Interpolation by Bounded Analytic Functions and Related Questions

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Abstract The paper investigates some interpolation questions related to the Khinchine–Ostrowski theorem, Zalcman's theorem on bounded approximation, and Rubel's problem on bounded analytic functions.

Keywords Bounded analytic functions • Bounded approximation • Fatou's interpolation theorem • G_δ set of measure zero

Mathematics Subject Classification: 30H05, 30H10

1 Introduction

Let $C(K)$ be the set of all continuous complex valued functions on a given subset K of $\mathbb{C}$. Let $\mathbb{D}$ and $\mathbb{T}$ be the open unit disk and the unit circle, respectively. As usual, H^∞ is the space of all bounded analytic functions on $\mathbb{D}$, and the familiar disc algebra A is the set of all elements of H^∞ that can be continuously extended to the closed unit disc.

The formulation and the proof of the following theorem of Khinchine and Ostrowski (in an even more general version) can be found in Privalov's book [9, p. 118].

Theorem A (Khinchine–Ostrowski) *Let $\{f_k\}$ be a sequence of functions analytic on $\mathbb{D}$ which satisfy the following conditions:*

(a) *there exists $M > 0$ such that $|f_k(z)| \leq M$ on $\mathbb{D}$, $k = 1, 2, \ldots$, and*
(b) *the sequence $\left\{f_k\left(e^{i\theta}\right)\right\}$ of radial boundary values of $f_k(z)$ converges at each point of some subset $E \subset \mathbb{T}$ of positive measure.*

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J. Mashreghi et al. (eds.), *New Trends in Approximation Theory*, Fields Institute Communications 81, https://doi.org/10.1007/978-1-4939-7543-3_11

Then $\{f_k\}$ *converges uniformly on compact subsets of* $\mathbb{D}$ *to a bounded analytic function* f, *and* $\{f_k(e^{i\theta})\}$ *converges almost everywhere on* E *to the radial boundary values* $f(e^{i\theta})$ *of* f.

The following theorem of Zalcman [10] is related to Theorem A.

Theorem B *Let* Γ *be a proper closed arc on* $\mathbb{T}$. *A function* $f \in C(\Gamma)$ *is uniformly approximable (on* Γ*) by polynomials* P_n *satisfying* $|P_n(z)| \leq M$ *on* $\mathbb{D}$, $n = 1, 2, \ldots,$ *if and only if there exists a function* g *analytic on* $\mathbb{D}$, $|g(z)| \leq M$ *on* $\mathbb{D}$, *such that*

$$f\left(e^{i\theta}\right) = \lim_{r\to 1} g\left(re^{i\theta}\right), \qquad e^{i\theta} \in \Gamma.$$

When the arc Γ is replaced by an arbitrary closed subset F of $\mathbb{T}$, we have (see [3]):

Theorem C *Let* F *be a closed subset of* $\mathbb{T}$. *In order that a function* $f \in C(F)$ *be uniformly approximable on* F *by polynomials* P_n *such that* $|P_n(z)| \leq M$ *on* $\mathbb{D}$, $n = 1, 2, \ldots,$ *it is necessary and sufficient that the following conditions be satisfied:*

(i) $|f(z)| \leq M$, $z \in F$, *and*
(ii) *There exists a function* g *analytic on* $\mathbb{D}$, $|g(z)| \leq M$ *on* $\mathbb{D}$, *such that*

$$f\left(e^{i\theta}\right) = \lim_{r\to 1} g\left(re^{i\theta}\right)$$

for almost all $e^{i\theta} \in F$.

Note that Theorem B is for proper arcs. Even Theorem C does not formally assume that the closed set F is proper, it becomes trivial when $F = \mathbb{T}$ (that is, together with Poisson integral representation formula, it gives the following well known fact: $f \in C(\mathbb{T})$ is uniformly approximable by polynomials if and only if f is in the disc algebra A).

Of course, Theorem A implies the necessity part of Theorem C, but it does not imply the necessity part of Theorem B. Indeed, the necessity part of Theorem B provides the equation

$$f\left(e^{i\theta}\right) = \lim_{r\to 1} g\left(re^{i\theta}\right)$$

everywhere on Γ including its endpoints, while Theorem A does not imply the same equation at the end points of Γ. Thus, the necessity part of Theorem B can be considered as a certain strengthening of the conclusion of Theorem A.

We are interested in particular in closed subsets of $\mathbb{T}$ which "behave" like closed arcs of $\mathbb{T}$. In this direction one can formulate the following open problem.

Problem 1 Describe all closed subsets F on $\mathbb{T}$ such that whenever $f \in C(F)$ is uniformly approximable on F by a sequence of polynomials, uniformly bounded on $\mathbb{T}$, then there exists a function $g \in H^\infty$ with radial limits existing and coinciding with f everywhere on F.

Note that the Rudin–Carleson theorem immediately implies that also the closed subsets of $\mathbb{T}$ of measure zero have the property mentioned in Problem 1.

The following recent (unpublished) theorem of Gardiner [Gardiner, S.J.: Response to a question of A. Danielyan. Private communication.] provides an answer to a question asked by the author.

Theorem 1 *Let $E \subset \mathbb{T}$ be a closed set that has positive lower Lebesgue density at every constituent point. If a function $f \in C(E)$ is uniformly approximable by polynomials P_n satisfying $|P_n(z)| \leq M$ on $\mathbb{D}$, $n = 1, 2, \ldots$, then there exists an analytic function g on $\mathbb{D}$ satisfying $|g(z)| \leq M$ on $\mathbb{D}$ and*

$$f\left(e^{i\theta}\right) = \lim_{r\to 1} g\left(re^{i\theta}\right) \qquad \left(e^{i\theta} \in E\right). \tag{1}$$

If $f \in C(E)$ and there exists an analytic function g on $\mathbb{D}$ satisfying $|g(z)| \leq M$ on $\mathbb{D}$ and

$$f\left(e^{i\theta}\right) = \lim_{r\to 1} g\left(re^{i\theta}\right) \qquad \left(e^{i\theta} \in E\right)$$

then, by Theorem C, the function f is uniformly approximable by polynomials P_n satisfying $|P_n(z)| \leq M$ on $\mathbb{D}$, $n = 1, 2, \ldots$.

Thus we have the following corollary (of Theorem C and Theorem 1) providing a new subclass of the class of sets which Problem 1 requires us to describe.

Corollary 1 *Let $E \subset \mathbb{T}$ be a closed set that has positive lower Lebesgue density at every constituent point. A function $f \in C(E)$ is uniformly approximable by polynomials P_n satisfying $|P_n(z)| \leq M$ on $\mathbb{D}$, $n = 1, 2, \ldots$, if and only if there is an analytic function g on $\mathbb{D}$ satisfying $|g(z)| \leq M$ on $\mathbb{D}$ and*

$$f\left(e^{i\theta}\right) = \lim_{r\to 1} g\left(re^{i\theta}\right) \qquad \left(e^{i\theta} \in E\right).$$

In Sect. 2 below we recall the definition of lower Lebesgue density. Such standard sets as closed (or open) intervals, of course, have positive lower Lebesgue density at every constituent point. As Buczolich [2] has shown there exist also Cantor sets with the same property.

If $f \in C(F)$ is uniformly approximable by a sequence of polynomials, uniformly bounded on $\mathbb{T}$, then by Theorem C (or by Theorem A) there exists a function $g \in H^\infty$ the radial limits of which are equal to f a.e. on F. This brings us to the following formulation:

Problem 2 Describe all closed subsets F on $\mathbb{T}$ such that whenever $f \in C(F)$ coincides a.e. on F with the radial limits of a function $g \in H^\infty$, then the radial limits of g exist on F and coincide with f at each point of F.

A further generalization of Problem 2 is the following problem.

Problem 3 Describe all G_δ subsets F on $\mathbb{T}$ such that whenever $f \in C(F)$ coincides a.e. on F with the radial limits of a function $g \in H^\infty$, then the radial limits of g exist on F and coincide with f at each point of F.

A simpler (still open) problem for a restricted set of functions defined on F is:

Problem 4 Describe all G_δ subsets F on $\mathbb{T}$ such that whenever $f \in C(\mathbb{T})$ coincides a.e. on F with the radial limits of a function $g \in H^\infty$, then the radial limits of g exist on F and coincide with f at each point of F.

It is easy to see that a closed subset of $\mathbb{T}$ of zero measure does not belong to the class of sets which Problem 2 (or, Problem 3 or 4) is requiring to describe as the class of sets mentioned contains "massive" sets only.[1]

However, for sets of measure zero too one can formulate appropriate interpolation problems. The most famous such problems have been solved, of course, by the classical Fatou (Theorem E below) and the Rudin–Carleson interpolation theorems. A further such interpolation problem proposed by Rubel (see [6, p. 168]) has been solved by the following recent result of the author [4].

Theorem D *Let F be a G_δ set of measure zero on $\mathbb{T}$. Then there exists a function $g \in H^\infty$ non-vanishing in $\mathbb{D}$ such that the radial limits of g exist everywhere on $\mathbb{T}$ and vanish precisely on F.*

If F is merely closed, the following result provides a more precise conclusion (see [7, p. 80]).

Theorem E *Let F be closed and of measure zero on $\mathbb{T}$. Then there exists an element in the disc algebra which vanishes precisely on F.*

It is well known that the existences of radial and angular limits of a function $g \in H^\infty$ at a point $t \in \mathbb{T}$ are equivalent. But, of course, the existence of the unrestricted limit at $t \in \mathbb{T}$ is a stronger requirement than the existence of the angular limit at $t \in \mathbb{T}$. (We say that g has an unrestricted limit at $t \in \mathbb{T}$ if the limit of g exists when $z \in \mathbb{D}$ approaches to t arbitrarily in $\mathbb{D}$.)

In the general case the function g in Theorem D cannot belong to the disc algebra. However, G_δ sets are sets of points of continuity, and this brings us to the idea of making the function g continuous on the set F at least. Below we show that this indeed is possible; we have the following new complement of Theorem D.

Theorem 2 *Let F be a G_δ set of measure zero on $\mathbb{T}$. Then there exists a function $g \in H^\infty$ non-vanishing in $\mathbb{D}$ such that:*

1) *g has non-zero radial limits everywhere on $\mathbb{T} \setminus F$; and*
2) *g has vanishing unrestricted limits at each point of F.*

[1] If a closed set E is of positive measure but has a portion of measure zero, then even such a set cannot belong to the class of sets which Problem 2 requires to describe. See Sect. 2 below for the definition of a portion of E.

The condition on F is not only sufficient, but necessary (cf. [4]). Both Theorems D and 2 extend Theorem E from closed sets to G_δ sets. The conclusion (2) makes Theorem 2 a better analogue of Theorem E.

As we noted, the condition on F in Theorem 2 is necessary as well. Thus, Theorem 2 describes all such sets F (on $\mathbb{T}$) for each of which there exists some $f \in H^\infty$ having vanishing unrestricted limits on F and non-zero radial limits on $\mathbb{T} \setminus F$ (the description is that F is a G_δ of measure zero).

Note that an arbitrary G_δ set on $\mathbb{T}$ is precisely the set of unrestricted limits for some $f \in H^\infty$ as Brown et al. [1] have shown (see [1, p. 52]). Their result is:

Theorem F *Let E be a G_δ set on $\mathbb{T}$. Then there exists a function $f \in H^\infty$ which has unrestricted limits at each point of E and at no point of $\mathbb{T} \setminus E$.*

Theorem E is a base for the Rudin–Carleson theorem; cf. [7, pp. 80–81]. (Note that the paper [5] derives the Rudin–Carleson theorem merely from Theorem E.) Similarly Theorems D and 2 bring us to the questions on the possibility of proving the appropriate versions of the Rudin–Carleson theorem for G_δ sets. The corresponding problem can be formulated in two parts as follows.

Problem 5

a) Let F be a G_δ set of measure zero on $\mathbb{T}$ and let either $f \in C(\mathbb{T})$ or, more generally, $f \in C(F)$. Then does there exist a function $g \in H^\infty$ such that the radial limits of g exist everywhere on $\mathbb{T}$ and coincide with f on F?
b) In addition to the requirement of part a), is it possible that g has unrestricted limits at each point of F?

Theorem 1 requires a closed set to be of positive lower Lebesgue density at every constituent point. In an attempt to relax this requirement, one can ask: Does any closed set with no portion[2] of measure zero belong to the class of sets which Problem 1 is asking to describe? The following result gives a negative answer to this question.

Theorem 3 *There exists a closed set $F \subset \mathbb{T}$ having no portion of measure zero and a function $f \in C(F)$ uniformly approximable on F by polynomials which are uniformly bounded on $\mathbb{T}$, such that no function $g \in H^\infty$ has radial limits equal to f at every point of F.*

2 Some Definitions, Auxiliary Results, and Remarks

The terminology used above is known, but we quickly mention some details just in case to avoid any possible confusion. Let $F \subset \mathbb{T}$ be closed; if $J \subset \mathbb{T}$ is an open arc containing a point of F, we call the intersection $F \cap J$ a portion of F. Buczolich [2]

[2] See Sect. 2 for the definition of portion of a closed set.

calls a nowhere dense perfect set with no portion of measure zero a *fat Cantor set*, but we do not use this term.

Let m be the (normalized) Lebesgue measure on $\mathbb{T}$. The lower density of F at $t \in \mathbb{T}$, denoted by $\underline{D}(t, F)$, is defined as

$$\underline{D}(t, F) = \liminf_{h \to 0} \frac{m(I_t(h) \cap F)}{2h},$$

where $I_t(h)$ is an open arc on $\mathbb{T}$ of length $2h$ and of midpoint at t (cf. [2, p. 497]). For a closed F, of course, $\underline{D}(t, F) = 0$ at all $t \in \mathbb{T} \setminus F$.

If a closed set F has a portion of measure zero, then obviously $\underline{D}(t, F) = 0$ at any $t \in F$ of that portion. But there are many closed sets which have positive lower Lebesgue density at every constituent point. In particular, as Buczolich [2, p. 499] has shown, there exist Cantor sets with lower Lebesgue density ≥ 0.5 at every constituent point; for any such set on $\mathbb{T}$ the above Theorem 1 is applicable.

The proof of Theorem 1 uses the standard lemma below (formulated in [**?**]), which follows from the classical theory (cf., e.g., Zygmund's book [11]).

For a Lebesgue integrable function u on $\mathbb{T}$ we denote by H_u the Poisson integral of u.

Lemma 1 *Let* $u : \mathbb{T} \to [-\infty, \infty]$ *be Lebesgue integrable. Then the Poisson integral* H_u *in* $\mathbb{D}$ *satisfies*

$$\liminf_{r \to 1} H_u(re^{i\theta}) \geq \liminf_{t \to 0+} \frac{1}{2t} \int_{[-t,t]} u(e^{i(\theta+\phi)}) d\phi \quad (0 \leq \theta \leq 2\pi).$$

For the convenience of the reader we present the proof of Lemma 1 in the next section.

The following lemma of Kolesnikov [8] is important for Theorem 2 (and Theorem D).

Lemma 2 *Let* G *be an open subset on* $\mathbb{T}$ *and let* $F \subset G$ *be a set of measure zero on* $\mathbb{T}$. *For any* $\epsilon > 0$ *there exists an open set* O, $F \subset O \subset G$, *and a function* $g \in H^\infty$ *such that:*

1) $|g(z)| < 2$, $0 < \Re g(z) < 1$ *for* $z \in \mathbb{D}$*;*
2) *the function* g *has a finite radial limit* $g(\zeta)$ *at each point* $\zeta \in \mathbb{T}$*;*
3) *at the points* $\zeta \in O$ *the function* g *is analytic and* $\Re g(\zeta) = 1$*;*
4) $|g(z)| \leq \epsilon$ *on every radius* R_{ζ_0} *with end-point at* $\zeta_0 \in \mathbb{T} \setminus G$.

3 Proofs

Proof (Lemma 1) The proof follows from Fatou's classical results presented in [11, pp. 99–101].

In this proof we identify the point $e^{i\theta} \in \mathbb{T}$ with $\theta \in [0, 2\pi]$ as usual. Let u be Lebesgue integrable function on $[0, 2\pi]$ (as in Lemma 1) and let U be the indefinite integral of u. Recall that the first symmetric derivative of U at x_0 is

$$D_1 U(x_0) := \lim_{h \to 0+} \frac{U(x_0 + h) - U(x_0 - h)}{2h}.$$

The appropriate upper and lower limits are called the upper and lower first symmetric derivatives, and are denoted by $\overline{D}_1 U(x_0)$ and $\underline{D_1} U(x_0)$, respectively (cf. [11, p. 99]).

The direct calculations imply

$$\underline{D_1} U(\theta) = \liminf_{t \to 0+} \frac{1}{2t} \int_{-t}^{t} u(\theta + \phi) d\phi \qquad (0 \le \theta \le 2\pi).$$

Since the right side of this equation is nothing else but the right side of the inequality in Lemma 1, to prove Lemma 1 one needs to verify that

$$\liminf_{r \to 1} H_u \left(re^{i\theta} \right) \ge \underline{D_1} U(\theta) \qquad (0 \le \theta \le 2\pi). \tag{2}$$

Consider the Fourier series $S[u]$ of the function u. As Zygmund notes immediately after the formulation of Theorem 7.9 in [11, p. 101], we may suppose that $S[u] = S'[U]$, where $S'[U]$ is the formally differentiated Fourier series of U. Thus for the series $S[u]$, part (i) of Theorem 7.9 of [11], can be strengthened by the second part of Theorem 7.2 (from [11, pp. 99–100]). The second part of Theorem 7.2 simply states that the limits of indetermination of Abel summation of $S'[U]$ as $r \to 1$ are contained between $\underline{D_1} U(\theta)$ and $\overline{D_1} U(\theta)$ for all $0 \le \theta \le 2\pi$. Since $S[u] = S'[U]$, the same is true for the limits of indetermination of Abel summation of $S[u]$. Thus

$$\overline{D_1} U(\theta) \ge \limsup_{r \to 1} H_u \left(re^{i\theta} \right) \ge \liminf_{r \to 1} H_u \left(re^{i\theta} \right) \ge \underline{D_1} U(\theta) \qquad (0 \le \theta \le 2\pi),$$

which obviously implies (2). Lemma 1 is proved.

We present the original proof of Theorem 1 from [**?**].

Proof (Theorem 1) Suppose $P_n \to f$ uniformly on E and $|P_n(z)| \le M$ on $\mathbb{D}$ for each n. Then, by subharmonicity,

$$\log |P_n - P_m| \le H_{\log|P_n - P_m|} \le \log^+(2M) + H_{\log|P_n - P_m|\chi_E} \text{ on } \mathbb{D}. \tag{3}$$

Since E has positive measure, $\{P_n\}$ is locally uniformly convergent on $\mathbb{D}$ to some analytic function g. (This follows from (3) using the estimate of $H_{\log|P_n - P_m|\chi_E}$ in terms of the harmonic measure of E; cf. [10, p. 379–380]. Of course, the same conclusion also follows from Theorem A.)

If $e^{i\theta} \in E$, then by hypothesis

$$\alpha_\theta := \liminf_{t\to 0+} \frac{1}{2t} \int_{[-t,t]} \chi_E\left(e^{i(\theta+\phi)}\right) d\phi > 0.$$

By Lemma 1 we can choose $r_\theta \in (0,1)$ such that

$$H_{\chi_E}\left(re^{i\theta}\right) \geq \frac{\alpha_\theta}{2} \qquad (r_\theta \leq r < 1)\,.$$

For large m, n it follows from (3) that

$$\log|P_n - P_m|\left(re^{i\theta}\right) \leq \log^+(2M) + \frac{\alpha_\theta}{2}\max_E \log|P_n - P_m| \qquad (r_\theta \leq r \leq 1)\,.$$

Thus $\{P_n\}$ converges uniformly on $\left\{re^{i\theta} : r_\theta \leq r \leq 1\right\}$, and (1) follows.

Theorem 1 is proved. □

Proof (Theorem 2) This proof is completely parallel to the proof of Theorem D in [4], and only contains an additional argument (observation), which we will indicate here. In [4] the above Lemma 2 has been used for the proof of Theorem D, which yields a function $g \in H^\infty$ with all needed properties except the property of having unrestricted vanishing limits at each point of F (the function g merely has vanishing radial limits at the points of F).

Without repeating the proof of Theorem D, we refer the reader to this proof in [4], where analytic (on $\mathbb{D}$) functions g_k are constructed such that their sum $\sum_{k=1}^{\infty} g_k(z) = h(z)$ is analytic as well.

As indicated in [4], the functions g_k in particular have such properties: $\Re g_k(z) > 0$ for $z \in \mathbb{D}$; at the points $\zeta \in O_k$ the function g_k is analytic and $\Re g_k(\zeta) = 1$, where O_k is an open set on $\mathbb{T}$ containing F $(k = 1, 2, \ldots)$.

We conclude that $\Re h(z) > 0$ for $z \in \mathbb{D}$. Also, since $\Re g_k(\zeta) = 1$ on O_k and $F \subset O_k$, obviously, not only the radial limit, but also the unrestricted limit of $\Re h(z)$ is $+\infty$ at each point of F.

The analytic function $f = 1/(1+h)$ is bounded by 1. Obviously it has vanishing unrestricted limits at each point of F, and as in [4], the function f also has all other necessary properties.

Theorem 2 is proved. □

Proof (Theorem 3) Let D be the "outer snake" domain (also known as the cornucopia) in the w-plane (D is a spiral domain around the unit circle $|w| = 1$). The circle $|w| = 1$ is an impression of a prime end R of the simply connected domain D. Let $w = \varphi(z)$ be the Riemann mapping function of $\mathbb{D}$ onto D. By Carathéodory's theorem, under the mapping $w = \varphi(z)$, the prime end R corresponds to a point A of the unit circle $\mathbb{T}$. Without loss of generality, we may assume A is 1. Then the radial limit of $\varphi(z)$ at the point 1 does not exist, because the image of the radius ending at 1 is a spiral surrounding the circle $|w| = 1$ infinitely many times. On the other hand,

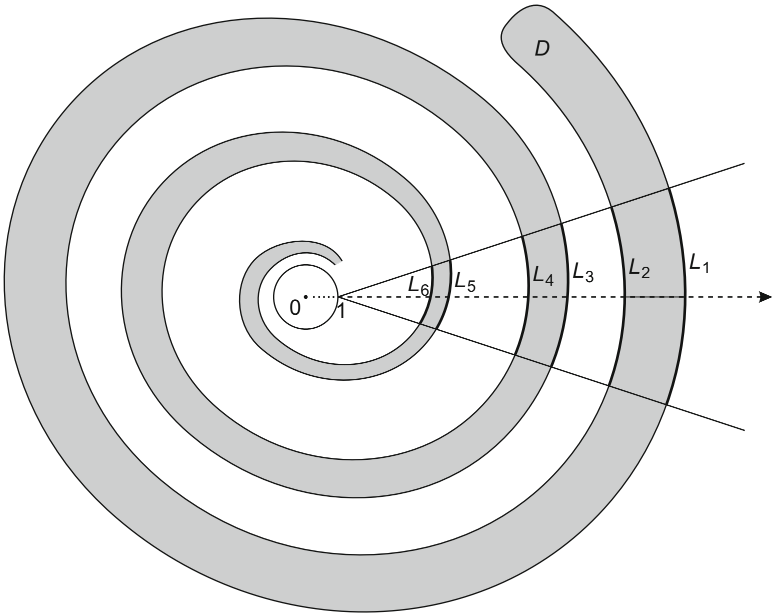

Fig. 1 The "outer snake" domain

at each point of the set $\mathbb{T} \setminus \{1\}$, the function $\varphi(z)$ has a radial limit; it can be defined by its radial limits on the set $\mathbb{T} \setminus \{1\}$, and the extended function is then continuous on $\mathbb{T} \setminus \{1\}$.

The inverse mapping function $z = \varphi^{-1}(w)$ can be extended continuously at each boundary point of D which does not belong to $|w| = 1$. This follows from the fact that each such point is an accessible boundary point of D.

In the w-plane consider an acute angle with vertex at $w = 1$ and such that the half line $[1, \infty)$ is a bisector for the angle; thus the angle lies outside of $|w| = 1$, and only its vertex is on the circle (see Fig. 1). This angle "cuts" countably many Jordan arcs on the boundary of the domain D, which we denote by $L_1, L_2, L_3, \ldots$ in the order from the right to left (so that L_1 is the farthest from $|w| = 1$). We take the set L_n to be closed so that L_n contains its endpoints. Under the mapping $z = \varphi^{-1}(w)$, the image of L_n is a closed arc Γ_n on $\mathbb{T}$. Clearly the arcs Γ_n are disjoint.

Recalling that the prime end R of D corresponds to the point $1 \in \mathbb{T}$, we conclude that the arcs Γ_n on $\mathbb{T}$ accumulate to 1 from either side (for this we also use the fact that the arcs L_n lie on "both" sides of the spiral domain D and they approach the circle $|w| = 1$). Thus, the set $F := \bigcup_{n=1}^{\infty} \Gamma_n \cup \{1\}$ is a closed set (on $\mathbb{T}$) and has no portion of measure zero. Let $f(z) = \varphi(z)$ if $z \in F \setminus \{1\}$ and $f(1) = 1$. Then f is continuous on F. Continuity on the arcs Γ_n is obvious, while continuity at the point 1 follows from the construction of the arcs L_n. Indeed, if a sequence $\{z_k\} \subset F$ approaches 1, then $z_k \in \Gamma_{n_k}$ for certain natural numbers n_k approaching infinity. Thus $f(z_k) \in L_{n_k}$; and because the arcs L_{n_k} approach the point 1 (the vertex of the above described angle), $f(z_k)$ approaches 1 as k tends to ∞.

The function f is equal to the radial limits of $\varphi(z)$ at all points of F except the point 1. Thus, by Theorem C, the function f is uniformly approximable on F by polynomials that are uniformly bounded on $\mathbb{D}$. On the other hand, no bounded analytic function can have radial limits equal to f at all points of F. Indeed, by the boundary uniqueness theorem, any such function must be identical with the function $\varphi(z)$, while as we have already seen, the radial limit of $\varphi(z)$ does not exist at $1 \in F$. This completes the proof of Theorem 3. □

Acknowledgements The author wishes to thank the referee for some mathematical and stylistic corrections, and D. Savchuk for preparation of the electronic file of Fig. 1. He also wishes to thank S. Gardiner, V. Totik, and L. Zalcman for valuable discussions related to the topic of this paper.

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