$.*

Remark 11.19 (a) Note that the definition $\bar{\alpha} \doteq (\gamma-d)/2\gamma$ implies $\bar{\alpha} \in (0,1/2)$.

(b) We refer the reader to [169] for examples of families $\{L(t)\}_{t\ge 0}$ that satisfy Condition 11.18.

(c) Using (11.16) and (11.17), it follows that for all $0 \le s < t \le T$ and $r \in O$,

$$\int_O |G(t,s,r,q)|^2 dq \le K^2(T)(t-s)^{-\frac{d}{\gamma}}. \tag{11.20}$$

Since $\gamma > d$, the estimate (11.20) says that

$$\sup_{(r,t)\in O\times[0,T]}\int_{[0,t]\times O}|G(t,s,r,q)|^2dq<\infty, \tag{11.21}$$

which in view of the linear growth assumption in (11.19) ensures that the stochastic integral in (11.13) is well defined.

(d) Lemma 4.1(ii) of [169] shows that under Condition 11.18, for every $\alpha<\bar{\alpha}$ there exists a constant $\tilde{K}(\alpha)$ such that for all $0\le t_1\le t_2\le T$ and all $r_1,r_2\in O$,

$$\int_0^T\int_O|G(t_1,s,r_1,q)-G(t_2,s,r_2,q)|^2dq\,ds\le\tilde{K}(\alpha)\rho\left((t_1,r_1),(t_2,r_2)\right)^{2\alpha},$$

where ρ is the Euclidean distance in $[0,T]\times O\subset\mathbb{R}^{d+1}$. This estimate will be used in the proof of Lemma 11.28.

The following theorem is due to Kotelenez (see Theorems 2.1 and 3.4 in [174]; see also Theorem 3.1 in [169]).

Theorem 11.20 *Assume Condition 11.18 and fix $\alpha\in(0,\bar{\alpha})$. There exists a measurable function*

$$\mathscr{G}^\varepsilon:\mathbb{B}_\alpha\times\mathbb{B}_0^T\to\mathscr{C}([0,T]:\mathbb{B}_\alpha)$$

such that for every filtered probability space $(\Omega,\mathscr{F},P,\{\mathscr{F}_t\})$ with a Brownian sheet B, $X_x^\varepsilon\doteq\mathscr{G}^\varepsilon(x,\sqrt{\varepsilon}B)$ is the unique mild solution of (11.12) (with initial condition x), and it satisfies $\sup_{0\le t\le T}E\|X_x^\varepsilon(t)\|_0^p<\infty$ for all $p\in[0,\infty)$.

For the rest of the section we consider only $\alpha\in(0,\bar{\alpha})$. For $f\in\mathscr{C}([0,T]:\mathbb{B}_\alpha)$, define

$$I_x(f)\doteq\inf_u\int_{[0,T]\times O}u^2(s,q)dsdq, \tag{11.22}$$

where the infimum is taken over all $u\in\mathscr{L}^2([0,T]\times O)$ such that

$$\begin{aligned}f(t,r)=&\int_O G(t,0,r,q)x(q)dq+\int_{[0,t]\times O}G(t,s,r,q)R(s,q,f(s,q))dsdq\\&+\int_{[0,t]\times O}G(t,s,r,q)A(s,q,f(s,q))u(s,q)dsdq.\end{aligned} \tag{11.23}$$

The following is the main result of this section, which is a uniform Laplace principle for $\{X_x^\varepsilon\}$. The definition of a uniform Laplace principle was given in Chap. 1. There the dependence on the parameter over which uniformity is considered was noted in the expectation operator. In this chapter, however, it will be more convenient to work with a common probability measure (instead of a collection parametrized by $x\in\mathbb{B}_\alpha$) and instead note the dependence on x in the collection of random variables, i.e., we write X_x^ε to note this dependence.

Theorem 11.21 *Assume Condition 11.18, let $\alpha\in(0,\bar{\alpha})$, and let X_x^ε be as in Theorem 11.20. Then I_x defined by (11.22) is a rate function on $\mathscr{C}([0,T]:\mathbb{B}_\alpha)$, and the* *family* $\{I_x, x \in \mathbb{B}_\alpha\}$ *of rate functions has compact level sets on compacts. Furthermore,* $\{X^\varepsilon_x\}$ *satisfies the Laplace principle on* $\mathscr{C}([0,T]:\mathbb{B}_\alpha)$ *with the rate function* I_x*, uniformly for* x *in compact subsets of* $\mathbb{B}_\alpha$*.*

Remark 11.22 (a) If part (b) of Condition 11.18 is weakened to merely the requirement that for every $\xi \in \mathbb{B}_\alpha$, $t \mapsto \int_O G(t,0,\cdot,q)\xi(q)dq$ be in $\mathscr{C}([0,T]:\mathbb{B}_\alpha)$, then the proof of Theorem 11.21 shows that for all $x \in \mathbb{B}_\alpha$, the large deviation principle for $\{X^\varepsilon_x\}$ on $\mathscr{C}([0,T]:\mathbb{B}_\alpha)$ holds (but not necessarily uniformly).

(b) The small noise LDP for a class of reaction–diffusion SPDEs, with $O=[0,1]$ and a bounded diffusion coefficient, has been studied in [235]. A difference in the conditions on the kernel G in [235] is that instead of (11.18), G satisfies the $\mathscr{L}^2$ estimate in Remark 11.19 (c) with $\bar\alpha = 1/4$. One finds that the proof of Lemma 11.28, which is at the heart of the proof of Theorem 11.21, uses only the $\mathscr{L}^2$ estimate rather than the condition (11.18). Using this observation and techniques in the proof of Theorem 11.21, one can extend results of [235] to the case in which the diffusion coefficient, instead of being bounded, satisfies the linear growth condition (11.19).

Since the proof of Theorem 11.21 relies on properties of the controlled process (11.14), the first step is to prove existence and uniqueness of solutions. This follows from a standard application of Girsanov's theorem. Following the convention used throughout the book, we denote the controlled version of the SPDE by an overbar.

Theorem 11.23 *Let* $\mathscr{G}^\varepsilon$ *be as in Theorem 11.20 and let* $u \in \bar{\mathscr{A}}_{b,N}$ *for some* $N \in \mathbb{N}$*, where* $\bar{\mathscr{A}}_{b,N}$ *is as defined in (11.9). For* $\varepsilon > 0$ *and* $x \in \mathbb{B}_\alpha$*, define*

$$\bar X^\varepsilon_x \doteq \mathscr{G}^\varepsilon\left(x, \sqrt{\varepsilon}B + \mathrm{Int}(u)\right),$$

where Int *is defined in (11.10). Then* $\bar X^\varepsilon_x$ *is the unique solution of (11.14).*

Proof Fix $u \in \bar{\mathscr{A}}_{b,N}$. Since

$$E\left[\exp\left\{-\frac{1}{\sqrt{\varepsilon}}\int_{[0,T]\times O} u(s,q)B(ds\times dq) - \frac{1}{2\varepsilon}\int_{[0,T]\times O} u^2(s,q)dsdq\right\}\right] = 1,$$

the measure $\gamma^{u,\varepsilon}$ defined by

$$d\gamma^{u,\varepsilon} = \exp\left\{-\frac{1}{\sqrt{\varepsilon}}\int_{[0,T]\times O} u(s,q)B(ds\times dq) - \frac{1}{2\varepsilon}\int_{[0,T]\times O} u^2(s,q)dsdq\right\} dP$$

is a probability measure on $(\Omega,\mathscr{F},P)$. Furthermore, $\gamma^{u,\varepsilon}$ is mutually absolutely continuous with respect to P, and by Girsanov's theorem (Theorem D.2), the process $B^{u/\sqrt{\varepsilon}} \doteq B + \varepsilon^{-1/2}\mathrm{Int}(u)$ on $(\Omega,\mathscr{F},\gamma^{u,\varepsilon},\{\mathscr{F}_t\})$ is a Brownian sheet. Thus by Theorem 11.20, $\bar X^\varepsilon_x = \mathscr{G}^\varepsilon\left(x,\sqrt{\varepsilon}B + \mathrm{Int}(u)\right)$ is the unique solution of (11.13), with B there replaced by $B^{u/\sqrt{\varepsilon}}$, on $(\Omega,\mathscr{F},\gamma^{u,\varepsilon},\{\mathscr{F}_t\})$. However, equation (11.13) with $B^{u/\sqrt{\varepsilon}}$ is precisely the same as equation (11.14), and since $\gamma^{u,\varepsilon}$ and P are mutually absolutely continuous, we get that $\bar X^\varepsilon_x$ is the unique solution of (11.14) on $(\Omega,\mathscr{F},P,\{\mathscr{F}_t\})$. This completes the proof. □

We next state two basic qualitative results regarding the processes $\bar{X}^{\varepsilon}_{x}$ that hold under Condition 11.18. The first is simply the controlled zero-noise version of the theorem just stated. Its proof follows from a simpler version of the arguments used in [174] to establish Theorem 11.20, and thus is omitted. The second is a standard convergence result, whose proof is given in Sect. 11.3.2.

Theorem 11.24 *Assume Condition 11.18, let $\alpha \in (0, \bar{\alpha})$, and fix $x \in \mathbb{B}_{\alpha}$ and $u \in \mathscr{L}^2([0, T] \times O)$. Then there is a unique function f in $\mathscr{C}([0, T] : \mathbb{B}_{\alpha})$ that satisfies equation (11.23).*

In analogy with the notation $\bar{X}^{\varepsilon}_{x}$ for the solution of (11.14), we denote the unique solution f given by Theorem 11.24 by $\bar{X}^{0}_{x}$.

Theorem 11.25 *Assume Condition 11.18 and let $\alpha \in (0, \bar{\alpha})$. Let $M < \infty$, and suppose that $x^{\varepsilon} \to x$ and $u^{\varepsilon} \to u$ in distribution as $\varepsilon \to 0$ with $\{u^{\varepsilon}\} \subset \bar{\mathscr{A}}_{b,M}$. Let $\bar{X}^{\varepsilon}_{x^{\varepsilon}}$ solve (11.14) with $u = u^{\varepsilon}$, and let $\bar{X}_{x}$ solve (11.14). Then $\bar{X}^{\varepsilon}_{x^{\varepsilon}} \to \bar{X}_{x}$ in distribution.*

Remark 11.26 As noted several times already in this book, the same analysis as that used to establish the large deviation bounds (and in particular the large deviation upper bound) typically yields compactness of level sets for the associated rate function. In the present setting, we note that the same argument used to prove Theorem 11.25 but with ε set to zero shows the following (under Condition 11.18). Suppose that $x_n \to x$ and $u_n \to u$ with $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{B}_{\alpha}$ and $\{u_n\}_{n\in\mathbb{N}} \subset S_M$, and that f_n solves (11.14) when (x, u) is replaced by (x_n, u_n). Then $f_n \to f$.

Proof (*of Theorem* 11.21) Define the map $\mathscr{G}^0 : \mathbb{B}_{\alpha} \times \mathbb{B}^{T}_{0} \to \mathscr{C}([0, T] : \mathbb{B}_{\alpha})$ as follows. If $x \in \mathbb{B}_{\alpha}$ and $\phi \in \mathbb{B}^{T}_{0}$ is of the form $\phi(t, x) \doteq \mathrm{Int}(u)(t, x)$ for some $u \in \mathscr{L}^2([0, T] \times O)$, we define $\mathscr{G}^0(x, \phi)$ to be the solution f to (11.23). Let $\mathscr{G}^0(x, \phi) = 0$ for all other $\phi \in \mathbb{B}^{T}_{0}$. In view of Theorem 11.16, it suffices to show that $(\mathscr{G}^{\varepsilon}, \mathscr{G}^0)$ satisfy Condition 11.15 with $\mathscr{X}$ and $\mathscr{E}$ there replaced by $\mathbb{B}_{\alpha}$ and $\mathscr{C}([0, T] : \mathbb{B}_{\alpha})$, respectively, and that for all $f \in \mathscr{E}$, the map $x \mapsto I_x(f)$ is lower semicontinuous. The latter property and the first part of Condition 11.15 follow directly from Theorem 11.24 and Remark 11.26. The second part of Condition 11.15 follows from Theorem 11.25. □

Thus all that remains to complete the proof is to verify Theorem 11.25.

11.3.2 Qualitative Properties of Controlled Stochastic Reaction–Diffusion Equations

This section is devoted to the proof of Theorem 11.25. Throughout this section we assume Condition 11.18 and consider any fixed $\alpha \in (0, \bar{\alpha})$, where $\bar{\alpha} \doteq (\gamma - d)/2\gamma$. Whenever a control u appears, the associated controlled SPDE is of the form (11.14), and its solution is denoted by $\bar{X}^{\varepsilon}_{x}$. Our first result shows that $\mathscr{L}^p$ bounds hold for controlled SDEs, uniformly when the initial condition and controls lie in compact sets and $\varepsilon \in [0, 1)$.

Lemma 11.27 *If K is any compact subset of $\mathbb{B}_\alpha$ and $M < \infty$, then for every $p \in [1, \infty)$,*

$$\sup_{u \in \bar{\mathscr{A}}_{b,M}} \sup_{x \in K} \sup_{\varepsilon \in [0,1)} \sup_{(t,r) \in [0,T] \times O} E\|\bar{X}^\varepsilon_x(t,r)\|^p < \infty.$$

Proof By Hölder's inequality, it suffices to establish the claim for all sufficiently large p. Using the standard bound for the pth power of a sum in terms of the pth powers of the summands and Doob's inequality (D.2) for the stochastic integral, there exists $c_1 \in (0, \infty)$ such that

$$\begin{aligned} E\|\bar{X}^\varepsilon_x(t,r)\|^p \le c_1 & \left\| \int_O G(t,0,r,q)x(q)dq \right\|^p \\ & + c_1 E \left\| \int_0^t \int_O G(t,s,r,q) R\left(s,q,\bar{X}^\varepsilon_x(s,q)\right) dq ds \right\|^p \\ & + c_1 E \left[\int_0^t \int_O |G(t,s,r,q)|^2 \left| A\left(s,q,\bar{X}^\varepsilon_x(s,q)\right)\right|^2 dq ds \right]^{\frac{p}{2}} \\ & + c_1 E \left[\int_0^t \int_O |G(t,s,r,q)| \left| A\left(s,q,\bar{X}^\varepsilon_x(s,q)\right)\right| |u(s,q)| dq ds \right]^p . \end{aligned}$$

Using (11.19) and the Cauchy-Schwarz inequality, the entire sum on the right-hand side above can be bounded by

$$c_2 \left[1 + E\left[\int_0^t \int_O |G(t,s,r,q)|^2 \|\bar{X}^\varepsilon_x(s,q)\|^2 dq\, ds \right]^{\frac{p}{2}} \right].$$

If $p > 2$, then Hölder's inequality yields

$$\Lambda_p(t) \le c_2 \left[1 + \left(\int_0^t \int_O |G(t,s,r,q)|^{2\tilde{p}} dq\, ds \right)^{\frac{p-2}{2}} \int_0^t \Lambda_p(s) ds \right],$$

where

$$\Lambda_p(t) \doteq \sup_{u \in \bar{\mathscr{A}}_{b,M}} \sup_{x \in K} \sup_{\varepsilon \in [0,1)} \sup_{r \in O} E\|\bar{X}^\varepsilon_x(t,r)\|^p$$

and $\tilde{p} \doteq p/(p-2)$. Recall that $\bar{\alpha} < 1/2$ (see Remark 11.19). Using (11.16) and (11.17), we obtain

$$\int_O |G(t,s,r,q)|^{2\tilde{p}} dq \le (K(T))^{2\tilde{p}} (t-s)^{-\frac{d}{\gamma}(2\tilde{p}-1)}.$$

Suppose p_0 is large enough that $(\frac{2p_0}{p_0-2} - 1)(1-2\bar{\alpha}) < 1$. Noting that $\bar{\alpha} \doteq (\gamma - d)/2\gamma$ implies $d/\gamma = 1 - 2\bar{\alpha}$, we have that for all $p \ge p_0$ and $t \in [0, T]$,

$$\left[\int_0^t \int_O |G(t,s,r,q)|^{2\tilde{p}} dq ds\right]^{\frac{p-2}{2}} \leq c_3 T^{[1-(2\tilde{p}-1)(1-2\bar{\alpha})]\frac{p-2}{2}}.$$

Thus for every $p \geq p_0$ there exists a constant c_4 such that

$$\Lambda_p(t) \leq c_4 \left[1 + \int_0^t \Lambda_p(s) ds\right].$$

The result now follows from Gronwall's lemma. □

The following lemma will be instrumental in proving tightness and weak convergence in Banach spaces such as $\mathbb{B}_\alpha$ and $\mathbb{B}_\alpha^T$. Recall that ρ denotes the Euclidean distance in $[0,T] \times O \subset \mathbb{R}^{d+1}$.

Lemma 11.28 *Let $\mathscr{V}$ be a collection of $\mathbb{R}^d$-valued predictable processes such that for all $p \in [2,\infty)$,*

$$\sup_{f\in\mathscr{V}} \sup_{(t,r)\in[0,T]\times O} \mathbb{E}\|f(t,r)\|^p < \infty. \tag{11.24}$$

Also, let $\mathscr{U} \subset \bar{\mathscr{A}}_{b,M}$ for some $M < \infty$. For $f \in \mathscr{V}$ and $u \in \mathscr{U}$, define

$$\Psi_1(t,r) \doteq \int_0^t \int_O G(t,s,r,q) f(s,q) B(dq \times ds),$$
$$\Psi_2(t,r) \doteq \int_0^t \int_O G(t,s,r,q) f(s,q) u(s,q) dq ds,$$

where the dependence on f and u is not made explicit in the notation. Then for all $\alpha < \bar{\alpha}$ and $i = 1, 2$,

$$\sup_{f\in\mathscr{V}, u\in\mathscr{U}} E\left[\sup_{\rho((t,r),(s,q))<1} \frac{\|\Psi_i(t,r) - \Psi_i(s,q)\|}{\rho((t,r),(s,q))^\alpha}\right] < \infty.$$

Proof We will prove the result for $i = 1$; the proof for $i = 2$ is identical (except for an additional application of the Cauchy-Schwarz inequality), and thus omitted. Henceforth we write, for simplicity, Ψ_1 as Ψ. We apply Theorem 6 of [157], according to which it suffices to show that there are $p \in (2,\infty)$, $c_p \in (0,\infty)$, and a function $\hat{\omega} : [0,\infty) \to [0,\infty)$ satisfying

$$\int_0^1 \frac{\hat{\omega}(u)}{u^{1+\alpha+(d+1)/p}} du < \infty \tag{11.25}$$

such that for all $0 \leq t_1 < t_2 \leq T$ and $r_1, r_2 \in O$, one has

$$\sup_{f\in\mathscr{V}, u\in\mathscr{U}} E\|\Psi(t_2,r_2) - \Psi(t_1,r_1)\|^p \leq c_p \left(\hat{\omega}(\rho((t_1,r_1),(t_2,r_2)))\right)^p. \tag{11.26}$$

We will show that (11.26) holds with $\hat{\omega}(u) = u^{\alpha_0}$ for some $\alpha_0 \in (\alpha, \bar{\alpha})$ and all p sufficiently large. With such choices (and p large enough), the integrand in (11.25) will be of the form u^{β} with $\beta \in (-1, 0)$. This will establish the result.

Fix α_1 such that $\alpha < \alpha_1 < \bar{\alpha}$ and let $t_1 < t_2, r_1, r_2 \in O$ and $p > 2$. We will need p to be sufficiently large, and the choice of p will be fixed in the course of the proof. By the Burkholder–Davis–Gundy inequality [Appendix D, (D.3)], there exists a constant c_1 such that

$$
\begin{aligned}
&E\,\|\Psi(t_2, r_2) - \Psi(t_1, r_1)\|^p \\
&\quad \le c_1 E\left[\int_0^T \int_O |G(t_2, s, r_2, q) - G(t_1, s, r_1, q)|^2 \, \|f(s, q)\|^2 dq\, ds\right]^{\frac{p}{2}}. \qquad (11.27)
\end{aligned}
$$

Let $\tilde{p} = p/(p-2)$ and $\delta = 4/p$. Note that $(2-\delta)\tilde{p} = \delta p/2 = 2$. Hölder's inequality (with parameters $p/(p-2)$ and $p/2$) and (11.24) give that the right-hand side of (11.27) is bounded above by

$$
\begin{aligned}
&c_1 \left[\int_0^T \int_O |G(t_2, s, r_2, q) - G(t_1, s, r_1, q)|^{(2-\delta)\tilde{p}}\, dq\, ds\right]^{\frac{p-2}{2}} \\
&\qquad \times \left[\int_0^T \int_O |G(t_2, s, r_2, q) - G(t_1, s, r_1, q)|^{\delta p/2}\, E\|f(s, q)\|^p dq\, ds\right] \\
&\qquad \le c_2 \left[\int_0^T \int_O |G(t_2, s, r_2, q) - G(t_1, s, r_1, q)|^2\, dq\, ds\right]^{\frac{p}{2}} \qquad (11.28)
\end{aligned}
$$

for a suitable constant c_2 that is independent of f. From part (d) of Remark 11.19, the expression in (11.28) can be bounded (for p large enough) by

$$
c_3 \rho\left((t_1, r_1), (t_2, r_2)\right)^{\alpha_1 p}.
$$

The result follows. ☐

The next lemma will be used to prove that the stochastic integral appearing in $\bar{X}^{\varepsilon}_x$ converges to 0 in $\mathscr{C}([0, T] \times O)$, a result that will be strengthened shortly.

Lemma 11.29 *Let $\mathscr{V}$ and Ψ_1 be as in Lemma 11.28, and for $f \in \mathscr{V}$, let $Z^{\varepsilon}_f \doteq \sqrt{\varepsilon}\, \Psi_1$. Then for every sequence $\{f_\varepsilon\} \subset \mathscr{V}$, $Z^{\varepsilon}_{f_\varepsilon} \to 0$ in $\mathscr{C}([0, T] \times O)$ and in probability as $\varepsilon \to 0$.*

Proof Note that for $t \in [0, T]$ and $r \in O$,

$$
\begin{aligned}
\sup_{f \in \mathscr{V}} E|\Psi_1(t, r)|^2 &= \sup_{f \in \mathscr{V}} \int_0^t \int_O |G(t, s, r, q)|^2 E\,\|f(s, q)\|^2\, dq ds \\
&\le c_1 \int_0^t \int_O |G(t, s, r, q)|^2 dq ds \\
&< \infty,
\end{aligned}
$$

where the last inequality is from (11.21). This shows that for such (t, r), $Z^{\varepsilon}_{f_\varepsilon}(t, r) \to 0$ in $\mathscr{L}^2$ and hence in probability. For $\delta \in (0, 1)$ and $x \in \mathscr{C}([0, T] \times O)$, define

$$\omega(x, \delta) \doteq \sup \left[\|x(t, r) - x(t', r')\| : \rho\left((t, r), (t', r')\right) \leq \delta \right].$$

Then $\omega(Z^{\varepsilon}_{f_\varepsilon}, \delta) \leq \sqrt{\varepsilon}\delta^{\alpha} M^{\varepsilon}_{f_\varepsilon}$, where

$$M^{\varepsilon}_{f_\varepsilon} \doteq \sup_{0<\rho((t,r),(s,q))<1} \frac{\|\Psi_1(t, r) - \Psi_1(s, q)\|}{\rho\left((t, r), (s, q)\right)^{\alpha}}.$$

Since $\alpha < \bar{\alpha}$, it follows by Lemma 11.28 that

$$\lim_{\delta \to 0} \sup_{\varepsilon \in (0,1)} E\omega(Z^{\varepsilon}_{f_\varepsilon}, \delta) = 0.$$

This establishes a form of uniform equicontinuity, and the result now follows from Theorem 14.5 of [167]. □

We now establish the main convergence result.

Proof (*of Theorem* 11.25) Consider sequences $\{x^{\varepsilon}\}$ and $\{u^{\varepsilon}\}$ as in the statement of Theorem 11.25. Letting $\bar{X}^{\varepsilon}_{x^\varepsilon}$ denote the corresponding controlled process, define

$$Z^{\varepsilon}_1(t, r) \doteq \int_O G(t, 0, r, q) x^{\varepsilon}(q) dq,$$

$$Z^{\varepsilon}_2(t, r) \doteq \int_0^t \int_O G(t, s, r, q) R(s, q, \bar{X}^{\varepsilon}_{x^\varepsilon}(s, q)) dq ds,$$

$$Z^{\varepsilon}_3(t, r) \doteq \sqrt{\varepsilon} \int_0^t \int_O G(t, s, r, q) A(s, q, \bar{X}^{\varepsilon}_{x^\varepsilon}(s, q)) B(dq \times ds),$$

$$Z^{\varepsilon}_4(t, r) \doteq \int_0^t \int_O G(t, s, r, q) A(s, q, \bar{X}^{\varepsilon}_{x^\varepsilon}(s, q)) u^{\varepsilon}(s, q) dq ds.$$

We first show that $\{Z^{\varepsilon}_i\}$ is tight in $\mathscr{C}([0, T] : \mathbb{B}_\alpha)$, for $i = 1, 2, 3, 4$. For $i = 1$, this follows from part (b) of Condition 11.18. Recall that the norm on $\mathbb{B}^T_\alpha$ is

$$\|\psi\|_{\alpha,T} \doteq \|\psi\|_{0,T} + \sup_{s,t \in [0,T], r,q \in O, s \neq t, r \neq q} \frac{|\psi(t, r) - \psi(s, q)|}{\rho\left((t, r), (s, q)\right)^{\alpha}},$$

with $\|\psi\|_{0,T} \doteq \sup_{t \in [0,T], q \in O} |\psi(t, r)|$. Since $\mathbb{B}^T_{\alpha^*}$ is compactly embedded in $\mathbb{B}^T_\alpha$ for $\bar{\alpha} > \alpha^* > \alpha$ (cf. [147, Lemma 6.33]), it suffices to show that for some $\alpha^* \in (\alpha, \bar{\alpha})$,

$$\sup_{\varepsilon \in (0,1)} P\left\{\|Z^{\varepsilon}_i\|_{\alpha^*,T} > K\right\} \to 0 \text{ as } K \to \infty \text{ for } i = 2, 3, 4. \tag{11.29}$$

For $i = 2, 4$, (11.29) is an immediate consequence of

$$\sup_{\varepsilon\in(0,1)} E\|Z_i^\varepsilon\|_{\alpha^*,T} < \infty,$$

which follows from Lemma 11.28, the linear growth condition (11.19), and Lemma 11.27. For $i = 3$, in view of Lemma 11.29, it suffices to establish

$$\sup_{\varepsilon\in(0,1)} E[Z_3^\varepsilon]_{\alpha^*,T} < \infty, \tag{11.30}$$

where for $z \in \mathbb{B}_\alpha^T$, $[z]_{\alpha^*,T} = \|z\|_{\alpha^*,T} - \|z\|_{0,T}$. From the linear growth condition (11.19) and Lemma 11.27, it follows that

$$\sup_{\varepsilon\in[0,1)} \sup_{(t,r)\in[0,T]\times O} E|A(t,r,\bar{X}^\varepsilon_{x^\varepsilon}(t,r))|^p < \infty.$$

The bound in (11.30) now follows on using Lemma 11.28, with $d = 1$ and

$$\mathscr{V} \doteq \{(t,r) \mapsto A(t,r,\bar{X}^\varepsilon_{x^\varepsilon}(t,r)),\ \varepsilon \in (0,1)\}.$$

Having shown tightness of Z_i^ε for $i = 1, 2, 3, 4$, we can extract a subsequence along which each of these processes and also $\bar{X}^\varepsilon_{x^\varepsilon}$ jointly converge in distribution, with $\bar{X}^\varepsilon_{x^\varepsilon}$ taking values in $\mathscr{C}([0,T] : \mathbb{B}_\alpha)$. Let Z_i and $\bar{X}_x$ denote the respective limits. We will show that

$$\begin{aligned}
Z_1(t,r) &= \int_O G(t,0,r,q)x(q)dq,\\
Z_2(t,r) &= \int_0^t \int_O G(t,s,r,q)R(s,q,\bar{X}_x(s,q))dqds,\\
Z_3(t,r) &= 0,\\
Z_4(t,r) &= \int_0^t \int_O G(t,s,r,q)A(s,q,\bar{X}_x(s,q))u(s,q)dqds.
\end{aligned} \tag{11.31}$$

The uniqueness result Theorem 11.24 will then complete the proof.

Convergence for $i = 1$ follows from part (b) of Condition 11.18. The case $i = 3$ follows from Lemmas 11.29, 11.27, and the linear growth condition. To deal with the cases $i = 2, 4$, we invoke the Skorohod representation theorem, which allows us to assume with probability one convergence for the purposes of identifying the limits. We give the proof of convergence only for the harder case $i = 4$. Denote the right side of (11.31) by $\hat{Z}_4(t,r)$. We have the bound

$$\begin{aligned}
&\left|Z_4^\varepsilon(t,r) - \hat{Z}_4(t,r)\right|\\
&\quad\le \int_0^t \int_O |G(t,s,r,q)| \left|A(s,q,\bar{X}^\varepsilon_{x^\varepsilon}(s,q)) - A(s,q,\bar{X}_x(s,q))\right| |u^\varepsilon(s,q)|\, dqds
\end{aligned}$$

$$+\left|\int_0^t \int_O G(t,s,r,q)A(s,q,\bar{X}_x(s,q))\,(u^\varepsilon(s,q)-u(s,q))\,dqds\right|. \quad (11.32)$$

Using the Cauchy-Schwarz inequality and the uniform Lipschitz property of A, for a suitable constant $c \in (0,\infty)$ the first term on the right side of (11.32) can be bounded above by

$$\sqrt{M}\left[\int_0^t \int_O |G(t,s,r,q)|^2 \left|A(s,q,\bar{X}^\varepsilon_{x^\varepsilon}(s,q)) - A(s,q,\bar{X}_x(s,q))\right|^2 dqds\right]^{1/2}$$
$$\le c\left(\sup_{(s,q)\in[0,T]\times O} \left\|\bar{X}^\varepsilon_{x^\varepsilon}(s,q) - \bar{X}_x(s,q)\right\|\right),$$

and thus it converges to 0 as $\varepsilon \to 0$. The second term in (11.32) converges to 0 as well, since $u^\varepsilon \to u$ as elements of $\bar{\mathscr{A}}_{b,M}$ and by (11.20) and the linear growth assumption (11.19),

$$\int_0^t \int_O |G(t,s,r,q)|^2 \left|A(s,q,\bar{X}_x(s,q))\right|^2 dqds < \infty.$$

By uniqueness of limits and noting that $\hat{Z}_4$ is a continuous random field, we see that $Z_4 = \hat{Z}_4$, and the proof is complete. □

11.4 Notes

Some general references for stochastic partial differential equations are [169, 175, 221, 243]. The material of this chapter is largely taken from [43]. The approach taken is different from that of [170, 235] and other early works on large deviations for SPDE [50, 52, 56, 60, 127, 139, 160, 209, 252, 261]. The arguments used in these papers, which build on the ideas of [7], proceed by approximating the original model by a suitable time and/or space discretization. First one establishes an LDP for the approximate system and then argues that an LDP continues to hold as one approaches the original infinite dimensional model. For the last step, one needs suitable exponential probability estimates. These are usually the most technical aspects of the proofs, and they often assume conditions stronger than those needed for the LDP. Examples of various models to which the approach has been applied can be found in the references listed at the beginning of Part III of this book.

An alternative approach, based on nonlinear semigroup theory and infinite dimensional Hamilton–Jacobi–Bellman (HJB) equations, has been developed in [131, 132]. This approach relies on a uniqueness result for the corresponding infinite dimensional nonlinear PDEs. The uniqueness requirement on the limit HJB equation is an extraneous artifact of the approach, and different models seem to require different methods for this, in general very hard, uniqueness problem. In contrast to the weak convergence approach, it requires an analysis of the model that goes significantly beyond the unique solvability of the SPDE.

One of the main reasons for proving a sample path LDP for any given stochastic system is as a step to validating the corresponding Freidlin–Wentzell large-time theory [140]. A key distinction between the cases of finite and infinite dimensional state is that open neighborhoods of points, which have compact closure in the finite dimensional case, are merely bounded in the infinite dimensional case. This means, unfortunately, that one should prove that the large deviation estimates are uniform for initial conditions in bounded sets in the latter case. As was discussed in Chap. 1, it is usually easy to establish a Laplace principle that is uniform with respect to initial conditions in compact sets using a straightforward argument by contradiction, which then gives the corresponding uniform LDP (Proposition 1.14). A different approach is needed for the infinite dimensional problem if one wants an LDP that is uniform over bounded sets, and one way to deal with the issue within the Laplace principle formalism is presented in [227].

The paper [38] studies large deviations for reaction–diffusion SPDE driven by a Poisson noise using representations for Poisson random measures of the form presented in Chap. 8.