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New Trends in Approximation Theory

In Memory of André Boivin

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Preface

The international conference entitled “New Trends in Approximation Theory” was held at the Fields Institute, in Toronto, from July 25 to July 29, 2016. The conference (which received financial support from the Fields Institute and CRM) was fondly dedicated to the memory of our unique friend and colleague André Boivin, who gave tireless service in Canada until the very last moment of his life in October 2014. The impact of his warm personality and his fine work on Complex Approximation Theory was reflected by the mathematical excellence and the wide research range of the 37 participants. In total there were 27 talks, delivered by well-established mathematicians and young researchers. In particular, 19 invited lectures were delivered by leading experts of the field, from 8 different countries (USA, France, Canada, Ireland, Greece, Spain, Israel, Germany). Videos and slides of the presentations can be found at the following link:

https://www.fields.utoronto.ca/video-archive/event/1996

The wide variety of presentations composed a mosaic of multiple aspects of Approximation Theory and highlighted interesting connections with important contemporary areas of analysis. In particular, the main topics that were discussed include the following:

1. Applications of Approximation Theory (isoperimetric inequalities, construction of entire order-isomorphisms, dynamical sampling);
2. Approximation by harmonic and holomorphic functions (and especially uniform and tangential approximation);
3. Polynomial and rational approximation;
4. Zeros of approximants and zero-free approximation;
5. Tools used in Approximation Theory (analytic capacities, Fourier and Markov inequalities);
6. Approximation on complex manifolds (Riemann surfaces), and approximation in product domains;
7. Approximation in function spaces (Hardy and Bergman spaces, disc algebra, de Branges–Rovnyak spaces);
8. Boundary behaviour and universality properties of Taylor and Dirichlet series.

Throughout the conference there was a very creative and friendly atmosphere, with many interesting discussions and mathematical interactions which, hopefully, will lead to future collaborations. The last talks of the conference were devoted to the main contributions of André Boivin in Approximation Theory and his collaborations which are presented in the first chapter in further detail.

Montréal, QC, Canada — Paul Gauthier
Tampa, FL, USA — Myrto Manolaki
Québec, QC, Canada — Javad Mashreghi

A Riemann surface, full of some (among many) good friends of André Boivin, during the conference “New Trends in Approximation Theory” which was held in his memory (Fields Institute, July 2016).

Contents

The Life and Work of André Boivin

Paul Gauthier, Myrto Manolaki, and Javad Mashreghi

Abstract André Boivin will be fondly remembered for many reasons. We shall attempt to convey the impact he has had on the authors of this note (and many others) by describing his wonderful personality and his important contributions in the field of Complex Approximation Theory.

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1 Instead of an Introduction…

Since “un bon croquis vaut mieux qu’un long discours”, and since André enjoyed reading comics, we thought of describing his wonderful personality via some representative pictures.

André Boivin, apart from being a brilliant mathematician, was also a polymath (meaning homo universalis). Indeed, his interests were very broad, ranging from literature and golf, to travelling and photography.

In particular, André had a beautiful collection of nature photographs. Here, with his camera in Winnipeg, during the CMS summer meeting (June 2014).

André bore his name Boivin [=Drinks Wine!] well. He was one of the best wine-experts. Apart from his very detailed knowledge about wine, he loved organizing wine and cheese parties in his house, for his friends and colleagues.

André Boivin was universally loved and respected by his students; not only because he was a very inspiring teacher, but also because he treated everyone as a member of his family. This picture is from one of the departmental parties that he used to host in his house.

Here, André with his doctoral student, Nadya Askaripour, after her Ph.D. defence.

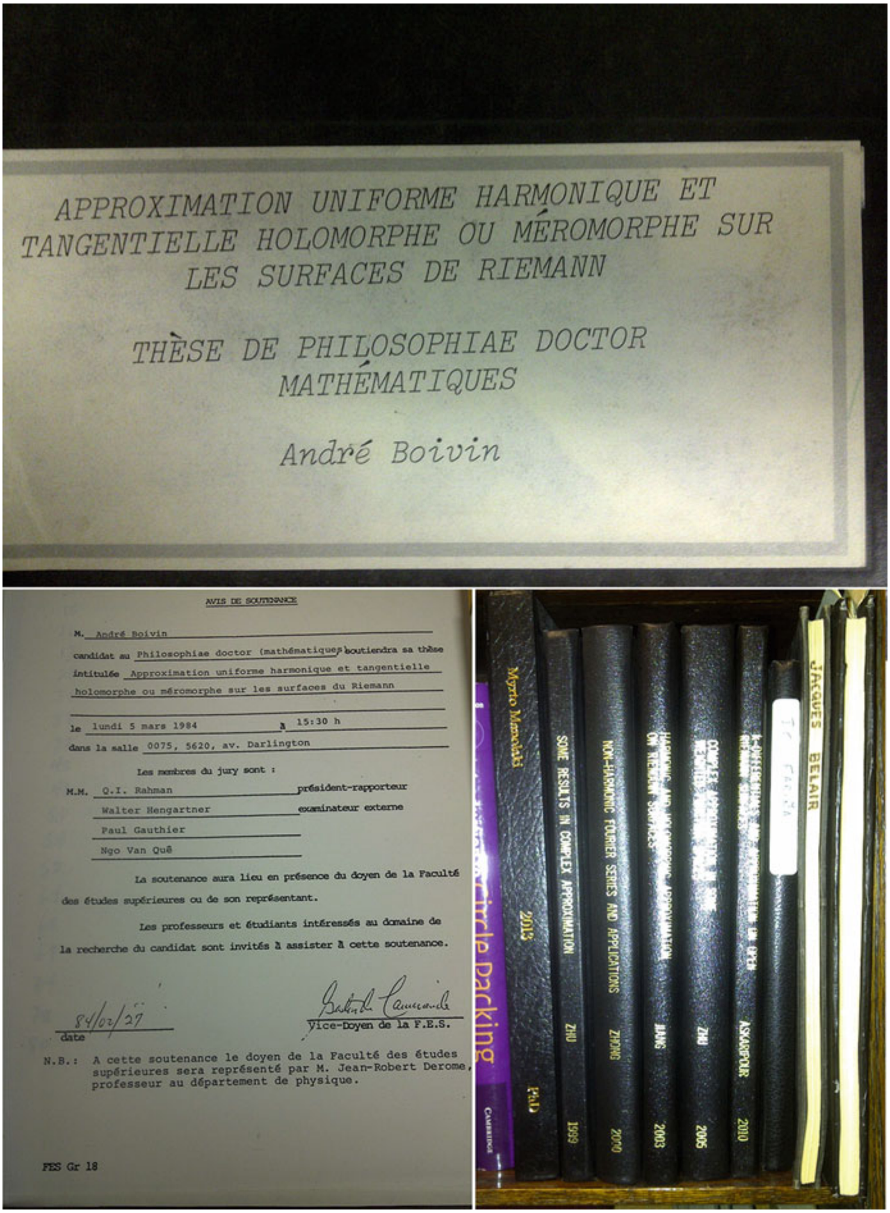
André playing with his two children, Alex and Mélanie.

André Boivin had many mathematical friends and participated in a great number of international conferences. Here with his postdoctoral student Myrto Manolaki and Vassili Nestoridis, during the conference “Universality weekend”, held in Kent State University, Ohio (April 2014).

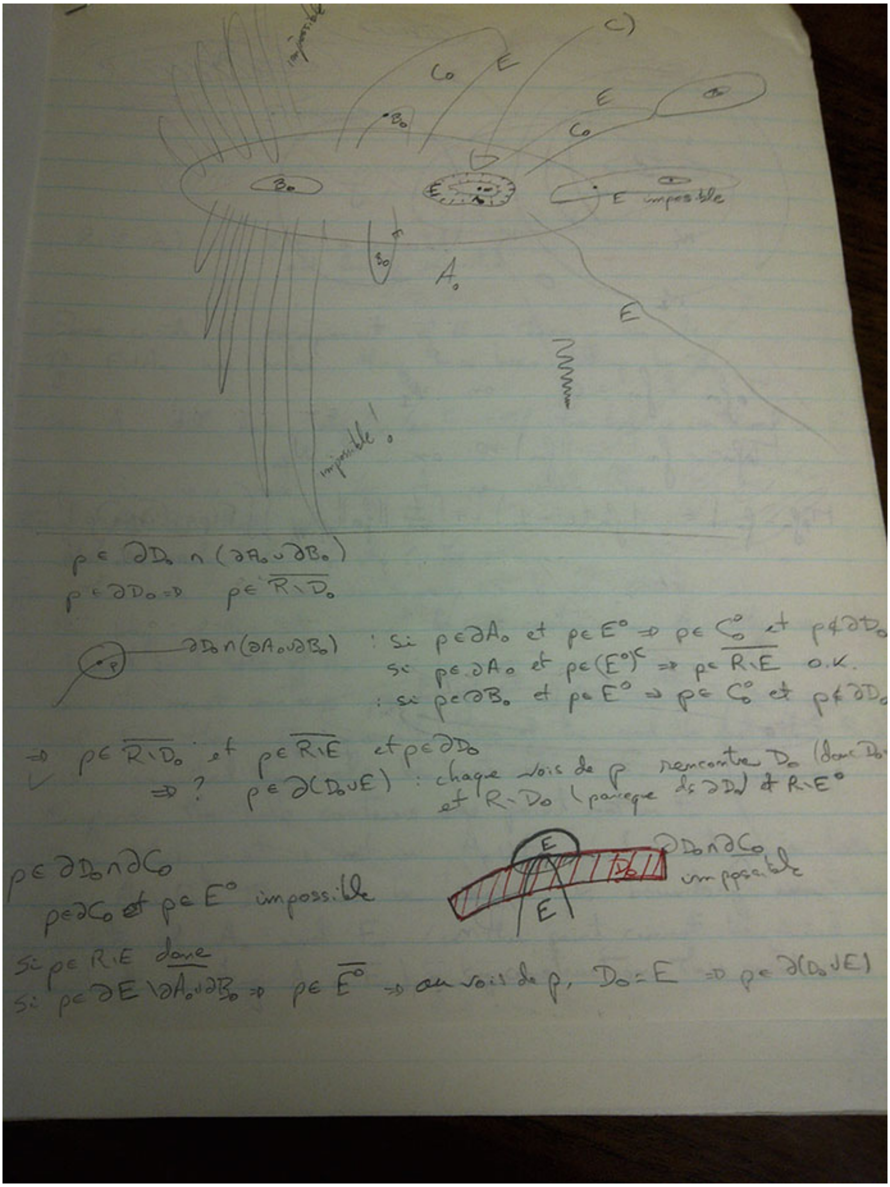
Two of the closest mathematical friends of André were Javad Mashreghi, and his doctoral advisor Paul Gauthier. Here, Paul Gauthier, André Boivin and Javad Mashreghi, during the conference “Complex Analysis and Potential Theory”, which was held in honour of Paul Gauthier and K. N. GowriSankaran, at Université de Montréal (June 2011).

André discussing with his unique vivid way with his colleague and good friend, Mashoud Khalhali.

Boivin served with distinction as Graduate Student Chair and Chair of the Department of Mathematics at the University of Western Ontario. His office was always full of life and positive energy, reflecting his very generous and warm personality.

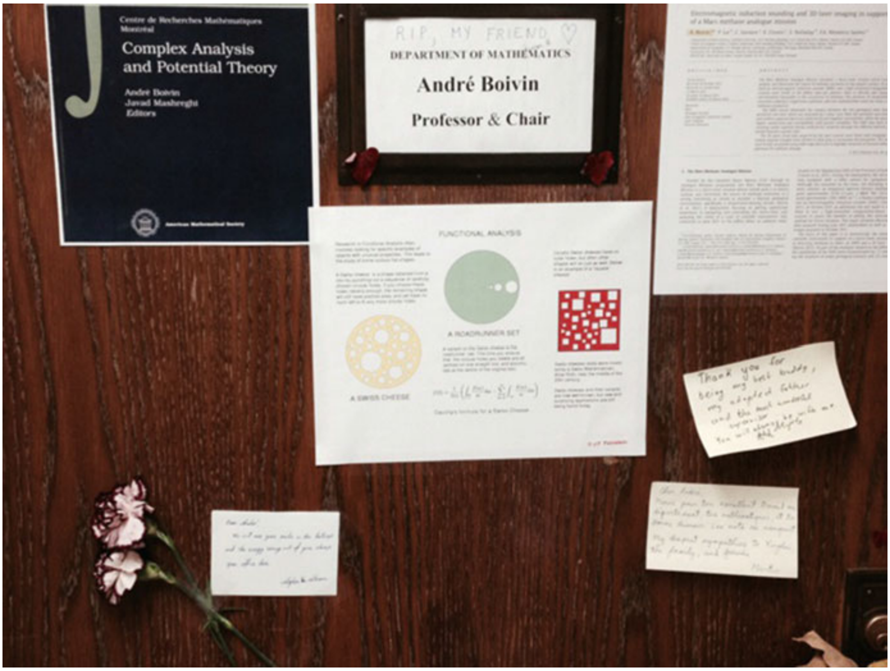


The Ph.D. thesis of André, and the Ph.D. theses of his students.



From his personal hand-written notes.

André with his beloved wife, Yinghui Jiang.



Our unique friend André, it was a great privilege meeting you. Thank you for infinitely many reasons!

2 The Life of André Boivin

André Boivin was born on August 7 1955 in Montréal where he was brought up in the warm environment of a bonded French Canadian family consisting of his parents Simonne Emard and Léon Boivin and his siblings Diane, Jean and François. André had a unique talent in being a true and devoted friend and it would not be an exaggeration to say that his friends were always part of his family. In particular, during his studies at l'Université de Montréal, he met Jacques Bélair, Jean-Pierre Dussault, Jacques Taillon and Monique Tanguay, with all of whom he developed a beautiful lifelong friendship. André obtained his B.Sc. in Mathematics in 1977 and, after completing his M.Sc. at the University of Toronto in 1979, he returned to l'Université de Montréal to pursue his doctoral studies under the supervision of Paul Gauthier. André and Paul developed a unique friendship which was based on mutual admiration, and which led to a very fruitful collaboration (they co-authored ten articles and they had several collaborators in common). He obtained his Ph.D. in 1984 for his thesis entitled "Approximation uniforme harmonique et tangentielle holomorphe ou méromorphe sur les surfaces de Riemann", which carved the main research path of his career. Subsequently he was awarded a 2-year

NSERC postdoctoral fellowship which was held at the University of California, Los Angeles (1984–1985) and at University College, London (1985–1986). During his postdoctoral studies he had the opportunity to interact with leading experts in Complex Analysis such as Theodore Gamelin, who influenced the directions of his future work.

In 1986 André was hired as an Assistant Professor in the Department of Mathematics at the University of Western Ontario and moved back to Canada along with his first wife Johanne Giroux, from London, England to London, Ontario. In 2004 he was promoted to Professor and in 2006 he was happily remarried to Yinghui Jiang. Thus André spent the rest of his life in London, together with his beloved family consisting of his wife Yinghui, his daughter Mélanie, his son Alex and his step son JP. His house was always open to his friends, colleagues and students. Anyone who has met André will know that as well as his passion for Mathematics and science, he enjoyed many of the finer things in life such as photography, cooking, music, literature and golf. Last but not least, as his surname indicates in French, André was quite the wine connoisseur!

His engaging and generous personality made him very popular among his colleagues and students, and he was often characterized as the heart and soul of the department. He served with distinction as Graduate Student Chair, and in 2011 was appointed Chair of the Department of Mathematics, a post that he served with remarkable devotion until the very last days of his life. André was, with no doubt, one of the most conscientious academic leaders. He was always trying to create a positive and creative atmosphere in the department, looking after every single detail. One of his many "invisible" contributions was the departmental Analysis seminar. Despite the small number of Analysis members in the department, André managed to keep the Analysis seminar series alive and of high quality by inviting some of the most prominent experts in the field to the department (and as always, by being an excellent host).

Among his many qualities, André was regarded as one of the most dedicated and influential lecturers and supervisors in the department and was always generous with his time. This was reflected in the great number of graduate students he successfully supervised. In total, he supervised more than 12 masters students, 5 Ph.D. students and 2 postdoctoral fellows. The four students who completed a Ph.D. under his supervision were Hua Liang Zhong (2000, "Non-harmonic Fourier series and applications"), Baoguo Jiang (2003, "Harmonic and holomorphic approximation on Riemann surfaces"), Chang Zhong Zhu (2005, "Complex approximation in some weighted function spaces") and Nadya Askaripour (2010, "Holomorphic k-differentials and holomorphic approximation on open Riemann surfaces"). The last (doctoral and postdoctoral) students of André were Fatemeh Sharifi and Myrto Manolaki, for whom André has been both an inspiring mentor and an affectionate father-figure.

As well as having a productive research career at the University of Western Ontario, he was an active participant in various Canadian committees including being a member of the Grant Selection Committee for Le Fonds de recherche du Québec-Nature et technologies (FRQNT). Adding to his many contributions to his field, in December 2009 he organized (with Tatyana Foth) a session on Complex Analysis at the Winter Meeting of the Canadian Mathematical Society in Windsor. In June 2011, together with Javad Mashreghi, he organized the international conference "Complex Analysis and Potential Theory" in honour of Paul Gauthier and K. N. GowriSankaran, which took place at the CRM, Montréal. Finally, André co-organized the 16th Annual Meeting of Chairs of Canadian Mathematics Departments, which was held at the University of Western Ontario 2 weeks after he suddenly passed away in October 2014.

André Boivin was known and respected not only for his important mathematical contributions and his tireless academic service but also for his uniquely generous and warm personality. He will be fondly remembered for his honesty and openness, for his endless positive energy and clever sense of humour, for his progressive and humanistic spirit, for being an influential teacher, a passionate mathematician, and, above all, for being a wonderful person and friend who appreciated life in all its dimensions. It is impossible to describe with words the impact he has had in our lives and the size of the gap he has left behind.

3 Work on Complex Analysis and Approximation Theory

André Boivin, being the mathematical son of Paul Gauthier and the mathematical great gra(n)dson of Constantin Carathéodory, worked in Complex Analysis and Approximation Theory. His main topics of investigation were approximation by holomorphic, meromorphic and harmonic functions on (open) Riemann surfaces, and in particular on the complex plane $\mathbb{C}$. Namely, he was interested in determining conditions, under which it is possible to have certain smooth extensions or approximations in various function spaces. He also worked on non-harmonic Fourier series and Cauchy-Riemann theory, collaborating with several leading experts. Boivin has written about 40 influential papers in these areas, with publications in prestigious journals (such as the Transactions of AMS and J. Anal. Math.).

3.1 Harmonic, Holomorphic and Meromorphic Approximation on Riemann Surfaces

One of the main components of the work of André Boivin, arising from his doctoral thesis [39], was concerned with harmonic, holomorphic and meromorphic approximation on Riemann surfaces. He wrote many important papers in this area [6, 13, 15, 21, 25, 28, 35–38, 40] and, in particular, he made significant contributions in the area of Carleman (or tangential) holomorphic and meromorphic approximation, which we shall describe below.

In 1927, Carleman, in his attempt to generalize the classical approximation theorem of Weierstrass, showed that, for each continuous function f on the real line $\mathbb{R}$ and for each positive continuous function ε on $\mathbb{R}$, there exists a holomorphic function g on $\mathbb{C}$ such that:

$$|f(z) - g(z)| < \varepsilon(z) \quad (z \in \mathbb{R}).$$

This important result gave rise to the following definition:

Definition 1 Let D be a domain in $\mathbb{C}$ (or, more generally, in a non-compact Riemann surface R), and let $E \subset D$ be a relatively closed set.

1. E is called a set of **Carleman holomorphic** (respectively **meromorphic**) approximation if, for each continuous function f on E which is holomorphic on E° and for each positive continuous function ε on E, there exists a holomorphic (respectively meromorphic) function g on D such that:

$$|f(z) - g(z)| < \varepsilon(z) \quad (z \in E).$$

2. If we replace above the phrase "each positive continuous function ε on E" by "each positive constant ε", we say that E is a set of **uniform holomorphic** (respectively **meromorphic**) approximation.

Obviously, each set of Carleman (holomorphic or meromorphic) approximation is a set of uniform (holomorphic or meromorphic) approximation. The characterization of compact subsets of $\mathbb{C}$ which are sets of uniform holomorphic approximation, follows by the celebrated theorem of Mergelyan: they are exactly the compact sets with connected complement in $\mathbb{C}$. In 1968, Arakelyan generalized this result, by providing a characterization of closed subsets of $\mathbb{C}$ which are sets of uniform approximation:

Theorem 1 (Arakelyan, 1968) *Let $D \subset \mathbb{C}$ be a domain, and $E \subset D$ be relatively closed set. The following are equivalent:*

1. *E is a set of uniform holomorphic approximation.*
2. *$D^* \setminus E$ is connected and locally connected (where D^* is the one-point compactification of D).*

It is striking that, although Runge's theorem has an analogue for non-compact Riemann surfaces (H. Behnke and K. Stein, Math. Ann. 120 (1949), 430–461), there is no known analogue for Arakelyan's theorem for a general non-compact Riemann surface. In fact, it can be shown that there is no topological characterization of closed sets of uniform holomorphic approximation. This difficult problem attracted the interest of André, who spent several years of his career trying to investigate it, and, in particular, he was working on this with his last doctoral student, Fatemeh Sharifi.

The contributions of Boivin in the area of Carleman approximation are of very significant importance. In 1971, Nersessian gave a complete characterization of sets of holomorphic Carleman approximation in the case of the complex plane, based on previous work of Gauthier. In 1986, in his paper in Math. Ann. [36], Boivin provided a complete characterization of the sets of holomorphic Carleman approximation on an arbitrary open Riemann surface.

The problem of characterizing the sets of meromorphic Carleman approximation still remains open (even in the case of the complex plane). Boivin's work has shed considerable light on this problem. For example he showed that the meromorphic analogue of the sufficient condition that appears in Nersessian's characterization of holomorphic Carleman approximation is not sufficient to characterize the sets of meromorphic Carleman approximation. He also provided a new sufficient condition (in terms of the Gleason parts) and one necessary condition (in terms of the fine topology) for sets to be sets of meromorphic Carleman approximation. Later, in a coauthored paper with Nersessian, he showed that the sufficient condition in terms of the Gleason parts, fails to be also necessary for this kind of approximation.

3.2 Approximation in Function Spaces

Apart from uniform approximation, Boivin also worked on approximation in various function spaces, including weighted H^p and L^p spaces, Lip_α, BMO, and C^m spaces (see [3, 8, 9, 26, 29, 31]).

A representative sample of this work can be found in [31], where Boivin and Verdera generalized to unbounded sets a wide range of important approximation results. In particular, for closed (or measurable) sets F in the complex plane, the authors considered $A_B(F)$, the set of holomorphic functions on the interior of F which belong to B, where B is any of the following function spaces: $L^p(F)$ (for $1 < p < \infty$), $\mathrm{Lip}_\alpha(F)$ (for $0 < \alpha < 1$), $\mathrm{BMO}(F)$ or $C^m(F)$. Inspired by the classical theorem of Vitushkin, Boivin and Verdera introduced some appropriate capacities to characterize the sets F, for which every function in $A_B(F)$ can be approximated, in the B-norm on F, by functions holomorphic in a neighbourhood of F. In the same spirit, in [29], the two authors, together with Joan Mateu, provided a Vitushkin-type theorem on approximation by holomorphic functions in a neighbourhood of a compact set E which additionally belong to some weighted L^p space on E. Subsequently, in [26], Boivin with Bonilla and Fariña, advanced further the theory of weighted L^p spaces by providing analogues of some of the most fundamental approximation theorems (Runge's theorem, fusion lemma and localization theorem). Finally, André Boivin investigated approximation properties of weighted Hardy spaces, in collaboration with Changzhong Zhu and Paul Gauthier ([3, 8] and [9]). Specifically, in [9] they studied expansion, moment and interpolation problems for Hardy spaces on the disc, with weight satisfying a certain technical condition (known as Muckenhoupt's condition).

3.3 Axiomatic Approximation (Extension) for Harmonic (Subharmonic) Functions and Elliptic Generalizations

The main thrust of André Boivin's work in approximation theory has been directed towards the existence of approximations more than with the computation of approximations. That is, given a space X of functions and a "nice" subspace Y, a fundamental question is whether, for every function f in X there exist functions in Y which approximate f. Finding an algorithm for actually computing the approximating functions has been less of a concern for him. For most programs which actually compute approximations, there is no proof that they actually converge. If one can prove that certain approximations do not exist, one can save engineers from wasting their time and resources in trying to compute an approximation which does not exist. The existence of nice approximations in Y of functions in X can be interpreted as the density of Y in X. Most work in approximation theory is from the point of view of functional analysis. The spaces X and Y are viewed as normed linear spaces. Moreover, the approximation of a function f is usually on a compact set. But in physics, unbounded models are frequently employed. Boivin has spent most of his career in trying to approximate functions on unbounded sets, such as the real line. One of the most natural forms of approximation is uniform approximation. But here, there is a fundamental difficulty in trying to employ functional analysis to approximate uniformly on unbounded sets. For example, while the space X of continuous functions on the real line, with the topology of uniform convergence, is a topological space and a vector space, it is not a topological vector space, because multiplication by scalars is not continuous. Boivin and his collaborators have had to summon considerable ingenuity to overcome this curse, but they have succeeded in discovering fundamental properties of functions spaces X and their subspaces Y which have allowed them to develop an axiomatic theory of approximation theory which applies to many of the most important function spaces on unbounded sets, for example C^m- or L^p-solutions to elliptic equations on unbounded sets. In Boivin's work approximating a function f by a nice function g usually means that g is defined on a larger set. The function f, if it is analytic, is usually not itself defined on a larger set, because of uniqueness properties of analytic functions. Boivin and his collaborators have also worked with subsolutions of elliptic equations. These are more flexible and sometimes allow extensions, which are perfect approximations, since the restriction of the approximation is actually *equal* to the original function. The papers in this direction are [2, 5, 16, 19, 22], and [23].

3.4 Cauchy-Riemann Theory

An independent component of the work of André Boivin was his work on Cauchy-Riemann (CR) Theory. In particular, he wrote three influential papers in this area ([24, 27] and [30]), co-authored with his former colleague and good friend, Roman Dwilewicz.

One of their papers [27], was concerned with the problem of classification of local CR mappings between CR manifolds (a problem that goes back to Poincaré). Moreover, they worked on uniform approximation of CR functions on tubular submanifolds in $\mathbb{C}^n = \mathbb{R}^n \times i\mathbb{R}^n$ (that is manifolds of the form $N \times i\mathbb{R}^n$, where $N \subset \mathbb{R}^n$ is a manifold). In particular, in their first paper [30], they showed that, if N is a compact connected manifold in $\mathbb{R}^2$, then, any CR function on the tubular submanifold $N \times i\mathbb{R}^2 \subset \mathbb{C}^2$, can be uniformly approximated, on compacts subsets, by holomorphic polynomials. In their last paper [24], published in 1998 in the Transactions of AMS, Boivin and Dwilewicz provided a complete generalization for CR functions of the classical Bochner tube theorem, which states that any holomorphic function on a tube $\tau(\Omega) = \Omega + i\mathbb{R}^n \subset \mathbb{C}^n$, with Ω a domain in $\mathbb{R}^n$, can be extended holomorphically to the tube over the convex hull of Ω. In particular, they showed the following general result, which replaces the assumption on Ω being a domain in $\mathbb{R}^n$ with a more general connected submanifold:

Theorem 2 (Boivin and Dwilewicz [30]) *Let N be a connected submanifold of $\mathbb{R}^n$ of class C^2. Then any continuous CR function on the tube $\tau(N) = N + i\mathbb{R}^n$ can be continuously extended to a CR function on $\tau(\text{ach}(N)) = \text{ach}(N) + i\mathbb{R}^n$, where $\text{ach}(N)$ denotes the almost convex hull of N (which is the union of N with the interior of the convex hull $\text{ch}(N)$ of N, taken in the smallest dimensional space which contains $\text{ch}(N)$). As a consequence, any CR function on $\tau(N)$ can be uniformly approximated on compact subsets by holomorphic polynomials.*

The above theorem of Boivin-Dwilewicz, which has received 15 citations, not only generalizes Bochner's tube theorem; it can also be considered as a version of the classical edge-of-the-wedge theorem of Bogolyubov (a theorem in complex analysis, which was originally proved as a tool to solve some problems in physics in connection with quantum field theory and dispersion relations).

3.5 Approximation by Systems of Exponentials

In recent years André Boivin became interested in the study of non-harmonic Fourier series; that is the study of approximation properties of systems of exponentials $\{e^{i\mu_n t}\}$ (for example, investigating when these systems form a basis or a frame). Such questions are of great importance since they have intimate connections with control theory and signal processing. Boivin investigated such (and similar) questions and published seven papers in this area [4, 7, 11, 12, 14, 18, 20], in collaboration with his two doctoral students, Hualiang Zhong and Changzhong Zhu, and with Terry Peters.

A component of this work was concerned with the following general problem:

Problem 1 Given a domain Ω in the complex plane and a set of functions $\{h_n\}$ in $L^2_a[\Omega]$ (the Hilbert space of square area-integrable analytic functions on Ω), find necessary and sufficient conditions on Ω, such that the system $\{h_n\}$ is complete in

$L_a^2[\Omega]$ (that is, if g is a function in $L_a^2[\Omega]$, then $\inf \|h - g\|_{L_a^2[\Omega]} = 0$, where the infimum is taken over all h in the linear span in $L_a^2[\Omega]$ of $\{h_n\}$).

For example, it is natural to ask to characterize all bounded domains Ω for which the monomials z^n, $n = 1, 2, \ldots$, constitute a complete system in $L_a^2[\Omega]$. Equivalently, this means to characterize the bounded domains Ω for which the polynomials are dense in $L_a^2[\Omega]$. Although there is a simple topological characterization of domains when we examine the density of polynomials with respect to the uniform norm, it turns out that the case of $L_a^2[\Omega]$ is a more delicate topological and geometrical problem. In fact, a complete answer is known only in a few special cases.

In [18], Boivin and Zhu studied the completeness of the system $\{z^{\tau_n}\}$ in $L_a^2[\Omega]$, where $\{\tau_n\}$ is a fixed sequence of complex numbers. In particular, for an unbounded simply connected domain Ω, they provided sufficient conditions (on Ω and on the sequence $\{\tau_n\}$), under which the system $\{z^{\tau_n}\}$ is complete in $L_a^2[\Omega]$. Problems of similar nature have been considered by Carleman, Dzhrbasian, Mergelyan, and Shen. Moreover, Boivin and Zhu studied the completeness in $L_a^2[\Omega]$ of more general systems, namely, of the form $\{f(\lambda_n z)\}$, where $\{\lambda_n\}$ is a sequence of complex numbers and f is either an entire function in the complex plane (see [20]), or an analytic function defined on the Riemann surface of the logarithm (see [20] and [7]). They also worked with incomplete systems of functions. In particular, in [4], they consider systems of the form $\{e^{-\lambda_k x}\}$ and $\{\psi_k(x)\}$ (where λ_k satisfies a certain Blaschke condition and $\psi_k(x)$ is given by a specific integral formula), which are known to be incomplete and bi-orthogonal in $L^2(0, +\infty)$. Using the Fourier transform as a tool, they managed to obtain bi-orthogonal expansions of each function in $L^2(0, +\infty)$, in terms of $\{e^{-\lambda_k x}\}$ and $\{\psi_k(x)\}$.

Boivin also worked on the stability of complex exponential frames $\{e^{i\lambda_n x}\}$ in the spaces $L^2(-\gamma, \gamma)$, where $\gamma > 0$ (see [12]). Moreover, in [11], André and his doctoral student Hualiang Zhong, studied completeness properties of exponential systems arising from the characteristic roots of the delay-differential equation $y'(t) = ay(t-1)$, where a is a real parameter. The main result of this paper is that such a system is complete in $L^2(-1/2, 1/2)$, but it does not form an unconditional basis.

3.6 Other Topics of Research

Apart from the above five main categories, André worked in various other topics in Analysis. For example in [17] he obtained results on the growth of entire functions representable as (generalized) Dirichlet series, in terms of their Dirichlet coefficients and exponents. Moreover, he worked on tensor approximation [32] and T-invariant algebras on Riemann surfaces [33, 34], extending the theory that was developed in the plane by Gamelin. Finally, two other independent topics that Boivin was interested in were the study of zero sets of harmonic and real analytic functions [1] and bounded pointwise approximation [10].

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A Note on the Density of Rational Functions in $A^\infty(\Omega)$

Javier Falcó, Vassili Nestoridis, and Ilias Zadik

Abstract We present a sufficient condition to ensure the density of the set of rational functions with prescribed poles in the algebra $A^\infty(\Omega)$.

Keywords Padé approximation • Rational functions • Rational approximation

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1 Introduction

Let K be a compact subset of $\mathbb{C}$ and K° the interior of K. As usual, $A(K)$ denotes the set of all analytic functions in K° that are continuous functions on K. When $A(K)$ is endowed with the supremum norm on K then $A(K)$ is a Banach algebra. The well-known Mergelyan's theorem claims that any function in $A(K)$ can be approximated by polynomials in the natural topology of $A(K)$ if the complement of K is connected.

Many times, approximation by polynomials of a function f is replaced with approximation by rational functions. In fact, Mergelyan's theorem can be extended to compact sets in $\mathbb{C}$ whose complement has a finite number of connected components if we substitute approximation by polynomials with approximation by rational functions [10, Exercise 1, Chapter 20]. The analogous multilinear version of this result can be found in [6].

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A known way of approximating a function f by rational functions is by considering the expansion of f as the ratio of two power series. Padé approximations are usually superior than Taylor series when functions contain poles because the use of rational functions allow to imitate these singularities. The theory of Padé approximants has been deeply studied during the last years, see for instance [3, 4, 9]. For a review of the classical theory of polynomial and rational approximation of functions in a complex domain we recommend [10, 11] and the references therein, and for a historical treatment of the theory of the class of best rational approximating functions called Padé approximants we recommend [1].

A natural subset of $A(K)$ is the set of all functions in $A(K)$ such that all of their derivatives also belong to $A(K)$. In general, if $\Omega \subset \mathbb{C}$ is an open set in $\mathbb{C}$, we say that a holomorphic function f defined on Ω belongs to $A^\infty(\Omega)$ if for every $l \in \{0, 1, 2, \ldots\}$ the l-th derivative $f^{(l)}$ extends continuously to $\overline{\Omega}$. The natural topology of $A^\infty(\Omega)$ is defined by the seminorms

$$\sup_{\substack{z\in\Omega \\ |z|\leq m}} \{|f^{(l)}(z)|\},$$

$l = 0, 1, 2, \ldots, m = 1, 2, 3, \ldots$ With this topology $A^\infty(\Omega)$ is a Fréchet algebra.

Naturally, every rational function with poles off the closure of Ω belongs to $A^\infty(\Omega)$. We denote by $X^\infty(\Omega)$ the closure of these rational functions in $A^\infty(\Omega)$.

Recent results, in one and several complex variables, in the study of rational approximation in $A^\infty(\Omega)$ with its natural topology can be find in [7–9]. Here we continue with this study by giving sufficient conditions to ensure that $X^\infty(\Omega) = A^\infty(\Omega)$.

In the paper "Padé approximants, density of rational functions in $A^\infty(\Omega)$ and smoothness of the integration operator" by Vassili Nestoridis and Ilias Zadik, the authors prove among other results that, under certain conditions, $X^\infty(\Omega) = A^\infty(\Omega)$.

Theorem 1 ([9, Theorem 5.7]) *Let $\Omega \subset \mathbb{C}$ be a bounded, connected, open set, such that:*

(a) $(\overline{\Omega})^\circ = \Omega$,
(b) $\{\infty\} \cup \left(\mathbb{C} \setminus \overline{\Omega}\right)$ *has exactly k connected components in the topology of the extended plane, $k \in \mathbb{N}$,*
(c) There exists $M > 0$ such that for all $a, b \in \Omega$, there exists a continuous function $\gamma : [0, 1] \to \Omega$ with $\gamma(0) = a, \gamma(1) = b$ and Length$(\gamma) \leq M$.

Now pick from each connected component of $\{\infty\} \cup \left(\mathbb{C} \setminus \overline{\Omega}\right)$ a point $a_i, i = 0, 1, 2, \ldots, k-1$ and set $S = \{a_0, \ldots, a_{k-1}\}$, where a_0 belongs to the unbounded component. Then the set of all rational functions with poles only in S is dense in $A^\infty(\Omega)$ and therefore $X^\infty(\Omega) = A^\infty(\Omega)$.

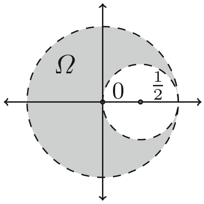
However, the proof of this theorem contains a gap. The proof presented in [9] uses that under the assumptions of the theorem there exist a finite set of Jordan curves $\gamma_0, \ldots, \gamma_{k-1}$ in Ω with the properties that they are closed, have finite length and

$$\operatorname{ind}(\gamma_i, a_j) = \begin{cases} 1 \text{ if } i = j, \\ 0 \text{ if } i \neq j. \end{cases}$$

Here we present an example of a domain Ω satisfying the hypothesis of the theorem. However, there exists no such Jordan curves in Ω satisfying the desired conditions. We denote by $D(z, r)$ the open disk of center z and radius r.

Example 1 Let

$$\Omega = D(0, 1) \setminus \overline{D(1/2, 1/2)}.$$



Clearly, Ω is a bounded open set with $\overline{\Omega}^\circ = \Omega$ and $\{\infty\} \cup (\mathbb{C} \setminus \overline{\Omega}) = D(1/2, 1/2) \cup \big(\mathbb{C} \setminus \overline{D(0, 1)}\big)$. Furthermore, any two points in Ω can be joined by a path in Ω of length at most 2π. However, for any point $a \in D(1/2, 1/2)$ and any closed Jordan curve γ in Ω we have that $\operatorname{ind}(\gamma, a) = 0$.

In the paper it is also suggested an alternative proof of Theorem 1 by using a Laurent decomposition, see [9, Remark 5.16] and [2] for the details. However, this alternative proof also requires the assumption of the existence of these rectifiable Jordan curves. We refer to [2, Section 4] for the details.

In the following section we present a sufficient condition to ensure that the theorem remains valid. We also partially answer a question asked by Nestoridis and Zadik, see [9, Remark 5.13].

2 Main Theorem

In this section we present a sufficient condition to ensure that the conclusion of Theorem 1 holds. This condition is weaker than the original conditions of Theorem 1 in the sense that it is not required that the open set Ω is connected. However it requires the existence of specific rectifiable Jordan curves. This, partially answers the question asked by Nestoridis and Zadik in [9, Remark 5.13].

Definition 1 (Separable by Curves Set) Let Ω be a bounded open set in $\mathbb{C}$ such that $\overline{\Omega}^\circ = \Omega$. Assume that $\{\infty\} \cup (\mathbb{C} \setminus \overline{\Omega})$ has a finite number of connected components $V_0, V_1, \ldots, V_{k-1}$ in the topology of the extended plane, where V_0 is the unbounded connected component of the complement of $\overline{\Omega}$. We say that Ω

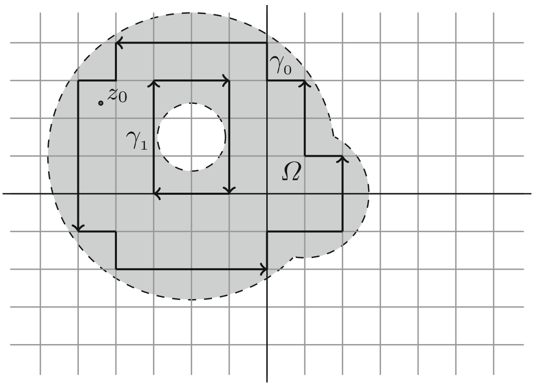
is **separable by curves** if there exist a point $z_0 \in \Omega$ and rectifiable Jordan curves $\gamma_0, \gamma_1, \ldots, \gamma_{k-1} \subset \overline{\Omega}$ with

$$\operatorname{ind}(\gamma_0, a) = \begin{cases} 1 \text{ if } a = z_0, \\ 1 \text{ if } a \in V_1 \cup \cdots \cup V_{k-1}, \end{cases} \quad \text{and} \quad \operatorname{ind}(\gamma_i, a) = \begin{cases} -1 \text{ if } a \in V_i, \\ 0 \text{ if } a = z_0, \\ 0 \text{ if } a \in V_s \text{ and } s \neq i, \end{cases}$$

for $i = 1, \ldots, k-1$, $s = 0, 1, \ldots, k-1$.
We note that the above curves do not have to be disjoint.

A natural example of a domain that satisfies the conditions of Definition 1 is a bounded domain whose complement has a finite number of connected components that are separated apart, i.e., there exists a positive number $\epsilon > 0$ such that the distance between any two connected components of the complement of $\overline{\Omega}$ is bigger than ϵ. For the sake of completeness, we provide here the construction of appropriate Jordan curves.

Fix any $z \in \Omega$. Consider a positive number $\tilde{\epsilon}$ smaller than ϵ and smaller than the distance from z to any of the connected components of the complement of $\overline{\Omega}$. Then, if we consider a subdivision of $\mathbb{C}$ into a grid of squares with diagonal smaller than $\tilde{\epsilon}$, by the compactness of $\overline{\Omega}$ and the compactness of the closure of the bounded components of the complement of $\overline{\Omega}$ there exist simply closed contours defined by the grid satisfying the conditions of Definition 1.



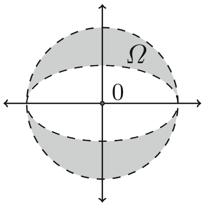
Another example of a separable by curves domain is a domain defined by a finite set of disjoint rectifiable Jordan curves. In this case, the Jordan curves that appear in Definition 1 can be chosen to be the same Jordan curves that define the domain, after possibly a reorientation. We also mention that Example 1 is a separable by curves domain.

Nestoridis and Zadik in [9, Remark 5.13] gave an example of an open set Ω that has more than one connected component, in which the density of rational functions in $A^\infty(\Omega)$ holds. Such example was the union of two open discs whose closures intersect on just one point. The authors used the fact that the intersection of the two closed disks is a singleton to prove the density of rational functions in $A^\infty(\Omega)$ for this particular open set Ω. However, it is not known under what conditions the density of rational functions can be ensured in $A^\infty(\Omega)$ for non-connected open sets Ω.

Example 2 Let consider

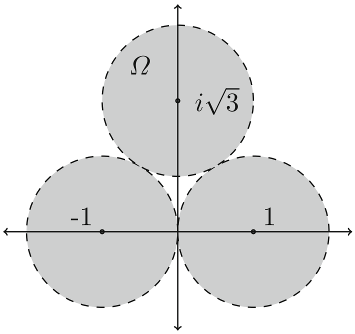
(a)

$$\Omega = \{z = x + iy \in \mathbb{C} : |z| < 1 \text{ and } x^2 + 2y^2 > 1\}.$$



(b)

$$\Omega = D(-1, 1) \cup D(1, 1) \cup D(i\sqrt{3}, 1).$$



Then, each Ω is a separable by curves open set that satisfies the conditions of Theorem 2 below.

Similar constructions to Example 2 can be done to find many examples of open sets Ω that satisfy the hypothesis of Theorem 2 below, have finitely many connected components and the intersection of the closures of the connected components consists of a finite number of points. This partially answers the question of Nestoridis and Zadik, see [9, Remark 5.13].

To prove our main result we will need a refinement of [9, Lemma 5.8] for functions in $A^\infty(\Omega)$. Our proof is modeled on the proof of that lemma.

Lemma 1 *Let Ω be an open set, $n \in \mathbb{N}$ and let $f \in A^\infty(\Omega)$. Let γ be any closed rectifiable curve in $\overline{\Omega}$ such that γ intersects $\partial\Omega$ at most at a finite number of points. Then, for $m \in \mathbb{N} \cup \{0\}$, $0 \leq m \leq n-1$, we have $\int_\gamma z^m f^{(n)}(z)dz = 0$.*

Proof We proceed by induction on n. Let γ be a closed rectifiable curve with $\gamma \subset \overline{\Omega}$ such that γ intersects $\partial\Omega$ at most at a finite number of points. Let $f \in A^\infty(\Omega)$; therefore, f and all its derivatives extend continuously on Ω closure.

For $n = 1$, we only need to prove that $\int_\gamma f'(z)dz = 0$. This follows from the fact that if γ is in Ω, then clearly the result holds. Also, if β is a polygonal simple curve parametrized by t in [0, 1] that only touches the boundary of Ω at most at one point, then by continuity of f on $\overline{\Omega}$,

$$\int_\beta f'(z)dz = f\big(\beta(1)\big) - f\big(\beta(0)\big).$$

Therefore, by continuity of f on $\overline{\Omega}$, for any closed rectifiable curve $\gamma \in \overline{\Omega}$ such that γ intersects $\partial\Omega$ at most at a finite number of points

$$\int_\gamma f'(z)dz = 0.$$

The case $n = 1$ is complete.

Suppose that the statement is true for $n = k$. We prove it now for $k+1$. For $m = 0$ we have that $\int_\gamma f^{(k+1)}(z)dz = \int_\gamma \big(f^{(k)}\big)' dz = 0$ by using the fact that $f^{(k)} \in A(\Omega)$ and the same argument that we used for the case $n = 1$. Now, let $m \in \{1, 2, \ldots, k\}$. Then, integrating by parts, we have that

$$\int_\gamma z^m f^{(k+1)}(z)dz = \int_\gamma z^m \big(f^k(z)\big)' dz = z^m f^{(k)}\big|_{\gamma(0)}^{\gamma(1)} - m\int_\gamma z^{m-1}\big(f^k(z)\big)' dz = 0$$

where we are using that $z^m f^{(k)}\big|_{\gamma(0)}^{\gamma(1)} = 0$ is zero because the curve γ is closed and $m\int_\gamma z^{m-1}\big(f^k(z)\big)'dz = 0$ is zero because of the hypothesis of induction. The proof is complete.

To continue, we present a revised version of [9, Theorem 5.7]. Since condition (d) in Theorem 2 has been weakened with respect to Theorem 1, we also present here the complete proof of the result, for the sake of completeness.

Theorem 2 *Let $\Omega \subset \mathbb{C}$ be a bounded open set such that:*

(a) $(\overline{\Omega})^\circ = \Omega$,

(b) $\{\infty\} \cup (\mathbb{C} \setminus \overline{\Omega})$ *has exactly k connected components in the topology of the extended plane, $V_0, V_1, \ldots, V_{k-1}$, $k \in \mathbb{N}$, with V_0 being the unbounded*

component, and the intersection of $\overline{V}_i$ *and* $\overline{V}_j$ *being at most a finite number of points, for* $0 \leq i, j \leq k-1$, $i \neq j$,

(c) Ω *is separable by curves,*

(d) There exists $M > 1$ *such that for all* $a, b \in \Omega$, *there exists a continuous function* $\gamma_{a,b} : [0, 1] \to \overline{\Omega}$ *with* $\gamma_{a,b}(0) = a, \gamma_{a,b}(1) = b$, *Length*$(\gamma_{a,b}) \leq M$ *and* $\Omega \subset B(0, M)$.

Now pick from each connected component of $\{\infty\} \cup (\mathbb{C} \setminus \overline{\Omega})$ *a point* $a_i, i = 0, 1, 2, \ldots, k-1$ *and set* $S = \{a_0, \ldots, a_{k-1}\}$, *where* a_0 *belongs to the unbounded component. Then the set of all rational functions with poles only in* S *is dense in* $A^{\infty}(\Omega)$ *and therefore* $X^{\infty}(\Omega) = A^{\infty}(\Omega)$.

Proof Let $f \in A^{\infty}(\Omega)$, $\epsilon > 0$ and $n \in \mathbb{N} \cup \{0\} = \{0, 1, 2, \ldots\}$. We need to find a rational function r with poles in S such that,

$$\sup_{w \in \Omega} |f^{(l)}(w) - r^{(l)}(w)| < \epsilon, \quad \text{for } l = 0, 1, \ldots, n.$$

First we treat the case $a_0 = \infty$. Since $f \in A^{\infty}(\Omega)$, it follows that $f^{(n)}$ is analytic in the open set Ω and continuous on $\overline{\Omega}$. As a consequence of Mergelyan's Theorem [10, Ch. 20, Ex. 1] there exists a rational function $\tilde{r}_n(z)$, with poles only in S, such that

$$\sup_{w \in \Omega} |f^{(n)}(w) - \tilde{r}_n(w)| < \min\left\{\frac{\epsilon}{2}, \frac{d^n \pi \epsilon}{n(k-1)(M+a)^{n-1}M}\right\},$$

where $d = \min\{1, d(a_1, \Omega), \ldots, d(a_{k-1}, \Omega)\} > 0$ and $a = \max_{i=1,\ldots,k-1} |a_i|$.

Since $\tilde{r}_n$ is a rational function with poles only in S, we can rewrite $\tilde{r}_n$ as

$$\tilde{r}_n(z) = r_n(z) + \sum_{i=1}^{k-1} \sum_{j=1}^{n} \frac{b_{i,j}}{(z - a_i)^j}$$

with $b_{i,j} \in \mathbb{C}$, $i = 1, 2, \ldots, k-1$ and $j = 1, 2, \ldots, n$, and r_n is a rational function such that

$$Res\big((z - a_i)^{j-1} r_n(z), a_i\big) = 0, \quad \text{for all } i = 1, 2, \ldots, k-1, \ j = 1, 2, \ldots, n.$$

By condition (c), Ω is separable by curves, hence there exists a point $z_0 \in \Omega$ and there exist rectifiable Jordan curves $\gamma_0, \gamma_1, \ldots, \gamma_{k-1} \subset \overline{\Omega}$ with

$$\text{ind}(\gamma_0, a) = \begin{cases} 1 \text{ if } a = z_0, \\ 1 \text{ if } a \in V_1 \cup \cdots \cup V_{k-1}, \end{cases} \quad \text{and} \quad \text{ind}(\gamma_i, a) = \begin{cases} -1 \text{ if } a \in V_i, \\ 0 \text{ if } a = z_0, \\ 0 \text{ if } a \in V_s \text{ for } s \neq i, \end{cases}$$

for $i = 1, \ldots, k-1$, $s = 0, 1, \ldots, k-1$.

Condition (b) ensures that the curves $\gamma_0, \gamma_1, \ldots, \gamma_{k-1}$ can be chosen to intersect $\partial\Omega$ at most at a finite number of points. Without loss of generality, by increasing M if necessary, we may assume that Length$(\gamma_i) \leq M$ for all $i = 1, 2, .., k-1$. Then, since all the poles of $\tilde{r}_n$ are in S, by Lemma 1, we have that

$$\begin{aligned}
|b_{i,j}| &= \Big|\frac{1}{2\pi i}\int_{\gamma_i}(z-a_i)^{j-1}\tilde{r}_n(z)dz\Big| \\
&= \Big|\frac{1}{2\pi i}\int_{\gamma_i}(z-a_i)^{j-1}(\tilde{r}_n(z) - f^{(n)}(z))dz\Big| \\
&\leq \frac{1}{2\pi}(M + |a_i|)^{j-1} M \sup_{w\in\Omega} |\tilde{r}_n(w) - f^{(n)}(w)| \\
&\leq \frac{d^n \epsilon}{2n(k-1)},
\end{aligned}$$

for $i = 1, 2, \ldots, k-1$ and $j = 1, 2, \ldots, n$.

Therefore,

$$\begin{aligned}
\sup_{w\in\Omega} |f^{(n)}(w) - r_n(w)| &\leq \sup_{w\in\Omega} \Big\{ |f^{(n)}(w) - \tilde{r}_n(w)| + \sum_{i=1}^{k-1}\sum_{j=1}^{n} \frac{b_{i,j}}{|z-a_i|^j} \Big\} \\
&\leq \sup_{w\in\Omega} |f^{(n)}(w) - \tilde{r}_n(w)| + \sum_{i=1}^{k-1}\sum_{j=1}^{n} \frac{d^{n-j}\epsilon}{2n(k-1)} \\
&\leq \epsilon.
\end{aligned}$$

Then, the function r_n is a rational function that has a Laurent expansion around each $a_i \in S \setminus \{\infty\}$, where the coefficients of $(z-a_i)^l$ are equal to zero for $l = -n, -n+1, \ldots, -1$. Thus, we can define recursively a sequence of rational functions $r_n, r_{n-1}, \ldots, r_1, r_0$ as

$$r_l(z) = f^{(l)}(z_0) + \int_{\gamma_{z_0,z}} r_{l+1}(z)dz$$

were $\gamma_{z_0,z}$ is a path in $\overline{\Omega}$ of length at most M joining the points z_0 and z. Setting $r = r_0$ finishes the proof for the case $a_0 = \infty$.

The case $a_0 \neq \infty$ follows from the previous case, [5, Lemma 2.2] and the triangular inequality. The proof is completed.

Remark 1 It is worth noting that Example 1 is a separable by curves domain that also satisfies the conditions of Theorem 2.

We don't know if the added condition that the open set Ω is separable by curves is a necessary condition to ensure that Theorem 1 holds. Furthermore, we don't even know if the condition that any two points of Ω can be joined by a path in $\overline{\Omega}$

whose length is bounded by a fixed constant M is a necessary condition to ensure the density of rational functions in $A^\infty(\Omega)$. We conclude this note with the following natural question.

Question 1 Does Theorem 2 remains valid if we remove any of the conditions (b)–(d)?

Notes and Comments A more general version of [9, Theorem 5.7] is stated in [9, Theorem 5.9]. Even though in [9, Theorem 5.9] it is required that the connected components of the complement of $\overline{\Omega}$ are separated apart, the proof of this result depends on [9, Theorem 5.7]. A stronger version of [9, Theorem 5.9] can be obtained as a consequence of Theorem 2 presented here.

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Approximation by Entire Functions in the Construction of Order-Isomorphisms and Large Cross-Sections

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Abstract A theorem of Hoischen states that given a positive continuous function $\varepsilon : \mathbb{R}^t \to \mathbb{R}$, a sequence $U_1 \subseteq U_2 \subseteq \ldots$ of open sets covering $\mathbb{R}^t$ and a closed discrete set $T \subseteq \mathbb{R}^t$, any C^∞ function $g : \mathbb{R}^t \to \mathbb{R}$ can be approximated by an entire function f so that for $k = 1, 2, \ldots$, for all $x \in \mathbb{R}^t \setminus U_k$ and for each multi-index α such that $|\alpha| \le k$,

(a) $|(D^\alpha f)(x) - (D^\alpha g)(x)| < \varepsilon(x)$;
(b) $(D^\alpha f)(x) = (D^\alpha g)(x)$ if $x \in T$.

This theorem has been useful in helping to analyze the existence of entire functions restricting to order-isomorphisms of everywhere non-meager subsets of $\mathbb{R}$, analogous to the Barth-Schneider theorem, which gives entire functions restricting to order-isomorphisms of countable dense sets, and the existence of entire functions f determining cross-sections $f \cap A$ through everywhere non-meager subsets A of $\mathbb{R}^{t+1} \cong \mathbb{R}^t \times \mathbb{R}$ whose projection $\{x \in \mathbb{R}^t : (x, f(x)) \in A\}$ onto $\mathbb{R}^t$ is everywhere non-meager, analogous to the Kuratowski-Ulam theorem which gives for residual sets A in $\mathbb{R}^{t+1}$, points $c \in \mathbb{R}$ so that the horizontal section of A determined by c has a residual projection $\{x \in \mathbb{R}^t : (x, c) \in A\}$ in $\mathbb{R}^t$. The insights gained from this work have also led to variations on the Hoischen theorem that incorporate the ability to require the values of the derivatives on a countable set to belong to given dense sets or to choose the approximating function so that the graphs of its derivatives cut a small section through a given null set or a given meager set. We discuss these results.

Keywords Complex approximation • Interpolation • Hoischen's theorem • Order-isomorphism • Piecewise monotone • Kuratowski-Ulam theorem • Sup-measurable • Oracle-cc forcing

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1 Introduction

In [11], it was shown that it is consistent with the ZFC axioms for set theory that any two subsets A and B of the real line that have cardinality $\aleph_1$ and have nonmeager intersection with each nonempty open interval are order-isomorphic via the restriction to the real line of an entire function g. The construction was quite flexible and the flexibility was indicated by building into the construction that if an order-isomorphism f of the real line of class C^n is given in advance, then g can be taken so that the derivatives of g up to order n approximate those of f as closely as desired. The tool used for arranging the approximation is a theorem of Hoischen [39]. Using this consistency theorem and absoluteness arguments, it was then possible to give a version of the Hoischen theorem that incorporated the ability to arrange for the approximating function (when increasing and onto) to also be an order isomorphism between two given countable dense sets. The ideas used in the one-variable setting in [11] were generalized for functions of several variables in [10] to give the consistency of the statement that for any everywhere nonmeager set $A \subseteq \mathbb{R}^{t+1}$ there is an entire function g of t variables, real-valued on $\mathbb{R}^t$, so that $\{x \in \mathbb{R}^t : (x, g(x)) \in A\}$ is everywhere nonmeager in $\mathbb{R}^t$. In other words, g cuts a large cross-section through A. Again, absoluteness arguments led to a version of the Hoischen theorem with new features. Subsequently the forcing arguments were eliminated from the proofs that originally used absoluteness arguments and a stronger version of the Hoischen theorem allowing control of the values of the derivatives on a countable set was obtained in [15]. In the present paper, we survey the historical context for the results and discuss the results themselves. Section 2 contains the basic approximation tools that we require. Section 3 begins the discussion of controlling the values of derivatives at countably many points. Section 4 deals with order-isomorphisms of dense subsets of $\mathbb{R}$. Section 5 concerns versions of the Kuratowski-Ulam theorem. Section 6 deals with measurability of superpositions $f(x, \varphi(x))$. Section 7 briefly discusses work in progress on piecewise monotone approximation.

We denote the α-th derivative of a function f of t variables by $D^\alpha f$. We use standard multi-index notation for the mixed partial derivatives of a function $f \colon \mathbb{R}^t \to \mathbb{R}$ or $f \colon \mathbb{C}^t \to \mathbb{C}$. If $\alpha = (\alpha_1, \dots, \alpha_t)$ is a sequence of nonnegative integers, then we write

$$|\alpha| = \alpha_1 + \cdots + \alpha_t,\ D^\alpha f = \frac{\partial^{\alpha_1 + \cdots + \alpha_t} f}{\partial^{\alpha_1} z_1 \dots \partial^{\alpha_t} z_t},\ z^\alpha = z_1^{\alpha_1} \dots z_t^{\alpha_t}\ (z \in \mathbb{C}^t)$$

We shall also need some standard measure theoretic and topological terminology. In a measure space $(X, \mathcal{B}, \mu)$, a set $A \subseteq X$ is called *thick* when $A \cap B \neq \emptyset$ for all $B \in \mathcal{B}$ such that $\mu(B) > 0$. If X is a probability space, this is equivalent to saying that A has outer measure one. Note that μ and its completion have the same thick sets. *Null* sets are sets of measure zero for the completion of μ. Recall that when X is a topological space, a set $A \subseteq X$ is called *nowhere dense* if the closure of A

has empty interior. A is said to be *meager* (or *a first category set*) if $A = \bigcup_{n=1}^{\infty} A_n$, where each A_n is nowhere dense. A is said to be *non-meager* (or *a second category set*) if A is not meager. A is *everywhere non-meager*[1] if $A \cap U$ is non-meager for each nonempty open set U. The terms meager, non-meager, and everywhere non-meager are topological analogs of the terms (describing sets in a complete measure space) null, non-null, and thick, respectively. There is also an analog of "measurable". For Lebesgue measure on $\mathbb{R}$, a set A is measurable if and only if $A \triangle B$ is null for some Borel set B. The topological analog is that $A \triangle B$ is meager for some Borel set B, which is equivalent to saying that $A \triangle U$ is meager for some open set U. In any topological space, sets with this property are said to have the *property of Baire*. For a set A having the property of Baire, the property that A is everywhere nonmeager is equivalent to the property that A is residual, i.e., the complement of A is meager. For a real-valued function f on a topological space, the topological analog of the notion of measurability is the property that $f^{-1}(U)$ has the property of Baire when U is open in $\mathbb{R}$. This is equivalent to saying that f has a continuous restriction to a residual G_δ set. We shall say that f is *BP-measurable* when this holds.

2 Approximating Derivatives

We begin with some basic approximation tools. We first introduce a lemma of Whitney which describes a type of approximation that we require frequently.

Lemma 2.1 ([66, Lemma 6]) *Let $U_1 \subseteq U_2 \subseteq \dots$ be open sets in $\mathbb{R}^t$ and write $U = \bigcup_{i=1}^{\infty} U_i$. Then if $f\colon U \to \mathbb{R}$ is of class C^k (k finite or infinite) in U, and $\varepsilon\colon U \to \mathbb{R}$ is a positive continuous function, then there is a real-analytic function $g\colon U \to \mathbb{R}$ such that $|D^\alpha g(x) - D^\alpha f(x)| < \varepsilon(x)$ when $x \in U \setminus U_i$, $|\alpha| \le i$ if k is infinite, $|\alpha| \le k$ if k is finite.*

We have modified Whitney's statement in two ways. Whitney requires that each U_i is bounded with $\operatorname{cl} U_i \subseteq U_{i+1}$, and he has a sequence of positive numbers $\varepsilon_1 \ge \varepsilon_2 \ge \dots$ instead of our function ε, the requirement on the derivatives being that $|D^\alpha g(x) - D^\alpha f(x)| < \varepsilon_i$ when $x \in U \setminus U_i$ and $|\alpha| \le i$ (when k is infinite) or $|\alpha| \le k$ (when k is finite). To see that this is equivalent to our version, let (I) denote Whitney's original lemma, (II) our version above.

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[1]The term is due to Lebesgue [50, p. 185] who applied it inside a "domain" D of $\mathbb{R}^n$. There are some problems with his definition associated primarily with the meaning of the word domain as explained on pp. 143–144. If we interpret domain as meaning "open domain" then Lebesgue's definition of "on D, A is everywhere non-meager in E" is equivalent to saying that as long as $D \cap E \neq \emptyset$ then $A \cap D$ is everywhere non-meager in $E \cap D$ in the sense given here. If we interpret domain as meaning "finite non-degenerate domain" (these are images of closed balls under homeomorphisms of $\mathbb{R}^t$), which is more in keeping with the suggestion on page 144, then domains D with Lebesgue's property do not exist when $A = E$ is a perfect nowhere dense set, contrary to his claim on p. 185.

Proof of Equivalence of (I) *and* (II) (I) $\Rightarrow$ (II) The conclusion of (II) only gets stronger if the sets U_i are smaller, so without loss of generality, we may assume that our given sequence U_i consists of bounded open sets with the closure of each term contained in the next.[2] Since ε is continuous and positive on the compact closure of U_i, $\varepsilon_i = \inf\{\varepsilon(x) : x \in U_{i+1}\}$ is positive, and clearly $\varepsilon_1 \geq \varepsilon_2 \geq \ldots$. If $x \in U \setminus U_i$, then let $m \geq i$ be least such that $x \in U_{m+1}$. Let the multi-index α satisfy $|\alpha| \leq i$ (so $|\alpha| \leq m$) if k is infinite, $|\alpha| \leq k$ if k is finite. The conclusion of (I) gives that $|D^\alpha g(x) - D^\alpha f(x)| < \varepsilon_m \leq \varepsilon(x)$.

(II) $\Rightarrow$ (I) This direction follows readily as long as we can define a continuous ε so that $\varepsilon(x) < \varepsilon_i$ when $x \in U \setminus U_i$. We can assume that the sequence ε_i is strictly decreasing. The assumptions on the sets U_i ensure that they have disjoint boundaries. Define ε to have the value ε_2 on U_1, and the value ε_{i+1} on $\operatorname{bd} U_i$ for $i = 1, 2, \ldots$. For each i, apply the Tietze Extension Theorem to get a continuous extension of ε to the set $\operatorname{cl} U_{i+1} \setminus U_i$, with values in $[\varepsilon_{i+2}, \varepsilon_{i+1}]$. The resulting function ε is as desired. □

In what follows, we have several theorems which assert the existence of a (usually entire) function g with some property P such that, in addition, g can be chosen to approximate smooth functions in the sense of Whitney's lemma above, sometimes additionally with interpolation on a closed discrete set. In order to simplify the statements of these results, we introduce the following terminology.

Definition 2.2 Let $U \subseteq \mathbb{R}^t$ be open. A class S of functions $U \to \mathbb{R}$ is said to *W-approximate*[3] smooth functions if the following holds. Let $f : U \to \mathbb{R}$ and let $\varepsilon : U \to \mathbb{R}$ be a positive continuous function.

(A) Let $k \geq 0$ be an integer. If f is a C^k function then there exists a function $g \in S$ such that $|D^\alpha g(x) - D^\alpha f(x)| < \varepsilon(x)$ for $x \in U$, $|\alpha| \leq k$.

(B) If f is a C^∞ function then for each sequence $U_0 \subseteq U_1 \subseteq \ldots$ of open sets with $U = \bigcup_{i=1}^\infty U_i$, there exists a function $g \in S$ such that for all $i = 0, 1, 2, \ldots$, $|D^\alpha g(x) - D^\alpha f(x)| < \varepsilon(x)$ for $x \in U \setminus U_i$, $|\alpha| \leq i$

We will say of the class S that it *W-approximates smooth functions with interpolation on closed discrete sets* if moreover, whenever we are given a $T \subseteq U$ which is closed discrete in U, we can ask in the conclusion to (A) and (B) that (under the same conditions on α and x) $D^\alpha g(x) = D^\alpha f(x)$ when $x \in T$.

Using this terminology, Lemma 2.1 says that the class of real-analytic functions W-approximates smooth functions on U. The next result, sometimes known as the

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[2]Write $U = \bigcup_{i=1}^\infty V_i$, where $V_1 = \emptyset$, the sets V_i are open and bounded, and $\operatorname{cl} V_i \subseteq V_{i+1}$. Then replace the sequence $U_1, U_2, \ldots$ by a sequence of the form $V_1, \ldots, V_1, V_2, \ldots, V_2, \ldots$, where V_1 is used as the nth term until the first n such that $V_2 \subseteq U_n$ (which exists because $\operatorname{cl} V_2$ is a compact subset of U). Then V_2 is the nth term from that point until the first n such that $V_3 \subseteq U_n$, and so on. Finally, modify this sequence by replacing each constant block $V_i, \ldots, V_i$ by a sequence of the form $V_i^k, \ldots, V_i^{k+m}, V_i$, where $V_i^j = \{x : d(x, V_i^c) > 1/j\}$, starting with k large enough so that $\operatorname{cl} V_{i-1} \subseteq V_i^k$. Note that $\operatorname{cl} V_i^k \subseteq \{x : d(x, V_i^c) \geq 1/k\} \subseteq V_i^{k+1}$.

[3]The W is for Whitney of course.

Walsh lemma, generalizes the well-know theorem of Weierstrass on approximation of continuous functions on compact intervals by polynomials.

Theorem 2.3 (See [27, Corollary 1.3] for Example) *Let $a < b$ be real numbers and let n be a nonnegative integer. Let $T \subseteq [a, b]$ be finite. Suppose $f: [a, b] \to \mathbb{R}$ is a function of class C^n and $\varepsilon > 0$. Then there exists a polynomial g such that for all $k = 0, \ldots, n$ and all $x \in [a, b]$, $|D^k g(x) - D^k f(x)| < \varepsilon$ and moreover, if $x \in T$ then $D^k f(x) = D^k g(x)$.*

Carleman [19] extended the Weierstrass theorem by showing that for any continuous function $f: \mathbb{R} \to \mathbb{R}$ and any continuous positive function $\varepsilon: \mathbb{R} \to \mathbb{R}$, there is an entire function $g: \mathbb{R} \to \mathbb{R}$ such that $|g(x) - f(x)| < \varepsilon(x)$ for all $x \in \mathbb{R}$. (This generalizes the Weierstrass theorem because the Taylor polynomials of g converge uniformly to g on compact intervals.) This was extended further by Hoischen to include approximation of the derivatives, when they exist.

Theorem 2.4 ([39], See also [33]) *The class of functions $g : \mathbb{R}^t \to \mathbb{R}$ which are the restriction of an entire function W-approximates smooth functions.*

The original statement in [39] is for approximating a complex-valued function $f: \mathbb{R}^t \to \mathbb{C}$ (and the approximating function g is then also complex-valued). It can be seen from the original proof (and was explicitly pointed out in [33]) that the argument produces a real-valued g if f is real-valued. Note that we can also, conversely, deduce the original version from the above version by decomposing $f(x) = f_1(x) + if_2(x)$ into its real and imaginary parts and applying the version above to f_1 and f_2. Similarly for Theorem 2.5 below.

For the case $t = 1$, this theorem is improved in [40] to give simultaneously approximation of the derivatives of a smooth function as well as interpolation of the restriction of the derivatives to a closed discrete set. Hoischen's method can be adapted to functions of several variables. The adaptation requires some effort, but it has been written out in detail in [13].

Theorem 2.5 ([40], See also [13]) *The class of functions $g : \mathbb{R}^t \to \mathbb{R}$ which are the restriction of an entire function W-approximates smooth functions with interpolation on closed discrete sets.*

In this paper, we deal only with functions defined on all of $\mathbb{R}$ or $\mathbb{R}^t$, but some of the results adapt to other domains. For functions of one variable, it was pointed out in [38, Theorem 3], that composing with natural analytic correspondences can be useful. For our purposes it matters also that these correspondences are order-preserving. As an example, we apply this procedure to adapt the Hoischen theorem to a bounded interval. Here one only has to be careful in transferring the function $\varepsilon(x)$ from a bounded interval to the real line.

Proposition 2.6 *Let $-\infty \leq a < b \leq \infty$. On the interval (a, b), the real-analytic functions W-approximate smooth functions with interpolation on closed discrete sets. Each real-analytic approximation g is obtained as $g(x) = \hat{g}(s(x))$ where $\hat{g}$ is the restriction to $\mathbb{R}$ of an entire function and s is a real-analytic order-isomorphism $(a, b) \to \mathbb{R}$. We can take*

- $s(x) = \tan(-(\pi/2) + \pi(x-a)/(b-a))$ *if* $-\infty < a < b < \infty$,
- $s(x) = \log(x-a)$ *if* $-\infty < a < b = \infty$,
- $s(x) = -\log(b-x)$ *if* $-\infty = a < b < \infty$,
- $s(x) = x$ *if* $-\infty = a < b = \infty$.

Remark 2.7 In [35, Theorem 1.3] it is shown that if $f\colon (a,b) \to \mathbb{R}$ is a function of class C^k, k finite, and $\varepsilon\colon (a,b) \to \mathbb{R}$ is a positive continuous function, then there exists a function g holomorphic in $\mathbb{C} \setminus ((-\infty,a] \cup [b,\infty))$ such that for all $i = 0,\ldots,k$ and all $x \in (a,b)$, $|D^i g(x) - D^i f(x)| < \varepsilon(x)$. The authors mention that Johanis [43] has shown that for each domain $\Omega \subseteq \mathbb{R}^t$, there is a domain $\tilde{\Omega} \subseteq \mathbb{C}^t$ depending only on Ω such that each C^k function on Ω can be approximated, along with its derivatives of order $\le k$, by functions holomorphic on $\tilde{\Omega}$. However, $\tilde{\Omega}$ is smaller than $\mathbb{C} \setminus ((-\infty,a] \cup [b,\infty))$ when $t = 1$ and $\Omega = (a,b)$ is an interval of $\mathbb{R}$. Proposition 2.6 adds interpolation to the result of [35], and when (a,b) is a bounded interval, we have that $g(z) = \hat{g}(s(z))$ is holomorphic anywhere s is holomorphic. If we take $s(z)$ to be the tangent function as suggested above, then g is holomorphic except at the points $a + j(b-a)$, $j \in \mathbb{Z}$.

Proof Let k be a nonnegative integer or ∞. Let $T \subseteq (a,b)$ be a closed discrete set. If k is infinite, let $U_0 \subseteq U_1 \subseteq \ldots$ be an increasing sequence of open sets which covers (a,b). Suppose $f\colon (a,b) \to \mathbb{R}$ is a function of class C^k and $\varepsilon\colon (a,b) \to \mathbb{R}$ is a positive continuous function. Below, we use (A) to mark the case k finite, (B) to mark the case k infinite.

As is well-known and easily verified by induction on $i = 0,1,\ldots$, there are polynomials $P^i_j(x_1,\ldots,x_i) \in \mathbb{Z}[x_1,\ldots,x_i]$, $j = 0,\ldots,i$, such that the i-th derivative of a composite function $u \circ v$ is given (when the required derivatives of u and v exist) by

$$D^i(u \circ v)(x) = \textstyle\sum_{j=0}^{i} D^j u(v(x)) P^i_j(D^1 v(x),\ldots,D^i v(x)).$$

Fix, as in the statement, an order-preserving real-analytic bijection $s\colon (a,b) \to \mathbb{R}$ which has a real-analytic inverse. Fix a continuous function $\hat{\varepsilon}\colon \mathbb{R} \to \mathbb{R}$ such that for all $x \in (a,b)$,[4]

$$\hat{\varepsilon}(s(x)) < \frac{\varepsilon(x)}{1 + \sum_{j=0}^{i} |P^i_j(D^1 s(x),\ldots,D^i s(x))|} \text{ when } \begin{cases} \text{(A)} & i = 0,\ldots,k \\ \text{(B)} & x \notin U_i, i = 0,1,2,\ldots \end{cases}$$

Define $\hat{f}\colon \mathbb{R} \to \mathbb{R}$ and a closed discrete set $\hat{T} \subseteq \mathbb{R}$ by $\hat{f}(x) = f(s^{-1}(x))$, $\hat{T} = s(T)$. In (B), define an increasing sequence $\hat{U}_1 \subseteq \hat{U}_2 \subseteq \ldots$ of open sets covering

[4] If we take a_ℓ, $\ell \in \mathbb{Z}$ so $\lim_{\ell\to-\infty} a_\ell = a$, $\lim_{\ell\to\infty} a_\ell = b$, then on each $[a_\ell, a_{\ell+1}]$ only finitely many values of i need to be considered and the corresponding continuous functions given by the right-hand side of the inequality have a common lower bound $\delta_\ell > 0$ on this interval, so $\hat{\varepsilon}$ can be taken, for example, to be a suitable continuous piecewise linear modification of $\sum_{\ell\in\mathbb{Z}} \delta_\ell \chi_{[s(a_\ell),s(a_{\ell+1}))}$.

$\mathbb{R}$ by $\hat{U}_i = s(U_i)$. Apply Theorem 2.5 to get a function $\hat{g}: \mathbb{R} \to \mathbb{R}$ which is the restriction to $\mathbb{R}$ of an entire function and satisfies the conclusion (corresponding to Definition 2.2 (A) or (B) as appropriate) of the Hoischen theorem. Of course we put $g(x) = \hat{g}(s(x))$ and g is then a real-analytic function. We also have $f(x) = \hat{f}(s(x))$, so for $x \in T$ and $i = 0, \ldots, k$ (for (A)) or $x \notin U_i$, $i = 0, 1, 2, \ldots$ (for (B)), we have

$$\begin{aligned} D^i g(x) &= \sum_{j=0}^{i} D^j \hat{g}(s(x)) P_j^i(D^1 s(x), \ldots, D^i s(x)) \\ &= \sum_{j=0}^{i} D^j \hat{f}(s(x)) P_j^i(D^1 s(x), \ldots, D^i s(x)) \\ &= D^i f(x) \end{aligned}$$

Finally, for $x \in (a, b)$, when $i = 0, \ldots, k$ (for (A)) or $i = 0, 1, 2, \ldots$ with $x \notin U_i$ (for (B)), we have that $s(x) \notin \hat{U}_i$ (for (B)), so

$$\begin{aligned} |D^i g(x) - D^i f(x)| &= |D^i(\hat{g} \circ s)(x) - D^i(\hat{f} \circ s)(x)| \\ &\leq \sum_{j=0}^{i} |D^j \hat{g}(s(x)) - D^j \hat{f}(s(x))||P_j^i(D^1 s(x), \ldots, D^i s(x))| \\ &\leq \hat{\varepsilon}(s(x)) \sum_{j=0}^{i} |P_j^i(D^1 s(x), \ldots, D^i s(x))| \\ &< \varepsilon(x) \end{aligned}$$

□

3 Controlling the Values of Derivatives on a Countable Set

According to Stäckel [64], Weierstrass produced in 1886 an example of a transcendental entire function f which takes rational values on rational arguments, thereby showing that a function with the latter property need not be a rational function. Stäckel generalized Weierstrass's result by proving the following.

Theorem 3.1 ([64]) *Let A be a countable subset of $\mathbb{C}$ and let B be dense in $\mathbb{C}$. Then there exist transcendental entire functions f such that $f(A) \subseteq B$.*

As Stäckel points out, $\mathbb{C}$ can be replaced in both places by $\mathbb{R}$. There is an enlightening discussion of this result in the context of analyzing the values of analytic functions at algebraic points in Chapter 3 of the text [51]. The function f is stated there to be $f(z) = \sum_{h=0}^{\infty} f_h z^h$ with rational coefficients f_h. That statement has a small error. If $0 \in A$ then we must have $f(0) \in B$ which might preclude the possibility that $f_0 = f(0)$ is rational. That the other coefficients f_h, or equivalently the derivatives $D^h f(0)$, $h \geq 1$, can be taken to be rational is correct (see Theorem 3.3 below), but it does seem to go beyond what Stäckel proved.

In response to a problem [37] regarding the existence of a differentiable function which takes rationals into rationals but whose derivative takes rationals into irrationals, W. Rudin proved the following theorem.

Theorem 3.2 ([59]) *Suppose that*

(1) A is a countable subset of $\mathbb{R}^t$, *and*
(2) for each multi-index α, B_α *is a dense subset of* $\mathbb{R}$.

Then there exists an $f \in C^\infty(\mathbb{R}^t)$ *such that* $D^\alpha f$ *maps A into* B_α, *for every* α.

This result can be improved in a couple of ways. A minor improvement is that the values of $D^\alpha f(p)$ can be restricted separately for each point $p \in A$. (The argument of [59] already gives this with only trivial changes.) A more significant improvement is that f can be taken to be an entire function, as in the theorem of Stäckel. For functions of one variable, this was done in [41].

Theorem 3.3 *Suppose that*

(1) A is a countable subset of $\mathbb{R}^t$, *and*
(2) for each $p \in A$ *and each multi-index* α, $B_{p,\alpha}$ *is a dense subset of* $\mathbb{R}$.

Then there exists a function $f : \mathbb{R}^t \to \mathbb{R}$ *which is the restriction to* $\mathbb{R}^t$ *of an entire function* $\mathbb{C}^t \to \mathbb{C}$ *and satisfies* $D^\alpha f(p) \in B_{p,\alpha}$ *for all* $p \in A$ *and for every multi-index* α.

Theorem 3.6 improves on this by incorporating approximation and interpolation and by adding a type of surjectivity condition stating that the derivatives $D^\alpha f$ can be required to take on many of their allowed values.

Proof For a fixed $p \in A$, list the pairs (p, α), where α is a multi-index, by increasing order of $|\alpha| = \alpha_1 + \cdots + \alpha_t$, without repetitions. Then interleave these orderings to get an enumeration of the pairs (p, α) consisting of a point $p \in A$ and a multi-index α as $\{(p^i, \alpha^i) : i \in \mathbb{N}\}$ in such a way that the following condition is satisfied. (Here $\alpha \leq \beta$ means $\alpha_k \leq \beta_k$ for all $k = 1, \ldots, t$.)

$$\text{If } i < j \text{ and } p^i = p^j \text{ then } \alpha^j \not\leq \alpha^i. \tag{1}$$

We shall build f as a sum

$$f(z) = \sum_{i=1}^{\infty} \lambda_i g_i(z)$$

where $\lambda_i > 0$ and $g_i(z)$ is the polynomial defined as follows. Let

$$S_i = \{j < i : p^j \neq p^i\},$$

and set

$$g_i(z) = (z - p^i)^{\alpha^i} \prod_{j \in S_i} \left(\sum_{k=1}^{t} (z_k - p^j_k)^2 \right)^{|\alpha^j|+1}.$$

All that matters regarding g_i is that

(a) $D^{\alpha^j} g_i(p^j) = 0$ for all $j < i$, whereas
(b) $D^{\alpha^i} g_i(p^i) \neq 0$.

For (a) when $p^j = p^i$, note that by the property (1) of our enumeration, there is at least one coordinate k for which $\alpha^j_k < \alpha^i_k$.

Write $f_i(z) = \sum_{j=1}^{i} \lambda_j g_j(z)$. Choose the coefficients λ_i recursively so that the following conditions are satisfied.

(c) When $|z| \leq i$, $|\lambda_i g_i(z)| \leq 2^{-i}$.
(d) $D^{\alpha^i} f_i(p^i) \in B_{p^i,\alpha^i}$.

To arrange these first choose $r_i > 0$ so that for any z such that $|z| \leq i$, $r_i|g_i(z)| \leq 2^{-i}$. Then from (b) and the fact that B_{p^i,α^i} is dense, we see that we can easily choose λ_i in the interval $(0, r_i)$ so that $D^{\alpha^i} f_i(p^i) = D^{\alpha^i} f_{i-1}(p^i) + \lambda_i D^{\alpha^i} g_i(p^i) \in B_{p^i,\alpha^i}$.

From (c) we get that $f = \sum_{i=1}^{\infty} \lambda_i g_i$ is an entire function. For any i, by (a) and (d) we have

$$\begin{aligned} D^{\alpha^i} f(p^i) &= D^{\alpha^i} f_i(p^i) + \sum_{k=i+1}^{\infty} \lambda_k D^{\alpha^i} g_k(p^i) \\ &= D^{\alpha^i} f_i(p^i) + 0 \in B_{p^i,\alpha^i}. \end{aligned}$$

This completes the proof. □

Note that f is guaranteed to not be a polynomial if for some $p \in A$ we remove 0 from each of the sets $B_{p,\alpha}$.

We now state one of the main theorems of [15]. We require the following definition.

Definition 3.5 A *fiber-preserving local homeomorphism* on $\mathbb{R}^{t+1} \cong \mathbb{R}^t \times \mathbb{R}$ is a homeomorphism $h: G^1_h \to G^2_h$ between two open sets $G^1_h, G^2_h \subseteq \mathbb{R}^{t+1}$ such that h has the form $h(x, y) = (x, h^*(x, y))$ for some continuous map $h^*: \mathbb{R}^{t+1} \to \mathbb{R}$.

Theorem 3.6 ([15, Theorem 3.2]) *Let $A \subseteq \mathbb{R}^t$ be a countable set and for each $p \in A$ and multi-index α, let $A_{p,\alpha} \subseteq \mathbb{R}$ be a countable dense set. Let $\mathcal{H}$ be a countable family of fiber-preserving local homeomorphisms. There exists a function $f: \mathbb{R}^t \to \mathbb{R}$ which is the restriction of an entire function on $\mathbb{C}^t$ and such that for all $k = 0, 1, 2, \ldots$*

(a) *for each $p \in A$ and multi-index α, $(D^\alpha f)(p) \in A_{p,\alpha}$;*
(b) *for each multi-index α, for any $q \in \mathbb{R}$, $h \in \mathcal{H}$ and any open ball $U \subseteq \mathbb{R}^t \setminus T$, if $(x, (D^\alpha f)(x)) \in G^1_h$ and $q = h^*(x, (D^\alpha f)(x))$ for some $x \in U \cap \operatorname{cl} Y_{h,q,\alpha}$, where $Y_{h,q,\alpha} = \{p \in A : \text{for some } q' \in A_{p,\alpha},\ (p, q') \in G^1_h \text{ and } q = h^*(p, q')\}$, then $q = h^*(p, (D^\alpha f)(p))$ for some $p \in U \cap A$.*

Furthermore, the class of such functions f W-approximates smooth functions with interpolation on closed discrete sets that are disjoint from A.

To get a sense of what clause (b) is saying, consider the case where $h = \mathrm{id}$ is the identity map, $h^*(x, y) = y$. We can write the statement of this case as follows. (These are, in condensed form, some comments made in [15].)

Clause (b) for $h = \mathrm{id}$. For each multi-index α, if $x \notin T$, $q = (D^\alpha f)(x)$, and there are points $p \in A$ arbitrarily close to x for which $q \in A_{p,\alpha}$, then there are points $p \in A$ arbitrarily close to x for which $q = (D^\alpha f)(p)$.

For an example where h is not the identity map, consider the case $t = 2$, $\alpha = 0$ (i.e., $\alpha = (0, 0)$). Let $A = \mathbb{Q} \times \mathbb{Q}$ and $A_{p,0} = \mathbb{Q}$ for all $p \in A$. Consider the function f given by the theorem. Let $q \in \mathbb{Q}$ and suppose that the equation

$$q = x_1 f(x_1, x_2) + {x_1}^2 + {x_2}^2 \tag{2}$$

has a solution $a = (a_1, a_2)$ with $a_1 \neq 0$ and that the fiber-preserving local homeomorphism h given by $h^*(x_1, x_2, y) = x_1 y + {x_1}^2 + {x_2}^2$, $x_1 \neq 0$, belongs to $\mathcal{H}$. The assumption that we can find values of $p \in A$ arbitrarily close to a for which there is a rational number q_p satisfying $q = p_1 q_p + {p_1}^2 + {p_2}^2$ is satisfied. (All $p \in A$ with $p_1 \neq 0$ have this property.) The conclusion then says f was constructed so that the points $p \in A$ which satisfy (2) are dense in the set of all solutions to (2).

4 Smooth Order-Isomorphisms

Cantor [17] characterized the rational numbers as the unique denumerable dense linear order without endpoints. It follows that any two countable dense subsets A and B of $\mathbb{R}$ are order-isomorphic. The order-isomorphism is easily seen to extend to an order-isomorphism of $\mathbb{R}$. More generally, we have the following. (Cf. the second paragraph of [31].)

Proposition 4.1 *If $K, L \subseteq \mathbb{R}$ are dense and $h: K \to L$ is an order isomorphism, then h extends to an order isomorphism of $\mathbb{R}$.*

Because order-isomorphisms of $\mathbb{R}$ are homeomorphisms, it follows that K and L must be indistinguishable topologically as subspaces of $\mathbb{R}$. In particular, if one of them is meager then the other must be meager as well.

The extension to an order-isomorphism of $\mathbb{R}$ of an isomorphism between countable dense sets as given by Cantor's theorem is in particular a monotone function and hence differentiable almost everywhere. The question of improving the smoothness of the isomorphism was examined by Franklin [31] who showed that it can be taken to be real-analytic and, on a bounded interval, can be taken to approximate the derivatives of a given real-analytic function.

Theorem 4.2 ([31], Theorem II and Its Corollary) *Given two open intervals (a, b) and (c, d) of $\mathbb{R}$, and any two sets A and B which are countable and dense in (a, b) and (c, d), respectively, there is a real-analytic function $f: (a, b) \to (c, d)$ with positive derivative and such that $f(A) = B$. If a, b, c, d are finite and g is a* *real-analytic function mapping* $[a, b]$ *onto* $[c, d]$ *and having positive derivative on* $[a, b]$*, then* f *can be chosen so that its first* m *derivatives (*m *being any number) approximate those of* g *uniformly.*

Motivated by the problem of finding order-isomorphisms of $[0, 1]$ which map each of the sets of rational, algebraic and transcendental numbers onto themselves, Melzak [53] observes that Franklin's methods show that if $\{A_i\}_{i\in\mathbb{N}}$ and $\{B_i\}_{i\in\mathbb{N}}$ are each a sequence of pairwise disjoint countable dense subsets of $(0, 1)$, then there is an analytic order-isomorphism f of $[0, 1]$ such that for each $i \in \mathbb{N}$ the function f maps A_i onto B_i. Moreover, given any order-preserving homeomorphism g of $[0, 1]$ of class C^n whose derivative is bounded away from zero, f can be chosen so that its first n derivatives are uniformly approximated by those of g.

Thinking of $f(x) = y$ as a two-place relation, Stäckel asked in [65] whether there is an analytic transcendental function f for which rational numbers are related (in either direction) only to rational numbers. In a similar vein, using somewhat ambiguous language, Erdős asked in [28, Problem 24]:

> Does there exist an entire function f, not of the form $f(x) = a_0 + a_1x$, such that the number $f(x)$ is rational or irrational according as x is rational or irrational? More generally, if A and B are two denumerable, dense sets, does there exist an entire function which maps A onto B?

The map in Franklin's result was improved to being the restriction to $\mathbb{R}$ of an entire function by Barth and Schneider [4], thereby solving Erdős's problem if "dense" is interpreted as meaning "dense in the real line". They also state without proof that their method gives the generalization to sequences of pairwise disjoint countable dense sets as obtained by Melzak for analytic functions, but that "the massive amount of bookkeeping involved in this proof is such as to make it impractical to include it in this paper". If in the problem of Erdős referred to above, we interpret "dense" as meaning "dense in the complex plane", then the problem was solved by Maurer [52]. For functions of several variables we have the following theorem of Rosay and Rudin.

Theorem 4.3 ([56, Theorem 2.2]) *Given any two countable dense subsets* X *and* Y *of* $\mathbb{C}^n$ $(n > 1)$*, there is an automorphism (bi-holomorphic map)* F *of* $\mathbb{C}^n$ *so that* $F(X) = Y$ *(and the Jacobian of* F *is identically equal to* 1*).*

An elegant proof of the Barth-Schneider result based on Maurer's work was given by Sato and Rankin [60]. (See also [54] which contains a variation on the same argument.) They make no comment about the result for sequences of pairwise disjoint countable dense sets, but their proof easily yields that version as well. Even though it is included in Theorem 3.6, we record here a direct proof of the Barth-Schneider theorem based on the aforementioned simplifications. These ideas were also used in the proof of Theorem 3.6.

Theorem 4.4 ([4]) *If* A *and* B *are countable dense subsets of* $\mathbb{R}$*, then there is an entire function* f *which restricts to an order-isomorphism of* A *onto* B*.*

Proof Fix one-to-one enumerations $A = \{a_j : j = 1, 2, \ldots\}$ and $B = \{b_j : j = 1, 2, \ldots\}$. Inductively define one-to-one re-enumerations $\{a_{k(n)} : n = 1, 2, \ldots\}$ and $\{b_{l(n)} : n = 1, 2, \ldots\}$ of A and B, respectively so that letting

- $P_1(z) = 1$
- $P_n(z) = (z - a_{k(1)}) \dots (z - a_{k(n-1)})$, n odd
- $P_n(z) = (z - a_{k(1)})^2 (z - a_{k(2)}) \dots (z - a_{k(n-1)})$, $n \geq 2$ even

(one factor is squared in the third clause so that P_n will have odd degree), we can take $f(z) = \lim_{n\to\infty} f_n(z)$ where

$$f_n(z) = c_1 P_1(z) + \cdots + c_n P_n(z)$$

for some numbers c_i chosen so that $f_2'(x) = c_2 P_2'(z) = c_2 > 1$ and for $n \geq 2$, $0 < c_n < r_n$ where r_n is chosen so that

(a) $r_n |P_n(z)| \leq 2^{-n}$ for $|z| \leq n$,
(b) $r_n P_n'(x) > -2^{-n}$ for $x \in \mathbb{R}$.

Let $k(1) = 1$, $l(1) = 1$, $c_1 = b_1$. We have $f_1(a_{k(1)}) = f_1(a_1) = c_1 = b_1 = b_{l(1)} \in B$. Let $l(2) = 2$ and choose any $k(2)$ so that the element $a_{k(2)}$ of A is such that the solution c_2 to the equation $f_2(a_{k(2)}) = b_{l(2)}$, namely

$$c_2 = \frac{b_{l(2)} - c_1}{a_{k(2)} - a_{k(1)}}$$

satisfies $c_2 > 1$.

At a stage $n \geq 2$, choose any r_n so that (a) and (b) hold. If n is even, let $k(n)$ be the least natural number such that $k(n) \neq k(i)$ for $i < n$, and then choose c_n so that $0 < c_n < r_n$ and the number

$$f_n(a_{k(n)} = f_{n-1}(a_{k(n)}) + c_n P_n(a_{k(n)})$$

belongs to B. Set $b_{l(n)} = f_n(a_{k(n)})$. Since $f_n' > 0$, $l(n) \neq l(i)$ for $i < n$.

If n is odd, let $l(n)$ be the least natural number such that $l(n) \neq l(i)$ for $i < n$. Temporarily fix any number c_n such that $0 < c_n < r_n$. Let x be the number such that

$$f_n(x) = f_{n-1}(x) + c_n P_n(x) = b_{l(n)}.$$

Note that x is not a zero of P_n since $f_{n-1}(a_{k(i)}) = b_{l(i)} \neq b_{l(n)}$ for $i = 1, \dots, n-1$. We have

$$c_n = \frac{b_{l(n)} - f_{n-1}(x)}{P_n(x)}.$$

Replace x by a nearby element $a_{k(n)}$ of A so that the condition $0 < c_n < r_n$ is preserved. We have

$$\begin{aligned} f_n'(x) &= c_1 P_1'(x) + c_2 P_2'(x) + \cdots + c_n P_n'(x) \\ &> 1 - 2^{-2} - 2^{-3} - \cdots - 2^{-n} > 1 - 2^{-1} > 0. \end{aligned}$$

□

Cohen [24, 25] showed that $\aleph_1 < \mathfrak{c}$ (the failure of the Continuum Hypothesis) is consistent with the axioms of ZFC (assuming that the axioms of ZFC are consistent). If we assume that $\aleph_1 < \mathfrak{c}$, then it is natural to inquire into the nature of subsets of $\mathbb{R}$ having cardinality $\aleph_1$, into whether, for example, are they measurable or meager. Baumgartner examined the validity of Cantor's isomorphism theorem for countable dense subsets of $\mathbb{R}$ if "countable dense" is replaced by "of cardinality $\aleph_1$ in every interval". In [7], a non-empty set S of real numbers is said to be $\aleph_1$-*dense* if S is without endpoints and there are exactly $\aleph_1$ members of S between any two distinct points of S. In particular, if $S \cap I$ has cardinality $\aleph_1$ for every nonempty open interval I, then S is $\aleph_1$-dense. We shall use the term only in this more restricted sense. Baumgartner proved the following theorem.

Theorem 4.5 ([7]) *If ZFC is consistent then so is the theory ZFC + "all $\aleph_1$-dense sets of reals are order-isomorphic".*

It is shown in [1] that the functions inducing the order-isomorphisms in Baumgartner's theorem cannot in general be taken to be smooth.

Proposition 4.6 ([1, Proposition 9.4]. See also [11, Proposition 1.2], [47, Theorem 1.5]) *There are $\aleph_1$-dense sets $A, B \subseteq \mathbb{R}$ such that for no nonconstant C^1 function $f : \mathbb{R} \to \mathbb{R}$ do we have $f[A] \subseteq B$.*

(The statement in [1] is stronger, but Kunen showed in [47] that the stronger version is false under the Proper Forcing Axiom.) The sets A and B given by the proof of Proposition 4.6 are meager. This leaves open the possibility that there may be a positive result for everywhere nonmeager sets. Shelah proved the following theorem as part of the proof of [61, Theorem 4.7], which states that if ZFC is consistent then so is ZFC + $\mathfrak{c} = \aleph_2$ + "There is a universal (linear) order of power $\aleph_1$."

Theorem 4.7 ([61]) *If ZFC is consistent, then so is ZFC + both of the following statements.*

(a) *There is a non-meager set in $\mathbb{R}$ of cardinality $\aleph_1$.*
(b) *Let A and B be everywhere non-meager subsets of $\mathbb{R}$ of cardinality $\aleph_1$. Then A and B are order-isomorphic.*

An examination of Shelah's model shows that the functions witnessing (b) fail to be differentiable at any constructible real. The main result of [11] builds on the construction of Theorem 4.7 to produce a model where the order-isomorphisms can be taken to be the restriction to $\mathbb{R}$ of an entire function.

Theorem 4.8 ([11, Theorem 1.7]) *If ZFC is consistent, then so is ZFC + $\mathfrak{c} = \aleph_2$ + the following statements.*

(a) *Every non-meager set in $\mathbb{R}$ has a non-meager subset of cardinality $\aleph_1$.*
(b) *For any two sequences, $(A_\alpha : \alpha < \omega_1)$ and $(B_\alpha : \alpha < \omega_1)$, each consisting of pairwise disjoint dense subsets of $\mathbb{R}$, if A_α and B_α are countable for $\alpha < \omega$ and are everywhere non-meager sets of cardinality $\aleph_1$ for $\omega \leq \alpha < \omega_1$, then there*

is an order-isomorphism of $\mathbb{R}$ *which is the restriction of an entire function and such that* $f[A_\alpha] = B_\alpha$ *for every* $\alpha < \omega_1$.

(c) *The class of functions satisfying the conclusion of* (b) *W-approximates smooth nondecreasing surjections* $\mathbb{R} \to \mathbb{R}$.

The proof uses Shelah's oracle-cc (cc = chain condition) forcing technique [62, Chapter IV]. The entire functions in the proof are constructed as the limit of a sequence, uniformly converging on compact sets in $\mathbb{C}$, of entire functions which are real linear combinations of products of the form

$$H(z) \prod_{a \in A} \sin\left(\frac{z-a}{n}\right),$$

where H is an entire function which on $\mathbb{R}$ is positive and such that $H(x)$ converges rapidly to zero as $x \to \pm\infty$, and A is a nonempty finite subset of $\mathbb{R}$ with $n \geq 4|A|$. The reason for choosing this particular family of functions, rather than polynomials as in the proof of Theorem 4.4, lies in the restrictions imposed by the oracle-cc method.

It was pointed out in [11] that from Theorem 4.8 and the Shoenfield Absoluteness Theorem [42, Theorem 98, page 530], we can deduce a version of the Barth-Schneider result with the ability to approximate derivatives.

Proposition 4.9 *For any two sequences,* $\langle A_n : n < \omega \rangle$ *and* $\langle B_n : n < \omega \rangle$*, each consisting of pairwise disjoint countable dense subsets of* $\mathbb{R}$*, there is an order-isomorphism* f *of* $\mathbb{R}$ *which is the restriction of an entire function and is such that* $f[A_n] = B_n$ *for every* $n < \omega$*. The class of such functions W-approximates smooth nondecreasing surjections of* $\mathbb{R}$ *onto itself.*

A proof that does not use forcing can be had by applying Theorem 3.6. (Take $A = \bigcup_{n=1}^{\infty} A_n$ and for $p \in A_n$, set $A_{p,0} = B_n$. By making f and its derivative approximate the identity function and its derivative, we get f so that $Df > 0$ and $f(A_n) = B_n$ for all n.) Another example of a consequence of Theorem 3.6 for isomorphisms of countable dense sets is the following.

Proposition 4.10 ([15, Corollary 1.13]) *For each* $n = 0, 1, 2, \ldots,$ *let* $\{A_{i,n}\}_{i=1}^{\infty}$ *and* $\{B_{i,n}\}_{i=1}^{\infty}$ *be sequences of pairwise disjoint countable dense subsets of* $\mathbb{R}$ *and* $(0, \infty)$*, respectively. Let* $N \in \mathbb{N}$ *and let* $U_1 \subseteq U_2 \subseteq \ldots$ *be a cover of* $\mathbb{R}$ *by open sets. Then there is a function* $f : \mathbb{R} \to (0, \infty)$ *which is the restriction of an entire function and is such that*

(1) *For* $n = 0, \ldots, N$ *and all* $x \in \mathbb{R}$, $D^n f(x) > 0$.
(2) *For* $n = 0, 1, 2, \ldots$ *and all* $x \in \mathbb{R}$ *such that* $x \notin U_n$, $D^n f(x) > 0$.
(3) *For* $n = 0, 1, 2, \ldots,$ $x \in \mathbb{R}$*, and* $y \in (0, \infty)$*, if* $D^n f(x) = y$ *then* $x \in A_{i,n}$ *if and only if* $y \in B_{i,n}$, $i = 1, 2, \ldots$.

5 Smooth Cross-Sections Through Non-meager Sets

The Kuratowski-Ulam theorem is a topological analog of the Fubini theorem for null sets in products of measure spaces.

Theorem 5.1 ([48]) *If X and Y are topological spaces and Y has a countable base, then for each meager set $A \subseteq X \times Y$, the vertical sections A_x, $x \in X$, are meager except for a meager set of points $x \in X$.*[5]

For the product space $\mathbb{R}^2$, the converses of the Kuratowski-Ulam and Fubini theorems are false without a measurability assumption on the set A. The converse of the Kuratowski-Ulam theorem holds when X and Y are Polish spaces (separable completely metrizable spaces) and A has the property of Baire. Reversing the roles of X and Y, we have that if X and Y are Polish spaces then for each set $A \subseteq X \times Y$ having the property of Baire, A is meager if and only if for all but a meager set of $c \in Y$, the section $A \cap (X \times \{c\})$ is meager in $X \times \{c\}$. By taking complements, this can be rephrased as saying that for each set $A \subseteq X \times Y$ having the property of Baire, A is everywhere non-meager if and only if for all but a meager set of $c \in Y$, the section $A \cap (X \times \{c\})$ is everywhere non-meager in X. If A does not have the property of Baire, then this theorem can fail dramatically. For example, using the Axiom of Choice, it is easy to construct sets $A \subseteq \mathbb{R}^{n+1}$, n a positive integer, such that A is everywhere non-meager but no two points of A have a coordinate in common, so that on any hyperplane perpendicular to one of the coordinate axes, A has at most one point. (The set A from the proof of Proposition 5.2 below is one example.) A natural attempt at an alternative to the Kuratowski-Ulam theorem for subsets of $\mathbb{R}^{n+1} \cong \mathbb{R}^n \times \mathbb{R}$ not having the property of Baire involves allowing the section to "bend" instead of having it go straight across. A section $A \cap (\mathbb{R}^n \times \{c\})$ can be thought of as the intersection of A with the (graph of the) constant function with value c. What if we look instead at $A \cap f$ where $f : \mathbb{R}^n \to \mathbb{R}$? If f is a polynomial, then we have the following result. The idea of the proof, from [22], is to take the coordinates of the points of A to be distinct elements of a transcendence base for $\mathbb{R}$ over $\mathbb{Q}$ which has nonempty intersection with every uncountable Borel set.

Proposition 5.2 ([10, Proposition 1.1], cf. [5]) *There is an everywhere non-meager set $A \subseteq \mathbb{R}^{n+1}$ such that for every polynomial function $f : \mathbb{R}^n \to \mathbb{R}$, $A \cap f$ is finite.*

Another limitation on getting a Kuratowski-Ulam theorem for everywhere non-meager sets is that, unlike the situation for sets having the property of Baire, there are sets A for which many translations $x \mapsto f(x) + c$ of any given real-analytic

[5]As pointed out in [55, Chapter 15], the proof of the Kuratowski-Ulam theorem requires only a countable π-base for Y (i.e., a countable collection of nonempty open sets so that each nonempty open sets contains one of them). In [32], a pair of spaces (X, Y) for which the conclusion of the Kuratowski-Ulam theorem holds is called a *K-U pair* and conditions under which a pair of spaces is a K-U pair are studied.

function f are disjoint from A, as in the next proposition which adapts arguments from [22] to a multivariate context.

Proposition 5.3 ([10, Proposition 1.2], cf. [22, Theorem 1 and Corollary 2]) *There is a set $A \subseteq \mathbb{R}$ intersecting every uncountable Borel set such that for any real-analytic function $f: \mathbb{R}^n \to \mathbb{R}$, the set $\{c \in \mathbb{R} : A^{n+1} \cap (f + c) = \emptyset\}$ intersects every uncountable Borel set.*
(It follows easily from the Kuratowski-Ulam theorem that for such a set A, A^{n+1} is everywhere non-meager in $\mathbb{R}^{n+1}$.)

Suppose there is a Lusin set $L \subseteq \mathbb{R}^{n+1}$, i.e., a set which is uncountable but has countable intersection with every meager set. The existence of such a set is independent of the axioms of ZFC, but can be established, for example, using the Continuum Hypothesis or in a model produced by adding uncountably many Cohen reals. By replacing L with the union of its translates by the members of a countable dense set, we can assume that L has uncountable intersection with every ball and hence is everywhere non-meager. We have that $L \cap (f + c)$ is countable for every continuous function $f: \mathbb{R}^n \to \mathbb{R}$ and every $c \in \mathbb{R}$ since the graph of $f + c$ is a closed nowhere dense set. Even if f is merely a Borel function, the Kuratowski-Ulam theorem shows that the graph of f is a meager set in $\mathbb{R}^{n+1}$ and hence the sections $L \cap (f + c)$ are still countable.

In spite of all these examples, it is consistent relative to ZFC that non-meager sets must have large continuous sections. The fundamental result in this direction was proven by Ciesielski and Shelah.

Theorem 5.4 ([21, Theorem 2]) *If ZFC is consistent, then so is ZFC + the following statement.*

Writing $C = \{0, 1\}^{\mathbb{N}}$ for the Cantor set, for every $A \subseteq C \times C$ for which the sets A and $A^c = (C \times C) \setminus A$ are everywhere non-meager in $C \times C$ there is a homeomorphism $f : C \to C$ such that the set $\{x \in C : (x, f(x)) \in A\}$ does not have the property of Baire in C.

Ciesielski and Natkaniec showed that for function on $\mathbb{R}$, the proof of Theorem 5.4 can be adapted to produce order-isomorphisms.

Theorem 5.5 ([22, Theorem 12A]) *If ZFC is consistent, then so is ZFC + $\mathfrak{c} = \aleph_2$ + the following statements.*

(a) *Every everywhere non-meager set in $\mathbb{R}$ has an everywhere non-meager subset of cardinality $\aleph_1$.*
(b) *For every family $\mathcal{A}$ consisting of $\aleph_1$ pairwise disjoint everywhere non-meager sets in $\mathbb{R}^2$, there is an increasing homeomorphism $f: \mathbb{R} \to \mathbb{R}$ such that $A \cap f$ is everywhere non-meager in f for every $A \in \mathcal{A}$.*

The main result of [10] is the following theorem, which shows that consistently for any everywhere non-meager set $A \subseteq \mathbb{R}^{n+1}$, we can find a function $f: \mathbb{R}^n \to \mathbb{R}$ which is the restriction of an entire function $\mathbb{C}^n \to \mathbb{C}$ such that $A \cap f$ is everywhere non-meager relative to the graph of f. The proof builds on the ideas in [21] and on the argument in [11], among other things extending the techniques from the latter to functions of several variables.

Theorem 5.6 ([10, Theorem 1.6]) *If ZFC is consistent, then so is ZFC +* $2^{\aleph_0} = \aleph_2$ *+ the following statements.*

(a) *Every non-meager set in* $\mathbb{R}$ *has a non-meager subset of cardinality* $\aleph_1$.
(b) *For every positive integer t and any everywhere non-meager subsets* E_α *of* $\mathbb{R}^{t+1}$, $\alpha < \omega_1$, *there is a function* $f\colon \mathbb{R}^t \to \mathbb{R}$ *which is the restriction of an entire function and is such that* $\{x \in \mathbb{R}^t : (x, f(x)) \in E_\alpha\}$ *is everywhere non-meager in* $\mathbb{R}^t$ *for every* $\alpha < \omega_1$.
(c) *Suppose that we are additionally given a countable dense set* $A \subseteq \mathbb{R}^n$, *and countable dense sets* $B_x \subseteq \mathbb{R}$, $x \in A$. *Then we may ask that for each* $x \in A$, $f(x) \in B_x$. *Moreover, for any dense* $A' \subseteq A$, *if the sets* B_x, $x \in A'$, *are all equal to some* $B \subseteq \mathbb{R}$ *and for all* $x \in A \setminus A'$ *we have* $B_x \cap B = \emptyset$, *then* $f[A']$ *is an interval of B.*
(d) *The class of functions f satisfying* (b), *or both* (b) *and* (c), *W-approximates smooth functions.*

In [14], it was shown that in (d) we can say “W-approximates smooth functions with interpolation on closed discrete sets that are disjoint from A.” Property (c) was also obtained in a stronger form. From Theorem 3.6 there follows a stronger result for a single everywhere non-meager set when that set is in fact residual. In this case, we prefer to state the result in terms of the meager complement.

Corollary 5.7 ([15, Corollary 1.9]) *For a given meager set E in* $\mathbb{R}^{t+1}$, *Theorem 3.6 holds with the additional clause* (c) *below.*

(c) *for every multi-index* α, $\{x \in \mathbb{R}^t : (x, D^\alpha f(x)) \in E\}$ *is meager.*

We do not know whether the measure theoretic analog of this result, replacing “meager” by “of Lebesgue measure zero,” is true.

Problem 5.8 ([15, Problem 1.10]) Let E be a set of Lebesgue measure zero in the plane. Is there an entire function $f(x) = \sum_{n=0}^{\infty} a_n x^n$ so that $\{x \in \mathbb{R} : (x, f(x)) \in E\}$ has Lebesgue measure zero in $\mathbb{R}$ if we require

(i) f has rational coefficients?
(ii) f takes rational values on rational numbers?

In [15, Theorem 5.2], we showed that Corollary 5.7 does hold for null in the place of meager if we weaken the requirement on the approximating functions to say that they are C^∞ rather than entire. If we give up the control of the values of the derivatives on a countable dense set (which leaves us with Hoischen’s theorem), then we may ask that the approximating function be entire with its derivatives satisfying that for a given null set E, for every multi-index α, $\{x \in \mathbb{R}^t : (x, (D^\alpha f)(x)) \in E\}$ is null. This follows essentially from Fubini’s theorem and the change of variable formula for the Lebesgue integral. (See [15, Theorem 2.6].)

In [57], Rosłanowski and Shelah proved a measure-theoretic analog of Theorem 5.4, showing that if ZFC is consistent then it remains consistent if we also assume the following statement. Here $\{0, 1\}^{\mathbb{N}}$ has its usual probability measure, and $\{0, 1\}^{\mathbb{N}} \times \{0, 1\}^{\mathbb{N}}$ has the product measure.

(RS) For any thick sets $E_\alpha \subseteq \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$, $\alpha < \omega_1$, there is a continuous $h: \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ such that

$$\{x \in \{0,1\}^{\mathbb{N}} : (x, h(x)) \in E_\alpha\}$$

is thick for all $\alpha < \omega_1$.

For our purposes, being able to find a continuous h which gives a large section through *two* sets E_0, E_1 simultaneously would suffice. In the model of [57], $\aleph_1 < \mathfrak{c}$ (specifically $\mathfrak{c} = \aleph_2$), and this is essential because under the Continuum Hypothesis there are thick sets $E \subseteq \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$ so that $E \cap N$ is countable whenever N is null. But the graph of a continuous (or even measurable) function h is null (by Fubini's theorem) so $\{x \in \{0,1\}^{\mathbb{N}} : (x, h(x)) \in E\}$ is countable. In Proposition 5.11, we extend the collection of spaces to which (RS) applies, showing in particular that it applies to subsets of $\mathbb{R}^t \times \mathbb{R}$ with Lebesgue measure on the factors. We shall use the following simple modification of the Tietze Extension Theorem.

Lemma 5.9 *Let F be a continuous real-valued function on a normal space X, and let M be a closed subspace of X. If $g: M \to \mathbb{R}$ is continuous, $\varepsilon > 0$, and $|g(x) - F(x)| < \varepsilon$, $x \in M$, then there is a continuous extension $G: X \to \mathbb{R}$ of g such that $|G(x) - F(x)| < \varepsilon$, $x \in X$.*

Proof The function $g - F$ takes values in $(-\varepsilon, \varepsilon)$ on M. Apply the Tietze Extension Theorem to extend $g - F$ to a continuous function $k: X \to (-\varepsilon, \varepsilon)$. Then $G = F + k$ is as desired. □

The next Proposition is standard. Recall that a Polish space X is a separable topological space which is completely metrizable. G_δ subsets of a Polish space are also Polish spaces [58, Proposition 33 p. 164]. We shall also need that the Jordan measurable open sets (i.e., those whose boundary has measure zero) form a base for the topology of X.[6]

Proposition 5.10 *Let μ be a non-atomic Borel probability measure on a Polish space X. Then for each $0 < \alpha < 1$, there is a Cantor set $K \subseteq X$ such that $\mu(K) = \alpha$ and for some homeomorphism $\varphi: \{0,1\}^{\mathbb{N}} \to K$, μ on K is a scaling (by α) of the image of the standard measure on $\{0,1\}^{\mathbb{N}}$.*

For the purposes of referring to it in the proof of the next proposition, let us say that a Cantor set K in X is *nice* if it is homeomorphic to $\{0,1\}^{\mathbb{N}}$ in such a way that μ on K is a scaling of the standard measure on $\{0,1\}^{\mathbb{N}}$. We have that μ is inner regular for the nice Cantor sets, because it is inner regular for the closed sets, and the proposition above can be applied to closed subspaces (upon re-scaling the measure).

[6]This is true in any completely regular topological probability space. See [6, p. 463] for example. For metric spaces, one can simply note that for each point $p \in X$, only for countably many $\varepsilon > 0$ can the sphere $S_\varepsilon(p) = \{x : d(x,p) = \varepsilon\}$ have positive measure. Hence, the balls $B_\varepsilon(x) = \{x : d(x,p) < \varepsilon\}$ for which $\mu(S_\varepsilon(p)) = 0$ form a base for the topology.

Proof Choosing countably many Jordan measurable open sets which form a base for the topology of X and discarding their boundaries, we are left with a zero-dimensional Polish space on which (the restriction of) μ is a non-atomic Borel probability measure. We now assume that X itself is zero-dimensional. Since Borel probability measures in Polish spaces are inner regular for the compact sets [44, Theorem 17.11], we can find a compact set $L \subseteq X$ with $\mu(L) > \alpha$. Subtracting from L the open sets which intersect L in a set of measure zero, we may assume that all relatively open subsets of L have positive measure, i.e., L is self-supporting. In particular, L has no isolated points, so L is homeomorphic to $\{0, 1\}^{\mathbb{N}}$ (for example see [67, Corollary 30.4]).

Now think of L as being the Cantor middle third set on $\mathbb{R}$, equipped with the image of μ under any homeomorphism of L with the Cantor middle third set. Fix positive numbers ε_n, $n = 0, 1, \ldots$, so that $\mu L = \alpha + \varepsilon_0$, $\varepsilon_{n+1} < \varepsilon_n$, and $\lim_{n\to\infty} \varepsilon_n = 0$.

The map $x \mapsto \mu(L \cap [0, x])$ is continuous since μ is non-atomic. We may therefore find closed subintervals $J_0 = [0, a] \cap L$, $J_1 = [b, 1] \cap L$ of L, with a having no immediate predecessor in L, and b having no immediate successor in L, such that $\mu(J_0) = \mu(J_1)$ and $\mu(J_0) + \mu(J_1) = \alpha + \varepsilon_1$. Note that J_0 and J_1 are disjoint. Continue, recursively choosing closed subintervals J_σ of L, for finite binary sequences σ, so that J_σ has no isolated points and the following hold.

(i) for each σ, $J_{\sigma^\frown 0}$ and $J_{\sigma^\frown 1}$ are pairwise disjoint closed subintervals of J_σ
(ii) for each n, the sets J_σ, $\sigma \in \{0, 1\}^n$, are all of the same measure
(iii) $\mu \bigcup\{J_\sigma : s \in \{0, 1\}^n\} = \alpha + \varepsilon_n$

Note that for any infinite binary sequence σ, we must have that the diameters of the intervals $J_{\sigma|n}$ of L converge to zero. Indeed, the properties above easily give that the measures of the $J_{\sigma|n}$ converge to zero. If the diameters do not, then $\bigcap_{n=1}^{\infty} J_{\sigma|n}$ is a nontrivial subinterval of L of measure zero, contradicting the fact that L is self-supporting. Then under the natural homeomorphism $\{0, 1\}^{\mathbb{N}} \to K = \bigcap_{n=1}^{\infty} \bigcup_{\sigma\in\{0,1\}^n} J_\sigma$, the image of the usual measure on $\{0, 1\}^{\mathbb{N}}$, scaled by α, equals μ. □

Proposition 5.11 *Assume* (RS). *Let X be a Polish space carrying a non-atomic σ-finite Borel measure μ. Equip $\mathbb{R}$ with Lebesgue measure and let $X \times \mathbb{R}$ have the product measure. Let $f: X \to \mathbb{R}$ be continuous, $\varepsilon > 0$. Then for any thick sets $E_\alpha \subseteq X \times \mathbb{R}$, $\alpha < \omega_1$, there is a continuous $h: X \to \mathbb{R}$ such that $|h(x) - f(x)| < \varepsilon$ for all $x \in X$ and*

$$\{x \in X : (x, h(x)) \in E_\alpha\}$$

is thick for all $\alpha < \omega_1$. If $L \subseteq X$ is a closed set of measure zero, we may also require that $h(x) = f(x)$ when $x \in L$.

Proof We shall work for simplicity with a single set $E \subseteq X\times\mathbb{R}$, but the general case is proven by simply rewording each claim about E to say the same thing about all sets E_α simultaneously. Since the notion of a thick set depends only on which Borel sets have measure zero, we may replace μ by a probability measure having the same null sets as μ. Hence we may assume that μ is a probability measure. (If μ is the zero measure then all sets are thick. If not, then we can write $X = \bigcup_{n=1}^{\infty} A_n$, where the sets A_n are pairwise disjoint and $0 < \mu(A_n) < \infty$. The probability measure ν given by $\nu(S) = \sum_{n=1}^{\infty} 2^{-n}\mu(S \cap A_n)/\mu(A_n)$ has the same null sets as μ.)

We shall construct a uniformly converging sequence of continuous functions $h_n: X \to \mathbb{R}$ as well as an increasing sequence of closed sets $M_n \subseteq X$ so that

(1) $M_0 = L, h_0 = f$
(2) $h_{n+1}(x) = h_n(x), x \in M_n$
(3) $|h_{n+1}(x) - h_n(x)| < \varepsilon/2^{n+1}, x \in X$
(4) $\mu(M_n) > 1 - 2^{-n}$
(5) $\mu^*\{x \in M_n : (x, h_n(x)) \in E\} = \mu(M_n)$

If we achieve this, then setting $h(x) = \lim_{n\to\infty} h_n(x)$ works. (3) ensures that $\{h_n(x)\}$ is a Cauchy sequence and $\{h_n\}$ converges uniformly to $h(x)$. By (2), $h(x) = h_n(x)$ for $x \in M_n$, so that

$$\mu^*\{x \in X : (x, h(x)) \in E\} \geq \mu^*\{x \in M_n : (x, h(x)) \in E\} = \mu(M_n)$$

and by (4) it follows that $\mu^*\{x \in X : (x, h(x)) \in E\} = 1$.

For the inductive step of the construction, let $U_n = X \setminus M_n$. Let $\{V_j\}$ be a maximal collection of disjoint Jordan measurable open subsets of U_n on each of which the variation of h_n is less than $\varepsilon/2^{n+2}$ and so that $\mu(\bigcup_j V_j) = \mu(U_n)$. (Cover U_n by countably many Jordan measurable B_j on each of which the variation of h_n is less than $\varepsilon/2^{n+2}$, and let V_j be the interior of $B_j \setminus \bigcup_{i<j} B_i$.)

Fix $N \in \mathbb{N}$ and $\varepsilon_j > 0, j = 1, \ldots, N$. By Proposition 5.10, there are nice Cantor sets $K_j \subseteq \operatorname{int} V_j$, $\mu(K_j) > \mu(V_j) - \varepsilon_j$. (4) will hold for $n + 1$ if we set $M_{n+1} = M_n \cup \bigcup_{j=1}^{N} K_j$, as long as we take N large enough and each of the $\varepsilon_j, j = 1, \ldots, N$, small enough. Fix any point $p_j \in K_j$ and choose a nice Cantor set $I_j \subseteq \mathbb{R}$ so that $h_n(p_j) \in I_j$ and the diameter of I_j is less than $\varepsilon/2^{n+2}$.

From (RS) we get a continuous function $u_j: K_j \to I_j$ so that $\mu^*(\{x \in K_j : (x, u_j(x)) \in E\} = \mu K_j$. We have $|u_j(x) - h_n(x)| < \varepsilon/2^{n+1}$ for each $x \in K_j$, because

$$|u_j(x) - h_n(x)| \leq |u_j(x) - h_n(p_j)| + |h_n(p_j) - h_n(x)|$$

and we have on the one hand that $u_j(x)$ and $h_n(p_j)$ both belong to I_j, and on the other hand p_j and x both belong to $K_j \subseteq V_j$. Use Lemma 5.9 to extend $(h_n|M_0) \cup \bigcup_{j=1}^{N} u_j$ to a continuous function $h_{n+1}: X \to \mathbb{R}$ so that $|h_{n+1}(x) - h_n(x)| < \varepsilon/2^{n+1}$ for all $x \in X$. This completes the construction. □

Unlike the proof of Theorem 5.4, the proof of (RS) has so far resisted attempts to produce an analog for subsets of the plane in which the functions h are smooth, or even monotone, let alone analytic.

Problem 5.12 Is there a thick subset E of the plane such that for no function $f\colon \mathbb{R} \to \mathbb{R}$ which is the restriction of an entire function on $\mathbb{C}$ do we have that $\{(x, f(x)) : x \in \mathbb{R}\}$ is thick in $\mathbb{R}$?

6 Measurability of $f(x, \varphi(x))$

One source of interest in the ability to find nice cross sections through subsets of $\mathbb{R} \times \mathbb{R}$ is the study of classes of functions $f(x, y)$ for which the differential equation

$$\frac{d\varphi(x)}{dx} = f(x, \varphi(x)) \tag{3}$$

has a solution in a suitable sense. The Cauchy-Peano existence theorem (see [23, Theorem 1.3]) states that if f is continuous on an open set $U \subseteq \mathbb{R}^2$ and $(x_0, y_0) \in U$ then there is a (not necessarily unique) function φ on an open interval I containing x_0 so that for $x \in I$, we have $(x, \varphi(x)) \in U$ and (3) holds. Since the composition $x \mapsto f(x, \varphi(x))$ is continuous, so is $D\varphi$. Hence, φ is a C^1 function. The condition that $x \mapsto f(x, \varphi(x))$ is continuous whenever $\varphi(x)$ is continuous implies that f is continuous. (See [45, Theorem 2]) Hence, it is natural to require that f is continuous if we want C^1 solutions. The functions φ needed in the proof of continuity of f can be taken to be C^∞ except at one point. The exceptional point is needed however as it is not true that continuity of $f(x, \varphi(x))$ when φ is differentiable implies continuity of f.

Example 6.2 Define $f\colon \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x, y) = \begin{cases} y^2/|x| & \text{if} |x| \geq y^2 \text{ and } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \\ 1 & \text{otherwise} \end{cases}$$

f is not continuous but the superpositions $f(x, \varphi(x))$ are continuous for all continuous functions φ such that $D\varphi(0)$ exists if $\varphi(0) = 0$.

Proof Since $f(x, y) = 1$ when $x = y^2$, except that $f(0, 0) = 0$, f is not continuous at $(0, 0)$. However, f is continuous on $\mathbb{R}^2 \setminus \{(0, 0)\}$ since on this set the formulas for f on $|x| \geq y^2$ and $|x| \leq y^2$ define continuous functions and they agree on the intersection $|x| = y^2$. Hence for continuous $\varphi(x)$, $f(x, \varphi(x))$ is continuous at any point $x \neq 0$ and even at $x = 0$ if $\varphi(0) \neq 0$. For any differentiable function φ such that $\varphi(0) = 0$, let M be any positive number larger than $|D\varphi(0)|$. Then for some $\delta > 0$ we have for $0 < |x| < \delta$ that

$$\frac{|\varphi(x)|}{|x|} = \left| \frac{\varphi(x) - \varphi(0)}{x - 0} \right| < M$$

and hence $|\varphi(x)| < M|x|$. Since $\varphi(x)$ is also continuous at 0, by shrinking δ we can also ensure that $|\varphi(x)| < 1/M$ for $0 < |x| < \delta$. The points $(x, \varphi(x))$ for $0 < |x| < \delta$ thus belong to the open set $U = \{(x, y) : |y| < M|x|, |y| < 1/M\}$. On U we have $|x| > |y|/M \geq y^2$, so

$$f(x, y) = \frac{y^2}{|x|} < \frac{M^2x^2}{|x|} = M^2|x| \to 0 \text{ as } x \to 0$$

Hence, f is continuous on $U \cup \{(0, 0)\}$ and therefore $f(x, \varphi(x))$ is continuous at $x = 0$. □

For the function of Example 6.2, there is a continuous function $\varphi(x)$ which satisfies $D\varphi(x) = f(x, \varphi(x))$ except at $x = 0$ (where it is not differentiable), namely

$$\varphi(x) = \begin{cases} -1/\log x & \text{if } x > 0 \\ 1/\log(-x) & \text{if } x < 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Carathéodory ([18, §§576–592], [29, §1], [46, §17]) examined less restrictive assumptions on f which are sufficient to ensure that when φ is measurable, the function $f(x, \varphi(x))$ will be integrable. He shows in particular that if $f(x, y)$ is continuous in y and measurable in x, then for each measurable function $\varphi(x)$, $x \mapsto f(x, \varphi(x))$ is measurable. By Lebesgue's proof that separately continuous functions on the plane are of Baire class one [50, page 201], it follows that f is also Lebesgue measurable.[7] If $x \mapsto f(x, \varphi(x))$ is measurable whenever φ is measurable, we say f is *superposition measurable*, or *sup-measurable*.[8] In [46, §17], superposition is studied as a (non-linear) operator on φ for a fixed f. In [46, §17.8], it is pointed out that, by Lusin's theorem, measurability of $x \mapsto f(x, \varphi(x))$ for continuous φ implies superposition measurability of f. There is a natural topological analog of the notion of sup-measurability, namely if X is a topological space then say that $f: X \times \mathbb{R} \to \mathbb{R}$ is *BP-sup-measurable* if whenever $\varphi: X \to \mathbb{R}$ is BP-measurable, so is the superposition $f(x, \varphi(x))$. We get an equivalent definition if we require BP-measurability of the superposition only when φ is a Borel function, but the definition is weaker if we require the condition only for continuous φ.

[7]Given $f(x, y)$ continuous in y and measurable in x, let $f_n(x, y)$ agree with $f(x, y)$ when $y = k/n$, $k \in \mathbb{Z}$, and interpolate linearly between adjacent points k/n, i.e., when $k/n \leq y \leq (k + 1)/n$, $f_n(x, y) = f(x, k/n) + n(y - k/n)(f(x, (k + 1)/n) - f(x, k/n))$. f_n is a measurable function since sums and products of measurable functions are measurable. Then $\lim_{n\to\infty} f_n(x, y) = f(x, y)$ is measurable.

[8]Carathéodory did not name the concept. The long form of the name, as "superpositionally measurable," is from (the Russian edition of) [46], the short form from [63]. For more on this concept see [26, Definition 2.5.25] and the results that follow it, or the papers [36, 49].

Example 6.3 ([3, Remark 3, p. 790]) There is a function $f: \mathbb{R}^2 \to \mathbb{R}$ such that the superpositions $f(x, \varphi(x))$ are BP-measurable whenever $\varphi: \mathbb{R} \to \mathbb{R}$ is continuous, but f is not BP-sup-measurable.

In this section we discuss conditions which ensure sup-measurability in the topological or in the measure-theoretic context. There are easy examples of measurable functions $f: \mathbb{R}^2 \to \mathbb{R}$ which are not sup-measurable. Examples of sup-measurable functions which are not measurable can be produced under the Continuum Hypothesis and under some weaker assumptions, but in [57] a model was constructed where all sup-measurable functions are measurable.

We begin with an observation regarding the possibility of solving a differential equation (3) when f is very pathological function. Our treatment is adapted from [45]. There is a thick set $A \subseteq \mathbb{R}^2$ which is the graph of a one-to-one function. Moreover, $A \cap \varphi$ has cardinality less than $\mathfrak{c}$ whenever $\varphi: \mathbb{R} \to \mathbb{R}$ is continuous.[9] Since A is the graph of a function, it has inner measure zero. Letting f denote the characteristic function of A, we have that f is a nonmeasurable function for which the following result holds.

Theorem 6.4 (Cf. [45, Theorem 4]) *With f as above, consider the differential equation $\varphi'(x) = f(x, \varphi(x))$.*

(1) *Each constant function φ satisfies the equation except possibly at one point $x \in \mathbb{R}$.*
(2) *Any differentiable function φ which satisfies this equation except possibly at less than $\mathfrak{c}$ points of $\mathbb{R}$ is constant.*
(3) *Any locally absolutely continuous function φ which satisfies this equation almost everywhere is constant.*

There are competing meanings of a solution to a differential equation $\varphi'(x) = f(x, \varphi(x))$ when f is discontinuous. See the introduction, as well as §4, of the text [29], and also [8], for a discussion of this point. In the Carathéodory theory, (3) above is the standard interpretation. (See [58, p. 108] for the theory of absolutely continuous functions on a compact interval. Locally absolutely continuous on $\mathbb{R}$ means absolutely continuous on compact subintervals of $\mathbb{R}$.) Note that it follows that if we fix a point $(x_0, y_0) \in \mathbb{R} \times \mathbb{R}$ then given the initial value condition $\varphi(x_0) = y_0$ there is a unique solution φ in the sense of (2) or (3) which satisfies the initial value condition. This solution, by (1), satisfies the equation except possibly at one point. (But the point $x = x_0$ itself might be the exceptional one.)

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[9] Here we identify φ with its graph in $\mathbb{R}^2$. Sketch of construction of A: Identify the cardinal $\mathfrak{c}$ of the continuum with the least ordinal of cardinality $\mathfrak{c}$. List the compact subsets of $\mathbb{R}^2$ of positive measure as $(K_\alpha : \alpha < \mathfrak{c})$, and the continuous functions $\mathbb{R} \to \mathbb{R}$ as $(\varphi_\alpha : \alpha < \mathfrak{c})$. By Fubini's Theorem, $m(E_\alpha) > 0$ where $E_\alpha = \{x \in \mathbb{R} : m((K_\alpha)_x) > 0\}$. ($m$ denotes Lebesgue measure on $\mathbb{R}$.) In particular, E_α has cardinality $\mathfrak{c}$, and $(K_\alpha)_x$ has cardinality $\mathfrak{c}$ for each $x \in E_\alpha$. Recursively choose points $(x_\alpha, y_\alpha) \in \mathbb{R}^2$ so that $x_\alpha \in E_\alpha \setminus \{x_\beta : \beta < \alpha\}$ and $y_\alpha \in (K_\alpha)_{x_\alpha} \setminus (\{y_\beta : \beta < \alpha\} \cup \{\varphi_\gamma(x_\alpha) : \gamma < \alpha\})$. Then set $A = \{(x_\alpha, y_\alpha) : \alpha < \mathfrak{c}\}$.

Proof

(1) Fix $k \in \mathbb{R}$ and let $\varphi(x) \equiv k$. Since A is the graph of a one-to-one function, it has at most one point on each horizontal line, and hence $f(x, \varphi(x)) = f(x, k) = 0 = \varphi'(x)$ except possibly for one value of x.
(2) Let φ be a differentiable function and suppose that $\varphi'(x) = f(x, \varphi(x))$ except possibly at less than $\mathfrak{c}$ points. By the choice of A, $f(x, \varphi(x)) = 0$ except for less than $\mathfrak{c}$ values of x. Thus, $\varphi'(x) = 0$ except at less than $\mathfrak{c}$ points of $\mathbb{R}$. Since derivatives are Darboux functions (i.e., they satisfy the intermediate value property), we must have $\varphi'(x) = 0$ for all $x \in \mathbb{R}$. (If φ' takes both zero and nonzero values, then by the intermediate value property it takes on continuum many values and hence is nonzero at continuum many points.[10]) Hence φ is constant.
(3) This time φ is absolutely continuous and we have $\varphi'(x) = f(x, \varphi(x))$ holding except on a set K of measure zero. By the choice of A, $f(x, \varphi(x)) = 0$ except on a set L of cardinality less than $\mathfrak{c}$. This gives that $M = \mathbb{R} \setminus \{x \in \mathbb{R} : \varphi'(x) = 0\} \subseteq K \cup L$. But M is measurable and cannot have positive measure or else $L \supseteq M \setminus K$ would have cardinality $\mathfrak{c}$. Thus, $\varphi'(x) = 0$ almost everywhere, and hence φ is constant (since absolutely continuity implies $\varphi(x) = \varphi(x_0) + \int_{x_0}^{x} \varphi'(t)\, dt$). □

We now turn to the question of whether sup-measurable functions must be measurable. The next proposition is a standard manipulation for which we cannot find a suitable reference, so we give a proof. Here, μ denotes Lebesgue measure on both the unit interval and the unit square, with the context distinguishing the two.

Proposition 6.5 *Let $A \subseteq [0, 1]^2$ be compact.*

(1) *The union B of all $\{x\} \times [a, b]$ where $0 \le a < b \le 1$ and $\mu(A_x \cap (a, b)) = 0$ is a $G_{\delta\sigma}$ set and $A \cap B$ has measure zero.*
(2) *The function $h\colon [0, 1]^2 \to [0, 1]^2$ defined by*

$$h(x, y) = (x, \mu(A_x \cap [0, y]))$$

is Borel measurable. The restriction $h|A$ is inverse measure preserving as a map from A onto $h(A)$, and maps $A \setminus B$ bijectively onto $h(A) \setminus h(B)$.

Note that $h(A)$ is the region of the square $[0, 1]^2$ on and under the graph of the Borel function $x \mapsto \mu(A_x)$. In particular, it is a Borel set. Then $h(A \setminus B)$ and $h(B)$ are disjoint analytic sets which partition $h(A)$ and hence, by the Lusin Separation Theorem [44, Theorem 14.7], they are Borel sets.

[10] Cf. Bruckner [9, Theorem 1.1]. In [20, Proposition 4] there is a direct proof that a differentiable function whose derivative is zero except at countably many points is constant. The proof works also for "less than $\mathfrak{c}$" instead of "countably many".

Proof

(1) If $[0, 1]^2 \setminus A = \bigcup_{n=1}^{\infty} U_n \times V_n$, U_n and V_n open intervals of $[0, 1]$, then writing (r, y) to mean the interval $r < t < y$ if $r < y$ and the interval $y < t < r$ if $y < r$, we have that

$$(x, y) \in B \Leftrightarrow \text{there are } a < b \text{ with } a \leq y \leq b \text{ and } \mu(A_x \cap (a, b)) = 0$$
$$\Leftrightarrow \text{there is a rational number } r \neq y \ \text{ with } \mu(A_x \cap (r, y)) = 0$$

$\mu(A_x \cap (y, r)) = 0$ says that for each rational $s > 0$, there is an N such that

$$\mu \bigcup_{n=1}^{N} \{V_n \cap (y, r) : x \in U_n\} > |y - r| - s.$$

The set of (x, y) satisfying this latter condition is open. The set of (x, y) satisfying $\mu(A_x \cap (y, r)) = 0$ is therefore G_δ and B is therefore $G_{\delta\sigma}$.

To compute the measure of $A \cap B$, note that for a given x, the union of all open intervals (a, b) with rational endpoints such that $\mu(A_x \cap (a, b)) = 0$ is an open set, and B_x is the union of the closures of its components. Hence $(A \cap B)_x$ has measure zero. By Fubini's theorem, $\mu(A \cap B) = 0$.

(2) Since $\mu(A_x \cap [0, y])$ is Borel measurable as a function of x and continuous as a function of y, by the argument of Lebesgue mentioned in the paragraph before Example 6.3, it is Borel measurable as a real-valued function on $[0, 1]^2$. Thus, the given function h is Borel measurable. h is fiber-preserving, and on A_x, if $y \notin B_x$ then $h(y') < h(y)$ whenever $y' < y$ and $h(y) < h(y')$ whenever $y < y'$, so f is one-to-one on $A \setminus B$. This also shows that $h(A \setminus B)$ is disjoint from $h(B)$. The inclusion $h(A) \setminus h(B) \subseteq h(A \setminus B)$ is true for any function and any sets, so we get $h(A \setminus B) = h(A) \setminus h(B)$.

As noted above, $h(A)$ is the region of the square $[0, 1]^2$ under the graph of the Borel function $x \mapsto \mu(A_x)$. Given a rectangle $[0, a] \times [0, b]$, its intersection E_{ab} with $h(A)$ has a section at x equal to the interval $[0, \min(b, \mu(A_x))]$. Hence the measure of this set, by Fubini's theorem is $\int_0^a \min(b, \mu(A_x))\, dx$. On the other hand, the pre-image of E_{ab} under $h|A$ is

$$A \cap \{(x, y) : 0 \leq x \leq a,\ \mu(A_x \cap [0, y]) \leq b\}.$$

By the Fubini theorem its measure is $\int_0^a \min(b, \mu(A_x))\, dx = \mu E_{ab}$. Since the collection of sets E_{ab} is closed under intersection, contains $E_{1,1} = h(A)$, and generates the Borel σ-algebra of $h(A)$, and the collection of Borel sets $E \subseteq h(A)$ for which $\mu E = \mu(A \cap h^{-1}(E))$ is a Dynkin system, $h|A$ is inverse measure preserving. $\square$

Theorem 6.6 (Cf. [57], the Proof of $(\boxtimes)^4_{\text{sup}} \Rightarrow (\boxtimes)^3_{\text{sup}}$ on p. 92) *Assume* (RS). *Let X be a Polish space carrying the completion μ of a non-atomic σ-finite Borel* *measure. Equip $\mathbb{R}$ with Lebesgue measure and let $X \times \mathbb{R}$ have the (complete) product measure. Then every sup-measurable function $f: X \times \mathbb{R} \to \mathbb{R}$ is measurable.*

Proof It is enough to prove the case where f is the characteristic function of a set E. (See [2, Proposition 1.5].) Suppose E is not measurable. We wish to find a Borel function $\varphi: X \to \mathbb{R}$ so that the superposition $f(x, \varphi(x))$ is not measurable, i.e., the set $\{x \in X : (x, \varphi(x)) \in E\}$ is not measurable. Since E is a non-measurable set, there is a compact set A in $X \times \mathbb{R}$ of positive measure such that $E \cap A$ and $A \setminus E$ are both thick in A. A has intersection of positive measure with one of the sets $X \times [n, n+1]$, $n \in \mathbb{Z}$, so we may assume that $A \subseteq X \times [0, 1]$. For some $\delta > 0$, we can choose a compact set $K_\delta \subseteq \{x \in X : \mu(A_x) > \delta\}$ of positive measure. Then by intersecting A with $K_\delta \times \mathbb{R}$, we may assume that all nonempty vertical sections of A have measure $> \delta$.

If we find a Borel function $\varphi: K_\delta \to [0, 1]$ such that $\{x \in K_\delta : (x, \varphi(x)) \in E\}$ is not measurable, then we are done by setting $\varphi(x) = 0$ when $x \in X \setminus K_\delta$. We may take K_δ to be a nice Cantor set by Proposition 5.10. We can re-scale the measure on K_δ so that it becomes homeomorphic in a measure preserving manner to $\{0, 1\}^{\mathbb{N}}$ with the usual product measure. Via the binary expansion map $k: \{0, 1\}^{\mathbb{N}} \to [0, 1]$ (which becomes bijective upon removing from $[0, 1]$ the set D of dyadic rational numbers and from $\{0, 1\}^N$ the set C of eventually constant sequences), we then have a measure preserving continuous surjection $g: K_\delta \times [0, 1] \to [0, 1]^2$ given by $g(x, y) = (k(x), y)$. A is carried to a compact set $g(A)$ whose vertical sections all have measure $> \delta$.

Let $E_0 = (E \cap A) \setminus (D \times [0, 1])$, $E_1 = (A \setminus E) \setminus (D \times [0, 1])$. E_0 and E_1 are disjoint thick subsets of A. Also, $g(E_0)$ and $g(E_1)$ are disjoint thick subsets of $g(A)$. Apply Proposition 6.5 to $g(A)$ to get $h: [0, 1]^2 \to [0, 1]^2$ and $B \subseteq [0, 1]^2$. Write $F_i = g(E_i) \setminus B$, $i = 0, 1$. Then $h(F_0)$ and $h(F_1)$ are disjoint thick subsets of $h(g(A))$, and $h(g(A))$ contains $[0, 1] \times [0, \delta]$. By Proposition 5.11, there is a continuous function $\varphi_0: [0, 1] \to [0, \delta]$ such that the sets

$$\{x \in [0, 1] : (x, \varphi_0(x)) \in h(F_0)\} \quad \text{and} \quad \{x \in [0, 1] : (x, \varphi_0(x)) \in h(F_1)\}$$

are both thick in $[0, 1]$. Note that the graph of φ_0 almost avoids the set $h(B)$ because the Borel set $U = \{x : (x, \varphi_0(x)) \in h(B)\}$ and the thick set $\{x : (x, \varphi_0(x)) \in h(F_0)\}$ are disjoint (the sets $h(F_0)$ and $h(B)$ are disjoint as $h(F_0) \subseteq h(g(A) \setminus B) = h(g(A)) \setminus h(B)$), so U has measure zero. Pulling $\varphi_0|([0, 1] \setminus U)$ back under h, we get a Borel subset φ_1 of $[0, 1]^2$ which is a function defined on $[0, 1] \setminus U$ (because h is fiber-preserving and its restriction to $A \setminus B$ is one-to-one) and hence is a Borel function. It has the property that

$$\{x \in [0, 1] \setminus U : (x, \varphi_1(x)) \in F_0\} \quad \text{and} \quad \{x \in [0, 1] \setminus U : (x, \varphi_1(x)) \in F_1\}$$

are both thick in $[0, 1]$. Pulling φ_1 back under g, we get a Borel subset φ_2 of $K_\delta \times [0, 1]$ which is a function defined on a Borel set S of measure one in K_δ and hence is a Borel function. It has the property that

$$S_0 = \{x \in S : (x, \varphi_2(x)) \in E_0\} \quad \text{and} \quad S_1 = \{x \in S : (x, \varphi_2(x)) \in E_1\}$$

are both thick in K_δ. Extend φ_2 to K_δ by setting $\varphi_2(x) = 0$ when $x \notin S$. Since $T = \{x \in K_\delta : (x, \varphi_2(x)) \in E\}$ contains S_0 and is disjoint from S_1, T is not measurable.

□

Let us introduce the following statement, which is analogous to (RS). It is similar to the conclusion of Theorem 5.4 and holds in the model of [21] with only minor changes to the construction. Cf. Lemma 5, p. 163 of [21].

(CS) For any everywhere non-meager sets $E_\alpha \subseteq \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$, $\alpha < \omega_1$, there is a continuous $h: \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ such that

$$\{x \in \{0,1\}^{\mathbb{N}} : (x, h(x)) \in E_\alpha\}$$

is everywhere non-meager for all $\alpha < \omega_1$.

As for Theorem 6.6, we have the following.

Theorem 6.7 (Cf. [21], the Proof of Corollary 3 on p. 161) *Assume* (CS). *Let X be a perfect Polish space. Then every BP-sup-measurable function $f: X \times \mathbb{R} \to \mathbb{R}$ is BP-measurable.*

It is necessary to assume that X is perfect, as can be seen by considering the case where X is a one-point space. Every function $f: X \times \mathbb{R} \to \mathbb{R}$ is then BP-sup-measurable.

Proof It is enough to prove the case where f is the characteristic function of a set E. (See [2, Proposition 1.5].) Suppose E does not have the property of Baire. We wish to find a Borel function $\varphi: X \to \mathbb{R}$ so that the superposition $f(x, \varphi(x))$ is not BP-measurable, i.e., the set $\{x \in X : (x, \varphi(x)) \in E\}$ does not have the property of Baire. Since E does not have the property of Baire, there are open sets $U \subseteq X$ and $V \subseteq \mathbb{R}$ such that E and its complement are both everywhere non-meager in $U \times V$.

Every perfect Polish space has a dense G_δ copy of the Baire space $\mathbb{N}^{\mathbb{N}}$. (See the proof of [16, Proposition 2.1], for example.) Because U and V are perfect Polish spaces, there are dense G_δ subspaces $U_0 \subseteq U$ and $V_0 \subseteq V$ homeomorphic to the Baire space. Then E and its complement are everywhere non-meager in $U_0 \times V_0$. If we find a Borel function $\varphi: U_0 \to V_0$ such that $\{x \in U_0 : (x, \varphi(x)) \in E\}$ and $\{x \in U_0 : (x, \varphi(x)) \notin E\}$ are everywhere non-meager in U_0, then we are done by setting $\varphi(x) = 0$ when $x \in X \setminus U_0$.

Upon removal of a countable set of points, the Cantor set $\{0,1\}^{\mathbb{N}}$ becomes homeomorphic to the Baire space. Hence we may think of our product $U_0 \times V_0$ as being a dense G_δ subset of $\{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$. Write K for the meager complement of this dense G_δ. From (CS) we then get a continuous function $\varphi: \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ such that

$$\{x \in \{0,1\}^{\mathbb{N}} : (x, \varphi(x)) \in E\} \quad \text{and} \quad \{x \in \{0,1\}^{\mathbb{N}} : (x, \varphi(x)) \notin E\}$$

are both everywhere non-meager. Note that the graph of φ almost avoids the meager set K because $\{x : (x, \varphi(x)) \in E\}$ and $\{x : (x, \varphi(x)) \in K\}$ are disjoint and the first is everywhere non-meager while the second is a Borel set, so $M = \{x : (x, \varphi_0(x)) \in K\}$ is meager. The restriction of φ to $U_0 \setminus M$, extended to U_0 by making it constant on $U_0 \cap M$, is the desired function. □

When $X = \mathbb{R}^t$, we get a stronger result in the model for Theorem 5.6. From that model we extract the following property.

$(*)_t$ For any everywhere non-meager sets $E_\alpha \subseteq \mathbb{R}^{t+1}$, $\alpha < \omega_1$, and any continuous function $g: \mathbb{R}^t \to \mathbb{R}$ and $\varepsilon > 0$, there is a function $h: \mathbb{R}^t \to \mathbb{R}$ which is the restriction of an entire function on $\mathbb{C}^t$ such that $|h(x) - g(x)| < \varepsilon$ for all $x \in \mathbb{R}^t$, and

$$\{x \in \mathbb{R}^t : (x, h(x)) \in E_\alpha\}$$

is everywhere non-meager for all $\alpha < \omega_1$.

Theorem 6.8 *Assume* $(*)_t$. *Let* $f: \mathbb{R}^t \times \mathbb{R} \to \mathbb{R}$. *Suppose* $f(x, \varphi(x))$ *is BP-measurable whenever* $\varphi: \mathbb{R}^t \to \mathbb{R}$ *is the restriction of an entire function. Then* f *is BP-measurable.*

Under $(*)_t$, our hypothesis that $f(x, \varphi(x))$ is BP-measurable whenever $\varphi: \mathbb{R}^t \to \mathbb{R}$ is an entire function thus sits between the statements that f is BP-sup-measurable and that f is BP-measurable. By Example 6.3, we could not include in our conclusion that f is BP-sup-measurable. (The function of Example 6.3 is easily seen directly to be BP-measurable because it is the characteristic function of a meager set.)

Proof If f is not BP-measurable, then for some $c \in \mathbb{R}$, the set $E = \{(x, y) \in \mathbb{R}^t \times \mathbb{R} : f(x, y) < c\}$ does not have the property of Baire. Therefore, there is a cube I in $\mathbb{R}^{t+1}$ such that E and its complement are both everywhere non-meager in I. By translating and scaling, we may take I to be the unit cube $[0, 1]^{t+1}$. Apply $(*)_t$ to the everywhere non-meager sets

$$(E \cap [0, 1]^{t+1}) \cup (\mathbb{R}^{t+1} \setminus [0, 1]^{t+1}) \quad \text{and} \quad ([0, 1]^{t+1} \setminus E) \cup (\mathbb{R}^{t+1} \setminus [0, 1]^{t+1})$$

with $g(x) \equiv 1/2$ and $\varepsilon = 1/2$ to get an entire function $\varphi: \mathbb{R}^t \to \mathbb{R}$ such that when $x \in [0, 1]^t$ we have $0 < \varphi(x) < 1$, and the sets

$$\{x \in [0, 1]^t : (x, \varphi(x)) \in E\} \quad \text{and} \quad \{x \in [0, 1]^t : (x, \varphi(x)) \notin E\}$$

are everywhere non-meager in $[0, 1]^t$ and hence

$$\{x \in \mathbb{R}^t : (x, \varphi(x)) \in E\} = \{x \in \mathbb{R}^t : f(x, \varphi(x)) < c\}$$

does not have the property of Baire. Thus, $f(x, \varphi(x))$ is not BP-measurable. □

7 Piecewise Monotone Approximation

Analytic functions on an interval of $\mathbb{R}$ are piecewise monotone. It seems natural that when they are being used to approximate piecewise monotone functions the approximating function and the functions being approximated will be "co-monotone," i.e., increasing and decreasing on the same intervals. This problem has been considered at length in the context of approximation by polynomials on compact intervals. For example, see [30], or [34, Chapter 1]. For monotone approximation, we have the following result.

Theorem 7.1 ([30, Theorem 2]) *Let $k_1 < k_2 < \ldots < k_p$ be fixed positive integers and let $\varepsilon_1, \ldots, \varepsilon_p$ be fixed signs (i.e., $\varepsilon_j = \pm 1$). Suppose $f \in C^k[a,b]$ and $k_p \leq k$. Assume*

$$\varepsilon_i D^{k_i} f(x) > 0 \text{ for } a < x < b \text{ and } i = 1, \ldots, p.$$

Suppose $m+1$ points are given so that

$$a \leq x_0 < x_1 < \cdots < x_m \leq b.$$

Then for n sufficiently large there are polynomials P_n, of degree less than or equal to n for which

$$\varepsilon_j D^{k_j} P_n(x) > 0 \text{ on } [a,b], j = 1, 2, \ldots, p,$$

$$P_n(x_i) = f(x_i),\ i = 0, l, \ldots, m,$$

$$\max_{a \leq x \leq b} |f(x) - P_n(x)| \leq (C/n^k)\omega(1/n),$$

where C is a constant depending only on $x_0, \ldots, x_m$, and ω is the modulus of continuity of $D^k f$ on $[a,b]$.

In [12], the following result was obtained.

Theorem 7.2 ([12, Theorem 1.2]) *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a nondecreasing continuous function with open range. Let $\varepsilon\colon \mathbb{R} \to \mathbb{R}$ be a positive continuous function. Let $T \subseteq R$ be a closed discrete set on which f is strictly increasing.*

(A) *Suppose that for some nonnegative integer k, f is a C^k function. Then there is a function $g\colon \mathbb{R} \to \mathbb{R}$ which is the restriction of an entire function and is such that the following properties hold.*

  (a) *For all $x \in \mathbb{R} \setminus T$ (and also for $x \in T$ if $k = 0$), $Dg(x) > 0$.*
  (b) *For $i = 0, \ldots, k$ and all $x \in \mathbb{R}$, $|D^i f(x) - D^i g(x)| < \varepsilon(x)$.*
  (c) *For $i = 0, \ldots, k$ and all $x \in T$, $D^i f(x) = D^i g(x)$.*

(B) *Suppose that* f *is a* C^∞ *function and* $U_1 \subseteq U_2 \subseteq \dots$ *is a sequence of open sets covering* $\mathbb{R}$. *Then there is a function* $g: \mathbb{R} \to \mathbb{R}$ *which is the restriction of an entire function and is such that the following properties hold.*

(a) *For all* $x \in \mathbb{R} \setminus T$, $Dg(x) > 0$
(b) *For* $i = 0, 1, 2, \dots$ *and all* $x \in \mathbb{R} \setminus U_i$, $|D^i f(x) - D^i g(x)| < \varepsilon(x)$.
(c) *For* $i = 0, 1, 2, \dots$ *and all* $x \in T \setminus U_i$, $D^i f(x) = D^i g(x)$.

In [12, Corollary 1.5] we also incorporated some control over the values of g on countable sets. The main technical challenge in the proof of Theorem 7.2 is maintaining the positive derivative for g in a neighborhood of a point $x \in E$ where f is flat in the sense that all derivatives which are known to exist are equal to zero. This problem was handled by temporarily tilting the graph of f upwards so that the derivative is strictly positive in a neighborhood of x, and then restoring the zero derivative after approximation by an entire function. The approximation problem still remains to be solved, but now the point x is no longer flat. The same method of proof gives the following result for functions on compact intervals.

Theorem 7.3 ([12, Theorem 1.3]) *Let* $a < b$ *be real numbers and let* k *be a nonnegative integer. Suppose* $f: [a, b] \to \mathbb{R}$ *is a nondecreasing function of class* C^k *and* $\varepsilon > 0$. *Let* $T \subseteq R$ *be a finite set on which* f *is strictly increasing. Then there is a polynomial* g *such that the following properties hold.*

(a) *For all* $x \in [a, b] \setminus T$ (and also for $x \in T$ if $n = 0$), $Dg(x) > 0$.
(b) *For* $i = 0, \dots, k$ *and all* $x \in \mathbb{R}$, $|D^i f(x) - D^i g(x)| < \varepsilon$.
(c) *For* $i = 0, \dots, k$ *and all* $x \in T$, $D^i f(x) = D^i g(x)$.

In work in preparation, the author has improved the technique to give a version of Theorem 3.6 for piecewise monotone functions. (On compact intervals we do not get the control of the derivatives on a countable set if we want polynomial approximations of course, but via Proposition 2.6 we can get it for real-analytic approximations.)

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Approximation by Solutions of Elliptic Equations and Extension of Subharmonic Functions

Paul Gauthier and Petr V. Paramonov

Dedicated to the memory of André Boivin

Abstract In this review we present the main results jointly obtained by the authors and André Boivin (1955–2014) during the last 20 years. We also recall some important theorems obtained with colleagues and give new applications of the above mentioned results. Several open problems are also formulated.

Keywords Elliptic operator • Banach space of distributions • Approximation on closed sets • L-analytic and L-meromorphic functions • Localization operator • C^m-extension of subharmonic functions

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1 Introduction

Let L be a homogeneous elliptic partial differential operator with constant *complex* coefficients (such as powers of the Cauchy-Riemann operator $\overline{\partial}$ in $\mathbf{C}$ or the Laplacean Δ in $\mathbf{R}^n$, $n \geq 2$). In [29] and [7], given a Banach space $(V, \|\cdot\|)$ of functions (distributions) on $\mathbf{R}^n$, $n \geq 2$, the problem of approximating solutions of the equation $Lu = 0$ on a closed subset F of $\mathbf{R}^n$, in the norm $\|\cdot\|$, by global (L-analytic or L-meromorphic) solutions of the equation was studied. Approximation

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theorems of Runge-, Roth-, Nersesyan- and Arakelyan-type (see [2, 24, 32, 34]) were obtained whenever the operator L and the Banach space V satisfied certain natural conditions.

In [3] the results of [29] and [7] were generalized to Banach spaces of functions (distributions) defined on any domain Ω of $\mathbf{R}^n$ ($n \geq 2$) and several new results based on recent new theorems of Vitushkin [37] type were formulated and discussed.

Using results on the solution of the Dirichlet problem for strongly elliptic equations in bounded smooth domains [1], we found in [3] (see Proposition 2 below) that an application of our theorems gave important new results in the theory of better-than-uniform approximation (see [3, Th. 4 (iii)]), which led to some very interesting examples on the possible boundary behaviour of solutions of homogeneous elliptic partial differential equations, analogous to those described in [11, Chapter IV, §5B] for holomorphic functions and in [12, §8] for harmonic functions.

In the present paper we also consider several settings of the C^m-subharmonic extension problem on domains in $\mathbf{R}^n$ and on open Riemann surfaces. The problem was completely solved (for all $m \in [0, +\infty)$) for the so-called *Runge*-type extensions. Several (in some sense sharp) sufficient conditions and counterexamples were found also for the *Walsh*-type extensions. As applications, these results allowed us to prove the existence of C^m-subharmonic extensions, automorphic with respect to some appropriate groups of automorphisms of an open Riemann surface.

Most of our results are based on Vitushkin's localization technique [37], generalized (here we cite only [25] and [36]) for approximations by solutions of a wide class of elliptic equations in different norms. When recalling previously published theorems, we shall say very little regarding the proofs. While this is mostly a survey of works closely related to the work of André Boivin, we shall also state some new theorems and formulate several related open problems in this topic. Some of our joint results and related problems are not included in the present survey (see [14] and it's extensions [15, 38]).

2 Definitions and Notations

For the reader's convenience, we summarize the definitions and main notations of [7, 29] and [3].

Let Ω be any fixed domain in $\mathbf{R}^n$, $n \geq 2$. We let $V = V(\Omega)$ stand for a Banach space, whose norm is denoted by $\| \quad \|$, which contains $C_0^\infty(\Omega)$, the set of test functions in Ω and is contained in $(C_0^\infty(\Omega))^*$, the space of distributions on Ω. We make some additional assumptions on V.

Conditions 1 and 2 *We assume that V is a topological $C_0^\infty(\Omega)$-submodule of $(C_0^\infty(\Omega))^*$, which means that for $f \in V$ and $\varphi \in C_0^\infty(\Omega)$, we have $\varphi f \in V$ with*

$$\|\varphi f\| \leq C(\varphi)\|f\| \tag{1}$$

and

$$|\langle f, \varphi \rangle| \leq C(\varphi)\|f\|, \tag{2}$$

where $\langle f, \varphi \rangle$ *denotes the action in* Ω *of the distribution* f *on the test function* φ *and* $C(\varphi)$ *is a constant independent of* f. *We note that this implies that the imbeddings* $C_0^\infty(\Omega) \hookrightarrow V$ *and* $V \hookrightarrow (C_0^\infty(\Omega))^*$ *are continuous (see [7, section 2.1]).*

Given a closed subset F in Ω, let $I(F)$ be the closure in V of (the family of) those $f \in V$ whose support in Ω in the sense of distributions (which will be denoted by $\text{supp}(f)$) is disjoint from F, and let $V(F) = V/I(F)$. The Banach space $V(F)$, endowed with the quotient norm, should be viewed as the natural (Whitney type) version of V on F (see [35, Chapter 6]). We shall write $\|f\|_F$ for the norm of the equivalence class (jet) $f_{(F)} := f + I(F)$ in $V(F)$ of the distribution $f \in V$:

$$\|f\|_F = \inf\{||g|| \ : \ g \in f_{(F)}\}\,.$$

For any open set D in Ω, let

$$V_{loc}(D) = \{f \in (C_0^\infty(D))^* \ : \ f\varphi \in V \ \text{ for each } \varphi \in C_0^\infty(D)\},$$

where φ and $f\varphi$ are extended to be identically zero in $\Omega \setminus D$. We endow $V_{loc}(D)$ with the projective limit topology of the spaces $V(K)$ partially ordered by inclusion of the compact sets $K \subset D$. For a closed set F in Ω, define $V_{loc}(F) = V_{loc}(\Omega)/J(F)$, where $J(F)$ is the closure in $V_{loc}(\Omega)$ of the family of those distributions in $V_{loc}(\Omega)$ whose support is disjoint from F. The topology on $V_{loc}(F)$ will be the quotient topology. Note that for compact sets K, the topological spaces $V(K)$ and $V_{loc}(K)$ are identical.

For $f \in V_{loc}(\Omega)$, we put $f_{(F),loc} := f + J(F)$. If D is a neighbourhood of F in Ω, then each $h \in V_{loc}(D)$ naturally defines an element (jet) $h_{(F),loc}$ in $V_{loc}(F)$ by taking $h_{(F),loc}$ to be the closure in $V_{loc}(\Omega)$ of the set of $f \in V_{loc}(\Omega)$ such that $f = h$ (as distributions) in some neighbourhood (depending on f) of F. In particular, this works for each $h \in C^\infty(D) \subset V_{loc}(D)$. For $f_{(F),loc} \in V_{loc}(F)$, we shall write $f_{(F),loc} \in V(F)$ (*or more briefly* $f \in V(F)$), if $V \cap f_{(F),loc} \neq \emptyset$. We shall then write $\|f_{(F),loc}\|_F$, or equivalently $\|f\|_F$, to mean $\|g\|_F$, where $g \in V \cap f_{(F),loc}$. Practically the same proof as in [7, section 2.1] shows that $V \cap J(F) = I(F)$ holds for each closed set F in Ω, which means that $\|f_{(F),loc}\|_F$ is well-defined.

For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$, with $\alpha_j \in \mathbf{Z}_+ (:= \{0, 1, 2, \ldots\})$, we let $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\alpha! = \alpha_1! \ldots \alpha_n!$, $x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$ for $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$ and $\partial^\alpha = (\partial/\partial x_1)^{\alpha_1} \ldots (\partial/\partial x_n)^{\alpha_n}$.

We denote by $B(a, \delta)$ (respectively $\overline{B}(a, \delta)$) the open (respectively closed) ball with center $a \in \mathbf{R}^n$ and radius $\delta > 0$. If $B = B(a, \delta)$ and $\theta > 0$ then $\theta B = B(a, \theta\delta)$ and $\theta\overline{B} = \overline{B}(a, \theta\delta)$.

Throughout this paper we let $L(\xi) = \sum_{|\alpha|=r} a_\alpha \xi^\alpha$, $\xi \in \mathbf{R}^n$, be a fixed homogeneous polynomial of degree r ($r \geq 1$) with *complex* constant coefficients and which satisfies the ellipticity condition $L(\xi) \neq 0$ for all $\xi \neq 0$. We associate to L the homogeneous elliptic operator of order r

$$L = L(\partial) = \sum_{|\alpha|=r} a_\alpha \partial^\alpha.$$

Let D be an open set in $\mathbf{R}^n$ and denote by $L(D)$ the set of distributions f in D such that $Lf = 0$ in D in the sense of distributions. It is well known [17, Theorem 4.4.1] that $L(D) \hookrightarrow C^\infty(D)$. Therefore if $D \subset \Omega$, then $L(D) \subset V_{loc}(D)$, and if $\{f_m\}$ is a sequence in $L(D)$ with $f_m \to f$ in $V_{loc}(D)$ as $m \to \infty$, then $f \in L(D)$, since convergence in $V_{loc}(D)$ is stronger than convergence in the sense of distributions, which preserves $L(D)$ [17, Theorem 4.4.2].

Functions from $L(D)$ will be called *L-analytic* in D. We shall also say that a distribution g in D is *L-meromorphic* in D if $\mathrm{supp}(Lg)$ is discrete in D and for each $a \in \mathrm{supp}(Lg)$ $(a \in D)$ there exist h, which is L-analytic in a neighbourhood of a, $k \in \mathbf{Z}_+$ and $\lambda_\alpha \in \mathbf{C}$, $0 \le |\alpha| \le k$, $\lambda_\alpha \ne 0$ for some α with $|\alpha| = k$, such that

$$g(x) = h(x) + \sum_{|\alpha|\le k} \lambda_\alpha \partial^\alpha \Phi(x-a)$$

in some neighbourhood of a, where Φ is a special fundamental solution of L as described in [17, Theorem 7.1.20]. The points $a \in \mathrm{supp}(Lg)$ will be called the *poles* of g.

We recall (see [10, p. 239] or [36, p. 163]) that there exists a $k > 1$ such that if T is a distribution with compact support contained in $B(a,\delta)$ and $f = \Phi * T$, then, for $|x-a| > k\delta$, we have the *Laurent-type expansion*:

$$f(x) = \langle T(y), \Phi(x-y)\rangle = \sum_{|\alpha|\ge 0} c_\alpha \partial^\alpha \Phi(x-a), \tag{3}$$

where $c_\alpha = (-1)^{|\alpha|}(\alpha!)^{-1}\langle T(y), (y-a)^\alpha\rangle$. The series converges in $C^\infty(\{|x-a| > k\delta\})$, which means that the series can be differentiated term by term and all such series converge uniformly on $\{|x-a| \ge k'\delta\}$, $k' > k$.

Let $\varphi \in C_0^\infty(\Omega)$. The *Vitushkin localisation operator* $\mathcal{V}_\varphi : (C_0^\infty(\Omega))^* \to (C_0^\infty(\Omega))^*$ associated to L and φ is defined as $\mathcal{V}_\varphi f = (\Phi * (\varphi Lf))|_\Omega$, where in the last equality $*$ denotes the convolution operator in $\mathbf{R}^n$.

Condition 3 *We require that for each $\varphi \in C_0^\infty(\Omega)$, the operator $\mathcal{V}_\varphi$ be invariant on $V_{loc}(\Omega)$, i.e. $\mathcal{V}_\varphi$ must send continuously $V_{loc}(\Omega)$ into $V_{loc}(\Omega)$. This means that if K is a compact subset of Ω and* $\mathrm{supp}(\varphi) \subset K$, *then for each $f \in V_{loc}(\Omega)$ one has $\mathcal{V}_\varphi f \in V_{loc}(\Omega)$ and*

$$\|\mathcal{V}_\varphi f\|_K \le C\|f\|_K, \tag{4}$$

where C is independent of f.

In connection with Condition 3 see the basic results [25].

We make one more assumption on V in relation with L.

Condition 4 *For each open ball B with $3\overline{B} \subset \Omega$, there exist $d > 0$ and $C > 0$ such that for each $h \in C^\infty(\mathbf{R}^n)$ satisfying $Lh = 0$ outside of B and $h(x) = O(|x|^{-d})$ as $|x| \to \infty$, one can find $v \in L(\Omega)$ with*

$$(h - v) \in V \quad \text{and} \quad \|h - v\| \leq C\|h\|_{3\overline{B}}. \tag{5}$$

In this assumption, instead of the constant 3, one can take any fixed real number greater than 1.

Here we recall some remarks on Conditions 1–4.

All Conditions 1–4 are satisfied by classical (non-weighted) spaces on any domain Ω in $\mathbf{R}^n$, for example $BC^m(\Omega)$, $BC^{m+\mu}(\Omega)$, $VMO(\Omega)$ and the Sobolev spaces $W^p_m(\Omega)$, $1 \leq p < \infty$. We shall give the definitions and precisely formulate this assertion only for the spaces $V = BC^m(\Omega)$ and $BC^{m+\mu}(\Omega)$.

For $m \in \mathbf{Z}_+$, let $BC^m(\Omega)$ be the space of all m - times continuously differentiable functions $f : \Omega \to \mathbf{C}$ with (finite) norm

$$||f||_{m,\Omega} = \max_{|\alpha| \leq m} \sup_{x \in \Omega} |\partial^\alpha f(x)|.$$

If $m \in \mathbf{Z}_+$ and $0 < \mu < 1$, then

$$BC^{m+\mu}(\Omega) = \{f \in BC^m(\Omega) \ : \ \omega^m_\mu(f, \infty) < \infty \ \text{ and } \omega^m_\mu(f, \delta) \to 0 \quad \text{as } \delta \to 0\},$$

where $\omega^m_\mu(f, \delta) = \sup \frac{|\partial^\alpha f(x) - \partial^\alpha f(y)|}{|x-y|^\mu}$, the supremum being taken over all multi-indices α such that $|\alpha| = m$ and all $x, y \in \Omega$ with $0 < |x - y| < \delta$. The norm in this space is defined as

$$||f||_{m+\mu,\Omega} = \max\{||f||_{m,\Omega}, \ \omega^m_\mu(f, \infty)\}.$$

We shall omit the index Ω in the latter norm whenever $\Omega = \mathbf{R}^n$. Finally, for any $m \geq 0$, we set $C^m(\Omega) = (BC^m(\Omega))_{loc}$.

Proposition 1 ([3, p. 949]) *Let Ω be a domain in $\boldsymbol{R}^n$, $n \geq 2$, and let $m \geq 0$. Then the pair $(L, V(\Omega))$ with $V(\Omega) = BC^m(\Omega)$ satisfies Conditions 1, 2, 3 and satisfies Condition 4 with $v = 0$.*

In [7, Corollary 1] (see also the brief discussion thereafter) and [29, Theorem 4] one sees how (whenever Conditions 1–3 are satisfied) Condition 4 can affect L-meromorphic and L-analytic approximation in the special case of weighted uniform holomorphic approximation ($n = 2, L = \overline{\partial}$).

The following proposition (see [3, p. 950]) provides us with another class of examples for which Conditions 1–4 are satisfied. These in turn allowed us to obtain (see Theorem 4 (iii–iv) below) new results on better-than-uniform approximation. Given m and q in $\mathbf{Z}_+$, with $q \leq m$, and a bounded domain Ω, set

$$BC^m_q(\Omega) = \{f \in BC^m(\Omega) \ \mid \ \text{for each } \alpha, \ |\alpha| \leq q, \ \lim_{x \to \partial\Omega} \partial^\alpha f(x) = 0\},$$

which is a Banach space with the norm $\|f\|_{m,\Omega}$.

Proposition 2 *Let L be a strongly elliptic operator of order $r = 2\ell$, $\ell \in \mathbf{Z}_+$, $\ell \geq 1$ (see [1, p.46]). Let $m, q \in \mathbf{Z}_+$, $m \geq \ell - 1$, $q \leq \ell - 1$. If Ω is bounded and $\partial\Omega$ is of class C^s, $s = max\{2\ell, [n/2] + 1 + m\}$ (see [1, p.128]), then the pair $(L, V = BC_q^m(\Omega))$ satisfies Conditions 1–4.*

In certain cases, we can weaken the restrictions on the smoothness of the domain Ω. Indeed, in the case $L = \Delta$ ($\ell = 1$) and $m = q = 0$ (uniform norm) the conclusion of the previous Proposition in fact holds for *every* bounded regular domain Ω; for $L = \Delta$ and $m = 1, q = 0$, it is sufficient that Ω be a Dini-Lyapunov domain (see [39, Theorems 2.2–2.5]).

3 Approximation Theorems

For an open set W in $\mathbf{R}^n$ denote by $W^* = W \cup \{*\}$ the one point compactification of W. As in [3, Section 4], a closed set F in a domain Ω will be called a Roth-Keldysh-Lavrent'ev set in Ω, or simply an Ω-*RKL set*, if $\Omega^* \setminus F$ is connected and locally connected. In this section we formulate our main approximation results [3], starting with sufficient conditions for approximation of Runge-type on closed sets.

Theorem 1 *Let Ω be a domain in $\boldsymbol{R}^n$, $n \geq 2$. Let $(L, V(\Omega))$ be a pair satisfying Conditions 1–4, F be a (relatively) closed subset of Ω, and f be L-analytic in some neighbourhood of F in Ω. Then, for each $\varepsilon > 0$, there exists an L-meromorphic function g on Ω with poles off F such that $(f_{(F),loc} - g_{(F),loc}) \in V(F)$ and*

$$\|f - g\|_F < \varepsilon.$$

Moreover, if F is an Ω-RKL set, then g can be chosen in $L(\Omega)$.

The next theorem deals with approximation of an *individual* function and shows that the problem is essentially local.

Theorem 2 *Let Ω be a domain in $\boldsymbol{R}^n$ $(n \geq 2)$, $(L, V(\Omega))$ be a pair satisfying Conditions 1–4, F be a (relatively) closed subset of Ω, and $f \in V_{loc}(\Omega)$. Then the following are equivalent:*

(i) *for each positive number ε, there exists an L-meromorphic function g in Ω with poles off F such that $(f_{(F),loc} - g_{(F),loc}) \in V(F)$ and $\|f - g\|_F < \varepsilon$;*
(ii) *for each ball B, $\overline{B} \subset \Omega$ and positive number ε, there exists g such that $Lg = 0$ on some neighbourhood of $F \cap \overline{B}$ and $\|f - g\|_{F \cap \overline{B}} < \varepsilon$;*
(iii) *the previous property is satisfied by each ball from some locally finite family of balls $\{B_j'\}$ covering F, where $\overline{B_j'} \subset \Omega$ for each j.*

For any subset X of $\mathbf{R}^n$, we let $L(X)$ stand for the collection of all functions f defined and L-analytic in some neighbourhood (depending on f) of X. For a closed set F in Ω we denote by $M_{LV}(F)$ (respectively $E_{LV}(F)$) the space of all $f_{(F),loc} \in V_{loc}(F)$ which satisfy the following property: for each $\varepsilon > 0$ there exists an L-

meromorphic function g in Ω with poles outside of F (respectively a function $g \in L(\Omega)$) such that $f - g \in V(F)$ and $||f - g||_F < \varepsilon$.

Problem 1 Given L and V (under Conditions 1–4), for which closed sets F does one have $M_{LV}(F) = E_{LV}(F)$?

We also introduce the space $V_L(F) = V_{loc}(F) \cap L(F^\circ)$, where F° means the interior of F. Whenever Conditions 1–4 hold, we have that by Theorem 1, $M_{LV}(F)$ is the closure in $V_{loc}(F)$ of the space $\{h_{(F),loc} \in V_{loc}(F) \mid h \in L(F)\}$. Moreover, if F is an Ω-RKL set, then $M_{LV}(F) = E_{LV}(F)$. On the other hand, in our paper [4] with Boivin, for $n = 2$ and $L = \overline{\partial}^2$, we give an example of a closed set F, which is not Ω-RKL, but $M_{LV}(F) = E_{LV}(F)$.

So, it is reasonable to discuss the *necessity* of F being a Ω-RKL set for the equality $M_{LV}(F) = E_{LV}(F)$ to be satisfied.

Let K be a compact set in Ω. Denote by $\widehat{K}$ the union of K and all the (connected) components of $\Omega \setminus K$ which are pre-compact in Ω. Obviously, the property $\widehat{K} = K$ means precisely that $\Omega^* \setminus K$ is connected, so that K is a Ω-RKL set.

Define

$$N(K) = N_{LV}(K) = \{a \in \widehat{K} \setminus K \ : \ (\Phi_a)_{(K)} \notin E_{LV}(K)\},$$

where $\Phi_a(x) = \Phi(x - a)$.

Condition N *We shall say that a pair $(L, V(\Omega))$ satisfies Condition N ("nonremovability of holes") if $N(K) \neq \emptyset$ for each compact set K with "holes", i.e. such that $K \neq \widehat{K}$.*

The same proof as in [7, Proposition 2] gives the following auxiliary result.

Proposition 3 *A pair $(L, V(\Omega))$ satisfies Condition N whenever all of the following conditions hold:*

(1) *$(L, V(\Omega))$ satisfies Conditions 1 and 2;*
(2) *$n = 2$ or $n \geq 3$ and L has the following symbol:*

$$L(\xi) = P_2(\xi) Q_{r-2}(\xi), \quad \xi \in \mathbf{R}^n,$$

where P_2 is some homogeneous (elliptic) polynomial of order two with real coefficients (so that P_2 has constant sign in $\mathbf{R}^n \setminus \{0\}$), and Q_{r-2} is some homogeneous polynomial of order $r - 2 \geq 0$;

(3) *$Ord(V) \geq r - 1$.*

For the definition of $\text{Ord}(V)$ when Ω is $\mathbf{R}^n$, see [7, Section 4.3]. Replacing $\mathbf{R}^n$ by Ω everywhere in that definition, we get the corresponding definition of $\text{Ord}(V(\Omega))$ for an arbitrary domain Ω.

One can also find in [7, Section 4.2] some informative examples concerning Condition N.

Problem 2 Study whether condition N holds or not for concrete pairs $(L, V(\Omega))$, that do not satisfy the requirements of Proposition 3(2). Is the property (3) always necessary?

In the proof of Proposition 3, Radó's theorem [31] (see also [33, Theorem 12.14] and [18]) is used. This theorem says that, *if a continuous function f in a domain D is holomorphic in D outside it's zeros, then f is holomorphic on all of D*. It is not difficult to find appropriate analogs of this result for operators L mentioned in Proposition 3(2).

Problem 3 To find analogues of Radó's theorem for other operators L.

Theorem 3 *If $(L, V(\Omega))$ satisfies Conditions 1–4, then the following statements are equivalent:*

(i) *for each (relatively) closed set $F \subset \Omega$ one has*

$$M_{LV}(F) = E_{LV}(F) \iff \{F \text{ is a } \Omega\text{-RKL set}\};$$

(ii) *for each compact set $K \subset \Omega$,*

$$M_{LV}(K) = E_{LV}(K) \iff \{\Omega^* \setminus K \text{ is connected}\};$$

(iii) *the pair $(L, V(\Omega))$ satisfies Condition N.*

Remark 1 Our proof of $(ii) \Rightarrow (iii)$ in fact shows that if for *some* compact set K in Ω there is a function $f \in L(K)$ which is not in $E_{LV}(K)$, then the same is true for some Φ_a, $a \in \widehat{K} \setminus K$.

3.1 Applications via Vitushkin-Type Approximation Theorems

From Theorems 2 and 3, it is not difficult to obtain the corresponding approximation (reduction) theorems for *classes* of functions (jets), analogous to that of [7, Proposition 1]. In this direction, we present only the following result which extends [7, Theorem 4]. Note that (iii) and (iv) are results on better-than-uniform approximation.

Theorem 4 *Let L (of order r) be as above, Ω be an arbitrary domain in $\boldsymbol{R}^n$ and F be a closed subset of Ω. Then*

(i) *for $V = BC^m(\Omega))$, where $m \in (r-2, r-1) \cup (r-1, r)$ ($m \in (r-1, r)$ if $r = 1$; see Sect. 3), the equality $V_L(F) = M_{LV}(F)$ holds if and only if there exists a constant $A \in (0, +\infty)$ such that for each ball B in Ω*

$$M_*^{n-r+m}(B \setminus F^\circ) \leq A M^{n-r+m}(B \setminus F)\,;$$

(ii) *for* $V = BC^m(\Omega)$*,* $m \geq r$*, the equality* $V_L(F) = M_{LV}(F)$ *holds if and only if* F° *is dense in* F*;*
(iii) *let* L*,* Ω *and* $V = BC_q^m(\Omega)$ *be as in Proposition 2, and additionally suppose that* $m \geq r$*, then the equality* $V_L(F) = M_{LV}(F)$ *holds if and only if* F° *is dense in* F*;*
(iv) *for each space* $V(\Omega)$*, which is mentioned in* (*i*) *and* (*ii*) *with* $m \geq r - 1$*, and in* (*iii*)*, and* L *satisfying property (2) of Proposition 3, the equality* $V_L(F) = E_{LV}(F)$ *holds if and only if* $V_L(F) = M_{LV}(F)$ *and (at the same time)* F *is a* Ω*-RKL set.*

Here and below $M^{n-r+m}(\cdot)$ and $M_*^{n-r+m}(\cdot)$ are the Hausdorff and lower Hausdorff *contents* of order $n - r + m$ respectively (cf. [36]).

The following result was proved in [19] for *compact* sets F in $\mathbf{R}^n$. It is deep and strong even for the operator $\bar{\partial}^2$ in $\mathbf{R}^2$ (the proof basically uses the Vitushkin technique). Our modest contribution is an additional application of Theorem 2.

Theorem 5 *For any elliptic operator* L *in* $\mathbf{R}^n$ *(as above), an arbitrary domain* Ω *in* $\mathbf{R}^n$ *and* $V = BC^0(\Omega)$ *(uniform norm), the following conditions are equivalent:*

(i) *for every closed set* F *in* Ω *one has* $M_{LV}(F) = V_L(F)$ *;*
(ii) $n = 2$ *and* L *has a locally bounded fundamental solution in* $\mathbf{R}^2$.

For each L and V under consideration we can pose the problem of approximation in the V-norm $\|\cdot\|$ (on compacta) by polynomial solutions or (on closed sets) by entire solutions of the equation $Lu = 0$. We restrict ourselves to the corresponding problem *for classes*.

Problem 4 Given L and V (under Conditions 1–4), for which closed sets F does one have $V_L(F) = E_{LV}(F)$?

By Theorem 5, for the case when $n = 2$ and L has a locally bounded fundamental solution and $V = BC^0(\Omega)$, Problems 1 and 4 are equivalent for an arbitrary domain Ω in $\mathbf{R}^2$. Under these restrictions, only partial answers are given to Problem 4: see [4, 9, 20] and [40].

We also can apply Theorem 2 to extend *individual* Vitushkin-type theorems from compact to closed sets in appropriate domains (see review [20] and the recent paper [21]). We formulate a corresponding theorem only for the main result of [21], regarding criteria for C^m-approximations by bianalytic (that is, $L = \bar{\partial}^2$ in $\mathbf{C}$) functions on compact sets in $\mathbf{C}$.

Theorem 6 *Let* Ω *be a domain in* $\mathbf{C}$*,* $L = \bar{\partial}^2$*,* $m \in (0,1) \cup (1,2)$ *and* $V = BC^m(\Omega)$*. For a closed set* F *in* Ω *and* $f \in V_{loc}(\Omega) = C^m(\Omega)$ *the following conditions are equivalent:*

(i) $f_{(F)} \in M_{LV}(F)$*;*
(ii) *there is an* $A > 0$ *and* $k \geq 1$ *such that for each open disk* $B = B(a, r)$ *with* $\overline{B(a, kr)} \subset \Omega$ *one has*

$$\left| \int_{\partial B} f(z)(z-a)dz \right| \le A\omega^{\mu}_{[m]B}(f)\, r^2 M^m \left(B(a,kr) \setminus F\right)$$

where $[m]$ *is the integer part of m,* $\mu = m - [m]$ *is the fractional part of m and*

$$\omega^{\mu}_{[m]B}(f) = \sup \frac{|\partial^{\alpha} f(x) - \partial^{\alpha} f(y)|}{|x-y|^{\mu}},$$

where the latter sup *is taken over all* $x \neq y \in \overline{B}$ *and all 2-indices* α *with* $|\alpha| = [m]$*;*

(iii) *the previous condition holds with* $k = 1$.

3.2 Applications: Asymptotic and Boundary Behaviour of L-Analytic Functions

Let $\pounds^n_r$ stand for the class of all homogeneous elliptic operators of order r in $\mathbf{R}^n$ ($n \geq 2$, $r \geq 1$) with constant complex coefficients (see Sect. 2 above).

As the first application of our approximation results, given an arbitrary $L \in \pounds^2_2$, we consider entire solutions of the equation $Lu = 0$ for which

$$\lim_{r \to \infty} u(re^{i\varphi}) =: U(e^{i\varphi})$$

exists for all $\varphi \in [0, 2\pi)$ as a finite limit in $\mathbf{C}$. We gave (see [8]) a complete characterization of the possible "radial limit functions" U. This is an analog of the work of A. Roth for entire holomorphic functions. The results seem new even for harmonic functions.

Problem 5 Formulate and study an analog of the previous result for other L (other $\pounds^n_r$; at least for $n = 2$).

As the second application (see [3, Section 6.1]), given $L \in \pounds^n_r$ and a domain Ω satisfying some mild conditions, we construct solutions in Ω of the equation $Lu = 0$ having some prescribed boundary behaviour.

Let Ω be a domain in $\mathbf{R}^n$, $n \geq 2$, $\Omega \neq \mathbf{R}^n$, and let $b \in \partial\Omega$. We shall say that a (continuous) path $\gamma : [0,1] \to \mathbf{R}^n$ is *admissible for* Ω *with end point b* if $\gamma : [0,1) \to \Omega$ and $\gamma(1) = b$. Given a continuous function f in Ω, denote by $\mathcal{C}_\gamma(f)$ the *cluster set* of f along γ at b, that is:

$$\mathcal{C}_\gamma(f) = \{w \in \mathbf{C} \cup \{\infty\} : \text{ there exists a sequence } \{t_n\} \subset [0,1) \text{ such that } t_n \to 1 \text{ and } f(\gamma(t_n)) \to w \text{ as } n \to \infty\}.$$

Theorem 7 *Let* $L \in \pounds^n_r$*, and let* $\Omega \subset \boldsymbol{R}^n$*,* $\Omega \neq \boldsymbol{R}^n$*, be a domain such that its boundary* $\partial\Omega$ *has no (connected) components that consist of a single point. Then* *there exists* $g \in L(\Omega)$ *with the property that for each* $b \in \partial\Omega$*, for each admissible path* γ *for* Ω *ending at* b *and for each* $\alpha \in \mathbf{Z}_+^n$*, one has*

$$\mathcal{C}_\gamma(\partial^\alpha g) = \mathbf{C} \cup \{\infty\}.$$

The following propositions [3] show that, at least for $L = \Delta$ in $\mathbf{R}^n$ and $L = \partial/\partial\bar{z}$ in $\mathbf{R}^2$, our theorem is close to being sharp.

Proposition 4 *If* Ω *is a domain in* $\boldsymbol{R}^n$ *such that* $\partial\Omega$ *has an isolated point* $b \in \boldsymbol{R}^n \cup \{\infty\}$*, then for each function* f *harmonic in* Ω *or (if* $n = 2$*) for each function* f *holomorphic in* Ω*, there exists an admissible path* γ *for* Ω *ending at* b *such that* $\mathcal{C}_\gamma(f)$ *is a single point in* $\mathbf{C} \cup \{\infty\}$.

Proposition 5 *Under the conditions of the previous proposition, for each* $\alpha \in \mathbf{Z}_+^n$ *there exists an admissible path* γ_α *for* Ω *ending at* b *such that* $\mathcal{C}_{\gamma_\alpha}(\partial^\alpha f)$ *is just a single point in* $\mathbf{C} \cup \{\infty\}$ *since the function* $\partial^\alpha f$ *is also harmonic (or holomorphic).*

Our third application (see [3, Theorem 6]) is in some sense in the opposite direction of the second one. Given a (smooth) domain Ω, we would like to prescribe (almost everywhere on $\partial\Omega$) the boundary values of an L-analytic function in Ω, together with the boundary values of a fixed number of its derivatives, as we approach the boundary of Ω in the normal direction (a "weakened" Dirichlet problem).

Theorem 8 *Let* $L \in \pounds_r^n$ *and let* Ω *be a domain of class* C^{r+1} *in* $\boldsymbol{R}^n$*. Let* h_k*,* $k = 0, 1, \ldots, r-1$*, be* σ*-measurable functions which are finite* σ*-almost everywhere, where* σ *is the* $n-1$ *dimensional Lebesgue measure on* $\partial\Omega$*. Then there exists* $h \in L(\Omega)$ *such that, for* $k = 0, \ldots, r-1$*, and for* σ*-almost all* $x \in \partial\Omega$*, the limit of* $(\partial^k h/\partial\vec{n}_x^k)(y)$ *is equal to* $h_k(x)$*, where the derivatives are taken in the direction of the outer normal at* x*, and* $y \in \Omega$ *tends to* $x \in \partial\Omega$ *along that normal direction.*

4 Extension of Subharmonic Functions

In this section we discuss two settings of the C^m-subharmonic extension problem on domains in $\mathbf{R}^n$, $n \geq 2$, and on arbitrary open Riemann surfaces (RS). The existence of these extensions is connected with the possibility of representing subharmonic functions by global Newtonian potentials of positive measures ("gravitational" potentials for $n = 3$), see [5, Sect. 3].

The problem is completely solved (for all $m \in [0, +\infty)$) for the so-called *Runge-type* extensions, which we now present. Let Ω be a domain in $\mathbf{R}^n$, $n \geq 2$, or an *open* RS, and let W be an open subset of Ω. Denote by $SH(W)$ the class of all *subharmonic* functions in W.

For a fixed $m \geq 0$, one says that (W, Ω) is a C^m*-subharmonic extension Runge pair* (a C^m*-SHER-pair*, for short), if for each *closed* (in Ω) set X, $X \subset W$, and for every function $f \in SH(W) \cap C^m(W)$ there is an $F \in SH(\Omega) \cap C^m(\Omega)$ such that $F|_X = f|_X$.

Problem 6 To describe all C^m-SHER-pairs.
This problem was completely solved in [13] and [12] for C^0-extensions, in [5] for all m and domains in $\mathbf{R}^n$, and in [6] for all m and RS .

Theorem 9 *Employing the previous notations,* (W, Ω) *is a* C^m*-SHER-pair if and only if* $\Omega^* \setminus W$ *is connected.*
As before, for an open set W in Ω, by $W^* = W \cup \{*\}$ we mean the one point compactification of W.

By the well known Gunning and Narasimhan theorem [16], for every open RS Ω, there exists a holomorphic mapping ρ of Ω into the complex plane $\mathbf{C}$, which is a local homeomorphism. In what follows we fix an open RS Ω and the atlas, induced by such a ρ, where the change of coordinates is the identity.

This result allows one to naturally define and deal with C^m-spaces and with distributions on domains in a RS Ω, as well as to properly define the Vitushkin localization operator (for the Laplacean in Ω), which preserves the classes of subharmonic and C^m-functions. The methods of [13] and [5] then were adapted in [6] to an arbitrary open RS. Below we give some natural applications of these results.

Let Ω be a RS and let $Aut(\Omega)$ be the group of biholomorphic mappings of Ω onto itself (the *automorphism group* of Ω). Let G be a subgroup of $Aut(\Omega)$. Denote by Ω/G the space of all G-orbits in Ω, with the quotient topology, induced by the projection $\pi : \Omega \to \Omega/G$ $(\pi : p \to [p])$. In what follows we require that G act on Ω *properly discontinuously* (that is, for each compact set K in Ω one has $g(K) \cap K \neq \emptyset$ for at most finitely many $g \in G$) and *freely* (when the only $g \in G$ having a fixed point is the identity map on Ω).

It is well known that in this case Ω/G becomes a RS, and $\pi : \Omega \to \Omega/G$ is locally biholomorphic.

The following application of Theorem 9 gives also a reasonable motivation for considering C^m-subharmonic extensions on RS.

Corollary 1 *Suppose* Ω *and* G *are as just above and the RS* Ω/G *is open. Let* W *be an open* G*-invariant subset of* Ω*. Then, for each* $m \geq 0$ *the following properties are equivalent:*

(a) *for each function* $f \in C^m(W) \cap SH(W)$*, which is* G*-invariant (that is,* $f(g(p)) = f(p)$ *for each* $p \in W$ *and* $g \in G$*), and for each closed (in* Ω*)* G*-invariant set* $X \subset W$ *one can find a* G*-invariant function* $F \in C^m(\Omega) \cap SH(\Omega)$ *such that* $F|_X = f$*;*
(b) $(\Omega/G)^* \setminus (W/G)$ *is connected.*

Example 1 For $\Omega = \mathbf{C}$ let G be the group of translations $\{g_s(z) = z + s : s \in \mathbf{Z}\}$. Then G is admissible and Ω_G is open (cylindrical surface; here we can take $\rho([z]) = \exp(2\pi iz)$). Let W be a G-invariant open subset of Ω. So, for each $m \geq 0$ the previous corollary can be applied.

In [6] we also studied another kind of C^m-subharmonic extension problem of Runge type closely related to that considered above. Let X be a closed subset in Ω and $m \geq 0$. We say that (X, Ω) is a *mixed* C^m*-subharmonic extension pair of Runge* *type* (a *mixed C^m-SHER-pair*, for short), if for each neighborhood W of X in Ω and for every function $f \in SH(W) \cap C^m(W)$ there is an $F \in SH(\Omega) \cap C^m(\Omega)$ such that $F = f$ on *some neighborhood* of X.

In the next theorem we give a purely topological characterization of mixed C^m-SHER-pairs analogous to that of [12, Chapter 6] for the case $m = 0$ (continuous subharmonic *strong* extension of Runge type).

Theorem 10 *Let X be a closed subset of an open Riemann surface Ω. The following are equivalent:*

(a) (X, Ω) *is a mixed C^m-SHER-pair;*
(b) $\Omega^* \setminus X$ *is connected and locally connected.*

Now we consider the *Walsh type* C^m-subharmonic extension problem on an open RS Ω (with fixed Gunning-Narasimhan atlas).

For an open set W in Ω and fixed $m \in \mathbf{Z}_+ = \{0, 1, 2, \dots\}$ we can naturally define the space $C^m(W)$ of all (real-valued) functions $f(p)$ on W that have continuous *derivatives* $\partial^\beta f(p)$ on W for all $\beta = (\beta_1, \beta_2) \in \mathbf{Z}_+^2$ with $|\beta| = \beta_1 + \beta_2 \le m$. We also need to consider the corresponding Banach C^m-space

$$BC^m(W) = \{f \in C^m(W) \ : \ \|f\|_{mW} = \max_{|\beta| \le m} \|\partial^\beta f\|_W < +\infty\},$$

where $\|f\|_E = \sup_{p \in E} |f(p)|$ is the uniform norm on a set E in Ω. Then $C^m(W)$ naturally becomes a Fréchet space. The Banach spaces $BC^m(W)$ (with finite norms $\|f\|_{mW}$), as well as the Fréchet spaces $C^m(W)$ can be defined for all $m \ge 0$ (see [6] for details).

For a *closed* set X in Ω and fixed $m \ge 0$ we shall deal with the *Whitney type* space $BC^m_{jet}(X)$ which consists of all (different) elements (jets) $F_X = \{\partial^\beta F|_X\}_{|\beta| \le m}$, $F \in BC^m(\Omega)$, with (finite) norm

$$\|F_X\|_{mX} = \inf\{||F^*||_{m\Omega} \ : \ F^* \in BC^m(\Omega) \ , \ F^*_X = F_X\}.$$

One then can naturally define the Fréchet spaces $C^m_{jet}(X)$. Notice that for each jet F_X in $C^m_{jet}(X)$ the condition $F_X \in SH(X^\circ)$ is well defined. Moreover, for $m \in [0, 1)$ or $X = \overline{X^\circ}$ the spaces $BC^m_{jet}(X)$ and $C^m_{jet}(X)$ can be considered as usual spaces of functions.

Given $m \ge 0$ and a closed set X in Ω, we say that the pair (X, Ω) is a *C^m-subharmonic extension Walsh pair* (briefly, *C^m-SHEW-pair*), if every jet $F_X \in C^m_{jet}(X) \cap SH(X^\circ)$ can be extended to a function $F^* \in C^m(\Omega) \cap SH(\Omega)$ in the sense that $F^*_X = F_X$.

Problem 7 To describe all C^m-SHEW-pairs ($X \neq \Omega$ and $X \neq \emptyset$).

This problem first appeared in [22] for C^1-subharmonic extensions from closed balls in $\mathbf{R}^n$. Several (in some sense sharp) sufficient conditions and counterexamples were found for the *Walsh*-type extensions from closed smooth domains onto $\mathbf{R}^n$ (see the survey [28]). Most of them are generalized and improved for RS in [6]. We shall formulate them, beginning with examples, when (X, Ω) *is not* a C^m-SHEW-pair [6].

First, if $\Omega^* \setminus X$ is not connected then (X, Ω) is not a C^m-SHEW-pair for any $m \geq 0$.

Moreover, for $m \in [0, 1)$ we know of *no nontrivial* C^m-SHEW-pair. For instance, if $\Omega^* \setminus X$ is connected and X° has some Ω-bounded component, then for $m \in [0, 1/2)$ the pair (X, Ω) *is not* a C^m-SHEW. If the mentioned component has smooth boundary, then the latter assertion holds for $m \in [0, 1)$. For any analytic Jordan arc X in Ω the pair (X, Ω) is not a C^m-SHEW.

If $m \geq 3$ then *for each* closed set X in Ω the pair (X, Ω) is not a C^m-SHEW.

If $m \in [2, 3)$ and $X \neq \overline{X^\circ}$ then (X, Ω) is not a C^m-SHEW-pair.

There is C^1-smooth closed Jordan domain X in $\mathbf{C}$ such that $(X, \mathbf{C})$ is not a C^1-SHEW-pair [23].

Proposition 6 *For* $\alpha \in (0, \pi)$ *let*

$$X_\alpha = \{z \in \mathbf{C} \ : \ |z| \leq 1 \,, \ \ arg(z) \in [-\pi + \alpha/2, \pi - \alpha/2]\} \,.$$

Then, for each $m \in (1, \pi/\alpha)$ *the pair* $(X_\alpha, \mathbf{C})$ *is not a* C^m*-SHEW.*

So, now we concentrate on the cases when $m \in [1, 3)$ and $X = \overline{X^\circ}$. We shall say that a *closed* set X in a Riemann surface Ω is *locally Jordan*, if $X = \overline{X^\circ}$ and for any point $q \in \partial X$ one can find an open neighborhood U of q in Ω such that $U \cap \partial X$ is an open Jordan arc (that is, Jordan arc without end points). The following localization result is specifically 2-dimensional: the proof of it‘s analog in $\mathbf{R}^n$ requires smoothness of ∂X.

Proposition 7 *Let* X *be locally Jordan closed set in* Ω *and* $m \in [1, 3)$. *Suppose that* $f \in C^m(X) \cap SH(X^\circ)$ *and for each* $q \in \partial X$ *one can find an (open)* Ω*-bounded neighborhood* U *of* q *and* $g \in C^m(U) \cap SH(U)$ *with* $g = f$ *on* $X \cap U$. *Then there is a neighborhood* W *of* X *in* Ω *and* $h \in C^m(W) \cap SH(W)$ *with* $h = f$ *on* X.

The next theorem has an analogue for domains Ω in $\mathbf{R}^n$, $n \geq 3$, and compact sets X in Ω (see [26, 27]).

Theorem 11 *Let* $m \in [1, 3)$ *and* $X = \overline{X^\circ}$ *be a closed subset of* Ω *with smooth (Dini-Lyapunov-type) boundary. The following are equivalent:*

(a) (X, Ω) *is a* C^m*-SHEW-pair;*
(b) $\Omega^* \setminus X$ *is connected.*

Example 2 Take $\Omega = \mathbf{C} \setminus \{0\}$, $J \in \mathbf{Z}$ $(J \geq 2)$, and let G be the group of rotations $\{g_j(z) = z \exp(2\pi i j/J) \ : \ j \in \{1, \ldots, J\}\}$ on Ω. Then G is admissible and Ω_G is open (here one can take $\rho([z]) = z^J$). (Notice that G is not admissible for $\Omega = \mathbf{C}$.) Let X be a locally Jordan G-invariant subset of Ω with smooth (Dini-Lyapunov) boundary. If $\Omega^*_G \setminus X_G$ is connected, then (by the previous Theorem) for each $m \in [1, 3)$, any G-invariant jet $F_X \in C^m_{jet}(X) \cap SH(X^\circ)$ can be extended to a G-invariant C^m-subharmonic function on Ω.

Problem 8 Which results on C^m-subharmonic extensions (among the above) can be generalized to higher dimensions and to solutions of other elliptic inequalities, and in what sense?

Let Ω be an n-dimensional C^∞-manifold and assume that $\rho : \Omega \to \mathbf{R}^n$ is a C^∞-mapping which is a local C^∞-diffeomorphism. We call the pair $\{\Omega, \rho\}$ a *spread N-manifold* and the mapping ρ a spread of Ω over $\mathbf{R}^n$. The Smale-Hirsch Theorem [30, Cor. 8.2] asserts that an open n-dimensional C^∞-manifold can be spread over $\mathbf{R}^n$ if and only if it is parallelizable.

Since the mapping ρ induces an atlas, where the change of coordinates is the identity, any differential operator (for instance, the Laplacian Δ) can be well defined in this atlas. Let L be any homogeneous elliptic operator in $\{\Omega, \rho\}$ with constant (complex) coefficients, and D be a domain in Ω. A function (distribution) f in D is called *L-subelliptic* if $Lf \geq 0$ in D in the distributional sense (usually one considers only real functions f if L is real). The problems of C^m-L-subelliptic extensions of Runge and Walsh types can then be naturally posed. (Moreover, given some reasonable Banach space V of functions on Ω, one can also consider V-L- subelliptic extensions).

We believe that Theorems 9 and 11 still hold for subharmonic functions (that is, when $L = \Delta$) in all dimensions $n \geq 3$ on all open spread n-manifolds, and that appropriate analogs of Propositions 6 and 7 can also be obtained.

On the other hand, to our knowledge, all the results obtained thus far on the relevant problems only apply to the case of the Laplacian $L = \Delta$ or the Cauchy-Riemann operator $L = \partial/\partial\bar{z}$ in $\mathbf{C}$ (the latter case is partially considered in [41]).

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Approximation in the Closed Unit Ball

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Abstract In this expository article, we present a number of classic theorems that serve to identify the closure in the sup-norm of various sets of Blaschke products, inner functions and their quotients, as well as the closure of the convex hulls of these sets. The results presented include theorems of Carathéodory, Fisher, Helson–Sarason, Frostman, Adamjan–Arov–Krein, Douglas–Rudin and Marshall. As an application of some of these ideas, we obtain a simple proof of the Berger–Stampfli spectral mapping theorem for the numerical range of an operator.

Keywords Approximation • Unit ball • Blaschke product • Inner function • Convex hull

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1 Introduction

Let X be a Banach space and let E be a subset of X. The *convex hull* of E, denoted by $\mathrm{conv}(E)$, is the set of all elements of the form $\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n$, where $x_j \in E$ and where $0 \le \lambda_j \le 1$ with $\lambda_1 + \lambda_2 + \cdots + \lambda_n = 1$. The *closed unit ball* of X is

$$\mathbf{B}_X = \{x \in X : \|x\| \le 1\}.$$

In this survey, we consider some Banach spaces of functions on the open unit disc $\mathbb{D}$ or on the unit circle $\mathbb{T}$, e.g., $L^\infty(\mathbb{T})$, $\mathcal{C}(\mathbb{T})$, H^∞ and the disc algebra $\mathcal{A}$, and explore the norm closure of some subsets of $\mathbf{B}_X$ and of their convex hulls.

The unimodular elements of the above function spaces enter naturally into our discussion. The unimodular elements of H^∞, denoted by $\mathbf{I}$, are a celebrated family that are called *inner functions*. For other function spaces we use the notation $\mathbf{U}_X$ to denote the family of unimodular elements of X, e.g.,

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$$\mathbf{U}_{\mathcal{C}(\mathbb{T})} := \{f \in \mathcal{C}(\mathbb{T}) : |f(\zeta)| = 1 \text{ for all } \zeta \in \mathbb{T}\}.$$

The prototype candidates for E are the set of finite Blaschke products (**FBP**), the set of all Blaschke products (**BP**), the inner functions (**I**), the measurable unimodular functions, as well as the quotients of pairs of functions in these families. We now summarize the main results. The formal statements and attributions will be detailed in the sections that follow.

Finite Blaschke products are elements of the disc algebra $\mathcal{A}$. In particular, when considered as functions on $\mathbb{T}$, they are elements of $\mathcal{C}(\mathbb{T})$. In this regard, for these elements and their quotients, we shall see that:

$$\boxed{\begin{aligned} \overline{\mathbf{FBP}} &= \mathbf{FBP} \\ \overline{\mathrm{conv}(\mathbf{FBP})} &= \mathbf{B}_{\mathcal{A}} \\ \overline{\mathbf{FBP}/\mathbf{FBP}} &= \mathbf{U}_{\mathcal{C}(\mathbb{T})} \\ \overline{\mathrm{conv}(\mathbf{FBP}/\mathbf{FBP})} &= \mathbf{B}_{\mathcal{C}(\mathbb{T})}. \end{aligned}}$$

Infinite Blaschke products are elements of the Hardy space H^∞. In particular, when considered as functions on $\mathbb{T}$, they are elements of $L^\infty(\mathbb{T})$. For these functions and their quotients, we shall see that:

$$\boxed{\begin{aligned} \overline{\mathbf{BP}} &= \overline{\mathbf{I}} = \mathbf{I} \\ \overline{\mathrm{conv}(\mathbf{BP})} &= \overline{\mathrm{conv}(\mathbf{I})} = \mathbf{B}_{H^\infty} \\ \overline{\mathbf{BP}/\mathbf{BP}} &= \overline{\mathbf{I}/\mathbf{I}} = \mathbf{U}_{L^\infty(\mathbb{T})} \\ \overline{\mathrm{conv}(\mathbf{BP}/\mathbf{BP})} &= \overline{\mathrm{conv}(\mathbf{I}/\mathbf{I})} = \mathbf{B}_{L^\infty(\mathbb{T})}. \end{aligned}}$$

In all the results above, we consider the norm topology. For the Hardy space H^∞, and thus a priori for the disc algebra $\mathcal{A}$, there is a weaker topology which is obtained via semi-norms

$$p_r(f) := \max_{|z| \le r} |f(z)|.$$

This is referred as the topology of uniform convergence on compact subsets (UCC) of $\mathbb{D}$. Naively speaking, it is easier to converge under the latter topology. Therefore, in some cases we will also study the UCC-closure of a set or its convex hull. We shall see that:

$$\boxed{\overline{\mathbf{FBP}}^{\mathrm{UCC}} = \mathbf{B}_{H^\infty}.}$$

Since on the one hand, **FBP** is the smallest approximating set in our discussion, and on the other hand $\mathbf{B}_{H^\infty}$ is the largest possible set that we can approximate, this last result closes the door on any further investigation regarding the UCC-closure.

2 Approximation on $\mathbb{D}$ by Finite Blaschke Products

$$\boxed{\text{Goal : } \overline{\mathbf{FBP}} = \mathbf{FBP}}$$

Let $f \in H^\infty$, and suppose that there is a sequence of finite Blaschke products that converges uniformly on $\mathbb{D}$ to f. Then, by continuity, we also have uniform convergence on $\overline{\mathbb{D}}$. Therefore f is necessarily a continuous function on $\overline{\mathbb{D}}$, and moreover it is a unimodular function on $\mathbb{T}$. It is an easy exercise to show that this function is necessarily a finite Blaschke product. A slightly more general version of this result is stated below.

Lemma 2.1 (Fatou [7]) *Let f be holomorphic in the open unit disc $\mathbb{D}$ and suppose that*

$$\lim_{|z|\to 1} |f(z)| = 1.$$

Then f is a finite Blaschke product.

Proof Since f is holomorphic on $\mathbb{D}$ and $|f|$ tends uniformly to 1 as we approach $\mathbb{T}$, it has a finite number of zeros in $\mathbb{D}$. Let B be the finite Blaschke product formed with the zeros of f. Then f/B and B/f are both holomorphic in $\mathbb{D}$, and their moduli uniformly tend to 1 as we approach $\mathbb{T}$. Hence, by the maximum principle, $|f/B| \leq 1$ and $|B/f| \leq 1$ on $\overline{\mathbb{D}}$. Thus f/B is constant on $\overline{\mathbb{D}}$, and the constant has to be unimodular. □

Lemma 2.1 immediately implies the following result.

Theorem 2.2 *The set* **FBP** *of finite Blaschke products is a closed subset of* $\mathbf{B}_{\mathcal{A}}$ *(and hence also a closed subset of* $\mathbf{B}_{H^\infty}$*).*

The following result is another simple consequence of Lemma 2.1. It will be needed in later approximation results in this article (see Theorem 5.2).

Corollary 2.3 *Let f be meromorphic in the open unit disc $\mathbb{D}$ and continuous on the closed unit disc $\overline{\mathbb{D}}$ (as a function into the Riemann sphere). Suppose that f is unimodular on the unit circle $\mathbb{T}$. Then f is the quotient of two finite Blaschke products.*

Proof Since f is unimodular on $\mathbb{T}$, meromorphic in $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$, it has a finite number of poles in $\mathbb{D}$. Let B_2 be the finite Blaschke product with zeros at the poles of f. Put $B_1 := B_2 f$. Then B_1 satisfies the hypotheses of Lemma 2.1, and so it is a finite Blaschke product. Thus $f = B_1/B_2$. □

3 Approximation on Compact Sets by Finite Blaschke Products

$$\boxed{\text{Goal} : \overline{\mathbf{FBP}}^{\text{UCC}} = \mathbf{B}_{H^\infty}}$$

If f is holomorphic on $\mathbb{D}$ and can be uniformly approximated on $\mathbb{D}$ by a sequence of finite Blaschke products, we saw that, by Lemma 2.1, f is itself a finite Blaschke product. A general element of $\mathbf{B}_{H^\infty}$ is far from being a finite Blaschke product and cannot be approached uniformly on $\mathbb{D}$ by finite Blaschke products. Nevertheless, a weaker type of convergence does hold. The following result says that, if we equip H^∞ with the topology of uniform convergence on compact subsets of $\mathbb{D}$, then the family of finite Blaschke products form a dense subset of $\mathbf{B}_{H^\infty}$. In a certain sense, this theorem circumscribes all the other results in this article.

Theorem 3.1 (Carathéodory) *Let $f \in \mathbf{B}_{H^\infty}$. Then there is a sequence of finite Blaschke products that converges uniformly to f on each compact subset of $\mathbb{D}$.*

Proof (This Proof is taken from [10, p. 5]) We construct a finite Blaschke product B_n such that the first $n+1$ Taylor coefficients of f and B_n are equal. Then, by Schwarz's lemma, we have

$$|f(z) - B_n(z)| \le 2|z|^n, \qquad (z \in \mathbb{D}),$$

and thus the sequence (B_n) converges uniformly to f on compact subsets of $\mathbb{D}$.

Let $c_0 := f(0)$. As f lies in the unit ball, $c_0 \in \overline{\mathbb{D}}$. If $|c_0| = 1$, then, by the maximum principle, f is a unimodular constant, and the result is obvious. So let us assume that $|c_0| < 1$. Writing

$$\tau_a(z) := \frac{a - z}{1 - \bar{a}z} \qquad (a, z \in \mathbb{D}),$$

let us set

$$B_0(z) := -\tau_{-c_0}(z) = \frac{z + c_0}{1 + \bar{c}_0 z}, \qquad (z \in \mathbb{D}). \tag{1}$$

Clearly, B_0 is a finite Blaschke product and its constant term is c_0.

The rest is by induction. Suppose that we can construct B_{n-1} for each element of $\mathbf{B}_{H^\infty}$. Set

$$g(z) := \frac{\tau_{c_0}(f(z))}{z}, \qquad (z \in \mathbb{D}). \tag{2}$$

By Schwarz's lemma, $g \in \mathbf{B}_{H^\infty}$. Hence, there is a finite Blaschke product B_{n-1} such that $g - B_{n-1}$ has a zero of order at least n at the origin. If B is a finite Blaschke product of degree n, and $w \in \mathbb{D}$, then it is easy to verify directly that $\tau_w \circ B$ and $B \circ \tau_w$ are also finite Blaschke products of order n. Hence

$$B_n(z) := \tau_{c_0}(zB_{n-1}(z)), \qquad (z \in \mathbb{D}), \tag{3}$$

is a finite Blaschke product. Since

$$f(z) = \tau_{c_0}(zg(z)), \qquad (z \in \mathbb{D}),$$

we naturally expect that B_n does that job. To establish this conjecture, it is enough to observe that

$$\tau_{c_0}(z_2) - \tau_{c_0}(z_1) = \frac{(1-|c_0|^2)(z_1 - z_2)}{(1-\bar{c}_0 z_1)(1-\bar{c}_0 z_2)}.$$

Hence, thanks to the presence of the factor $z(g(z) - B_{n-1}(z))$, the difference

$$f(z) - B_n(z) = \tau_{c_0}(zg(z)) - \tau_{c_0}(zB_{n-1}(z))$$

is divisible by z^{n+1}. □

Remark Equation (3) is perhaps a bit misleading, as if we have a recursive formula for the sequence $(B_n)_{n\geq 0}$. A safer way is to write the formula as

$$B_{n,f}(z) := \tau_{c_0}(zB_{n-1,g}(z)), \qquad (n \geq 1),$$

where g is related to f via (2). Let us compute an example by finding $B_1 := B_{1,f}$ We know that

$$B_{1,f}(z) = \tau_{c_0}(zB_{0,g}(z)).$$

Write $f(z) = c_0 + c_1 z + \cdots$ and observe that

$$g(z) = \frac{\tau_{c_0}(f(z))}{z} = \frac{-c_1}{1-|c_0|^2} + O(z).$$

Then, by (1), we have

$$B_{0,g}(z) = -\tau_{\frac{c_1}{1-|c_0|^2}}(z),$$

and so we get

$$B_1(z) = \tau_{c_0}(-z\tau_{\frac{c_1}{1-|c_0|^2}}(z)).$$

One may directly verify that

$$B_1(z) = c_0 + c_1 z + O(z^2),$$

as required.

4 Approximation on $\mathbb{D}$ by Convex Combinations of Finite Blaschke Products

$$\boxed{\text{Goal} : \overline{\text{conv}(\mathbf{FBP})} = \mathbf{B}_{\mathcal{A}}}$$

As we saw in Sect. 2, if a function $f \in H^\infty$ can be uniformly approximated by a sequence of finite Blaschke products on $\mathbb{D}$, then f is continuous on $\overline{\mathbb{D}}$. The same result holds if we can approximate f by elements that are convex combinations of finite Blaschke products. The only difference is that, in this case, f is not necessarily unimodular on $\mathbb{T}$. We can just say that $\|f\|_\infty \leq 1$. More explicitly, the uniform limit of convex combinations of finite Blaschke products is a continuous function in the closed unit ball of H^∞. It is rather surprising that the converse is also true.

Theorem 4.1 (Fisher [8]) *Let $f \in \mathbf{B}_{\mathcal{A}}$, and let $\varepsilon > 0$. Then there are finite Blaschke products B_j and convex weights $(\lambda_j)_{1\leq j\leq n}$ such that*

$$\|\lambda_1 B_1 + \lambda_2 B_2 + \cdots + \lambda_n B_n - f\|_\infty < \varepsilon.$$

Proof For $0 \leq t \leq 1$, let $f_t(z) := f(tz)$, $z \in \mathbb{D}$. Since f is continuous on $\overline{\mathbb{D}}$, we have

$$\lim_{t\to 1} \|f_t - f\|_\infty = 0. \tag{4}$$

By Theorem 3.1, there is a sequence of finite Blaschke products that converges uniformly to f on compact subsets of $\mathbb{D}$. Based on our notation, this means that, given $\varepsilon > 0$ and $t < 1$, there is a finite Blaschke product B such that

$$\|f_t - B_t\|_\infty < \varepsilon/2.$$

Therefore, by (4), there is a finite Blaschke product B such that

$$\|f - B_t\|_\infty < \varepsilon.$$

If we can show that B_t itself is actually a convex combination of finite Blaschke products, the proof is done.

Firstly, note that $(gh)_t = g_t h_t$ for all g and h, and that the family of convex combinations of finite Blaschke products is closed under multiplication. Hence it is enough only to consider a Blaschke factor

$$B(z) = \frac{\alpha - z}{1 - \overline{\alpha} z}.$$

Secondly, it is easy to verify that

$$B_t(z) = \frac{\alpha - tz}{1 - \overline{\alpha} tz} = \frac{t(1 - |\alpha|^2)}{1 - |\alpha|^2 t^2} \times \frac{\alpha t - z}{1 - \overline{\alpha} tz} + \frac{|\alpha|(1 - t^2)}{1 - |\alpha|^2 t^2} \times e^{i \arg \alpha}. \tag{5}$$

The combination on the right side is almost good. More precisely, it is a combination of a Blaschke factor and a unimodular constant (a special case of a finite Blaschke product), with positive coefficients, but the coefficients do not add up to one. Indeed, we have

$$1 - \frac{t(1 - |\alpha|^2)}{1 - |\alpha|^2 t^2} - \frac{|\alpha|(1 - t^2)}{1 - |\alpha|^2 t^2} = \frac{(1 - t)(1 - |\alpha|)}{1 + |\alpha| t}.$$

But this obstacle is easy to overcome. We can simply add

$$0 = \frac{(1 - t)(1 - |\alpha|)}{2(1 + |\alpha| t)} \times 1 + \frac{(1 - t)(1 - |\alpha|)}{2(1 + |\alpha| t)} \times (-1)$$

to both sides of (5) to obtain a convex combination of finite Blaschke products. Of course, the factor 1 in the last identity can be replaced by any other finite Blaschke product. □

In technical language, Theorem 4.1 says that the closed convex hull of finite Blaschke products is precisely the closed unit ball of the disc algebra $\mathcal{A}$.

5 Approximation on $\mathbb{T}$ by Quotients of Finite Blaschke Products

$$\boxed{\text{Goal} : \overline{\mathbf{FBP}/\mathbf{FBP}} = \mathbf{U}_{\mathcal{C}(\mathbb{T})}}$$

If B_1 and B_2 are finite Blaschke products, then B_1/B_2 is a continuous unimodular function on $\mathbb{T}$. Helson and Sarason showed that the family of all such quotients is uniformly dense in the set of continuous unimodular functions [11, p. 9].

To prove the Helson–Sarason theorem, we need an auxiliary lemma.

Lemma 5.1 *Let $f \in \mathbf{U}_{\mathcal{C}(\mathbb{T})}$. Then there exists $g \in \mathbf{U}_{\mathcal{C}(\mathbb{T})}$ such that either*

$$f(\zeta) = g^2(\zeta)$$

or

$$f(\zeta) = \zeta g^2(\zeta)$$

for all $\zeta \in \mathbb{T}$.

Proof Since $f : \mathbb{T} \longrightarrow \mathbb{T}$ is uniformly continuous, we can take N so big that $|\theta - \theta'| \leq 2\pi/N$ implies $|f(e^{i\theta}) - f(e^{i\theta'})| < 2$. Now, we divide $\mathbb{T}$ into N arcs

$$T_k = \left\{ e^{i\theta} : \frac{2(k-1)\pi}{N} \leq \theta \leq \frac{2k\pi}{N} \right\}, \qquad (1 \leq k \leq N).$$

Then $f(T_k)$ is a closed arc in a semicircle, and thus there is a continuous function $\phi_k(\theta)$ on the interval $\frac{2(k-1)\pi}{N} \leq \theta \leq \frac{2k\pi}{N}$ such that

$$f(e^{i\theta}) = \exp(i\phi_k(\theta)), \qquad (e^{i\theta} \in T_k).$$

These functions are uniquely defined up to additive multiples of 2π. We adjust those additive constants so that

$$\phi_k(2k\pi/N) = \phi_{k+1}(2k\pi/N)$$

for $k = 1, 2, \ldots, N-1$. Define $\phi(\theta) := \phi_k(\theta)$ for $\frac{2(k-1)\pi}{N} \leq \theta \leq \frac{2k\pi}{N}$, $k = 1, 2, \ldots, N$. Then we get a continuous function $\phi(\theta)$ on $[0, 2\pi]$ such that

$$f(e^{i\theta}) = \exp(i\phi(\theta)), \qquad (e^{i\theta} \in \mathbb{T}).$$

Since

$$\exp(i(\phi(2\pi) - \phi(0))) = f(e^{2\pi i})/f(e^{0i}) = 1,$$

$\phi(2\pi) - \phi(0)$ is an integer multiple of 2π. If $(\phi(2\pi) - \phi(0))/2\pi$ is *even*, then set

$$g(e^{i\theta}) := \exp(i\phi(\theta)/2),$$

and if $(\phi(2\pi) - \phi(0))/2\pi$ is *odd*, then set

$$g(e^{i\theta}) := \exp(i(\phi(\theta) - \theta)/2).$$

Then g is a continuous unimodular function on $\mathbb{T}$ such that either $f(e^{i\theta}) = g^2(e^{i\theta})$ or $f(e^{i\theta}) = e^{i\theta} g^2(e^{i\theta})$ for all $e^{i\theta} \in \mathbb{T}$. $\square$

Theorem 5.2 (Helson–Sarason [11]) *Let* $f \in \mathbf{U}_{\mathcal{C}(\mathbb{T})}$ *and let* $\varepsilon > 0$. *Then there are finite Blaschke products* B_1 *and* B_2 *such that*

$$\left\| f - \frac{B_1}{B_2} \right\|_{\mathcal{C}(\mathbb{T})} < \varepsilon.$$

Proof According to Lemma 5.1, it is enough to prove the result for unimodular functions of the form $f = g^2$ (note that $b(e^{i\theta}) := e^{i\theta}$ is a Blaschke factor). Without loss of generality, assume that $\varepsilon < 1$.

By Weierstrass's theorem, there is a trigonometric polynomial $p(z)$ such that

$$\|g - p\|_{\mathcal{C}(\mathbb{T})} < \varepsilon.$$

The restriction $\varepsilon < 1$ ensures that p has no zeros on $\mathbb{T}$. Let $p^*(z) := \overline{p(1/\overline{z})}$, and consider the quotient p/p^*. Since p is a good approximation to g, we expect that p/p^* should be a good approximation to $g/g^* = g^2 = f$. More precisely, on the unit circle $\mathbb{T}$, we have

$$\frac{g}{g^*} - \frac{p}{p^*} = \frac{(g-p)p^* + (p^* - g^*)p}{g^* p^*},$$

which gives

$$|f - p/p^*| \leq |g - p| + |p^* - g^*| \leq 2\varepsilon.$$

It is enough now to note that p/p^* is a meromorphic function that is unimodular and continuous on $\mathbb{T}$, and thus, according to Corollary 2.3, it is the quotient of two finite Blaschke products. □

If we allow approximation by quotients of general Blaschke products, then it turns out that we can approximate a much larger class of functions. This is the subject of the Douglas–Rudin theorem, to be established in Sect. 9 below.

6 Approximation on $\mathbb{D}$ by Convex Combination of Quotients of Finite Blaschke Products

$$\boxed{\text{Goal} : \overline{\text{conv}(\mathbf{FBP}/\mathbf{FBP})} = \mathbf{B}_{\mathcal{C}(\mathbb{T})}}$$

The quotient of two finite Blaschke products is a continuous unimodular function on $\mathbb{T}$. Hence a convex combination of such fractions stays in the closed unit ball of $\mathcal{C}(\mathbb{T})$. As the first step in showing that this set is dense in $\mathbf{B}_{\mathcal{C}(\mathbb{T})}$, we consider the larger set of all unimodular elements of $\mathcal{C}(\mathbb{T})$, and then pass to the special subclass of quotients of finite Blaschke products.

Lemma 6.1 *Let $f \in \mathbf{B}_{\mathcal{C}(\mathbb{T})}$ and let $\varepsilon > 0$. Then there are $u_j \in \mathbf{U}_{\mathcal{C}(\mathbb{T})}$ and convex weights $(\lambda_j)_{1\le j\le n}$ such that*

$$\|\lambda_1 u_1 + \lambda_2 u_2 + \cdots + \lambda_n u_n - f\|_{\mathcal{C}(\mathbb{T})} < \varepsilon.$$

Proof Let $w \in \mathbb{D}$. Then, by the Cauchy integral formula,

$$w = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{\zeta + w}{\zeta(1+\overline{w}\zeta)}\,d\zeta = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+w}{1+\overline{w}e^{i\theta}}\,d\theta.$$

In a sense, the integral on the right side is an infinite convex combination of unimodular elements. We shall approximate it by a Riemann sum and thereby obtain an ordinary finite convex combination. Since

$$\left|\frac{e^{i\theta}+w}{1+\overline{w}e^{i\theta}} - \frac{e^{i\theta'}+w}{1+\overline{w}e^{i\theta'}}\right| \le \frac{1+|w|}{1-|w|}|\theta-\theta'|,$$

for

$$\Delta_N := \left|w - \frac{1}{N}\sum_{k=1}^N \frac{e^{i2k\pi/N}+w}{1+\overline{w}e^{i2k\pi/N}}\right|,$$

we obtain the estimation

$$\begin{aligned}
\Delta_N &= \left|\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+w}{1+\overline{w}e^{i\theta}}\,d\theta - \frac{1}{N}\sum_{k=1}^N \frac{e^{i2k\pi/N}+w}{1+\overline{w}e^{i2k\pi/N}}\right| \\
&\le \frac{1}{2\pi}\sum_{k=1}^N \int_{2(k-1)\pi/N}^{2k\pi/N} \left|\frac{e^{i\theta}+w}{1+\overline{w}e^{i\theta}} - \frac{e^{i2k\pi/N}+w}{1+\overline{w}e^{i2k\pi/N}}\right| d\theta \\
&\le \frac{1}{2\pi}\sum_{k=1}^N \int_{2(k-1)\pi/N}^{2k\pi/N} \frac{1+|w|}{1-|w|}\frac{2\pi}{N}\,d\theta \\
&= \frac{1+|w|}{1-|w|}\frac{2\pi}{N}.
\end{aligned}$$

As $\|f\|_\infty \le 1$, we have $|f(e^{i\theta})| \le 1$ for all $e^{i\theta} \in \mathbb{T}$. Hence, by the estimate above,

$$\left|(1-\varepsilon)f(e^{i\theta}) - \frac{1}{N}\sum_{k=1}^N \frac{e^{i2k\pi/N} + (1-\varepsilon)f(e^{i\theta})}{1+(1-\varepsilon)\overline{f(e^{i\theta})}e^{i2k\pi/N}}\right| \le \frac{1+|(1-\varepsilon)f(e^{i\theta})|}{1-|(1-\varepsilon)f(e^{i\theta})|}\frac{2\pi}{N}.$$

Thus, for each $e^{i\theta} \in \mathbb{T}$,

$$\left| f(e^{i\theta}) - \frac{1}{N} \sum_{k=1}^{N} \frac{e^{i2k\pi/N} + (1-\varepsilon) f(e^{i\theta})}{1 + (1-\varepsilon)\overline{f(e^{i\theta})} e^{i2k\pi/N}} \right| \leq \varepsilon + \frac{4\pi}{\varepsilon N}.$$

But, each

$$u_k(e^{i\theta}) = \frac{e^{i2k\pi/N} + (1-\varepsilon) f(e^{i\theta})}{1 + (1-\varepsilon)\overline{f(e^{i\theta})} e^{i2k\pi/N}}$$

is in fact a unimodular continuous function on $\mathbb{T}$. Thus, given $\varepsilon > 0$, it is enough to choose N so large that $4\pi/(\varepsilon N) < \varepsilon$, to get

$$\left| f(e^{i\theta}) - \frac{1}{N} \sum_{k=1}^{N} u_k(e^{i\theta}) \right| \leq 2\varepsilon$$

for all $e^{i\theta} \in \mathbb{T}$. □

In the light of Theorem 5.2, it is now easy to pass from an arbitrary unimodular element to the quotient of two finite Blaschke products.

Theorem 6.2 *Let $f \in \mathbf{B}_{\mathcal{C}(\mathbb{T})}$ and let $\varepsilon > 0$. Then there are finite Blaschke products B_{ij}, $1 \leq i, j \leq n$ and convex weights $(\lambda_j)_{1 \leq j \leq n}$ such that*

$$\left\| \lambda_1 \frac{B_{11}}{B_{12}} + \lambda_2 \frac{B_{21}}{B_{22}} + \cdots + \lambda_n \frac{B_{n1}}{B_{n2}} - f \right\|_{\mathcal{C}(\mathbb{T})} < \varepsilon.$$

Proof By Lemma 6.1, there are $u_j \in \mathbf{U}_{\mathcal{C}(\mathbb{T})}$ and convex weights $(\lambda_j)_{1 \leq j \leq n}$ such that

$$\|\lambda_1 u_1 + \lambda_2 u_2 + \cdots + \lambda_n u_n - f\|_{\mathcal{C}(\mathbb{T})} < \varepsilon/2.$$

For each k, by Theorem 5.2, there are finite Blaschke products B_{k1} and B_{k2} such that

$$\|u_k - B_{k1}/B_{k2}\|_\infty < \varepsilon/2.$$

Hence

$$\left\| f - \sum_{k=1}^{n} \lambda_k B_{k1}/B_{k2} \right\|_\infty \leq \left\| f - \sum_{k=1}^{n} \lambda_k u_k \right\|_\infty + \sum_{k=1}^{n} \lambda_k \|u_k - B_{k1}/B_{k2}\|_\infty < \varepsilon.$$

This completes the proof. □

7 Approximation on $\mathbb{D}$ by Infinite Blaschke Products

$$\boxed{\text{Goal}: \overline{\mathbf{BP}} = \bar{\mathbf{I}} = \mathbf{I}}$$

We start to describe our approximation problem as in the beginning of Sect. 2. But extra care is needed here, since we are dealing with infinite Blaschke products and they are not continuous on $\overline{\mathbb{D}}$. Let $f \in H^\infty$ and assume that there is a sequence of infinite Blaschke products that converges uniformly on $\mathbb{D}$ to f. First of all, we surely have $\|f\|_\infty \le 1$. But we can say more. For each Blaschke product in the sequence, there is an exceptional set of Lebesgue measure zero such that on the complement the product has radial limits. The union of all these exceptional sets still has Lebesgue measure zero, and, at all points outside this union, each infinite Blaschke product has a radial limit. Therefore, the function f itself must have a radial limit of modulus one almost everywhere. In technical language, f is an inner function. Hence, in short, if we can uniformly approximate an $f \in H^\infty$ by a sequence of infinite Blaschke products, then f is necessarily an inner function. Frostman showed that the converse is also true.

Let ϕ be an inner function for the open unit disc. Fix $w \in \mathbb{D}$ and consider $\phi_w = \tau_w \circ \phi$, i.e.,

$$\phi_w(z) = \frac{w - \phi(z)}{1 - \overline{w}\phi(z)}, \qquad (z \in \mathbb{D}).$$

Since τ_w is an automorphism of the open unit disc and ϕ maps $\mathbb{D}$ into itself, then clearly so does ϕ_w, i.e. ϕ_w is also an element of the closed unit ball of H^∞. Moreover, for almost all $e^{i\theta} \in \mathbb{T}$,

$$\lim_{r \to 1} \phi_w(re^{i\theta}) = \frac{w - \phi(e^{i\theta})}{1 - \overline{w}\phi(e^{i\theta})} = -\overline{\phi(e^{i\theta})}\frac{w - \phi(e^{i\theta})}{\overline{w} - \overline{\phi(e^{i\theta})}} \in \mathbb{T}.$$

Therefore, for each $w \in \mathbb{D}$, the function ϕ_w is in fact an inner function. What is much less obvious is that ϕ_w has a good chance of being a Blaschke product. More precisely, the exceptional set

$$\mathcal{E}(\phi) := \{w \in \mathbb{D} : \phi_w \text{ is not a Blaschke product}\}$$

is small. Frostman showed that the Lebesgue measure of $\mathcal{E}(\phi)$ is zero. In fact, there is even a stronger version saying that the logarithmic capacity of $\mathcal{E}(\phi)$ is zero. But the simpler version with measure is enough for our approximation problem. We start with a technical lemma.

Lemma 7.1 *Let ϕ be an inner function in the open unit disc $\mathbb{D}$. Then the limit*

$$\lim_{r\to 1}\int_0^{2\pi} \log\left|\phi(re^{i\theta})\right| d\theta$$

exists. Moreover, ϕ is a Blaschke product if and only if

$$\lim_{r\to 1}\int_0^{2\pi} \log\left|\phi(re^{i\theta})\right| d\theta = 0.$$

Proof Considering the canonical decomposition $\phi = BS_\sigma$, where B is a Blaschke product and S_σ is a singular inner function with measure σ, we have

$$\log\left|\phi(re^{i\theta})\right| = \log\left|B(re^{i\theta})\right| - \frac{1}{2\pi}\int_{\mathbb{T}} \frac{1-r^2}{1+r^2-2r\cos(\theta-t)}\, d\sigma(t).$$

Using Fubini's theorem, we obtain

$$\int_0^{2\pi} \log\left|\phi(re^{i\theta})\right| d\theta = \int_0^{2\pi} \log\left|B(re^{i\theta})\right| d\theta - \int_{\mathbb{T}} d\sigma(t). \tag{6}$$

Thus the main task is to deal with Blaschke products.

First of all, we have

$$\frac{1}{2\pi}\int_0^{2\pi} \log\left|B(re^{i\theta})\right| d\theta \le 0 \tag{7}$$

for all r with $0 \le r < 1$. Now, without loss of generality, we assume that $B(0) \ne 0$, since otherwise we can divide B by z^m, where m is the order of the zero of B at the origin, and this modification does not change the limit. Then, by Jensen's formula,

$$\log|B(0)| = \sum_{|z_n|<r} \log\left(\frac{|z_n|}{r}\right) + \frac{1}{2\pi}\int_0^{2\pi} \log|B(re^{i\theta})|\, d\theta$$

for all r, $0 < r < 1$. Since $B(0) = \prod_{n=1}^{\infty} |z_n|$, we thus obtain

$$\frac{1}{2\pi}\int_0^{2\pi} \log\left|B(re^{i\theta})\right| d\theta = \sum_{|z_n|<r} \log\left(\frac{r}{|z_n|}\right) - \sum_{n=1}^{\infty} \log\left(\frac{1}{|z_n|}\right).$$

Given $\varepsilon > 0$, choose N so large that

$$\sum_{n=N+1}^{\infty} \log\left(\frac{1}{|z_n|}\right) < \varepsilon.$$

Then, for $r > |z_N|$,

$$\frac{1}{2\pi}\int_0^{2\pi} \log\left|B(re^{i\theta})\right| d\theta \geq \sum_{n=1}^{N} \log\left(\frac{r}{|z_n|}\right) - \sum_{n=1}^{N} \log\left(\frac{1}{|z_n|}\right) - \varepsilon.$$

Therefore,

$$\liminf_{r\to 1} \frac{1}{2\pi}\int_0^{2\pi} \log\left|B(re^{i\theta})\right| d\theta \geq -\varepsilon,$$

and, since ε is an arbitrary positive number,

$$\liminf_{r\to 1} \frac{1}{2\pi}\int_0^{2\pi} \log\left|B(re^{i\theta})\right| d\theta \geq 0.$$

Finally, (7) and the last inequality together imply that

$$\lim_{r\to 1} \frac{1}{2\pi}\int_0^{2\pi} \log\left|B(re^{i\theta})\right| d\theta = 0.$$

Returning now to (6), we see that

$$\lim_{r\to 1} \int_0^{2\pi} \log\left|\phi(re^{i\theta})\right| d\theta = \lim_{r\to 1^-} \int_0^{2\pi} \log\left|B(re^{i\theta})\right| d\theta - \int_{\mathbb{T}} d\sigma(t) = -\sigma(\mathbb{T}).$$

This formula also shows that

$$\lim_{r\to 1} \int_0^{2\pi} \log\left|\phi(re^{i\theta})\right| d\theta = 0,$$

if and only if $\sigma(\mathbb{T}) = 0$, and, since σ is a positive measure, this holds if and only if $\sigma \equiv 0$. Therefore, the above limit is zero if and only if ϕ is a Blaschke product. □

In view of the following result, the functions ϕ_w, $w \in \mathbb{D}$, are called the *Frostman shifts* of ϕ.

Lemma 7.2 (Frostman [9]) *Let ϕ be an inner function for the open unit disc. Fix $0 < \rho < 1$, and define*

$$\mathcal{E}_\rho(\phi) := \{e^{i\theta} \in \mathbb{T} : \phi_{\rho e^{i\theta}} \text{ is not a Blaschke product}\}.$$

Then $\mathcal{E}_\rho(\phi)$ has Lebesgue measure zero.

Note that this theorem implies that the two-dimensional Lebesgue measure of $\mathcal{E}(\phi)$ is also zero.

Proof For each $\alpha \in \mathbb{D}$, we have

$$\frac{1}{2\pi}\int_0^{2\pi}\log\left|\frac{\rho e^{i\theta}-\alpha}{1-\rho e^{-i\theta}\alpha}\right|d\theta=\max(\log\rho,\log|\alpha|). \tag{8}$$

Since ϕ is inner, we can replace α by $\phi(re^{it})$ and then integrate with respect to t. This gives

$$\frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi}\log\left|\frac{\rho e^{i\theta}-\phi(re^{it})}{1-\rho e^{-i\theta}\phi(re^{it})}\right|d\theta dt=\int_0^{2\pi}\max(\log\rho,\log|\phi(re^{it})|)\,dt.$$

Since ρ is fixed and $|\phi|\leq 1$, the family

$$f_r(e^{it})=\max(\log\rho,\log|\phi(re^{it})|),\qquad (e^{it}\in\mathbb{T}),$$

where the parameter r runs through $[0,1)$, is *uniformly* bounded in modulus by the positive constant $-\log\rho$, and

$$\lim_{r\to 1}f_r(e^{it})=\max(\log\rho,\lim_{r\to 1}\log|\phi(re^{it})|)=\max(\log\rho,0)=0$$

for almost all $e^{it}\in\mathbb{T}$. Hence, by the dominated convergence theorem,

$$\lim_{r\to 1}\int_0^{2\pi}f_r(e^{it})\,dt=0,$$

which we rewrite as

$$\lim_{r\to 1}\int_0^{2\pi}\left(\int_0^{2\pi}\log\left|\frac{\rho e^{i\theta}-\phi(re^{it})}{1-\rho e^{-i\theta}\phi(re^{it})}\right|d\theta\right)dt=0.$$

But, considering the fact that the integrand is negative, the Fubini theorem gives

$$\lim_{r\to 1}\int_0^{2\pi}\left(\int_0^{2\pi}\log\left|\frac{\rho e^{i\theta}-\phi(re^{it})}{1-\rho e^{-i\theta}\phi(re^{it})}\right|dt\right)d\theta=0.$$

Set

$$M(r,\theta):=\int_0^{2\pi}-\log\left|\frac{\rho e^{i\theta}-\phi(re^{it})}{1-\rho e^{-i\theta}\phi(re^{it})}\right|dt.$$

Then $M(r,\theta)\geq 0$ for all r,θ, and

$$\lim_{r\to 1}\int_0^{2\pi}M(r,\theta)\,d\theta=0. \tag{9}$$

Now, we put together two facts. First, according to Lemma 7.1, we know that, for each θ,

$$\lim_{r\to 1} M(r,\theta)$$

exists. Second, by Fatou's lemma,

$$\int_0^{2\pi}\left(\liminf_{r\to 1} M(r,\theta)\right) d\theta \le \liminf_{r\to 1}\int_0^{2\pi} M(r,\theta)\, d\theta.$$

Hence, by (9) and the fact that $M(r,\theta)\ge 0$, we conclude that

$$\int_0^{2\pi}\left(\lim_{r\to 1} M(r,\theta)\right) d\theta = 0.$$

In particular, we must have $\lim_{r\to 1} M(r,\theta) = 0$ for almost all $\theta\in[0,2\pi]$, i.e.,

$$\lim_{r\to 1}\int_0^{2\pi} \log\left|\phi_{\rho e^{i\theta}}(re^{it})\right| dt = 0$$

for almost all $\theta\in[0,2\pi]$. Therefore, again by Lemma 7.1, $\phi_{\rho e^{i\theta}}$ is indeed a Blaschke product for almost all $\theta\in[0,2\pi]$. In other words, $\mathcal{E}_\rho(\phi)$ has Lebesgue measure zero. □

The preceding result immediately implies the approximation theorem that we are seeking. It shows that the set **BP** of Blaschke products is uniformly dense in the set of all inner functions **I**.

Theorem 7.3 (Frostman [9]) *Let ϕ be an inner function in the open unit disc. Then, given $\varepsilon > 0$, there is a Blaschke product B such that*

$$\|\phi - B\|_\infty < \varepsilon.$$

Proof Take $\rho\in(0,1)$ small enough so that $2\rho/(1-\rho) < \varepsilon$. According to Lemma 7.2, on the circle $\{|z| = \rho\}$ there are many candidates $\rho e^{i\theta}$ such that $\phi_{\rho e^{i\theta}}$ is a Blaschke product. Pick any one of them. Then, we have

$$|\phi(z) + \phi_{\rho e^{i\theta}}(z)| = \left|\frac{\rho e^{i\theta} - \rho e^{-i\theta}\phi^2(z)}{1-\rho e^{-i\theta}\phi(z)}\right| \le \frac{2\rho}{1-\rho} < \varepsilon$$

for all $z\in\mathbb{D}$. This simply means that $\|\phi + \phi_{\rho e^{i\theta}}\|_\infty < \varepsilon$. Now take $B := -\phi_{\rho e^{i\theta}}$. □

Frostman's approximation result (Theorem 7.3) should be compared with Carathéodory theorem (Theorem 3.1). On one hand, the approximation in Frostman's result is stronger. The convergence is uniform on $\mathbb{D}$, and not just on a fixed compact subset of $\mathbb{D}$. But, on the other hand, it only applies to a smaller class of functions (inner functions) in the closed unit disc of H^∞.

Theorem 7.3 may also be considered as a generalization of Theorem 2.2. In the latter, we consider a small set of Blaschke products (just finite Blaschke products) and thus we are not able to approximate all inner functions. But Frostman says that, if we enlarge our set and consider all Blaschke products, then we can approximate all inner functions.

However, though this interpretation is true, it is not the whole truth. Theorem 2.2 says that the set of finite Blaschke products is a closed subset of $\partial \mathbf{B}_{H^\infty}$. Then Theorem 7.3 says that its complement in the family of inner functions, i.e., $\mathbf{I} \setminus \mathbf{FBP}$, is also a closed subset of $\partial \mathbf{B}_{H^\infty}$ such that infinite Blaschke products are uniformly dense in $\mathbf{I} \setminus \mathbf{FBP}$. In fact, by considering zeros, it is easy to see that

$$\mathrm{dist}_\infty(\mathbf{FBP},\ \mathbf{I} \setminus \mathbf{FBP}) \geq 1,$$

i.e., both parts are well separated on the boundary of $\mathbf{B}_{H^\infty}$.

8 Existence of Unimodular Functions in the Coset $f + H^\infty(\mathbb{T})$

To study duality on Hardy spaces, we recall some well-known facts from functional analysis. Let X be a Banach space, and let X^* denote its dual space. Let A be a closed subspace of X. The *annihilator* of A is

$$A^\perp := \{\Lambda \in X^* :\ \Lambda(a) = 0 \text{ for all } a \in A\},$$

which is a closed subspace of X^*. The canonical projection of X onto the quotient space X/A is defined by

$$\begin{aligned} \pi : X &\longrightarrow X/A \\ x &\longmapsto x + A. \end{aligned}$$

For each $x \in X$, by the definition of norm in the quotient space X/A, we have

$$\mathrm{dist}(x, A) = \inf_{a \in A} \|x - a\| = \|\pi(x)\|_{X/A}. \tag{10}$$

Using the Hahn–Banach theorem from functional analysis, we have

$$\mathrm{dist}(x, A) = \sup_{\Lambda \in A^\perp, \|\Lambda\|_{X^*} = 1} |\Lambda(x)|.$$

Moreover, the supremum is attained, i.e., there is $\Lambda_0 \in A^\perp$ with $\|\Lambda_0\|_{X^*} = 1$ such that

$$\mathrm{dist}(x, A) = \Lambda_0(x).$$

Thanks to these remarks, we obtain the dual identifications

$$(X/A)^* = A^{\perp} \quad \text{and} \quad A^* = X^*/A^{\perp}.$$

For a Banach space X of functions defined on the unit circle $\mathbb{T}$, we define X_0 to be the family of all functions $e^{i\theta}f(e^{i\theta})$ such that $f \in X$. In all cases that we consider below, X_0 is a closed subspace of X. If $f \in X$ has a holomorphic extension to the open unit disc, then the holomorphic extension of $e^{i\theta}f(e^{i\theta})$ would be $zf(z)$, a function having a zero at the origin. This fact explains the notation X_0.

The following lemma summarizes a number of dual identifications of interest to us.

Lemma 8.1 ([10, §IV.1]) *Let* $1 \le p < \infty$*, and let* $1/p + 1/q = 1$. *Then:*

(a) $(L^p/H^p)^* = H_0^q$,
(b) $(L^p/H_0^p)^* = H^q$,
(c) $(H^p)^* = L^q/H_0^q$,
(d) $(H_0^p)^* = L^q/H^q$.

We can apply this method to study $\operatorname{dist}(f, H^p(\mathbb{T}))$, where f is an element of $L^p(\mathbb{T})$ and $1 \le p \le \infty$. In the following, we just need the case $p = \infty$.

Theorem 8.2 *Let* $f \in L^\infty(\mathbb{T})$. *Then the following hold.*

(a) *There exists* $g \in H^\infty(\mathbb{T})$ *such that*

$$\operatorname{dist}(f, H^\infty(\mathbb{T})) = \|f - g\|_\infty.$$

(b) *We have*

$$\operatorname{dist}(f, H^\infty(\mathbb{T})) = \sup_{h \in H_0^1(\mathbb{T}), \|h\|_1 = 1} \left| \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta})h(e^{i\theta})\, d\theta \right|.$$

Proof (a) By (10), there are $g_n \in H^\infty(\mathbb{T})$, $n \ge 1$, such that

$$\operatorname{dist}(f, H^\infty(\mathbb{T})) = \lim_{n\to\infty} \|f - g_n\|_\infty.$$

Hence, $(\|g_n\|_1)_{n\ge1}$ is a bounded sequence in $H^\infty(\mathbb{T})$. By Lemma 8.1(b), $H^\infty(\mathbb{T})$ is the dual of $L^1(\mathbb{T})/H_0^1(\mathbb{T})$. Hence, looking at the sequence $(g_n)_{n\ge1}$ as a family of uniformly bounded linear functionals on $L^1(\mathbb{T})/H_0^1(\mathbb{T})$, by the Banach–Alaoglu theorem, we can extract a subsequence that is convergent in the weak* topology of $H^\infty(\mathbb{T})$. More explicitly, there exists $g \in H^\infty(\mathbb{T})$ and a subsequence $(n_k)_{k\ge1}$ such that

$$\lim_{k\to\infty} \frac{1}{2\pi} \int_0^{2\pi} h(e^{i\theta})g_{n_k}(e^{i\theta})\, d\theta = \frac{1}{2\pi} \int_0^{2\pi} h(e^{i\theta})g(e^{i\theta})\, d\theta$$

for all $h \in L^1(\mathbb{T})$. By Hölder's inequality,

$$\left| \frac{1}{2\pi} \int_0^{2\pi} h(e^{i\theta})(f(e^{i\theta}) - g_{n_k}(e^{i\theta}))\, d\theta \right| \le \|h\|_1 \|f - g_{n_k}\|_\infty.$$

Let $k \longrightarrow \infty$ to get

$$\left| \frac{1}{2\pi} \int_0^{2\pi} h(e^{i\theta})(f(e^{i\theta}) - g(e^{i\theta}))\, d\theta \right| \le \|h\|_1 \operatorname{dist}(f, H^\infty(\mathbb{T}))$$

for all $h \in L^1(\mathbb{T})$.

If $L^1(\mathbb{T})$ were the dual of $L^\infty(\mathbb{T})$, then we would have been able to use duality techniques and the reasoning would have been easier. But, since $L^1(\mathbb{T})$ is a proper subclass of the dual of $L^\infty(\mathbb{T})$, we have to proceed differently.

If E is a measurable subset of $\mathbb{T}$, then its characteristic function $h = \chi_E$ is integrable. Hence, with this choice, we obtain

$$\left| \frac{1}{|E|} \int_E (f(e^{i\theta}) - g(e^{i\theta}))\, d\theta \right| \le \operatorname{dist}(f, H^\infty(\mathbb{T}))$$

for all measurable sets $E \subset \mathbb{T}$ with $|E| = \int_E d\theta \neq 0$. This is enough to conclude $\|f - g\|_\infty \le \operatorname{dist}(f, H^\infty(\mathbb{T}))$. Note that the reverse inequality $\operatorname{dist}(f, H^\infty(\mathbb{T})) \le \|f - g\|_\infty$ is a direct consequence of the definition of $\operatorname{dist}(f, H^\infty(\mathbb{T}))$.

(b) By definition,

$$\operatorname{dist}(f, H^\infty(\mathbb{T})) = \|f + H^\infty(\mathbb{T})\|_{L^\infty(\mathbb{T})/H^\infty(\mathbb{T})},$$

and, by Lemma 8.1(d), we have $L^\infty(\mathbb{T})/H^\infty(\mathbb{T}) = (H_0^1(\mathbb{T}))^*$. Hence

$$\begin{aligned} \operatorname{dist}(f, H^\infty(\mathbb{T})) &= \|f + H^\infty(\mathbb{T})\|_{(H_0^1(\mathbb{T}))^*} \\ &= \sup_{h \in H_0^1(\mathbb{T}), \|h\|_1 = 1} \left| \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) h(e^{i\theta})\, d\theta \right|. \end{aligned}$$

This completes the proof. □

Let $f \in L^\infty(\mathbb{T})$. If the coset $f + H^\infty(\mathbb{T})$ contains a unimodular element, then necessarily $\operatorname{dist}(f, H^\infty(\mathbb{T})) \le 1$. A profound result of Adamjan–Arov–Krein says that, under the slightly more restrictive condition $\operatorname{dist}(f, H^\infty(\mathbb{T})) < 1$, the reverse implication holds. In this section, we discuss this result, which will be needed in studying the closed convex hull of Blaschke products. We start with a technical lemma.

Lemma 8.3 *Let* $f_n \in H^1(\mathbb{D})$ *with* $\|f_n\|_1 \le 1$, $n \ge 1$. *Suppose that there is a measurable subset* E *of* $\mathbb{T}$ *with* $|E| \neq 0$ *such that*

$$\lim_{n\to\infty} \int_E |f_n(e^{i\theta})|\, d\theta = 0.$$

Then

$$\lim_{n\to\infty} f_n(0) = 0.$$

Proof If $|E| = 2\pi$, then the result is an immediate consequence of the identity

$$f_n(0) = \frac{1}{2\pi}\int_{\mathbb{T}} f_n(e^{i\theta})\, d\theta = \frac{1}{2\pi}\int_E f_n(e^{i\theta})\, d\theta.$$

If $0 < |E| < 2\pi$, then, on the one hand,

$$\frac{1}{|E|}\int_E \log|f_n(e^{i\theta})|\, d\theta \le \log\left(\frac{1}{|E|}\int_E |f_n(e^{i\theta})|\, d\theta\right) \longrightarrow -\infty,$$

as $n \longrightarrow \infty$, and, on the other hand,

$$\begin{aligned}\frac{1}{|\mathbb{T}\setminus E|}\int_{\mathbb{T}\setminus E} \log|f_n(e^{i\theta})|\, d\theta &\le \log\left(\frac{1}{|\mathbb{T}\setminus E|}\int_{\mathbb{T}\setminus E} |f_n(e^{i\theta})|\, d\theta\right)\\ &\le \log\left(\frac{\|f_n\|_1}{|\mathbb{T}\setminus E|}\right) \le \log\left(\frac{1}{|\mathbb{T}\setminus E|}\right).\end{aligned}$$

Therefore,

$$\lim_{n\to\infty}\int_0^{2\pi} \log|f_n(e^{i\theta})|\, d\theta = -\infty.$$

Finally, since $\log|f_n|$ is a subharmonic function, we have

$$|f_n(0)| \le \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log|f_n(e^{i\theta})|\, d\theta\right),$$

and thus $f_n(0) \longrightarrow 0$ as $n \longrightarrow \infty$. □

The space $H^\infty(\mathbb{T})$ contains all inner functions, which are elements of modulus one. The following result shows that if we slightly perturb $H^\infty(\mathbb{T})$ in $L^\infty(\mathbb{T})$, it still contains unimodular elements.

Theorem 8.4 (Adamjan–Arov–Krein [1–3]) *Let* $f \in L^\infty(\mathbb{T})$ *be such that*

$$\operatorname{dist}(f, H^\infty(\mathbb{T})) < 1.$$

Then there exists an $\omega \in f + H^\infty(\mathbb{T})$ *with*

$$|\omega(e^{i\theta})| = 1$$

for almost all $e^{i\theta} \in \mathbb{T}$.

Proof (Garnett [13, p. 150]) The proof is long and thus we divide it into several steps.

Step 1: Definition of ω as the solution of an extremal problem.

Since $\operatorname{dist}(f, H^\infty(\mathbb{T})) < 1$, the set

$$\mathcal{E} := \{\omega : \omega \in f + H^\infty(\mathbb{T}), \|\omega\|_\infty \le 1\}$$

is not empty. Let

$$\alpha := \sup_{\omega \in \mathcal{E}} \left| \frac{1}{2\pi} \int_0^{2\pi} \omega(e^{i\theta})\, d\theta \right|.$$

We show that the supremum is attained. There are $\omega_n = f + g_n \in \mathcal{E}$, $n \ge 1$, such that

$$\lim_{n\to\infty} \left| \frac{1}{2\pi} \int_0^{2\pi} \omega_n(e^{i\theta})\, d\theta \right| = \alpha.$$

Since $g_n \in H^\infty(\mathbb{T})$ and $\|g_n\|_\infty \le 1 + \|f\|_\infty$, there exist $g \in H^\infty(\mathbb{T})$ and a subsequence $(n_k)_{k\ge1}$ such that

$$\lim_{k\to\infty} \frac{1}{2\pi} \int_0^{2\pi} h(e^{i\theta}) g_{n_k}(e^{i\theta})\, d\theta = \frac{1}{2\pi} \int_0^{2\pi} h(e^{i\theta}) g(e^{i\theta})\, d\theta \tag{11}$$

for all $h \in L^1(\mathbb{T})$. Indeed, since the g_n are uniformly bounded, we can say that a subsequence g_{n_k} converges weak* to an element $g \in L^\infty(\mathbb{T})$. But the weak*-convergence implies $\hat{g}(n) = 0$, $n \le -1$, so in fact we have $g \in H^\infty(\mathbb{T})$.

Set $\omega := f + g$. By (11),

$$\lim_{k\to\infty} \frac{1}{2\pi} \int_0^{2\pi} h(e^{i\theta}) \omega_{n_k}(e^{i\theta})\, d\theta = \frac{1}{2\pi} \int_0^{2\pi} h(e^{i\theta}) \omega(e^{i\theta})\, d\theta$$

for all $h \in L^1(\mathbb{T})$. This fact implies

$$\|\omega\|_\infty \le \liminf_{k\to\infty} \|\omega_{n_k}\|_\infty \le 1,$$

which ensures that $\omega \in \mathcal{E}$. Moreover, taking $h \equiv 1$, we get

$$\left| \frac{1}{2\pi} \int_0^{2\pi} \omega(e^{i\theta})\, d\theta \right| = \alpha,$$

and thus we can write

$$\frac{1}{2\pi} \int_0^{2\pi} \omega(e^{i\theta})\, d\theta = \alpha e^{i\theta_0}. \tag{12}$$

Step 2: $\|\omega\|_\infty = 1$.

Let $\omega_1 := \omega + (1 - \|\omega\|_\infty)e^{i\theta_0}$. Thus $\omega_1 \in \mathcal{E}$ and, by the definition of α,

$$\left| \frac{1}{2\pi} \int_0^{2\pi} \omega_1(e^{i\theta})\, d\theta \right| \leq \alpha.$$

But, by (12),

$$\left| \frac{1}{2\pi} \int_0^{2\pi} \omega_1(e^{i\theta})\, d\theta \right| = |\alpha e^{i\theta_0} + (1 - \|\omega\|_\infty)e^{i\theta_0}| = \alpha + 1 - \|\omega\|_\infty.$$

Hence $\|\omega\|_\infty \geq 1$. We already know that $\|\omega\|_\infty \leq 1$, and thus $\|\omega\|_\infty = 1$.

Step 3: $\operatorname{dist}(\omega, H_0^\infty(\mathbb{T})) = 1$.

Let $\varepsilon > 0$, let $g \in H_0^\infty(\mathbb{T})$, and set $\omega_1 := \omega - g + \varepsilon e^{i\theta_0}$. Then $\omega_1 \in f + H^\infty(\mathbb{T})$, and, by (12),

$$\left| \frac{1}{2\pi} \int_0^{2\pi} \omega_1(e^{i\theta})\, d\theta \right| = |\alpha e^{i\theta_0} + \varepsilon e^{i\theta_0}| = \alpha + \varepsilon > \alpha.$$

Thus, according to the definition of α, we have $\omega_1 \notin \mathcal{E}$. Thus

$$\|\omega - g + \varepsilon e^{i\theta_0}\|_\infty > 1$$

for all $\varepsilon > 0$ and all $g \in H_0^\infty(\mathbb{T})$. Let $\varepsilon \to 0$ to get

$$\|\omega - g\|_\infty \geq 1$$

for all $g \in H_0^\infty(\mathbb{T})$. However, when $g \equiv 0$, we also know that $\|\omega\|_\infty = 1$. Hence $\operatorname{dist}(\omega, H_0^\infty(\mathbb{T})) = 1$.

Before moving on to Step 4, we remark that Theorem 8.2(b), applied to the function $e^{-i\theta}\omega(e^{i\theta})$, implies that there are $h_n \in H^1(\mathbb{T})$, $n \geq 1$, with $\|h_n\|_1 = 1$, such that

$$\operatorname{dist}(\omega, H_0^\infty(\mathbb{T})) = \lim_{n\to\infty} \frac{1}{2\pi}\int_0^{2\pi} \omega(e^{i\theta})h_n(e^{i\theta})\,d\theta = 1. \tag{13}$$

The extension of h_n to the open unit disc is also denoted by h_n.

Step 4: For all measurable sets $E \subset \mathbb{T}$ with $|E| \neq 0$, we have

$$\liminf_{n\to\infty} \int_E |h_n(e^{i\theta})|\,d\theta > 0.$$

Since $h_n - h_n(0) \in H_0^1(\mathbb{T})$, by (the easy part of) Theorem 8.2(b),

$$\|h_n - h_n(0)\|_1 \operatorname{dist}(\omega, H^\infty(\mathbb{T})) \geq \left|\frac{1}{2\pi}\int_0^{2\pi} \omega(e^{i\theta})(h_n(e^{i\theta}) - h_n(0))\,d\theta\right|,$$

and, by (12),

$$\frac{1}{2\pi}\int_0^{2\pi} \omega(e^{i\theta})(h_n(e^{i\theta}) - h_n(0))\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} \omega(e^{i\theta})h_n(e^{i\theta})\,d\theta - h_n(0)\alpha e^{i\theta_0}.$$

Thus, we have

$$(1 + |h_n(0)|) \operatorname{dist}(\omega, H^\infty(\mathbb{T})) \geq \left|\frac{1}{2\pi}\int_0^{2\pi} \omega(e^{i\theta})h_n(e^{i\theta})\,d\theta\right| - \alpha|h_n(0)|,$$

which yields

$$|h_n(0)| \geq \frac{\left|\frac{1}{2\pi}\int_0^{2\pi} \omega(e^{i\theta})h_n(e^{i\theta})\,d\theta\right| - \operatorname{dist}(\omega, H^\infty(\mathbb{T}))}{\alpha + \operatorname{dist}(\omega, H^\infty(\mathbb{T}))}.$$

Let $n \longrightarrow \infty$ to get, by (13),

$$\liminf_{n\to\infty} |h_n(0)| \geq \frac{1 - \operatorname{dist}(\omega, H^\infty(\mathbb{T}))}{\alpha + \operatorname{dist}(\omega, H^\infty(\mathbb{T}))}.$$

But $\operatorname{dist}(\omega, H^\infty(\mathbb{T})) = \operatorname{dist}(f, H^\infty(\mathbb{T})) < 1$. Hence, $\liminf_{n\to\infty} |h_n(0)| > 0$. Now, apply Lemma 8.3.

Step 5: ω is unimodular.

We know that $|\omega(e^{i\theta})| \leq 1$ for almost all $e^{i\theta} \in \mathbb{T}$. Let $0 \leq \lambda < 1$ and set

$$E_\lambda := \{e^{i\theta} \in \mathbb{T} : |\omega(e^{i\theta})| \leq \lambda\}.$$

Then

$$\left|\int_0^{2\pi} \omega(e^{i\theta})h_n(e^{i\theta})d\theta\right| \le \lambda \int_{E_\lambda} |h_n(e^{i\theta})|\,d\theta + \int_{\mathbb{T}\setminus E_\lambda} |h_n(e^{i\theta})|\,d\theta.$$

Since $\|h_n\|_1 = 1$,

$$\frac{1}{2\pi}\int_{\mathbb{T}\setminus E_\lambda} |h_n(e^{i\theta})|\,d\theta = 1 - \frac{1}{2\pi}\int_{E_\lambda} |h_n(e^{i\theta})|\,d\theta,$$

and thus

$$\frac{1-\lambda}{2\pi}\int_{E_\lambda} |h_n(e^{i\theta})|\,d\theta \le 1 - \left|\frac{1}{2\pi}\int_0^{2\pi} \omega(e^{i\theta})h_n(e^{i\theta})\,d\theta\right|.$$

By (13), this inequality implies that

$$\lim_{n\to\infty}\int_{E_\lambda} |h_n(e^{i\theta})|\,d\theta = 0.$$

Therefore, by Step 4, $|E_\lambda| = 0$ for all $0 \le \lambda < 1$. □

Theorem 8.4 has a geometric interpretation. Let $\mathcal{U}(\mathbb{T})$ denote the family of all unimodular functions in $L^\infty(\mathbb{T})$. Then Theorem 8.4 says that the open unit ball of $L^\infty(\mathbb{T})$ is a subset of $H^\infty(\mathbb{T}) + \mathcal{U}(\mathbb{T})$.

Corollary 8.5 *Let $f \in H^\infty(\mathbb{T})$ with $\|f\|_\infty < 1$, and let ω be an inner function. Then $f + \omega H^\infty(\mathbb{T})$ contains an inner function.*

Proof Consider $g := f/\omega$. Then $g \in L^\infty(\mathbb{T})$ and

$$\operatorname{dist}(g, H^\infty(\mathbb{T})) \le \|g\|_\infty = \|f\|_\infty < 1.$$

Thus, by Theorem 8.4, $g + H^\infty(\mathbb{T})$ contains a unimodular function. Therefore, upon multiplying by the inner function ω, the set $f + \omega H^\infty(\mathbb{T})$ also contains a unimodular function. But $f + \omega H^\infty(\mathbb{T}) \subset H^\infty(\mathbb{T})$, and thus any unimodular function in this set has to be inner. □

9 Approximation on $\mathbb{T}$ by Quotients of Inner Functions

$$\boxed{\text{Goal}: \overline{\mathbf{BP}/\mathbf{BP}} = \overline{\mathbf{I}/\mathbf{I}} = \mathbf{U}_{L^\infty(\mathbb{T})}}$$

If ω_1 and ω_2 are inner functions, then the quotient ω_1/ω_2 is unimodular on $\mathbb{T}$. But how much of the family of all unimodular functions on $\mathbb{T}$ do these quotients

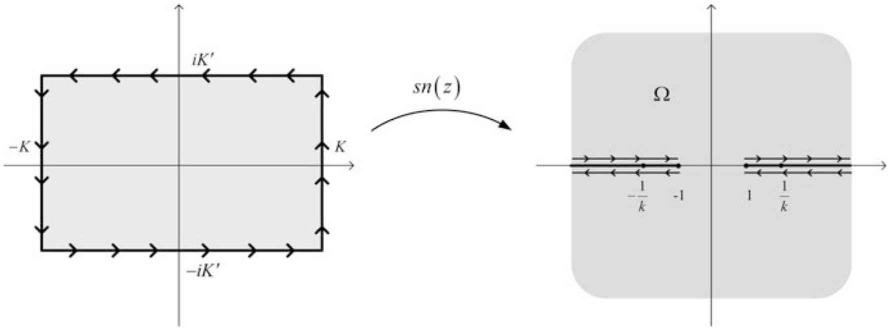

Fig. 1 The elliptic function sn(z)

occupy? The Douglas–Rudin theorem provides a satisfactory answer. To study this result, we need to examine closely some special conformal mappings.

Fix the parameter k, where $0 < k < 1$. Let

$$K := \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}, \tag{14}$$

$$K' := \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k'^2t^2)}}, \tag{15}$$

where $k' := \sqrt{1-k^2}$.

Let $\Omega := \mathbb{C} \setminus (-\infty, -1] \cup [1, \infty)$. Then the *Jacobi elliptic function* $\mathrm{sn}(z)$, or more precisely $\mathrm{sn}(z, k)$, is the conformal mapping shown in Fig. 1.

The parameter k is free to be any number in the interval $(0, 1)$, and thus we have a family of elliptic functions. The elliptic function $\mathrm{sn}(z)$ *continuously* maps the boundaries of the rectangle to the boundary of Ω in the Riemann sphere, i.e. $(-\infty, -1] \cup [1, \infty) \cup \{\infty\}$. We emphasize that sn *continuously* maps the closed rectangle $[-K, K] \times [-iK', iK']$ to $\mathbb{C} \cup \{\infty\}$. In particular, it maps $\pm iK'$ continuously to ∞, i.e., if we approach to $\pm iK'$, then $\mathrm{sn}(z)$ tends to infinity. However, sn is not injective on the boundaries of the rectangle. If we traverse the path

$$-iK' \longrightarrow (K - iK') \longrightarrow K \longrightarrow (K + iK') \longrightarrow iK'$$

on the boundary of the rectangle (naively speaking, half of the boundary on the right side), then its image under sn is the interval $[1, \infty]$, which is traversed twice in the following manner:

$$\infty \longrightarrow \frac{1}{k} \longrightarrow 1 \longrightarrow \frac{1}{k} \longrightarrow \infty.$$

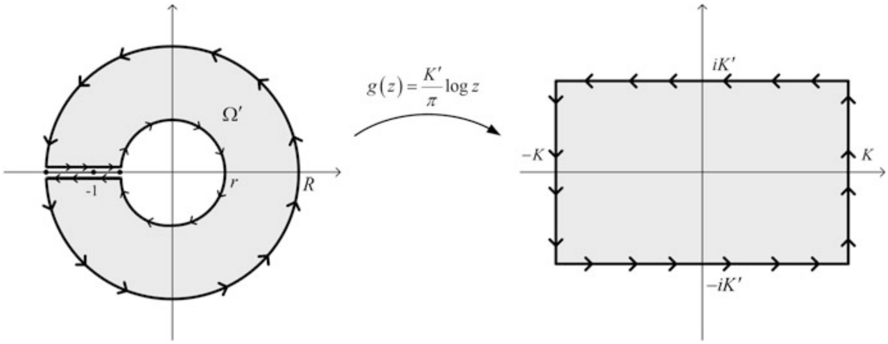

Fig. 2 The main branch of logarithm

If we continue on the boundary of the rectangle on the path

$$iK' \longrightarrow (-K+iK') \longrightarrow -K \longrightarrow (-K-iK') \longrightarrow -iK',$$

then its image under sn is the interval $[\infty,-1]$, which is traversed twice as

$$\infty \longrightarrow -\frac{1}{k} \longrightarrow -1 \longrightarrow -\frac{1}{k} \longrightarrow \infty.$$

Let

$$\Omega' := \{z : r < |z| < R\} \setminus (-R,-r),$$

where

$$r := \exp(-\pi K/K') \qquad \text{and} \qquad R := \exp(\pi K/K').$$

Then $g(z) := (K'/\pi)\log z$ maps Ω' onto the rectangle $[-K,K]\times[-iK',iK']$. Here, log is the principal branch of the logarithm. To better demonstrate the behavior of g, we put a thin slot on the interval $[-R,-r]$ and study g above and below this slot. See Fig. 2.

The conformal mapping g has a continuous extension to the closed annulus $\{z : r \le |z| \le R\}$ in the following special manner. It is continuous at all points of the circles $\{|z| = r\}$ and $\{|z| = R\}$ except at $z=-r$ and $z=-R$. If we start from $z=-R$ and traverse counterclockwise the circle $\{|z|=R\}$ until we reach this point again, then the image of this path under g is the segment $\{K\}\times[K-iK',K+iK']$. We emphasize that

$$\lim_{\theta\to-\pi} g(Re^{i\theta}) = K-iK' \qquad \text{and} \qquad \lim_{\theta\to\pi} g(Re^{i\theta}) = K+iK'.$$

Similarly, if we start from $z = -r$ and traverse clockwise the circle $\{|z| = r\}$ until we reach this point again, then the image of this path under g is the segment $\{-K\} \times [-K + iK', -K - iK']$. Note that,

$$\lim_{\theta \to -\pi} g(re^{i\theta}) = -K - iK' \qquad \text{and} \qquad \lim_{\theta \to \pi} g(re^{i\theta}) = -K + iK'.$$

Understanding the behavior of g at the points of $[-R, -r]$ is very delicate. It depends on the way we approach these points. If we approach them from the upper half plane, then g continuously and bijectively maps $[-R, -r]$ into the segment $[K, -K] \times \{iK'\}$. But, if we approach them from the lower half plane, then g continuously and bijectively maps $[-r, -R]$ into the segment $[-K, K] \times \{-iK'\}$. Therefore, for each $-R \le x \le -r$, we have

$$\lim_{\substack{z \to x \\ \operatorname{Im} z > 0}} g(z) = \frac{K'}{\pi} \log|x| + iK' \qquad \text{and} \qquad \lim_{\substack{z \to x \\ \operatorname{Im} z < 0}} g(z) = \frac{K'}{\pi} \log|x| - iK'.$$

In particular,

$$\lim_{\substack{z \to -1 \\ \operatorname{Im} z > 0}} g(z) = iK' \qquad \text{and} \qquad \lim_{\substack{z \to -1 \\ \operatorname{Im} z < 0}} g(z) = -iK'.$$

At this point, we combine the last two mappings by defining

$$h := \operatorname{sn} \circ g.$$

At first glance, h is a conformal mapping from Ω' onto Ω. But h maps continuously and bijectively $(-R, -1)$ onto $(1/k, \infty)$, and $(-1, -r)$ onto $(-\infty, -1/k)$, and it also maps continuously $\{-1\}$ to ∞. Therefore, by Riemann's theorem, h is indeed conformal at all points of $(-R, -r)$ with a simple pole at $\{-1\}$. Thus h is a conformal mapping form the annulus $\{z : r < |z| < R\}$ onto $\mathbb{C} \cup \{\infty\} \setminus [-1/k, -1] \cup [1, 1/k]$. See Fig. 3.

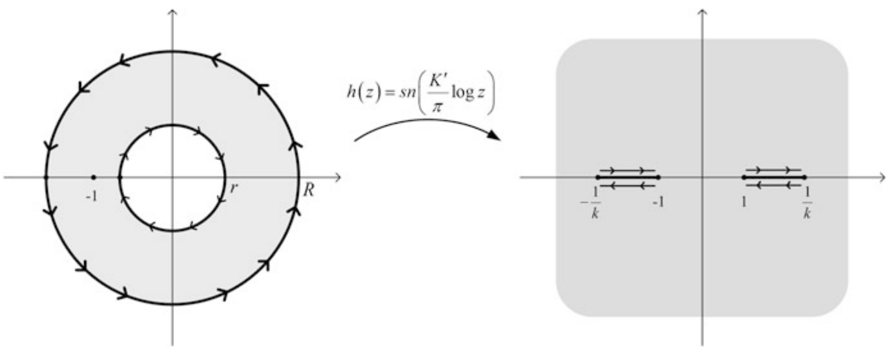

Fig. 3 The conformal mapping h

We are now ready to define our main conformal mapping. Fix $0 < \theta_0 < \pi$, and fix ε with

$$0 < \varepsilon < \min\{\theta_0,\ \pi - \theta_0\}.$$

Pick $k \in (0, 1)$ such that

$$\frac{(k-1)^2}{4k} = \frac{\tan(\frac{\theta_0+\varepsilon}{2})}{\tan(\frac{\theta_0}{2})} - 1.$$

Set

$$\ell := \tan(\theta_0/2) \qquad \text{and} \qquad \ell' := \tan((\theta_0+\varepsilon)/2) = \ell\left(1 + \frac{(k-1)^2}{4k}\right). \tag{16}$$

Then the Möbius transformation

$$z \longmapsto \frac{k(i-\alpha)z + (i\beta - \alpha)}{k(i+\alpha)z + (i\beta + \alpha)},$$

where

$$\alpha = \frac{(1+k)\ell\tan(\varepsilon/2)}{\ell(1-k) + 2\tan(\varepsilon/2)} \qquad \text{and} \qquad \beta = \frac{-(1-k)\ell + 2k\tan(\varepsilon/2)}{\ell(1-k) + 2\tan(\varepsilon/2)},$$

maps the real line into the unit circle in such a way that

$$-1/k \mapsto 1, \qquad -1 \mapsto e^{-i\varepsilon}, \qquad 1 \mapsto e^{i(\theta_0+\varepsilon)}, \qquad 1/k \mapsto e^{i\theta_0}.$$

Moreover,

$$\infty \mapsto \frac{k(i-\alpha)}{k(i+\alpha)}, \qquad \frac{-(i\beta+\alpha)}{k(i+\alpha)} \mapsto \infty.$$

Therefore

$$\Phi(z) := \frac{k(i-\alpha)h(z) + (i\beta - \alpha)}{k(i+\alpha)h(z) + (i\beta + \alpha)}$$

is a conformal mapping from the annulus $\{z : r < |z| < R\}$ to $\mathbb{C} \cup \{\infty\} \setminus \Gamma$, where Γ consists of two arcs of the unit circle:

$$\Gamma = \{e^{i\theta} : -\varepsilon \le \theta \le 0\} \cup \{e^{i\theta} : \theta_0 \le \theta \le \theta_0 + \varepsilon\}.$$

Figure 4 describes how the boundaries of the annulus are mapped.

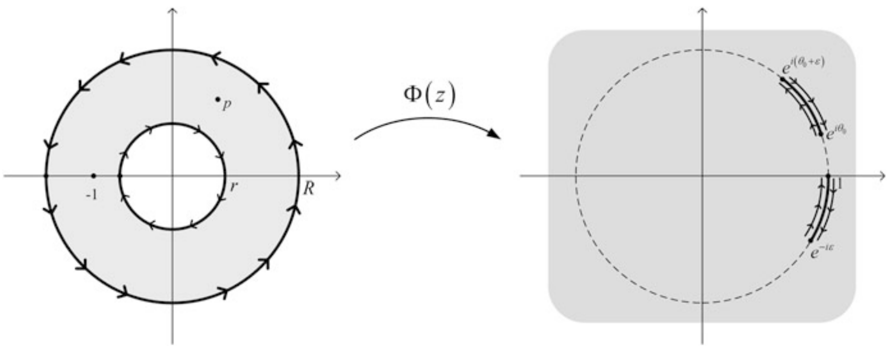

Fig. 4 The conformal mapping Φ

Note that Φ is conformal at -1 with

$$\Phi(-1) = \frac{i-\alpha}{i+\alpha} \in \mathbb{T},$$

and there is a unique point in the annulus, p say, such that

$$h(p) = -\frac{i\beta + \alpha}{k(i+\alpha)},$$

and thus $\Phi(p) = \infty$. This point is a simple pole of Φ. Since p is a simple pole and since Φ is a conformal mapping, it follows that $(z-p)\Phi(z)$ is a *bounded* holomorphic function on the annulus, i.e.,

$$|(z-p)\Phi(z)| \le C < \infty \tag{17}$$

for all z in the annulus.

The conformal mapping Φ plays a crucial rule in the proof of the following result of Douglas and Rudin.

Theorem 9.1 (Douglas–Rudin [5]) *Let $\phi \in \mathbf{U}_{L^\infty(\mathbb{T})}$, i.e., a measurable unimodular function on $\mathbb{T}$, and let $\varepsilon > 0$. Then there are inner functions ω_1 and ω_2 (even Blaschke products) such that*

$$\left\| \phi - \frac{\omega_1}{\omega_2} \right\|_{L^\infty(\mathbb{T})} < \varepsilon.$$

Proof First we consider a special class of unimodular functions. Let E be a measurable subset of $\mathbb{T}$, and let $0 < \theta_0 < \pi$. Set

$$\phi(e^{i\theta}) := \begin{cases} e^{i\theta_0}, \text{ if } e^{i\theta} \in E, \\ \ \ 1, \ \ \text{if } e^{i\theta} \in \mathbb{T} \setminus E. \end{cases}$$

Thus ϕ is a unimodular function that takes only two different values on $\mathbb{T}$. Given $\varepsilon > 0$, pick k such that (16) holds. Then K and K' are defined respectively by (14) and (15). Set

$$u(e^{i\theta}) := \begin{cases} \ \ \pi K/K', \ \text{if } e^{i\theta} \in E, \\ -\pi K/K', \text{if } e^{i\theta} \in \mathbb{T} \setminus E, \end{cases}$$

and let $U = P_r * u$ be its harmonic extension to the open unit disc with the harmonic conjugate $V = Q_r * u$. Since $-\pi K/K' < U < \pi K/K'$, the holomorphic function $F = \exp(U + iV)$ maps the unit disc into the annulus

$$\{z : \exp(-\pi K/K') < |z| < \exp(\pi K/K')\}.$$

Moreover,

$$F(e^{i\theta}) = \lim_{r\to 1} F(re^{i\theta}) \in \{z : |z| = \exp(\pi K/K')\} \tag{18}$$

for almost all $e^{i\theta} \in E$, and

$$F(e^{i\theta}) = \lim_{r\to 1} F(re^{i\theta}) \in \{z : |z| = \exp(-\pi K/K')\} \tag{19}$$

for almost all $e^{i\theta} \in \mathbb{T} \setminus E$.

Let $\Psi := \Phi \circ F$, where Φ is the conformal mapping depicted in Fig. 4. Then Ψ is a meromorphic function with poles at the points $\{z \in \mathbb{D} : F(z) = p\}$. Since Φ has a simple pole at p, the order of Ψ at a pole z_0 is *equal* the order of z_0 as a zero of $F(z) - p$. Moreover, since $F(z) - p \in H^\infty(\mathbb{D})$, the zeros of $F(z) - p$ form a Blaschke sequence in $\mathbb{D}$, and, by the canonical factorization theorem, $F(z) - p$ can be decomposed as

$$F(z) - p = B(z)S(z)O(z), \tag{20}$$

where B is a Blaschke product, S is a singular inner function and O is an outer function. We shall show that $\omega(z) := B(z)S(z)\Psi(z)$ is an inner function (note that the product is inner, not $\Psi(z)$ alone).

First of all, since the poles of Ψ are canceled by the zeros of B, the function ω is holomorphic on $\mathbb{D}$. Secondly,

$$|F(e^{i\theta}) - p| = |B(e^{i\theta})S(e^{i\theta})O(e^{i\theta})| = |O(e^{i\theta})|$$

for almost all $e^{i\theta} \in \mathbb{T}$. Moreover, by (18),

$$|O(e^{i\theta})| = |F(e^{i\theta}) - p| \geq \exp(\pi K/K') - |p| > 0$$

for almost all $e^{i\theta} \in E$, and, by (19),

$$|O(e^{i\theta})| = |F(e^{i\theta}) - p| \geq |p| - \exp(-\pi K/K') > 0$$

for almost all $e^{i\theta} \in \mathbb{T} \setminus E$. Thus $|O|$ is bounded away from zero on $\mathbb{T}$, which, by Smirnov's theorem, implies that

$$\frac{1}{O} \in H^\infty(\mathbb{D}).$$

Finally, by (17) and (20),

$$\begin{aligned} |\omega(z)| &= |B(z)||S(z)||\Psi(z)| \\ &= |B(z)||S(z)||\Phi(F(z))| \\ &\leq |B(z)||S(z)|\frac{C}{|F(z)-p|} \\ &\leq \frac{C}{|O(z)|} \leq C' \end{aligned}$$

for all $z \in \mathbb{D}$. Thus $\omega \in H^\infty(\mathbb{D})$. Moreover, for almost all $e^{i\theta} \in \mathbb{T}$,

$$\omega(e^{i\theta}) = B(e^{i\theta})S(e^{i\theta})\Psi(e^{i\theta}) \in \mathbb{T}.$$

Therefore, ω is indeed an inner function.

Turning back to Ψ, we note that

$$\Psi = \frac{\omega}{BS}$$

is the quotient of two inner functions. Also, by (18) and the behavior of Φ on the circle $\{z : |z| = \exp(\pi K/K')\}$, we have

$$|\phi(e^{i\theta}) - \Psi(e^{i\theta})| = |e^{i\theta_0} - \Phi(F(e^{i\theta}))| \leq \varepsilon$$

for almost all $e^{i\theta} \in E$, and, by (19) and the behavior of Φ on the circle $\{z : |z| = \exp(-\pi K/K')\}$, we also have

$$|\phi(e^{i\theta}) - \Psi(e^{i\theta})| = |1 - \Phi(F(e^{i\theta}))| \leq \varepsilon$$

for almost all $e^{i\theta} \in \mathbb{T} \setminus E$. This means that

$$\|\phi - \Psi\|_\infty \leq \varepsilon.$$

To show that an arbitrary measurable unimodular function can be uniformly approximated by the quotient of inner functions, we use a simple approximation technique. Let ϕ be a measurable unimodular function. Given $\varepsilon > 0$, choose $N \geq 1$ such that $2\pi/N < \varepsilon$. Let

$$E_k := \{e^{i\theta} : 2\pi(k-1)/N \leq \arg\phi(e^{i\theta}) < 2\pi k/N\}, \qquad (1 \leq k \leq N),$$

and let

$$\phi_k(e^{i\theta}) := \begin{cases} e^{i2\pi k/N}, & \text{if } e^{i\theta} \in E_k, \\ 1, & \text{if } e^{i\theta} \neq E_k. \end{cases}$$

Then each ϕ_k is unimodular and takes *only* two different values on $\mathbb{T}$, and

$$\|\phi - \phi_1\phi_2\cdots\phi_N\|_\infty \leq \varepsilon.$$

According to the first part of the proof, there are inner functions ω_{k1} and ω_{k2} such that

$$\|\phi_k - \omega_{k1}/\omega_{k2}\|_\infty < \varepsilon/N, \qquad (1 \leq k \leq N).$$

Since

$$\begin{aligned} \phi - \frac{\omega_{11}}{\omega_{12}}\frac{\omega_{21}}{\omega_{22}}\cdots\frac{\omega_{N1}}{\omega_{N2}} &= \phi - \phi_1\phi_2\phi_3\cdots\phi_N \\ &+ (\phi_1 - \frac{\omega_{11}}{\omega_{12}})\phi_2\phi_3\cdots\phi_N \\ &+ \frac{\omega_{11}}{\omega_{12}}(\phi_2 - \frac{\omega_{21}}{\omega_{22}})\phi_3\cdots\phi_N \\ &\vdots \\ &+ \frac{\omega_{11}}{\omega_{12}}\frac{\omega_{21}}{\omega_{22}}\cdots(\phi_N - \frac{\omega_{N1}}{\omega_{N2}}), \end{aligned}$$

we thus have

$$\left\| \phi - \frac{\omega_{11}\omega_{21}\cdots\omega_{N1}}{\omega_{12}\omega_{22}\cdots\omega_{N2}} \right\|_\infty \leq 2\varepsilon.$$

In the light of Frostman's theorem, ω_1 and ω_2 can be replaced by Blaschke products. This concludes the proof. □

10 Approximation on $\mathbb{D}$ by Convex Combinations of Quotients of Blaschke Products

$$\boxed{\text{Goal}: \overline{\text{conv}(\mathbf{BP}/\mathbf{BP})} = \overline{\text{conv}(\mathbf{I}/\mathbf{I})} = \mathbf{B}_{L^\infty(\mathbb{T})}}$$

Clearly, a unimodular measurable function on $\mathbb{T}$ is in the closed unit ball of $L^\infty(\mathbb{T})$. In the first step in studying the closed convex hull of quotients of Blaschke products, we show that the family of all unimodular measurable functions on $\mathbb{T}$ is a large set in $L^\infty(\mathbb{T})$, in the sense that the closed convex hull of this family is precisely the closed unit ball of $L^\infty(\mathbb{T})$. The results in this section are taken from [5].

Lemma 10.1 *Let $f \in \mathbf{B}_{L^\infty(\mathbb{T})}$ and let $\varepsilon > 0$. Then there are $u_j \in \mathbf{U}_{L^\infty(\mathbb{T})}$ and convex weights $(\lambda_j)_{1\le j\le n}$ such that*

$$\|\lambda_1 u_1 + \lambda_2 u_2 + \cdots + \lambda_n u_n - f\|_{L^\infty(\mathbb{T})} < \varepsilon.$$

Proof Proceeding precisely as in the proof of Lemma 6.1, we obtain

$$\left| f(e^{i\theta}) - \frac{1}{N}\sum_{k=1}^{N} \frac{e^{i2k\pi/N} + (1-\varepsilon)f(e^{i\theta})}{1+(1-\varepsilon)\overline{f(e^{i\theta})}e^{i2k\pi/N}} \right| \le \varepsilon + \frac{4\pi}{\varepsilon N}$$

for each $e^{i\theta} \in \mathbb{T}$. But each

$$u_k(e^{i\theta}) = \frac{e^{i2k\pi/N} + (1-\varepsilon)f(e^{i\theta})}{1+(1-\varepsilon)\overline{f(e^{i\theta})}e^{i2k\pi/N}}$$

is in fact a unimodular function on $\mathbb{T}$. Thus, given $\varepsilon > 0$, it is enough to choose N so large that $4\pi/(\varepsilon N) < \varepsilon$ to get

$$\left| f(e^{i\theta}) - \frac{1}{N}\sum_{k=1}^{N} u_k(e^{i\theta}) \right| \le 2\varepsilon$$

for all $e^{i\theta} \in \mathbb{T}$. □

Theorem 9.1 and Lemma 10.1 together show that the closed convex hull in $L^\infty(\mathbb{T})$ of the set

$$\left\{ \frac{\omega_1}{\omega_2} : \omega_1 \text{ and } \omega_2 \text{ are inner} \right\}$$

is precisely the closed unit ball of $L^\infty(\mathbb{T})$.

Theorem 10.2 (Douglas–Rudin [5]) *Let $f \in \mathbf{B}_{L^\infty(\mathbb{T})}$ and let $\varepsilon > 0$. Then there are inner functions ω_{ij}, $1 \le i,j \le n$ (even Blaschke products) and convex weights $(\lambda_j)_{1\le j\le n}$ such that*

$$\left\| \lambda_1 \frac{\omega_{11}}{\omega_{12}} + \lambda_2 \frac{\omega_{21}}{\omega_{22}} + \cdots + \lambda_n \frac{\omega_{n1}}{\omega_{n2}} - f \right\|_{L^\infty(\mathbb{T})} < \varepsilon.$$

Proof By Lemma 10.1, there are $0 \le \lambda_1, \lambda_2, \ldots, \lambda_n \le 1$ with $\lambda_1 + \cdots + \lambda_n = 1$, and unimodular functions $u_1, u_2, \ldots, u_n$ such that

$$\|\lambda_1 u_1 + \lambda_2 u_2 + \cdots + \lambda_n u_n - f\|_\infty < \varepsilon/2.$$

Also, for each k, by Theorem 9.1, there are inner functions ω_{k1} and ω_{k2} such that

$$\|u_k - \omega_{k1}/\omega_{k2}\|_\infty < \varepsilon/2.$$

Then

$$\left\| f - \sum_{k=1}^n \lambda_k \omega_{k1}/\omega_{k2} \right\|_\infty \le \left\| f - \sum_{k=1}^n \lambda_k u_k \right\|_\infty + \sum_{k=1}^n \lambda_k \|u_k - \omega_{k1}/\omega_{k2}\|_\infty < \varepsilon.$$

This completes the proof. □

Remark Since the product of two inner functions is an inner function, in the quotients appearing in Theorem 10.2, we can take a common denominator and thus, without loss of generality, assume that all the ω_{k2} are equal. Hence, under the conditions of Theorem 10.2, there are inner functions ω and $\omega_1, \ldots, \omega_n$ such that

$$\left\| \lambda_1 \frac{\omega_1}{\omega} + \lambda_2 \frac{\omega_2}{\omega} + \cdots + \lambda_n \frac{\omega_n}{\omega} - f \right\|_\infty < \varepsilon.$$

The same remark obviously applies to quotients of Blaschke products.

11 Approximation on $\mathbb{D}$ by Convex Combination of Infinite Blaschke Products

$$\boxed{\text{Goal} : \overline{\text{conv}(\mathbf{BP})} = \overline{\text{conv}(\mathbf{I})} = \mathbf{B}_{H^\infty}}$$

To study convex combinations of Blaschke products, we need the following variant of Theorem 10.2.

Lemma 11.1 *Let $f \in H^\infty$ and let $\varepsilon > 0$. Then there are real constants a_j and inner functions ω and ω_j such that*

$$a_1 \frac{\omega_1}{\omega} + a_2 \frac{\omega_2}{\omega} + \cdots + a_n \frac{\omega_n}{\omega} \in H^\infty$$

and

$$\left\| a_1 \frac{\omega_1}{\omega} + a_2 \frac{\omega_2}{\omega} + \cdots + a_n \frac{\omega_n}{\omega} - f \right\|_\infty < \varepsilon.$$

Proof The result is clear if $f \equiv 0$, so let us assume that $f \not\equiv 0$. By the remark following Theorem 10.2, there are $0 \le \lambda_1, \lambda_2, \ldots, \lambda_m \le 1$ with $\lambda_1 + \lambda_2 + \cdots + \lambda_m = 1$, and inner functions $\omega_1, \omega_2, \ldots, \omega_m$ and ω such that

$$\left\| \lambda_1 \frac{\omega_1}{\omega} + \lambda_2 \frac{\omega_2}{\omega} + \cdots + \lambda_m \frac{\omega_m}{\omega} - \frac{f}{\|f\|_\infty} \right\|_\infty < \varepsilon', \tag{21}$$

where $\varepsilon' = \varepsilon/(2\|f\|_\infty)$. Put

$$F := \frac{1}{\varepsilon'} \left(\lambda_1 \frac{\omega_1}{\omega} + \lambda_2 \frac{\omega_2}{\omega} + \cdots + \lambda_m \frac{\omega_m}{\omega} \right).$$

Then $F \in L^\infty(\mathbb{T})$, and the last inequality shows that

$$\operatorname{dist}(F, H^\infty(\mathbb{T})) < 1.$$

Hence, by Theorem 8.4, there are $G \in H^\infty(\mathbb{T})$ and a unimodular function I such that $F = I + G$. But, since ωF is in $H^\infty(\mathbb{T})$, the function $\omega_0 := \omega I = \omega F - \omega G$ is a unimodular function in $H^\infty(\mathbb{T})$. In other words, ω_0 is an inner function, and thus $I = \omega_0/\omega$ is the quotient of two inner functions. Moreover,

$$\lambda_1 \frac{\omega_1}{\omega} + \lambda_2 \frac{\omega_2}{\omega} + \cdots + \lambda_m \frac{\omega_m}{\omega} - \varepsilon' \frac{\omega_0}{\omega} = \varepsilon'(F - I) = \varepsilon' G \in H^\infty(\mathbb{T}),$$

and, by (21),

$$\left\| \lambda_1 \frac{\omega_1}{\omega} + \lambda_2 \frac{\omega_2}{\omega} + \cdots + \lambda_m \frac{\omega_m}{\omega} - \varepsilon' \frac{\omega_0}{\omega} - \frac{f}{\|f\|_\infty} \right\|_\infty < 2\varepsilon'.$$

Therefore,

$$\left\| a_1 \frac{\omega_1}{\omega} + a_2 \frac{\omega_2}{\omega} + \cdots + a_m \frac{\omega_m}{\omega} + a_{m+1} \frac{\omega_{m+1}}{\omega} - f \right\|_\infty < 2\|f\|_\infty \varepsilon' = \varepsilon,$$

where $a_k := \lambda_k \|f\|_\infty$, for $1 \le k \le m$, and $a_{m+1} := -\varepsilon' \|f\|_\infty$ and $\omega_{m+1} := \omega_0$. □

Now we are able to show that the closed convex hull of the family of all inner functions on $\mathbb{T}$ is precisely the closed unit ball of $H^\infty(\mathbb{T})$.

Theorem 11.2 (Marshall [14]) *Let $f \in \mathbf{B}_{H^\infty}$ and let $\varepsilon > 0$. Then there are inner functions ω_j (even Blaschke products) and convex weights $(\lambda_j)_{1\le j\le n}$ such that*

$$\|\lambda_1\omega_1 + \lambda_2\omega_2 + \cdots + \lambda_n\omega_n - f\|_\infty < \varepsilon.$$

Proof By Lemma 11.1, there are real constants $a_1, \dots, a_n$ and inner functions $\omega, \omega_1, \dots, \omega_n$ such that

$$g := a_1\frac{\omega_1}{\omega} + a_2\frac{\omega_2}{\omega} + \cdots + a_n\frac{\omega_n}{\omega} \in H^\infty(\mathbb{T}),$$

and $\|g - (1-2\varepsilon)f\|_\infty < \varepsilon$. Hence it is enough to approximate g by convex combination of inner functions. Note that $\|g\|_\infty < 1-\varepsilon$.

Set

$$\omega_0 := \omega_1\omega_2\cdots\omega_n.$$

Since

$$\overline{g} = \overline{a}_1\frac{\overline{\omega}_1}{\overline{\omega}} + \overline{a}_2\frac{\overline{\omega}_2}{\overline{\omega}} + \cdots + \overline{a}_n\frac{\overline{\omega}_n}{\overline{\omega}} = a_1\frac{\omega}{\omega_1} + a_2\frac{\omega}{\omega_2} + \cdots + a_n\frac{\omega}{\omega_n},$$

we clearly have

$$\omega_0\overline{g} \in H^\infty(\mathbb{T}).$$

This property is the main advantage of g over f. Now we follow a similar procedure to that in the proof of Lemma 6.1.

Let $w \in \mathbb{D}$ and $\gamma \in \mathbb{T}$. Then, by the Cauchy integral formula,

$$w = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{\gamma\zeta + w}{\zeta(1+\overline{w}\gamma\zeta)}\,d\zeta = \frac{1}{2\pi}\int_0^{2\pi} \frac{\gamma e^{i\theta} + w}{1+\overline{w}\gamma e^{i\theta}}\,d\theta.$$

Since

$$\left|\frac{\gamma e^{i\theta} + w}{1+\overline{w}\gamma e^{i\theta}} - \frac{\gamma e^{i\theta'} + w}{1+\overline{w}\gamma e^{i\theta'}}\right| \le \frac{1+|w|}{1-|w|}|\theta-\theta'|,$$

for

$$\Delta_N := \left|w - \frac{1}{N}\sum_{k=1}^{N}\frac{\gamma e^{i2k\pi/N} + w}{1+\overline{w}\gamma e^{i2k\pi/N}}\right|,$$

we have the estimation

$$\begin{aligned}
\Delta_N &= \left| \frac{1}{2\pi} \int_0^{2\pi} \frac{\gamma e^{i\theta} + w}{1 + \overline{w}\gamma e^{i\theta}}\, d\theta - \frac{1}{N} \sum_{k=1}^{N} \frac{\gamma e^{i2k\pi/N} + w}{1 + \overline{w}\gamma e^{i2k\pi/N}} \right| \\
&\leq \frac{1}{2\pi} \sum_{k=1}^{N} \int_{2(k-1)\pi/N}^{2k\pi/N} \left| \frac{\gamma e^{i\theta} + w}{1 + \overline{w}\gamma e^{i\theta}} - \frac{\gamma e^{i2k\pi/N} + w}{1 + \overline{w}\gamma e^{i2k\pi/N}} \right| d\theta \\
&\leq \frac{1}{2\pi} \sum_{k=1}^{N} \int_{2(k-1)\pi/N}^{2k\pi/N} \frac{1 + |w|}{1 - |w|} \frac{2\pi}{N}\, d\theta \\
&= \frac{1 + |w|}{1 - |w|} \frac{2\pi}{N}.
\end{aligned}$$

Hence, for almost all $e^{i\theta} \in \mathbb{T}$, setting $w := g(e^{i\theta})$ and $\gamma := \omega_0(e^{i\theta})$, we get

$$\left| g(e^{i\theta}) - \frac{1}{N} \sum_{k=1}^{N} \frac{\omega_0(e^{i\theta}) e^{i2k\pi/N} + g(e^{i\theta})}{1 + \overline{g(e^{i\theta})}\omega_0(e^{i\theta}) e^{i2k\pi/N}} \right| \leq \frac{1 + |g(e^{i\theta})|}{1 - |g(e^{i\theta})|} \frac{2\pi}{N}.$$

Thus, for almost all $e^{i\theta} \in \mathbb{T}$,

$$\left| f(e^{i\theta}) - \frac{1}{N} \sum_{k=1}^{N} \frac{\omega_0(e^{i\theta}) e^{i2k\pi/N} + g(e^{i\theta})}{1 + \overline{g(e^{i\theta})}\omega_0(e^{i\theta}) e^{i2k\pi/N}} \right| \leq 3\varepsilon + \frac{4\pi}{\varepsilon N}.$$

But, for each k,

$$\omega_k(e^{i\theta}) := \frac{\omega_0(e^{i\theta}) e^{i2k\pi/N} + g(e^{i\theta})}{1 + \overline{g(e^{i\theta})}\omega_0(e^{i\theta}) e^{i2k\pi/N}}$$

is in fact an inner function, since in the first place it is a unimodular function, and besides $g, \omega_0, \overline{g}\omega_0 \in H^\infty(\mathbb{T})$ and $|1 + \overline{g(e^{i\theta})}\omega_0(e^{i\theta}) e^{i2k\pi/N}| \geq \varepsilon$, for almost all $e^{i\theta} \in \mathbb{T}$. Therefore, given $\varepsilon > 0$, it is enough to choose N so large that $4\pi/(\varepsilon N) < \varepsilon$ to get

$$\left| f(e^{i\theta}) - \frac{1}{N} \sum_{k=1}^{N} \omega_k(e^{i\theta} \right| \leq 4\varepsilon$$

for almost all $e^{i\theta} \in \mathbb{T}$.

By Frostman's theorem, there are Blaschke products $B_1, \ldots, B_n$ such that $\|\omega_k - B_k\|_\infty < \varepsilon/2$, for each $1 \leq k \leq n$. Hence

$$\left\| f - \sum_{k=1}^{n} \lambda_k B_k \right\|_\infty \le \left\| f - \sum_{k=1}^{n} \lambda_k \omega_k \right\|_\infty + \sum_{k=1}^{n} \lambda_k \|B_k - \omega_k\|_\infty < \varepsilon.$$

This completes the proof. □

12 An Application: The Halmos Conjecture

Let H be a complex Hilbert space and T be a bounded linear operator on H. The *numerical range* of T is defined by

$$W(T) := \{\langle Tx, x\rangle : x \in H,\ \|x\| = 1\}.$$

It is a convex set whose closure contains the spectrum of T. If $\dim H < \infty$, then $W(T)$ is compact. The *numerical radius* of T is defined by

$$w(T) := \sup\{|\langle Tx, x\rangle| : x \in H,\ \|x\| = 1\}.$$

It is related to the operator norm via the double inequality

$$\|T\|/2 \le w(T) \le \|T\|. \tag{22}$$

If further T is self-adjoint, then $w(T) = \|T\|$. In contrast with spectra, it is not true in general that $W(p(T)) = p(W(T))$ for polynomials p, nor is it true if we take convex hulls of both sides. However, some partial results do hold. Perhaps the most famous of these is the power inequality: for all $n \ge 1$, we have

$$w(T^n) \le w(T)^n.$$

This was conjectured by Halmos and, after several partial results, was established by Berger using dilation theory. An elementary proof was given by Pearcy in [15]. A more general result was established by Berger and Stampfli in [4]. They showed that, if $w(T) \le 1$, then, for all f in the disc algebra with $f(0) = 0$, we have

$$w(f(T)) \le \|f\|_\infty.$$

Again their proof used dilation theory. We give an elementary proof of this result along the lines of Pearcy's proof of the power inequality.

We require two folklore lemmas about finite Blaschke products.

Lemma 12.1 *Let B be a finite Blaschke product. Then $\zeta B'(\zeta)/B(\zeta)$ is real and strictly positive for all $\zeta \in \mathbb{T}$.*

Proof We can write

$$B(z) = c \prod_{k=1}^{n} \frac{a_k - z}{1 - \overline{a}_k z},$$

where $a_1, \dots, a_n \in \mathbb{D}$ and $c \in \mathbb{T}$. Then

$$\frac{B'(z)}{B(z)} = \sum_{k=1}^{n} \frac{1 - |a_k|^2}{(z - a_k)(1 - \overline{a}_k z)}.$$

In particular, if $\zeta \in \mathbb{T}$, then

$$\frac{\zeta B'(\zeta)}{B(\zeta)} = \sum_{k=1}^{n} \frac{1 - |a_k|^2}{|\zeta - a_k|^2},$$

which is real and strictly positive. □

Lemma 12.2 *Let B be a Blaschke product of degree n such that $B(0) = 0$. Then, given $\gamma \in \mathbb{T}$, there exist $\zeta_1, \dots, \zeta_n \in \mathbb{T}$ and $c_1, \dots, c_n > 0$ such that*

$$\frac{1}{1 - \overline{\gamma} B(z)} = \sum_{k=1}^{n} \frac{c_k}{1 - \overline{\zeta}_k z}. \tag{23}$$

Proof Given $\gamma \in \mathbb{T}$, the roots of the equation $B(z) = \gamma$ lie on the unit circle, and by Lemma 12.1 they are simple. Call them $\zeta_1, \dots, \zeta_n$. Then $1/(1 - \overline{\gamma} B)$ has simple poles at the ζ_k. Also, as $B(0) = 0$, we have $B(\infty) = \infty$ and so $1/(1 - \overline{\gamma} B)$ vanishes at ∞. Expanding it in partial fractions gives (23), for some choice of $c_1, \dots, c_n \in \mathbb{C}$.

The coefficients c_k are easily evaluated. Indeed, from (23) we have

$$c_k = \lim_{z \to \zeta_k} \frac{1 - \overline{\zeta}_k z}{1 - \overline{\gamma} B(z)} = \lim_{z \to \zeta_k} \frac{(\zeta_k - z)/\zeta_k}{(B(\zeta_k) - B(z))/B(\zeta_k)} = \frac{B(\zeta_k)}{\zeta_k B'(\zeta_k)}.$$

In particular $c_k > 0$ by Lemma 12.1. □

Theorem 12.3 (Berger–Stampfli [4]) *Let H be a complex Hilbert space, let T be a bounded linear operator on H with $w(T) \le 1$, and let f be a function in the disc algebra such that $f(0) = 0$. Then $w(f(T)) \le \|f\|_\infty$.*

Proof (Klaja–Mashreghi–Ransford [12]) Suppose first that f is a finite Blaschke product B. Suppose also that the spectrum $\sigma(T)$ of T lies within the open unit disc $\mathbb{D}$. By the spectral mapping theorem $\sigma(B(T)) = B(\sigma(T)) \subset \mathbb{D}$ as well. Let $x \in H$ with $\|x\| = 1$. Given $\gamma \in \mathbb{T}$, let $\zeta_1, \dots, \zeta_n \in \mathbb{T}$ and $c_1, \dots, c_n > 0$ as in Lemma 12.2. Then we have

$$\begin{aligned} 1 - \overline{\gamma}\langle B(T)x, x\rangle &= \langle (I - \overline{\gamma} B(T))x, x\rangle \\ &= \langle y, (I - \overline{\gamma} B(T))^{-1} y\rangle \qquad \text{where } y := (I - \overline{\gamma} B(T))x \end{aligned}$$

$$
\begin{aligned}
&= \Big\langle y, \sum_{k=1}^{n} c_k (I - \overline{\zeta}_k T)^{-1} y \Big\rangle && \text{by (23)} \\
&= \sum_{k=1}^{n} c_k \langle (I - \overline{\zeta}_k T) z_k, z_k \rangle && \text{where } z_k := (I - \overline{\zeta}_k T)^{-1} y \\
&= \sum_{k=1}^{n} c_k (\|z_k\|^2 - \overline{\zeta}_k \langle T z_k, z_k \rangle).
\end{aligned}
$$

Since $w(T) \leq 1$, we have $\operatorname{Re}(\|z_k\|^2 - \overline{\zeta}_k \langle T z_k, z_k \rangle) \geq 0$, and as $c_k > 0$ for all k, it follows that

$$
\operatorname{Re}(1 - \overline{\gamma} \langle B(T)x, x \rangle) \geq 0.
$$

As this holds for all $\gamma \in \mathbb{T}$ and all x of norm 1, it follows that $w(B(T)) \leq 1$.

Next we relax the assumption on f, still assuming that $\sigma(T) \subset \mathbb{D}$. We can suppose that $\|f\|_\infty = 1$. Then, by Carathéodory's theorem (Theorem 3.1), there exists a sequence of finite Blaschke products B_n that converges locally uniformly to f in $\mathbb{D}$. Moreover, as $f(0) = 0$, we can also arrange that $B_n(0) = 0$ for all n. By what we have proved, $w(B_n(T)) \leq 1$ for all n. Also $B_n(T)$ converges in norm to $f(T)$, because $\sigma(T) \subset \mathbb{D}$. It follows that $w(f(T)) \leq 1$, as required.

Finally we relax the assumption that $\sigma(T) \subset \mathbb{D}$. By what we have already proved, $w(f(rT)) \leq \|f\|_\infty$ for all $r < 1$. Interpreting $f(T)$ as $\lim_{r \to 1^-} f(rT)$, it follows that $w(f(T)) \leq \|f\|_\infty$, *provided* that this limit exists. In particular this is true when f is holomorphic in a neighborhood of $\overline{\mathbb{D}}$. To prove the existence of the limit in the general case, we proceed as follows. Given $r, s \in (0, 1)$, the function $g_{rs}(z) := f(rz) - f(sz)$ is holomorphic in a neighborhood of $\overline{\mathbb{D}}$ and vanishes at 0, so, by what we have already proved, $w(g_{rs}(T)) \leq \|g_{rs}\|_\infty$. Therefore,

$$
\|f(rT) - f(sT)\| = \|g_{rs}(T)\| \leq 2w(g_{rs}(T)) \leq 2\|g_{rs}\|_\infty.
$$

The right-hand side tends to zero as $r, s \to 1^-$, so, by the usual Cauchy-sequence argument, $f(rT)$ converges as $r \to 1^-$. This completes the proof. □

Remark The assumption that $f(0) = 0$ is essential in the Berger–Stampfli theorem. Without this assumption, the situation becomes more complicated. The best result in this setting is Drury's teardrop theorem [6]. See also [12] for an alternative proof.

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A Thought on Approximation by Bi-Analytic Functions

Dmitry Khavinson

Dedicated to the memory of André Boivin, a kind and gentle friend.

Abstract A different approach to the problem of uniform approximations by the module of bi-analytic functions is outlined. This note follows the ideas from Khavinson (On a geometric approach to problems concerning Cauchy integrals and rational approximation. PhD thesis, Brown University, Providence, RI (1983), Proc Am Math Soc 101(3):475–483 (1987), Michigan Math J 34(3):465–473 (1987), Contributions to operator theory and its applications (Mesa, AZ, 1987). Birkhäuser, Basel (1988)), Gamelin and Khavinson (Am Math Mon 96(1):18–30 (1989)) and the more recent paper (Abanov et al. A free boundary problem associated with the isoperimetric inequality. arXiv:1601.03885, 2016 preprint), regarding approximation of $\bar{z}$ by analytic functions.

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1 Introduction

The ideas sketched in this note were inspired by the talk of J. Verdera at the Approximation Theory Conference dedicated to A. Boivin held at the Fields Institute in Toronto in July 2016. Denote by $R_2(K)$ the uniformly closed rational module generated by functions $f(\zeta) + \bar{z}g(\zeta)$, with f and g analytic in the neighborhood of a compact set K in $\mathbb{C}$. Equivalently, $R_2(K)$ is the uniform closure on K of functions $f(\zeta) + \bar{z}(\zeta)$, with f, g being rational functions with poles off K, i.e., $f, g \in R(K)$.

The bi-analytic rational module R_2, and more generally $R_N(K)$ generated by $f_1(\zeta) + \bar{z}f_2(\zeta) + \cdots + \bar{z}^{N-1}f_N(\zeta), f_j \in R(K)$ have been studied intensely in the 1970s

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and 1980s—cf., e.g., [3, 15–20]. The subject remained dormant after that for almost two decades until a remarkable and deep paper of Mazalov [14].

Here, we want to suggest a different point of view on the approximation by bi-analytic functions based on extending the notion of "analytic content" in [1, 2, 7, 10] to this setting. Namely, let us accept the following definition:

Definition 1 Let $\lambda_2(K) := \inf_{\varphi \in R_2(K)} \left\| \frac{\bar{z}^2}{2} - \varphi \right\|_{C(K)}$, and call λ_2 the bi-analytic content of K.

(From now on, $\| \cdot \| = \| \cdot \|_{C(K)}$ unless otherwise specified.) The analogy with $\lambda(K)$, the analytic content defined first in [12], is clear. Indeed, $\lambda(K) := \inf_{\varphi \in R(K)} \|\bar{z} - \varphi(z)\|$, $R(K) = \overline{\text{Ker } \bar{\partial}_{\|\cdot\|}}$ while $\frac{\partial}{\partial \bar{z}}(\bar{z}) = 1$, making $\bar{z}$ the simplest non-analytic function. $R_2(K) = \overline{\text{Ker } \bar{\partial}^2_{\|\cdot\|}}$ and $\bar{\partial}^2 \left(\frac{\bar{z}^2}{2} \right) = 1$.

2 An "Analogue" of the Stone–Weierstrass Theorem

The following simple proposition supports the introduction of $\lambda_2(K)$—cf. [9, 11].

Proposition 1 $\lambda_2(K) = 0$ *iff* $R_2(K) = C(K)$.

Proof (Sketch) The necessity is obvious. To see the sufficiency, note that $\lambda_2(K) = 0$ yields $\bar{z}^2 \in R_2(K)$. Hence one can approximate $\bar{z}^2$ by functions $r_1(z) + \bar{z}\, r_2$. Thus, for any $r \in R(K)$ we have $r\bar{z}^2 \sim rr_1 + \bar{z} r r_2$, where we put "$\sim$" for "approximate uniformly". Hence, $\bar{z}^3 \sim \bar{z}\,(r_1 + \bar{z}\, r_2) \sim \bar{z} r_1 + (r_3 + r_4\, \bar{z}) \in R_2(K)$, where all $r_j \in R(K)$. Hence, $\bar{z}^3 \sim r_5 + \bar{z} r_6$ and then $\bar{z}^2\,(r_7 + \bar{z}\, r_8) \in R_2(K)$ since $\bar{z}^3 r_8 \sim (r_5 + \bar{z}\, r_6)\, r_8 \in R_2(K)$. A straightforward induction yields that all monomials $\bar{z}^n z^m$ are approximable by $R_2(K)$. Weierstrass' approximation theorem finishes the argument.

3 Green's Formula and Duality

As is well-known, the fundamental solution for $\bar{\partial}^2$ is $\frac{1}{\pi} \frac{\bar{z}}{z}$. Hence, Green's formula yields immediately the following (cf. [17]).

Lemma 1 *For any* $\varphi \in C_0^\infty$ *and any* $z \in \mathbb{C}$, *we have*

$$\varphi(z) = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{\partial^2 \varphi(\zeta)}{\partial \bar{\zeta}^2} \frac{\overline{\zeta - z}}{\zeta - z} \, dA(\zeta). \tag{1}$$

(Here and onward, $dA(\zeta)$ denotes the normalized area measure $\frac{1}{\pi}\, dx\, dy$.)

Lemma 2

$$\lambda_2(K) = \max_{z\in K}\left|\int_K \frac{\overline{\zeta - z}}{\zeta - z}\, dA(\zeta)\right|. \tag{2}$$

Proof (A Sketch, Since the Argument is Standard, cf., e.g., [7]) Extend $\frac{1}{2}\bar{z}^2$ to $\varphi_0 \in C_0^\infty$ with the support in a fixed disk $D = \{z : |z| \le R < \infty\}$. For $\epsilon > 0$, let Ω_ϵ be a smoothly bounded ϵ-neighborhood of K. For $z \in K$, split the integral in (1) into three parts:

$$\int_{D\setminus\Omega_\epsilon} + \int_{\Omega_\epsilon\setminus K} + \int_K =: I + II + III.$$

$I \in R_2(K)$, $\|II\| \le const(\varphi_0)\mathrm{Area}(\Omega_\epsilon \setminus K)$, and the statement follows since $\bar{\partial}_z^2\varphi_0 \equiv 1$ on K.

Set

$$F(z) := \int_K \frac{\overline{\zeta - z}}{\zeta - z}\, dA(\zeta). \tag{3}$$

Clearly, $F(z) \in C^1\left(\mathbb{R}^2\right)$. Thus, $\max_{z\in K}|F(z)|$ occurs somewhere on K.

Let $R_a = R_a(K) = \sqrt{\frac{\mathrm{Area}(K)}{\pi}}$ denote the "area radius" of a disk with the same area as K.

Lemma 3 *$\lambda_2(K) \le R_a^2$. Moreover,* $\sup\{\lambda_2(K) : \mathrm{Area}(K)$ *is fixed*$\} = R_a^2$*, although there is no "extremal" set K for which equality occurs.*

Proof The first statement follows from Lemma 2 since the integrand in (3) is bounded by 1. The rest follows at once if one considers a sequence of "cigar-shaped" domains Ω_n with a fixed area symmetric with respect to the x-axis and tangent to the y-axis at the origin. Then, $F(0) \to R_a^2$ since $\frac{\bar{\zeta}}{\zeta} \to 1$ pointwise on Ω_n and is bounded by 1, so the Lebesgue bounded convergence theorem applies.

Remark 1 Recall that unlike the bi-analytic content, the analytic content ($\lambda(K) := \mathrm{dist}_{C(K)}\left(\bar{z}, R(K)\right)$ is bounded above by R_a and the equality holds for disks and only for disks modulo sets of area zero(cf. [2, 7]).

4 Bi-Analytic Content of Disks

Proposition 2 *Let $\overline{D} = \{z : |z| \le R\}$. Then, $\lambda_2\left(\overline{D}\right) = \frac{1}{2}R^2$.*

Proof By taking $\varphi \equiv 0 \subset R_2\left(\overline{D}\right)$, we see that $\lambda_2\left(\overline{D}\right) \le \frac{1}{2}R^2$. To obtain the converse inequality, note that for any polynomials P_1, P_2 we have

$$\begin{aligned}\left\| \frac{1}{2}\bar{z}^2 - P_1 - \bar{z}P_2 \right\|_D &\ge \left\| \frac{1}{2}\bar{z}^2 - P_1 - \bar{z}P_2 \right\|_{\partial D} \\ &\ge \inf_{P_1,P_2} \frac{1}{R^2} \left\| \frac{1}{2}R^4 - z^2P_1 - P_2R^2z \right\|_{\partial D} \\ &= \inf_{P:P(0)=0} \left\| \frac{1}{2}R^2 - P(z) \right\|_{\partial D} = \frac{1}{2}R^2.\end{aligned} \tag{4}$$

(The latter infimum is a trivial extremal problem in $H^\infty(D)$-setting (cf. [4], [13, Ch. 8]) and is easily computable, e.g., by duality:

$$\inf_{\substack{f\in H^\infty(D)\\ f(0)=0}} \|C-f\|_{\partial D} = C \sup_{\substack{f\in H^1(D)\\ \|f\|_{H^1}=1}} \left| \int_{\partial D} f\,ds \right| = C,$$

for any constant $C > 0$.) Since P_1, P_2 were arbitrary, the proposition follows.

5 Bounds for λ_2

The following statement is obvious.

Corollary 1 *Let K be a compact subset of $\mathbb{C}$ and the outer and inner radii R_o, R_i denote, respectively, the minimal radius of a disk containing K (i. e., R_o), and the maximal radius of a disk contained in K. Then,*

$$\frac{1}{2}R_i^2 \le \lambda_2(K) \le \frac{1}{2}R_o^2. \tag{5}$$

(Of course, here, we tacitly used Runge's theorem in its simplest form: $R\left(\overline{D}\right) =$ uniform closure of polynomials, for any disk D.)

Corollary 2 ([17]) *$R_2(K) = C(K)$ if and only if K is nowhere dense.*

The necessity follows at once from the lower bound in (5) and Proposition 1. The proof of sufficiency, given by Trent and Wang in [17], cannot be shortened or simplified any further. Thus, for the reader's convenience, we only indicate the outline.

(i) By the Hahn–Banach theorem it suffices to check that μ annihilating $R_2(K)$ must be zero, i.e., annihilates all C_0^∞-functions.

(ii) By Lemma 1 and Fubini's theorem, it suffices to check that an R_2-analogue of the Cauchy transform for μ

$$\check{\mu}(z) := \int_{\mathbb{C}} \frac{\overline{\zeta - z}}{\zeta - z}\, d\mu(\zeta) \tag{6}$$

vanishes a.e. wrt dA.

(iii) The Lebesgue bounded convergence theorem yields that $\check{\mu}$ is continuous in $\mathbb{C}$ except at atoms of μ, i.e., at at most countably many points.
(iv) If K is nowhere dense, $\check{\mu}$ vanishes in $\mathbb{C} \setminus K$, and by (iii) in all of $\mathbb{C}$ except for a countable set and the proof is finished.

6 Concluding Remarks

1. The referee suggested an elegant short cut in the proof of Proposition 2: in the second term in (4) use for the lower bound the L^2-norm on the circle and apply the Pythagoras' theorem. This simplification might prove useful in more general domains.
2. Undoubtedly, the above scheme can be extended to more general "rational modules" associated with the operator $\overline{\partial_z}^N$, i.e., to $R_N(K)$.
3. Most likely, one may consider the bi-analytic content or, more generally, N-analytic content for other norms than the uniform norm, e.g., Bergman L^p-norms, Hardy norms, etc. The recent results in that direction for the analytic content [5, 6, 8] yield some interesting connections and the latter continue forthcoming.
4. It would be interesting to tighten the inequality (5), perhaps obtaining sharper bounds that might involve deeper geometric characteristics of K, e.g., perimeter, capacity, torsional rigidity. For the analytic content this line of inquiry proved to be quite fruitful (cf. [5, 6, 8], cited above).

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Chebyshev Polynomials Associated with a System of Continua

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Abstract We establish estimates from above for the uniform norm of the Chebyshev polynomials associated with a system of continua $K \subset \mathbb{C}$ by constructing monic polynomials with small norms on K. The estimates are exact (up to a constant factor) in the case where K has a piecewise quasiconformal boundary and its complement $\Omega = \overline{\mathbb{C}} \setminus K$ has no outward pointing cusps.

Keywords Chebyshev polynomials • Equilibrium measure • Green's function • Quasiconformal curve • System of continua

Msc codes: 30C10, 30C20, 30C62, 30C85, 31A15, 31A20

1 Introduction and Results

For a compact set $K \subset \mathbb{C}$ with infinitely many points, the n-th Chebyshev polynomial on K is the monic polynomial of degree n, $T_n(z, K) = T_n(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$, $a_j \in \mathbb{C}$, which minimizes the supremum norm, $||p||_K := \sup_{z \in K} |p(z)|$, among all monic polynomials $p(z)$ of degree n.

It is well known [9, p. 155,Theorem 5.5.4 and Corollary 5.5.5] that for every $n \in \mathbb{N}$, we have

$$||T_n||_K \geq \text{cap}(K)^n \quad \text{and} \quad \lim_{n\to\infty} ||T_n||_K^{1/n} = \text{cap}(K),$$

where $\text{cap}(K)$ is the logarithmic capacity of K.

Throughout this paper, we assume that $K \subset \mathbb{C}$ is a compact set consisting of finitely many disjoint nonempty closed connected sets (continua) $K^j, j = 1, \dots, m$, i.e.

$$K = \cup_{j=1}^m K^j; \quad K^j \cap K^k = \emptyset, \text{ for } j \neq k; \quad \text{diam}(K^j) > 0,$$

where $\text{diam}(S) := \sup_{u,v \in S} |u - v|$, $S \subset \mathbb{C}$. We call such K a *system of continua*.

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Since the maximum modulus principle implies that $|T_n(z)|$ takes its maximum value on K on the outer boundary of K, we may assume that K coincides with its polynomially convex hull. Let $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ denote the extended complex plane. Let $\Omega := \overline{\mathbb{C}} \setminus K$ denote the complement of K in $\overline{\mathbb{C}}$. Then Ω is a domain with $\infty \in \Omega$ and $\mathrm{cap}(K) > 0$, so there exists a Borel probability measure μ supported on ∂K and a function

$$g(z) = \int \log|z-\zeta|\, d\mu(\zeta) - \log \mathrm{cap}(K) = \log|z| - \log \mathrm{cap}(K) + O\left(|z|^{-1}\right)$$

which is subharmonic in $\mathbb{C}$ and harmonic in $\Omega \setminus \{\infty\}$ [8, p. 205,Theorem 9.7]. This is the Green's function of Ω with logarithmic pole at ∞ and μ is the equilibrium measure of K (see [9]).

To study the relationship between $||T_n||_K$ and $\mathrm{cap}(K)^n$ we consider the *Widom factors* of K

$$W_n = W_n(K) := \frac{||T_n||_K}{\mathrm{cap}(K)^n}.$$

It is not known if the Widom factors are bounded for an arbitrary system of continua, but several examples of classes of sets with bounded Widom factors are known. Widom [18] showed that W_n is bounded for a system of smooth Jordan arcs/curves and Andrievskii [3] showed that W_n is bounded for a system of continua with quasismooth boundary. For the complete survey of results concerning this problem and further citations, see [3, 6, 10–12], and [14–18].

We establish estimates from above for the Widom factors of a system of continua. Our estimates are exact (up to a constant factor) in the case where each K^j has a piecewise quasiconformal boundary and its complement $\overline{\mathbb{C}} \setminus K^j$ is a John domain.

The sequence $(W_n)_{n=1}^{\infty}$ is logarithmically subadditive, i.e. $W_{n+m} \leq W_n W_m$. To establish an upper bound for all $n \in \mathbb{N}$, we find an arithmetic subsequence (n_k) for which W_{n_k} is bounded.

1.1 Localization and Conformal Mappings

For $s > 0$, we denote the s-level set of the Green's function by

$$K_s := \{z \in \Omega : g(z) = s\}.$$

Since K is contained in the interior of the polynomially convex hull of K_s, Frostman's theorem [9, p. 59, Theorem 3.3.4] implies

$$\log \operatorname{cap}(K_s) = \int \log|\zeta - z|\, d\mu_s(\zeta), \quad z \in K,$$

where μ_s is the equilibrium measure of K_s.

We choose $0 < s_0 < 1$ so small that $K_s = \cup_{j=1}^m K_s^j$ where the K_s^j are mutually disjoint smooth Jordan curves and

$$\operatorname{dist}(\zeta, K) = \operatorname{dist}(\zeta, K^j), \;\; \zeta \in K_s^j, \;\; 0 < s \leq s_0,$$

where $\operatorname{dist}(S, S') := \inf_{\sigma\in S, \sigma'\in S'} |\sigma - \sigma'|$ is the distance from $S \subset \mathbb{C}$ to $S' \subset \mathbb{C}$.

We let $\tilde{g}(z)$ be the (multiple-valued) harmonic conjugate of $g(z)$ [1, p. 250, Lemma 3] and define the analytic function $\Phi : \Omega \to \mathbb{D}^* := \{w \in \overline{\mathbb{C}} : |w| > 1\}$ by

$$\Phi(z) := \exp(g(z) + i\tilde{g}(z)).$$

The total change in argument of Φ around the curve K_s^j is given by the net-change of $\tilde{g}$ around K_s^j, which by Gauss' theorem [13, p. 83] equals

$$\Delta_{K_s^j}\tilde{g} := \int_{K_s^j} \frac{\partial \tilde{g}}{\partial t_\zeta}|d\zeta| = \int_{K_s^j} \frac{\partial g}{\partial n_\zeta}|d\zeta| = 2\pi\mu(K^j) =: 2\pi\omega_j,$$

where $|d\zeta|$ is the linear (arc length) measure on the curve K_s^j. For future reference, we note that $0 < \omega_j \leq 1 = \sum_{j=1}^m \omega_j$.

Following [3], we consider the conformal and univalent mapping $\varphi_j(z) := \Phi^{1/\omega_j}(z)$ of $\Omega_0^j := \operatorname{inn}(K_{s_0}^j)\setminus K^j$ onto the annulus $A_0^j := \{w \in \mathbb{D}^* : 1 < |w| < e^{s_0/\omega_j}\}$ where $\operatorname{inn}(K_{s_0}^j)$ denotes the Jordan domain with boundary curve $K_{s_0}^j$. Thus, we have

$$K_s^j = \{z \in \Omega_0^j : |\varphi_j(z)| = e^{s/\omega_j}\}, \; 0 < s < s_0.$$

We denote by $\psi_j := \varphi_j^{-1}$ the inverse mapping.

For technical reasons, we choose $0 < s_1 < b\log(\frac{1}{2}(e^{s_0} + 1))$ so small that for $\zeta \in K_s^j$, $0 < s \leq s_1$, we simultaneously have

$$\operatorname{dist}(\zeta, \partial\Omega_0^j) = \operatorname{dist}(\zeta, K) \quad \text{and} \quad \operatorname{dist}(\varphi_j(\zeta), \partial A_0^j) = |\varphi_j(\zeta)| - 1.$$

We will use the following terminology throughout.

A Jordan arc $L \subset \mathbb{C}$ is called a *quasiconformal arc* [8, p. 107] if there is a constant $\Lambda_L > 0$, depending only on L, such that for any $z_1, z_2 \in L$,

$$\operatorname{diam}(L(z_1, z_2)) \leq \Lambda_L|z_1 - z_2|, \tag{1}$$

where $L(z_1, z_2) \subset L$ is the subarc joining z_1 and z_2.

A simply connected domain $G \subset \overline{\mathbb{C}}$ is called a *John domain* [8, p. 96] if there is a constant $M > 0$ such that for any crosscut γ of G,

$$\operatorname{diam}(H) \leq M \operatorname{diam}(\gamma) \tag{2}$$

holds for one of the components H of $G \setminus \gamma$.

We call $K = \cup_{j=1}^{m} K^j$ a *piecewise quasiconformal system* if it is a system of continua such that each K^j has a piecewise quasiconformal boundary and the complement $\overline{\mathbb{C}} \setminus K^j$ is a John domain.

Throughout, we let $c_1, c_2, \ldots$ denote positive constants that are either absolute or depend only on K; otherwise, the dependence on other parameters is explicitly stated. We will use the conventions

$$a := 3240\pi/b, \quad b := \min_j \omega_j \log 2.$$

1.2 Results for Widom Factors

The main objective of this paper is to prove the following results.

Theorem 1 *Let K be a system of continua. For $n = Nq$, $N, q \in \mathbb{N}$, $aq < Ns_1$, we have*

$$W_n(K) \leq \exp\left(c_1 q^2 + \frac{c_2 q}{2^q} \max_{z \in K} \int_{K_{aq/N}} \frac{\operatorname{dist}(\zeta, K)^q}{|z - \zeta|^{q+1}} |d\zeta| \right). \tag{3}$$

This estimate appears in [3] with $q = 1$.

Theorem 2 *Let K be a piecewise quasiconformal system. Then for every $n \in \mathbb{N}$ we have*

$$||T_n||_K \leq c_3 \operatorname{cap}(K)^n. \tag{4}$$

Our constructions below are based on the discretization of the equilibrium measure due to Totik [14–16], representation of the Green's function via special conformal mappings due to Widom [18], distortion properties of conformal mappings near the boundary of their domains, [5] or [8], and the special polynomials associated with a continuum due to Andrievskii and Nazarov.

For the remainder of this paper, we will write $A \preceq B$ to mean that $A \geq 0$ and $B \geq 0$ are real-valued functions such that $A \leq c_4 B$. We write $A \asymp B$ when $A \preceq B$ and $B \preceq A$ simultaneously.

2 System of Continua

We begin by partitioning each K_s^j uniformly into N_j subarcs I_k^j according to the equilibrium measure μ_s of K_s. For $N \in \mathbb{N}$, $N > 1/b$, $j = 1, \dots, m$, we define

$$N_j := [N\omega_j] \text{ or } [N\omega_j] + 1, \quad \text{so that} \quad N = \sum_{j=1}^{m} N_j.$$

Let $I_k^j := \psi_j(J_k^j)$, $k = 1, \dots, N_j$, and

$$J_k^j := \left\{ e^{s/\omega_j + i\theta} : \frac{2\pi k}{N_j} \le \theta \le \frac{2\pi(k+1)}{N_j} \right\}.$$

Denote the endpoints of these arcs by $\xi_k^j := \psi_j(w_k^j)$ and $w_k^j := e^{s/\omega_j + 2\pi i k/N_j}$, respectively. Then, $K_s^j = \cup_{k=1}^{N_j} I_k^j$ and we have for $j = 1, \dots, m$, $k = 1, \dots, N_j$,

$$\mu_s(I_k^j) = \frac{\omega_j}{N_j} \asymp \frac{1}{N}. \tag{5}$$

2.1 Main Lemma

We will use the following lemma to prove our general estimate (3). This lemma appears in [3] with $q = 1$.

Lemma 1 *Let $s = \frac{aq}{N}$, where $N, q \in \mathbb{N}$, $0 < s < s_1$. Then for $j = 1, \dots, m$, $k = 1, \dots, N_j$, we have*

$$\frac{\operatorname{dist}(I_k^j, K)}{q} \preceq |\xi_{k+1}^j - \xi_k^j| \le \operatorname{diam}(I_k^j) \le |I_k^j| \le \frac{\operatorname{dist}(I_k^j, K)}{10q}, \tag{6}$$

where $|I_k^j| := \int_{I_k^j} |d\zeta|$ is the linear (arc length) measure of I_k^j.

Proof By an immediate consequence of Koebe's one-quarter theorem [5, p. 23, Lemma 2.3], for $s = \frac{aq}{N} < s_1$, $w \in A_0^j$, $|w| \le e^{s/\omega_j}$, and $\zeta = \psi_j(w)$, we have

$$\frac{\operatorname{dist}(\zeta, K)}{4(|w| - 1)} \le |\psi_j'(w)| \le \frac{4\operatorname{dist}(\zeta, K)}{|w| - 1}. \tag{7}$$

Moreover, if $|u - w| \le (|w| - 1)/2$ and $\xi = \psi_j(u)$, then

$$\frac{|u - w|}{16(|w| - 1)} \le \frac{|\xi - \zeta|}{\operatorname{dist}(\zeta, K)} \le \frac{16|u - w|}{|w| - 1}. \tag{8}$$

Let $w \in J_k^j$. Since w_k^j, w_{k+1}^j are the endpoints of the circular arc J_k^j, we have

$$|w - w_k^j| \leq |w_{k+1}^j - w_k^j| \leq \frac{s}{90q}. \tag{9}$$

For $\zeta := \psi_j(w) \in I_k^j$, (8) together with (9) implies

$$\frac{|\zeta - \xi_k^j|}{\operatorname{dist}(\xi_k^j, K)} \leq \frac{16s}{90q(e^{s/\omega_j} - 1)} < \frac{1}{2}.$$

We denote by ζ^* any point of K^j such that $|\zeta - \zeta^*| = \operatorname{dist}(\zeta, K)$. Then, we obtain

$$\operatorname{dist}(\xi_k^j, K) \leq |\xi_k^j - \zeta^*| \leq |\zeta - \zeta^*| + |\xi_k^j - \zeta| \leq \operatorname{dist}(\zeta, K) + \frac{1}{2}\operatorname{dist}(\xi_k^j, K),$$

which implies

$$\operatorname{dist}(\zeta, K) \leq \frac{3}{2}\operatorname{dist}(\xi_k^j, K) \leq 3\operatorname{dist}(I_k^j, K). \tag{10}$$

Then by (7), we have

$$|\psi_j'(w)| \leq \frac{4\operatorname{dist}(\psi_j(w), K)}{e^{s/\omega_j} - 1} \leq \frac{4\operatorname{dist}(\psi_j(w), K)}{s}. \tag{11}$$

Combining (10) and (11), we get the last inequality in (6)

$$|I_k^j| = \int_{J_k^j} |\psi_j'(w)||dw| \leq \frac{8}{s} \int_{\frac{2\pi k}{N_j}}^{\frac{2\pi(k+1)}{N_j}} \operatorname{dist}(\psi_j(e^{s/\omega_j + i\theta}), K)\, d\theta \leq \frac{\operatorname{dist}(I_k^j, K)}{10q}.$$

The first inequality in (6) follows from (8) and trivial estimates

$$\frac{|\xi_{k+1}^j - \xi_k^j|}{\operatorname{dist}(I_k^j, K)} \geq \frac{|\xi_{k+1}^j - \xi_k^j|}{\operatorname{dist}(\xi_k^j, K)} \geq \frac{|w_{k+1}^j - w_k^j|}{16(e^{s/\omega_j} - 1)} \geq \frac{\omega_j}{8sN_j} \succeq \frac{1}{q}.$$

2.2 Proof of Theorem 1

Proof We let $N, q \in \mathbb{N}$, $s = \frac{aq}{N} < s_1$, and N_j as above. We use the uniform partition according to μ_s, $K_s^j = \cup_{k=1}^{N_j} I_k^j$, as above. For $p = 1, \ldots, q$, define the quantities

$$m_{k,p}^j := \frac{1}{\mu_s(I_k^j)} \int_{I_k^j} (\zeta - \xi_k^j)^p \, d\mu_s(\zeta).$$

Now consider the numbers $r^j_{k,l} = r^j_{k,l}(I^j_k)$, $l = 1, \dots, q$, the solutions of the system

$$\sum_{l=1}^{q} (r^j_{k,l})^p = q m^j_{k,p} =: \tilde{m}^j_{k,p}, \quad p = 1, \dots, q,$$

defined as the zeros of the monic polynomial of degree q,

$$z^q + a_{q-1} z^{q-1} + \cdots + a_0 = 0, \tag{12}$$

with coefficients satisfying Newton's identities

$$\tilde{m}^j_{k,p} + a_{q-1}\tilde{m}^j_{k,p-1} + \cdots + a_{q-p+1}\tilde{m}^j_{k,1} = -p a_{q-p}, \quad p = 1, \dots, q. \tag{13}$$

Since $|m^j_{k,p}| \le (d_{j,k})^p$ and thus $|\tilde{m}^j_{k,p}| \le q(d_{j,k})^p$, where $d_{j,k} := \operatorname{diam}(I^j_k)$, we can use induction and (13) to obtain

$$|a_{q-p}| \le (q d_{j,k})^p, \quad p = 1, \dots, q. \tag{14}$$

As a consequence, if $r^j_{k,l} \neq 0$, using (12) and (14), we have

$$1 \le \frac{|a_{q-1}|}{|r^j_{k,l}|} + \cdots + \frac{|a_0|}{|r^j_{k,l}|^q} \le \frac{q d_{j,k}}{|r^j_{k,l}|} + \cdots + \left(\frac{q d_{j,k}}{|r^j_{k,l}|}\right)^q \le \sum_{p=1}^{\infty} \left(\frac{q d_{j,k}}{|r^j_{k,l}|}\right)^p,$$

which implies for $l = 1, \dots, q$,

$$|r^j_{k,l}| \le 2 q d_{j,k}. \tag{15}$$

We define the numbers

$$\zeta^j_{k,l} := \xi^j_k + r^j_{k,l}$$

and for $n = Nq = q \sum_j N_j$, we construct the monic polynomial of degree n

$$P_n(z) = \prod_{j=1}^{m} \prod_{k=1}^{N_j} \prod_{l=1}^{q} (z - \zeta^j_{k,l}).$$

Fix $z \in \partial K$. First, we estimate the expression

$$N \log \operatorname{cap}(K_s) = \sum_{j=1}^{m} \sum_{k=1}^{N_j} \left(N - \frac{N_j}{\omega_j}\right) \int_{I_k^j} \log|z-\zeta|\, d\mu_s(\zeta)$$
$$+ \sum_{j=1}^{m} \sum_{k=1}^{N_j} \frac{1}{\mu_s(I_k^j)} \int_{I_k^j} \log|z-\zeta|\, d\mu_s(\zeta)$$

by defining the quantities

$$\Sigma_1(z) := \sum_{j=1}^{m} \sum_{k=1}^{N_j} \left(N - \frac{N_j}{\omega_j}\right) \int_{I_k^j} \log|z-\zeta|\, d\mu_s(\zeta),$$

$$\Sigma_2(z) := \sum_{j=1}^{m} \sum_{k=1}^{N_j} \frac{1}{\mu_s(I_k^j)} \int_{I_k^j} \log|z-\zeta|\, d\mu_s(\zeta).$$

By (5), we have $N - \frac{N_j}{\omega_j} \preceq 1, j = 1, \ldots, m$. Thus, we have the following estimate

$$|\Sigma_1(z)| \preceq \int_{K_s} |\log|z-\zeta||\, d\mu_s(\zeta)$$
$$\leq |\log \operatorname{diam}(K_s)| + \int_{K_s} \log \frac{\operatorname{diam}(K_s)}{|z-\zeta|}\, d\mu_s(\zeta)$$
$$\leq 2\log^+ \operatorname{diam}(K_s) - \log \operatorname{cap}(K_s) \preceq 1.$$

This gives an estimate for the main quantity of interest

$$\begin{aligned} \log|P_n(z)| - n\log \operatorname{cap}(K_s) &= \log|P_n(z)| - qN\log \operatorname{cap}(K_s) \\ &\preceq q + \log|P_n(z)| - q\Sigma_2(z). \end{aligned} \tag{16}$$

We need to estimate the following expression

$$\log|P_n(z)| - q\Sigma_2(z) = \sum_{j,k,l} \frac{1}{\mu_s(I_k^j)} \int_{I_k^j} \log\left|\frac{z-\zeta_{k,l}^j}{z-\zeta}\right| d\mu_s(\zeta). \tag{17}$$

To this end, we consider the expansion (for any local branch of the logarithm we have $\Re \log(1-u) = \log|1-u|$)

$$\begin{aligned}\log\left(\frac{z-\zeta_{k,l}^j}{z-\zeta}\right) &= \log\left(1-\frac{\zeta_{k,l}^j-\xi_k^j}{z-\xi_k^j}\right)-\log\left(1-\frac{\zeta-\xi_k^j}{z-\xi_k^j}\right)\\ &= \sum_{p=1}^{q}\frac{1}{p}\left(\left(\frac{\zeta-\xi_k^j}{z-\xi_k^j}\right)^p-\left(\frac{\zeta_{k,l}^j-\xi_k^j}{z-\xi_k^j}\right)^p\right)+B_k^j(\zeta).\end{aligned} \tag{18}$$

By (15), the error term is bounded by

$$|B_k^j(\zeta)| = |B_k^j(z,\zeta,\zeta_{k,l}^j)| \preceq \frac{1}{q}\left(\frac{2qd_{j,k}}{\mathrm{dist}(z,I_k^j)}\right)^{q+1}. \tag{19}$$

Now using (18), (19), and the definition of the $r_{k,l}^j$, we can estimate (17) by

$$\sum_{j,k,l}\frac{1}{\mu_s(I_k^j)}\int_{I_k^j}\log\left|\frac{z-\zeta_{k,l}^j}{z-\zeta}\right|\,d\mu_s(\zeta) \preceq \sum_{j,k}\left(\frac{2qd_{j,k}}{\mathrm{dist}(z,I_k^j)}\right)^{q+1}. \tag{20}$$

Combining (16) and (20), we get

$$\log|P_n(z)| - n\log\mathrm{cap}(K_s) \preceq q + \sum_{j,k}\left(\frac{2qd_{j,k}}{\mathrm{dist}(z,I_k^j)}\right)^{q+1}.$$

Noting that $\mathrm{cap}(K_s) = e^s\mathrm{cap}(K)$, we obtain

$$\begin{aligned}|\log|P_n(z)| - n\log\mathrm{cap}(K)| &\le ns + |\log|P_n(z)| - n\log\mathrm{cap}(K_s)|\\ &\preceq q^2 + \sum_{j,k}\left(\frac{2qd_{j,k}}{\mathrm{dist}(z,I_k^j)}\right)^{q+1}.\end{aligned} \tag{21}$$

Applying (6), we can bound the sum by an integral over K_s^j as follows

$$\int_{K_s^j}\frac{\mathrm{dist}(\zeta,K)^q}{|\zeta-z|^{q+1}}|d\zeta| \ge \sum_{k=1}^{N_j}\frac{(10qd_{j,k})^q|I_k^j|}{(\mathrm{dist}(z,I_k^j)+d_{j,k})^{q+1}} \ge \frac{2^q}{2q}\sum_{k=1}^{N_j}\left(\frac{2qd_{j,k}}{\mathrm{dist}(z,I_k^j)}\right)^{q+1}.$$

Finally, by the last inequality and (21), we get (3)

$$W_n(K) \le \frac{||P_n||_K}{\mathrm{cap}(K)^n} \le \exp\left(c_1q^2 + \frac{c_2q}{2^q}\max_{z\in K}\int_{K_s}\frac{\mathrm{dist}(\zeta,K)^q}{|\zeta-z|^{q+1}}|d\zeta|\right).$$

3 Piecewise Quasiconformal System

All necessary background material can be found in [2, 5, 7], and [8]. A crucial fact for our method is that every conformal mapping onto a quasidisk (domain with quasiconformal boundary) can be extended to a quasiconformal homeomorphism $\overline{\mathbb{C}} \to \overline{\mathbb{C}}$ whose maximal dilatation depends only on the quasidisk [8, p. 114, Theorem 5.17].

In this section, we assume $K = \cup_j K^j$ is a piecewise quasiconformal system. By (2), ψ_j extends continuously to the unit circle $\mathbb{T}$. By K^j having a piecewise quasiconformal boundary, we mean that there is a partition $\{e^{i\theta_k^j}\}_{k=1}^{\alpha_j}$ of $\mathbb{T}$ with $\theta_1^j < \cdots < \theta_{\alpha_j}^j < \theta_{\alpha_j+1}^j := \theta_1^j + 2\pi$ so that $L_k^j := \psi_j(T_k^j) \subset \partial K^j$ is a quasiconformal arc, $k = 1, \ldots, \alpha_j$, where $T_k^j := \{e^{i\theta} : \theta_k^j \le \theta \le \theta_{k+1}^j\} \subset \mathbb{T}$.

Once and for all, we fix $0 < s < s_1$. We will use the following notations in what follows:

$$A^j := \{w \in \mathbb{D}^* : 1 < |w| < e^{s/\omega_j}\}, \quad A_k^j := \{w \in A^j : \theta_k^j < \arg(w) < \theta_{k+1}^j\},$$

$$\tilde{\Gamma}_k^j := \{e^{s/\omega_j + i\theta} : \theta_k^j \le \theta \le \theta_{k+1}^j\}, \quad \tilde{\gamma}_k^j := [\tau_k^j, w_k^j],$$

$$\tau_k^j := e^{i\theta_k^j} \in T_k^j, \quad w_k^j := e^{s/\omega_j + i\theta_k^j} \in \tilde{\Gamma}_k^j,$$

$$\Omega^j := \psi_j(A^j), \quad \Omega_k^j := \psi_j(A_k^j), \quad \Gamma_k^j := \psi_j(\tilde{\Gamma}_k^j),$$

$$\gamma_k^j := \psi_j(\tilde{\gamma}_k^j), \quad z_k^j := \psi_j(\tau_k^j) \in L_k^j, \quad \xi_k^j := \psi_j(w_k^j) \in \Gamma_k^j.$$

Note that $\Gamma_k^j = \overline{\Omega_k^j} \cap K_s^j$ and $\{z_k^j\} = \gamma_k^j \cap L_k^j$. Also $\partial A_k^j = \tilde{\gamma}_k^j \cup \tilde{\Gamma}_k^j \cup \tilde{\gamma}_{k+1}^j \cup T_k^j$ and $\partial \Omega_k^j = \gamma_k^j \cup \Gamma_k^j \cup \gamma_{k+1}^j \cup L_k^j$.

By [4, Lemma 2, (4.14)], it follows that

$$|\zeta - z_k^j| \preceq \text{dist}(\zeta, K), \quad \zeta \in \gamma_k^j, \tag{22}$$

and

$$|\gamma_k^j(\zeta_1, \zeta_2)| \preceq |\zeta_1 - \zeta_2|, \quad \zeta_1, \zeta_2 \in \gamma_k^j.$$

Thus, by the Ahlfors criterion [7, p. 100], γ_k^j and Γ_k^j are quasiconformal arcs which do not meet at a cusp. By (22), γ_k^j and L_k^j do not meet at a cusp point and by assumption, L_k^j is a quasiconformal arc. Hence, $\partial\Omega_k^j$ is a quasiconformal curve. Thus, we can extend $\psi_j|_{A_k^j}$ to a Q_k^j−quasiconformal homeomorphism $\psi_{j,k} : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ for some $Q_k^j \ge 1$ [7, p. 98] such that $\overline{\Omega_k^j} = \psi_{j,k}\left(\overline{A_k^j}\right)$. The inverse mapping $\varphi_{j,k} :=$

$\psi_{j,k}^{-1}$ is also Q_k^j-quasiconformal. Each $\varphi_{j,k}$, $\psi_{j,k}$ is a Q-quasiconformal automorphism of $\overline{\mathbb{C}}$, where $Q := \max_{1\le j\le m, 1\le k\le \alpha_j} Q_k^j$. The following result describes the distortion properties of Q-quasiconformal automorphisms of $\overline{\mathbb{C}}$.

Lemma 2 ([5], p. 29) *Suppose the function $\eta = F(\zeta)$ is a Q-quasiconformal mapping of the extended plane onto itself with $F(\infty) = \infty$. Let $\zeta_j \in \mathbb{C}$, $\eta_j = F(\zeta_j)$, $j = 1, 2, 3$. Then:*

(i) the conditions $|\zeta_1 - \zeta_2| \preceq |\zeta_1 - \zeta_3|$ and $|\eta_1 - \eta_2| \preceq |\eta_1 - \eta_3|$ are equivalent and the constants are mutually dependent and depend on Q but not on the ζ_j, η_j;
(ii) if $|\zeta_1 - \zeta_2| \preceq |\zeta_1 - \zeta_3|$, then

$$\left|\frac{\eta_1 - \eta_3}{\eta_1 - \eta_2}\right|^{1/Q} \preceq \left|\frac{\zeta_1 - \zeta_3}{\zeta_1 - \zeta_2}\right| \preceq \left|\frac{\eta_1 - \eta_3}{\eta_1 - \eta_2}\right|^{Q}, \tag{23}$$

where the constants are mutually dependent and depend on Q but not on the ζ_j, η_j.

Now consider $\zeta \in \Gamma_k^j$ and let $\zeta^* \in K$ be such that $|\zeta - \zeta^*| = \operatorname{dist}(\zeta, K)$. Using Lemma 2 with the quasiconformal mapping $\varphi_{j,k}$ and the points ζ, ζ^*, z_l^j, $l = k$ or $k + 1$, we conclude that

$$|\zeta - z_l^j| \preceq |\zeta - \zeta^*| = \operatorname{dist}(\zeta, K), \quad \zeta \in \Gamma_k^j, \quad l = k \text{ or } l = k+1. \tag{24}$$

We claim that

$$\operatorname{dist}(z, L_k^j) \preceq \operatorname{dist}(z, \Omega_k^j), \quad z \in \partial K^j \setminus L_k^j. \tag{25}$$

Indeed, let $z' = z'(j,k) \in \partial\Omega_k^j = \gamma_k^j \cup \Gamma_k^j \cup \gamma_{k+1}^j \cup L_k^j$ be such that $|z - z'| = \operatorname{dist}(z, \Omega_k^j)$. The nontrivial cases occur when $z' \notin L_k^j$, i.e. $z' \in \gamma_l^j$, $l = k, k+1$, or $z' \in \Gamma_k^j$. By (22) or (24), we have

$$\operatorname{dist}(z, L_k^j) \le |z - z_l^j| \le |z - z'| + |z' - z_l^j| \preceq |z - z'| = \operatorname{dist}(z, \Omega_k^j),$$

which implies (25).

For $z \in \partial K^j$ we denote by $z_{j,k}^*$ any point of L_k^j satisfying the property:

$$|z - z_{j,k}^*| = \operatorname{dist}(z, L_k^j).$$

We claim that

$$|\zeta - z_{j,k}^*| \preceq |\zeta - z|, \quad \zeta \in \Omega_k^j, \; z \in \partial K^j \setminus L_k^j. \tag{26}$$

Indeed, for $\zeta \in \Omega_k^j$, (25) implies

$$|\zeta - z_{j,k}^*| \le |\zeta - z| + |z - z_{j,k}^*| \preceq |\zeta - z| + \mathrm{dist}(z, \Omega_k^j) \preceq |\zeta - z|,$$

and (26) follows.

For $\zeta \in \overline{\Omega_k^j}$, we consider the point $\zeta_K \in L_k^j$ defined by

$$\zeta \mapsto \zeta_K := \psi_{j,k}\left(\frac{\varphi_{j,k}(\zeta)}{|\varphi_{j,k}(\zeta)|}\right).$$

Now take any $\zeta \in \Gamma_k^j$ and $z \in L_k^j$. Using Lemma 2, we have

$$|\varphi_{j,k}(\zeta) - \varphi_{j,k}(z)| \preceq |\varphi_{j,k}(\zeta) - \varphi_{j,k}(\zeta_K)|,$$

and (23) implies

$$\frac{\mathrm{dist}(\zeta, K)}{|\zeta - z|} \le \left|\frac{\zeta - \zeta_K}{\zeta - z}\right| \preceq \left(\frac{|\varphi_j(\zeta)| - 1}{|\varphi_{j,k}(\zeta) - \varphi_{j,k}(z)|}\right)^{1/Q}. \tag{27}$$

3.1 Proof of Theorem 2

Proof Fix $q = Q$. Let $z \in \partial K^j \setminus L_k^j$, $z_{j,k}^*$ as above, $\tau_{j,k}^* := \varphi_{j,k}(z_{j,k}^*) \in T_k^j$, and $s = \frac{aq}{N} < s_1$. We rewrite the integral (3) using the above decomposition of K_s

$$\int_{K_s} \frac{\mathrm{dist}(\zeta, K)^q}{|\zeta - z|^{q+1}} |d\zeta| = \sum_{j=1}^{m} \sum_{k=1}^{\alpha_j} \int_{\Gamma_k^j} \frac{\mathrm{dist}(\zeta, K)^q}{|\zeta - z|^{q+1}} |d\zeta|.$$

Since $\psi_{j,k}(\tau_{j,k}^*) = z_{j,k}^*$, by (7), (26), (27), and a change of variables, we obtain

$$\begin{aligned}
\int_{\Gamma_k^j} \frac{\mathrm{dist}(\zeta, K)^q}{|\zeta - z|^{q+1}} |d\zeta| &\preceq \int_{\Gamma_k^j} \frac{\mathrm{dist}(\zeta, K)^q}{|\zeta - z_{j,k}^*|^{q+1}} |d\zeta| \\
&\preceq \frac{1}{s} \int_{|\tau| = e^{s/\omega_j}} \left(\frac{\mathrm{dist}(\psi_{j,k}(\tau), K)}{|\psi_{j,k}(\tau) - \psi_{j,k}(\tau_{j,k}^*)|}\right)^{q+1} |d\tau| \\
&\preceq \frac{1}{s} \int_{|\tau| = e^{s/\omega_j}} \frac{s^{(q+1)/Q} |d\tau|}{|\tau - \tau_{j,k}^*|^{(q+1)/Q}} \\
&\preceq s^{(q+1)/Q - 1} \int_0^{2\pi} |e^{s/\omega_j + i\theta} - 1|^{-(q+1)/Q}\, d\theta \preceq 1
\end{aligned}$$

As $N \to \infty$ ($s \to 0$) the integrals in (3) remain bounded. This proves (4).

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Constrained L^2-Approximation by Polynomials on Subsets of the Circle

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Abstract We study best approximation to a given function, in the least square sense on a subset of the unit circle, by polynomials of given degree which are pointwise bounded on the complementary subset. We show that the solution to this problem, as the degree goes large, converges to the solution of a bounded extremal problem for analytic functions which is instrumental in system identification. We provide a numerical example on real data from a hyperfrequency filter.

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1 Introduction

This paper deals with best approximation to a square summable function, on a finite union I of arcs of the unit circle $\mathbb{T}$, by a polynomial of fixed degree which is bounded by 1 in modulus on the complementary system of arcs $J = \mathbb{T} \setminus I$. This we call, for short, the polynomial problem. We are also concerned with the natural limiting version when the degree goes large, namely best approximation in $L^2(I)$ by a Hardy function of class H^2 which is bounded by 1 on J. To distinguish this issue from the polynomial problem, we term it the analytic problem. The latter is a variant, involving mixed norms, of constrained extremal problems for analytic functions considered in [2, 3, 12, 13, 18]. As we shall see, solutions to the polynomial problem converge to those of the analytic problem as the degree tends to infinity, in a sense to be made precise below. This is why solving for high degree the polynomial problem (which is finite-dimensional) is an interesting way to regularize and approximately solve the analytic problem (which is infinite-dimensional). This is the gist of the present work.

Constrained extremal problems for analytic functions, in particular the analytic problem defined above, can be set up more generally in the context of weighted

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approximation, i.e. seeking best approximation in $L^2(I, w)$ where w is a weight on I. In fact, that kind of generalization is useful for applications as we shall see. As soon as w is invertible in $L^\infty(I)$, though, such a weighted problem turns out to be equivalent to another one with unit weight, hence the present formulation warrants most practical situations. This property allows one to carry the analytic problem over to more general curves than the circle. In particular, in view of the isomorphism between Hardy spaces of the disk and the half-plane arising by composition with a Möbius transform [10, ch. 10], best approximation in $L^2(I)$ from H^2 of the disk can be converted to weighted best approximation in $L^2(\mathfrak{I}, w)$ from the Hardy space $\mathfrak{h}^2$ of a half-plane with $\mathfrak{I}$ a finite union of bounded intervals on the line and w a weight arising from the derivative of the Möbius transform. Since this weight is boundedly invertible on $\mathfrak{I}$, it follows that the analytic problem on the circle and its analog on the line are equivalent. One may also define another Hardy space $\mathcal{H}^2$, say of the right half-plane as the space of analytic functions whose L^2-means over vertical lines are uniformly bounded. Then, best approximation in $L^2(I)$ from H^2 is equivalent to best approximation from $\mathcal{H}^2$ in $L^2(\mathfrak{I})$, i.e. weight is no longer needed. Of course, such considerations hold for many other domains and boundary curves than the half-plane and the line, but the latter are of special significance to us as we now explain.

Indeed, on the line, constrained extremal problems for analytic functions naturally arise in Engineering when studying deconvolution issues, in particular those pertaining to system identification and design. This motivation is stressed in [2, 4, 5, 12, 19], whose results are effectively used today to identify microwave devices [1, 14]. More precisely, recall that a linear time-invariant dynamical system is just a convolution operator, hence the Fourier-Laplace transform of its output is that of its input times the Fourier-Laplace transform of its kernel. The latter is called the transfer-function. Now, by feeding periodic inputs to a stable system, one can essentially recover the transfer function pointwise on the line, but typically in a restricted range of frequencies only, corresponding to the passband of the system, say $\mathfrak{I}$ [9]. Here, the type of stability under consideration impinges on the smoothness of the transfer function as well as on the precise kind of recovery that can be achieved, and we refer the reader to [6, Appendix 2] for a more thorough analysis. For the present discussion, it suffices to assume that the system is stable in the L^2 sense, i.e. that it maps square summable inputs to square summable outputs. Then, its transfer function lies in H^∞ of the half-plane [15], and to identify it we are led to approximate the measurements on $\mathfrak{I}$ by a Hardy function with a bound on its modulus. Still, on $\mathfrak{I}$, a natural criterion from the stochastic viewpoint is $L^2(\mathfrak{I}, w)$, where the weight w is the reciprocal of the pointwise covariance of the noise assumed to be additive [16]. Since this covariance is boundedly invertible on I, we face an analytic problem on the line upon normalizing the bound on the transfer function to be 1. This stresses how the analytic problem on the line, which can be mapped back to the circle, connects to system identification. Now, this analytic problem is convex but infinite-dimensional. Moreover, as Hardy functions have no discontinuity of the first kind on the boundary [11, ch. II, ex. 7] and since the solution to an analytic problem generically has exact modulus 1 on J, as we prove later on, it will typically oscillate at the endpoints of I, J which is unsuited. One way around these difficulties is to solve the polynomial problem for sufficiently high degree, as a means to regularize and approximately solve the analytic one. This was an initial motivation by the authors to write the present paper, and we provide the reader in Sect. 5 with a numerical example on real data from a hyperfrequency filter. It must be said that the polynomial problem itself has numerical issues: though it is convex in finitely many variables, bounding the modulus on J involves infinitely many convex constraints which makes it of so-called semi-infinite programming type. A popular technique to handle such problems is through linear matrix inequalities, but we found it easier to approximate from below the polynomial problem by a finite-dimensional one with finitely many constraints, in a demonstrably convergent manner as the number of these constraints gets large.

The organization of the paper is as follows. In Sect. 2 we set some notation and we recall standard properties of Hardy spaces. We state the polynomial and analytic problems in Sect. 3, where we also show they are well-posed. Section 4 deals with the critical point equations characterizing the solutions, and with convergence of the polynomial problem to the analytic one. Finally, we report on some numerical experiment in Sect. 5.

2 Notations and Preliminaries

Throughout we let $\mathbb{T}$ be the unit circle and $I \subset \mathbb{T}$ a finite union of nonempty open arcs whose complement $J = \mathbb{T}\setminus I$ has nonempty interior. If h_1 (resp. h_2) is a function defined on a set containing I (resp. J), we put $h_1 \vee h_2$ for the concatenated function, defined on the whole of $\mathbb{T}$, which is h_1 on I and h_2 on J.

For $E \subset \mathbb{T}$, we let ∂E and $\mathring{E}$ denote respectively the boundary and the interior of E when viewed as a subset of $\mathbb{T}$; we also let χ_E for the characteristic function of E and $h_{|_E}$ for the restriction of h to E. Lebesgue measure on $\mathbb{T}$ is just the image of Lebesgue measure on $[0, 2\pi)$ under the parametrization $\theta \mapsto e^{i\theta}$. We denote by $|E|$ the measure of a measurable subset $E \subset \mathbb{T}$, and if $1 \le p \le \infty$ we write $L^p(E)$ for the familiar Lebesgue space of (equivalence classes of a.e. coinciding) complex-valued measurable functions on E with norm

$$\|f\|_{L^p(E)} = \left(\frac{1}{2\pi}\int_E |f(e^{i\theta})|^p \, d\theta\right)^{1/p} < \infty \quad \text{if } 1 \le p < \infty,$$

$$\|f\|_{L^\infty(E)} = \operatorname{ess.}\sup_{\theta\in E} |f(e^{i\theta})| < \infty.$$

We sometimes indicate by $L^p_{\mathbb{R}}(E)$ the real subspace of real-valued functions. We also set

$$\langle f, g\rangle_E = \frac{1}{2\pi}\int_E f(e^{i\theta})\overline{g(e^{i\theta})}\, d\theta \tag{1}$$

whenever $f \in L^p(E)$ and $g \in L^q(E)$ with $1/p + 1/q = 1$. If f and g are defined on a set containing E, we write for simplicity $\langle f, g\rangle_E$ to mean $< f_{|_E}, g_{|_E} >$ and $\|f\|_{L^p(E)}$ to mean $\|f_{|_E}\|_{L^p(E)}$. Hereafter $C(E)$ stands for the space of bounded complex-valued continuous functions on E endowed with the sup norm, while $C_{\mathbb{R}}(E)$ indicates real-valued continuous functions.

Recall that the Hardy space H^p is the closed subspace of $L^p(\mathbb{T})$ consisting of functions whose Fourier coefficients of strictly negative index do vanish. We refer the reader to [11] for standard facts on Hardy spaces, in particular those recorded hereafter. Hardy functions are the nontangential limits a.e. on $\mathbb{T}$ of functions holomorphic in the unit disk $\mathbb{D}$ having uniformly bounded L^p means over all circles centered at 0 of radius less than 1:

$$\|f\|_{H^p} = \sup_{0\le r<1}\left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p\, d\theta\right)^{1/p} \quad \text{if } 1 \le p < \infty, \qquad \|f\|_{H^\infty} = \sup_{z\in\mathbb{D}} |f(z)|. \tag{2}$$

The correspondence between such a holomorphic function f and its non tangential limit $f^\sharp$ is one-to-one and even isometric, namely the supremum in (2) is equal to $\|f^\sharp\|_p$, thereby allowing us to identify f and $f^\sharp$ and to drop the superscript $\sharp$. Under this identification, we regard members of H^p both as functions in $L^p(\mathbb{T})$ and as holomorphic functions in the variable $z \in \mathbb{D}$, but the argument (which belongs to $\mathbb{T}$ in the former case and to $\mathbb{D}$ in the latter) helps preventing confusion. It holds in fact that $f_r(e^{i\theta}) = f(re^{i\theta})$ converges as $r \to 1^-$ to $f(e^{i\theta})$ in $L^p(\mathbb{T})$ when $f \in H^p$ and $1 \le p < \infty$. It follows immediately from (2) and Hölder's inequality that, whenever $g_1 \in H^{p_1}$ and $g_2 \in H^{p_2}$, we have $g_1 g_2 \in H^{p_3}$ if $1/p_1 + 1/p_2 = 1/p_3$.

Given $f \in H^p$, its values on $\mathbb{D}$ are obtained from its values on $\mathbb{T}$ through a Cauchy as well as a Poisson integral [17, ch. 17, thm 11], namely:

$$f(z) = \frac{1}{2i\pi}\int_{\mathbb{T}} \frac{f(\xi)}{\xi - z}\, d\xi, \quad \text{and also } f(z) = \frac{1}{2\pi}\int_{\mathbb{T}} \mathrm{Re}\left\{\frac{e^{i\theta}+z}{e^{i\theta}-z}\right\} f(e^{i\theta})\, d\theta, \quad z \in \mathbb{D}, \tag{3}$$

where the right hand side of the first equality in (3) is a line integral. The latter immediately implies that the Fourier coefficients of a Hardy function on the circle are the Taylor coefficients of its power series expansion at 0 when viewed as a holomorphic function on $\mathbb{D}$. In this connection, the space H^2 is especially simple to describe: it consists of those holomorphic functions g in $\mathbb{D}$ whose Taylor coefficients at 0 are square summable, namely

$$g(z) = \sum_{k=0}^{\infty} a_k z^k : \quad \|g\|_{H^2}^2 := \sum_{k=0}^{\infty} |a_k|^2 < +\infty, \qquad g(e^{i\theta}) = \sum_{k=0}^{\infty} a_k e^{ik\theta}, \tag{4}$$

where the convergence of the last Fourier series holds in $L^2(\mathbb{T})$ by Parseval's theorem (and also pointwise a.e. by Carleson's theorem but we do not need this deep result). Incidentally, let us mention that for no other value of p is it known how to characterize H^p in terms of the size of its Fourier coefficients.

By the Poisson representation (i.e. the second integral in (3)), a Hardy function g is also uniquely represented, up to a purely imaginary constant, by its real part h on $\mathbb{T}$ according to:

$$g(z) = i\mathrm{Im}g(0) + \frac{1}{2\pi}\int_{\mathbb{T}} \frac{e^{i\theta}+z}{e^{i\theta}-z}\, h(e^{i\theta})\, d\theta\,, \quad z \in \mathbb{D}. \tag{5}$$

The integral in (5) is called the *Riesz-Herglotz transform* of h and, whenever $h \in L^1_{\mathbb{R}}(\mathbb{T})$, it defines a holomorphic function in $\mathbb{D}$ which is real at 0 and whose nontangential limit exists a.e. on $\mathbb{T}$ with real part equal to h. Hence the Riesz-Herglotz transform (5) assumes the form $h(e^{i\theta}) + i\widetilde{h}(e^{i\theta})$ a.e. on $\mathbb{T}$, where the real-valued function $\widetilde{h}$ is said to be *conjugate* to h. It is a theorem of M. Riesz [11, chap. III, thm 2.3] that if $1 < p < \infty$, then $\widetilde{h} \in L^p_{\mathbb{R}}(\mathbb{T})$ when $h \in L^p_{\mathbb{R}}(\mathbb{T})$. This neither holds for $p = 1$ nor for $p = \infty$.

A nonzero $f \in H^p$ can be uniquely factored as $f = jw$ where

$$w(z) = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \log|f(e^{i\theta})|\, d\theta\right\} \tag{6}$$

belongs to H^p and is called the *outer factor* of f, while $j \in H^\infty$ has modulus 1 a.e. on $\mathbb{T}$ and is called the *inner factor* of f. That $w(z)$ in (6) is well-defined rests on the fact that $\log|f| \in L^1$ if $f \in H^1 \setminus \{0\}$; it entails that a H^p function cannot vanish on a subset of strictly positive Lebesgue measure on $\mathbb{T}$ unless it is identically zero. For simplicity, we often say that a function is outer (resp. inner) if it is equal, up to a unimodular multiplicative constant, to its outer (resp. inner) factor.

Closely connected to Hardy spaces is the Nevanlinna class N^+, consisting of holomorphic functions in $\mathbb{D}$ that can be factored as jE, where j is an inner function and E an outer function of the form

$$E(z) = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \log\rho(e^{i\theta})\, d\theta\right\}\,, \tag{7}$$

with ρ a positive function such that $\log\rho \in L^1(\mathbb{T})$ (though ρ itself may not be summable). Such a function has nontangential limits of modulus ρ a.e. on $\mathbb{T}$. The Nevanlinna class is instrumental in that $N^+ \cap L^p(\mathbb{T}) = H^p$, see [10, thm 2.11] or [11, 5.8, ch.II]. Thus, formula (7) defines a H^p-function if and only if $\rho \in L^p(\mathbb{T})$.

Let $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere. The Hardy space $\bar{H}^p$ of $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ can be given a treatment parallel to H^p upon changing z into $1/z$. Specifically, $\bar{H}^p$ consists of functions in $L^p(\mathbb{T})$ whose Fourier coefficients of strictly positive index do vanish; these are, a.e. on $\mathbb{T}$, the complex conjugates of H^p-functions, and they can also be viewed as nontangential limits of functions analytic in $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ having uniformly bounded L^p means over all circles centered at 0 of radius bigger than 1. We further single out the subspace $\bar{H}^p_0$ of $\bar{H}^p$, consisting of functions vanishing at infinity or, equivalently, having vanishing mean on $\mathbb{T}$. Thus, a function belongs to $\bar{H}^p_0$ if, and only if it is of the form $e^{-i\theta}\overline{g(e^{i\theta})}$ for some $g \in H^p$. For $G \in \bar{H}^p_0$, the Cauchy formula assumes the form :

$$G(z) = \frac{1}{2\,i\,\pi} \int_{\mathbb{T}} \frac{G(\xi)}{z-\xi}\, d\xi\,, \quad z \in \hat{\mathbb{C}} \setminus \overline{\mathbb{D}}. \tag{8}$$

It follows at once from the Cauchy formula that the duality product $\langle\ ,\ \rangle_{\mathbb{T}}$ makes H^p and $\bar{H}^q_0$ orthogonal to each other, and it reduces to the familiar scalar product when $p = q = 2$. In particular, we have the orthogonal decomposition :

$$L^2(\mathbb{T}) = H^2 \oplus \bar{H}^2_0. \tag{9}$$

For $f \in C(\mathbb{T})$ and $\nu \in \mathcal{M}$, the space of complex Borel measures on $\mathbb{T}$, we set

$$\nu.f = \int_{\mathbb{T}} f(e^{i\theta})\, d\nu(\theta) \tag{10}$$

and this pairing induces an isometric isomorphism between $\mathcal{M}$ (endowed with the norm of the total variation) and the dual of $C(\mathbb{T})$ [17, thm 6.19]. If we let $\mathcal{A} \subset H^\infty$ designate the disk algebra of functions analytic in $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$, and if $\mathcal{A}_0$ indicates those functions in $\mathcal{A}$ vanishing at zero, it is easy to see that $\mathcal{A}_0$ is the orthogonal space under (10) to those measures whose Fourier coefficients of strictly negative index do vanish. Now, it is a fundamental theorem of F. and M. Riesz that such measures are absolutely continuous, that is have the form $d\nu(\theta) = g(e^{i\theta})\, d\theta$ with $g \in H^1$. The Hahn-Banach theorem implies that H^1 is dual *via* (10) to the quotient space $C(\mathbb{T})/\mathcal{A}_0$ [11, chap. IV, sec. 1]. Equivalently, $\bar{H}^1_0$ is dual to $C(\mathbb{T})/\overline{\mathcal{A}}$ under the pairing arising from the line integral :

$$(\dot{f}, F) = \frac{1}{2i\pi} \int_{\mathbb{T}} f(\xi)F(\xi)\, d\xi\,, \tag{11}$$

where F belongs to $\bar{H}^1_0$ and $\dot{f}$ indicates the equivalence class of $f \in C(\mathbb{T})$ modulo $\overline{\mathcal{A}}$. Therefore, contrary to $L^1(\mathbb{T})$, the spaces H^1 and $\bar{H}^1_0$ enjoy a weak-* compactness property of their unit ball.

We define the analytic and anti-analytic projections $\mathbf{P}_+$ and $\mathbf{P}_-$ on Fourier series by :

$$\mathbf{P}_+\left(\sum_{n=-\infty}^{\infty} a_n e^{in\theta}\right) = \sum_{n=0}^{\infty} a_n e^{in\theta}, \quad \mathbf{P}_-\left(\sum_{n=-\infty}^{\infty} a_n e^{in\theta}\right) = \sum_{n=-\infty}^{-1} a_n e^{in\theta}\,.$$

It is a theorem of M. Riesz theorem [11, ch. III, sec, 1] that $\mathbf{P}_+ : L^p \to H^p$ and $\mathbf{P}_- : L^p \to \bar{H}^p_0$ are bounded for $1 < p < \infty$, in which case they coincide with the Cauchy projections:

$$\mathbf{P}_+(h)(z) = \frac{1}{2i\pi}\int_{\mathbb{T}} \frac{h(\xi)}{\xi - z}\, d\xi, \quad z \in \mathbb{D}, \qquad \mathbf{P}_-(h)(s) = \frac{1}{2i\pi}\int_{\mathbb{T}} \frac{h(\xi)}{s - \xi}\, d\xi, \quad s \in \hat{\mathbb{C}} \backslash \overline{\mathbb{D}}. \tag{12}$$

When restricted to $L^2(\mathbb{T})$, the projections $\mathbf{P}_+$ and $\mathbf{P}_-$ are just the orthogonal projections onto H^2 and $\bar{H}_0^2$ respectively. Although $\mathbf{P}_\pm(h)$ needs not be the Fourier series of a function when h is merely in $L^1(\mathbb{T})$, it is Abel summable almost everywhere to a function lying in $L^s(\mathbb{T})$ for $0 < s < 1$ and it can be interpreted as a function in the Hardy space of exponent s that we did not introduce [10, cor. to thm 3.2]. To us it will be sufficient, when $h \in L^1$, to regard $\mathbf{P}_\pm(f)$ as the Fourier series of a distribution.

Finally, we let P_n denote throughout the space of complex algebraic polynomials of degree at most n. Clearly, $P_n \subset H^p$ for all p.

3 Two Extremal Problems

We first state the polynomial problem discussed in Sect. 1. We call it *PBEP*(n) for "Polynomial Bounded Extremal Problem":

PBEP(n)

For $f \in L^2(I)$, find $k_n \in P_n$ such that $|k_n(e^{i\theta})| \leq 1$ for a.e. $e^{i\theta} \in J$ and

$$\|f - k_n\|_{L^2(I)} = \inf_{\substack{g \in P_n \\ |g| \leq 1 \text{ a.e. on } J}} \|f - g\|_{L^2(I)} \,. \tag{13}$$

Next, we state the analytic problem from Sect. 1 that we call *ABEP* for "Analytic Bounded Extremal Problem":

ABEP

Given $f \in L^2(I)$, find $g_0 \in H^2$ such that $|g_0(e^{i\theta})| \leq 1$ a.e. on J and

$$\|f - g_0\|_{L^2(I)} = \inf_{\substack{g \in H^2 \\ |g| \leq 1 \text{ a.e. on } J}} \|f - g\|_{L^2(I)} \,. \tag{14}$$

Note that, in *ABEP*, the constraint $|g| \leq 1$ on J could be replaced by $|g| \leq \rho$ where ρ is a positive function in $L^2(J)$. For if $\log \rho \in L^1(J)$ then, denoting by $w_{1\vee(1/\rho)}$ the outer factor having modulus 1 on I and $1/\rho$ on J, we find that $g \in H^2$ satisfies $|g| \leq \rho$ on J if and only if $h = g w_{1\vee(1/\rho)}$ lies in H^2 and satisfies $|h| \leq 1$ on J. It is so because, for g as indicated, h lies in the Nevanlinna class by construction and $|h|_{|_I} = |g|_{|_I}$ while $|h|_{|_J} = |g|_{|_J}/\rho$. If, however, $\log \rho \notin L^1(J)$, then we must have $\int_J \log \rho = -\infty$ because $\rho \in L^2(J)$, consequently the set of candidate approximants reduces to $\{0\}$ anyway because a nonzero Hardy function has summable log-modulus. Altogether, it is thus equivalent to consider *ABEP* for the product f times $(w_{1\vee\rho^{-1}})_{|_I}$. A similar argument shows that we could replace the error criterion $\|.\|_{L^2(I)}$ by a weighted norm $\|.\|_{L^2(I,w)}$ for some weight w which is non-negative and invertible in $L^\infty(I)$. Then, the problem reduces to *ABEP* for $f(w_{\rho^{1/2}\vee 0})_I$.

Such equivalences do not hold for *PBEP*(n) because the polynomial character of k_n is not preserved under multiplication by outer factors. Still, the results to come continue to hold if we replace in *PBEP*(n) the constraint $|k_n| \leq 1$ by $|k_n| \leq \rho$ on J and the criterion $\|.\|_{L^2(I)}$ by $\|.\|_{L^2(I,w)}$, provided that $\rho \in C(J)$ and that w is invertible in $L^\infty(I)$. Indeed, we leave it to the reader to check that proofs go through with obvious modifications.

After these preliminaries, we are ready to state a basic existence and uniqueness result.

Theorem 1 *Problems PBEP*(n) *and ABEP have a unique solution. Moreover, the solution* g_0 *to ABEP satisfies* $|g_0| = 1$ *almost everywhere on J, unless* $f = g_{|_I}$ *for some* $g \in H^2$ *such that* $\|g\|_{L^\infty(J)} \leq 1$.

Proof Consider the sets

$$E_n = \{g_{|_I} :\ g \in P_n,\ \|g\|_{L^\infty(J)} \leq 1\},$$

$$F = \{g_{|_I} :\ g \in H^2,\ \|g\|_{L^\infty(J)} \leq 1\}.$$

Clearly $E_n \subset F$ are convex and nonempty subsets of $L^2(I)$, as they contain 0. To prove existence and uniqueness, it is therefore enough to show they are closed, for we can appeal then to well-known properties of the projection on a closed convex set in a Hilbert space. Since $E_n = P_n \cap F$, it is enough in fact to show that F is closed. For this, let g_m be a sequence in H^2 with $|g_m|_{|_J} \leq 1$ and such that $(g_m)_{|_I}$ converges in $L^2(I)$. Obviously g_m is a bounded sequence in $L^2(\mathbb{T})$, some subsequence of which converges weakly to $h \in H^2$. We continue to denote this subsequence with g_m. The restrictions $(g_m)_{|_I}$ a fortiori converge weakly to $h_{|_I}$ in $L^2(I)$, and since the strong and the weak limit must coincide when both exist we find that $(g_m)_{|_I}$ converges to $h_{|_I}$ in $L^2(I)$. Besides, $(g_m)_{|_J}$ is contained in the unit ball of $L^\infty(J)$ which is dual to $L^1(J)$, hence some subsequence (again denoted by $(g_m)_{|_J}$) converges weak-* to some $h_1 \in L^\infty(J)$ with $\|h_1\|_{L^\infty(J)} \leq 1$. But since $(g_m)_{|_J}$ also converges weakly to $h_{|_J}$ in $L^2(J)$, we have that

$$\langle h_1, \varphi\rangle_J = \lim_{m\to\infty} \langle g_m, \varphi\rangle_J = \langle h_{|_J}, \varphi\rangle_J$$

for all $\varphi \in L^2(J)$ which is dense in $L^1(J)$. Consequently $h_1 = h_{|_J}$, thereby showing that $\|h\|_{L^\infty(J)} \leq 1$, which proves that F is closed.

Assume now that f is not the trace on I of an H^2-function which is less than 1 in modulus on I. To prove that $|g_0| = 1$ a.e. on J, we argue by contradiction. If not, there is a compact set K of positive measure, lying interior to J, such that $\|g_0\|_{L^\infty(K)} \leq 1 - \delta$ for some $0 < \delta < 1$; it is so because, by hypothesis, J must consist of finitely many closed arcs, of which one at least has nonempty interior. For K' an arbitrary subset of K, consider the Riesz-Herglotz transform of its characteristic function:

$$h_{K'}(z) = \frac{1}{2\pi}\int_{K'} \frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\theta\,, \quad z \in \mathbb{D}, \tag{15}$$

and put $w_t = \exp(th_{K'})$ for $t \in \mathbb{R}$, which is the outer function with modulus $\exp t$ on K' and 1 elsewhere. By construction, $g_0 w_t$ is a candidate approximant in *ABEP* for all $t < -\log(1-\delta)$. Thus, the map $t \mapsto \|f - g_0 w_t\|^2_{L^2(I)}$ attains a minimum at $t = 0$. Because K is at strictly positive distance from I, we may differentiate this expression with respect to t under the integral sign and equate the derivative at $t = 0$ to zero which gives us $2\mathrm{Re}\langle f - g_0, h_{K'} g_0\rangle_I = 0$. Replacing $g_0 w_t$ by $ig_0 w_t$, which is a candidate approximant as well, we get a similar equation for the imaginary part so that

$$0 = \langle f - g_0\,,\, h_{K'} g_0\rangle_I = \langle (f - g_0)\bar{g}_0\,,\, h_{K'}\rangle_I. \tag{16}$$

Let e^{it_0} be a density point of K and I_l the arc centered at e^{it_0} of length l, so that $|I_l \cap K)|/l \to 1$ as $l \to 0$. Since

$$\left|\frac{e^{it}+e^{i\theta}}{e^{it}-e^{i\theta}} - \frac{e^{it_0}+e^{i\theta}}{e^{it_0}-e^{i\theta}}\right| \le \frac{2l}{\mathrm{dist}^2(K,I)} \quad \text{for } e^{it} \in I_l \cap K, \;\; e^{i\theta} \in I, \tag{17}$$

it follows by dominated convergence that

$$\lim_{l\to 0} \frac{1}{|I_l \cap K|}\int_{I_l\cap K}\left|\frac{e^{it}+e^{i\theta}}{e^{it}-e^{i\theta}} - \frac{e^{it_0}+e^{i\theta}}{e^{it_0}-e^{i\theta}}\right| dt = 0, \quad \text{uniformly w.r. to } e^{i\theta} \in I,$$

and therefore that

$$\lim_{l\to 0}\frac{h_{I_l\cap K}(e^{i\theta})}{|I_l\cap K|} = \frac{e^{it_0}+e^{i\theta}}{e^{it_0}-e^{i\theta}} \quad \text{uniformly w.r. to } e^{i\theta} \in I.$$

Applying now (16) with $K' = I_l \cap K$ and taking into account that $(e^{it_0} + e^{i\theta})/(e^{it_0} - e^{i\theta})$ is pure imaginary on I, we find in the limit, as $l \to 0$ that

$$\frac{1}{2\pi}\int_I \frac{e^{it_0}+e^{i\theta}}{e^{it_0}-e^{i\theta}}\Big((f-g_0)\bar{g}_0\Big)(e^{i\theta})\,d\theta = 0. \tag{18}$$

Next, let us consider the function

$$F(z) = \frac{1}{2\pi}\int_I \frac{e^{i\theta}+z}{e^{i\theta}-z}\Big((f-g_0)\bar{g}_0\Big)(e^{i\theta})\,d\theta$$

$$= -\frac{1}{2\pi}\int_I \Big((f-g_0)\bar{g}_0\Big)(e^{i\theta})d\theta + \frac{1}{i\pi}\int_I \frac{\Big((f-g_0)\bar{g}_0\Big)(\xi)d\xi}{\xi - z}$$

which is the sum of a constant and of twice the Cauchy integral of $(f-(g_0)_{|_I})(\bar{g}_0)_{|_I} \in L^1(I)$, hence is analytic in $\hat{\mathbb{C}}\setminus I$. Equation (18) means that F vanishes at every density point of K, and since a.e. point in K is a density point F must vanish identically because its zeros accumulate in the interior of J. Denoting by F^+ and F^- the nontangential limits of F from sequences of points in $\mathbb{D}$ or $C \setminus \overline{\mathbb{D}}$ respectively, we now get from the Plemelj-Sokhotski formulas [11, ch. III] that

$$0 = F^+(\xi) - F^-(\xi) = (f - g_0)(\xi)\overline{g_0(\xi)}, \qquad a.e.\ \xi \in I.$$

Thus, either g_0 is nonzero a.e. on I, in which case $f = (g_0)_{|_I}$ and we reach the desired contradiction, or else $g_0 \equiv 0$. In the latter case, if we put id for the identity map on $\mathbb{T}$, we find that $t \mapsto \|f - t\,\mathrm{id}^k\|^2_{L^2(I)}$ has a minimum at $t = 0$ for each integer $k \geq 0$, since $e^{i\theta} \mapsto te^{ik\theta}$ is a candidate approximant for $t \in [-1, 1]$. Differentiating with respect to t and expressing that the derivative at $t = 0$ is zero, we deduce that all Fourier coefficients of non-negative index of $(f - (g_0)_{|_I}) \vee 0$ do vanish. This means this last function lies in $\bar{H}^2$, but as it vanishes on J it is identically zero, therefore $f = (g_0)_{|_I}$ in all cases. ■

Remark the theorem shows that the constraint $|g_0| \leq 1$ on J is saturated in a very strong sense for problem *ABEP*, namely $|g_0| = 1$ a.e. on J unless f is already the trace of the solution on I. In contrast, it is not true that $\|k_n\|_{L^\infty(J)} = 1$ unless $f = g_{|_I}$ for some $g \in P_n$ such that $\|g\|_{L^\infty(J)} < 1$. To see this, observe that the set E_n is not only closed but compact. Indeed, if we pick distinct points $\xi_1, \cdots, \xi_{n+1}$ in J and form the Lagrange interpolation polynomials $L_j \in P_n$ such that $L_j(\xi_j) = 1$ and $L_j(\xi_\ell) = 0$ if $\ell \neq j$, we get a basis of P_n in which the coordinates of every $g \in P_n$ meeting $\|g\|_{L^\infty(J)} \leq 1$ are bounded by 1 in modulus. Hence E_n is bounded in P_n, and since it is closed by the proof of Theorem 1 it is compact. Thus, each $f \in L^2(I)$ has a best approximant from E_n, and if $(p_n)_I$ is a best approximant to f with $p_n \in P_n$, then for $\lambda > \|p_n\|_{L^\infty(J)}$ we find that p_n/λ is a best approximant to f/λ in $L^2(I)$ which is strictly less than 1 on J. This justifies the remark.

4 Critical Point Equations and Convergence of Approximants

At this point, it is worth recalling informally some basic principles from convex optimization, for which the reader may consult [7]. The solution to a strictly convex minimization problem is characterized by a variational inequality expressing that the *criterium* increases under admissible increments of the variable. If the problem is smooth enough, such increments admit a tangent space at the point under consideration (i.e. the solution) in the variable space. We term it the tangent space to the constraints, and its orthogonal in the dual space to the variable space is called the orthogonal space to the constraints (at the point under consideration). The variation of the objective function must vanish on the tangent space to the constraints to the first order, thereby giving rise to the so-called *critical point equation*. It says that the gradient of the objective function, viewed as an element of the dual space to the variable space, lies in the orthogonal space to the constraints. If a basis of the latter is chosen, the coordinates of the gradient in this basis are known as the *Lagrange parameters*. More generally, one can form the Lagrangian which is a function of the variable and of the Lagrange parameters, not necessarily optimal ones. It is obtained by adding the gradient of the criterion, at the considered value of the variable, with the member of the orthogonal space to the constraints defined by the chosen Lagrange parameters. By what precedes, the Lagrangian must vanish at the solution for appropriate values of the Lagrange parameters. One can further define a function of the Lagrange parameters only, by minimizing the Lagrangian with respect to the variable. This results in a concave function which gets maximized at the optimal value of the Lagrange parameters for the original problem. This way, one reduces the original constrained convex minimization problem to an unsconstrained concave maximization problem, called the dual problem. In an infinite-dimensional context, the arguments needed to put this program to work may be quite subtle.

Below we derive the critical point equation for *PBEP*(n) described in (13). For $g \in P_n$ define

$$E(g) = \{x \in J,\ |g(x)| = ||g||_{L^\infty(J)}\},$$

which is the set of extremal points of g on J.

Theorem 2 *A polynomial $g \in P_n$ is the solution to PBEP(n) iff the following two conditions hold:*

- $||g||_{L^\infty(J)} \leq 1$,
- *there exists a set of r distinct points $x_1, \cdots, x_r \in E(g)$ and non-negative real numbers $\lambda_1, \cdots, \lambda_r$, with $0 \leq r \leq 2n+2$, such that*

$$\langle g - f, h\rangle_I + \sum_{j=1}^{r} \lambda_j g(x_j)\overline{h(x_j)} = 0, \qquad \forall h \in P_n. \tag{19}$$

Moreover the λ_j's meet the following bound

$$\sum_{j=1}^{r} \lambda_j \leq 2||f||^2_{L^2(I)}. \tag{20}$$

We emphasize that the set of extremal points $\{x_j,\ j = 1, \ldots, r\}$ is possibly empty (i.e $r = 0$).

Proof Suppose g verifies the two conditions and differs from the solution k_n. Set $h = k_n - g \in P_n$ and observe that

$$\mathrm{Re}\left(g(x_i)\overline{h(x_i)}\right) = \mathrm{Re}\left(g(x_i)\overline{k_n(x_i)} - 1\right) \leq 0, \;\; i = 1 \ldots r. \tag{21}$$

From the uniqueness and optimality of k_n we deduce that

$$\begin{aligned} ||k_n - f||^2_{L^2(I)} &= ||g - f + h||^2_{L^2(I)} \\ &= ||g - f||^2_{L^2(I)} + ||h||^2_{L^2(I)} + 2\mathrm{Re}\langle g - f, h\rangle_I \\ &< ||g - f||^2_{L^2(I)}. \end{aligned}$$

Consequently $\mathrm{Re}\langle g - f, h\rangle_{L^2(I)} < 0$ which, combined with (21), contradicts (19).

Conversely, suppose that g is the solution to $PBEP(n)$ and let ϕ_0 be the $\mathbb{R}$-linear forms on P_n given by

$$\phi_0(h) = \mathrm{Re}\langle g - f, h\rangle_I, \qquad h \in P_n.$$

For each extremal point $x \in E(g)$, define further a $\mathbb{R}$-linear form ϕ_x by

$$\phi_x(h) = \mathrm{Re}\left(g(x)\overline{h(x)}\right), \quad h \in P_n.$$

Put K for the union of these forms:

$$K = \{\phi_0\} \cup \{\phi_x, \; x \in E(g)\}.$$

If we let $P_n^{\mathbb{R}}$ indicate P_n viewed as a real vector space, K is a subset of the dual $(P_n^{\mathbb{R}})^*$. As J is closed by definition, simple inspection shows that K is closed and bounded in $(P_n^{\mathbb{R}})^*$ (it is in fact finite unless g is a constant), hence it is compact and so is its convex hull $\hat{K}$ as $(P_n^{\mathbb{R}})^*$ is finite-dimensional. Suppose for a contradiction that $0 \notin \hat{K}$. Then, since $(P_n^{\mathbb{R}})^{**} = P_n^{\mathbb{R}}$ because $P_n^{\mathbb{R}}$ is finite-dimensional, there exists by the Hahn-Banach theorem an $h_0 \in P_n$ such that,

$$\phi(h_0) \geq \tau > 0, \quad \forall \phi \in \hat{K}.$$

The latter and the continuity of g and h_0 ensure the existence of a neighborhood V of $E(g)$ on $\mathbb{T}$ such that for x in $U = J \cap V$ we have $\mathrm{Re}\left(g(x)\overline{h_0(x)}\right) \geq \frac{\tau}{2} > 0$, whereas for x in $J \setminus U$ it holds that $|g(x)| \leq 1 - \delta$ for some $\delta > 0$. Clearly, for $\epsilon > 0$ with $\epsilon ||h_0||_{L^\infty(J)} < \delta$, we get that

$$\sup_{J \setminus U} |g(x) - \epsilon h_0(x)| \leq 1. \tag{22}$$

Moreover, assuming without loss of generality that $\epsilon < 1$, it holds for $x \in U$ that

$$\begin{aligned} |g(x) - \epsilon h_0(x)|^2 &= |g(x)|^2 - 2\mathrm{Re}\left(g(x)\overline{h_0(x)}\right) + \epsilon^2|h_0(x)|^2 \\ &\leq |g(x)|^2 - 2\mathrm{Re}\left(\epsilon g(x)\overline{h_0(x)}\right) + \epsilon^2|h_0(x)|^2 \\ &\leq 1 - \epsilon\tau + \epsilon^2||h_0||^2_{L^\infty(J)}. \end{aligned}$$

The latter combined with (22) shows that, for ϵ sufficiently small, we have

$$||g - \epsilon h_0||_{L^\infty(J)} \leq 1. \tag{23}$$

However, since

$$\begin{aligned} ||f - g - \epsilon h_0||^2_{L^2(J)} &= ||f - g||^2_{L^2(J)} - 2\epsilon\phi_0(h_0) + \epsilon^2||h_0||^2_{L^2(J)} \\ &\leq ||f - g||^2_{L^2(J)} - 2\epsilon\tau + \epsilon^2||h_0||^2_{L^2(J)}, \end{aligned} \tag{24}$$

we deduce in view of (23) that for ϵ small enough the polynomial $g - \epsilon h_0$ performs better than g in *FBEP*, thereby contradicting optimality. Hence $0 \in \hat{K}$, therefore by Carathedory's theorem [8, ch. 1, sec. 5] there are r' elements γ_j of K, with $1 \leq r' \leq 2(n+1)+1$ (the real dimension of $P_n^{\mathbb{R}}$ plus one), such that

$$\sum_{j=1}^{r'} \alpha_j \gamma_j = 0 \tag{25}$$

for some positive α_j satisfying $\sum \alpha_j = 1$. Of necessity ϕ_0 is a γ_j, otherwise evaluating (25) at g yields the absurd conclusion that

$$0 = \sum_{j=1}^{r'} \alpha_j \gamma_j(g) = \sum_{j=1}^{r'} \alpha_j |g(x_j)|^2 = 1.$$

Equation (25) can therefore be rewritten as

$$\alpha_1 \mathrm{Re}\langle f - g, h\rangle_I + \sum_{j=2}^{r'} \alpha_j \mathrm{Re}(g(x_j)\overline{h(x_j)}) = 0 \quad \forall h \in P_n, \quad \alpha_1 \neq 0.$$

Dividing by α_1 and noting that the last equation is also true with ih instead of h yields (19) with $r = r' - 1$. Finally, replacing h by g in (19) we obtain

$$\sum_{j=1}^{r} |\lambda_j| = \sum_{j=1}^{r} \lambda_j| = \langle f-g, g\rangle_I \leq \ \langle f-g, f-g\rangle_I + |\langle f-g, f\rangle_I|$$
$$\leq ||f-g||^2_{L^2(I)} + ||f-g||_{L^2(I)}||f||_{L^2(I)}$$
$$\leq 2||f||^2_{L^2(I)}$$

where the next to last majorization uses the Schwarz inequality and the last that 0 is a candidate approximant for *PBEP*(n) whereas g is the optimum. ■

The next result describes the behavior of k_n when n goes to infinity, in connection with the solution g_0 to *ABEP*.

Theorem 3 *Let k_n be the solution to PBEP(n) defined in (13), and g_0 the solution to ABEP described in (14). When $n \to \infty$, the sequence $(k_n)_{|_I}$ converges to $(g_0)_{|_I}$ in $L^2(I)$, and the sequence $(k_n)_{|_J}$ converges to $(g_0)_{|_J}$ in the weak-* topology of $L^\infty(J)$, as well as in $L^p(J)$-norm for $1 \leq p < \infty$ if f is not the trace on I of a H^2-function which is at most 1 in modulus on J. Altogether this amounts to:*

$$\lim_{n\to\infty} ||g_0 - k_n||_{L^p(\mathbb{T})} = 0, \quad 1 \leq p \leq 2, \tag{26}$$

$$\lim_{n\to\infty} \langle k_n, h\rangle_J = \langle g_0, h\rangle_J \quad \forall h \in L^1(J), \tag{27}$$

$$\text{if } f \neq g_0 \text{ on } I, \ \lim_{n\to\infty} ||g_0 - k_n||_{L^p(J)} = 0, \quad 1 \leq p < \infty. \tag{28}$$

Proof Our first objective is to show that g_0 can be approximated arbitrary close in $L^2(I)$ by polynomials that remain bounded by 1 in modulus on J. By hypothesis I is the finite union of $N \geq 1$ open disjoint sub-arcs of $\mathbb{T}$. Without loss of generality, it can thus be written as

$$I = \bigcup_{i=1}^{N} (e^{ia_i}, e^{ib_i}), \qquad 0 = a_1 \leq b_1 \leq a_2 \cdots \leq b_N \leq 2\pi.$$

Let (ϵ_n) be a sequence of positive real numbers decreasing to 0. We define a sequence (v_n) in H^2 by

$$v_n(z) = g_0(z) \exp\left(-\frac{1}{2\pi}\left(\sum_{i=1}^{N} \int_{a_i}^{a_i+\epsilon_n} \frac{e^{it}+z}{e^{it}-z} \log|g_0| dt + \int_{b_i-\epsilon_n}^{b_i} \frac{e^{it}+z}{e^{it}-z} \log|g_0| dt\right)\right)$$

Note that indeed $v_n \in H^2$ for n large enough because then it has the same modulus as g_0 except over the arcs $(a_i, a_i + \epsilon_n)$ and $(b_i - \epsilon_n, b_i)$ where it has modulus 1. *We claim* that $(v_n)_{|_I}$ converges to g_0 in $L^2(I)$ as $n \to \infty$. To see this, observe that v_n converges a.e. on I to g_0, for each $z \in I$ remains at some distance from the sub-arcs $(a_i, a_i + \epsilon_n)$ and $(b_i, b_i + \epsilon_n)$ for all n sufficiently large, hence the argument of the exponential in (29) converges to zero as $n \to \infty$ by absolute continuity of $\log|g_0|dt$. Now, we remark that by construction $|v_n| \le |g_0| + 1$, hence by dominated convergence, we get that

$$\lim_{n\to\infty} ||g_0 - v_n||_{L^2(I)} = 0.$$

This proves the claim. Now, let $\epsilon > 0$ and $0 < \alpha < 1$ such that $||g_0 - \alpha g_0||_{L^2(I)} \le \frac{\epsilon}{4}$. Let also n_0 be so large that $||v_{n_0} - g_0||_{L^2(I)} \le \frac{\epsilon}{4}$. For $0 < r < 1$ define $u_r \in \mathcal{A}$ (the disk algebra) by $u_r(z) = v_{n_0}(rz)$ so that, by Poisson representation,

$$u_r(e^{i\theta}) = \int_{\mathbb{T}} P_r(\theta - t) v_{n_0}(re^{it})dt,$$

where P_r is the Poisson kernel. Whenever $e^{i\phi} \in J$, we note by construction that $|v_n| = 1$ a.e on the sub-arc $(e^{i(\phi-\epsilon_{n_0})}, e^{i(\phi+\epsilon_{n_0})})$. This is to the effect that

$$\begin{aligned} |u_r(e^{i\phi})| &\le \int_{\mathbb{T}} P_r(\phi - t)|v_{n_0}(re^{it})|dt \\ &\le P_r(\epsilon_{n_0}) \int_{\mathbb{T}} |v_{n_0}(re^{it})|dt + \int_{-\epsilon_{n_0}}^{+\epsilon_{n_0}} P_r(t)dt \\ &\le P_r(\epsilon_{n_0})||v_{n_0}||_{L^1(\mathbb{T})} + 1 \le P_r(\epsilon_{n_0})||v_{n_0}||_{L^2(\mathbb{T})} + 1 \end{aligned}$$

by Hölder's inequality. Hence, for r sufficiently close to 1, we certainly have that $|u_r| \le 1/\alpha^2$ on J and otherwise that $||u_r - v_{n_0}||^2_{L^2(I)} \le \frac{\epsilon}{4}$ since $u_r \to v_{n_0}$ in H^2. Finally, call q the truncated Taylor expansion of u_r (which converges uniformly to the latter on $\mathbb{T}$), where the order of truncation has been chosen large enough to ensure that $|q| \le 1/\alpha$ on J and that $||q - u_r||^2_{L^2(I)} \le \frac{\epsilon}{4}$. Then, we have that

$$\begin{aligned} ||\alpha q - g_0||_{L^2(I)} &\le \alpha \left(||q - u_r||_{L^2(I)} + ||u_r - v_{n_0}||_{L^2(I)} + ||v_{n_0} - g_0||_{L^2(I)}\right) \\ &\quad + ||g_0 - \alpha g_0||_{L^2(I)} \le \epsilon. \end{aligned}$$

Thus, we have found a polynomial (namely αq) which is bounded by 1 in modulus on J and close by ϵ to g_0 in $L^2(I)$. By comparison, this immediately implies that

$$\lim_{n\to\infty} ||f - k_n||_{L^2(I)} = ||f - g_0||_{L^2(I)}, \tag{29}$$

from which (26) follows by Hölder's inequality. Moreover, being bounded in H^2, the sequence (k_n) has a weakly convergent sub-sequence. The traces on J of this subsequence are in fact bounded by 1 in $L^\infty(J)$-norm, hence up to another subsequence we obtain (k_{n_m}) converging also in the weak-* sense on J. Let g be the weak limit (H^2 sense) of k_{n_m}, and observe that $g_{|J}$ is necessarily the weak-* limit of $(k_{n_m})_{|J}$ in $L^\infty(J)$, as follows by integrating against functions from $L^2(J)$ which is dense in $L^1(J)$. Since balls are weak-* closed in $L^\infty(J)$, we have that $||g||_{L^\infty(J)} \leq 1$, and it follows from (29) that $||f - g||_{L^2(I)} = ||f - g_0||_{L^2(I)}$. Thus, $g = g_0$ by the uniqueness part of Theorem 1. Finally, if $f \neq g_0$ on J, then we know from Theorem 1 that $|g_0| = 1$ a.e. on J. In this case, (29) implies that $\limsup ||k_{n_m}||_{L^2(\mathbb{T})} \leq ||g_0||_{L^2(\mathbb{T})}$, and since the norm of the weak limit is no less than the limit of the norms it follows that $(k_{n_m})_{|J}$ converges strongly to $(g_0)_{|J}$ in the strictly convex space $L^2(J)$. The same reasoning applies in $L^p(J)$ for $1 < p < \infty$. Finally we remark that the preceding arguments hold true when k_n is replaced by any subsequence of itself; hence k_n contains no subsequence not converging to g_0 in the sense stated before, which achieves the proof. ■

We come now to an analog of Theorem 2 in the infinite dimensional case. We define $H_J^{2,\infty}$ and $H_I^{2,1}$ to be the following vector spaces:

$$H_J^{2,\infty} = \{h \in H^2,\ ||h||_{L^\infty(J)} < \infty\},$$

$$H_I^{2,1} = \{h \in H^1,\ ||h||_{L^2(I)} < \infty\},$$

endowed with the natural norms. We begin with an elementary lemma.

Lemma 1 *Let $v \in L^1(J)$ such that $\mathbf{P}_+(0 \vee v) \in H_I^{2,1}$. Then:*

$$\forall h \in H_J^{2,\infty},\ \langle \mathbf{P}_+(0 \vee v), h\rangle_{\mathbb{T}} = \langle v, h\rangle_J.$$

Proof Let u be the function defined on $\mathbb{T}$ by

$$u = (0 \wedge v) - \mathbf{P}_+(0 \vee v).$$

By assumption $u \in L^1(\mathbb{T})$, and by its very definition all Fourier coefficients of u of non-negative index vanish. Hence $u \in \bar{H}_0^1$, and since it is L^2 integrable on I where it coincides with $-\mathbf{P}_+(0 \vee v)$, .we conclude that $\overline{u} \in H_I^{2,1}$ and that $\overline{u}(0) = 0$ Now, for $h \in H_J^{2,\infty}$ we have that

$$\begin{aligned} \langle v\chi_J, h\rangle_{\mathbb{T}} = &\ \langle u, h\rangle_{\mathbb{T}} + \langle \mathbf{P}_+(0 \vee v), h\rangle_{\mathbb{T}} \\ = &\ \overline{u}(0)h(0) + \langle \mathbf{P}_+(0 \vee v), h\rangle_{\mathbb{T}} \\ = &\ \langle \mathbf{P}_+(0 \vee v), h\rangle_{\mathbb{T}} \end{aligned} \tag{30}$$

where the second equality follows from the Cauchy formula because $(\overline{u}h) \in H^1$. ■

Theorem 4 *Suppose that $f \in L^2(I)$ is not the trace on I of a H^2-function of modulus less or equal to 1 a.e on J. Then, $g \in H^2$ is the solution to ABEP iff the following two conditions hold.*

- $|g(e^{i\theta})| = 1$ *for a.e. $e^{i\theta} \in J$,*
- *there exists a nonnegative real function $\lambda \in L^1_{\mathbb{R}}(J)$ such that,*

$$\forall h \in H_J^{2,\infty},\ \langle g-f,h\rangle_I + \langle \lambda g,h\rangle_J = 0. \tag{31}$$

Proof Suppose g verifies the two conditions and differs from g_0. Set $h = (g_0 - g) \in H_J^{2,\infty}$ and observe that

$$\mathrm{Re}\langle \lambda g,h\rangle_J = \frac{1}{2\pi}\int_J \lambda(\mathrm{Re}(\overline{g}g_0)-1) \le 0. \tag{32}$$

In another connection, since $-h$ is an admissible increment from g_0, the variational inequality characterizing the projection onto a closed convex set gives us (*cf.* Theorem 1) $\mathrm{Re}\langle g_0 - f,h\rangle_I \le 0$, whence

$$\mathrm{Re}\langle g-f,h\rangle_I = \mathrm{Re}\langle g_0-f,h\rangle_I - \langle h,h\rangle_I < 0$$

which, combined with (32), contradicts (31).

Suppose now that g is the solution of *ABEP*. The property that $|g| = 1$ on J has been proven in Theorem 1. In order to let n tend to infinity, we rewrite (19) with self-explaining notations as

$$\langle k_n - f, e^{im\theta}\rangle_I + \sum_{j=1}^{r(n)} \lambda_j^n k_n(e^{i\theta_j^n})\overline{e^{im\theta_j^n}} = 0, \quad \forall m \in \{0\ldots n\},\ . \tag{33}$$

We define (Λ_n), $n \in \mathbb{N}$, to be a family of linear forms on $C(J)$ defined as

$$\Lambda_n(u) = \sum_{j=1}^{r(n)} \lambda_j^n k_n(e^{i\theta_j^n})u(e^{\theta_j^n}), \quad \forall u \in C(J).$$

Equation (20) shows that (Λ_n) is a bounded sequence in the dual $C(J)^*$ which by the Banach-Alaoglu theorem admits a weak-* converging subsequence whose limit we call Λ. Moreover, the Riesz representation theorem ensures the existence of a complex measure μ to represent Λ so that, appealing to Theorem 3 and taking the limit in (33), we obtain

$$\langle g_0 - f, e^{im\theta}\rangle_I + \int_J \overline{e^{im\theta}}\,d\mu = 0, \quad \forall m \in \mathbb{N}.. \tag{34}$$

Now, the F. and M. Riesz theorem asserts that the measure which is μ on J and $(g_0 - f)d\theta$ on I is absolutely continuous with respect to Lebesgue measure, because its Fourier coefficients of nonnegative index do vanish, by (34). Therefore there is $v \in L^1(J)$ such that,

$$\langle g_0 - f, e^{im\theta}\rangle_I + \langle v, e^{im\theta}\rangle_J = 0, \quad \forall m \in \mathbb{N},$$

which is equivalent to

$$\langle g_0 - f, e^{im\theta}\rangle_I + \langle \lambda g_0, e^{im\theta}\rangle_J = 0, \quad \forall m \in \mathbb{N}, \tag{35}$$

where we have set $\lambda(z) = v(z)\overline{g_0(z)}$ $\forall z \in J$. Equation (35) means that

$$\mathbf{P}_+((g_0 - f)\chi_I) = -\mathbf{P}_+(0 \vee \lambda g_0),$$

which indicates that $\mathbf{P}_+(0 \vee \lambda g_0)$ lies in H^2. Thus, thanks to Lemma 1, we get that

$$\langle g_0 - f, u\rangle_I + \langle \lambda g_0, u\rangle_J = 0, \quad \forall u \in H_J^{2,\infty}. \tag{36}$$

In order to prove the realness as well as the nonnegativity of λ, we pick $h \in C^\infty_{c,\mathbb{R}}(I)$, the space of smooth real-valued functions with compact support on I, and we consider its Riesz-Herglotz transform

$$b(z) = \frac{1}{2\pi}\int_I \frac{e^{it}+z}{e^{it}-z}\, h(e^{it})\, dt = \frac{1}{2\pi}\int_{\mathbb{T}} \frac{e^{it}+z}{e^{it}-z}\, \chi_I(e^{it})\, h(e^{it})\, dt\,. \tag{37}$$

It is standard that b is continuous on $\overline{\mathbb{D}}$ [11, ch. III, thm. 1.3]. For $t \in \mathbb{R}$, define $\omega_t = \exp(tb)$ which is the outer function whose modulus is equal to $\exp th$ on I and 1 on J. The function $g_0\,\omega_\lambda$ is a candidate approximant in problem *ABEP*, hence $t \mapsto \|f - g_0\,\omega_t\|^2_{L^2(I)}$ reaches a minimum at $t = 0$. By the boundedness of b, we may differentiate this function with respect to t under the integral sign, and equating the derivative to 0 at $t = 0$ yields

$$0 = \mathrm{Re}\langle (f - g_0)\overline{g_0}, b\rangle_I = \mathrm{Re}\langle (f - g_0), bg_0\rangle_I.$$

In view of (36), it implies that

$$0 = \mathrm{Re}\langle \lambda g_0, bg_0\rangle_J = \mathrm{Re}\langle \lambda, b\rangle_J,$$

where we used that $|g_0| \equiv 1$ on J. Remarking that b is pure imaginary on J, this means

$$\langle Im(\lambda), b\rangle_{L^2(J)} = 0, \quad \forall h \in C^\infty_{c,\mathbb{R}}(I).$$

Letting $h = h_m$ range over a sequence of smooth positive functions which are approximate identies, namely of unit $L^1(I)$-norm and supported on the arc $[\theta - 1/m, \theta + 1/m]$ with $e^{i\theta} \in I$, we get in the limit, as $m \to \infty$, that

$$\langle Im(\lambda), (e^{i\theta} + .)/(e^{i\theta} - .)\rangle_J = 0, \quad e^{i\theta} \in I.$$

Then, appealing to he Plemelj-Sokhotski formulas as in the proof of Theorem 1, this time on J, we obtain that $Im(\lambda) = 0$ which proves that λ is real-valued. Note that the argument based on the Plemelj-Sokhotski formulas and the Hahn-Banach theorem together imply that the space generated by $\xi \mapsto (e^{i\theta} + \xi)/(e^{i\theta} - \xi)$, as $e^{i\theta}$ ranges over an infinite compact subset lying interior to J, is dense in $L^p(I)$ for $1 < p < \infty$. In fact using the F. and M. Riesz theorem and the Plemelj-Sokhoski formulas, it is easy to see that such functions are also uniformly dense in $C(\bar{I})$. Then, using that *ABEP* is a convex problem, we obtain upon differentiating once more that

$$\mathrm{Re}\langle (g_0 - f)\bar{g}_0, b^2\rangle_I \ \geq 0,$$

which leads us by (36) to

$$\mathrm{Re}\langle \lambda, ((e^{i\theta} + .)/(e^{i\theta} - .))^2\rangle_J = \mathrm{Re}\langle \lambda g_0, g_0((e^{i\theta} + .)/(e^{i\theta} - .))^2\rangle_J \ \leq 0, \qquad e^{i\theta} \in I.$$

By the density property just mentioned this implies that $((e^{i\theta} + .)/(e^{i\theta} - .))^2_{|_{\bar{I}}}$ is dense in the set of nonpositive continuous functions on $\bar{I}$, therefore $\lambda \geq 0$. Note also that (35) implies $(f - g_0) \vee \lambda g_0 \in \bar{H}^1$, hence it cannot vanish on a subset of $\mathbb{T}$ of positive measure unless it is the zero function. But this would imply $f = g$ a.e on I which contradicts the hypothesis. This yields $\lambda > 0$ a.e on J. ■

5 A Numerical Example

For practical applications the continuous constraint of *PBEP* on the arc J is discretized in $m + 1$ points. Suppose that $J = \{e^{it}, t \in [-\theta, \theta]\}$, for some $\theta \in [0, \pi]$. Call J_m the discrete version of the arc J defined by

$$J_m = \{e^{it}, t \in \{-\theta + \frac{2k\theta}{m}, k \in \{0 \ldots m\}\}$$

we define following auxiliary extremal problem:

DBEP(n,m)
For $f \in L^2(I)$, find $k_{n,m} \in P_n$ such that $\forall t \in J_m |k_{n,m}(t)| \leq 1$ and

$$\|f - k_{n,m}\|_{L^2(I)} = \min_{\substack{g \in P_n \\ |g| \leq 1 \text{ a.e. on } J_m}} \|f - g\|_{L^2(I)} \,. \tag{38}$$

For the discretized problem **DBEP(n,m)**, the following holds.

Theorem 5 *For* $\lambda = (\lambda_0, \ldots, \lambda_m) \in \mathbb{R}^{m+1}$ *and* $g \in P_n$ *define the Lagrangian*

$$L(\lambda, g) = \|f - g\|_{L^2(I)} + \sum_{k=0}^{m} \lambda_k (|g(e^{i(-\theta + \frac{2k\theta}{m})}|^2 - 1),$$

then

- *Problem* **DBEP(n,m)** *has a unique solution* $k_{n,m}$,
- $k_{n,m}$ *is also the unique solution of the concave maximisation problem:*

$$\text{to find } g_{opt} \text{ and } \lambda_{opt} \text{ solving for} \quad \max_{\lambda \geq 0} \min_{g \in P_n} L(\lambda, g), \tag{39}$$

  where $\lambda \geq 0$ *means that each component of* λ *is non negative.*
- *For a fixed n,* $\lim_{m\to\infty} k_{n,m} = k_n$ *in* P_n.

The proof of Theorem 5 follows from standard convex optimization theory, using in addition that the *sup*-norm of the derivative of a polynomial of degree n on $\mathbb{T}$ is controlled by the values it assumes at a set of $n + 1$ points. This depends on Bernstein's inequality and on the argument using Lagrange interpolation polynomials used in the Remark after Theorem 1.

In the minmax problem (39) , the minimization is a quadratic convex problem. It can be tackled efficiently by solving the critical point equation which is a linear system of equations similar to (19). Eventually, an explicit expression of the gradient and of the hessian of the concave maximization problem (39) allows us for a fast converging computational procedure to estimate $k_{n,m}$.

Figure 1 represents a solution to problem **DBEP(n,m)**, where f is obtained from partial measurement of the scattering reflexion parameter of a wave-guide microwave filter by the CNES (French Space Agency). The problem is solved for $n = 400$ and $m = 800$, while the constraint on J has been renormalized to 0.96 (instead of 1). The modulus of $k_{400,800}$ is plotted as a blue continuous line while the measurements $|f|$ appear as red dots. As the reader can see, the fit is extremely good.

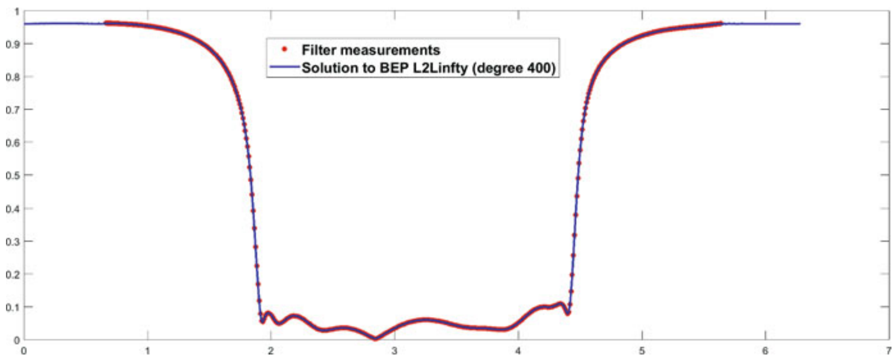

Fig. 1 Solution of DBEP at hand of partial scattering measurements of a microwave filter

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Extremal Bounds of Teichmüller-Wittich-Belinskiĭ Type for Planar Quasiregular Mappings

Anatoly Golberg

Abstract The theorems of TWB (Teichmüller-Wittich-Belinskiĭ) type imply the local conformality (or weaker properties) of quasiconformal mappings at a prescribed point under assumptions of the finiteness of appropriate integral averages of the quantity $K_\mu(z) - 1$, where $K_\mu(z)$ stands for the real dilatation coefficient. We establish the extremal bounds for distortions of the moduli of annuli in terms of integrals in TWB theorems under quasiconformal and quasiregular mappings and illustrate their sharpness by several examples. Some local conditions weaker than the conformality are also discussed.

Keywords Module of families of curves • Moduli of annuli • Local conformality • Local weak conformality • Quasiconformal and quasiregular mappings • Extremal bounds

Msc codes: Primary 30C62, 30C75; Secondary 30C65

1 Introduction

The notions of quasiconformality and quasiregularity in a domain are natural extensions of the notion of conformality. The homeomorphic mapping $f(z) = z(\sqrt{|z|} + 1)$ provides a simple example of quasiconformality at any domain, and this mapping is conformal at the origin. The interest to a question whether global quasiconformality or its generalizations can guarantee for a mapping to be conformal at a prescribed point has been raised about 80 years ago starting from the papers by Menshoff [28] and Teichmüller [34].

There exist several equivalent definitions of quasiconformal and quasiregular mappings. Each of them involves certain tools. Among the most powerful methods for studying geometric features of quasiconformal and of quasiregular mappings is

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the method of extremal lengths (moduli) which goes back to the classical work of Ahlfors-Beurling. The goal of our paper is to present new inequalities for moduli of the families of curves (paths) involving the integrals containing measurable functions and arbitrary admissible metrics. We simultaneously use the families of joining and separating curves and establish extremal bounds for the moduli of annuli. The main new results are Theorems 7 and 9, and the inequalities (12) and (13) which lead to improvements of an estimate of Belinskiĭ (Theorem 4). The proofs of these inequalities and Theorem 9 related to nonhomeomorphic case are here sketched. An additional purpose of the paper is to discuss the local conditions that are weaker than conformal and present a wide spector of illustrating examples. All such results can be regarded as Teichmüller-Wittich-Belinskiĭ type theorems.

1.1 Quasiconformality in the Plane

The analytic definition of quasiconformality implies that if f is K-quasiconformal in a domain $G \subset \mathbb{C}$, then it has L^2-derivatives, $|f_z(z)| > 0$ almost everywhere in G and therefore the complex dilatation $\mu_f(z) = f_{\bar{z}}(z)/f_z(z)$ is a well-defined measurable function in G. The definition of the complex dilatation naturally leads us to the first order linear elliptic partial differential equation $f_{\bar{z}}(z) = \mu(z) f_z(z)$; where $\mu(z)$ is a measurable function satisfying $||\mu||_\infty = \operatorname{ess\,sup}|\mu(z)| \leq k < 1$. This equation, known as the Beltrami equation, has a rich history; see e.g. [21], [22].

The quantity

$$K_\mu(z) = \frac{1 + |\mu|}{1 - |\mu|}$$

is called the real characteristic of quasiconformality (Lavrentiev's characteristic) or the maximal dilatation of the mapping f.

1.2 Quasiregularity in the Plane

Quasiregular mappings lie in the core of Geometry and Analysis and have strong connections to certain differential equations; see, e.g. [4, 10]. Stoilow's decomposition theorem says that any quasiregular (nonhomeomorphic) mapping in the plane is a composition $f = g \circ h$, where g is the analytic function and h is quasiconformal. Thus, any analytic function in $\mathbb{C}$ is simply a two dimensional quasiregular mapping whose dilatation coefficient equals 1. Branching of the covering mappings involves the existence of sets on which the mappings fail to be locally homeomorphic, and this causes topological complication and a strong difference between quasiregularity and quasiconfomality.

2 Theorems of Teichmüller-Wittich-Belinskiĭ Type

The theorems of TWB (Teichmüller-Wittich-Belinskiĭ) type play an important role in various questions of Complex Analysis. They provide the conformality of mappings from the integral closeness of their real dilatation of quasiconformality to 1.

2.1 Teichmüller-Wittich-Belinskiĭ Theorem

The first result relates to Teichmüller [34], who established

Theorem 1 *Let* $w = f(z)$ *be a quasiconformal mapping such that* $K_\mu(z) \leq c(|z|)$ *and*

$$\int_{|z|>r_0} \frac{c(|z|) - 1}{|z|^2} dxdy \; < \; \infty$$

Then there exists a number $0 < A < \infty$ *such that*

$$|f(z)| = A|z|(1 + \varepsilon(|z|)), \quad \textit{with} \quad \lim_{|z|\to\infty} \varepsilon(|z|) = 0.$$

In his proof Teichmüller used a notion of the module set introduced by himself. Ten years later, using quite different methods, namely Ahlfors' method of differential inequalities, Wittich [37] published a slightly weaker form of Theorem 1.

Theorem 2 *Let* $w = f(z)$ *be a quasiconformal mapping and*

$$\int_{|z|>r_0} \frac{K_\mu(z) - 1}{|z|^2} dxdy \; < \; \infty.$$

Then there exists a number $0 < A < \infty$ *such that*

$$|f(z)| = A|z|(1 + \varepsilon(|z|)), \quad \textit{with} \quad \lim_{|z|\to\infty} \varepsilon(|z|) = 0.$$

The existence of $\lim(|f(z)|/|z|)$ provides a necessary condition of conformality for the mapping $f(z)$. It is called asymptotic dilation of f. Six years after Wittich's paper, Belinskiĭ [6] established that the finiteness of the integrals in the TW (Teichmüller-Wittich) theorems implies the conformality of mappings.

Theorem 3 *Let* $w = f(z)$ *be a quasiconformal mapping of a punctured disc* $0 < |z| \leq 1$ *such that*

$$\int\limits_{0<|z|\leq 1} \frac{K_\mu(z)-1}{|z|^2} dxdy = A < \infty.$$

Then there exist both

$$\lim_{z\to 0} w = w_0, \quad \textit{and} \quad \lim_{z\to 0} \frac{w-w_0}{z} \neq 0, \infty.$$

Reich and Walczak [30] extended TW theorems using the directional dilatations in two appropriate directions. Let us also mention the results of the TWB theorems by Lehto [24], Martio and Gutlyanskiĭ [17], Brakalova and Jenkins [9] and others.

2.2 Grötzsch and Belinskiĭ's Estimates

The main ingredient in Belinskiĭ's result on conformality is the following statement; see e.g. [7].

Theorem 4 *Let an annulus* $\{r \leq |z| \leq 1\}$ $(0 < r < 1)$ *be mapped quasiconformally on another annulus* $\{\rho \leq |w| \leq 1\}$ $(0 < \rho < 1)$. *Then the following double inequality*

$$-\frac{1}{2\pi} \iint\limits_{r\leq|z|\leq 1} \frac{K_\mu(z)-1}{|z|^2} dxdy \leq \log\frac{\rho}{r} \leq \frac{\log\rho}{2\pi\log r} \iint\limits_{r\leq|z|\leq 1} \frac{K_\mu(z)-1}{|z|^2} dxdy \tag{1}$$

holds.

This estimate generalizes an inequality of Grötzsch [15], which holds for K-quasiconformal mappings ($K_\mu(z) \leq K < \infty$) and states the following sharp estimate for bounded dilatations,

$$r^K \leq \rho \leq r^{\frac{1}{K}}. \tag{2}$$

In both estimates (1) and (2), the bounds are attained by the radial stretchings $f(z) = z|z|^{K-1}$ and $f(z) = z|z|^{1/K-1}$, $K > 1$. For refined versions of such inequalities we refer to the monographs [23, 25].

A point is that the quantity $\log \rho/r$ measures the distortion of module of annuli in both image and inverse image under K-quasiconformal mappings that plays a crucial role in studying various properties of mappings. One of fruitful applications relates to Picard type theorems and the value distribution theory as well; see [11, 32]. In order to get sharper estimates we have to consider much flexible tools and establish extremal bounds in estimates of a Grötzsch-Belinskiĭ type. For recent results concerning the extremal estimates of such a type we refer to [5, 13].

3 Directional Dilatations and Corresponding Estimates for Moduli

Let us first mention that various directional dilatations for the planar mappings have been exploited by Andreian Cazacu [2], Reich and Walczak [30], Gutlyanskiĭ et al. [18], Ryazanov et al. [33]. We recall these quantities following the authors of the indicated papers. Consider, for example, the **tangential** and **radial dilatations** provided, respectively, by

$$K^T_\mu(z,z_0) = \frac{\left|1 - \frac{\overline{z-z_0}}{z-z_0}\mu(z)\right|^2}{1-|\mu(z)|^2}, \tag{3}$$

and

$$K^r_\mu(z,z_0) = \frac{1-|\mu(z)|^2}{\left|1 + \frac{\overline{z-z_0}}{z-z_0}\mu(z)\right|^2}.$$

Here z_0 is an arbitrary point of $\mathbb{C}$, while z runs over a given domain.

The idea to use the complex dilatation μ instead of $|\mu|$ goes back to Andreian Cazacu [1]. She introduced the following quantity

$$d(z) = \frac{|\partial_\tau f(z)|^2}{J_f(z)}, \tag{4}$$

where $\partial_\tau f(z)$ denotes the derivative of f in the tangential direction τ at a regular point z, i.e. f is differentiable and its Jacobian $J_f(z)$ is positive at z (see [2]).

The quantity d is represented by the Lavrentiev characteristics K_μ and α at z as follows:

$$d(z) = \frac{\cos^2\alpha(z)}{K_\mu(z)} + K_\mu(z)\sin^2\alpha(z).$$

Recall that $K_\mu(z)$ is equal to the ratio of the greatest to the smallest semi-axes of the characteristic ellipse centered at z, and $\alpha(z)$ denotes the angle between its greatest axis and the direction τ.

The function d is defined almost everywhere (a.e.) in G and is measurable. Obviously,

$$\frac{1}{K_\mu(z)} \le d(z) \le K_\mu(z)$$

a.e. in G, and the both bounds are attained.

The quantity

$$D_{\mu,z_0}(z) = K_\mu^T(z, z_0)$$

where K_μ^T is defined by (3), was regarded in Gutlyanskiĭ-Martio-Sugawa-Vuorinen [18] as a dilatation of μ at z with respect to $z_0 \in \hat{\mathbb{C}}$.

This dilatation $D_{\mu,z_0}(z)$ is called the **angular dilatation** and arises from the following relation: if $f \in W_{loc}^{1,1}$ satisfies the Beltrami equation and $z = z_0 + re^{i\theta}$, then for almost all z we have the equality

$$\left|\frac{\partial f}{\partial \theta}(z)\right|^2 = r^2 D_{\mu,z_0}(z) J_f(z).$$

In the view of the relation

$$\left|\frac{\partial f}{\partial r}(z)\right|^2 = D_{-\mu,z_0}(z) J_f(z),$$

the quantity $D_{-\mu,z_0}$ is called the **radial dilatation** of μ at z_0.

Both quantities $D_{\mu,z_0}(z)$ and $D_{-\mu,z_0}(z)$ are called the **directional dilatations**. These dilatations coincide with the dilatation (4) for $\tau = \theta + \pi/2$ and $\tau = \theta$, respectively, $z - z_0 = r \exp i\theta$. As was mentioned above, the relation (3) yields a representation of the directional dilatation D_{μ,z_0} in terms of the Beltrami coefficient μ.

Note that $D_{\mu,z_0}(z)$ is also a measurable function in G and satisfies a.e. for each point $z_0 \in \hat{\mathbb{C}}$ the inequalities

$$\frac{1}{K_\mu(z)} \le D_{\mu,z_0}(z) \le K_\mu(z).$$

It is proved in [18] that $D_{\mu,0}(z) = K_\mu(z)$ holds a.e. if and only if $\mu(z)$ has the form $-\rho(z)z/\bar{z}$, where ρ is a non-negative measurable function.

Using these dilatations, the authors of [18] proved the existence theorem in the case, when the usual dilatation K_μ fails to satisfy the known integrability conditions, and describe sufficient conditions which ensure Hölder continuity of f.

Another new dilatation was introduced in [33]. Let a mapping $f: G \to \mathbb{C}$ be differentiable at z, and $J_f(z) \neq 0$. Given $\omega \in \mathbb{C}$, with $|\omega| = 1$, the **derivative in the direction** ω of the mapping f at the point z is defined by

$$\partial_\omega f(z) = \lim_{t \to 0+} \frac{f(z + t\omega) - f(z)}{t}.$$

Accordingly, the **radial direction** at a point $z \in G$ with respect to the center $z_0 \in \mathbb{C}$, $z_0 \neq z$, is

$$\omega_0 = \omega_0(z, z_0) = \frac{z - z_0}{|z - z_0|}.$$

The **radial and tangential dilatations** of f at z with respect to z_0 are defined by

$$K^r(z, z_0, f) = \frac{|J_f(z)|}{|\partial_r^{z_0} f(z)|^2}, \qquad K^T(z, z_0, f) = \frac{|\partial_\tau^{z_0} f(z)|^2}{|J_f(z)|},$$

respectively; here $\partial_r^{z_0} f(z)$ and $\partial_\tau^{z_0} f(z)$ denote the derivatives of f at z in the directions ω_0 and in $\tau = i\omega_0$, respectively.

The **big radial dilatation** of f at z with respect to z_0 is defined by

$$K^R(z, z_0, f) = \frac{|J_f(z)|}{|\partial_R^{z_0} f(z)|^2},$$

where

$$|\partial_R^{z_0} f(z)| = \min_{|\omega|=1} \frac{|\partial_\omega f(z)|}{|\Re(\omega \overline{\omega_0})|}.$$

It is not hard to verify that

$$K^r(z, z_0, f) \leq K^R(z, z_0, f) \leq K_\mu(z).$$

The following result of [33] describes the relations between the directional dilatations.

Theorem 5 *Let z be a regular point of a homeomorphism $f : G \to \mathbb{C}$ with complex dilatation $\mu(z)$ such that $|\mu(z)| < 1$. Then*

$$K^r(z, z_0, f) = K_\mu^r(z, z_0), \quad K^T(z, z_0, f) = K_\mu^T(z, z_0) = K^R(z, z_0, f).$$

If, in addition, f and f^{-1} belong to $W_{\text{loc}}^{1,2}$, then f is a Q-homeomorphism at every point $z_0 \in G$ with $Q(z) = K_\mu^T(z, z_0)$.

For the main features of Q-homeomorphisms, we refer to the book [27].

Now consider the multidimensional case. The classical dilatations of quasiconformality (inner and outer) at a regular point $x \in \mathbb{R}^n$ are defined by

$$L_f(x) = \frac{|J_f(x)|}{l^n(f'(x))}, \quad K_f(x) = \frac{||f'(x)||^n}{|J_f(x)|}.$$

Here

$$||f'(x)|| = \max_{|h|=1} |f'(x)h|, \quad l(f'(x)) = \min_{|h|=1} |f'(x)h|, \quad J_f(x) = \det f'(x).$$

Two directional characteristics in $\mathbb{R}^n$, using the derivative of f in a direction $h, h \neq 0$, at x, given by

$$\partial_h f(x) = \lim_{t \to 0+} \frac{f(x+th) - f(x)}{t},$$

have been introduced in [16] and [12].

For a point $x_0 \in \mathbb{R}^n$, the **angular dilatation** of the mapping f at the point $x \neq x_0$ **with respect to** x_0 is defined by

$$D_f(x, x_0) = \frac{|J_f(x)|}{\ell_f^n(x, x_0)}$$

respectively, the **normal dilatation**

$$T_f(x, x_0) = \left(\frac{\mathscr{L}_f^n(x, x_0)}{|J_f(x)|} \right)^{\frac{1}{n-1}}.$$

Here

$$\ell_f(x, x_0) = \min_{|h|=1} \frac{|\partial_h f(x)|}{|\langle h, u \rangle|}, \qquad \mathscr{L}_f(x, x_0) = \max_{|h|=1} \big(|\partial_h f(x)| |\langle h, u \rangle| \big),$$

and $u = (x - x_0)/|x - x_0|$. The dilatations $D_f(x, x_0)$ and $T_f(x, x_0)$ both are measurable in a domain $G \subset \mathbb{R}^n$, and one concludes from the relations

$$l(f'(x)) \leq \ell_f(x, x_0) \leq |\partial_u f(x)| \leq \mathscr{L}_f(x, x_0) \leq ||f'(x)||,$$

(which are true for each x_0) that

$$K_f^{-1}(x) \leq D_f(x, x_0) \leq L_f(x).$$

The normal dilatation $T_f(x, x_0)$ has the same bounds as $D_f(x, x_0)$, since

$$K_f^{-1}(x) \leq L_f^{-\frac{1}{n-1}}(x) \leq T_f(x, x_0) \leq K_f^{\frac{1}{n-1}}(x) \leq L_f(x).$$

In the case of mapping of planar domains,

$$K_\mu^{-1}(z) \leq D_f(z, z_0) \leq K_\mu(z), \quad K_\mu^{-1}(z) \leq T_f(z, z_0) \leq K_\mu(z).$$

Note that both angular and normal dilatations range between 0 and ∞, while the classical dilatations (including the multidimensional case) are always greater than or equal to 1.

In the case when $z_0 = 0$, we will write $D_f(z)$ and $T_f(z)$ instead of $D_f(z, 0)$ and $T_f(z, 0)$.

Let $A = \{z \in \mathbb{C} : a < |z| < b\}$ be an annulus centered at the origin. Denote by Γ_A and by Σ_A the families of all curves which join and separate the boundary of A in A, respectively. In addition to a standard admissibility condition for a family of curves Γ ($\rho \in \operatorname{adm}\Gamma$) defining for the conformal module, we also use the **extended admissibility** ($\rho \in \operatorname{extadm}\Gamma$) which means that ρ is admissible except for a subfamily $\tilde{\Gamma}$ whose conformal module vanishes ($\mathcal{M}(\tilde{\Gamma}) = 0$).

The following result formulated now for the planar case can be found in [14].

Theorem 6 *Let $f : D \to \mathbb{C}$ be a quasiconformal mapping. Suppose that the directional dilatations $D_f(z)$ and $T_f(z)$ are locally integrable in the annulus $A \subset G$. Then the following double inequalities*

$$\inf_{\rho \in extadm\, \Gamma_A} \int_A T_f^{-1}(z)\rho^2(|z|)\, dxdy \le \mathcal{M}(f(\Gamma_A)) \le \inf_{\rho \in adm\, \Gamma_A} \int_A D_f(z)\varrho^2(|z|)\, dxdy\,, \tag{5}$$

$$\inf_{\rho \in extadm\, \Sigma_A} \int_A D_f^{-1}(z)\rho^2(z/|z|)\, dxdy \le \mathcal{M}(f(\Sigma_A)) \le \inf_{\rho \in adm\, \Sigma_A} \int_A T_f(z)\varrho^2(z/|z|)\, dxdy\,, \tag{6}$$

hold.

The following statement can be regarded as a Grötzsch-Belinskiĭ type estimate.

Theorem 7 *Let $f : D \to \mathbb{C}$ be a quasiconformal mapping. Suppose that the directional dilatations $D_f(z)$ and $T_f(z)$ are locally integrable in the annulus $A \subset G$. Then*

$$-\frac{1}{2\pi}\int_A \frac{T_f(z) - 1}{|z|^2}\, dxdy \le \operatorname{mod} A - \operatorname{mod} f(A) \le \frac{\log\frac{b}{a}}{\int_A \frac{D_f(z)}{|z|^2}\, dxdy}\int_A \frac{D_f(z) - 1}{|z|^2}\, dxdy\,. \tag{7}$$

Proof The functions

$$\rho_1(z) = \frac{1}{|z|\log\frac{b}{a}} \quad \text{and} \quad \rho_2(z) = \frac{1}{2\pi|z|}$$

are admissible for the families Γ_A and Σ_A, respectively. Substituting these functions into the right-hand sides of (5) and (6) using the relation

$$\operatorname{mod} f(A) = \frac{2\pi}{\mathscr{M}(f(\Gamma_A))} = 2\pi \mathscr{M}(f(\Sigma_A)),$$

one derives

$$\frac{2\pi \log \frac{b}{a}}{\int\limits_A \frac{D_f(z)}{|z|^2}\, dxdy} \le \operatorname{mod} f(A) \le \frac{1}{2\pi} \int\limits_A \frac{T_f(z)}{|z|^2}\, dxdy\,.$$

Now taking into account that $\operatorname{mod} A = \log \frac{b}{a}$, we arrive at (7). □

4 Extremal Bounds for Moduli Under Homeomorphic Mappings

The following double inequality provides a slightly different variant of the inequality (7) whose right-hand side can be obtained similarly to the inequality (2.15) in [8],

$$-\frac{1}{2\pi} \iint\limits_{r\le|z|\le 1} \frac{T_f(z)-1}{|z|^2} dxdy \le \log\frac{\rho}{r} \le \frac{\log\rho}{2\pi\log r} \iint\limits_{r\le|z|\le 1} \frac{D_f(z)-1}{|z|^2} dxdy. \tag{8}$$

It also provides an improvement of (1) that will be illustrated by several examples. It is interesting to compare with the inequality which involves the dilatations $D_\mu(z)$ and $D_{-\mu}(z)$

$$\begin{aligned} -\frac{1}{2\pi} \iint\limits_{r\le|z|\le 1} \frac{D_{-\mu}(z)-1}{|z|^2} dxdy &\le \operatorname{mod} A - \operatorname{mod} f(A) \\ &\le \frac{\operatorname{mod} f(A)}{\operatorname{mod} A} \cdot \frac{1}{2\pi} \iint\limits_{r\le|z|\le 1} \frac{D_\mu(z)-1}{|z|^2} dxdy; \end{aligned} \tag{9}$$

see [20].

Two more estimates

$$-\frac{1}{2\pi} \iint\limits_{r\le|z|\le 1} \frac{T_f(z)-1}{|z|^2} dxdy \le \log\frac{\rho}{r} \le \frac{1}{2\pi} \iint\limits_{r\le|z|\le 1} \frac{D_f(z)-1}{D_f(z)|z|^2} dxdy, \tag{10}$$

and

$$-\frac{\log\frac{1}{r}}{\iint\limits_{r\le|z|\le 1}\frac{dxdy}{T_f(z)|z|^2}}\iint\limits_{r\le|z|\le 1}\frac{T_f(z)-1}{T_f(z)|z|^2}dxdy\le\log\frac{\rho}{r}$$

$$\le\frac{\log\frac{1}{r}}{\iint\limits_{r\le|z|\le 1}\frac{D_f(z)dxdy}{|z|^2}}\iint\limits_{r\le|z|\le 1}\frac{D_f(z)-1}{|z|^2}dxdy \tag{11}$$

essentially refine Belinskiĭ's estimate (1). All these inequalities are obtained from the right inequalities (5) and (6) choosing suitable admissible functions. But the sharpness of these estimates remains still open. Moreover, in the left inequalities (5) and (6) the infima cannot be removed by substitution of arbitrary admissible metrics. So, we only must use the functions on which these infima are attained.

To this end, we apply the results of [27]. Namely, for the families of curves that join the boundary components of the annulus A, one obtains

$$\frac{2\pi}{\int\limits_a^b\frac{d\rho}{\rho\int\limits_0^{2\pi}T_f^{-1}(z)d\theta}}\le\mathscr{M}(f(\Gamma_A))\le\frac{2\pi}{\int\limits_a^b\frac{d\rho}{\rho\int\limits_0^{2\pi}D_f(z)d\theta}}.$$

This estimate is based on the assertions of Lemma 7.4 in [27] for the integrals which are involved in (5).

Next, we apply the relationship between the conformal moduli of Γ_A and Σ_A, and obtain

$$\frac{1}{2\pi}\int\limits_a^b\frac{d\rho}{\rho\int\limits_0^{2\pi}D_f(z)d\theta}\le\mathscr{M}(f(\Sigma_A))\le\frac{1}{2\pi}\int\limits_a^b\frac{d\rho}{\rho\int\limits_0^{2\pi}T_f^{-1}(z)d\theta}.$$

Thus,

$$\int\limits_a^b\frac{d\rho}{\rho\int\limits_0^{2\pi}D_f(z)d\theta}\le\operatorname{mod}f(A)\le\int\limits_a^b\frac{d\rho}{\rho\int\limits_0^{2\pi}T_f^{-1}(z)d\theta},$$

which finally yields

$$\int\limits_a^b\frac{\frac{1}{2\pi}\int\limits_0^{2\pi}T_f^{-1}(z)d\theta-1}{\frac{1}{2\pi}\int\limits_0^{2\pi}T_f^{-1}(z)d\theta}\frac{d\rho}{\rho}\le\operatorname{mod}A-\operatorname{mod}f(A)\le\int\limits_a^b\frac{\frac{1}{2\pi}\int\limits_0^{2\pi}D_f(z)d\theta-1}{\frac{1}{2\pi}\int\limits_0^{2\pi}D_f(z)d\theta}\frac{d\rho}{\rho}. \tag{12}$$

For the family of separating curves, we apply Theorem 2 from [27] and rewrite the inequality (6) by

$$\int_0^{2\pi}\int_a^b \frac{D_f(z)\,d\rho d\theta}{\rho\left(\int\limits_0^{2\pi} D_f(z)d\theta\right)^2} \le \mathscr{M}(f(\Sigma_A)) \le \int_0^{2\pi}\int_a^b \frac{d\rho d\theta}{\rho T_f(z)\left(\int\limits_0^{2\pi} T_f^{-1}(z)d\theta\right)^2},$$

or in the terms of moduli,

$$2\pi\int_0^{2\pi}\int_a^b \frac{D_f(z)\,d\rho d\theta}{\rho\left(\int\limits_0^{2\pi} D_f(z)d\theta\right)^2} \le \operatorname{mod} f(A) \le 2\pi\int_0^{2\pi}\int_a^b \frac{d\rho d\theta}{\rho T_f(z)\left(\int\limits_0^{2\pi} T_f^{-1}(z)d\theta\right)^2}.$$

Finally, in the terms of Belinskiĭ's estimate, it takes the form

$$\frac{1}{2\pi}\int_0^{2\pi}\int_a^b \frac{T_f(z)\left(\frac{1}{2\pi}\int\limits_0^{2\pi} T_f^{-1}(z)d\theta\right)^2 - 1}{T_f(z)\left(\frac{1}{2\pi}\int\limits_0^{2\pi} T_f^{-1}(z)d\theta\right)^2}\frac{d\rho d\theta}{\rho}$$
$$\le \operatorname{mod} A - \operatorname{mod} f(A) \le \frac{1}{2\pi}\int_0^{2\pi}\int_a^b \frac{\left(\frac{1}{2\pi}\int\limits_0^{2\pi} D_f(z)d\theta\right)^2 - D_f(z)}{\left(\frac{1}{2\pi}\int\limits_0^{2\pi} D_f(z)d\theta\right)^2}\frac{d\rho d\theta}{\rho}. \tag{13}$$

5 Local Weak Conformality Conditions

In this section, we discuss several local conditions that are weaker than the conformality at a point. Let us assume (for convenience) the following normalization $f(0) = 0$.

5.1 Conformality

The **conformality** at the origin is characterized by existence of the limit

$$\lim_{z\to 0}\frac{f(z)}{z} = C \ne 0, \infty.$$

5.2 Asymptotic Homogeneity

A mapping f is called **asymptotically homogeneous** at the origin [19] if

$$f(\zeta z) \sim \zeta f(z), \quad \forall \zeta \in \mathbb{C}, \quad z \to 0.$$

5.3 Asymptotic Dilation

The quantity

$$\lim_{|z|\to 0} \frac{|f(z)|}{|z|} = |C|, \quad C \neq 0, \infty$$

is called **asymptotic dilation at the origin**.

5.4 Circle-Like

A mapping is called **circle-like** at the origin if

$$\lim_{r\to 0} \frac{\max\limits_{|z|=r} |f(z)|}{\min\limits_{|z|=r} |f(z)|} = 1.$$

This notion was introduced by Teichmüller [34]; see also [28]. Due to Menshoff's paper [28], such a mapping is said to **preserve infinitesimal circles** at the origin.

5.5 Weak Conformality

A mapping **preserves angles on circles** at the point $z = 0$ if for an appropriate choice of a branch of the argument

$$\arg f(re^{i\theta_2}) - \arg f(re^{i\theta_1}) - (\theta_2 - \theta_1) \to 0 \quad \text{as} \quad r \to 0,$$

uniformly in θ_1, θ_2. Following [17], a homeomorphism $f : B \to B$, which is both circle-like and preserves angles on circles at $z = 0$, is called **weakly conformal** at 0.

5.6 Local Quasiconformality

A mapping is H-**locally quasiconformal** ($H \geq 1$) at the origin if

$$\limsup_{r\to 0} \frac{\max\limits_{|z|=r} |f(z)|}{\min\limits_{|z|=r} |f(z)|} \leq H.$$

The classes of mappings obeying the conditions mentioned in the corresponding Sect. 5.i by $\mathscr{F}_i$. Thus the narrowest class, i.e. of mappings conformal at the origin, is denoted by $\mathscr{F}_1$, whereas the widest one (locally quasiconformal at 0) corresponds to $\mathscr{F}_6$. The following statements describe the relations between the classes $\mathscr{F}_i$.

Theorem 8 *There are the following relations between the classes* $\mathscr{F}_i$*:*

(1) $\mathscr{F}_1 \subset \mathscr{F}_3 \subset \mathscr{F}_4 \subset \mathscr{F}_6$;
(2) $\mathscr{F}_1 \subset \mathscr{F}_2 \subset \mathscr{F}_5 \subset \mathscr{F}_6$.

Proof Almost all relations given in the theorem are trivial. Let us show that every mapping of class $\mathscr{F}_2$ asymptotically preserves angles on circles and is circle-like, and therefore belongs to $\mathscr{F}_5$. Indeed, given two points z_1, z_2, such that $|z_1| = |z_2| = r$, or equivalently $z_1 = re^{i\theta_1}, z_2 = re^{i\theta_2}$, one can write by the definition of the asymptotic homogeneity

$$f\left(re^{i\theta_2}\right) \sim e^{i(\theta_2-\theta_1)} f\left(re^{i\theta_1}\right) \quad \text{for} \quad r \to 0.$$

This implies $\arg f\left(re^{i\theta_2}\right) - \arg f\left(re^{i\theta_1}\right) - (\theta_2 - \theta_1) \to 0$ as $r \to 0$.

To show the second implication, we note by the continuity of f that both the minimum and maximum of $|f(z)|$ are attained on $|z| = r$ at say z_1 and z_2, respectively. Then

$$\max_{|z|=r} |f(z)| = \left|f\left(re^{i\theta_2}\right)\right| = \left|f\left(re^{i\theta_1} e^{i(\theta_2-\theta_1)}\right)\right| \sim \left|f\left(re^{i\theta_1}\right)\right| = \min_{|z|=r} |f(z)|.$$

□

6 Examples

The purpose of this section is both to provide a wide range of examples of quasiconformal and quasiregular mappings which satisfy the weakened conformality conditions described in Sect. 5 and to illustrate the sharpness of estimates given in Sect. 4.

6.1 Example

The radial stretching

$$f(z) = z|z|$$

can be rewritten in the form $f(z) = z^{\frac{3}{2}}\bar{z}^{\frac{1}{2}}$, and therefore,

$$f_z = \frac{3}{2}z^{\frac{1}{2}}\bar{z}^{\frac{1}{2}}, \quad f_{\bar{z}} = \frac{1}{2}z^{\frac{3}{2}}\bar{z}^{-\frac{1}{2}}, \quad \mu(z) = \frac{z}{3\bar{z}}.$$

Further,

$$|\partial f_h(z)| = |f_z h + f_{\bar{z}}\bar{h}| = |f_z||1 + \mu(z)e^{-2i\theta}| = \frac{3}{2}|z|\left|1 + \frac{1}{3}e^{-2i(\theta-\tau)}\right|,$$

$$|\partial f_h(z)| \leq 2|z| = ||f'(z)||, \quad |\partial f_h(z)| \geq |z| = l(f'(z)), \quad J_f(z) = 2|z|^2,$$

and

$$\begin{aligned}\ell_f(z,0) &= \min_\theta \frac{|\partial f_h(z)|}{|\cos\theta|} = \min_\theta \frac{\frac{3}{2}\left|1 + \frac{1}{3}e^{-2i\theta}\right||z|}{|\cos\theta|} = 3|z| \min_\theta \frac{\left|1 + \frac{1}{3}e^{-2i\theta}\right|}{|1 + e^{-2i\theta}|} \\ &= |z| \min_\theta \frac{|3 + e^{-2i\theta}|}{|1 + e^{-2i\theta}|} = |z|\langle 2, 1\rangle = 2|z|.\end{aligned}$$

In a similar way, we obtain

$$\begin{aligned}\mathscr{L}_f(z,0) &= \max_\theta |\partial f_h(z)||\cos\theta| = \max_\theta \frac{3}{2}|z|\left|1 + \frac{1}{3}e^{-2i\theta}\right||\cos\theta| \\ &= \max_\theta \frac{3}{4}|z|\left|1 + \frac{1}{3}e^{-2i\theta}\right||1 + e^{-2i\theta}| = \frac{3}{4}|z|2\big(1 + \frac{1}{3}\big) = 2|z|.\end{aligned}$$

By (3),

$$\begin{aligned}D_\mu(z) &= \frac{\left|1 - \mu(z)\frac{\bar{z}}{z}\right|^2}{1 - |\mu(z)|^2} = \frac{\left|1 - \frac{1}{3}\right|^2}{1 - \frac{1}{9}} = \frac{1}{2}, \quad D_{-\mu}(z) = \frac{\left|1 + \mu(z)\frac{\bar{z}}{z}\right|^2}{1 - |\mu(z)|^2} \\ &= \frac{\left|1 + \frac{1}{3}\right|^2}{1 - \frac{1}{9}} = 2.\end{aligned}$$

Thus,

$$K_\mu(z) = 2, \quad D_f(z,0) = \frac{1}{2}, \quad T_f(z,0) = 2, \quad D_\mu(z) = \frac{1}{2}, \quad D_{-\mu}(z) = 2.$$

Note that this mapping carries out the annulus with radii r and 1 onto annulus with radii r^2 and 1. Thus, the Belinskiĭ double inequality (1) becomes

$$-\log\frac{1}{r}\le \log r\le 2\log\frac{1}{r},$$

i.e. only the left bound is attained, although all the estimates (8)–(11), which involve the directional dilatations, became equalities

$$-\log\frac{1}{r}\le \log r\le -\log\frac{1}{r},\quad \text{or}\quad -\log\frac{b}{a}\le -\log\frac{b}{a}\le -\log\frac{b}{a}.$$

The extremal estimates (12) and (13) are both sharp, since the left and right bounds coincide

$$-\log\frac{b}{a}=-\log\frac{b}{a}=-\log\frac{b}{a}.$$

Using the definitions of Sect. 5, one concludes that this mapping preserves angles on circles, is circle-like and weakly conformal at the origin, and therefore belongs to $\mathscr{F}_6$. But it fails to be conformal and asymptotic homogeneous at 0 as well.

6.2 Example

The homeomorphism

$$f(z)=ze^{-i\log|z|}$$

can be regarded as a "quick rotation" around the origin. For this mapping

$$f(z)=ze^{-\frac{1}{2}i(\log z+\log\bar z)},\ f_z=\big(1-\frac{i}{2}\big)e^{-i\log|z|},\ f_{\bar z}=-\frac{iz}{2\bar z}e^{-i\log|z|},\ \mu(z)=\frac{1}{1+2i}\frac{z}{\bar z},$$

$$|\partial f_h(z)|=|f_z h+f_{\bar z}\bar h|=|f_z||1+\mu(z)e^{-2i\theta}|=\left|1-\frac{i}{2}\right|\left|1+\frac{1}{1+2i}e^{-2i(\theta-\tau)}\right|,$$

and the stretchings and Jacobian are equal to

$$|\partial f_h(z)|\le\frac{\sqrt5+1}{2}=||f'(z)||,\quad |\partial f_h(z)|\ge\frac{\sqrt5-1}{2}=l(f'(z)),\quad J_f(z)=1.$$

Next we get

$$D_\mu(z) = \frac{\left|1-\mu(z)\frac{\bar z}{z}\right|^2}{1-|\mu(z)|^2} = \frac{\left|1-\frac{1}{1+2i}\right|^2}{1-\left|\frac{1}{1+2i}\right|^2} = 1,\ D_{-\mu}(z) = \frac{\left|1+\mu(z)\frac{\bar z}{z}\right|^2}{1-|\mu(z)|^2}$$

$$= \frac{\left|1+\frac{1}{1+2i}\right|^2}{1-\left|\frac{1}{1+2i}\right|^2} = 2,$$

$$\ell_f(z,0) = \min_\theta \frac{|\partial f_h(z)|}{|\cos\theta|} = \min_\theta \frac{\frac{\sqrt5}{2}\left|1+\frac{1}{1+2i}\frac{z}{\bar z}e^{-2i\theta}\right|}{|\cos\theta|} = \min_\theta \frac{\left|1+2i+\varepsilon^2\right|}{|\mathrm{Re}\,\varepsilon|}$$

$$= \min_{|w|=1} \frac{|1+2i+w|}{|1+w|} = \langle\psi(1),\mathbf{n}\rangle = \langle 1+i, i\rangle = 1,$$

$$\mathscr{L}_f(z,0) = \max_\theta |\partial f_h(z)||\cos\theta| = \max_\theta \frac{\sqrt5}{2}\left|1+\frac{1}{1+2i}\frac{z}{\bar z}e^{-2i\theta}\right||\cos\theta|$$

$$= \max_{|w|=1}\frac14|1+2i+w||1+w| = \frac12\max_\alpha\sqrt{(3+\cos\alpha+2\sin\alpha)(1+\cos\alpha)}$$

$$= \frac{\sqrt{1+\sqrt[3]{4}}}{2+\sqrt[3]{4}-2\sqrt[3]{2}},$$

where $\psi(z) = (c+z)/(1+z), \mathbf{n} = (c-1)/|c-1|, \varepsilon = e^{-2i(\theta-\tau)}, h = e^{i\theta}, z = |z|e^{i\tau}$. Finally, we have

$$K_\mu(z) = \frac{(\sqrt5+1)^2}{4},\quad D_f(z,0) = 1,\quad T_f(z,0) = \frac{1+\sqrt[3]{4}}{(2+\sqrt[3]{4}-2\sqrt[3]{2})^2} \approx 2.27,$$

$$D_\mu(z) = 1,\quad D_{-\mu}(z) = 2.$$

This homeomorphism preserves any annulus centered at the origin, therefore both quantities $\log\rho/r$ and $\operatorname{mod} A - \operatorname{mod} f(A)$ vanish. By (1), one obtains

$$-1.62\log\frac1r \le 0 \le 1.62\log\frac1r.$$

The estimates (8)–(11) are sharper than (1) since the right bound is attained in all of them. The same situation occurs with the extremal bounds (12) and (13), namely

$$-1.27\log\frac ba \le 0 \le 0.$$

This mapping belongs to all classes $\mathscr{F}_i$ except for $\mathscr{F}_2$ (asymptotic homogeneous), and obviously is not conformal at the origin.

6.3 Example

Consider another rotation

$$f(z) = |z|e^{i2\theta}.$$

For this mapping,

$$f(z) = z^{\frac{3}{2}}\bar{z}^{-\frac{1}{2}}, \quad f_z = \frac{3}{2}z^{\frac{1}{2}}\bar{z}^{-\frac{1}{2}}, \quad f_{\bar{z}} = -\frac{1}{2}z^{\frac{3}{2}}\bar{z}^{-\frac{3}{2}}, \quad \mu(z) = -\frac{1}{3}\frac{z}{\bar{z}},$$

$$|\partial f_h(z)| = |f_z h + f_{\bar{z}}\bar{h}| = |f_z||1 + \mu(z)e^{-2i\theta}| = \frac{3}{2}\left|1 - \frac{1}{3}e^{-2i(\theta-\tau)}\right|,$$

$$|\partial f_h(z)| \leq 2 = ||f'(z)||, \quad |\partial f_h(z)| \geq 1 = l(f'(z)), \quad J_f(z) = 2,$$

$$D_\mu(z) = \frac{\left|1 - \mu(z)\frac{\bar{z}}{z}\right|^2}{1 - |\mu(z)|^2} = \frac{\left|1 + \frac{1}{3}\right|^2}{1 - \frac{1}{9}} = 2, \quad D_{-\mu}(z) = \frac{\left|1 + \mu(z)\frac{\bar{z}-\bar{z}_0}{z-z_0}\right|^2}{1 - |\mu(z)|^2}$$

$$= \frac{\left|1 - \frac{1}{3}\right|^2}{1 - \frac{1}{9}} = \frac{1}{2}.$$

$$\ell_f(z,0) = \min_\theta \frac{|\partial f_h(z)|}{|\cos\theta|} = \min_\theta \frac{\frac{1}{2}\left|1 - \frac{1}{3}\frac{z}{\bar{z}}e^{-2i\theta}\right|}{|\cos\theta|} = \min_\theta \frac{\left|-3 + \varepsilon^2\right|}{|\operatorname{Re}\varepsilon|}$$

$$= \min_{|w|=1} \frac{|-3 + w|}{|1 + w|} = \langle\psi(1), \mathbf{n}\rangle = \langle -1, -1\rangle = 1,$$

$$\mathscr{L}_f(z,0) = \max_\theta |\partial f_h(z)||\cos\theta| = \max_\theta \frac{\sqrt{3}}{2}\left|1 - \frac{1}{3}\frac{z}{\bar{z}}e^{-2i\theta}\right||\cos\theta|$$

$$= \max_{|w|=1} \frac{1}{4}|3 - w||1 + w| = \frac{1}{4}\max_\alpha \sqrt{(10 - 6\cos\alpha)(2 + 2\cos\alpha)}$$

$$= \frac{2}{\sqrt{3}},$$

where $\psi(z) = (c+z)/(1+z), \mathbf{n} = (c-1)/|c-1|, \varepsilon = e^{-2i(\theta-\tau)}, h = e^{i\theta}, z = |z|e^{i\tau}$. Thus,

$$K_\mu(z) = 2, \quad D_f(z,0) = 2, \quad T_f(z,0) = \frac{2}{3}, \quad D_\mu(z) = 2, \quad D_{-\mu}(z) = \frac{1}{2}.$$

This mapping f is not homeomorphic; it belongs to $\mathscr{F}_3$, $\mathscr{F}_4$, $\mathscr{F}_6$ but $f \notin \mathscr{F}_1$, $\mathscr{F}_2$, $\mathscr{F}_5$.

Regarding the estimates (8)–(13) related to the directional dilatations, they do not hold (including the extremal bounds). Only Belinskiĭ's inequality (1) is valid, however, it is not sharp,

$$-\log\frac{1}{r} \le 0 \le \log\frac{1}{r}.$$

This mapping carries out any annulus centered at the origin onto itself.

6.4 Example

Consider now a homeomorphism of the unit disc $\mathbb{B}$,

$$f(z) = |z|(\sin\theta + 2)e^{i\theta}.$$

This mapping can be written in the form

$$f(z) = \frac{1}{2i}z^{\frac{3}{2}}\bar{z}^{-\frac{1}{2}} - \frac{1}{2i}z^{\frac{1}{2}}\bar{z}^{\frac{1}{2}} + 2z,$$

and by a direct calculation,

$$f_z = \frac{3}{4i}z^{\frac{1}{2}}\bar{z}^{-\frac{1}{2}} - \frac{1}{4i}z^{-\frac{1}{2}}\bar{z}^{\frac{1}{2}} + 2, \quad f_{\bar{z}} = -\frac{1}{4i}z^{\frac{3}{2}}\bar{z}^{-\frac{3}{2}} - \frac{1}{4i}z^{\frac{1}{2}}\bar{z}^{-\frac{1}{2}},$$

and

$$|f_z| = \frac{|3e^{i\theta} - e^{-i\theta} + 8i|}{4}, \quad |f_{\bar{z}}| = \frac{|e^{2i\theta} + 1|}{4},$$

$$K_\mu(z) = \frac{|3e^{i\theta} - e^{-i\theta} + 8i| + |e^{2i\theta} + 1|}{|3e^{i\theta} - e^{-i\theta} + 8i| - |e^{2i\theta} + 1|}.$$

Obviously, this mapping preserves angles on circles but it does not be even circle-like. However, $f \in \mathscr{F}_6$ with $H = 3$, but $f \notin \mathscr{F}_1$, $\mathscr{F}_2$, $\mathscr{F}_3$ $\mathscr{F}_4$, $\mathscr{F}_6$.

6.5 Example

A "slow rotation" (see, e.g. [7])

$$f(z) = ze^{i\sqrt{-\log|z|}}$$

is not conformal at the origin, but $f \in \mathscr{F}_i$ for all $i = 2, 3, 4, 5, 6$. A direct calculation yields

$$|f_z| = \sqrt{\frac{-16 \log |z| + 1}{-16 \log |z|}}, \quad |f_{\bar{z}}| = \frac{1}{4\sqrt{-\log |z|}}$$

and

$$K_\mu(z) = \frac{-8 \log |z| + 1 + \sqrt{-16 \log |z| + 1}}{-8 \log |z|}.$$

6.6 Example

Another "slow rotation"

$$f(z) = z e^{i \log \log \frac{e}{|z|}},$$

with

$$\mu(z) = \frac{1}{1 + 2i \log(e/|z|)} \frac{z}{\bar{z}}$$

is also not conformal at the origin, but $f \in \mathscr{F}_i$ for all $i = 2, 3, 4, 5, 6$. For this mapping,

$$K_\mu(z) = \frac{3\sqrt{|z|} + 2}{2\sqrt{|z|} + 2}.$$

6.7 Example

The stretching

$$f(z) = z(\sqrt{|z|} + 1) = z^{\frac{5}{4}} \bar{z}^{\frac{1}{4}} + z$$

provides a quasiconformal mapping of the unit disc $|z| < 1$, which is conformal at the origin. Thus, $f \in \mathscr{F}_i$ for all $i = 1, 2, \ldots, 6$. Indeed, by a direct calculation,

$$f_z = \frac{5}{4} z^{\frac{1}{4}} \bar{z}^{\frac{1}{4}} + 1, \quad f_{\bar{z}} = \frac{1}{4} z^{\frac{5}{4}} \bar{z}^{-\frac{3}{4}},$$

and hence,

$$\mu(z) = \frac{\sqrt{|z|}}{5\sqrt{|z|}+4}\frac{z}{\bar{z}}.$$

6.8 Example

The stretching

$$f(z) = z\left(\log\frac{1}{|z|}\right)^2$$

defined in a punctured unit disc has the derivatives

$$f_z = \log\frac{1}{|z|}\left(\log\frac{1}{|z|}-1\right), \quad f_{\bar{z}} = -\log\frac{1}{|z|}\frac{z}{\bar{z}},$$

and

$$\mu(z) = \frac{1}{1-\log\frac{1}{|z|}}\frac{z}{\bar{z}}.$$

It does not have a finite asymptotic dilation at the origin, and, therefore, is not conformal at 0, although $f \in \mathscr{F}_2, \mathscr{F}_4, \mathscr{F}_5, \mathscr{F}_6$.

6.9 Example

The mapping

$$f(z) = ze^{i\varepsilon\log(1/|z|)}$$

transforms any ray $re^{i\theta}$, $0 < r < 1$, into the logarithmic spiral with wind about the origin infinitely many times. The limit $\lim_{z\to 0} \arg f(z)/z$ does not exist and hence f is not conformal at the origin, nevertheless $|f(z)| = |z|$ in any neighborhood of 0. Note that f is also not asymptotically homogeneous.

6.10 Example

Consider a mapping

$$f(z) = ze^{i\varphi(\theta)}$$

with a nonlinear real-valued continuously differentiable function φ satisfying $\varphi(0) = 0$ and $\varphi(2\pi) = 2\pi$. This mapping fails to be conformal, asymptotically homogeneous, preserving angles on circles and, therefore, is not weakly conformal. But it has asymptotic dilation which equal to 1 and hence $f \in \mathscr{F}_4, \mathscr{F}_6$.

6.11 Example

Consider the radial stretching od the unit disc $\mathbb{B}$ defined by

$$f(z) = z(1 - \log |z|), \quad z \neq 0, \quad f(0) = 0.$$

Its derivatives are equal to $f_z = (1 - 2\log |z|)/2, f_{\bar{z}} = -z/2\bar{z}$ and

$$|\partial_h f| = \frac{1}{2}\left(1 + 2\log \frac{1}{|z|}\right)\left|1 - \frac{z}{\bar{z}} \frac{e^{-2i\theta}}{1 + 2\log \frac{1}{|z|}}\right|.$$

Letting $z = re^{i\psi}$ and $h = e^{i\theta}$, one derives

$$|\partial_h f| \leq 1 + \log \frac{1}{|z|} = ||f'(z)||, \quad |\partial_h f| \geq \log \frac{1}{|z|} = l(f'(z));$$

the equalities here occur for $\psi = \theta + \pi/2$ and $\psi = \theta$, respectively. Thus,

$$J_f(z) = \left(1 + \log \frac{1}{r}\right) \log \frac{1}{r}, \quad K_\mu(z) = 1 + \frac{1}{\log \frac{1}{r}}.$$

A calculation of the directional dilatations is much more complicated. We first find the quantity $\mathscr{L}_f(z)$.

$$\begin{aligned}\mathscr{L}_f(z) &= \max_\theta |\partial f_h(r) \cos \theta| = \max_\theta \left(|\partial f_z(r)||1 + \mu(r)e^{-2i\theta}|| \cos \theta|\right) \\ &= \frac{1 + 2\log \frac{1}{r}}{4} \max_\theta |(1 + ke^{-2i\theta})(1 + e^{-2i\theta})|,\end{aligned}$$

where $\mu(z) = f_{\bar{z}}/f_z = kz/\bar{z}$, $|f_z(r)| = (1 + 2\log(1/r))/2$, $k = -1/(1 + 2\log(1/r))$. A straightforward computation implies

$$\mathcal{L}_f(z,0) = \log\frac{1}{r} \quad \text{for} \quad \log\frac{1}{r} \geq 1 + \sqrt{2},$$

$$\mathcal{L}_f(z,0) = \frac{\left(1 + \log\frac{1}{r}\right)^2}{2\sqrt{1 + 2\log\frac{1}{r}}} \quad \text{for} \quad \log\frac{1}{r} \leq 1 + \sqrt{2}.$$

Thus,

$$T_f(z) = \frac{\log\frac{1}{r}}{1 + \log\frac{1}{r}}, \qquad \log\frac{1}{r} \geq 1 + \sqrt{2},$$

$$T_f(z) = \frac{\left(1 + \log\frac{1}{r}\right)^{\frac{n+1}{n-1}}}{2^{\frac{n}{n-1}}\left(\log\frac{1}{r}\right)^{\frac{1}{n-1}}\left(1 + 2\log\frac{1}{r}\right)^{\frac{n}{2(n-1)}}}, \qquad \log\frac{1}{r} \leq 1 + \sqrt{2}.$$

The dilatation $D_f(z)$ can be calculated using a technique related to functions of one complex variable and presented in [16]. The result is

$$D_f(z) = 1 + \frac{1}{\log\frac{1}{r}}$$

Indeed,

$$\begin{aligned}
\ell_f(z,0) &= \min_\theta \frac{|\partial f_h(r)|}{|\cos\theta|} = \min_\theta \frac{|\partial f_z(r)||1 + \mu(r)e^{-2i\theta}|}{|\cos\theta|} \\
&= \left(1 + 2\log\frac{1}{r}\right)\min_\theta \frac{|1 + ke^{-2i\theta}|}{|1 + e^{-2i\theta}|} \\
&= \left(1 + 2\log\frac{1}{r}\right)\left\langle\frac{1+k}{2}, 1\right\rangle = \log\frac{1}{r}.
\end{aligned}$$

Finally, let us remark that f is not conformal and does not have asymptotic dilation at the origin, but $f \in \mathcal{F}_2, \mathcal{F}_4, \mathcal{F}_5$, and, therefore, $f \in \mathcal{F}_6$.

6.12 Example

For the radial stretch in $\mathbb{C}$ of the form

$$f(z) = z|z|^{K-1}, \quad K \geq 1,$$

all the dilatations have been calculated in [12, 16] (also for higher dimensions). Regarding to the plane, we obtain

$$K_\mu(z) = K, \quad D_f(z) = \frac{1}{K}, \quad T_f(z) = K.$$

On the estimates and relationship of this mappings with the classes $\mathscr{F}_i$, we refer to Sect. 6.1, since it relates to this example with $K = 2$.

6.13 Example

Consider the radial stretch in $\mathbb{C}$ of the form

$$f(z) = z|z|^{\frac{1}{K}-1}, \;\; K \geq 1 \;\; f(0) = 0\,.$$

For this mapping, the calculations similar to above imply

$$K_\mu(z) = K, \quad D_f(z) = K;$$

see, e.g. [12, 16]. The computation of $T_f(z)$ splits into two cases $K \leq \sqrt{2}$ and $K \geq \sqrt{2}$ and results in

$$T_f(z) = \frac{1}{K}, \;\; K \leq \sqrt{2} \quad \text{and} \quad T_f(z) = \frac{K^3}{4(K^2-1)}, \;\; K \geq \sqrt{2}.$$

This homeomorphism maps any annulus with radii a and b onto the annulus with the radii $a^{1/K-1}$ and $b^{1/K-1}$. By (1) the right bound is sharp,

$$-(K-1)\log\frac{1}{r} \leq \left(\frac{1}{K}-1\right)\log r \leq \frac{K-1}{K}\log\frac{1}{r}.$$

The inequalities (8), (10)–(13) become sharp,

$$\frac{K-1}{K}\log\frac{b}{a} \leq \log\frac{b}{a} - \frac{1}{K}\log\frac{b}{a} \leq \frac{K-1}{K}\log\frac{b}{a}.$$

The relations with the classes $\mathscr{F}_i$ remain the same as in the previous example.

7 Extremal Bounds for Moduli Under Nonhomeomorphic Mappings

We have mentioned in Sect. 6.3 that the estimates (12) and (13) do not hold if the mapping is not homeomorphic. In order to correct these inequalities we have to involve the multiplicity function.

For $A \subset G$ and $y \in \overline{\mathbb{R}^n}$, we write

$$N(y,f,A) = \operatorname{card} f^{-1}(y) \cap A \quad \text{and} \quad N(f,A) = \sup_y N(y,f,A).$$

The estimates for spatial quasiregular mappings with classical dilatation coefficients can be found in [29, 31, 32, 35, 36].

The following result provides an extension of the estimate (5) to the class of planar quasiregular mappings.

Theorem 9 *Let $f : D \to \mathbb{C}$ be a quasiregular mapping. Suppose that the directional dilatations $D_f(z)$ and $T_f(z)$ are locally integrable in the annulus $A \subset G$. Then the following double inequality*

$$\begin{aligned} \frac{1}{N(f,A)} \inf_{\rho \in extadm\, \Gamma_A} \int_A T_f^{-1}(z)\rho^2(|z|)\, dxdy \;&\le\; \mathscr{M}(f(\Gamma_A)) \\ &\le\; \inf_{\rho \in adm\, \Gamma_A} \int_A D_f(z)\varrho^2(|z|)\, dxdy \end{aligned} \tag{14}$$

hold.

Proof We sketch the proof; it is valid also for higher dimensions. The function

$$\rho(x) = \rho^*(f(x))\mathscr{L}_f(x,x_0)$$

is admissible for the family Γ_A (of curves joining the boundary components of the spherical ring with radii a and b and centered at x_0); see [14].

Now arguing similarly to [3, 26] and [14], one can obtain

$$\int_{\mathbb{R}^n} \rho^{*n}(y)\, dm(y) = \int_A \frac{\rho(x)}{N(y,f,A)} \frac{|J_f(x)|}{\mathscr{L}_f^n(x,x_0)}\, dm(x) \ge \frac{1}{N(f,A)} \int_A \frac{\rho^n(x)}{T_f^{n-1}(x,x_0)}\, dm(x).$$

Taking the infimum over all $\rho \in \operatorname{extadm} \Gamma_A$, we have

$$\mathscr{M}(f(\Gamma_A)) \ge \frac{1}{N(f,A)} \inf_{\rho \in \operatorname{extadm} \Gamma_A} \int_A \frac{\rho^n(x)}{T_f^{n-1}(x,x_0)}\, dm(x),$$

which completes the proof, since the right-hand side of the inequality (14) remains the same as in (5); cf. [29] and [32]. □

Applying the arguments given in the proof of the double inequality (12), we similarly arrive at the following extremal bounds

$$\int_a^b \frac{\frac{1}{2\pi}\int_0^{2\pi} T_f^{-1}(z)d\theta - N(f,A)}{\frac{1}{2\pi}\int_0^{2\pi} T_f^{-1}(z)d\theta}\frac{d\rho}{\rho} \leq \operatorname{mod} A - \operatorname{mod} f(A)$$
$$\leq \int_a^b \frac{\frac{1}{2\pi}\int_0^{2\pi} D_f(z)d\theta - 1}{\frac{1}{2\pi}\int_0^{2\pi} D_f(z)d\theta}\frac{d\rho}{\rho}. \tag{15}$$

Now it not hard to see that the double inequality (15) holds for the rotation defined in Sect. 6.3. Indeed, by a direct calculation we obtain

$$-\frac{1}{3}\log\frac{b}{a} \leq 0 \leq \frac{1}{2}\log\frac{b}{a}.$$

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Families of Universal Taylor Series Depending on a Parameter

Evgeny Abakumov, Jürgen Müller, and Vassili Nestoridis

Abstract We construct families of universal Taylor series on Ω depending on a parameter $w \in G$, where Ω and G are planar simply connected domains. The functions to be approximated depend on the parameter w, $w \in G$. The partial sums implementing the universal approximation are one variable partial sums with respect to $z \in \Omega$ for each fixed value of the parameter $w \in G$. The universal approximation extends to mixed partial derivatives. This phenomenon is generic in $H(\Omega \times G)$.

Keywords Universal Taylor series • Baire's Theorem • Runge's Theorem • Generic property • Mixed partial derivatives

A.M.S. Classification: Primary 30K05; Secondary 32A30

1 Introduction

The first result concerning the existence of universal Taylor series was established before 1914 by Fekete (see [18]). He proved the existence of a real power series $\sum_{n=1}^{\infty} a_n x^n$, whose partial sums approximate uniformly on $[-1, 1]$ every continuous function $h : [-1, 1] \to \mathbb{R}$ with $h(0) = 0$. In the early 1950s Seleznev proved the existence of complex power series $\sum_{n=0}^{\infty} a_n z^n$ with radius of convergence 0,

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whose partial sums approximate every polynomial uniformly on each compact set $K \subset \mathbb{C} \setminus \{0\}$ with connected complement [21]. In the early 1970s Luh [9] and independently Chui and Parnes [2] proved the existence of universal Taylor series with positive radius of convergence defining a function holomorphic in a simply connected domain $\Omega \subset \mathbb{C}$ and whose partial sums approximate every polynomial uniformly on each compact set $K \subset \mathbb{C}$ with connected complement such that $K \cap \overline{\Omega} = \emptyset$.

In the latter result the universal approximation does not necessarily hold on the boundary of the domain of definition Ω. In 1996 a stronger result was obtained, where the universal approximation was valid on the boundary $\partial\Omega$, as well [15]. The universal approximation was initially implemented by partial sums of the Taylor expansion of the universal function with respect to a fixed center $\zeta \in \Omega$. However, it was soon realized that the result persists when the center in Ω is varied [16]. After some years [12, 14] it was proved that the class of universal functions remains unchanged, whether the center of expansion in a simply connected domain Ω is varied or not. Thus, possible definitions of universal Taylor series are the following [12, 16].

Definition 1.1 Let $\Omega \subset \mathbb{C}$ be a domain and $f : \Omega \to \mathbb{C}$ a holomorphic function.

1. For $\zeta_0 \in \Omega$ fixed, the function f belongs to the class $U(\Omega, \zeta_0)$ if the sequence of the partial sums

$$S_N(f, \zeta_0)(z) = \sum_{j=0}^{N} \frac{f^{(j)}(\zeta_0)}{j!}(z - \zeta_0)^j,$$

$N = 0, 1, 2, \ldots$, of the Taylor development of f with center ζ_0 satisfies the following: For every compact set $K \subset \mathbb{C}, K \cap \Omega = \emptyset$ with connected complement K^c and for every function $h : K \to \mathbb{C}$ continuous on K and holomorphic in K°, there exists a sequence (λ_n) of positive integers such that

$$\sup_{z \in K} \left| S_{\lambda_n}(f, \zeta_0)(z) - h(z) \right| \to 0, \quad \text{as } n \to +\infty.$$

2. The function f belongs to the class $U(\Omega)$, if the partial sums

$$S_N(f, \zeta)(z) = \sum_{j=0}^{N} \frac{f^{(j)}(\zeta)}{j!}(z - \zeta)^j,$$

$\zeta \in \Omega, N = 0, 1, 2, \ldots$ satisfy the following condition: For every compact set $K \subset \mathbb{C} \setminus \Omega$ with connected complement and every function $h : K \to \mathbb{C}$ continuous on K and holomorphic in K°, there exists a sequence (λ_n) of positive integers such that for every compact set $L \subset \Omega$ we have

$$\sup_{\zeta \in L} \sup_{z \in K} \left| S_{\lambda_n}(f, \zeta)(z) - h(z) \right| \to 0 \quad \text{as } n \to +\infty.$$

Obviously $U(\Omega, \zeta_0) \supset U(\Omega)$. Further, if Ω is simply connected, both classes $U(\Omega, \zeta_0)$ and $U(\Omega)$ are G_δ and dense in $H(\Omega)$ endowed with the topology of uniform convergence on compact subsets of Ω [12, 16]. Actually, in this case $U(\Omega, \zeta_0) = U(\Omega)$ ([14], see also [12]).

In this paper, we consider a parameter $w \in G$, where G is some simply connected domain in $\mathbb{C}$, and for every $w \in G$ we find functions $f(\cdot, w)$ in $U(\Omega)$ having the property that a function $h(\cdot, w)$ defined on a compact set $K \subset \mathbb{C}$ and depending on the parameter $w \in G$ can be approximated by the partial sums of $f(\cdot, w)$ with the same sequence (λ_n) for all $w \in G$. Furthermore, the approximation extends to partial derivatives with respect to the parameter w and to mixed partial derivatives with respect to z and w (cf. [17]). It is possible to consider one fixed center of expansion $b(w)$ for every $w \in \Omega$, which is given by a holomorphic function $b : G \to \Omega$, or one may consider all possible centers $\zeta \in \Omega$. In the latter case, partial derivatives with respect to ζ are also allowed. In this way, functions f holomorphic in $\Omega \times G$ can be constructed in such a way that for every fixed $w \in G$, the partial sums implementing the universal approximation are those of the functions of one variable $\Omega \ni z \to f(z, w) \in \mathbb{C}$.

We prove that the corresponding universality phenomenon is generic in the space $H(\Omega \times G)$ of holomorphic functions on $\Omega \times G$ endowed with the topology of uniform convergence on compacta. Towards this end, we use Baire's Category Theorem. For the role of Baire's theorem in Analysis we refer to [4] and [7].

2 Main Results

Let (μ_n) be a strictly increasing sequence of positive integers and (c_j) a sequence of complex numbers. We say that (c_j) has Ostrowski-gaps relative to (μ_n) if a sequence (q_n) exists with $0 < q_n \to 0$ as $n \to \infty$ and so that

$$\sup_{q_n \mu_n \leq j \leq \mu_n} |c_j|^{1/j} \to 0 \qquad (n \to \infty)$$

(see e.g. [13], cf. also [6, p. 311]). Moreover, if (λ_n) is a sequence of positive integers with $\lambda_n = q_n \mu_n$ as above, we say that the sequence (c_j) has Ostrowski-gaps (λ_n, μ_n).

The starting point of our considerations is the following observation:

Proposition 2.1 *Let $\Omega \subset \mathbb{C}$ be a simply connected domain, $f \in U(\Omega) = U(\Omega, \zeta_0)$, K a compact set in $\mathbb{C} \setminus \Omega$ with connected complement, and $h : K \to \mathbb{C}$ a function continuous on K and holomorphic in K°. Let (λ_n) be a sequence as in Definition 1.1. Then for every fixed $z \in K$ we have*

$$\frac{\partial}{\partial \zeta} S_{\lambda_n}(f, \zeta)(z) \to 0 = \frac{\partial}{\partial \zeta} h(z) \quad \textit{as } n \to +\infty$$

uniformly on compact subsets of Ω. *Furthermore, the sequence* (λ_n) *may be chosen so that in addition for every compact set* $L \subset \Omega$ *we have*

$$\sup_{\zeta\in L}\sup_{z\in K}\left|\frac{\partial}{\partial\zeta}S_{\lambda_n}(f,\zeta)(z)\right| \to 0 \quad \textit{as } n \to +\infty.$$

Proof a) For fixed $z \in K$ the function $\Omega \ni \zeta \to S_{\lambda_n}(f,\zeta)(z) \in \mathbb{C}$ is holomorphic in Ω. According to Definition 1.1 this sequence of elements of $H(\Omega)$ converges uniformly on compacta (with respect to $\zeta \in \Omega$) to the constant function $h(z)$. Weierstrass' theorem implies that $\frac{\partial}{\partial\zeta}S_{\lambda_n}(f,\zeta)(z) \to \frac{\partial h}{\partial\zeta}(z) = 0$ for each $\zeta \in \Omega$ and even uniformly in each compact subset of Ω. Thus we have

$$\sup_{\zeta\in L}\left|\frac{\partial}{\partial\zeta}S_{\lambda_n}(f,\zeta)(z)\right| \to 0, \quad \text{as } n \to +\infty$$

for every fixed $z \in K$.

b) By a straightforward computation we find

$$\begin{aligned}\frac{\partial}{\partial\zeta}S_{\lambda_n}(f,\zeta)(z) &= S_{\lambda_n}(f',\zeta)(z) - S_{\lambda_n-1}(f',\zeta)(z)\\ &= \frac{((f')^{(\lambda_n)}(\zeta)}{\lambda_n!}(z-\zeta)^{\lambda_n} =: A_{\lambda_n}(\zeta,z).\end{aligned}$$

It is known [3, 12, 14] that for any $f \in U(\Omega,\zeta_0) = U(\Omega)$ the sequence of Taylor coefficients of f' with center ζ_0 has Ostrowski gaps relative to some sequence (μ_n) and that the sequence (λ_n) may be chosen so that $q_n\mu_n = \lambda_n - 1$. Then

$$\left|\frac{(f')^{(\lambda_n)}(\zeta_0)}{\lambda_n!}\right|^{1/\lambda_n} \to 0 \quad (n \to \infty)$$

and therefore, since $\lambda_n \to +\infty$, we have $\sup_{z\in K}\left|A_{\lambda_n}(\zeta_0,z)\right| \to 0$, as $n \to +\infty$. It follows that

$$\sup_{z\in K}\left|S_{\lambda_n}(f',\zeta_0)(z) - S_{\lambda_n-1}(f',\zeta_0)(z)\right| \to 0, \quad \text{as } n \to +\infty.$$

Since the sequence of Taylor coefficients of f' with center ζ_0 has Ostrowski gaps (λ_n,μ_n) and (λ_n-1,μ_n), it follows from [12, Lemma 9.2] that for every compact subset L of Ω we have

$$\sup_{\zeta\in L}\sup_{z\in K}\left|S_{\lambda_n}(f',\zeta_0)(z) - S_{\lambda_n}(f',\zeta)(z)\right| \to 0$$

and

$$\sup_{\zeta \in L} \sup_{z \in K} \left| S_{\lambda_{n-1}}(f', \zeta_0)(z) - S_{\lambda_n - 1}(f', \zeta)(z) \right| \to 0$$

as $n \to +\infty$. Putting things together it is easily seen that

$$\sup_{\zeta \in L} \sup_{z \in K} \left| \frac{\partial}{\partial \zeta} S_{\lambda_n}(f, \zeta)(z) \right| \to 0, \quad \text{as } n \to +\infty.$$

This completes the proof.

Let Ω and G be two simply connected domains in $\mathbb{C}$. For $f \in H(\Omega \times G)$ and $w \in G, \zeta \in \Omega$ and $z \in \mathbb{C}$ we denote

$$\widetilde{S}_N(f, w, \zeta)(z) = \sum_{j=0}^{N} \frac{\partial^j f}{\partial u^j}(u, w)\Big|_{u=\zeta} \cdot \frac{1}{j!} (z - \zeta)^j$$

and we consider the following classes of functions.

Definition 2.2 Let $b : G \to \Omega$ be a holomorphic function. The class $U(\Omega, G, b)$ contains all functions $f \in H(\Omega \times G)$ such that the sequence $\widetilde{S}_N(f, w, \zeta)$ satisfies the following: For every compact set $K \subset \mathbb{C}, K \cap \Omega = \emptyset$, K^c connected, and any holomorphic function h in an open neighborhood of $K \times G$ ($h \in H(K \times G)$), there exists a sequence (λ_n) of positive integers such that for every compact set $F \subset G$

$$\sup_{w \in F} \sup_{z \in K} \left| \widetilde{S}_{\lambda_n}\big(f, w, b(w)\big)(z) - h(z, w) \right| \to 0, \quad \text{as } n \to +\infty.$$

Definition 2.3 The class $U(\Omega, G)$ contains all functions $f \in H(\Omega \times G)$ such that the sequence $\widetilde{S}_N(f, w, \zeta)$ satisfies the following.

For every compact set $K \subset \mathbb{C}, K \cap \Omega = \emptyset$, K^c connected and any $h \in H(K \times G)$, there exists a sequence (λ_n) of positive integers such that for all compact sets $F \subset G$, $L \subset \Omega$

$$\sup_{w \in F} \sup_{\zeta \in L} \sup_{z \in K} \left| \widetilde{S}_{\lambda_n}(f, w, \zeta)(z) - h(z, w) \right| \to 0, \quad \text{as } n \to +\infty.$$

Theorem 2.4 *For all holomorphic functions* $b : G \to \Omega$ *we have*

$$U(\Omega, G) = U(\Omega, G, b).$$

We need parameter modifications of several known results. For potential theoretic notions as for example that of Green's functions and (non-)thinness, we refer to [19]. Let $||f||_M$ denote the sup-norm of a bounded function f on M.

Lemma 2.5 *Let $F \subset \mathbb{C}$ compact and let $P_n : \mathbb{C} \times F \to \mathbb{C}$ be continuous and such that $P_n(\cdot, w)$ is a polynomial of degree $\leq \mu_n$. If $E \subset \mathbb{C}$ is closed and non-thin at ∞ with*

$$\limsup_{n\to\infty}(||P_n(z,\cdot)||_F)^{1/\mu_n} \leq 1 \quad \textit{for all } z \in E,$$

then

$$\limsup_{n\to\infty}(||P_n||_{M\times F})^{1/\mu_n} \leq 1 \quad \textit{for all compact } M \subset \mathbb{C}.$$

For sake of completeness, we sketch the proof which is based on Bernstein's lemma (see e.g. [19, Theorem 5.5.7]) and the following characterization of non-thinness at ∞ in terms of Green's functions (see [13, Lemma 1]).

Let $E \subset \mathbb{C}$ be closed and suppose that $E_R := \{w \in E : |w| \leq R\}$ has positive capacity for $R > 0$ sufficiently large. If D_R denotes the component of $\mathbb{C}_\infty \setminus E_R$ containing ∞ then E is non-thin at ∞ if and only if the Green's functions g_{D_R} for D_R satisfy

$$g_{D_R}(z, \infty) \to 0 \quad \text{as } R \to \infty.$$

In the first step, one can reduce the proof to the case of $\mathbb{C} \setminus E$ having no bounded components (cf. [13], proof of Lemma 1). The functions $v_n : \mathbb{C} \to \mathbb{C}$, defined by

$$v_n(z) := \max\left(\frac{1}{\mu_n}\log||P_n(z,\cdot)||_F, 0\right) \quad \text{for } z \in \mathbb{C},$$

are subharmonic in $\mathbb{C}$ [19, Theorem 2.4.7] and from Bernstein's lemma it can be deduced that

$$\limsup_{n\to\infty} v_n(z) \leq g_{D_R}(z, \infty) \quad \text{for } z \in D_R$$

(cf. [13], proof of Lemma 1). Then, from the above characterization of non-thinness at ∞, we obtain that $v_n \to 0$ in $\mathbb{C} \setminus E$, as $n \to \infty$. According to the assumption, this implies $v_n \to 0$ in $\mathbb{C}$, where the convergence turns out to be locally uniformly in $\mathbb{C}$. This is equivalent to the statement of Lemma 2.5.

For $F \subset G$ compact and $j = 0, 1, \ldots$ we define

$$a_j(F) := \frac{1}{j!}\sup_{w\in F}\left|\frac{\partial^j f}{\partial u^j}(u, w)\big|_{u=b(w)}\right|.$$

As an application of Cauchy's estimates we then get (cf. for example the proof of the Lemma in [3])

Lemma 2.6 *Let $F \subset G$ be compact. If (μ_n) is a sequence of integers with*

$$\limsup_{n\to\infty} \sup_{w\in F} (||\tilde{S}_{\mu_n}(f, w, b(w))||_M)^{1/\mu_n} \leq 1$$

for all compact $M \subset \mathbb{C}$, then the sequence $(c_j) = (a_j(F))$ has Ostrowski-gaps relative to (μ_n).

A more sophisticated application of Cauchy's estimates in conjunction with the three circles theorem or the two constants theorem yields

Lemma 2.7 *Suppose that $(a_j(F))$ has Ostrowski-gaps (λ_n, μ_n). Then*

$$\sup_{w\in F} \sup_{\zeta\in L} ||\tilde{S}_{\lambda_n}(f, w, \zeta) - \tilde{S}_{\lambda_n}(f, w, b(w))||_M \to 0 \quad \textit{as } n \to \infty,$$

for all compact $L \subset \Omega$ and all compact $M \subset \mathbb{C}$.

The proof is similar to the proof of Theorem 1 of [10]; see also [12, Lemma 9.2].

Proof of Theorem 2.4 Obviously, we have $U(\Omega, G) \subset U(\Omega, G, b)$. Let $f \in U(\Omega, G, b)$. We show that $f \in U(\Omega, G)$. To this aim consider $h \in H(K \times G)$, where K is as in Definition 2.2, and $F \subset G$ compact. Moreover, suppose that (K_n) is an increasing sequence of compact sets in Ω^c with K_n^c connected, $K \cap K_n = \emptyset$ and so that $E := \bigcup_n K_n$ is closed and non-thin at ∞ (such a sequence exists; cf. Lemma 1 in [14]). We define $g_n : K \cup K_n \to \mathbb{C}$ by

$$g_n(z.w) := \begin{cases} h(z, w), & (z, w) \in K \times G \\ 0, & (z, w) \in K_n \times G \end{cases}.$$

The definition of $U(\Omega, G, b)$ implies that a (strictly increasing) sequence (μ_n) exists with

$$\sup_{w\in F} \sup_{z\in K\cup K_n} |\widetilde{S}_{\mu_n}\big(f, w, b(w)\big)(z) - g_n(z, w)| < 1/n$$

for all n. Then

$$P_n(z, w) := \widetilde{S}_{\mu_n}\big(f, w, b(w)\big)(z)$$

satisfies the assumptions of Lemma 2.5. Thus, from Lemmas 2.5 and 2.6 we obtain that $(a_j(F))$ has Ostrowski-gaps (λ_n, μ_n). From the definition of Ostrowski-gaps it follows that

$$\sup_{w\in F} \sup_{z\in K} \left|\widetilde{S}_{\mu_n}(f, w, b(w))(z) - \widetilde{S}_{\lambda_n}(f, w, b(w))(z)\right| \to 0, \quad \text{as } n \to +\infty.$$

But then the equiconvergence property of Lemma 2.7 implies that

$$\sup_{w\in F}\sup_{\zeta\in L}\sup_{z\in K}\left|\widetilde{S}_{\lambda_n}(f,w,\zeta)(z)-h(z,w)\right|\to 0,\quad \text{as } n\to+\infty.$$

This shows that $f\in U(\Omega,G)$. □

Remark 2.8 We consider the class $\widetilde{U}(\Omega,G)$. Its definition is the same as the definition of the class $U(\Omega,G)$ but in addition we require the following: For all compact sets $R\subset\Omega$ and $S\subset G$ we have

$$\sup_{z\in R}\sup_{w\in S}\sup_{\zeta\in R}\left|\widetilde{S}_{\lambda_n}(f,w,\zeta)(z)-f(z,w)\right|\to 0\qquad \text{as } n\to+\infty.$$

As in [14, Corollary 1] it is seen that from Ostrowski's classical results on overconvergence and the above proof of Theorem 2.4 it follows that also

$$\widetilde{U}(\Omega,G)=U(\Omega,G).$$

We shall show that the class $\widetilde{U}(\Omega,G)$ is residual in $H(\Omega\times G)$. Actually, we prove this for a subclass of $U(\Omega,G)$.

Definition 2.9 Let $b:G\to\Omega$ be a holomorphic function. The class $U'(\Omega,G,b)$ contains all functions $f\in H(\Omega\times G)$ such that the sequence $\widetilde{S}_N(f,w,\zeta)$ satisfies the following: For every compact set $K\subset\mathbb{C}, K\cap\Omega=\emptyset$, K^c connected, and any holomorphic function h in an open neighborhood of $K\times G$ ($h\in H(K\times G)$), there exists a sequence (λ_n) of positive integers such that the following holds: For every compact set $F\subset G$ and every differential operator $D_{\alpha_1,\alpha_2}=\frac{\partial^{\alpha_1}}{\partial z^{\alpha_1}}\frac{\partial^{\alpha_2}}{\partial w^{\alpha_1}}$, $\alpha_1,\alpha_2\in\{0,1,2,\ldots\}$ it holds

$$\sup_{w\in F}\sup_{z\in K}\left|D_{\alpha_1,\alpha_2}\widetilde{S}_{\lambda_n}\big(f,w,b(w)\big)(z)-D_{\alpha_1,\alpha_2}h(z,w)\right|\to 0,\quad \text{as } n\to+\infty.$$

Definition 2.10 The class $U'(\Omega,G)$ contains all functions $f\in H(\Omega\times G)$ such that the sequence $\widetilde{S}_N(f,w,\zeta)$ satisfies the following: For every compact set $K\subset\mathbb{C}, K\cap\Omega=\emptyset$, K^c connected and any $h\in H(K\times G)$, there exists a sequence (λ_n) of positive integers such that the following holds: For all compact sets $F\subset G$, $L\subset\Omega$ and for every differential operator $D_{\alpha_1,\alpha_2,\alpha_3}=\frac{\partial^{\alpha_1}}{\partial z^{\alpha_1}}\frac{\partial^{\alpha_2}}{\partial w^{\alpha_2}}\frac{\partial^{\alpha_3}}{\partial \zeta^{\alpha_2}}$, $\alpha_1,\alpha_2,\alpha_3\in\{0,1,2,\ldots\}$, it holds

$$\sup_{w\in F}\sup_{\zeta\in L}\sup_{z\in K}\left|D_{\alpha_1,\alpha_2,\alpha_3}\widetilde{S}_{\lambda_n}(f,w,\zeta)(z)-D_{\alpha_1,\alpha_2,\alpha_3}h(z,w)\right|\to 0,\quad \text{as } n\to+\infty.$$

Proposition 2.11 *For all holomorphic functions $b:G\to\Omega$ we have*

$$U'(\Omega,G)\subset U'(\Omega,G,b).$$

Proof Let $f \in U'(\Omega, G)$. Then, according to the Proposition 2.1 we have

$$\sup_{w\in F}\sup_{\zeta\in L}\sup_{z\in K}\left|D_{\alpha_1,\alpha_2,\alpha_3}\widetilde{S}_{\lambda_n}(f,w,\zeta)(z)\right| \to 0, \quad \text{as } n\to+\infty,$$

provided that $\alpha_3 \neq 0$.

We choose L compact such that $b(F) \subset L \subset \Omega$. Then,

$$\begin{aligned}
&D_{\alpha_1,1}\widetilde{S}_{\lambda_n}\big(f,w,b(w)\big)(z)\\
&= D_{\alpha_1,1,0}\widetilde{S}_{\lambda_n}(f,w,b(w))(z) + D_{\alpha_1,0,1}\widetilde{S}_{\lambda_n}(f,w,b(w))(z)\cdot b'(w)\\
&\to D_{\alpha_1,1,0}h(z,w) + 0 \equiv D_{\alpha_1,1}h(z,w),
\end{aligned}$$

as $n \to +\infty$, uniformly on $F \times K$, because b' is bounded on the compact set L containing $b(F)$.

For general α_2 the proof follows by induction.

Theorem 2.12 *The class $U'(\Omega, G)$ is a residual subset of $H(\Omega \times G)$ endowed with the topology of uniform convergence on compacta.*

Proof It is known (see e.g. [5]) that polynomials in two variables are dense in the space of holomorphic functions defined on the product of two simply connected planar open sets endowed with the topology of uniform convergence on compacta. Thus, the function h can be taken to be a polynomial of two variables.
For compact sets $K \subset \mathbb{C}, F \subset G, L \subset \Omega$, a polynomial h, a finite subset I of $\{0,1,2,\ldots\}^3$, $s \in \{1,2,\ldots\}$ and $n \in \{0,1,2,\ldots\}$ we consider the set $E(K,F,L,h,I,s,n)$ of all $g \in H(\Omega\times G)$ such that

$$\sup_{w\in F}\sup_{\zeta\in L}\sup_{z\in K}\left|D_{\alpha_1,\alpha_2,\alpha_3}\widetilde{S}_n(g,w,\zeta)(z) - D_{\alpha_1,\alpha_2,\alpha_3}h(z,w)\right| < \frac{1}{s}$$

for all $(\alpha_1,\alpha_2,\alpha_3) \in I$.

It is known [12] that there exists a sequence $K_m, m = 1,2,\ldots$, of compact subsets of $\mathbb{C}\setminus\Omega$ with K_m^c connected, such that for every compact set $K \subset \mathbb{C}\setminus\Omega$ with K^c connected there exists $m \in \{1,2,\ldots\}$ so that $K \subset K_m$.

We also consider F_τ, $\tau = 1,2,\ldots$ and L_ρ, $\rho = 1,2,\ldots$, two exhausting families of compact sets in G and Ω, respectively. Since G and Ω are simply connected we may assume that F_τ and L_ρ have connected complements [20]. Finally, let $h_j, j = 1,2,\ldots$, be an enumeration of the polynomials in two variables with coefficients in $\mathbb{Q}+i\mathbb{Q}$.

One can easily see that

$$U'(\Omega,G) = \bigcap_{I,m,\tau,\rho,j,s}\ \bigcup_{n} E(K_m,F_\tau,L_\rho,h_j,I,s,n)$$

where I varies in the set of finite subsets of $\{0,1,2,\ldots\}^3$, which is a denumerable set.

If we show that each $E(K, F, L, h, I, s, n)$ is open in $H(\Omega \times G)$, then it will follow that $U(\Omega, G)$ is a G_δ set. Further, if we show in addition that $\bigcup_n E(K_m, F_\tau, L_\rho, h_j, I, s, n)$ is dense in $H(\Omega \times G)$ for every fixed m, τ, ρ, j, I and s then Baire's Category Theorem would imply that $U(\Omega, G)$ is a dense G_δ subset of the Fréchet space $H(\Omega \times G)$.

We consider compact sets M, M_1, T and T_1, such that $L \subset M^\circ \subset M \subset M_1^\circ \subset M_1 \subset \Omega$ and $F \subset T^\circ \subset T \subset T_1^\circ \subset T_1 \subset G$. Let also V be an open set in $\mathbb{C}$ containing K. We consider another two compact sets S and S_1 such that $K \subset S^\circ \subset S \subset S_1^\circ \subset S_1 \subset V$. Then $\mathrm{dist}(M \times T \times S, (M_1^\circ \times T_1^\circ \times S_1^\circ)^c) > r$ for some $r > 0$. Suppose $g \in E(K, F, L, h, I, s, n)$. We show that each $\varphi \in H(\Omega \times G)$ which is sufficiently (uniformly) close to g on the compact set $M_1 \times T_1 \subset \Omega \times G$ belongs to $E(K, F, L, h, I, s, n)$.

By Cauchy estimates on discs with radius r centered on points of $M \times T \supset M^\circ \times T^\circ$ we conclude that $\widetilde{S}_n(\varphi, w, \zeta)(z)$ and $\widetilde{S}_n(g, w, \zeta)(z)$ are close on the open set $M^\circ \times T^\circ \times S^\circ$ if φ is uniformly close to g on the compact set $M_1 \times T_1$. Since $D_{\alpha_1,\alpha_2,\alpha_3}$ is a continuous operator on $H(M^\circ \times T^\circ \times S^\circ)$ it follows that $D_{\alpha_1,\alpha_2,\alpha_3}\widetilde{S}_n(\varphi, w, \zeta)(z)$ and $D_{\alpha_1,\alpha_2,\alpha_3}\widetilde{S}_n(g, w, \zeta)(z)$ are uniformly close on the compact set $L \times F \times K \subset M^\circ \times T^\circ \times S^\circ$. Therefore, $\varphi \in E(K, F, L, h, I, s, n)$ and this set is open.

Next we will show that the sets $\bigcup_n E(K_m, F_\tau, L_\rho, h_j, I, s, n)$ are dense in $H(\Omega \times G)$.

Let $f \in H(\Omega \times G)$, let $\widetilde{L} \subset \Omega$ a compact set, $\widetilde{F} \subset G$ another compact set and $\varepsilon > 0$. Without loss of generality we may assume that $L_\rho \subset \widetilde{L}$ and that $\widetilde{L}^c$ is connected and $F_\tau \subset \widetilde{F}$. We have to find $n \in \{0, 1, 2, \ldots\}$ and $g \in E(K_m, F_\tau, L_\rho, h_j, I, s, n)$ such that

$$\sup_{z\in\widetilde{L}} \sup_{w\in\widetilde{F}} \left| g(z, w) - f(z, w) \right| < \varepsilon.$$

We consider the sets $\widetilde{L} \times \widetilde{F}$ and $K_m \times F_\tau$. Since $\widetilde{L}$ and K_m are disjoint compact sets in $\mathbb{C}$ with connected complements we can find two disjoint simply connected open sets Ω_1 and Ω_2 such that $\widetilde{L} \subset \Omega_1 \subset \Omega$ and $K_m \subset \Omega_2 \subset \mathbb{C}$. We also recall that the open set G contains $\widetilde{F}$ and G is simply connected. The open sets $\Omega_1 \times G$ and $\Omega_2 \times G$ in $\mathbb{C}^2$ are disjoint and $(\Omega_1 \times G) \cup (\Omega_2 \times G) = (\Omega_1 \cup \Omega_2) \times G$ is a product of two simply connected planar open sets. Therefore, Runge's theorem (see e.g. [5]) can be applied to this set.

We consider the holomorphic function $\varphi : (\Omega_1 \cup \Omega_2) \times G \to \mathbb{C}$ defined by $\varphi(z, w) = f(z, w)$ on $\Omega_1 \times G$ and $\varphi(z, w) = h_j(z, w)$ on $\Omega_2 \times G$. Runge's theorem yields a sequence of polynomials $g_\lambda(z, w), \lambda = 1, 2, \ldots$ converging to $\varphi(z, w)$ uniformly on each compact set of the open set $(\Omega_1 \cup \Omega_2) \times G$. Weierstrass' theorem implies that $D_{\alpha_1,\alpha_2} g_\lambda(z, w) \underset{\lambda\to\infty}{\longrightarrow} D_{\alpha_1,\alpha_2}\varphi(z, w)$ uniformly on compacta of $(\Omega_1 \cup \Omega_2) \times G$. Thus, we can find λ so that, if we set $g = g_\lambda$, we have

$$\sup_{z\in\widetilde{L}} \sup_{w\in\widetilde{F}} \left| g(z, w) - f(z, w) \right| < \varepsilon$$

and

$$\sup_{w\in F_\tau}\sup_{z\in K_m}\left|D_{\alpha_1,\alpha_2}g(z,w)-D_{\alpha_1,\alpha_2}h(z,w)\right|<1/s$$

for all α_1,α_2 with $(\alpha_1,\alpha_2,0)\in I$. Now, since g is a polynomial

$$\widetilde{S_n}(g,w,\zeta)(z)=g(w,z)\qquad\text{for all }\zeta,$$

provided n is bigger than the degree of g. Thus

$$D_{\alpha_1,\alpha_2,0}\widetilde{S_n}(g,w,\zeta)(z)=D_{\alpha_1,\alpha_2}g(z,w)$$

and therefore, since $D_{\alpha_1,\alpha_2,0}h(z,w)=D_{a_1,a_2}h(z,w)$,

$$\sup_{w\in F_\tau}\sup_{z\in K_m}\sup_{\zeta\in L_\rho}\left|D_{\alpha_1,\alpha_2,0}\widetilde{S_n}(g,w,\zeta)(z)-D_{\alpha_1,\alpha_2,0}h(z,w)\right|<\frac{1}{s}.$$

If $\alpha_3\neq 0$ then $D_{\alpha_1,\alpha_2,\alpha_3}\widetilde{S_n}(g,w,\zeta)(z)=D_{\alpha_1,\alpha_2,\alpha_3}g(z,w)=0$ as well as $D_{\alpha_1,\alpha_2,\alpha_3}h(z,w)=0$. It follows that

$$\sup_{w\in F_\tau}\sup_{z\in K_m}\sup_{\zeta\in L_\rho}\left|D_{\alpha_1,\alpha_2,\alpha_3}\widetilde{S_n}(g,w,\zeta)(z)-D_{\alpha_1,\alpha_2,\alpha_3}h(z,w)\right|=0<\frac{1}{s}.$$

Therefore $\bigcup_n E(K_m,F_\tau,L_\rho,h_j,I,s,n)$ is open and dense in the complete metrizable space $H(\Omega\times G)$. Baire's theorem yields that their denumerable intersection is also G_δ and dense. This proves that $U'(\Omega,G)$ is G_δ and dense.

Remark 2.13

- The classes $U'(\Omega,G,b)$ and $U'(\Omega,G)$ are subsets of $H(\Omega\times G)$. We can consider analogous classes in $A^\infty(\Omega\times G)$ (see also [8, 11]). We remind that a holomorphic function $f\in H(\Omega,G)$ belongs to $A^\infty(\Omega\times G)$, iff $D_{\alpha_1,\alpha_2}f$ extends continuously to $\overline{\Omega\times G}$ for all differential operators $D_{\alpha_1,\alpha_2}=\frac{\partial^{\alpha_1}}{\partial z^{\alpha_1}}\frac{\partial^{\alpha_2}}{\partial w^{\alpha_2}}$, $\alpha_1,\alpha_2\in\{0,1,2,\ldots,\}$. The topology of $A^\infty(\Omega\times G)$ is defined by the seminorms

$$\sup_{(z,w)\in\overline{\Omega\times G},\|(z,w)\|\le n}\left|D_{\alpha_1,\alpha_2}f(z,w)\right|,\ n,\alpha_1,\alpha_2\in\{0,1,2,\ldots\}.$$

  In the new definitions the supremuma with respect to z,w,ζ will be calculated on compact subsets of $\overline{\Omega},\overline{G}$ and $\overline{\Omega}$ respectively, but the universal approximation will be required on compact subsets $K\times G$, $K\cap\overline{\Omega}\neq\emptyset$, K^c connected only. These new classes will be residual in $A^\infty(\Omega\times G)$. The proof is similar to the proof of Theorem 2.12 mainly because the function g, which is a polynomial, obviously belongs to $A^\infty(\Omega\times G)$.

- In all the above results the set $\Omega \times G$ can be replaced by $\Omega \times G_1 \times \cdots \times G_d$, where $\Omega, G_1, \ldots, G_d$ are planar simply connected domains. The proofs are largely the same, because every function $f \in H(\Omega \times G_1, \times \ldots \times G_d)$ can be approximated uniformly on compacta by polynomials [5].
- Consider any infinite subset μ of the set of natural numbers. Then in the definition of the class $U(\Omega, G)$ if we require that $\lambda_j \in \mu$ for all $j = 1, 2, \ldots$, then we find another class $U^{\mu}(\Omega, G)$. This class is also residual. The proof is similar to the proof of the main result of Theorem 2.12. It suffices to mention two points. First, in the description of the class as intersection of a union, the union this time will be taken only for $n \in \mu$. Second, in the density argument we find a polynomial $g(\cdot, \cdot)$ and then we choose a natural number n greater than the degree of g. Certainly we can choose $n \in \mu$, because μ is an infinite subset of the set of natural numbers. Thus $U^{\mu}(\Omega, G)$ is also residual and hence dense. This implies in a standard way [1] algebraic genericity. That is, $U(\Omega, G) \cup \{0\}$ contains a vector subspace dense in $H(\Omega \times G)$.

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Interpolation by Bounded Analytic Functions and Related Questions

Arthur A. Danielyan

Abstract The paper investigates some interpolation questions related to the Khinchine–Ostrowski theorem, Zalcman's theorem on bounded approximation, and Rubel's problem on bounded analytic functions.

Keywords Bounded analytic functions • Bounded approximation • Fatou's interpolation theorem • G_δ set of measure zero

Mathematics Subject Classification: 30H05, 30H10

1 Introduction

Let $C(K)$ be the set of all continuous complex valued functions on a given subset K of $\mathbb{C}$. Let $\mathbb{D}$ and $\mathbb{T}$ be the open unit disk and the unit circle, respectively. As usual, H^∞ is the space of all bounded analytic functions on $\mathbb{D}$, and the familiar disc algebra A is the set of all elements of H^∞ that can be continuously extended to the closed unit disc.

The formulation and the proof of the following theorem of Khinchine and Ostrowski (in an even more general version) can be found in Privalov's book [9, p. 118].

Theorem A (Khinchine–Ostrowski) *Let $\{f_k\}$ be a sequence of functions analytic on $\mathbb{D}$ which satisfy the following conditions:*

(a) *there exists $M > 0$ such that $|f_k(z)| \leq M$ on $\mathbb{D}$, $k = 1, 2, \ldots$, and*
(b) *the sequence $\{f_k(e^{i\theta})\}$ of radial boundary values of $f_k(z)$ converges at each point of some subset $E \subset \mathbb{T}$ of positive measure.*

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Then $\{f_k\}$ *converges uniformly on compact subsets of* $\mathbb{D}$ *to a bounded analytic function* f, *and* $\{f_k(e^{i\theta})\}$ *converges almost everywhere on* E *to the radial boundary values* $f(e^{i\theta})$ *of* f.

The following theorem of Zalcman [10] is related to Theorem A.

Theorem B *Let* Γ *be a proper closed arc on* $\mathbb{T}$. *A function* $f \in C(\Gamma)$ *is uniformly approximable (on* Γ*) by polynomials* P_n *satisfying* $|P_n(z)| \leq M$ *on* $\mathbb{D}$, $n = 1, 2, \ldots$, *if and only if there exists a function* g *analytic on* $\mathbb{D}$, $|g(z)| \leq M$ *on* $\mathbb{D}$, *such that*

$$f\left(e^{i\theta}\right) = \lim_{r\to 1} g\left(re^{i\theta}\right), \qquad e^{i\theta} \in \Gamma.$$

When the arc Γ is replaced by an arbitrary closed subset F of $\mathbb{T}$, we have (see [3]):

Theorem C *Let* F *be a closed subset of* $\mathbb{T}$. *In order that a function* $f \in C(F)$ *be uniformly approximable on* F *by polynomials* P_n *such that* $|P_n(z)| \leq M$ *on* $\mathbb{D}$, $n = 1, 2, \ldots$, *it is necessary and sufficient that the following conditions be satisfied:*

(i) $|f(z)| \leq M$, $z \in F$, *and*
(ii) *There exists a function* g *analytic on* $\mathbb{D}$, $|g(z)| \leq M$ *on* $\mathbb{D}$, *such that*

$$f\left(e^{i\theta}\right) = \lim_{r\to 1} g\left(re^{i\theta}\right)$$

for almost all $e^{i\theta} \in F$.

Note that Theorem B is for proper arcs. Even Theorem C does not formally assume that the closed set F is proper, it becomes trivial when $F = \mathbb{T}$ (that is, together with Poisson integral representation formula, it gives the following well known fact: $f \in C(\mathbb{T})$ is uniformly approximable by polynomials if and only if f is in the disc algebra A).

Of course, Theorem A implies the necessity part of Theorem C, but it does not imply the necessity part of Theorem B. Indeed, the necessity part of Theorem B provides the equation

$$f\left(e^{i\theta}\right) = \lim_{r\to 1} g\left(re^{i\theta}\right)$$

everywhere on Γ including its endpoints, while Theorem A does not imply the same equation at the end points of Γ. Thus, the necessity part of Theorem B can be considered as a certain strengthening of the conclusion of Theorem A.

We are interested in particular in closed subsets of $\mathbb{T}$ which "behave" like closed arcs of $\mathbb{T}$. In this direction one can formulate the following open problem.

Problem 1 Describe all closed subsets F on $\mathbb{T}$ such that whenever $f \in C(F)$ is uniformly approximable on F by a sequence of polynomials, uniformly bounded on $\mathbb{T}$, then there exists a function $g \in H^\infty$ with radial limits existing and coinciding with f everywhere on F.

Note that the Rudin–Carleson theorem immediately implies that also the closed subsets of $\mathbb{T}$ of measure zero have the property mentioned in Problem 1.

The following recent (unpublished) theorem of Gardiner [Gardiner, S.J.: Response to a question of A. Danielyan. Private communication.] provides an answer to a question asked by the author.

Theorem 1 *Let $E \subset \mathbb{T}$ be a closed set that has positive lower Lebesgue density at every constituent point. If a function $f \in C(E)$ is uniformly approximable by polynomials P_n satisfying $|P_n(z)| \leq M$ on $\mathbb{D}$, $n = 1, 2, \ldots$, then there exists an analytic function g on $\mathbb{D}$ satisfying $|g(z)| \leq M$ on $\mathbb{D}$ and*

$$f\left(e^{i\theta}\right) = \lim_{r \to 1} g\left(re^{i\theta}\right) \qquad \left(e^{i\theta} \in E\right). \tag{1}$$

If $f \in C(E)$ and there exists an analytic function g on $\mathbb{D}$ satisfying $|g(z)| \leq M$ on $\mathbb{D}$ and

$$f\left(e^{i\theta}\right) = \lim_{r \to 1} g\left(re^{i\theta}\right) \qquad \left(e^{i\theta} \in E\right)$$

then, by Theorem C, the function f is uniformly approximable by polynomials P_n satisfying $|P_n(z)| \leq M$ on $\mathbb{D}$, $n = 1, 2, \ldots$.

Thus we have the following corollary (of Theorem C and Theorem 1) providing a new subclass of the class of sets which Problem 1 requires us to describe.

Corollary 1 *Let $E \subset \mathbb{T}$ be a closed set that has positive lower Lebesgue density at every constituent point. A function $f \in C(E)$ is uniformly approximable by polynomials P_n satisfying $|P_n(z)| \leq M$ on $\mathbb{D}$, $n = 1, 2, \ldots$, if and only if there is an analytic function g on $\mathbb{D}$ satisfying $|g(z)| \leq M$ on $\mathbb{D}$ and*

$$f\left(e^{i\theta}\right) = \lim_{r \to 1} g\left(re^{i\theta}\right) \qquad \left(e^{i\theta} \in E\right).$$

In Sect. 2 below we recall the definition of lower Lebesgue density. Such standard sets as closed (or open) intervals, of course, have positive lower Lebesgue density at every constituent point. As Buczolich [2] has shown there exist also Cantor sets with the same property.

If $f \in C(F)$ is uniformly approximable by a sequence of polynomials, uniformly bounded on $\mathbb{T}$, then by Theorem C (or by Theorem A) there exists a function $g \in H^\infty$ the radial limits of which are equal to f a.e. on F. This brings us to the following formulation:

Problem 2 Describe all closed subsets F on $\mathbb{T}$ such that whenever $f \in C(F)$ coincides a.e. on F with the radial limits of a function $g \in H^\infty$, then the radial limits of g exist on F and coincide with f at each point of F.

A further generalization of Problem 2 is the following problem.

Problem 3 Describe all G_δ subsets F on $\mathbb{T}$ such that whenever $f \in C(F)$ coincides a.e. on F with the radial limits of a function $g \in H^\infty$, then the radial limits of g exist on F and coincide with f at each point of F.

A simpler (still open) problem for a restricted set of functions defined on F is:

Problem 4 Describe all G_δ subsets F on $\mathbb{T}$ such that whenever $f \in C(\mathbb{T})$ coincides a.e. on F with the radial limits of a function $g \in H^\infty$, then the radial limits of g exist on F and coincide with f at each point of F.

It is easy to see that a closed subset of $\mathbb{T}$ of zero measure does not belong to the class of sets which Problem 2 (or, Problem 3 or 4) is requiring to describe as the class of sets mentioned contains "massive" sets only.[1]

However, for sets of measure zero too one can formulate appropriate interpolation problems. The most famous such problems have been solved, of course, by the classical Fatou (Theorem E below) and the Rudin–Carleson interpolation theorems. A further such interpolation problem proposed by Rubel (see [6, p. 168]) has been solved by the following recent result of the author [4].

Theorem D *Let F be a G_δ set of measure zero on $\mathbb{T}$. Then there exists a function $g \in H^\infty$ non-vanishing in $\mathbb{D}$ such that the radial limits of g exist everywhere on $\mathbb{T}$ and vanish precisely on F.*

If F is merely closed, the following result provides a more precise conclusion (see [7, p. 80]).

Theorem E *Let F be closed and of measure zero on $\mathbb{T}$. Then there exists an element in the disc algebra which vanishes precisely on F.*

It is well known that the existences of radial and angular limits of a function $g \in H^\infty$ at a point $t \in \mathbb{T}$ are equivalent. But, of course, the existence of the unrestricted limit at $t \in \mathbb{T}$ is a stronger requirement than the existence of the angular limit at $t \in \mathbb{T}$. (We say that g has an unrestricted limit at $t \in \mathbb{T}$ if the limit of g exists when $z \in \mathbb{D}$ approaches to t arbitrarily in $\mathbb{D}$.)

In the general case the function g in Theorem D cannot belong to the disc algebra. However, G_δ sets are sets of points of continuity, and this brings us to the idea of making the function g continuous on the set F at least. Below we show that this indeed is possible; we have the following new complement of Theorem D.

Theorem 2 *Let F be a G_δ set of measure zero on $\mathbb{T}$. Then there exists a function $g \in H^\infty$ non-vanishing in $\mathbb{D}$ such that:*

1) *g has non-zero radial limits everywhere on $\mathbb{T} \setminus F$; and*
2) *g has vanishing unrestricted limits at each point of F.*

---

[1] If a closed set E is of positive measure but has a portion of measure zero, then even such a set cannot belong to the class of sets which Problem 2 requires to describe. See Sect. 2 below for the definition of a portion of E.

The condition on F is not only sufficient, but necessary (cf. [4]). Both Theorems D and 2 extend Theorem E from closed sets to G_δ sets. The conclusion (2) makes Theorem 2 a better analogue of Theorem E.

As we noted, the condition on F in Theorem 2 is necessary as well. Thus, Theorem 2 describes all such sets F (on $\mathbb{T}$) for each of which there exists some $f \in H^\infty$ having vanishing unrestricted limits on F and non-zero radial limits on $\mathbb{T} \setminus F$ (the description is that F is a G_δ of measure zero).

Note that an arbitrary G_δ set on $\mathbb{T}$ is precisely the set of unrestricted limits for some $f \in H^\infty$ as Brown et al. [1] have shown (see [1, p. 52]). Their result is:

Theorem F *Let E be a G_δ set on $\mathbb{T}$. Then there exists a function $f \in H^\infty$ which has unrestricted limits at each point of E and at no point of $\mathbb{T} \setminus E$.*

Theorem E is a base for the Rudin–Carleson theorem; cf. [7, pp. 80–81]. (Note that the paper [5] derives the Rudin–Carleson theorem merely from Theorem E.) Similarly Theorems D and 2 bring us to the questions on the possibility of proving the appropriate versions of the Rudin–Carleson theorem for G_δ sets. The corresponding problem can be formulated in two parts as follows.

Problem 5

a) Let F be a G_δ set of measure zero on $\mathbb{T}$ and let either $f \in C(\mathbb{T})$ or, more generally, $f \in C(F)$. Then does there exist a function $g \in H^\infty$ such that the radial limits of g exist everywhere on $\mathbb{T}$ and coincide with f on F?
b) In addition to the requirement of part a), is it possible that g has unrestricted limits at each point of F?

Theorem 1 requires a closed set to be of positive lower Lebesgue density at every constituent point. In an attempt to relax this requirement, one can ask: Does any closed set with no portion[2] of measure zero belong to the class of sets which Problem 1 is asking to describe? The following result gives a negative answer to this question.

Theorem 3 *There exists a closed set $F \subset \mathbb{T}$ having no portion of measure zero and a function $f \in C(F)$ uniformly approximable on F by polynomials which are uniformly bounded on $\mathbb{T}$, such that no function $g \in H^\infty$ has radial limits equal to f at every point of F.*

2 Some Definitions, Auxiliary Results, and Remarks

The terminology used above is known, but we quickly mention some details just in case to avoid any possible confusion. Let $F \subset \mathbb{T}$ be closed; if $J \subset \mathbb{T}$ is an open arc containing a point of F, we call the intersection $F \cap J$ a portion of F. Buczolich [2]

[2] See Sect. 2 for the definition of portion of a closed set.

calls a nowhere dense perfect set with no portion of measure zero a *fat Cantor set*, but we do not use this term.

Let m be the (normalized) Lebesgue measure on $\mathbb{T}$. The lower density of F at $t \in \mathbb{T}$, denoted by $\underline{D}(t, F)$, is defined as

$$\underline{D}(t, F) = \liminf_{h \to 0} \frac{m(I_t(h) \cap F)}{2h},$$

where $I_t(h)$ is an open arc on $\mathbb{T}$ of length $2h$ and of midpoint at t (cf. [2, p. 497]). For a closed F, of course, $\underline{D}(t, F) = 0$ at all $t \in \mathbb{T} \setminus F$.

If a closed set F has a portion of measure zero, then obviously $\underline{D}(t, F) = 0$ at any $t \in F$ of that portion. But there are many closed sets which have positive lower Lebesgue density at every constituent point. In particular, as Buczolich [2, p. 499] has shown, there exist Cantor sets with lower Lebesgue density ≥ 0.5 at every constituent point; for any such set on $\mathbb{T}$ the above Theorem 1 is applicable.

The proof of Theorem 1 uses the standard lemma below (formulated in [**?**]), which follows from the classical theory (cf., e.g., Zygmund's book [11]).

For a Lebesgue integrable function u on $\mathbb{T}$ we denote by H_u the Poisson integral of u.

Lemma 1 *Let* $u : \mathbb{T} \to [-\infty, \infty]$ *be Lebesgue integrable. Then the Poisson integral* H_u *in* $\mathbb{D}$ *satisfies*

$$\liminf_{r \to 1} H_u(re^{i\theta}) \geq \liminf_{t \to 0+} \frac{1}{2t} \int_{[-t,t]} u(e^{i(\theta+\phi)}) d\phi \quad (0 \leq \theta \leq 2\pi).$$

For the convenience of the reader we present the proof of Lemma 1 in the next section.

The following lemma of Kolesnikov [8] is important for Theorem 2 (and Theorem D).

Lemma 2 *Let* G *be an open subset on* $\mathbb{T}$ *and let* $F \subset G$ *be a set of measure zero on* $\mathbb{T}$*. For any* $\epsilon > 0$ *there exists an open set* O*,* $F \subset O \subset G$*, and a function* $g \in H^\infty$ *such that:*

1) $|g(z)| < 2$, $0 < \Re g(z) < 1$ *for* $z \in \mathbb{D}$*;*
2) *the function* g *has a finite radial limit* $g(\zeta)$ *at each point* $\zeta \in \mathbb{T}$*;*
3) *at the points* $\zeta \in O$ *the function* g *is analytic and* $\Re g(\zeta) = 1$*;*
4) $|g(z)| \leq \epsilon$ *on every radius* R_{ζ_0} *with end-point at* $\zeta_0 \in \mathbb{T} \setminus G$*.*

3 Proofs

Proof (Lemma 1) The proof follows from Fatou's classical results presented in [11, pp. 99–101].

In this proof we identify the point $e^{i\theta} \in \mathbb{T}$ with $\theta \in [0, 2\pi]$ as usual. Let u be Lebesgue integrable function on $[0, 2\pi]$ (as in Lemma 1) and let U be the indefinite integral of u. Recall that the first symmetric derivative of U at x_0 is

$$D_1 U(x_0) := \lim_{h\to 0+} \frac{U(x_0 + h) - U(x_0 - h)}{2h}.$$

The appropriate upper and lower limits are called the upper and lower first symmetric derivatives, and are denoted by $\overline{D_1} U(x_0)$ and $\underline{D_1} U(x_0)$, respectively (cf. [11, p. 99]).

The direct calculations imply

$$\underline{D_1} U(\theta) = \liminf_{t\to 0+} \frac{1}{2t} \int_{-t}^{t} u(\theta + \phi) d\phi \qquad (0 \le \theta \le 2\pi).$$

Since the right side of this equation is nothing else but the right side of the inequality in Lemma 1, to prove Lemma 1 one needs to verify that

$$\liminf_{r\to 1} H_u \left(re^{i\theta}\right) \ge \underline{D_1} U(\theta) \qquad (0 \le \theta \le 2\pi). \tag{2}$$

Consider the Fourier series $S[u]$ of the function u. As Zygmund notes immediately after the formulation of Theorem 7.9 in [11, p. 101], we may suppose that $S[u] = S'[U]$, where $S'[U]$ is the formally differentiated Fourier series of U. Thus for the series $S[u]$, part (i) of Theorem 7.9 of [11], can be strengthened by the second part of Theorem 7.2 (from [11, pp. 99–100]). The second part of Theorem 7.2 simply states that the limits of indetermination of Abel summation of $S'[U]$ as $r \to 1$ are contained between $\underline{D_1} U(\theta)$ and $\overline{D_1} U(\theta)$ for all $0 \le \theta \le 2\pi$. Since $S[u] = S'[U]$, the same is true for the limits of indetermination of Abel summation of $S[u]$. Thus

$$\overline{D_1} U(\theta) \ge \limsup_{r\to 1} H_u \left(re^{i\theta}\right) \ge \liminf_{r\to 1} H_u \left(re^{i\theta}\right) \ge \underline{D_1} U(\theta) \qquad (0 \le \theta \le 2\pi),$$

which obviously implies (2). Lemma 1 is proved.

We present the original proof of Theorem 1 from [**?**].

Proof (Theorem 1) Suppose $P_n \to f$ uniformly on E and $|P_n(z)| \le M$ on $\mathbb{D}$ for each n. Then, by subharmonicity,

$$\log |P_n - P_m| \le H_{\log|P_n - P_m|} \le \log^+(2M) + H_{\log|P_n - P_m|\chi_E} \text{ on } \mathbb{D}. \tag{3}$$

Since E has positive measure, $\{P_n\}$ is locally uniformly convergent on $\mathbb{D}$ to some analytic function g. (This follows from (3) using the estimate of $H_{\log|P_n - P_m|\chi_E}$ in terms of the harmonic measure of E; cf. [10, p. 379–380]. Of course, the same conclusion also follows from Theorem A.)

If $e^{i\theta} \in E$, then by hypothesis

$$\alpha_\theta := \liminf_{t\to 0+} \frac{1}{2t} \int_{[-t,t]} \chi_E\left(e^{i(\theta+\phi)}\right) d\phi > 0.$$

By Lemma 1 we can choose $r_\theta \in (0,1)$ such that

$$H_{\chi_E}\left(re^{i\theta}\right) \geq \frac{\alpha_\theta}{2} \qquad (r_\theta \leq r < 1)\,.$$

For large m, n it follows from (3) that

$$\log|P_n - P_m|\left(re^{i\theta}\right) \leq \log^+(2M) + \frac{\alpha_\theta}{2}\max_E \log|P_n - P_m| \qquad (r_\theta \leq r \leq 1)\,.$$

Thus $\{P_n\}$ converges uniformly on $\left\{re^{i\theta} : r_\theta \leq r \leq 1\right\}$, and (1) follows.

Theorem 1 is proved. □

Proof (Theorem 2) This proof is completely parallel to the proof of Theorem D in [4], and only contains an additional argument (observation), which we will indicate here. In [4] the above Lemma 2 has been used for the proof of Theorem D, which yields a function $g \in H^\infty$ with all needed properties except the property of having unrestricted vanishing limits at each point of F (the function g merely has vanishing radial limits at the points of F).

Without repeating the proof of Theorem D, we refer the reader to this proof in [4], where analytic (on $\mathbb{D}$) functions g_k are constructed such that their sum $\sum_{k=1}^{\infty} g_k(z) = h(z)$ is analytic as well.

As indicated in [4], the functions g_k in particular have such properties: $\Re g_k(z) > 0$ for $z \in \mathbb{D}$; at the points $\zeta \in O_k$ the function g_k is analytic and $\Re g_k(\zeta) = 1$, where O_k is an open set on $\mathbb{T}$ containing F $(k = 1, 2, \ldots)$.

We conclude that $\Re h(z) > 0$ for $z \in \mathbb{D}$. Also, since $\Re g_k(\zeta) = 1$ on O_k and $F \subset O_k$, obviously, not only the radial limit, but also the unrestricted limit of $\Re h(z)$ is $+\infty$ at each point of F.

The analytic function $f = 1/(1+h)$ is bounded by 1. Obviously it has vanishing unrestricted limits at each point of F, and as in [4], the function f also has all other necessary properties.

Theorem 2 is proved. □

Proof (Theorem 3) Let D be the "outer snake" domain (also known as the cornucopia) in the w-plane (D is a spiral domain around the unit circle $|w| = 1$). The circle $|w| = 1$ is an impression of a prime end R of the simply connected domain D. Let $w = \varphi(z)$ be the Riemann mapping function of $\mathbb{D}$ onto D. By Carathéodory's theorem, under the mapping $w = \varphi(z)$, the prime end R corresponds to a point A of the unit circle $\mathbb{T}$. Without loss of generality, we may assume A is 1. Then the radial limit of $\varphi(z)$ at the point 1 does not exist, because the image of the radius ending at 1 is a spiral surrounding the circle $|w| = 1$ infinitely many times. On the other hand,

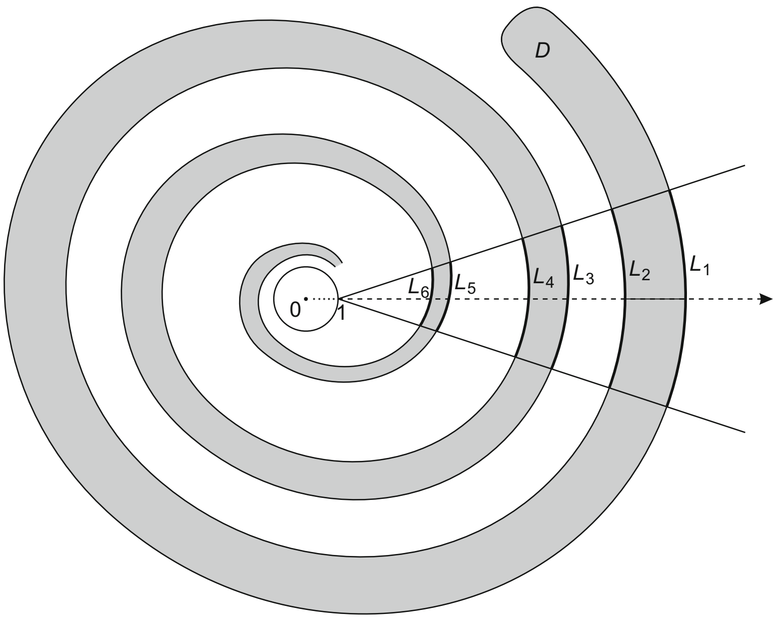

Fig. 1 The "outer snake" domain

at each point of the set $\mathbb{T} \setminus \{1\}$, the function $\varphi(z)$ has a radial limit; it can be defined by its radial limits on the set $\mathbb{T} \setminus \{1\}$, and the extended function is then continuous on $\mathbb{T} \setminus \{1\}$.

The inverse mapping function $z = \varphi^{-1}(w)$ can be extended continuously at each boundary point of D which does not belong to $|w| = 1$. This follows from the fact that each such point is an accessible boundary point of D.

In the w-plane consider an acute angle with vertex at $w = 1$ and such that the half line $[1, \infty)$ is a bisector for the angle; thus the angle lies outside of $|w| = 1$, and only its vertex is on the circle (see Fig. 1). This angle "cuts" countably many Jordan arcs on the boundary of the domain D, which we denote by $L_1, L_2, L_3, \ldots$ in the order from the right to left (so that L_1 is the farthest from $|w| = 1$). We take the set L_n to be closed so that L_n contains its endpoints. Under the mapping $z = \varphi^{-1}(w)$, the image of L_n is a closed arc Γ_n on $\mathbb{T}$. Clearly the arcs Γ_n are disjoint.

Recalling that the prime end R of D corresponds to the point $1 \in \mathbb{T}$, we conclude that the arcs Γ_n on $\mathbb{T}$ accumulate to 1 from either side (for this we also use the fact that the arcs L_n lie on "both" sides of the spiral domain D and they approach the circle $|w| = 1$). Thus, the set $F := \bigcup_{n=1}^{\infty} \Gamma_n \cup \{1\}$ is a closed set (on $\mathbb{T}$) and has no portion of measure zero. Let $f(z) = \varphi(z)$ if $z \in F \setminus \{1\}$ and $f(1) = 1$. Then f is continuous on F. Continuity on the arcs Γ_n is obvious, while continuity at the point 1 follows from the construction of the arcs L_n. Indeed, if a sequence $\{z_k\} \subset F$ approaches 1, then $z_k \in \Gamma_{n_k}$ for certain natural numbers n_k approaching infinity. Thus $f(z_k) \in L_{n_k}$; and because the arcs L_{n_k} approach the point 1 (the vertex of the above described angle), $f(z_k)$ approaches 1 as k tends to ∞.

The function f is equal to the radial limits of $\varphi(z)$ at all points of F except the point 1. Thus, by Theorem C, the function f is uniformly approximable on F by polynomials that are uniformly bounded on $\mathbb{D}$. On the other hand, no bounded analytic function can have radial limits equal to f at all points of F. Indeed, by the boundary uniqueness theorem, any such function must be identical with the function $\varphi(z)$, while as we have already seen, the radial limit of $\varphi(z)$ does not exist at $1 \in F$. This completes the proof of Theorem 3. □

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On Two Interpolation Formulas for Complex Polynomials

Richard Fournier and Stephan Ruscheweyh

Abstract We discuss, from various points of view (for example the unicity of nodes), two recent interpolation formulas for algebraic polynomials leading to various Bernstein-Markov type inequalities. We also show that each formula contains, as a special case, the Marcel Riesz interpolation formula for trigonometric polynomials.

Keywords Interpolation formulas for polynomials • Bernstein-Markov inequalities

2000 *Mathematics Subject Classification*: 30H05, 30H10

1 Introduction and Statement of the Results

Let $\mathbb{D}$ denote the unit disc of the complex plane and let $\mathcal{P}_n$ be the class of complex polynomials of degree at most n. In this note we compare two general interpolation formulas for the class $\mathcal{P}_n$.

Given a system of angles

$$\Theta := \{\theta_j : 0 \leq \theta_0 < \theta_1 < \ldots \theta_{n-1} < \theta_n \leq \pi\} \tag{1}$$

we define

$$w(z) := \prod_{j=0}^{n} (z - \cos\theta_j)$$

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and

$$L_j(z) = \frac{w(z)}{(z - \cos\theta_j)w'(\cos\theta_j)} := \sum_{k=0}^{n} a_{k,j} T_k(z),\ j = 0, \dots, n,$$

where T_k stands for the k-th Chebyshev polynomial. We also set

$$l_j(z) := \sum_{k=0}^{n} a_{k,j} z^k,\ j = 0, \dots, n.$$

1.1 The First Interpolation Formula

It has been shown in [3] that for any linear functional $\mathfrak{L}$ over $\mathcal{P}_n$ we have

$$\mathfrak{L}(p) = \sum_{j=0}^{n} \mathfrak{L}(l_j) \frac{p(e^{i\theta_j}) + p(e^{-i\theta_j})}{2}, \quad p \in \mathcal{P}_n. \tag{2}$$

The known proofs of (2) depend on quadrature formulae or else on the Lagrange interpolation formula, see [1–3] for details.

1.2 A Special Case

The functionals

$$\mathfrak{S}_t(p) := \frac{p(e^{it}) - p(e^{-it})}{e^{it} - e^{-it}}, \quad 0 \le t \le \pi,\ p \in \mathcal{P}_n,$$

are of particular interest. In [1–3] it has been shown that

$$\mathfrak{S}_t(p) = \sum_{j=0}^{n} c(t,j) \frac{p(e^{ij\pi/n}) + p(e^{-ij\pi/n})}{2}, \tag{3}$$

where

$$c(t,j) = \begin{cases} \dfrac{(-1)^j}{n} \dfrac{\cos(j\pi) - \cos(nt)}{\cos(j\pi/n) - \cos(t)},\ 1 \le j \le n-1, \\ \dfrac{(-1)^j}{2n} \dfrac{\cos(j\pi) - \cos(nt)}{\cos(j\pi/n) - \cos(t)},\ j \in \{0, n\}, \end{cases}$$

and

$$\sum_{j=0}^{n} |c(t,j)| \le n, \quad 0 \le t \le \pi. \tag{4}$$

Remark Equation (3) at $t = 0$ for arbitrary $p \in \mathcal{P}_n$ but applied to the polynomial $p(e^{it}z)$ combined with (4) contains the inequality

$$\left| e^{it} p'(e^{it}) - \frac{n}{2} p(e^{it}) \right| \le \frac{n}{4} \max_{\substack{0 \le j \le n \\ j \text{ odd}}} \left| p(e^{i(t+j\pi/n)}) + p(e^{i(t-j\pi/n)}) \right|, \ t \in \mathbb{R},$$

which refines the famous Bernstein inequality for complex polynomials in $\mathbb{D}$. It has also been shown in [1, 2] that (3) contains the Duffin-Schaeffer improvement of Markov's inequality for the first derivative of polynomials $p \in \mathcal{P}_n$ on the unit interval $[-1, 1]$. We refer the reader to the book of Rahman and Schmeisser [6] concerning polynomial inequalities and interpolation formulae.

For an arbitrary set of nodes Θ as in (1) we write

$$\mathfrak{S}_t(p) = \sum_{j=0}^{n} d(t,j) \frac{p(e^{i\theta_j}) + p(e^{-i\theta_j})}{2}, \quad p \in \mathcal{P}_n. \tag{5}$$

The following result has been established in [3].

Theorem 1 *Let n be an odd integer and $d(t,j)$ as in (5). Then*

$$\max_{0 \le t \le \pi} \sum_{j=0}^{n} |d(t,j)| \le n \tag{6}$$

if, and only if, $\Theta = \{j\pi/n, \ j = 0, \ldots, n\}$.

The following theorem fills a gap left by Theorem 1.

Theorem 2 *For n even the conclusion of Theorem 1 is not generally valid.*

This result is implied by the following

Counterexample

Let $n = 4$ *and* $\Theta = \{0, \ \pi/4, \ 2\pi/5, \ 3\pi/4, \ \pi\}$. *Then*

$$d(t,0) = \frac{(1+\cos(t))\cos(t)(-1+\sqrt{5}-4\cos(t))}{-5+\sqrt{5}}$$

$$d(t,1) = \frac{(-2-\sqrt{8}+\sqrt{20})\cos(t) + (-4-\sqrt{2}+\sqrt{10})\cos(2t) - \sqrt{8}\cos(3t)}{2+\sqrt{32}-\sqrt{20}}$$

$$d(t,2) = (8 - \frac{24}{\sqrt{5}})\cos(t)\sin(t)^2$$

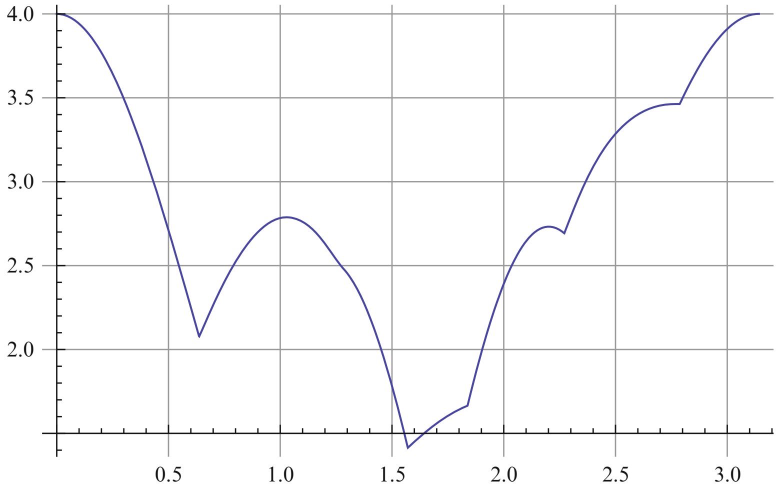

Fig. 1 $S(t)$ ($0 \leq t \leq \pi$); $S(0) = S(\pi) = 4$

$$d(t,3) = \frac{(-2+\sqrt{8}+\sqrt{20})\cos(t) + (-4+\sqrt{2}-\sqrt{10})\cos(2t) + \sqrt{8}\cos(3t)}{-2+\sqrt{32}+\sqrt{20}}$$

$$d(t,4) = \frac{(1-\cos(t))\cos(t)(-1+\sqrt{5}-4\cos(t))}{3+\sqrt{5}}$$

and

$$S(t) := \sum_{j=0}^{n} |d(t,j)| \leq 4, \quad 0 \leq t \leq \pi,$$

with equality if $t = 0$ *and* $t = \pi$*, see Fig. 1.*

However, the following result gives a necessary condition on Θ for the relation (6) to hold.

Theorem 3 *Let* Θ *be as in* (1) *and let the* $d(t,j)$, $0 \leq j \leq n$, *be as in* (5). *Then* (6) *can hold only if*

$$\{\cos(\frac{j\pi}{n}) \,:\, 0 \leq j \leq n\} \subseteq \{\cos(\theta_j \pm \theta_k) \,:\, 0 \leq j,k \leq n\}. \tag{7}$$

Note that (2) can also be written in the form

$$(p * H)(z) = \sum_{j=0}^{n} (l_j * H)(1) \, \frac{p(e^{i\theta_j}z) + p(e^{-i\theta_j}z)}{2} \tag{8}$$

where $*$ denotes the Hadamard product and H is an arbitrary member of $\mathcal{P}_n$.

1.3 The Second Interpolation Formula

Let $\mathcal{P}$ denote the class of analytic functions f in the unit disk $\mathbb{D}$ satisfying $f(0) = 1$ and $\operatorname{Re} f(z) > \frac{1}{2}$. We wish to compare (8) with the following identity:

$$(p * Q)(z) = \sum_{j=1}^{2n} \lambda_j p(w_j \zeta^{1/n} z), \tag{9}$$

which holds for all $z, \zeta \in \partial\mathbb{D}, p \in \mathcal{P}_n$ and $Q \in \mathcal{P} \cap \mathcal{P}_{n-1}$, and where $w_j := e^{ij\pi/n}$ and

$$\lambda_j := \frac{1}{2n}\left(2\operatorname{Re} Q(\overline{w_j \zeta^{1/n}}) - 1\right) \geq 0, \quad \sum_{j=1}^{2n} \lambda_j = 1.$$

This interpolation formula is due to Frappier et al. [5] and is, as has recently been shown (see [4]), actually equivalent to an older result (see [7, Cor. 4.3]), namely

$$|(p * Q)(z)| + |(p * \tilde{Q})(z)| \leq ||p||, \quad z \in \mathbb{D}, \tag{10}$$

for $p \in \mathcal{P}_n$ and $Q \in \mathcal{P} \cap \mathcal{P}_{n-1}$. Here $\tilde{Q}(z) := z^n \overline{Q(1/\bar{z})}$ and $||\cdot||$ denotes the uniform norm in $\mathbb{D}$.

We note in passing that (9) contains a discrete refinement of the Bernstein polynomial inequality on the unit disk:

$$||p'|| \leq n \max_{1 \leq j \leq 2n} |p(w_j)|, \quad p \in \mathcal{P}_n. \tag{11}$$

On the other hand the formulas (8) and (9) are of a very different nature since (9) contains $\zeta \in \partial\mathbb{D}$ as an essentially free parameter which is the key to obtain (11) by setting $Q(z) := \sum_{k=0}^{n}(1 - \frac{k}{n})z^k$ so that Q satisfies the conditions set for Q in the context of (9).

In spite of some similarity the identities (8) and (9) do not seem to be comparable. For instance (8) does not seem to contain (11) while (8) is more flexible in other ways. Still there is some overlap between these two formulas:

Theorem 4 *For n even both the formulas* (3) *and* (9) *imply the Marcel Riesz interpolation formula for trigonometric polynomials.*

Another property of the coefficients in formula (9) concerns the uniqueness of the interpolating nodes under certain circumstances

Theorem 5 *Let $\zeta \in \partial\mathbb{D}, Q \in \mathcal{P}_{n-1} \cap \mathcal{P}$ and assume that for some $k \leq 2n$ a set of distinct nodes $\{V_j : j = 1, \ldots, k\} \subset \partial\mathbb{D}$ has the property that*

$$(Q * p)(z) = \sum_{j=1}^{k} \Lambda_j p(\zeta^{1/n} V_j z), \quad p \in \mathcal{P}_n, \ z \in \mathbb{D}, \tag{12}$$

for complex coefficients $\{\Lambda_j : j = 1, \ldots, k\}$. Then all coefficients Λ_j are positive and $V_j^{2n} = 1$ for $j = 1, \ldots, k$.

We mention that the cases $1 \leq k < 2n$ of Theorem 5 with equally spaced nodes $\{V_j : j = 1, \ldots, k\} \subset \partial\mathbb{D}$ follow from previous work of Frappier et al. [5, Thm. 8].

2 Proof of Theorem 3

With a polynomial $P(z) = \sum_{k=0}^{n} a_k T_k(z)$ we associate $p(z) = \sum_{k=0}^{n} a_k z^k$; the identification $P \leftrightarrow p$ is an isomorphism of $\mathcal{P}_n$ with

$$P(\cos\theta) = \frac{1}{2}(p(e^{i\theta}) + p(e^{-i\theta})).$$

Let

$$\mathfrak{S}_\theta(p) = \frac{p(e^{i\theta}) - p(e^{-i\theta})}{e^{i\theta} - e^{-i\theta}} = \sum_{j=0}^{n} d(\theta, j) \frac{p(e^{i\theta_j}) + p(e^{-i\theta_j})}{2}, \quad p \in \mathcal{P}_n.$$

with

$$\max_\theta \sum_{j=0}^{n} |d(\theta, j)| \leq n.$$

Then

$$p'(1) = \sum_{j=0}^{n} d(0, j) \frac{p(e^{i\theta_j}) + p(e^{-i\theta_j})}{2}$$

and, more generally, for any real φ

$$e^{i\varphi}p'(e^{i\varphi}) = \sum_{j=0}^{n} d(0,j)\frac{p(e^{i(\varphi+\theta_j)}) + p(e^{i(\varphi-\theta_j)})}{2}.$$

Hence

$$\begin{aligned} P'(\cos\varphi) &= \frac{e^{i\varphi}p(e^{i\varphi}) - e^{-i\varphi}p(e^{-i\varphi})}{e^{i\varphi} - e^{-i\varphi}} \\ &= \frac{1}{2}\sum_{j=0}^{n} d(0,j)\left(\frac{p(e^{i(\theta_j+\varphi)}) - p(e^{i(\theta_j-\varphi)})}{e^{i\varphi} - e^{-i\varphi}}\right. \\ &\qquad\qquad\left. + \frac{p(e^{i(-\theta_j+\varphi)}) - p(e^{i(-\theta_j-\varphi)})}{e^{i\varphi} - e^{-i\varphi}}\right) \\ &= \frac{1}{2}\sum_{j,k=0}^{n} d(0,j)d(\varphi,k)\left(\frac{p(e^{i(\theta_j+\theta_k)}) + p(e^{-i(\theta_j+\theta_k)})}{2}\right. \\ &\qquad\qquad\left. + \frac{p(e^{i(\theta_j-\theta_k)}) + p(e^{i(-\theta_j+\theta_k)})}{2}\right) \end{aligned}$$

and therefore, for arbitrary φ,

$$\begin{aligned} |P'(\cos\varphi)| &\le \frac{1}{2}\sum_{j,k=0}^{n} |d(0,j)d(\varphi,k)||P(\cos(\theta_j+\theta_k)) + P(\cos(\theta_j-\theta_k))| \\ &\le n^2 \max_{0\le j,k\le n} |P(\cos(\theta_j \pm \theta_k))|. \end{aligned}$$

Since the last inequality holds for any $P \in \mathcal{P}_n$ it follows by the unicity result of Duffin and Schaeffer (see [6, p. 574]) that

$$\{\cos(\frac{j\pi}{n}) \,:\, 0 \le j \le n\} \subseteq \{\cos(\theta_j \pm \theta_k) \,:\, 0 \le j,k \le n\},$$

and the proof is complete. □

3 Proof of Theorem 4

We first deal with formula (3). For $N \in \mathbb{N}$ let $n = 2N$ and $p \in \mathcal{P}_n$. Then (3) with $t = 0$ gives

$$\begin{aligned}\mathfrak{S}_0(p) &= p'(1) - \frac{n}{2}p(1)\\ &= \sum_{\substack{j=1\\ j \text{ odd}}}^{n-1} \frac{(-1)^j}{n}\frac{\cos(j\pi)-1}{\cos(j\pi/n)-1}\frac{p(e^{ij\pi/n})+p(e^{-ij\pi/n})}{2}\\ &= \frac{-1}{N}\sum_{\substack{j=1\\ j \text{ odd}}}^{2N-1}\frac{p(e^{ij\pi/(2N)})+p(e^{-ij\pi/(2N)})}{2(1-\cos(j\pi/(2N))}.\end{aligned} \tag{13}$$

Let also $u(\theta)$ be a trigonometric polynomial of degree at most N. Then the polynomial $p(e^{i\theta}) := e^{iN\theta}u(\theta)$ belongs to $\mathcal{P}_n$ with

$$\begin{aligned}-iu'(0) &= p'(1) - Np(1)\\ &= \frac{-1}{N}\sum_{\substack{j=1\\ j \text{ odd}}}^{2N-1}\frac{e^{ij\pi/2}u(j\pi/(2N)) + e^{-ij\pi/2}u(-j\pi/(2N))}{2(1-\cos(j\pi/(2N))}\\ &= \frac{-1}{N}\sum_{k=1}^{N}\frac{i(-1)^{k-1}(u((2k-1)\pi/(2N)) - u(-(2k-1)\pi/(2N)))}{2(1-\cos((2k-1)\pi/(2N)))},\end{aligned}$$

and finally

$$\begin{aligned}u'(0) &= \sum_{k=1}^{N}\frac{(-1)^{k-1}}{4N\sin^2((2k-1)\pi/(4N))}\left(u(\frac{(2k-1)\pi}{2N}) - u(\frac{4N-2k+1}{2N}\pi)\right)\\ &= \sum_{k=1}^{2N}\frac{(-1)^{k-1}}{4N\sin^2((2k-1)\pi/(4N))}u(\frac{(2k-1)\pi}{2N}).\end{aligned}$$

This is of course (compare [6, p. 559]) the important Marcel Riesz interpolation formula.

Let us now consider formula (9) with $n = 2N$ even, $\zeta = 1$, and $Q(z) = \sum_{k=0}^{n}(1-\frac{k}{n})z^k$. It is a well-known property of the Fejér kernel that $\operatorname{Re} Q(e^{i\theta}) \geq \frac{1}{2}$ for all θ and therefore $Q \in \mathcal{P}_{n-1} \cap \mathcal{P}$. It follows from (9) that for any $p \in \mathcal{P}_{2N}$ we have

$$(p * Q)(1) = p(1) - \frac{p'(1)}{2N} = \sum_{j=1}^{4N}\frac{2\operatorname{Re} Q(\overline{w_j}) - 1}{4N}p(w_j)$$

with $w_j = e^{ij\pi/(2N)}$, $j = 1, \ldots, 4N$, and

$$\frac{2\,\mathrm{Re}\,Q(\overline{w_j}) - 1}{4N} = \frac{1}{8N^2}\left(\frac{\sin(j\pi/2)}{\sin(j\pi/(4N))}\right)^2.$$

We therefore obtain

$$p(1) - \frac{p'(1)}{2N} = \sum_{j=1}^{4N-1} \frac{1}{8N^2}\left(\frac{\sin(j\pi/2)}{\sin(j\pi/(4N))}\right)^2 p(w_j) + \frac{p(1)}{2}$$

and

$$\begin{aligned} p'(1) - N\,p(1) &= \sum_{\substack{j=1 \\ j \text{ odd}}}^{4N-1} \frac{-1}{4N}\frac{1}{\sin^2(j\pi/(4N)}\,p(e^{ij\pi/(2N)}) \\ &= \sum_{\substack{j=1 \\ j \text{ odd}}}^{2N-1} \frac{-1}{N}\frac{1}{1-\cos(j\pi/(2N))}\frac{p(e^{ij\pi/(2N)})}{2} \\ &\quad + \sum_{\substack{j=2N+1 \\ j \text{ odd}}}^{4N-1} \frac{-1}{N}\frac{1}{1-\cos(j\pi/(2N))}\frac{p(e^{ij\pi/(2N)})}{2}. \end{aligned}$$

It is easy to see that this last identity coincides with (13) and therefore formula (9) also contains the Marcel Riesz interpolation formula.

4 Proof of Theorem 5

Assume that (12) holds. Then $p \to Q * p$ is a bound preserving operator on $\mathcal{P}_n$ and we have the representation

$$(Q * p)(z) = \sum_{j=1}^{k} \Lambda_j p(\zeta^{1/n} V_j z) = \int_{\partial\mathbb{D}} p(t\zeta^{1/n} z)\,d\mu(t), \quad p \in \mathcal{P}_n,$$

where μ is a complex Borel measure with $\int_{\partial\mathbb{D}} |d\mu(t)| \le 1$. Setting $p \in \mathcal{P}_0$ we obtain

$$1 = \int_{\partial\mathbb{D}} d\mu(t) = \left|\int_{\partial\mathbb{D}} d\mu(t)\right| \le \int_{\partial\mathbb{D}} |d\mu(t)| \le 1$$

so that equality must hold everywhere in this chain of (in)equalities. In other words μ is a probability measure on $\partial\mathbb{D}$ with jumps steps $\Lambda_j \geq 0,\ j = 1, \dots, k$. For $|z| = 1$ it follows that

$$(Q * \tilde{p})(z) = \widetilde{(\tilde{Q} * p)}(z) = z^n \overline{(\tilde{Q} * p)(z)} = \sum_{j=1}^{k} \Lambda_j V_j^n \zeta z^n p(V_j \zeta^{1/n} z)$$

i.e.,

$$\zeta\, (\tilde{Q} * p)(z) = \sum_{j=1}^{k} \Lambda_j \overline{V_j}^n p(V_j \zeta^{1/n} z)$$

and

$$((Q + \zeta\tilde{Q}) * p)(z) = \sum_{j=1}^{k} \Lambda_j (1 + \overline{V_j}^n) p(V_j \zeta^{1/n} z)$$

is valid for $|z| \leq 1$. By (10) we conclude that the operator $p \to (Q + \zeta\tilde{Q})(p)$ is bound preserving over $\mathcal{P}_n$ with $((Q + \zeta\tilde{Q})) * p(0) = p(0)$ for $p \in \mathcal{P}_n$. As before we conclude that for the coefficients

$$\Lambda_j (1 + \overline{V_j}^n) \geq 0, \quad j = 1, \dots, k,$$

so that $\overline{V_j}^n$ can only equal ± 1, the assertion.

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Operators with Simple Orbital Behavior

Gabriel T. Prăjitură

Abstract In this paper we consider two similarity-invariant classes of operators on a complex Hilbert space. A complete description, in terms of properties of various parts of the spectrum, is obtained for the operators in the closure and for the operators in the interior of each of these classes.

Keywords Orbital behavior • Fredholm • Closure • Interior

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1 Introduction

Throughout this paper $\mathcal{H}$ will denote an infinite dimensional complex Hilbert space and $\mathcal{B}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. For an operator $T \in \mathcal{B}(\mathcal{H})$ we will use $\sigma(T)$ to denote the spectrum of T, $\sigma_e(T)$ for the essential spectrum of T, and $\sigma_p(T)$ for the point spectrum of T (that is, the set of eigenvalues of T). For an operator T we will denote by $\sigma_{lre}(T)$ the *left and right essential spectrum* of T, that is the intersection of the right essential spectrum and left essential spectrum. It is also known as the Wolf spectrum.

Recall that $T \in \mathcal{B}(\mathcal{H})$ is called a semi-Fredholm operator if it has closed range and either $\operatorname{nul} T = \dim \ker T$ or $\operatorname{nul} T^* = \dim \ker T^*$ is finite. When this is the case we define the Fredholm index of T by

$$\operatorname{ind} T = \operatorname{nul} T - \operatorname{nul} T^*.$$

We will use

$$\rho_{SF}(T) = \{\lambda \in \mathbb{C} : T - \lambda \quad \text{is semi - Fredholm}\}$$

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for the semi-Fredholm domain of T. We refer to the last chapter in [5] for the basics of Fredholm theory. We only point that the semi-Fredholm domain is an open set, it contains the resolvent set of the operator and parts of the spectrum. On the resolvent set the Fredholm index is 0 but it may be 0 on some parts of the spectrum as well. It may also contain parts of the essential spectrum, in which case the index will be either ∞ or $-\infty$.

We will use $\rho_{sF+}(T)$ for the semi Fredholm domain with strictly positive Fredholm index, $\rho_{sF-}(T)$ for the semi Fredholm domain with strictly negative Fredholm index, and $\rho_{sF}(T)$ for the semi Fredholm domain inside the spectrum.

We will denote by $\sigma_{p0}(T)$ the set of normal eigenvalues of T (i.e. the set of isolated eigenvalues of T with the property that the corresponding Riesz spectral invariant subspace is of finite dimension). We want to point out the fact that the elements of $\sigma_{p0}(T)$ are boundary points of $\sigma(T)$ which do not belong to $\sigma_e(T)$ but to the Fredholm domain and having, in fact, Fredholm index 0. Thus if an operator has an one-point spectrum which also happens to be an eigenvalue then that point *is not* a normal eigenvalue regardless of the dimension of the corresponding kernel. We will use $\rho_{sF\cdot}(T)$ for the semi Fredholm domain inside the spectrum except for the normal eigenvalues. Finally, $\rho_{sF0\cdot}(T)$ will stand for the semi Fredholm domain inside the spectrum with index zero except for the normal eigenvalues.

In this way we are seeing the spectrum of an operator as the disjoint union

$$\sigma(T) = \rho_{sF+}(T) \cup \rho_{sF-}(T) \cup \rho_{sF0\cdot}(T) \cup \sigma_{p0}(T) \cup \sigma_{lre}(T)$$

out of which only the last one is necessarily non empty.

We also have

$$\rho_{sF\cdot}(T) = \rho_{sF+}(T) \cup \rho_{sF-}(T) \cup \rho_{sF0\cdot}(T)$$

all sets being open and the first two in the union being stable under small perturbations.

We will denote by $\mathbb{D}$ the open unit disc in the complex plane and by $\overline{\mathbb{D}}$ its closure.

For a class C of operators in $\mathcal{B}(\mathcal{H})$ we will use cl C to denote its closure and int C to denote its interior.

2 The Classes

In [9] we started the study of orbital behavior of operators in terms of the oscillation properties of the norms of the vectors in the orbit. There are many classes of operators that can be defined in terms of this behavior. We will start with the simplest two:

$$C_1(\mathcal{H}) = \{T \in \mathcal{B}(\mathcal{H}) : T^n x \to 0 \ \text{ for all } \ x \in \mathcal{H}\}$$

and

$$C_2(\mathcal{H}) = \{T \in \mathcal{B}(\mathcal{H}) : ||T^n x|| \to \infty \text{ for all } x \in \mathcal{H}, x \neq 0\}.$$

It is simple to see that both classes are similarity invariant.

The following proposition lists some of the properties of $C_1(\mathcal{H})$ (or its complement).

Proposition 1 *Let $\mathcal{H}$ be a Hilbert space.*

(i) If $\sigma(T) \subset \mathbb{D}$ then $T \in C_1(\mathcal{H})$.
(ii) If $\sigma(T) \setminus \overline{\mathbb{D}} \neq \emptyset$ then $T \notin C_1(\mathcal{H})$.
(iii) If $\sigma_p(T) \setminus \mathbb{D} \neq \emptyset$ then $T \notin C_1(\mathcal{H})$.

Proof

(i) This follows from the spectral radius formula which implies that

$$\lim_n ||T^n|| = 0.$$

(ii) See [8]
(iii) Let

$$\lambda \in \sigma_p(T) \setminus \mathbb{D}$$

and $x \neq 0$ a corresponding eigenvalue.
Since

$$||T^n x|| = |\lambda|^n ||x|| \geq ||x||$$

this orbit does not have limit 0. □

We have a similar statement for C_2.

Proposition 2 *Let $\mathcal{H}$ be a Hilbert space.*

(i) If $\sigma(T) \cap \overline{\mathbb{D}} = \emptyset$ then $T \in C_2(\mathcal{H})$
(ii) If $\sigma_p(T) \cap \overline{\mathbb{D}} \neq \emptyset$ then $T \notin C_2(\mathcal{H})$.
(iii) If $\lambda \in \sigma_p(T^*)$ and x is not orthogonal on $\ker(T^* - \lambda)$, *then*

$$\lim_n ||T^n x|| = \infty.$$

Proof

(i) In this case T is invertible and $\sigma(T^{-1}) \subset \mathbb{D}$ which implies, as above, that

$$\lim_n ||T^{-n}|| = 0$$

Then, for $x \in \mathcal{H}$,

$$||x|| = ||T^{-n}T^n x \leq ||T^{-n}||||T^n x|| \implies \frac{||x||}{||T^{-n}||} \leq ||T^n x||$$

from where the result is obvious.

(ii) Similar to (iii) in the previous proposition.

(iii) Let

$$T^* = \begin{pmatrix} \lambda & A \\ 0 & B \end{pmatrix}$$

with respect to $\mathcal{H} = \ker(T^* - \lambda) \oplus \ker(T^* - \lambda)^{\perp}$.

Then, with respect to the same decomposition of $\mathcal{H}$, $x = x_1 \oplus x_2$, with $x_1 \neq 0$, and

$$T = \begin{pmatrix} \bar{\lambda} & 0 \\ A^* & B^* \end{pmatrix} \quad \text{and} \quad T^n = \begin{pmatrix} \bar{\lambda}^n & 0 \\ * & * \end{pmatrix}$$

Therefore

$$||T^n x|| \geq |\lambda|^n ||x_1|| \to \infty.$$

□

Proposition 3 *Let $2 \leq m \leq \infty$, $(\mathcal{H}_k)_{k=1}^m$ be Hilbert spaces and T a bounded linear operator on $\mathcal{H} = \oplus_{k=1}^m \mathcal{H}_k$.*

(i) If $m < \infty$, T has an upper triangular form with respect to the decomposition $\mathcal{H} = \oplus_{k=1}^m \mathcal{H}_k$ and if each diagonal entry of T has all nonzero orbits with limit infinity then so does T.

(ii) If T has a lower triangular form with respect to the decomposition $\mathcal{H} = \oplus_{k=1}^m \mathcal{H}_k$ and if each diagonal entry of T has all nonzero orbits with limit infinity then so does T.

Proof

(i) Let $x \neq 0 \in \mathcal{H}$. Then $x = \oplus_{k=1}^m x_k$, with $x_k \in \mathcal{H}_k$. There is $1 \leq j \leq m$ such that x_j is the last nonzero component of x. Let T_j be the diagonal entry of T corresponding to $\mathcal{H}_j$. Then

$$||T^n x|| \geq ||T_j^n x_j|| \to \infty$$

(ii) Goes about the same way except that this time j is chosen to correspond to the first non zero component of x.

□

Remark 1 If, in (i) of the previous proposition, the direct sum of spaces is infinite then the implication is not necessarily true.

Proof Let

$$T = \begin{pmatrix} \sqrt{2} & -1 & -\frac{1}{\sqrt{2}} & -\frac{1}{2} & \frac{1}{\sqrt{8}} & \dots \\ 0 & \sqrt{2} & -1 & -\frac{1}{\sqrt{2}} & -\frac{1}{2} & \dots \\ 0 & 0 & \sqrt{2} & -1 & -\frac{1}{\sqrt{2}} & \dots \\ 0 & 0 & 0 & \sqrt{2} & -1 & \dots \\ \dots \end{pmatrix}$$

Since

$$T = \sqrt{2}I - S - \frac{1}{\sqrt{2}}S^2 - \frac{1}{2}S^3 - \frac{1}{\sqrt{8}}S^4 - \dots$$

where S is the backward shift, it is easy to see that T is bounded.

All nonzero orbits of the diagonal entries are going to infinity. In the same time, if

$$x = \left(1, \frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{\sqrt{8}}, \dots\right)^t$$

then $Tx = 0$ and thus

$$||T^n x|| \to 0.$$

□

3 Apostol-Morrel Simple Models

Apostol-Morrel simple models were developed in [4] based on earlier work on Constantin Apostol [1, 2] and were a byproduct of the work to solve Problem 7 in [6]. Section 6.1 in [7] has a simplified version of the construction and also has some comments after the Corollary 6.2 (on page 160) and in Remark 6.3 (on page 162) about the possibility of replacing each of the pieces of the original construction by other types of operators.

Roughly speaking, an Apostol-Morrel simple model is a diagonal operator with a simple spectral picture, having a spectrum with a finite number of components, each a nice set (topologically speaking), and whose similarity orbit gets close to any operator with a close enough spectral picture.

Thus, in order to characterize the closure of a similarity invariant class of operators in terms of spectral properties it suffices to build Apostol-Morrel simple models that are in the closure and have the required spectral properties.

One can build custom Apostol-Morrel simple models starting from the Closure of similarity orbit theorem (Theorem 9.2 in [3]). We agree that the Theorem is hard to grasp and even harder to use, but we invite the reader to notice that the theorem becomes quite simple in the case of operators for which the essential spectrum has no isolated points. This, combined with the general principle that one does not have to approximate something in a certain way but only to approximate an approximation of it in that way will give us the possibility of avoiding all the technicalities of the special cases of the theorem and take advantage of its full force.

If K is a compact set of complex numbers and $\varepsilon > 0$ we will denote by

$$(K)_\varepsilon = \{z \in \mathbb{C} : d(z, K) \leq \varepsilon\}$$

If T is an operator then, as the normal eigenvalues can only accumulate near the left and right essential spectrum, there are only at most a finite number of such eigenvalues that will not belong to $(\sigma_{lre}(T))_\varepsilon$. If there are none, we skip this step, but if there are some, then we first use Riesz spectral decomposition theorem to write T as $T = T_1 \oplus T_0$, where T_0 is an operator on a finite dimensional space with

$$\sigma(T_0) = \sigma_{p0}(T) \setminus (\sigma_{lre}(T))_\varepsilon .$$

Then, rather than applying Theorem 6.1 in [7] to T, we will apply it to T_1. That will give us the possibility of making better and from a wider pool choices for one of the pieces of the approximant.

Notice that by enlarging $\sigma_{lre}(T)$ we were left not only with at most a finite number of normal eigenvalues but also with at most a finite number of components of the semi Fredholm domain, slightly smaller. Moreover, the set $(\sigma_{lre}(T))_\varepsilon$ has only a finite number of components.

To each component of the $(\sigma_{lre}(T))_\varepsilon$ we will associate either a point belonging to it or a closed disc included in it.

The next step will be to make a slight enlargement of each former component of the semi Fredholm domain by placing an analytic Cauchy domain in between the original component U and the remains of it in $U \setminus (\sigma_{lre}(T))_\varepsilon$ which will not contain the points chosen above (if that is what we chose) and will not intersect the closed discs chosen above (if that is what we chose).

The points may be on the boundary of the Cauchy domains and that boundary may go along the boundary of a disc if that is convenient. This can always be arranged by making the discs large enough or replacing them by closures on analytic Cauchy domains.

We will ignore the analytic Cauchy domains corresponding to components of index 0.

To each remaining analytic Cauchy domain we will associate either an inflation of the Bergman shift on the Bergaman space of the domain (if the index was strictly negative), an inflation of the adjoint of the Bergman shift on the Bergaman space of the domain (if the index was strictly positive). In each case the number of summands in the direct sum is equal to the Fredholm index.

To each disc in a component we will associate any operator having that disks as spectrum and such that the spectrum is entirely left and right essential spectrum. For example we can use the multiplication by z operator on L^2 of the disk, if we do not want eigenvalues, or we can choose a countably dense set of complex numbers from the disc and consider an infinite inflation of the diagonal operator with diagonal entries those numbers, if eigenvalues are desirable.

To all points selected we will associate a normal operator with spectrum exactly those points and with the spectrum equal the left and right essential spectrum. It can be even refined to be algebraic.

Let M_+ be the direct sum of all adjoints of the Bergman shifts operators from above, if any, M_- the direct sum of all Bergman shifts operators from above, M the direct sum of all operators on closed discs from above, if any, and N the normal operator with finite spectrum, if any. The first two may be absent but at least one of the last two should exist. Of course all these operators will depend on ε and T.

The following result is a combination of Proposition 2.1 and Theorem 2.3 in [4] and the way they are presented in Section 6.1 in [7].

Theorem 1 *Let T be an operator as above. For every $\delta > 0$ there is an operator S similar to*

$$M_+ \oplus M_- \oplus M \oplus N \oplus T_0$$

such that

$$||T - S|| < \delta$$

The actual construction above is done with $\varepsilon = \frac{\delta}{8}$.

An operator of the type

$$M_+ \oplus M_- \oplus M \oplus N \oplus T_0$$

is called a simple model. Thus the theorem says that operators similar to simple models are dense in $\mathcal{B}(\mathcal{H})$.

Notice that

$$\rho_{sF+}(S) \subset \rho_{sF+}(T) \qquad \rho_{sF-}(S) \subset \rho_{sF+}(T) \qquad \text{and} \qquad \sigma_{p0}(S) \subset \sigma_{p0}(T).$$

The theorem can be modified to get an operator S similar only to $M_+ \oplus M_- \oplus M \oplus N$ such that

$$||T - S \oplus T_0|| < \delta.$$

Moreover, T_0 can be replaced by a finite dimensional operator obtained by perturbating the eigenvalues of T_0 by numbers less than δ.

The general form of the models can be changed in different ways and it can include direct sums of operators with overlapping spectra. As far as we know, this is the first time when such models are used.

In order to use such models we need the Closure of similarity orbit theorem. The reduced form, Theorem 9.1 in [3] will be sufficient.

For this we need to introduce the concept of spectral domination and of spectral equivalence. We refer to the top of page 5 in [3] for the full definition of the concept. Since we will only use it for operators without normal eigenvalues and no isolated points in the essential spectrum we will only give a particular form of the concept.

In this case spectral equivalence means that the two operators have the same semi Fredholm domain, the same index and the same minimal index. Recall that for an operator T and $\lambda \in \rho_{sF}(T)$, the minimal index is

$$\min\{\dim\ker(T-\lambda), \dim\ker(T-\lambda)^*\}.$$

4 Closures and Interiors

We will start by describing the closure of C_1.

Theorem 2 $\operatorname{cl} C_1$ *consists of all operators* $T \in \mathcal{B}(\mathcal{H})$ *such that*

(a) $\rho_{sF}(T) \subset \mathbb{D}$
(b) $\sigma_{p0}(T) \subset \overline{\mathbb{D}}$
and
(c) If K *is a component of* $\sigma_{lre}(T)$ *then* $K \cap \overline{\mathbb{D}} \neq \emptyset$.

Proof We will show first that the condition are necessary.

If (b) or (c) are not satisfied then $\sigma(T)$ has a component outside $\overline{\mathbb{D}}$ and so will have all operators in some neighborhood of T. Therefore, by (iii) of Proposition 1, T cannot be in $\operatorname{cl} C_1$.

If (a) is not satisfied then, since $\rho_{sF}(T)$ is an open set, there is $z \in \rho_{sF}(T)$ such that $|z| > 1$. Then, because the semi Fredholm domain is stable, $z \in \rho_{sF}(S)$ for all S in some neighborhood of T and the conclusion follows as above.

To see that the conditions are sufficient we will construct an Apostol Morrel simple model which is the closure of the class and such that something similar to it is close to T.

Condition (a) ensures that the spectra of M_+ and M_- are in the open unit disc. Condition (c) implies the possibility of choosing M or N (no need for both) with spectrum in the open unit disc. Finally,

$$\left(1-\frac{1}{n}\right)T_0$$

also has the spectrum in the open unit disc. □

The interior of C_1 is easy to characterize.

Proposition 4

$$\operatorname{int} C_1 = \{T \in \mathcal{B}(\mathcal{H}) : \sigma(T) \subset \mathbb{D}\}$$

Proof "$\subset$"

Taking into account (ii) of Proposition 1, it suffices to prove that if there is $z \in \sigma(T)$ such that $|z| = 1$, there is a sequence of operators $T_n \to T$ such that $T_n \notin C_1$. It is easy to see that

$$T_n = \left(1 + \frac{1}{n}\right) T$$

has this property.

"$\supset$"

Follows from (i) of Proposition 1 and the upper semi continuity of the spectrum. □

We will look now at the closure of C_2.

Theorem 3 $\operatorname{cl} C_2$ *consists of all operators* $T \in \mathcal{B}(\mathcal{H})$ *such that*

(a) no component of the spectrum is included in $\mathbb{D}$.
(b) $\rho_{sF+}(T) \cap \mathbb{D} = \emptyset$.

Proof The necessity of (a) follows from (i) of Proposition 1 and the necessity of (b) follows from (ii) of Proposition 2.

For the sufficiency, it suffices to justify separate each piece of an Apostol-Morrel simple model.

$$\left(1 + \frac{1}{n}\right) T_0$$

is approximating T_0 and the spectrum is completely outside the closed unit disc.

M_+ has spectrum outside the closed unit disc, so that is OK.

The spectrum of M (or N, again, no need for both) can also be placed outside the closed unit disc.

The only difficult piece is M_-. Because of direct sum properties we only have to justify it for one Bergman shift.

If the analytic Cauchy domain goes outside the closed unit disc then we can use (iii) of Proposition 2 and the fact that the kernels corresponding to eigenvalues of the adjoint that are outside the disc span the space.

If the boundary of the analytic Cauchy domain touches the disc then

$$\left(1 + \frac{1}{n}\right) M_-$$

is in the closure, which is enough to imply the result.

The problem case is when the closure of the analytic Cauchy domain is included in the open unit disc. For this we need to come up with a completely new type of Apostol-Morrel model.

The analytic Cauchy domain in case, call it V, was initially included in a component of the spectrum and this component reached at least to the unit circle. Therefore $(\sigma_{lre}(T))_\varepsilon$ has a component that goes outside the closed unit disc and shares boundary with this analytic Cauchy domain. In this case, instead of a circle, we can choose an analytic Cauchy domain as well (say W) which goes outside the closed unit disc and which, together with the other one and the common boundary, form an analytic Cauchy domain, U.

We consider now the Bergman shift on U, $M_-(U)$ and the multiplication by z on $L^2(W)$, $M(W)$. Let $M_-(V)$ be the Bergman shift on V.

The operators $M_-(V) \oplus M(W)$ and $M_-(U) \oplus M(W)$ have the same spectral picture. Therefore they have the same semi Fredholm domain, the same index, -1, and the same minimal index, 0.

Therefore each of them is the limit of a sequence of similarities of the other one. As we saw before, $M_-(U)$ and $M(W)$ are in the closure of the class on their corresponding spaces. Therefore the direct sum is in the closure of the sum and so is $M_-(V) \oplus M(W)$.

This completes the proof. □

Before we characterize the interior we need to discuss a certain result in spectral theory.

Let T be an operator in $\mathcal{B}(\mathcal{H})$ with U a component of $\rho_{sF+}(T)$ with the minimal index 0.

If the space is spanned by

$$\{\ker(T - \lambda) : \lambda \in U\}$$

then the operator is in one of the Cowen Douglas classes. We want to see what can we say about the operator when it is not.

Let $\mathcal{H}_1$ be the subspace spanned by the kernels above. With respect to $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_1^\perp$ we can write

$$T = \begin{pmatrix} T_1 & A \\ 0 & B \end{pmatrix}$$

It is clear that T_1 is in a Cowen-Douglas class. We will look at B.

It is easy to see in an upper triangular form as here that if T and T_1 are invertible then B is invertible. Applying this to the Calkin algebra we get that $U \subset \rho_{sF}(B)$. It is clear that for $\lambda \in U$, $\ker(B-\lambda)^* = (0)$. Therefore either U is not in the spectrum of B or $U \subset \rho_{sF+}(B)$. In the first case we are done. In the second we repeat the procedure for B.

There are two possibilities here. Either the procedure stops after a finite number of steps or it continues without stop.

In both cases we end up with

$$T = \begin{pmatrix} T' & A \\ 0 & C \end{pmatrix}$$

where, if the C part is not absent, $U \cap \sigma(C) = \emptyset$.

In the first case, we get

$$T' = \begin{pmatrix} T_1 & * & * & * \ldots & * \\ 0 & T_2 & * & * \ldots & * \\ 0 & 0 & T_3 & * \ldots & * \\ \ldots \\ 0 & 0 & 0 & 0 \ldots & T_p \end{pmatrix}$$

where

$$\sigma(T_1) = \sigma(T_2) = \ldots \sigma(T_p) = \operatorname{cl} U$$

$$\rho_{sF+}(T_1) = \rho_{sF+}(T_2) = \cdots = \rho_{sF+}(T_p) = U$$

and each of them acts on a space generated by their kernels.

In the second case we get

$$T' = \begin{pmatrix} T_1 & * & * & * \ldots \\ 0 & T_2 & * & * \ldots \\ 0 & 0 & T_3 & * \ldots \\ \ldots \end{pmatrix}$$

where, for every n

$$\sigma(T_n) = \operatorname{cl} U, \qquad \rho_{sF+}(T_n) = U$$

and each T_n acts on a space generated by its kernels.

Theorem 4

$$\operatorname{int} C_2 = \{T \in \mathcal{B}(\mathcal{H}) : \sigma(T) \cap \bar{\mathbb{D}} = \emptyset\}$$

$$\bigcup \{T \in \mathcal{B}(\mathcal{H}) : \bar{\mathbb{D}} \subset \rho_{sF-}(T), \text{min. ind}(T - \lambda) = 0 \ \textit{if} \ |\lambda| \leq 1\}$$

Proof "$\supset$"

By the upper semi continuity of the spectrum and of the minimal index and the stability of the Fredholm index, both sets on the right are open. The first is included in the interior of C_2 by (i) of Proposition 2.

We can write T^* in one of the two formes discussed before the theorem. By (i) and (iii) of Proposition 2, each diagonal entry of T has all nonzero orbits going to infinity. Then we use (ii) of Proposition 3 to conclude that T is in C_2.

"$\subset$"

Suppose not. This means first that $\sigma(T) \cap \bar{\mathbb{D}} \neq \emptyset$. Since $T \in C_2$, there is no point spectrum (so no strictly positive semi Fredholm index, no 0 index, no strictly negative index with minimal index greater than 0, no normal eigenvalues). Therefore $\sigma(T) \cap \bar{\mathbb{D}}$ can only contain left and right essential spectrum and semi Fredholm domain with strictly negative Fredholm index and minimal index 0. Since $\bar{\mathbb{D}}$ is not included in the second,

$$\sigma_{lre}(T) \cap \bar{\mathbb{D}} \neq \emptyset.$$

Now, as we showed in the approximation by Apostol-Morrel models, we can approximate T by operators having eigenvalues placed in any part we choose of the left and right essential spectrum. Thus we can choose the eigenvalues in $\sigma_{lre}(T) \cap \bar{\mathbb{D}}$, which will imply that we approximate T by operators not in C_2. Therefore T cannot be in the interior of C_2.

This contradiction completes the proof. □

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Taylor Series, Universality and Potential Theory

Stephen J. Gardiner

Abstract Universal approximation properties of Taylor series have been intensively studied over the past 20 years. This article highlights the role that potential theory has played in such investigations. It also briefly discusses potential theoretic aspects of universal Laurent series, universal Dirichlet series, and universal polynomial expansions of harmonic functions.

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1 Introduction

Universality refers to the phenomenon where a single object, when subjected to a countable process, yields approximations to all members of some universal collection. An example of this is hypercyclicity, since a hypercyclic vector is one that has dense orbit under repeated application of a certain operator. The main focus of this article is on another example, namely that of universal Taylor series, which are defined below. An excellent overview of universal series and hypercyclicity may be found in Grosse-Erdmann [27].

Let $\mathrm{Hol}(\Omega)$ denote the space of functions which are holomorphic on a domain $\Omega \subset \mathbb{C}$, endowed with the topology of local uniform convergence. Given $f \in \mathrm{Hol}(\Omega)$ and $\zeta \in \Omega$, we will study the partial sums of the Taylor series about ζ, namely

$$S_m(f,\zeta)(z) = \sum_{n=0}^{m} \frac{f^{(n)}(\zeta)}{n!}(z-\zeta)^n \quad (z \in \mathbb{C}; m \geq 0).$$

The complement of a set A will be denoted by A^c.

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Definition 1 Let $f \in \mathrm{Hol}(\Omega)$ and $\zeta \in \Omega$. Then f is said to have a *universal Taylor series* about ζ if, for any compact $K \subset \Omega^c$ with K^c connected, and for any $g \in C(K) \cap \mathrm{Hol}(K^\circ)$, there is a subsequence $(S_{m_k}(f,\zeta))$ which converges uniformly to g on K. The collection of all functions f with this property will be denoted by $\mathcal{U}(\Omega,\zeta)$.

Remarkably, such universal approximation properties of a Taylor series turn out to be generic for holomorphic functions on simply connected domains, as the following result of Nestoridis (see [38, 39]) shows. A set (in some Baire space) is called *residual* if its complement is of first Baire category.

Theorem 1 *If $\Omega \subset \mathbb{C}$ is a simply connected domain and $\zeta \in \Omega$, then $\mathcal{U}(\Omega,\zeta)$ is a residual subset of* $\mathrm{Hol}(\Omega)$.

Many further results on universal Taylor series may be found in [34], and an axiomatic approach to the subject is provided in [7]. The purpose of this article is to highlight how potential theoretic methods have recently shed light on the existence and properties of functions in $\mathcal{U}(\Omega,\zeta)$, and also on universal Laurent series, universal Dirichlet series, and universal polynomial expansions of harmonic functions. The reader is referred to the books [2] and [41] for accounts of the various potential theoretic notions that arise below.

2 Existence of Universal Taylor Series

There are two main situations where the existence (and abundance) of universal Taylor series is well understood. One of these is where Ω is simply connected, as we saw in Theorem 1 above. The other is where Ω^c is compact and connected, which is covered by the following result of Melas [33].

Theorem 2 *If $\Omega \subset \mathbb{C}$ is a domain such that Ω^c is compact and connected and $\zeta \in \Omega$, then $\mathcal{U}(\Omega,\zeta)$ is a residual subset of* $\mathrm{Hol}(\Omega)$.

The existence question for universal Taylor series on more general domains Ω remains largely unresolved. However, potential theory has shed significant light on it, as we will now describe.

A basic tool here is Bernstein's lemma (see [41]). This says that, if $L \subset \mathbb{C}$ is non-polar and compact, and p is a polynomial of degree $m \geq 1$, then

$$\frac{1}{m}\log\frac{|p(z)|}{\sup_L |p|} \leq G_{\widehat{\mathbb{C}}\setminus L}(z,\infty) \quad (z \in \mathbb{C}),$$

where $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and $G_\omega(\cdot,\cdot)$ is the Green function for ω (interpreted as 0 outside $\omega \times \omega$). It is natural to consider the subharmonic functions

$$u_m = \frac{1}{m}\log|S_m(f,\zeta) - f| \;\text{ on }\; \Omega, \quad \text{and} \quad v_m = \frac{1}{m}\log|S_m(f,\zeta)| \;\text{ on }\; \mathbb{C}.$$

For a given sequence (m_k), we may further consider the upper semicontinuous regularizations u^* and v^* of the functions

$$u = \limsup_{k\to\infty} u_{m_k} \text{ on } \Omega, \quad \text{and} \quad v = \limsup_{k\to\infty} v_{m_k} \text{ on } \mathbb{C}, \tag{1}$$

respectively. These are again subharmonic functions (by Corollary 5.7.2 of [2]) since the sequences (u_{m_k}) and (v_{m_k}) are locally uniformly bounded above in their respective domains, by Bernstein's lemma.

Less immediately obvious is the relevance to this question of the notion of thinness. We recall that a set $A \subset \mathbb{C}$ is *thin* at a point w if and only if

$$\sum_{n=1}^{\infty} \frac{n}{\log(1/c^*(A_n))} < \infty \quad \text{(Wiener's criterion)}, \tag{2}$$

where $A_n = \{z \in A : 2^{-n-1} \le |z - w| \le 2^{-n}\}$ and c^* denotes outer logarithmic capacity. Thinness of a set at ∞ is defined by means of inversion.

If s is subharmonic on a neighbourhood of w, then there is a set A, thin at w, such that $s(z) \to s(w)$ as $z \to w$ outside A. Also, a boundary point w of an open set ω is regular for the Dirichlet problem on ω if and only if ω^c is non-thin at w. If A is thin at w, then there are arbitrarily small circles centred at w which do not intersect A. Thus, if $\Omega \subsetneq \mathbb{C}$ is simply connected, then Ω^c is certainly non-thin at infinity.

The following result is due to Müller et al. [36] (see also [35]).

Theorem 3 *If Ω is a multiply connected domain and Ω^c is non-thin at ∞, then $\mathcal{U}(\Omega, \zeta) = \emptyset$ for all $\zeta \in \Omega$.*

For an alternative to the original proof of this result we can use the following lemma, which is based on Theorem 7.6.7 of [2] (see the proof of Lemma 4.2.1 in [32] for a more detailed explanation).

Lemma 1 *Let s be a subharmonic function on $\mathbb{C}$ satisfying $s(z) \le a + b\log^+|z|$, where $a \in \mathbb{R}$ and $b > 0$. If $s \le c$ on a set A which is non-thin at ∞, then s is constant and $s \le c$ on $\mathbb{C}$.*

Theorem 3 can be deduced as follows. Suppose there exists f in $\mathcal{U}(\Omega, \zeta)$, choose a point w in a bounded component L of Ω^c, and let $|l| < 1$. Then we can find (m_k) such that $|S_{m_k}(f, \zeta)| \le 1$ on $\{z \in \Omega^c : |z| \le k\}$ and $S_{m_k}(f, \zeta)(w) \to l$. It follows from Bernstein's lemma that the function v^* (see (1)) satisfies the hypotheses of Lemma 1 with $c = 0$ and $b = 1$, whence $u^* \le 0$ on Ω. Since $u^* < 0$ on the disc of convergence of the Taylor series, it follows from the maximum principle that $u^* < 0$ on Ω. Hence $(S_{m_k}(f, \zeta))$ converges locally uniformly on Ω, and so on $\Omega \cup L$ by the maximum principle. The value of $\lim_{k\to\infty} S_{m_k}(f, \zeta)(w)$ is thus uniquely determined by the holomorphic extension of f to L, contradicting the arbitrary choice of l.

The above argument also yields part (i) of the next theorem, which is again taken from [35] and [36]. We recall that the series $\sum a_n(z - \zeta)^n$ is said to have *Ostrowski gaps* (m_k, p_k) if

$$1 \le m_1 < p_1 \le m_2 < p_2 \le \ldots, \text{ where } p_k/m_k \to \infty, \text{ and}$$

$$|a_n|^{1/n} \to 0 \text{ as } n \to \infty \text{ through } \bigcup_k \{m_k + 1 \le n \le p_k\}.$$

Theorem 4 *Let $f \in \mathcal{U}(\Omega, \zeta)$, where Ω is simply connected and $\zeta \in \Omega$. Then the sequence (m_k) in Definition 1 can be chosen so that*

(i) $S_{m_k}(f, \zeta) \to f$ locally uniformly in Ω, and
(ii) the Taylor series of f about ζ has Ostrowski gaps (m_k, p_k).

Theorem 3 shows that, for multiply connected domains Ω, thinness of Ω^c at ∞ is necessary for the existence of universal Taylor series. It was conjectured in [36] that the same condition is also sufficient. However, this was disproved in [24], where it was shown that $\mathcal{U}(\Omega, \zeta)$ can be empty when Ω^c is the union of a non-degenerate continuum L and an additional point ξ. Further, for a particular choice of L, the location of ζ and ξ turn out to be crucial to the existence question. (This was the first known example of a domain Ω for which the existence of functions in $\mathcal{U}(\Omega, \zeta)$ depends on the choice of ζ.) Let $D(\zeta, r)$ denote the open disc of centre ζ and radius r. The existence and non-existence assertions below are drawn from [16] and [24], respectively.

Example 1 If $\Omega = (\overline{D}(3, 1) \cup \{1\})^c$, then $\mathcal{U}(\Omega, 5) \neq \emptyset$ and $\mathcal{U}(\Omega, 0) = \emptyset$.

Subsequently, such examples were seen to be special cases of the following result from [17], in which thinness again plays a role.

Theorem 5 *Let $r = \text{dist}(\zeta, \Omega^c)$, where Ω is a domain and $\zeta \in \Omega$, and suppose that $\Omega^c \backslash \overline{D}(\zeta, r)$ is non-polar. If Ω^c is thin at a point $\xi \in \partial\Omega \cap \partial D(\zeta, r)$, then $\mathcal{U}(\Omega, \zeta) = \emptyset$.*

Although Theorems 3 and 5 both feature thinness, they do so in contrasting ways: in the former result, thinness of Ω^c at ∞ is necessary for the existence of universal Taylor series on multiply connected domains Ω, whereas, in the latter result, *non*-thinness of Ω^c at the *closest* points of $\partial\Omega$ to ζ is necessary. Since a polar set is everywhere thin, we immediately deduce:

Corollary 1 *Let Ω be a domain and $\zeta \in \Omega$. If $D(\zeta, R) \backslash \Omega$ is non-empty and polar for some $R > 0$, and Ω^c is non-polar, then $\mathcal{U}(\Omega, \zeta) = \emptyset$.*

The non-existence assertion in Example 1 clearly follows from this corollary, as does the next observation.

Example 2 If $\Omega = (\overline{D}(0, 1) \cup E)^c$, where

$$E = \bigcup_{k=1}^{\infty} \left\{ \left(1 + 2^{-k}\right) e^{m2^{1-k}\pi i} : 1 \le m \le 2^k \right\},$$

then $\mathcal{U}(\Omega, \zeta) = \emptyset$ for every $\zeta \in \Omega$.

Next we mention a more subtle, and intriguing, example.

Example 3 Let $\Omega = (\overline{D}(\xi, r) \cup \{\xi_0\})^c$, where $0 \in \Omega$ and $\xi_0 \notin \overline{D}(\xi, r)$. Then

(a) $\mathcal{U}(\Omega, 0) \neq \emptyset$ if $|\xi_0| \geq |\xi| + r$;
(b) $\mathcal{U}(\Omega, 0) = \emptyset$ if $|\xi_0| < \sqrt{|\xi|^2 - r^2}$.

Part (a) follows by combining results from [16] and [44]. Part (b) is a special case of a more general result in [17].

Open Question What happens if $\sqrt{|\xi|^2 - r^2} \leq |\xi_0| < |\xi| + r$?

The critical case here, where $|\xi_0| = \sqrt{|\xi|^2 - r^2}$, corresponds to the situation where the circles $\partial D(0, |\xi_0|)$ and $\partial D(\xi, r)$ meet at right angles. In this case, the harmonic measures for

1. the inner domain $D(0, |\xi_0|)\backslash\overline{D}(\xi, r)$ and the point 0, and
2. the outer domain $\left[\overline{D}(0, |\xi_0|) \cup \overline{D}(\xi, r)\right]^c$ and the point ∞,

are comparable along their common boundary $\partial D(0, |\xi_0|)\backslash\overline{D}(\xi, r)$. This reflects the fact that both Theorem 5 and Example 3(b) rely on an inner-outer comparison of harmonic measures (or, equivalently, of Green functions), as we will now explain.

When studying universal Taylor series, we have a function $f \in \text{Hol}(\Omega)$ and a subsequence $(S_{m_k}(f, \zeta))$ which is uniformly bounded on some compact set K. Let U be the largest domain containing ζ on which this sequence is locally uniformly convergent. Usually we have $K \cap U = \emptyset$. What can we say about the set U when $\Omega\backslash U \neq \emptyset$? Clearly it is simply connected. Further, we can adapt our sketch proof of Theorem 3 to see that it is bounded. However, we can say much more [17].

Proposition 1 *Let U and K be as above, where $K \cap U = \emptyset$ and $\Omega\backslash U \neq \emptyset$. Then the upper semicontinuous regularization s^* of the function*

$$s(z) = \begin{cases} -G_U(z, \zeta) & (z \in U) \\ G_V(z, \infty) & (z \in V) \\ 0 & (\textit{elsewhere in } \mathbb{C}) \end{cases}, \tag{3}$$

where $V = \widehat{\mathbb{C}}\backslash(\overline{U} \cup K)$, is subharmonic on $\Omega \cup (\partial U)^c$ and continuously vanishes on ∂U.

Roughly speaking, the first two parts of formula (3) arise from consideration of the functions u and v in (1). One can show that $u^* \leq s^*$ on Ω, and then deduce that U would not be maximal if the function s^* were not subharmonic on $\Omega \cup (\partial U)^c$.

Finally in this section, we mention that Costakis and Tsirivas [15] have used potential theory to investigate the existence of holomorphic functions f on the unit disc $\mathbb{D}$ and sequences (λ_n) in $\mathbb{N}$ with the following doubly universal approximation property: for any compact $K \subset \mathbb{D}^c$ with K^c connected and any $g_1, g_2 \in C(K) \cap \text{Hol}(K^\circ)$, there is a sequence (m_k) such that $S_{m_k}(f, \zeta) \to g_1$ and $S_{\lambda_{m_k}}(f, \zeta) \to g_2$ uniformly on K. They show that such functions f exist if and only if $\limsup \lambda_n/n = \infty$. Further, if this last condition holds, then such functions form a residual subset of Hol($\mathbb{D}$). The inspiration for such a result was drawn from concepts in dynamical systems and ergodic theory. Vlachou [45] has recently generalized this theorem to cover multiply universal approximation on simply connected domains (see also [10]).

3 Boundary Behaviour I

We will now discuss the boundary behaviour of functions in $\mathcal{U}(\Omega, \zeta)$, beginning with the case where $\Omega = \mathbb{D}$. Bayart [6] showed that functions f in $\mathcal{U}(\mathbb{D}, 0)$ must be unbounded near every point of the unit circle $\mathbb{T}$. Costakis and Melas [13] showed that such functions assume every complex value, with at most one exception, infinitely often on $\mathbb{D}$ (cf. Picard's theorem). Both of these results are contained in the next theorem, taken from [20]. It says that f has a Picard-type property near each point of $\mathbb{T}$, whence universal Taylor series must have "universal boundary behaviour".

Theorem 6 *Let $f \in \mathcal{U}(\mathbb{D}, 0)$. Then, for every $w \in \mathbb{T}$ and $r > 0$, the function f assumes every complex value, with at most one exception, infinitely often on $D(w, r) \cap \mathbb{D}$.*

In fact, if f omits at least two values in $U := D(w, r) \cap \mathbb{D}$ for some $w \in \mathbb{T}$ and r, then Schottky's theorem tells us that f has restricted growth near $\mathbb{T}$ in U. This allows Theorem 6 to be deduced from the following [20].

Theorem 7 *Let $\psi : [0, 1) \to (0, \infty)$ be an increasing function such that*

$$\int_0^1 \log^+ \log^+ \psi(t) dt < \infty. \tag{4}$$

If $f \in$ Hol($\mathbb{D}$) and $|f(z)| \leq \psi(|z|)$ on $D(w, r) \cap \mathbb{D}$ for some $w \in \mathbb{T}$ and $r > 0$, then $f \notin \mathcal{U}(\mathbb{D}, 0)$.

The proof of this result uses the following equiconvergence property [34] (cf. [29]), which holds when $(S_m(f, 0))$ has Ostrowski gaps (m_k, p_k):

$$S_{m_k}(f, \zeta)(z) - S_{m_k}(f, 0)(z) \to 0 \quad (k \to \infty) \text{ locally uniformly on } \{(\zeta, z) \in \mathbb{D} \times \mathbb{C}\}. \tag{5}$$

The hypothesized local growth restriction on f allows us to estimate $S_{m_k}(f, \zeta)$ near w when ζ is suitably chosen in U, and a normal families argument can then be applied to the sequence $(S_{m_k}(f, 0))$ on a neighbourhood of w, in view of (5).

The above reasoning relies on certain geometric properties of the boundary. It thus does not apply to general simply connected domains Ω, where $\partial\Omega$ may not be locally connected, or may have a fractal nature. In this more general context it was established in [36] that functions in $\mathcal{U}(\Omega, \zeta)$ are not holomorphically extendable beyond Ω, but it was not known until recently whether they need be unbounded. Indeed they must be, but even more can be said [18], as follows.

Theorem 8 *Let $f \in \mathcal{U}(\Omega, \zeta)$, where Ω is simply connected and $\zeta \in \Omega$. Then, for every $w \in \partial\Omega$ and $r > 0$, and every component U of $D(w,r) \cap \Omega$, the set $\mathbb{C}\backslash f(U)$ is polar.*

What is actually shown in [18] is that the subharmonic function $\log|f|$ cannot have a positive harmonic majorant on any such component U. The stated conclusion follows because, otherwise, Myrberg's theorem would show that $\log^+|z|$ has a harmonic majorant h on the open set $f(U)$, whence $\log|f|$ would have the positive harmonic majorant $h \circ f$ on U.

The proof of Theorem 8 relies on Martin boundary theory (a generalization of Carathéodory's theory of prime ends) and minimal thinness. We will discuss the role of the latter concept further in the next section. Similar tools were combined with Proposition 1 to establish the following extremely general result [21]. (Of course, the theorem becomes vacuous when $\mathcal{U}(\Omega, \zeta) = \emptyset$.)

Theorem 9 *For any domain Ω and any point $\zeta \in \Omega$, every function f in $\mathcal{U}(\Omega, \zeta)$ is unbounded.*

Unlike Theorem 8, which only concerned simply connected domains, this result does not assert the unboundedness of f near every boundary point. Indeed, as Melas [33] has observed for domains with discrete complement, it is even possible for f to have removable singularities at some isolated boundary points.

We will revisit the boundary behaviour of universal Taylor series later, following a discussion of some recent results concerning Taylor series in general.

4 Recent Results for General Taylor Series

We next describe some recent results for the disc which connect the boundary behaviour of a Taylor series with its behaviour on the boundary itself. Applications to universal Taylor series will be given in the following section. We denote by $\operatorname{nt}\lim_{z\to w} f(z)$ the non-tangential limit of a function f on $\mathbb{D}$ at a point w of $\mathbb{T}$, whenever it exists.

A classical result of this nature is Abel's Limit Theorem. It says that, if $f \in \mathrm{Hol}(\mathbb{D})$ and $(S_m(f,0))$ converges at some $w \in \mathbb{T}$, then $\operatorname{nt}\lim_{z\to w} f(z)$ exists and equals $\lim_{m\to\infty}(S_m(f,0))(w)$. Theorem 1 shows that nothing similar can be deduced from the convergence of a subsequence $(S_{m_k}(f,0))$, even on a subarc of $\mathbb{T}$. However, the following result of Beise et al. [8] raises an interesting question. We recall that a closed subset F of $\mathbb{T}$ is called a *Dirichlet set* if some subsequence of (z^n) tends to 1 uniformly on F.

Theorem 10 *Let $\Omega \subset \mathbb{C}$ be a domain containing 0 such that each component of $\widehat{\mathbb{C}}\backslash\Omega$ meets $\mathbb{T}$, and let $F \subset \mathbb{T}\cap\Omega$ be a Dirichlet set. Then there is a residual subset of functions f in $\mathrm{Hol}(\Omega)$ with the property that, for every $g \in C(F)$, there is a subsequence $(S_{m_k}(f,0))$ converging to g uniformly on F.*

The most interesting case of this result is where $\mathbb{D} \subset \Omega$. The above universal approximation occurs within the domain where f is holomorphic, whereas functions in $\mathcal{U}(\mathbb{D}, 0)$ have no holomorphic extension beyond $\mathbb{D}$. Since Dirichlet sets can have

Hausdorff dimension 1 but not positive arclength, the question naturally arises whether such universal approximation can occur on sets $F \subset \mathbb{T}$ of positive arclength where f is holomorphic. The next result [22] shows that this cannot happen, even where f merely has a finite non-tangential limit.

Theorem 11 *Given $f \in \mathrm{Hol}(\mathbb{D})$ and an increasing sequence (m_k), let*

$$E = \{w \in \mathbb{T} : S(w) := \lim_{k\to\infty} S_{m_k}(f,0)(w) \text{ exists}\},$$

$$F = \{w \in \mathbb{T} : f(w) := \mathrm{nt}\lim_{z\to w} f(z) \text{ exists}\}.$$

Then $S = f$ almost everywhere on $E \cap F$ (with respect to arclength).

Thus knowledge of $\lim_{k\to\infty} S_{m_k}(f,0)$ on a non-negligible subset of $\mathbb{T}$ where f is well-behaved does give information about the function f itself. This result fails if we replace non-tangential limits by radial limits. Indeed, Costakis [11] has shown that there are functions f in $\mathcal{U}(\mathbb{D},0)$ with radial limit 0 on any given closed nowhere dense set $F \subset \mathbb{T}$, and such a set F can be chosen to have positive arclength.

By a *fat approach region* to $1 \in \mathbb{T}$ we mean a set of the form

$$\omega(1) = \{x + iy \in \mathbb{D} : |y| < a \text{ and } b < x < 1 - \phi(y)\},$$

for some $a > 0$ and $b \in (0,1)$, where $\phi : [-a,a] \to [0,\infty)$ is Lipschitz and

$$\int_{-a}^{a} y^{-2}\phi(y)dy < \infty. \tag{6}$$

(For example, we could choose $\phi(y) = |y|^\alpha$ for some $\alpha > 1$.) It is easy to see that $\mathrm{nt}\lim_{z\to 1} \chi_{\omega(1)}(z) = 1$. Fat approach regions $\omega(e^{it})$ to other points e^{it} of $\mathbb{T}$ are formed by rotation. The next result is taken from [19].

Theorem 12 *Suppose that $f \in \mathrm{Hol}(\mathbb{D})$, where*

(i) f is bounded on two fat approach regions $\omega(e^{it_1}), \omega(e^{it_2})$ $(t_1 < t_2 < t_1 + 2\pi)$, and
(ii) $(S_{m_k}(f,0))$ is uniformly bounded on an open arc I containing $J = \{e^{it} : t_1 \leq t \leq t_2\}$.

Then f is bounded on the sector $\{rw : 0 < r < 1, w \in J\}$.

A further result, from [23], concerns boundary behaviour at a single point:

Theorem 13 *Suppose that the Taylor series of $f \in \mathrm{Hol}(\mathbb{D})$ has Ostrowski gaps (m_k, p_k). If*

(i) f' is bounded on a fat approach region to $w \in \mathbb{T}$, and
(ii) $(S_{m_k}(f,0))$ is uniformly bounded on an open arc containing w, then

$$\lim_{k\to\infty} S_{m_k}(f,0)(w) = \mathrm{nt}\lim_{z\to w} f(z). \tag{7}$$

The above non-tangential limit automatically exists, by hypothesis (i). The theorem asserts both the existence of the first limit in (7) and the equality of the two limits. Although this is reminiscent of Abel's Limit Theorem, the existence of the non-tangential limit now forms part of the hypothesis rather than the conclusion. The result fails if the Ostrowski gap condition is omitted, as can easily be seen by considering the function $z \longmapsto (1-z)^{-1}$ and $w \in \mathbb{T}\backslash\{1\}$ and a suitable subsequence (S_{m_k}).

We will now give a brief indication of the potential theory underlying the above theorems. Suppose that s is subharmonic and bounded above on a domain ω and let $w \in \partial\omega$. (For simplicity we will assume that ω is bounded.) In general, we cannot say much about the boundary behaviour of s at w. However, in the special case where ω is the punctured disc $\mathbb{D}\backslash\{0\}$ and $w = 0$, the point w is a removable singularity for s. Thus there exists $l \in [-\infty, \infty)$ such that $s(z) \to l$ as $z \to w$ outside some set A which is thin at w. In such circumstances we say that s has *fine limit* l at w and write $\operatorname{f\,lim}_{z\to w} s(z) = l$. More generally, and less trivially, if ω^c is thin at w, then $\operatorname{f\,lim}_{z\to w} s(z)$ still exists.

Now let $\mu_z^{\omega}(A)$ denote the harmonic measure of a set $A \subset \partial\omega$ with respect to ω and $z \in \omega$. If s is subharmonic and bounded above on a neighbourhood of $\overline{\omega}$, then certainly $s(z) \le \int_{\partial\omega} s d\mu_z^{\omega}$. However, it is possible to extend the notion of harmonic measure to the case where z is replaced by a point $w \in \partial\omega$ at which ω^c is thin (that is, w is an irregular boundary point of ω). If s is subharmonic and bounded above on a neighbourhood of $\overline{\omega}\backslash\{w\}$, then the fine limit of the preceding paragraph satisfies

$$\operatorname{f}\lim_{z\to w} s(z) \le \int_{\partial\omega} s d\mu_w^{\omega}. \tag{8}$$

Now suppose that (s_k) is a decreasing sequence of such functions on some neighbourhood of $\overline{\omega}\backslash\{w\}$ with a negative limit. It follows from (8) that $\operatorname{f\,lim}_{z\to w} s_{k_0}(z) < 0$ for some k_0.

We now proceed by analogy. There is a related notion of a set $A \subset \mathbb{D}$ being *minimally thin* at a point $w \in \mathbb{T}$. If $\omega \subset \mathbb{D}$ is an approach region to $w \in \mathbb{T}$ that is bounded by the graph of a Lipschitz function, then $\mathbb{D}\backslash\omega$ is minimally thin at w if and only if ω is a fat approach region. Thus, for such sets, the condition (6) plays the role of the Wiener criterion for thinness (2). We write $\operatorname{mf\,lim}_{z\to w} g(z) = l$ if $g(z) \to l$ as $z \to w$ outside a set A which is minimally thin at w.

If s is subharmonic on $\mathbb{D}$, and $s(z)/(-\log|z|)$ is bounded above near $w \in \mathbb{T}$, then $\operatorname{mf\,lim}_{z\to w} s(z)/(-\log|z|)$ exists and lies in $[-\infty, \infty)$. This is the boundary analogue of the earlier fine limit assertion for an upper bounded subharmonic function on a punctured disc. More generally, the same conclusion is known to hold if s is subharmonic merely on a fat approach region ω to $w \in \mathbb{T}$ and $s(z)/(-\log|z|)$ is bounded above there. Further, if (s_k) is a decreasing sequence of such functions with negative limit, then $\operatorname{mf\,lim}_{z\to w} s_{k_0}(z)/(-\log|z|) < 0$ for some k_0.

We can now sketch a proof of Theorem 12. For simplicity we assume that the bounds in hypotheses (i) and (ii) are both 1. By Bernstein's lemma,

$$\frac{1}{m_k}\log|S_{m_k}(f,0)(z)| \leq G_{\widehat{\mathbb{C}}\setminus\bar{I}}(z,\infty) \leq c\log\frac{1}{|z|} \quad \text{on } \omega(e^{it_1})\cup\omega(e^{it_2})$$

for some $c > 0$, so

$$s_k(z) \leq c\log\frac{1}{|z|} \quad \text{on } \omega(e^{it_1})\cup\omega(e^{it_2}), \text{ where } s_k = \frac{1}{m_k}\log\frac{|S_{m_k}(f,0) - f|}{2}.$$

It is easily seen that $\limsup_{k\to\infty} s_k(z) \leq \log|z| < 0$ on $\mathbb{D}$, so we can apply our previous observations to find k_0 and a set A, minimally thin at both e^{it_1} and e^{it_2}, such that

$$s_{k_0} < 0 \Rightarrow |S_{m_k}(f,0) - f| < 2 \Rightarrow |S_{m_k}(f,0)| < 3 \text{ on } [\omega(e^{it_1})\cup\omega(e^{it_2})]\setminus A \ \ (k \geq k_0).$$

(Strictly speaking, (s_k) need not be decreasing, so there is a little extra work to do here.) It follows from the maximum principle that f is bounded on the stated sector.

Remark Hypothesis (i) in Theorem 12 can be relaxed to require merely that $\log|f| \leq h$ on the two fat approach regions, where h is some positive harmonic function on $\mathbb{D}$. The conclusion must then be slightly weakened to say that f is bounded on all sectors of the form $\{re^{it} : 0 < r < 1, t_1 + \varepsilon \leq t \leq t_2 - \varepsilon\}$, where $\varepsilon > 0$.

Corollary 2 *Let* $f \in \mathrm{Hol}(\mathbb{D})$ *and suppose that* $\log|f(z)| \leq \psi(|z|)$ *on* $D(w,r)\cap\mathbb{D}$ *for some* $w \in \mathbb{T}$ *and* $r > 0$, *where* $\psi : [0,1) \to (0,\infty)$ *is an increasing continuous function such that*

$$\int_0^1 \sqrt{\frac{\psi(\sigma)}{1-\sigma}}\,d\sigma < \infty.$$

If, on the arc $I = D(w,r)\cap\mathbb{T}$, *a subsequence* (S_{m_k}) *is uniformly bounded and pointwise convergent almost everywhere, then*

$$\operatorname{nt}\lim_{z\to w} f(z) \textit{ exists and equals } \lim_{k\to\infty} S_{m_k}(w) \textit{ almost everywhere on } I.$$

To see this, we note from an argument of Rippon (Section 3 of [42], cf. [43]) that, at each point $w \in I$, there is a fat approach region on which $\log|f|$ is majorized by a positive harmonic function on $\mathbb{D}$. From Theorem 12 and the above remark we see that f must be bounded near each point of I. Hence, by Fatou's theorem, f has a finite non-tangential limit at almost every point of I. The desired equality of the limits almost everywhere now follows from Theorem 11.

Corollary 2 complements recent universality results of Beise and Müller [9] concerning Taylor series that lie in Bergman spaces.

5 Boundary Behaviour II

We will now apply Theorems 11–13 to obtain additional results about the boundary behaviour of universal Taylor series. The first two corollaries below originally appeared in [19], and the third in [23].

Corollary 3 *Let $f \in \mathcal{U}(\mathbb{D}, 0)$. Then, at almost every point w in $\mathbb{T}$, the image $f(\Gamma)$ of every Stolz angle Γ at w is dense in $\mathbb{C}$.*

Proof Plessner's theorem states that, at almost every point w of $\mathbb{T}$, either $\operatorname{nt}\lim_{z\to w} f(z)$ exists or $\overline{f(\Gamma)} = \mathbb{C}$ for every Stolz angle Γ at w. Theorem 11 says that, for any given (m_k), the value of $\operatorname{nt}\lim_{z\to w} f(z)$ determines the value of $\lim_{k\to\infty} S_{m_k}(f, 0)$ almost everywhere that it exists on $\mathbb{T}$. Thus, if $f \in \mathcal{U}(\mathbb{D}, 0)$, the first possibility in Plessner's theorem must fail almost everywhere.

Corollary 4 *Suppose that $f \in \mathrm{Hol}(\mathbb{D})$ and $\log|f| \le h$ on fat approach regions to two distinct points, e^{it_1} and e^{it_2}, where h is a positive harmonic function on $\mathbb{D}$. Then $f \notin \mathcal{U}(\mathbb{D}, 0)$.*

Proof If $(S_{m_k}(f, 0))$ were to converge on an open arc I containing e^{it_1} and e^{it_2}, then Theorem 12 (and the remark preceding Corollary 2) would imply the boundedness of f near certain subarcs of I. This is impossible for functions f in $\mathcal{U}(\mathbb{D}, 0)$.

Corollary 5 *If $f \in \mathrm{Hol}(\mathbb{D})$ and f' is bounded on a fat approach region to a point of $\mathbb{T}$, then $f \notin \mathcal{U}(\mathbb{D}, 0)$.*

Proof This follows by combining Theorem 13 with conclusion (ii) of Theorem 4.

In connection with Corollary 4 we note from the following result [19] that it is possible for a function f in $\mathcal{U}(\mathbb{D}, 0)$ to satisfy the inequality $\log|f| \le h$ on a fat approach region to a single point $w \in \mathbb{T}$, provided h tends to infinity at w.

Proposition 2 *Let $A \subset \mathbb{D}$, where $\overline{A} \cap \mathbb{T} = \{1\}$, and let $g : \mathbb{D} \to (1, \infty)$ be a continuous function such that $g(z) \to \infty$ as $z \to 1$. Then there exists $f \in \mathcal{U}(\mathbb{D}, 0)$ such that $|f| \le g$ on A.*

Example 4 Let D be a disc that is internally tangent to $\mathbb{T}$ at the point 1. As noted in [34] (see the proof of Proposition 5.6 there), no member of $\mathcal{U}(\mathbb{D}, 0)$, when restricted to D, can have a limit at 1. However, by the above proposition, there exists f in $\mathcal{U}(\mathbb{D}, 0)$ satisfying $|f(z)| \le |z-1|^{-1/3}$ on D. It follows that the function $z \mapsto (z-1)f(z)$ does not belong to $\mathcal{U}(\mathbb{D}, 0)$, and so the universality property of Taylor series is not preserved under multiplication by non-constant polynomials. Similarly, no antiderivative of this function f can belong to $\mathcal{U}(\mathbb{D}, 0)$. It is an open question whether derivatives of universal Taylor series are again universal.

Since universal Taylor series have mainly been investigated on simply connected domains, it is natural to ask if the universality property of Taylor series is conformally invariant. That is, given a conformal map $\Phi : \Omega_0 \to \Omega$ between simply connected domains and $f \in \mathcal{U}(\Omega, \zeta)$, does it follow that $f \circ \Phi \in \mathcal{U}(\Omega_0, \Phi^{-1}(\zeta))$?

The above results enable us to show that the answer is negative [19]. Let $\mathbb{S}$ denote the strip $\{-1 < \mathrm{Re} z < 1\}$.

Theorem 14 *There is a function* $f \in \mathcal{U}(\mathbb{S}, 0)$ *such that, for any conformal mapping* $\Phi : \mathbb{D} \to \mathbb{S}$*, the function* $f \circ \Phi$ *does not belong to* $\mathcal{U}(\mathbb{D}, \Phi^{-1}(0))$.

To see why this is the case, we remark that the construction used for Proposition 2 also yields the following analogue for the strip [19]. (The key difference here is that no continuum in $\mathbb{S}^c$ contains both 1 and -1.)

Proposition 3 *Let* $A \subset \mathbb{S}$ *be bounded, where* $\overline{A} \cap \partial\mathbb{S} = \{\pm 1\}$*, and let* $g : \mathbb{S} \to (1, \infty)$ *be a continuous function such that* $g(z) \to \infty$ *as* $z \to \pm 1$*. Then there exists* $f \in \mathcal{U}(\mathbb{S}, 0)$ *such that* $|f| \leq g$ *on* A*.*

To see why Theorem 14 holds, we choose the set A above to contain fat approach regions to both 1 and -1, and the function g to be of the form e^h, where h is a positive harmonic function on $\mathbb{S}$ with limit ∞ at ± 1. We next note that fat approach regions in $\mathbb{S}$ to two distinct points of $\partial\mathbb{S}$ are images under Φ of fat approach regions in $\mathbb{D}$ to two distinct points of $\mathbb{T}$. (More generally, the notion of minimal thinness can be formulated purely in terms of harmonic and superharmonic functions, and so is conformally invariant.) In view of Corollary 4, and Theorem 15 below, we now see that $f \circ \Phi$ cannot belong to $\mathcal{U}(\mathbb{D}, \Phi^{-1}(0))$.

6 Dependence on the Centre of Expansion

The next result [36] is a consequence of Theorem 4(ii) and the equiconvergence property (5).

Theorem 15 *If* Ω *is simply connected, then the collection* $\mathcal{U}(\Omega, \zeta)$ *is independent of the centre of expansion* ζ*.*

The situation when Ω^c is compact and connected (cf. Theorem 2) is more subtle. Bayart [4] established the following result, which contains work of Costakis [12] for the special case where Ω^c is a polygon.

Theorem 16 *If* $\Omega \subset \mathbb{C}$ *is a domain such that* Ω^c *is compact and connected, then* $\cap_{\zeta \in \Omega} \mathcal{U}(\Omega, \zeta)$ *is a residual subset of* $\mathrm{Hol}(\Omega)$.

This leaves open the question of whether $\mathcal{U}(\Omega, \zeta)$ can depend on ζ in this setting. It turns out that this is indeed the case [21].

Theorem 17 *If* Ω *is the exterior domain of a Dini-smooth Jordan curve, then, for every* $\zeta \in \Omega$*, there exists* ζ_1 *such that* $\mathcal{U}(\Omega, \zeta) \backslash \mathcal{U}(\Omega, \zeta_1) \neq \emptyset$*.*

We will briefly outline why this result holds in the case where $\Omega = \left(\overline{\mathbb{D}}\right)^c$. Let $w \in \mathbb{T} \backslash \{-1\}$. There is an analogue of Proposition 2 in this context which says that, if $A \subset \Omega$, where $\overline{A} \cap \mathbb{T} = \{-1, w\}$, and $g : \Omega \to (1, \infty)$ is a continuous function with limit ∞ at both -1 and w, then there exists $f_1 \in \mathcal{U}(\Omega, 2)$ such that $|f_1| \leq g$ on A. We can take g to be of the form e^h here, where h is positive and harmonic on Ω. On the other hand, there is an analogue of Corollary 4 in this context which says that no function f_2 in $\mathcal{U}(\Omega,-2)$ can satisfy $\log|f_2| \leq h$ on fat approach regions to -1 and w provided we take $w \in D(-2,\sqrt{3}) \cap (\mathbb{T}\backslash\{-1\})$. (The $\sqrt{3}$ here is closely related to Example 3(b).) Hence $f_1 \in \mathcal{U}(\Omega,2)\backslash\mathcal{U}(\Omega,-2)$ in this case, as required.

7 Universal Laurent Series

Theorem 3 shows that, for many multiply connected domains, there is no hope of finding any universal Taylor series. The picture is completely different, however, if we turn our attention to Laurent series. Let $\Omega \subset \mathbb{C}$ be a domain of the form $\Omega = \mathbb{C}\backslash\left(\bigcup_{j=0}^{k} A_j\right)$, where $k \geq 1$, the sets A_j are pairwise disjoint continua in $\widehat{\mathbb{C}}$, and $\infty \in A_0$. (At least one of these continua should be non-degenerate, to avoid triviality.) Each function f in $\mathrm{Hol}(\Omega)$ has a unique decomposition of the form

$$f = \sum_{j=0}^{k} f_j, \text{ where } f_j \in \mathrm{Hol}\left(\widehat{\mathbb{C}}\backslash A_j\right) \ (j = 0,\ldots,k) \text{ and } f_j(\infty) = 0 \ (j = 1,\ldots,k).$$

We fix $\alpha_j \in A_j$ for each $j = 1,\ldots,k$. Then f_j has a Laurent expansion outside some closed disc centred at α_j, and the coefficient of $(z-\alpha_j)^{-n}$ in this expansion will be denoted by $b_n(f_j,\alpha_j)$. We define, for $\zeta \in A_0^c$,

$$M_m(f,\zeta)(z) = \sum_{n=0}^{m} \frac{f_0^{(n)}(\zeta)}{n!}(z-\zeta)^n + \sum_{j=1}^{k}\sum_{n=1}^{m} \frac{b_n(f_j,\alpha_j)}{(z-\alpha_j)^n} \quad (z \in \mathbb{C}\backslash\{\alpha_1,\ldots,\alpha_k\}).$$

Definition 2 We say that f has a *universal Laurent series* with respect to $\{\alpha_1,\ldots,\alpha_k\}$ if, for every compact set $K \subset (\Omega \cup \{\alpha_1,\ldots,\alpha_k\})^c$ with K^c connected, and every function $g \in C(K) \cap \mathrm{Hol}(K^\circ)$, there is a sequence (m_n) in $\mathbb{N}$ such that

$$\sup_{\zeta\in J}\sup_{z\in K} |M_{m_n}(f,\zeta)(z) - g(z)| \to 0 \quad (k \to \infty)$$

for every compact set $J \subset A_0^c$. The collection of functions with this property will be denoted by $\mathcal{U}_L(\Omega;\alpha_1,\ldots,\alpha_k)$.

This notion was introduced by Costakis et al. [14], who showed that $\mathcal{U}_L(\Omega;\alpha_1,\ldots,\alpha_k)$ is a residual subset of $\mathrm{Hol}(\Omega)$. The theory was further developed by Müller et al. in [36] and [37]. For example, [37] contains a version for Laurent series of Theorem 4(i) above. (See also [26].)

The following analogue of Theorem 8 for the boundary behaviour of universal Laurent series was established in [21].

Theorem 18 *Let $f \in \mathcal{U}_L(\Omega;\alpha_1,\ldots,\alpha_k)$, where Ω is as above. Then, for any disc D centred at a point of $\partial\Omega\backslash\{\alpha_1,\ldots,\alpha_k\}$, the set $\mathbb{C}\backslash f(D\cap\Omega)$ is polar.*

It was also shown in [21] that the collection $\mathcal{U}_L(\Omega;\alpha_1,\ldots,\alpha_k)$ can depend on the choice of $\alpha_1,\ldots,\alpha_k$. This is true even when $k=1$ and Ω is an exterior Jordan domain:

Theorem 19 *Let $f\in\mathcal{U}_L(\Omega;\alpha_1)$, where Ω is an exterior Jordan domain and $\alpha_1\in\partial\Omega$. Then $\mathcal{U}_L(\Omega;\alpha_1)\backslash\mathcal{U}_L(\Omega;\alpha)\neq\emptyset$ for every $\alpha\in\Omega^c\backslash\{\alpha_1\}$.*

The proof of this result is related to that of Corollary 4 above.

8 Universal Dirichlet Series

Let $\mathbb{C}_+$ denote the half-plane $\{s=\sigma+it:\sigma>0\}$. Further, let $D(\mathbb{C}_+)$ denote the space of holomorphic functions on $\mathbb{C}_+$ which are representable there as an absolutely convergent general Dirichlet series, $f(s)=\sum_1^\infty a_n e^{-\lambda_n s}$, where (λ_n) is an unbounded strictly increasing sequence in $[0,\infty)$.

Theorems 11–13 have recently [23] been generalized to cover functions in $D(\mathbb{C}_+)$. We give a sample below. Let $T_m(s)$ denote the partial sum $\sum_1^m a_n e^{-\lambda_n s}$.

Theorem 20 *Given $f\in D(\mathbb{C}_+)$ and an increasing sequence (m_k), let*

$$E=\{w\in i\mathbb{R}:T(w):=\lim_{k\to\infty}T_{m_k}(w)\ \textit{exists}\}$$

and

$$F=\{w\in i\mathbb{R}:f(w):=\mathrm{nt}\lim_{s\to w}f(s)\ \textit{exists}\}.$$

Then $f=S$ almost everywhere on $E\cap F$.

We now restrict our attention to ordinary Dirichlet series, that is, where $\lambda_n=\log n$.

Definition 3 We say that a member $\sum_1^\infty a_n n^{-s}$ of $D(\mathbb{C}_+)$ is a *universal Dirichlet series* if, for every compact set $K\subset\{\sigma\le 0\}$ with K^c connected, and every function $g\in C(K)\cap\mathrm{Hol}(K^\circ)$, there is a subsequence (T_{m_k}) of the partial sums that converges uniformly to g on K. The collection of all universal Dirichlet series will be denoted by $\mathcal{U}_D(\mathbb{C}_+)$.

Aron et al. [3] have recently shown that such series are topologically generic in the space of absolutely convergent ordinary Dirichlet series on $\mathbb{C}_+$, endowed with the topology induced by the semi-norms $\left\|\sum_1^\infty a_n n^{-s}\right\|_\sigma=\sum_1^\infty|a_n|\,n^{-\sigma}$. Bayart [5] had previously established this for approximation on a more restricted collection of compact sets K. However, almost nothing was known about the boundary behaviour of universal Dirichlet series, including even the question of whether universal Dirichlet series could have a holomorphic extension beyond $\mathbb{C}_+$. We now list some consequences of Theorem 20 taken from [23], where they are stated in greater generality. (The obvious analogue of Corollary 4 also holds.)

Corollary 6 *Let $f \in \mathcal{U}_D(\mathbb{C}_+)$. Then, for almost every $w \in i\mathbb{R}$, the set $f(\Gamma)$ is dense in $\mathbb{C}$ for every Stolz angle $\Gamma \subset \mathbb{C}_+$ with vertex at w.*

Corollary 7 *Let $f \in \mathcal{U}_D(\mathbb{C}_+)$. Then, for any disc D centred on $i\mathbb{R}$, the set $\mathbb{C} \backslash f(D \cap \mathbb{C}_+)$ is polar.*

Corollary 8 *Let $f \in \mathcal{U}_D(\mathbb{C}_+)$. Then there is a residual set $Z \subset \mathbb{R}$ such that $\{f(\sigma + it) : 0 < \sigma < 1\}$ is dense in $\mathbb{C}$ for every $t \in Z$.*

Corollary 6 follows from Theorem 20 just as Corollary 3 was deduced from Theorem 11. Corollary 7 holds because, otherwise, $\log |f|$ would have a positive harmonic majorant on $D \cap \mathbb{C}_+$ (see the paragraph following Theorem 8) and then Fatou's theorem would yield a contradiction to Corollary 6. Corollary 8 follows from Corollary 7 because, by the Collingwood maximality theorem, it is enough to show that f has a maximal unrestricted cluster set at each point of $i\mathbb{R}$.

The obvious analogues of Theorem 12 and Corollary 2 hold for general Dirichlet series. From the latter we can immediately deduce the following.

Corollary 9 *Let $\psi : (0, 1] \to (0, \infty)$ be a decreasing continuous function such that*

$$\int_0^1 \sqrt{\frac{\psi(\sigma)}{\sigma}} d\sigma < \infty.$$

If $f \in D(\mathbb{C}_+)$ and $\log |f(\sigma + it)| \leq \psi(\sigma)$ on $D(w, r) \cap \mathbb{C}_+$ for some $w \in i\mathbb{R}$ and $r > 0$, then $f \notin \mathcal{U}_D(\mathbb{C}_+)$.

This result is significantly weaker than Theorem 7 for universal Taylor series.

9 Universal Polynomial Expansions of Harmonic Functions

Let $B(x_0, r)$ denote the open ball of centre x_0 and radius r in Euclidean space $\mathbb{R}^N$ $(N \geq 2)$, and let $\mathbb{B} = B(0, 1)$. If h is a harmonic function on $B(x_0, r)$, then it has an expansion there of the form $h(x) = \sum_{n=0}^{\infty} H_n(x - x_0)$, where H_n is a homogeneous harmonic polynomial of degree n. We denote by $\mathbf{S}_m(h, x_0)$ the partial sum, up to degree m, of this series.

Definition 4 Let h be a harmonic function on a domain $\Omega \subset \mathbb{R}^N$ and let $x_0 \in \Omega$. We say that h has a *universal polynomial expansion* about x_0 if, for any compact $K \subset \Omega^c$ with K^c connected, and for any function g which is harmonic on a neighbourhood of K, there is a subsequence $(\mathbf{S}_{m_k}(h, x_0))$ which converges uniformly to g on K. The collection of all such functions h will be denoted by $\mathcal{U}_H(\Omega, x_0)$.

Gauthier and Tamptse [25] have shown that, if $\left(\mathbb{R}^N \cup \{\infty\}\right) \backslash \Omega$ is connected and $x_0 \in \Omega$, then $\mathcal{U}_H(\Omega, x_0)$ is a residual subset of the space of all harmonic functions on Ω, endowed with the topology of local uniform convergence. (See also Armitage [1] for an earlier, related result.) A substantial advance in the theory was made by Manolaki [31] who proved, among other things, the following.

Theorem 21 *Let $\Omega \subset \mathbb{R}^N$ be a domain such that $\mathbb{R}^N \backslash \Omega$ contains an infinite cone, and let $x_0 \in \Omega$. Then*

(a) the existence of functions in $\mathcal{U}_H(\Omega, x_0)$ requires $\left(\mathbb{R}^N \cup \{\infty\}\right) \backslash \Omega$ to be connected;
(b) the collection $\mathcal{U}_H(\Omega, x_0)$ is independent of the choice of x_0;
(c) no function in $\mathcal{U}_H(\Omega, x_0)$ can be extended harmonically to any larger domain.

This result is related to a previous paper [30], in which she established analogues for harmonic functions of celebrated results of Ostrowski [40] on the relationship between overconvergence and gap structure for complex power series. The proof also uses a process of inductive complexification, which relies on Ω omitting an infinite cone C: the significance of this hypothesis lies in the facts that the intersection with Ω^c of any line parallel to the axis of C contains a half-line, and that half-lines embedded in a plane are non-thin at infinity.

Logunov [28] has proved the following analogue of Theorem 7 for harmonic functions, which strengthens a result in [20].

Theorem 22 *Let $\psi : [0, 1) \to (0, \infty)$ be an increasing function such that (4) holds. If h is harmonic on $\mathbb{B}$ and $|h(x)| \le \psi(|x|)$ on $B(y, r) \cap \mathbb{B}$ for some $y \in \partial\mathbb{B}$ and $r > 0$, then $f \notin \mathcal{U}_H(\mathbb{B}, 0)$.*

Finally, Golitsyna [26] has recently developed a theory of universal Laurent expansions for harmonic functions.

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Subharmonic Images of a Convergent Sequence

Paul Gauthier and Myrto Manolaki

Abstract In this paper we characterize the sequences of possible values of a subharmonic function along a convergent sequence of points. We also discuss some related open questions and possible generalizations.

Keywords Subharmonic interpolation

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1 Introduction

A classical interpolation theorem of Weierstrass (see, for example [9, Ch. 15]) asserts that, if (a_n) is a sequence of distinct points in a domain G of the complex plane $\mathbb{C}$ without accumulation points in G and (b_n) is any sequence in $\mathbb{C}$, then there is a holomorphic function f in G such that $f(a_n) = b_n$ for all n. For example, if we choose as (b_n) an enumeration of all complex numbers with rational coordinates, one can obtain a function f such that the sequence $(f(a_n))$ is dense in $\mathbb{C}$. On the other hand, if (a_n) is a sequence of distinct points in the domain G with accumulation point in G, and (b_n) is any sequence in $\mathbb{C}$, with the property that there is a holomorphic function f in G such that $f(a_n) = b_n$ for all n, then (b_n) has to satisfy certain conditions. First of all it has to be convergent. Secondly, the function f with the above property is uniquely determined by the identity principle. A complete characterization of pairs of convergent sequences (a_n) and (b_n) in $\mathbb{C}$,

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with the a_n distinct, such that there exists an entire function f, for which $f(a_n) = b_n$ for all n, was discovered independently by Bendixson [2] in 1887 and a century later by Peretz in 1983 (in [8] and in his unpublished manuscript "Analytic image of a convergent sequence"). (See also [7].) The condition for this characterization is given in terms of the convergence of one specific series involving the sequences (a_n) and (b_n).

This paper is concerned with similar interpolation problems for subharmonic (or continuous subharmonic) functions. This is rather different from the known extension theorems for subharmonic functions (see [5] and [6]), where a subharmonic function is given in a neighbourhood of a set A and we wish to extend it to a given, larger domain.

Since subharmonic functions take their values in $[-\infty, +\infty)$ and are upper semicontinuous, if (x_n) is a sequence of distinct points converging to a point x_0 and (y_n) is a sequence of values in $[-\infty, +\infty)$, such that $y_n = u(x_n), n = 1, 2, \dots,$ for some function u subharmonic in a neighbourhood of x_0, then

$$\limsup_{n\to\infty} u(x_n) \leq u(x_0) < +\infty.$$

In this note, we show that this obvious necessary condition is in fact also sufficient to characterize such "interpolation pairs" $((x_n), (y_n))$ for subharmonic functions. More precisely, we show the following:

Theorem 1 *Let B_{x_0} be an open ball in $\mathbb{R}^N (N \geq 2)$ with center x_0, and let (x_n) be a sequence of distinct points in $B_{x_0} \setminus \{x_0\}$ which converges to x_0. If (y_n) is a sequence of values in $[-\infty, +\infty)$ with $\limsup_{n\to\infty} y_n =: b < +\infty$, then, there exist a subharmonic function u on B_{x_0}, satisfying the following:*

(i) $u(x_n) = y_n$ *for each* $n = 1, 2, \dots;$
(ii) $\limsup_{x\to x_0} u(x) = b;$
(iii) $u : B_{x_0} \setminus \{x_0\} \to [-\infty, +\infty)$ *is continuous (in the extended sense) and finite-valued on* $B_{x_0} \setminus \{x_n : n \in \mathbb{N}\}$.

Remark 1 If we do not require condition (ii) to be valid and we only require the interpolating function u to be continuous and subharmonic on $B_{x_0} \setminus \{x_0\}$, then the result could follow by applying known extension theorems for subharmonic functions (such as Theorem 6.1 of [5]). However, these extension theorems do not provide estimates for the growth of the subharmonic extension u, and therefore, we cannot conclude that x_0 is a removable singularity for u. To overcome this difficulty, we provide a constructive proof by carefully "gluing" specific Dirichlet solutions on spherical shells around x_0.

We express our deep affection for our dear friend André Boivin (respectively doctoral student and postdoctoral supervisor), who would have collaborated in this work had he not tragically left us far too soon.

2 Preliminaries

For $y \in \mathbb{R}^N$ and $0 < r < R$ we denote by $S_y[r,R]$ the closed shell $\{x : r \le \|x-y\| \le R\}$ and by $S_y(r,R)$ the open shell $\{x : r < \|x-y\| < R\}$. We also write $S_y[r]$ to denote the sphere $\{x : \|x-y\| = r\}$. If $y = 0$ we simply write $S[r,R]$, $S(r,R)$ and $S[r]$ (instead of $S_0[r,R]$, $S_0(r,R)$ and $S_0[r]$). Moreover we denote by $U_y(x)$ the fundamental solution for the Laplacian with pole at y:

$$U_y(x) = \begin{cases} -\ln|x-y| & \text{for } N = 2, \\ \|x-y\|^{2-N} & \text{for } N > 2. \end{cases}$$

Throughout this paper, for a radial function L and $r > 0$, we (abusively) denote by $L(r)$ the (common) value of $L(x)$ for $\|x\| = r$. In particular, we write $U_0(r)$ to signify $U_0(x)$, for some (hence every) x for which $\|x\| = r$.

It is easy to see that the function

$$L_{r,R}^{m,M}(x) := m\frac{U_0(x)}{U_0(r)} + \left(\frac{U_0(r) - U_0(x)}{U_0(r) - U_0(R)}\right)\left(M - m\frac{U_0(R)}{U_0(r)}\right) \tag{1}$$

is the solution of the Dirichlet problem on $S[r,R]$, with boundary values m on $S[r]$ and M on $S[R]$.

One of the main tools we will use is the following classical theorem:

Lemma 1 (Gluing Principle [Cor. 3.2.4, [1]]) *Let ω be an open subset of an open set Ω in $\mathbb{R}^N$. Let s be subharmonic on ω and S subharmonic on Ω, and suppose that* $\limsup_{x\to y, x\in\omega} s(x) \le S(y)$ *for all $y \in \partial\omega \cap \Omega$. Then the function*

$$v(x) = \begin{cases} \max\{s(x), S(x)\} & (x \in \omega) \\ S(x) & (x \in \Omega \setminus \omega) \end{cases}$$

is subharmonic on Ω.

Proof The proof follows immediately by verifying the submeanvalue property for the function v for all points in $\partial\omega \cap \Omega$.

We can apply the gluing principle to prove the following lemma, which is of fundamental importance for the proof of Theorem 1.

Lemma 2 *Fix $0<r_1<R_2<R_1$ and $m_2<m_1<M_1$, and let $M_2 \in (m_1, L_{r_1,R_1}^{m_1,M_1}(R_2))$. For each $r_2 \in (0, r_1)$, we consider the function*

$$L_{21}(x) := \begin{cases} L_{r_2,R_2}^{m_2,M_2}(x) & \textit{for } r_2 \le \|x\| \le r_1, \\ \max\{L_{r_2,R_2}^{m_2,M_2}(x), L_{r_1,R_1}^{m_1,M_1}(x)\} & \textit{for } r_1 \le \|x\| \le R_2, \\ L_{r_1,R_1}^{m_1,M_1}(x) & \textit{for } R_2 \le \|x\| \le R_1. \end{cases}$$

Then, for all sufficiently small r_2, the function L_{21} is continuous on $S[r_2, R_1]$ and subharmonic on $S(r_2, R_1)$.

Proof For simplicity we write L_{r_i} instead of $L^{m_i,M_i}_{r_i,R_i}$ for $i = 1, 2$. For each fixed $r \in (r_2, R_2]$,

$$L_{r_2} \nearrow M_2 \quad \text{uniformly on} \quad S(r, R_2],$$

as $r_2 \searrow 0$. This follows easily by using the formula (1). In particular, for $r = r_1$, and sufficiently small r_2,

$$\max\{L_{r_2}(x), L_{r_1}(x)\} = L_{r_2}(x), \quad \text{for} \quad \|x\| = r_1, \tag{2}$$

so L_{21} is continuous on $\|x\| = r_1$.

On $\|x\| = R_2$, we have $L_{r_2}(R_2) = M_2 < L_{r_1}(R_2)$. Hence

$$\max\{L_{r_2}(x), L_{r_1}(x)\} = L_{r_1}(x), \quad \text{for} \quad \|x\| = R_2, \tag{3}$$

so L_{21} is continuous on $\|x\| = R_2$.

The function L_{21} is obviously continuous elsewhere, hence it is continuous on the whole $S[r_2, R_1]$. Finally, we can conclude that L_{21} is subharmonic on $S(r_2, R_1)$ for sufficiently small r_2. To see this, we use (2) and (3) to apply twice the gluing principle (Lemma 1): for the functions $S_1 := L_{r_2}$ and $s_1 := L_{r_1}$ which are subharmonic on the sets $\Omega_1 := S(r_2, R_2)$ and $\omega_1 := S(r_1, R_2)$ respectively; and for the functions $S_2 := L_{r_1}$ and $s_2 := L_{r_2}$ which are subharmonic on the sets $\Omega_2 := S(r_1, R_1)$ and $\omega_2 := S(r_1, R_2)$ respectively.

Lemma 3 *Let $A = \{a_1, \ldots, a_n\}$ and $P = \{p_1, \ldots, p_m\}$ be two disjoint sets in the open shell $S(r, R)$. Then there exists a continuous (in the extended sense) superharmonic function $U_{P,A}$ on $\mathbb{R}^N$, such that $U_{P,A}(a_j) = 0$ for all $a_j \in A$ and $U^{-1}_{P,A}(\infty) \cap S[r, R] = P$.*

Proof We consider the superharmonic function $U_P(x) := \sum_{i=1}^m U_{p_i}(x)$. Let $c_j := U_P(a_j)$. To finish the proof it suffices to find a finite-valued superharmonic and continuous function V on $\mathbb{R}^N$ such that $V(a_j) = -c_j$ for all $j = 1, 2, \ldots, n$, and define $U_{P,A}(x) := V(x) + U_P(x)$. One way to see that the construction of such a function V is possible is, by using Theorem 6.1 of [5]. Indeed, let E be the union of n pairwise disjoint, closed balls D_j in $S(r, R)$ with respective centres at a_j (where $j = 1, 2, \ldots, n$). Then, by Theorem 6.1 of [5], the function which equals to $-c_j$ on each D_j, can be extended continuously and superharmonically on $\mathbb{R}^N$ (because the complement of E in the one-point compactification of $\mathbb{R}^N$ is connected and locally connected).

Lemma 4 *Let $A = \{a_1, \ldots, a_n\}$ and $P = \{p_1, \ldots, p_m\}$ be two disjoint sets in the open shell $S(r, R)$ and let $U_{P,A}$ be as in Lemma 3. For each $j = 1, 2, \ldots, n$ and sufficiently small $\lambda > 0$, we consider the closed balls $K_{j,\lambda}$ with centres at a_j and* *radii equal to* λ*, such that they are all contained in* $S(r,R)$ *and they are pairwise disjoint. We put* $K_\lambda = K_{1,\lambda} \cup \cdots \cup K_{n,\lambda}$ *and let* $S_\lambda := S(r,R) \setminus K_\lambda$ *and* $S_0 := S(r,R) \setminus A$.

Let $m < M$ *and suppose that we are given finitely many real values* $b_1, b_2, \dots, b_n$*, with* $b_j < m$*, for all* $j = 1, 2, \dots, n$.

(i) For $\mu > 0$*, we denote by* h_μ *the solution of the Dirichlet problem on* $S[r,R]$ *with boundary values* $m + \mu U_{P,A}$ *and* $M + \mu U_{P,A}$ *on* $S[r]$ *and* $S[R]$ *respectively. As* $\mu \to 0$*, the directed family of subharmonic functions* $-\mu U_{P,A} + h_\mu$ *converges uniformly on compact subsets of* $S(r,R) \setminus P$ *to* $L^{m,M}_{r,R}$ *(the solution of the Dirichlet problem on* $S(r,R)$ *with boundary values* m *and* M *on* $S[r]$ *and* $S[R]$ *respectively).*

(ii) For $\lambda > 0$*, let* $h_{\lambda,\mu}$ *be the solution of the Dirichlet problem on* S_λ*, with boundary values* $m + \mu U_{P,A}$ *on* $S[r]$*,* $M + \mu U_{P,A}$ *on* $S[R]$ *and* b_j *on each* $\partial K_{j,\lambda}$*, and equal to* b_j *on* $K_{j,\lambda}$.

We consider the function $u_{\lambda,\mu} : S[r,R] \to [-\infty, +\infty)$ *defined by*

$$u_{\lambda,\mu} := -\mu U_{P,A} + h_{\lambda,\mu}. \tag{4}$$

Then, $u_{\lambda,\mu}$ *is continuous on* $S[r,R]$ *and converges uniformly on compact subsets of* $S_0 \setminus P$ *to* $-\mu U_{P,A} + h_\mu$*, as* $\lambda \to 0$*. Moreover, for small* μ *and* λ*, the function* $u_{\lambda,\mu}$ *is subharmonic on* $S(r,R)$.

Proof Let $\lambda_0 > 0$ be such that for all $\lambda \le \lambda_0$ the hypothesis in the first paragraph of the lemma is satisfied.

It is obvious that, as $\mu \to 0$, the functions $m + \mu U_{P,A}$ and $M + \mu U_{P,A}$ converge uniformly to m and M on $S[r]$ and $S[R]$ respectively, and so by the maximum principle, we conclude that h_μ converges to $L^{m,M}_{r,R}$ uniformly on $S[r,R]$. Moreover, as $\mu \to 0$, the function $\mu U_{P,A}$ tends uniformly to zero on compact subsets of $\mathbb{R}^N \setminus P$, and so part (i) follows immediately. Also we may choose μ is so small that

$$\mu U_{P,A}(x) < \max\{|m|, |M|\}, \quad (x \in \partial S(r,R) \cup K_\lambda,\ \lambda \le \lambda_0).$$

Therefore, since h_μ converges uniformly on $S[r,R]$ to $L^{m,M}_{r,R}$, we may assume that μ is so small that

$$|h_\mu(x)| < 2\max\{|m|, |M|\}, \quad \forall x \in S(r,R). \tag{5}$$

The function $u_{\lambda,\mu} : S[r,R] \to [-\infty, +\infty)$ is continuous on $S[r,R]$ by the regularity of the Dirichlet problem (since K_j is non-thin at each $x \in \partial K_j$).

Let K be a compact subset of $S_0 \setminus P$. For all $z \in K$ and all sufficiently small λ, we have that $z \in S_\lambda$ and

$$-\mu U_{P,A}(z) + h_\mu(z) - u_{\lambda,\mu}(z) = \int_{\partial S_\lambda} [h_\mu(\zeta) - h_{\lambda,\mu}(\zeta)] d\omega_\lambda(z,\zeta),$$

where $\omega_\lambda(z, E)$ is the harmonic measure for S_λ of a Borel set $E \subset \partial S_\lambda$ evaluated at a point $z \in S_\lambda$. Since $h_\mu = h_{\lambda,\mu}$ on $\partial S(r, R)$ we may write

$$-\mu U_{P,A}(z) + h_\mu(z) - u_{\lambda,\mu}(z) = \sum_{j=1}^{n} \int_{\partial K_{j,\lambda}} [h_\mu(\zeta) - h_{\lambda,\mu}(\zeta)] d\omega_\lambda(z, \zeta).$$

Therefore, since $h_{\lambda,\mu} = b_j$ on $K_{j,\lambda}$, we can use (5) to deduce that

$$|-\mu U_{P,A}(z) + h_\mu(z) - u_{\lambda,\mu}(z)| < \sum_{j=1}^{n} (2 \max\{|m|, |M|\} + |b_j|)\omega_\lambda(z, \partial K_{j,\lambda}). \tag{6}$$

Now, we will estimate $\omega_\lambda(z, \partial K_{j,\lambda})$. Choose $\Lambda > 0$ so large that, for all small λ and each $j = 1, \ldots, n$, the set S_λ is contained in the open shell $S_{a_j}(\lambda, \Lambda)$ and denote by $\omega_{j,\lambda}$ the harmonic measure for $S_{a_j}(\lambda, \Lambda)$. Then,

$$\omega_\lambda(z, \partial K_{j,\lambda}) < \omega_{j,\lambda}(z, \partial K_{j,\lambda}) = 1 - \frac{U_{a_j}(\lambda) - U_{a_j}(z)}{U_{a_j}(\lambda) - U_{a_j}(\Lambda)}.$$

It is now clear that $\omega_\lambda(\cdot, \partial K_{j,\lambda})$ converges uniformly to zero on compact subsets of S_0, as $\lambda \to 0$ and, from (4) and (6), it follows that $u_{\lambda,\mu}$ converges uniformly to $-\mu U_{P,A} + h_\mu$, on compact subsets of $S_0 \setminus P$.

We have that $h_\mu \to L^{m,M}_{r.R}$ uniformly on $S[r, R]$ as $\mu \to 0$ and, for every compact subset Q of $S(r, R)$, we have $\min\{L^{m,M}_{r,R}(x) : x \in Q\} > m$ (minimum principle). Thus, for all sufficiently small μ, we have $h_\mu > m$ on Q. Hence, for every compact subset Q of $S_0 \setminus P$, we have $u_{\lambda,\mu} > m$ on Q for small λ and μ. In particular, let Q_j be a closed ball centred at a_j, which (as we may assume) contains all of the balls $K_{j,\lambda}$ in its interior. Then, for sufficiently small λ, it follows from the minimum principle that $u_{\lambda,\mu} > b_j$ on $Q_j \setminus K_{j,\lambda}$ (because $b_j < m$ for all $j = 1, 2, \ldots, n$). Consequently, $u_{\lambda,\mu}$ satisfies the submeanvalue inequality on $\partial K_{j,\lambda}$. Since $u_{\lambda,\mu}$ is upper semi-continuous and obviously satisfies the submeanvalue inequality elsewhere, it follows that $u_{\lambda,\mu}$ is indeed subharmonic. This completes the proof of part (ii).

3 Proof of Theorem 1

Throughout the proof of Theorem 1 we will use the same notation as in the statements of the previous lemmas. Also, by abuse of notation, we use the same letter to denote a sequence of distinct points and the set of all terms of the sequence.

Proof We may (and shall) assume that $x_0 = 0$. Our goal can be rephrased in the following equivalent form. Suppose we are given a sequence $A = (a_1, a_2, \ldots)$ of distinct non-zero points, tending to 0 in $\mathbb{R}^N$, and a corresponding sequence $B = (b_1, b_2, \ldots)$ of real values such that $\limsup_{j\to\infty} b_j =: b < +\infty$. We are also given a bounded sequence $P = (p_1, p_2, \ldots)$ of distinct points, distinct from 0 and from the points of A, having no limit points except possibly 0, and we wish to construct a corresponding subharmonic function u, in a neighbourhood of 0, such that $u(a_j) = b_j$, $j = 1, 2, \ldots$, $u^{-1}(-\infty) = P$ and $\limsup_{x\to 0} u(x) = b$. (If we construct such an interpolating function in a ball with center at 0, we can then extend it subharmonically and continuously to any given ball of center at 0 using Theorem 6.1 of [5].)

Let (m_j) be a sequence strictly decreasing to b such that $m_1 > \sup\{b_j : j=1,2,\ldots\}$ and $m_j \notin B$ for each j. (This is possible because $\limsup_{j\to\infty} b_j =: b < +\infty$.) Choose $R_1 > \max\{\|x\| : x \in A \cup P\}$ and $r_1 \in (0, \min\{1, R_1\})$, with $(A \cup P) \cap S[r_1] = \emptyset$ and $A \cap S(r_1, R_1) \neq \emptyset$. Since $\limsup_{j\to\infty} b_j = b$, we may choose r_1 so small that $\max\{b_j : \|a_j\| < r_1\} < m_2$. Choose $M_1 > m_1$; choose R_2 with $r_1 < R_2 < R_1$ such that $(A \cup P) \cap S[r_1, R_2] = \emptyset$.

We now apply Lemma 4 for $r=r_1, R=R_1, m=m_1, M=M_1, A_1=A \cap S[r_1, R_1]$, $B_1=\{b_j : a_j \in A_1\}$, $P_1=P \cap S[r_1, R_1]$ and the compact set $Q_1 = S[R_2]$. By Lemma 4, noting that $-\mu_1 U_{P_1,A_1} + h_{\mu_1} < L_{r_1,R_1}^{m_1,M_1}$ on $S(r_1, R_1)$, we may choose μ_1 and λ_1 sufficiently small so that u_{λ_1,μ_1} is subharmonic on $S(r_1, R_1)$ and

$$m_1 < \min_{\|x\|=R_2} u_{\lambda_1,\mu_1}(x) \equiv M_2 < L_{r_1,R_1}^{m_1,M_1}(R_2).$$

We set $u_1 = u_{\lambda_1,\mu_1}$ on $S(r_1, R_1)$.

All sufficiently small r_2 satisfy the conclusion of Lemma 2, so we may choose such an $r_2 < \min\{1/2, r_1\}$, for which $(A \cup P) \cap S[r_2] = \emptyset$ and $A \cap S(r_2, r_1) \neq \emptyset$. Moreover, we may choose r_2 so small that $\max\{b_j : \|a_j\| < r_2\} < m_3$.

Choose R_3 with $r_2 < R_3 < r_1$ such that $(A \cup P) \cap S[r_2, R_3] = \emptyset$. We now apply Lemma 4 for $r = r_2, R = R_2, m = m_2, M = M_2, A_2 = A \cap S[r_2, R_2]$, $B_2 = \{b_j : a_j \in A_2\}$, $P_2 = P \cap S[r_2, R_2]$ and the compact set $Q_2 = S[R_3] \cup S[r_1]$. By Lemma 4, and since $m_2 < L_{r_2,R_2}^{m_2,M_2}$ on $S(r_2, R_2)$, we may choose μ_2 and λ_2 sufficiently small so that u_{λ_2,μ_2} is subharmonic on $S(r_2, R_2)$,

$$m_2 < \min_{\|x\|=R_3} u_{\lambda_2,\mu_2}(x) \equiv M_3 < L_{r_2,R_2}^{m_2,M_2}(R_3)$$

and also such that

$$m_1 < \min_{\|x\|=r_1} u_{\lambda_2,\mu_2}(x).$$

Then,

$$u_{\lambda_2,\mu_2}(x) > m_1 = u_1(x) \quad \text{for} \quad \|x\| = r_1$$

and

$$u_{\lambda_2,\mu_2}(x) = M_2 \leq u_1(x) \quad \text{for} \quad \|x\| = R_2.$$

The function

$$u_2(x) = \begin{cases} u_{\lambda_2,\mu_2}(x) & \text{for } r_2 \le \|x\| \le r_1, \\ \max\{u_{\lambda_2,\mu_2}(x), u_1(x)\} & \text{for } r_1 \le \|x\| \le R_2, \\ u_1(x) & \text{for } R_2 \le \|x\| \le R_1, \end{cases}$$

is continuous on $S[r_2, R_1]$ and subharmonic on $S(r_2, R_1)$ (since r_2 was chosen sufficiently small to satisfy the conclusion of Lemma 2).

To define u_3, we repeat the process. All sufficiently small r_3 satisfy the conclusion of Lemma 2, so we may choose such an $r_3 < \min\{1/3, r_2\}$, for which $(A \cup P) \cap S[r_3] = \emptyset$ and $A \cap S(r_3, r_2) \neq \emptyset$. Moreover, we may choose r_3 so small that $\max\{b_j : \|a_j\| < r_3\} < m_4$.

Choose R_4 with $r_3 < R_4 < r_2$ such that $(A \cup P) \cap S[r_3, R_4] = \emptyset$. We now apply Lemma 4 for $r = r_3, R = R_3, m = m_3, M = M_3, A_3 = A \cap S[r_3, R_3]$, $B_3 = \{b_j : a_j \in A_3\}$, $P_3 = P \cap S[r_3, R_3]$ and the compact set $Q_3 = S[R_4] \cup S[r_2]$. By Lemma 4, we may choose λ_3 and μ_3 sufficiently small so that u_{λ_3,μ_3} is subharmonic on $S(r_3, R_3)$,

$$m_3 < \min_{\|x\|=R_4} u_{\lambda_3,\mu_3}(x) \equiv M_4 < L^{m_3,M_3}_{r_3,R_3}(R_4)$$

and

$$m_2 < \min_{\|x\|=r_2} u_{\lambda_3,\mu_3}(x).$$

Then,

$$u_{\lambda_3,\mu_3}(x) > m_2 = u_2(x) \quad \text{for} \quad \|x\| = r_2$$

and

$$u_{\lambda_3,\mu_3}(x) = M_3 \le u_2(x) \quad \text{for} \quad \|x\| = R_3.$$

The function

$$u_3(x) = \begin{cases} u_{\lambda_3,\mu_3}(x) & \text{for } r_3 \le \|x\| \le r_2, \\ \max\{u_{\lambda_3,\mu_3}(x), u_2(x)\} & \text{for } r_2 \le \|x\| \le R_3, \\ u_2(x) & \text{for } R_3 \le \|x\| \le R_1, \end{cases}$$

is continuous on $S[r_3, R_1]$ and subharmonic on $S(r_3, R_1)$ (since r_3 was chosen sufficiently small to satisfy the conclusion of Lemma 2).

We continue in this manner to define a sequence (u_j) of functions continuous on $S[r_j, R_1]$ and subharmonic on $S(r_j, R_1)$.

The sequence u_j is eventually stable on compact subsets of the punctured ball $S(0, R_1)$, and so it converges to a limit function $u : S(0, R_1) \to [-\infty, +\infty)$

which is continuous and subharmonic. Moreover, $\limsup_{x\to 0} u(x) = b < +\infty$. Hence, the origin is a removable singularity; thus we may extend u to a function subharmonic on the ball $S[0, R_1)$, centred at the origin and of radius R_1. By construction $u^{-1}(-\infty) = P$ and $u(a_j) = b_j$ for all $j = 1, 2, \ldots$. This concludes the proof.

4 Concluding Remarks and Questions

In this final section we list some remarks and questions for further directions on similar interpolation problems.

1. We note that the interpolating function of Theorem 1 can be chosen to be subharmonic and continuous (in the extended sense) on all of $\mathbb{R}^N$. To see this, let u be the subharmonic function on the ball $B = B(x_0, R)$ obtained in Theorem 1 and choose $0 < \rho_2 < \rho_1 < R$, such that the points x_j lie in $B(x_0, \rho_2)$ and denote $B_j = B(x_0, \rho_j)$. By the Riesz decomposition [1, Cor. 4.4.3],

$$u = -\int_{B_1} U_y d\mu_u(y) + h \quad \text{on} \quad B_1,$$

where h is harmonic on B_1 and μ_u is the Riesz measure of u on B. The potential is well-defined and superharmonic on all of $\mathbb{R}^N$ and, using Theorem 6.1 of [5], we can extend the restriction of h on B_2 to a function H, continuous and subharmonic on all of $\mathbb{R}^N$. (See also [6].) Thus,

$$v = -\int_{B_1} U_y d\mu_u(y) + H$$

is a subharmonic function on all of $\mathbb{R}^N$ which performs the required interpolation. We note that v is continuous in the extended sense, that is, as a mapping from $\mathbb{R}^N$ to $[-\infty, +\infty)$. Indeed, v is continuous in the extended sense on B_2, since it agrees with u, which is continuous in the extended sense. The potential is continuous in the extended sense on B_1, since it differs from u by a harmonic function. By the construction of the proof, we may assume that u is harmonic on an open neighbourhood of $S[\rho_2, \rho_1]$ and so the restriction of μ_u to B_1 has support in a compact subset K of B_2. Hence the potential is harmonic (and so continuous) outside K. Since the subharmonic function v differs from the potential by the continuous function H, it follows that v is continuous outside of K. Thus, v is continuous in the extended sense on all of $\mathbb{R}^N$.
2. It is obvious that Theorem 1 would not remain valid in the case of convex functions in $\mathbb{R}$ (which are the 1-dimensional analogue of subharmonic functions). For example for $x_n = 1/n$ and $y_n = (-1)^n$ it is impossible to find a convex function u in $\mathbb{R}$ with $u(x_n) = y_n$ for all n (not even for $n = 1, 2, 3$).

Subharmonic functions in higher dimensions have less rigid behaviour, which makes interpolation easier. For example, a key-ingredient of the proof of our theorem was Lemma 2, which again would not be valid in dimension 1.

3. Harmonic functions have much more rigid behaviour than (continuous) subharmonic functions. In particular, given a sequence of distinct points x_n converging to a point x_0, if h is a harmonic function on a neighbourhood of x_0 containing these points, the possible sequences $y_n = h(x_n)$ are extremely limited. It would be natural to ask under which conditions harmonic interpolation of a convergent sequence would be valid; that is, given a convergent sequence (x_n) in $\mathbb{R}^N$ and a convergent sequence (y_n) in $\mathbb{R}$, under which conditions on (x_n) and (y_n) can we have that there is a harmonic function h in $\mathbb{R}^N$ for which $h(x_n) = y_n$ for all n?

   It is quite easy to give simple examples of pairs of sequences (x_n) and (y_n), for which harmonic interpolation is not possible. The heuristic ontological principle that, if there is at most one of a certain type of object, then there are "probably" none, suggests a connection between uniqueness and non-existence. We present an example in this spirit. Let C be a cone in $\mathbb{R}^N$ with vertex at 0 and let D be a dense subset of $S[1] \cap C$. We construct a sequence $X = (x_n)$, convergent to 0, such that, for each $x \in D$, the set $X \cap [0, x]$ contains a sequence of points of the form (x_{2k}) that accumulate at 0. We consider the sequence (y_n) which equals to $1/n$ if n is odd and equals to 0 in n is even. We claim that there does not exist a harmonic function h on a neighbourhood of 0 for which $h(x_n) = y_n$ for all n. Indeed, if there was such a function h, we would have that, for all n even numbers, $h(x_n) = y_n = 0$. Using the real analyticity of h, we can conclude that $h = 0$ on each segment of the form $X \cap [0, x]$, for all x in D. Since D is dense in $S[1] \cap C$, we conclude (using the continuity of h) that $h \equiv 0$ on C and so (by the identity principle) h is identically 0 in $\mathbb{R}^N$. This contradicts the fact that for n odd $h(x_n) = y_n = 1/n \neq 0$. Hence, there is no harmonic function h in a neighbourhood of 0 for which $h(x_n) = y_n$.

   The sequence (x_n) we constructed here is an example of an "analytic uniqueness sequence" (for more details see [3]). We note that, for each such sequence (x_n), we would be able to construct analogous counterexamples where harmonic interpolation fails.

4. Theorem 1 can be extended if we replace the assumption that the sequence (x_n) has only one accumulation point, with finitely many accumulation points (by preserving the condition of $\limsup_{n\to\infty} y_n < +\infty$). It is natural to ask how far we can push the conditions on a sequence (x_n) so that (continuous) subharmonic interpolation is possible under the simple condition of $\limsup_{n\to\infty} y_n < +\infty$. More generally, what can be said if we replace the sequences (x_n) by more general closed sets A (and the corresponding interpolating values (y_n) with an upper semicontinuous function $B : A \to [-\infty, +\infty)$)? Since subharmonic functions take the value $-\infty$ only on a polar set, it makes sense to examine the more general case only for the finite setting, i.e. when the data function B takes only finite values (or, if it takes the value $-\infty$, this happens only on a polar subset of A). (Of course if the set A does not have empty interior we must have a submeanvalue property for our data function.) For example, does the method of

our proof extend to polar closed sets A? The answer is yes (without continuity) if all values $B(a) = -\infty$ for all $a \in A$ since, by the definition of polar sets, we can find a subharmonic function on a neighbourhood of A, which takes the value $-\infty$ on A. However, this function can be highly discontinuous (see p. 69 of [1]).

5. As discussed above, the techniques of the proof of Theorem 1 can be used to obtain the following:

   Corollary 1 *Let A be a compact set in* $\mathbb{R}^N$ *with a finite number of accumulation points, and let* $f \in C(A)$*. Then there exists a function* $F \in C(\mathbb{R}^N)$*, which is subharmonic on* $\mathbb{R}^N$*, such that* $F = f$ *on A. That is, A is an interpolation set for continuous entire subharmonic functions.*

   One further direction would be to examine interpolation by smooth subharmonic functions; that is, to find conditions on the sequences (x_n) and (y_n) (or, more generally on the set A and the function f on A) under which the interpolating function can be C^∞. This relates to the work in [4].

6. Although we do not in general have uniqueness for interpolating subharmonic functions along a sequence of points, perhaps uniqueness can be attained by imposing additional constraints. For example, what can be said about interpolation where, besides interpolating at a sequence of points (x_n) converging to 0 we are additionally interpolating the mean values around 0? To be more precise:

   Problem We consider a sequence (y_n) of points in $[-\infty, +\infty)$ with $b := \limsup_{n\to\infty} y_n < +\infty$. We also fix a function $M : (0, 1] \to \mathbb{R}$ which is increasing, such that $M(r)$ is a convex function of the fundamental solution $U_0(r)$ and

$$\lim_{r\to 0^+} M(r) = b.$$

   Is it possible for the spherical mean values $\mathcal{M}(u; 0, r)$ of (some of) the interpolating function(s) u on the sphere $S[r]$, obtained by Theorem 1, to coincide with $M(r)$ for all $r \in (0, 1]$?

   The conditions we put on M are necessary from Corollary 3.2.6 and Theorem 3.5.6 of [1].

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