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Gregory G. Smith
Bernd Sturmfels
Editors

Combinatorial Algebraic Geometry

Selected Papers From the 2016 Apprenticeship Program

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Preface

In 2016, the July–December Thematic Program at the Fields Institute focused on Combinatorial Algebraic Geometry. From a modern perspective, this branch of algebraic geometry is devoted to the study of *combinatorial varieties*: those algebraic varieties, schemes, spaces, or stacks whose geometric structures have a concrete combinatorial description. Toric varieties and Schubert varieties might be the most prominent members of this class. However, many other spaces, such as the Deligne–Mumford compactification of the moduli space of curves with marked points, the toroidal compactifications of the moduli space of abelian varieties, and the Hilbert scheme of points, also belong within this conceptual framework. Despite being special, combinatorial varieties nevertheless account for a disproportionately large number of the spaces arising in applied algebraic geometry, combinatorial optimization, commutative algebra, representation theory, mathematical physics, and other fields. Developments in tropical geometry and Newton–Okounkov bodies are also expanding the class of algebraic spaces with a recognized combinatorial structure.

Among the training activities for new researchers in the Combinatorial Algebraic Geometry Program, the Apprenticeship Program served as the flagship. Partially funded by the Clay Mathematics Institute and the Fields Institute, the Apprenticeship Program brought together over 35 early-career mathematicians for an intense 2 weeks (21 August–3 September) of mathematics. Led by Bernd Sturmfels, the participants explored new problems, stressed their computers with calculations, discovered the roots of the field by reading historic papers, networked with peers, and developed the skills of the trade. The overall vision was to encourage young researchers to become active in the field and to promote new collaborations.

This volume of the Fields Institute Communications series consolidates selected articles from the Apprenticeship Program. Written primarily by junior mathematicians (± 5 years from their PhD), the articles cover a range of topics in combinatorial algebraic geometry: curves, surfaces, Grassmannians, convexity, abelian varieties, and moduli spaces. Almost all of the participants in the Apprenticeship Program are coauthors on at least one paper which was initiated at the Fields Institute.

A few articles include junior coauthors who did not participate directly in the Apprenticeship Program, or more senior mathematicians. This book bridges the gap between graduate courses and cutting-edge research by connecting historical sources, computation, explicit examples, and new results.

We are grateful to many readers who provided valuable feedback. These include Enrique Arrondo, Matthew Baker, Cristiano Bocci, Rob Eggermont, Daniel Erman, Ghislain Fourier, Naoki Fujita, Paul Alexander Helminck, Nathan Ilten, Gary Kennedy, Kaie Kubjas, Antonio Laface, Christian Liedtke, Diane Maclagan, Madhusudan Manjunath, Chris Manon, Emilia Mezzetti, Marta Panizzut, Kristian Ranestad, Jenna Rajchgot, Rainer Sinn, Jenia Tevelev, and Timo De Wolff.

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Leipzig, Germany Bernd Sturmfels
29 June 2017

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Fitness, Apprenticeship, and Polynomials

Bernd Sturmfels

Abstract This article discusses the design of the Apprenticeship Program at the Fields Institute, held 21 August–3 September 2016. Six themes from combinatorial algebraic geometry were selected for the 2 weeks: curves, surfaces, Grassmannians, convexity, abelian combinatorics, parameters and moduli. The activities were structured into fitness, research and scholarship. Combinatorics and concrete computations with polynomials (and theta functions) empowers young scholars in algebraic geometry, and it helps them to connect with the historic roots of their field. We illustrate our perspective for the threefold obtained by blowing up six points in $\mathbb{P}^3$.

MSC 2010 codes: 14Q15, 05E40, 14-01

1 Design

A thematic program on *Combinatorial Algebraic Geometry* took place at the Fields Institute, Toronto, Canada, during the Fall Semester 2016. The program organizers were David Cox, Megumi Harada, Diane Maclagan, Gregory Smith, and Ravi Vakil.

As part of this semester, the Clay Mathematics Institute funded the "Apprenticeship Weeks", held from 21 August 2016 to 3 September 2016. This article discusses the design and mathematical scope of this fortnight. The structured activities took place in the mornings and afternoons on Monday, Wednesday, and Friday, as well as the mornings on Tuesday and Thursday. The posted schedule was identical for both weeks; see Table 1.

The term "fitness" is an allusion to physical exercise. In order to improve physical fitness, many of us go to the gym. A personal trainer can greatly enhance that experience. The trainer develops your exercise plan and pushes you beyond

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Table 1 Weekly schedule of activities

Days	Times	Activities
Monday–Wednesday–Friday	09:00–09:30	Introduction to today's theme
Monday–Wednesday–Friday	09:30–11:15	Working on fitness problems
Monday–Wednesday–Friday	11:15–12:15	Solutions to fitness problems
Monday–Wednesday–Friday	14:00–14:30	Dividing into research teams
Monday–Wednesday–Friday	14:30–17:00	Team work on projects
Monday–Wednesday–Friday	17:00–18:00	Teams present findings
Tuesday–Thursday	09:00–12:00	Discussion of the scholarship theme

previously perceived limits. He makes you sweat a lot, ensures that you use exercise equipment correctly, and helps you to feel good about yourself afterwards. In the context of team sports, the coach plays that role. She works towards the fitness of the entire team, where every player will contribute to the best of their abilities.

The six fitness sessions were designed to be as intense as those in sports. Ten problems were posted for each session, and these were available online two or three days in advance. By design, these demanding problems were open-ended and probed a different aspect of the theme. Section 3 of this article contains the complete list of problems, along with a brief discussion and references that contain some solutions.

The "apprentices" were about 40 early-career mathematicians, graduate students and postdocs, coming from a wide range of backgrounds. An essential feature of the Apprenticeship Weeks was the effort to build teams, and to promote collaboration as much as possible. This created an amazing sense of community within the group.

At 9:00 a.m. on each Monday, Wednesday or Friday, a brief introduction was given to each fitness question. We formed ten teams to work on the problems. At 11:15 a.m. we got together again, and one person from each team gave a brief presentation on what had been discussed and discovered. Working on a challenging problem, with a group of new collaborators, for less than two hours created a very intense and stimulating experience. A balanced selection process ensured that each participant had the opportunity to present for their team at least once.

At 2:00 p.m. the entire group re-assembled and they discussed research-oriented problems for the afternoons. This was conducted in the style of the American Institute for Mathematics (AIM), whereby one of the participants serves as the discussion leader, and only that person is allowed to touch the blackboard. This led to an ample supply of excellent questions, some a direct continuation of the morning fitness problems, and others only vaguely inspired by these. Again, groups were formed for the afternoon, and they engaged in learning and research. Computations and literature search played a big role, and a lot of teaching went on in the groups.

Tuesdays and Thursdays were discussion days. Here the aim was to create a sense of scholarship among the participants. The morning of these days involved studying various software packages, classical research papers from the 19th and early 20th centuries, and the diverse applications of combinatorial algebraic geometry. The prompts are given in Sect. 2. The afternoons on discussion days were unstructured to allow the participants time to ponder, probe, and write up their many new ideas.

2 Scholarship Prompts

Combinatorial algebraic geometry is a field that, by design, straddles mathematical boundaries. One aim is to study algebraic varieties with special combinatorial features. At its roots, this field is about systems of polynomial equations in several variables, and about symmetries and other special structures in their solution sets.

Section 5 offers a concrete illustration of this perspective for a system of polynomials in 32 variables. The objects of combinatorial algebraic geometry are amenable to a wide range of software tools, which are now used widely among the researchers.

Another point we discussed is the connection to problems outside of pure mathematics. A new field, *Applied Algebraic Geometry*, has arisen in the past decade. The techniques used there often connect back to 19th and 20th century work in algebraic geometry, which is much more concrete and combinatorial than many recent developments. And, even for her study of current abstract theories, an apprentice may benefit from knowing the historic origins that have inspired the development of algebraic geometry. Understanding these aspects, by getting hands-on experiences and by studying original sources, was a focus in this part of the program.

In what follows, we replicate the hand-outs for the four Tuesday and Thursday mornings. The common thread can be summarized as: back to the roots. These hand-outs were given to the participants as prompts for explorations and discussions. For several of the participants, it was their first experience with software for algebraic geometry. For others, it offered a first opportunity to read an article that was published over 100 years ago.

Tuesday, 23 August 2016: Software

- Which software tools are most useful for performing computations in *Combinatorial Algebraic Geometry*? Why?
- Many of us are familiar with *Macaulay2*. Some of us are familiar with *Singular*. What are your favorite packages within these systems?
- Lots of math is supported by general-purpose computer algebra systems such as *Sage*, *Maple*, *Mathematica*, or *Magma*. Do you use any of these regularly? For research or for teaching? How often and in which context?
- Other packages that are useful for our community include *Bertini*, *PHCpack*, *4ti2*, *Polymake*, *Normaliz*, *GFan*. What are these and what do they do? Who developed them and why?
- Does visualization matter in algebraic geometry? Have you tried software like *Surfex*?
- Which software tool do you want to learn today?

Thursday, 25 August 2016: The Nineteenth Century

Algebraic Geometry has a deep and distinguished history that goes back hundreds of years. Combinatorics entered the scene a bit more recently.

Young scholars interested in algebraic geometry are strongly encouraged to familiarize themselves with the literature from the nineteenth century. Dig out papers from that period and *read them*! Go for the original sources. Some are in English. Do not be afraid of languages like French, German, Italian.

Today we form groups. Each group will explore the life and work of one mathematician, with focus on what he has done in algebraic geometry. Identify one key paper written by that author. Then present your findings.

Here are some suggestions, listed alphabetically:

- Alexander von Brill
- Arthur Cayley
- Michel Chasles
- Luigi Cremona
- Georges Halphen
- Otto Hesse
- Ernst Kummer
- Max Noether
- Julius Plücker
- Bernhard Riemann
- Friedrich Schottky
- Hermann Schubert
- Hieronymus Zeuthen

Tuesday, 30 August 2016: Applications

The recent years have seen a lot of interest in applications of algebraic geometry, outside of core pure mathematics. An influential event was a research year 2006–07 at the Institute for Mathematics and its Applications (IMA) in Minneapolis. Following a suggestion by Doug Arnold (then IMA director and SIAM president), it led to the creation of the Society for Industrial and Applied Mathematics (SIAM) activity group in Algebraic Geometry and (ultimately) to the SIAM Journal on Applied Algebra and Geometry. The reader is referred to these resources for further information. These interactions with the sciences and engineering have been greatly enhanced by the interplay with Combinatorics and Computation seen here at the Fields Institute. However, the term "Algebraic Geometry" has to be understood now in a broad sense.

Today we form groups. Each group will get familiar with one field of application, and they will select one paper in Applied Algebraic Geometry that represent an interaction with that field. Read your paper and then present your findings. Here are some suggested fields, listed alphabetically:

- Approximation Theory
- Bayesian Statistics
- Chemical Reaction Networks
- Coding Theory
- Combinatorial Optimization
- Computer Vision
- Cryptography
- Game Theory
- Geometric Modeling
- Machine Learning
- Maximum Likelihood Inference
- Neuroscience
- Phylogenetics
- Quantum Computing
- Semidefinite Programming
- Systems Biology

Thursday, 1 September 2016: The Early Twentieth Century

One week ago we examined the work of some algebraic geometers from the nineteenth century. Today, we move on to the early twentieth century, to mathematics that was published prior to World War II. You are encouraged to familiarize yourselves with the literature from the period 1900–1939. Dig out papers from that period and *read them*! Go for the original sources. Some are written in English. Do not be afraid of languages like French, German, Italian, Russian.

Each group will explore the life and work of one mathematician, with focus on what (s)he has done in algebraic geometry during that period. Identify one key paper written by that author. Then present your findings.

Here are some suggestions, listed alphabetically:

- Eugenio Bertini
- Guido Castelnuovo
- Wei-Liang Chow
- Arthur B. Coble
- Wolfgang Gröbner
- William V.D. Hodge
- Wolfgang Krull
- Solomon Lefschetz
- Frank Morley
- Francis S. Macaulay
- Amalie Emmy Noether
- Ivan Georgievich Petrovsky
- Virginia Ragsdale
- Gaetano Scorza
- Francesco Severi

3 Fitness Prompts

This section presents the six worksheets for the morning sessions on Mondays, Wednesdays and Fridays. These prompts inspired most of the articles in this volume. Specific pointers to dates refer to events that took place at the Fields Institute. The next section contains notes for each problem, offering references and solutions.

Monday, 22 August 2016: Curves

1. Which genus can a smooth curve of degree 6 in $\mathbb{P}^3$ have? Give examples.
2. Let $f(x) = (x-1)(x-2)(x-3)(x-6)(x-7)(x-8)$ and consider the genus 2 curve $y^2 = f(x)$. Where is it in the moduli space M_2 ? Compute the Igusa invariants. Draw the Berkovich skeleton for the field of 5-adic numbers.
3. The *tact invariant* of two plane conics is the polynomial of bidegree $(6, 6)$ in the $6 + 6$ coefficients which vanishes when the conics are tangent. Compute this invariant explicitly. How many terms does it have?
4. *Bring's curve* lives in a hyperplane in $\mathbb{P}^4$. It is defined by $x_0^i+x_1^i+x_2^i+x_3^i+x_4^i = 0$ for $1 \le i \le 3$. What is its genus? Determine all tritangent planes of this curve.
5. Let X be a curve of degree d and genus g in $\mathbb{P}^3$. The Chow form of X defines a hypersurface in the Grassmannian $\mathrm{Gr}(1, \mathbb{P}^3)$. Points are lines that meet X. Find the dimension and (bi)degree of its singular locus.
6. What are the equations of the secant varieties of *elliptic normal curves*?

7. Let X_P be the *toric variety* defined by a three-dimensional lattice polytope P, as in Milena Hering's 18–22 July 2016 course. Intersect X_P with two general hyperplanes to get a curve. What is the degree and genus of that curve?
8. A 2009 article by Sean Keel and Jenia Tevelev presents *Equations for* $\overline{\mathrm{M}}_{0,n}$. Write these equations in *Macaulay2* format for $n = 5$ and $n = 6$. Can you see the ψ-classes (seen in Renzo Cavalieri's 18–22 July 2016 course) in these coordinates?
9. Review the statement of *Torelli's Theorem* for genus 3. Using *Sage* or *Maple*, compute the 3×3 Riemann matrix of the Fermat quartic $\{x^4 + y^4 + z^4 = 0\}$. How can you recover the curve from that matrix?
10. The moduli space M_7 of genus 7 curves has dimension 18. What is the codimension of the locus of plane curves? Hint: Singularities are allowed.

Wednesday, 24 August 2016: Surfaces

1. A nondegenerate surface in $\mathbb{P}^n$ has degree at least $n - 1$. Prove this fact and determine all surfaces of degree $n - 1$. Give their equations.
2. How many lines lie on a surface obtained by intersecting two quadratic hypersurfaces in $\mathbb{P}^4$? Find an instance where all lines are defined over $\mathbb{Q}$.
3. What is the maximum number of singular points on an irreducible quartic surface in $\mathbb{P}^3$? Find a surface and compute its *projective dual*.
4. Given a general surface of degree d in $\mathbb{P}^3$, the set of its *bitangent lines* is a surface in $\mathrm{Gr}(1, \mathbb{P}^3)$. Determine the cohomology class (or bidegree) of that surface.
5. Pick two random circles C_1 and C_2 in $\mathbb{R}^3$. Compute both their *Minkowski sum* $C_1 + C_2$ and their *Hadamard product* $C_1 \star C_2$. Try other curves.
6. Let X be the surface obtained by blowing up five general points in the plane. Compute the *Cox ring* of X. Which of its ideals describe points on X?
7. The incidences among the 27 lines on a cubic surface defines a 10-regular graph. Compute the complex of independent sets in this graph.
8. The Hilbert scheme of points on a smooth surface is smooth. Why? How many torus-fixed points are there on the Hilbert scheme of 20 points in $\mathbb{P}^2$? What can you say about the graph that connects them?
9. State the *Hodge Index Theorem*. Verify this theorem for cubic surfaces in $\mathbb{P}^3$, by explicitly computing the matrix for the intersection pairing.
10. List the equations of one *Enriques surface*. Verify its Hodge diamond.

Friday, 26 August 2016: Grassmannians

1. Find a point in $\mathrm{Gr}(3, \mathbb{C}^6)$ with precisely 16 non-zero Plücker coordinates. As in June Huh's 18–22 July 2016 course, determine the Chow ring of its *matroid*.
2. The coordinate ring of the Grassmannian $\mathrm{Gr}(3, \mathbb{C}^6)$ is a *cluster algebra* of finite type. What are the cluster variables? List all the clusters.
3. Consider two general surfaces in $\mathbb{P}^3$ whose degrees are d and e respectively. How many lines in $\mathbb{P}^3$ are *bitangent* to both surfaces?
4. The *rotation group* $\mathrm{SO}(n, \mathbb{R})$ is an affine variety in the space of real $n \times n$-matrices. Can you find a formula for the degree of this variety?

5. The *flag variety* for $GL_4(\mathbb{C})$ is a six-dimensional subvariety of $\mathbb{P}^3 \times \mathbb{P}^5 \times \mathbb{P}^3$. Compute its ideal and determine its tropicalization.
6. Classify all toric ideals that arise as initial ideals for the flag variety above. For each such toric degeneration, compute the *Newton–Okounkov body*.
7. The Grassmannian $\mathrm{Gr}(4, 7)$ has dimension 12. Four *Schubert cycles* of codimension 3 intersect in a finite number of points. How large can that number be? Exhibit explicit cycles whose intersection is reduced.
8. The *affine Grassmannian* and the *Sato Grassmannian* are both infinite-dimensional versions of the Grassmannian. How are they related?
9. The coordinate ring of the Grassmannian $\mathrm{Gr}(2, \mathbb{C}^7)$ is $\mathbb{Z}^7$-graded. Determine the Hilbert series and the multidegree of $\mathrm{Gr}(2, \mathbb{C}^7)$ for this grading.
10. The *Lagrangian Grassmannian* parametrizes n-dimensional isotropic subspaces in $\mathbb{C}^{2n}$. Find a Gröbner basis for its ideal. What is a 'doset'?

Monday, 29 August 2016: Convexity

1. The set of nonnegative binary sextics is a closed full-dimensional convex cone in $\mathrm{Sym}_6(\mathbb{R}^2) \cong \mathbb{R}^7$. Determine the face poset of this convex cone.
2. Consider *smooth* projective toric fourfolds with eight invariant divisors. What is the maximal number of torus-fixed points of any such variety?
3. Choose three general ellipsoids in $\mathbb{R}^3$ and compute the convex hull of their union. Which algebraic surfaces contribute to the boundary?
4. Explain how the Alexandrov–Fenchel Inequalities (for convex bodies) can be derived from the Hodge Index Theorem (for algebraic surfaces).
5. The blow-up of $\mathbb{P}^3$ at six general points is a threefold that contains 32 special surfaces (exceptional classes). What are these surfaces? Which triples intersect? Hint: Find a six-dimensional polytope that describes the combinatorics.
6. Prove that every face of a spectrahedron is an exposed face.
7. How many combinatorial types of reflexive polytopes are there in dimension 3? In dimension 4? Draw pictures of some extreme specimen.
8. A 4×4-matrix has six off-diagonal 2×2-minors. Their binomial ideal in 12 variables has a unique toric component. Determine the f-vector of the polytope (with 12 vertices) associated with this toric variety.
9. Consider the Plücker embedding of the real Grassmannian $\mathrm{Gr}(2, \mathbb{R}^5)$ in the unit sphere in $\mathbb{R}^{10}$. Describe its convex hull. Hint: Calibrations, Orbitopes.
10. Examine Minkowski sums of three tetrahedra in $\mathbb{R}^3$. What is the maximum number of vertices such a polytope can have? How to generalize?

Wednesday, 31 August 2016: Abelian Combinatorics

1. The intersection of two quadratic surfaces in $\mathbb{P}^3$ is an *elliptic curve*. Explain its group structure in terms of geometric operations in $\mathbb{P}^3$.
2. A 2006 paper by Keiichi Gunji gives explicit equations for all *abelian surfaces* in $\mathbb{P}^8$. Verify his equations in *Macaulay2*. How to find the group law?
3. Experiment with Swierczewski's *Sage* code for the numerical evaluation of the *Riemann theta function* $\theta(\tau; z)$. Verify the functional equation.
4. *Theta functions with characteristics* $\theta[\epsilon, \epsilon'](\tau; z)$ are indexed by two binary vectors $\epsilon, \epsilon' \in \{0, 1\}^g$. They are odd or even. How many each?

5. Fix the symplectic form $\langle x, y\rangle := x_1y_4 + x_2y_5 + x_3y_6 + x_4y_1 + x_5y_2 + x_6y_3$ on the 64-element vector space $\mathbb{F}_2^6$. Determine all isotropic subspaces.
6. Explain the combinatorics of the root system of type E_7. How would you choose coordinates? How many pairs of roots are orthogonal?
7. In 1879, Cayley published a paper in Crelle's journal titled *Algorithms for* ... What did he do? How does it relate the previous two exercises?
8. The *regular matroid* R_{10} defines a degeneration of abelian 5-folds. Describe its periodic tiling on $\mathbb{R}^5$ and secondary cone in the 2nd Voronoi decomposition. Explain the application to Prym varieties due to Gwena.
9. Consider the Jacobian of the plane quartic curve defined over $\mathbb{Q}_2$ by

$$41x^4+1530x^3y+3508x^3z+1424x^2y^2+2490x^2yz-2274x^2z^2+470xy^3+680xy^2z$$
$$-\,930xyz^2 + 772xz^3 + 535y^4 - 350y^3z - 1960y^2z^2 - 3090yz^3 - 2047z^4$$

Compute its limit in *Alexeev's moduli space* for the 2-adic valuation.
10. Let Θ be the *theta divisor* on an abelian threefold X. Find $n = \dim H^0(X, k\Theta)$. What is the smallest integer k such that $k\Theta$ is very ample? Can you compute (in *Macaulay2*) the ideal of the corresponding embedding $X \hookrightarrow \mathbb{P}^{n-1}$?

Friday, 2 September 2016: Parameters and Moduli

1. Write down (in *Macaulay2* format) the two generators of the *ring of invariants* for ternary cubics. For which plane cubics do both invariants vanish?
2. Fix a $\mathbb{Z}$-grading on the polynomial ring $S = \mathbb{C}[a, b, c, d]$ defined by $\deg(a) = 1$, $\deg(b) = 4$, $\deg(c) = 5$, and $\deg(d) = 9$. Classify all homogeneous ideals I such that S/I has Hilbert function identically equal to 1.
3. Consider the Hilbert scheme of eight points in affine 4-space $\mathbb{A}^4$. Identify a point that is not in the main component. List its ideal generators.
4. Let X be the set of all symmetric 4×4-matrices in $\mathbb{R}^{4\times4}$ that have an eigenvalue of multiplicity ≥ 2. Compute the $\mathbb{C}$-Zariski closure of X.
5. Which cubic surfaces in $\mathbb{P}^3$ are stable? Which ones are semi-stable?
6. In his second lecture on August 15, Valery Alexeev used six lines in $\mathbb{P}^2$ to construct a certain moduli space of K3 surfaces with 15 singular points. List the most degenerate points in the boundary of that space.
7. Find the most singular point on the Hilbert scheme of 16 points in $\mathbb{A}^3$.
8. The polynomial ring $\mathbb{C}[x, y]$ is graded by the 2-element group $\mathbb{Z}/2\mathbb{Z}$ where $\deg(x) = 1$ and $\deg(y) = 1$. Classify all Hilbert functions of homogeneous ideals.
9. Consider all threefolds obtained by blowing up six general points in $\mathbb{P}^3$. Describe their Cox rings and Cox ideals. How can you compactify this moduli space?
10. The moduli space of tropical curves of genus 5 is a polyhedral space of dimension 12. Determine the number of i-faces for $i = 0, 1, 2, \ldots, 12$.

4 Notes and Solutions

Solutions to several of the 60 fitness problems can be found in the articles of this volume. In this section, we present the relevant pointers to these articles and offer references for the other problems that did not lead to articles in this book.

Notes on Curves

1. Castelnuovo classified the degree and genus pairs (d, g) for all smooth curves in $\mathbb{P}^n$. This was extended to characteristic p by Ciliberto [25]. For $n = 3$ and $d = 6$, the possible genera are $g = 0, 1, 2, 3, 4$. The *Macaulay2* package *RandomCurves* can compute examples. The Hartshorne–Rao module [50] plays a key role.
2. See Sect. 2 in the article by Bolognese, Brandt and Chua [1]. An approach using Igusa invariants was developed by Helminck in [32].
3. The tact invariant has 3210 terms by [57, Example 2.7].
4. See Sect. 2.1 in the article by Harris and Len [2]. The analogous problem for bitangents of plane quartics is discussed by Chan and Jiradilok [3].
5. This is solved in the article by Kohn, Nødland, and Tripoli [4].
6. Following Fisher [29], elliptic normal curves are defined by the 4×4-subpfaffians of the Klein matrix and their secant varieties are defined by its larger subpfaffians.
7. The degree of a projective toric variety X_P is the volume of its lattice polytope P. The genus of a complete intersection in X_P was derived by Khovanskii in 1978. We recommend the tropical perspective by Steffens and Theobald in [53, Sect. 4.1].
8. See the article by Monin and Rana [5] for a solution up to $n = 6$.
9. See [26] for how to compute the forward direction of the Torelli map of an arbitrary plane curve. For the backward direction in genus 3, see [61, Sect. 5.2].
10. Trinodal sextics form a 16-dimensional family and their codimension in M_7 is two. This is a result is originally due to Severi, but also derived by Castryck and Voight in [24, Theorem 2.1].

Notes on Surfaces

1. This was solved by Del Pezzo in 1886. Eisenbud and Harris [27] give a beautiful introduction to *varieties of minimal degree* including their equations.
2. This is a *del Pezzo surface* of degree 4. It has 16 lines. To make them rational, map $\mathbb{P}^2$ into $\mathbb{P}^3$ via a $\mathbb{Q}$-basis for the cubics vanishing at five $\mathbb{Q}$-points in $\mathbb{P}^2$.
3. The winner, with 16 singular points, is the *Kummer surface* [34]. It is self-dual.
4. This is solved in the article by Kohn, Nødland, and Tripoli [4].
5. See Sect. 5 in the article by Friedenberg, Oneto, and Williams [6].
6. This is the del Pezzo surface in Problem 2. Its Cox ring is a polynomial ring in 16 variables modulo an ideal generated by 20 quadrics. Ideal generators that are universal over the base $\overline{\mathrm{M}}_{0,5}$ are listed in [47, Proposition 2.1]. Ideals of points

on the surface are torus translates of the toric ideal of the five-dimensional demicube D_5. For six points in $\mathbb{P}^2$ we refer to Bernal, Corey, Donten-Bury, Fujita, and Merz [7].

7. This is the clique complex of the *Schläfli graph*. The f-vector of this simplicial complex is $(27, 216, 720, 1080, 648, 72)$. The Schläfli graph is the edge graph of the *E_6-polytope*, denoted 2_{21}, which is a cross section of the *Mori cone* of the surface.
8. The torus-fixed points on $\mathrm{Hilb}^{20}(\mathbb{P}^2)$ are indexed by ordered triples of partitions $(\lambda_1, \lambda_2, \lambda_3)$ with $|\lambda_1| + |\lambda_2| + |\lambda_3| = 20$. The number of such triples equals 341,649. The graph that connects them is a variant of the graph for the Hilbert scheme of points in the affine plane. The latter was studied by Hering and Maclagan in [33].
9. The signature of the intersection pairing is $(1, r - 1)$ where r is the rank of the Picard group. This is $r = 7$ for the cubic surface. From the analysis in Problem 7, we can get various symmetric matrices that represent the intersection pairing.
10. See the article by Bolognese, Harris, and Jelisiejew [8].

Notes on Grassmannians

1. See the article by Wiltshire-Gordon, Woo, and Zajackowska [9].
2. In addition to the 20 Plücker coordinates p_{ijk}, one needs two more functions: $p_{123}p_{456} - p_{124}p_{356}$ and $p_{234}p_{561} - p_{235}p_{461}$. The six boundary Plücker coordinates $p_{123}, p_{234}, p_{345}, p_{456}, p_{561}, p_{612}$ are frozen. The other 16 coordinates are the *cluster variables* for $\mathrm{Gr}(3, \mathbb{C}^6)$. This was derived by Scott in [51, Theorem 6].
3. This is worked out in the article by Kohn, Nødland and Tripoli [4].
4. This is the main result of Brandt, Bruce, Brysiewicz, Krone and Robeva [10].
5. See the article by Bossinger, Lamboglia, Mincheva, and Mohammadi [11].
6. See the article by Bossinger, Lamboglia, Mincheva, and Mohammadi [11].
7. The maximum number is 8. This is obtained by taking the partition $(2, 1)$ four times. For this problem, and many other Schubert problems, instances exist where all solutions are real; see the works of Sottile such as [52, Theorem 3.9].
8. The Sato Grassmannian is more general than the affine Grassmannian. These are studied in *integrable systems* and *geometric representation theory*, respectively.
9. A formula for the $\mathbb{Z}^n$-graded Hilbert series of $\mathrm{Gr}(2, \mathbb{C}^n)$ is given by Witaszek [63, Sect. 3.3]. For an introduction to multidegrees, see [40, Sect. 8.5]. Try the *Macaulay2* commands `Grassmannian` and `multidegree`. Escobar and Knutson [12] find the multidegree of an important variety from computer vision.
10. The coordinate ring of the Lagrangian Grassmannian is an algebra with straightening law over a double poset or *doset*. See the exposition in [48, Sect. 3].

Notes on Convexity

1. The face lattice of the cone of non-negative binary forms of degree d is described in Barvinok's textbook [20, Sect. II.11]. In more variables, this is much more difficult.
2. This seems to be an open problem. For seven invariant divisors, this was resolved by Gretenkort et al. [30]. Note the conjecture stated in the last line of that paper.
3. We refer to Nash, Pir, Sottile, and Ying [13] and to the *youtube* video *The Convex Hull of Ellipsoids* by Nicola Geismann, Michael Hemmer, and Elmar Schömer.
4. We refer to Ewald's textbook, specifically [28, Sect. IV.5 and Sect. VII.6].
5. The relevant polytope is the six-dimensional demicube; its 32 vertices correspond to the 32 special divisors. See the notes for Problem 9 in Parameters and Moduli.
6. This was first proved by Ramana and Goldman in [43, Corollary 1].
7. Kreuzer and Skarke [37] classified such reflexive polytopes up to lattice isomorphism. There are 4319 in dimension 3 and 473,800,776 in dimension 4. Lars Kastner classified the list of 4319 into combinatorial types. He found that there are 558 combinatorial types of reflexive 3-polytopes and they have up to 14 vertices.
8. This six-dimensional polytope is obtained from the direct product of two identical regular tetrahedra by removing the four pairs of corresponding vertices. It is the convex hull of the points $e_i \oplus e_j$ in $\mathbb{R}^4 \oplus \mathbb{R}^4$ where $i, j \in \{1, 2, 3, 4\}$ with $i \neq j$. Using the software *Polymake*, we find its f-vector to be $(12, 54, 110, 108, 52, 12)$.
9. The faces of the Grassmann orbitopes $\operatorname{conv}\big(\mathrm{Gr}(2, \mathbb{C}^n)\big)$, for $n \geq 5$, are described in [49, Theorem 7.3]. It is best to start with the case $n = 4$ in [49, Example 7.1].
10. The maximum number of vertices is 38 by the formula of Karavelas et al. [35, Sect. 6.1, Eq. (49)]. A definitive solution to the problem of characterizing face numbers of Minkowski sums of polytopes was given by Adiprasito and Sanyal [17].

Notes on Abelian Combinatorics

1. A beautiful solution was written up by Qiaochu Yuan when he was a high school student; see [62]. The idea is to simultaneously diagonalize the two quadrics, project their intersection curve into the plane, and obtain an *Edwards curve*.
2. This is a system of nine quadrics and three cubics, derived from Coble's cubic following Theorem 3.2 in [45]. Using theta functions as in Lemma 3.3 of [45], one obtains the group law.
3. See [61] and compare with Problem 9 in Curves.

4. For the 2^{2g} pairs (ϵ, ϵ'), we check whether $\epsilon \cdot \epsilon'$ is even or odd. There are $2^{g-1}(2^g + 1)$ even ones and $2^{g-1}(2^g - 1)$ odd ones.
5. The number of isotropic subspaces of $\mathbb{F}_2^6$ is 63 of dimension 1, 315 in dimension 2, and 135 in dimension 3. The latter are the Lagrangians [46, Sect. 6].
6. The root system of type E_7 has 63 positive roots. They are discussed in [46, Sect. 6].
7. Cayley gives a bijection between the 63 positive roots of E_7 with the 63 nonzero vectors in $\mathbb{F}_2^6$. Two roots have inner product zero if and only if the corresponding vectors in $\mathbb{F}_2^6 \backslash \{0\}$ are orthogonal in the setting of Problem 5. See [46, Table 1].
8. This problem implicitly refers to Gwena's article [31]. Since the matroid R_{10} is not cographic, the corresponding tropical abelian varieties are not in the Schottky locus of Jacobians.
9. This fitness problem is solved in the article by Bolognese, Brandt, and Chua [1]. Chan and Jiradilok [3] also study an important special family of plane quartics.
10. The divisor $k\Theta$ is very ample for $k = 3$. This embeds any abelian threefold into $\mathbb{P}^{26}$. For products of three planar cubic curves, this gives the Segre embedding.

Notes on Parameters and Moduli

1. The solution can be found, for instance, on the website math.stanford.edu/∼notzeb/aronhold.html. The two generators have degree 4 and 6. The quartic invariant is known as the *Aronhold invariant* and it vanishes when the ternary cubic is a sum of three cubes of linear forms. Both invariants vanish when the cubic curve has a cusp.
2. This refers to extra irreducible components in toric Hilbert schemes [42]. These schemes were first introduced by Arnold [19], who coined the term *A-graded algebras*. Theorem 10.4 in [54] established the existence of an extra component for $A = [\,1\;3\;4\;7\,]$. We ask to verify the second entry in Table 10-1 in [54, p. 88].
3. Cartwright et al. [22] showed that the Hilbert scheme of eight points in $\mathbb{A}^4$ has two irreducible components. An explicit point in the non-smoothable component is given in the article by Douvropoulos, Jelisiejew, Nødland, and Teitler [14].
4. At first, it is surprising that X has codimension 2. The point is that we work over $\mathbb{R}$. The analogous set over $\mathbb{C}$ is the hypersurface of a sum-of-squares polynomial. The $\mathbb{C}$-Zariski closure of X is a nice variety of codimension 2. The defining ideal and its Hilbert–Burch resolution are explained in [56, Sect. 7.1].
5. This is an exercise in Geometric Invariant Theory [41]. A cubic surface is stable if and only if it has at most ordinary double points (A_1 singularities). For semi-stable surfaces, A_2 singularities are allowed. For an exposition, see E. Reinecke's Bachelor thesis [44, Theorem 3.6] written under the supervision of D. Huybrechts.
6. This is the moduli space of stable hyperplane arrangements [18] for the case of six lines in $\mathbb{P}^2$ and a choice of parameters [18, Sect. 5.7]. For some parameters, this is the tropical compactification associated with the tropical Grassmannian $\mathrm{trop}\big(\mathrm{Gr}(3, \mathbb{C}^6)\big)$, so the most degenerate points correspond to the seven generic types of tropical planes in 5-space; see [38, Fig. 5.4.1].

7. See Theorem 2.3 in [55].
8. For each partition, representing a monomial ideal in $\mathbb{C}[x, y]$, we count the odd and even boxes in its Young diagram. The resulting Hilbert functions $h: \mathbb{Z}/2\mathbb{Z} \to \mathbb{N}$ are

$$\big(h(\text{even}), h(\text{odd})\big) = (k^2 + m, k(k+1) + m) \text{ or } \big((k+1)^2 + m, k(k+1) + m\big).$$

   where $k, m \in \mathbb{N}$ This was contributed by Dori Bejleri [21, Sect. 1.3].
9. The blow-up of $\mathbb{P}^{n-3}$ at n points is a Mori dream space. Its Cox ring has 2^{n-1} generators, constructed explicitly by Castravet and Tevelev in [23]. These form a Khovanskii basis [36], by [60, Theorem 7.10]. The Cox ideal is studied in [59]. Each point on its variety represents a rank two stable quasiparabolic vector bundle on $\mathbb{P}^1$ with n marked points. The relevant moduli space is $\overline{\mathrm{M}}_{0,n}$.
10. The moduli space of tropical curves of genus 5 serves as the first example in the article by Lin and Ulirsch [15]. The article by Kastner, Shaw, and Winz [16] discusses state-of-the-art software for computing with such polyhedral spaces.

5 Polynomials

The author of this article holds the firm belief that algebraic geometry concerns the study of solution sets to systems of polynomial equations. Historically, geometers explored curves and surfaces that are zero sets of polynomials. It is the insights gained from these basic figures that have led, over the course of centuries, to the profound depth and remarkable breadth of contemporary algebraic geometry. However, many of the current theories are now far removed from explicit varieties, and polynomials are nowhere in sight. We are advocating for algebraic geometry to take an outward-looking perspective. Our readers should be aware of the wealth of applications in the sciences and engineering and be open to a "back to the basics" approach in both teaching and scholarship. From this perspective, the interaction with combinatorics can be particularly valuable. Indeed, combinatorics is known to some as the "nanotechnology of mathematics". It is all about explicit objects—those that can be counted, enumerated, and dissected with laser precision. And, these objects include some beautiful polynomials and the ideals they generate.

The following example serves as an illustration. We work in a polynomial ring $\mathbb{Q}[p]$ in 32 variables, one for each subset of $\{1, 2, 3, 4, 5, 6\}$ whose cardinality is odd:

$$p_1, p_2, \ldots, p_6,\ p_{123}, p_{124}, p_{125}, \ldots, p_{356}, p_{456},\ p_{12345}, p_{12346}, \ldots, p_{23456}\,.$$

The polynomial ring $\mathbb{Q}[p]$ is $\mathbb{Z}^7$-graded by setting $\deg(p_\sigma) = e_0 + \sum_{i\in\sigma} e_i$, where $e_0, e_1, \ldots, e_6$ is the standard basis of $\mathbb{Z}^7$. Let X be a 5×6-matrix of variables, and let I be the kernel of the ring map $\mathbb{Q}[p] \to \mathbb{Q}[X]$ that takes the variable p_σ to the determinant of the submatrix of X with column indices σ and row indices $1, 2, \ldots, |\sigma|$.

Table 2 Degrees and minimal generators for the ideal I

Degree	Generators
$(2,0,0,1,1,1,1)$	$p_3p_{456} - p_4p_{356} + p_5p_{346} - p_6p_{345}$
$(2,0,1,0,1,1,1)$	$p_2p_{456} - p_4p_{256} + p_5p_{246} - p_6p_{245}$
$\vdots$	$\vdots$
$(2,1,1,1,1,0,0)$	$p_1p_{234} - p_2p_{134} + p_3p_{124} - p_4p_{123}$
$(2,0,1,1,1,1,2)$ $\vdots$	$p_{256}p_{346} - p_{246}p_{356} + p_{236}p_{456}$ $\vdots$
$(2,2,1,1,1,1,0)$	$p_{125}p_{134} - p_{124}p_{135} + p_{123}p_{145}$
$(2,1,1,1,1,2,2)$	$p_{156}p_{23456} - p_{256}p_{13456} + p_{356}p_{12456} - p_{456}p_{12356}$
$\vdots$	$\vdots$
$(2,2,2,1,1,1,1)$	$p_{123}p_{12456} - p_{124}p_{12356} + p_{125}p_{12346} - p_{126}p_{12345}$

The ideal I is prime and $\mathbb{Z}^7$-graded. It has multiple geometric interpretations. First of all, it describes the partial flag variety of points in 2-planes in hyperplanes in $\mathbb{P}^5$. This flag variety lives in $\mathbb{P}^5 \times \mathbb{P}^{19} \times \mathbb{P}^5$, thanks to the Plücker embedding. Its projection into the factor $\mathbb{P}^{19}$ is the Grassmannian $\mathrm{Gr}(3, \mathbb{C}^6)$ of 2-planes in $\mathbb{P}^5$. Flag varieties are studied by Bossinger, Lamboglia, Mincheva, and Mohammadi in [11].

But, let the allure of polynomials now speak for itself. Our ideal I has 66 minimal quadratic generators. Sixty generators are unique up to scaling in their degree; see Table 2. The other six minimal generators live in degree $(2,1,1,1,1,1,1)$ and are 4-term Grassmann–Plücker relations like $p_{126}p_{345} - p_{125}p_{346} + p_{124}p_{356} - p_{123}p_{456}$.

Here is an alternate interpretation of the ideal I. It defines a variety of dimension $15 = \binom{6}{2}$ in $\mathbb{P}^{31}$ known as the *spinor variety*. In this guise, I encodes the algebraic relations among the principal subpfaffians of a skew-symmetric 6×6-matrix. Such subpfaffians are indexed with the subsets of $\{1,2,3,4,5,6\}$ of even cardinality. The trick is to fix a natural bijection between even and odd subsets. This variety is similar to the *Lagrangian Grassmannian* seen in Problem 10 on Grassmannians.

At this point, readers who like combinatorics and computations may study I. Can you compute the tropical variety of I? Which of its maximal cones are *prime* in the sense of Kaveh and Manon [36, Theorem 1]? These determine *Khovanskii bases* for $\mathbb{Q}[p]/I$ and hence toric degenerations of the spinor variety in $\mathbb{P}^{31}$. Their combinatorics is recorded in a list of *Newton-Okounkov polytopes* with 32 vertices.

Each of these polytopes comes with a linear projection to the six-dimensional demicube, which is the convex hull in $\mathbb{R}^7$ of the 32 points $\deg(p_\sigma)$. We saw this demicube in Problem 5 on Convexity, whose theme we turn to shortly.

It is the author's opinion that Khovanskii bases deserve more attention than the Newton-Okounkov bodies they give rise to. The former are the algebraic manifestation of a toric degeneration. These must be computed and verified. Looking at a Khovanskii basis through the lens of convexity reveals the Newton–Okounkov body.

We now come to a third, and even more interesting, geometric interpretation of our 66 polynomials. It has to do with *Cox rings*, and their Khovanskii bases, similar to those in the article by Bernal, Corey, Donten-Bury, Fujita, and Merz [7]. We begin by replacing the generic 5×6-matrix X by one that has the special form in [23, Eq. (1.2)]:

$$X = \begin{bmatrix} u_1^2 x_1 & u_2^2 x_2 & u_3^2 x_3 & u_4^2 x_4 & u_5^2 x_5 & u_6^2 x_6 \\ u_1 y_1 & u_2 y_2 & u_3 y_3 & u_4 y_4 & u_5 y_5 & u_6 y_6 \\ u_1 v_1 x_1 & u_2 v_2 x_2 & u_3 v_3 x_3 & u_4 v_4 x_4 & u_5 v_5 x_5 & u_6 v_6 x_6 \\ v_1 y_1 & v_2 y_2 & v_3 y_3 & v_4 y_4 & v_5 y_5 & v_6 y_6 \\ v_1^2 x_1 & v_2^2 x_2 & v_3^2 x_3 & v_4^2 x_4 & v_5^2 x_5 & v_6^2 x_6 \end{bmatrix}.$$

Now, the polynomial ring $\mathbb{Q}[X]$ gets replaced by $\mathbb{k}[x_1, x_2, \ldots, x_6, y_1, y_2, \ldots, y_6]$ where $\mathbb{k}$ is the field extension of $\mathbb{Q}$ generated by the entries of a 2×6-matrix of scalars:

$$U = \begin{bmatrix} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \\ v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{bmatrix}. \qquad (1)$$

We assume that the 2×2-minors of U are nonzero. Let J denote the kernel of the odd-minors map $\mathbb{k}[p] \to \mathbb{k}[X]$ as before. The ideal J is also $\mathbb{Z}^7$-graded and it strictly contains the ideal I. Castravet and Tevelev [23, Theorem 1.1] proved that $\mathbb{k}[p]/J$ is the *Cox ring* of the blow-up of $\mathbb{P}^3_{\mathbb{k}}$ at six points. These points are Gale dual to U. We refer to J as the *Cox ideal* of that rational threefold whose Picard group $\mathbb{Z}^7$ furnishes the grading. The affine variety in $\mathbb{A}^{32}_{\mathbb{k}}$ defined by J is ten-dimensional (it is the *universal torsor*). Quotienting by a seven-dimensional torus action yields our threefold. The same story for blowing up five points in $\mathbb{P}^2_{\mathbb{k}}$ is Problem 6 on Surfaces.

In [59], we construct the Cox ideal by duplicating the ideal of the spinor variety:

$$J = I + \mathbf{u} * I. \qquad (2)$$

Here $\mathbf{u}$ is a vector in $(\mathbb{k}^*)^{32}$ that is derived from U. The ideal $\mathbf{u} * I$ is obtained from I by scaling the variables f_σ with the coordinates of $\mathbf{u}$. In particular, the Cox ideal J is minimally generated by 132 quadrics. Now, there are two generators in each of the sixty $\mathbb{Z}^7$-degrees in our table and there are 12 generators in degree $(2, 1, 1, 1, 1, 1, 1)$.

Following [60, Example 7.6], we fix the rational function field $\mathbb{K} = \mathbb{Q}(t)$ and set

$$U = \begin{bmatrix} 1 & t & t^2 & t^3 & t^4 & t^5 \\ t^5 & t^4 & t^3 & t^2 & t & 1 \end{bmatrix}.$$

The ring map $\mathbb{K}[p] \to \mathbb{K}[X]$ now maps the variables p_σ like this:

$$p_1 \mapsto \underline{x_1}$$

$$p_{123} \mapsto \underline{x_1 y_2 x_3} t^6 - (x_1 x_2 y_3 + y_1 x_2 x_3) t^7 + (y_1 x_2 x_3 + x_1 x_2 y_3) t^9 - x_1 y_2 x_3 t^{10}$$

$$p_{12345} \mapsto \underline{x_1 y_2 x_3 y_4 x_5} t^{10} - (y_1 x_2 x_3 y_4 x_5 + x_1 y_2 x_3 x_4 y_5 + \cdots + x_1 x_2 y_3 y_4 x_5) t^{11} + \cdots$$

Table 3 Degrees and minimal generators for the ideal J

Degree	Generators
$(2, 1, 1, 1, 1, 0, 0)$	$p_1p_{234} - p_2p_{134} + p_3p_{124} - p_4p_{123}$
$(2, 1, 1, 1, 1, 0, 0)$	$t^4p_1p_{234} - t^2p_2p_{134} + t^2p_1p_{234} + p_4p_{123}$

Table 4 Degrees and minimal generators for the ideal in(J)

Degree	Generators	
$(2, 0, 0, 1, 1, 1, 1)$	$p_3p_{456} - p_4p_{356}$	$p_5p_{346} - p_6p_{345}$
$(2, 0, 1, 0, 1, 1, 1)$	$p_2p_{456} - p_4p_{256}$	$p_5p_{246} - p_6p_{245}$
$\vdots$	$\vdots$	$\vdots$
$(2, 1, 1, 1, 1, 0, 0)$	$p_1p_{234} - p_2p_{134}$	$p_3p_{124} - p_4p_{123}$
$(2, 0, 1, 1, 1, 1, 2)$	$p_6p_{23456} - p_{236}p_{456}$	$p_{246}p_{356} - p_{256}p_{346}$
$\vdots$	$\vdots$	$\vdots$
$(2, 2, 1, 1, 1, 1, 0)$	$p_1p_{12345} - p_{123}p_{145}$	$p_{124}p_{135} - p_{125}p_{134}$
$(2, 1, 1, 1, 1, 2, 2)$	$p_{156}p_{23456} - p_{256}p_{13456}$	$p_{356}p_{12456} - p_{456}p_{12356}$
$\vdots$	$\vdots$	$\vdots$
$(2, 2, 2, 1, 1, 1, 1)$	$p_{123}p_{12456} - p_{124}p_{12356}$	$p_{125}p_{12346} - p_{126}p_{12345}$

A typical example of a $\mathbb{Z}^7$-degree with two minimal generators appears in Table 3.

The algebra generators p_σ form a Khovanskii basis for $\mathbb{K}[p]/J$ with respect to the t-adic valuation. The toric algebra resulting from this flat family is generated by the underlined monomials. Its toric ideal in(J) is generated by 132 binomial quadrics; see Table 4. These 132 binomials define a toric variety that is a degeneration of our universal torsor. The ideal in(J) is relevant in both biology and physics. It represents the *Jukes-Cantor model* in phylogenetics [58] and the *Wess-Zumino-Witten model* in conformal field theory [39]. Beautiful polynomials can bring the sciences together.

Let us turn to another fitness problem. The past three pages offered a capoeira approach to Problem 9 in Parameters and Moduli. The compactification is that given by the tropical variety of the universal Cox ideal, to be computed as in [45, 47]. The base space is $\overline{\mathrm{M}}_{0,6}$ with points represented by 2×6-matrices U as in (1). We encountered several themes that are featured in other articles in this book: flag varieties, Grassmannians, $\mathbb{Z}^n$-gradings, Cox rings, Khovanskii bases, and toric ideals. The connection to spinor varieties was developed in the article [59] with Mauricio Velasco. The formula (2) is derived in [59, Theorem 7.4] for the blow-up of $\mathbb{P}^{n-3}$ at n points when $n \leq 8$. It is still a conjecture for $n \geq 9$. On your trail towards solving such open problems, fill your backpack with polynomials. They will guide you.

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From Curves to Tropical Jacobians and Back

Barbara Bolognese, Madeline Brandt, and Lynn Chua

Abstract For a curve over an algebraically closed field that is complete with respect to a nontrivial valuation, we study its tropical Jacobian. We first tropicalize the curve and then use the weighted metric graph to compute the tropical Jacobian. Finding the abstract tropicalization of a general curve defined by polynomial equations is difficult, because an embedded tropicalization may not be faithful, and there is no known algorithm for carrying out semistable reduction. We solve these problems for hyperelliptic curves by using admissible covers. We also calculate the period matrix from a weighted metric graph, which gives the tropical Jacobian and tropical theta divisor. Lastly, we look at how to compute a curve that has a given period matrix.

MSC 2010 codes: 14T05

1 Introduction

Let X be a nonsingular curve of genus g over an algebraically closed field $\mathbb{K}$ that is complete with respect to a non-archimedean valuation. Let R be the valuation ring of $\mathbb{K}$ with maximal ideal $\mathfrak{m}$, and let $\Bbbk := R/\mathfrak{m}$ be its residue field. We associate to X its *abstract tropicalization*: the dual weighted metric graph Γ of the special fibre of a semistable model of X. Finding the abstract tropicalization of a general

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curve is difficult and there is no known algorithm; see [13, Remark 3]. In this paper, we solve this problem for hyperelliptic curves and discuss strategies for finding abstract tropicalizations of all curves.

Given a weighted metric graph Γ, we compute its *period matrix* Q_Γ, which corresponds to the *tropical Jacobian* of the curve X. By taking the Voronoi decomposition dual to the Delaunay subdivision associated to Q_Γ, we obtain the *tropical theta divisor*. This process can also be inverted. The set of period matrices that arise as the tropical Jacobian of a curve is the *tropical Schottky locus*. Starting with a principally polarized tropical abelian variety whose period matrix Q is known to lie in the tropical Schottky locus, we give a procedure to compute a curve whose tropical Jacobian corresponds to Q.

This process of associating a tropical Jacobian to a curve can also be accomplished via the classical Jacobian. Jacobians of curves are principally polarized abelian varieties; among all abelian varieties, they are the most extensively studied. Both algebraic curves and abelian varieties have extremely rich geometries. Jacobians provide a link between such geometries and often reveal hidden features of algebraic curves. To associate a tropical Jacobian to a complex algebraic curve X, one first constructs its classical Jacobian $J(X) := H^0(X, \omega_X)^*/H_1(X, \mathbb{Z})$, where ω_X denotes the cotangent bundle of the curve. This complex torus admits a natural principal polarization Θ, called the *theta divisor*, such that the pair $\big(J(X), \Theta\big)$ is a principally polarized abelian variety. We can then obtain the tropical Jacobian by taking the Berkovich skeleton of the classical Jacobian. Baker–Rabinoff [9] and Viviani [37] independently prove that this alternative path gives the same result as ours. However, the classical process conceals computational difficulties and proves much more challenging to carry out in explicit examples. Methods are implemented in the *Maple* package *algcurves* [16] for computing Jacobians numerically over $\mathbb{C}$.

The structure of this paper is depicted in Fig. 1. Dashed arrows indicate steps which are not yet algorithmic. In Sect. 2, we find the abstract tropicalization of all hyperelliptic curves. Section 3 discusses issues with embedded tropicalizations, states some known results about certifying the faithfulness of an embedded tropicalization, and outlines the process of semistable reduction. This step of the procedure is far from being algorithmic, and this section focuses on examples which illustrate

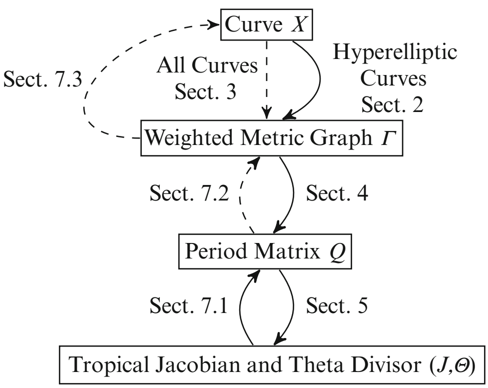


Fig. 1 The process of associating a tropical Jacobian to a curve

obstacles. In Sect. 4, we describe how to find the period matrix of a weighted metric graph. We then define and give examples of the tropical Jacobian and its theta divisor in Sect. 5. In Sect. 6, we discuss the tropical Schottky problem. Finally, Sect. 7 describes the challenges in reversing this process.

2 Hyperelliptic Curves

This section finds the abstract tropicalization of hyperelliptic curves. For elliptic curves, the tropicalization can be completely described in terms of the j-invariant [24]. Similarly, tropicalizations of genus 2 curves (all of which are hyperelliptic) can be described by *tropical Igusa invariants* [22]. This problem was also solved in genus 2 by studying the curve as a double cover of $\mathbb{P}^1$ ramified at six points; see [34, Sect. 5]. In this section, we generalize the latter method to find tropicalizations of all hyperelliptic curves. Let X be a nonsingular hyperelliptic curve of genus g over $\mathbb{K}$, an algebraically closed field which is complete with respect to a nontrivial non-archimedean valuation. Our goal is to find Γ, the abstract tropicalization of X.

Let $\mathrm{M}_{g,n}$ be the moduli space of genus g curves with n marked points; see [23]. The space $\mathrm{M}_{0,2g+2}$ maps surjectively onto the hyperelliptic locus inside M_g by identifying each hyperelliptic curve of genus g with a double cover of $\mathbb{P}^1$ ramified at $2g+2$ marked points. In characteristic other than 2, the normal form for the equation of a hyperelliptic curve is $y^2 = f(x)$, where $f(x)$ has degree $2g + 2$, and the roots of f are distinct. The roots of f correspond to the ramification points of the double cover.

The space $\mathrm{M}^{\mathrm{trop}}_{0,2g+2}$ is the tropicalization of $\mathrm{M}_{0,2g+2}$. A *phylogenetic tree* is a metric tree with leaves labelled $\{1, 2, \ldots, m\}$ and no vertices of degree 2. Such a tree is determined by the distances $d_{i,j}$ between the leaves. Following [26, Chap. 4.3], we see that $\mathrm{M}^{\mathrm{trop}}_{0,2g+2}$ parametrizes the space of phylogenetic trees with $2g + 2$ leaves using the Plücker embedding to map $\mathrm{M}_{0,2g+2}$ into the Grassmannian $\mathrm{Gr}(2, \mathbb{K}^{2g+2})$:

$$\big\{(a_i : b_i) : 1 \le i \le 2g + 2\big\} \mapsto (p_{1,2} : p_{1,3} : \cdots : p_{2g+1,2g+2}) \quad \text{where } p_{i,j} = a_ib_j - a_jb_i.$$

As in [34, Sect. 5], the distances are $d_{i,j} = -2\,\mathrm{val}(p_{i,j}) + n$ for a suitable constant n. Using the Neighbour Joining Algorithm [33, Algorithm 2.41] , one can construct the unique tree, along with the lengths of its interior edges, using only the leaf distances $d_{i,j}$ as input. Since the lengths of leaf edges can only be defined up to adding a constant length to each leaf, we think of this tree as a metric graph where the leaves have infinite length and the interior edges have prescribed lengths. This process realizes $\mathrm{M}^{\mathrm{trop}}_{0,2g+2}$ as a $(2g-1)$-dimensional fan inside $\mathbb{TP}^{\binom{2g+2}{2}-1}$; see [26, Sect. 2.5]. The space $\mathrm{M}^{\mathrm{trop}}_{0,2g+2}$ can be computed as a tropical subvariety of $\mathbb{TP}^{\binom{2g+2}{2}-1}$, because it has a tropical basis given by the Plücker relations for $\mathrm{Gr}(2, \mathbb{K}^{2g+2})$; see [26, Chap. 4.4]. Each cone corresponds to a combinatorial type of tree and the dimension

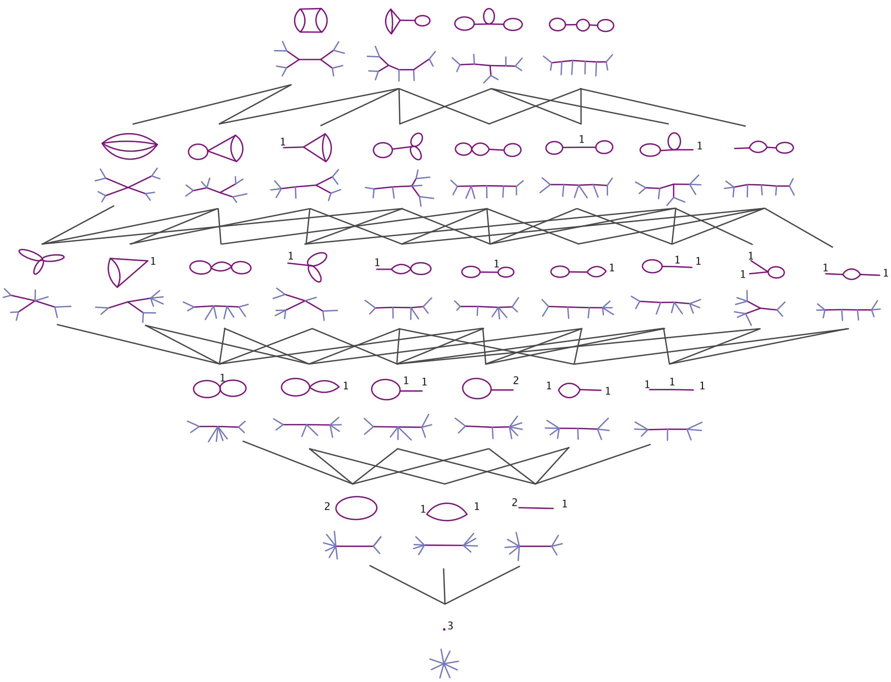


Fig. 2 The poset of unlabelled trees with eight leaves

of each cone corresponds to the number of interior edges in the tree; see Fig. 2 in the case $g = 3$. The next step is to take the corresponding point in $\mathrm{M}_{0,2g+2}^{\mathrm{trop}}$, as a tree on $2g + 2$ leaves, and compute a weighted metric graph in $\mathrm{M}_g^{\mathrm{trop}}$; again see Fig. 2.

To describe this correspondence for general g, we collect some definitions related to metric graphs; compare with [12].

Definition 2.1 A *metric graph* is a metric space Γ, together with a graph G and a length function $\ell \colon E(G) \to \mathbb{R}_{>0} \cup \{\infty\}$ such that Γ is obtained by gluing intervals e of length $\ell(e)$, or by gluing rays to their endpoints, according to how they are connected in G. In this case, the pair (G, ℓ) is a *model* for Γ. A *weighted metric graph* is a metric graph Γ with a weight function on its points $w : \Gamma \to \mathbb{Z}_{\geq 0}$, such that $\sum_{v \in \Gamma} w(v)$ is finite.

Edges of infinite length are *infinite leaves*, and these only meet the rest of the graph in one endpoint. A *bridge* is an edge whose deletion increases the number of connected components. The *genus* of a weighted metric graph (Γ, w) is

$$\sum_{v \in \Gamma} w(v) + |E(G)| - |V(G)| + 1\,,$$

where G is any model of Γ. Two weighted metric graphs of genus at least 2 are isomorphic if one can be obtained from the other via graph automorphisms, or by removing infinite leaves or leaf vertices v with $w(v) = 0$, together with the edge connected to it. In this way, every weighted metric graph has a *minimal skeleton*.

A model is *loopless* if there is no vertex with a loop edge. The *canonical loopless model* of Γ, with genus at least 2, is the graph G with vertices

$$V(G) := \left\{ x \in \Gamma : \begin{array}{l} \text{valence of } x \text{ is at least 2, } w(x) > 0, \\ \text{or } x \text{ is the midpoint of a loop} \end{array} \right\} .$$

If (G, ℓ) and (G', ℓ') are loopless models for the metric graphs Γ and Γ', then a *morphism of loopless models* $\varphi: (G, \ell) \to (G', \ell')$ is a map $V(G) \cup E(G) \to V(G') \cup E(G')$ of sets such that

- all vertices of G map to vertices of G';
- if $e \in E(G)$ maps to $v \in V(G')$, then the endpoints of e also map to v;
- if $e \in E(G)$ maps to $e' \in E(G')$, then the endpoints of e map to vertices of e';
- infinite leaves in G map to infinite leaves in G'; and
- if $\varphi(e) = e'$, then $\ell'(e')/\ell(e)$ is an integer. (These integers must be specified if the edges are infinite leaves.)

An edge $e \in E(G)$ is *vertical* if φ maps e to a vertex of G'. The morphism φ is *harmonic* if, for all $v \in V(G)$, the *local degree*

$$d_v := \sum_{\substack{e: v \in e \\ \varphi(e) = e'}} \frac{\ell'(e')}{\ell(e)}$$

is the same for all choices of $e' \in E(G')$. If the local degree is positive, then φ is *nondegenerate*. The *(global) degree* of a harmonic morphism is defined as

$$\sum_{\substack{e \in E(G) \\ \varphi(e) = e'}} \frac{\ell'(e')}{\ell(e)} .$$

We also say that φ satisfies the *local Riemann–Hurwitz condition* if:

$$2 - 2w(v) = d_v \Big(2 - 2w'\big(\varphi(v)\big)\Big) - \sum_{e: v \in e} \frac{\ell'\big(\varphi(e)\big)}{\ell(e)} - 1 .$$

If φ satisfies this condition at every vertex v in the canonical loopless model of Γ, then φ is called an *admissible cover* [14].

Definition 2.2 ([12, Theorem 1.3]) Let Γ be a weighted metric graph, and let (G, ℓ) denote its canonical loopless model. We say that Γ is *hyperelliptic* if there exists a nondegenerate harmonic morphism of degree 2 from G to a tree.

A hyperelliptic curve will always tropicalize to a hyperelliptic weighted metric graph, however not every hyperelliptic weighted metric graph is the tropicalization of a hyperelliptic curve.

Theorem 2.3 ([1, Corollary 4.15]) *If Γ is a minimal weighted metric graph of genus at least 2, then there is a smooth proper hyperelliptic curve X of genus g having Γ as its minimal skeleton if and only if Γ is hyperelliptic and, for all $p \in \Gamma$, the number of bridge edges adjacent to p is at most $2w(p) + 2$.*

Lemma 2.4 and its proof give an algorithm for taking a tree with $2g + 2$ infinite leaves and obtaining a metric graph which is an admissible cover of the tree.

Lemma 2.4 *Every tree T with $2g + 2$ infinite leaves has an admissible cover φ by a unique hyperelliptic metric graph Γ of genus g, and φ is harmonic of degree 2.*

Proof Let T be a tree with $2g + 2$ infinite leaves. If all infinite leaves are deleted, then a finite tree T' remains. Let $v_1, v_2, \ldots, v_k$ be the vertices of T', ordered such that, for $i < j$, the distance from v_i to v_k is at least the distance from v_j to v_k.

We construct Γ iteratively by building the preimage of each vertex v_i, asserting along the way that the local Riemann–Hurwitz condition holds. We begin with v_1, which has a positive number n_1 of leaf edges in T. Since φ has degree 2, it must be locally of degree 1 or 2 at every vertex of Γ. Since the preimage of an infinite leaf must be an infinite leaf, attach n_1 infinite leaves at the preimage $\varphi^{-1}(v_1)$ in Γ. At any vertex in Γ with infinite leaves, the morphism φ has local degree 2, so we attach to Γ an infinite leaf e such that $\ell(\varphi(e))/\ell(e) = 2$. There is a unique vertex in the preimage $\varphi^{-1}(v_1)$; otherwise, there would need to be another edge in the preimage of each leaf, so the degree of the morphism would be greater than 2. Let e_1 be the edge connecting v_1 to some other v_i. There are two possibilities:

1. The preimage of e_1 is two edges in Γ, each with length $\ell(e_1)$, and the local Riemann–Hurwitz equation is $2 - 2w\big(\varphi^{-1}(v_1)\big) = 2(2 - 0) - (n_1 + 0 + 0)$, which is only possible if n_1 is even and $\varphi^{-1}(v_1)$ has weight $(n_1 - 2)/2$.
2. The preimage of e_1 is one edge in Γ, with length $\ell(e_1)/2$, and the local Riemann–Hurwitz equation is $2-2w(\varphi^{-1}(v_1)) = 2(2-0)-(n_1+1)$, which is only possible if n_1 is odd, and $\varphi^{-1}(v_1)$ has weight $(n_1 - 1)/2$.

Now, we proceed to the other vertices. As long as the order of the vertices is respected, at each vertex v_i, there will be at most one edge e_i whose preimage in Γ we do not know. What happens at v_i is determined by the local Riemann–Hurwitz data. For $i > 1$, let n_i be the number of infinite leaves at v_i plus the number of edges $e \in T$ such that $e = \{v_i, v_j\}$, with $j < i$, and $\varphi^{-1}(e)$ is a bridge in Γ. If $n_i > 0$, then one of the two above possibilities holds. However, it is possible that $n_i = 0$, which yields the third possibility:

3. If $n_i = 0$ and $v_i' \in \varphi^{-1}(v_i)$, then the local Riemann–Hurwitz equation becomes $2 - 2w(v_i') = d_{v_i}(2 - 0) - (0)$. It follows that $d_{v_i} = 1$ and $w(v_i') = 0$, which implies that there are two vertices in $\varphi^{-1}(v_i)$.

Finally, we glue the pieces of Γ as specified by T, and contract the leaf edges on Γ. The fact that Γ has genus g is a consequence of the local Riemann–Hurwitz condition. □

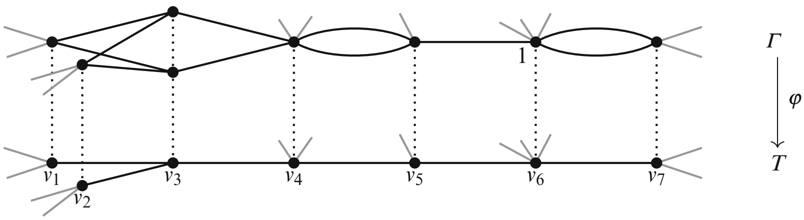


Fig. 3 The tree T and the hyperelliptic weighted metric graph Γ that admissibly covers it

This process did not require a tree with an even number of leaves. Applying this procedure for a tree with an odd number of leaves one obtains a hyperelliptic metric graph. However, this graph is not the tropicalization of a hyperelliptic curve.

Example 2.5 A tree with vertices labelled $v_1, v_2, \ldots, v_7$ appears in Fig. 3. Beginning with v_1, we observe that $n_1 = 2$, which means that the edge from v_1 to v_3 has two edges in its preimage. The same is true for v_2. Moving on to v_3, we see that $n_3 = 0$, so v_3 has two points in Γ which map to it. We can connect the edges from $\varphi^{-1}(v_1)$ and $\varphi^{-1}(v_2)$ to the two points in $\varphi^{-1}(v_3)$. Since $\varphi^{-1}(v_3)$ consists of two points, the edge from v_3 to v_4 corresponds to two edges in Γ. Since $n_4 = 2$, the edge from v_4 to v_5 also splits. Next, we have $n_5 = 1$, which means that the edge from v_5 to v_6 corresponds to a bridge in Γ. As $n_6 = 4$, the edge v_6 to v_7 splits, and the vertex mapping to v_6 has genus 1. Lastly, we have $n_7 = 2$, so the point mapping to v_7 has genus 0. All edges depicted in the image have the same length as the corresponding edges in the tree, except for the bridge, which has length equal to half the length of the corresponding edge in the tree.

The next theorem shows that the metric graph from Lemma 2.4 is the tropicalization of a hyperelliptic curve.

Theorem 2.6 *Fix a positive integer g. Let X be a hyperelliptic curve of genus g, given by the double cover of $\mathbb{P}^1$ ramified at $2g+2$ points $p_1, p_2, \ldots, p_{2g+2}$. If T is the tree corresponding to the tropicalization of $\mathbb{P}^1$ with the marked points $p_1, p_2, \ldots, p_{2g+2}$, and Γ is the unique hyperelliptic weighted metric graph that admits an admissible cover to T, then Γ is the abstract tropicalization of X.*

Proof This follows from Remark 20 and Theorem 4 in [14]. The hyperelliptic locus of M_g is the space $\overline{H}_{g\to 0,2}\big((2)^{2g+2}\big)$ of admissible covers with $2g+2$ ramification points of order 2. The space $\overline{H}^{\mathrm{an}}_{g\to 0,2}\big((2)^{2g+2}\big)$ is the Berkovich analytification of $\overline{H}_{g\to 0,2}\big((2)^{2g+2}\big)$, so a point X is represented by an admissible cover with $2g+2$ ramification points of order 2. By Theorem 4 in [14], the diagram

$$\begin{array}{ccccc}
\overline{\mathrm{M}}^{\mathrm{an}}_{0,2g+2} & \xleftarrow{br^{\mathrm{an}}} & \overline{H}^{\mathrm{an}}_{g\to 0,2}\big((2)^{2g+2}\big) & \xrightarrow{src^{\mathrm{an}}} & \overline{\mathrm{M}}^{\mathrm{an}}_{g} \\
\downarrow{\scriptstyle \mathrm{trop}} & & \downarrow{\scriptstyle \mathrm{trop}} & & \downarrow{\scriptstyle \mathrm{trop}} \\
\overline{\mathrm{M}}^{\mathrm{trop}}_{0,2g+2} & \xleftarrow{br^{\mathrm{trop}}} & \overline{H}^{\mathrm{trop}}_{g\to 0,2}\big((2)^{2g+2}\big) & \xrightarrow{src^{\mathrm{trop}}} & \overline{\mathrm{M}}^{\mathrm{trop}}_{g}
\end{array}$$

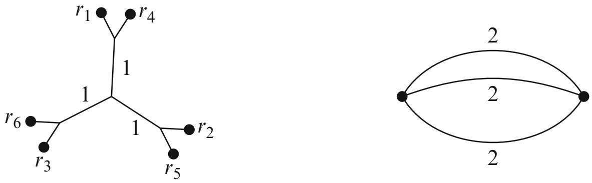


Fig. 4 This is the tree and metric graph for Problem 2 on curves

commutes. The morphisms *src* take a cover to its source curve, marked at the entire inverse image of the branch locus, and the morphisms *br* take a cover to its base curve, marked at its branch points. Starting with an element X of $\overline{H}^{\text{an}}_{g\to 0,2}\big((2)^{2g+2}\big)$, we seek $\text{trop}\big(src^{\text{an}}(X)\big) \in \overline{\text{M}}^{\text{trop}}_g$. Lemma 2.4 enables us to find an inverse for br^{trop}. If $T = \text{trop}(br^{\text{an}}(X))$, then the commutativity of the diagram establishes that $\text{trop}\big(src^{\text{an}}(X)\big) = src^{\text{trop}}\big((br^{\text{trop}})^{-1}(T)\big) = \Gamma$. □

Example 2.7 ([36, Problem 2 on Curves]) Consider the curve

$$y^2 = (x-1)(x-2)(x-3)(x-6)(x-7)(x-8)$$

with the 5-adic valuation. In $\text{M}^{\text{trop}}_{0,6}$, this gives us the point

$$(p_{1,2}, p_{1,3}, \ldots, p_{5,6}) = (0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0)\,.$$

When n is any integer, this gives us a tree metric with distances given by

$$(d_{1,2}, d_{1,3}, \ldots, d_{5,6}) = (n, n, n-2, n, n, n, n, n-2, n, n, n, n-2, n, n, n)\,.$$

This is a metric for the tree on the left of Fig. 4 and the metric graph on the right.

3 Other Curves

Beyond the hyperelliptic case, finding the abstract tropicalization of a curve is very hard. In this section, we highlight some of the difficulties and discuss two approaches to this problem: faithful tropicalization and semistable reduction. We offer the following example as an illustration of the difficulties.

Example 3.1 ([36, Problem 9 on Abelian Combinatorics]) Consider the curve in $\mathbb{P}^2$ given by the zero locus of

$$\begin{aligned} f(x,y,z) = {} & 41x^4 + 1530x^3y + 3508x^3z + 1424x^2y^2 + 2490x^2yz - \\ & 2274x^2z^2 + 470xy^3 + 680xy^2z - 930xyz^2 + 772xz^3 + \\ & 535y^4 - 350y^3z - 1960y^2z^2 - 3090yz^3 - 2047z^4\,, \end{aligned}$$

defined over $\mathbb{Q}_2$. The induced regular subdivision of the Newton polygon will be trivial because the 2-adic valuation of the coefficients on the x^4, y^4, z^4 terms is 0. Therefore, no information about the structure of the abstract tropicalization can be detected from this embedded tropicalization. In Example 3.4, we pick another coordinate system that allows us to find a faithful tropicalization.

Faithful Tropicalization To discuss the *Berkovich skeleton* of a curve, consider an algebraically closed field $\mathbb{K}$ that is complete with respect to a nontrivial non-archimedean valuation, and let X be a nonsingular curve defined over $\mathbb{K}$. The *Berkovich analytification* X^{an} is a topological space whose underlying set consists of all multiplicative seminorms on the coordinate ring $\mathbb{K}[X]$ that are compatible with the valuation val on $\mathbb{K}$; see [5]. It has the coarsest topology such that, for all $f \in \mathbb{K}[X]$, the map sending a seminorm $|\cdot|$ to $|f|$ is continuous; this is different from the metric structure, see [7, Sect. 5.3]. When X is a smooth proper geometrically integral curve of genus greater than or equal to 1, the space X^{an} has a strong deformation retraction onto finite metric graphs called *skeletons* of X^{an}, and there is a unique *minimal skeleton*; see [4].

The minimal skeleton of the Berkovich analytification is the same metric graph as the dual graph of the special fibre of a stable model of a curve X. Suppose that $i: X \hookrightarrow \mathbb{A}^n$ is an embedding of X and the polynomials $f_1, f_2, \ldots, f_m$ generate the ideal of the image. Let $\mathrm{trop}(X, i)$ denote the embedded tropicalization; details including how to find the metric on $\mathrm{trop}(X, i)$ can be found in [26]. The Berkovich analytification is related to embedded tropicalizations as follows.

Theorem 3.2 ([30, Theorem 1.1]) *If X is an affine variety over $\mathbb{K}$, then there is a homeomorphism $X^{an} \to \varprojlim \mathrm{trop}(X, i)$.*

The homeomorphism is given by the inverse limit of maps $\pi_i: X^{\mathrm{an}} \to \mathbb{A}^m$ defined by $\pi_i(x) = \big(-\log|f_1|_x, -\log|f_2|_x, \ldots, -\log|f_m|_x\big)$, where $|\cdot|_x$ denotes the norm corresponding to the point $x \in X^{\mathrm{an}}$. The image of this map is equal to $\mathrm{trop}(X, i)$.

Given one embedded tropicalization, we want to extract information about the Berkovich skeleton. In some cases, the embedded tropicalization contains enough information to recover the structure of the skeleton of X^{an}. Identifying these cases is the problem of *certifying faithfulness*; see [8, Subsect. 5.23].

Theorem 3.3 *Let X be a smooth curve in $\mathbb{P}^n_{\mathbb{K}}$ of genus g. If $\dim H_1\big(\mathrm{trop}(X), \mathbb{R}\big) = g$, all vertices of $\mathrm{trop}(X) \subset \mathbb{R}^{n+1}/\mathbb{R}\mathbf{1}$ are trivalent, and all edges have multiplicity 1, then the minimal skeleta of $\mathrm{trop}(X)$ and X^{an} are isometric. In particular, if X is a smooth curve in $\mathbb{P}^2_{\mathbb{K}}$ whose Newton polygon and subdivision form a unimodular triangulation, then the minimal skeleta of $\mathrm{trop}(X)$ and X^{an} are isometric.*

Proof This follows from a result of Baker, Payne, and Rabinoff [8, Corollary 5.2.8] who assume that all vertices of $\mathrm{trop}(X) \subset \mathbb{R}^{n+1}/\mathbb{R}1$ are trivalent, all edges have multiplicity 1, Σ has no leaves, and $\dim H_1(\mathrm{trop}(X), \mathbb{R}) = \dim H_1(\Sigma, \mathbb{R})$. We are detecting information about the minimal skeleton Σ of X^{an}, so we may ignore their hypothesis on Σ. Thus, we have that $g \geq \dim H_1(\Sigma, \mathbb{R}) \geq \dim H_1\big(\mathrm{trop}(X), \mathbb{R}\big)$. When $g = \dim H_1\big(\mathrm{trop}(X), \mathbb{R}\big)$, we also have $g = \dim H_1(\Sigma, \mathbb{R})$. Furthermore, it is only possible that Σ has leaves when $g > \dim H_1(\Sigma, \mathbb{R})$. □

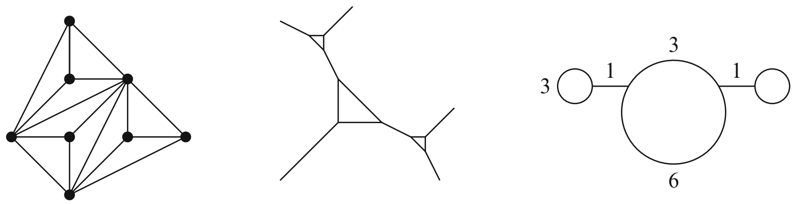


Fig. 5 The Newton polygon from Example 3.4 together with its unimodular triangulation on the left, the embedded tropicalization in the centre, and the metric graph on the right

Fig. 6 The Newton polygon from Example 3.5 with its regular subdivision on the left, and the embedded tropicalization on the right

The next example illustrates how to apply Theorem 3.3 to find the metric graph of the curve given in Problem 9.

Example 3.4 (Problem 9 continued) If $x := \frac{1}{12}X + \frac{1}{2}Y - \frac{1}{12}Z$, $y := \frac{1}{2}X - \frac{1}{2}Y$, and $z := -\frac{5}{12}X - \frac{1}{12}Z$, then we obtain

$$0 = -256X^3Y - 2X^2Y^2 - 256XY^3 - 8X^2YZ - 8XY^2Z - XYZ^2 - 2XZ^3 - 2YZ^3.$$

We calculate the regular subdivision of the Newton polygon in *polymake* [21], weighted by the 2-adic valuations of the coefficients. The embedded tropicalization and corresponding metric graph, with edge lengths, are depicted in Fig. 5. Since all vertices are trivalent, all edges have multiplicity 1, and $\dim H_1\big(\mathrm{trop}(X), \mathbb{R}\big) = 3$, Theorem 3.3 establishes that this is the abstract tropicalization of the curve.

We offer another example to illustrate some of the shortcomings of Theorem 3.3. One ultimately needs to use semistable reduction to solve the problem in this case.

Example 3.5 Consider the curve X in $\mathbb{P}^2$ over the Puiseux series field $\mathbb{C}\{\{t\}\}$ defined by $xyz^2 + x^2y^2 + 29t(xz^3 + yz^3) + 17t^2(x^3y + xy^3) = 0$. Tropicalizing X with this embedding, we obtain the embedded tropicalization in Fig. 6. Since this is not a unimodular triangulation, Theorem 3.3 does not allow us to draw any conclusions. However, the next subsection shows that this is a faithful tropicalization.

Semistable Reduction If we fail to certify the faithfulness of a tropicalization, then we find the metric graph Γ by taking the dual graph of the special fibre of a semistable model for X; see [7]. To this end, we outline the process of finding the semistable model of a curve X.

Let X be a reduced nodal curve over $\mathbb{K}$. For each irreducible component C of X, let $\varphi\colon \widetilde{C} \to C$ be the normalization of C. The curve X is *semistable* if every smooth rational component meets the rest of the curve in at least two points, or every component of $\widetilde{C}$ has at least two points x such that $\varphi(x)$ is a singularity in

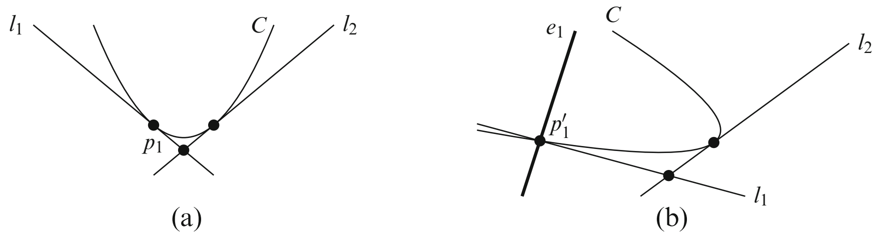


Fig. 7 Illustrations of the special fibres in Example 3.7

X. If R is the valuation ring of $\mathbb{K}$, then the scheme $\mathrm{Spec}(R)$ contains two points: one corresponding to the zero ideal (0) and another corresponding to the maximal ideal $\mathfrak{m}$ of R. If $\mathscr{X}$ is a scheme over $\mathrm{Spec}(R)$, we call the fibre of the the point corresponding to (0) the *generic fibre*, and the fibre over the point corresponding to $\mathfrak{m}$ the *special fibre*.

Definition 3.6 If X is any finite type scheme over $\mathbb{K}$, then a *model* for X is a flat and finite type scheme $\mathscr{X}$ over R whose generic fibre is isomorphic to X. This model is *semistable* if the special fibre $\mathscr{X}_{\mathbb{k}} = X \times_R \mathbb{k}$ is a semistable curve over $\mathbb{k} := R/\mathfrak{m}$.
The Semistable Reduction Theorem shows that X always admits a semistable model. However, the proofs of this theorem only contain hints towards an algorithmic approach; see [15] and [3].

Given a model for X, we now describe a procedure for finding a semistable model; see [23]. The first step is to blow up the total space $\mathscr{X}$, removing any singularities in the special fibre, to arrive at a family whose special fibre is a nodal curve. At this point, our work is not yet done because the resulting curve may be nonreduced.

Example 3.7 (Example 3.5 continued) The special fibre is a conic with two tangent lines, depicted in Fig. 7a. We denote the conic by C and the two lines by l_1 and l_2. By blowing up the total space at the point p_1, the result, depicted in Fig. 7b, is that l_1 and C are no longer tangent, but they do intersect in the exceptional divisor e_1. The exceptional divisor e_1 has multiplicity 2, coming from the multiplicity of the point p_1. In the figures, we encode the multiplicities of the components by their thickness. Next, we blow up the total space at the point labelled p_1' to get Fig. 8a. The new exceptional divisor e_1' has multiplicity 4, and the curves l_1 and C no longer intersect. All points except $l_2 \cap C$ are either smooth or have nodal singularities, so blowing up these two points gives the configuration in Fig. 8b.

To eliminate the nonreduced components in the special fibre, we make successive base changes of prime order p. Explicitly, we take the pth cover of the family branched along the special fibre. If D is a component of multiplicity q in the special fibre, either p does not divide q, in which case D is in the branch locus, or else we obtain p copies of D branched along the points where D meets the branch locus, and the multiplicity is reduced by $1/p$.

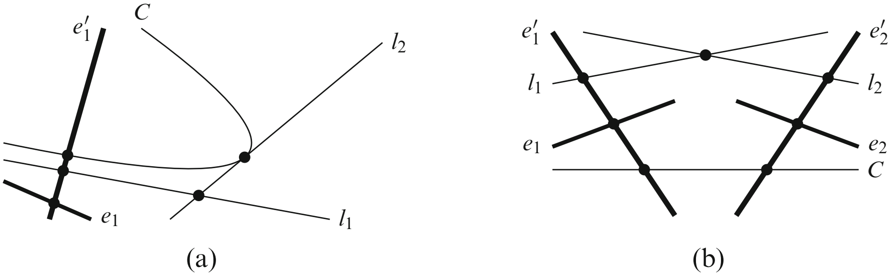


Fig. 8 More illustrations of the special fibres in Example 3.7

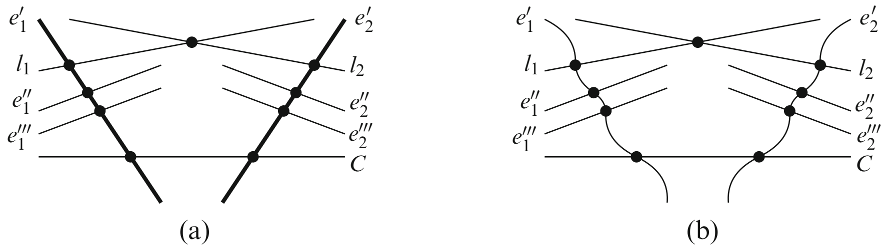
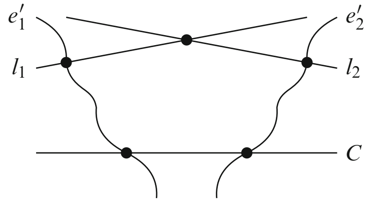


Fig. 9 Illustrations of the special fibres in Example 3.8

Fig. 10 The special fibre of the semistable model



Example 3.8 (Example 3.5 continued) We must make two base changes of order 2. Starting with Fig. 8b, we see that l_1, l_2, and C are in the branch locus. The curves e_1' and e_2' are replaced by their respective double covers branched at two points, which is again a rational curve. We continue to call these e_1' and e_2', and they each have multiplicity 2. The curves e_1 and e_2 are disjoint from the branch locus, so each one is replaced by two disjoint rational curves. The result is depicted in Fig. 9a. In the second base change of order 2, all components except e_1' and e_2' are in the branch locus. The curves e_1' and e_2' each meet the branch locus in four points, which, by the Riemann–Hurwitz theorem, means they will be replaced by genus 1 curves. The result is depicted in Fig. 9b.

The last step is to blow down all rational curves which meet the rest of the fibre exactly once, depicted in Fig. 10. This gives us a semistable model of X.

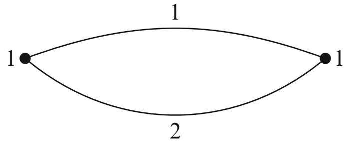


Fig. 11 The metric graph from Example 3.10

Weighted Metric Graphs From a faithful tropicalization, the abstract tropicalization of a curve X is obtained simply by taking the minimal skeleton of $\mathrm{trop}(X)$. Given a semistable model $\mathscr{X}$ of X, this coincides with the *dual graph* of $\mathscr{X}_{\Bbbk}$; see [7].

Definition 3.9 Let $C_1, C_2, \ldots, C_n$ be the irreducible components of $\mathscr{X}_{\Bbbk}$, the special fibre of a semistable model of X. The *dual graph* G of $\mathscr{X}_{\Bbbk}$ has vertices v_i corresponding to the components C_i with $w(v_i) = g(C_i)$, and an edge $e_{i,j}$ between v_i and v_j if the corresponding components C_i and C_j intersect in a node q. The completion of the local ring $\mathscr{O}_{\mathscr{X},q}$ is isomorphic to $R[[x, y]]/(xy - f)$, where R is the valuation ring of $\mathbb{K}$, and f lies the maximal ideal $\mathfrak{m}$ of R, so we define $\ell(e_{i,j}) = \mathrm{val}(f)$.

Example 3.10 (Example 3.5 continued) Taking the dual graph of the semistable model described in the previous subsection, we obtain a cycle with two vertices of weight 1. Hence, Theorem 5.24 in [8] shows that the cycle arising from embedded tropicalization was a faithful tropicalization. Hence, the metric graph is as depicted in Fig. 11. Since the vertices corresponding to l_1 and l_2 each have valence 2, we do not depict these in the model in Fig.11.

4 Period Matrices of Weighted Metric Graphs

A weighted metric graph $\Gamma := (G, w, \ell)$ is a triple consisting of a metric graph G, a function w on $V(G)$ assigning nonnegative weights to the vertices, and a function ℓ on $E(G)$ assigning positive lengths to the edges. Given a weighted metric graph Γ, we describe a procedure to compute its period matrix, following [6, 11, 28].

Fix an orientation of the edges of G. For any edge $e \in E(G)$, let $s(e)$ denote the source vertex and let $t(e)$ denote the target vertex. The spaces of 0-chains and 1-chains of G with coefficients in $\mathbb{Z}$ are defined as

$$C_0(G,\mathbb{Z}) = \Big\{ \sum_{v\in V(G)} a_v\, v : a_v \in \mathbb{Z}\Big\}, \quad \text{and} \quad C_1(G,\mathbb{Z}) = \Big\{ \sum_{e\in E(G)} a_e\, e : a_e \in \mathbb{Z}\Big\}.$$

The module $C_1(G,\mathbb{Z})$ is equipped with the inner product

$$\Big\langle \sum_{e\in E(G)} a_e\, e, \sum_{e\in E(G)} b_e\, e\Big\rangle := \sum_{e\in E(G)} a_e\, b_e\, \ell(e)\,.$$

The *boundary map* $\partial\colon C_1(G,\mathbb{Z}) \to C_0(G,\mathbb{Z})$ acts linearly on 1-chains by mapping an edge e to $t(e)-s(e)$. To change the coefficients, we tensor this map with the relevant commutative ring, so $C_i(G,\mathbb{R}) := C_i(G,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$ for $0 \le i \le 1$. The kernel of this map is the first homology group $H_1(G,\mathbb{Z})$ of G, whose rank is $g(G) = |E(G)| - |V(G)| + 1$. Let $|w| := \sum_{v\in V(G)} w(v)$, and let $g := g(G) + |w|$ be the genus of Γ. Consider the positive semidefinite form Q_Γ on $H_1(G,\mathbb{Z}) \oplus \mathbb{Z}^{|w|}$, which vanishes on the second summand $\mathbb{Z}^{|w|}$ and is defined on $H_1(G,\mathbb{Z})$ by $Q_\Gamma\left(\sum_{e\in E(G)} \alpha_e\, e\right) := \sum_{e\in E(G)} \alpha_e^2\, \ell(e)$.

Definition 4.1 If $\omega_1, \omega_2, \dots, \omega_{g(G)}$ is a basis of $H_1(G,\mathbb{Z})$, then we identify the free module $H_1(G,\mathbb{Z}) \oplus \mathbb{Z}^{|w|}$ with $\mathbb{Z}^g$. In this situation, we may express Q_Γ as a positive semidefinite $g \times g$ matrix, called the *period matrix* of Γ. Choosing a different basis gives another matrix related by an action of $\mathrm{GL}(g,\mathbb{Z})$.

To calculate the period matrix of a given weighted metric graph Γ, fix an arbitrary orientation of the edges of the underlying graph G and choose a spanning tree T of G. Set $m := |E(G)|$. Label the edges of G such that $e_1, e_2, \dots, e_{g(G)}$ are not in the spanning tree T, and $e_{g(G)+1}, e_{g(G)+2}, \dots, e_m$ are. It follows that each subgraph $T \cup \{e_i\}$, for $1 \le i \le g(G)$, contains a unique cycle ω_i in G, and the cycles $\omega_1, \omega_2, \dots, \omega_{g(G)}$ form a basis of the lattice $H_1(G,\mathbb{Z})$. Traverse each cycle ω_i in the direction specified by e_i. Consider the vector $b_i \in \mathbb{Z}^m$ representing the cycle ω_i:

$$j\text{th entry of } b_i := \begin{cases} 1 & \text{if } e_j \text{ belongs to } \omega_i \text{ and has the same orientation,}\\ 0 & \text{if } e_j \text{ does not belong to } \omega_i,\\ -1 & \text{if } e_j \text{ belongs to } \omega_i \text{ and has the opposite orientation.} \end{cases}$$

Let B be the $\big(g(G) \times m\big)$-matrix whose ith row is b_i. The matrix B is the totally unimodular matrix representing the cographic matroid of G; see [29]. Suppose that all vertices in G have weight zero, so that $g(G) = g$. If D is the diagonal $(m \times m)$-matrix with nonzero entries $\ell(e_1), \ell(e_2), \dots, \ell(e_m)$, then the period matrix is $Q_\Gamma := BDB^{\mathsf{T}}$. If we label the columns of B by $u_1, u_2, \dots, u_m$, the period matrix equals $Q_\Gamma = \ell(e_1)u_1u_1^{\mathsf{T}} + \ell(e_2)u_2u_2^{\mathsf{T}} + \cdots + \ell(e_m)u_mu_m^{\mathsf{T}}$. Thus, the cone of all matrices that are period matrices of G, allowing the edge lengths to vary, is the rational open polyhedral cone $\sigma_G := \mathbb{R}_{>0}\langle u_1u_1^{\mathsf{T}}, u_2u_2^{\mathsf{T}}, \dots, u_mu_m^{\mathsf{T}}\rangle$. Finally, if Γ has vertices of nonzero weight, then the period matrix is given by the same construction with $g - g(G)$ additional rows and columns with zero entries.

Example 4.2 Consider the complete graph on four vertices in Fig. 12. We indicate in the figure an arbitrary choice of the edge orientations. If we choose the spanning tree T consisting of the edges $\{e_4, e_5, e_6\}$, then the corresponding cycle basis is $\omega_1 := e_1 + e_5 + e_4$, $\omega_2 := e_2 + e_6 - e_5$, and $\omega_3 := e_3 - e_4 - e_6$, which yields

$$B = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0\\ 0 & 1 & 0 & 0 & -1 & 1\\ 0 & 0 & 1 & -1 & 0 & -1 \end{bmatrix}.$$

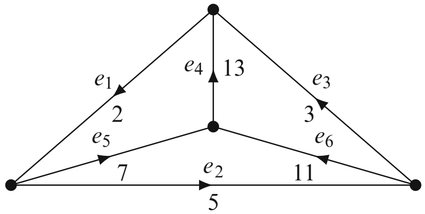


Fig. 12 The metric graph and edge orientation used in Example 4.2

Hence, the period matrix is

$$Q_\Gamma = B \begin{bmatrix} 2&0&0&0&0&0\\ 0&5&0&0&0&0\\ 0&0&3&0&0&0\\ 0&0&0&13&0&0\\ 0&0&0&0&7&0\\ 0&0&0&0&0&11 \end{bmatrix} B^\mathsf{T} = \begin{bmatrix} 22&-7&-13\\ -7&23&-11\\ -13&-11&27 \end{bmatrix}.$$

5 Tropical Jacobians

In this section, we use period matrices to define and study tropical Jacobians of curves as principally polarized tropical abelian varieties.

Let S^g_+ be the set of symmetric positive semidefinite $(g\times g)$-matrices with rational nullspace (their kernels have bases defined over $\mathbb{Q}$). The group $\mathrm{GL}(g,\mathbb{Z})$ acts on S^g_+ by the map that sends $(X, Q) \in \mathrm{GL}(g,\mathbb{Z}) \times \mathrm{S}^g_+$ to $X^\mathsf{T} Q X$. For any weighted metric graph Γ of genus g, the period matrix Q_Γ belongs to S^g_+. A *tropical torus* of dimension g is the quotient $X = \mathbb{R}^g/\Lambda$, where Λ is a lattice of rank g in $\mathbb{R}^g$. A *polarization* on X is given by a quadratic form Q on $\mathbb{R}^g$. Following [6, 11], the pair $(\mathbb{R}^g/\Lambda, Q)$ is called a *principally polarized tropical abelian variety* whenever $Q \in \mathrm{S}^g_+$. Two principally polarized tropical abelian varieties are isomorphic if there is some $X \in \mathrm{GL}(g,\mathbb{R})$ that maps one lattice to the other, and sends one quadratic form to the other. By choosing a representative of each isomorphism class of $(\mathbb{R}^g/\mathbb{Z}^g, Q)$, we obtain the *moduli space* A_g^{trop} *of principally polarized tropical abelian varieties*. The structure of A_g^{trop} is described in Sect. 6.

Definition 5.1 The *tropical Jacobian* of a curve is the principally polarized tropical abelian variety $(\mathbb{R}^g/\mathbb{Z}^g, Q)$, where Q is the period matrix of the curve.

The period matrix also induces a Delaunay subdivision of $\mathbb{R}^g$ which gives rise to the *tropical theta divisor* on the tropical Jacobian. To be more explicit, fix $Q \in \mathrm{S}^g_+$ and consider the map $\psi_Q\colon \mathbb{Z}^g \longrightarrow \mathbb{Z}^g \times \mathbb{R}$ defined by $x \mapsto (x, x^\mathsf{T} Q x)$. By taking the convex hull of the image of ψ_Q in $\mathbb{R}^g \times \mathbb{R} \cong \mathbb{R}^{g+1}$ and projecting away from the last coordinate, we obtain a periodic decomposition of the lattice $\mathbb{Z}^g \subset \mathbb{R}^g$, called the *Delaunay subdivision* of Q. Naively, this operation corresponds to looking at the polyhedron from below and recording the visible faces on the lattice. This is an infinite periodic analogue of the regular subdivision of a polytope. The *tropical theta*

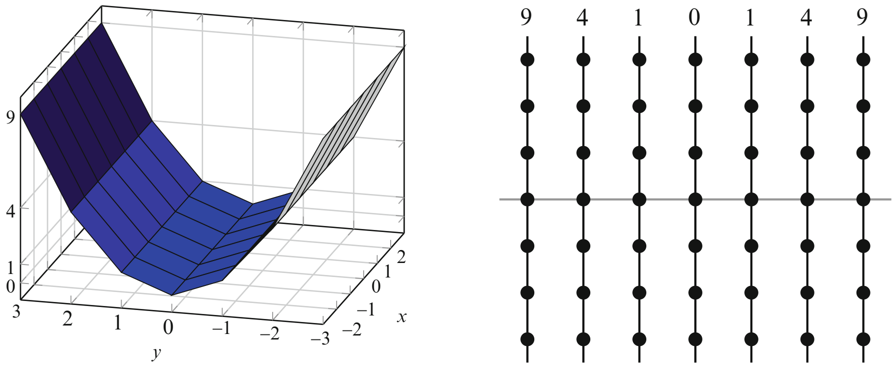


Fig. 13 Convex hull and Delaunay subdivision for the quadratic form in Example 5.2

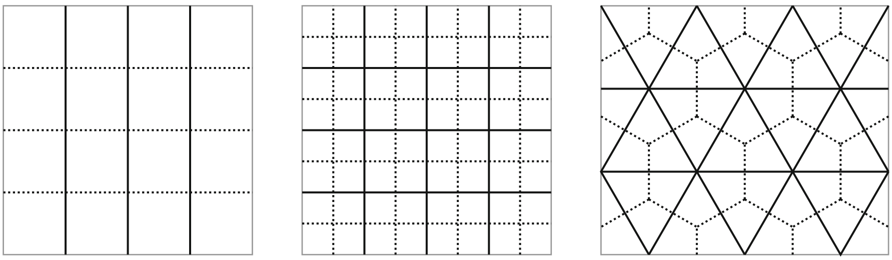


Fig. 14 Delaunay decompositions of $\mathbb{R}^2$ and their associated Voronoi decompositions

divisor on a principally polarized tropical abelian variety $(\mathbb{R}^g/\Lambda, Q)$ is the tropical hypersurface of the *theta function*

$$\Theta\colon \mathbb{R}^g \to \mathbb{R} \text{ where } \Theta(x) = \max_{\lambda\in\Lambda}\left\{\lambda^\mathsf{T} Q x - \tfrac{1}{2}\lambda^\mathsf{T} Q\lambda\right\}.$$

Example 5.2 For $Q := \left[\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right] \in \mathrm{S}^2_+$, the function $\psi_Q\colon \mathbb{Z}^2 \to \mathbb{Z}^2 \times \mathbb{R}$ is given by $(x, y) \mapsto (x, y, x^2)$. The lower faces of the convex hull of the image of ψ_Q appear on the left of Fig. 13, and the associated Delaunay subdivision appears on the right.

Dualizing the Delaunay subdivision produces the *Voronoi decomposition*. For $g = 2$, we illustrate a few Delaunay subdivisons (with solid lines) together with their associated Voronoi decomposition (with dotted lines) in Fig. 14.

As with algebraic curves, there is a natural map from a tropical curve to its tropical Jacobian. Let $\Gamma := (G, w, \ell)$ be a weighted metric graph, let $p_0 \in \Gamma$ be a fixed basepoint, and let $\omega_1, \omega_2, \ldots, \omega_g$ be a basis of $H_1(G, \mathbb{Z})$. For any point p in Γ, we encode a path from p_0 to p by a divisor $c(p) = \sum_i a_i e_i$, where the coefficient $a_i \in \mathbb{Z}$ records the orientation (and number of occurrences) of the edge e_i in the given path. By identifying the tropical torus $\mathbb{R}^g/\Lambda$ with our chosen cycle basis, we obtain the *tropical Abel–Jacobi map* $\mu\colon \Gamma \to \mathbb{R}^g/\Lambda$ which sends $p \in \Gamma$ to $\big(\langle c(p), \omega_1\rangle, \langle c(p), \omega_2\rangle, \ldots, \langle c(p), \omega_g\rangle\big)$; see [28]. This map is independent of the

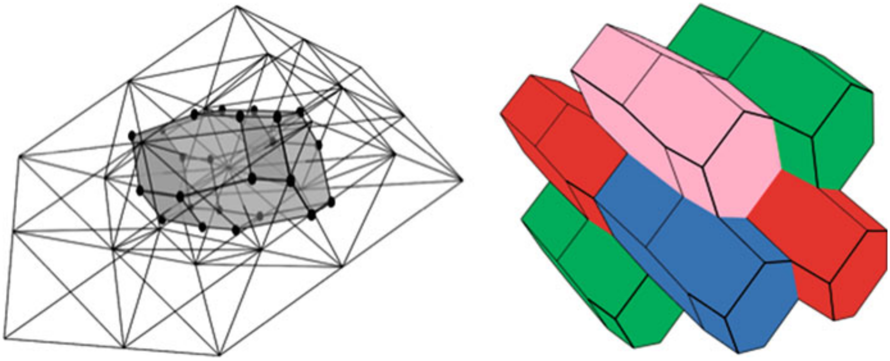


Fig. 15 The left shows the Delaunay subdivision by tetrahedra and a dual permutohedron in grey and the right illustrates a tiling of $\mathbb{R}^3$ by permutohedra

choice of path from p_0 to p and extends linearly to a map on all divisors on Γ. The *degree* of a tropical divisor $D = \sum_i a_i p_i$ is $\sum_i a_i \in \mathbb{Z}$. Moreover, a tropical divisor $D = \sum_i a_i p_i$ is *effective* if $a_i \geq 0$ for all i. Let W_{g-1} be the image of effective divisors of degree $g-1$ under the tropical Abel–Jacobi map.

Theorem 5.3 ([28, Corollary 8.6]) *The set W_{g-1} is the tropical theta divisor up to translation.*

Example 5.4 (Example 4.2 continued) The *polyhedral* [17] package in *GAP* [20] show that the fundamental region of the Delaunay subdivision for the quadratic form in Example 4.2 consists of six tetrahedra in the unit cube which share the main diagonal as an edge. The associated Voronoi decomposition gives a tiling of $\mathbb{R}^3$ by permutohedra; see Fig. 15. The tropical theta divisor has f-vector $(6, 12, 7)$. In Fig. 16, we illustrate the correspondence between W_2 and the tropical theta divisor. Each vertex of the permutohedron corresponds to a divisor supported on the vertices of Γ. The square faces correspond to divisors supported on the interiors of edges of Γ that do not meet in a vertex. Each hexagonal face corresponds to a divisor that is supported on edges of Γ and is adjacent to a fixed vertex. The edges correspond to keeping one point of the divisor fixed, and moving the other point along an edge of Γ. The curves depicted above represent the embedding of Γ into its Jacobian under the Abel–Jacobi map, which is again K_4.

6 Tropical Schottky Problem

In this section, we describe the structure of the moduli space A_g^{trop} using Voronoi reduction theory. Given a Delaunay subdivision D of $\mathbb{R}^g$, let σ_{D} denote the set of all matrices $Q \in \mathrm{S}_+^g$ that produce the same subdivision. The *secondary cone* of D is the Euclidean closure $\overline{\sigma_{\mathrm{D}}}$ of σ_{D} in $\mathbb{R}^{\binom{g+1}{2}}$; it is a closed rational polyhedral cone. The $\mathrm{GL}(g, \mathbb{Z})$-action on S_+^g extends to an action on the set of secondary cones.

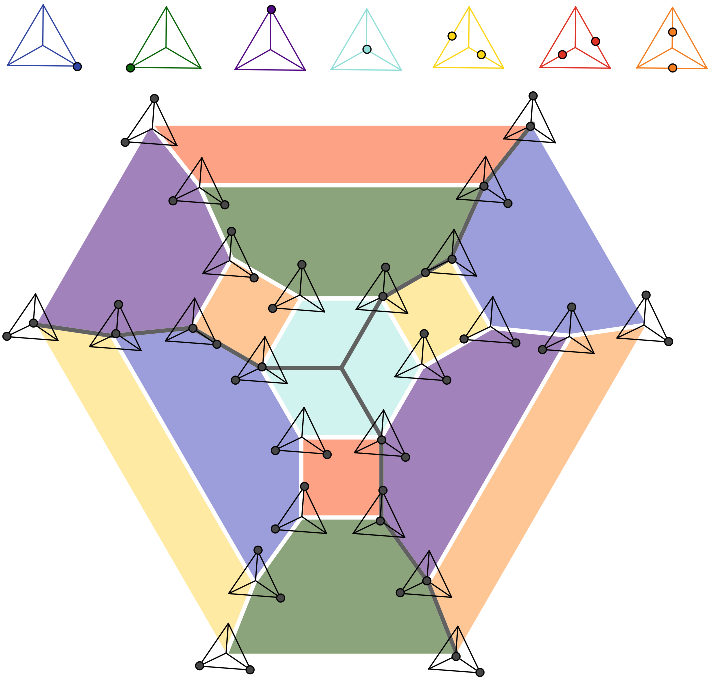


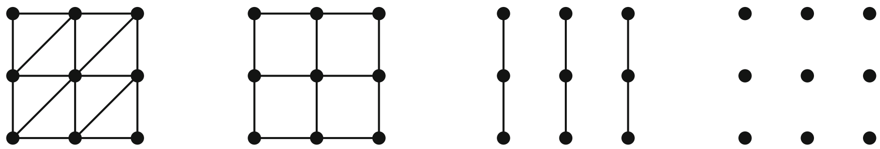
Fig. 16 Correspondence between W_2 and the tropical theta divisor

Theorem 6.1 ([38]) *The set of secondary cones forms an infinite polyhedral* $\mathrm{GL}(g,\mathbb{Z})$*-periodic decomposition of the cone* S^g_+*, known as the* second Voronoi decomposition.

As a consequence of this theorem, one may choose Delaunay subdivisions $\mathrm{D}_1, \mathrm{D}_2, \ldots, \mathrm{D}_k$ of $\mathbb{R}^g$ such that the corresponding secondary cones are representatives for $\mathrm{GL}(g,\mathbb{Z})$-equivalence classes of secondary cones. The moduli space A_g^{trop} is a *stacky fan* whose cells correspond to these classes; see [6, 11]. For each Delaunay subdivision D, consider the stabilizer $\mathrm{Stab}(\sigma_{\mathrm{D}}) := \{X \in \mathrm{GL}(g,\mathbb{Z}) : \sigma_{\mathrm{D}} \cdot X = \sigma_{\mathrm{D}}\}$. If the cell $C(\mathrm{D}) := \overline{\sigma_{\mathrm{D}}}/\,\mathrm{Stab}(\sigma_{\mathrm{D}})$ is the quotient of the secondary cone by the stabilizer, then we have $A_g^{\mathrm{trop}} = \bigsqcup_{i=1}^{k} C(\mathrm{D}_i)/\sim$, where we take the disjoint union of the cells $C(\mathrm{D}_1), C(\mathrm{D}_2), \ldots, C(\mathrm{D}_k)$ and quotient by the equivalence relation $\sim$ induced by $\mathrm{GL}(g,\mathbb{Z})$-equivalence of matrices in S^g_+ (which corresponds to gluing the cones).

Example 6.2 For $g = 2$, we may choose the Delaunay subdivisions $\mathrm{D}_1, \mathrm{D}_2, \mathrm{D}_3, \mathrm{D}_4$ as shown in Fig. 17. These have the property that their secondary cones give representatives for $\mathrm{GL}(g,\mathbb{Z})$-equivalence classes of secondary cones. The corresponding secondary cones are

$$\overline{\sigma_{D_1}} = \left\{ \begin{bmatrix} a+c & -c \\ -c & b+c \end{bmatrix} : a,b,c \in \mathbb{R} \right\}, \qquad \overline{\sigma_{D_2}} = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : a,b \in \mathbb{R} \right\},$$



D_1 D_2 D_3 D_4

Fig. 17 Delaunay subdivisions for $g = 2$

$$\overline{\sigma_{D_3}} = \left\{ \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} : a \in \mathbb{R} \right\}, \qquad \overline{\sigma_{D_4}} = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}.$$

The *tropical Torelli map* $t_g^{\text{trop}}\colon \mathrm{M}_g^{\text{trop}} \to A_g^{\text{trop}}$ sends a weighted metric graph of genus g to its tropical Jacobian, which is the element of A_g^{trop} corresponding to its period matrix. The image of this map is the *tropical Schottky locus*. To characterize this locus, consider the *cographic matroid* $M^*(G)$ associated to a graph G: it is representable by the totally unimodular matrix B constructed in Sect. 4; see [29]. The $\mathrm{GL}(g, \mathbb{Z})$-equivalence class of the corresponding secondary cone σ_G is independent of the choice of totally unimodular matrix representing $M^*(G)$. Hence, we associate to $M^*(G)$ a unique cell $C\big(M^*(G)\big)$ of A_g^{trop}. A matroid is *simple* if it has no loops and no parallel elements. We define the following stacky subfan of A_g^{trop} corresponding to simple cographic matroids,

$$A_g^{\text{cogr}} := \{C(M) : M \text{ a simple cographic matroid of rank at most } g\}.$$

The image of the tropical Torelli map t_g^{trop} is A_g^{cogr}. We call A_g^{cogr} the *tropical Schottky locus*; see [6, 11]. When $g \leq 3$, we have $A_g^{\text{cogr}} = A_g^{\text{trop}}$ and every element of S_+^g is a period matrix of a weighted metric graph. However, when $g \geq 4$, this inclusion is proper. For example, computations in [11] show that A_4^{cogr} has 25 cells while A_4^{trop} has 61 cells, and A_5^{cogr} has 92 cells whereas A_5^{trop} has 179,433 cells.

7 ... and Back

So far, we have a process that produces the tropical Jacobian of a curve given its defining equations. In this section, we examine whether it is possible to take a principally polarized tropical abelian variety X in the tropical Schottky locus, and produce a curve whose tropical Jacobian is precisely X. Several of the steps described in the previous sections are far from being one-to-one. Many algebraic curves have the same abstract tropicalization; all curves with a smooth stable model tropicalize to a single weighted vertex. In the same fashion, the non-injectivity of the tropical Torelli map implies that the same positive semidefinite matrix can be associated to more than one weighted metric graph; see [6, 11]. The purpose of this section is therefore to construct an arbitrary curve with a given tropical Jacobian.

From Tropical Jacobians to Positive Semidefinite Matrices Let $(\mathbb{R}^g/\Lambda, \Theta)$ be a tropical Jacobian, and fix an isomorphism $\Lambda \cong \mathbb{Z}^g$. The tropical theta divisor Θ corresponds to a Voronoi decomposition dual to a Delaunay subdivision D. We can describe D by a collection of hyperplanes $\{H_1, H_2, \ldots, H_k\}$ such that the lattice translates by $\mathbb{Z}^g$ of these hyperplanes cut out the polytopes in D. Following [27, Fact 4.1.4] and [19], one may choose these hyperplanes with normal vectors $u_1, u_2, \ldots, u_k \in \mathbb{R}^g$ such that the matrix with $u_1, u_2, \ldots, u_k$ as its columns is unimodular. The secondary cone of D is then $\sigma_{\mathrm{D}} = \mathbb{R}_{>0}\langle u_1u_1^{\mathsf{T}}, u_2u_2^{\mathsf{T}}, \ldots, u_ku_k^{\mathsf{T}}\rangle$. Thus, any quadratic form lying in the positive span of the rank-one forms $u_iu_i^{\mathsf{T}}$, for $1 \le i \le k$, will have the Delaunay subdivision D, so we can take $Q = u_1u_1^{\mathsf{T}} + u_2u_2^{\mathsf{T}} + \cdots + u_ku_k^{\mathsf{T}}$.

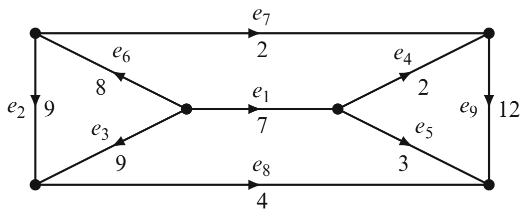
From Positive Semidefinite Matrices to Weighted Metric Graphs Fix $g \ge 0$ and let Q be a $(g \times g)$-matrix in S^g_+. If Q is not positive definite, there exists a change of basis such that Q has a positive definite $(g' \times g')$-submatrix and remaining entries zero. This corresponds to adding $|w| = g - g'$ weights on the vertices of the graph, which can be done arbitrarily as long as every weight zero vertex has degree at least 3. Hence, without loss of generality, we assume that Q is positive definite. Our goal is to determine if Q corresponds to an element of the tropical Schottky locus and, if so, to find a weighted metric graph that has Q as its period matrix.

Consider all combinatorial types of simple graphs with genus at most g. We compute their corresponding secondary cones. Let S be the set of these secondary cones. The *polyhedral* [17] package in *GAP* [20] allows one to compute the secondary cone σ_Q of Q; the underlying theory is described in [35]. Again, using the *polyhedral* package, with external calls to the program *ISOM* [31, 32], one may check if σ_Q is $\mathrm{GL}(g, \mathbb{Z})$-equivalent to any cone in S; see [18, Sect. 4] for the implementation details. If σ_Q is not equivalent to any cone in S, then Q is not the period matrix of a weighted metric graph. Otherwise, there exists a cone σ in S that belongs to the same $\mathrm{GL}(g, \mathbb{Z})$-equivalence class as σ_Q. Let G be the graph associated to σ, and find a matrix $X \in \mathrm{GL}(g, \mathbb{Z})$ which maps σ_Q to σ and sends Q to a matrix Q' in σ. Write $\sigma_G = \mathbb{R}_{>0}\langle u_1u_1^{\mathsf{T}}, u_2u_2^{\mathsf{T}}, \ldots, u_mu_m^{\mathsf{T}}\rangle$, so $Q' = \alpha_1u_1u_1^{\mathsf{T}} + \alpha_2u_2u_2^{\mathsf{T}} + \cdots + \alpha_mu_mu_m^{\mathsf{T}}$ for positive $\alpha_1, \alpha_2, \ldots, \alpha_m \in \mathbb{R}$. It follows that Q' is the period matrix of G with edge lengths $\alpha_1, \alpha_2, \ldots, \alpha_m$. Since $X \in \mathrm{GL}(g, \mathbb{Z})$ corresponds to a different choice of cycle basis, we have constructed a metric graph with Q as its period matrix.

Example 7.1 Consider the positive definite matrix

$$Q = \begin{bmatrix} 17 & 5 & 3 & 5 \\ 5 & 19 & 7 & 11 \\ 3 & 7 & 23 & 16 \\ 5 & 11 & 16 & 29 \end{bmatrix}.$$

Fig. 18 Weighted metric graph in Example 7.1



Using the *polyhedral* package, we see that σ_Q is $\mathrm{GL}(4,\mathbb{Z})$-equivalent to the cone $\sigma \in S$ corresponding to the weighted metric graph in Fig. 18, via the transformation

$$X = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & -1 & 0 & 0 \end{bmatrix}, \qquad Q' = X^{\mathsf{T}} Q X = \begin{bmatrix} 26 & 9 & -9 & 0 \\ 9 & 20 & 7 & -2 \\ -9 & 7 & 23 & 3 \\ 0 & -2 & 3 & 17 \end{bmatrix}.$$

Hence Q is in the tropical Schottky locus, and Q' is the period matrix of the metric graph in Fig. 18, with the cycle basis consisting of $e_2 + e_6 - e_3$, $-e_1 + e_6 + e_7 - e_4$, $-e_1 + e_3 + e_8 - e_5$, and $e_4 + e_9 - e_5$.

Currently, this algorithm is impractical for genus greater than 5, as the classification of Delaunay subdivisions is known only up to dimension 5; see [18].

From Weighted Metric Graphs to Algebraic Curves Given a weighted metric graph Γ, we want to produce equations defining a curve which tropicalizes to Γ. Any weighted metric graph Γ arises through tropicalization; see [2, Theorem 1.2.1]. Given a smooth curve X with Γ as its tropicalization, there exists a rational map $f\colon X \to \mathbb{P}^3$ such that the restriction of $\mathrm{trop}(f)$ to the skeleton Γ is an isometry onto its image; see [9, Theorem 8.2]. Since f is not required to be a closed immersion, this does not necessarily give a faithful tropicalization, see [9, Remark 8.5].

The paper [10] also studies this question for a specific class of metric graphs. They give a method for producing curves over $\mathbb{C}((t^{1/l}))$, embedded in a toric scheme and with a faithful tropicalization to the input metric graph Γ. They start by defining a suitable nodal curve whose dual graph is a model for Γ, and use deformation theory to show that the nodal curve can be lifted to a proper flat semistable curve over R with the nodal curve as its special fibre, which tropicalizes to Γ.

We now describe a procedure for finding a nodal curve over $\mathbb{C}$ whose dual graph is a model for Γ. Let G be a weighted stable graph of genus g with n infinite edges; a stable graph G is a connected graph such that each vertex of weight zero has valence at least three. The dual graph of a stable curve is always a stable graph. The original idea for this procedure is due to Kollár [25], and works in a much more general setup. Suppose that the stable graph $\Gamma := (G, w, \ell)$ is such that:

- for each vertex $v \in V(G)$, the weight $w(v)$ is of the form $w(v) = \binom{d(v)-1}{2}$ for some integer $d(v)$, and

- for each two vertices $v_1, v_2 \in V(G)$, one has $|E(v, w)| \leq d(v_1)d(v_2)$, where $|E(v_1, v_2)|$ denotes the number of edges between v_1 and v_2.

It follows that every component of G is realizable by a curve in $\mathbb{P}^2$, and it is possible to achieve the right number of intersection points between every two components. More precisely, one proceeds as follows:

1. Label the vertices as $\{v_1, v_2, \ldots, v_n\} = V(G)$. For each $1 \leq i \leq n$, take a general smooth plane curve C_i of degree $d_i := d(v_i)$.
2. We have now a reducible plane curve C, whose irreducible components are the curves C_i of degree d_i and, by the genus–degree formula, of genus $w(v_i)$. Any two components C_i and C_j intersect in $d_i d_j$ points, by Bézout's formula. We choose any $k_{i,j} = d_i d_j - |E(v_i, v_j)|$ of those, and set $r = \sum_{i,j} k_{i,j}$.
3. Take the blow up $X = \mathrm{Bl}_{p_1,p_2,\ldots,p_r} \mathbb{P}^2$ of $\mathbb{P}^2$ at the chosen points $p_1, p_2, \ldots, p_r$, and consider the proper transform $\widetilde{C}$ of C in X.
4. The curve X lives in the product $\mathbb{P}^2 \times \mathbb{P}^1$. Embed X in $\mathbb{P}^{2\times 3-1} = \mathbb{P}^5$ via a Segre embedding, and take the image of $\widetilde{C}$. This gives a projective curve with components of the correct genera (as the genus is a birational invariant), and any two components will intersect precisely at the correct number of points. Hence, its dual graph will be G.

Example 7.2 Consider the graph in Fig. 19. It has two components of genus 0 and two components of genus 1, which we realize as a pair of lines and cubics in general position in $\mathbb{P}^2$. The two lines intersect in a point, the two cubics intersect in nine points and each cubic will intersect each line in three points. The corresponding curve arrangement appears in Fig. 20. We blow up eight of the nine intersection points between the two cubics, because they correspond to edges between the components of genus 1 in the graph. Moreover, the two components of genus 0 do not share an edge, so the unique intersection point between the two lines must be blown up, as well as the three intersection points of a chosen cubic with a line, two out of the three intersection point with the remaining line, and one of the three intersection points of the first cubic with the second line. In Fig. 20, these points are marked. The result is a curve in $\mathbb{P}^2 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^5$, whose components have the correct genera and intersect at the correct number of points, and whose equations can be explicitly computed.

Remark 7.3 The theory shows that it is, in principle, possible to find a smooth curve over $\mathbb{K}$ with a prescribed metric graph as its tropicalization. For certain types

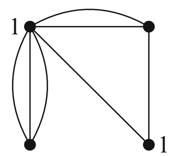


Fig. 19 The weighted graph discussed in Example 7.2

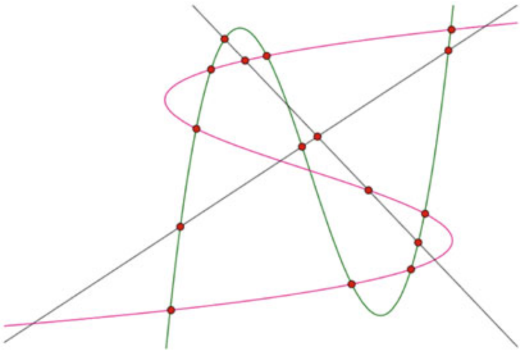


Fig. 20 Arrangement of curves in $\mathbb{P}^2$

of graphs, more work has been done in this direction; see [10]. Nevertheless, this problem is far from being solved in full generality in an algorithmic way.

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Tritangent Planes to Space Sextics: The Algebraic and Tropical Stories

Corey Harris and Yoav Len

Abstract We discuss the classical problem of counting planes tangent to general canonical sextic curves at three points. We determine the number of real tritangents when the curve is real. We also revisit a curve constructed by Emch with the greatest known number of real tritangents and, conversely, construct a curve with very few real tritangents. Using recent results on the relation between algebraic and tropical theta characteristics, we show that the tropicalization of a canonical sextic curve has 15 tritangent planes.

MSC 2010 codes: 14T05, 14H50, 14N10, 14P25

1 Introduction

In this article, we study tritangent planes to general sextic curves in three-dimensional projective space which, in particular, are not hyperelliptic. By general, we mean that the curve is the intersection of a smooth quadric and a smooth cubic. A plane in space is determined by three parameters and, when chosen generically, meets a sextic in six points. Requiring that two contact points coincide to form a tangent imposes a single condition on the parameters. Therefore, a finite number of planes are expected to be tangent to the curve at three points. Making this argument precise and finding the exact number of tritangent planes is more subtle and dates back to the mid-nineteenth century with the work of Clebsch [11]. Sixty years later, an understanding of these tritangents was the impetus and principal goal for Coble [10].

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The tritangent planes to a sextic are closely related with other classical problems such as the 27 lines on a cubic surface [15, Chapter V.4] and the 28 bitangents to a quartic curve [12, Chapter 6]. The projection from a general point of a cubic surface is a double cover of a plane branched along a quartic. The image of each of the 27 lines is bitangent to this quartic with an additional bitangent given by blowing up the indeterminacy locus of the projection. Quite similarly, a del Pezzo surface of degree one forms a double cover of a quadric cone, branched along a smooth sextic of genus 4. The (-1)-curves are mapped to conics, each meeting the sextic in three points, and the planes containing these conics are tritangent planes—they have intersection multiplicity two at each of the three points.

In a lecture given by Arnold [2] at his 60th birthday conference at the Fields Institute, he referred to this as one of his mathematical trinities. Other examples of trinities are the exceptional Lie algebras E_6, E_7, E_8, the rings $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and the three polytopes tetrahedron, cube, and dodecahedron.

> The next dream I want to present is an even more fantastic set of theorems and conjectures. Here I also have no theory and actually the ideas form a kind of religion rather than mathematics. The key observation is that in mathematics one encounters many trinities … I mean the existence of some "functorial" constructions connecting different trinities. The knowledge of the existence of these diagrams provides some new conjectures which might turn to be true theorems … I have heard from John MacKay that the straight lines on a cubical surface, the tangents of a quartic plane curve, and the tritangent planes of a canonic sextic curve of genus 4 form a trinity parallel to E_6, E_7, and E_8.

Our interest stems from two variations of the classical problem: the real case and the tropical case. For a real space sextic, one may ask how many of the tritangent planes are real. In Sect. 3, we appeal to the theory of real theta characteristics to show that the answer depends on the topology of the real curve: the number of connected components and how they are arranged on the Riemann surface of the complex curve. In some cases, all the tritangents may be real, but their three points of tangency may include a complex-conjugate pair; see Theorem 3.1. We also explore this phenomenon through explicit examples. We reexamine a construction of Arnold Emch, in which he claimed to find 120 tritangents, and show that he overcounted.

Theorem 3.2 *Emch's curve has only* 108 *planes tritangent at three real points.*
On the other extreme, we construct a real space sextic with only one connected component and find its eight real tritangent planes.

In Sect. 4, we set up a tropical formulation of the problem. Once the notion of a tropical tritangent plane is established, we ask how many such planes are carried by a tropical sextic curve. This is a natural sequel to earlier tropical counting problems, such as the number of lines on a tropical cubic surface [23] and the number of bitangents to a tropical plane quartic [5, 9].

Theorem 5.2 *A smooth tropical sextic curve Γ in $\mathbb{R}^3$ has at most* 15 *classes of tritangent planes. If it is the tropicalization of a sextic C on a smooth quadric in $\mathbb{P}^3$, then it has exactly* 15 *equivalence classes of tritangent planes.*
In Lemma 5.5, we show that the question can actually be replaced by the simpler problem of counting tritangents to tropical curves of bidegree $(3, 3)$ in tropical $\mathbb{P}^1 \times \mathbb{P}^1$. This result paves the way for a computational study of tropical tritangents.

2 Algebraic Space Sextics

Throughout this section, we work over an algebraically closed field $\mathbb{k}$ of characteristic different from 2. For simplicity, the reader may assume that the field is $\mathbb{C}$.

Let $C \subset \mathbb{P}^3$ be the intersection of a quadric and a cubic surface. Such a curve is a smooth canonical sextic [15, Proposition IV.6.3]. The intersection of a hyperplane with C is a divisor of degree 6 and rank 3 and is the canonical divisor K_C of C. In particular, the genus of C is 4. It follows that, whenever H is tangent to C at three points, those points form a divisor D such that $2D \simeq K_C$.

Definition 2.1 A *theta characteristic* is a divisor class $[D]$ such that $2D \simeq K_C$. A theta characteristic is *odd* or *even* depending on the parity of $\dim H^0(C, D)$.

Theorem 2.2 *If C is the sextic defined by the intersection of a smooth cubic and a smooth quadric in $\mathbb{P}^3$, then it has* 120 *tritangent planes that are in bijection with its odd theta characteristics.*

Proof Let D be a theta characteristic of C obtained from a hyperplane section, and set $h := \dim H^0(C, D)$. We claim that $h = 1$. To begin with, Clifford's Theorem [1, Chapter 3.1] implies that h is strictly smaller than 3. As D is obtained from intersecting a curve with a plane, it is effective, so h also cannot be 0. The geometric version of the Riemann–Roch Theorem states that, in the canonical embedding, the support of a divisor of degree d with h global sections spans a subspace of dimension $d - h$. It follows that $h = 2$ if and only if the contact points are colinear. To see that this is impossible, consider the quadric Q containing C. Since Q is smooth, it is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. The intersection of Q with a plane H is a $(1, 1)$-curve on Q. If it contains a line of one of the rulings, it also contains a line in the other. Therefore, H intersects C at points not in the support of D and is not a tritangent.

On the other hand, given an odd theta characteristic $p_1 + p_2 + p_3$, a plane H through p_1, p_2, p_3 intersects C at a divisor of the form $p_1 + p_2 + p_3 + q_1 + q_2 + q_3$ for some points q_1, q_2, q_3. Since $p_1 + p_2 + p_3 + q_1 + q_2 + q_3$ and $2(p_1 + p_2 + p_3)$ are both canonical, we get an equivalence of divisors $p_1 + p_2 + p_3 \simeq q_1 + q_2 + q_3$. Since $p_1 + p_2 + p_3$ is an odd theta characteristic, Clifford's Theorem implies that the rank of $p_1 + p_2 + p_3$ is zero. It follows that these equivalent divisors are equal. We deduce that $p_1 + p_2 + p_3 + q_1 + q_2 + q_3 = 2p_1 + 2p_2 + 2p_3$ and H is tritangent to C at p_1, p_2, p_3.

We conclude that tritangent planes are in bijection with the odd theta characteristics of C. The number of odd characteristics of a curve of genus g is $2^{g-1}(2^g - 1)$; see [20, Sect. 4]. As $g = 4$ in this case, we have 120 tritangent planes. □

A canonical sextic does have two classes of colinear divisors of degree 3. Indeed, those correspond to intersections of the sextic with the rulings of the ambient quadratic surface. However, as seen in the proof above, such a divisor is never obtained as the intersection of a hyperplane with the curve.

Remark 2.3 A smooth quadratic surface in $\mathbb{P}^3$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ via the Segre embedding [18, Lemma 3.31]. Under this isomorphism, the sextic corresponds to a curve of bidegree $(3, 3)$ on $\mathbb{P}^1 \times \mathbb{P}^1$, and a tritangent plane corresponds to a tritangent $(1, 1)$-curve. It follows that a $(3, 3)$-curve on a quadratic surface has 120 tritangent $(1, 1)$-curves as well. This result can also be deduced directly; the argument is similar to the proof of Theorem 2.2.

Our initial interest in tritangent planes came from studying tritangent planes to Bring's curve, as part of the apprenticeship workshop at the Fields Institute [22, Problem 4 on Curves]. Bring's curve is a space sextic, traditionally written in supernumerary coordinates by considering a special plane in $\mathbb{P}^4$. In particular, it is the intersection of the quadric given by $x^2 + y^2 + z^2 + t^2 + u^2 = 0$ and the cubic given by $x^3 + y^3 + z^3 + t^3 + u^3 = 0$ in the plane $x + y + z + t + u = 0$.

Edge [13] found equations for all 120 tritangent planes to Bring's curve. They come in two types. Type (i) tritangent planes are determined by three stalls of the curve. These are the points at which the osculating plane has order of contact higher than expected. In this case, the general point on the curve has order 3 contact with the osculating plane, and the stall points have order 4. Plücker's formulas for space curves tell us that there are exactly 60 stalls on Bring's curve.

Let α, β, γ be the three distinct roots of $\theta^3 + 2\theta^2 + 3\theta + 4 = 0$. Each of the equations $\gamma t - \beta u = 0$, $\alpha u - \gamma z = 0$, $\beta z - \alpha t = 0$ defines a plane that is tritangent to our curve and contains the tangent line at the stall point $[1 : 1 : \alpha : \beta : \gamma]$. The rest of the type (i) planes are given by replacing $\{z, t, u\}$ with any of the 60 ordered triples in $\{x, y, z, t, u\}$. Each of these contains the tangent line at a stall point given by an appropriate permutation of the coordinates of $[1 : 1 : \alpha : \beta : \gamma]$. The construction yields three tritangent planes through each of the 60 stall points of the curve, and every plane contains three stall points. In other words, the containment relation between type (i) tritangents and stalls determines the edges of a bipartite graph such that every vertex has valence 3. Thus, there are 60 such tritangent planes.

The type (ii) tritangent planes each contain exactly one stall point. One of them is given by $(\alpha - 1)(\alpha + 4)z + (\beta - 1)(\beta + 4)t + (\gamma - 1)(\gamma + 4)u = 0$, and the rest are obtained, again, by replacing $\{z, t, u\}$ with ordered triples in $\{x, y, z, t, u\}$. These results are summarized in the following theorem.

Theorem 2.4 ([13]) *Bring's curve has* 60 *tritangent planes of type (i) and* 60 *tritangent planes of type (ii).* □

3 Tritangents to Real Space Curves

In this section, we restrict ourselves to smooth curves that are defined over the real numbers with non-empty real part. The real part of a curve consists of a disjoint union of ovals, where by *oval* we mean a simple closed loop. For a curve of genus g, the number of these ovals cannot exceed $g+1$. See [24] and [7] for nice introductions to real algebraic geometry.

We say that a tritangent plane is *real* if it is defined over the real numbers, and *totally-real* if in addition the tangency points are all real. For a real tritangent that is not totally-real, the tangency points consist of a real point and a pair of complex conjugate points. For a smooth sextic on a smooth quadric in $\mathbb{P}^3$, real tritangent planes are in bijection with real odd theta characteristics.

Proposition 3.1 ([17]) *Let C be a real curve of genus g such that its real part $C(\mathbb{R})$ consists of s distinct ovals.*

1. *If $C(\mathbb{R})$ separates C, then there are $2^{g-1}(2^{s-1}+1)$ real even theta characteristics and $2^{g-1}(2^{s-1}-1)$ real odd ones.*
2. *If $C(\mathbb{R})$ doesn't separate C, then there are 2^{g+s-2} real even and 2^{g+s-2} real odd theta characteristics.*

A real curve of genus g with exactly $g+1$ ovals is referred to as an *M-curve*. Any disjoint union of $g+1$ cycles on a Riemann surface of genus g separates the surface, so an M-curve always corresponds to the first case in Proposition 3.1. In particular, a canonical space sextic with five ovals has 120 real tritangent planes. The question remains, how many of them are totally-real?

In [14], Emch claimed that the tritangents of a real space sextic with five ovals are all totally-real and constructed an example of such a curve and its tritangents. However, several of the planes were overcounted, and only 108 of its tritangents are totally-real. We are not aware of any previous literature that has addressed this issue.

We begin by considering a union of three lines in $\mathbb{P}^2$, so that they bound an equilateral triangle with incentre at the origin. For instance, if we choose one line to be of the form $x = a$, we find that the other lines should have slope $\pm 1/\sqrt{3}$. Choosing $a = -\sqrt{3}$ yields $p(x, y) = (x + \sqrt{3})(x - y\sqrt{3} - 3)(x + y\sqrt{3} - 3)$, and $\mathrm{V}\big(p(x, y)\big) \subset \mathbb{A}^2$ is our union of lines (Fig. 1).

The set of points $\{(x, y) \in \mathbb{A}^2 : p(x, y) = 2\}$ is a smooth cubic curve with four real branches, one of which is an oval bounded by our triangle. The polynomial $c(x, y, z) := p(x, y) - 2$ has zeros along a cubic cylinder in $\mathbb{A}^3$. A sphere centred at the origin of sufficiently large radius meets each of the components of the cubic and meets the central component twice. Therefore, the intersection of the cubic surface

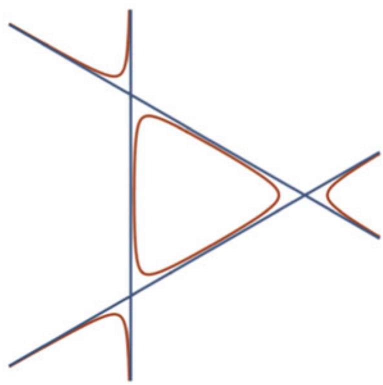

Fig. 1 Union of lines and a smooth cubic

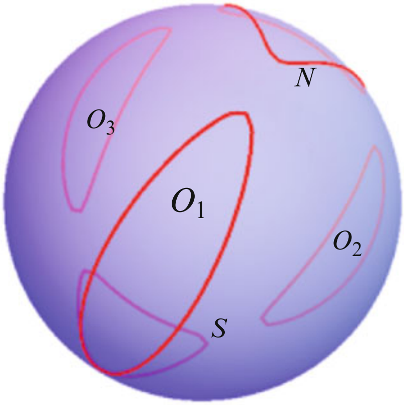

Fig. 2 Real sextic with five ovals

with a sphere yields a space sextic with five ovals. We refer to the top and bottom ovals as N and S, and the other three as O_1, O_2, O_3 (Fig. 2).

Theorem 3.2 *The real sextic space curve defined by the equations* $c(x, y, z) = 0$ *and* $x^2 + y^2 + z^2 = 25$ *has* 108 *totally-real tritangent planes.*

Proof We break the proof into parts based on the type of tritangent plane. There are tritangents that touch three distinct ovals, tritangents that touch an oval twice and another oval once, and tritangents that touch a single oval three times. We label them as the $(1, 1, 1)$-, $(2, 1)$-, and (3)-tritangents respectively.

80 $(1, 1, 1)$-*Tritangents* Given any three ovals of the curve, there exist $2^3 = 8$ classes of planes that separate them; a given plane has some of the ovals 'above' it and some 'below'. Such a plane can be moved to touch the three ovals each at one point in a unique way. (For an analogue in the plane, consider two general non-concentric ellipses and their four bitangents.) This yields $8\binom{5}{3} = 80$ tritangents, and there are no other tritangent planes meeting each of three ovals once.

$12 + 18$ $(2, 1)$-*Tritangents* For each O_i, there are four tritangents that touch it twice. To find them, consider the plane tangent to O_i at its northernmost point and southernmost point. The plane can be rotated to keep two points tangent to O_i. As it rotates, it meets the other two O_j each once yielding two tritangents for a total of $(3)(2) = 6$ such tritangents. The projection of these two tritangents to the xy-plane is pictured in Fig. 3. Similarly, for the oval N (resp. S), there are nine tritangents that touch it twice. To see them, pick a side of the triangle of N (resp. S), and consider the opposing O_i. There is a tritangent that touches N (resp. S) at two points along this side and touches O_i at its northernmost point, and similarly one which touches the O_i's southernmost point. Finally, there is a tritangent that touches the opposing point of S (resp. N). This yields $(9)(2) = 18$ tritangents.

There are no additional $(2, 1)$-tritangents meeting N, S twice. We claim that there are no additional $(2, 1)$-tritangents meeting O_i twice. The oval O_i has two reflectional symmmetries, one through the 'equator' and one through the great circle

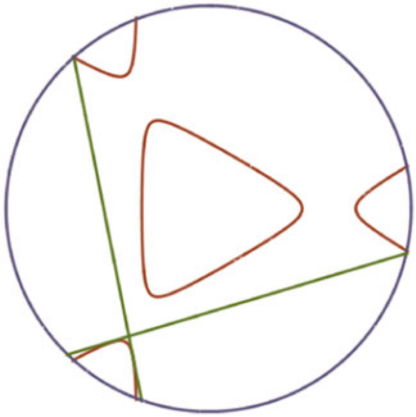

Fig. 3 Two (2, 1)-tritangents projected to xy-plane

determined by the northernmost and southernmost points. If $p \in O_i$ is a point that is not fixed by either reflection, then the images under reflection, denoted p' and p'', each share a tangent plane to O_i with p. Hence, the tangent line T_pO_i to O_i at p intersects $T_{p'}O_i$ and $T_{p''}O_i$. The two planes determined by these lines are the only bitangent planes to O_i at p. If p' is given by reflecting p through the xy-plane, then the corresponding bitangent is a tritangent only if the projection is a bitangent line. From Fig. 3, it is apparent that we have already counted all these. If p'' is the other reflection, then the bitangent to p and p'' cuts out a circle on the sphere which is contained in O_i. Therefore, it cannot be tangent at a point on another oval.

4 (3)-*Tritangents* The oval N has three maxima with respect to height in the z-direction. There is a plane which touches the oval at these three points. Similarly, it has three minima, and there is another tritangent plane there. The same is true for S. We thus have $(2)(2) = 4$ four more tritangent planes.

It is easy to see that there are no more (3)-tritangent planes meeting N or S only. A tritangent plane also cannot meet O_i only as, by symmetry, such a plane would have to touch O_i at either its northernmost or southernmost point, but there are not other points sharing a bitangent with either of these. □

We have shown that Emch's curve has fewer totally-real tritangents than was previously thought. A natural question is thus reopened.

Question 3.3 Does there exist a canonically embedded real space sextic with 120 totally-real tritangent planes?

We now consider a curve with significantly fewer totally-real tritangent planes (Fig. 4). Let C be the sextic determined by the intersection of the unit sphere S_2 defined by $x^2 + y^2 + z^2 = 1$ and the Clebsch diagonal cubic S_3 defined by

$$81(x^3 + y^3 + z^3) - 189(x^2y + x^2z + xy^2 + y^2z + xz^2 + yz^2) + 54xyz$$
$$+ 126(xy + xz + yz) - 9(x^2 + y^2 + z^2) - 9(x + y + z) + 1 = 0\,.$$

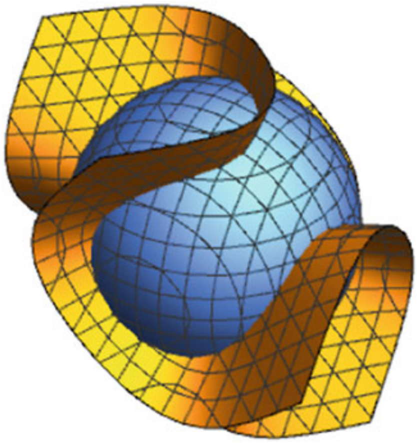

Fig. 4 The intersection of a cubic and a quadric yields a sextic

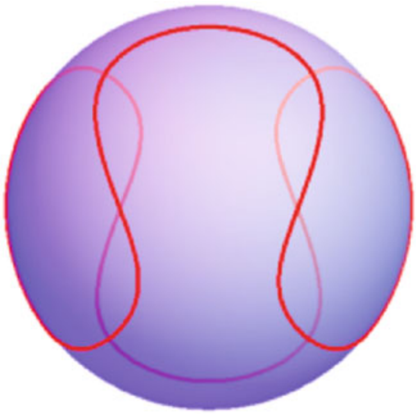

Fig. 5 A real sextic curve with a single connected component

This cubic surface has a threefold rotational symmetry about the axis $x = y = z$. In Fig. 5, this corresponds to the $\frac{2\pi}{3}$ rotation about the north pole p_N and south pole p_S. Since C has real points, Proposition 3.1 implies that it has eight real tritangent planes. As the following theorem shows, these planes exist and are all totally-real.

Theorem 3.4 *The curve C has exactly eight totally-real tritangent planes.*

Proof Let q denote a point on C that minimizes the distance to p_N. This point lies on a circle in $\mathbb{R}^3$ which is the intersection of S_2 with a sphere centred at p_N of radius $d(p_N, q)$. By the threefold rotational symmetry, there are at least three distinct points at which C touches the circle. Thus, the plane containing this circle is tritangent to C and there are exactly three tangency points. The same argument for p_S gives a second tritangent plane to C.

The curve C has a reflectional symmetry through the plane determined by p_N, p_S, and q. The three points associated to any tritangent to C must lie on a circle on S_2 which cannot cross C. By the reflectional symmetry of C, this circle must meet C at one of the points q from the two known tritangents. Such a circle, of a sufficiently small radius, meets C at no other points. With a sufficiently large radius, it meets C transversely at multiple points. Thus, there are circles on either side of C, which touch but don't cross the curve at some point other than q. Since the total intersection number of the two curves cannot exceed $(3)(2) = 6$, this circle is tangent at exactly three points. Hence, we find one additional circle for each of the six points q, for a total of 8 tritangents planes. By Proposition 3.1, this is the maximum possible. □

4 Tropical Space Sextics

In this section, we consider tropical space sextics of genus 4 and show that they have similar enumerative properties. We begin with a brief overview of tropical curves. We focus on the notions that are necessary for defining and studying tropical tritangent planes. The interested reader may find a more thorough treatment in [19].

Definition 4.1 A tropical curve is a metric graph Γ embedded in $\mathbb{R}^n$, together with an integer weight function on the edges, such that

- The direction vector of each edge is rational.
- At each vertex, the weighted sum of the primitive integral vectors of the edges around the vertex is zero.

The *genus* of a tropical curve is the first Betti number $\dim H^1(\Gamma, \mathbb{Z})$ of the graph. We assume throughout that the weights on the edges are all one.

Definition 4.2 A tropical curve is of *degree d* if it has d infinite ends in each of the directions $-e_1, -e_2, \ldots, -e_n, e_1+e_2+\cdots+e_n$. A plane curve is of *bidegree* (d_1, d_2) if it has d_1 ends in each of the directions $e_2, -e_2$, and d_2 ends in the directions $e_1, -e_1$.

Example 4.3 The graph in Fig. 6 is a tropical plane curve of degree 3 and genus 1.

Definition 4.4 A *tropical plane* in $\mathbb{R}^3$ is a two-dimensional polyhedral complex with a unique vertex v whose 1-skeleton consists of the rays $v + \mathbb{R}_{\geq 0}(-e_1)$, $v +$

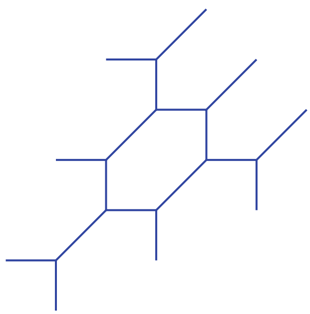

Fig. 6 A tropical elliptic curve

$\mathbb{R}_{\geq 0}(-e_2)$, and $v + \mathbb{R}_{\geq 0}(e_1 + e_2 + e_3)$. The maximal faces are the cones generated by each pair of rays. In other words, it is a translation of the 2-skeleton of the fan of the toric variety $\mathbb{P}^3$.

More generally, a *tropical variety* is a balanced polyhedral complex in $\mathbb{R}^n$; see [19, Definition 3.3.1]. Tropical hypersurfaces, namely tropical varieties of codimension 1, are simply constructed by taking the dual complex of a subdivision of a polytope with integer vertices. The hypersurface is *tropically smooth* if the subdivision is a unimodular triangulation.

For the rest of this section, we assume that $\Bbbk$ is an algebraically closed field endowed with a non-trivial non-archimedean valuation val. For example, $\Bbbk$ could be the field of Puiseux series over $\mathbb{C}$ consisting of all elements of the form

$$x = a_k\, t^{\frac{k}{n}} + a_{k+1}\, t^{\frac{k+1}{n}} + a_{k+2}\, t^{\frac{k+2}{n}} + \cdots$$

for all choices of $k \in \mathbb{Z}$, $n \in \mathbb{N}$, and coefficients $a_i \in \mathbb{C}$. In this case, the valuation is simply given by $\operatorname{val}(x) = \frac{k}{n}$.

Let X be a variety in $(\Bbbk^*)^n$. The *tropicalization* map $\operatorname{trop}\colon X(\Bbbk) \to \mathbb{R}^n$ is defined by $\operatorname{trop}(x_1, x_2, \ldots, x_n) = \big(-\operatorname{val}(x_1), -\operatorname{val}(x_2), \ldots, -\operatorname{val}(x_n)\big)$. The tropicalization of X is the closure in $\mathbb{R}^n$ of $\operatorname{trop} X(\Bbbk)$. The reader will be relieved to know that the tropicalization of a variety is a tropical variety of the same dimension. Moreover, the tropicalization of a generic curve of degree d (resp. bidegree (d_1, d_2)) is a tropical curve of degree d (resp. bidegree (d_1, d_2)). Similarly, the tropicalization of a plane in $(\Bbbk^*)^3$ is a tropical plane in $\mathbb{R}^3$.

Example 4.5 The tropicalization of the degree 3 plane curve defined by

$$t + x + y + xy + t\,x^2 + t\,y^2 + t^2\,x^2y + t^2\,xy^2 + t^4\,x^3 + t^4\,y^3$$

is the tropical curve of degree 3 appearing in Fig. 6. The tropicalization of the curve of bidegree $(1, 2)$ defined by $1 + x + y + t\,xy + t^3\,xy^2 + t^3\,y^2$ is the tropical curve of bidegree $(1, 2)$ depicted in Fig. 7.

In tropical geometry, as in classical algebraic geometry, the intersection of two varieties may have higher than expected dimension. This can happen even when the tropical varieties are tropicalizations of algebraic varieties that do intersect in

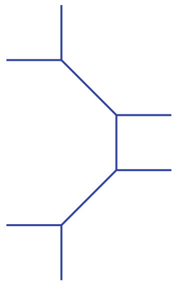

Fig. 7 A tropical $\mathbb{P}^1 \times \mathbb{P}^1$ curve of bidegree (1,2)

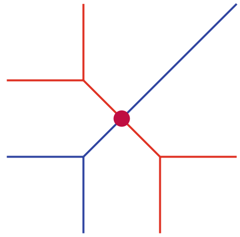

Fig. 8 Two tropical curves with a multiplicity 2 intersection

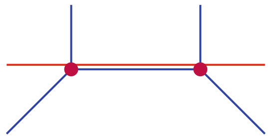

Fig. 9 Two tropical curves with a stable intersection consisting of two points

the expected dimension. The stable intersection of two tropical varieties provides a solution to this problem. Roughly speaking, the stable intersection involves generically perturbing the tropical varieties and taking the limit of their intersection as the perturbations tend to zero. More precisely, whenever cells σ_1, σ_2 of tropical varieties Σ_1, Σ_2 span $\mathbb{R}^n$, their set-theoretic intersection will be a cell in the stable intersection. To assign a weight to this cell, consider the lattices N_1 and N_2 obtained by intersecting σ_1 and σ_2 with the lattice $\mathbb{Z}^n$ in $\mathbb{R}^n$ respectively. The weight assigned to their set-theoretic intersection of σ_1 and σ_2 equals $m(\sigma_1)m(\sigma_2)[\mathbb{Z}^n : N_1 + N_2]$, where $m(\sigma_i)$ denotes the weight of the cell $\sigma_i \in \Sigma_i$, for $1 \leq i \leq 2$, and $[\mathbb{Z}^n : N_1 + N_2]$ denotes the index of the sublattice generated by N_1 and N_2.

Example 4.6 Consider the two tropical curves depicted in Fig. 8. The primitive vectors $(1, 1)$ and $(1, -1)$ span the sublattice obtained by intersecting the appropriate curve with the lattice $\mathbb{Z}^2$. Hence, the multiplicity of the intersection point equals $\det \left[\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right] = 2$, which is the index of the relevant sublattice. This is consistent with tropical Bézout's Theorem because this is an intersection of a line with a $(1, 1)$-curve.

The two tropical curves in Fig. 9 do not intersect properly. However, by slightly perturbing the horizontal line, the two tropical curves intersect at two points with multiplicity 1. Taking the limit as the red curve returns to its original position, we see that the stable intersection has multiplicity 2.

Definition 4.7 Two tropical varieties are *tangent* at a point q if their intersection at q has weight at least 2 or if q is in the interior of a bounded segment of their set-theoretic intersection. They are tritangent to each other if they are tangent at three disjoint places (either points or segments) counted with multiplicity. Two tritangents are *equivalent* if the tangency points are linearly equivalent divisors.

As a consequence, tritangent tropical varieties may meet at three places with multiplicity 2 each, at two places with multiplicity 4 and 2, or at one place with multiplicity 6. For various examples of curves that are tritangent to each other, see Fig. 11. When Γ is the tropicalization of a curve C, then any tangent of C tropicalizes to a tangent of Γ, and a tritangent tropicalizes to a tritangent; see [21, Theorem 6.4].

5 Tropical Divisors, Theta Characteristics, and Tritangent Planes

Divisors on tropical curves are defined analogously to algebraic curves. A *divisor* D on Γ is a finite formal sum $D := a_1 p_1 + a_2 p_2 + \cdots + a_k p_k$, where each a_i is an integer and each p_i is a point of the curve. The *degree* of D is the sum of coefficients $a_1 + a_2 + \cdots + a_k$, and we say that D has a_i *chips* at p_i. A divisor is *effective* if $a_i \geq 0$ for all i. In analogy with the algebraic case, there is a suitable equivalence relation between divisors, and a notion of rank which, roughly speaking, reflects the dimension in which the divisor moves. The curve has a canonical divisor class K_Γ which fits in a tropical Riemann–Roch Theorem

$$r(D) - r(K_\Gamma - D) = \deg(D) - g + 1,$$

where r is the rank of a divisor and g the genus of the tropical curve; see [6, Theorem 1.12]. For a lucid introduction to tropical divisor theory also see [4].

A *tropical theta characteristic* on a tropical curve Γ is defined in exactly the same way as algebraic theta characteristic. It is a divisor class $[D]$ such that $2D \simeq K_\Gamma$. The Jacobian of a tropical curve of genus g is isomorphic to a g-dimensional real torus $\mathbb{R}^g/\mathbb{Z}^g$; see [3, Theorem 3.4] and [8, Sect. 5]. Since theta characteristics are in bijection with its 2-torsion points, there are 2^g theta characteristics. One of them is not effective and the rest are effective. They are easily computed via the following algorithm introduced by Zharkov [25].

To get an effective theta characteristic, fix a cycle γ in Γ. At every point p that locally maximizes the distance from γ, place $a-1$ chips at p, where a is the number of incoming edges at p from the direction of γ. The process is often described pictorially as follows. A fire spreads along the graph at equal speed away from γ. If a is the number of incoming fires at a point p, we place $a - 1$ chips at that point. To obtain the unique non-effective theta characteristic, repeat the same process, but replace γ with the set of vertices of the graph, and place a negative chip at each vertex.

Example 5.1 Let Γ be the curve in Fig. 10, where the infinite ends are omitted. As the genus is 2, we expect the theta characteristics to have degree 1. For the picture on the left, the middle of the bottom horizontal edge is the unique local maximum from the chosen cycle (marked with a point). The corresponding theta

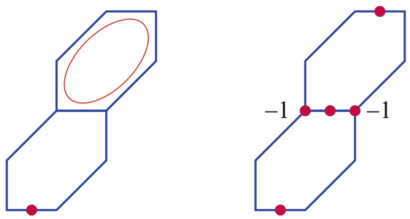

Fig. 10 Two theta characteristics on a curve of genus 2

characteristic has a single chip at that point. For the picture on the right, the middle of each horizontal edge locally maximizes the distance from the vertices. The non-effective theta characteristic of this curve therefore consists of a negative chip at each of the three vertices and a chip at the middle of each horizontal edge. Each of these divisors, when multiplied by two is equivalent to the canonical divisor, so they are half canonical.

Let Γ be a tropical curve of degree 6 and genus 4 in $\mathbb{R}^3$. Its stable intersection with a tropical plane Π is an effective divisor of degree 6. We claim that its rank is 3. Indeed, we can find a tropical plane through any three general points, and any pair of divisors obtained this way is linearly equivalent. By the tropical Riemann–Roch Theorem, a divisor of degree 6 and rank 3 has to be equivalent to the canonical divisor. Consequently, every tritangent plane gives rise to an effective theta characteristic on Γ. By definition, non-equivalent tritangent planes correspond to different theta characteristics. It follows that the number of equivalence classes of tritangent planes is bounded above by the number of effective theta characteristics which is $2^4 - 1 = 15$.

Theorem 5.2 *A smooth tropical sextic curve Γ in $\mathbb{R}^3$ has at most* 15 *classes of tritangent planes. If it is the tropicalization of a sextic C on a smooth quadric in $\mathbb{P}^3$, then it has exactly* 15 *equivalence classes of tritangent planes.*

Proof The first statement follows from the discussion preceding theorem. By Theorem 2.2, the curve C has 120 tritangent planes. If the plane H is tritangent to C, then tropical plane trop(H) is tritangent to Γ. Moreover, by [16, Theorem 1.1], each tropical effective theta characteristic of Γ is the tropicalization of eight odd theta characteristics of C. Since different theta characteristics correspond to non-equivalent tritangent planes, there are 15 distinct classes of planes tritangent to Γ. □

Remark 5.3 It is possible for a tropical sextic to have an infinite continuous family of tritangent planes. However, the tangency points of such a family will consist of linearly equivalent divisors, and as such the corresponding tritangent planes are equivalent. If the sextic is the tropicalization of an algebraic sextic, then each equivalence consists of eight tritangent planes (counted with multiplicity) that can be lifted to tritangent planes of the algebraic sextic.

The proof of Theorem 5.2 relies on the fact that the given tropical curve arises as the tropicalization of an algebraic curve. Nevertheless, we expect this result to be true more generally.

Conjecture 5.4 Every tropical sextic curve of genus 4 in $\mathbb{R}^3$ has exactly 15 equivalence classes of tritangent planes.

We now explore the tropical analogue of the relation between quadric surfaces and $\mathbb{P}^1 \times \mathbb{P}^1$ described in Remark 2.3. To examine the analogous statement in tropical geometry, recall that a smooth tropical quadric in $\mathbb{R}^3$ is dual to a unimodular triangulation of the 3-simplex with vertices $(0, 0, 0)$, $(2, 0, 0)$, $(0, 2, 0)$, and $(0, 0, 2)$. From the proof of [19, Theorem 4.5.8], such a triangulation has a unique interior edge corresponding to a unique bounded face of the quadric. This face can be seen as a model for tropical $\mathbb{P}^1 \times \mathbb{P}^1$. More precisely, we have the following result.

Lemma 5.5 *Let Σ be a tropical smooth quadric surface in $\mathbb{R}^3$. If R is a rectangle in $\mathbb{R}^2$, Then there is an affine linear map from R onto a parallelogram in the bounded face of Σ inducing a bijection between curves of bidegree (d, d) in $\mathbb{R}^2$, whose bounded edges are all contained in R, and curves of degree $2d$ in $\mathbb{R}^3$ that are contained in Σ and have their bounded edges contained in the parallelogram.*

Proof Let F be the unique bounded face of Σ. By [19, Theorem 4.5.8], there are two tropical lines through each point of F that are fully contained in Σ. The bounded edge of each is fully contained in F and is parallel to one of two directions. We denote these two directions by u_1 and u_2. These directions are determined by F and do not depend on the point. Every ray in F that is parallel to u_i, for $1 \leq i \leq 2$, can be extended past the boundary of F by attaching infinite ends in two of the directions $\{-e_1, -e_2, -e_3, e_1 + e_2 + e_3\}$ and a ray parallel to $-u_i$ can be extended by attaching ends in the two remaining directions.

Let $\varphi \colon R \to F$ be the given affine linear map. The map φ sends a vertex of R to point $p \in F$ and the two adjacent vertices to $p + \lambda u_1, p + \lambda u_2$ where $\lambda \in \mathbb{R}$. If Γ is the (d, d)-curve in $\mathbb{R}^2$ whose bounded edges are contained in R, then each of the infinite end is mapped by φ to rays that are parallel to $\pm u_1$ or $\pm u_2$. By extending all the rays emanating from $\varphi(\Gamma \cap R)$, we get a curve of degree $2d$ in $\mathbb{R}^3$ contained in Σ. ☐

This lemma leads to an alternative formulation of Conjecture 5.4.

Corollary 5.6 *For tropical sextics whose bounded edges are contained in the bounded face of a smooth tropical quadric, Conjecture 5.4 is equivalent to saying that every $(3, 3)$-curve in $\mathbb{R}^2$ has 15 classes of tritangent $(1, 1)$-curves.*

Proof As in Lemma 5.5, let Γ be a $(3, 3)$-curve in $\mathbb{R}^2$ mapping to a sextic in $\mathbb{R}^3$ under an affine linear map φ. Every $(1, 1)$-curve that is tritangent to Γ maps to a conic curve in $\mathbb{R}^3$ that is tritangent to $\varphi(\Gamma)$ and contained in Σ. By construction, the conic curve is not contained in a tropical line or in any of the standard planes in $\mathbb{R}^3$. Therefore, we can find three general points on it that span a unique tropical plane. This plane contains the conic curve and is tritangent to $\varphi(\Gamma)$. ☐

Example 5.7 Figure 11 shows 15 equivalence classes of tritangent $(1, 1)$-curves to a tropical $(3, 3)$-curve in $\mathbb{R}^2$. By Corollary 5.6, this curve corresponds to a tropical sextic in $\mathbb{R}^3$ reaching the maximal number of tritangent planes. To find each tritangent curve, we choose a non-trivial cycle in the graph, compute the corresponding theta characteristic via Zharkov's algorithm, and find a $(1, 1)$-curve through it.

We stress that some of the odd theta characteristics, in fact, give rise to infinitely many tritangent $(1, 1)$-curves, for instance the third tritangent in the second row in Fig. 11 when counting from the top left. However, as the tangency points of these different $(1, 1)$-curves are equivalent divisors, they are all in the same equivalence class; see Remark 5.3.

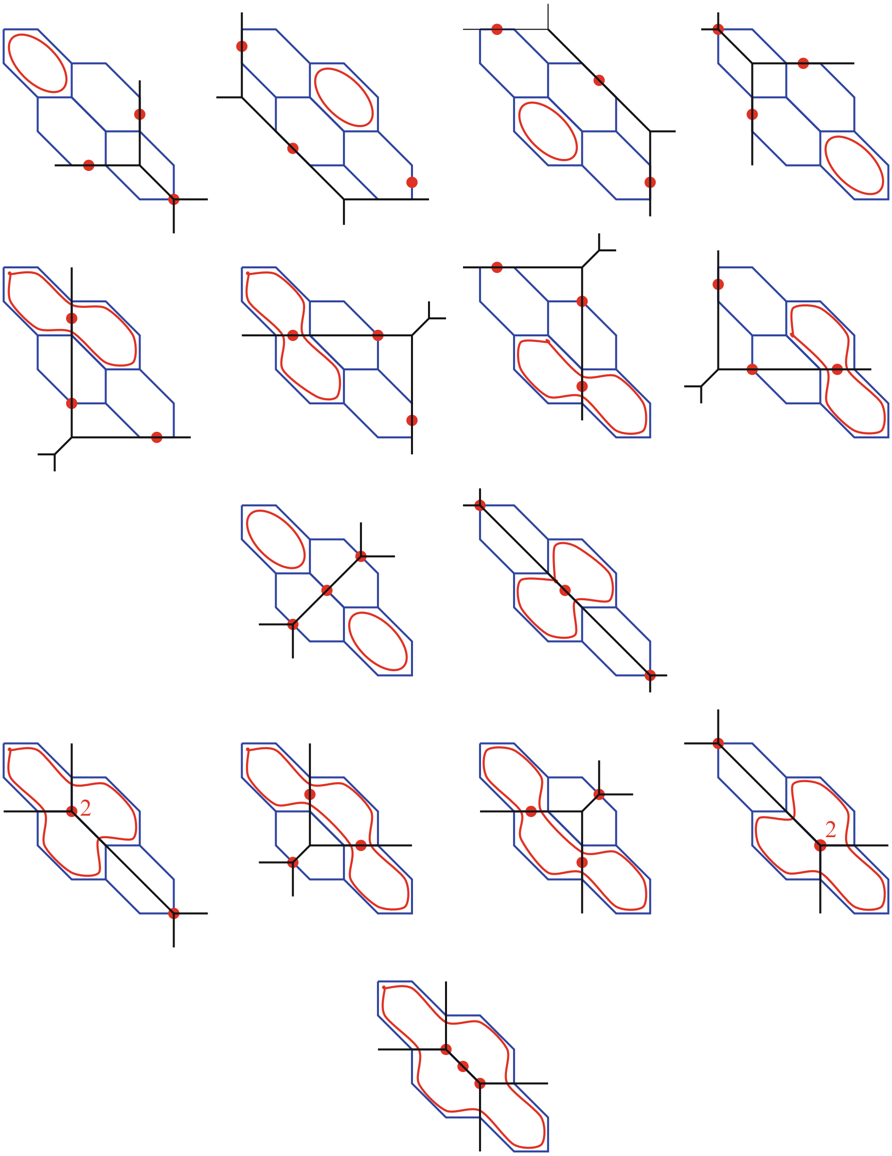

Fig. 11 The 15 tritangents to a tropical sextic

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Theta Characteristics of Tropical K_4-Curves

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Abstract A K_4-curve is a smooth proper curve X of genus 3 over a field with valuation whose Berkovich skeleton Γ is a complete graph on four vertices. The curve X has 28 effective theta characteristics—the 28 bitangents to a canonical embedding—while Γ has exactly seven effective tropical theta characteristics, as shown by Zharkov. We prove that the 28 effective theta characteristics of a K_4-curve specialize to the theta characteristics of its minimal skeleton in seven groups of four.

MSC 2010 codes: 14T05 (primary), 14C20, 14H45, 14H50 (secondary)

1 Introduction

This paper provides a rigorous link between the classical and tropical theories of theta characteristics for a special class of algebraic curves that we call K_4-curves. Fix an algebraically closed field K, complete with respect to a nontrivial nonarchimedean valuation. A K_4-*curve* is an algebraic curve over K whose Berkovich skeleton is a metric complete graph on four vertices. These curves provide a convenient window into the study of theta characteristics and their tropicalizations.

It is well known that every smooth plane quartic has exactly 28 distinct bitangents. Abstractly, these correspond to the 28 effective theta characteristics on a genus 3 nonhyperelliptic curve under its canonical embedding. In [33], building on [27], Zharkov developed a theory of theta characteristics in tropical geometry. In this framework, a tropical curve (also known as a metric graph) of genus 3 has exactly seven effective theta characteristics. Zharkov's theory, while compelling, is "synthetic": it predated a precise connection to classical algebraic curves, as far as

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we know. The specialization theorem in tropical geometry [1] has since provided a connection, showing that theta characteristics of a curve X do indeed tropicalize to theta characteristics of its skeleton Γ. Our main theorem provides a new rigorous connection between the two theories of theta characteristics past the general setup of specialization:

Theorem 1.1 *If X is a K_4-curve with Berkovich skeleton Γ, then the* 28 *effective theta characteristics of X specialize to the effective theta characteristics of Γ in seven groups of four.*

This answers a question in [3] in the special case of K_4-curves. It complements the purely combinatorial analysis in that paper of smooth plane tropical quartics and their bitangents, which are shown to fall in seven equivalence classes. The techniques that we use span both abstract tropical curves (notably the nonarchimedean Slope Formula of [5]) and embedded tropical curves (notably tropical intersection theory [28, 29]). Better yet, these two realms interact in an interesting way in some of our arguments. In addition, the key input that we use from classical algebraic geometry is a calculation of the limits of the 28 bitangents in a family of plane quartics specializing to a union of four lines, carried out by Caporaso–Sernesi [12, Sect. 3.4].

In fact, in the language of classical algebraic geometry, our theorem amounts to the following. Suppose we have a one-parameter family of plane quartics degenerating to a union of four non-concurrent lines in $\mathbb{P}^2$. This is exactly the data of a K_4-curve; see Theorem 3.2. Sometimes it is the case that a bitangent contact point will specialize into one of the six nodes of the special fibre. In that case, Theorem 1.1 gives refined information about that specialization. It says that after a sequence of blowups replacing the node with a chain of rational curves, such that the bitangent contact point now specializes to a smooth point of the special fibre, Zharkov's algorithm describes where on that chain the specialization occurs.

This particular interpretation aside, it is natural to ask whether Theorem 1.1 should be true for all curves of genus 3. The answer is yes. Since this paper first appeared as a preprint, Jensen and Len proved a significant generalization of Theorem 1.1, showing in particular that for a curve X of genus g, every effective theta characteristic of Γ lifts to 2^{g-1} even and 2^{g-1} odd theta characteristics on X [20]; this was previously obtained in the case of hyperelliptic curves by Panizzut [30]. Our techniques are different: Jensen–Len use the Weil pairing on the Jacobian of the curve, while we use the interaction betwen abstract and embedded tropical curves, and tropical intersection theory. We also refer the reader to the article [17] which contains a higher-dimensional analogue to Theorem 1.1, counting tropical tritangent planes to space sextic curves, and to forthcoming work of Len–Markwig that deals with liftings of tropical bitangents to plane tropical curves.

There are some interesting combinatorics and computations specific to the case of K_4-curves, as shown in Sects. 4–5. In Theorem 4.2, we prove a statement that is stronger than Theorem 1.1 for a generic honeycomb curve; see Definition 4.1. The *bitangents* of such a curve tropicalize exactly in seven groups of four. This is stronger than Theorem 1.1 in the sense that two bitangents can have different tropicalizations in $\mathbb{R}^2$ even while their contact points retract to the same place on the skeleton. Indeed, in Sect. 5, we give exactly such an example, of a nongeneric honeycomb plane quartic whose bitangents tropicalize in groups of 2 and 4. Furthermore, we compute the beginnings of the Puiseux expansions of the 28 bitangents in this example. This section may be of independent interest to computational algebraic geometers, because it showcases some of the difficulties of computing over the field of Puiseux series, and how we used tropical techniques to carry out a computation which a priori is not tropical at all.

2 Preliminaries

In this section, we develop preliminary notions on tropical curves and tropicalization, semistable models, theta characteristics, and specialization, all of which interact in our results. We refer the reader who is interested in more details on these topics to the survey article [2] and the textbook [25].

Graphs, Metric Graphs, and Tropical Curves All graphs in this paper are finite and connected; multiple edges and loops are allowed. We write $G := (V, E)$ for a graph with vertices $V = V(G)$ and edges $E = E(G)$. The *degree* of a vertex $v \in V(G)$, denoted $\deg(v)$, is the number of edges in $E(G)$ incident to v. For $n \in \mathbb{N}$, the *complete graph* K_n is the graph on n vertices in which each pair of distinct vertices is connected by an edge.

A *metric graph* Γ is a graph G together with a length function $\ell\colon E(G) \to \mathbb{R}_{>0}$. By identifying an edge $e \in E(G)$ with the interval $[0, \ell(e)] \subset \mathbb{R}$, we view Γ as a one-dimensional CW complex, and we will freely identify Γ with this topological space. The *genus* of Γ is $b_1(\Gamma) := |E(G)| - |V(G)| + 1$. An *abstract tropical curve* is a vertex-weighted metric graph, that is a metric graph (G, ℓ) together with a weight function $w\colon V(G) \to \mathbb{Z}_{\geq 0}$. A *stable* tropical curve is an abstract tropical curve satisfying the condition that $\deg(v) \geq 2w(v) - 2$ for all $v \in V(G)$. If Γ is a metric graph all of whose edge lengths lie in some subgroup $\Lambda \subseteq \mathbb{R}$, then Γ is Λ*-rational*. In this situation, we say that $p \in \Gamma$ is a Λ-rational point if the distance from p to any (equivalently every) vertex $v \in \Gamma$ is in Λ.

There is a combinatorial theory of divisors on metric graphs that mirrors, and has a precise relationship with, the classical theory of divisors on algebraic curves. This theory was first developed by Baker–Norine [4]. Here we only give a bare-bones account of the part that we need; see [4, 16, 27] for more information. A *divisor* on a metric graph Γ is an element of the free abelian group $\operatorname{Div}\Gamma$ generated by the points $p \in \Gamma$, where Γ is regarded as a topological space, so the point p may lie in the interior of an edge. There is a *degree* map $\operatorname{Div}\Gamma \to \mathbb{Z}$ sending $\sum a_p p$ to $\sum a_p$. Write $\operatorname{Div}^d \Gamma$ for the set of divisors of degree d. We say $D = \sum a_p p$ is *effective* if $a_p \geq 0$ for all p, and the set of all the effective divisors is $\operatorname{Div}_{\geq 0}\Gamma$. The canonical divisor on Γ is $K_\Gamma := \sum_{p\in\Gamma} \big(\deg(p) - 2\big)\, p$, where $\deg(p) := 2$ if p is not a vertex.

A *rational function* on Γ is a continuous function f on Γ that is piecewise-linear with integer slopes on each edge. The divisor of f is

$$\operatorname{div} f := \sum_{p \in \Gamma} (\text{sum of outgoing slopes at } p)\, p\,.$$

Such a divisor is called *principal*. We say that $D \sim D'$ are *linearly equivalent* divisors if $D - D'$ is principal, and we define the *Picard group* $\operatorname{Pic} \Gamma := \operatorname{Div} \Gamma / \sim$. The *rank* of a divisor D is

$$r(D) := \max \left\{ k \in \mathbb{Z} : \begin{array}{l} \text{for all } E \in \operatorname{Div}^k_{\geq 0} \Gamma, \text{ there exists } E' \in \operatorname{Div}_{\geq 0} \Gamma \\ \text{such that } D - E \sim E' \end{array} \right\}.$$

Since the condition holds vacuously for $k = -1$, we see that $r(D) = -1$ if and only if D is not equivalent to any effective divisor. We say that an effective divisor E on a metric graph Γ is *rigid* if it is the unique effective divisor in its linear equivalence class. The following lemma characterizes rigidity; we will use it and Corollary 2.2 to construct lifts of canonical divisors in Sect. 3.

Lemma 2.1 *Let D be an effective divisor on a metric graph Γ. If $D = \sum_{i=1}^k a_i p_i$ for some $a_i > 0$, then D is rigid if and only if, for every nonempty closed subset $S \subseteq \Gamma$ with $\partial S \subseteq \{p_1, p_2, \ldots, p_k\}$, there is a $p_i \in S$ with $\operatorname{outdeg}_S(p_i) > a_i$, where $\operatorname{outdeg}_S(p)$ is the number of edges at p that leave S.*

Proof If there is a closed subset $S \subseteq \Gamma$ such that $\partial S \subseteq \{p_1, p_2, \ldots, p_k\}$ and $\operatorname{outdeg}_S(p_i) \leq a_i$ for all $p_i \in S$, then for sufficiently small $\epsilon > 0$, there is a rational function f on Γ taking on value 0 on S, decreasing to value $-\epsilon$ along each edge leaving S, and taking on value $-\epsilon$ everywhere else. It follows that $D + \operatorname{div}(f)$ is effective. Conversely, suppose that D is not rigid. If we choose $f \neq 0$ such that $D + \operatorname{div}(f)$ is effective, then $S := \{p \in \Gamma : f \text{ is maximized at } p\}$ has $\partial S \subseteq \{p_1, p_2, \ldots, p_k\}$ and $\operatorname{outdeg}_S(p_i) \leq a_i$ for all $1 \leq i \leq k$. □

The next corollary is an immediate consequence of Lemma 2.1.

Corollary 2.2 *Let D be an effective divisor on a metric graph Γ.*

1. *If $D := a\,p$ for some $a > 0$ and the space $\Gamma \setminus \{p\}$ is connected, then the divisor D is rigid if and only if $a < \deg(p)$.*
2. *If $D := p_1 + p_2 + \cdots + p_d$ for points $p_1, p_2, \ldots, p_d$ lying on the interiors of d distinct edges whose removal does not disconnect Γ, then D is rigid.* □

Remark 2.3 For readers who are familiar with the notion of *q-reducedness*, we observe that D is rigid if and only if it is q-reduced for all $q \in \Gamma$. The only if direction is clear from the definition. For the other direction, suppose that D is not rigid. If we choose $f \neq 0$ such that $D + \operatorname{div}(f)$ is effective, then D is not reduced with respect to any q in the boundary of $\{p \in \Gamma : f \text{ is minimized at } p\}$.

Semistable Models Throughout this paper, K denotes an algebraically closed field that is complete with respect to a nontrivial nonarchimedean valuation $\operatorname{val}: K^* \to \mathbb{R}$. Let $\Lambda := \operatorname{val}(K^*) \subseteq \mathbb{R}$ denote the value group of K, let R denote the valuation ring of K with maximal ideal m, and let $k := R/m$ be the residue field. The corresponding polynomial rings are $S_K := K[x_0, x_2, \dots, x_n]$, $S_R := R[x_0, x_2, \dots, x_n]$, and $S_k := k[x_0, x_2, \dots, x_n]$. For $a \in R$, we write $\overline{a}$ for the reduction of a, that is the image of a under the canonical quotient map $R \to k$, and we write $\overline{f} \in S_k$ for the coefficient-wise reduction of a polynomial $f \in S_R$.

Let X be a finite-type scheme over K. By an *algebraic model* for X, we mean a flat and finite type scheme $\mathfrak{X}$ over R whose generic fibre is isomorphic to X. We next describe the basic source of algebraic models for subvarieties of projective space.

Lemma 2.4 *Let $I \subset S_K$ be a homogeneous ideal and let $X := \operatorname{Proj} S_K/I$ be the corresponding projective scheme. If $I_R := I \cap S_R$, then $\mathfrak{X} := \operatorname{Proj} S_R/I_R$ is an algebraic model for X. Furthermore, if I is generated by the polynomial $f \in S_R$ which does not reduce to 0 in S_k, then $I \cap S_R = fS_R$, so $I \cap S_R$ is the principal ideal generated by f.*

Proof Given a closed subscheme of $\mathbb{A}^n_K$ defined by the ideal $I \subset K[x_1, x_2, \dots, x_n]$, Proposition 4.4 in [19] shows that $I \cap R[x_1, x_2, \dots, x_n]$ defines a flat scheme with general fibre isomorphic to X. Hence, it suffices to check the first part locally on the affine open subsets defined by $x_i \neq 0$ for $0 \leq x \leq n$; see [19, Remark 4.6]. On such an affine open subset, the defining ideal of X is $I|_{x_i=1} \subset K[x_1, x_2, \dots, x_{i-1}, x_{i+1}, x_{i+2}, \dots, x_n]$, for which $I|_{x_i=1} \cap R[x_1, x_2, \dots, x_{i-1}, x_{i+1}, x_{i+2}, \dots, x_n]$ is the equation of a model. Since we have $I|_{x_i=1} \cap R[x_1, x_2, \dots, x_{i-1}, \hat{x}_i, x_{i+1}, x_{i+2}, \dots, x_n] = I_R|_{x_i=1}$, we deduce that $\operatorname{Proj} S_R/I_R$ gives a model for X.

For the second part, consider an ideal I generated by $f \in S_R$ with $\overline{f} \neq 0$ and suppose $g \in S_K$ satisfies $fg \in S_R$. We claim that $g \in S_R$. Suppose not, and pick some scalar $a \in K$ with positive valuation such that $ag \in S_R$ and $\overline{ag} \neq 0$. Since $\overline{f} \neq 0$ and $\overline{ag} \neq 0$, we have $\overline{afg} \neq 0$. However, we have $\overline{a} = 0$ which gives a contradiction. □

Let X be a smooth proper curve of genus g over K. A *semistable model* for X is a proper model $\mathfrak{X}$ over R whose special fibre $\mathfrak{X}_k := \mathfrak{X} \times_R k$ is a nodal curve over k. If the special fibre $\mathfrak{X}_k$ is a stable curve, then we say that $\mathfrak{X}$ is a *stable model*. For all $g \geq 2$, stable models exist; see [10].

To any semistable model $\mathfrak{X}$ over R, we associate a tropical curve $\Gamma_{\mathfrak{X}}$ as follows. The vertices v_i of $\Gamma_{\mathfrak{X}}$ correspond to the irreducible components C_i of $\mathfrak{X}$ and there is an edge between two vertices if and only if the corresponding components intersect. For a node p of $\mathfrak{X}_k$, lying on components C_i and C_j, the completion of the local ring $\mathscr{O}_{\mathfrak{X},p}$ is isomorphic to $R[\![x, y]\!]/(xy - \alpha)$ for some $\alpha \in m$, and $\operatorname{val}(\alpha) \in \Lambda_{>0}$ is independent of all choices. Hence, the length of the edge between the vertices v_i and v_j is simply $\operatorname{val}(\alpha)$. Thus, the metric graph $\Gamma_{\mathfrak{X}}$ has edge lengths in the value group.

Classical and Tropical Theta Characteristics Let X be a smooth proper curve of genus g over K. A *theta characteristic* of X is a divisor class $[D] \in \operatorname{Pic}^{g-1}(X)$ such that $2[D] = [K_X]$. It is *effective* if D is linearly equivalent to an effective divisor. The theta characteristic $[D]$ is *odd* or *even* if $\dim_K H^0\big(X, \mathrm{O}_X(D)\big)$ is odd or even. It is well-known that X has exactly 2^{2g} theta characteristics, and of these, exactly $(2^g - 1)2^{g-1}$ are odd; see [15, Sect. 5.1.1].

When X is a nonhyperelliptic curve of genus 3, the effective theta characteristics of X are precisely its odd theta characteristics, of which there are $28 = (2^3 - 1)(2^2)$. These are precisely divisor classes representable as $[P + Q]$ such that $2P + 2Q \sim K_X$. In other words, the points P and Q are contact points of the 28 bitangent lines to the curve X under its canonical embedding as a smooth plane quartic in $\mathbb{P}^2$.

Now, suppose that Γ is a metric graph of genus g. A *theta characteristic* of Γ is a divisor class $[D] \in \mathrm{Pic}^{g-1}\,\Gamma$ such that $2[D] = [K_\Gamma]$. Zharkov [33] gives an algorithm for computing theta characteristics of Γ and shows that Γ has exactly 2^g theta characteristics, all but one of which are effective. His description of the effective theta characteristics can be reformulated as follows. The effective theta characteristics of Γ are in bijection with the $2^g - 1$ nonempty Eulerian subgraphs of Γ, that is, the subgraphs of Γ that have everywhere even valence. More precisely, if $S \subseteq \Gamma$ is an Eulerian subgraph, then the distance function $d(S, -)$ produces an orientation of $\Gamma \setminus S$ in which segments are directed away from S. This partial orientation, taken together with a cyclic orientation on S, orients Γ. With this orientation, the divisor $\sum_{p\in\Gamma} \big(\mathrm{val}_+(p) - 1\big)\, p$ represents an effective theta characteristic, where $\mathrm{val}_+(p)$ is indegree. The divisor arising from different Eulerian subgraphs are shown to be pairwise linearly inequivalent.

Specialization The theory of specializing divisors from curves to graphs was developed in [1]. We recall the relevant facts here, starting with skeletons of Berkovich curves [7, 8, 32].

Let X be a smooth proper curve of genus $g \geq 2$ over K. Fix a semistable model $\mathfrak{X}$ for X and write $\Gamma_{\mathfrak{X}}$ for its associated metric graph. When there is no ambiguity, we simply write Γ for the associated metric graph. There is a canonical embedding of $\Gamma_{\mathfrak{X}}$ in the Berkovich analytification X^{an} of X such that X^{an} admits a retraction $\tau\colon X^{\mathrm{an}} \to \Gamma_{\mathfrak{X}}$. We call $\Gamma_{\mathfrak{X}}$ a *skeleton* for X. If $g \geq 2$, then X has a *minimal skeleton* $\Gamma_{\mathfrak{X}}$ corresponding to a stable model $\mathfrak{X}$ for X; see [5, Sect. 4.16].

There is a natural inclusion $X(K) \hookrightarrow X^{\mathrm{an}}$. The composition $X(K) \hookrightarrow X^{\mathrm{an}} \twoheadrightarrow \Gamma$ induces, by linearity, a *specialization map* $\tau_*\colon \mathrm{Div}(X) \to \mathrm{Div}(\Gamma)$. The specialization map is, by construction, a degree-preserving group homomorphism that sends effective divisors to effective divisors and principal divisors to principal divisors; see [32] or [5, Theorem 5.15]. Hence, τ_* descends to a map $\mathrm{Pic}(X) \to \mathrm{Pic}(\Gamma)$ which, by a slight abuse of notation, will also be denoted by τ_*.

Lemma 2.5 *The specialization map τ_* takes effective theta characteristics of X to effective theta characteristics of Γ.*

Proof Since Lemma 4.19 in [1] shows that the canonical class on X specializes to the canonical class on Γ, the claim follows from the stated properties of τ_*. □

Tropicalizations Suppose $V \subseteq (K^*)^n$ is a subvariety of an algebraic torus. The *tropicalization* of V is the closure in the usual Euclidean topology on $\mathbb{R}^n$ of the set $\big\{\big(\mathrm{val}(x_1), \mathrm{val}(x_2), \ldots, \mathrm{val}(x_n)\big) : (x_1, x_2, \ldots, x_n) \in V\big\} \subseteq \mathbb{R}^n$. This set can be equipped with the structure of a polyhedral complex and positive integer multiplicities on top-dimensional faces, so that the result is a balanced complex of dimension equal to the dimension of V; see [25] for the details.

If $V \subseteq \mathbb{P}^n$, then we may consider the torus $T = \{(x_0 : x_1 : \cdots : x_{n-1} : 1)\} \subset \mathbb{P}^n$ and, when we refer to the tropicalization of V, we mean the tropicalization of $V \cap T$, the part inside the torus. For example, consider a line $Ax + By + z = 0$ in $\mathbb{P}^2$, with $A, B \in K^*$. The tropicalization of this line is the one-dimensional polyhedral complex consisting of a point at $\big(-\operatorname{val}(A), -\operatorname{val}(B)\big) \in \mathbb{R}^2$ and three rays emanating from this point in directions $(1, 0)$, $(0, 1)$, and $(-1, -1)$. In this situation, we say that $\big(-\operatorname{val}(A), -\operatorname{val}(B)\big)$ is the *centre* of the tropical line.

3 Plane Quartics in K_4-Form

Let $X := \mathrm{V}(f) \subset \mathbb{P}^2$ be a smooth quartic curve over K. After scaling, we may assume that the minimum valuation of the 15 coefficients of f is zero. Hence, Lemma 2.4 shows that the quartic polynomial $f \in R[x, y, z]$ defines an algebraic model $\mathfrak{X}$ for X. We start with the main definition in this section.

Definition 3.1 We say that a smooth quartic $X := \mathrm{V}(f)$ is in K_4*-form* if the special fibre of the model $\mathfrak{X}$ for X described in Lemma 2.4 is $xyz(x + y + z) = 0$.

The following theorem characterizes the curves that have embeddings in K_4-form.

Theorem 3.2 *If X is a smooth proper curve of genus* 3 *over K, then the following are equivalent:*

1. *X has an embedding as a smooth plane quartic in K_4-form.*
2. *X has an embedding as a smooth plane quartic whose tropicalization in $\mathbb{R}^2$ has a strong deformation retract to a metric K_4.*
3. *The minimal skeleton of X is a metric K_4.*

If these equivalent conditions hold, we will say that X is a K_4-curve.
We highlight the differences between the parts: the first is an assertion about the existence of a certain algebraic model for X, the second is an assertion about the existence of a tropicalization satisfying a topological criterion, and the third is an assertion about the topology of its minimal Berkovich skeleton (an intrinsic property).

Proof

2. $\Longrightarrow$ 1.: Suppose that X has an embedding as a smooth plane quartic $C := \mathrm{V}(f) \subset \mathbb{P}^2_K$ whose tropicalization $\operatorname{trop}(C)$ strongly deformation retracts to a K_4. It follows that $\mathbb{R}^2 \setminus \operatorname{trop}(C)$ has exactly three bounded regions and each pair of bounded regions is separated by an edge of $\operatorname{trop}(C)$. Let $\tilde{f} := f|_{z=1}$ be the dehomogenization, write $\tilde{f} = \sum c_{i,j} x^i y^j$ for some $c_{i,j} \in K$, and consider the Newton polygon $\operatorname{Newt}(\tilde{f}) := \operatorname{conv}\{(i, j) \in \mathbb{Z}^2 : c_{i,j} \neq 0\}$ of $\tilde{f}$. Since $\deg(f) = 4$, we see that $\operatorname{Newt}(\tilde{f})$ is a subpolytope of the triangle $\operatorname{conv}\{(0, 0), (4, 0), (0, 4)\}$.

The Newton subdivision $\Delta(\tilde{f})$ is obtained by projecting the lower faces of the polytope $P(\tilde{f}) = \operatorname{conv}\{(i, j, \operatorname{val}(c_{ij}))\} \subset \mathbb{R}^3$ onto the polytope $\operatorname{Newt}(\tilde{f}) \subset \mathbb{R}^2$. Proposition 3.1.6 in [25] proves that $\Delta(\tilde{f})$ is a polyhedral complex that is dual to the tropical curve $\operatorname{trop}(\mathrm{V}(\tilde{f}))$. The three bounded regions of $\mathbb{R}^2 \setminus \operatorname{trop}(C)$ correspond to interior vertices of $\operatorname{Newt}(\tilde{f})$, so these interior vertices must be $(1, 1)$, $(2, 1)$, and $(1, 2)$. Each pair of bounded regions in $\mathbb{R}^2 \setminus \operatorname{trop}(C)$ is separated by an edge in $\operatorname{trop}(C)$, so there is an edge joining each pair of interior vertices in $\Delta(\tilde{f})$. Hence, the triangle $T := \operatorname{conv}\{(1, 1), (2, 1), (1, 2)\}$ is a face of $\Delta(\tilde{f})$. After a suitable projective transformation of f, we may assume that the triangle T is the unique lowest face of $P(\tilde{f})$; all coefficients of f have nonnegative valuation and the coefficients with valuation 0 are precisely $c_{1,1}$, $c_{2,1}$, and $c_{1,2}$. Lemma 2.4 implies that the polynomial f defines a model for X whose special fibre has the form $ax^2yz + bxy^2z + cxyz^2 = xyz(ax + by + cz) = 0$ for some $a, b, c \in k^*$. By rescaling, we may assume $a = b = c = 1$.

1. $\Longrightarrow$ 3.: A plane embedding of X in K_4-form gives, by definition, a stable model $\mathfrak{X}$ of X whose skeleton is a metric K_4.

3. $\Longrightarrow$ 2.: Given a stable model $\mathfrak{X}$ for X, the special fibre $\mathfrak{X}_k$ has four irreducible components C_1, C_2, C_3, C_4, each isomorphic to $\mathbb{P}^1_k$, that intersect pairwise. Let Γ be the minimal skeleton of X. If $\Omega_{\mathfrak{X}/R}$ is the pushforward to $\mathfrak{X}$ of the sheaf of relative Kähler differentials on the smooth locus of $\mathfrak{X}$ over R, then Lemma 2.11 in [22] shows that $\Omega_{\mathfrak{X}/R}$ is a line bundle whose restriction to X is the canonical bundle and whose restriction to $\mathfrak{X}_k$ is the relative dualizing sheaf. Since hyperellipticity is preserved under passing to the skeleton [1] and Γ is not hyperelliptic [13], we see that X is not hyperelliptic. Thus, the general fibre of $\mathfrak{X}$ is canonically embedded as a smooth plane quartic over K, and the special fibre is embedded as a stable curve in $\mathbb{P}^2_k$. We deduce that the restriction of $\Omega_{\mathfrak{X}/R}$ to each component C_i is isomorphic to $\omega_{C_i}(p_1^i + p_2^i + p_3^i) \cong \mathscr{O}_{C_i}(1)$, where p_1^i, p_2^i, and p_3^i are the nodes on C_i. Hence, $\mathfrak{X}_k$ is the union of four lines in $\mathbb{P}^2_k$, no three of which are concurrent. All such quadruples of lines are projectively equivalent, we may assume they are $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 0$. It follows that $\mathfrak{X}$ is defined by a homogeneous quartic polynomial $f \in R[x, y, z]$ and $\mathfrak{X}_k = \mathrm{V}(x^2yz + xy^2z + xyz^2)$. Moreover, the triangle $T = \operatorname{conv}\{(1, 1), (2, 1), (1, 2)\}$ is a face of the Newton subdivision $\Delta(\tilde{f})$, where $\tilde{f} = f|_{z=1}$ is the dehomogenization of f and $\Delta(\tilde{f})$ is the subdivision of its Newton polygon $\operatorname{Newt}(\tilde{f})$.

Let v be the vertex of $\operatorname{trop}(\mathrm{V}(f))$ dual to T. Since $\mathrm{V}(f)$ is smooth, it does not contain the coordinate lines $x = 0$, $y = 0$, or $z = 0$ as a component, so $\operatorname{Newt}(\tilde{f})$ meets the lines $i = 0$, $j = 0$, and $i + j = 4$. The polynomial f has at least one of the three terms x^4, x^3y, and x^3z in its support, otherwise $\mathrm{V}(f)$ would be singular at $(1 : 0 : 0)$. Thus, the Newton polygon $\operatorname{Newt}(\tilde{f})$ contains at least one of the points $(4, 0)$, $(3, 1)$, or $(3, 0)$. By symmetry, we see that $\operatorname{Newt}(\tilde{f})$ contains at least one of the points $(0, 4)$, $(1, 3)$ or $(0, 3)$, and at least one of the points $(0, 0)$, $(1, 0)$, or $(0, 1)$. As this Newton polygon contains these three other points, the triangle T must lie in the interior of $\operatorname{Newt}(\tilde{f})$. It follows that a cycle remains when v is deleted from $\operatorname{trop}(\mathrm{V}(f))$. Furthermore, v must be attached to that cycle along the three edges of $\operatorname{trop}(\mathrm{V}(f))$ dual to the three edges of T. Therefore, the tropical curve $\operatorname{trop}(\mathrm{V}(f))$ contains a metric K_4 inside its bounded subcomplex. It has the homotopy type of a K_4, because the rank of $H_1(\operatorname{trop}(\mathrm{V}(f)), \mathbb{Z})$ cannot exceed the number of interior lattice points of $\operatorname{Newt}(\tilde{f})$, which is exactly 3. □

Remark 3.3 As Proposition 2.3 in [3] establishes, not all metric graphs of genus 3 can be realized as a subcomplex of the dual complex to a Newton subdivision of a quartic, so graph K_4 is crucial in Theorem 3.2. Moreover, we are not claiming that the embedding of $X \cong C \subset \mathbb{P}^2_K$ may be chosen so that $\operatorname{trop}(C)$ is tropically smooth. In fact, [11] gives inequalities on the edge lengths of a metric K_4 that are necessary and sufficient for it to be embeddable in $\mathbb{R}^2$ as part of a tropically smooth plane quartic.

Remark 3.4 Theorem 3.2 suggests the following algorithmic problems. Given a smooth plane quartic $f := \sum c_{i,j,k}x^i y^j z^k$, how can we tell whether f defines a K_4-curve. If it does, how can one read off its six edge lengths? In principle, one could attempt to compute a semistable model and local equations for the nodes in the special fibre. However, Theorem 3.2 opens up the possibility of finding a more explicit algorithm, using tropical techniques, in the special case of K_4-curves. Indeed, the theorem shows that being a K_4-curve is equivalent to having a projective reembedding in $\mathbb{P}^2_K$ whose Newton subdivision contains the triangle $T = \operatorname{conv}\{(1,1),(2,1),(1,2)\}$, and this property may be encoded explicitly as a system of inequalities on the valuations of the 15 coefficients defining a quartic curve. The general algorithmic question of computing the abstract tropical curve or minimal Berkovich skeleton associated to a nonarchimedean curve is an interesting one; see [9] for more on the status of this problem as well as its relationship with computing tropical Jacobians.

Proposition 3.5 *Let $C := \mathrm{V}(f) \subset \mathbb{P}^2_K$ be a smooth quartic in K_4-form. Suppose the minimum valuation of the coefficients of f is 0, so that f defines a model $\mathfrak{C}$ for C over R. Consider the 28 bitangents $l_1, l_2, \ldots, l_{28}$ of C, and let $L_1, L_2, \ldots, L_{28} \subset \mathbb{R}^2$ denote their tropicalizations. If $P_1, P_2, \ldots, P_{28} \in \mathbb{R}^2$ are the centres of the 28 tropical lines L_i, then we have the following:*

1. *Four of the P_i's lie in the region $\{(a,b) : a > 0, a > b\}$.*
2. *Four of the P_i's lie in the region $\{(a,b) : b > 0, b > a\}$.*
3. *Four of the P_i's lie in the region $\{(a,b) : a < 0, b < 0\}$.*
4. *Four of the P_i's are $(0,0)$.*
5. *Four of the P_i's lie in the region $\{(a,a) : a > 0\}$.*
6. *Four of the P_i's lie in the region $\{(0,-a) : a > 0\}$.*
7. *Four of the P_i's lie in the region $\{(-a,0) : a > 0\}$.*

Proof Write $l_i := \mathrm{V}(\alpha_i x + \beta_i y + \gamma_i z)$, where $\alpha_i, \beta_i, \gamma_i \in R$, such that the minimum valuation of α_i, β_i, and γ_i is 0. Lemma 2.4 implies that the special fibre of l_i is the line $\overline{\alpha_i}x + \overline{\beta_i}y + \overline{\gamma_i}z = 0 \subset \mathbb{P}^2_k$. As shown by Caporaso–Sernesi [12, 3.4.11 and Lemma 2.3.1], the 28 bitangents l_i limit to the seven lines $x = 0$, $y = 0$, $z = 0$, $x+y+z = 0$, $x+y = 0$, $x+z = 0$, and $y+z = 0$, each with multiplicity four. This

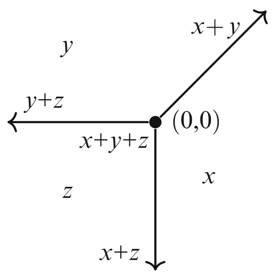


Fig. 1 The seven regions of $\mathbb{R}^2$ in Proposition 3.5; Each region supports four centres of the 28 tropicalized bitangents, and their limiting equations are as shown

means that the closure over the space of all plane quartics of the incidence variety of bitangents over smooth quartics is flat and the fibre over the singular quartic $xyz(x + y + z)$ is as claimed. The lines $x + y = 0$, $x + z = 0$, and $y + z = 0$ are simply the three additional lines that are spanned by the six points of pairwise intersection of x, y, z, and $x + y + z$. For the four lines $l_i = \mathrm{V}(\alpha_i x + \beta_i y + \gamma_i z)$ with limit $x = 0$, we have $\mathrm{val}(\alpha_i) = 0$ and $\mathrm{val}(\beta_i), \mathrm{val}(\gamma_i) > 0$. The centre P_i of the tropicalized line L_i is at $\big(-\mathrm{val}(\alpha_i/\gamma_i), -\mathrm{val}(\beta_i/\gamma_i)\big) \in \mathbb{R}^2$, so it follows that $P_i = (a, b)$ where $a > 0$ and $a > b$. The other six cases are similar. □

Figure 1 illustrates the regions of $\mathbb{R}^2$ corresponding to the seven cases.

To prove the main result in this section, we need a general condition under which a divisor can be lifted pointwise. The idea in the subsequent lemma, which uses rigidity to lift tropical divisors, is reminiscent of arguments in [21, Sect. 6]. If E and E' are effective divisors, say that E' is a *subdivisor* of E if $E - E'$ is again effective.

Lemma 3.6 *Let X be a smooth proper curve of genus g, at least 2, over K and let Γ denote a skeleton for X. Suppose $[\tilde{E}] \in \mathrm{Pic}^d(X)$ is a divisor on X with $\dim_K H^0\big(X, \mathscr{O}_X(\tilde{E})\big) \geq r + 1$, and let $[E] := \tau_*([\tilde{E}]) \in \mathrm{Pic}^d(\Gamma)$. If $D \in [E]$ is a effective divisor on Γ such that support of D is Λ-rational and D has a rigid subdivisor of degree $d - r$, then D lifts pointwise to an effective divisor $\tilde{D} := \tilde{p}_1 + \tilde{p}_2 + \cdots + \tilde{p}_d \in [\tilde{E}]$ with $\tau(\tilde{p}_i) = p_i$.*

Proof Write $D = p_1 + p_2 + \cdots + p_d \in [E]$, and suppose $p_{r+1} + p_{r+2} + \cdots + p_d$ is rigid. By [1, Sect. 2.3], the retraction map $\tau\colon X(K) \to \Gamma$ is surjective on Λ-rational points, so we may pick arbitrary lifts $\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_r \in X(K)$ for the points $p_1, p_2, \ldots, p_r$. Since $\dim_K H^0\big(X, \mathscr{O}_X(\tilde{E})\big) > r$, there exist $\tilde{q}_{r+1}, \tilde{q}_{r+2}, \ldots, \tilde{q}_d$ such that

$$\tilde{D} := \tilde{p}_1 + \tilde{p}_2 + \cdots + \tilde{p}_r + \tilde{q}_{r+1} + \tilde{q}_{r+2} + \cdots + \tilde{q}_d \in [\tilde{E}].$$

Setting $q_i := \tau(\tilde{q}_i)$, we have $\tau_*(\tilde{D}) = p_1 + p_2 + \cdots + p_r + q_{r+1} + q_{r+2} + \cdots + q_d \sim D$, so $q_{r+1} + q_{r+2} + \cdots + q_d \sim p_{r+1} + p_{r+2} + \cdots + p_d$. Since $p_{r+1} + p_{r+2} + \cdots + p_d$ is rigid, we conclude that $\tau_*(\tilde{D}) = D$. □

Corollary 3.7 *If $D \in [K_\Gamma]$ is an effective and Λ-rational divisor on a metric graph Γ of genus g, and D has a rigid subdivisor of degree $g - 1$, then the divisor D lifts pointwise to a canonical divisor on X.* □

Remark 3.8 The results of Lemma 3.6 and Corollary 3.7 do not follow from the main lifting criterion established in [6, Theorem 1.1]; our rigidity assumptions allow a pointwise lift, whereas the lift in [6] may involve up to g extra pairs of zeros and poles which cancel under specialization. Our lifting results also fail to follow directly from the Mikhalkin correspondence results [26], because it is important for us to be able to lift the given tropical plane quartic curve together with a tropical canonical embedding of it to an algebraic curve.

We now prove the main theorem in the section.

Theorem 3.9 *Let X be a genus 3 smooth proper curve over K whose minimal skeleton Γ is a metric K_4. The 28 odd theta characteristics of X are sent to the seven odd theta characteristics of Γ in groups of four.*

Proof Given an effective theta characteristic $[P + Q]$ on Γ, we show that at least four effective theta characteristics on X specialize to it. Since X has 28 effective theta characteristics, it follows that *exactly* four of them specialize to each effective theta characteristic on Γ. We start by using two rational functions on Γ as coordinate functions to give an embedding of Γ in $\mathbb{R}^2$ that can be completed to a balanced tropical curve. We then argue that this embedding can be lifted to a canonical embedding of $X \subset \mathbb{P}^2_K$ as a plane quartic in K_4-form. Finally, we use Proposition 3.5 to find four bitangents to X whose contact points specialize to P and Q on Γ.

Let V_1, V_2, V_3, and V_4 denote the vertices of Γ, let $E_{i,j}$ denote the edge between V_i and V_j, and let a, b, c, d, e, and f denote the lengths of the edges $E_{1,2}$, $E_{1,3}$, $E_{1,4}$, $E_{3,4}$, $E_{2,4}$, and $E_{2,3}$ respectively. As in [33], the seven effective theta characteristics of Γ are in bijection with the seven nonempty Eulerian subgraphs of Γ: the four 3-cycles and the three 4-cycles in Γ.

Suppose that $[P + Q]$ corresponds to a 3-cycle. After relabelling, the cycle is $V_1V_2V_3$ and, after permuting the labels V_1, V_2, and V_3, we may assume $c = \min(c, d, e)$. Let $x := \min(a, e, f)$ and $y := \min(b, d, f)$. With this notation, we have the following explicit embedding of Γ in $\mathbb{R}^2$ such that the coordinate functions are rational functions on Γ; see Fig. 2. Put the vertices V_1, V_2, V_3, and V_4 at $(0, 0)$, $(x, 0)$, $(0, y)$, and $(-c, -c)$ respectively. Embed the edge $E_{1,4}$ in $\mathbb{R}^2$ as a straight line segment between V_1 and V_4. If $a = \min(a, e, f)$, then we embed $E_{1,2}$ by simply joining V_1 and V_2 with a straight line segment. However, when $a \neq \min(a, e, f)$, we embed $E_{1,2}$ as the piecewise-linear path $(0, 0) \to (\frac{a+x}{2}, 0) \to (x, 0)$; in particular, $E_{1,2}$ contributes multiplicity 2 to the segment between $(x, 0)$ and $(\frac{a+x}{2}, 0)$ in the image of this embedding in $\mathbb{R}^2$. Similarly, if $b = \min(b, d, f)$, then we embed $E_{1,3}$ by simply joining V_1 and V_3 with a straight line segment; when $b \neq \min(b, d, f)$, we embed $E_{1,3}$ as the piecewise-linear path $(0, 0) \to (0, \frac{b+y}{2}) \to (0, y)$. Finally, we embed

- $E_{2,3}$ as the piecewise-linear path $(x, 0) \to \left(\frac{f+x}{2}, \frac{f-x}{2}\right) \to \left(\frac{f-y}{2}, \frac{f+y}{2}\right) \to (0, y)$,
- $E_{3,4}$ as the piecewise-linear path $(0, y) \to \left(\frac{y-d}{2}, y\right) \to \left(\frac{-c-d}{2}, -c\right) \to (-c, -c)$,
- $E_{2,4}$ as the piecewise-linear path $(-c, -c) \to \left(-c, \frac{-c-e}{2}\right) \to \left(x, \frac{x-e}{2}\right) \to (x, 0)$.

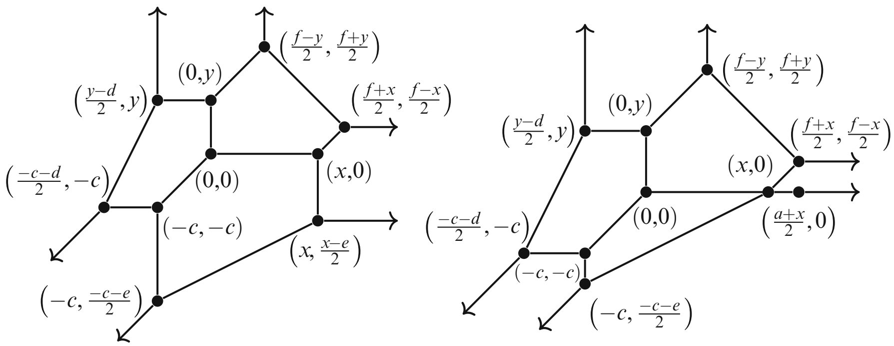


Fig. 2 An embedding of Γ using tropical rational coordinate functions. The six infinite rays have multiplicity 2 and the bounded segments have multiplicity 1 except for the edge between $(x,0)$ and $(a+x/2,0)$ which has multiplicity 2. On the left, we depict the case with $\min(a,e,f)=a$ and $\min(b,d,f)=b$. On the right, we depict the case with $\min(a,e,f)=e$ and $\min(b,d,f)=b$

To describe some rational functions on Γ, let $p_z, q_z \in \Gamma$ be the points mapping to $\left(\frac{-c-f}{2},-c\right)$ and $\left(-c,\frac{-c-e}{2}\right)$. Similarly, let $p_x, q_x \in \Gamma$ be the points mapping to

$$\begin{cases} \left(\frac{f+a}{2},\frac{f-a}{2}\right) \text{ and } \left(a,\frac{a-e}{2}\right) & \text{if } a=\min(a,e,f), \\ \left(\frac{f+a}{2},\frac{f-a}{2}\right) \text{ and } \left(\frac{a+e}{2},0\right) & \text{if } e=\min(a,e,f), \\ \left(\frac{a+f}{2},0\right) \text{ and } \left(f,\frac{f-e}{2}\right) & \text{if } f=\min(a,e,f). \end{cases}$$

It follows that the x-coordinate of this embedding is a rational function F on Γ with $\operatorname{div}(F)=2p_z+2q_z-2p_x-2q_x$. Furthermore, we claim that $2p_z+2q_z \sim K_\Gamma$. Indeed, if $S \subset \Gamma$ is the cycle $V_1V_2V_3$ and $d(S,-)\colon \Gamma \to \mathbb{R}$ is the distance function, regarded as a rational function on Γ, then we have

$$\operatorname{div} d(S,-) = 2p_z + 2q_z - V_1 - V_2 - V_3 - V_4 = 2p_z + 2q_z - K_\Gamma.$$

Similarly, the y-coordinate of this embedding of Γ is a rational function G with $\operatorname{div} G = 2p_z + 2q_z - 2p_y - 2q_y$ and $2p_y + 2q_y$ is also in the canonical class of Γ. Thus, the image of Γ under (F,G) becomes a tropical plane curve in $\mathbb{R}^2$ by adding six infinite rays each of multiplicity two; we add two rays in direction $(1,0)$ at p_x and q_x, two rays in direction $(0,1)$ at p_y and q_y, and two in direction $(-1,-1)$ at p_z and q_z.

All of the edge lengths a, b, c, d, e, f lie in the value group Λ, so each point p_x, q_x, p_y, q_y, p_z, q_z is a Λ-rational point of Γ; the value group Λ is divisible because K is algebraically closed. We claim that each divisor $D_0 = 2p_x + 2q_x$, $D_1 = 2p_y + 2q_y$, and $D_z = 2p_z + 2q_z$ has a rigid subdivisor of degree 2. If $i \in \{x,y,z\}$ and p_i is a vertex of Γ, then Corollary 2.2 shows that $2p_i$ is rigid. By symmetry, the analogous statement holds for q_i. If neither p_i nor q_i is a vertex of Γ, then they are on the interiors of distinct edges in Γ, and $p_i + q_i$ is again rigid by Corollary 2.2. Applying Corollary 3.7, we see that the divisors D_0, D_1, D_2 can be lifted pointwise to canonical divisors $\tilde{D}_0$, $\tilde{D}_1$, $\tilde{D}_2$ on X. Choosing global sections $s_0, s_1, s_2 \in H^0(X, K_X)$ with zeros $\tilde{D}_0$, $\tilde{D}_1$, $\tilde{D}_2$ respectively, the Slope Formula for tropical curves (see either [5, Theorem 5.15] or [32, Proposition 3.3.15]) implies that the embedding $(s_0, s_1, s_2)\colon X \hookrightarrow \mathbb{P}^2$ is, up to a shift in $\mathbb{R}^2$, a lift of $(F, G)\colon \Gamma \to \mathbb{R}^2$. This shift can be corrected by rescaling coordinates on $\mathbb{P}^2$. Hence, Theorem 3.2 establishes that X is canonically embedded in $\mathbb{P}^2$ as a smooth plane quartic in K_4-form.

It remains to analyze the bitangents in this case. Proposition 3.5 shows that there are four bitangents of X whose tropicalizations L_1, L_2, L_3, and L_4 are centred in the open southwest quadrant of $\mathbb{R}^2$. For $1 \leq j \leq 4$, let $C_j = (u_j, v_j) \in \mathbb{R}^2$ denote the centre of L_j. We claim that the point C_j cannot lie to the left of the piecewise-linear path $(x, 0) \to \left(x, \frac{x-e}{2}\right) \to \left(-c, \frac{-c-e}{2}\right)$. If it did, then the rightward ray of L_i would intersect trop(X) in a point on one of these segments with stable intersection multiplicity 1, but this contradicts the main theorem of Osserman–Rabinoff [29] along with the fact that each bitangent meets X in two points of multiplicity 2 (or one point of multiplicity 4). Similarly, C_j cannot lie below $(0, y) \to \left(\frac{y-f}{2}, y\right) \to \left(\frac{-c-f}{2}, -c\right)$. It follows that $u_j \leq \frac{-c-f}{2}$ and $v_j \leq \frac{-c-e}{2}$, so L_i intersects trop(X) at two points, each with multiplicity 2, which retract to $p_z, q_z \in \Gamma$. This is precisely the theta characteristic associated to the cycle $V_1V_2V_3$. By symmetry, we have therefore proved the main claim for the four effective theta characteristics of Γ corresponding under the Zharkov bijection to the four 3-cycles in Γ. In other words, we have proved that at least four effective theta characteristics of X specialize to each of these effective theta characteristic of Γ.

Finally, an analogous argument shows the claim for the three effective theta characteristics of Γ that correspond to 4-cycles in Γ. For instance, Proposition 3.5 proves that there are four bitangents of X whose tropicalizations are tropical lines centred on the open ray $\{(\alpha, 0) : \alpha < 0\}$. Moreover, the centres lie to the left of $\left(\frac{y-f}{2}, y\right) \to \left(\frac{-c-f}{2}, -c\right)$, again by [29]. This means that the two bitangent contact points tropicalize to the midpoint of $E_{3,4}$ and somewhere on the edge $E_{1,2}$. Hence, those four theta characteristics on X specialize to the theta characteristic associated to the cycle $V_1V_3V_2V_4$. $\square$

4 The 28 Bitangents of Honeycomb Quartics

Following [23], we say that a smooth plane quartic curve $X := V(f) \subset \mathbb{P}^2_K$ is a *honeycomb curve* if the regular subdivision of the triangle $\Delta_4 := \text{conv}\{(0,0), (4,0), (0,4)\}$ induced by f is the standard one obtained by slicing Δ_4 by the lines $x = i$, $y = i$, $x + y = i$ for all $1 \leq i \leq 3$; see Fig. 3. Honeycomb curves play a special role in tropical geometry [14, 31]. In this section, we will determine almost exactly where the 28 bitangents of honeycomb quartic curves go under tropicalization. In fact, our description is exact for *generic* honeycomb curves.

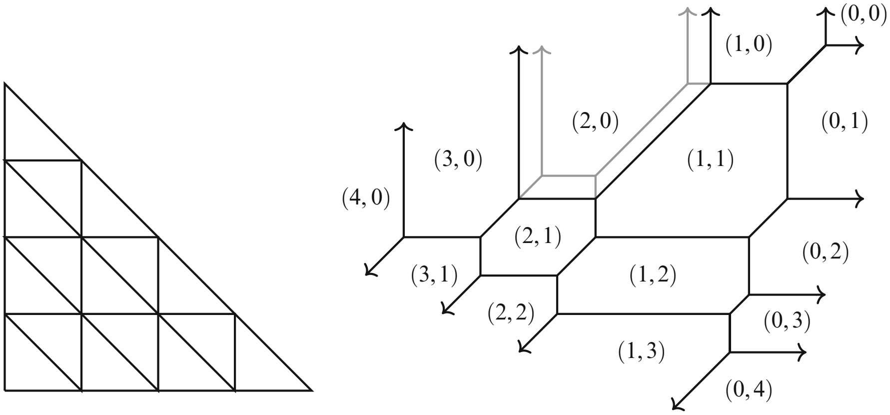


Fig. 3 The subdivision Δ_4 and a tropical honeycomb plane quartic trop $X \subset \mathbb{R}^2$; the curve shown in grey illustrates Remark 4.4

In the final section of the paper, we will give an interesting explicit calculation of the 28 bitangents of a honeycomb plane quartic over $\mathbb{C}\{\{t\}\}$, giving an idea of the behaviour that can arise in the non-generic case. The tools we use in this section are tropical intersection theory [28, 29], the census of bitangents and their limits provided in [12], and the elementary geometry of tropical plane curves.

The curve trop(X) divides $\mathbb{R}^2$ into 15 regions, one for each lattice point in Δ_4. Let $R_{i,j}$ denote the region corresponding to the lattice point (i, j); see Fig. 3. The three bounded hexagonal regions are $R_{1,1}, R_{1,2}, R_{2,1}$ and their unique common vertex is O. For adjacent regions $R_{i,j}$ and $R_{k,l}$, write $E_{i,j;k,l}$ for the unique edge that they share, and write $\ell(E_{i,j;k,l})$ for the line spanned by $E_{i,j;k,l}$. The line $\ell(E_{1,1;1,2})$ intersects the boundary of $R_{2,1}$ in two points: O and another point S_x. Similarly, the line $\ell(E_{1,1;2,1})$ intersects the boundary of $R_{1,2}$ at O and S_y, and the line $\ell(E_{1,2;2,1})$ intersects the boundary of $R_{1,1}$ at O and S_z. Finally, we label some of the pairwise intersections of these lines: set $T_x := \ell(E_{0,1;1,1}) \cap \ell(E_{0,3;1,2}) \in \mathbb{R}^2$, set $T_y := \ell(E_{1,0;1,1}) \cap \ell(E_{3,0;2,1})$, and set $T_z := \ell(E_{2,1;3,1}) \cap \ell(E_{1,2;1,3})$; see Fig. 4.

Definition 4.1 A honeycomb curve $X := \mathrm{V}(f)$ with $f := \sum_{i+j\leq 4} c_{i,j}x^i y^j z^{4-i-j}$ is *generic* if we have $a_{3,1} + a_{1,1} - a_{3,0} - a_{1,2} \neq 0$, $a_{0,3} + a_{2,1} - a_{1,3} - a_{1,1} \neq 0$ and $a_{1,0} + a_{1,2} - a_{0,1} - a_{2,1} \neq 0$ where $a_{i,j} := \mathrm{val}(c_{i,j})$.

Theorem 4.2 *Let* $X := \mathrm{V}(f) \subset \mathbb{P}^2_K$ *be a honeycomb plane quartic curve and let* $L_1, L_2, \ldots, L_{28}$ *be the tropicalizations of its* 28 *bitangents. If* $P_i \in \mathbb{R}^2$ *is the centre of the tropical line* L_i *for* $1 \leq i \leq 28$*, then we have the following.*

1. *Four of the* P_i*'s are* T_x.
2. *Four of the* P_i*'s are* T_y.
3. *Four of the* P_i*'s are* T_z.
4. *Four of the* P_i*'s are* O.
5. *Four of the* P_i*'s lie on the closed ray in direction* $(1, 1)$ *based at* S_z.

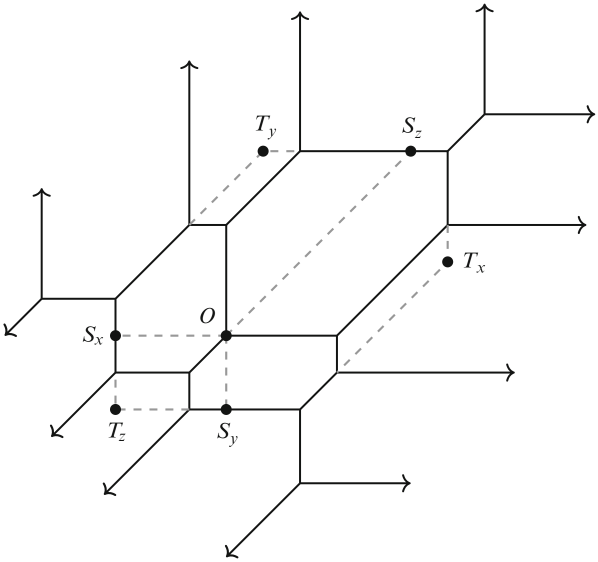


Fig. 4 The seven centres of the bitangents to a generic honeycomb quartic. Exactly four bitangents are centred at each of the locations

6. *Four of the P_i's lie on the closed ray in direction* $(-1, 0)$ *based at* S_x.
7. *Four of the P_i's lie on the closed ray in direction* $(0, -1)$ *based at* S_y.

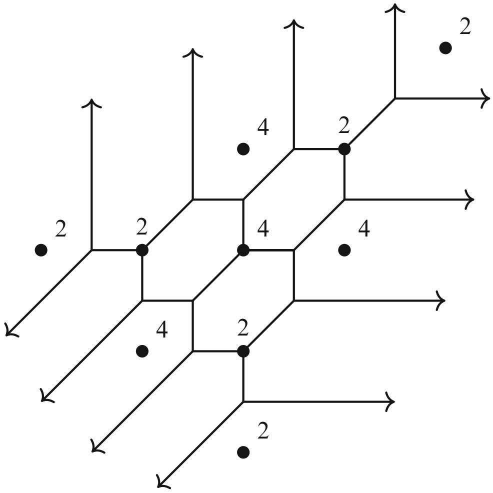
Moreover, if X is a generic honeycomb curve, then the points in parts 5–7 must be exactly S_z, S_x, and S_y, respectively.

Proof If necessary, change coordinates so that $O = (0, 0)$. Proposition 3.5 shows that each of the seven regions in Fig. 1 supports exactly four of the P_i's. Thus, part 4 follows immediately.

Since four of the P_i's lie southwest of O, each of the north and east rays of the corresponding lines L_i intersect trop(X) is a single connected component with stable intersection multiplicity of 2; see [29]. It follows that P_i must lie on or below the closed segment $E_{2,1;3,1}$, and it must lie on or to the left of the closed segment $E_{1,2;1,3}$. Therefore, the four points are exactly T_z which proves part 3. Similar arguments establishes part 1 and part 2.

Proposition 3.5 also guarantees that four of the P_i lie in the direction $(1, 1)$ from O. If P_i lies within the open region $R_{1,1}$, then $L_i \cap \text{trop}(X)$ would have at least three distinct connected components, contradicting the fact that L_i is a tropicalization of a bitangent. This proves that part 5 holds. Moreover, if X is generic, then the only way that the connected component of $L_i \cap \text{trop}(X)$ that contains S_z can have stable intersection multiplicity 2 is if $P_i = S_z$; otherwise $L_i \cap \text{trop}(X)$ would have more than two connected components. This proves that four of the P_i are S_z when X is a generic honeycomb curve. Again, similar arguments establish part 6 and part 7. □

Fig. 5 The 28 centres of the bitangents to the honeycomb quartic (NHC)



Remark 4.3 It is straightforward to calculate the coordinates of each of these points exactly in terms of the $a_{i,j}$'s, using the duality between tropical plane curves and their lifted Newton subdivisions [25]. Thus, for almost all honeycomb quartics X, we can produce an explicit formula for the tropicalizations of the 28 bitangents of X in terms of the coefficients of its defining equation.

For non-generic honeycomb curves, the bitangent centres need not be grouped in fours and may even appear in the regions $R_{0,0}$, $R_{4,0}$, and $R_{0,4}$; see Fig. 5.

Remark 4.4 Classically, a smooth plane quartic curve is uniquely determined by its 28 bitangents [12, 24]. If X and $X' \subset \mathbb{P}^2$ are plane quartics with smooth tropicalization and the 28 tropicalizations of their bitangents agree, do the tropicalizations of X and X' agree? Theorem 4.2 shows that the answer is no. More dramatically, it yields infinite families of tropical honeycomb quartics such that all lifts of them to classical curves have the same 28 tropicalized bitangents. For instance, any two curves tropicalizing to the honeycomb curve in Fig. 3 and the one obtained from it by shrinking region $R_{2,0}$ (as shown in Fig. 3 in grey) have the same 28 tropicalized bitangents.

5 Computing the 28 Bitangents of a Honeycomb Quartic

In the final section of this paper, we compute of the Puiseux expansions for the 28 bitangents of a specific K_4-quartic. Our calculation illustrates how tropical geometry may be used in computations that are not a priori tropical. Throughout this section, let $K := \mathbb{C}\{\{t\}\}$. The smooth plane quartic X defined by the equation

$$f := xyz(x+y+z) + t(x^2y^2 + x^2z^2 + y^2z^2) \\ +t^2(x^3y + xy^3 + x^3z + xz^3 + y^3z + yz^3) + t^5(x^4 + y^4 + z^4) \tag{NHC}$$

is a non-generic honeycomb curve, in the sense of Definition 4.1. The tropicalization of X is a smooth tropical plane curve in which every bounded segment has lattice length 1. The computations in this section were carried out in *Macaulay2* [18], and the results are summarized in Table 1.

Let $\mathscr{B}$ be the set of all $(A, B) \in \mathbb{C}\{\{t\}\}^2$ such that $Ax + By + z = 0$ is a bitangent to the honeycomb curve X. Since X admits no bitangent of the form $Ax + By = 0$, we have $|\mathscr{B}| = 28$. Observe that $Ax + By + z = 0$ is a bitangent to X if and only if the polynomial $f(x, y, -Ax - By)$ is a perfect square. To exploit this condition, we introduce a square-detecting ideal in the next lemma.

Lemma 5.1 *Let K be an algebraically closed field of characteristic 0. If the ideal $J \subseteq K[X_0, X_1, \ldots, X_4]$ is generated by the seven cubic polynomials*

$$8X_1X_4^2 - 4X_2X_3X_4 + X_3^3, \qquad 8X_0^2X_3 - 4X_0X_1X_2 + X_1^3,$$

$$16X_0X_4^2 + 2X_1X_3X_4 - 4X_2^2X_4 + X_2X_3^2, \qquad 16X_0^2X_4 + 2X_0X_1X_3 - 4X_0X_2^2 + X_1^2X_2,$$

$$8X_0X_1X_4 - 4X_0X_2X_3 + X_1^2X_3, \qquad 8X_0X_3X_4 - 4X_1X_2X_4 + X_1X_3^2,$$

$$X_0X_3^2 - X_1^2X_4.$$

then we have $(c_0, c_1, \ldots, c_4) \in \mathrm{V}(J)$ if and only if the polynomial

$$s(x, y) := c_4x^4 + c_3x^3y + c_2x^2y^2 + c_1xy^3 + c_0y^4 \in K[x, y]$$

is a perfect square.

Proof Since K is algebraically closed, the polynomial $s(x, y)$ is a perfect square if and only if there exist $C, D, C', D' \in K$ such that $s(x, y) = (Cx + Dy)^2(C'x + D'y)^2$. Expanding and eliminating C, D, C', and D' produces the ideal J above. □

To identify the bitangents to X, we expand $f(x, y, -Ax - By)$ as a homogeneous quartic polynomial in x and y, whose five coefficients are polynomials in A and B. Substituting these five polynomials for $X_0, X_1, \ldots, X_4$ in J yields an ideal $I \subset K[A, B]$ generated by seven polynomials whose variety is $\mathscr{B}$, the bitangents of X. Because of their length, we avoid displaying the explicit generators for I.

Our goal is to compute $\mathrm{V}(I)$. Even though this variety is just 28 points, it is not simple to carry out its computation over the field of Puiseux series. We now explain our strategy for carrying it out. First, we will determine the 28 valuations $(\mathrm{val}(A), \mathrm{val}(B)) \in \mathbb{R}^2$ of the bitangent coefficients, i.e. we will compute the locations of the 28 tropicalized bitangents. For this we use elimination theory and Newton polygons to determine the valuations of $\mathrm{V}(I)$ under specially chosen projections. Second, we will bound the denominators of the exponents of t that show up in the Puiseux expansions of pairs (A, B), allowing us to pass from Puiseux series to power series (after an appropriate base change). Finally, we will use repeated specialization to $t = 0$ to successively compute the the Puiseux expansions of the bitangent coefficients at each of the determined locations.

The first step is accomplished in the proposition below, whose proof makes use of computations in *Macaulay2*; see Fig. 5. Note that a bitangent $(A, B) \in \mathscr{B}$ tropicalizes to a line centred at $\big(-\operatorname{val}(A), -\operatorname{val}(B)\big) \in \mathbb{R}^2$.

Proposition 5.2 *If* $A, B \in K = \mathbb{C}\{\{t\}\}$ *with* $a := \operatorname{val}(A)$ *and* $b := \operatorname{val}(B)$, *then the* 28 *bitangents* $Ax + By + z = 0$ *of the non-generic honeycomb curve* $X = \mathrm{V}(f)$ *include*

- *exactly* 4 *with* $(a, b) = (0, 0)$,
- *exactly* 4 *with* $(a, b) = (-2, 0)$,
- *exactly* 4 *with* $(a, b) = (0, -2)$,
- *exactly* 4 *with* $(a, b) = (2, 2)$,
- *exactly* 2 *with* $(a, b) = (-2, -2)$,
- *exactly* 2 *with* $(a, b) = (-4, -4)$,
- *exactly* 2 *with* $(a, b) = (2, 0)$,
- *exactly* 2 *with* $(a, b) = (4, 0)$,
- *exactly* 2 *with* $(a, b) = (0, 2)$,
- *exactly* 2 *with* $(a, b) = (0, 4)$.

Proof The first four statements all follow from Theorem 4.2. By symmetry, it suffices to prove that there are exactly two tropicalized bitangents centred at $(2, 2)$ and two centred at $(4, 4)$. Let $(A, B) \in \mathscr{B}$. Following Proposition 3.4.8 in [25], we use tropical geometry in dimension 1, together with a special change of coordinates, get information about the possible values of $A+B$ and $A+B+1$. Rewriting the ideal $I \subset K[A, B]$ in terms of coordinates $A' := A + B$ and B and eliminating B produces an ideal that is principally generated by a polynomial $p \in K[A']$ such that $p(A') = 0$ if and only if $A' = A + B$ for some $(A, B) \in \mathscr{B}$. Because of the symmetry of f, there are fewer possible values of $A + B$, which makes this elimination calculation doable in *Macaulay2*.

Before completing the proof, we collect four simple claims which are verified by computation in *Macaulay2*.

Claim The polynomial p is a squarefree polynomial of degree 16. The roots of p have valuations $-4, -2, 0$, and 2, and these valuations are attained with multiplicity $1, 5, 7$, and 3 respectively.

Proof We calculated p in *Macaulay2*. To check that it is squarefree, we computed a specialization, say $t = 1$, of the resultant of p and p' and observed that it is nonzero; the actual resultant of p and p' is much too large to compute exactly. The valuations of the roots of p are determined, with multiplicity, by the valuations of the 17 coefficients of p, via the method of the Newton polygon. □

Claim If $(A, B) \in \mathscr{B}$, then we have $\operatorname{val}(A + B + 1) \leq 0$.

Proof Using *Macaulay2*, we rewrote I in terms of coordinates $A'' := A + B + 1$ and B. We then eliminated B. The result is a polynomial whose Newton polygon is comprised of segments of nonnegative slope. □

Claim There are exactly four points of $\mathscr{B}$ of the form (A, A) and their valuations are $(0, 0)$ and $(2, 2)$ with multiplicity 2 each.

Proof We substituted $B = A$ into I and computed a Gröbner basis. The result is an ideal principally generated by a degree four polynomial. Analyzing its Newton polygon yields the claim about the valuations. □

Claim If $(A, B) \in \mathscr{B}$ such that $\big(\mathrm{val}(A), \mathrm{val}(B)\big) = (-a, -a)$ for some $a \geq 2$, then we have $\mathrm{val}(A + B) = -a$, so $\big(\mathrm{val}(A), \mathrm{val}(B)\big)$ is either $(-4, -4)$ or $(-2, -2)$.

Proof The second part follows from the first by our initial claim. To prove the first, suppose instead $\mathrm{val}(A + B) > \mathrm{val}(A) = \mathrm{val}(B)$. By symmetry, if $Ax + By + z = 0$ is a bitangent, then $Ax + y + Bz = 0$ is also a bitangent and $\left(\frac{A}{B}, \frac{1}{B}\right) \in \mathscr{B}$. Therefore, the second claim shows that $\mathrm{val}\left(\frac{A}{B} + \frac{1}{B} + 1\right) \leq 0$, but $\mathrm{val}\left(\frac{1}{B}\right) > 0$ and $\mathrm{val}\left(\frac{A+B}{B}\right) > 0$ yields a contradiction. □

The first claim shows that exactly 16 distinct values of $A + B$ occur. The third claim implies that there are four bitangents of the form (A, A), and the symmetry of f implies that the remaining 24 bitangents form twelve pairs (A, B) and (B, A). If two points $(A, B) \neq (A', B') \in \mathscr{B}$ satisfy $A + B = A' + B'$, then we see that $(A', B') = (B, A)$. Since p has a unique root of valuation -4 by the first claim, there is exactly one pair of bitangents $(A, B) \neq (B, A) \in \mathscr{B}$ with $\mathrm{val}(A + B) = -4$. The fourth claim establishes that these must be the unique pair of bitangents that have valuation $(-4, -4)$; all other cases given in Theorem 4.2 give pairs (A, B) with $\mathrm{val}(A + B) > -4$. Therefore, by combining the fourth claim and Theorem 4.2, we conclude that there are exactly two bitangents that have valuation $(-2, -2)$. □

Remark 5.3 Let $(A, B) \in \mathscr{B}$ be one of m bitangents that has the same tropicalization. If n is the smallest positive integer such that $A, B \in \mathbb{C}(\!(t^{1/n})\!)$, then we have $n \leq m$. Indeed, the automorphisms $\mathbb{C}(\!(t^{1/n})\!)/\mathbb{C}(\!(t)\!)$ preserve valuations, so we already produce n bitangents with the same tropicalization.

Fix a positive integer n and consider a point $(a, b) \in \mathbb{R}^2$ that is a possible tropicalization of $(A, B) \in \mathscr{B}$ as determined in Proposition 5.2. We now explain how to compute, at least in principle, the Puiseux expansions of the bitangents (A, B) that lie in $\mathbb{C}(\!(t^{1/n})\!)^2$ and whose tropicalizations are (a, b). After a base change $s^n = t$, we may assume $n = 1$ and, after a change of coordinates, we may assume $(a, b) = (0, 0)$. Proposition 4.4 in [19] shows that the variety

$$\left\{(a, b) \in \mathbb{C}^2 : \text{there exist } (A, B) \in \mathscr{B} \text{ such that } (a, b) = \big(\mathrm{val}(A), \mathrm{val}(B)\big)\right\}$$

is cut out by the ideal $(I \cap R[A, B])|_{t=0}$. For each $(a, b) \in (\mathbb{C}^*)^2$ in this variety, we make a change of coordinates $A := a + tA'$, $B := b + tB'$ and repeat the process to obtain the next coefficient in the expansion. We obtain the desired precision by making sufficient iterations. To find all 28 bitangents, we make this computation at each possible location found in Proposition 5.2 and for each $n \leq m$.

In practice, finding the saturation $(I : t^\infty)$ is much too slow, so when we run this algorithm, we simply use $I|_{t=0}$ rather than $(I : t^\infty)|_{t=0}$. The scheme-theoretic inclusion $\mathrm{V}(I|_{t=0}) \supseteq \mathrm{V}\big((I : t^\infty)|_{t=0}\big)$ implies that $\deg \mathrm{V}(I|_{t=0}) \geq \deg \mathrm{V}\big((I : t^\infty)|_{t=0}\big)$ when $\dim \mathrm{V}(I|_{t=0}) = 0$. In other words, using $I|_{t=0}$ at each stage gives us only a set-theoretic upper bound on the possible bitangents $(A, B) \in \mathscr{B}$, computed to arbitrary precision. However, if the sum of the degrees of all of the zero-dimensional

Table 1 Puiseux series expansion of $(A, B) \in \mathscr{B}$

val	(A, B)
$(0, 0)$	$(1, 1), (1 + 4t + 4t^3 - 24t^4 + \cdots, 1 + 4t + 4t^3 - 24t^4 + \cdots),$ $(1, 1 - 4t + 16t^2 - 68t^3 + \cdots), (1 - 4t + 16t^2 - 68t^3 + \cdots, 1)$
$(2, 2)$	$(t^2 + 2it^{\frac{5}{2}} - 2t^3 - 5it^{\frac{7}{2}} + \cdots, t^2 + 2it^{\frac{5}{2}} - 2t^3 - 5it^{\frac{7}{2}} + \cdots),$ $(t^2 + 2it^{\frac{5}{2}} - 2t^3 - 5it^{\frac{7}{2}} + \cdots, t^2 - 2it^{\frac{5}{2}} - 2t^3 + 5it^{\frac{7}{2}} + \cdots),$ $(t^2 - 2it^{\frac{5}{2}} - 2t^3 + 5it^{\frac{7}{2}} + \cdots, t^2 - 2it^{\frac{5}{2}} - 2t^3 + 5it^{\frac{7}{2}} + \cdots),$ $(t^2 - 2it^{\frac{5}{2}} - 2t^3 + 5it^{\frac{7}{2}} + \cdots, t^2 + 2it^{\frac{5}{2}} - 2t^3 - 5it^{\frac{7}{2}} + \cdots)$
$(0, -2)$	$(1, t^{-2} + 2it^{-\frac{3}{2}} - 2t^{-1} - 5it^{-\frac{1}{2}} + \cdots), (1, t^{-2} - 2it^{-\frac{3}{2}} - 2t^{-1} + 5it^{-\frac{1}{2}} + \cdots),$ $(1 + 4it^{\frac{1}{2}} - 8t - 18it^{\frac{3}{2}} + \cdots, t^{-2} + 2it^{-\frac{3}{2}} - 2t^{-1} - 5it^{-\frac{1}{2}} + \cdots),$ $(1 - 4it^{\frac{1}{2}} - 8t + 18it^{\frac{3}{2}} + \cdots, t^{-2} - 2it^{-\frac{3}{2}} - 2t^{-1} + 5it^{-\frac{1}{2}} + \cdots)$
$(-2, 0)$	$(t^{-2} + 2it^{-\frac{3}{2}} - 2t^{-1} - 5it^{-\frac{1}{2}} + \cdots, 1), (t^{-2} - 2it^{-\frac{3}{2}} - 2t^{-1} + 5it^{-\frac{1}{2}} + \cdots, 1),$ $(t^{-2} + 2it^{-\frac{3}{2}} - 2t^{-1} - 5it^{-\frac{1}{2}} + \cdots, 1 + 4it^{\frac{1}{2}} - 8t - 18it^{\frac{3}{2}} + \cdots),$ $(t^{-2} - 2it^{-\frac{3}{2}} - 2t^{-1} + 5it^{-\frac{1}{2}} + \cdots, 1 - 4it^{\frac{1}{2}} - 8t + 18it^{\frac{3}{2}} + \cdots)$
$(2, 0)$	$(4t^2 + 4it^{\frac{5}{2}} - 12t^3 - 18it^{\frac{7}{2}} + \cdots, 1 + 2it^{\frac{1}{2}} - 2t - 5it^{\frac{3}{2}} + \cdots),$ $(4t^2 - 4it^{\frac{5}{2}} - 12t^3 + 18it^{\frac{7}{2}} + \cdots, 1 - 2it^{\frac{1}{2}} - 2t + 5it^{\frac{3}{2}} + \cdots)$
$(0, 2)$	$(1 + 2it^{\frac{1}{2}} - 2t - 5it^{\frac{3}{2}} + \cdots, 4t^2 + 4it^{\frac{5}{2}} - 12t^3 - 18it^{\frac{7}{2}} + \cdots),$ $(1 - 2it^{\frac{1}{2}} - 2t + 5it^{\frac{3}{2}} + \cdots, 4t^2 - 4it^{\frac{5}{2}} - 12t^3 + 18it^{\frac{7}{2}} + \cdots)$
$(-2, -2)$	$(\frac{1}{4}t^{-2} + \frac{i}{4}t^{-\frac{3}{2}} + \frac{1}{2}t^{-1} + \frac{i}{8}t^{-\frac{1}{2}} + \cdots, \frac{1}{4}t^{-2} - \frac{i}{4}t^{-\frac{3}{2}} + \frac{1}{2}t^{-1} - \frac{i}{8}t^{-\frac{1}{2}} + \cdots),$ $(\frac{1}{4}t^{-2} - \frac{i}{4}t^{-\frac{3}{2}} + \frac{1}{2}t^{-1} - \frac{i}{8}t^{-\frac{1}{2}} + \cdots, \frac{1}{4}t^{-2} + \frac{i}{4}t^{-\frac{3}{2}} + \frac{1}{2}t^{-1} + \frac{i}{8}t^{-\frac{1}{2}} + \cdots)$
$(4, 0)$	$(2t^4 + 2it^{\frac{9}{2}} - 10t^5 - 13it^{\frac{11}{2}} + \cdots, 1 + 2it^{\frac{1}{2}} - 2t - 5it^{\frac{3}{2}} + \cdots),$ $(2t^4 - 2it^{\frac{9}{2}} - 10t^5 + 13it^{\frac{11}{2}} + \cdots, 1 - 2it^{\frac{1}{2}} - 2t + 5it^{\frac{3}{2}} + \cdots)$
$(0, 4)$	$(1 + 2it^{\frac{1}{2}} - 2t - 5it^{\frac{3}{2}} + \cdots, 2t^4 + 2it^{\frac{9}{2}} - 10t^5 - 13it^{\frac{11}{2}} + \cdots),$ $(1 - 2it^{\frac{1}{2}} - 2t + 5it^{\frac{3}{2}} + \cdots, 2t^4 - 2it^{\frac{9}{2}} - 10t^5 + 13it^{\frac{11}{2}} + \cdots)$
$(-4, -4)$	$(\frac{1}{2}t^{-4} + \frac{i}{2}t^{-\frac{7}{2}} + 2t^{-3} + \frac{5}{4}it^{-\frac{5}{2}} + \cdots, \frac{1}{2}t^{-4} - \frac{i}{2}t^{-\frac{7}{2}} + 2t^{-3} - \frac{5}{4}it^{-\frac{5}{2}} + \cdots),$ $(\frac{1}{2}t^{-4} - \frac{i}{2}t^{-\frac{7}{2}} + 2t^{-3} - \frac{5}{4}it^{-\frac{5}{2}} + \cdots, \frac{1}{2}t^{-4} + \frac{i}{2}t^{-\frac{7}{2}} + 2t^{-3} + \frac{5}{4}it^{-\frac{5}{2}} + \cdots)$

schemes in question drops to 28, then we know that the 28 Puiseux expansions we have computed up to that point do correspond to true bitangents. This termination condition does indeed happen in our example. The results of our computations are shown in Table 1.

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Secants, Bitangents, and Their Congruences

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Abstract A congruence is a surface in the Grassmannian $\mathrm{Gr}(1, \mathbb{P}^3)$ of lines in projective 3-space. To a space curve C, we associate the Chow hypersurface in $\mathrm{Gr}(1, \mathbb{P}^3)$ consisting of all lines which intersect C. We compute the singular locus of this hypersurface, which contains the congruence of all secants to C. A surface S in $\mathbb{P}^3$ defines the Hurwitz hypersurface in $\mathrm{Gr}(1, \mathbb{P}^3)$ of all lines which are tangent to S. We show that its singular locus has two components for general enough S: the congruence of bitangents and the congruence of inflectional tangents. We give new proofs for the bidegrees of the secant, bitangent and inflectional congruences, using geometric techniques such as duality, polar loci and projections. We also study the singularities of these congruences.

MSC 2010 codes: 14M15, 14H50, 14J70, 14N10, 14N15, 51N35, 14C17

1 Introduction

The aim of this article is to study subvarieties of Grassmannians which arise naturally from subvarieties of complex projective 3-space $\mathbb{P}^3$. We are mostly interested in threefolds and surfaces in $\mathrm{Gr}(1, \mathbb{P}^3)$. These are classically known as *line complexes* and *congruences*. We determine their classes in the Chow ring of

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$\mathrm{Gr}(1,\mathbb{P}^3)$ and their singular loci. Throughout the paper, we use the phrase 'singular points of a congruence' to simply refer to its singularities as a subvariety of the Grassmannian $\mathrm{Gr}(1,\mathbb{P}^3)$. In the older literature, this phrase refers to points in $\mathbb{P}^3$ lying on infinitely many lines of the congruence; nowadays, these are called *fundamental points*.

The *Chow hypersurface* $\mathrm{CH}_0(C) \subset \mathrm{Gr}(1,\mathbb{P}^3)$ of a curve $C \subset \mathbb{P}^3$ is the set of all lines in $\mathbb{P}^3$ that intersect C, and the *Hurwitz hypersurface* $\mathrm{CH}_1(S) \subset \mathrm{Gr}(1,\mathbb{P}^3)$ of a surface $S \subset \mathbb{P}^3$ is the Zariski closure of the set of all lines in $\mathbb{P}^3$ that are tangent to S at a smooth point. Our main results are consolidated in the following theorem.

Theorem 1.1 *Let $C \subset \mathbb{P}^3$ be a nondegenerate curve of degree d and geometric genus g having only ordinary singularities $x_1, x_2, \ldots, x_s$ with multiplicities $r_1, r_2, \ldots, r_s$. If* $\mathrm{Sec}(C)$ *denotes the locus of secant lines to C, then the singular locus of* $\mathrm{CH}_0(C)$ *is* $\mathrm{Sec}(C) \cup \bigcup_{i=1}^{s}\{L \in \mathrm{Gr}(1,\mathbb{P}^3) : x_i \in L\}$*, the bidegree of* $\mathrm{Sec}(C)$ *is*

$$\left(\tfrac{1}{2}(d-1)(d-2) - g - \sum_{i=1}^{s} \tfrac{1}{2} r_i(r_i-1), \tfrac{1}{2}d(d-1)\right),$$

and the singular locus of $\mathrm{Sec}(C)$*, when C is smooth, consists of all lines that intersect C with total multiplicity at least* 3.

Let $S \subset \mathbb{P}^3$ be a general surface of degree d with $d \geq 4$. If $\mathrm{Bit}(S)$ *denotes the locus of bitangents to S and* $\mathrm{Infl}(S)$ *denotes the locus of inflectional tangents to S, then the singular locus of* $\mathrm{CH}_1(S)$ *is* $\mathrm{Bit}(S) \cup \mathrm{Infl}(S)$*, the bidegree of* $\mathrm{Bit}(S)$ *is*

$$\left(\tfrac{1}{2}d(d-1)(d-2)(d-3), \tfrac{1}{2}d(d-2)(d-3)(d+3)\right),$$

the bidegree of $\mathrm{Infl}(S)$ *is* $\big(d(d-1)(d-2), 3d(d-2)\big)$*, and the singular locus of* $\mathrm{Infl}(S)$ *consists of all lines that are inflectional tangents at at least two points of S or intersect S with multiplicity at least* 4 *at some point.*

The bidegree of $\mathrm{Infl}(S)$ also appears in [22, Prop. 4.1]. The bidegrees of $\mathrm{Bit}(S)$, $\mathrm{Infl}(S)$, and $\mathrm{Sec}(C)$, for smooth C, already appear in [2], a paper to which we owe a great debt. Nevertheless, we give new, more geometric, proofs not relying on Chern class techniques. The singular loci of $\mathrm{Sec}(C)$, $\mathrm{Bit}(S)$, and $\mathrm{Infl}(S)$ are partially described in Lemma 2.3, Lemma 4.3, and Lemma 4.6 in [2].

Using duality, we establish some relationships of the varieties in Theorem 1.1.

Theorem 1.2 *If C is a nondegenerate smooth space curve, then the secant lines of C are dual to the bitangent lines of the dual surface $C^\vee$ and the tangent lines of C are dual to the inflectional tangent lines of $C^\vee$.*

Congruences and line complexes have been actively studied both in the nineteenth century and in modern times. The study of congruences goes back to Kummer [20], who classified those of order 1; the order of a congruence is the number of lines in the congruence that pass through a general point in $\mathbb{P}^3$. The Chow hypersurfaces of space curves were introduced by Cayley [4] and generalized to arbitrary varieties by Chow and van der Waerden [5]. Many results from the second half of the nineteenth century are detailed in Jessop's monograph [16]. Hurwitz hypersurfaces and further generalizations known as higher associated or coisotropic hypersurfaces are studied in [11, 19, 28]. Catanese [3] shows that Chow hypersurfaces of space curves and Hurwitz hypersurfaces of surfaces are exactly the self-dual hypersurfaces in the Grassmannian $\mathrm{Gr}(1, \mathbb{P}^3)$. Ran [25] studies surfaces of order 1 in general Grassmannians and gives a modern proof of Kummer's classification. Congruences play a role in algebraic vision and multi-view geometry, where cameras are modeled as maps from $\mathbb{P}^3$ to congruences [24]. The multidegree of the image of several of those cameras is computed by Escobar and Knutson in [9].

In Sect. 2, we collect basic facts about the Grassmannian $\mathrm{Gr}(1, \mathbb{P}^3)$ and its subvarieties. Section 3 studies the singular locus of the Chow hypersurface of a space curve and computes its bidegree. Section 4 describes the singular locus of a Hurwitz hypersurface and Sect. 5 uses projective duality to calculate the bidegree of its components. In Sect. 6, we connect the intersection theory in $\mathrm{Gr}(1, \mathbb{P}^3)$ to Chow and Hurwitz hypersurfaces. Finally, Sect. 7 analyzes the singular loci of secant, bitangent, and inflectional congruences.

This article provides complete solutions to Problem 5 on Curves, Problem 4 on Surfaces, and Problem 3 on Grassmannians in [29].

2 The Degree of a Subvariety in $\mathrm{Gr}(1, \mathbb{P}^3)$

In this section, we provide the geometric definition for the degree of a subvariety in $\mathrm{Gr}(1, \mathbb{P}^3)$. An alternative approach, using coefficients of classes in the Chow ring, can be found in Sect. 6. For information about subvarieties of more general Grassmannians, we recommend [1].

The Grassmannian $\mathrm{Gr}(1, \mathbb{P}^3)$ of lines in $\mathbb{P}^3$ is a four-dimensional variety that embeds into $\mathbb{P}^5$ via the Plücker embedding. In particular, the line in 3-space spanned by two distinct points $(x_0 : x_1 : x_2 : x_3), (y_0 : y_1 : y_2 : y_3) \in \mathbb{P}^3$ is identified with the point $(p_{0,1} : p_{0,2} : p_{0,3} : p_{1,2} : p_{1,3} : p_{2,3}) \in \mathbb{P}^5$, where $p_{i,j}$ is the minor formed of ith and jth columns of the matrix $\left[\begin{smallmatrix} x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \end{smallmatrix}\right]$. The Plücker coordinates $p_{i,j}$ satisfy the relation $p_{0,1}p_{2,3} - p_{0,2}p_{1,3} + p_{0,3}p_{1,2} = 0$. Moreover, every point in $\mathbb{P}^5$ satisfying this relation is the Plücker coordinates of some line. Dually, a line in $\mathbb{P}^3$ is the intersection of two distinct planes. If the planes are given by the equations $a_0x_0 + a_1x_1 + a_2x_2 + a_3x_3 = 0$ and $b_0x_0 + b_1x_1 + b_2x_2 + b_3x_3 = 0$, then the minors $q_{i,j}$ of the matrix $\left[\begin{smallmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \end{smallmatrix}\right]$ are the dual Plücker coordinates and also satisfy $q_{0,1}q_{2,3} - q_{0,2}q_{1,3} + q_{0,3}q_{1,2} = 0$. The map given by $p_{0,1} \mapsto q_{2,3}$, $p_{0,2} \mapsto -q_{1,3}$, $p_{0,3} \mapsto q_{1,2}$, $p_{1,2} \mapsto q_{0,3}$, $p_{1,3} \mapsto -q_{0,2}$, and $p_{2,3} \mapsto q_{0,1}$ allows one to conveniently pass between these two coordinate systems.

A *line complex* is a threefold $\Sigma \subset \mathrm{Gr}(1, \mathbb{P}^3)$. For a general plane $H \subset \mathbb{P}^3$ and a general point $v \in H$, the degree of Σ is the number of points in Σ corresponding to a line $L \subset \mathbb{P}^3$ such that $v \in L \subset H$. For instance, if $C \subset \mathbb{P}^3$ is a curve, then the Chow hypersurface $\mathrm{CH}_0(C) := \{L \in \mathrm{Gr}(1, \mathbb{P}^3) : C \cap L \neq \varnothing\}$ is a line complex. A general plane H intersects C in $\deg(C)$ many points, so there are $\deg(C)$ many lines in H that pass through a general point $v \in H$ and intersect C; see Fig. 1. Hence, the degree of the Chow hypersurface is equal to the degree of the curve.

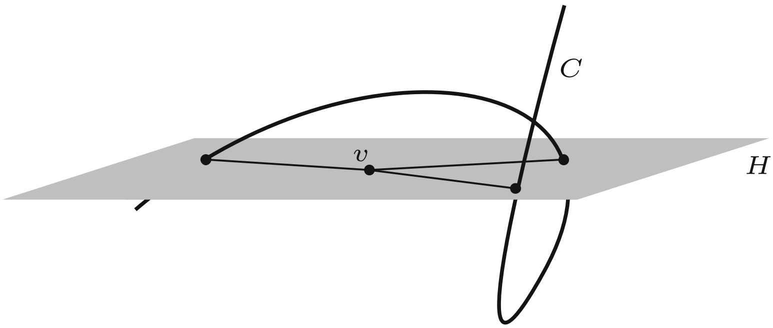

Fig. 1 The degree of the Chow hypersurface

A *congruence* is a surface $\Sigma \subset \mathrm{Gr}(1, \mathbb{P}^3)$. For a general point $v \in \mathbb{P}^3$ and a general plane $H \subset \mathbb{P}^3$, the bidegree of a congruence is a pair (α, β), where the *order* α is the number of points in Σ corresponding to a line $L \subset \mathbb{P}^3$ such that $v \in L$ and the *class* β is the number of points in Σ corresponding to lines $L \subset \mathbb{P}^3$ such that $L \subset H$. For instance, consider the congruence of all lines passing through a fixed point x. Given a general point v, this congruence contains a unique line passing through v, namely the line spanned by x and v. Given a general plane H, we have $x \notin H$, so this congruence does not contain any line that lies in H. Hence, the set of lines passing through a fixed point is a congruence with bidegree $(1, 0)$. A similar argument shows that the congruence of lines lying in a fixed plane has bidegree $(0, 1)$.

The degree of a curve $\Sigma \subset \mathrm{Gr}(1, \mathbb{P}^3)$ is the number of points in Σ corresponding to a line $L \subset \mathbb{P}^3$ that intersects a general line in $\mathbb{P}^3$. Equivalently, it is the number of points in the intersection of Σ with the Chow hypersurface of a general line. For instance, the set of all lines in $\mathbb{P}^3$ that lie in a fixed plane $H \subset \mathbb{P}^3$ and contain a fixed point $v \in H$ forms a curve in $\mathrm{Gr}(1, \mathbb{P}^3)$. This curve has degree 1, because a general line has a unique intersection point with H and there is a unique line passing through this point and v. In other words, this curve is a line in $\mathrm{Gr}(1, \mathbb{P}^3)$.

Finally, the degree of a zero-dimensional subvariety is simply the number of points in the variety.

3 Secants of Space Curves

This section describes the singular locus of the Chow hypersurface of a space curve. For a curve with mild singularities, we also compute the bidegree of its secant congruence.

A curve $C \subset \mathbb{P}^3$ is defined by at least two homogeneous polynomials in the coordinate ring of $\mathbb{P}^3$, and these polynomials are not uniquely determined. However, there is a single equation that encodes the curve C. Specifically, its Chow hypersurface $\mathrm{CH}_0(C) := \{L \in \mathrm{Gr}(1, \mathbb{P}^3) : C \cap L \neq \varnothing\}$ is determined by a single polynomial in the Plücker coordinates on $\mathrm{Gr}(1, \mathbb{P}^3)$. This equation, known as the *Chow form* of C, is unique up to rescaling and the Plücker relation. For more on Chow forms; see [6].

Example 3.1 ([6, Prop. 1.2]) The twisted cubic is a smooth rational curve of degree 3 in $\mathbb{P}^3$. Parametrically, this curve is the image of the map $\nu_3 : \mathbb{P}^1 \to \mathbb{P}^3$ defined by $(s : t) \mapsto (s^3 : s^2t : st^2 : t^3)$. The line L, which is determined by the two equations $a_0x_0 + a_1x_1 + a_2x_2 + a_3x_3 = 0$ and $b_0x_0 + b_1x_1 + b_2x_2 + b_3x_3 = 0$, intersects the twisted cubic if and only if there exists a point $(s : t) \in \mathbb{P}^1$ such that

$$a_0s^3 + a_1s^2t + a_2st^2 + a_3t^3 = 0 = b_0s^3 + b_1s^2t + b_2st^2 + b_3t^3 \,.$$

The resultant for these two cubic polynomials, which can be expressed as a determinant of an appropriate matrix with entries in $\mathbb{Z}[a_0, a_1, a_2, a_3, b_0, b_1, b_2, b_3]$, vanishes exactly when they have a common root. It follows that the line L meets the twisted cubic if and only if

$$0 = \det \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & 0 & 0 \\ 0 & a_0 & a_1 & a_2 & a_3 & 0 \\ 0 & 0 & a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 & 0 & 0 \\ 0 & b_0 & b_1 & b_2 & b_3 & 0 \\ 0 & 0 & b_0 & b_1 & b_2 & b_3 \end{bmatrix} = -\det \begin{bmatrix} q_{0,1} & q_{0,2} & q_{0,3} \\ q_{0,2} & q_{0,3} + q_{1,2} & q_{1,3} \\ q_{0,3} & q_{1,3} & q_{2,3} \end{bmatrix} ,$$

where $q_{i,j}$ are the dual Plücker coordinates. Hence, the Chow form of the twisted cubic is $q_{0,3}^3 + q_{0,3}^2 q_{1,2} - 2q_{0,2}q_{0,3}q_{1,3} + q_{0,1}q_{1,3}^2 + q_{0,2}^2 q_{2,3} - q_{0,1}q_{0,3}q_{2,3} - q_{0,1}q_{1,2}q_{2,3}$.

We next record a technical lemma. If I_X is the saturated homogeneous ideal defining the subvariety $X \subset \mathbb{P}^n$, then the tangent space $T_x(X)$ at the point $x \in X$ can be identified with $\left\{ y \in \mathbb{P}^n : \sum_{i=0}^n \frac{\partial f}{\partial x_i}(x) y_i = 0 \text{ for all } f(x_0, x_1, \ldots, x_n) \in I_X \right\}$.

Lemma 3.2 *Let $f : X \to Y$ be a birational finite surjective morphism between irreducible projective varieties and let $y \in Y$. The variety Y is smooth at the point y if and only if the fibre $f^{-1}(y)$ contains exactly one point $x \in X$, the variety X is smooth at the point x, and the differential $d_xf : T_x(X) \to T_y(Y)$ is an injection.*

Proof First, suppose that Y is smooth at the point y. Since Y is normal at the point y, the Zariski Connectedness Theorem [21, Sect. III.9.V] proves that the fibre $f^{-1}(y)$ is a connected set in the Zariski topology. As f is a finite morphism, its fibres are finite and we deduce that $f^{-1}(y) = \{x\}$. If Y_0 is the open set of smooth points in Y and let $X_0 := f^{-1}(Y_0)$, then Zariski's Main Theorem [21, Sect. III.9.I] implies that the restriction of f to X_0 is an isomorphism of X_0 with Y_0. In particular, we have that $x \in X_0 \subset X$ is a smooth point. Moreover, Theorem 14.9 in [13] shows that the differential d_xf is injective.

For the other direction, suppose that $f^{-1}(y) = \{x\}$ for some smooth point $x \in X$ with injective differential d_xf. Let Y_1 be an open neighbourhood of y

containing points in Y with one-element fibres and injective differentials. Combining Lemma 14.8 and Theorem 14.9 in [13] produces an isomorphism of $X_1 := f^{-1}(Y_1)$ with Y_1. Since $x \in X_1$ is smooth, we conclude that $y \in Y_1 \subset Y$ is smooth. □

When the curve C has degree at least two, the set of lines that meet it in two points forms a surface $\mathrm{Sec}(C) \subset \mathrm{Gr}(1, \mathbb{P}^3)$ called the *secant congruence* of C. More precisely, $\mathrm{Sec}(C)$ is the closure in $\mathrm{Gr}(1, \mathbb{P}^3)$ of the set of points corresponding to a line in $\mathbb{P}^3$ which intersects the curve C at two smooth points. A line meeting C at a singular point might not belong to $\mathrm{Sec}(C)$, even though it has intersection multiplicity at least two with the curve; see Remark 3.4.

The following theorem is the main result in this section.

Theorem 3.3 *Let $C \subset \mathbb{P}^3$ be an irreducible curve of degree at least* 2*. If* $\mathrm{Sing}(C)$ *denotes the singular locus of the curve C, then the singular locus of the Chow hypersurface for C is* $\mathrm{Sec}(C) \cup \left(\bigcup_{x \in \mathrm{Sing}(C)} \{L \in \mathrm{Gr}(1, \mathbb{P}^3) : x \in L\} \right)$.

Proof We first show that the incidence variety $\Phi_C := \{(v, L) : v \in L\} \subset C \times \mathrm{Gr}(1, \mathbb{P}^3)$ is smooth at the point (v, L) if and only if the curve C is smooth at the point $v \in C$. Let $f_1, f_2, \ldots, f_k \in \mathbb{C}[x_0, x_1, x_2, x_3]$ be generators for the saturated homogeneous ideal of C in $\mathbb{P}^3$. Consider the affine chart of $\mathbb{P}^3 \times \mathrm{Gr}(1, \mathbb{P}^3)$ where $x_0 \neq 0$ and $p_{0,1} \neq 0$. We may assume that $v = (1 : \alpha : \beta : \gamma)$ and the line L is spanned by the points $(1 : 0 : a : b)$ and $(0 : 1 : c : d)$. We have that $v \in L$ if and only if the line L is given by the row space of matrix

$$\begin{bmatrix} 1 & \alpha & \beta & \gamma \\ 0 & 1 & c & d \end{bmatrix} = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \beta - \alpha c & \gamma - \alpha d \\ 0 & 1 & c & d \end{bmatrix},$$

which is equivalent to $a = \beta - \alpha c$ and $b = \gamma - \alpha d$. Hence, in the chosen affine chart, Φ_C can be written as

$$\left\{ (\alpha, \beta, \gamma, a, b, c, d) : f_i(1, \alpha, \beta, \gamma) = 0 \text{ for } 1 \leq i \leq k,\ a = \beta - \alpha c,\ b = \gamma - \alpha d \right\}.$$

As $\dim \Phi_C = 3$, it is smooth at the point (v, L) if and only if its tangent space has dimension three or, equivalently, the Jacobian matrix

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1}(1,\alpha,\beta,\gamma) & \frac{\partial f_1}{\partial x_2}(1,\alpha,\beta,\gamma) & \frac{\partial f_1}{\partial x_3}(1,\alpha,\beta,\gamma) & 0 & 0 & 0 & 0 \\ \frac{\partial f_2}{\partial x_1}(1,\alpha,\beta,\gamma) & \frac{\partial f_2}{\partial x_2}(1,\alpha,\beta,\gamma) & \frac{\partial f_2}{\partial x_3}(1,\alpha,\beta,\gamma) & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_k}{\partial x_1}(1,\alpha,\beta,\gamma) & \frac{\partial f_k}{\partial x_2}(1,\alpha,\beta,\gamma) & \frac{\partial f_k}{\partial x_3}(1,\alpha,\beta,\gamma) & 0 & 0 & 0 & 0 \\ -c & 1 & 0 & -1 & 0 & -\alpha & 0 \\ -d & 0 & 1 & 0 & -1 & 0 & -\alpha \end{bmatrix}$$

has rank four. We see that this Jacobian matrix has rank four if and only if the Jacobian matrix of C has rank two, in which case $v \in C$ is smooth. Therefore, we deduce that Φ_C is smooth at the point (v, L) exactly when C is smooth at the point v.

By Lemma 14.8 in [13], the projection $\pi\colon \Phi_C \to \mathrm{CH}_0(C)$ defined by $(v, L) \mapsto L$ is finite; otherwise C would contain a line contradicting our assumptions. Moreover, the general fibre of π has cardinality 1 because the general line $L \in \mathrm{CH}_0(C)$ intersects C in a single point. Hence, π is birational. Applying Lemma 3.2 shows that $\mathrm{CH}_0(C)$ is smooth at L if and only if $\pi^{-1}(L) = \{(v, L)\}$ where $v \in C$ is a smooth point and the differential $d_{(v,L)}\pi$ is injective. Using our chosen affine chart, we see that the differential $d_{(v,L)}\pi$ sends every element in the kernel of the Jacobian matrix to its last four coordinates. This map is not injective if and only if the kernel contains an element of the form $\begin{bmatrix} * & * & * & 0 & 0 & 0 & 0 \end{bmatrix}^{\mathsf{T}} \neq 0$. Such an element belongs to the kernel if and only if it is equal to $\begin{bmatrix} \lambda & c\lambda & d\lambda & 0 & 0 & 0 & 0 \end{bmatrix}^{\mathsf{T}}$ for some $\lambda \in \mathbb{C} \setminus \{0\}$ and

$$\frac{\partial f_i}{\partial x_1}(1, \alpha, \beta, \gamma) + c\frac{\partial f_i}{\partial x_2}(1, \alpha, \beta, \gamma) + d\frac{\partial f_i}{\partial x_3}(1, \alpha, \beta, \gamma) = 0$$

for all $1 \leq i \leq k$. Hence, for a smooth point $v \in C$, the differential $d_{(v,L)}\pi$ is not injective if and only if L is the tangent line of C at v. Since we have that $|\pi^{-1}(L)| = 1$ if and only if L is not a secant line and all tangent lines to C are contained in $\mathrm{Sec}(C)$, we conclude that $\mathrm{CH}_0(C)$ is smooth at L if and only if $L \notin \mathrm{Sec}(C)$ and L meets C at a smooth point. □

Remark 3.4 Local computations show that the secant congruence of C generally does not contain all lines through singular points of C. To be more explicit, let $x \in C$ be an *ordinary singularity*; the point x is the intersection of r branches of C with $r \geq 2$, and the r tangent lines of the branches at x are pairwise different. We claim that a line L intersecting C only at the point x is contained in $\mathrm{Sec}(C)$ if and only if L lies in a plane spanned by two of the r tangent lines at x. The union of all those lines forms the tangent star of C at x; see [17, 27].

Suppose that $x = (1 : 0 : 0 : 0)$ and $L \in \mathrm{Sec}(C)$ intersects the curve C only at the point x. The line L must be the limit of a family of lines L_t that intersect C at two distinct smooth points. Without loss of generality, the line L is not one of the tangent lines of the curve C at the point x and each line L_t intersects at least two distinct branches of C. Since there are only finitely many branches, we can also assume that each line L_t in the family intersects the same two branches of the curve C. These two branches are parametrized by $\big(1 : f_1(s) : f_2(s) : f_3(s)\big)$ and $\big(1 : g_1(s) : g_2(s) : g_3(s)\big)$ with $f_i(0) = 0 = g_j(0)$ for $1 \leq i, j \leq 3$. It follows that tangent lines to these branches are spanned by x and $\big(1 : f_1'(0) : f_2'(0) : f_3'(0)\big)$ or $\big(1 : g_1'(0) : g_2'(0) : g_3'(0)\big)$. Parametrizing intersection points, we see that the line L_t intersects the first branch at $\big(1 : f_1\big(\varphi(t)\big) : f_2\big(\varphi(t)\big) : f_3\big(\varphi(t)\big)\big)$ and the second branch at $\big(1 : g_1\big(\psi(t)\big) : g_2\big(\psi(t)\big) : g_3\big(\psi(t)\big)\big)$ where $\varphi(0) = 0 = \psi(0)$. Hence, the Plücker coordinates for L_t are

$$\left(\tfrac{g_1(\psi(t))-f_1(\varphi(t))}{t} : \tfrac{g_2(\psi(t))-f_2(\varphi(t))}{t} : \cdots : \tfrac{f_2(\varphi(t))g_3(\psi(t))-f_3(\varphi(t))g_2(\psi(t))}{t}\right).$$

Taking the limit as $t \to 0$, we obtain the line L with Plücker coordinates

$$\left(g_1'(0)\psi'(0) - f_1'(0)\varphi'(0) : g_2'(0)\psi'(0) - f_2'(0)\varphi'(0) : \cdots : 0\right).$$

This line is spanned by the point x and

$$\left(1 : g_1'(0)\psi'(0) - f_1'(0)\varphi'(0) : g_2'(0)\psi'(0) - f_2'(0)\varphi'(0) : g_3'(0)\psi'(0) - f_3'(0)\varphi'(0)\right),$$

so it lies in the plane spanned by the two tangent lines. From this computation, we also see that all lines passing through x and lying in the plane spanned by the tangent lines can be approximated by lines that intersect both of the branches at points different from x. For this, one need only choose $\varphi(t) = \lambda t$ and $\psi(t) = \mu t$ for all possible $\lambda, \mu \in \mathbb{C} \setminus \{0\}$.

Using Chern classes, Proposition 2.1 in [2] calculates the bidegree of the secant congruence of a smooth curve. We give a geometric description of this bidegree and extend it to curves with ordinary singularities.

Theorem 3.5 *If $C \subset \mathbb{P}^3$ is a nondegenerate irreducible curve of degree d and genus g having only ordinary singularities $x_1, x_2, \ldots, x_s$ with multiplicities $r_1, r_2, \ldots, r_s$, then the bidegree of the secant congruence* $\mathrm{Sec}(C)$ *is*

$$\left(\binom{d-1}{2} - g - \sum_{i=1}^{s} \binom{r_i}{2}, \binom{d}{2}\right).$$

Proof Let $H \subset \mathbb{P}^3$ be a general plane. The intersection of H with C consists of d points. Any two of these points define a secant line lying in H; see Fig. 2. Hence, there are $\binom{d}{2}$ secant lines contained in H, which gives the class of $\mathrm{Sec}(C)$.

To compute the order of $\mathrm{Sec}(C)$, let $v \in \mathbb{P}^3$ be a general point. Projecting away from v defines a rational map $\pi_v : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$. Set $C' := \pi_v(C)$. The map π_v sends a line passing through v and intersecting C at two points to a simple node of the plane curve C'; see Fig. 6. Moreover, every ordinary singularity of C is sent to an ordinary singularity of C' with the same multiplicity, and the plane curve C' has the same

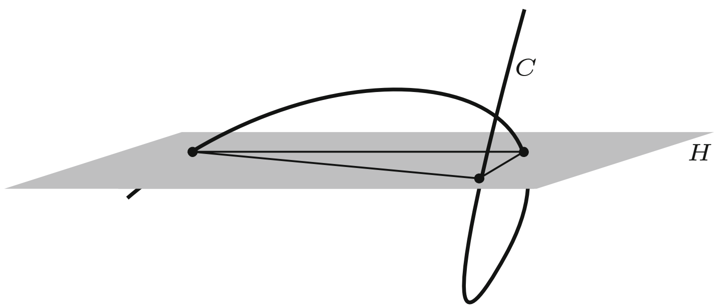

Fig. 2 The class of the secant congruence

degree as the space curve C. As the geometric genus is invariant under birational transformation, it also has the same genus; see [14, Theorem II.8.19]. Thus, the genus-degree formula for plane curves [26, p. 54, Eq. (7)] shows that the genus of C is equal to $\binom{d-1}{2} - \sum_{i=1}^{s} \binom{r_i}{2}$ minus the number of secants of C passing through v. □

Remark 3.6 If $C \subset \mathbb{P}^3$ is a curve of degree at least 2 that is contained in a plane, then its secant congruence consists of all lines in that plane and has bidegree $(0, 1)$.

Problem 5 on Curves in [29] asks to compute the dimension and bidegree of $\mathrm{Sing}(\mathrm{CH}_0(C))$. When C is not a line, Theorem 3.3 establishes that $\mathrm{Sing}(\mathrm{CH}_0(C))$ is two-dimensional. For completeness, we also state its bidegree explicitly.

Corollary 3.7 *If $C \subset \mathbb{P}^3$ is an irreducible curve of degree $d \geq 2$ and geometric genus g having only ordinary singularities $x_1, x_2, \ldots, x_s$ with multiplicities $r_1, r_2, \ldots, r_s$, then the bidegree of* $\mathrm{Sing}\left(\mathrm{CH}_0(C)\right)$ *equals* $\left(\binom{d-1}{2} - g - \sum_{i=1}^{s} \binom{r_i}{2} + s, \binom{d}{2}\right)$ *if C is nondegenerate, and $(s, 1)$ if C is contained in a plane.*

Proof The bidegree of each congruence $\{L \in \mathrm{Gr}(1, \mathbb{P}^3) : x_i \in L\}$ is $(1, 0)$. Hence, combining Theorem 3.3, Theorem 3.5, and Remark 3.6 proves the corollary. □

4 Bitangents and Inflections of a Surface

This section describes the singular locus of the Hurwitz hypersurface of a surface in $\mathbb{P}^3$. For a surface $S \subset \mathbb{P}^3$ that is not a plane, the Hurwitz hypersurface $\mathrm{CH}_1(S)$ is the Zariski closure of the set of all lines in $\mathbb{P}^3$ that are tangent to S at a smooth point. Its defining equation in Plücker coordinates is known as the Hurwitz form of S; see [28].

In analogy with the secant congruence of a curve, we associate two congruences to a surface $S \subset \mathbb{P}^3$. Specifically, the Zariski closure in $\mathrm{Gr}(1, \mathbb{P}^3)$ of the set of lines tangent to a surface S at two smooth points forms the *bitangent congruence*;

$$\mathrm{Bit}(S) := \overline{\left\{L \in \mathrm{Gr}(1, \mathbb{P}^3) : x, y \in L \subset T_x(S) \cap T_y(S) \text{ for distinct smooth points } x, y \in S\right\}}.$$

The *inflectional locus* associated to S is the Zariski closure in $\mathrm{Gr}(1, \mathbb{P}^3)$ of the set of lines that intersect the surface S at a smooth point with multiplicity at least 3;

$$\mathrm{Infl}(S) := \overline{\left\{L \in \mathrm{Gr}(1, \mathbb{P}^3) : L \text{ intersects } S \text{ at a smooth point with multiplicity at least } 3\right\}}.$$

A general surface of degree d in $\mathbb{P}^3$ is a surface defined by a polynomial corresponding to a general point in $\mathbb{P}(\mathbb{C}[x_0, x_1, x_2, x_3]_d)$. For a general surface, the inflectional locus is a congruence. However, this is not always the case, as Remark 5.8 demonstrates.

In parallel with Sect. 3, the main result in this section describes the singular locus of the Hurwitz hypersurface of S.

Theorem 4.1 *If $S \subset \mathbb{P}^3$ is an irreducible smooth surface of degree at least 4 which does not contain any lines, then we have* $\operatorname{Sing}\big(\mathrm{CH}_1(S)\big) = \mathrm{Bit}(S) \cup \mathrm{Infl}(S)$.

Proof We first show that the incidence variety

$$\Phi_S := \{(v, L) : v \in L \subset T_v(S)\} \subset S \times \mathrm{Gr}(1, \mathbb{P}^3)$$

is smooth. Let $f \in \mathbb{C}[x_0, x_1, x_2, x_3]$ be the defining equation for S in $\mathbb{P}^3$. Consider the affine chart in $\mathbb{P}^3 \times \mathrm{Gr}(1, \mathbb{P}^3)$ where $x_0 \neq 0$ and $p_{0,1} \neq 0$. We may assume that $v = (1 : \alpha : \beta : \gamma)$ and the line L is spanned by the points $(1 : 0 : a : b)$ and $(0 : 1 : c : d)$. In this affine chart, S is defined by $g_0(x_1, x_2, x_3) := f(1, x_1, x_2, x_3)$. As in the proof of Theorem 3.3, we have that $v \in L$ if and only if $a = \beta - \alpha c$ and $b = \gamma - \alpha d$. For such a pair (v, L), we also have that $L \subset T_v(S)$ if and only if $(0 : 1 : c : d) \in T_v(S)$. Setting $g_1 := \frac{\partial g_0}{\partial x_1} + c\frac{\partial g_0}{\partial x_2} + d\frac{\partial g_0}{\partial x_3}$, we have $L \subset T_v(S)$ if and only if $g_1(\alpha, \beta, \gamma) = 0$. Hence, in the chosen affine chart, Φ_S can be written as

$$\left\{(\alpha, \beta, \gamma, a, b, c, d) : g_j(\alpha, \beta, \gamma) = 0 \text{ for } 0 \le j \le 1,\ a = \beta - \alpha c,\ b = \gamma - \alpha d\right\}.$$

As $\dim \Phi_S = 3$, it is smooth at the point (v, L) if and only if its tangent space has dimension three or, equivalently, its Jacobian matrix

$$\begin{bmatrix} \frac{\partial g_0}{\partial x_1}(\alpha,\beta,\gamma) & \frac{\partial g_0}{\partial x_2}(\alpha,\beta,\gamma) & \frac{\partial g_0}{\partial x_3}(\alpha,\beta,\gamma) & 0 & 0 & 0 & 0 \\ \frac{\partial g_1}{\partial x_1}(\alpha,\beta,\gamma) & \frac{\partial g_1}{\partial x_2}(\alpha,\beta,\gamma) & \frac{\partial g_1}{\partial x_3}(\alpha,\beta,\gamma) & 0 & 0 & \frac{\partial g_0}{\partial x_2}(\alpha,\beta,\gamma) & \frac{\partial g_0}{\partial x_3}(\alpha,\beta,\gamma) \\ -c & 1 & 0 & -1 & 0 & -\alpha & 0 \\ -d & 0 & 1 & 0 & -1 & 0 & -\alpha \end{bmatrix}$$

has rank four. Since S is smooth, we deduce that this Jacobian matrix has rank four, so Φ_S is also smooth.

Since S does not contain any lines, all fibres of the projection $\pi \colon \Phi_S \to \mathrm{CH}_1(S)$ defined by $(v, L) \mapsto L$ are finite, so Lemma 14.8 in [13] implies that π is finite. Moreover, the general fibre of π has cardinality 1, so π is birational. Applying Lemma 3.2 shows that $\mathrm{CH}_1(S)$ is smooth at the point (v, L) if and only if the fibre $\pi^{-1}(L)$ consists of one point (v, L) and the differential $d_{(x,L)}\pi$ is injective. In particular, we have $|\pi^{-1}(L)| = 1$ if and only if L is not a bitangent. It remains to show that the differential $d_{(v,L)}\pi$ is injective if and only if L is a simple tangent of S at v. Using our chosen affine chart, we see that the differential $d_{(v,L)}\pi$ projects every element in the kernel of the Jacobian matrix on its last four coordinates. This map is not injective if and only if the kernel contains an element of the form $\begin{bmatrix} * & * & * & 0 & 0 & 0 & 0\end{bmatrix}^{\mathsf{T}} \neq 0$. Such an element belongs to the kernel if and only if it is equal to $\begin{bmatrix}\lambda & c\lambda & d\lambda & 0 & 0 & 0 & 0\end{bmatrix}^{\mathsf{T}}$ for some $\lambda \in \mathbb{C}\backslash\{0\}$ and $g_1(\alpha, \beta, \gamma) = 0 = g_2(\alpha, \beta, \gamma)$ where $g_2 := \frac{\partial g_1}{\partial x_1} + c\frac{\partial g_1}{\partial x_2} + d\frac{\partial g_1}{\partial x_3}$. Parametrizing the line L by

$$\ell(s,t) := (s : s\alpha + t : s\beta + tc : s\gamma + td)$$

for $(s : t) \in \mathbb{P}^1$ shows that the line L intersects the surface S with multiplicity at least 3 at v if and only if $f\big(\ell(s,t)\big)$ is divisible by t^3. This is equivalent to the conditions that $g_1(\alpha,\beta,\gamma) = \frac{\partial}{\partial t}\big[f\big(\ell(s,t)\big)\big]\big|_{(1,0)} = 0$ and $g_2(\alpha,\beta,\gamma) = \frac{\partial^2}{\partial^2 t}\big[f\big(\ell(s,t)\big)\big]\big|_{(1,0)} = 0$. □

Remark 4.2 If S is a surface of degree at most 3 and the line L is bitangent to S, then L is contained in S. Indeed, if L is not contained in S, then the intersection $L \cap S$ consists of at most three points, counted with multiplicity, so L cannot be a bitangent. On the other hand, when the degree of S is at least four, the hypothesis that S does not contain any lines is relatively mild. For example, a general surface of degree at least 4 in $\mathbb{P}^3$ does not contain a line; see [32].

5 Projective Duality

This section uses projective duality to compute the bidegrees of the components of the singular locus of the Hurwitz hypersurface of a surface in $\mathbb{P}^3$, and to relate the secant congruence of a curve to the bitangent congruence of its dual surface.

Let $\mathbb{P}^n$ be the projectivization of the vector space $\mathbb{C}^{n+1}$. If $(\mathbb{P}^n)^*$ denotes the projectivization of the dual vector space $(\mathbb{C}^{n+1})^*$, then the points in $(\mathbb{P}^n)^*$ correspond to hyperplanes in $\mathbb{P}^n$. Given a projective subvariety $X \subset \mathbb{P}^n$, a hyperplane in $\mathbb{P}^n$ is tangent to X at a smooth point $x \in X$ if it contains the embedded tangent space $T_x(X) \subset \mathbb{P}^n$. The *dual variety* $X^\vee$ is the Zariski closure in $(\mathbb{P}^n)^*$ of the set of all hyperplanes in $\mathbb{P}^n$ that are tangent to X at some smooth point.

Example 5.1 If V is a linear subspace of $\mathbb{C}^{n+1}$ and $X := \mathbb{P}(V)$, then the dual variety $X^\vee$ is the set of all hyperplanes containing $\mathbb{P}(V)$, which is exactly the projectivization of the orthogonal complement $V^\perp \subset (\mathbb{C}^{n+1})^*$ with respect to the nondegenerate bilinear form $(x,y) \mapsto \sum_{i=0}^{n} x_i y_i$. In particular, $X^\vee$ is not the projectivization of V^*, and $(\mathbb{P}^n)^\vee = \varnothing$.

Remark 5.2 The dual of a line in $\mathbb{P}^2$ is a point, and the dual of a plane curve of degree at least 2 is again a plane curve. The dual of a line in $\mathbb{P}^3$ is a line, and the dual of a curve in $\mathbb{P}^3$ of degree at least 2 is a surface. The dual of plane in $\mathbb{P}^3$ is a point and the dual of a surface in $\mathbb{P}^3$ of degree at least 2 can be either a curve or a surface.

From our perspective, the key properties of dual varieties are the following. If X is irreducible, then its dual $X^\vee$ is also irreducible; see [11, Proposition I.1.3]. Moreover, the Biduality Theorem shows that, if $x \in X$ is smooth and $H \in X^\vee$ is smooth, then H is tangent to X at the point x if and only if the hyperplane in $(\mathbb{P}^n)^*$ corresponding to x is tangent to $X^\vee$ at the point H; see [11, Theorem I.1.1]. In particular, any irreducible variety $X \subset \mathbb{P}^n$ is equal to its double dual $(X^\vee)^\vee \subset \mathbb{P}^n$; again see [11, Theorem I.1.1].

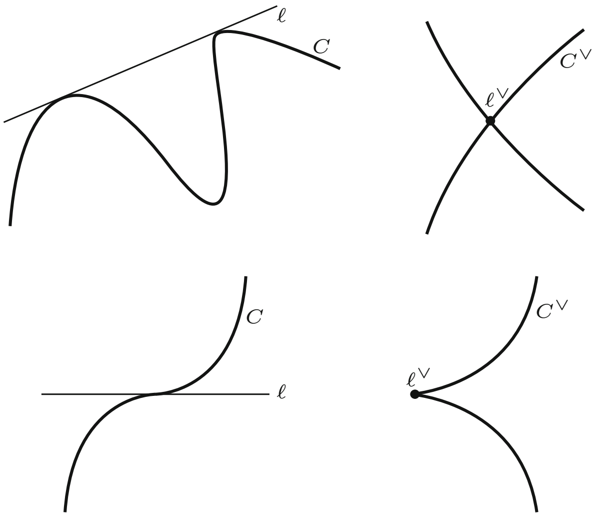

Fig. 3 A bitangent and an inflectional line corresponding to a node and a cusp of the dual curve

The next lemma, which relates the number and type of singularities of a plane curve to the degree of its dual variety, plays an important role in calculating the bidegrees of the bitangent and inflectional congruences. A point v on a planar curve C is a *simple node* or a *cusp* if the formal completion of $\mathscr{O}_{C,v}$ is isomorphic to $\mathbb{C}[\![z_1, z_2]\!]/(z_1^2 + z_2^2)$ or $\mathbb{C}[\![z_1, z_2]\!]/(z_1^3 + z_2^2)$ respectively; see Fig. 3. Both singularities have multiplicity 2; nodes have two distinct tangents and cusps have a single tangent.

Lemma 5.3 (Plücker's Formula [7, Example 1.2.8]) *If $C \subset \mathbb{P}^2$ is an irreducible curve of degree d with exactly κ cusps, δ simple nodes, and no other singularities, then the degree of the dual curve $C^\vee$ is $d(d-1) - 3\kappa - 2\delta$.*

Proof (Sketch) Let $f \in \mathbb{C}[x_0, x_1, x_2]$ be the defining equation for C in $\mathbb{P}^2$, so we have $\deg(f) = d$. To begin, assume that C is smooth. The degree of its dual $C^\vee \subset (\mathbb{P}^2)^*$ is the number of points of $C^\vee$ lying on a general line $L \subset (\mathbb{P}^2)^*$. By duality, the degree equals the number of tangent lines to C passing through a general point $y \in \mathbb{P}^2$. Such a tangent line at the point $v \in C$ passes through the point y if and only if $g := y_0 \frac{\partial f}{\partial x_0}(v) + y_1 \frac{\partial f}{\partial x_1}(v) + y_2 \frac{\partial f}{\partial x_2}(v) = 0$. Hence, the degree of $C^\vee$ is the number of points in $\mathrm{V}(f, g)$; the vanishing set of f and g. Since $\deg(g) = d - 1$, this finite set contains $d(d-1)$ points.

If C is singular, then the degree of $C^\vee$ is the number of lines that are tangent to C at a smooth point and pass through the general point y. Those smooth points are contained in the set $\mathrm{V}(f, g)$, but all of the singular points also lie in $\mathrm{V}(f, g)$. The curve $\mathrm{V}(g)$ passes through each node of C with intersection multiplicity two and through each cusp of C with intersection multiplicity 3. Therefore, we conclude that $\deg(C^\vee) = d(d-1) - 3\kappa - 2\delta$. □

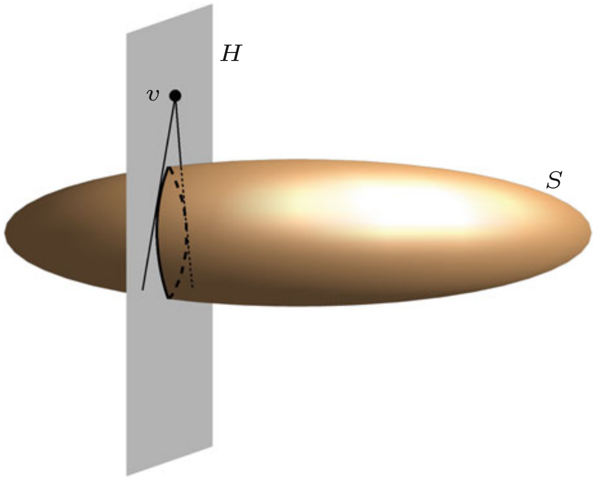

Fig. 4 The degree of the Hurwitz hypersurface

Using Lemma 5.3, we can compute the degree of the Hurwitz hypersurface of a smooth surface; this formula also follows from Theorem 1.1 in [28].

Proposition 5.4 *For an irreducible smooth surface* $S \subset \mathbb{P}^3$ *of degree* d *with* $d \geq 2$, *the degree of the Hurwitz hypersurface* $\mathrm{CH}_1(S)$ *is* $d(d-1)$.

Proof Let $H \subset \mathbb{P}^3$ be a general plane and $v \in H$ be a general point. The degree of $\mathrm{CH}_1(S)$ is the number of tangent lines L to S such that $v \in L \subset H$. Bertini's Theorem [13, Theorem 17.16] implies that the intersection $S \cap H$ is a smooth plane curve of degree d. The degree of $\mathrm{CH}_1(S)$ is the number of tangent lines to $S \cap H$ passing through the general point v; see Fig. 4. By definition, this is equal to the degree of the dual plane curve $(S \cap H)^\vee$, so Lemma 5.3 shows $\deg\big(\mathrm{CH}_1(S)\big) = d(d-1)$. □

Using Lemma 5.3, we can also count the number of bitangents and inflectional tangents to a general smooth plane curve.

Proposition 5.5 *A general smooth irreducible curve in* $\mathbb{P}^2$ *of degree* d *has exactly* $\frac{1}{2}d(d-2)(d-3)(d+3)$ *bitangents and* $3d(d-2)$ *inflectional tangents.*

Proof Let $C \subset \mathbb{P}^2$ be a general smooth irreducible curve of degree d. A bitangent to C corresponds to a node of $C^\vee$, and an inflectional tangent to C corresponds to a cusp of $C^\vee$; see Fig. 3 and [12, pp. 277–278]. Lemma 5.3 shows that $C^\vee$ has degree $d(d-1)$. Let κ and δ be the number of cusps and nodes of $C^\vee$, respectively. Applying Lemma 5.3 to the plane curve $C^\vee$ yields

$$d = \deg(C) = \deg\big((C^\vee)^\vee\big) = d(d-1)\big(d(d-1)-1\big) - 3\kappa - 2\delta\,.$$

The dual curves C and $C^\vee$ have the same geometric genus; see [31, Proposition 1.5]. Hence, the genus-degree formula [26, p. 54, Eq. (7)] gives

$$\tfrac{1}{2}(d-1)(d-2) = \text{genus}(C) = \text{genus}(C^\vee) = \tfrac{1}{2}\big(d(d-1)-1\big)\big(d(d-1)-2\big)-\kappa-\delta\,.$$

Solving this system of two linear equations in κ and δ, we obtain $\kappa = 3d(d-2)$ and $\delta = \frac{1}{2}d(d-2)(d-3)(d+3)$. □

The next result is the main theorem in this section and solves Problem 4 on Surfaces in [29]. The bidegrees of the bitangent and the inflectional congruence for a general smooth surface appear in [2, Proposition 3.3], and the bidegree of the inflectional congruence also appears in [22, Proposition 4.1].

Theorem 5.6 *Let $S \subset \mathbb{P}^3$ be a general smooth irreducible surface of degree d with $d \geq 4$. The bidegree of* Bit(S) *is* $\big(\frac{1}{2}d(d-1)(d-2)(d-3), \frac{1}{2}d(d-2)(d-3)(d+3)\big)$, *and the bidegree of* Infl(S) *is* $\big(d(d-1)(d-2), 3d(d-2)\big)$.

Proof For a general plane $H \subset \mathbb{P}^3$, Bertini's Theorem [13, Theorem 17.16] implies that the intersection $S \cap H$ is a smooth plane curve of degree d. By Proposition 5.5, the number of bitangents to S contained in H is $\frac{1}{2}d(d-2)(d-3)(d+3)$, which is the class of Bit(S). Similarly, the number of inflectional tangents to S contained in H is $3d(d-2)$, which is the class of Infl(S).

It remains to calculate the number of bitangents and inflectional lines of the surface S that pass through a general point $y \in \mathbb{P}^3$. Following the ideas in [23, p. 230], let $f \in \mathbb{C}[x_0, x_1, x_2, x_3]$ be the defining equation for S in $\mathbb{P}^3$, and consider the polar curve $C \subset S$ with respect to the point y; the set C consists of all points $x \in S$ such that the line through y and x is tangent to S at the point x; see Fig. 5. The condition that the point x lies on the curve C is equivalent to saying that the point y belongs to $T_x(S)$. As in the proof for Lemma 5.3, we have $C = \mathrm{V}(f, g)$ where $g := y_0 \frac{\partial f}{\partial x_0} + y_1 \frac{\partial f}{\partial x_1} + y_2 \frac{\partial f}{\partial x_2} + y_3 \frac{\partial f}{\partial x_3}$. Thus, the curve C has degree $d(d-1)$.

Projecting away from the point y gives the rational map $\pi_y \colon \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$. Restricted to the surface S, this map is generically finite, with fibres of cardinality d, and is

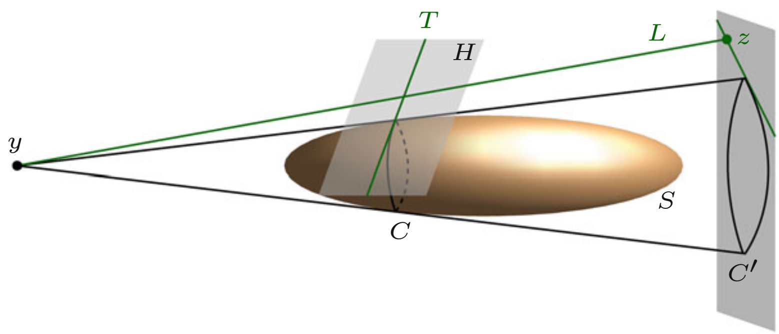

Fig. 5 Polar curve

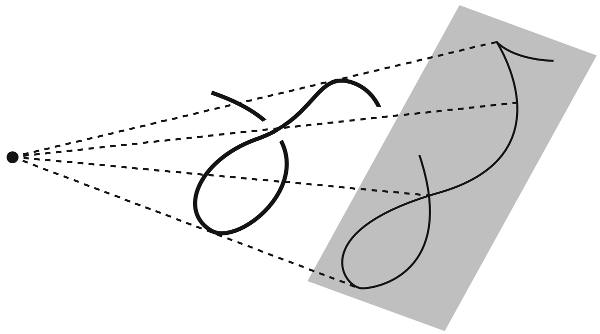

Fig. 6 A secant projecting onto a node and a tangent projecting to a cusp

ramified over the curve C. If C' is the image of C under π_y, then a bitangent to the surface S that passes through y contains two points of C and these points are mapped to a simple node in C'; see Fig. 6. All of these nodes in C' have two distinct tangent lines because no bitangent line passing through y is contained in a bitangent plane that is tangent at the same two points as the line; the bitangent planes to S form a one-dimensional family, so the union of bitangent lines they contain is a surface in $\mathbb{P}^3$ that does not contain the general point y.

We claim that the inflectional lines to S passing through the point y are exactly the tangent lines of C passing through y. The line between a point $x \in S$ and the point y is parametrized by the map $\ell\colon \mathbb{P}^1 \to \mathbb{P}^3$ which sends the point $(s:t) \in \mathbb{P}^1$ to the point $(sx_0 + ty_0 : sx_1 + ty_1 : sx_2 + ty_2 : sx_3 + ty_3) \in \mathbb{P}^3$. It follows that this line is an inflectional tangent to S if and only if $f\big(\ell(s,t)\big)$ is divisible by t^3. This is equivalent to the conditions that $\frac{\partial}{\partial t}\big[f\big(\ell(s,t)\big)\big]\big|_{(1,0)} = 0$ and $\frac{\partial^2}{\partial t^2}\big[f\big(\ell(s,t)\big)\big]\big|_{(1,0)} = 0$, which means that $x \in C$ and $y_0\frac{\partial g}{\partial x_0} + y_1\frac{\partial g}{\partial x_1} + \cdots + y_3\frac{\partial g}{\partial x_3} = 0$, or in other words $y \in T_x(C)$. Therefore, the inflectional lines to S passing through y are the tangents to C passing through y, and are mapped to the cusps of C'; again see Fig. 6.

Since the bitangent and inflectional lines to S passing through y correspond to nodes and cusps of C', it suffices to count the number κ' of cusps and the number δ' of simple nodes in the plane curve C'. We subdivide these calculations as follows.

$\kappa' = d(d-1)(d-2)$ From our parametrization of the line through points $x \in S$ and y, we see that this line is an inflectional tangent to S if and only if $x \in \mathrm{V}(f,g,h)$ where $h := y_0\frac{\partial g}{\partial x_0} + y_1\frac{\partial g}{\partial x_1} + \cdots + y_3\frac{\partial g}{\partial x_3}$. Since $\deg(h) = d-2$ and S is general, the set $\mathrm{V}(f,g,h)$ consists of $d(d-1)(d-2)$ points.

$\deg\big((C')^\vee\big) = \deg(S^\vee)$ By duality, the degree d' of the curve $(C')^\vee$ is the number of tangent lines to $C' \subset \mathbb{P}^2$ passing through a general point $z \in \mathbb{P}^2$. The preimage of z under the projection π_y is a line $L \subset \mathbb{P}^3$ containing y; see Fig. 5. Hence, d' is the number of tangent lines to C intersecting L in a point different from y. For every line T that is tangent to C at a point x and intersects the line L, it follows that the pair L and T spans the tangent plane of S at the point x. On the other hand, given any plane H which is tangent to S at the point x and contains L, we deduce that x must lie on the polar curve C and H is spanned by L and the tangent line to C at x, so this tangent line intersects L. Therefore, d' is the number of tangent planes to S containing L, which is the degree of the dual surface $S^\vee$.

$\deg(S^\vee) = d(d-1)^2$ By duality, the degree of $S^\vee$ is the number of tangent planes to the surface S containing a general line, or the number of tangent planes to S containing two general points $y, z \in \mathbb{P}^3$. Thus, this is the number of intersection points of the two polar curves of S determined by y and z, which is the cardinality of the set $\mathrm{V}(f, g, \tilde{g})$ where $\tilde{g} := z_0 \frac{\partial f}{\partial x_0} + z_1 \frac{\partial f}{\partial x_1} + z_2 \frac{\partial f}{\partial x_2} + z_3 \frac{\partial f}{\partial x_3}$. Since $\deg(\tilde{g}) = d-1$, we conclude that $\deg(S^\vee) = d(d-1)^2$.

Finally, both the surface S and the point y are general, so Lemma 5.3 implies that $d(d-1)^2 = \deg\big((C')^\vee\big) = \deg(C')\big(\deg(C') - 1\big) - 3d(d-1)(d-2) - 2\delta'$. Since $\deg(C') = \deg(C) = d(d-1)$, we have $\delta' = \frac{1}{2}d(d-1)(d-2)(d-3)$. □

We end this section by proving that the secant locus of an irreducible smooth curve is isomorphic to the bitangent congruence of its dual surface via the natural isomorphism between $\mathrm{Gr}(1, \mathbb{P}^3)$ and $\mathrm{Gr}\big(1, (\mathbb{P}^3)^*\big)$. A subvariety $\Sigma \subset \mathrm{Gr}(1, \mathbb{P}^3)$ is sent under this isomorphism to the variety $\Sigma^\perp \subset \mathrm{Gr}\big(1, (\mathbb{P}^3)^*\big)$ consisting of the dual lines $L^\vee$ for all $L \in \Sigma$. For every congruence $\Sigma \subset \mathrm{Gr}(1, \mathbb{P}^3)$ with bidegree (α, β), the bidegree of $\Sigma^\perp$ is (β, α).

Theorem 5.7 *If $C \subset \mathbb{P}^3$ is a nondegenerate irreducible smooth curve, then we have* $\mathrm{Sec}(C)^\perp = \mathrm{Bit}(C^\vee)$*, the inflectional lines of $C^\vee$ are dual to the tangent lines of C, and* $\mathrm{Infl}(C^\vee) \subset \mathrm{Bit}(C^\vee)$.

Proof We first show that $\mathrm{Sec}(C)^\perp = \mathrm{Bit}(C^\vee)$. Consider a line L that intersects C at two distinct points x and y, but is equal to neither $T_x(C)$ nor $T_y(C)$. Together the line L and $T_x(C)$ span a plane corresponding to a point $a \in C^\vee$. Similarly, the span of the lines L and $T_y(C)$ corresponds to a point $b \in C^\vee$. Without loss of generality, we may assume that both a and b are smooth points in $C^\vee$. By the Biduality Theorem, the points $a, b \in C^\vee$ must be distinct with tangent planes corresponding to x and y. Thus, the line $L^\vee$ is tangent to $C^\vee$ at the points a, b, and $\mathrm{Sec}(C)^\perp \subset \mathrm{Bit}(C^\vee)$. To establish the other inclusion, let L' be a line that is tangent to $C^\vee$ at two distinct smooth points $a, b \in C^\vee$. The tangent planes at the points a, b correspond to two points x, $y \in C$. If $x \neq y$, then $(L')^\vee$ is the secant to C through these two points. If $x = y$, then the Biduality Theorem establishes that $(L')^\vee$ is the tangent line of C at x. In either case, we see that $\mathrm{Bit}(C^\vee) \subset \mathrm{Sec}(C)^\perp$, so $\mathrm{Sec}(C)^\perp = \mathrm{Bit}(C^\vee)$.

For the second part, let L be an inflectional line at a smooth point $a \in C^\vee$. A point $y \in L^\vee \setminus C$ corresponds to a plane H such that $L = T_a(C^\vee) \cap H$, so the line L is also an inflectional line to the plane curve $C^\vee \cap H \subset H$. Regarding L as a subvariety of the projective plane H, its dual variety is a cusp on the plane curve $(C^\vee \cap H)^\vee \subset H^*$; see Fig. 3. If $\pi_y \colon \mathbb{P}^3 \dashrightarrow \mathbb{P}^2 \cong H^*$ denotes the projection away from the point y, then we claim that $(C^\vee \cap H)^\vee$ equals $\pi_y(C)$; for a more general version see [15, Proposition 6.1]. Indeed, a smooth point $z \in \pi_y(C)$ is the projection of a point of C whose tangent line does not contain y. Together this tangent line and the point y span a plane such that its dual point w is contained in the curve $C^\vee \cap H$. Thus, the tangent line $T_z\big(\pi_y(C)\big)$ equals $\pi_y(w^\vee)$; the latter is the line in H^* dual to the point $w \in H$. In other words, we have $\big(\pi_y(C)\big)^\vee \subset C^\vee \cap H$. Since both curves are irreducible, this inclusion must be an equality. Hence, when considering L in the projective plane H, its dual point is a cusp of $\pi_y(C)$. It follows that $L^\vee$ is the tangent line $T_x(C)$, where $x \in C$ is the point corresponding to the tangent plane $T_a(C^\vee)$; see Fig. 6. Reversing these arguments shows that the dual of a tangent line to C is an inflectional line to $C^\vee$. Since every tangent line to C is contained in $\mathrm{Sec}(C)$, we conclude that $\mathrm{Infl}(C^\vee) \subset \mathrm{Bit}(C^\vee)$. □

Proof of Theorem 1.2 This result is a restatement of Theorem 5.7. □

Remark 5.8 Theorem 5.7 shows that $\mathrm{Infl}(C^\vee)$ is a curve, as $\mathrm{Infl}(C^\vee)^\perp$ is the set of tangent lines to C, so the inflectional locus of a surface in $\mathbb{P}^3$ is not always a congruence.

Remark 5.9 For a curve $C \subset \mathbb{P}^3$ with dual surface $C^\vee \subset (\mathbb{P}^3)^*$, Theorem 20 in [19] establishes that $\mathrm{CH}_0(C)^\perp = \mathrm{CH}_1(C^\vee)$. Combined with Theorem 5.7, we see that the singular locus of the Hurwitz hypersurface $\mathrm{CH}_1(C^\vee)$, for smooth C, has just one component, namely the bitangent congruence.

Remark 5.10 For a surface $S \subset \mathbb{P}^3$ with dual surface $S^\vee \subset (\mathbb{P}^3)^*$, Theorem 20 in [19] also establishes that $\mathrm{CH}_1(S)^\perp = \mathrm{CH}_1(S^\vee)$. If both S and $S^\vee$ have mild singularities, then the proof of Lemma 5.1 in [2] shows that $\mathrm{Bit}(S)^\perp = \mathrm{Bit}(S^\vee)$.

6 Intersection Theory on $\mathrm{Gr}(1, \mathbb{P}^3)$

In this section, we recast the degree of a subvariety in $\mathrm{Gr}(1, \mathbb{P}^3)$ in terms of certain products in the Chow ring.

Consider a smooth irreducible variety X of dimension n. For each $j \in \mathbb{N}$, the group $Z^j(X)$ of codimension-j cycles is the free abelian group generated by the closed irreducible subvarieties of X having codimension j. Given a variety W of codimension $j-1$ and a nonzero rational function f on W, we have the cycle $\mathrm{div}(f) := \sum_Z \mathrm{ord}_Z(f)\, Z$ where the sum runs over all subvarieties Z of W with codimension 1 in W and $\mathrm{ord}_Z(f) \in \mathbb{Z}$ is the order of vanishing of f along Z. The group of cycles rationally equivalent to zero is the subgroup generated by the cycles $\mathrm{div}(f)$ for all codimension-$(j-1)$ subvarieties W of X and all nonzero rational functions f on W. The Chow group $A^j(X)$ is the quotient of $Z^j(X)$ by the subgroup of cycles rationally equivalent to zero. We typically write $[Z]$ for the class of a subvariety Z in the appropriate Chow group. Since X is the unique subvariety of codimension 0, we see that $A^0(X) \cong \mathbb{Z}$. We also have $A^1(X) \cong \mathrm{Pic}(X)$. Crucially, the direct sum $A^*(X) := \bigoplus_{j=0}^n A^j(X)$ forms a commutative $\mathbb{Z}$-graded ring called the *Chow ring* of X. The product arises from intersecting cycles: for subvarieties V and W of X having codimension j and k and intersecting transversely, the product $[V][W] \in A^{j+k}(X)$ is the sum of the irreducible components of $V \cap W$. More generally, intersection theory aims to construct an explicit cycle to represent the product $[V][W]$.

Example 6.1 The Chow ring of $\mathbb{P}^n$ is isomorphic to $\mathbb{Z}[H]/(H^{n+1})$ where H is the class of a hyperplane. In particular, any subvariety of codimension d is rationally equivalent to a multiple of the intersection of d hyperplanes.

To a given a vector bundle $\mathscr{E}$ of rank r on X, we associate its *Chern classes* $c_i(\mathscr{E}) \in A^i(X)$ for $0 \leq i \leq r$; see [30]. When $\mathscr{E}$ is globally generated, these classes are represented by degeneracy loci; the class $c_{r+1-j}(\mathscr{E})$ is associated to the locus of points $x \in X$ where j general global sections of $\mathscr{E}$ fail to be linearly independent. In particular, $c_r(\mathscr{E})$ is represented by the vanishing locus of a single general global section. Given a short exact sequence $0 \to \mathscr{E}' \to \mathscr{E} \to \mathscr{E}'' \to 0$ of vector bundles, the Whitney Sum Formula asserts that $c_k(\mathscr{E}) = \sum_{i+j=k} c_i(\mathscr{E}')c_j(\mathscr{E}'')$; see [10, Theorem 3.2]. Moreover, if $\mathscr{E}^* := \mathscr{H}om(\mathscr{E}, \mathscr{O}_X)$ denotes the dual vector bundle, then we have $c_i(\mathscr{E}^*) = (-1)^i c_i(\mathscr{E})$ for $0 \leq i \leq r$; see [10, Remark 3.2.3].

Example 6.2 Given nonnegative integers $a_1, a_2, \dots, a_n$, consider the vector bundle $\mathscr{E} := \mathscr{O}_{\mathbb{P}^n}(a_1) \oplus \mathscr{O}_{\mathbb{P}^n}(a_2) \oplus \cdots \oplus \mathscr{O}_{\mathbb{P}^n}(a_n)$. Since each $\mathscr{O}_{\mathbb{P}^n}(a_i)$ is globally generated, the Chern class $c_1\big(\mathscr{O}_{\mathbb{P}^n}(a_i)\big)$ is the vanishing locus of a general homogeneous polynomial $\mathbb{C}[x_0, x_1, \dots, x_n]$ of degree a_i, so $c_1\big(\mathscr{O}_{\mathbb{P}^n}(a_i)\big) = a_i H$ in $A^*(\mathbb{P}^n)$. Hence, the Whitney Sum Formula implies that $c_n(\mathscr{E}) = \prod_{i=1}^n c_1\big(\mathscr{O}(a_i)\big) = \prod_{i=1}^n (a_i H)$.

Example 6.3 If $\mathscr{T}_{\mathbb{P}^n}$ is the tangent bundle on $\mathbb{P}^n$, then we have the short exact sequence $0 \to \mathscr{O}_{\mathbb{P}^n} \to \mathscr{O}_{\mathbb{P}^n}(1)^{\oplus(n+1)} \to \mathscr{T}_{\mathbb{P}^n} \to 0$; see [14, Example 8.20.1]. The Whitney Sum Formula implies that

$$c_1(\mathscr{T}_{\mathbb{P}^n}) = (n+1)c_1\big(\mathscr{O}_{\mathbb{P}^n}(1)\big) - c_1(\mathscr{O}_{\mathbb{P}^n}) = (n+1)H$$

and $c_2(\mathscr{T}_{\mathbb{P}^n}) = c_2\big(\mathscr{O}_{\mathbb{P}^n}(1)^{\oplus(n+1)}\big) = \binom{n+1}{2}H^2$.

Example 6.4 Let $Y \subset \mathbb{P}^n$ be a smooth hypersurface of degree d. If $\mathscr{T}_Y$ is the tangent bundle of Y, then we have the exact sequence $0 \to \mathscr{T}_Y \to \mathscr{T}_{\mathbb{P}^n}|_Y \to \mathscr{O}_{\mathbb{P}^n}(d)|_Y \to 0$; see [14, Proposition 8.20]. Setting $h := H|_Y$ in $A^*(Y)$, the Whitney Sum Formula implies that $c_1(\mathscr{T}_Y) = c_1(\mathscr{T}_{\mathbb{P}^n}|_Y) - c_1\big(\mathscr{O}_{\mathbb{P}^n}(d)|_Y\big) = (n+1)h - dh = (n+1-d)h$ and $c_2(\mathscr{T}_Y) = c_2(\mathscr{T}_{\mathbb{P}^n}|_Y) - c_1(\mathscr{T}_Y)c_1\big(\mathscr{O}_{\mathbb{P}^n}(d)|_Y\big) = \big(\binom{n+1}{2} - (n+1-d)d\big)h^2$.

We next focus on the Chow ring of $\mathrm{Gr}(1, \mathbb{P}^3)$; see [1, 30]. Fix a complete flag $v_0 \in L_0 \subset H_0 \subset \mathbb{P}^3$ where the point v_0 lies in the line L_0, and the line L_0 is contained in the plane H_0. The Schubert varieties in $\mathrm{Gr}(1, \mathbb{P}^3)$ are the following subvarieties:

$$\begin{aligned} \Sigma_0 &:= \mathrm{Gr}(1, \mathbb{P}^3)\,, & \Sigma_1 &:= \{L : L \cap L_0 \neq \varnothing\} \subset \mathrm{Gr}(1, \mathbb{P}^3)\,, \\ \Sigma_{1,1} &:= \{L : L \subset H_0\} \subset \mathrm{Gr}(1, \mathbb{P}^3)\,, & \Sigma_2 &:= \{L : v_0 \in L\} \subset \mathrm{Gr}(1, \mathbb{P}^3)\,, \\ \Sigma_{2,1} &:= \{L : v_0 \in L \subset H_0\} \subset \mathrm{Gr}(1, \mathbb{P}^3)\,, & \Sigma_{2,2} &:= \{L_0\} \subset \mathrm{Gr}(1, \mathbb{P}^3)\,. \end{aligned}$$

The corresponding classes $\sigma_I := [\Sigma_I]$, called the *Schubert cycles*, form a basis for the Chow ring $A^*(\mathrm{Gr}(1, \mathbb{P}^3))$; see [8, Theorem 5.26]. Since the sum of the subscripts gives the codimension, we have

$$A^0\big(\mathrm{Gr}(1,\mathbb{P}^3)\big) \cong \mathbb{Z}\sigma_0\,, \quad A^1\big(\mathrm{Gr}(1,\mathbb{P}^3)\big) \cong \mathbb{Z}\sigma_1\,, \quad A^2\big(\mathrm{Gr}(1,\mathbb{P}^3)\big) \cong \mathbb{Z}\sigma_{1,1} \oplus \mathbb{Z}\sigma_2\,,$$
$$A^3\big(\mathrm{Gr}(1,\mathbb{P}^3)\big) \cong \mathbb{Z}\sigma_{2,1}\,, \quad A^4\big(\mathrm{Gr}(1,\mathbb{P}^3)\big) \cong \mathbb{Z}\sigma_{2,2}\,.$$

To understand the product structure, we use the transitive action of $\mathrm{GL}(4,\mathbb{C})$ on $\mathrm{Gr}(1,\mathbb{P}^3)$. Specifically, Kleiman's Transversality Theorem [18] shows that, for two subvarieties V and W in $\mathrm{Gr}(1,\mathbb{P}^3)$, a general translate U of V under the $\mathrm{GL}(4,\mathbb{C})$-action is rationally equivalent to V and the intersection of U and W is transversal at the generic point of any component of $U \cap W$. Hence, we have $[V][W] = [U \cap W]$. To determine the product $\sigma_{1,1}\sigma_2$, we intersect general varieties representing these classes: $\sigma_{1,1}$ consists of all lines L contained in a fixed plane H_0, and σ_2 is all lines L containing a fixed point v_0. Since a general point does not lie in a general plane, we see that $\sigma_{1,1}\sigma_2 = 0$. Similar arguments yield all products:

$$\sigma_{1,1}^2 = \sigma_{2,2}\,, \qquad \sigma_2^2 = \sigma_{2,2}\,, \qquad \sigma_{1,1}\sigma_2 = 0\,, \qquad \sigma_1\sigma_{2,1} = \sigma_{2,2}\,,$$
$$\sigma_1\sigma_{1,1} = \sigma_{2,1}\,, \qquad \sigma_1\sigma_2 = \sigma_{2,1}\,, \qquad \sigma_1^2 = \sigma_2 + \sigma_{1,1}\,.$$

The degree of a subvariety in $\mathrm{Gr}(1,\mathbb{P}^3)$, introduced in Sect. 2, can be interpreted as certain coefficients of its class in the Chow ring. Geometrically, the order α of a surface $X \subset \mathrm{Gr}(1,\mathbb{P}^3)$ is the number of lines in X passing through the general point v_0. Since we may intersect X with a general variety representing σ_2, it follows that α equals the coefficient of σ_2 in $[X]$. Similarly, the class β of X is the coefficient of $\sigma_{1,1}$ in $[X]$, the degree of a threefold $\Sigma \subset \mathrm{Gr}(1,\mathbb{P}^3)$ is the coefficient of σ_1 in $[\Sigma]$, and the degree of a curve $C \subset \mathrm{Gr}(1,\mathbb{P}^3)$ is the coefficient of $\sigma_{2,1}$ in $[C]$.

The degree of a subvariety in $\mathrm{Gr}(1,\mathbb{P}^3)$ also has a useful reinterpretation via Chern classes of tautological vector bundles. Let $\mathscr{S}$ denote the tautological subbundle, the vector bundle whose fibre over the point $W \in \mathrm{Gr}(1,\mathbb{P}^3)$ is the two-dimensional vector space $W \subseteq \mathbb{C}^4$. Similarly, let $\mathscr{Q}$ be the tautological quotient bundle whose fibre over W is $\mathbb{C}^4/W$. Both $\mathscr{S}^*$ and $\mathscr{Q}$ are globally generated; $H^0\big(\mathrm{Gr}(1,\mathbb{P}^3),\mathscr{S}^*\big) \cong (\mathbb{C}^4)^*$ and $H^0\big(\mathrm{Gr}(1,\mathbb{P}^3),\mathscr{Q}\big) \cong \mathbb{C}^4$; see [1, Proposition 0.5]. A global section of $\mathscr{S}^*$ corresponds to a nonzero map $\varphi\colon \mathbb{C}^4 \to \mathbb{C}$, where its value at the point W is $\varphi|_W\colon W \to \mathbb{C}$. The Chern class $c_2(\mathscr{S}^*)$ is represented by the vanishing locus of φ, so we have $c_2(\mathscr{S}^*) = \sigma_{1,1} = c_2(\mathscr{S})$. For two general sections $\varphi, \psi\colon \mathbb{C}^4 \to \mathbb{C}$ of $\mathscr{S}^*$, the Chern class $c_1(\mathscr{S}^*)$ is represented by the locus of points W where $\varphi|_W$ and $\psi|_W$ fail to be linearly independent or $W \cap \ker(\varphi) \cap \ker(\psi) \neq \{0\}$. Generality ensures that $\ker(\varphi) \cap \ker(\psi)$ is a two-dimensional subspace of $\mathbb{C}^4$, so $c_1(\mathscr{S}^*) = -c_1(\mathscr{S}) = \sigma_1$. Similarly, a global section of $\mathscr{Q}$ corresponds to a point $v \in \mathbb{C}^4$; its value at W is simply the image of the point in $\mathbb{C}^4/W$. Thus, $c_2(\mathscr{Q})$ is represented by the locus of those W containing v, which is σ_2. Two global sections of $\mathscr{Q}$ are linearly dependent at W when the two-dimensional subspace of $\mathbb{C}^4$ spanned by the points intersects W nontrivially, so $c_1(\mathscr{Q}) = \sigma_1$. Finally, for a surface $X \subset \mathrm{Gr}(1,\mathbb{P}^3)$ with $[X] = \alpha\sigma_2 + \beta\sigma_{1,1}$, we obtain

$$c_2(\mathscr{Q})\,[X] = \sigma_2(\alpha\sigma_2 + \beta\sigma_{1,1}) = \alpha\sigma_{2,2}\,,$$

$$c_2(\mathscr{S})\,[X] = \sigma_{1,1}(\alpha\sigma_2 + \beta\sigma_{1,1}) = \beta\sigma_{2,2}\,,$$

so computing the bidegree is equivalent to calculating the products $c_2(\mathscr{Q})\,[X]$ and $c_2(\mathscr{S})\,[X]$ in the Chow ring.

We close this section with three examples demonstrating this approach.

Example 6.5 Given a smooth surface S in $\mathbb{P}^3$, we recompute the degree of $\mathrm{CH}_1(S)$; compare with Proposition 5.4. Theorem 9 in [19] implies that this degree equals the degree $\delta_1(S)$ of the first polar locus $M_1(S) = \{x \in S : y \in T_xS\}$, where y is a general point of $\mathbb{P}^3$ (this locus is the polar curve in the proof of Theorem 5.6). Letting T_S be the tangent bundle of S, Example 14.4.15 in [10] shows that $\delta_1(S) = \deg\big(3h - c_1(T_S)\big)$. Hence, Example 6.4 gives $\delta_1(S) = \deg(3h - h(3+1-d)) = (d-1)\deg(h)$. Since S is a degree d surface, the degree of the hyperplane h equals d, so $\delta_1(S) = d(d-1)$.

Example 6.6 (Problem 3 on Grassmannians in [29]) Let $S_1, S_2 \subset \mathbb{P}^3$ be general surfaces of degree d_1 and d_2, respectively, with $d_1, d_2 \geq 4$. To find the number of lines bitangent to both surfaces, it suffices to compute the cardinality of $\mathrm{Bit}(S_1) \cap \mathrm{Bit}(S_2)$. Theorem 5.6 establishes that, for all $1 \leq i \leq 2$, we have $[\mathrm{Bit}(S_i)] = \alpha_i\sigma_2 + \beta_i\sigma_{1,1}$ where $\alpha_i := \frac{1}{2}d_i(d_i-1)(d_i-2)(d_i-3)$ and $\beta_i := \frac{1}{2}d_i(d_i-2)(d_i-3)(d_i+3)$. It follows that $[\mathrm{Bit}(S_1) \cap \mathrm{Bit}(S_2)] = [\mathrm{Bit}(S_1)][\mathrm{Bit}(S_2)] = (\alpha_1\alpha_2 + \beta_1\beta_2)\sigma_{2,2}$, so the number of lines bitangent to S_1 and S_2 is

$$\tfrac{1}{4}d_1(d_1-1)(d_1-2)(d_1-3)d_2(d_2-1)(d_2-2)(d_2-3) \\ + \tfrac{1}{4}d_1(d_1-2)(d_1-3)(d_1+3)d_2(d_2-2)(d_2-3)(d_2+3)\,.$$

Example 6.7 Let $S \subset \mathbb{P}^3$ be a general surface of degree d_1 with $d_1 \geq 4$, and let $C \subset \mathbb{P}^3$ be a general curve of degree d_2 and geometric genus g with $d_2 \geq 2$. To find the number of lines bitangent to S and secant to C, it suffices to compute the cardinality of $\mathrm{Bit}(S) \cap \mathrm{Sec}(C)$. Theorem 5.6 and Theorem 3.5 imply that

$$\begin{aligned}
[\mathrm{Bit}(S)] &= \tfrac{1}{2}d_1(d_1-1)(d_1-2)(d_1-3)\,\sigma_2 + \tfrac{1}{2}d_1(d_1-2)(d_1-3)(d_1+3)\,\sigma_{1,1}\,,\\
[\mathrm{Sec}(C)] &= \big(\tfrac{1}{2}(d_2-1)(d_2-2) - g\big)\,\sigma_2 + \tfrac{1}{2}d_2(d_2-1)\,\sigma_{1,1}\,.
\end{aligned}$$

It follows that $[\mathrm{Bit}(S) \cap \mathrm{Sec}(C)] = [\mathrm{Bit}(S)][\mathrm{Sec}(C)] = \gamma\sigma_{2,2}$ where

$$\gamma := \tfrac{1}{4}d_1(d_1-1)(d_1-2)(d_1-3)\big((d_2-1)(d_2-2)-2g\big) \\ + \tfrac{1}{4}d_1(d_1-2)(d_1-3)(d_1+3)d_2(d_2-1)\,,$$

so the number of lines bitangent to S and secant to C is γ.

7 Singular Loci of Congruences

This section investigates the singular points of the secant, bitangent, and inflectional congruences. We begin with the singularities of the secant locus of a smooth irreducible curve.

Proposition 7.1 *Let C be a nondegenerate smooth irreducible curve in $\mathbb{P}^3$. If L is a line that intersects the curve C in three or more distinct points, then the line L corresponds to a singular point in* $\mathrm{Sec}(C)$.

Proof The symmetric square $C^{(2)}$ is the quotient of $C \times C$ by the action of the symmetric group $\mathfrak{S}_2$, so points in this projective variety are unordered pairs of points on C; see [13, pp. 126–127]. The map $\varpi\colon C^{(2)} \to \mathrm{Sec}(C)$, defined by sending $\{x, y\}$ to the line spanned by the points x and y if $x \neq y$ or to the tangent line $T_x(C)$ if $x = y$, is a birational morphism. Since $|L \cap C| \geq 3$, the fibre $\varpi^{-1}(L)$ is a finite set containing more than one element. Hence, $\varpi^{-1}(L)$ is not connected and the Zariski Connectedness Theorem [21, Sect. III.9.V] proves that $\mathrm{Sec}(C)$ is singular at L. ◻

Lemma 7.2 *If $f \in \mathbb{C}[\![z, w]\!]$ satisfies $f(z, w) = -f(w, z)$, then the linear form $z - w$ divides the power series f.*

Proof We write the formal power series f as a sum of homogeneous polynomials $f = \sum_{i\in\mathbb{N}} f_i$. Since we have $f(z, w) + f(w, z) = 0$, it follows that, in each degree i, we have $f_i(z, w) + f_i(w, z) = 0$. In particular, we see that $f_i(w, w) = 0$. If we consider $f_i(w, z)$ as a polynomial in the variable z with coefficients in $\mathbb{C}[w]$, it follows that w is a root of f_i. Thus, we conclude that $z - w$ divides f_i for all $i \in \mathbb{N}$. ◻

Theorem 7.3 *Let C be a nondegenerate smooth irreducible curve in $\mathbb{P}^3$. If a point in* $\mathrm{Sec}(C)$ *corresponds to a line L that intersects C in a single point x, then the intersection multiplicity of L and C at x is at least 2. Moreover, the line L corresponds to a smooth point of* $\mathrm{Sec}(C)$ *if and only if the intersection multiplicity is exactly* 2.

We thank Jenia Tevelev for help with the following proof.

Proof Suppose the line L intersects the curve C at the point x with multiplicity 2. Without loss of generality, we may work in the affine open subset with $x_3 \neq 0$, and we assume that $x = (0 : 0 : 0 : 1)$ and $L = \mathrm{V}(x_1, x_2)$. Since C is smooth, there is a local analytic isomorphism φ from a neighbourhood of the origin in $\mathbb{A}^1$ to a neighbourhood of the point x in C. The map φ will have the form $\varphi(z) = \big(\varphi_0(z), \varphi_1(z), \varphi_2(z)\big)$ for some $\varphi_0, \varphi_1, \varphi_2 \in \mathbb{C}[\![z]\!]$. We have $\varphi_0'(0) \neq 0$ and $\varphi_1'(0) = \varphi_2'(0) = 0$ because L is the tangent to the curve C at x. After making an analytic change of coordinates, we may assume that $\varphi(z) = \big(z, \varphi_1(z), \varphi_2(z)\big)$. As L is a simple tangent, at least one of φ_1 and φ_2 must vanish at 0 with order exactly 2. Hence, we may assume that $\varphi_1(z) = z^2 + z^3 f(z)$ and $\varphi_2(z) = z^2 g(z)$ for some $f, g \in \mathbb{C}[\![z]\!]$. The line spanned by the distinct points $\varphi(z)$ and $\varphi(w)$ on the curve C is given by the row space of the matrix

$$\begin{bmatrix} z & z^2 + z^3 f(z) & z^2 g(z) & 1 \\ w & w^2 + w^3 f(w) & w^2 g(w) & 1 \end{bmatrix} .$$

The Plücker coordinates are skew-symmetric power series, so Lemma 7.2 implies that they are divisible by $z - w$. In particular, if $f(z) = \sum_i a_i z^i$, then we have $p_{0,3} = z - w$,

$$p_{0,1} = z\big(w^2 + w^3 f(w)\big) - w\big(z^2 + z^3 f(z)\big) = -zw(z-w)\Big(1 + \sum_i a_i \sum_{j=0}^{i+1} w^j z^{i+1-j}\Big) ,$$

$$p_{1,3} = z^2 + z^3 f(z) - w^2 - w^3 f(w) = (z-w)\Big(z + w + \sum_i a_i \sum_{j=0}^{i+2} z^j w^{i+2-j}\Big) .$$

The symmetric square $(\mathbb{A}^1)^{(2)}$ of the affine line $\mathbb{A}^1$ is a smooth surface isomorphic to the affine plane $\mathbb{A}^2$; see [13, Example 10.23]. Consider the map $\varpi\colon (\mathbb{A}^1)^{(2)} \to \mathrm{Sec}(C)$ defined by sending the pair $\{z, w\}$ of points in $\mathbb{A}^1$ to the line spanned by the points $\varphi(z)$ and $\varphi(w)$ if $z \neq w$ or to the tangent line of C at $\varphi(z)$ if $z = w$. In other words, the map ϖ sends $\{z, w\}$ to $\big(-zw + h_1(z,w) : \frac{p_{0,2}}{z-w} : 1 : \frac{p_{1,2}}{z-w} : z + w + h_2(z,w) : \frac{p_{2,3}}{z-w}\big)$ where

$$h_1(z,w) := -zw \sum_i a_i \sum_{j=0}^{i+1} w^j z^{i-j+1} \qquad \text{and} \qquad h_2(z,w) := \sum_i a_i \sum_{j=0}^{i+2} z^j w^{i+2-j} .$$

Since the forms zw and $z + w$ are local coordinates of $(\mathbb{A}^1)^{(2)}$ in a neighbourhood of the origin, we conclude that ϖ is a local isomorphism and $\mathrm{Sec}(C)$ is smooth at the point corresponding to L.

Suppose the line L intersects the curve C at the point x with multiplicity at least 3. It follows that the line L is contained in the Zariski closure of the set of lines that intersect C in at least three points or that intersect C in two points, one with multiplicity at least 2. By Proposition 7.1 and Lemma 2.3 in [2], we conclude that the line is singular in $\mathrm{Sec}(C)$. □

Corollary 7.4 *Let C be a nondegenerate smooth irreducible curve in $\mathbb{P}^3$. If the line L corresponds to a point in* $\mathrm{Sec}(C)$*, then L corresponds to a singular point of* $\mathrm{Sec}(C)$ *if and only if one of the following three conditions is satisfied:*

- *the line L intersects the curve C in three or more distinct points,*
- *the line L intersects the curve C in exactly two points and L is the tangent line to one of these two points,*
- *the line L intersects the curve C at a single point with multiplicity at least* 3.

Proof Combine Proposition 7.1, Lemma 2.3 in [2], and Theorem 7.3. □

Analogously, we want to describe the singularities of the inflectional locus $\mathrm{Infl}(S)$ and the bitangent locus $\mathrm{Bit}(S)$ of a surface $S \subset \mathbb{P}^3$.

Theorem 7.5 *If $S \subset \mathbb{P}^3$ is an irreducible smooth surface of degree at least* 4 *which does not contain any lines, then the singular locus of* Infl(S) *corresponds to lines which either intersect S with multiplicity at least* 3 *at two or more distinct points, or intersect S with multiplicity at least* 4 *at some point.*

Proof We consider the incidence variety

$$\Psi_S := \overline{\{(x, L) : L \text{ intersects } S \text{ at } x \text{ with multiplicity } 3\}} \subset S \times \mathrm{Gr}(1, \mathbb{P}^3)\,.$$

The projection $\pi\colon \Psi_S \to \mathrm{Infl}(S)$, defined by $(x, L) \mapsto L$, is a surjective morphism. Since S does not contain any lines, all fibres of π are finite and Lemma 14.8 in [13] implies that the map π is finite. Moreover, the general fibre of π has cardinality one, so π is birational. To apply Lemma 3.2, we need to examine the singularities of Ψ_S and the differential of π.

Let $f \in \mathbb{C}[x_0, x_1, x_2, x_3]$ be the defining equation for S in $\mathbb{P}^3$. Consider the affine chart in $\mathbb{P}^3 \times \mathrm{Gr}(1, \mathbb{P}^3)$ where $x_0 \neq 0$ and $p_{0,1} \neq 0$. We may assume $x = (1 : \alpha : \beta : \gamma)$ and the line L is spanned by the points $(1 : 0 : a : b)$ and $(0 : 1 : c : d)$. In this affine chart, S is defined by $g_0(x_1, x_2, x_3) := f(1, x_1, x_2, x_3)$. As in the proof of Theorem 3.3, we have $x \in L$ if and only if $a = \beta - \alpha c$ and $b = \gamma - \alpha d$. Parametrizing the line L by $\ell(s, t) := (s : s\alpha + t : s\beta + tc : s\gamma + td)$ for $(s : t) \in \mathbb{P}^1$ shows that L intersects S with multiplicity at least m at x if and only if $f\big(\ell(s, t)\big)$ is divisible by t^m. This is equivalent to

$$\tfrac{\partial}{\partial t}\big[f\big(\ell(s,t)\big)\big]\Big|_{(1,0)} = \tfrac{\partial^2}{\partial t^2}\big[f\big(\ell(s,t)\big)\big]\Big|_{(1,0)} = \cdots = \tfrac{\partial^{m-1}}{\partial t^{m-1}}\big[f\big(\ell(s,t)\big)\big]\Big|_{(1,0)} = 0\,.$$

Setting $g_k := \big[\frac{\partial}{\partial x_1} + c\frac{\partial}{\partial x_2} + d\frac{\partial}{\partial x_3}\big]^k g_0$ for $k \geq 1$, the incidence variety Ψ_S can be written on the chosen affine chart as

$$\big\{(\alpha, \beta, \gamma, a, b, c, d) : g_k(\alpha, \beta, \gamma) = 0 \text{ for } 0 \leq k \leq 2,\ a = \beta - \alpha c,\ b = \gamma - \alpha d\big\}\,.$$

As $\dim \Psi_S = 2$, it is smooth at the point (x, L) if and only if its tangent space has dimension 2 or, equivalently, its Jacobian matrix

$$\begin{bmatrix}
\frac{\partial g_0}{\partial x_1}(\alpha,\beta,\gamma) & \frac{\partial g_0}{\partial x_2}(\alpha,\beta,\gamma) & \frac{\partial g_0}{\partial x_3}(\alpha,\beta,\gamma) & 0 & 0 & 0 & 0 \\
\frac{\partial g_1}{\partial x_1}(\alpha,\beta,\gamma) & \frac{\partial g_1}{\partial x_2}(\alpha,\beta,\gamma) & \frac{\partial g_1}{\partial x_3}(\alpha,\beta,\gamma) & 0 & 0 & \frac{\partial g_0}{\partial x_2}(\alpha,\beta,\gamma) & \frac{\partial g_0}{\partial x_3}(\alpha,\beta,\gamma) \\
\frac{\partial g_2}{\partial x_1}(\alpha,\beta,\gamma) & \frac{\partial g_2}{\partial x_2}(\alpha,\beta,\gamma) & \frac{\partial g_2}{\partial x_3}(\alpha,\beta,\gamma) & 0 & 0 & 2\frac{\partial g_1}{\partial x_2}(\alpha,\beta,\gamma) & 2\frac{\partial g_1}{\partial x_3}(\alpha,\beta,\gamma) \\
-c & 1 & 0 & -1 & 0 & -\alpha & 0 \\
-d & 0 & 1 & 0 & -1 & 0 & -\alpha
\end{bmatrix}$$

has rank five. Since S is smooth, the first two and the last two rows of the Jacobian matrix are linearly independent. If Ψ_S is singular at (x, L), then the third row is a linear combination of the others; specifically, there exist scalars $\lambda, \mu \in \mathbb{C}$ such that $\frac{\partial g_2}{\partial x_j}(\alpha, \beta, \gamma) = \lambda \frac{\partial g_1}{\partial x_j}(\alpha, \beta, \gamma) + \mu \frac{\partial g_0}{\partial x_j}(\alpha, \beta, \gamma)$ for $1 \leq j \leq 3$. It follows that

$g_3(\alpha, \beta, \gamma) = \lambda g_2(\alpha, \beta, \gamma) + \mu g_1(\alpha, \beta, \gamma) = 0$. Thus, the line L intersects the surface S at the point x with multiplicity at least 4 if Ψ_S is singular at (x, L).

It remains to show that the differential $d_{(x,L)}\pi : T_{(x,L)}(\Psi_S) \to T_L\big(\operatorname{Infl}(S)\big)$ is not injective if and only if the line L intersects the surface S at the point x with multiplicity at least 4. The differential $d_{(x,L)}\pi$ sends every element in the kernel of the Jacobian matrix to its last four coordinates. This map is not injective if and only if the kernel contains an element of the form $\begin{bmatrix} * & * & * & 0 & 0 & 0 & 0 \end{bmatrix}^{\mathsf{T}} \neq 0$. Such an element belongs to the kernel if and only if it equals $\begin{bmatrix} \lambda & c\lambda & d\lambda & 0 & 0 & 0 & 0 \end{bmatrix}^{\mathsf{T}}$ for some $\lambda \in \mathbb{C} \setminus \{0\}$ and $g_1(\alpha, \beta, \gamma) = g_2(\alpha, \beta, \gamma) = g_3(\alpha, \beta, \gamma) = 0$. This shows that the line L intersects the surface S at the point x with multiplicity at least 4 if and only if $d_{(x,L)}$ is not injective.

Finally, the fibre $\pi^{-1}(L)$ consists of more than one point if and only if L intersects S with multiplicity at least 3 at two or more distinct points, so Lemma 3.2 completes the proof. □

Proof of Theorem 1.1 The first part related to the curve C is an amalgamation of Theorem 3.3, Theorem 3.5, Theorem 7.3, and Corollary 7.4. Similarly, the second part related to the surface S is an amalgamation of Theorem 4.1, Theorem 5.6, and Theorem 7.5. □

Proposition 7.6 *Let $S \subset \mathbb{P}^3$ be a general irreducible surface of degree at least* 4. *If L is a line that is tangent to S at three or more distinct points, then the line L corresponds to a singular point of* $\operatorname{Bit}(S)$.

Proof As in the proof of Proposition 7.1, the symmetric square $S^{(2)}$ is the quotient of $S \times S$ by the action of the symmetric group $\mathfrak{S}_2$. The projection ϖ from

$$\overline{\big\{(\{x, y\}, L) : x \neq y,\ x, y \in L \subset T_x(S) \cap T_y(S)\big\}} \subset S^{(2)} \times \operatorname{Gr}(1, \mathbb{P}^3)$$

onto $\operatorname{Bit}(S)$, defined by sending the pair $(\{x, y\}, L) \mapsto L$ is a birational morphism. The fibre $\varpi^{-1}(L)$ is a finite set containing more than one element if L is tangent to S in at least three distinct points. Hence, $\varpi^{-1}(L)$ is not connected and the Zariski Connectedness Theorem [21, Sect. III.9.V] proves that $\operatorname{Bit}(S)$ is singular at L. □

We do not yet have a full understanding of points in $\operatorname{Bit}(S)$ for which the corresponding lines have an intersection multiplicity greater than 4 at a point of S. We know that a line L that is tangent to the surface S at exactly two points corresponds to a smooth point in $\operatorname{Bit}(S)$ if and only if the intersection multiplicity of L and S at both points is exactly 2. Moreover, given a line L that is tangent to S at a single point, the intersection multiplicity of L and S at this point is at least 4, and the line L corresponds to a smooth point of $\operatorname{Bit}(S)$ when the multiplicity is exactly four; see [2, Lemma 4.3]. To complete this picture, we make the following prediction.

Conjecture 7.7 Let $S \subset \mathbb{P}^3$ be a general irreducible surface of degree at least 4. If a point in the bitangent congruence $\operatorname{Bit}(S)$ corresponds to a line L that is tangent to S at a single point x such that the intersection multiplicity of L and S at x is at least 5, then L corresponds to a singular point of $\operatorname{Bit}(S)$.

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Equations of $\overline{\mathrm{M}}_{0,n}$

Leonid Monin and Julie Rana

Abstract We study the moduli space $\overline{\mathrm{M}}_{0,n}$ of genus 0 curves with n marked points. Following Keel and Tevelev, we give explicit polynomials in the Cox ring of $\mathbb{P}^1 \times \mathbb{P}^2 \times \cdots \times \mathbb{P}^{n-3}$ that, conjecturally, determine $\overline{\mathrm{M}}_{0,n}$ as a subscheme. Using *Macaulay2*, we prove that these equations generate the ideal for $5 \leq n \leq 8$. For $n \leq 6$, we also give a cohomological proof that these polynomials realize $\overline{\mathrm{M}}_{0,n}$ as a subvariety of $\mathbb{P}^{(n-2)!-1}$ embedded by the complete log canonical linear system.

MSC 2010 codes: 14H10 (primary), 13D02 (secondary)

1 Introduction

Let $\mathrm{M}_{0,n}$ denote the moduli space of genus 0 curves with n marked points, and let $\overline{\mathrm{M}}_{0,n}$ be its Deligne–Mumford–Knudsen compactification [4]. In [14], Kapranov constructs $\overline{\mathrm{M}}_{0,n}$ as a Chow quotient of the Grassmannian $\mathrm{Gr}(2,n)$, which allowed him to present $\overline{\mathrm{M}}_{0,n}$ as a sequence of blowups of $\mathbb{P}^{n-3}$. Following this, Keel and Tevelev [16] described an embedding of $\overline{\mathrm{M}}_{0,n}$ into the space of sections of particular characteristic classes on $\overline{\mathrm{M}}_{0,n}$, and an embedding $\varphi\colon \overline{\mathrm{M}}_{0,n} \to \mathbb{P}^1 \times \mathbb{P}^2 \times \cdots \times \mathbb{P}^{n-3}$. We obtain equations satisfied by $\varphi(\overline{\mathrm{M}}_{0,n})$ in the Cox ring of $\mathbb{P}^1 \times \mathbb{P}^2 \times \cdots \times \mathbb{P}^{n-3}$.

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Lemma 1.1 *Consider the embedding* $\varphi\colon \overline{\mathrm{M}}_{0,n} \to \mathbb{P}^1 \times \mathbb{P}^2 \times \cdots \times \mathbb{P}^{n-3}$ *and let* $w_0^{(i)}, w_1^{(i)}, \ldots, w_i^{(i)}$ *be the homogeneous coordinates on the ith factor of the target. The image of* φ *satisfies the* $\binom{n-1}{4}$ *equations given by the* 2*-minors of the matrices*

$$\begin{bmatrix} w_0^{(i)}\left(w_0^{(j)} - w_{i+1}^{(j)}\right) & w_1^{(i)}\left(w_1^{(j)} - w_{i+1}^{(j)}\right) & \cdots & w_i^{(i)}\left(w_i^{(j)} - w_{i+1}^{(j)}\right) \\ w_0^{(j)} & w_1^{(j)} & \cdots & w_i^{(j)} \end{bmatrix}$$

for all $1 \le i < j \le n-3$.

Conjecture 1.2 Let I_n be the prime ideal in the Cox ring of $\mathbb{P}^1 \times \mathbb{P}^2 \times \cdots \times \mathbb{P}^{n-3}$ that defines the embedding $\varphi(\overline{\mathrm{M}}_{0,n})$ scheme-theoretically. Let J_n be the ideal in the Cox ring of $\mathbb{P}^1 \times \mathbb{P}^2 \times \cdots \times \mathbb{P}^{n-3}$ generated by the equations of Lemma 1.1. Then I_n is the unique B-saturation of J_n, where $B = \cap_{i=1}^{n-3} \langle w_0^{(i)}, w_1^{(i)}, \ldots, w_i^{(i)} \rangle$. The ideal I_n is minimally generated by $\binom{n-1}{d+1}$ polynomials of degree d, for $3 \le d \le n-2$. The degree of I_n is $(2n-7)!!$, the number of trivalent phylogenetic trees on $n-1$ leaves. The lexicographic initial monomial ideals are square-free and Cohen–Macaulay.

We verified Conjecture 1.2 for $n = 5, 6, 7, 8$ using *Macaulay2*. In particular, we partially answered the question posed in [20, Problem 8 on Curves].

In Sect. 4, we provide a list of degree 4 polynomials contained in I_n and describe a conjectural method to count the number of minimal generators of I_n in any degree. The combinatorial description of the ideal of relations of the invariant ring of $\overline{\mathrm{M}}_{0,n}$ presented in [12] offers another possibly promising method.

In a slightly different direction, we consider the embedding Φ of $\overline{\mathrm{M}}_{0,n}$ into $\mathbb{P}^{(n-3)!-1}$ given by the κ class. By work of Keel and Tevelev [16], the ideal defining $\overline{\mathrm{M}}_{0,n}$ as a subscheme of $\mathbb{P}^{(n-3)!-1}$ is generated by the polynomials defining the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^2 \times \cdots \times \mathbb{P}^{n-3} \to \mathbb{P}^{(n-3)!-1}$, together with the preimages in the Cox ring of $\mathbb{P}^{(n-3)!-1}$ of the polynomials in I_n of degree $(2, 2, \ldots, 2)$. For example, in the case of $\overline{\mathrm{M}}_{0,5}$, we have

Corollary 1.3 *The ideal of the embedding of* $\overline{\mathrm{M}}_{0,5}$ *into* $\mathbb{P}^5$ *via the* κ *class is generated by the five quadrics*

$$t_0t_1 - t_0t_4 + t_2t_3 - t_1t_2,\ t_0t_4 - t_3t_4 + t_3t_5 - t_1t_5,\ t_1t_3 - t_0t_4,\ t_2t_3 - t_0t_5,\ t_2t_4 - t_1t_5. \tag{1}$$

Using the Chow quotient description of $\overline{\mathrm{M}}_{0,n}$ due to Kapranov [13], Gibney and Maclagan [8] provide equations for $\overline{\mathrm{M}}_{0,5} \subset \mathbb{P}^{21}$. Many of the listed equations in [8] are linear, so they have effectively given an embedding of $\overline{\mathrm{M}}_{0,5}$ into $\mathbb{P}^5$. Using *Macaulay2*, we were able to eliminate variables in such a way that the resulting embedding is a nonsingular variety of dimension 2 given by five quadrics in $\mathbb{P}^5$. Beyond this, it is not yet clear how these equations relate to ours.

The result of Keel and Tevelev motivates the following conjecture.

Conjecture 1.4 The ideal J_n contains all polynomials of degree $(2, 2, \ldots, 2)$ in I_n, and is therefore enough to determine the equations of $\Phi(\overline{\mathrm{M}}_{0,n})$ in the Cox ring of $\mathbb{P}^{(n-3)!-1}$ ideal-theoretically.

In Sect. 5, we prove Conjecture 1.4 for $\overline{M}_{0,5}$ and $\overline{M}_{0,6}$ using cohomological techniques developed in [16]. We hope that these techniques can be extended to larger values of n. In particular, we have

Theorem 1.5 *Let J_6 be the ideal generated by the five polynomials of Lemma 4.1. Then the ideal $\tilde{I}_6$ generated by polynomials of degree (d,d,d) in I_6 is generated by the degree $(2,2,2)$ polynomials in J_6. Equivalently, the embedding of $\overline{M}_{0,n}$ in $\mathbb{P}^{23}$ defined by the κ class is generated by the polynomials of degree $(2,2,2)$ in J_6 and the Segre relations.*

The paper is structured as follows. In Sect. 2, we give some background about the moduli spaces $\overline{M}_{0,n}$. In Sect. 3, we describe in detail the embedding $\varphi\colon \overline{M}_{0,n} \to \mathbb{P}^1 \times \mathbb{P}^2 \times \cdots \times \mathbb{P}^{n-3}$ which we will use in later sections to find equations of $\overline{M}_{0,n}$. In particular, away from the boundary of $\overline{M}_{0,n}$, we give a parametrization of φ that extends to the full moduli space, and provide a geometric interpretation of φ in the cases $n = 4$ and $n = 5$. In Sect. 4, we give explicit equations satisfied by $\varphi(\overline{M}_{0,n})$ that are forced by the parametrization. In Sect. 5 and 6 we recall the cohomological machinery developed in [16] and use it to prove that J_5 and J_6 contain all polynomials of degree $(2,2,\ldots,2)$ in the respective Cox rings.

2 Background on the Moduli Space $\overline{M}_{0,n}$

We begin with a brief introduction to the moduli space of pointed rational curves. For more details, we recommend the lectures and lecture notes of Cavalieri [1]. For those new to the theory of stable curves, we recommend [11].

For $n \geq 3$, the moduli space $M_{0,n}$ parametrizes ordered n-tuples of distinct points on $\mathbb{P}^1$. Two n-tuples $(p_1, p_2, \ldots, p_n)$ and $(q_1, q_2, \ldots, q_n)$ are equivalent if there exists a projective transformation $g \in \mathrm{PGL}(2,\mathbb{C})$ such that

$$(q_1, q_2, \ldots, q_n) = \big(g(p_1), g(p_2), \ldots, g(p_n)\big)\,.$$

Since a projective transformation can map three points in $\mathbb{P}^1$ to any other three points and is uniquely determined by their image, the dimension of $M_{0,n}$ equals $n-3$.

The space $M_{0,n}$ is not compact because the points p_i are distinct. There are a number of compactifications of $M_{0,n}$, including those described by Losev-Manin [17] and Keel [15]. But the first and most well-known is $\overline{M}_{0,n}$, the Deligne-Mumford-Knudsen compactification, described explicitly by Kapranov [13, 14]. The moduli space $\overline{M}_{0,n}$ parametrizes *stable n-pointed rational curves*.

Definition 2.1 A stable n-pointed rational curve is a tuple $(C, p_1, p_2, \ldots, p_n)$, where

1. C is a connected curve of arithmetic genus 0 with at most simple nodal singularities;
2. $p_1, p_2, \ldots, p_n$ are distinct nonsingular points on C;
3. each irreducible component of C has at least three special points (either marked points or nodes).

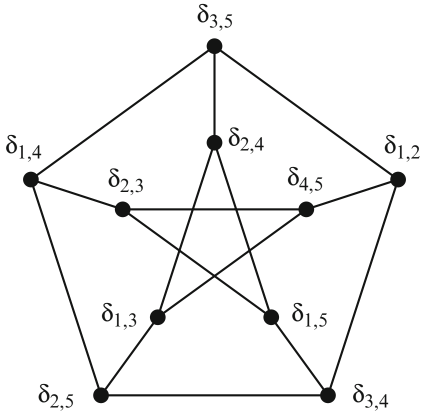

Fig. 1 The Petersen graph describes the boundary complex of $\overline{\mathrm{M}}_{0,5}$

The *dual graph* of a stable curve $(C, p_1, p_2, \ldots, p_n)$ is defined as a graph having a vertex for each irreducible component, an edge between two vertices for each point of intersection of the corresponding components, and for each marked point, a labelled half edge attached to the appropriate vertex. Since C has arithmetic genus 0, the dual graph is a tree.

The boundary $\overline{\mathrm{M}}_{0,n} \setminus \mathrm{M}_{0,n}$ is a normal crossing divisor with a natural stratification by the dual graphs. The codimension of the stratum $\delta(\Gamma)$ in $\overline{\mathrm{M}}_{0,n}$ corresponding to the dual graph Γ is one less than the number of vertices of Γ:

$$\operatorname{codim}(\delta(\Gamma)) = \#V(\Gamma) - 1.$$

Thus, each divisorial component of the boundary of $\overline{\mathrm{M}}_{0,n}$ corresponds to stable curves with dual graph Γ corresponding to a partition of $\{1, 2, \ldots, n\}$ into two disjoint sets I and I^c, each of cardinality at least 2. Given such a partitition, we denote the corresponding irreducible boundary divisor of $\overline{\mathrm{M}}_{0,n}$ by δ_I. Note that two divisors δ_I and δ_J intersect in $\overline{\mathrm{M}}_{0,n}$ if and only if $I \subset J$, $I \subset J^c$, $J \subset I$, or $J \subset I^c$.

Example 2.2 Consider the graph whose vertices correspond to the irreducible boundary divisors in $\overline{\mathrm{M}}_{0,5}$. Two vertices are joined by an edge if the corresponding boundary divisors intersect (see Fig. 1). Originally a purely combinatorial construction, the Petersen graph shown in Fig. 1 is in fact the link of the tropical moduli space $M_{0,5}^{\mathrm{trop}}$. For a brief introduction to the moduli space $M_{g,n}^{\mathrm{trop}}$, we recommend the lectures and lecture notes of Melody Chan [2]. For a thorough introduction to tropical geometry, see [18].

Remark 2.3 A general philosophy is that birational models of a given compactified moduli space should provide alternate compactifications which themselves have modular interpretations. From this perspective, describing the birational geometry of $\overline{\mathrm{M}}_{0,n}$ is not only interesting in its own right, but also provides a window into current research in moduli theory. As a first step in this direction, in [10], Harris and Mumford proved that the moduli spaces $\overline{\mathrm{M}}_{g,n}$ are of general type for large enough g.

Thus, understanding the birational geometry of $\overline{\mathrm{M}}_{g,n}$ boils down to describing all ample divisors on the moduli space. A long-standing conjecture of Fulton and Faber, the so-called "F-conjecture", describes the ample cone of $\overline{\mathrm{M}}_{g,n}$; see for example [7] in which the F-conjecture is reduced to the genus 0 case. Notably, in [9], Hu and Keel conjectured that $\overline{\mathrm{M}}_{0,n}$ is a Mori Dream Space, a result that would have implied the F-conjecture. This was recently disproved for $n > 133$ by Castravet and Tevelev [3]; their techniques were quickly extended to $n > 13$ in [6]. We recommend the excellent survey [5] for those interested in learning more about birational models and alternative compactifications of $\overline{\mathrm{M}}_{g,n}$.

3 The Embedding of $\overline{\mathrm{M}}_{0,n}$ in $\mathbb{P}^1 \times \mathbb{P}^2 \times \cdots \times \mathbb{P}^{n-3}$

We begin with a description of an embedding $\varphi\colon \overline{\mathrm{M}}_{0,n} \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^2 \times \cdots \times \mathbb{P}^{n-3}$, followed by a parametrization of φ on the interior of $\overline{\mathrm{M}}_{0,n}$ which extends to the full moduli space. We then give a geometric interpretation of φ in the cases $\overline{\mathrm{M}}_{0,4}$ and $\overline{\mathrm{M}}_{0,5}$. This realizes $\overline{\mathrm{M}}_{0,5}$ in Theorem 3.8 as a pencil of conics in $\mathbb{P}^1 \times \mathbb{P}^2$. Finally, we use the parametrization to list a set of equations satisfied by the image $\varphi(\overline{\mathrm{M}}_{0,n}) \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^{n-3}$.

We recall two well-studied maps. First, the forgetful map $\pi_n\colon \overline{\mathrm{M}}_{0,n} \to \overline{\mathrm{M}}_{0,n-1}$, is given by forgetting the last point of $[C, p_1, p_2, \ldots, p_n]$ and stabilizing the curve. That is, π_n contracts the components of C that have less than three special points among $p_1, p_2, \ldots, p_{n-1}$, and remembers the points of intersection.

For the second, let $\mathbb{L}_i$ be the line bundle on $\overline{\mathrm{M}}_{0,n}$ whose fibre over a point $[C, p_1, p_2, \ldots, p_n]$ is the cotangent space of $\mathbb{P}^1$ at p_i. Define $\psi_i = c_1(\mathbb{L}_i)$ to be the first Chern class of $\mathbb{L}_i$. The Kapranov map ψ_n is the rational map given by the linear system $|\psi_n|$. This map was first described in detail by Kapranov [14], who proved in particular that $\psi_n : \overline{\mathrm{M}}_{0,n} \to \mathbb{P}^{n-3}$.

Theorem 3.1 ([16, Corollary 2.7]) *The map*

$$\Phi = (\pi_n, \psi_n) : \overline{\mathrm{M}}_{0,n} \to \overline{\mathrm{M}}_{0,n-1} \times \mathbb{P}^{n-3}$$

is a closed embedding.

Corollary 3.2 *We have a closed embedding* $\varphi\colon \overline{\mathrm{M}}_{0,n} \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^2 \times \cdots \times \mathbb{P}^{n-3}$.

Proof Apply Theorem 3.1 successively. □

Let us describe the map φ of Corollary 3.2 explicitly. Since φ is a closed embedding, it is enough to describe it only on the smooth part $\mathrm{M}_{0,n}$ of $\overline{\mathrm{M}}_{0,n}$, which is an open, dense subset of $\overline{\mathrm{M}}_{0,n}$. To this end, consider the restriction of π_n to $\mathrm{M}_{0,n}$ and let $F = \pi_n^{-1}([\mathbb{P}^1, p_1, p_2, \ldots, p_{n-1}]) \simeq \mathbb{P}^1 \setminus \{p_1, p_2, \ldots, p_{n-1}\}$ be the fibre over a point in $\mathrm{M}_{0,n-1}$. We have the following description of the line bundle $\mathbb{L}_n$ over the fibre F.

Lemma 3.3 ([16]) *We have* $\mathbb{L}_n|_F = \omega_{\mathbb{P}^1}(p_1+p_2+\cdots+p_{n-1})$. *In particular,* $\psi_n|_F = K_{\mathbb{P}^1} + p_1 + p_2 + \cdots + p_{n-1}$.

Following Lemma 3.3, we obtain a basis of $H^0(F, K_F + p_1 + p_2 + \cdots + p_{n-1})$.

Lemma 3.4 *The vector space* $H^0(F, K_F + p_1 + p_2 + \cdots + p_{n-1})$ *has dimension* $n-2$ *and a basis is given by the one-forms* $\left\{\frac{dx}{(x-p_1)(x-p_2)}, \frac{dx}{(x-p_1)(x-p_3)}, \ldots, \frac{dx}{(x-p_1)(x-p_{n-1})}\right\}$.

Proof Since the canonical class of $F \simeq \mathbb{P}^1$ is $K_F = -2[\mathrm{pt}]$, the dimension of the space of global sections follows from, for example, Riemann-Roch. Since the $n-2$ one-forms listed have two poles, each at different pairs of points, they are linearly independent, and so form a basis. □

We thus obtain an explicit parametrization of the map $\psi_n|_F : F \hookrightarrow \mathbb{P}^{n-3}$.

Proposition 3.5 *Let* $F \simeq \mathbb{P}^1 \backslash \{p_1, p_2, \ldots, p_{n-1}\}$ *be the fibre of the map* π_n *over the point* $[\mathbb{P}^1, p_1, p_2, \ldots, p_{n-1}] \in \mathrm{M}_{0,n-1}$. *The restriction* $\psi_n|_F : F \hookrightarrow \mathbb{P}^{n-3}$ *is given in coordinates by* $x \mapsto \left[\frac{p_1-p_2}{x-p_2} : \frac{p_1-p_3}{x-p_3} : \cdots : \frac{p_1-p_{n-1}}{x-p_{n-1}}\right]$.

Proof Using a properly rescaled basis from Lemma 3.4, we can write the map as

$$x \mapsto \left[\frac{p_1-p_2}{(x-p_1)(x-p_2)} : \frac{p_1-p_3}{(x-p_1)(x-p_3)} : \cdots : \frac{p_1-p_{n-1}}{(x-p_1)(x-p_{n-1})}\right].$$

Multiplying through by $x - p_1$ gives the result. □

By Corollary 3.2, we can extend ψ_n uniquely to all of $\overline{\mathrm{M}}_{0,n}$. By our choice of basis, the points $p_1, p_2, \ldots, p_{n-1} \in \bar{F}$ map to the coordinate points of $\mathbb{P}^{n-3}$, in particular to points in general position. Thus, $\psi_n(\bar{F})$ is a degree $n-3$ curve in $\mathbb{P}^{n-3}$, i.e. a (generically) smooth rational normal curve passing through these $n-1$ fixed points. Taking the parameter x to be p_n, the map $\varphi : \overline{\mathrm{M}}_{0,n} \to \mathbb{P}^1 \times \mathbb{P}^2 \times \cdots \times \mathbb{P}^{n-3}$ on the ith factor of the target is given by

$$[C, p_1, p_2, \ldots, p_n] \mapsto \left[\frac{p_1-p_2}{p_{i+3}-p_2} : \frac{p_1-p_3}{p_{i+3}-p_3} : \cdots : \frac{p_1-p_{i+2}}{p_{i+3}-p_{i+2}}\right].$$

Example 3.6 We describe the embedding $\varphi : \overline{\mathrm{M}}_{0,4} \to \mathbb{P}^1$ explicitly. Away from p_2, p_3, p_4, we have $\varphi : [\mathbb{P}^1, p_1, p_2, p_3, p_4] \mapsto \left[\frac{p_1-p_2}{p_4-p_2} : \frac{p_1-p_3}{p_4-p_3}\right]$. Thus, away from the boundary of $\overline{\mathrm{M}}_{0,4}$, we see that φ is an isomorphism mapping a 4-tuple of distinct points on $\mathbb{P}^1$ to their cross-ratio. The boundary of $\overline{\mathrm{M}}_{0,4}$ consists of the three points $\delta_{1,2}$, $\delta_{1,3}$, and $\delta_{1,4}$; see Fig. 2. By taking limits $p_1 \to p_2$, $p_1 \to p_3$, and $p_1 \to p_4$, we see that these boundary points map under φ to the points $[0 : 1]$, $[1 : 0]$, and $[1 : 1]$, respectively.

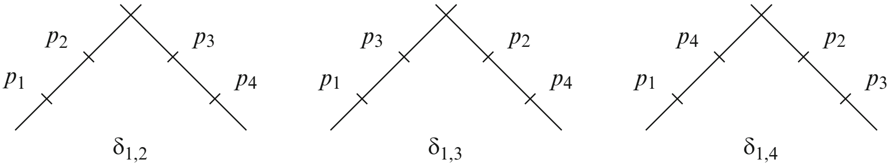

Fig. 2 The boundary divisors of $\overline{\mathrm{M}}_{0,4}$

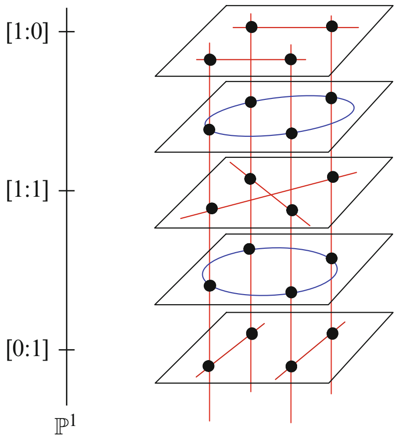

Fig. 3 The embedding $\varphi: \overline{\mathrm{M}}_{0,5} \to \mathbb{P}^1 \times \mathbb{P}^2$. The red lines correspond to boundary divisors

Example 3.7 For $n = 5$, the embedding $\varphi: \overline{\mathrm{M}}_{0,5} \to \mathbb{P}^1 \times \mathbb{P}^2 \simeq \overline{\mathrm{M}}_{0,4} \times \mathbb{P}^2$ has the form

$$[\mathbb{P}^1, p_1, p_2, \ldots, p_5] \mapsto \left(\left[\frac{p_1 - p_2}{p_4 - p_2} : \frac{p_1 - p_3}{p_4 - p_3} \right], \left[\frac{p_1 - p_2}{p_5 - p_2} : \frac{p_1 - p_3}{p_5 - p_3} : \frac{p_1 - p_4}{p_5 - p_4} \right] \right).$$

The forgetful map π_5 restricts to $\overline{\mathrm{M}}_{0,5}$ the projection of $\overline{\mathrm{M}}_{0,4} \times \mathbb{P}^2$ onto the first factor, so the fibre of π_5 over any point in $\mathrm{M}_{0,4} \simeq \mathbb{P}^1 \setminus \{[0:1], [1:0], [1:1]\}$ is a smooth conic passing through four fixed points $[1:0:0]$,$[0:1:0]$,$[0:0:1]$, and $[1:1:1]$ in a copy of $\mathbb{P}^2$. The fibres over $[0:1]$, $[1:0]$, $[1:1]$ are the singular conics passing through these four fixed points. In particular, six boundary divisors of $\overline{\mathrm{M}}_{0,5}$ map to the components of three singular conics, and the remaining four map to the fibres $\mathbb{P}^1 \times \{[1:0:0]\}$, $\mathbb{P}^1 \times \{[0:1:0]\}$, $\mathbb{P}^1 \times \{[0:0:1]\}$, and $\mathbb{P}^1 \times \{[1:1:1]\}$; see Fig. 3. Since $\overline{\mathrm{M}}_{0,5}$ is the pencil of conics in $\mathbb{P}^2$ passing through these given four points, we easily write down an embedding of $\overline{\mathrm{M}}_{0,5}$ in $\mathbb{P}^1_{[a_0:a_1]} \times \mathbb{P}^2_{[b_0:b_1:b_2]}$. Its equation is $a_0 b_1 (b_0 - b_2) - a_1 b_0 (b_1 - b_2)$.

We observe that $\psi: \overline{\mathrm{M}}_{0,5} \to \mathbb{P}^2$ is a blowdown. Indeed, since there is unique conic passing through any five points in $\mathbb{P}^2$, the map $\psi: \overline{\mathrm{M}}_{0,5} \to \mathbb{P}^2$ is 1-to-1 away from $[1:0:0]$, $[0:1:0]$, $[0:0:1]$, and $[1:1:1]$. The fibres over these points are all isomorphic to $\mathbb{P}^1$.

Proposition 3.8 *With the choice of coordinates above, the ideal which defines the moduli space* $\overline{\mathrm{M}}_{0,5}$ *as a projective variety in* $\mathbb{P}^1_{[a_0:a_1]} \times \mathbb{P}^2_{[b_0:b_1:b_2]}$ *is generated by* $a_0b_1(b_0 - b_2) - a_1b_0(b_1 - b_2)$.

Proof This is a direct computation consisting of checking that the variety described by this principal ideal in the multigraded ring $\mathbb{C}[a_0, a_1, b_0, b_1, b_2]$ is two-dimensional, smooth, and irreducible. We verified this using *Macaulay2*. □

4 Equations of $\overline{\mathrm{M}}_{0,n}$

We provide a list of equations contained in the ideal defining the scheme $\varphi(\overline{\mathrm{M}}_{0,n})$ in the Cox ring of $\mathbb{P}^1 \times \mathbb{P}^2 \times \cdots \times \mathbb{P}^{n-3}$. Let $w_0^{(i)}, w_1^{(i)}, \cdots, w_i^{(i)}$ be homogeneous coordinates on the ith factor of $\mathbb{P}^1 \times \mathbb{P}^2 \times \cdots \times \mathbb{P}^{n-3}$. By Proposition 3.5, the embedding $\varphi \colon \overline{\mathrm{M}}_{0,n} \to \mathbb{P}^1 \times \mathbb{P}^2 \times \cdots \times \mathbb{P}^{n-3}$ is given in coordinates by $w_j^{(i)} = p_{1,j+2}/p_{i+3,j+2}$ where $p_{i,j} := p_i - p_j$.

Lemma 4.1 *The image of* φ *in* $\mathbb{P}^1 \times \mathbb{P}^2 \times \cdots \times \mathbb{P}^{n-3}$ *satisfies the* $\binom{n-1}{4}$ *polynomials given by the* 2×2 *minors of the matrices*

$$\begin{bmatrix} w_0^{(i)}\left(w_0^{(j)} - w_{i+1}^{(j)}\right) & w_1^{(i)}\left(w_1^{(j)} - w_{i+1}^{(j)}\right) & \cdots & w_i^{(i)}\left(w_i^{(j)} - w_{i+1}^{(j)}\right) \\ w_0^{(j)} & w_1^{(j)} & \cdots & w_i^{(j)} \end{bmatrix}$$

for all $1 \leq i < j \leq n-3$. *Let* J_n *be the ideal generated by these polynomials. No proper subset of these polynomials forms a basis of* J_n.

Proof The proof is direct calculation. Choose columns r and s. We show that

$$\left(w_r^{(j)} - w_{i+1}^{(j)}\right) w_s^{(j)} w_r^{(i)} = \left(w_s^{(j)} - w_{i+1}^{(j)}\right) w_r^{(j)} w_s^{(i)} .$$

Indeed, the following equalities hold:

$$\begin{aligned} w_r^{(j)} - w_{i+1}^{(j)} &= \frac{p_{1,r+2}}{p_{j+3,r+2}} - \frac{p_{1,i+3}}{p_{j+3,r+3}} \\ &= \frac{p_{1,r+2}p_{j+3,i+3} - p_{1,i+3}p_{j+3,r+2}}{p_{j+3,r+2}p_{j+3,i+3}} = \frac{p_{1,j+3}p_{r+2,i+3}}{p_{j+3,r+2}p_{j+3,i+3}}. \end{aligned}$$

So we have

$$\begin{aligned} \left(w_r^{(j)} - w_{i+1}^{(j)}\right) w_s^{(j)} w_r^{(i)} &= \frac{p_{1,j+3}p_{r+2,i+3}}{p_{j+3,r+2}p_{j+3,r+2}} \cdot \frac{p_{1,s+2}}{p_{j+3,s+2}} \cdot \frac{p_{1,r+2}}{p_{i+3,r+2}} \\ &= -\frac{p_{1,j+3}p_{1,s+2}p_{1,r+2}}{p_{j+3,r+2}p_{j+3,s+2}p_{j+3,i+3}}. \end{aligned}$$

A similar calculation gives that $\left(w_s^{(j)} - w_{i+1}^{(j)}\right) w_r^{(j)} w_s^{(i)}$ equals the same expression.

For each $i \in \{1, 2, \ldots, n-4\}$, there are $n-3-i$ matrices with $i+1$ columns each. Thus, the number of 2×2 minors is given by

$$\sum_{i=1}^{n-4} (n-3-i)\binom{i+1}{2} = \binom{n-1}{4}.$$

The polynomials have the same total degree and different initial terms under lex order, so are linearly independent over $\mathbb{C}$. This proves the final statement. □

Example 4.2 We list the polynomials of Lemma 4.1 satisfied by $\overline{\mathrm{M}}_{0,7}$ as a subscheme of $\mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^3 \times \mathbb{P}^4$. The Cox ring of the product of projective spaces is $\mathbb{C}[a_0, a_1, b_0, b_1, b_2, c_0, c_1, c_2, c_3, d_0, d_1, d_2, d_3, d_4]$. The six matrices of Lemma 4.1 are

$$\begin{bmatrix} a_0(b_0-b_2) & a_1(b_1-b_2) \\ b_0 & b_1 \end{bmatrix}, \quad \begin{bmatrix} b_0(c_0-c_3) & b_1(c_1-c_3) & b_2(c_2-c_3) \\ c_0 & c_1 & c_2 \end{bmatrix},$$
$$\begin{bmatrix} a_0(c_0-c_2) & a_1(c_1-c_2) \\ c_0 & c_1 \end{bmatrix}, \quad \begin{bmatrix} b_0(d_0-d_3) & b_1(d_1-d_3) & b_2(d_2-d_3) \\ d_0 & d_1 & d_2 \end{bmatrix},$$
$$\begin{bmatrix} a_0(d_0-d_2) & a_1(d_1-d_2) \\ d_0 & d_1 \end{bmatrix}, \quad \begin{bmatrix} c_0(d_0-d_4) & c_1(d_1-d_4) & c_2(d_2-d_4) & c_3(d_3-d_4) \\ d_0 & d_1 & d_2 & d_3 \end{bmatrix}.$$

Taking all 2-minors gives the following 15 polynomials in the ideal defining $\overline{\mathrm{M}}_{0,7}$ as a subscheme of $\mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^3 \times \mathbb{P}^4$:

$$\begin{aligned}
&c_2d_2d_3 - c_3d_2d_3 + c_3d_2d_4 - c_2d_3d_4, && c_1d_1d_3 - c_3d_1d_3 + c_3d_1d_4 - c_1d_3d_4, \\
&c_0d_0d_3 - c_3d_0d_3 + c_3d_0d_4 - c_0d_3d_4, && c_1d_1d_2 - c_2d_1d_2 + c_2d_1d_4 - c_1d_2d_4, \\
&b_1d_1d_2 - b_2d_1d_2 + b_2d_1d_3 - b_1d_2d_3, && c_0d_0d_2 - c_2d_0d_2 + c_2d_0d_4 - c_0d_2d_4, \\
&b_0d_0d_2 - b_2d_0d_2 + b_2d_0d_3 - b_0d_2d_3, && c_0d_0d_1 - c_1d_0d_1 + c_1d_0d_4 - c_0d_1d_4, \\
&b_0d_0d_1 - b_1d_0d_1 + b_1d_0d_3 - b_0d_1d_3, && a_0d_0d_1 - a_1d_0d_1 + a_1d_0d_2 - a_0d_1d_2, \\
&b_1c_1c_2 - b_2c_1c_2 + b_2c_1c_3 - b_1c_2c_3, && b_0c_0c_2 - b_2c_0c_2 + b_2c_0c_3 - b_0c_2c_3, \\
&b_0c_0c_1 - b_1c_0c_1 + b_1c_0c_3 - b_0c_1c_3, && a_0c_0c_1 - a_1c_0c_1 + a_1c_0c_2 - a_0c_1c_2, \\
&a_0b_0b_1 - a_1b_0b_1 + a_1b_0b_2 - a_0b_1b_2.
\end{aligned}$$

Conjecture 4.3 Let $\mathbb{C}[w_j^{(i)} : 1 \le i \le n-3, 0 \le j \le i]$ be the Cox ring of the product $\mathbb{P}^1 \times \mathbb{P}^2 \times \cdots \times \mathbb{P}^{n-3}$ and let $B = \bigcap_{i=1}^{n-3} \langle w_0^{(i)}, w_1^{(i)}, \ldots, w_i^{(i)} \rangle$ be its irrelevant ideal. If I_n is the B-saturated ideal that defines the subscheme $\varphi(\overline{\mathrm{M}}_{0,n})$ of $\mathbb{P}^1 \times \mathbb{P}^2 \times \cdots \times \mathbb{P}^{n-3}$

and G is a minimal Gröbner basis of I_n, then the number of polynomials of degree d in G is $\binom{n-1}{d+1}$.

Computations in *Macaulay2* support Conjecture 4.3 for $4 \leq n \leq 8$. The following lemma gives additional evidence.

Lemma 4.4 *For each choice of* $0 \leq i < j \leq k < l < m \leq n-3$*, the embedding of* $\overline{\mathrm{M}}_{0,n}$ *in* $\mathbb{P}^1 \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^{n-3}$ *satisfies the following* $\binom{n-1}{5}$ *linearly independent equations of degree* 4 *in the Cox ring:*

$$w_{k+1}^{(m)} w_{l+1}^{(m)} \left(w_i^{(k)} w_j^{(l)} - w_j^{(k)} w_i^{(l)}\right) + w_{l+1}^{(m)} w_j^{(m)} \left(w_j^{(k)} w_i^{(l)} - w_i^{(k)} w_i^{(l)}\right) + w_j^{(m)} w_{k+1}^{(m)} \left(w_i^{(k)} w_i^{(l)} - w_i^{(k)} w_j^{(l)}\right) = 0\,. \tag{2}$$

Proof The number of equations with $j = k$ is $\binom{n-2}{5}$ and the number with $j \neq k$ is $\binom{n-2}{4}$, so there are $\binom{n-2}{5} + \binom{n-2}{4} = \binom{n-1}{5}$ equations in total. Since all of the equations have different initial terms with respect to lex order, they are linearly independent.

The rest of the proof is direct calculation. For simplicity, we first shift the indices:

$$\begin{aligned} &w_{k-2}^{(m-3)} w_{l-2}^{(m-3)} \left(w_{i-2}^{(k-3)} w_{j-2}^{(l-3)} - w_{j-2}^{(k-3)} w_{i-2}^{(l-3)}\right) \\ &\quad + w_{l-2}^{(m-3)} w_{j-2}^{(m-3)} \left(w_{j-2}^{(k-3)} w_{i-2}^{(l-3)} - w_{i-2}^{(k-3)} w_{i-2}^{(l-3)}\right) \\ &\quad\quad + w_{j-2}^{(m-3)} w_{k-2}^{(m-3)} \left(w_{i-2}^{(k-3)} w_{i-2}^{(l-3)} - w_{i-2}^{(k-3)} w_{j-2}^{(l-3)}\right) = 0\,. \end{aligned}$$

Then one checks that the following identities hold:

$$w_{k-2}^{(m-3)} w_{l-2}^{(m-3)} \left(w_{i-2}^{(k-3)} w_{j-2}^{(l-3)} - w_{j-2}^{(k-3)} w_{i-2}^{(l-3)}\right) = A \frac{p_{k,l}}{p_{m,l} p_{m,k} p_{k,j} p_{l,j}}$$

$$w_{l-2}^{(m-3)} w_{j-2}^{(m-3)} \left(w_{j-2}^{(k-3)} w_{i-2}^{(l-3)} - w_{i-2}^{(k-3)} w_{i-2}^{(l-3)}\right) = A \frac{1}{p_{m,l} p_{m,j} p_{k,j}}$$

$$w_{j-2}^{(m-3)} w_{k-2}^{(m-3)} \left(w_{i-2}^{(k-3)} w_{i-2}^{(l-3)} - w_{i-2}^{(k-3)} w_{j-2}^{(l-3)}\right) = -A \frac{1}{p_{m,j} p_{m,k} p_{l,j}}$$

where $A := \frac{p_{1,i} p_{1,j} p_{1,l} p_{1,k} p_{j,i}}{p_{l,i} p_{k,i}}$. Furthermore the following is true:

$$\begin{aligned} \frac{1}{p_{m,j} p_{m,k} p_{l,j}} - \frac{1}{p_{m,l} p_{m,j} p_{k,j}} &= \frac{1}{p_{m,j}} \frac{p_{m,l} p_{k,j} - p_{m,k} p_{l,j}}{p_{m,l} p_{k,j} p_{m,k} p_{l,j}} \\ &= \frac{p_{m,j}}{p_{m,j}} \frac{p_{k,l}}{p_{m,l} p_{m,k} p_{k,j} p_{l,j}} = \frac{p_{k,l}}{p_{m,l} p_{m,k} p_{k,j} p_{l,j}}, \end{aligned}$$

which completes the proof. □

The equations in Lemma 4.4 are equivalent to the following, modulo the ideal J_n:

$$w_i^{(k)}w_i^{(l)}w_j^{(m)}w_{k+1}^{(m)} - w_i^{(k)}w_{k+1}^{(l)}w_j^{(m)}w_{k+1}^{(m)} - w_i^{(k)}w_i^{(l)}w_j^{(m)}w_{l+1}^{(m)} + w_j^{(k)}w_i^{(l)}w_j^{(m)}w_{l+1}^{(m)} + w_i^{(k)}w_{k+1}^{(l)}w_j^{(m)}w_{l+1}^{(m)} - w_j^{(k)}w_i^{(l)}w_{k+1}^{(m)}w_{l+1}^{(m)} . \qquad (3)$$

Indeed, the difference of the equations (2) and (3) is

$$w_i^{(k)}\left(w_{k+1}^{(l)}w_j^{(m)}\left(w_{k+1}^{(m)} - w_{l+1}^{(m)}\right) - w_j^{(l)}w_{k+1}^{(m)}\left(w_j^{(m)} - w_{l+1}^{(m)}\right)\right) \in J_n .$$

We expect that the polynomials (3) are the minimal generators of I_n in degree 4.

Example 4.5 Conjecture 4.3 predicts that there are $\binom{7-1}{4+1} = 6$ polynomials of degree 4 in a reduced Gröbner basis of I_7. Using *Macaulay2*, we compute the unique B-saturation of J_7, which gives the six polynomials of degree 4

$$b_1c_1d_2d_3 - b_1c_3d_2d_3 - b_1c_1d_2d_4 + b_2c_1d_2d_4 + b_1c_3d_2d_4 - b_2c_1d_3d_4 ,$$
$$b_0c_0d_2d_3 - b_0c_3d_2d_3 - b_0c_0d_2d_4 + b_2c_0d_2d_4 + b_0c_3d_2d_4 - b_2c_0d_3d_4 ,$$
$$b_0c_0d_1d_3 - b_0c_3d_1d_3 - b_0c_0d_1d_4 + b_1c_0d_1d_4 + b_0c_3d_1d_4 - b_1c_0d_3d_4 ,$$
$$a_0c_0d_1d_2 - a_0c_2d_1d_2 - a_0c_0d_1d_4 + a_1c_0d_1d_4 + a_0c_2d_1d_4 - a_1c_0d_2d_4$$
$$a_0b_0d_1d_2 - a_0b_2d_1d_2 - a_0b_0d_1d_3 + a_1b_0d_1d_3 + a_0b_2d_1d_3 - a_1b_0d_2d_3$$
$$a_0b_0c_1c_2 - a_0b_2c_1c_2 - a_0b_0c_1c_3 + a_1b_0c_1c_3 + a_0b_2c_1c_3 - a_1b_0c_2c_3 ,$$

in $\mathbb{C}[a_0, a_1, b_0, b_1, b_2, c_0, c_1, c_2, c_3, d_0, d_1, d_2, d_3, d_4]$. These polynomials correspond to the equations (3). The last equation coincides with the unique equation of degree 4 in a Gröbner basis of I_6, and the structure of each equation is similar. In general, we expect that equations of degree d in a Gröbner basis for I_n have structure similar to the unique equation of top degree in a Gröbner basis for I_{d+2}. Moreover, for each pair $0 \le i < j \le 2$, there is a unique polynomial of degree $(0, 1, 1, 2)$ containing b_i and b_j.

Let G be a reduced Gröbner basis of I_n under lex order, and G_d the subset of G consisting of polynomials of degree d. Fix a $\mathbb{P}^i$ and choose two $w_j^{(i)}, w_k^{(i)}$. For each choice of $d-2$ of the remaining spaces $\mathbb{P}^{i+1}, \mathbb{P}^{i+2}, \ldots, \mathbb{P}^{n-3}$, we conjecture that there is a unique polynomial in G_d of degree one in the chosen variables, other than the last occurring projective space, in which the polynomial has degree two.

If the polynomials of degree d can be counted in this way, then Conjecture 4.3 will be true, upon application of the following combinatorial fact:

Lemma 4.6 *We have*

$$\sum_{i=1}^{n-3} \binom{n-3-i}{d-2}\binom{i+1}{2} = \binom{n-1}{d+1} .$$

Proof We rewrite the left hand side as a hypergeometric series and apply the Chu-Vandermonde Identity, see e.g. [19]. Let ${}_2F_1\left(\begin{smallmatrix} a & b \\ & c \end{smallmatrix}\right)$ denote the series

$$\sum_{k=0}^{\infty} \frac{a(a+1)\cdots(a+k-1)\cdot b(b+1)\cdots(b+k-1)}{k!c(c+1)\cdots(c+k-1)}.$$

Note that if either a or b is negative then the series is finite.

Now, changing the index of the series on the left hand side of the desired identity to $k = i - 1$, we let C_k be the k^{th} term. Expanding factorials and cancellation gives

$$\frac{C_{k+1}}{C_k} = \frac{(k+d+2-n)(k+3)}{(n-k-4)(k+1)}.$$

This means that the left hand side of the desired identity can be written as

$$\sum_{k=0}^{n-4} \binom{n-4-k}{d-2}\binom{k+2}{2} = \binom{n-4}{d-2} {}_2F_1\begin{pmatrix} -(n-d-2) & 3 \\ & 4-n \end{pmatrix}.$$

The Chu-Vandermonde Identity [19, Equation (2.7)] gives the desired identity:

$$\begin{aligned}
\binom{n-4}{d-2} {}_2F_1\begin{pmatrix} -(n-d-2) & 3 \\ & 4-n \end{pmatrix} &= \binom{n-4}{d-2} \frac{(1-n)(2-n)\cdots(-(d+2))}{(4-n)(5-n)\cdots(-(d-1))} \\
&= \binom{n-4}{d-2} \frac{(n-1)!}{(d+1)!} \frac{(d-2)!}{(n-4)!} \\
&= \frac{(n-1)(n-2)(n-3)(n-4)!}{(d+1)d(d-1)(d-2)!(n-4-d+2)!} \\
&= \binom{n-1}{d+1}.
\end{aligned}$$

□

5 The Number of Equations of $\overline{\mathrm{M}}_{0,n}$ in $\overline{\mathrm{M}}_{0,n-1} \times \mathbb{P}^{n-3}$

We recall some cohomological tools developed in [16]. Working in the ideal defining $\overline{\mathrm{M}}_{0,n}$ as a subscheme of $\overline{\mathrm{M}}_{0,n-1} \times \mathbb{P}^{n-3}$, these tools allow us to realize the number of equations of a given bidegree as the dimension of the space of global sections of a certain sheaf on $\overline{\mathrm{M}}_{0,n-1}$. We apply this to the case $n = 5$ to prove that J_5 contains all polynomials in I_5 of degree (d, d). We do the same for J_6 and I_6 in the next section.

Let $\sigma\colon U \to \overline{\mathrm{M}}_{0,n}$ be the universal curve over $\overline{\mathrm{M}}_{0,n}$ and ω be the relative dualizing sheaf. The *κ-class* on $\overline{\mathrm{M}}_{0,n}$ is the pushforward of the first Chern class of ω, namely $\kappa := \sigma_*\big(c_1(\omega)\big)$. If $K_{\overline{\mathrm{M}}_{0,n}}$ is the canonical class on $\overline{\mathrm{M}}_{0,n}$ and δ_I the classes of the boundary divisors, then Keel and Tevelev [16] prove that $\kappa \sim K_{\overline{\mathrm{M}}_{0,n}} + \sum \delta_I$, κ is very ample, and the composition of φ with the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^2 \times \cdots \times \mathbb{P}^{n-3} \hookrightarrow \mathbb{P}^{(n-2)!-1}$ is the embedding of $\overline{\mathrm{M}}_{0,n}$ via the κ class. Proposition 3.8 implies:

Corollary 5.1 *The ideal of the embedding of $\overline{\mathrm{M}}_{0,5}$ into $\mathbb{P}^5$ via the κ class is generated by the five quadrics in* (1).

Proof The first two equations are given by first multiplying the equation of Proposition 3.8 by x_0 and x_1 to obtain equations homogeneous of the same degree in x_i and y_j, then mapping into $\mathbb{P}^5$ by the Segre embedding. The final three are the Segre relations. By [16], the resulting map from $\overline{\mathrm{M}}_{0,5} \to \mathbb{P}^5$ is given by the κ class. □

We will prove that, for $5 \le n \le 6$, the ideals generated by polynomials in I_n of degree $(d, d, \ldots, d)$ are generated by polynomials in J_n. The following theorem tells us that this is enough to understand the ideal of $\Phi(\overline{\mathrm{M}}_{0,n})$ in $\mathbb{P}^{(n-2)!-1}$ via the κ class.

Theorem 5.2 ([16]) *The ideal that defines $\Phi(\overline{\mathrm{M}}_{0,n})$ as a subscheme of $\mathbb{P}^{(n-2)!-1}$ is generated by quadrics. Equivalently, if $\tilde{I}_n$ is the ideal generated by all polynomials of degree $(d, d, \ldots, d)$ contained in the ideal I_n in the Cox ring of $\mathbb{P}^1 \times \mathbb{P}^2 \times \cdots \times \mathbb{P}^{n-3}$, then $\tilde{I}_n$ is generated by polynomials of degree $(2, 2, \ldots, 2)$.*

Let V_{ψ_n} be the vector bundle on $\overline{\mathrm{M}}_{0,n}$ defined by the exact sequence

$$0 \to V_{\psi_n} \to H^0(\overline{\mathrm{M}}_{0,n}, \psi_n) \otimes \mathscr{O}_{\overline{\mathrm{M}}_{0,n}} \to \psi_n \to 0, \tag{4}$$

and consider the map $\Phi = (\pi_n, \psi_n)\colon \overline{\mathrm{M}}_{0,n} \to \overline{\mathrm{M}}_{0,n-1} \times \mathbb{P}^{n-3}$. By [16, Lemma 4.1], we have the following resolution of the structure sheaf of $\Phi(\overline{\mathrm{M}}_{0,n})$ in $\overline{\mathrm{M}}_{0,n-1} \times \mathbb{P}^{n-3}$:

$$0 \to \mathscr{M}_n^{n-4} \boxtimes \mathscr{O}(3-n) \to \cdots \to \mathscr{M}_n^1 \boxtimes \mathscr{O}(-2) \to \mathscr{O}_{\overline{\mathrm{M}}_{0,n-1} \times \mathbb{P}^{n-3}} \to \Phi_* \mathscr{O}_{\overline{\mathrm{M}}_{0,n}} \to 0, \tag{5}$$

where $\mathscr{M}_n^p = R^1 \pi_{n*}(\wedge^{p+1} V_{\psi_n})$.

To gain control of the sheaves in (5), we have

Theorem 5.3 ([16, Theorem 4.3]) *There exists a vector bundle Q on $\overline{\mathrm{M}}_{0,n}$ and exact sequences*

$$0 \to \pi_n^* \mathscr{M}_{n-1}^p \to \mathscr{M}_n^p \to Q \to 0 \tag{6}$$

$$0 \to V_{\psi_n} \to Q \to \mathscr{M}_n^{p-1} \to 0. \tag{7}$$

Remark 5.4 The vector bundle Q is defined explicitly in [16].

Thus, by tensoring (5) with the correct line bundles one can compute the expected number of equations for $\overline{\mathrm{M}}_{0,n} \subset \overline{\mathrm{M}}_{0,n} \times \mathbb{P}^{n-3}$ in particular degrees.

Lemma 5.5 *For all integers $n \ge 5$ and $a > 0$, the ideal I_n defining $\overline{\mathrm{M}}_{0,n}$ as a subscheme of $\overline{\mathrm{M}}_{0,n-1} \times \mathbb{P}^{n-3}$ contains exactly $h^0(\overline{\mathrm{M}}_{0,n-1}, \mathscr{M}_{n-1}^1 \otimes \kappa_{n-1}^{\otimes a})$ linearly independent equations of bidegree $(a, 2)$. Additionally, we have the short exact sequence*

$$0 \to H^0(\overline{\mathrm{M}}_{0,n-1}, \pi_{n-1}^* \mathscr{M}_{n-2}^1 \otimes \kappa_{n-1}^{\otimes a}) \to H^0(\overline{\mathrm{M}}_{0,5}, \mathscr{M}_{n-1}^1 \otimes \kappa_{n-1}^{\otimes a}) \\ \to H^0(\overline{\mathrm{M}}_{0,n-1}, V_{\psi_{n-2}} \otimes \kappa_{n-1}^{\otimes a}) \to 0\,.$$

Proof We use the resolution of the structure sheaf of $\Phi_* \mathscr{O}_{\overline{\mathrm{M}}_{0,n}}$ given by the exact sequence (5). Tensoring this with $\kappa_{n-1}^{\otimes a} \boxtimes \mathscr{O}(2)$, we obtain the following exact sequence on $\overline{\mathrm{M}}_{0,n-1} \times \mathbb{P}^{n-3}$:

$$0 \to (\mathscr{M}_{n-1}^{n-4} \otimes \kappa_{n-1}^{\otimes a}) \boxtimes \mathscr{O}(1-n) \to (\mathscr{M}_{n-1}^{n-3} \otimes \kappa_{n-1}^{\otimes a}) \boxtimes \mathscr{O}(2-n) \\ \to \cdots \to (\mathscr{M}_{n-2}^1 \otimes \kappa_{n-1}^{\otimes a}) \boxtimes \mathscr{O}(-1) \to (\mathscr{M}_{n-1}^1 \otimes \kappa_{n-1}^{\otimes a}) \boxtimes \mathscr{O} \\ \to \kappa_{n-1}^{\otimes a} \boxtimes \mathscr{O}(2) \to \Phi_* \mathscr{O}_{\overline{\mathrm{M}}_{0,n}} \otimes (\kappa^{\otimes a} \boxtimes \mathscr{O}(2)) \to 0\,.$$

By the Künneth Formula, and since $\mathscr{O}(k)$ is acyclic for $1-n \le k \le -1$, we have

$$H^i\big(\overline{\mathrm{M}}_{0,n-1} \times \mathbb{P}^{n-3}, (\mathscr{M}_{n-1}^{k+2n-5} \otimes \kappa_{n-1}^{\otimes a}) \boxtimes \mathscr{O}(k)\big) = 0$$

for all $1-n \le k \le -1$ and $i \ge 0$. Moreover, by Lemma 6.5 of [16], we have

$$H^1\big(\overline{\mathrm{M}}_{0,n-1} \times \mathbb{P}^{n-3}, (\mathscr{M}_{n-1}^1 \otimes \kappa_{n-1}^{\otimes a}) \boxtimes \mathscr{O}\big) = 0\,.$$

Thus, we obtain a short exact sequence in cohomology:

$$0 \to H^0\big(\overline{\mathrm{M}}_{0,n-1} \times \mathbb{P}^{n-3}, (\mathscr{M}_{n-1}^1 \otimes \kappa_{n-1}^{\otimes a}) \boxtimes \mathscr{O}\big) \\ \to H^0\big(\overline{\mathrm{M}}_{0,n-1} \times \mathbb{P}^{n-3}, \kappa_{n-1}^{\otimes a} \boxtimes \mathscr{O}(2)) \\ \xrightarrow{\tau} H^0\Big(\overline{\mathrm{M}}_{0,n-1} \times \mathbb{P}^{n-3}, \Phi_* \mathscr{O}_{\overline{\mathrm{M}}_{0,n}} \otimes \big(\kappa^{\otimes a} \boxtimes \mathscr{O}(2)\big)\Big) \to 0\,.$$

In particular, the number of equations of bidegree $(a, 2)$ in the ideal defining $\overline{\mathrm{M}}_{0,n}$ as a subvariety of $\overline{\mathrm{M}}_{0,n-1} \times \mathbb{P}^{n-3}$ is given by the dimension of the kernel of τ, which is $H^0(\overline{\mathrm{M}}_{0,n-1} \times \mathbb{P}^{n-3}, (\mathscr{M}_{n-1}^1 \otimes \kappa_{n-1}^{\otimes a}) \boxtimes \mathscr{O})$. By the Künneth Formula, this vector space is isomorphic to $H^0(\overline{\mathrm{M}}_{0,n-1}, \mathscr{M}_{n-1}^1 \otimes \kappa_{n-1}^{\otimes a})$.

Using the short exact sequences (6) and (7), and noting that $\mathscr{M}_n^0 = 0$ for all n, we have the short exact sequence $0 \to \pi_{n-1}^* \mathscr{M}_{n-2}^1 \to \mathscr{M}_{n-1}^1 \to V_{\psi_{n-2}} \to 0$. We tensor with $\kappa_{n-1}^{\otimes a}$. Then $H^i(\overline{\mathrm{M}}_{0,n-1}, \pi_{n-1}^* \mathscr{M}_{n-2}^1 \otimes \kappa_{n-1}^{\otimes a}) = 0$ for $i > 0$, by Lemma 6.5 in [16], and so taking cohomology gives the short exact sequence. □

Example 5.6 For a reality check, recall from Proposition 3.8 that I_5 is generated by one equation of degree $(1, 2)$, and Lemma 4.1 gives one equation satisfied by $\overline{\mathrm{M}}_{0,5}$ in $\mathbb{P}^1 \times \mathbb{P}^2$. On the other hand, Lemma 5.5 tells us that $\overline{\mathrm{M}}_{0,5}$ has $h^0(\overline{\mathrm{M}}_{0,4}, \mathscr{M}_4^1 \otimes \kappa_4^{\otimes a})$ linearly independent equations of degree $(a, 2)$.

Theorem 5.7 *We have* $h^0(\overline{\mathrm{M}}_{0,4}, \mathscr{M}_4^1 \otimes \kappa_4) = 1$ *and* $h^0(\overline{\mathrm{M}}_{0,4}, \mathscr{M}_4^1 \otimes \kappa_4^{\otimes 2}) = 2$.

Proof Under the isomorphism $\overline{\mathrm{M}}_{0,4} \simeq \mathbb{P}^1$, we have $\kappa_4 = \mathscr{O}_{\mathbb{P}^1}(1)$. We show that on $\overline{\mathrm{M}}_{0,4}$, we have $\mathscr{M}_4^1 = \psi_4^{-1} = \mathscr{O}_{\mathbb{P}^1}(1)$. Since $\mathscr{M}_3^1 = 0$, the exact sequence (6) becomes $0 \to 0 \to \mathscr{M}_4^1 \to Q \to 0$ and tells us that $\mathscr{M}_4^1 \simeq Q$. Since $\mathscr{M}_n^0 = 0$ for all n, the sequence (7) becomes $0 \to V_{\psi_4} \to Q \to 0 \to 0$, and therefore $\mathscr{M}_4^1 \simeq Q \simeq V_{\psi_4}$. We can determine the bundle V_{ψ_4} by analyzing the exact sequence (4). On $\overline{\mathrm{M}}_{0,4}$, this is $0 \to V_{\psi_4} \to H^0(\overline{\mathrm{M}}_{0,4}, \psi_4) \otimes \mathscr{O}_{\overline{\mathrm{M}}_{0,4}} \to \psi_4 \to 0$. Taking determinants gives

$$\mathscr{O}_{\overline{\mathrm{M}}_{0,4}} = \det\left(H^0(\overline{\mathrm{M}}_{0,4}, \psi_4) \otimes \mathscr{O}_{\overline{\mathrm{M}}_{0,4}}\right) = \det(V_{\psi_4}) \otimes \det(\psi_4) = V_{\psi_4} \otimes \psi_4 .$$

In particular, $\mathscr{M}_4^1 \simeq V_{\psi_4}$ is a line bundle dual to ψ_4. Recall that $\overline{\mathrm{M}}_{0,4} \simeq \mathbb{P}^1$ and $\psi_4 \simeq \mathscr{O}_{\mathbb{P}^1}(1)$, so the dimension of $H^0(\overline{\mathrm{M}}_{0,4}, \mathscr{M}_4^1 \otimes \mathscr{O}_{\mathbb{P}^1}(2))$ is given by

$$h^0\left(\overline{\mathrm{M}}_{0,4}, \mathscr{M}_4^1 \otimes \mathscr{O}_{\mathbb{P}^1}(2)\right) = h^0\left(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(-1) \otimes \mathscr{O}_{\mathbb{P}^1}(2)\right) = h^0\left(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(1)\right) = 2 .$$

Similarly we compute the dimension of $H^0\left(\overline{\mathrm{M}}_{0,4}, \mathscr{M}_4^1 \otimes \mathscr{O}_{\mathbb{P}^1}(1)\right)$ to be

$$h^0\left(\overline{\mathrm{M}}_{0,4}, \mathscr{M}_4^1 \otimes \mathscr{O}_{\mathbb{P}^1}(1)\right) = h^0\left(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(-1) \otimes \mathscr{O}_{\mathbb{P}^1}(1)\right) = h^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}) = 1 . \quad \square$$

6 Equations for $\overline{\mathrm{M}}_{0,6}$

We apply the tools from the previous section to prove Theorem 6.3, which states that $\tilde{I}_6$ is generated by polynomials in J_6. Lemma 4.1 gives five polynomials satisfied by $\overline{\mathrm{M}}_{0,6}$ in $\mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^3$:

$$\begin{aligned} f_1 &:= b_1c_1c_2 - b_2c_1c_2 + b_2c_1c_3 - b_1c_2c_3 , \\ f_2 &:= b_0c_0c_2 - b_2c_0c_2 + b_2c_0c_3 - b_0c_2c_3 , \\ f_3 &:= b_0c_0c_1 - b_1c_0c_1 + b_1c_0c_3 - b_0c_1c_3 , \\ f_4 &:= a_0c_0c_1 - a_1c_0c_1 + a_1c_0c_2 - a_0c_1c_2 , \\ f_5 &:= a_0b_0b_1 - a_1b_0b_1 + a_1b_0b_2 - a_0b_1b_2 . \end{aligned}$$

Let J_6 be the ideal in $\mathbb{C}[a_0, a_1, b_0, b_1, b_2, c_0, c_1, c_2, c_3]$ generated by $f_1, f_2, \ldots, f_5$, and let I_6 be the unique B-saturated ideal defining $\overline{\mathrm{M}}_{0,6}$ scheme-theoretically, where the irrelevant ideal is $B = \langle a_0, a_1\rangle \cap \langle b_0, b_1, b_2\rangle \cap \langle c_0, c_1, c_2, c_3\rangle$. Using *Macaulay2*, we verified that I_6 is prime, and is generated by J_6 and

$$f_6 := a_0b_0c_1c_2 - a_0b_2c_1c_2 - a_0b_0c_1c_3 + a_1b_0c_1c_3 + a_0b_2c_1c_3 - a_1b_0c_2c_3 .$$

Proposition 6.1 *The ideal J_6 is properly contained in the ideal I_6, but the parts of homogeneous degrees $(2,2,2)$ of the ideals J_6 and I_6 coincide.*

Proof Let $I_{(i,j,k)}$, respectively $J_{(i,j,k)}$, be the vector space of polynomials of degree (i,j,k) in I_6, respectively J_6. Multiplying the polynomials $f_1, f_2, \dots, f_5$ by all monomials of the correct degree gives a spanning set of $J_{(i,j,k)}$. Computing the dimension of $J_{(i,j,k)}$ involves determining which of the resulting polynomials are redundant. We used *Macaulay2* to show that $\dim J_{(1,1,2)} = 9 < \dim I_{(1,1,2)} = 10$ and $\dim J_{(2,2,2)} = \dim I_{(2,2,2)} = 55$. □

Remark 6.2 As we will see, the second part of Proposition 6.1 implies that the homogeneous parts of the ideals I_6 and J_6 coincide. Corollary 6.6 shows that these ideals contain the correct number of homogeneous equations of degree $(2,2,2)$.

Theorem 6.3 *Let $\tilde{I}_6$ be the ideal generated by the polynomials of degree (d,d,d) in I_6. Then $\tilde{I}_6$ is generated by the polynomials of degree $(2,2,2)$ contained in J_6. Equivalently, the embedding $\Phi(\overline{\mathrm{M}}_{0,n})$ in $\mathbb{P}^{23}$ defined by the κ class is generated by the homogeneous polynomials of degree $(2,2,2)$ in J_6 and the Segre relations.*

The proof of Theorem 6.3 requires the following two lemmas.

Lemma 6.4 *We have $h^0(\overline{\mathrm{M}}_{0,5}, V_{\psi_4} \otimes \kappa_5^{\otimes 2}) = 24$ and $h^0(\overline{\mathrm{M}}_{0,5}, V_{\psi_4} \otimes \kappa_5) = 11$.*

Proof We have on $\overline{\mathrm{M}}_{0,5}$ the exact sequence $0 \to V_{\psi_4} \to \mathbb{C}^3 \otimes \mathscr{O}_{\overline{\mathrm{M}}_{0,5}} \to \psi_4 \to 0$. We tensor with $\kappa_5^{\otimes 2}$. Noting that $H^i(\overline{\mathrm{M}}_{0,5}, V_{\psi_4} \otimes \kappa_5^{\otimes 2}) = 0$ for $i > 0$ by [16, Lemma 6.5], the long exact sequence in cohomology gives the short exact sequence

$$0 \to H^0(\overline{\mathrm{M}}_{0,5}, V_{\psi_4} \otimes \kappa_5^{\otimes 2}) \to H^0(\overline{\mathrm{M}}_{0,5}, \mathbb{C}^3 \otimes \kappa_5^{\otimes 2}) \to H^0(\overline{\mathrm{M}}_{0,5}, \psi_4 \otimes \kappa_5^{\otimes 2}) \to 0.$$

We shall determine the dimensions of the middle term and the last term.

For the middle term, we note first that any global section of $\mathbb{C}^3 \otimes \kappa_5^{\otimes 2}$ is of the form $\alpha \otimes \beta$ where $\alpha \in \mathbb{C}^3$ and $\beta \in H^0(\overline{\mathrm{M}}_{0,5}, \kappa_5^{\otimes 2})$. Using that $\overline{\mathrm{M}}_{0,5} \simeq \mathrm{Bl}_{q_1,q_2,\dots,q_4}(\mathbb{P}^2)$, we let $\sigma\colon \mathrm{Bl}_{q_1,q_2,\dots,q_4}(\mathbb{P}^2) \to \mathbb{P}^2$ be the blowup, $E_1, E_2, \dots, E_4$ the exceptional divisors, and $L_{i,j}$ the proper transform of the line passing through q_i and q_j on $\mathbb{P}^2$. The κ class is given by $\kappa_5 = K_{\overline{\mathrm{M}}_{0,5}} + \sum \delta_I$, where the sum is taken over all boundary divisors δ_I of $\overline{\mathrm{M}}_{0,5}$, so we have the linear equivalence

$$\kappa_5 \sim \left(\sigma^*(K_{\mathbb{P}^2}) + \sum_{i=1}^{4} E_i\right) + \left(\sum_{i=1}^{4} E_i + \sum_{1 \le i < j \le 4} L_{i,j}\right).$$

Since $L_{i,j} \sim \sigma^* H - E_i - E_j$, where H is the class of a hyperplane section on $\mathbb{P}^2$, we can write $\kappa_5^{\otimes 2}$ as $(\kappa_5)^{\otimes 2} \sim \sigma^*(6H) - 2\sum_{i=1}^{4} E_i$. In particular, this gives

$$H^0(\overline{\mathrm{M}}_{0,5}, \kappa_5^{\otimes 2}) \simeq H^0\left(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(6H) \otimes \left(\bigotimes_{i=1}^{4} \mathscr{I}_{q_i}^{\otimes 2}\right)\right),$$

where $\mathscr{I}_q$ denotes the skyscraper sheaf at q. By explicitly writing out equations, one can check that the conditions that a curve C of degree 6 on $\mathbb{P}^2$ attain nodes at four points in general position are linearly independent. Thus, this latter vector space has dimension $\binom{8}{2}-12 = 16$, and therefore $h^0(\overline{\mathrm{M}}_{0,5}, \mathbb{C}^3 \otimes \kappa_5^{\otimes 2}) = 3 \cdot 16 = 48$. Repeating the above with $\kappa_5^{\otimes 2}$ replaced by κ_5, we have $h^0(\overline{\mathrm{M}}_{0,5}, \mathbb{C}^3 \otimes \kappa_5) = 3\left(\binom{5}{2} - 4\right) = 18$. Since we have $\psi_4 \sim \delta_{1,2} + \delta_{3,5} + \delta_{4,5} \sim H$, it follows that

$$\psi_4 \otimes \kappa_5^{\otimes 2} \sim \sigma^*(7H) - 2(E_1 + E_2 + E_3 + E_4)\,.$$

With this, we compute the dimension of $H^0(\overline{\mathrm{M}}_{0,5}, \psi_4 \otimes \kappa_5^{\otimes 2})$ to be

$$h^0(\overline{\mathrm{M}}_{0,5}, \psi_4 \otimes \kappa_5^{\otimes 2}) = h^0\left(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(7) \otimes \left(\bigotimes_{i=1}^{4} \mathscr{I}_{q_i}^{\otimes 2}\right)\right) = \binom{9}{2} - 12 = 24.$$

Finally, we repeat the above computation with $\kappa_5^{\otimes 2}$ replaced by κ_5, and we find

$$h^0(\overline{\mathrm{M}}_{0,5}, \psi_4 \otimes \kappa_5) = h^0\left(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(4) \otimes \left(\bigotimes_{i=1}^{4} \mathscr{I}_{q_i}\right)\right) = \binom{6}{2} - 4 = 11\,.$$

□

Lemma 6.5 *We have* $h^0(\overline{\mathrm{M}}_{0,5}, \pi^*\mathscr{M}_4^1 \otimes \kappa_5^{\otimes 2}) = 11$ *and* $h^0(\overline{\mathrm{M}}_{0,5}, \pi^*\mathscr{M}_4^1 \otimes \kappa_5) = 3$.

Proof We note first that on $\overline{\mathrm{M}}_{0,4} \simeq \mathbb{P}^1$, we have $\mathscr{M}_4^1 = \psi_4^{-1}$ and $\psi_4 = \mathscr{O}(-1)$, so $\pi^*(\mathscr{M}_4^1) = \pi^*(\psi_4^{-1})$. Thus, we have $\pi^*(\mathscr{M}_4^1) = \pi^*\big(\mathscr{O}(1)\big) = -2H + \sum_{i=1}^4 E_i$. This allows us to write the class $\pi^*\mathscr{M}_4^1 \otimes \kappa_5^{\otimes 2}$ as

$$\pi^*\mathscr{M}_4^1 \otimes \kappa_5^{\otimes 2} \sim \sigma^*(-2H) + \sum_{i=1}^{4} E_i + \sigma^*(6H) - 2\sum_{i=1}^{4} E_i = \sigma^*(4H) - \sum_{i=1}^{4} E_i.$$

Using the push-pull formula, we can now compute the desired dimensions:

$$h^0(\overline{\mathrm{M}}_{0,5}, \pi^*\mathscr{M}_4^1 \otimes \kappa_5^{\otimes 2}) = h^0\left(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(4) \otimes \bigotimes_{i=1}^{4} \mathscr{I}_{q_i}\right) = \binom{6}{2} - 4 = 11\,,$$

$$h^0(\overline{\mathrm{M}}_{0,5}, \pi^*\mathscr{M}_4^1 \otimes \kappa_5) = h^0\big(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(1)\big) = 3\,.$$

□

Applying Lemmas 6.4 and 6.5 to the short exact sequence of Lemma 5.5 gives

Corollary 6.6 *The number of equations defining* $\overline{\mathrm{M}}_{0,6}$ *in the line bundle* $\kappa_5^{\otimes 2} \boxtimes \mathscr{O}(2)$ *on* $\overline{\mathrm{M}}_{0,5} \times \mathbb{P}^3$ *is* 35. *Similarly, for the line bundle* $\kappa_5 \boxtimes \mathscr{O}(2)$, *the number is* 10.

Proof For $n = 6$ and $a = 1, 2$, Lemma 5.5 gives the short exact sequences

$$0 \to H^0(\overline{\mathrm{M}}_{0,5}, \pi^*\mathscr{M}_4^1 \otimes \kappa_5^{\otimes 2}) \to H^0(\overline{\mathrm{M}}_{0,5}, \mathscr{M}_5^1 \otimes \kappa_5^{\otimes 2}) \to H^0(\overline{\mathrm{M}}_{0,5}, V_{\psi_4} \otimes \kappa_5^{\otimes 2}) \to 0$$

$$0 \to H^0(\overline{\mathrm{M}}_{0,5}, \pi^*\mathscr{M}_4^1 \otimes \kappa_5) \to H^0(\overline{\mathrm{M}}_{0,5}, \mathscr{M}_5^1 \otimes \kappa_5) \to H^0(\overline{\mathrm{M}}_{0,5}, V_{\psi_4} \otimes \kappa_5) \to 0\,.$$

The result follows from these, and Lemmas 6.4 and 6.5. □

Proof of Theorem 6.3 There are no equations in I_5 of degree $(1, 1)$, so the ideal generated by equations of degree $(1, 1, 2)$ in I_6 has number of linearly independent polynomials coinciding with the number of linearly independent polynomials defining $\overline{\mathrm{M}}_{0,6}$ in the line bundle $\kappa_5 \boxtimes \mathscr{O}(2)$ on $\overline{\mathrm{M}}_{0,5} \times \mathbb{P}^3$. Since there are two equations in I_5 of degree $(2, 2)$ in the Cox ring of $\mathbb{P}^1 \times \mathbb{P}^2$, both of which must be homogenized, we see that I_6 contains 20 linearly independent polynomials of degree $(2, 2, 2)$ in the Cox ring of $\mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^3$. By Corollary 6.6, the number of linearly independent polynomials of degree $(2, 2, 2)$ is $h^0(\overline{\mathrm{M}}_{0,5}, \mathscr{M}_5^1 \otimes \kappa_5^{\otimes 2}) + 20 = 55$. The number of linearly independent polynomials in I_6 of degree $(1, 1, 2)$ equals 10. Together with Proposition 6.1, this completes the proof. □

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Appendix

We include the *Macaulay2* code used to verify Conjecture 1.2.

```
R = QQ[a0,a1,b0,b1,b2,c0,c1,c2,c3,d0,d1,d2,d3,d4,
        e0,e1,e2,e3,e4,e5,f0,f1,f2,f3,f4,f5,f6];
```

The rows of the following matrix are coordinates on $\mathbb{P}^1, \mathbb{P}^2, \mathbb{P}^3, \mathbb{P}^4, \mathbb{P}^5, \mathbb{P}^6$

```
M = matrix{{0,0,0,0,0,0,0},        {a0,a1,0,0,0,0,0},
           {b0,b1,b2,0,0,0,0},    {c0,c1,c2,c3,0,0,0},
           {d0,d1,d2,d3,d4,0,0},  {e0,e1,e2,e3,e4,e5,0},
           {f0,f1,f2,f3,f4,f5,f6} };

M05 = {{2,2}};
M06 = {{2,2},{3,2},{3,3}};
M07 = {{2,2},{3,2},{3,3},{4,2},{4,3},{4,4}};
M08 = {{2,2},{3,2},{3,3},{4,2},{4,3},{4,4},{5,2},{5,3},
      {5,4},{5,5}};
M09 = {{2,2},{3,2},{3,3},{4,2},{4,3},{4,4},{5,2},{5,3},
      {5,4},{5,5},{6,2},{6,3},{6,4},{6,5},{6,6}};
```

Select your desired *n* here:

```
L = M07;
```

Lemma 4.1 involves the 2×2-minors of the matrices

```
Q = apply(L, l -> {submatrix(M,{l_1-1,l_0}, 0..l_1-1),
        M_(l_1)_(l_0) } )
S = apply(Q, T -> matrix{ apply(entries transpose T_0,
            x -> x_0 * (x_1-T_1) ), (entries T_0)_1 })
```

We form the ideal $J = J_n$ of all such 2-minors, and compute the prime ideal I by saturation.

```
J = sum apply(S,N -> minors(2,N));
J = saturate(J,ideal(a0,a1));
J = saturate(J,ideal(b0,b1,b2));
J = saturate(J,ideal(c0,c1,c2,c3));
J = saturate(J,ideal(d0,d1,d2,d3,d4));
J = saturate(J,ideal(e0,e1,e2,e3,e4,e5));
I = saturate(J,ideal(f0,f1,f2,f3,f4,f5,f6));
```

The following are used to determine whether the dimension of I is correct, as well as compute the degree of I, and the minimal number of generators in each degree.

```
codim I, degree I, betti mingens I
```

We finally note that the initial ideal is square-free and Cohen–Macaulay:

```
M = monomialIdeal leadTerm I;
betti mingens M
```

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Minkowski Sums and Hadamard Products of Algebraic Varieties

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Abstract We study Minkowski sums and Hadamard products of algebraic varieties. Specifically, we explore when these are varieties and examine their properties in terms of those of the original varieties. This project was inspired by Problem 5 on Surfaces in [13].

MSC 2010 codes: 14M99, 14N05, 14Q15, 14R99

1 Introduction

In algebraic geometry, we have several constructions to build new algebraic varieties from given ones. Examples of classical, well-studied constructions are joins, secant varieties, rational normal scrolls, and Segre products. In these cases, it is very interesting to relate geometric properties of the constructed variety to those of the original varieties. In this article, we focus on the *Minkowski sum* and the *Hadamard product* of algebraic varieties. These are constructed by considering the entry-wise sum and multiplication, respectively, of points on the varieties. Due to the nature of these operations, there is a remarkable difference between the affine and the projective case.

The entrywise sum is not well-defined over projective spaces. For this reason, we consider only Minkowski sums of affine varieties. However, in the case of affine

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cones, the Minkowski sum corresponds to the classical join of the corresponding projective varieties. Conversely, we focus on Hadamard products of projective varieties and, in particular, of varieties of matrices with fixed rank. This is because these Hadamard products parametrize interesting problems related to algebraic statistics and quantum information.

Our original motivating question was the following.

Question 1.1 Which properties do the Minkowski sum and the Hadamard product have with respect to the properties of the original varieties? In particular, what are their dimensions and degrees?

We now introduce these constructions. We work over an algebraically closed field $\Bbbk$. We use the notation $\Bbbk^\times := \Bbbk \setminus \{0\}$.

The Minkowski sum of a pair of algebraic varieties is constructed from the coordinatewise sums of their points. More precisely, the *Minkowski sum* $X + Y$ of two affine subvarieties $X, Y \subset \mathbb{A}^n$ in the same ambient space is the Zariski closure of the image of the product $X \times Y$ under the morphism $\phi_+ \colon \mathbb{A}^n \times \mathbb{A}^n \to \mathbb{A}^n$ defined by $\big((a_1, a_2, \ldots, a_n), (b_1, b_2, \ldots, b_n)\big) \mapsto (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$. We focus on affine subvarieties because the coordinatewise sum is not well-defined for points in projective space. From the definition, we see that $\dim(X + Y) \le \dim(X) + \dim(Y)$ and, when X and Y are both irreducible, the variety $X + Y$ is also irreducible. As Example 3.1 illustrates, the image $\varsigma(X \times Y)$ may not be closed.

As far as we know, there is no literature on the Minkowski sums of varieties. We compute the dimension and degree of Minkowski sums of generic affine varieties.

Theorem 3.9 *Given two affine varieties $X, Y \subset \mathbb{A}^n$ in general position, we have* $\dim(X + Y) = \min(\dim X + \dim Y, n)$.

Corollary 3.12 *Assume that the characteristic of the underlying field $\Bbbk$ is not equal to* 2. *If the affine varieties $X, Y \subset \mathbb{A}^n$ are contained in disjoint nonparallel affine subspaces, then we have* $\deg(X + Y) = \deg(X)\deg(Y)$.

A crucial observation in our computations is that the Minkowski sum of affine varieties disjoint at infinity can be described in terms of the join of their projectivizations, see Proposition 3.5 and Remark 3.6. This is a construction inspired by the combinatorial Cayley trick used to construct Minkowski sums of polytopes.

In analogy with Minkowski sums, the Hadamard product of a pair of algebraic varieties is constructed from the coordinatewise products of their points. Specifically, the *Hadamard product* $X \star Y$ of two projective varieties $X, Y \subset \mathbb{P}^n$ is the Zariski closure of the image of $X \times Y$ under the rational map $\phi_* \colon \mathbb{P}^n \times \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ defined by $([a_0 : a_1 : \cdots : a_n], [b_0 : b_1 : \cdots : b_n]) \mapsto [a_0 b_0 : a_1 b_1 : \cdots : a_n b_n]$. The indeterminacy locus of this rational map ϖ is the union of all products of complementary coordinate subspaces. In other words, if $\Bbbk$ is a field and $\mathbb{P}^n := \operatorname{Proj}(\Bbbk[x_0, x_1, \ldots, x_n])$, then the domain of ϖ consists of all points in $\mathbb{P}^n$ except for $\bigcup_{\mathscr{I}} \mathrm{V}(x_i : i \in \mathscr{I}) \times \mathrm{V}(x_i : i \notin \mathscr{I})$, where $\mathscr{I}$ is any subset of $\{0, 1, \ldots, n\}$. As with Minkowski sums, this definition implies that $\dim(X \star Y) \le \dim(X) + \dim(Y)$ and, when X and Y are both irreducible, the variety $X \star Y$ is also irreducible. Example 4.1 shows that the image $\varpi(X \times Y)$ need not be closed.

In [1], the authors studied the geometry of Hadamard products, with a particular focus on the case of linear spaces. This work has been continued in [2].

In particular, we are interested in studying Hadamard products of varieties of matrices. The Hadamard product of matrices is a classical operation in matrix analysis [7]. Its most relevant property is that it is closed on positive matrices. The Hadamard product of tensors appeared more recently in quantum information [8] and in statistics [4, 11]. In the latter, the authors studied restricted Boltzmann machines which are statistical models for binary random variables where some are hidden. From a geometric point of view, this reduces to studying *Hadamard powers* of the first secant variety of Segre products of copies of $\mathbb{P}^1$. An interesting question is to understand how to express matrices as Hadamard products of small rank matrices. We call these expressions *Hadamard decompositions*. We define *Hadamard ranks* of matrices by using a multiplicative version of the usual definitions used for additive tensor decompositions. The study of Hadamard ranks is related to the study of Hadamard powers of secant varieties of Segre products of projective spaces.

In Sect. 4, we focus in particular on the dimension of these Hadamard powers. We define the expected dimension and, consequently, we define the expected rth Hadamard generic rank, i.e., the expected number of rank r matrices needed to decompose the generic matrix of size $m \times n$ as their Hadamard product. It is

$$\text{exp. Hrk}_r^{\circ}(m, n) = \left\lceil \frac{\dim \mathbb{P}(\text{Mat}_{m,n}) - \dim(X_1)}{\dim(X_r) - \dim(X_1)} \right\rceil = \left\lceil \frac{mn - (m + n - 1)}{r(m + n - r) - m - n + 1} \right\rceil .$$

We confirm this is correct for square matrices of small size using *Macaulay2*.

The paper is structured as follows. In Sect. 2, we present some explicit computations of these varieties. We use both *Macaulay2* [5] and *Sage* [12]. These computations allowed us to conjecture some geometric properties of Minkowski sums and Hadamard products of algebraic varieties. In Sect. 3, we analyze Minkowski sums of affine varieties. In particular, we prove that, under genericity conditions, the dimension of the Minkowski sum is the sum of the dimensions and we investigate the degree of the Minkowski sum. In Sect. 4, we study Hadamard products and Hadamard powers of projective varieties. In particular, we focus on the case of Hadamard powers of projective varieties of matrices of given rank. We introduce the notion of Hadamard decomposition and Hadamard rank of a matrix. These concepts may be viewed as the multiplicative versions of the well-studied *additive decomposition* of tensors and *tensor ranks*.

2 Experiments

Problem 5 on Surfaces in [13] requests: *Pick two random circles C_1 and C_2 in $\mathbb{R}^3$. Compute their Minkowski sum $C_1 + C_2$ and their Hadamard product $C_1 \star C_2$. Try other curves.* To accomplish this, we use the algebra softwares *Macaulay2* and *Sage* to obtain equations and nice graphics. Via elimination theory, we can compute the ideals of Minkowski sums and Hadamard products in *Macaulay2* as follows:

```
 R = QQ[z_1..z_n,
        x_1..x_n,y_1..y_n];
 I = ideal( ... ); -- ideal of X in variables x_i;
 J = ideal( ... ); -- ideal of Y in variables y_i;

---- construct the ideals of graphs of entry-wise
   addition and multiplication maps
---- phi_+ and phi_star
 S = I + J + ideal(z_1-x_1-y_1,...,z_n-x_n-y_n);
 P = I + J + ideal(z_1-x_1*y_1,...,z_n-x_n*y_n);
 Msum = eliminate(toList{x_1..x_n | y_1..y_n}, S);
 Hprod = eliminate(toList{x_1..x_n | y_1..y_n}, P);
```

With *Sage*, we produced graphics of the real parts of Minkowski sums and Hadamard products of curves in $\mathbb{A}^3$. This is the script we used.

```
 A.<x1,x2,x3,y1,y2,y3,z1,z2,z3>=QQ[]
 I=( ... )*A # ideal of X in the variables x;
 J=( ... )*A # ideal of Y in the variables y;

# construct the ideals defining the graphs of entry-
   wise addition and multiplication maps
# phi_+ and phi_star
 S = I + J + (z1-(x1+y1),z2-(x2+y2),z3-(x3+y3))*A
 P = I + J + (z1-(x1*y1),z2-(x2*y2),z3-(x3*y3))*A

 MSum = S.elimination_ideal([x1,x2,x3,y1,y2,y3])
 HProd = P.elimination_ideal([x1,x2,x3,y1,y2,y3])

# Assuming we get a surface, take the one generator of
# each ideal.
 MSumGen=MSum.gens()[0]
 HProdGen=HProd.gens()[0]

# We plot these surfaces.
# Because MSumGen and HProdGen are considered as
# elements of A, which has 9 variables, they take
# 9 arguments.
 var('z1,z2,z3')
 implicit_plot3d(MSumGen(0,0,0,0,0,0,z1,z2,z3)==0,
                (z1, -3, 3), (z2, -3,3), (z3, -3,3))
 implicit_plot3d(HProdGen(0,0,0,0,0,0,z1,z2,z3)==0,
                (z1, -3, 3), (z2, -3,3), (z3, -3,3))
```

In Figs. 1, 2, 3, and 4 are some of the pictures we obtained. These experiments gave us a first idea about the properties of Minkowski sums and Hadamard products.

The fact that $X+Y$ and $X \star Y$ are linear projections of $X \times Y \subset \mathbb{A}^n \times \mathbb{A}^n$ and of $X \times Y \subset \mathbb{P}^n \times \mathbb{P}^n \subset \mathbb{P}^{n^2+2n}$, respectively, leads us to expect certain geometric properties of $X+Y$ and $X \star Y$.

Because the projection of a variety $Z \subset \mathbb{P}^N$ in generic position from a linear space L with $\dim(Z)+\dim(L) < N-1$ is generically one-to-one, we naively expect that, for X and Y in general position with $\dim(X)+\dim(Y) < n$, we have

$$\dim(X+Y) = \dim(X) + \dim(Y)\,,$$
$$\deg(X+Y) = \deg(X\times Y) = \deg(X)\deg(Y), \quad (X\times Y \subset \mathbb{A}^n)$$
$$\dim(X\star Y) = \dim(X) + \dim(Y)\,,$$
$$\deg(X\star Y) = \deg(X\times Y) = \binom{\dim(X)+\dim(Y)}{\dim(X)}\deg(X)\deg(Y),\ (X\times Y\subset \mathbb{P}^{n^2+2n})\,.$$

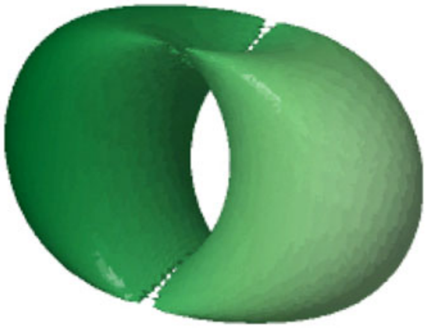

Fig. 1 Minkowski sum of a circle of radius 1 in the x_0, x_1-plane and circle of radius 2 in the x_0, x_2-plane

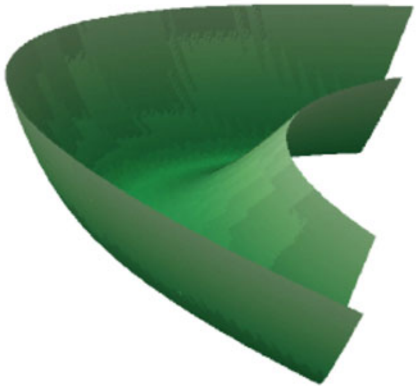

Fig. 2 Minkowski sum of the two parabolas $x_0 = x_1$ in the x_0, x_1-plane and $x_1 = x_2$ in the x_1, x_2-plane

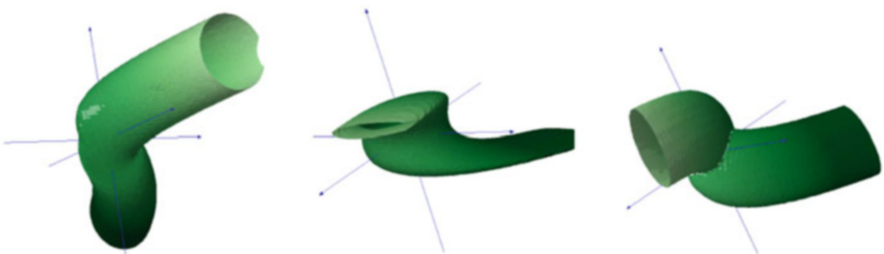

Fig. 3 Minkowski sum of the twisted cubic with the unit circle in the x_0, x_1-plane, x_1, x_2-plane, and x_0, x_2-plane from left to right, respectively

Fig. 4 The real part of the Hadamard product of (left) the unit circle in the $x_2 = 1$ plane and the unit circle in the $x_1 = 1$ plane and (right) the circles $x_0 + (x_1 + x_2)^2 = 1, x_2 - x_1 = 1$ and $x_0 + (x_2 - x_1)^2 = 1, x_1 + x_2 = 1$

These expectations, however, do not follow directly from the projections of the varieties in general position because, even for X and Y in general position, $X \times Y$ is not in general position. Hence, we need further analysis as in the following sections.

3 Minkowski Sums of Affine Varieties

In this section, we analyze the Minkowski sum of two affine varieties. To begin, we demonstrate that the image of the coordinatewise-sum map $\varsigma\colon \mathbb{A}^n \times \mathbb{A}^n \to \mathbb{A}^n$ need not be closed.

Example 3.1 Consider the plane curves $X := \mathrm{V}(x_1x_2 - 1)$ and $Y := \mathrm{V}(x_1x_2 + 1)$ in $\mathbb{A}^2 := \mathrm{Spec}(\Bbbk[x_1, x_2])$. We claim that the image $\varsigma(X \times Y)$ is not closed in the Zariski topology. For $a, b \in \Bbbk^\times$, the map ς sends the point $\big((a, 1/a), (-b, 1/b)\big) \in X \times Y$ to the point $(p, q) \in \mathbb{A}^2$ if and only if we have $a-b = p$ and $\frac{1}{a} + \frac{1}{b} = q$ or, equivalently, $a = p+b$ and $qb^2 + (pq-2)b-p = 0$. For $pq \neq 0$, we see that there is at least one point in $X \times Y$ mapping to $(p, q) \in \mathbb{A}^2$, because $\Bbbk$ is an algebraically closed field. Suppose char $\Bbbk \neq 2$. If $p = 0$ then we have $qa = qb = 2$; if $p \neq 0 = q$, then we have $2a = 2b = -p$. Hence, the origin $(0, 0) \in \mathbb{A}^2$ does not lie in $\varsigma(X \times Y)$, so the image is not closed in the Zariski topology.

One of our main tools for proving results about the Minkowski sum is an alternative description of it in terms of the join of the two varieties.

For X, Y subvarieties of $\mathbb{A}^n$ or $\mathbb{P}^n$, we let $J_{\mathrm{set}}(X, Y)$ be the *setwise join* of X and Y, i.e., the union of the lines connecting distinct points $x \in X$ and $y \in Y$. This space is usually not closed and its Zariski closure $J(X, Y)$ is the classical *join* of X and Y.

Our analysis of the Minkowski sum of affine algebraic sets X and Y via a join will involve hyperplanes positioned as in Lemma 3.2 below. For an intuitive sense of the statement of the lemma, one may consider the case where L, M, and N are the projectivizations of parallel affine hyperplanes.

Lemma 3.2 *Let $L, M, N \subset \mathbb{P}^n$ be three distinct hyperplanes with a common pairwise intersection; $E := L \cap M = L \cap N = M \cap N$, and consider two nonempty disjoint subvarieties $X \subset M$ and $Y \subset N$. Say $X \subset M$ and $Y \subset N$ are nonempty disjoint subvarieties. Let $X^a = X \setminus E$, $Y^a = Y \setminus E$, $\partial X = X \cap E$, and $\partial Y = Y \cap E$. Then:*

(i) $J(X, Y) = J_{\text{set}}(X, Y)$,
(ii) $J(X, Y) \cap L = (J_{\text{set}}(X^a, Y^a) \cap L) \cup J_{\text{set}}(\partial X, \partial Y) \cup \partial X \cup \partial Y$, *and*
(iii) $J(X, Y) \cap L \setminus E = J_{\text{set}}(X^a, Y^a) \cap L$.

In particular, if X and Y have positive dimension then

$$J(X, Y) \cap L = (J_{\text{set}}(X^a, Y^a) \cap L) \cup J_{\text{set}}(\partial X, \partial Y)\,.$$

Proof

(i) Because X and Y are disjoint, we have $J_{\text{set}}(X, Y)$ is Zariski closed, so $J(X, Y) = J_{\text{set}}(X, Y)$ [see Example 6.17 on p. 70 of [6]].

(ii) From the first part, we have

$$J(X, Y) = J_{\text{set}}(X^a, Y^a) \cup J_{\text{set}}(X^a, \partial Y) \cup J_{\text{set}}(\partial X, Y^a) \cup J_{\text{set}}(\partial X, \partial Y).$$

So to get the claimed expression for $J(X, Y) \cap L$ it suffices to show

(a) $J_{\text{set}}(X^a, \partial Y) \cap L, J_{\text{set}}(\partial X, Y^a) \cap L \subset \partial X \cup \partial Y$ and
(b) $J_{\text{set}}(\partial X, \partial Y) \cup \partial X \cup \partial Y \subset L$.

(a) By symmetry it is enough to show that $J_{\text{set}}(X^a, \partial Y) \cap L \subset \partial Y$.
Say $x \in X^a$ and $y \in \partial Y$. So $y \in L$ but $x \notin L$. Thus, the line between x and y intersects L in exactly $\{y\} \subset \partial Y$.

(b) We show that $J_{\text{set}}(\partial X, \partial Y) \cup \partial X \cup \partial Y \subset E$.
By definition, $\partial X, \partial Y \subset E$. So, because E is a linear space, for any $x \in \partial X$ and $y \in \partial Y$, the line between x and y is contained in E.

(iii) First, note that because $J_{\text{set}}(\partial X, \partial Y) \cup \partial X \cup \partial Y \subset E$, we have

$$(J(X, Y) \cap L) \setminus E \subset (J_{\text{set}}(X^a, Y^a) \cap L) \setminus E.$$

Hence, we just need to show that $J_{\text{set}}(X^a, Y^a) \cap L$ is disjoint from E.

Considering any $x \in X^a$ and $y \in Y^a$, it suffices to show that the line ℓ between x and y does not meet E. If we assume, towards a contradiction, that there is some $z \in \ell \cap E$, then z and x would be distinct points on the hyperplane M, so the line ℓ between them would be contained in M. But $y \in Y^a \subset N \setminus E = N \setminus M$, so ℓ cannot be contained in M.

Finally, if X and Y are positive dimensional then $\partial X = X \cap L$ and $\partial Y = Y \cap L$ are nonempty, so $\partial X, \partial Y \subset J_{\text{set}}(\partial X, \partial Y)$. □

Our alternative description of the Minkowski sum will give us cases in which $X +_{\text{set}} Y$ is already closed. Recall from Example 3.1 that for the two plane curves $X := V(x_1x_2 - 1)$ and $Y := V(x_1x_2 + 1)$, $X +_{\text{set}} Y$ is not Zariski closed. Note that in this example, X and Y have a common asymptote, or equivalently, that their projective closures meet at the line at infinity. We will see that when the characteristic of the base field is not 2, all cases where $X +_{\text{set}} Y$ is not closed share an analogous property.

Definition 3.3 Let $X, Y \subset \mathbb{A}^n$ be varieties and denote the projective closures of X and Y in $\mathbb{P}^n$ by $\overline{X}$ and $\overline{Y}$, respectively. Let $H_0 = \{[x_0 : x_1 : \cdots : x_n] \in \mathbb{P}^n : x_0 = 0\}$ be the *hyperplane at* ∞. We say that X and Y are *disjoint at infinity* if $\overline{X} \cap \overline{Y} \cap H_0 = \varnothing$.

Remark 3.4 If X and Y are disjoint at infinity then $\dim(X \cap Y) < 1$, thus we get that $\dim X + \dim Y \leq n$.

Proposition 3.5 *Assume that the characteristic of* $\Bbbk$ *is not 2. Suppose* $X, Y \subset \mathbb{A}^n$ *are varieties that are disjoint at infinity. Let* z_0, z_1 *be distinct scalars and let* $\tilde{X}, \tilde{Y} \subset \mathbb{P}^{n+1}$ *be the projective closures of* $X \times \{z_0\}$ *and* $Y \times \{z_1\}$*, respectively. Let* $x_0, x_1, \ldots, x_n, z$ *be the coordinates on* $\mathbb{P}^{n+1}$.

If we identify $S = \{z = \frac{z_0+z_1}{2} x_0\} \subset \mathbb{P}^{n+1}$ *with* $\mathbb{P}^n$ *and* $S \setminus H_0$ *with* $\mathbb{A}^n$*, then:*

(i) $J_{\text{set}}(\tilde{X}, \tilde{Y}) = J(\tilde{X}, \tilde{Y})$*;*
(ii) $\frac{1}{2}(X + Y) = J(\tilde{X}, \tilde{Y}) \cap S \setminus H_0$*;*
(iii) $X +_{\text{set}} Y = X + Y$*, namely,* $X +_{\text{set}} Y$ *is Zariski closed.*

Proof

(i) Let $E = \{x_0 = 0\} \subset \mathbb{P}^{n+1}$ be the hyperplane at ∞ in $\mathbb{P}^{n+1}$. Note that

$$E \cap \{z = z_0x_0\} = E \cap \left\{z = \frac{z_0 + z_1}{2} x_0\right\} = E \cap \{z = z_1x_0\} = \{z = 0, x_0 = 0\},$$

which is identified with H_0. Therefore the statement that X and Y are disjoint at infinity is equivalent to

$$\tilde{X} \cap \tilde{Y} \cap E = \varnothing.$$

On the other hand, $\tilde{X} \setminus E = X \times \{z_0\}$ and $\tilde{Y} \setminus E = Y \times \{z_1\}$, and so we see that $\tilde{X} \cap \tilde{Y} = \varnothing$. So, by Lemma 3.2 applied to $S = \{z = \frac{z_0+z_1}{2} x_0\}$, $\tilde{X} \subset \{z = z_0x_0\}$, and $\tilde{Y} \subset \{z = z_1x_0\}$, we find that

$$J_{\text{set}}(\tilde{X}, \tilde{Y}) = J(\tilde{X}, \tilde{Y}) \quad \text{and} \quad J(\tilde{X}, \tilde{Y}) \cap S \setminus H_0 = J_{\text{set}}(X \times \{z_0\}, Y \times \{z_1\}) \cap S.$$

(ii) & (iii) For any $x \in X$ and $y \in Y$, the line between the points $(x, z_0) \in X \times \{z_0\}$ and $(y, z_1) \in Y \times \{z_1\}$ meets the affine hyperplane $S \setminus E = \{z = \frac{z_0+z_1}{2}\}$ in exactly the point $(\frac{x+y}{2}, \frac{z_0+z_1}{2})$. So, we have shown that $J(\tilde{X}, \tilde{Y}) \cap S \setminus$

$H_0 = \frac{1}{2}(X +_{\text{set}} Y)$. In particular, because $J(\tilde{X}, \tilde{Y})$ is closed, this tells us that $X +_{\text{set}} Y$ is a closed subset of $S \setminus H_0 \cong \mathbb{A}^n$. Hence, we have $\frac{1}{2}(X + Y) = \frac{1}{2}(X +_{\text{set}} Y) = J(\tilde{X}, \tilde{Y}) \cap S \setminus H_0$. □

Remark 3.6 We call the construction $\frac{1}{2}(X + Y) = J(\tilde{X}, \tilde{Y}) \cap S \setminus H_0$ the *Cayley trick*, as the underlying idea is exactly the same as that of the Cayley trick used to construct Minkowski sums of polytopes.

As a consequence of the following lemma, if we restrict to $\dim X + \dim Y \leq n$, then the hypothesis that X and Y are disjoint at infinity is a *genericity condition*.

Lemma 3.7 *Let $X, Y \subset \mathbb{P}^n$ be varieties with $\dim X + \dim Y < n$. Then, we have that the set $\{g \in \mathrm{GL}(n+1, \Bbbk) : gX \cap Y = \varnothing\}$ is a nonempty open subset of $\mathrm{GL}(n+1, \Bbbk)$. That is, for generic $g \in \mathrm{GL}(n+1, \Bbbk)$, gX and Y do not intersect.*

Proof First, note that for any point $p \in \mathbb{P}^n$ the stabilizer of p in $\mathrm{GL}(n+1, \Bbbk)$ has dimension $n^2 + n + 1$. This is because any two point stabilizers in $\mathrm{GL}(n+1, \Bbbk)$ are conjugate and the stabilizer of $[1 : 0 : 0 : \cdots : 0] \in \mathbb{P}^n$ is the set of all $g \in \mathrm{GL}(n+1, \Bbbk)$ with first column of the form $\begin{bmatrix} * & 0 & 0 & \cdots & 0 \end{bmatrix}^{\mathsf{T}}$, which has dimension $(n+1)n + 1$.

Let $Z = \{(g, x, y) \in \mathrm{GL}(n+1, \Bbbk) \times X \times Y : gx = y\}$ which is a subvariety of $\mathrm{GL}(n+1, \Bbbk) \times X \times Y$. Let $\pi_1 : Z \to \mathrm{GL}(n+1, \Bbbk)$ and $\pi_2 : Z \to X \times Y$ be the restrictions of the canonical projections from $\mathrm{GL}(n+1, \Bbbk) \times X \times Y$. Note that π_2 is surjective, because for any $x, y \in \mathbb{P}^n$ there exists some $g \in \mathrm{GL}(n+1, \Bbbk)$ taking x to y. Further, the fibre over any point $(x, y) \in X \times Y$ is a left coset of a point stabilizer in $\mathrm{GL}(n+1, \Bbbk)$ and so has dimension $n^2 + n + 1$. Thus, $\dim Z = n^2 + n + 1 + \dim X + \dim Y < n^2 + 2n + 1 = \dim \mathrm{GL}(n+1, \Bbbk)$.

Because $X \times Y$ is projective, the projection $\mathrm{GL}(n+1, \Bbbk) \times X \times Y \to \mathrm{GL}(n+1, \Bbbk)$ is a closed map, so $\pi_1(Z)$ is a closed subset of $\mathrm{GL}(n+1, \Bbbk)$. Since $\dim \pi_1(Z) \leq \dim Z < \dim \mathrm{GL}(n+1, \Bbbk)$, $\pi_1(Z)$ is a proper closed subset of $\mathrm{GL}(n+1, \Bbbk)$. So,

$$\{g \in \mathrm{GL}(n+1, \Bbbk) : gX \cap Y = \varnothing\} = \mathrm{GL}(n+1, \Bbbk) \setminus \pi_1(Z)$$

is a nonempty open subset of $\mathrm{GL}(n+1, \Bbbk)$. □

Now, we claim that if $X, Y \subset \mathbb{A}^n$ are varieties with $\dim X + \dim Y \leq n$, then

for general $g \in GL(n, \Bbbk)$, gX and Y are disjoint at infinity.

To see this, note that, considering $H_0 = \{x_0 = 0\}$,

$$\dim(\overline{X} \cap H_0) + \dim(\overline{Y} \cap H_0) \leq \dim X - 1 + \dim Y - 1 < \dim X + \dim Y - 1 \leq n - 1\,.$$

The action of $\mathrm{GL}(n, \Bbbk)$ on $\mathbb{A}^n$ extends to an action on $\mathbb{P}^n$, and the identification $H_0 \cong \mathbb{P}^{n-1}$ is $\mathrm{GL}(n, \Bbbk)$-equivariant. So we have

$$\overline{gX} \cap \overline{Y} \cap H_0 = (\overline{gX} \cap H_0) \cap (\overline{Y} \cap H_0) = g(\overline{X} \cap H_0) \cap (\overline{Y} \cap H_0).$$

By Lemma 3.7, for general $g \in \mathrm{GL}(n, \Bbbk)$ this is empty.

Remark 3.8 One could use the group of affine transformations $\mathrm{Aff}_n = \mathbb{A}^n \rtimes \mathrm{GL}(n, \Bbbk)$, because shifting an affine variety does not change the part at infinity of its projective closure.

When a result holds under the same conditions as Lemma 3.7, i.e., if we fix X and Y then it holds for gX and Y, for general $g \in \mathrm{Aff}_n$, we shall say that the result holds for X and Y *in general position*.

We are now ready to compute the dimension of Minkowski sums. Based on the examples in Sect. 2, it seems that for $\dim X + \dim Y \leq n$, we get $\dim(X + Y) = \dim X + \dim Y$. This does happen generically.

Theorem 3.9 *For affine varieties $X, Y \subset \mathbb{A}^n$ in general position, we have*

$$\dim(X + Y) = \min(\dim X + \dim Y, n)\,.$$

Proof Since for any $X, Y \subset \mathbb{A}^n$, we have $\dim(X + Y) \leq \min\{\dim(X) + \dim(Y), n\}$, it suffices to prove the converse for X, Y in general position. Set $k := \dim(X)$ and $l := \dim(Y)$. For any vector v, we have $(X + v) + Y = (X + Y) + v$, so it suffices to show that for general $g \in \mathrm{GL}(n, \Bbbk)$, $\dim(gX + Y) \geq \min\{\dim(X) + \dim(Y), n\}$. We consider the case $\dim(X) + \dim(Y) \leq n$, the case $\dim(X) + \dim(Y) \geq n$ being analogous. By just looking at full-dimensional irreducible components of X and Y, we may assume without loss of generality that X and Y are irreducible.

We denote by $T_p(X)$ the *tangent space* to the variety X at the point p. For now fix $g \in \mathrm{GL}(n, \Bbbk)$. If $(p, q) \in \mathbb{A}^n \times \mathbb{A}^n$ then

$$(d\varsigma)_{(p,q)} : T_p\mathbb{A}^n \times T_q\mathbb{A}^n \to T_{p+q}\mathbb{A}^n$$

is simply the addition map ς, and so we see that if $p \in gX$ and $q \in Y$ then $T_p(gX) + T_qY \subseteq T_{p+q}(gX + Y)$. So to conclude that $\dim(gX + Y) \geq \dim(X) + \dim(Y)$ it suffices to show that there is a dense subset Ξ of $gX + Y$ such that for each $\xi \in \Xi$ there exist $p \in gX$ and $q \in Y$ with $\xi = p + q$ and $T_p(gX) \cap T_qY = 0$, for then

$$\begin{aligned}\dim(T_\xi(gX + Y)) &\geq \dim(T_p(gX) + T_qY) \\ &= \dim(T_p(gX)) + \dim(T_qY) \geq \dim(X) + \dim(Y),\end{aligned}$$

and because Ξ is dense some $\xi \in \Xi$ is a smooth point of $gX + Y$. Also, because the image of a dense subset under a continuous function is a dense subset of the image, we see that it suffices to show that there is a nonempty open subset of $gX \times Y$ such that for (p, q) in this set, $T_p(gX) \cap T_qY = 0$.

For any variety $Z \subset \mathbb{A}^n$ let Z_{sm} denote the smooth locus of Z. So we have the morphism $\psi_Z : Z_{\mathrm{sm}} \to \mathrm{Gr}(\dim(Z), \mathbb{A}^n)$, $p \mapsto T_pZ$, and we let Ψ_Z denote the image of this morphism inside the Grassmannian.

Consider $U = \{(V, W) \in \mathrm{Gr}(k, \mathbb{A}^n) \times \mathrm{Gr}(l, \mathbb{A}^n) : V \cap W \neq 0\}$, which is an open subset of $\mathrm{Gr}(k, \mathbb{A}^n) \times \mathrm{Gr}(l, \mathbb{A}^n)$. In particular, if we let

$$\varphi_g := \psi_{gX} \times \psi_Y : gX_{\mathrm{sm}} \times Y_{\mathrm{sm}} \to \mathrm{Gr}(k, \mathbb{A}^n) \times \mathrm{Gr}(l, \mathbb{A}^n),$$

then $\varphi_g^{-1}(U)$ is a (possibly empty) open subset of $gX \times Y$, and if $(p, q) \in \varphi_g^{-1}(U)$ then $T_p(gX) \cap T_qY = 0$. Also, $\varphi_g^{-1}(U)$ is nonempty if and only if $(\Psi_{gX} \times \Psi_Y) \cap U$ is nonempty. So we conclude that to show that $\dim(gX + Y) \geq \dim(X) + \dim(Y)$, it suffices to show that $(\Psi_{gX} \times \Psi_Y) \cap U \neq \varnothing$.

Now we let $g \in \mathrm{GL}(n, \Bbbk)$ vary. Fix $p \in X_{\mathrm{sm}}$ and $q \in Y_{\mathrm{sm}}$. So for $g \in \mathrm{GL}(n, \Bbbk)$, $gp \in (gX)_{\mathrm{sm}}$ with $T_{gp}(gX) = g(T_pX)$. Now T_qY is an l-dimensional subspace of $\mathbb{A}^n$ and so because $k + l \leq n$, $\{V \in \mathrm{Gr}(k, \mathbb{A}^n) : V \cap T_qY = 0\}$ is a nonempty open subset of the Grassmannian. So because $\mathrm{GL}(n, \Bbbk)$ acts transitively on $\mathrm{Gr}(k, \mathbb{A}^n)$ we conclude that for generic $g \in \mathrm{GL}(n, \Bbbk)$, $T_{gp}(gX) \cap T_qY = g(T_pX) \cap T_qY = 0$. Thus $(gp, q) \in (\Psi_{gX} \times \Psi_Y) \cap U$, and so $\dim(gX + Y) \geq \dim(X) + \dim(Y)$.

For the case where $\dim(X) + \dim(Y) \geq n$ the same proof works upon replacing the condition that tangent spaces intersect trivially with the condition that they intersect transversely. □

Further, when the characteristic of the base field is not 2 we can use the Cayley trick to show that the condition of *disjoint at infinity* is sufficient to have additivity of dimension.

Theorem 3.10 *Assume the characteristic of the base field is not* 2*. If* $X, Y \subset \mathbb{A}^n$ *are varieties that are disjoint at infinity, then we have* $\dim(X + Y) = \dim X + \dim Y$.

Proof For any $X, Y \subset \mathbb{A}^n$, we have $\dim(X + Y) \leq \dim(X) + \dim(Y)$. If either X or Y has dimension zero then $X + Y$ is a union of finitely many shifts of the other and so has dimension $\dim X + \dim Y$.

Assume X and Y have both positive dimension. Then, by Proposition 3.5 (with any z_0, z_1) and Lemma 3.2, we have that $J(\tilde{X}, \tilde{Y}) \cap S = \frac{1}{2}(X + Y) \cup J(\partial X, \partial Y)$ where $\partial X = \overline{X} \cap H_0$ and $\partial Y = \overline{Y} \cap H_0$ are the parts at infinity of the projective closures of X and Y, and $\frac{1}{2}(X + Y)$ is an open subset of $J(\tilde{X}, \tilde{Y})$ while $J(\partial X, \partial Y)$ is closed. Hence, we have

$$\begin{aligned}\dim X + \dim Y = \dim J(\tilde{X}, \tilde{Y}) - 1 &\leq \dim(J(\tilde{X}, \tilde{Y}) \cap S) \\ &= \dim\left(\tfrac{1}{2}(X + Y) \cup J(\partial X, \partial Y)\right) \\ &= \max\{\dim(X + Y), \dim J(\partial X, \partial Y)\} \\ &= \max\{\dim(X + Y), \dim X + \dim Y - 1\},\end{aligned}$$

where the last equality follows since

$$\dim J(\partial X, \partial Y) = \dim X - 1 + \dim Y - 1 + 1 = \dim X + \dim Y - 1\,.$$

So $\dim(X + Y) = \dim X + \dim Y$. □

We now consider the degree of Minkowski sums. The *degree* of a variety X of dimension d in $\mathbb{A}^n$ or $\mathbb{P}^n$ is the number of points in the intersection of X and a general linear subspace of dimension $n - d$.

Proposition 3.11 *Let $\Bbbk$ be the ground field with characteristic other than 2. Let $X, Y \subset \mathbb{A}^n$ be varieties which are disjoint at infinity. Then, for generic $\alpha \in \Bbbk^\times$, in the same notation as in Proposition 3.5, we have that* $\deg(\alpha X + Y) = \deg\left(J(\tilde{X}, \tilde{Y})\right)$.

Proof The proof will go in three main steps.

(i) Show that, up to projective equivalence, dilating X by a generic $\alpha \in \Bbbk^\times$ and then applying the Cayley trick is the same as intersecting $J(\tilde{X}, \tilde{Y})$ with a generic hyperplane whose affine part is parallel to $S \backslash H_0$.
(ii) Prove that for generic α the corresponding hyperplane intersects $J(\tilde{X}, \tilde{Y})$ generically transversely.
(iii) Apply Bézout's theorem and show that the part of the intersection that is at infinity does not contribute to the degree.

Once again we use our Cayley trick and, to simplify computations, we fix $z_0 = 0$ and $z_1 = 1$. Note that, for any $\alpha \in \Bbbk^\times$, αX and Y are disjoint at infinity so we still get the conclusions of Proposition 3.5 and Theorem 3.10.
(i) For $\alpha \in \Bbbk^\times$, let

$$\Phi_\alpha = \begin{bmatrix} 1 & 0 & \cdots & 0 & \alpha - 1 \\ 0 & & & & 0 \\ \vdots & & \alpha I_n & & \vdots \\ 0 & & & & 0 \\ 0 & 0 & \cdots & 0 & \alpha \end{bmatrix} \in \mathrm{GL}(n+2, \Bbbk)\,.$$

We consider $\mathrm{GL}(n+2, \Bbbk)$ as acting on $\mathbb{P}^{n+1}$ with coordinates $x_0, x_1, \ldots, x_n, z$. For $\alpha, \beta \in \Bbbk^\times$ we have

$$\Phi_\alpha \Phi_\beta = \begin{bmatrix} 1 & & \alpha - 1 \\ & \alpha I_n & \\ & & \alpha \end{bmatrix} \begin{bmatrix} 1 & & \beta - 1 \\ & \beta I_n & \\ & & \beta \end{bmatrix} = \begin{bmatrix} 1 & & \beta - 1 + \beta(\alpha - 1) \\ & \alpha\beta I_n & \\ & & \alpha\beta \end{bmatrix} = \Phi_{\alpha\beta},$$

so $\alpha \mapsto \Phi_\alpha$ is a group homomorphism $\Bbbk^\times \to GL(n+2, \Bbbk)$.

Note that Φ_α acts on the hyperplane $\{z = 0\}$ as

$$\Phi_\alpha \begin{bmatrix} 1 \\ \underline{x} \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ \alpha\underline{x} \\ 0 \end{bmatrix} \quad \text{and} \quad \Phi_\alpha \begin{bmatrix} 0 \\ \underline{x} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \alpha\underline{x} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \underline{x} \\ 0 \end{bmatrix}.$$

Similarly, Φ_α fixes the hyperplane $\{z = x_0\}$ pointwise. Thus, $\Phi_\alpha(\tilde{X}) = \widetilde{\alpha X}$ and $\Phi_\alpha(\tilde{Y}) = \tilde{Y}$. Since Φ_α acts as a projective transformation, and so takes lines to lines, $\Phi_\alpha(J(\tilde{X}, \tilde{Y})) = J(\widetilde{\alpha X}, \tilde{Y})$. In particular, we have $\deg J(\widetilde{\alpha X}, \tilde{Y}) = \deg J(\tilde{X}, \tilde{Y})$. We know that $\frac{1}{2}(\alpha X + Y) = J\left(\widetilde{\alpha X}, \tilde{Y}\right) \cap S \setminus H_0$, so we consider $\Phi_\alpha^{-1}(S \setminus H_0)$. We get

$$S \setminus H_0 = \left\{z = \tfrac{1}{2}x_0\right\} \setminus \{z = 0, x_0 = 0\} = \left\{x_0 = 1, z = \tfrac{1}{2}\right\},$$

and we find that, for $\alpha \neq -1$,

$$\Phi_\alpha^{-1} \begin{bmatrix} 1 \\ \underline{w} \\ 1/2 \end{bmatrix} = \begin{bmatrix} 1 & & \alpha^{-1} - 1 \\ & \alpha^{-1} I_n & \\ & & \alpha^{-1} \end{bmatrix} \begin{bmatrix} 1 \\ \underline{w} \\ 1/2 \end{bmatrix} = \begin{bmatrix} \frac{\alpha^{-1}+1}{2} \\ \alpha^{-1}\underline{w} \\ \alpha^{-1}/2 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{2\alpha^{-1}}{\alpha^{-1}+1}\underline{w} \\ \frac{\alpha^{-1}}{\alpha^{-1}+1} \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{2}{1+\alpha}\underline{w} \\ \frac{1}{1+\alpha} \end{bmatrix}.$$

Thus $\Phi_\alpha^{-1}(S \setminus H_0) = \{x_0 = 1, z = \frac{1}{1+\alpha}\} = \{z = \frac{1}{1+\alpha}x_0\} \setminus H_0$.
(ii) We claim that, for generic α, $\left\{z = \frac{1}{1+\alpha}x_0\right\}$ intersects $J(\tilde{X}, \tilde{Y})$ generically transversely. First, it suffices to only consider the *affine points* of $J(\tilde{X}, \tilde{Y})$, i.e. those with $x_0 = 1$, because

$$\begin{aligned} \dim\left(J(\tilde{X}, \tilde{Y}) \cap \left\{z = \tfrac{1}{1+\alpha}x_0\right\} \setminus H_0\right) &= \dim\left(J(\widetilde{\alpha X}, \widetilde{Y}) \cap S \setminus H_0\right) \\ &= \dim(\alpha X + Y) = \dim X + \dim Y \\ &> \dim J(\partial X, \partial Y) = \dim(J(\tilde{X}, \tilde{Y}) \cap H_0), \end{aligned}$$

But $\mathbb{A}^{n+1} \subset \mathbb{P}^{n+1}$ is the disjoint union of $\{x_0 = 1, z = a\}$ as a ranges over $\Bbbk$, so for all but finitely many $a \in k$, $\{x_0 = 1, z = a\} \cap J(\tilde{X}, \tilde{Y})$ must not be contained in the singular locus of $J(\tilde{X}, \tilde{Y})$. So, for all but finitely many $\alpha \in \Bbbk^\times$, we have that $\{x_0 = 1, z = \frac{1}{1+\alpha}\} \cap J(\tilde{X}, \tilde{Y})$ must not be contained in the singular locus of $J(\tilde{X}, \tilde{Y})$. So for generic $\alpha \in \Bbbk^\times$, the general point of $\{z = \frac{1}{1+\alpha}x_0\} \cap J(\tilde{X}, \tilde{Y})$ is a smooth point of $J(\tilde{X}, \tilde{Y})$. In order to check transversality, we need another description of this intersection, which we compute now. Namely, we find

$$\begin{aligned} J(\tilde{X}, \tilde{Y}) \cap \left\{z = \tfrac{1}{1+\alpha}\right\} &= \Phi_\alpha^{-1}\left(J\left(\widetilde{\alpha X}, \tilde{Y}\right) \cap S \setminus H_0\right) \\ &= \Phi_\alpha^{-1}\left(J(\alpha X \times \{0\}, Y \times \{1\}) \cap S \setminus H_0\right) \\ &= J\left(X \times \{0\}, Y \times \{1\}\right) \cap \left\{z = \tfrac{1}{1+\alpha}\right\} \end{aligned}$$

where the second equality follows from Lemma 3.2.

Thus, considering $p \in J(\tilde{X}, \tilde{Y}) \cap \{z = \frac{1}{1+\alpha}\}$, we have that p is on the line between the points $(x, 0)$ and $(y, 1)$, for some $x \in X$ and $y \in Y$. Since this line intersects $S = \{z = \frac{1}{1+\alpha}x_0\}$ transversely and $T_pJ(\tilde{X}, \tilde{Y})$ contains this line, if p is a smooth point of $J(\tilde{X}, \tilde{Y})$ then we have that $\{z = \frac{1}{1+\alpha}x_0\}$ and $J(\tilde{X}, \tilde{Y})$ intersect transversely at p. Thus, for generic α, $\{z = \frac{1}{1+\alpha}x_0\}$ intersects $J(\tilde{X}, \tilde{Y})$ transversely.
(iii) For such an α, applying Bézout's theorem gives us that

$$\deg\left(J\left(\tilde{X}, \tilde{Y}\right) \cap \left\{z = \tfrac{1}{1+\alpha}x_0\right\}\right) = \deg\left(J(\tilde{X}, \tilde{Y})\right).$$

We can write $J(\tilde{X}, \tilde{Y}) \cap \{z = \frac{1}{1+\alpha} x_0\}$ as the disjoint union of the open subset $J(\tilde{X}, \tilde{Y}) \cap \{x_0 = 1, z = \frac{1}{1+\alpha}\}$ and the closed subset $J(\tilde{X}, \tilde{Y}) \cap H_0$. Now,

$$\begin{aligned} J(\tilde{X}, \tilde{Y}) \cap \left\{x_0 = 1, z = \tfrac{1}{1+\alpha}\right\} &= \Phi_\alpha^{-1}\left(J(\widetilde{\alpha X}, \tilde{Y}) \cap S \setminus H_0\right) \\ &= \Phi_\alpha^{-1}\left(\tfrac{1}{2}(\alpha X + Y)\right) \end{aligned}$$

has dimension $\dim X + \dim Y$ and $J(\tilde{X}, \tilde{Y}) \cap H_0 = J(\partial X, \partial Y)$ has dimension $\dim X + \dim Y - 1$. Therefore,

$$\begin{aligned} \deg\left(J\left(\tilde{X}, \tilde{Y}\right) \cap \left\{x_0 = 1, z = \tfrac{1}{1+\alpha}\right\}\right) &= \deg\left(J(\tilde{X}, \tilde{Y}) \cap \left\{z = \tfrac{1}{1+\alpha} x_0\right\}\right) \\ &= \deg\left(J(\tilde{X}, \tilde{Y})\right). \end{aligned}$$

Finally, since we have $\frac{1}{2}(\alpha X + Y) = \Phi_\alpha\left(J(\tilde{X}, \tilde{Y}) \cap \left\{x_0 = 1, z = \frac{1}{1+\alpha}\right\}\right)$, we obtain $\deg(\alpha X + Y) = \deg\left(J(\tilde{X}, \tilde{Y})\right)$. □

Corollary 3.12 *Suppose* $\Bbbk$ *has characteristic other than* 2*. Let* $X, Y \subset \mathbb{A}^n$ *be varieties whose projective closures* $\overline{X}, \overline{Y} \subset \mathbb{P}^n$ *are contained in complementary linear subspaces; equivalently,* X, Y *are contained in disjoint affine subspaces which are not parallel. Then for generic* $\alpha \in \Bbbk^\times$*,* $\deg(\alpha X + Y) = \deg(X)\deg(Y)$.

Proof Since $\overline{X}$ and $\overline{Y}$ are contained in complementary linear spaces they are disjoint, so, in particular, X and Y are disjoint at infinity.

By Proposition 3.11, for a generic $\alpha \in \Bbbk^\times$, we have $\deg(\alpha X + Y) = \deg\left(J(\tilde{X}, \tilde{Y})\right)$. Moreover, the projective closures $\overline{X}$ and $\overline{Y}$ being contained in complementary linear spaces implies that $\tilde{X}$ and $\tilde{Y}$ are also contained in complementary linear spaces, so $\deg\left(J(\tilde{X}, \tilde{Y})\right) = \deg(\tilde{X})\deg(\tilde{Y})$; see [6, Example 18.17]. Thus, for a generic $\alpha \in \Bbbk^\times$, we obtain $\deg(\alpha X + Y) = \deg J(\tilde{X}, \tilde{Y})) = \deg(\tilde{X})\deg(\tilde{Y}) = \deg(X)\deg(Y)$. □

4 Hadamard Products of Projective Varieties

We defined the Hadamard product of projective varieties $X, Y \subset \mathbb{P}^n$ as

$$X \star Y := \overline{\{p \star q : p \in X,\ q \in Y,\ p \star q \text{ is defined}\}} \subset \mathbb{P}^n,$$

where $p \star q$ is the point obtained by entry-wise multiplication of the points p, q.

Also in this case the operation of closure is crucial.

Example 4.1 Consider the Hadamard product between the rational normal curve $\mathscr{C}_3 = \{[a^3 : a^2 b : ab^2 : b^3] : [a : b] \in \mathbb{P}^1\}$ in $\mathbb{P}^3$ and the point $P = [0 : 1 : 1 : 0]$. Now, we obviously have $\mathscr{C}_3 \star P \subset \{z_0 = z_3 = 0\}$. The equality follows because, if

$ab \neq 0$, then we have that $[0 : a : b : 0] = [a^3 : a^2b : ab^2 : b^3] \star [0 : 1 : 1 : 0]$. However, in this case the operation of taking the closure is needed in order to get the entire line; indeed, the points $[0 : 1 : 0 : 0]$ and $[0 : 0 : 1 : 0]$ cannot be written as the Hadamard product of a point in $\mathscr{C}_3$ and the point P.

Another useful way to describe the Hadamard product of projective varieties is as a linear projection of the Segre product of X and Y, i.e., the variety obtained as the image of $X \times Y$ under the map

$$\begin{array}{cccc} \psi_{n,n} : & \mathbb{P}^n \times \mathbb{P}^n & \longrightarrow & \mathbb{P}^{n^2+2n}, \\ & ([a_0 : \ldots : a_n], [b_0 : \ldots : b_n]) & \mapsto & [a_0b_0 : a_0b_1 : a_0b_2 : \ldots : a_nb_{n-1} : a_nb_n]. \end{array}$$

If z_{ij}, with $i = 0, \ldots, n, j = 0, \ldots, n$, are the coordinates of the ambient space of the Segre product $\mathbb{P}^{n^2+2n}$, then the Hadamard product $X \star Y$ is the projection of $X \times Y$ with respect to the linear space $\{z_{ii} = 0 : 0 \leq i \leq n\}$.

Therefore, if X and Y are irreducible, then $X \star Y$ is irreducible and the dimension of their Hadamard product is at most the sum of the dimensions of the original varieties, i.e., $\dim(X \star Y) \leq \dim(X) + \dim(Y)$.

Example 4.2 It is easy to find examples where equality does not hold. Actually, the dimension of the Hadamard product of two varieties can be arbitrarily small. E.g., consider two skew lines in $\mathbb{P}^3$ as $H_{01} = H_0 \cap H_1 = \{[0 : 0 : a : b] : [a : b] \in \mathbb{P}^1\}$ and $H_{23} = H_2 \cap H_3 = \{[c : d : 0 : 0] : [c : d] \in \mathbb{P}^1\}$. Then $H_{01} \star H_{23}$ is empty.

A classic approach to compute the dimension of projective varieties is to look at their tangent space. From now, we consider $\mathbb{C}$ as the ground field in order to avoid fuzzy behaviours caused by positive characteristics or non algebraically closed fields. Also, this is the case we want to consider in our applications.

In the case of joins, there is a result by A. Terracini [14] which describes the tangent space of the join at a generic point in terms of the tangent spaces of the original varieties. In [1], the authors proved a version of this result for Hadamard products of projective varieties.

Lemma 4.3 ([1, Lemma 2.12]) *Let $p \in X$ and $q \in Y$ be generic points, then the tangent space to the Hadamard product $X \star Y$ at the point $p \star q$ is given by*

$$T_{p\star q}(X \star Y) = \left\langle p \star T_qY, T_pX \star q \right\rangle .$$

Another powerful tool to study Hadamard products of projective varieties is tropical geometry. In particular, we have the following relation. Since we are not using tropical geometry elsewhere, here we assume the reader to be familiar with the concept of *tropicalization of a variety*. For the inexperienced reader, we suggest to read [10] for an introduction of the topic.

Proposition 4.4 ([10, Proposition 5.5.11]) *Given two irreducible varieties $X, Y \subset \mathbb{P}^n$, the tropicalization of the Hadamard product of X and Y is the Minkowski sum of their tropicalizations:* $\mathrm{trop}(X \star Y) = \mathrm{trop}(X) + \mathrm{trop}(Y)$.

Applying this result, in [1], the authors gave an upper-bound for the dimension of the Hadamard product of two varieties.

Proposition 4.5 ([1, Proposition 5.4]) *Let $X, Y \subset \mathbb{P}^n$ be irreducible varieties. If $H \subset (\mathbb{C}^*)^{n+1}/\mathbb{C}^*$ be the maximal subtorus acting on X and Y and $G \subset (\mathbb{C}^*)^{n+1}/\mathbb{C}^*$ is the smallest subtorus having a coset containing X and a coset containing Y, then we have* $\dim(X \star Y) \leq \min\{\dim(X) + \dim(Y) - \dim(H), \dim(G)\}$.

We call this upper bound *expected dimension* and denote it $\exp.\dim(X \star Y)$. However, this is not always the correct dimension. In [1], the authors present an example of a Hadamard product of two projective varieties with dimension strictly smaller than the expected dimension.

From the definition of the Hadamard product of two varieties, it makes sense also to analyze self Hadamard products of a projective variety. We call them *Hadamard powers* of a projective variety.

Definition 4.6 We define the *sth Hadamard power* of a projective variety X as

$$X^{\star s} := X \star X^{\star(s-1)}, \text{ for } s \geq 0,$$

where $X^{\star 0} := [1 : \ldots : 1]$.

In general, a projective variety is not contained in its Hadamard powers. However, if $\mathbf{1}_n = [1 : \ldots : 1] \in \mathbb{P}^n$ lies in the variety X, we get the following chain of not necessary strict inclusions

$$X \subset X^{\star 2} \subset \cdots \subset X^{\star s} \subset \cdots \subset \mathbb{P}^n. \tag{1}$$

Therefore, it becomes very natural to check if the Hadamard powers of a projective variety X eventually fill the ambient space. In general, the answer is no.

Proposition 4.7 *If X is a toric variety in $\mathbb{P}^n$, then we have $X = X^{\star 2}$.*

Proof Since any toric variety contains the point $[1 : \ldots : 1]$, it follows that $X \subset X^{\star 2}$. The other inclusion follows by applying Proposition 4.5 to the case $X = Y = H$. □

Remark 4.8 Recently, C. Bocci and E. Carlini gave a necessary and sufficient condition for a plane irreducible curve $C \subset \mathbb{P}^2$ to have its tth Hadamard power equal to the curve itself. This result has been shared with us in private communication and will appear in [3].

Remark 4.9 Proposition 4.7 can be proved directly by recalling that the ideals defining toric varieties are given by *binomial ideals*, namely ideals whose generators are differences of monomials as $f_{\alpha,\beta} = x^\alpha - x^\beta$, where $\alpha, \beta \in \mathbb{N}^{n+1}$ and we use the multi-index notation $x^\alpha := x_0^{\alpha_0} \cdots x_n^{\alpha_n}$.

Now, consider two points of X, $p = [p_0 : \ldots : p_n]$ and $q = [q_0 : \ldots : q_n]$. For any generator $f_{\alpha,\beta}$ of the ideal defining X, we have $p^\alpha - p^\beta = q^\alpha - q^\beta = 0$. Therefore,

$$\begin{aligned}(p \star q)^\alpha - (p \star q)^\beta &= p^\alpha q^\alpha - p^\beta q^\beta = p^\alpha q^\alpha - p^\alpha q^\beta + p^\alpha q^\beta - p^\beta q^\beta \\ &= p^\alpha (q^\alpha - q^\beta) - q^\beta (p^\alpha - p^\beta) = 0;\end{aligned}$$

hence, $p \star q \in X$.

Remark 4.10 Given a projective variety $X \subset \mathbb{P}^n$, the *sth secant variety* $\sigma_s(X)$ is the Zariski closure of the union of linear spaces spanned by s points lying on X. This is a very classical object that has been studied since the second half of nineteenth century. In particular, we have a chain of not necessary strict inclusions given by

$$X \subset \sigma_2(X) \subset \cdots \subset \sigma_s(X) \subset \cdots \subset \mathbb{P}^n.$$

Therefore, we can ask if the secant varieties of a variety X eventually fill the ambient space. It is not difficult to prove that the answer is no. Indeed, if H is a linear space, then $\sigma_2(H) = H$ and, therefore, if X is degenerate, i.e., it is contained in a proper linear subspace of $\mathbb{P}^n$, then its secant varieties do not fill the ambient space.

Hadamard powers of projective varieties may be viewed as the multiplicative version of the classical notion of secant varieties where instead of looking at the linear span of points lying on a variety we consider their Hadamard product. Moreover, by Proposition 4.7, we have that the role played by linear spaces in the case of secant varieties is taken by toric varieties in the case of Hadamard products.

Example 4.11 A concrete example satisfying the assumptions of Proposition 4.7 is the variety $X_1 \subset \mathbb{P}(\mathrm{Mat}_{m,n})$ of rank 1 matrices of size $m \times n$. Indeed, it is generated by the 2×2 minors of the generic matrix $(z_{ij})_{i=1,\dots,m}^{j=1,\dots,n}$. Therefore, $X_1^{\star 2} = X_1$. This gives another proof of the well-known fact that the Hadamard product of two rank 1 matrices is still of rank 1.

The latter example raises a very interesting question.

Question 4.12 What if we consider matrices of rank higher than 1? Can we decompose *all* matrices as Hadamard products of rank $r > 1$ matrices?

The answer is positive, as we show in the following proposition.

Proposition 4.13 *Let M be a matrix of size $m \times n$ and fix $2 \le r \le \min\{m, n\}$. Then, M can be written as the Hadamard product of at most $\left\lceil \frac{\min\{m,n\}}{r-1} \right\rceil$ matrices of rank less than or equal to r.*

Proof Without loss of generality, we may assume that $m \le n$ and let $\{v_1, \dots, v_m\}$ be the rows of the matrix M. Then, consider the following matrices $\left(N = \left\lceil \frac{m}{r-1} \right\rceil\right)$:

$$A_1 = \begin{bmatrix} v_1 \\ \vdots \\ v_{r-1} \\ \mathbb{1}_{n-r+1,n} \end{bmatrix}, \; A_2 = \begin{bmatrix} \mathbb{1}_{r-1,n} \\ v_r \\ \vdots \\ v_{2r-1} \\ \mathbb{1}_{n-2r+1,n} \end{bmatrix}, \dots, A_N = \begin{bmatrix} \mathbb{1}_{(N-1)r-1,n} \\ v_{(N-1)r} \\ \vdots \\ v_n \end{bmatrix}.$$

Then, it is easy to check that $M = A_1 \star A_2 \star \cdots \star A_N$.

If $n \leq m$, we do the same constructions, considering columns instead of rows. □

Therefore, it makes sense to give the following definitions.

Definition 4.14 Let M be a matrix and fix $r \geq 2$. We call an *rth Hadamard decomposition* of M an expression of the type $M = A_1 \star \ldots \star A_s$, where $\operatorname{rank}(A_i) \leq r$. We define the *rth Hadamard rank* of M as the smallest length of such a decomposition, i.e.,

$$\operatorname{Hrk}_r(M) = \min \left\{ s : \begin{array}{l} \text{there exist } A_1, A_2, \ldots, A_s \text{ with } \operatorname{rank}(A_i) \leq r \\ \text{and } M = A_1 \star A_2 \star \cdots \star A_s \end{array} \right\}.$$

We define the *generic rth Hadamard rank* of matrices of size $m \times n$ as

$$\operatorname{Hrk}_r^{\circ}(m, n) = \min\{s : X_r^{\star s} = \mathbb{P}(\mathrm{Mat}_{m,n})\},$$

and the *maximal rth Hadamard rank* of matrices of size $m \times n$ as

$$\operatorname{Hrk}_r^{\max}(m, n) = \max\{\operatorname{Hrk}_r(M) : M \in \mathrm{Mat}_{m,n}\}.$$

These definitions may be seen as the multiplicative versions of the more common notion of *tensor ranks*, where we consider *additive decompositions* of tensors as sums of decomposable tensors. In terms of matrices, we look at decomposition as sums of rank 1 matrices. A massive amount of work has been devoted to problems related to tensor ranks during the last few decades, especially due to their applications to statistics, data analysis, signal process, and others. See [9] for a complete exposition of the current state of the art.

The Hadamard product of matrices, i.e., the entrywise product, is the naive definition for matrix multiplication that any school student would hope to study. Even if it is not the standard multiplication we have been taught, it is a very interesting operation, with nice properties and applications in matrix analysis, statistics and physiscs. As mentioned in the introduction, the generalization to the case of tensors has been used in data mining and quantum information [4, 8]. We look at it from a geometric point of view, by studying Hadamard powers of varieties of matrices.

For a fixed positive integer $r \leq \min\{m, n\}$, let $X_r \subset \mathbb{P}(\mathrm{Mat}_{m,n})$ be the variety of $(m \times n)$-matrices with rank at most r. In other words, X_r is the rth secant variety of the Segre product $\mathbb{P}^{m-1} \times \mathbb{P}^{n-1}$. These are well-studied classic objects. Since $\mathbb{1}_{m,n}$, the matrix of all 1's, which is the identity element for the Hadamard product, is contained in the variety X_r, we have a chain of inclusions as in (1).

Remark 4.15 Our aim is to study Hadamard powers of the varieties X_r of matrices with rank at most r. As we observed before, we can view the Hadamard power $X_r^{\star 2}$ as a linear projection of the Segre product $X_r \times X_r$. In terms of matrices, this is the geometric translation of the well-known fact that *the Hadamard product of two matrices is a submatrix of their Kronecker product*. Indeed, if $M = (m_{i,j}) \in \mathrm{Mat}_{m,n}$ and $N = (n_{i,j}) \in \mathrm{Mat}_{m,n}$, we define the Kronecker product as $M \otimes N = (m_{i,j}n_{h,k}) \in \mathrm{Mat}_{m^2,n^2}$. Then, $M \star N = (M \otimes N)|_{I,J}$, where $(M \otimes N)|_{I,J}$ denotes the restriction on the indexes $I = \{1, m+2, 2m+3, \ldots, m^2\}$ and $J = \{1, n+2, 2n+3, \ldots, n^2\}$.

Hadamard powers of a specific space of tensors has been considered in [4] as the geometric interpretation of a particular statistical model. Therefore, we believe that the definitions of Hadamard ranks of matrices, and more generally of tensors, are very natural and may be an interesting area of research from several perspectives.

Proposition 4.13 gives us an upper bound on the rth Hadamard rank, i.e.,

$$\mathrm{Hrk}_r^{\max}(m,n) \leq \left\lceil \frac{\min\{m,n\}}{r-1} \right\rceil.$$

We can also give a lower bound on the generic rank as a straightforward application of the following well-known property of the Hadamard product of matrices.

Lemma 4.16 *Given two matrices A, B, we have that* $\mathrm{rank}(A \star B) \leq \mathrm{rank}(A)\,\mathrm{rank}(B)$.

Proof Say that $\mathrm{rank}(A) = r_1$ and $\mathrm{rank}(B) = r_2$. Consider the additive decomposition of A and B as sums of rank 1 matrices, i.e.,

$$A = \sum_{i=1}^{r_1} a_i \cdot b_i^{\mathrm{T}} \text{ and } B = \sum_{j=1}^{r_2} c_j \cdot d_j^{\mathrm{T}},$$

where a_i, b_i, c_j, d_j are column vectors. Then, we get that

$$A \star B = \sum_{i=1}^{r_1} \sum_{j=1}^{r_2} (a_i \star c_j) \cdot (b_i \star d_j)^{\mathrm{T}}.$$

Therefore, we have that $\mathrm{rank}(A \star B) \leq r_1 r_2$. □

As an immediate consequence of this lemma we see that that $X_r^{\star 2} \subset X_{r^2}$, for any r. In particular, we obtain a lower bound on the generic Hadamard rank.

Corollary 4.17 *For $r \geq 2$, the generic rth Hadamard rank of $(m \times n)$-matrices is at least* $\lceil \log_r(\min\{m,n\}) \rceil$.

Proof If $s < \lceil \log_r(\min\{m,n\}) \rceil$, then $r^s < \min\{m,n\}$. Hence, the Hadamard product of s matrices of rank r cannot have maximal rank and, therefore, it cannot be enough to cover the entire space of matrices of size $m \times n$. □

Therefore, we have the following chain of inequalities.

$$\lceil \log_r(\min\{m,n\}) \rceil \leq \mathrm{Hrk}_r^{\circ}(m,n) \leq \mathrm{Hrk}_r^{\max}(m,n) \leq \left\lceil \frac{\min\{m,n\}}{r-1} \right\rceil. \quad (2)$$

By this chain of inclusions we get the following result.

Proposition 4.18 *If $m \leq n$ and $r = m - 1$, then we have*

$$\mathrm{Hrk}^{\circ}_{m-1}(m, n) = \mathrm{Hrk}^{\max}_{m-1}(m, n) = 2.$$

Proof On the left hand side of (2) we have $\lceil \log_{m-1}(m) \rceil = 2$.

On the right hand side, we have $\left\lceil \frac{m}{m-2} \right\rceil$, which is equal to 2 if $m \geq 4$. Then, in order to conclude, we just need to prove the case $m = 3$.

Let $m = 3$. If we consider a matrix M of rank ≤ 2, then it lies on X_2. Assume that M has rank 3 and let $v_i = (v_{i,1}, \dots, v_{i,n})$, for $i = 1, 2, 3$, be the rows of M. Consider the first two rows. If $v_{1,j}$ and $v_{2,j}$ are not both equal to zero, for all $j = 1, \dots, n$, then there exists a linear combination of $\lambda v_1 + \mu v_2$ with all entries different from zero and, therefore, we can decompose M as follows

$$M = \begin{bmatrix} v_{1,1} & \dots & v_{1,n} \\ v_{2,1} & \dots & v_{2,n} \\ \lambda v_{1,1} + \mu v_{2,1} & \dots & \lambda v_{1,n} + \mu v_{2,n} \end{bmatrix} \star \begin{bmatrix} 1 & \dots & 1 \\ 1 & \dots & 1 \\ \frac{v_{3,1}}{\lambda v_{1,1} + \mu v_{2,1}} & \dots & \frac{v_{3,n}}{\lambda v_{1,n} + \mu v_{2,n}} \end{bmatrix}.$$

If we have $v_{1,j} = v_{2,j} = 0$, for some $j = 1, \dots, n$, any linear combination of v_1 and v_2 will have the jth entry equal to zero. Therefore, we cannot use the previous algorithm. Hence, we define $\widetilde{v}_i$, for $i = 1, 2$, as

$$\widetilde{v}_{i,j} = \begin{cases} v_{i,j} & \text{if } v_{1,j} \neq 0 \text{ or } v_{2,j} \neq 0; \\ 1 & \text{if } v_{1,j} = v_{2,j} = 0. \end{cases}$$

Now, there exists a linear combination of $\lambda \widetilde{v}_1 + \mu \widetilde{v}_2$ with all entries different from zero. Therefore, if we define a row u as

$$u_i = \begin{cases} 1 & \text{if } v_{1,j} \neq 0 \text{ or } v_{2,j} \neq 0; \\ 0 & \text{if } v_{1,j} = v_{2,j} = 0, \end{cases}$$

we can decompose M as

$$M = \begin{bmatrix} \widetilde{v}_{1,1} & \dots & \widetilde{v}_{1,n} \\ \widetilde{v}_{2,1} & \dots & \widetilde{v}_{2,n} \\ \lambda \widetilde{v}_{1,1} + \mu \widetilde{v}_{2,1} & \dots & \lambda \widetilde{v}_{1,n} + \mu \widetilde{v}_{2,n} \end{bmatrix} \star \begin{bmatrix} u_1 & \dots & u_n \\ u_1 & \dots & u_n \\ \frac{v_{3,1}}{\lambda \widetilde{v}_{1,1} + \mu \widetilde{v}_{2,1}} & \dots & \frac{v_{3,n}}{\lambda \widetilde{v}_{1,n} + \mu \widetilde{v}_{2,n}} \end{bmatrix}.$$

Therefore, we conclude that $\mathrm{Hrk}^{\max}_2(3, n) = 2$. □

Example 4.19 If we have

$$M = \begin{bmatrix} 1 & 2 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 \end{bmatrix}, \quad \widetilde{v}_1 = (1, 2, 1, 1), \quad \widetilde{v}_2 = (-1, 1, 1, 0), \quad \text{and} \quad u = (1, 1, 0, 1),$$

then we obtain

$$M = \begin{bmatrix} 1 & 2 & 1 & 1 \\ -1 & 1 & 1 & 0 \\ 1 & 5 & 3 & 2 \end{bmatrix} \star \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & \frac{1}{5} & \frac{1}{3} & 1 \end{bmatrix}.$$

Remark 4.20 We proved that, for $r = \min\{m, n\} - 1$, the rth Hadamard rank is equal to 2. Actually, the upper-bound in (2) lets us be more precise. Indeed, we can say that for any $\frac{\min\{m,n\}+2}{2} < r < \min\{m, n\}$, we get $\mathrm{Hrk}^{\circ}_r(m, n) = 2$.

In other cases, we need a more geometric approach in order to understand the generic Hadamard rank. By using Proposition 4.5, we can define the expected dimension for the sth Hadamard power of the variety X_r of rank r matrices.

Proposition 4.21 *In the same above notation,*

$$\dim(X_r^{\star s}) \leq \min\left\{s\dim(X_r) - (s-1)\dim(X_1), \dim\, \mathbb{P}(\mathrm{Mat}_{\mathrm{m,n}})\right\}. \tag{3}$$

Proof We proceed by induction on s. For $s = 1$, it follows trivially from definitions. Consider $s > 1$. Then, since $X_r^{\star s} = X_r^{\star(s-1)} \star X_r$, by Proposition 4.5 and by the inductive hypothesis, we get

$$\begin{aligned}\dim(X_r^{\star s}) &\leq \min\left\{\dim(X_r^{\star(s-1)}) + \dim(X_r) - \dim(X_1), \dim \mathbb{P}(\mathrm{Mat}_{\mathrm{m,n}})\right\} \\ &= \min\left\{s\dim(X_r) - (s-1)\dim(X_1), \dim \mathbb{P}(\mathrm{Mat}_{\mathrm{m,n}})\right\}.\end{aligned}$$

□

We refer to the formula on the right hand side of (3) as the *expected dimension* of $X_r^{\star s}$. More precisely, we have the following

$$\begin{aligned}\exp.\dim(X_r^{\star s}) &= \min\left\{s\dim(X_r) - (s-1)\dim(X_1), \dim \mathbb{P}(\mathrm{Mat}_{\mathrm{m,n}})\right\} \\ &= \min\left\{sr(n+m-r) - (s-1)(n+m-1), mn\right\} - 1.\end{aligned}$$

Therefore, the *expected generic rth Hadamard rank* is

$$\exp.\mathrm{Hrk}^{\circ}_r(m, n) = \left\lceil \frac{\dim \mathbb{P}(\mathrm{Mat}_{m,n}) - \dim(X_1)}{\dim(X_r) - \dim(X_1)} \right\rceil = \left\lceil \frac{mn - (m+n-1)}{r(m+n-r) - m - n + 1} \right\rceil. \tag{4}$$

Remark 4.22 A very important concept in the world of additive decomposition of tensors is the idea of *identifiability*, namely, we say that a tensor is *identifiable* if it has a unique decomposition as sum of decomposable tensors. Since we are viewing Hadamard decomposition as a multiplicative version of tensor decomposition, we might look for identifiability also in this set up. However, in this case, we cannot have identifiability for any matrix. For a rth Hadamard decomposition of a matrix M, we have $M = A_1 \star A_2 \star \cdots \star A_s$, with $\mathrm{rank}(A_i) = r$. Hence, for any $(s-1)$-tuple of rank 1 matrices $R_1, \ldots, R_{s-1}$, all with non-zero entries, we can construct a different rth Hadamard decomposition as

$$M = \left(R_1 \star A_1\right) \star \cdots \star \left(R_{s-1} \star A_{s-1}\right) \star \left((R_1 \star \cdots \star R_{s-1})^{\star(-1)} \star A_s\right),$$

where $R^{\star(-1)}$ denotes the Hadamard inverse of the matrix R. Here, we have to recall that $\operatorname{rank}(R_i \star A_i) \leq \operatorname{rank}(A_i)$, for any $i = 1, \ldots, s-1$, by Lemma 4.16, and, similarly, we see that $\operatorname{rank}\left((R_1 \star \cdots \star R_{s-1})^{\star(-1)} \star A_s\right) \leq \operatorname{rank}(A_s)$, because we have $\operatorname{rank}(R_1 \star \cdots \star R_{s-1})^{\star(-1)} = 1$.

We can check that (3) is the actual dimension and, consequently, (4) gives the correct generic rth Hadamard rank for matrices of small size.

Here we describe an algorithm written with *Macaulay2* to compute the dimensions of Hadamard powers of varieties of square matrices of given rank. This allows us to compute the corresponding generic Hadamard ranks (Table 1). We reduced to square matrices for simplicity of exposition, but the code can be easily generalized.

The key point is to use Lemma 4.3 which states that the tangent space to $X_r^{\star s}$ at a generic point $A_1 \star \cdots \star A_s$ is given by

$$T_{A_1 \star \cdots \star A_s}(X_r^{\star s}) = \langle T_{A_1}(X_r) \star A_2 \star \cdots \star A_s, \ldots, A_1 \star \cdots \star A_{s-1} \star T_{A_s}(X_r)\rangle \tag{5}$$

Hence, we first need to construct the tangent spaces at s random points of X_r.

Recall that, if A is a matrix of rank r written as $A = \sum_{i=1}^{r} u_i \cdot v_i^T$, $u_i, v_i \in \mathbb{C}^n$, the tangent space of X_r at A is given by

$$T_A\left(X_r\right) = \left\langle u_1 \cdot (\mathbb{C}^n)^T + (\mathbb{C}^n) \cdot v_1^T, \ldots, u_r \cdot (\mathbb{C}^n)^T + (\mathbb{C}^n) \cdot v_r^T \right\rangle .$$

Here is *Macaulay2* code.

```
INPUT:  n = sizes of matrices;
        r = rank of matrices;
        s = Hadamard power to compute;
OUTPUT: D = dimension of the sth Hadamard power of
           the variety of rank r matrices of size nxn.

S := QQ[z_(1,1)..z_(n,n), a_(1,1)..a_(n,r),
    b_(1,1)..b_(n,r), c_(1,1)..c_(2*r,n)];
---- Construct s random matrices of rank r
u = for i from 1 to s list
    for j from 1 to 2*r list
        random(S^n,S^{0});
A = for i from 0 to (s-1) list sum (
    for j from 0 to (r-1) list
        u_i_(2*j) * transpose(u_i_(2*j+1)));
---- Construct their tangent spaces
C = for i from 1 to 2*r list
    genericMatrix(S,c_(i,1),n,1);
TA = for i from 0 to (s-1) list sum
    for j from 0 to (r-1) list
        u_i_(2*j) * transpose C_(2*j) +
            C_(2*j+1) * transpose(u_i_(2*j+1));
```

Now, we construct the vector spaces spanning the tangent space of $X_r^{\star s}$ as in (5). First, we define a function HP to compute the Hadamard product of two matrices.

```
-- Method to construct the Hadamard product of a
-- list of matrices of same size;
HP = method();
HP List := L -> (
    s := #L;
    r := numRows(L_0);
    c := numColumns(L_0);
    for i from 1 to (s-1) do
        if (numRows(L_i)!=r or numColumns(L_i)!=c) then
            return << "error";
    H := for i from 0 to (r-1) list
        for j from 0 to (c-1) list product (
            for h from 0 to (s-1) list (L_h)_j_i);
    return matrix H)
-- Construct the two vector spaces spanning the tangent
-- space of the Hadamard power and find their equations
-- in the space of matrices
TAstar = for i from 0 to (s-1) list
    HP(toList(set{TA_i}+set(A)-set{A_i}));
M = genericMatrix(S,z_(1,1),n,n);
H = for i from 0 to (s-1) list
    ideal flatten entries (M - TAstar_i);
H1 = for i from 0 to (s-1) list
    eliminate(toList(c_(1,1)..c_(2*r,n)),H_i);
T = QQ[z_(1,1)..z_(n,n)];
E = for i from 0 to (s-1) list sub(H1_i,T);
```

In E, we have the list of the equations of the tangent spaces to the variety X_r at the s random points. From these, we can construct a vector basis for each tangent space. Now, in order to compute the dimension of their span it is enough to compute the rank of the matrix obtained by collecting all these vector bases together.

```
K = for i from 0 to (s-1) list
    kernel transpose
        contract(transpose vars(T),mingens E_i);
tt = mingens K_0 | mingens K_1;
if s >= 3 then (
    for i from 2 to (s-1) do tt = tt | mingens K_i);
D = rank tt
```

In the following table, we list the generic rth Hadamard ranks that we have computed for square matrices of small size.

Table 1 Generic rth Hadamard ranks of square matrices of size $n \times n$ with $n \leq 14$

n	r	rth Hadamard rank	n	r	rth Hadamard rank	n	r	rth Hadamard rank
3	2	2	9	4	2	12	5	2
4	2	2		5	2		6	2
5	2	3	10	2	5	13	2	7
	3	2		3	3		3	4
6	2	3		4	2		4	3
	3	2		5	2		5	2
7	2	4	11	2	6		6	2
	3	2		3	3		7	2
	4	2		4	2	14	2	7
8	2	4		5	2		3	4
	3	3		6	2		4	3
	4	2	12	2	6		5	2
9	2	5		3	4		6	2
	3	3		4	3		7	2

By Remark 4.20, we could restrict to the cases $r < \frac{n+2}{2}$; for $r \geq \frac{n+2}{2}$, we know that $\mathrm{Hrk}^{\circ}_{r}(n, n) = 2$. This computation required less than 9 min on a laptop with a processor 2.2GHs Intel Core i7 processor

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Khovanskii Bases of Cox–Nagata Rings and Tropical Geometry

Martha Bernal Guillén, Daniel Corey, Maria Donten-Bury, Naoki Fujita, and Georg Merz

Abstract The Cox ring of a del Pezzo surface of degree 3 has a distinguished set of 27 minimal generators. We investigate conditions under which the initial forms of these generators generate the initial algebra of this Cox ring. Sturmfels and Xu provide a classification in the case of degree 4 del Pezzo surfaces by subdividing the tropical Grassmannian $\mathrm{TGr}(2, \mathbb{Q}^5)$. After providing the necessary background on Cox–Nagata rings and Khovanskii bases, we review the classification obtained by Sturmfels and Xu. We then describe our classification problem in the degree 3 case and its connections to tropical geometry. In particular, we show that two natural candidates, $\mathrm{TGr}(3, \mathbb{Q}^6)$ and the Naruki fan, are insufficient to carry out the classification.

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1 Introduction

The starting point for this article is the following problem proposed by Sturmfels and Xu as [27, Problem 5.4]: *determine all equivalence classes of three-dimensional sagbi subspaces of* $\mathbb{k}^6$. Let us begin with clarifying two important aspects of our notation. First, instead of the name *sagbi bases* (resp. *sagbi subspaces*) where *sagbi*, first used in [21], stands for "subalgebra analogue to Gröbner bases for ideals", we will use the name *Khovanskii bases* (resp. *Khovanskii subspaces*). This new name was introduced in a much more general setting in a recent article [13]. Second, we make some assumptions on the underlying field $\mathbb{k}$. We usually take $\mathbb{k}$ to be the field of rational functions $\mathbb{Q}(t)$, but to formulate and work on this problem one may consider any other field with a nontrivial valuation. The residue field of $\mathbb{k}$ for the considered valuation will be denoted by k.

The fundamental objects for this chapter are Khovanskii bases and moneric sets. We repeat their definitions after Sturmfels and Xu; see [27, Sect. 3] for more details and comments on their properties. By val: $\mathbb{k}^\times \to \mathbb{Z}$, we denote a valuation map of $\mathbb{k}$. If $\mathbb{k} = F(t)$ for some field F, we use the following valuation: $\mathrm{val}(p) \in \mathbb{Z}$ is the unique integer $\omega \in \mathbb{Z}$ such that $t^{-\omega}p(t)$ takes a nonzero value at $t = 0$. When $f \in \mathbb{k}[x_1, x_2, \dots, x_n]$, we can compute its *initial form* $\mathrm{in}(f)$. If ω_0 is the minimum of val for coefficients of all monomials in f, then $\mathrm{in}(f) = (t^{-\omega_0}f)|_{t=0} \in k[x_1, x_2, \dots, x_n]$. That is, $\mathrm{in}(f)$ identifies all monomials of f whose coefficients have smallest valuation.

Definition 1.1 A subset $\mathscr{F} \subset \mathbb{k}[x_1, x_2, \dots, x_n]$ is *moneric* if $\mathrm{in}(f)$ is a monomial for all $f \in \mathscr{F}$.

For a $\mathbb{k}$-subalgebra $U \subseteq \mathbb{k}[x_1, x_2, \dots, x_n]$, we define the *initial algebra* $\mathrm{in}(U)$ as the k-subalgebra generated by $\mathrm{in}(f)$ for all $f \in U$.

Definition 1.2 A subset $\mathscr{F} \subset U$ is a *Khovanskii basis* of a $\mathbb{k}$-subalgebra $U \subseteq \mathbb{k}[x_1, x_2, \dots, x_n]$ if

- $\mathscr{F}$ is moneric, and
- the initial algebra $\mathrm{in}(U)$ is generated by $\{\mathrm{in}(f) : f \in \mathscr{F}\}$ as a $\mathbb{k}$-algebra.

We are interested in Khovanskii bases of Cox–Nagata rings, which will be described in Sect. 2. After they are introduced, we will be able to explain how a three-dimensional subspace of $\mathbb{k}^6$ determines a basis, possibly a Khovanskii basis, of the Cox ring of a del Pezzo surface of degree 3. We say that such a subspace is moneric (resp. Khovanskii) if the corresponding basis is moneric (resp. Khovanskii), see Definition 2.4. We look at moneric subspaces up to an equivalence relation which respects the property of being a Khovanskii subspace, see Definition 2.5.

We suggest that the reader treat this text as an introduction to the concept of Khovanskii bases and related research problems. For us, understanding the geometric motivation and connections was as important as solving the combinatorial classification problem itself. This is the reason why, besides presenting our approach to answering the main question, we also spend a significant amount of time on exploring its background.

In Sect. 2, we define the Cox ring and explain its construction for del Pezzo surfaces. We also introduce the Nagata action, which provides a link between linear subspaces of $\mathbb{k}^n$ and choices of initial forms of generators of Cox rings of del Pezzo surfaces (i.e. candidates for moneric or Khovanskii bases of the Cox ring).

Section 3 is dedicated to explaining the geometric consequence of a Khovanskii basis in terms of degenerations. Roughly speaking, a Khovanskii basis of a (finitely generated) subalgebra U of the polynomial ring yields a degeneration of $\mathrm{Spec}(U)$ to a toric variety. We show that we obtain even more if we choose a Khovanskii basis of the Cox ring $\mathrm{Cox}(X)$ of a variety X. We do not only obtain a toric degeneration of $\mathrm{Spec}(\mathrm{Cox}(X))$, but also toric degenerations of X with respect to all possible embeddings.

In Sect. 4, we explain and give examples for the problem which motivated Sturmfels and Xu to study Khovanskii bases of Cox–Nagata rings. It turns out that a Khovanskii basis allows us to compute the Hilbert function of a del Pezzo surface with respect to a specific embedding by counting lattice points in dilations of a rational convex polytope.

Finally, Sects. 5–6 describe our first attempts to classify three-dimensional Khovanskii subspaces of $\mathbb{k}^6$. First we describe two tropical varieties which we expect to be related to the problem: the tropical Grassmannian $\mathrm{TGr}(3,6)$ and the tropical moduli space of del Pezzo surfaces of degree 3. We then explain how we tried to use them as parametrizing spaces for moneric and Khovanskii subspaces. The conclusion is that neither of these models has the combinatorial structure suitable to play this role.

2 Cox–Nagata Rings

Let G be a linear group acting on a polynomial ring R over a field $\mathbb{k}$. Hilbert's 14th problem asks whether the ring of invariants R^G is a finitely generated $\mathbb{k}$-algebra. The answer is affirmative when the group G is reductive or $G = \mathbb{G}_a$. Nagata considered the action of a codimension 3 linear subspace $G \subset \mathbb{C}^n$ acting on the polynomial ring $R = \mathbb{C}[x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n]$ via $x_i \mapsto x_i$ and $y_i \mapsto y_i + \lambda_i x_i$, where $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in G$. He proved that the ring of invariants R^G is, in general, not finitely generated for $n = 16$, see [18]. Mukai realized the ring of invariants R^G as a certain Rees algebra and as such, it is isomorphic to the Cox ring of a blow-up [17]. Mukai's description of R^G provides conditions for it to be finitely generated and a way of computing its generators, at least for $\mathrm{codim}\, G \leq 3$.

In this section, we review Mukai's description of R^G. We recall the definition of Cox rings and study with some more detail the isomorphism between R^G when $\mathrm{codim}\, G = 3$ and the Cox ring of the blow-up of $\mathbb{P}^2$ at n points in general position. Next we specialize to the blow-up of six points and give a description of the invariants that generate R^G.

Nagata's Action Let $R := \Bbbk[x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n]$ be the polynomial ring with a $\mathbb{Z}^n$-grading induced by setting $\deg(x_i) = \deg(y_i) = e_i$, where $e_1, e_2, \ldots, e_n$ is the standard basis of $\mathbb{Z}^n$. Let $G \subset \Bbbk^n$ be a linear subspace of codimension r given by the equations

$$\begin{aligned} a_{1,1}t_1 + a_{1,2}t_2 + \cdots + a_{1,n}t_n &= 0 \\ a_{2,1}t_1 + a_{2,2}t_2 + \cdots + a_{2,n}t_n &= 0 \\ &\vdots \\ a_{r,1}t_1 + a_{r,2}t_2 + \cdots + a_{r,n}t_n &= 0\,. \end{aligned}$$

We consider Nagata's action of G on R. As x_i is invariant for all $1 \le i \le n$, we can extend the action to the localization

$$R_{\mathbf{x}} = R[x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}] = \Bbbk[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}, y_1 x_1^{-1}, y_2 x_2^{-1}, \ldots, y_n x_n^{-1}].$$

The grading on R extends naturally to a grading on $R_{\mathbf{x}}$ with $\deg(x_i^{-1}) = -e_i$. Now, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in G$ acts on $R_{\mathbf{x}}$ by $x_i \mapsto x_i$ and $\frac{y_i}{x_i} \mapsto \frac{y_i}{x_i} + \lambda_i$. If $y_i' = \frac{y_i}{x_i}$, then the element $\lambda \in G$ acts on $\Bbbk[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}, y_1', y_2', \ldots, y_n']$ by $x_i \mapsto x_i$ and $y_i' \mapsto y_i' + \lambda_i$. A direct computation shows that the invariant ring $R_{\mathbf{x}}^G$ is generated over $\Bbbk[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$ by the linear polynomials $\ell_i' := a_{i,1}y_1' + a_{i,2}y_2' + \cdots + a_{i,n}y_n'$ for $1 \le i \le r$. Let $x_0 = \prod_{j=1}^n x_j$ and

$$\ell_i = x_0 \ell_i' = x_0 \left(a_{i,1}\frac{y_1}{x_1} + a_{i,2}\frac{y_2}{x_2} + \cdots + a_{i,n}\frac{y_n}{x_n} \right). \tag{1}$$

We define the algebra $U := \Bbbk[\ell_1, \ell_2, \ldots, \ell_r] \subset R^G$. Let V be the $\Bbbk$-vector space spanned by $\ell_1, \ell_2, \ldots, \ell_r$. Then U is a $\mathbb{Z}$-graded ring and V is its degree one part. We also let $V_i \subset V$ be the polynomials in V that do not have $y_i \prod_{i \neq j} x_j$ as a monomial and $I_i \subset U$ the ideal generated by V_i. Then we have the following:

Proposition 2.1 *The invariant algebra R^G is the extended multi-Rees algebra*

$$U[x_1, x_2, \ldots, x_n] + \sum_{d \in \mathbb{Z}^n} \left(I_1^{d_1} \cap \cdots \cap I_n^{d_n}\right) x_1^{-d_1} \cdots x_n^{-d_n} \subset U[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}].$$

Proof A proof is found in either [17] or [1, Sect. 4.3.4]. □

Cox Rings The Cox ring of a smooth projective variety X over the field $\Bbbk$, with finitely generated torsion free divisor class group $\mathrm{Cl}(X)$, is the ring

$$\mathrm{Cox}(X) = \bigoplus_{(a_1, a_2, \ldots, a_r) \in \mathbb{Z}^r} H^0\left(X, \mathscr{O}_X(a_1 D_1 + a_2 D_2 + \cdots + a_r D_r)\right),$$

where $D_1, D_2, \ldots, D_r$ is a fixed basis of $\mathrm{Cl}(X) \simeq \mathbb{Z}^r$. This ring has the structure of a $\mathbb{k}$-algebra. When it is finitely generated the variety X is called a Mori Dream Space. This is the case for smooth del Pezzo surfaces of degree $1 \le d \le 9$, for which generators and relations among them are known.

We let A be an $r \times n$ matrix with entries in $\mathbb{k}$ such that G is the kernel of A. We denote by $a^{(i)}$ the ith column vector of A and assume that they are pairwise linearly independent. Denote by X_G the del Pezzo surface resulting from the blow-up of $\mathbb{P}^2$ at n different points with homogeneous coordinates $a^{(i)}$. The del Pezzo surface X_G is determined by G only up to isomorphism: an isomorphism of $\mathbb{P}^2$ as a linear map leaves the rowspace of A, and therefore also the kernel G, invariant and induces an isomorphism of the corresponding del Pezzo surfaces. The Picard group $\mathrm{Pic}(X_G)$ is isomorphic to $\mathbb{Z}^{n+1}$ and is generated by the proper transform of the hyperplane class H and the classes of the exceptional divisors E_i for $1 \le i \le n$. Thus the Cox ring of X_G is:

$$\mathrm{Cox}(X_G) = \bigoplus_{(d_0, d_1, \ldots, d_n) \in \mathbb{Z}^{n+1}} H^0\big(X_G, \mathscr{O}(d_0 H + d_1 E_1 + d_2 E_2 + \cdots + d_n E_n)\big)\,.$$

Given a divisor class $D := d_0 H + d_1 E_1 + d_2 E_2 + \cdots + d_n E_n$, the corresponding homogeneous part $\mathrm{Cox}(X_G)_D$ is the space $H^0\big(X_G, \mathscr{O}(D)\big)$. If $d_0 \ge 0$ then D is the class of the proper transform of a degree d_0 hypersurface that has multiplicity $-d_i$ in the point $a^{(i)}$. Thus we can identify $H^0(X_G, \mathscr{O}(D))$ with the space of homogeneous polynomials of degree d_0 in $\mathbb{k}[z] = \mathbb{k}[z_1, z_2, \ldots, z_r]$ that have multiplicity at least $-d_i$ at $a^{(i)}$. Let I_i' be the vanishing ideal in $\mathbb{k}[z]$ of the point $a^{(i)}$. The latter vector space is precisely

$$\big((I_1')^{-d_1} \cap (I_2')^{-d_2} \cap \cdots \cap (I_n')^{-d_n}\big)_{d_0}, \tag{2}$$

where $(I_i')^{-d_i} = \mathbb{k}[z]$ if $-d_i \le 0$. If $d_0 < 0$ then we have $H^0(X_G, \mathscr{O}(D)) = 0$.

Consider the map $\mathrm{Cox}(X_G)_D \simeq H^0(X_G, \mathscr{O}(D)) \longrightarrow R^G_d$ given by

$$g(z_1, z_2, \ldots, z_r) \mapsto g(\ell_1, \ell_2, \ldots, \ell_r) x_1^{d_1} x_2^{d_2} \cdots x_n^{d_n}\,,$$

where $d = (d_0 + d_1, d_0 + d_2, \ldots, d_0 + d_n)$, and $\ell_i, 1 \le i \le r$ are as in (1). Recall that $\ell_i = x_0 \ell_i'$ where $\ell_i' \in R^G_{\mathbf{x}}$ are the invariants in $R_{\mathbf{x}}$ of degree $0 \in \mathbb{Z}^n$. As g is homogeneous of degree d_0, then $g(\ell_1, \ell_2, \ldots, \ell_r) = x_0^{d_0} g(\ell_1', \ell_2', \ldots, \ell_r')$ is an invariant of degree $(d_0, d_0, \ldots, d_0) \in \mathbb{Z}^n$. Thus $g(\ell_1, \ell_2, \ldots, \ell_r) x_1^{d_1} x_2^{d_2} \cdots x_n^{d_n}$ is indeed an element of R^G of degree $(d_0 + d_1, d_0 + d_2, \ldots, d_0 + d_n)$. Now we notice that

$$R^G = U[x_1^{\pm 1}, x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \cap R = \mathbb{k}[\ell_1, \ell_2, \ldots, \ell_r][x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}] \cap R.$$

Given $d \in \mathbb{Z}^n$, any homogeneous element $f \in R^G_d$ admits a presentation of the form

$$f = \sum_{v \in \mathbb{Z}^n} h_v(\ell_1, \ell_2, \ldots, \ell_r) x_1^{v_1} x_2^{v_2} \cdots x_n^{v_n}$$

where the h_v are homogeneous of degree $d - v$. On the other hand, the ℓ_i are homogeneous of degree $(1, 1, \dots, 1) \in \mathbb{Z}^n$ and therefore $d - v = (d_0, d_0, \dots, d_0)$ for some $d_0 \geq 0$. Thus, $h_v(z_1, z_2, \dots, z_r) \in \Bbbk[z_1, z_2, \dots, z_r]_{d_0}$. Moreover, by Proposition 2.1, we may assume that $h_v(\ell_1, \ell_2, \dots, \ell_r) \in (I_i)^{-v_i}$ and therefore $h_v(z_1, z_2, \dots, z_r) \in (I_i')^{-v_i}$. Thus, given $d \in \mathbb{Z}^n$ fixed, we have an isomorphism

$$\bigoplus_{\substack{D=(d_0,d_1,\dots,d_n)\in\mathbb{Z}^n \\ d=(d_0+d_1,d_0+d_2,\dots,d_0+d_n)}} \operatorname{Cox}(X_G)_D \longrightarrow (R^G)_d \tag{3}$$

$$g(z) \longmapsto g(\ell_1, \ell_2, \dots, \ell_r) x_1^{d_1} x_2^{d_2} \cdots x_n^{d_n} \tag{4}$$

where $d = (d_0 + d_1, d_0 + d_2, \dots, d_0 + d_n)$. This and the previous proposition prove the following:

Proposition 2.2 *The Cox ring of X_G is isomorphic to R^G.*

We should observe that the ideals I_i' in (2) do not change if we rescale the columns of A, yet the image of a polynomial g under the isomorphism (3) can be different.

The Cox Ring of a del Pezzo Surface In [2] it was proven that the Cox ring of a del Pezzo surface of degree at least 2 is generated by the global sections over the exceptional curves. An exceptional curve is one with self-intersection -1. Such a curve has only one global section (up to scalar multiplication). We use this knowledge and the isomorphism of the previous part to compute a set of generators for R^G.

Example 2.3 Before we move to the case of del Pezzo surfaces of degree 3, most important for us, let us say what the Cox ring of a del Pezzo surface of degree 4 looks like. This is a sketch of a solution to Problem 6 on Surfaces in [25].

We need to identify all exceptional curves on the blow-up of $\mathbb{P}^2$ in 5 points $P_1, P_2, \dots, P_5$ in general position. First, there are 5 exceptional divisors of the blow-up, $E_1, E_2, \dots, E_5$. Then one checks that strict transforms of lines through two points P_i, P_j are exceptional curves. As divisors, they are linearly equivalent to $H - E_i - E_j$. Finally, there is one conic through all five chosen points, and its strict transform also is an exceptional curve, linearly equivalent to $2H - E_1 - E_2 - E_3 - E_4 - E_5$. Thus we have 16 generators of the Cox ring in total.

Relations between them come, roughly speaking, from the possibility of decomposing a divisor class as sums of the ones given above in a few different ways. For instance, $2H - E_1 - E_2 - E_3 - E_4$ can be written as:

$$\begin{aligned}(H - E_1 - E_2) + (H - E_3 - E_4) &= (H - E_1 - E_3) + (H - E_2 - E_4) \\ &= (H - E_1 - E_4) + (H - E_2 - E_3)\,.\end{aligned}$$

This leads to relations of corresponding sections which generate the Cox ring. A good explanation of these computations (also for del Pezzo surfaces of smaller degree) can be found in the MSc thesis of J.C. Ottem [20]. It is worth noting that different choices of points give different relations, but the Cox rings are isomorphic.

Let G and A be as before with $r = 3$, $n = 6$ and suppose that the points $a^{(i)} \in \mathbb{P}^{r-1}$ are in general position, that is, no three of them lie on a line and no six on a conic. Then X_G is a del Pezzo surface of degree 3 and it has 27 exceptional curves, determined in a very similar way as in Example 2.3. These are the classes of:

- the exceptional divisors E_i, $1 \leq i \leq 6$,
- the proper transforms of lines which pass through pairs of the blown-up points L_{ij}, $1 \leq i < j \leq 6$, and
- the proper transforms of conics through five of these points, Q_i with $1 \leq i \leq 6$.

The classes in $\mathrm{Pic}(X_G)$ of these curves are E_i, $H - E_i - E_j$ and $2H - \sum_{j \neq i} E_j$. This means that R^G is generated by the images under (3) of the unique polynomials g in $\Bbbk[z_1, z_2, \ldots, z_r]$ having the prescribed multiplicity on the blown-up points. Now we compute these images explicitly. For simplicity we will denote $[6] = \{1, 2, \ldots, 6\}$.

Let us start with the exceptional divisors E_i. We have that the only monomials of degree 0 in $\Bbbk[z]$ are the non-zero constants and they all belong to $(I_i')^{-1} = \Bbbk[z]$. Thus, by (3) we get

$$\left((I_i')^{-1}\right)_0 = \Bbbk[z]_0 \simeq \mathrm{Cox}(X_G)_{E_i} \simeq (R^G)_{e_i},$$

where $e_i \in \mathbb{Z}^6$ is the ith standard basis vector, and this isomorphism maps $1 \mapsto 1 \cdot x_i$. Thus the elements $\{x_i : 1 \leq i \leq 6\}$ are part of the chosen generating set of R^G.

For each class of the form $H - E_i - E_j$, there is a polynomial of degree one in $I_i' \cap I_j'$, namely, the equation of the unique line through the points $a^{(i)}$ and $a^{(j)}$. This is

$$g(z_1, z_2, z_3) = \left((a_{2j}a_{3i} - a_{2i}a_{3j})z_1 + (a_{1i}a_{3j} - a_{1j}a_{3i})z_2 + (a_{1j}a_{2i} - a_{1i}a_{2j})z_3\right).$$

The image of this polynomial in R^G is

$$g(\ell_1, \ell_2, \ell_3) \cdot (x_i x_j)^{-1} = -\sum_{k \neq i,j} p_{ijk} y_k \Big(\prod_{s \notin \{i,j,k\}} x_s \Big),$$

where the p_{ijk} are the Plücker coordinates of A, and it has degree $\sum_{k \neq i,j} e_k \in \mathbb{Z}^6$.

Finally, for the class $2H - \sum_{j \neq m} E_j$ there is also a unique polynomial of degree 2 in $\cap_{j \neq m} I_j'$: the defining polynomial of the unique conic through the five points different from $a^{(m)}$. A direct computation shows that the image of this conic has the form

$$G_m = (x_m)^2 \sum_{i<j, i,j \in [6] \setminus m} p_{([6] \setminus i,j,m)} \; y_i y_j \prod_{k \in [6] \setminus \{i,j,m\}} p_{ijk} x_k + (y_m) \cdot \sum_{i \in [6]} (u_i - v_i) y_i \prod_{k \neq i} x_k$$

where $u_i - v_i$ is a binomial of degree 4 in the Plücker coordinates of A. This conic generator has degree $e_m + \sum_{i \in [6]} e_i$

It is worth noting that even when it is difficult to write the exact expression of the polynomials G_m, its computation is straightforward. Also, we observe that the generators of R^G are determined up to scalar multiple by G since the Plücker coordinates of the matrix A are. Yet, as observed after Proposition 2.2, R^G is not itself an invariant of the isomorphism class of X_G.

Moneric and Khovanskii Subspaces The preceding paragraphs show how a codimension 3 vector subspace $G \subset \mathbb{k}^n$, or a matrix containing its basis, gives a minimal generating set of the Cox ring of a del Pezzo surface of degree $9 - n$. Having covered this, we can finally introduce Khovanskii and moneric subspaces.

Definition 2.4 We say that a codimension 3 subspace $G \subset \mathbb{k}^n$ is *Khovanskii* (resp. *moneric*) if the corresponding minimal generating set of the Cox ring of a del Pezzo surface of degree $9 - n$ is a Khovanskii (resp. moneric) basis of R^G.

We would like to consider moneric and Khovanskii bases up to the following equivalence relation:

Definition 2.5 Codimension 3 subspaces $G, G' \subset \mathbb{k}^n$ will be called *equivalent* if the corresponding initial algebras of the Cox ring of a del Pezzo surface are equal.

If G and G' are Khovanskii and determine the same initial terms of the minimal generating set of corresponding Cox rings then they are equivalent.

3 Khovanskii Basis and Degeneration of the Cox Ring

Degeneration of varieties is a powerful tool in algebraic geometry, used on many different occasions. The idea behind it is to introduce a notion of a “limit” of a family of algebraic varieties. However, since the Zariski topology on an algebraic variety is not well behaved in this sense (it is for example almost never Hausdorff), it turns out that a better replacement for an arbitrary family of varieties is the notion of a flat family. This notion has the desirable feature that limit points exist and are unique if we parametrize over a one dimensional variety. It also ensures that the points in the family, including the limit point have the same Hilbert function, and thus share many invariants such as e.g. the degree and the genus. Degenerations thus motivate the following approach: to compute properties of a given variety X first degenerate the variety to a more accessible variety X' and then do the computations on this variety. This idea can be realized in the notion of a Khovanskii basis.

Toric Degenerations The following definition makes precise what we mean by a degeneration of a variety.

Definition 3.1 Let $(\mathbb{k}^\circ, \mathfrak{m})$ be a discrete valuation ring and let X be a variety over $\mathbb{k} = \mathrm{Quot}(\mathbb{k}^\circ)$. A *degeneration* of the variety X is a flat family $\tilde{X} \to \mathrm{Spec}(\mathbb{k}^\circ)$ such that $\tilde{X} \times_{\mathbb{k}^\circ} \mathrm{Spec}(\mathbb{k}) \cong X$. A degeneration is *toric* if the special fibre $\tilde{X} \times_{\mathbb{k}^\circ} \mathrm{Spec}(\mathbb{k}^\circ/\mathfrak{m})$ is a toric variety.

In this section, we provide a method for degenerating a variety with respect to all possible embeddings at once. The idea is to degenerate the Cox ring of the given variety which contains information about all possible embeddings of the variety. In order to talk about degenerations of a projective variety with respect to a specific embedding, we need to take the choice of a very ample line bundle into account.

Definition 3.2 Let $(\Bbbk^\circ, \mathfrak{m})$ be a discrete valuation ring and X be a projective variety over $\Bbbk$, together with a very ample line bundle L. A family $\tilde{X} \to \mathrm{Spec}(\Bbbk^\circ)$ together with a line bundle $\tilde{L}$ is called a *toric degeneration of X with respect to the embedding given by L* if it is a toric degeneration, $\tilde{L}$ is flat over $\mathrm{Spec}(\Bbbk^\circ)$, we have $\tilde{L}_{|\tilde{X}\times \mathrm{Spec}(\Bbbk)} \cong L$ and the line bundle $\tilde{L}_{|\tilde{X}\times \mathrm{Spec}(\Bbbk^\circ/\mathfrak{m})}$ is ample.

In the above definition, we did not assume that $\tilde{L}_{|\tilde{X}\times \mathrm{Spec}(\Bbbk^\circ/\mathfrak{m})}$ is very ample. However, if we consider the Veronese embedding of the embedded variety X, we may assume that X as well as the special fibre are embedded in the same $\mathbb{P}^N$. More concretely by replacing L with a high enough multiple $L^{\otimes k}$, we can make sure that $L_{|\tilde{X}\times \mathrm{Spec}(\Bbbk^\circ/\mathfrak{m})}$ is also very ample.

Degenerations of del Pezzo Surfaces via the Cox Ring Let $\Bbbk = F(t)$ for a field F of characteristic 0. We often assume $F = \mathbb{Q}$. As in Sect. 2, given $n \in \{1, 2, \ldots, 8\}$ we can associate to a matrix $A \in Mat_{\Bbbk}(3, n)$ which has maximal rank, with kernel G, the variety X_G. The variety X_G is the blow-up of $\mathbb{P}^2$ at the points represented by A. Proposition 2.2 gives us the following identity

$$\mathrm{Cox}(X_G) \simeq \Bbbk[x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n]^G =: R^G\,.$$

By varying the variable t we can interpret X_G as a family of del Pezzo surfaces over F and $\mathrm{Cox}(X_G)$ as the corresponding family of Cox rings. Note however that the only property we are using in this section about the variety X_G is that its Cox ring R^G is a subalgebra of a polynomial ring $\Bbbk[x_1, x_2, \ldots, x_n]$.

Let $\Bbbk^\circ$ be the corresponding valuation ring of $\Bbbk$, i.e. the set of all elements having nonnegative valuation.

Theorem 3.3 *Let* $R \subset \Bbbk[x_1, x_2, \ldots, x_n]$ *be a* $\Bbbk$*-subalgebra. A finite Khovanskii basis* $\mathscr{F}$ *of* R *induces a toric degeneration of* $\mathrm{Spec}(R)$.

Proof Let $\mathscr{F}$ be a finite Khovanskii basis. For $f \in \mathscr{F}$, let us denote by $\mathrm{trop}(f)(\mathbf{0})$ the minimum of the valuation of the coefficients of f. Consider the $\Bbbk^\circ$-algebra generated by

$$\{t^{-\,\mathrm{trop}(f)(\mathbf{0})}f : f \in R^G\} \subset \Bbbk^\circ[x_1, x_2, \ldots, x_n].$$

We claim that $\mathrm{Spec}(R^G_{\Bbbk^\circ}) \to \mathrm{Spec}(\Bbbk^\circ)$ is a toric degeneration of $\mathrm{Spec}(R)$.

It is a flat morphism since $R^G_{\Bbbk^\circ}$ is a torsion free module over the discrete valuation ring $\Bbbk^\circ$. Now, the general fibre is given by $\mathrm{Spec}(R^G_{\Bbbk^\circ} \otimes_{\Bbbk^\circ} \Bbbk) \cong \mathrm{Spec}(R)$, and the special fibre is $\mathrm{Spec}\left(R^G_{\Bbbk^\circ} \otimes_{\Bbbk^\circ} \Bbbk^\circ/(t)\right) \cong \mathrm{Spec}\left(\mathrm{in}(R)\right)$. The last thing to prove is that the algebra $\mathrm{in}(R^G_{\Bbbk^\circ})$ is an affine semigroup algebra. But this follows easily from the fact that it is a finitely generated algebra generated by monomials. □

As a consequence of the above theorem we conclude that a finite Khovanskii basis $\mathscr{F}$ of R induces a toric degeneration of $\mathrm{Spec}(R)$. Now we want to show

how this toric degeneration gives a toric degeneration of X_G with respect to any embedding. For this purpose the following lemma is helpful.

Lemma 3.4 *Let $\mathscr{F}$ be a finite Khovanskii basis of R^G. Let L be a very ample line bundle on X_G and $T := \bigoplus T_q := \bigoplus_{q\in\mathbb{N}_0} H^0(X, L^{\otimes q}) \subset R^G$ be its graded section ring. Then* in(T) *is finitely generated.*

Proof Let $f_1, f_2, \ldots, f_w \in R^G$ be homogenous elements which form a Khovanskii basis of R^G. For each $\beta \in \mathbb{N}_0^w$ consider the set of all polynomials $f_\beta := \prod_{i=1}^w f_i^{\beta_i}$ such that there is a non-negative integer $p \in \mathbb{N}$ for which we have

$$\sum_{i=1}^{w} \beta_i \cdot \deg(f_i) = p \cdot \deg(L). \tag{5}$$

By a slight abuse of notation, we use deg for the function which assigns to a section as well as to a divisor the corresponding integer vector under the isomorphism $\mathrm{Pic}(X_G) \cong \mathbb{Z}^{n+1}$. Our first claim is that this set forms a (possibly non-finite) Khovanskii basis of T. Indeed, let $f \in T_q$ be a homogeneous element. Using the assumption that the f_i's form a Khovanskii basis for R^G, we deduce that there are finitely many $\alpha_j \in \mathbb{N}_0^w$, and $c_j \in k$ which satisfy $\mathrm{in}(f) = \sum_j c_j \cdot \mathrm{in}(f_{\alpha_j})$. Since f was homogeneous, the degrees of all the f_{α_j} match the degree of f, and we deduce that all the f_{α_j} fulfill the above prescribed property of (5).

Next, we want to prove that finitely many f_β suffice to form a Khovanskii basis. The question can be reformulated into the question of the finite generation of the following semigroup:

$$S := \left\{(\beta_1, \beta_2, \ldots, \beta_w, k) \in \mathbb{N}_0^w \times \mathbb{N} : \sum_{i=1}^{w} \beta_i \cdot \deg(f_i) = k \cdot \deg(L)\right\}.$$

Consider the cone $C(S)$ generated by S in $\mathbb{R}^{w+1}$. Then $C(S) \cap (\mathbb{Z}^{w+1} \setminus \{0\}) = S$, hence by Gordan's Lemma (see e.g. [5, Sect. 1.2]), S is finitely generated. □

Theorem 3.5 *A finite Khovanskii basis of $R^G = \mathrm{Cox}(X_G)$ induces a toric degeneration of X_G with respect to all possible embeddings.*

Proof Let L be a very ample line bundle on X and let $T := \bigoplus_{q\in\mathbb{N}_0} H^0(X, L^{\otimes q})$ be its graded section algebra. Define the algebra

$$T_{\Bbbk^\circ} = \{t^{-\mathrm{trop}(f)(\mathbf{0})} f \mid f \in T\} \subset \Bbbk^\circ[x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n],$$

where again $\mathrm{trop}(f)(\mathbf{0})$ denotes the minimum of the valuation of the coefficients of f. The $\mathbb{N}_0$-grading on T defines a natural grading on $T_{\Bbbk^\circ}$. As L is very ample the section ring T is finitely generated. Hence, the same follows for the graded algebra $T_{\Bbbk^\circ}$. The flatness of $T_{\Bbbk^\circ}$ can easily be derived from the torsion freeness over the discrete valuation ring $\Bbbk^\circ$. Thus, we get an induced flat morphism

$$\mathrm{Proj}(\bigoplus_{q\in\mathbb{N}_0}(T_{\Bbbk^\circ})_q) \to \mathrm{Spec}(\Bbbk^\circ)$$

and an induced line bundle $\tilde{L} = \mathcal{O}_{T_{\Bbbk^\circ}}(1)$. For the computations of the fibres, we use the following two identities on the graded pieces: $(T_{\Bbbk^\circ})_q \otimes_{\Bbbk^\circ} k \cong \mathrm{in}(T_q)$ and $(T_{\Bbbk^\circ})_q \otimes_{\Bbbk^\circ} \Bbbk \cong T_q$, where $\mathrm{in}(T_q)$ is the k-vector space generated by $\mathrm{in}(f)$ for all $f \in T_q$. Therefore we get the following:

$$\mathrm{Proj}(\bigoplus_{q\in\mathbb{N}_0}(T_{\Bbbk^\circ})_q) \times \mathrm{Spec}(\Bbbk) = \mathrm{Proj}(\bigoplus_{q\in\mathbb{N}_0}(T_{\Bbbk^\circ})_q \otimes \Bbbk) = \mathrm{Proj}(\bigoplus_{q\in\mathbb{N}_0} T_q) \cong X,$$

$$\mathrm{Proj}(\bigoplus_{q\in\mathbb{N}_0}(T_{\Bbbk^\circ})_q) \times \mathrm{Spec}(k) = \mathrm{Proj}(\bigoplus_{q\in\mathbb{N}_0}(T_{\Bbbk^\circ})_q \otimes k) = \mathrm{Proj}(\bigoplus_{q\in\mathbb{N}_0} \mathrm{in}(T_q)) =: X_T.$$

The previous lemma implies that the graded algebra $\mathrm{in}(T) = \bigoplus_{q\in\mathbb{N}_0}\mathrm{in}(T_q)$ is finitely generated by monomials, and can be seen as a quotient of the semigroup algebra of

$$S := \bigoplus_{q\in\mathbb{N}_0} S_q := \bigoplus_{q\in\mathbb{N}_0}\{(\beta_1,\beta_2,\ldots,\beta_w,q) \in \mathbb{N}_0^w \times \{q\} \mid \sum \beta_i \cdot \deg(f_i) = q\cdot\deg(L)\}.$$

This shows that $X_T = \mathrm{Proj}(\bigoplus_{q\in\mathbb{N}_0}\mathrm{in}(T_q))$ is a toric variety and $\tilde{L}_{|X_T} = \mathcal{O}_{X_T}(1)$ is the induced ample line bundle. □

4 Hilbert Functions of Del Pezzo Surfaces

The original motivation of the paper [27] is to give an interpretation of the Hilbert function of the Cox–Nagata ring R^G as a counting function of the numbers of lattice points in slices of some explicit rational convex polyhedral cone. If we focus on a specific embedding of the variety X_G into a projective space, then this interpretation induces a realization of the Hilbert function of X_G with respect to the embedding as the Ehrhart function of an explicit rational convex polytope.

Such an Ehrhart-type formula has appeared in many areas of mathematics: Berenstein-Zelevinsky's description of tensor product multiplicities for representations [3], Holtz-Ron's work on zonotopal algebras [11], the theory of Newton–Okounkov bodies [12, 15], and so forth. Having an Ehrhart-type formula for a mathematical object enables us to relate it with many areas of mathematics through convex geometry. One more important point is that an Ehrhart-type formula can be easier to evaluate since a polytope is bounded and given by a finite number of inequalities.

The theory of Khovanskii bases gives a systematic way to construct an Ehrhart-type formula for the Hilbert function of a graded ring under some assumptions. We explain this construction following [27]. Let $\mathbb{k}$ be the rational function field $\mathbb{Q}(t)$, $\mathbb{k}[x_1, x_2, \ldots, x_m]$ the polynomial ring over $\mathbb{k}$ in m variables, and $\mathbf{x}^{\mathbf{a}} := x_1^{a_1} x_2^{a_2} \cdots x_m^{a_m}$ for $\mathbf{a} = (a_1, a_2, \ldots, a_m) \in \mathbb{Z}_{\geq 0}^m$. Note that the residue field k is identical to the field $\mathbb{Q}$ of rational numbers. Fix $\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_m \in \mathbb{Z}^n$, and define a $\mathbb{Z}^n$-graded $\mathbb{k}$-algebra structure on $\mathbb{k}[x_1, x_2, \ldots, x_m]$ by $\deg(x_i) := \mathbf{d}_i$ for $1 \leq i \leq m$. We assume that the homogeneous parts $\mathbb{k}[x_1, x_2, \ldots, x_m]_{\mathbf{d}}$, $\mathbf{d} \in \mathbb{Z}^n$, are finite-dimensional. Let U be a $\mathbb{Z}^n$-graded $\mathbb{k}$-subalgebra of $\mathbb{k}[x_1, x_2, \ldots, x_m]$ with a finite Khovanskii basis $\mathscr{F} \subset U$. The $\mathbb{Z}^n$-grading of U induces a $\mathbb{Z}^n$-graded k-algebra structure on $\mathrm{in}(U)$. The *Hilbert function* of U is a map $\psi : \mathbb{Z}^n \to \mathbb{Z}_{\geq 0}$ given by $\psi(\mathbf{d}) := \dim_{\mathbb{k}}(U_{\mathbf{d}})$ for $\mathbf{d} \in \mathbb{Z}^n$. Let $\mathbb{Z}_{\geq 0}(\mathrm{in}(\mathscr{F}))$ (resp. $\mathbb{Z}(\mathrm{in}(\mathscr{F}))$) be the subsemigroup (resp. the subgroup) of $\mathbb{Z}^m$ generated by $\{\mathbf{a} \in \mathbb{Z}_{\geq 0}^m : \mathbf{x}^{\mathbf{a}} \in \mathrm{in}(\mathscr{F})\}$ and the zero vector. Denote by $\varGamma \subset \mathbb{R}^m$ the smallest real closed convex cone containing $\mathbb{Z}_{\geq 0}(\mathrm{in}(\mathscr{F}))$. The $\mathbb{Z}^n$-graded $\mathbb{k}$-algebra structure on $\mathbb{k}[x_1, x_2, \ldots, x_m]$ induces a $\mathbb{Z}^n$-grading of the semigroup $\varGamma \cap \mathbb{Z}(\mathrm{in}(\mathscr{F}))$. We observe that $\mathrm{in}(U)$ is identical to the semigroup algebra of $\mathbb{Z}_{\geq 0}(\mathrm{in}(\mathscr{F}))$, which is regarded as a $\mathbb{Z}^n$-graded k-subalgebra of the semigroup algebra of $\varGamma \cap \mathbb{Z}(\mathrm{in}(\mathscr{F}))$.

Proposition 4.1 *If the initial algebra* $\mathrm{in}(U)$ *is normal, then the value* $\psi(\mathbf{d})$ *for all* $\mathbf{d}\mathbb{Z}^n$ *equals the cardinality of* $\{\mathbf{a} \in \varGamma \cap \mathbb{Z}(\mathrm{in}(\mathscr{F})) : \deg(\mathbf{a}) = \mathbf{d}\}$.

Proof Fix $\mathbf{d} \in \mathbb{Z}^n$ such that $U_{\mathbf{d}} \neq \{0\}$, and take a $\mathbb{k}$-basis $\{f_1, f_2, \ldots, f_r\}$ of $U_{\mathbf{d}}$. If the initial forms $\mathrm{in}(f_1), \ldots, \mathrm{in}(f_r)$ are linearly dependent, then the definition of initial forms implies that there exist $c_1, c_2, \ldots, c_r \in \mathbb{k}$ such that $\mathrm{in}(c_1 f_1 + c_2 f_2 + \cdots + c_r f_r)$ does not belong to the k-linear space spanned by $\mathrm{in}(f_1), \mathrm{in}(f_2), \ldots, \mathrm{in}(f_r)$. Then by replacing f_i for some $1 \leq i \leq r$ with $c_1 f_1 + c_2 f_2 + \cdots + c_r f_r$, we can increase the dimension of the k-linear space spanned by $\mathrm{in}(f_1), \mathrm{in}(f_2), \ldots, \mathrm{in}(f_r)$. Repeating this procedure, we obtain a $\mathbb{k}$-basis $\{\tilde{f}_1, \tilde{f}_2, \cdots, \tilde{f}_r\}$ of $U_{\mathbf{d}}$ such that the initial forms $\mathrm{in}(\tilde{f}_1), \mathrm{in}(\tilde{f}_2), \cdots, \mathrm{in}(\tilde{f}_r)$ are linearly independent.

Then it follows that these form a k-basis of $\mathrm{in}(U)_{\mathbf{d}}$. In particular, the $\mathbb{k}$-algebra U and its initial algebra $\mathrm{in}(U)$ share the same Hilbert function. Since $\mathrm{in}(U)$ is identical to the semigroup algebra of $\mathbb{Z}_{\geq 0}(\mathrm{in}(\mathscr{F}))$, the group $\mathbb{Z}(\mathrm{in}(\mathscr{F}))$ is regarded as a subset of the field of fractions of $\mathrm{in}(U)$. Hence the normality assumption on $\mathrm{in}(U)$ implies that the semigroup $\mathbb{Z}_{\geq 0}(\mathrm{in}(\mathscr{F}))$ is saturated in $\mathbb{Z}(\mathrm{in}(\mathscr{F}))$, and hence that $\mathbb{Z}_{\geq 0}(\mathrm{in}(\mathscr{F})) = \varGamma \cap \mathbb{Z}(\mathrm{in}(\mathscr{F}))$. In particular, the initial algebra $\mathrm{in}(U)$ is identical to the semigroup algebra of $\varGamma \cap \mathbb{Z}(\mathrm{in}(\mathscr{F}))$. This proves the proposition. □

Remark 4.2 Our proof of Theorem 3.5 in Sect. 3 also uses initial forms. In the case $U = R^G$, the description of ψ in Proposition 4.1 reflects the toric degeneration of X_G constructed in the theorem. Let us fix a very ample line bundle L on X_G, and take a multi-degree $\mathbf{d}$ such that $(R^G)_{\mathbf{d}} = H^0(X_G, L)$. Then the Hilbert polynomial of X_G with respect to the corresponding embedding is identical to the polynomial in l given by $\psi(l\mathbf{d})$ for $l \gg 0$. In addition, by [10, Chap. I, Theorem 9.9], the Hilbert polynomial of X_G is identical to that of the resulting toric variety from the toric degeneration. From these observations and our proof of Theorem 3.5, we obtain an Ehrhart-type description of $\psi(l\mathbf{d})$ for $l \gg 0$, which is identical to the formula in Proposition 4.1.

Normality (or saturatedness) is a key to an Ehrhart-type formula in general. In the case of in(U), the theory of Gröbner bases can be applied to prove the normality as follows. Since $\mathscr{F}$ is moneric, we deduce that in(U) is generated by a finite number of monomials, and hence that the ideal I of relations is spanned by a set of binomials [26, Lemma 4.1]. Then we obtain a useful sufficient condition for the normality of in(U) in terms of a Gröbner basis of I (see [26, Proposition 13.15]).

Example 4.3 (Elementary Symmetric Function) Following [27, Example 3.2], set

$$e_l(t, x_1, x_2, \ldots, x_m) := \sum_{1 \le j_1 < j_2 < \cdots < j_l \le m} t^{(j_1-1)+(j_2-2)+\cdots+(j_l-l)} x_{j_1} x_{j_2} \cdots x_{j_l}$$

for $1 \le l \le m$, and $\mathscr{F} := \{e_l(t, x_1, x_2, \ldots, x_m) : 1 \le l \le m\} \subset \Bbbk[x_1, x_2, \ldots, x_m]$.

We obtain the elementary symmetric functions by specializing at $t = 1$. Let U be the $\Bbbk$-subalgebra of $\Bbbk[x_1, x_2, \ldots, x_m]$ generated by $\mathscr{F}$. It is easily checked that the initial algebra in(U) is identical to the k-subalgebra of $k[x_1, x_2, \ldots, x_m]$ generated by $\{x_1, x_1x_2, \ldots, x_1x_2 \cdots x_m\}$, and hence that $\mathscr{F}$ is a Khovanskii basis. In addition, the initial algebra in(U) is normal since $x_1, x_1x_2, \ldots, x_1x_2 \cdots x_m$ are algebraically independent.

Since in$(\mathscr{F}) = \{x_1, x_1x_2, \ldots, x_1x_2 \cdots x_m\}$, we have $\mathbb{Z}(\text{in}(\mathscr{F})) = \mathbb{Z}^m$ and

$$\varGamma \cap \mathbb{Z}^m = \{(a_1, a_2, \ldots, a_m) \in \mathbb{Z}^m : a_1 \ge a_2 \ge \cdots \ge a_m \ge 0\}. \tag{6}$$

Regard U as a $\mathbb{Z}_{\ge 0}$-graded $\Bbbk$-algebra by the total degree in variables $x_1, x_2, \ldots, x_m$. Let $\psi\colon \mathbb{Z}_{\ge 0} \to \mathbb{Z}_{\ge 0}$ denote the Hilbert function. We deduce by Proposition 4.1 and equation (6) that the value $\psi(r)$ for $r \in \mathbb{Z}_{\ge 0}$ equals the cardinality of

$$\{(a_1, a_2, \ldots, a_m) \in \mathbb{Z}^m : a_1 \ge a_2 \ge \cdots \ge a_m \ge 0,\ a_1 + \cdots + a_m = r\};$$

this is identical to the set of partitions of r with at most m parts.

Let us come back to our situation of interest.

Example 4.4 ([27, Theorem 3.5 and Proposition 3.6]) Let $G \subset \Bbbk^n$ be a generic subspace of dimension 1, and $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \Bbbk^n$ a nonzero element of G. We consider a $(2 \times n)$-matrix

$$\begin{bmatrix} \alpha_1 x_1 & \alpha_2 x_2 & \cdots & \alpha_n x_n \\ y_1 & y_2 & \cdots & y_n \end{bmatrix},$$

and, for $1 \le i < j \le n$, denote by $p_{i,j}$ the (2×2)-minor of this matrix with column indices i, j, that is, $p_{i,j} = \alpha_i x_i y_j - \alpha_j x_j y_i$. If we regard $\alpha_i x_i$ as an indeterminate, then the $\Bbbk$-subalgebra of R^G generated by $\{p_{i,j} : 1 \le i < j \le n\}$ is identical to the homogeneous coordinate ring of the Grassmannian of lines in the $(n-1)$-dimensional projective space over $\Bbbk$ with respect to the usual Plücker embedding. In particular, the minors $p_{i,j}$, for $1 \le i < j \le n$, satisfy the Plücker relation:

$p_{i,l}p_{j,m} - p_{i,m}p_{j,l} = p_{i,j}p_{l,m}$ for $1 \le i < j < l < m \le n$. This is a key to the fact that $\mathscr{F} := \{p_{ij} : 1 \le i < j \le n\} \cup \{x_i : 1 \le i \le n\}$ is a Khovanskii basis of R^G. The normality of the initial algebra $\mathrm{in}(R^G)$ follows from the criterion explained before Example 4.3. Thus we can apply Proposition 4.1 to obtain an Ehrhart-type formula for the Hilbert function ψ of R^G. We may assume without loss of generality that $\mathrm{val}(\alpha_1) < \mathrm{val}(\alpha_2) < \cdots < \mathrm{val}(\alpha_n)$. Then since $\mathrm{in}(p_{ij}) \in k^* x_i y_j$, we deduce that the value of ψ at $(r, u_1, u_2, \ldots, u_n) \in \mathbb{Z}^{n+1}$ equals the number of $(a_{i,j})_{i=1,2,\, j=1,\ldots,n} \in \mathbb{Z}^{2n}_{\ge 0}$ satisfying the following conditions:

$$a_{2,1} = 0,\ a_{2,2} + \cdots + a_{2,l+1} \le a_{1,1} + \cdots + a_{1,l},\ 1 \le l \le n-1,$$
$$a_{1,l} + a_{2,l} = u_l,\ 1 \le l \le n,\ a_{2,1} + \cdots + a_{2,n} = r.$$

By [27, Theorem 6.1], for $4 \le n \le 8$, there exists a generic $\mathbb{k}$-subspace $G \subset \mathbb{k}^n$ of codimension 3 such that R^G has a finite Khovanskii basis $\mathscr{F}$ and $\mathrm{in}(R^G)$ is normal. Hence Proposition 4.1 produces an Ehrhart-type formula for the Hilbert function ψ of the $\mathbb{Z}^{n+1}$-graded algebra R^G. The case of degree 5 del Pezzo surfaces is included in Example 4.4. In the case of del Pezzo surfaces of degree 3 and 4, Sturmfels and Xu gave a system of explicit linear inequalities defining the corresponding rational convex polytope for a specific subspace G, see [27, Example 1.3 and Corollary 5.2].

Since G is generic, the function ψ is independent of the choice of G. Hence we obtain a system of Ehrhart-type formulas for the same function ψ. If G is a different generic subspace, then the induced Ehrhart-type formula may be different, that is, the corresponding rational convex polytope may not be unimodularly equivalent. In order to compute ψ rapidly, we want to determine a generic subspace G such that the number of linear inequalities defining the corresponding rational convex polytopes is as small as possible.

In case of del Pezzo surfaces of degree 4, Sturmfels and Xu proved that the optimal number of linear inequalities is 12. Their proof relies on giving the complete classification of the subspaces G which produce Khovanskii bases [27, Theorem 4.1]. In addition, they conjectured that in the case of degree 3 the number 21 of linear inequalities in [27, Corollary 5.2] is minimal. One motivation of this research is to generalize their argument for degree 4 del Pezzo surfaces to the case of degree 3, and to prove the conjecture by giving a complete characterization of all Khovanskii subspaces G.

5 Tropicalization

Tropicalization is a procedure that associates to a very affine variety X (i.e. a closed subvariety of an algebraic torus) a rational polyhedral complex $\mathrm{trop}(X)$ in $\mathbb{R}^N$. Of the many ways of characterizing $\mathrm{trop}(X)$, there are two descriptions that will be useful for our purposes. In terms of initial degenerations, $\mathrm{trop}(X)$ is the set of all $w \in \mathbb{R}^N$ such that $\mathrm{in}_w X$ is nonempty (note that the ideal of $\mathrm{in}_w X$ coincides with the initial ideal as defined in [4] in the case where the valuation on $\mathbb{K}$ is trivial; for the definition of $\text{in}_w X$ in general, see [8, Sect. 5]). This allows us to compute $\text{trop}(X)$ using computer algebra software such as *gfan* [6]. When X is defined over an algebraically closed field with a nontrivial valuation, $\text{trop}(X)$ is the closure of the set of coordinatewise valuations. As this is the description we use for our classification problem, we will provide a more precise formulation of this characterization.

Let $\mathbb{K}$ be a field with a (possibly trivial) valuation $\text{val} : \mathbb{K}^* \to \mathbb{R}$, and X a closed subvariety of the algebraic torus $\mathbb{G}_m^N(\mathbb{K})$. We define the Bieri–Groves set $\mathscr{A}(X)$ of X to be $\mathscr{A}(X) = \{\big(\text{val}(x_1), \text{val}(x_2), \ldots, \text{val}(x_N)\big) \in \mathbb{R}^N : (x_1, x_2, \ldots, x_N) \in X\}$. Now, suppose $\mathbb{L}$ is an algebraically closed field extension of $\mathbb{K}$ with a nontrivial valuation extending the valuation on $\mathbb{K}$. By abuse of notation, we will also call this valuation $\text{val} : \mathbb{L}^* \to \mathbb{R}$. Let $X_{\mathbb{L}}$ denote the extension of X to a closed subvariety of $\mathbb{G}_m^N(\mathbb{L})$. Tropicalization is unchanged under field extension, i.e. $\text{trop}(X_{\mathbb{L}}) = \text{trop}(X)$, see [16, Thm 3.2.4]. By the Fundamental Theorem of Tropical Geometry [16, Thm 3.2.3], the closure of $\mathscr{A}(X_{\mathbb{L}})$ in $\mathbb{R}^N$ is $\text{trop}(X_{\mathbb{L}})$. Moreover, if the valuation on $\mathbb{K}$ is trivial, then $\text{trop}(X)$ is a rational polyhedral *fan* in $\mathbb{R}^N$.

Now let us specialize to the case of the tropical Grassmannian. We let $\mathbb{K} = \mathbb{Q}$ and $\mathbb{L}$ will denote Puiseux series over $\mathbb{C}$. The Grassmannian $\text{Gr}(d, \mathbb{Q}^n)$ can be viewed as a subvariety of $\mathbb{P}^{N-1}(\mathbb{Q})$ via its Plücker embedding, where $N = \binom{n}{d}$. Let $\text{Gr}_0(d, \mathbb{Q}^n)$ be the intersection of the affine cone of $\text{Gr}(d, \mathbb{Q}^n)$ with the dense torus (the locus where all Plücker coordinates are nonzero). This gives us a closed subvariety of $\mathbb{G}_m^N(\mathbb{Q})$, so we may form the tropicalization $\text{trop}(\text{Gr}_0(d, \mathbb{Q}^n))$. Let us abbreviate this by $\text{TGr}(d, \mathbb{Q}^n)$. This is a rational polyhedral fan in $\mathbb{R}^N$. We index the coordinates of $\mathbb{R}^N$ by the d-tuples of the numbers 1 through n. In [24], Speyer and Sturmfels give a combinatorial description of $\text{TGr}(2, \mathbb{Q}^n)$ in terms of the space of phylogenetic trees on n leaves (up to sign). In particular, they show that $d = (d_{ij})$ is a point in $\text{TGr}(2, \mathbb{Q}^n)$ if and only if for each 4-tuple $1 \leq i < j < k < l \leq n$, the maximum of

$$d_{ij} + d_{kl}, \quad d_{ik} + d_{jl}, \quad d_{il} + d_{jk}$$

is attained at least twice.

In the classification of Khovanskii subspaces G of $\Bbbk^5$, G can be viewed as a $\Bbbk$-valued point of $\text{Gr}_0(2, \mathbb{Q}^5)$, where $\Bbbk = \mathbb{Q}(t)$. If (p_{ij}) are the Plücker coordinates of G, then the valuations of the Plücker coordinates $d_{ij} = -\text{val}(p_{ij})$ are integers. This means that the Bieri–Groves set of the $\Bbbk$-valued points of $\text{Gr}_0(2, \mathbb{Q}^5)$ is the set of integer points in $\text{TGr}(2, \mathbb{Q}^5)$.

The Naruki fan is a fan structure on the tropicalization of the moduli space of marked del Pezzo surfaces of degree 3. Let Y^6 be the moduli space of degree 3 marked del Pezzo surfaces. We can express Y^6 as an open subvariety of the space of configurations of six labelled points in $\mathbb{P}^2$ in linear general position; call this space X^6. By the Gelfand–MacPherson correspondence, we can recover X^6 from the space of 3×6 matrices by taking appropriate quotients. The Grassmannian $\text{Gr}(3, \mathbb{K}^6)$ is identified with the quotient of $\text{Mat}_{\mathbb{K}}(3, 6)$ by the left-multiplication action of GL_3, i.e. $\text{Gr}(3, \mathbb{K}^6) = \text{GL}_3 \setminus \text{Mat}_{\mathbb{K}}(3, 6)$. Now let $\text{Gr}_0(3, \mathbb{K}^6)$ be the points in $\text{Gr}(3, \mathbb{K}^6)$ with a representative in $\text{Mat}_{\mathbb{K}}(3, \mathbb{K}^6)$ whose maximal minors do not vanish (in fact, this will hold for any representative). The torus acts on $\mathrm{Gr}_0(3, \mathbb{K}^6)$. The action on $\mathrm{Mat}_{\mathbb{K}}(3, 6)$ by right multiplication of diagonal invertible 6×6 matrices induces an action of the torus $\mathbb{G}_m^6(\mathbb{K})$ on $\mathrm{Gr}_0(3, \mathbb{K}^6)$. The Gelfand-MacPherson correspondence (see [14, Sect. 2.6]) provides the identification $X^6 = \mathrm{Gr}_0(3, \mathbb{K}^6)/\mathbb{G}_m^6(\mathbb{K})$. Here, we view the columns of the matrix representative as the points in the configuration. The Plücker embedding induces an embedding of X^6 into the torus $\mathbb{G}_m^{20}(\mathbb{K})/\mathbb{G}_m^6(\mathbb{K}) \cong \mathbb{G}_m^{14}$ as a closed subvariety. Under this correspondence, the six points in $\mathbb{P}^2(\mathbb{K})$ lie on a conic if and only if the Plücker coordinates satisfy

$$C := p_{1,3,4}p_{1,5,6}p_{2,3,5}p_{2,4,6} - p_{1,3,5}p_{1,4,6}p_{2,3,4}p_{2,5,6} = 0\,.$$

To see this, note that there is only one conic up to projective transformation, e.g. take $xz = y^2$. So the points lie on a conic if and only if this configuration can be represented by a matrix of the form

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_1^2 & a_2^2 & a_3^2 & a_4^2 & a_5^2 & a_6^2 \end{bmatrix}$$

for $a_1, a_2, \ldots, a_6$ in $\mathbb{k}$. It suffices to check the above Plücker identity for this matrix. The vanishing locus of C corresponds to an irreducible Weil divisor of X^6. By the description of degree 3 del Pezzo surfaces as blow-ups of $\mathbb{P}^2$ at six points in general position, we may identify Y^6 with $X^6 \setminus V(C)$. Under this identification, we see that Y^6 is a very affine variety and can be realized as a closed subvariety of $\mathbb{G}_m^{15}$ (this follows from [9, Lemma 6.1]). Therefore, the tropicalization of Y^6 may be viewed as the underlying set of a pure four dimensional fan in $\mathbb{R}^{15}$. By [9], $\mathrm{trop}(Y^6)$ admits a unique coarsest fan structure called the Naruki fan. The coordinates of this fan compute the possible valuations of the Plücker coordinates p_{ijk} and C (up to the action of $\mathbb{G}_m^6$).

6 The Search for a Combinatorial Structure

In order to classify three-dimensional Khovanskii subspaces of $\mathbb{k}^6$ we are looking for a combinatorial structure which parametrizes equivalence classes of such subspaces. When we identify a right structure (probably a fan of convex polyhedral cones), the next, and the last, step will be to subdivide it such that each chamber in the subdivision corresponds to a different class of moneric bases, some of them Khovanskii.

Degree 4 del Pezzo Surfaces and $\mathbf{TGr(2,\mathbb{Q}^5)}$ In the case of the Cox ring of a del Pezzo surface of degree 4, i.e. G being represented by a 2×5 matrix, this role was played by the tropical Grassmannian $\mathrm{TGr}(2, \mathbb{Q}^5)$, introduced in Sect. 5. It is a seven-dimensional fan in the ten-dimensional space, a product of a five-dimensional lineality space and the cone over the Petersen graph. It is worth noting that this two-dimensional part is also the tropicalization of (the very affine part of) the moduli space of degree 4 del Pezzo surfaces, see [23].

The map from the set of equivalence classes of subspaces G to $\mathrm{TGr}(2, \mathbb{Q}^5)$ is given by the tropical Plücker coordinates $d_{ij} = -\operatorname{val}(p_{ij})$ for $1 \leq i < j \leq 5$. In this way $\mathrm{TGr}(2, \mathbb{Q}^5)$, or its set of integral points, becomes a *good parametrizing set* for equivalence classes of moneric and Khovanskii subspaces (see Definition 2.5). This means that it satisfies the conditions of the following important definition.

Definition 6.1 For a set M to be a *good parametrizing set* for moneric and Khovanskii subspaces of $\mathbb{k}^n$ we require that for any subspaces G and G' mapped to the same point of M, if G is moneric (resp. Khovanskii), then G' is also moneric (resp. Khovanskii).

The reason for this property is that all coefficients in generators of the Cox ring in this case (see [27, Thm 4.1]) are monomials in Plücker coordinates. Thus if G and G' have the same sequence (d_{ij}) then they determine the same initial forms of all generators. In particular, if one of them is moneric or Khovanskii, then the second one also is, and obviously they are equivalent.

Degree 3 del Pezzo Surfaces, $\mathbf{TGr(3, \mathbb{Q}^6)}$ and the Naruki Fan To find a fan parametrizing moneric subspaces G for the case of del Pezzo surfaces of degree 3 (which can be embedded in $\mathbb{P}^3$ as smooth cubic surfaces), we tested two natural candidates. The first one is the tropical Grassmannian $\mathrm{TGr}(3, \mathbb{Q}^6)$.

Example 6.2 Take the subspace represented by the matrix G written below. Its sequence of (negatives of) tropical Plücker's coordinates is

$$(d_{i,j,k}) = (5, 11, 10, 4, 13, 15, 9, 18, 12, 15, 4, 10, 1, 9, 3, 6, 14, 8, 11, 14).$$

We modify G slightly to the matrix G' by changing the sign of the fourth term in the first row.

$$G = \begin{bmatrix} t^4 & t & t^8 & t^3 & t^9 & 1 \\ t^{11} & t^7 & t & t^7 & t^6 & 1 \\ t^9 & 1 & t^5 & t^9 & t^{11} & t^6 \end{bmatrix} \qquad G' = \begin{bmatrix} t^4 & t & t^8 & -t^3 & t^9 & 1 \\ t^{11} & t^7 & t & t^7 & t^6 & 1 \\ t^9 & 1 & t^5 & t^9 & t^{11} & t^6 \end{bmatrix}$$

One checks that the modification does not affect the tropical Plücker coordinates. That is, G and G' are mapped to the same point of $\mathrm{TGr}(3, \mathbb{Q}^6)$. Thus, if a coefficient in the formula for a generator is a monomial in Plücker coordinates, it will also take the same value for G and G'. All coefficients of generators corresponding to lines, and also some coefficients of generators corresponding to conics, have this form.

However, generators corresponding to conics have also some coefficients which are binomials in Plücker coordinates, and it turns out that they are the reason for $\mathrm{TGr}(3, \mathbb{Q}^6)$ being insufficient for our task. Look at the generator corresponding to the conic G_6 through points 1, 2, 3, 4, 5 (in [27, p. 443]):

$$
\begin{aligned}
G_6 = {} & p_{1,2,3}p_{1,2,4}p_{1,2,5}p_{3,4,5}y_1y_2x_3x_4x_5x_6^2 + p_{1,2,3}p_{1,3,5}p_{1,3,4}p_{2,4,5}y_1y_3x_2x_4x_5x_6^2 + \\
& p_{1,2,4}p_{1,3,4}p_{1,4,5}p_{2,3,5}y_1y_4x_2x_3x_5x_6^2 + p_{1,2,5}p_{1,3,5}p_{1,4,5}p_{2,3,4}y_1y_5x_2x_3x_4x_6^2 + \\
& p_{1,2,3}p_{2,3,4}p_{2,3,5}p_{1,4,5}y_2y_3x_1x_4x_5x_6^2 + p_{1,2,4}p_{2,3,4}p_{2,4,5}p_{1,3,5}y_2y_4x_1x_3x_5x_6^2 + \\
& p_{1,2,5}p_{2,3,5}p_{2,4,5}p_{1,3,4}y_2y_5x_1x_3x_4x_6^2 + p_{1,3,4}p_{2,3,4}p_{3,4,5}p_{1,2,5}y_3y_4x_1x_2x_5x_6^2 + \\
& p_{1,3,5}p_{2,3,5}p_{3,4,5}p_{1,2,4}y_3y_5x_1x_2x_4x_6^2 + p_{1,4,5}p_{2,4,5}p_{3,4,5}p_{1,2,3}y_4y_5x_1x_2x_3x_6^2 + \\
& (p_{1,2,4}p_{2,3,5}p_{1,3,6}p_{1,4,5} - p_{1,2,3}p_{2,4,5}p_{1,4,6}p_{1,3,5})y_1y_6x_2x_3x_4x_5x_6 + \\
& (p_{1,2,4}p_{1,3,5}p_{2,3,6}p_{2,4,5} - p_{1,2,3}p_{1,4,5}p_{2,4,6}p_{2,3,5})y_2y_6x_1x_3x_4x_5x_6 + \\
& (p_{1,3,4}p_{1,2,5}p_{2,3,6}p_{3,4,5} + p_{1,2,3}p_{1,4,5}p_{3,4,6}p_{2,3,5})y_3y_6x_1x_2x_4x_5x_6 + \\
& (p_{1,2,4}p_{1,3,5}p_{3,4,6}p_{2,4,5} - p_{1,3,4}p_{1,2,5}p_{2,4,6}p_{3,4,5})y_4y_6x_1x_2x_3x_5x_6 + \\
& (p_{1,2,5}p_{1,3,4}p_{3,5,6}p_{2,4,5} - p_{1,3,5}p_{1,2,4}p_{2,5,6}p_{3,4,5})y_5y_6x_1x_2x_3x_4x_6 + \\
& (p_{1,2,4}p_{1,3,5}p_{2,3,6}p_{4,5,6} - p_{1,2,3}p_{1,4,5}p_{2,4,6}p_{3,5,6})y_6^2x_1x_2x_3x_4x_5 \,.
\end{aligned}
$$

The signs are different from those in [27], which is a result of permuting the indices in Plücker coordinates. We compute the valuation of its second binomial coefficient, $p_{1,2,4}p_{1,3,5}p_{2,3,6}p_{2,4,5} - p_{1,2,3}p_{1,4,5}p_{2,4,6}p_{2,3,5}$: for G it is 36, but for G' it is 37. Both monomials have valuation 36, as shown by the sequence $(d_{i,j,k})$, but for G' the coefficients are such that the lowest terms cancel.

Moreover, computation of the remaining coefficients for G_6 show that in both cases this is the minimal valuation. Only for G it is the smallest one, and for G' there are more coefficients with valuation 37. We obtain that initial forms of G_6 are $-2x_1x_3x_4x_5x_6y_2y_6$ and $-x_1x_4x_5(x_6^2y_2y_3 + 2x_3x_6y_2y_6 + x_2x_3y_6^2)$ for G and G' respectively. That is, G' is *not moneric*, and one can check by computing other generators that G is. This example was constructed using *Macaulay2* [7].

Thus we have two subspaces mapped to a single point of $\mathrm{TGr}(3, \mathbb{Q}^6)$, such that one is moneric and the second is not. This shows that the tropical Grassmannian is too coarse to be a good parametrizing set: the property of being moneric is not well-defined for its points. It is worth noting that the point corresponding to G and G' does not lie in the interior of a maximal cone of $\mathrm{TGr}(3, \mathbb{Q}^6)$, but we expect the same phenomenon to appear also at interior points of maximal cones.

The second candidate for the parametrizing space is, as suggested in [27, Problem 5.4], the tropical moduli space of (smooth, marked) del Pezzo surfaces of degree 3. Its combinatorial structure is the Naruki fan, described in Sect. 5 (see also [9, 19] and [22, Sect. 6]). Recall that a 3×6 matrix G corresponds to a sequence of coordinates which are either monomials in Plücker coordinates or products of such monomials and a binomial $C = p_{1,3,4}p_{1,5,6}p_{2,3,5}p_{2,4,6} - p_{1,3,5}p_{1,4,6}p_{2,3,4}p_{2,5,6}$, which encodes the condition for six points lying on a conic.

Example 6.3 We compute the value of the binomial C for both G and G' and obtain the result that they both have valuation 37. These matrices are mapped to the same point in $\operatorname{trop}(Y^6)$, so this is not a good space for parametrizing moneric classes.

To summarize, Examples 6.2 and 6.3 prove the following result.

Proposition 6.4 *Neither the tropical Grassmannian* $\mathrm{TGr}(3, \mathbb{Q}^6)$ *nor the Naruki fan is a good parametrizing set for moneric classes of three-dimensional subspaces of* $\Bbbk^6$ *in the sense of Definition 6.1.*

The conclusion is that to find a good parametrizing set for our problem we should probably look for an another variety (maybe a different embedding of Y^6), whose coordinates are more closely related to binomials which appear in the Cox ring conic generators. Of 36 binomials appearing in 6 conic generators, 6 are equivalent (up to a Plücker relation) to C, and the remaining 30 are different, and also pairwise different. Hence our strategy will be to consider a variety embedded in a projective space using all 31 equivalence classes of binomials, tropicalize it and subdivide the fan structure obtained in this way to parametrize three-dimensional moneric and Khovanskii subspaces of $\Bbbk^6$.

We finish with a remark that the tropical moduli space of cubic surfaces is not sufficient for one more reason: it requires being enlarged by adding a lineality space to the fan.

Example 6.5 Consider

$$G'' = \begin{bmatrix} 1 & t & t^8 & t^3 & t^9 & 1 \\ t^7 & t^7 & t & t^7 & t^6 & 1 \\ t^5 & 1 & t^5 & t^9 & t^{11} & t^6 \end{bmatrix}$$

which comes from G by multiplying the first column by $1/t^4$. Note that if we treat columns of a matrix as coordinates of points in $\mathbb{P}^2$, then G and G'' represent the same choice of six points, so the same marked del Pezzo surface. Note also that if we looked at the kernels of G and G'' as choices of six points in $\mathbb{P}^2$, we would also get the same sets, because multiplying the first column of G by $1/t^4$ corresponds to multiplying the first row of a matrix representing $\ker G$ by t^4. However, one can compute the conic generator G_6 for G'' and learn that it has a binomial leading term, so G'' is not moneric.

This shows that the property of being moneric is not well-defined for a marked del Pezzo surface—its behaviour varies in the set of matrices representing the same choice of six points on the plane. The same phenomenon can be observed also in the case of degree 4 del Pezzo surfaces, where $\mathrm{TGr}(2, \mathbb{Q}^5)$ was used to parametrize moneric subspaces. This is one of the reasons for considering the full $\mathrm{TGr}(2, \mathbb{Q}^5)$, not only the tropicalization of the moduli space of degree 4 del Pezzo surfaces, i.e. the cone over the Petersen graph. The lineality space is equally important. The subdivision determining equivalence classes of moneric subspaces is not a pull-back of a subdivision of the cone over the Petersen graph to $\mathrm{TGr}(2, \mathbb{Q}^5)$ via the projection along the lineality space, it cuts through fibres of this projection. Thus, by analogy, we expect that to use some variant of the tropical moduli space of del Pezzo surfaces of degree 3 to parametrize three-dimensional moneric subspaces of $\mathbb{k}^6$ we should also enlarge it by adding a lineality space.

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Equations and Tropicalization of Enriques Surfaces

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Abstract In this article, we explicitly compute equations of an Enriques surface via the involution on a K3 surface. We also discuss its tropicalization and compute the tropical homology, thus recovering a special case of the result of [18], and establish a connection between the dimension of the tropical homology groups and the Hodge numbers of the corresponding algebraic Enriques surface.

MSC 2010 codes: 14T05, 14J28, 14N10

1 Introduction

In the classification of algebraic surfaces, Enriques surfaces comprise one of four types of minimal surfaces of Kodaira dimension 0. There are a number of surveys on Enriques surfaces. For those new to the theory, we recommend the excellent exposition found in [2] and [3], and for a more thorough treatment, the book [10]. Another recommended source is Dolgachev's brief introduction to Enriques surfaces [11].

The first Enriques surface was constructed in 1896 by Enriques himself [12] to answer negatively a question posed by Castelnuovo (1895): *Is every surface with* $p_g = q = 0$ *rational?*; see Sect. 2 for the meaning of p_q and q. Enriques' original surface has a beautiful geometric construction: the normalization of a degree 6 surface in $\mathbb{P}^3$ with double lines given by the edges of a tetrahedron. Another

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construction, the Reye congruence, defined a few years earlier by Reye [25], was later proved by Fano [13] to be an Enriques surface. Since these first constructions, there have been many examples of Enriques surfaces, most often as quotients of K3 surfaces by a fixed-point-free involution. In [9], Cossec describes all birational models of Enriques surfaces given by complete linear systems.

As we recall in Sect. 2, every Enriques surface has an unramified double cover given by a K3 surface. Often exploiting this double cover, topics of particular interest relate to lattice theory, moduli spaces and their compactifications, automorphism groups of Enriques surfaces, and Enriques surfaces in characteristic 2. While there are many constructions of Enriques surfaces, none give explicit equations for an Enriques surface embedded in a projective space. In this paper, by interpreting the work of Cossec–Verra, we give explicit ideals for all Enriques surfaces.

Theorem 1.1 *Let Y be the toric fivefold of degree 16 in $\mathbb{P}^{11}$ that is obtained by taking the join of the Veronese surface in $\mathbb{P}^5$ with itself. The intersection of Y with a general linear subspace of codimension 3 is an Enriques surface, and every Enriques surface arises in this way.*

By construction, the Enriques surface in Theorem 1.1 is arithmetically Cohen–Macaulay. Its homogeneous prime ideal in the polynomial ring with 12 variables is generated by the twelve binomial quadrics that define Y and three additional linear forms. Code for producing this Enriques surface in *Macaulay2* appears in Sect. 3.

After having constructed Enriques surfaces explicitly, we focus on their tropicalizations, with the purpose of studying their combinatorial properties. For this we choose a different K3 surface, namely a hypersurface $S \subset (\mathbb{P}^1)^3$ with an involution σ, see Example 4.3. We get a fairly complete picture for its tropicalization (Fig. 1). In particular, we recover its Hodge numbers and, conjecturally, the Hodge numbers of S/σ, which was [28, Problem 10 on Surfaces]; this was the starting point of this work.

Proposition 1.2 (Example 4.3, Propositions 5.7–5.8) *The dimensions of tropical homology groups of the tropicalization of the K3 surface S agree with the Hodge numbers of S. The dimensions of the σ-invariant parts of tropical homology groups agree with the Hodge numbers of the Enriques surface S/σ.*

Finally, we discuss an analogue of Castelnuovo's question on the tropical and analytic level. Since the analytifications of rational varieties are contractible by

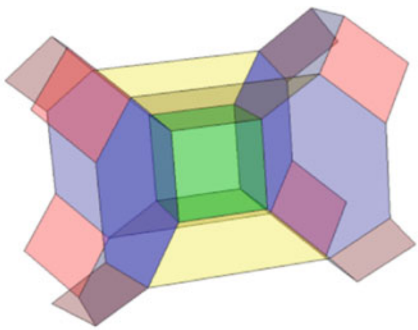

Fig. 1 A tropical K3 surface in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ that is fixed under the involution

Corollary 1.1.4 in [6], we ask: are the analytifications of K3 or Enriques surfaces contractible? We give a negative answer to this question, the counterexample being the analytification of S from Example 4.3.

Theorem 1.3 *The analytification S^{an} of the K3 surface S is homotopy equivalent to a two-dimensional sphere. The surface S has a fixed-point-free involution σ and the analytification of the Enriques surface S/σ retracts onto the real projective plane $\mathbb{RP}^2$. In particular, neither S^{an} nor $(S/\sigma)^{\mathrm{an}}$ is contractible.*

The contents of the paper are as follows. In Sect. 2, we give some background about Enriques surfaces. Next, in Sect. 3, we exploit a classical construction to obtain an Enriques ideal in a codimension 3 linear space in $\mathbb{P}^{11}$ and prove Theorem 1.1. In Sect. 4, we discuss the basics of tropical geometry and analytic spaces in the sense of Berkovich. Example 4.3 provides an Enriques surface S/σ arising from a K3 surface $S \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ with an involution σ. The surface S is suitable from the tropical point of view (its tropical variety is schön and multiplicity one everywhere) and is used throughout the paper. In Sect. 5, we compute the tropical homology groups of $\operatorname{trop}(S)$ and, conjecturally, of $\operatorname{trop}(S/\sigma)$. We also prove Proposition 1.2. In Sect. 6, we discuss the topology of analytifications of S and S/σ and prove Theorem 1.3.

2 Background

Apart from the code snippets, we work over an algebraically closed field of characteristic zero. An *Enriques surface* X is a smooth projective surface such that $q(X) := h^1(X, \mathscr{O}_X) = 0$, $\omega_X^{\otimes 2} \simeq \mathscr{O}_X$ and $\omega_X \not\simeq \mathscr{O}_X$, where $\omega_X = \bigwedge^2 \Omega_X^1$ is the canonical bundle of X. It follows that X is minimal, see [3], and its geometric genus is $p_g(X) := h^2(X, \mathscr{O}_X) = 0$. Enriques surfaces are defined the same way over any field of characteristic other than 2. By Lemma 15.1 in [2], the Hodge diamond of an Enriques surface X appears in Fig. 2. An Enriques surface admits an unramified double cover $f\colon Y \to X$, where Y is a K3 surface, see [2, Lemma 15.1] or [3, Proposition VIII.17]. The Hodge diamond of Y appears in Fig. 3. Since Y is simply connected, the fundamental group of an Enriques surface is $\mathbb{Z}/2\mathbb{Z}$; see [2, Sect. 15]. The cover $Y \to X$ is a quotient of Y by an involution σ that exchanges the two points of each fibre. Conversely, for a K3 surface Y with a fixed-point-free involution σ, the quotient Y/σ is an Enriques surface.

Fig. 2 Hodge diamond of an Enriques surface

$$\begin{array}{ccccc} & & h^{0,0} & & \\ & h^{1,0} & & h^{0,1} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{2,1} & & h^{1,2} & \\ & & h^{2,2} & & \end{array} = \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 10 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Fig. 3 Hodge diamond of a K3 surface

$$\begin{matrix} & & h^{0,0} & & \\ & h^{1,0} & & h^{0,1} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{2,1} & & h^{1,2} & \\ & & h^{2,2} & & \end{matrix} = \begin{matrix} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{matrix}$$

3 Enriques Surfaces via K3 Complete Intersections in $\mathbb{P}^5$

In this section, we construct Enriques surfaces via K3 surfaces in $\mathbb{P}^5$. One cannot hope for especially simple equations—for instance, an Enriques surface cannot be a hypersurface in $\mathbb{P}^3$.

Proposition 3.1 *If $X \subset \mathbb{P}^N_{\mathbb{C}}$ is a smooth toric threefold and $S = X \cap H$ is a smooth hyperplane section, then S is simply connected, and is not an Enriques surface.*

Proof Since X is a smooth projective toric variety, it is simply connected; see [14, Sect. 3.2]. A homotopical version of Lefschetz' theorem asserts that the fundamental groups of $X \cap H$ and X are isomorphic via the natural map; see [1] and [4, 2.3.10]. Thus, S is simply connected. Since an Enriques surface admits a non-trivial étale double cover, it is never simply connected. □

Remark 3.2 This proof generalizes to other complete intersections inside smooth toric varieties, provided that intermediate complete intersections are smooth.

Following [3, Example VIII.18], we construct an Enriques surface from a K3 surface that is an intersection of quadrics in $\mathbb{P}^5$. Fix $\mathbb{P}^5 := \operatorname{Proj}(\mathbb{C}[x_0, x_1, x_2, y_0, y_1, y_2])$. The fixed point set of the involution $\sigma : \mathbb{P}^5 \to \mathbb{P}^5$ given by $\sigma(x_i) = x_i$ and $\sigma(y_i) = -y_i$, for all $0 \le i \le 2$, is equal to the union of $\mathbb{P}^2 = \mathrm{V}(y_0, y_1, y_2)$ and $\mathbb{P}^2 = \mathrm{V}(x_0, x_1, x_2)$. Fix quadrics $F_i \in \mathbb{C}[x_0, x_1, x_2]$ and $G_i \in \mathbb{C}[y_0, y_1, y_2]$, where $0 \le i \le 2$ and set $Q_i := F_i + G_i$. By construction, these quadrics are fixed by σ. Choose Q_0, Q_1, Q_2 so that they form a complete intersection. For the surface $S = S_{\mathbf{Q}} := \mathrm{V}(Q_0, Q_1, Q_2)$, the Adjunction Formula gives $K_S = \mathscr{O}_S(-6+2+2+2) = \mathscr{O}_S$. Since the surface S is a complete intersection of quadrics in $\mathbb{P}^5$, it follows that $h^1(\mathscr{O}_S) = 0$; see [3, Lemma VIII.9]. Thus, if S is smooth, then it is a K3 surface fixed under the involution σ. We now formalize exactly which assumptions must be satisfied by the three quadrics to obtain a smooth Enriques surface.

Definition 3.3 Let $\mathbf{Q} := (Q_0, Q_1, Q_2)$ be a triple of quadrics where $Q_i := F_i + G_i$ for some $F_i \in \mathbb{C}[x_0, x_1, x_2]$ and $G_i \in \mathbb{C}[y_0, y_1, y_2]$. We say that the quadrics $\mathbf{Q}$ are *enriquogeneous* if the following conditions are satisfied:

1. the forms $\mathbf{Q} = (Q_0, Q_1, Q_2)$ are a complete intersection,
2. the surface $S = \mathrm{V}(Q_0, Q_1, Q_2)$ is smooth,
3. the surface $S = \mathrm{V}(Q_0, Q_1, Q_2)$ does not intersect the fixed-point set of σ.

The third condition is equivalent to F_0, F_1, F_2 having no common zeros in $\mathbb{C}[x_0, x_1, x_2]$ and G_0, G_1, G_2 having no common zeros in $\mathbb{C}[y_0, y_1, y_2]$, so it is an open condition. For a choice of enriquogeneous quadrics $\mathbf{Q}$, we obtain an Enriques surface as $S_{\mathbf{Q}}/\sigma$. The set of enriquogeneous quadrics is open inside $(\mathbb{A}^{6+6})^3$, so that a *general* choice of forms gives an Enriques surface. In [9], Cossec shows that every complex Enriques surface may be obtained in this way if one allows $\mathbf{Q}$ not satisfying the smoothness condition; see also [30]. Notably, Lietdke proves that the same is true for Enriques surfaces over any characteristic [21]. To develop our intuition, we demonstrate that, over $\mathbb{C}$, these surfaces give at most a ten-dimensional space of Enriques surfaces.

Each Q_i is chosen from the same 12-dimensional affine space and $S_{\mathbf{Q}}$ depends only on their span, which is an element of the 27-dimensional variety $\mathrm{Gr}\left(3, \mathbb{C}^{12}\right)$. Since we have fixed σ, the quadrics Q_i yield an isomorphic K3 surface (with an isomorphic involution) if we act on $\mathbb{P}^5$ by an automorphism that commutes with σ. Such automorphisms are given by block matrices in $PGL(6)$ of the form

$$C = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \qquad \text{or} \qquad C = \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$$

where A and B are matrices in $\mathrm{GL}(3, \mathbb{C})$, up to scaling. Thus, the space of automorphisms preserving the σ-invariant quadrics has dimension $(2)(9) - 1 = 17$. Modulo these automorphisms, we obtain a ten-dimensional projective space of K3 surfaces with an involution. The condition that $\mathbf{Q}$ be enriquogeneous is an open condition, so the space of Enriques surfaces is also ten-dimensional.

We now aim to make the Enriques surfaces constructed as $S_{\mathbf{Q}}/\sigma$ explicit. In other words, we want to present them as embedded into a projective space. The first step is to identify the quotient of $\mathbb{P}^5$ by the involution σ. Let $S = \mathbb{C}[x_0, x_1, x_2, y_0, y_1, y_2]$ be the homogeneous coordinate ring, so the quotient is $\mathrm{Proj}\,(S^\sigma) = \mathrm{Proj}\left(\mathbb{C}[x_i, y_i y_j]\right)$. The Enriques surface $S_{\mathbf{Q}}$ is cut out of $\mathrm{Proj}\left(\mathbb{C}[x_i, y_i y_j]\right)$ by the quadrics $\mathbf{Q}$ such that $S_{\mathbf{Q}} = \mathrm{Proj}\left(\mathbb{C}[x_i, y_i y_j]/\mathbf{Q}\right)$. This does not give us an embedding into $\mathbb{P}^8$, because the variables x_i and $y_i y_j$ have different degrees. Rather we obtain an embedding into a weighted projective space $\mathbb{P}(1^3, 2^6)$. Therefore, we replace $\mathbb{C}[x_i, y_i y_j]$ by the Veronese subalgebra $\mathbb{C}[x_i x_j, y_i y_j]/\mathbf{Q}$. This algebra is generated by the 12 elements $x_i x_j$, $y_i y_j$, where $0 \le i, j \le 2$, which implies that $S_{\mathbf{Q}}$ is embedded into a $\mathbb{P}^{11}$. The relations $\mathbf{Q}$ are *linear* in the variables $x_i x_j$ and $y_i y_j$, so $S_{\mathbf{Q}}$ is embedded into a $\mathbb{P}^8$.

Let us rephrase this geometrically. Consider the second Veronese embedding $v\colon \mathbb{P}^5 \to \mathbb{P}^{20}$. The coordinates of $\mathbb{P}^{20}$ are forms of degree two in x_i and y_i. The involution σ extends to an involution on $\mathbb{P}^{20}$ and the invariant coordinate ring is generated by the *linear forms* corresponding to the products $x_i x_j$ and $y_i y_j$. Thus, the quotient is embedded into $\mathbb{P}^{11}$; we have

$$\begin{array}{ccc} \mathbb{P}^5 & \xrightarrow{\;v\;} & \mathbb{P}^{20} \\ & \searrow^{\pi} & \downarrow \\ & & \mathbb{P}^{11} \end{array}$$

where π denotes the quotient by the involution σ. The image $\pi(\mathbb{P}^5)$ is cut out by 12 binomial quadrics: the six usual equations between $x_i x_j$ and the six corresponding equations for $y_i y_j$. It is the join of two Veronese surfaces which constitute its singular locus. Quadrics in $\mathbb{C}[x_i, y_i]$ of the form $F_i + G_i$ for $F_i \in \mathbb{C}[x_i]$ and $G_i \in \mathbb{C}[y_i]$ correspond bijectively to linear forms on the above $\mathbb{P}^{11}$. A choice of enriquogeneous quadrics $\mathbf{Q}$ corresponds to a general choice of three linear forms on $\mathbb{P}^{11}$. We obtain the corresponding Enriques surface $S_{\mathbf{Q}}$ as a linear section of $\pi(\mathbb{P}^5)$. Summing up, we have the chain of inclusions $V \cap \pi(\mathbb{P}^5) \subset \pi(\mathbb{P}^5) \subset \mathbb{P}(1^3, 2^6) \subset \mathbb{P}^{11}$, where V is a codimension three linear section. Although $V \cap \pi(\mathbb{P}^5)$ is a complete intersection in $\pi(\mathbb{P}^5)$, this does not contradict (the natural generalisation of) Proposition 3.1, because $\pi(\mathbb{P}^5)$ is singular. Since sufficiently ample embeddings of varieties are always cut out by quadrics, see [23, 27], this suggests that our embedding is sufficiently good.

Proof of Theorem 1.1 The surfaces obtained from enriquogeneous quadrics are arithmetically Cohen–Macaulay of degree 16 as they are linear sections of $\pi(\mathbb{P}^5)$ possessing those properties. Every Enriques surface can be obtained by this procedure if one allows $\mathbf{Q}$ not satisfying the smoothness condition by [9]. □

We provide *Macaulay2* [15] code for finding the equations of $S_{\mathbf{Q}}$. To simplify the computation, we work over a finite field.

```
kk = ZZ/1009;
P5 = kk[x0,x1,x2,y0,y1,y2];
P11 = kk[z0,z1,z2,z3,z4,z5,z6,z7,z8,z9,z10,z11];
pii = map(P5, P11, {x0^2, x0*x1, x0*x2, x1^2,
                    x1*x2, x2^2, y0^2, y0*y1,
                    y0*y2, y1^2, y1*y2, y2^2});
```

The kernel of `pii` is generated by 12 binomial quadrics and has degree 16.

```
assert(kernel pii == ideal(
      z10^2-z9*z11, z8*z10-z7*z11, z8*z9-z7*z10,
      z8^2-z6*z11,  z7*z8-z6*z10,  z7^2-z6*z9,
      z4^2-z3*z5,   z2*z4-z1*z5,   z2*z3-z1*z4,
      z2^2-z0*z5,   z1*z2-z0*z4,   z1^2-z0*z3))
assert(degree kernel pii == 16)
```

We next generate an Enriques surface from a random set of linear forms named `linForms`. To see the quadrics in $\mathbb{P}^5$, compute `pii(linForms)`.

```
linForms = random(P11^3, P11^{-1})
```

```
randomEnriques = (kernel pii) + ideal linForms
```

We now verify that this is in fact an Enriques surface. Computationally, it is much easier to check this for the associated K3 surface, because we need only check that `K3` is a smooth surface (first two assertions below) and that the involution is fixed-point-free on `K3` (last two assertions).

```
K3 = ideal pii(linForms)
assert (dim K3 == 3)
assert (dim saturate ideal singularLocus K3 == -1)
```

```
assert (dim saturate (K3 + ideal(y0,y1,y2)) == -1)
assert (dim saturate (K3 + ideal(x0,x1,x2)) == -1)
```

If the `K3` passes all the assertions, then `randomEnriques` is an Enriques surface. Its ideal is given by 12 binomial quadrics listed above and three linear forms in `P11`.

Example 3.4 Over $\Bbbk = \mathbb{F}_{1009}$, the choice of

```
linForms = matrix{{2*z2+z6+5*z7+8*z11,
                   2*z0+8*z4+z9,
                   5*z1+4*z3+4*z5+6*z8}}
```

in the above algorithm gives an Enriques surface.

Finally, we check that $\pi(\mathbb{P}^5)$ is arithmetically Cohen-Macaulay. Using `betti res kernel pii`, we obtain its Betti table.

1	.	.	.	.	.	.
.	12	16	6	.	.	.
.	.	36	96	100	48	9

The projective dimension of $\pi(\mathbb{P}^5)$ (the number of columns) is equal to the codimension, thus $\pi(\mathbb{P}^5) \subset \mathbb{P}^{11}$ is arithmetically Cohen-Macaulay; see [26, Sect. 10.2]. Therefore, all its linear sections are also arithmetically Cohen-Macaulay.

4 Analytified and Tropical Enriques Surfaces

This section discusses the basics of tropical and analytic geometry and constructs a K3 surface whose tropicalization is nice enough for computations of tropical homology. In Example 4.3, we present a K3 surface with an involution, which on the tropical side is the antipodal map. As an excellent reference for tropical varieties, we recommend [22], especially Sect. 6.2. For analytic spaces in the sense of Berkovich, we recommend [5, 17].

Let $\Bbbk$ be a field extension of $\mathbb{C}$ with a nontrivial valuation $\mathrm{val}\colon \Bbbk^* \to \mathbb{R}$ such that $\mathrm{val}(\mathbb{C}^*) = \{0\}$. We assume that $\Bbbk$ is algebraically closed, so the image $\mathrm{val}(\Bbbk^*)$ is dense in $\mathbb{R}$. Without much loss of generality, one could simply consider the field $\Bbbk = \mathbb{C}\{\{z\}\} = \bigcup_{n\in\mathbb{N}} \mathbb{C}((z^{1/n}))$ of Puiseux series, with valuation yielding the lowest exponent of z appearing in the series. For every point $p = (p_1, p_2, \ldots, p_n) \in (\Bbbk^*)^n$, its valuation is $\mathrm{val}(p) = \big(\mathrm{val}(p_1), \mathrm{val}(p_2), \ldots, \mathrm{val}(p_n)\big)$.

Definition 4.1 Let X be a toric variety with torus $(\Bbbk^*)^n$ and $Y \subset X$ be a closed subvariety. The tropical variety of Y, denoted by $\mathrm{trop}(Y \subset X)$ or briefly $\mathrm{trop}(Y)$, is the closure of the set $\{\mathrm{val}(p) : p \in (\Bbbk^*)^n \cap Y\} \subset \mathbb{R}^n$.
The tropical variety $\mathrm{trop}(Y \subset X)$ is a polyhedral complex of dimension $\dim Y$ with rich combinatorial structure; see [22, Chapter 3].

A morphism of tori $\varphi\colon (\Bbbk^*)^n \to (\Bbbk^*)^m$ is given by $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_m)$ where $\varphi_i(t) = b_i \cdot t^{a_i}$ for $1 \le i \le m$. For each such φ, there is a *tropicalized map*

$\mathrm{trop}(\varphi)\colon \mathbb{R}^n \to \mathbb{R}^m$ given by $\mathrm{trop}(\varphi)_i(v) = \mathrm{val}(b_i) + (a_i \cdot v)$ for $1 \le i \le m$. One verifies that the following diagram commutes:

$$\begin{array}{ccc} (\Bbbk^*)^n & \xrightarrow{\varphi} & (\Bbbk^*)^m \\ \downarrow \mathrm{val} & & \downarrow \mathrm{val} \\ \mathbb{R}^n & \xrightarrow{\mathrm{trop}\,(\varphi)} & \mathbb{R}^m \end{array}$$

This naive tropicalization is not a functor—it is known how to tropicalize a map only when it is monomial. This problem is solved by passing to Berkovich spaces. We will not discuss Berkovich spaces in detail: we invite the reader to see [5, 17] or [24] for a slightly more elementary introduction.

For every finite-type scheme X over a valued field $\Bbbk$, its Berkovich analytification X^{an} is the analytic space which best approximates X; see [5, Chap. 3]. The space X^{an} is locally ringed (in the usual sense, see [29, 4.3.6]) and there is a morphism $\pi\colon X^{\mathrm{an}} \to X$ such that every other map from an analytic space factors through π. If $X = \operatorname{Spec} A$ is affine, then the points of X^{an} are in bijection with the multiplicative semi-norms on A which extend the norm on $\Bbbk$. Most importantly, the analytification is *functorial*: for every map $f\colon X \to Y$, we get an induced map $f^{\mathrm{an}}\colon X^{\mathrm{an}} \to Y^{\mathrm{an}}$. If $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ are affine, then f induces $f^{\#}\colon B \to A$ and the map f^{an} takes a seminorm $|\cdot|$ on A to the seminorm $b \to |f^{\#}(b)|$ on B.

The analytification of an affine variety X is the limit of its tropicalizations by [24]. To be more precise, let X be an affine variety. For two embeddings $i\colon X \to \mathbb{A}^n$ and $j\colon X \to \mathbb{A}^m$, and a toric morphism $\varphi\colon \mathbb{A}^n \to \mathbb{A}^m$ satisfying $j = \varphi \circ i$, we obtain a tropicalized map $\mathrm{trop}(X \subset \mathbb{A}^n) \to \mathrm{trop}(X \subset \mathbb{A}^m)$. For every embedding $X \subset \mathbb{A}^n$, there is an associated map $X^{\mathrm{an}} \to \mathrm{trop}(X \subset \mathbb{A}^n)$, sending a multiplicative seminorm $|\cdot|$ to the valuation $-\log|\cdot|$, see [24, p. 544]. The main result in [24] is that the inverse limit is homeomorphic to the Berkovich analytification. Hence, one has $X^{\mathrm{an}} = \varprojlim \mathrm{trop}(X \subset \mathbb{A}^n)$.

We now return to the case of Enriques surfaces. We are interested in finding an Enriques surface S/σ with a K3 cover S suitable for tropicalization. Specifically, we would like σ to be an involution acting without fixed points on the tropical side. In this sense, the examples obtained as in Sect. 3 are not suitable.

Example 4.2 Consider the K3 surface $S_{\mathbf{Q}}$ defined via the enriquogeneous quadrics in Sect. 3 with $\sigma(x_0, x_1, x_2, y_0, y_1, y_2) = (x_0, x_1, x_2, -y_0, -y_1, -y_2)$. Since we have $\mathrm{val}(-1) = 0$, the tropicalized involution $\mathrm{trop}(\sigma)$ is the identity map on $\mathbb{R}^6$.

To obtain a K3 surface with an involution σ tropicalizing to a fixed-point-free involution, we consider embeddings into products of $\mathbb{P}^1$. Consider the involution $\tau\colon \mathbb{P}^1 \to \mathbb{P}^1$ given by $\tau([x:y]) = [y:x]$ and the involution $\sigma\colon (\mathbb{P}^1)^3 \to (\mathbb{P}^1)^3$ given by applying τ to every coordinate. The map τ restricts to the torus $\mathbb{C}^*$ and is given by $\mathbb{C}^* \ni t \to t^{-1} \in \mathbb{C}^*$. Therefore, we have $\mathrm{trop}(\tau)(v) = -v$. Consequently, the tropicalization $\mathrm{trop}(\sigma)\colon \mathbb{R}^3 \to \mathbb{R}^3$ is given by $\mathrm{trop}(\sigma)(v) = -v$. This map is non-trivial and has only one fixed point.

Example 4.3 (A K3 Surface with a Fixed-Point-Free Involution) Let $S \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ be a smooth surface given by a section of the anticanonical divisor of $(\mathbb{P}^1)^3$, namely a triquadratic polynomial. The Newton polytope of S is the three-dimensional cube $[0, 2]^3$. We introduce the following assumptions on S:

1. S is smooth;
2. S is invariant under the involution σ; and
3. the subdivision induced by S on its Newton polytope $[0, 2]^3$ is a unimodular triangulation, that is, the polytopes in the triangulation are tetrahedra of volume equal to $1/6$; see [22, p. 13].

Each such S is a K3 surface. Under our assumptions, the point $(0, 0, 0)$ is not in the tropical variety of S. Indeed, if it were in $\mathrm{trop}(S)$, that variety would not be locally linear at $(0, 0, 0)$. But $\mathrm{trop}(S)$ is coming from a unimodular triangulation, so it is locally linear everywhere. Hence, the point $(0, 0, 0)$ is outside and $\mathrm{trop}(\sigma)$ is a fixed-point-free involution on $\mathrm{trop}(S)$. The map $\sigma : S \to S$ induces also an involution $\sigma^{\mathrm{an}} : S^{\mathrm{an}} \to S^{\mathrm{an}}$ which is compatible with $\mathrm{trop}(\sigma)$ under the projection π; the following diagram commutes.

$$\begin{array}{ccc} S^{\mathrm{an}} & \xrightarrow{\sigma^{\mathrm{an}}} & S^{\mathrm{an}} \\ \downarrow \pi & & \downarrow \pi \\ \mathrm{trop}(S) & \xrightarrow{\mathrm{trop}(\sigma)} & \mathrm{trop}(S) \end{array}$$

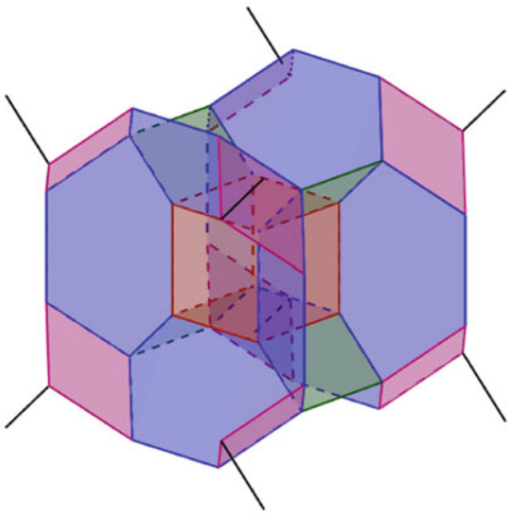

Fig. 4 The bounded part of a tropical K3 surface in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$

5 The Tropical Homology

In this section, we explicitly calculate the tropical homology of a tropical K3 surface and a tropical Enriques surface. We use the construction in Example 4.3 in order to obtain tropicalizations which are locally linear (locally look like tropicalizations of linear spaces), and then compute their tropical cohomology groups. In accordance with the results in [18], the dimensions of such homology groups coincide with the Hodge numbers of the surfaces themselves. We carry out the calculation by hand for some curves, a tropical K3 surface and also for an object, which we believe to be the associated tropical Enriques surface; see [20] for computation of tropical homology using *polymake*.

Theorem 5.1 ([18, Special Case of Theorem 2]) *If $X \subset \mathbb{P}^N$, and its tropicalization* $\mathrm{trop}(X) \subset \mathrm{trop}(\mathbb{P}^N)$ *has multiplicities all equal to* 1 *and is locally linear, then the tropical Hodge numbers agree with Hodge numbers of X:* $\dim H_{p,q}(\mathrm{trop}(X)) = \dim H^{p,q}(X, \mathbb{R})$.

For the definition of multiplicities, we refer to [22, Chap. 3]. A tropical variety is *locally linear* if a Euclidean neighbourhood of each point is isomorphic to a Euclidean open subset of the tropicalization of a linear subspace $\mathbb{P}^n \subset \mathbb{P}^m$; see [31]. A hypersurface in $\mathbb{P}^N$ is locally linear if and only if the subdivision of its Newton polygon is a triangulation. It has multiplicities one if and only if this triangulation is unimodular.

In Theorem 5.1, we do not assume that X intersects the torus of $\mathbb{P}^N$. Therefore, this theorem applies to $X \subset (\mathbb{P}^1)^3 \subset \mathbb{P}^7$ or more generally to X in any projective toric variety with fixed embedding. Moreover, one might wonder whether Theorem 5.1 identifies not only dimensions but homology classes. This is possible when the appropriate spectral sequence degenerates at the E_2 page. This E_2 page is $H^q(X, \mathscr{F}^p)$, where $\mathscr{F}^p = \mathrm{Hom}(\mathscr{F}_p, \mathbb{R})$; see the discussion after Corollary 2 in [18] or [8].

We provide generalities about tropical homology and compute some examples of interest; for a more detailed introduction see [7, 18]. We compute the dimensions of the tropical homology groups and show how Theorem 5.1 holds. The last part of the paper is dedicated to showing a particular instance of this theorem for a special tropical K3 surface with involution and for its quotient.

Tropical projective space $\mathrm{trop}(\mathbb{P}^n) = \mathbb{TP}^n$ is homeomorphic to an n-simplex; see [22, Chap. 6.2]. It is covered by $n+1$ copies of $\mathbb{T}^n = \mathrm{trop}(\mathbb{A}^n) = (\{-\infty\} \cup \mathbb{R})^n$, that are complements of torus invariant divisors. Let X be a tropical subvariety of $\mathbb{TP}^n$. The definitions of sheaves $\mathscr{F}_p$ and groups $C_{p,q}$ computing the homology are all local, so we assume that $X \subset \mathrm{trop}(\mathbb{A}^n)$ is contained in one of the distinguished open subsets. We denote by $\mathbb{T}^J = \{x \in \mathbb{T}^n : x_i = -\infty \text{ for all } i \notin J\}$, for $J \subseteq \{1, 2, \dots, n\}$, the tropicalization of smaller torus orbits. Let $X \in \mathbb{T}^n$ be a polyhedral complex. The *sedentarity* $I(x)$ of a point $x \in X$ is the set of coordinates of x which are equal to $-\infty$, and we set $J(x) := \{1, \dots, n\} \setminus I(x)$. We denote by $\mathbb{R}^{J(x)} = \mathbb{R}^n/\mathbb{R}^{I(x)}$ the interior of $\mathbb{T}^{J(x)}$. For a face $E \subset X \cap \mathbb{R}^{J(x)}$ adjacent to x, we let $T_x(E) \subset T_x(\mathbb{R}^{J(x)})$ be the cone spanned by the tangent vectors to E starting at x and directed towards E. Fix the following terminology:

1. The *tropical tangent space* $\mathscr{F}_1(x) \subset T_x(\mathbb{R}^{J(x)})$ is the vector space generated by all $T_x(E)$ for all E adjacent faces to x;
2. The *tropical multitangent space* $\mathscr{F}_p(x) \subset \bigwedge^p T_x(\mathbb{R}^{J(x)})$ is the vector space generated by all vectors of the form $v_1 \wedge v_2 \wedge \cdots \wedge v_p$ where $v_1, v_2, \ldots, v_p \in T_x(E)$ for all E adjacent faces to x (this implies $\mathscr{F}_0(x) \cong \mathbb{R}$)

The multitangent vector space $\mathscr{F}_p(x)$ for $x \in X$ only depends on the minimal face $\Delta \subset X$ containing x. Hence, we can write $\mathscr{F}_p(\Delta) := \mathscr{F}_p(x)$ for each $x \in \Delta$. We have the following group of (p, q)-chains

$$C_{p,q}(X) := \bigoplus_{\Delta\ q-\text{dim face of } X} \mathscr{F}_p(\Delta)$$

giving rise to the chain complex

$$C_{p,\bullet} = \cdots \longrightarrow C_{p,q+1}(X) \xrightarrow{\partial} C_{p,q}(X) \xrightarrow{\partial} C_{p,q-1}(X) \longrightarrow \cdots$$

where the differential ∂ is the usual simplicial differential (we choose orientation for each face) composed with inclusion maps given by $\iota \colon \mathscr{F}_p(\Delta) \to \mathscr{F}_p(\Delta')$ for $\Delta' \prec \Delta$. Even when Δ' and Δ have different sedentarities, we have $I(\Delta') \supset I(\Delta)$ so we get a natural map $\mathbb{R}^{J(x)} = \mathbb{R}^n/\mathbb{R}^{I(x)} \twoheadrightarrow \mathbb{R}^n/\mathbb{R}^{I(x')} = \mathbb{R}^{J(x')}$ inducing the map $\iota \colon \mathscr{F}_p(\Delta) \to \mathscr{F}_p(\Delta')$.

Definition 5.2 The *(p, q)th tropical homology group* $H_{p,q}(X)$ of X is the qth homology group of the complex $C_{p,\bullet}$.

In the light of Theorem 5.1, if $X = \text{trop}(X')$ is a tropicalization of suitable variety X', then $\dim H_{p,q}(X)$ are the Hodge numbers of X'. For all X, the tropical Poincaré duality holds: $\dim H_{d-p,d-q}(X) = \dim H_{p,q}(X)$, see [19].

Example 5.3 Let's compute the tropical homology of a tropical line L; see Fig. 5.

$p = 0$: From the discussion above, we see that $C_{0,0}(L) = \mathbb{R}^4$ and $C_{0,1} = \mathbb{R}^3$ injects into $C_{0,0}$. Thus, we have $\dim H_{0,0}(X) = 1$ and $H_{0,1}(X) = 0$.

$p = 1$: The chain complex is $0 \to C_{1,1}(X) \to C_{1,0}(X) \to 0$. As in the previous case, we see that $C_{1,0}(X) = \mathscr{F}_1(v_1) = \mathbb{R}\langle e_1, e_2\rangle$, where $e_1 = (-1, 0)$ and $e_2 = (0, -1)$ are the standard basis vectors of $\mathbb{R}^2$ up to a sign. Moreover, we have

$$C_{1,1}(X) = \mathscr{F}_1(p) \oplus \mathscr{F}_1(q) \oplus \mathscr{F}_1(r) = \mathbb{R}\langle e_1\rangle \oplus \mathbb{R}\langle e_2\rangle \oplus \mathbb{R}\langle -e_1 - e_2\rangle\,.$$

The differential $\mathbb{R}\langle e_1\rangle \oplus \mathbb{R}\langle e_2\rangle \oplus \mathbb{R}\langle -e_1 - e_2\rangle \xrightarrow{\partial} \mathbb{R}\langle e_1, e_2\rangle$ is given by the natural inclusion $e_1 \mapsto e_1$, $e_2 \mapsto e_2$, and $-e_1 - e_2 \mapsto -e_1 - e_2$. Hence, the kernel of the differential is one-dimensional, generated by the sum $\langle e_1\rangle + \langle e_2\rangle + \langle -e_1 - e_2\rangle$, so we conclude that $\dim H_{1,0}(X) = 0$ and $\dim H_{1,1}(X) = 1$.

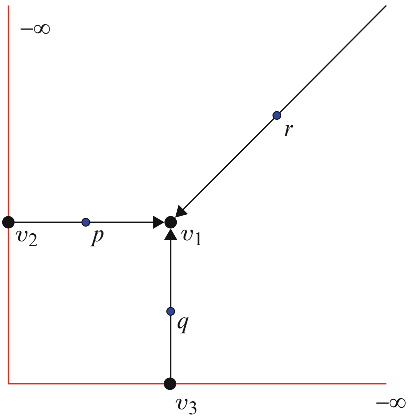

Fig. 5 A tropical line

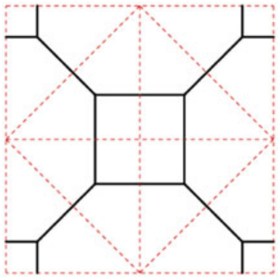

Fig. 6 A tropical elliptic curve in $\mathbb{P}^1 \times \mathbb{P}^1$

Remark 5.4 By definition, we have $\mathscr{F}_0(x) = \mathbb{R}$, so the complex $C_{0,\bullet}$ is the singular homology complex for the subdivision of X by polyhedra. Thus, the tropical homology group $H_{0,q}(X)$ is identified with the singular homology group $H_q(X, \mathbb{R})$.

Example 5.5 (Elliptic Curve) We next compute the tropical homology of an elliptic curve in $\mathbb{P}^1 \times \mathbb{P}^1$. Its tropicalization is shown in Fig. 6. From the isomorphism $H_{0,q}(X) \cong H_q(X, \mathbb{R})$, it follows that $H_{0,0}(X) \cong \mathbb{R}$ and $H_{0,1}(X) \cong \mathbb{R}$. We can compute $H_{1,1}(X)$ directly from the complex $C_{1,1}(X) \to C_{1,0}(X)$. It follows that $C_{1,1}(X) \cong \mathbb{R}^E$ and $C_{1,0}(X) \cong \mathbb{R}^{2V}$, where $E = 16$ (respectively, $V = 8$) denotes the number of edges (respectively, of interior vertices). The kernel of the map $C_{1,1}(X) \to C_{1,0}(X)$ is generated by the boundary of the square, hence $H_{1,1}(X) \cong \mathbb{R}$.

del Pezzo in $(\mathbb{P}^1)^3$ Consider a surface S in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ given by a section of $\mathscr{O}(1,1,1) := \mathscr{O}(1) \boxtimes \mathscr{O}(1) \boxtimes \mathscr{O}(1)$; this is a del Pezzo surface, its anticanonical divisor is, by adjunction, the restriction of $\mathscr{O}(1,1,1)$, so the anticanonical degree is 6. The equation for S can be written as $F = \sum_{0 \le i,j,k \le 1} a_{i,j,k} x^i y^j z^k$, where x, y, z are local coordinates on the three projective lines. Suppose that we are over a valued field and that $a_{i,j,k} = a_{1-i,1-j,1-k}$ for all indices and that $a_{1,0,0} > \max(a_{0,1,0}, a_{0,0,1})$. Hence, the induced subdivision of a cube is regular, as seen in Fig. 8. From the picture, we see that there are 6 points, 18 edges, and 19 faces in the non-sedentary part of trop(S). The tropical variety trop $\left((\mathbb{P}^1)^3\right) \simeq (\mathbb{R} \cup \{\pm\infty\})^3$ is homeomorphic

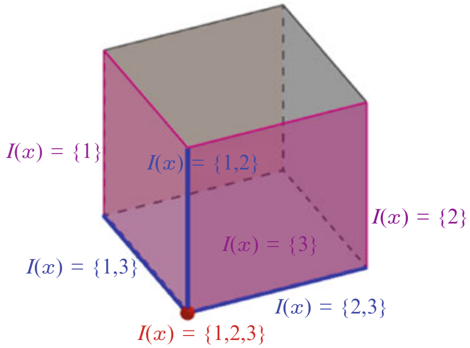

Fig. 7 A tropicalization of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ with the sedentarities of the faces at infinity

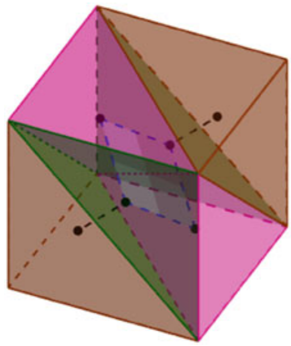

Fig. 8 Regular subdivision of the cube and tropical del Pezzo

to the cube, see Fig. 7. Its faces correspond to torus-invariant divisors in $(\mathbb{P}^1)^3$. The boundary $\mathrm{trop}(S) \setminus \mathbb{R}^3$ decomposes into six components, the intersections of $\mathrm{trop}(S)$ with those faces. To understand the sedentary points, we use the following result.

Theorem 5.6 ([22, Theorem 6.2.18]) *If $Y \subset T$ and $\overline{Y}$ is the closure of Y in a toric variety X, then the tropical variety* $\mathrm{trop}(\overline{Y})$ *is the closure of* $\mathrm{trop}(Y)$ *in* $\mathrm{trop}(X)$.

Applying Theorem 5.6 to $Y = \bar{S}$, we see that the boundary of the tropicalization is the tropicalization of the boundary, so we have $\mathrm{trop}(S) \cap \mathrm{trop}(D) = \mathrm{trop}(S \cap D)$ for each torus-invariant divisor. The torus-invariant divisors are defined by $x^{\pm 1}$, $y^{\pm 1}$, $z^{\pm 1}$. Without loss of generality, assume $D = \mathrm{V}(x)$. By restricting the element F to D, we obtain the quadric $\sum_{0 \le j,k \le 1} a_{0,j,k} y^j z^k$ whose tropicalization is given in Fig. 9. In particular, it has five edges, two mobile points, and four sedentary points. Table 1 summarizes the strata.

This information enables us to compute the $C_{p,q}$ without analyzing the maps, because our del Pezzo is locally linear: near each vertex the tropical structure looks like the tropicalization of $\mathbb{P}^2 \subset \mathbb{P}^3$, as shown in Fig. 4. The complexes are

$$\begin{aligned}
C_{0,2} &= \mathbb{R}^{19} \to C_{0,1} = \mathbb{R}^{18} \oplus \mathbb{R}^{30} \to C_{0,0} = \mathbb{R}^{30} \\
C_{1,2} &= \mathbb{R}^{2\cdot 19} \to C_{1,1} = \mathbb{R}^{3\cdot 18} \oplus \mathbb{R}^{30} \to C_{1,0} = \mathbb{R}^{3\cdot 6} \oplus \mathbb{R}^{2\cdot 12} \\
C_{2,2} &= \mathbb{R}^{19} \to C_{2,1} = \mathbb{R}^{2\cdot 18} \qquad \to C_{2,0} = \mathbb{R}^{3\cdot 6}.
\end{aligned}$$

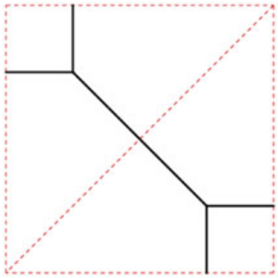

Fig. 9 A tropical quadric in $\mathbb{P}^1 \times \mathbb{P}^1$

Table 1 Strata in tropical del Pezzo

Sedentarity	0	1	2
Points	6	12	12
Edges	18	30	–
Faces	19	–	–

By comparing $H_{0,\bullet}$ with singular homology and using Poincaré duality, we obtain $H_{0,0} \simeq H_{2,2} \simeq \mathbb{R}$ $H_{0,1} = H_{0,2} = H_{2,0} = H_{2,1} = 0$, so the interesting part is the homology of $C_{1,\bullet}$. It is not impossible to compute this homology by hand. However, to save space, we only outline a series of reductions. Each of these reductions involves to finding an exact subcomplex $D \subset C_{1,\bullet}$ and reducing to computing homology of $C_{1,\bullet}/D$. Consider a sedentary point p on the face of a cube. This point has two edges e_1, e_2 going towards the boundary of this face (and a third edge, which is irrelevant here). In $C_{1,\bullet}$, these polyhedra give an exact subcomplex $\mathbb{R}[e_1] \oplus \mathbb{R}[e_2] \to \mathbb{R}^2[p]$, so the homology of $C_{1,\bullet}$ is the homology of the quotient C' by all these subcomplexes for 12 choices of p. The quotient is $\mathbb{R}^{(2)(19)} \to \mathbb{R}^{(3)(18)} \oplus \mathbb{R}^6 \to \mathbb{R}^{(3)(6)}$. Next, consider one of the two corner vertices in Fig. 8 and all its adjacent faces (three edges, three faces, one simplex). In the tropical variety, those correspond to one point p, three edges e_i, and three faces f_i that glue together to form on tropical $\mathbb{A}^2$. Such an $\mathbb{A}^2$ has no higher homology, so the sequence $\bigoplus \mathbb{R}^2[f_i] \to \bigoplus \mathbb{R}^3[e_i] \to \bigoplus \mathbb{R}^3[p]$ is an exact subcomplex of C'. Dividing C' by the subcomplexes given by two corner vertices, we see that C'' equal to $\mathbb{R}^{(2)(13)} \to \mathbb{R}^{(3)(12)} \oplus \mathbb{R}^6 \to \mathbb{R}^{(3)(4)}$. The module $\mathbb{R}^{(3)(4)}$ corresponds to four multitangent spaces at four vertices of the square in the interior; see Fig. 8. Since none of the edges adjacent to them was modified in the process, the right map is surjective. Hence, we have $H_{1,0} = 0$. By Poincaré duality, we deduce that $H_{1,2} = 0$ and $\dim H_{1,1} = 36 + 6 - 26 - 12 = 4$, as expected from the Hodge diamond of a del Pezzo of anticanonical degree 6.

A K3 Surface in $(\mathbb{P}^1)^3$ Let $S \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ be a K3 surface over a valued field $\Bbbk$ as in Example 4.3. This subsection discusses its tropical homology and relations to its Hodge classes; using tropical homology, we recover the expected Hodge numbers and an anti-symplectic involution.

As explained in [22, Definition 2.3.8, Fig. 1.3.3], the polyhedral decomposition of the tropicalization is dual to the subdivision induced on the $2 \times 2 \times 2$ cube by the

Table 2 Strata in tropical K3

Sedentarity	0	1	2
Points	48	48	24
Edges	120	96	–
Faces	98	–	–

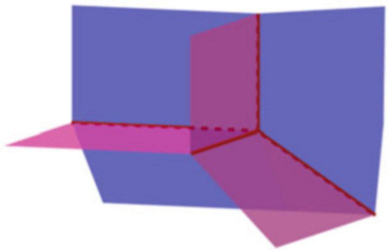

Fig. 10 Tropicalization of $\mathbb{P}^2$

coefficients of S; see Fig. 4. Restrict to the torus and consider polyhedra with empty sedentarity. The tropical variety trop(S) comes from a regular subdivision into 48 simplices, so it has 48 distinguished points. Each face of the subdivision (or edge in the tropicalization) is either "inner", shared by two tetrahedra, or "outer", adjacent to only one of them. There are 48 outer faces and each tetrahedron has four faces, so there is a total of $\big((48)(4) + 48\big)/2 = 120$ faces. As seen in the del Pezzo case, there are 19 edges in a subdivision of a unit cube. In the $2 \times 2 \times 2$ cube, we have (8)(19) of those segments; 36 of them are adjacent to exactly two cubes, six of them are adjacent to four cubes, and the others stick to one cube. Therefore, there are $(8)(19) - 36 - (3)(6) = 98$ segments. The boundary of trop(S) is the intersection of trop(S) with the boundary of this cube. Pick a face $\mathscr{F}$ of the cube. It is the tropicalization of one of the six toric divisors $x_i^{\pm 1}$ for $1 \le i \le 3$, say to x_1. Theorem 5.6 implies that $\operatorname{trop}(S) \cap \mathscr{F} = \operatorname{trop}\big(S \cap \mathrm{V}(x_1)\big)$. But $S \cap \mathrm{V}(x_1)$ is an elliptic curve in $\mathbb{P}^1 \times \mathbb{P}^1$ and Sect. 5.5 shows that its tropicalization has 16 edges, 8 mobile points, and 8 sedentary points; see Fig. 6. Table 2 enumerates the strata.

Once again, this information enables us to compute the $C_{p,q}$ without analyzing the maps. Near each vertex the tropical structure looks like the tropicalization of $\mathbb{P}^2 \subset \mathbb{P}^3$ see Fig. 10 and compare with Fig. 4. The complexes are

$$\begin{aligned}
C_{0,2} = \mathbb{R}^{98} &\to C_{0,1} = \mathbb{R}^{120} \oplus \mathbb{R}^{96} &&\to C_{0,0} = \mathbb{R}^{120}\\
C_{1,2} = \mathbb{R}^{2\cdot 98} &\to C_{1,1} = \mathbb{R}^{3\cdot 120} \oplus \mathbb{R}^{96} &&\to C_{1,0} = \mathbb{R}^{3\cdot 48} \oplus \mathbb{R}^{2\cdot 48}\\
C_{2,2} = \mathbb{R}^{98} &\to C_{2,1} = \mathbb{R}^{2\cdot 120} &&\to C_{2,0} = \mathbb{R}^{3\cdot 48}.
\end{aligned}$$

In particular, we see that $\chi(C_{1,\bullet}) = 2\cdot 98 - 3\cdot 120 - 96 + 3\cdot 48 + 2\cdot = -20$ as expected. Moreover, one can show that $H_{1,0} = 0$, roughly because the classes of sedentary edges surject to classes of sedentary points and other points can be analyzed directly by Fig. 4. By Poincaré duality, we have $H_{1,2} = 0$, so we obtain

$$-20 = \chi(C_{1,\bullet}) = \dim H_{1,0} - \dim H_{1,1} + \dim H_{1,2} = -\dim H_{1,1}\,.$$

We now consider $(0, q)$-classes. The homology of $C_{0,\bullet}$ is just the singular homology of the tropical variety by Remark 5.4. The tropical variety is contractible to the boundary of the cube. Thus, $C_{0,\bullet}$ is exact in the middle and its homology groups are the homology groups of the sphere: $H_{0,0} \simeq \mathbb{R}$, $H_{0,1} = 0$, and $H_{0,2} = \mathbb{R}$. In other words, this gives an explicit proof for our special case of Theorem 5.1.

Proposition 5.7 *The tropical Hodge numbers of tropical variety* trop(S) *agree with the Hodge numbers of the surface S.* □

We expose an explicit generator of $H_{0,2}$ and analyze the action of σ on this space. Briefly speaking, this class is obtained as the boundary of the interior of the cube. To expand this, consider the boundary of the cube and the complex $C_2' \to C_1' \to C_0'$ computing its singular homology. This boundary can be embedded into a full cube and the complex C' becomes part of the complex C'' computing the homology of the cube $0 \to C_3'' \to C_2'' \to C_1'' \to C_0''$. Since the cube is contractible, the complex C'' is exact. Hence, the unique class ω in $H^2(C')$ is the boundary of the class Ω in C_3''. Consider the action of σ on the $\mathbb{R}^3$ containing the tropical variety. We have $\sigma(\mathbf{x}) = -\mathbf{x}$ in $\mathbb{R}^3$, so σ changes orientation and $\sigma(\Omega) = -\Omega$. It follow that $\sigma(\omega) = \sigma(\partial\Omega) = -\omega$.

Finally, we investigate the σ-invariant part $C^\sigma_{p,\bullet}$ of the complexes $C_{p,\bullet}$. Since we work over characteristic different from two, the functor $(-)^\sigma$ is exact and the homology of $C^\sigma_{p,\bullet}$ is the invariant part of the homology of $C_{p,\bullet}$. Since trop$(S)/\sigma$ is a tropical manifold, $C^\sigma_{\bullet,\bullet}$ computes its tropical homology; see [7, Chap. 7]. In particular, the homology groups $H^\sigma_{p,q} = H^q(C^\sigma_{p,\bullet})$ satisfy $H^\sigma_{p,q} = H^\sigma_{2-p,2-q}$. We believe, although we have not proved it formally, that the manifold trop$(S)/\sigma$ is a tropicalization of the Enriques surface S/σ. If this is the case, the homology classes of $C^\sigma_{\bullet,\bullet}$ compute the tropical homology of Enriques surface S/σ. It is straightforward to compute the dimensions of $C^\sigma_{p,q}$, because trop(S) does not contain the origin. As a consequence, every face F of trop(S) is mapped by σ to a unique face F' so that the action of σ on the space spanned by $[F]$ and $[F']$ always decomposes into an invariant subspace $[F] + [F']$ and an anti-invariant space $[F] - [F']$. Therefore, we have $\dim C^\sigma_{p,q} = \frac{1}{2} \cdot \dim C_{p,q}$, for all p, q, and the sequences are

$$
\begin{aligned}
C^\sigma_{0,2} &= \mathbb{R}^{49} &\to C^\sigma_{0,1} &= \mathbb{R}^{60} \oplus \mathbb{R}^{48} &\to C^\sigma_{0,0} &= \mathbb{R}^{60}\\
C^\sigma_{1,2} &= \mathbb{R}^{2\cdot 49} &\to C^\sigma_{1,1} &= \mathbb{R}^{3\cdot 60} \oplus \mathbb{R}^{48} &\to C^\sigma_{1,0} &= \mathbb{R}^{3\cdot 24} \oplus \mathbb{R}^{2\cdot 24}\\
C^\sigma_{2,2} &= \mathbb{R}^{49} &\to C^\sigma_{2,1} &= \mathbb{R}^{2\cdot 60} &\to C^\sigma_{2,0} &= \mathbb{R}^{3\cdot 24}.
\end{aligned}
$$

The generator ω of $H_{0,2}$ does not lie in $H^\sigma_{0,2}$. Thus, we have $H^\sigma_{0,2} = H^\sigma_{0,1} = 0$ and $H^\sigma_{0,0} \simeq \mathbb{R}$. By symmetry, we also have $H^\sigma_{2,0} = H^\sigma_{2,1} = 0$ and $H^\sigma_{2,2} \simeq \mathbb{R}$, which yields

$$
-\dim H_{1,1} = \dim H_{1,0} - \dim H_{1,1} + \dim H_{1,2} = -\chi\left(C^\sigma_{1,\bullet}\right) = -\tfrac{1}{2}\chi(C_{1,\bullet}) = -10\,.
$$

Summarizing these calculations, we obtain the following counterpart of Proposition 5.7.

Proposition 5.8 *The dimensions of the σ-invariant parts of tropical homology groups of S agree with the Hodge numbers of S/σ.* □

6 Topology of Analytifications of Enriques Surfaces

In this section, we analyze the analytification of an Enriques surface that is the quotient of the K3 surface from Example 4.3. Fix a valued field $\Bbbk$ and a K3 surface $S \subset (\mathbb{P}^1)^3$ over $\Bbbk$ together with an involution $\sigma: S \to S$, as in Example 4.3. We first analyze the topology of S^{an} itself.

Proposition 6.1 *The topological space S^{an} has a strong deformation retraction onto a two-dimensional sphere C. More precisely, there exist continuous maps $s: C \to S^{\mathrm{an}}$ and $e: S^{\mathrm{an}} \to C$, so that $e \circ s = \mathrm{id}_C$ and $s \circ e$ is homotopic to $\mathrm{id}_{S^{\mathrm{an}}}$. The maps s and e may be chosen to be σ-equivariant, where σ acts on C antipodally.*

Proof We consider the tropicalization $\mathrm{trop}(S) \subset (\mathbb{R} \cup \{\pm\infty\})^3$ with the antipodal involution $\mathrm{trop}(\sigma)$. We abbreviate $\mathrm{trop}(\sigma)$ as σ. There is a cube $C \subset \mathrm{trop}(S)$ fixed under the involution, see Fig. 4. This cube is a strong deformation retract of $\mathrm{trop}(S)$ and the retraction can be chosen to be σ-equivariant. In the following we identify C with a two-dimensional sphere.

It remains to prove that the tropical variety $\mathrm{trop}(S)$ is a strong deformation retract of S^{an} under the map $\pi: S^{\mathrm{an}} \to \mathrm{trop}(S)$. The tropical variety $\mathrm{trop}(S)$ is schön; its intersection with every torus orbit is smooth; see [22, Definition 6.4.19]. Moreover, all multiplicities of top degree polyhedra are equal to one, so the multiplicity at each point is equal to one by semicontinuity; see [22, Lemma 3.3.6]. Therefore, π has a section $\mathrm{trop}(S) \to S^{\mathrm{an}}$ whose image is equal to a skeleton $S(\mathscr{S}, H)$ of a suitable semistable model $(\mathscr{S}, H)$ of S; see [16, Remark 9.12]. The skeleton $S(\mathscr{S}, H)$ is a proper strong deformation retract of S^{an} by [17, Sect. 4.9]. The retraction map $S^{\mathrm{an}} \to \mathrm{trop}(S)$ is equal to π, so σ-equivariant as discussed in Example 4.3. The retraction s in the claim of the theorem is the composition of retractions from S^{an} to $\mathrm{trop}(S)$ and from $\mathrm{trop}(S)$ to the cube constructed above. □

Corollary 6.2 *The analytified K3 surface S^{an} is homotopy equivalent to a two-dimensional sphere.* □

Remark 6.3 From Proposition 6.1, it does not follow that the homotopy between $s \circ e$ and $\mathrm{id}_{S^{\mathrm{an}}}$ can be chosen σ-equivariantly. This is most likely true, but presently there seems to be no reference for this fact.

We now analyze the topology of the analytification of the Enriques surface S/σ using our knowledge about S^{an}. The quotient map $q: S \to S/\sigma$ analytifies to $q^{\mathrm{an}}: S^{\mathrm{an}} \to (S/\sigma)^{\mathrm{an}}$. For any X, we denote $\pi: X^{\mathrm{an}} \to X$ the natural map. Summarizing, we consider the following diagram.

$$
\begin{array}{ccccc}
S^{\mathrm{an}} & \xrightarrow{\sigma^{\mathrm{an}}} & S^{\mathrm{an}} & \xrightarrow{q^{\mathrm{an}}} & (S/\sigma)^{\mathrm{an}} \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\
S & \xrightarrow{\sigma} & S & \xrightarrow{q} & S/\sigma
\end{array}
$$

It is crucial that q^{an} is a quotient by σ^{an}, as we now prove.

Proposition 6.4 *We have* $(S/\sigma)^{\mathrm{an}} = S^{\mathrm{an}}/\sigma^{\mathrm{an}}$ *as topological spaces.*

Proof First, we prove the equality of sets $(S/\sigma)^{\mathrm{an}} = S^{\mathrm{an}}/\sigma^{\mathrm{an}}$. Consider $x \in (S/\sigma)^{\mathrm{an}}$ and its image $\pi(x) \in S/\sigma$. If $U = \mathrm{Spec}(A)$ is an affine neighbourhood of the point $\pi(x)$, then the point x corresponds to a semi-norm $|\cdot|_x$ on A and $\pi(x)$ corresponds to the prime ideal $\mathfrak{p}_x = \{f \in A : |f|_x = 0\}$; see [5, Remark 1.2.2]. We denote by $\mathscr{H}(x)$ the completion of the fraction field of $A/\mathfrak{p}_x = \kappa(\pi(x))$. We have the equality of fibres $S^{\mathrm{an}}_x = \left(S_x \times_{\kappa(\pi(x))} \mathscr{H}(x)\right)^{\mathrm{an}}$; see [5, p. 65]. In down-to-earth terms, the set S^{an}_x consists of multiplicative seminorms on the $\mathscr{H}(x)$-algebra $R = H^0(S_x, \mathscr{O}_{S_x}) \otimes_{\kappa(\pi(x))} \mathscr{H}(x)$ which extend the norm $|\cdot|_x$ on $\mathscr{H}(x)$. Using Proposition 1.3.5 in [5], we may assume $\mathscr{H}(x)$ is algebraically closed. Since $H^0(S_x, \mathscr{O}_{S_x})^{\sigma} = \kappa(\pi(x))$, we have $R^{\sigma} = \mathscr{H}(x)$. Similarly, the ring R is a free $\mathscr{H}(x)$-module of rank 2. It follows that R is isomorphic to either $\mathscr{H}(x)^{\times 2}$ with σ permuting the coordinates or to $\mathscr{H}(x)[\varepsilon]/\varepsilon^2$. Given a multiplicative seminorm $|\cdot|_y$ on R, its kernel $\mathfrak{q} = \{f \in R : |f|_y = 0\}$ is a prime ideal in R. In both cases, we have $R/\mathfrak{q} = \mathscr{H}(x)$. Since $|\cdot|_y$ agrees with $|\cdot|_x$ on $\mathscr{H}(x)$, we see that $|\cdot|_y$ is determined uniquely by its kernel. The involution σ acts transitively on those, hence σ^{an} acts transitively on the set S^{an}_x and the equality is proven.

Second, we prove that $(S/\sigma)^{\mathrm{an}} = S^{\mathrm{an}}/\sigma^{\mathrm{an}}$ as topological spaces; in other words, the topology on $(S/\sigma)^{\mathrm{an}}$ is induced from this of S^{an}. Take an open subset $U \subset S^{\mathrm{an}}$. We want to show that $q^{\mathrm{an}}(U)$ is open. Clearly, the union $U \cup \sigma^{\mathrm{an}}(U) \subset S^{\mathrm{an}}$ is open and a union of fibres, so its complement $Z \subset S^{\mathrm{an}}$ is closed and a union of fibres. The map q^{an} is finite, so it proper and hence closed; see [5, 3.4.7 and 3.3.6]. In particular, the image $q^{\mathrm{an}}(Z) \subset (S/\sigma)^{\mathrm{an}}$ is closed, so its complement $\pi(U) = (S/\sigma)^{\mathrm{an}} \setminus q^{\mathrm{an}}(Z)$ is open. This proves that $(S/\sigma)^{\mathrm{an}} = S^{\mathrm{an}}/\sigma^{\mathrm{an}}$ as topological spaces. □

Corollary 6.5 *There exists a retraction from* $(S/\sigma)^{\mathrm{an}}$ *onto* $\mathbb{RP}^2$*. In particular* $(S/\sigma)^{\mathrm{an}}$ *is not contractible.*

Proof The argument follows formally from Proposition 6.1 and Proposition 6.4. If C is a two dimensional sphere with an antipodal involution σ and $C/\sigma \simeq \mathbb{RP}^2$, then Proposition 6.1 provides the σ-invariant map $e\colon S^{\mathrm{an}} \to C$ and its section $s\colon C \to S^{\mathrm{an}}$. We now produce equivalents of s and e on the level of $S^{\mathrm{an}}/\sigma^{\mathrm{an}} \simeq (S/\sigma)^{\mathrm{an}}$.

$$
\begin{array}{ccccc}
S^{\mathrm{an}} & \xrightarrow{\sigma^{\mathrm{an}}} & S^{\mathrm{an}} & \xrightarrow{q^{\mathrm{an}}} & (S/\sigma)^{\mathrm{an}} \\
{\scriptstyle s}\uparrow\downarrow{\scriptstyle e} & & {\scriptstyle s}\uparrow\downarrow{\scriptstyle e} & & {\scriptstyle s}\uparrow\downarrow{\scriptstyle e} \\
C & \xrightarrow{\mathrm{trop}\,(\sigma)} & C & \xrightarrow{q} & C/\sigma = \mathbb{RP}^2
\end{array}
$$

The map $q \circ e \colon S^{\text{an}} \to s(C)/\sigma = \mathbb{RP}^2$ satisfies $q \circ e \circ \sigma^{\text{an}} = q \circ e$ so, by Proposition 6.4, it induces a unique map $e \colon S^{\text{an}}/\sigma^{\text{an}} = (S/\sigma)^{\text{an}} \to \mathbb{RP}^2$. Similarly, the map $q^{\text{an}} \circ s$ satisfies $q^{\text{an}} \circ s \circ \text{trop}(\sigma) = q^{\text{an}} \circ s$, so induces a unique map $s \colon \mathbb{RP}^2 \to (S/\sigma)^{\text{an}}$. It follows that $e \circ s \colon \mathbb{RP}^2 \to \mathbb{RP}^2$ is the unique map induced σ-invariant map $q \circ e \circ s = q$. Therefore, we have $e \circ s = \text{id}_{\mathbb{RP}^2}$ and $s \circ e$ is a retraction of $(S/\sigma)^{\text{an}}$ onto $s(\mathbb{RP}^2) \simeq \mathbb{RP}^2$. □

Remark 6.6 If the difficulty presented in Remark 6.3 was removed, a similar argument would show that $(S/\sigma)^{\text{an}}$ strongly deformation retracts onto $\mathbb{RP}^2$.

Proof of Theorem 1.3 Follows from Proposition 6.1 and Corollary 6.5. □

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Specht Polytopes and Specht Matroids

John D. Wiltshire-Gordon, Alexander Woo, and Magdalena Zajaczkowska

Abstract The generators of the classical Specht module satisfy intricate relations. We introduce the Specht matroid, which keeps track of these relations, and the Specht polytope, which also keeps track of convexity relations. We establish basic facts about the Specht polytope: the symmetric group acts transitively on its vertices and irreducibly on its ambient real vector space. A similar construction builds a matroid and polytope for a tensor product of Specht modules, giving Kronecker matroids and Kronecker polytopes instead of the usual Kronecker coefficients. We call this process of upgrading from numbers to matroids and polytopes "matroidification". In the course of describing these objects, we also give an elementary account of the construction of Specht modules. Finally, we provide code to compute with Specht matroids and their Chow rings.

MSC 2010 codes: 05E10

1 Overview

The irreducible representations of the symmetric group S_n were worked out by Young and Specht in the early twentieth century, and they remain omnipresent in algebraic combinatorics. The symmetric group S_n has a unique irreducible representation for each partition of $n \in \mathbb{N}$. For example, S_4 has exactly five irreducible

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representations corresponding to the integer partitions (4), (3, 1), (2, 2), (2, 1, 1) and (1, 1, 1, 1). Young and Specht constructed these irreducible representations, which are now called *Specht modules*. Young [20] gave a matrix representation and Specht [18] gave a combinatorial spanning set. Garnir [8] later explained how to rewrite Specht's spanning set in terms of Young's basis, which are now called *Garnir relations*. Modern accounts can be found in Sagan [16] or James and Kerber [10].

The classical approach, however, privileges Young's basis over other bases and the Garnir relations over other linear dependencies. Focusing on Young's basis and the Garnir relations immediately breaks the symmetry of Specht's spanning set and ignores its other combinatorial properties. Certainly, there are linear relations other than those given by Garnir and bases other than those given by Young to investigate!

To this end, we introduce the *Specht matroid*, which encodes all the linear dependencies among the vectors of Specht's spanning set. We also introduce the *Specht polytope*, which provides a way to visualize the Specht module, since the polytope sits inside the corresponding real vector space with positive volume, and the action of the symmetric group takes the polytope to itself. In the case of the partition $(n-1, 1)$, we recover both classical constructions and objects of current research interest. The corresponding Specht polytope is essentially the *root polytope* of type A_n. In Theorem 7.5, we record a result of Ardila, Beck, Hoşten, Pfeifle and Seashore [2] describing the faces of this polytope. The Specht matroid for the partition $(n-1, 1)$ is the matroid of the braid arrangement, and hence its Chow ring is the cohomology ring for the moduli space $\overline{\mathscr{M}_{0,n}}$ of n marked points on the complex projective line [4, 6]. We compute further examples of Chow rings in Sect. 5, including the solution to Problem 1 on Grassmannians in [19], which partially inspired this project. We state a combinatorial conjecture for the graded dimensions of the Chow rings for the Specht matroid for the partition $(2, 1^{n-1})$. However, we do not study any further connections with moduli spaces.

Our approach allows us to upgrade familiar combinatorial coefficients to matroids and polytopes. By analogy with *categorification*, which sometimes upgrades numbers to vector spaces, we call this process *matroidification*, or *polytopification* when working over the reals. This is the subject of Sect. 8. In Theorem 8.3, we polytopify the Kronecker coefficients, building the *Kronecker polytopes*. We also construct the *Kronecker matroids* encoding the Garnir-style rewriting rules that govern linear dependence in a tensor product of Specht modules. An analogue of Young's basis for the Kronecker matroid would provide a combinatorial rule for computing Kronecker coefficients. In Theorems 8.6 and 8.9, we give similar results for Littlewood–Richardson coefficients and plethysm coefficients.

The outline of the article is as follows. We begin in Sect. 2 with a self-contained construction of the Specht module. This construction is a bit unusual in that it makes no mention of tabloids, polytabloids, standard tableaux, or the group algebra. In Sect. 4, 5, and 6, we introduce respectively the Specht matroid, its Chow ring, and the Specht polytope; we then prove some basic general facts about them. Section 7 is devoted to the partitions $(n-1, 1)$ and $(2, 1^{n-1})$, for which the Specht matroids and polytopes coincide with other well-studied objects. We describe Kronecker, Littlewood–Richardson, and plethysm matroids and polytopes in Sect. 8. Section 9 includes code for calculating the objects described in this paper.

2 Introduction to Specht Modules

This section gives an exposition of the representation theory of the symmetric group motivated by elementary combinatorics. The reader who wishes to start with the main statements should first look at Definitions 2.14 and 2.16, and Theorem 2.18.

We begin with an elementary combinatorics problem: In how many distinct ways can the letters in a word TENNESSEE be rearranged? There are 9! ways to move the letters around, but since some letters are repeated, some of these rearrangements give the same string. For example, the four Es can be rearranged to appear in any order without affecting the string. This reasoning gives the following answer:

$$\#\{\text{rearrangements of }\mathtt{TENNESSEE}\} = \tfrac{9!}{1!\,4!\,2!\,2!}\,.$$

The idea of rearranging letters can be formalized as an action of the symmetric group. In our example, the symmetric group S_9 acts on the word TENNESSEE. The stabilizer subgroup of S_9 with respect to the word TENNESSEE is isomorphic to $S_1 \times S_4 \times S_2 \times S_2$. Hence, the given argument provides an isomorphism of S_9-*sets*, which is to say, a bijection that commutes with the group action:

$$\{\text{rearrangements of }\mathtt{TENNESSEE}\} \simeq \tfrac{S_9}{S_1\times S_4\times S_2\times S_2}\,.$$

Using the orbit-stabilizer theorem, we recover the numerical answer above.

Now, we add a layer of complexity. Suppose we wish to understand the S_9-set

$$\{\text{rearrangements of }\mathtt{TENNESSEE}\} \times \{\text{rearrangements of }\mathtt{SASSAFRAS}\},$$

where S_9 acts diagonally on the two factors. This diagonal action makes sense because SASSAFRAS has the same number of letters as TENNESSEE. Each factor in this Cartesian product has a single S_9-orbit, but the product certainly does not. The pairs

$$\begin{pmatrix}\mathtt{EEEENNSST}\\ \mathtt{SSSSAAAFR}\end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix}\mathtt{TENNESSEE}\\ \mathtt{SASSAFRAS}\end{pmatrix}$$

cannot be in the same orbit because the upper Es and lower Ss always appear together in the first pair but not in the second. Alternatively, their orbits have different sizes. If we consider a column, such as $\substack{\mathtt{E}\\ \mathtt{S}}$, as a compound letter, then the total number of distinct rearrangements of the first compound word is $9!/(4!\,2!\,1!\,1!\,1!) = 7560$, which does not equal the number $9!/(1!\,3!\,2!\,1!\,1!\,1!) = 30240$ of rearrangements of the second compound word.

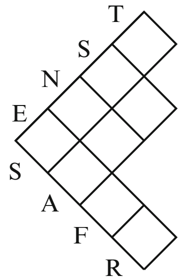


Fig. 1 A simultaneous histogram

For the construction of the Specht module, we are interested in *free orbits*; an orbit is free if each of its points has trivial stabilizer. In our context, a non-trivial stabilizer comes from repeated columns, so a pair is in a free orbit if and only if all of its columns are distinct. For example, the pair

$$\begin{pmatrix}\texttt{SNETNEESE}\\ \texttt{SASSSAFAR}\end{pmatrix}$$

has no repeated columns, so its orbit is free. We claim the following:

- There is only one free orbit, and we have already found it.
- The proof is basically a picture, and the proof-picture-idea is strong enough to construct a complete set of irreducible representations over $\mathbb{C}$ for the symmetric group S_n. These are the Specht modules. (The story would be the same for any field of characteristic zero.)

In Fig. 1, the boxes provide a simultaneous histogram tallying the letter multiplicities for each word. From the picture, we see that the letter `E` from the word `TENNESSEE` must appear once with each of the letters `S`, `A`, `F`, and `R`. Indeed, in order to keep distinct the four columns in which `E` appears, `E` must be paired with each of the four available letters in the bottom row. Removing the four `E`s from the pool along with one copy of each of the letters `S`, `A`, `F`, and `R`, we may proceed to pair off `N` with the two letters `S` and `A`. Continuing inductively, we see that the combinations that appear in a valid pair of rearrangements give the boxes in the diagram.

We give some definitions that encode these pictures.

Definition 2.1 A *partition* of $n \in \mathbb{N}$ is an integer vector $\lambda := (\lambda_1, \lambda_1, \ldots, \lambda_\ell)$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0$ with $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n$. The number $\ell := \ell(\lambda)$ is the *length* of the partition.

Definition 2.2 A *diagram* is a finite subset of $\mathbb{N}^2_{\geq 1}$. The elements of a diagram are called *boxes*. The *diagram* associated to a partition λ is $D(\lambda) := \{(i,j) : 1 \leq j \leq \lambda_i\}$ where, by convention, $\lambda_i = 0$ if $i > \ell(\lambda)$.
The partition in our running example is $(4, 3, 1, 1)$; its associated diagram is

$$\{(1,1),(1,2),(1,3),(1,4),(2,1),(2,2),(2,3),(3,1),(4,1)\}.$$

Throughout, we use matrix coordinates, so $(2, 3)$ denotes the box in the second row and third column.

The following proposition is immediate.

Proposition 2.3 *A diagram D corresponds to a partition λ if and only if D is closed under natural coordinatewise partial order. In other words, $D = D(\lambda)$ for some partition λ if and only if, for any $(i,j) \in D$ and any (a,b) with $1 \leq a \leq i$ and $1 \leq b \leq j$, we have $(a,b) \in D$.*

Definition 2.4 We say two words w_1, w_2 of the same length n *have complementary rearrangements* if the diagonal action of S_n on the product

$$\{\text{rearrangements of } w_1\} \times \{\text{rearrangements of } w_2\}$$

has a unique free orbit. Furthermore, if (w_1, w_2) is in this free orbit, then we say w_1 and w_2 are *complementary*.

As Fig. 1 shows, the words `TENNESSEE` and `SASSAFRAS` have complementary rearrangements; these words are not complementary, but the rearrangements `TENENEESS` and `SASSAFRAS` are.

Theorem 2.5 *Two words w_1, w_2 have complementary rearrangements if and only if there exists a parititon diagram D with the* simultaneous histogram *property:*

$$\#\,\textit{occurrences in } w_i \textit{ of its jth most-common letter} = \#\{(d_1, d_2) \in D : d_i = j\}\,.$$

Proof If we have a partition diagram D with the simultaneous histogram property, then we can put in the box (d_1, d_2) the d_1-th most common letter in w_1 and the d_2-th most common letter in w_2 breaking ties arbitrarily. The boxes have distinct pairs, so there exists at least one free orbit; this orbit is unique by the iterative argument below Fig. 1. In the other direction, rewrite the words using positive integers, so that in each word k appears at least as often as $k + 1$, and take

$$D = \{(d_1, d_2) : d_1 \text{ appears in a column with } d_2 \text{ in the unique free orbit}\}\,. \qquad \square$$

We use the idea of complementary words to construct irreducible representations of the symmetric group S_n. Before doing so, we collect some basic definitions in representation theory.

Definition 2.6 Let G be a group. A *(complex) representation* of G is a $\mathbb{C}$-vector space V along with a linear action of G on V: for any vectors $v, w \in V$, any scalar $c \in \mathbb{C}$, and any element $g \in G$, we have $g\cdot v + g\cdot w = g\cdot(v+w)$, and $c(g\cdot v) = g\cdot(cv)$. Alternatively, the data of a representation can be encoded in a group homomorphism $G \to \mathrm{GL}(V)$, where $\mathrm{GL}(V)$ is the group of invertible linear automorphisms of V.

Definition 2.7 For representations V and W of the group G, a linear transformation $\varphi\colon V \to W$ is a *map of G representations* if φ commutes with the action of G: for all $g \in G$ and all $v \in V$, we have $\varphi(g \cdot v) = g \cdot (\varphi(v))$.

Definition 2.8 If V and W are representations of G, then the *tensor product* $V \otimes W$ is a representation of G under the action $g \cdot (v \otimes w) = (g \cdot v) \otimes (g \cdot w)$.

Definition 2.9 If V is a representation of G, then its *dual* $V^\vee := \mathrm{Hom}(V, \mathbb{C})$ has the linear action of G such that, for $f \in V^\vee$ and $g \in G$, the functional $g \cdot f$ satisfies $(g \cdot f)(v) = f(g^{-1} \cdot v)$ for any $v \in V$.

We also need two definitions specific to the symmetric group S_n.

Definition 2.10 For any $n \in \mathbb{N}$, the representation ε is the one-dimensional vector space on which S_n acts by sign. More precisely, for $v \in \varepsilon$, we have $g \cdot v = v$ if g is an even permutation and $g \cdot v = -v$ if g is an odd permutation. If V is any representation of S_n, then $V \otimes \varepsilon$ is isomorphic to V as a vector space, but the action of S_n differs in that the action of an odd permutation picks up a sign.

Definition 2.11 Given a word w, the representation $V(w)$ is the vector space with basis given by the rearrangements of w, with S_n acting by permuting our basis according to how it rearranges words.

The representation $V(w)$ is special in that the action of S_n is actually induced from a combinatorial action of S_n on a basis of $V(w)$. This property has a useful consequence.

Lemma 2.12 *Given any word w of length n, we have a canonical isomorphism of S_n representations $V(w) \simeq V(w)^\vee$ given by identifying our basis of rearrangements with its dual basis.*

Proof Since $V(w_1)$ has a canonical basis $\{v_r\}$, where r is an arbitrary rearrangement, the vector space $V(w_1)^\vee$ has a dual basis $\{f_r\}$, and

$$(\sigma \cdot f_r)(v_{r'}) = \begin{cases} 1 \text{ if } r' = \sigma \cdot r, \\ 0 \text{ if } r' \neq \sigma \cdot r. \end{cases}$$

Since $\sigma \cdot f_r = f_{\sigma \cdot r}$, the map $v_r \mapsto f_r$ is an isomorphism of S_n representations. □

We are now ready to construct representations of S_n using the combinatorics of complementary words; see Definition 2.4.

Corollary 2.13 *If w_1 and w_2 are words of length n with complementary rearrangements, then there is a unique-up-to-scaling map $\varphi\colon V(w_2) \otimes \varepsilon \longrightarrow V(w_1)$ of S_n representations and the image of φ is an irreducible representation.*

Proof Consider an arbitrary map $\Psi\colon V(w_2) \otimes V(w_1)^\vee \to \epsilon$ of S_n representations; the linear map Ψ satisfies $\sigma \cdot \Psi(v) = \Psi(\sigma \cdot v)$ for all v. Any such map must factor through the quotient $V(w_2) \otimes V(w_1)^\vee / W$, where W is the subspace spanned by elements of the form $\left(\mathrm{sign}(\sigma) v - \sigma v\right)$ for all v. In this quotient, any element of S_n acts on the image of any vector by sign. Pairs of rearrangements form a basis for the tensor product, so the images of these basis vectors still span the quotient. If some pair of rearrangements has a repeated column, then swapping those columns fixes the pair. The vectors indexed by such pairs become zero in the quotient because transpositions are odd.

By Theorem 2.5, the action on pairs has a unique free orbit. Any two vectors in the free orbit are related by a unique permutation, and so any vector spans the quotient, which must therefore be one-dimensional. Using tensor-hom adjunction,

$$\begin{aligned}\operatorname{Hom}\left(V(w_2)\otimes V(w_1)^\vee/W,\varepsilon\right) &\simeq \operatorname{Hom}\left(V(w_2)\otimes V(w_1)^\vee,\varepsilon\right)\\ &\simeq \operatorname{Hom}\left(V(w_2)\otimes \varepsilon, V(w_1)\right),\end{aligned}$$

where the first space is one-dimensional by the first paragraph. All isomorphisms are natural. We may take φ to be any nonzero vector in the last hom-space. This shows we have a unique-up-to-scaling linear map $\varphi\colon V(w_2)\otimes\varepsilon \longrightarrow V(w_1)$.

Now, we show that $V = \operatorname{Im}\varphi$ is irreducible. Suppose $U \subset V$ is a proper subrepresentation. By Maschke's Theorem, there exists a complementary subrepresentation $U' \subseteq V$ with the property that $V = U \oplus U'$. Let $\pi\colon V \to V$ denote the projection with kernel U' and image U. But the composite $V(w_2)\otimes\varepsilon \longrightarrow V \xrightarrow{\pi} V \longrightarrow V(w_1)$ cannot be a scalar multiple of φ because it is nonzero and has a different image. □

The next definition helps us write an explicit example of the linear map φ.

Definition 2.14 Let w_1, w_2 be fixed words of length n, and let r_1 and r_2 be arbitrary rearrangements of w_1 and w_2 respectively. Define *Young's character* to be

$$\mathrm{Y}_{w_1,w_2}(r_1,r_2) := \sum_{\sigma} \operatorname{sign}(\sigma),$$

where $\sigma \in S_n$ ranges over all permutations such that $\sigma\cdot w_1 = r_1$ and $\sigma\cdot w_2 = r_2$.

Proposition 2.15 *Young's character takes values in* $\{-1,0,1\}$. *Whenever writing* r_2 *on top of* r_1 *has a repeated column, we have* $\mathrm{Y}_{w_1,w_2}(r_1,r_2) = 0$. *If* w_1 *and* w_2 *are complementary, then* $\mathrm{Y}_{w_1,w_2}(r_1,r_2) \neq 0$ *exactly when all columns are distinct.*

Proof If there is a repeated column, then interchanging those columns does not change the value of Y, but it also introduces a sign change (since interchanging two columns is an odd permutation). It follows that $\mathrm{Y} = 0$ in this case. If all the columns are distinct, then there is at most one permutation carrying each row back to w_i, and so the sum either is empty or has a single term. In the event that w_1 and w_2 are complementary, the sum is nonempty. □

Definition 2.16 If w_1, w_2 are complementary words of length n, the *Specht matrix* $\varphi(w_1,w_2)$ is the $\{\text{rearrangements of } w_1\} \times \{\text{rearrangements of } w_2\}$ matrix with (r_1,r_2)-entry equal to $\mathrm{Y}_{w_1,w_2}(r_1,r_2)$. If w_1 and w_2 are not complementary but have complementary rearrangements, then we choose complementary rearrangements w_1' and w_2' of w_1 and w_2 respectively and define the Specht matrix $\varphi(w_1,w_2)$ to be $\varphi(w_1',w_2')$. The column-span of the Specht matrix (as a subspace of $V(w_1)$) is the *Specht module* $V(w_1,w_2)$. The symmetric group acts on $V(w_1,w_2)$ by permuting the rearrangements of w_1.

Table 1 Specht matrix $\varphi(1\,1\,2\,2, 1\,2\,1\,2)$

	1 1 2 2	1 2 1 2	1 2 2 1	2 1 1 2	2 2 1 1	2 1 2 1
1 2 1 2	1	0	−1	−1	1	0
1 1 2 2	0	−1	1	1	0	−1
1 2 2 1	−1	1	0	0	−1	1
2 1 2 1	1	0	−1	−1	1	0
2 2 1 1	0	−1	1	1	0	−1
2 1 1 2	−1	1	0	0	−1	1

When w_1 and w_2 are not themselves complementary, the Specht matrix $\varphi(w_1, w_2)$ is only defined up to a global choice of sign depending on which complementary rearrangements are chosen, but the Specht module is independent of this choice.

Example 2.17 For the words $w_1 = 1\,1\,2\,2$ and $w_2 = 1\,2\,1\,2$, the Specht matrix $\varphi(1\,1\,2\,2, 1\,2\,1\,2)$ is shown in Table 1. We now describe the action of S_n on the rows of this Specht matrix. Consider the action of $\sigma = (1\,2\,3)$ on the row word w_1. This action changes the order of rows to 2 1 1 2, 1 2 1 2, 2 2 1 1, 1 2 2 1, 2 1 2 1 and 1 1 2 2. What effect does it have on the columns of the Specht matrix? Let us look, for example, at the first column, labelled by 1 1 2 2. With the new row order this column becomes $c = (-1, 1, 0, -1, 1, 0)$, which is the original column for $r_2 = 2\,1\,1\,2$. If we consider also the action of σ on the rearrangements of w_2, we see that the rearrangement $r_2 = 2\,1\,1\,2$ of w_2 becomes $\sigma \cdot 2112 = 1\,1\,2\,2$, which is the label of the first column. The permutation σ acts this way on the Specht module.

We have the following fundamental fact about this representation.

Theorem 2.18 *The Specht module $V(w_1, w_2)$ is an irreducible representation of S_n.*

Proof By the proof of Corollary 2.13, we see that having a unique free orbit gives a unique-up-to-scaling map $\varphi\colon V(w_2) \otimes \varepsilon \to V(w_1)$ whose image is irreducible. It remains only to show that the Specht matrix provides an explicit choice for φ. This was accomplished in Proposition 2.15, which shows that Y provides a map $\varepsilon \to V(w_1) \otimes V(w_2)^\vee$, where we have used the fact that the actions of S_n on $V(w_2)$ and $V(w_2)^\vee$ are canonically equal. □

The representation does not actually depend on the words but only on the partition diagram, so we make the following definition:

Definition 2.19 If λ is a partition of n, then the *Specht module* $V(\lambda)$ is the Specht module $V(w_1, w_2)$ for any choice of complementary w_1 and w_2 such that the diagram showing w_1 and w_2 are complementary is $D(\lambda)$. We call w_1 the *row word* and w_2 the *column word*.

For example, the matrix in Example 2.17 is the Specht matrix $V(2, 2)$.

Remark 2.20 Since every entry of the Specht matrix is a 0, 1, or −1, the Specht module can be similarly defined over any field. However, over a field of positive characteristic, Maschke's Theorem does not hold. Nevertheless, over any field of characteristic other than 2, the statements above show that the Specht module is *indecomposable*, meaning that $V(w_1, w_2)$ cannot be written as the direct sum of two subrepresentations.

If w_1 and w_2 are complementary words with associated diagram D, then w_2 and w_1 are also complementary, with an associated diagram which is the transpose of D. It will be useful to have a definition describing this relationship.

Definition 2.21 Given a partition λ, the *conjugate partition* λ^* is the partition whose diagram $D(\lambda^*)$ is the transpose of the diagram $D(\lambda)$. Formally, we have $\lambda_i^* = \#\{k : \lambda_k \geq i\}$.
For example, if $\lambda = (4, 3, 1, 1)$, then $\lambda^* = (4, 2, 2, 1)$. This natural combinatorial construction has the following representation-theoretic meaning.

Theorem 2.22 *We have an isomorphism of S_n representations $V(\lambda^*) \simeq V(\lambda)^\vee \otimes \varepsilon$.*

Proof Transposing the Specht matrix $\phi(w_1, w_2)$ gives the Specht matrix $\phi(w_2, w_1)$, which is the Specht matrix for the conjugate partition. After transposing, the symmetric group acts on $\phi(w_2, w_1)$ by rearranging the *column word* w_1. This is not the correct action of the symmetric group on the Specht matrix; the action is off only by a sign and a dual because, by the identity $\Upsilon(w_2, \sigma w_1) = (-1)^\sigma \cdot \Upsilon(\sigma^{-1} w_2, w_1)$, the symmetric group acts on the row word w_2 by σ^{-1} and picks up a sign with odd permutations. □

Remark 2.23 We have not needed it, but it is actually the case that Specht modules are self-dual in the sense that there is an abstract isomorphism $V(\lambda)^\vee \simeq V(\lambda)$. Choosing a basis from the columns of the Specht matrix, it would be possible to write matrices for the action of the symmetric group. Evidently, these matrices would contain only real numbers—in fact, only rational numbers—and so their traces would be real as well. It follows that the character of a Specht module is real, and so its dual, whose character is given by complex conjugation, is the same. With this fact in mind, Theorem 2.22 gives that $V(\lambda^*) \cong V(\lambda) \otimes \varepsilon$.

We remark briefly on the relationship between the construction above and the more traditional presentation found in James and Kerber [10, Chap. 7.1] or Sagan [16, Chap. 2.3]. In the usual construction, one defines a *column-strict filling* of λ to be a labelling of $D(\lambda)$ by the integers $\{1, 2, \ldots, n\}$ such that every column is strictly increasing. Then, for each column-strict filling T, one associates an element v_T in an abstractly defined vector space, and the Specht module is the span of the vectors v_T as one takes all possible fillings T. In the definition of Specht module $V(\lambda)$ used here (Definition 2.19), we start with a word w_2, which we can take to be $w_2 = 1^{\mu_1} 2^{\mu_2} \cdots k^{\mu_k}$, where $\mu = \lambda^*$ and $k = \lambda_1$. Each rearrangement r of the word w_2 gives a column of the corresponding Specht matrix, which we can interpret as a vector v_r, and $V(\lambda)$ is defined as the span of the vectors v_r as one takes all possible rearrangements r. For each rearrangement r of w_2, one can define an associated filling T_r: the one where the labels in column i are the positions of the appearances of i in r. For example, if $\lambda = (4, 3, 1, 1)$, so $w_2 = 111122334$, and we take $r = 131243112$ (or $r =$ `ESENTSEEN` if we let $w_2 =$ `TENNESSEE`), then T_r is

$T_r =$

1	4	2	5
3	9	6	
7			
8			

This correspondence between column-strict fillings and rearrangements essentially gives the correspondence between our version and the usual version. Our version, however, sometimes differs by a sign (when the minimal rearrangement of w_2 to r is by an odd permutation). This sign turns out to be useful; for instance, Theorem 6.2 would be harder to state otherwise.

3 A Complete Set of Irreducible Representations

The Specht modules $V(\lambda)$ as λ varies over all partitions of n form a complete set of finite-dimensional irreducible representations for S_n. The usual proof, found in [16, Sect. 2.4], shows that $V(\lambda) \not\cong V(\mu)$ for $\lambda \neq \mu$. Since there are as many conjugacy classes of S_n as there are partitions of n, we have all the irreducible representations. We give a new and different argument that the Specht modules are a complete set of irreducibles assuming an unproven combinatorial conjecture.

Definition 3.1 In a diagram D, the *hook* of a box $d \in D$ consists of the box d, all boxes directly to the right of d, and all boxes directly below d. In other words, if $d = (i, j) \in D$, the hook of d is the set of all $(a, b) \in D$ such that $a \geq i$ and $b = j$ or $a = i$ and $b \geq j$. The *hook length* of a box $\Gamma(d)$ is the number of boxes in its hook.

Definition 3.2 The *dimension* of a diagram D with n boxes is $\dim D := \frac{n!}{\prod_{d \in D} \Gamma(d)}$.

By a beautiful result of Frame, Robinson, and Thrall [7], the dimension of a diagram equals the number of standard Young tableaux, which is the dimension of its Specht module. A bijective proof of this result was later given by Novelli, Pak, and Stoyanovskii [13]. Consequently, $\dim D$ is always an integer (Fig. 2).

An *ordered set partition* of a finite set S is a sequence $(P_1, P_2, \ldots, P_\ell)$ of subsets of S such that each P_i is nonempty, $P_i \cap P_j = \varnothing$ for all $i \neq j$, and $\bigcup_i P_i = S$. An ordered set partition is *properly ordered* if the parts are nonincreasing in size, so $\#P_i \geq \#P_{i+1}$ for all i. For each properly ordered set partition $P = (P_1, P_2, \ldots, P_\ell)$

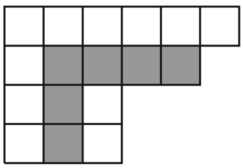


Fig. 2 A hook with six boxes inside the diagram $D(\lambda)$ for $\lambda = (6, 5, 3, 3)$

of the set $\{1, 2, \ldots, n\}$, pick a word w_P of length n so that the ith and jth letters of w_P match if and only if i and j are in the same P_k. For each P, choose also a word w'_P so that w_P and w'_P are complementary. Every properly ordered set partition P gives rise to an underlying partition $\lambda(P)$ with $\lambda(P)_i = \#P_i$. A set partition with parts of distinct sizes will have only one proper ordering, but a set partition with parts of equal sizes will have more. For notational simplicity, set $d(P) := \dim D\big(\lambda(P)\big)$.

While searching for a Specht matrix proof that every irreducible representation of the symmetric group is isomorphic to some Specht module, we uncovered the following conjecture, which has been checked for $n \leq 5$.

Conjecture 3.3 If $\sigma, \tau \in S_n$ are two permutations, then Young's character in Definition 2.14 satisfies

$$\sum_P \sum_r d(P)^2\, \Upsilon(\sigma w_P, r)\, \Upsilon(\tau w_P, r) = \begin{cases} (n!)^2 & \text{if } \sigma = \tau \\ 0 & \text{if } \sigma \neq \tau, \end{cases}$$

where the first sum is over all properly ordered set partitions of $\{1, 2, \ldots, n\}$, and the second sum is over all rearrangements of the word w'_P.

For our application, we desire a proof of the conjecture that does *not* make use of the following theorem.

Theorem 3.4 *Every irreducible representation of S_n arises exactly once from a diagram with n boxes.*

Proof This proof assumes Conjecture 3.3. There are three parts: first, we build a block matrix whose blocks are built from Specht matrices; second, we use the conjecture to show that this matrix has full rank; finally, we conclude that the regular representation $\mathbb{C}S_n$ is spanned by a sum of Specht modules.

Build a block matrix M with a single block row and a block column for every properly ordered set partition P of the set $\{1, 2, \ldots, n\}$. The block M_P associated to P is a matrix whose rows are indexed by S_n and whose columns are indexed by the rearrangements of w'_P. The (σ, r)-entry of M_P is $\Upsilon_{w_P, w'_P}(\sigma w_P, r)$. The rows of M_P come directly from the Specht matrix associated to the complementary pair (w_P, w'_P), so the column-span of M_P is isomorphic to the Specht module $V(\lambda_P)$.

Conjecture 3.3 asserts that the rows of this block matrix are orthogonal under the inner product given by the diagonal inner product $\langle u, v\rangle = \sum_{P;r} d(P)^2 \cdot (u_{P;r} v_{P;r})$, where $u_{P;r}$ denotes the entry of u in the column indexed by P and r. Consequently, the block matrix M has full rank.

The natural action of S_n by permuting the rows of this matrix is the regular representation $\mathbb{C}S_n$. Since M has full rank, the image of M must be the regular representation. On the other hand, the image of M is the span of the images of M_P, and the image of each M_P is isomorphic to some Specht module $V(\lambda_P)$. Hence, the regular representation is spanned by a sum of Specht modules.

The regular representation always contains a copy of every irreducible representation, so every irreducible representation of S_n is isomorphic to the Specht module

$V(\lambda)$ for some partition λ. Since there are as many conjugacy classes of S_n as there are partitions of n, and the number of distinct irreducible representations is always equal to the number of conjugacy classes, the Specht modules must be distinct. □

4 Specht Matroids

A *matroid* is a combinatorial generalization of the dependence relations among a finite set of vectors in a vector space. This structure can be defined in many equivalent ways, each with an axiomatization on some collection of subsets of a *ground set E*, which is the abstraction of our original set of vectors. Because of the presence of many equivalent definitions, each useful in a different context, it has become customary not to define the word "matroid" but instead to only give definitions of the *bases*, *independent sets*, *circuits*, *rank function*, or other linear algebra notion associated to the underlying object M, the matroid. Since we will only work with matroids that come from a set of vectors in a $\mathbb{C}$-vector space (so called $\mathbb{C}$-*representable matroids*), we do not give any of these abstract definitions. We refer the interested reader to [15].

Let E be a finite set, which we take to be $\{1, 2, \ldots, k\}$ for convenience, and let $\{v_i : i \in E\}$ be a set of vectors spanning a vector space $\mathbb{C}^n$. A subset $B \subseteq E$ is a *basis* of the matroid $M(v_1, v_2, \ldots, v_k)$ if $\{v_i : i \in I\}$ is a basis of $\mathbb{C}^n$. A subset $C \subseteq E$ is a *circuit* of M if it is a *minimal dependent set*: $v(C) = \{v_i : i \in C\}$ is dependent but any proper subset of $v(C)$ is independent. Given some subset $A \subset E$, the *rank* of A, denoted $r(A)$, is the dimension of the subspace spanned by $\{v_i : i \in A\}$. A *flat* of M is a maximal subset of E of a given rank; in other words, $F \subseteq E$ is a flat if $r(F \cup \{i\}) > r(F)$ for all $i \notin F$. One can think of each flat F as representing the subspace spanned by $\{v_i : i \in F\}$, which gives a bijection between subspaces spanned by a subset of $\{v_1, v_2, \ldots, v_k\}$ and flats. This correspondence shows that the flats of a matroid M form a lattice under inclusion, called the *lattice of flats* of M.

Given a partition λ, we define the *Specht matroid* $M(\lambda)$ to be the matroid formed from the columns of the Specht matrix for λ. The Specht matrix depends on a global choice of sign coming from the complementary words chosen, but the matroid is independent of this choice.

Example 4.1 We describe the matroid $M(2, 2)$, which is the matroid represented by the columns of the Specht matrix in Example 2.17. The circuits are $\{1122, 2211\}$, $\{1212, 2121\}$, and $\{1221, 2112\}$ and all sets of three vectors not containing one of the first three sets. The bases are the 12 different sets of 2 vectors not containing a circuit. One can choose any of the six vectors as the first vector in the basis, which leaves four choices for the second vector, but this procedure chooses every basis twice. The five flats are $\varnothing$, the three circuits of size 2, and the set of all six vectors.

We characterize the possible circuits of size 2, which also characterizes the flats of rank 1.

Theorem 4.2 *The Specht matroid $M(\lambda)$ has a circuit with two elements if and only if the diagram of λ has two columns of the same size.*

Proof For simplicity, we let $\mu = \lambda^*$ and let our complementary words be x and w, where $w = 1^{\mu_1}2^{\mu_2}\cdots k^{\mu_k}$ (with $k = \lambda_1$) and $x = 12\cdots\mu_1 1\cdots\mu_2\cdots 1\cdots\mu_k$. The pair (x, w) is part of the free orbit on the rearrangements of (x, w). We could give a proof describing the circuits involving any arbitrary rearrangement r of w. However, since the symmetric group S_n acts transitively on the matroid $M(\lambda)$, we may choose our favorite rearrangement of w, which is w itself. For any other rearrangement r, we have a circuit of two elements involving r if and only if there is a circuit of two elements involving w.

Suppose λ has two columns of the same size, so there exists some i such that $\mu_i = \mu_{i+1}$. Consider w and the rearrangement r where all the i's and $(i + 1)$'s have been switched, so $r = 1^{\mu_1}2^{\mu_2}\cdots(i-1)^{\mu_{i-1}}(i+1)^{\mu_i}(i)^{\mu_i}(i+2)^{\mu_{i+2}}(i+3)^{\mu_{i+3}}\cdots k^{\mu_k}$. For any rearrangement s of the row word, we have $Y(s, w) = (-1)^{\mu_i}Y(s, r)$. Hence, $v_w - (-1)^{\mu_i}v_r = 0$, and $\{w, r\}$ forms a circuit.

Now, suppose all the columns of λ are distinct. Suppose r is some rearrangement of w with $r \neq w$. We show there is some rearrangement s of the row word such that $Y(s, w) \neq Y(s, r)$. Let k be the smallest integer such that $w_k \neq r_k$, and let $i = w_k$ and $j = r_k$. By our construction of w, $j > i$, and since the columns of λ are distinct, $\mu_j < \mu_i$. We also have $r_M = i$ for some $M > m$, where $m = \mu_1 + \mu_2 + \cdots + \mu_i$ is the position of the last occurrence of the letter i in w. Now, consider the rearrangement s of the row word switching x_k and x_m. The pair (s, w) is part of the free orbit, and, in fact, $Y(s, w) = -1$. However, we have $Y(s, r) = 0$ because $(s_M, r_M) = (a, i)$ for some $a < x_k$ and $(s_{k-x_k-a}, r_{k-x_k-a}) = (a, i)$. Therefore, the vectors v_r and v_w do not form a circuit for any rearrangement r. □

Matroids have a number of interesting invariants. One is the characteristic polynomial, a generalization of the chromatic polynomial for a graphical matroid. The characteristic polynomial $p_M(t)$ for a matroid M can be calculated recursively by deletion and contraction, so it is a specialization of the Tutte polynomial $T_M(x, y)$. It would be interesting to find formulas or characterizations of the Tutte or characteristic polynomials of Specht matroids.

Example 4.3 For the Specht matroid $M(2, 1, 1, 1)$, we use *Sage* to compute the Tutte polynomial: $T_{M(2,1,1,1)}(x, y) = x^4 + x^3 + x^2 + x + y$. The characteristic polynomial is related to the Tutte polynomial by the formula $p_M(t) = (-1)^{r(M)}T_M(1-t, 0)$. Since $r\big(M(2, 1, 1, 1)\big) = 4$, we get $p_{M(2,1,1,1)}(t) = t^4 - 5t^3 + 10t^2 - 10t + 4$. Similar computations produce $p_{M(3,2)}(t) = t^5 - 15t^4 + 90t^3 - 260t^2 + 350t - 166$ and $p_{M(2,2,1)}(t) = t^5 - 10t^4 + 45t^3 - 105t^2 + 120t - 51$.

5 Chow Rings

Given a matroid M, Feichtner and Yuzvinksy [6] (following DeConcini and Procesi [4] in the representable case) define the *Chow ring* $A^*(M) \simeq \mathbb{Q}[x_F]/I_M$ of M as follows. There is one generator x_F for each nonempty flat F, and the ideal

I_M is generated by the following two types of relations: $x_F x_G \in I_M$ whenever F and G are incomparable, and $\sum_{F \supset \{e\}} x_F \in I_M$ for every element e in the ground set.

Remark 5.1 The definition of Feichtner and Yuzvinsky also requires as input a *building set*, which is a subset of the flats satisfying some combinatorial properties with respect to the lattice. The definition we have given here is the case of the *maximal* building set, which is the one containing every nonempty flat. A slightly different presentation of the Chow ring appears also in the literature. In [1], Adiprasito, Huh, and Katz use a presentation that differs from the one of Feichtner and Yuzvinksy in not using the generator (which can be rewritten in terms of other generators) corresponding to the entire ground set of M.

The next example gives a solution to Problem 1 on Grassmannians in [19].

Example 5.2 Let us consider the matroid M which is formed from the columns of the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 2 & 3 & 4 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix},$$

where the columns are labelled $0, 1, \ldots, 5$. This matrix represents a point in $\mathrm{Gr}(3, \mathbb{C}^6)$ with 16 nonzero Plücker coordinates. To determine the Chow ring of M, first we list all nonempty flats of M:

$$\{0\}, \{0,1,2,3,4,5\}, \{0,1,3,4\}, \{0,2\}, \{0,5\}, \{1\}, \{1,2\}, \{1,5\}, \{2\}, \{2,3\}, \{2,4\}, \{2,5\}, \{3\}, \{3,5\}, \{4\}, \{4,5\}, \{5\}.$$

There is one generator x_F of the Chow ring $A^*(M)$ for each nonempty flat F. The monomial generators in I_M coming from pairs of incomparable flats are

$$x_0x_{12}, x_0x_{15}, x_0x_{23}, x_0x_{24}, x_0x_{25}, x_0x_{35}, x_0x_{45}, x_1x_{02}, x_1x_{05}, x_1x_{23}, x_1x_{24}, x_1x_{25}, x_1x_{35}, x_1x_{45}, x_2x_{05}, x_2x_{15}, x_2x_{35}, x_2x_{45}, x_2x_{0134}, x_3x_{02}, x_3x_{12}, x_3x_{05}, x_3x_{15}, x_3x_{24}, x_3x_{25}, x_3x_{45}, x_4x_{02}, x_4x_{12}, x_4x_{05}, x_4x_{15}, x_4x_{23}, x_4x_{25}, x_4x_{35}, x_5x_{02}, x_5x_{12}, x_5x_{23}, x_5x_{24}, x_5x_{0134}.$$

The relations in I_M of the second type are

$$x_0 + x_{02} + x_{05} + x_{0134} + x_{012345}, x_1 + x_{12} + x_{15} + x_{0134} + x_{012345}, x_2 + x_{02} + x_{12} + x_{23} + x_{24} + x_{25} + x_{012345}, x_3 + x_{23} + x_{35} + x_{0134} + x_{012345}, x_4 + x_{24} + x_{45} + x_{0134} + x_{012345}, x_5 + x_{05} + x_{15} + x_{25} + x_{35} + x_{45} + x_{012345}.$$

Copying these generators and relations into *Macaulay2* (either by hand or using the *Sage* code in Sect. 9), we obtain that the Hilbert series of $A^*(M)$ is $1 + 11T + T^2$.

Table 2 Dimensions of Chow groups of $M(\lambda)$

λ	0	1	2	3	4	5
(4)	1					
(3, 1)	1	8	1			
(2, 2)	1	1				
(2, 1, 1)	1	7	1			
(1, 1, 1, 1)	1					
(5)	1					
(4, 1)	1	41	41	1		
(3, 1, 1)	1	303	2552	2552	303	1
(2, 2, 1)	1	151	541	151	1	
(3, 2)	1	256	1026	256	1	
(2, 1, 1, 1)	1	21	21	1		
(1, 1, 1, 1, 1)	1					

Table 2 lists the dimensions of $A^i(M(\lambda))$ for all partitions λ of $n = 4$ and $n = 5$. Every row of Table 2 is palindromic, so $\dim A^i\big(M(\lambda)\big) = \dim A^{d-1-i}\big(M(\lambda)\big)$ for all i, where $d = \dim V(\lambda)$. In fact, the Chow ring $A^*(M)$ satisfies an algebraic version of Poincaré duality for any matroid M. Feichtner and Yuzvinsky [6, Corollary 2] prove this fact for a representable matroid M by showing that $A^*(M)$ is the cohomology ring of a smooth, proper algebraic variety. This equality of dimensions is extended to non-representable matroids by Adiprasito, Huh, and Katz [1, Theorem 6.19] as the first major step in their proof of the log-concavity of coefficients of the characteristic polynomial of an arbitrary matroid. Finding a combinatorial interpretation of these dimensions remains an interesting open problem.

By finding a Gröbner basis for I_M and determining the standard monomials, Feichtner and Yuzvinsky describe a monomial basis for $A^*(M)$ as follows [6, Corollary 1].

Theorem 5.3 (Paraphrased from [6], Corollary 1) *The ring $A^*(M)$ has a basis consisting of the monomials 1 and $\prod_{i=1}^{k} x_{F_i}^{d_i}$ such that $F_1 \subseteq F_2 \subseteq \cdots \subseteq F_k$ and $0 < d_i < \operatorname{rank}(F_i) - \operatorname{rank}(F_{i-1})$ for all i (considering $\operatorname{rank}(F_0) = 0$ by convention).*

The next example illustrates how to use Theorem 5.3.

Example 5.4 Consider the Specht matroid $M(2, 1, 1, 1)$, which is a uniform matroid on five elements. Denote the elements of its ground set by $\{0, 1, 2, 3, 4\}$. Using *Sage*, we get the list of nonempty flats of $M(2, 1, 1, 1)$ in Table 3. By considering one element sequences of flats, we get one monomial of degree 1 from each flat of rank greater than 1. This gives 21 monomials of degree 1. Flats of rank 1 do not contribute to the list of monomials because there is no integer d such that $0 < d < 1$.

We can get a monomial of degree 2 in two ways. Quadratic monomials which are a square of only one variable are obtained from one element sequences consisting of flats of rank greater than 2. There are 11 such flats. For the other quadratic monomials, we need to consider sequences of flats of length 2 such that the rank of

Table 3 Nonempty flats of $M(2,1,1,1)$

Rank	Flats
1	$\{0\},\{1\},\{2\},\{3\},\{4\}$
2	$\{0,1\},\{0,2\},\{0,3\},\{0,4\},\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}$
3	$\{0,1,2\},\{0,1,3\},\{0,1,4\},\{0,2,3\},\{0,2,4\},\{0,3,4\},\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\}$
4	$\{0,1,2,3,4\}$

the first element and the difference between the ranks of the two elements are each at least 2. In our case we get 10 such sequences. They are of the form $F_k \subset F_{\{0,1,2,3,4\}}$, where F_k is a flat of rank 2.

The only way to obtain a monomial of degree 3 is from the one element sequence $F_{\{0,1,2,3,4\}}$. Since the biggest rank of a flat in our matroid is 4, there are no monomials of degree 4 of higher. The dimensions we obtained are $1, 21, 21, 1$, which agrees with the next-to-last row of Table 2.

In the case where a matroid M is a Specht matroid, Theorem 5.3 has an appealing consequence. We say that a finite-dimensional representation V of a finite group G is a *permutation representation* if the group action on V arises from an action of G on a basis of V. In other words, the vector space V has a basis $v_1, v_2, \ldots, v_d$ such that, for all i with $1 \leq i \leq d$ and all $g \in G$, we have $g \cdot v_i = v_j$ for some basis element v_j. These representations are particularly easy to understand because one only has to understand the combinatorics of a group acting on a finite set.

Given an element v_r of the ground set of the Specht matroid, the action of any permutation $\sigma \in S_n$ on $V(w_1)$ takes v_r to $(-1)^\sigma v_{\sigma^{-1}r}$. Therefore, given a flat F, which we think of as the subspace spanned by $\{v_{r_1}, v_{r_2}, \ldots, v_{r_k}\}$, a permutation σ sends F to the flat $\sigma^{-1}F$ corresponding to the subspace spanned by $\{v_{\sigma^{-1}r_1}, v_{\sigma^{-1}r_2}, \ldots, sv_{\sigma^{-1}r_k}\}$. This action on flats induces an action on the Chow ring $A^*\big(M(\lambda)\big)$ sending a monomial $\prod_{i=1}^k x_{F_i}^{d_i}$ to $\prod_{i=1}^k x_{\sigma^{-1}F_i}^{d_i}$. Since a monomial satisfies the conditions of Theorem 5.3 if and only if its image under the action of σ does, the action of S_n on $A^*(M(\lambda))$ is a permutation action. If one understood the action of S_n on the set of flags of flats of $M(\lambda)$, one would be able to easily determine the graded character of $A^*\big(M(\lambda)\big)$ by substituting characters for dimensions in the computations of Feichtner and Yuzvinsky [6, p. 526]

6 Specht Polytopes

Given a partition λ, $\binom{n}{\lambda}$ we define the *Specht polytope* $P(\lambda)$ to be the convex hull in $\mathbb{R}^N$, where $N = \binom{n}{\lambda}$, of the columns of the Specht matrix. This polytope is defined only up to a global sign; this choice will be irrelevant for our purposes because any polytope is projectively equivalent to its negative.

Example 6.1 Consider the partition $(2,1,1)$. The row and column words for this partition are 1 1 2 3 and 1 2 1 1, respectively. The Specht matrix appears in Table 4.

Table 4 Specht matrix for (2, 1, 1)

	1 2 1 1	1 1 2 1	1 1 1 2	1 2 1 1
1 1 2 3	1	0	0	−1
1 1 3 2	−1	0	0	1
1 2 1 3	0	−1	0	1
1 2 3 1	0	0	1	−1
1 3 1 2	0	1	0	−1
1 3 2 1	0	0	−1	1
2 1 1 3	−1	1	0	0
2 1 3 1	1	0	−1	0
2 3 1 1	0	−1	1	0
3 1 1 2	1	−1	0	0
3 1 2 1	−1	0	1	0
3 2 1 1	0	1	−1	0

Using *Macaulay2*, we can verify that the polytope in $\mathbb{R}^{12}$ which is the convex hull of the columns of the matrix in Table 4 is a three-dimensional simplex.

Theorem 6.2 *Every column of the Specht matrix is a vertex of* $P(\lambda)$.

Proof Suppose some column of the Specht matrix can be written as a non-trivial convex combination of the others. Since S_n acts transitively on the columns of the Specht matrix, this would mean that every column can be written as a convex combination of the others, which would mean $P(\lambda)$ has no vertices, which is impossible. □

Theorem 6.3 *Every Specht polytope other than* $P(1, 1, \ldots, 1)$ *contains the origin.*

Proof First, we show that each row of a Specht matrix corresponding to a partition different from $(1, 1, \ldots, 1)$ contains the same number of 1's as of -1's. Let r be some permutation of a row word, and let $\mathrm{Stab}(r)$ be the set of all permutations preserving the word r. Since we excluded the partition $(1, 1, \ldots, 1)$, the stabilizer $\mathrm{Stab}(r)$ is a nontrivial direct product of symmetric groups. Nonzero entries in the row corresponding to r in the Specht matrix have values 1 or -1 depending on the signature of an element in $\mathrm{Stab}(r)$. Since $\mathrm{Stab}(r)$ has an equal number of odd and even permutations, the entries add up to 0. If $c_1, c_2, \ldots, c_m$ are the columns of the Specht matrix, then we have $\sum_{i=1}^{m} \frac{1}{m} c_i = 0$, which finishes the proof. □

Theorem 6.4 *The dimension of the Specht polytope matches the dimension of the Specht module for any partition other than* $(1, 1, \ldots, 1)$.

Proof Since the Specht module is the span of the columns of a Specht matrix, its dimension is equal to the rank of this matrix. By Theorem 6.3, the corresponding polytope contains the origin, so its dimension matches the dimension of the linear span of its vertices. □

Table 5 Dimensions and f-vectors of Specht polytopes

λ	Dimension	f-vector
(3,1)	3	(1, 12, 24, 14, 1)
(2,2)	2	(1, 3, 3, 1)
(2,1,1)	3	(1, 4, 6, 4, 1)
(4,1)	4	(1, 20, 60, 70, 30, 1)
(3,2)	5	(1, 15, 60, 80, 45, 12, 1)
(3,1,1)	6	(1, 20, 120, 290, 310, 144, 24, 1)
(2,2,1)	5	(1, 10, 45, 90, 75, 22, 1)
(2,1,1,1)	4	(1, 5,10,10,5,1)

Table 6 Dimensions of $A^k\big(M(2,1^{n-1})\big)$

n	k				
	0	1	2	3	4
2	1				
3	1	1			
4	1	7	1		
5	1	21	21	1	
6	1	51	161	51	1

We conclude this section with Table 5, which gives f-vectors and dimensions of some of the Specht polytopes. We have not found yet any interpretation of these data.

7 Examples: The Partitions $(2, 1^{n-1})$ and $(n-1, 1)$

In the previous section, we saw that the Specht polytope for the partition $(2, 1, 1)$ is a simplex. In fact, any partition of the form $(2, 1^{n-1})$ corresponds to a simplex.

Theorem 7.1 *The Specht polytope $P(2, 1^{n-1})$ is an $(n-1)$-dimensional simplex.*

Proof The Specht module $M(2, 1^{n-1})$ of size n has dimension $n-1$. By Theorem 6.4, the Specht polytope has the same dimension. The partition $(2, 1^{n-1})$ has the column word $1\,2\,1\,1\cdots 1$, which has n rearrangements. Hence, the Specht polytope $P(2, 1^{n-1})$ has at most n vertices and dimension $n-1$, so it must be an $(n-1)$-simplex. □

Correspondingly, for $\lambda = (2, 1^{n-1})$, the matroid $M(\lambda)$ is the generic matroid on n elements of total rank $n-1$. The Hilbert series for $A^*(M)$ for any generic matroid M is calculated by Feichtner and Yuzvinsky. It is not too difficult to modify their computation to include information on the action of S_n. The dimensions of the Chow groups of this Specht matroid appear in Table 6. We make the following conjecture with the help of the *OEIS* [14].

Conjecture 7.2 The dimension of $A^k\big(M(2, 1^{n-1})\big)$ is the number of permutations in S_n with no fixed points and $k+1$ excedances (*OEIS* A046739).

A *fixed point* of a permutation $\sigma \in S_n$ is an index i such that $\sigma(i) = i$, and an *excedance* is an index i such that $\sigma(i) > i$. Consider the cyclic subgroup $C_n \subset S_n$ generated by the n-cycle $c = (12\cdots n)$. Conjugation by c preserves the number of fixed points and the number of excedances, so, for any fixed n and k, the cyclic group C_n acts on the set of permutations of S_n with no fixed points and $k+1$ excedances. As in Sect. 5, the symmetric group S_n acts on the Feichtner–Yuzvinsky basis of $A^k\big(M(2, 1^{n-1})\big)$, and C_n acts by restricting this action. A possible refinement of the conjecture asserts that the orbit structures of these two actions coincide; we have checked this refinement for $n \leq 6$. For $n = 6$ and $k = 2$, both actions have 1 orbit of size 1, 2 orbits of size 2, 4 orbits of size 3, and 24 orbits of size 6.

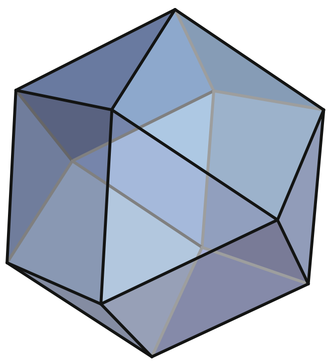
We switch our attention to partitions of the form $(n-1, 1)$. We start by describing the Specht matrices coming from this partition.

Proposition 7.3 *Each column of a Specht matrix for the partition* $(n-1, 1)$ *is of the form* $e_i - e_j$*, where* e_i *is a standard unit vector in* $\mathbb{R}^n$*, and, for* $n \geq 4$*,* $e_i - e_j$ *and* $e_j - e_i$ *are both columns of the Specht matrix for any* i, j *with* $1 \leq i < j \leq n$*.*

Proof For the partition $(n-1, 1)$, a choice of row word is $w_1 = 1\,1\cdots1\,1\,2$ and a choice of column word is $w_2 = 1\,2\,3\cdots(n-1)\,1$. Let r_2 be a rearrangement of the column word w_2. In the column for r_2, there are exactly two nonzero entries, namely the entries corresponding to the rearrangements of w_1 in which the 2 is in the same position as one of the 1's in r_2. Let us call these rearrangements r_1 and r_1'. If σ is a permutation such that $\sigma w_1 = r_1$ and $\sigma w_2 = r_2$, and σ' is a permutation such that $\sigma' w_1 = r_1'$ and $\sigma' w_2 = r_2$, then σ and σ' differ by a transposition, so we have $Y(r_1, r_2) = -Y(r_1', r_2)$. Given i and j, to obtain $e_i - e_j$ and $e_j - e_i$, use the column corresponding to some rearrangement r of w_2 with the 1's in the ith and jth positions and the column corresponding to rearrangement r' obtained from switching the positions of the 2 and 3 in r. □

For the partition $(3, 1)$, the Specht polytope naturally lives in a four-dimensional space, but as it is a three-dimensional object, it can be drawn in a 3-space; see Fig. 3.

Fig. 3 Specht polytope $(3, 1)$



In what follows, we denote a vertex of a Specht polytope $P(n-1,1)$ by $v_{i,j}$ if the corresponding column in the Specht matrix has 1 in the ith position and -1 in the jth position. It turns out that the Specht polytopes $P(n-1,1)$ have already been studied by Ardila, Beck, Hoşten, Pfeifle, and Seashore [2].

Definition 7.4 A *root polytope* P_{A_n} of type A_n is the convex hull of the points $e_i - e_j$ for $1 \leq i \neq j \leq n$ where $i, j \in \{1, 2, \ldots, n\}$.
This definition is different from the definition of Gelfand, Graev, and Postnikov [9], which uses only the positive roots and zero. For this class of polytopes, Ardila, Beck, Hoşten, Pfeifle and Seashore [2, Proposition 8] give the following description of their edges and facets.

Theorem 7.5 *The polytope $P_{A_n} \in \mathbb{R}^{n+1}$ has dimension n and is contained in the hyperplane $H_0 = \{x \in \mathbb{R}^{n+1} : \sum_{i=0}^{n} x_i = 0\}$. It has $(n-1)n(n+1)$ edges, which are of the form $v_{ij}v_{ik}$ and $v_{ik}v_{jk}$ for i, j, k distinct. It has $2^{n+1}-2$ facets, which can be labelled by the proper subsets S of $[0, n] := \{0, 1, \ldots, n\}$. The facet F_S is defined by the hyperplane $H_S := \left\{x \in \mathbb{R}^{n+1} : \sum_{i \in S} x_i = 1\right\}$, and it is congruent to the product of simplices $\Delta_S \times \Delta_T$, where $T = [0, n] - S$.*
The main idea in the proof of this theorem is that, if f is a linear functional and i, j, k, l are all different, then $f(v_{i,j}) + f(v_{k,l}) = f(v_{i,l}) + f(v_{k,j})$. Hence f cannot be maximized at only one of the line segments $v_{i,j}v_{k,l}$ or $v_{i,l}v_{k,j}$, so neither can be an edge. A similar argument works both for ruling out pairs of vertices of the form $v_{i,j}$ and $v_{j,k}$ as edges and for determining the facets.

Ardila, Beck, Hoşten, Pfeifle and Seashore also gave the following description of the lattice points inside P_{A_n}.

Theorem 7.6 *The only lattice points in P_{A_n} are its vertices and the origin.*

Proof A polytope P_{A_n} is contained in an $(n-1)$-sphere with radius $\sqrt{2}$ and centre 0. The only lattice points in this sphere are $\pm e_i$ and $\pm e_i \pm e_j$ for $1 \leq i, j \leq n$. Since P_{A_n} is contained in the hyperplane $H_0 = \{x \in \mathbb{R}^{n+1} : \sum_{i=0}^{n} x_i = 0\}$, the only lattice points contained in P_{A_n} are the vertices and the origin. □

The matroid $M(n-1,1)$ is the matroid for the braid arrangement of S_n. This was one of the original motivations of DeConcini and Procesi [4] for studying Chow rings of representable matroids. Moreover, $A^*(M)$ is the cohomology ring for the moduli space $\overline{\mathrm{M}}_{0,n}$ of n marked points on the complex projective line, which DeConcini and Procesi [3] show can be realized as the successive blowup of $\mathbb{P}^{n-2}$ at all the subspaces in the intersection lattice of the braid arrangement. For more information on $\overline{\mathrm{M}}_{0,n}$, see [12].

This matroid is also the graphical matroid on the complete graph K_n on n vertices. By the usual translation between graphical matroids and graphs, vectors in the matroid correspond to (directed) edges of the graph, and a basis of the matroid corresponds to a spanning tree for the graph. To be precise, we can label the vertices of K_n by $\{1, 2, \ldots, n\}$, and, with this labelling, the edge from i to j corresponds to the vector $e_j - e_i$. If we start with the column word $w_2 = 1\,1\,2\,3 \cdots n$, then the vector $e_j - e_i$ is v_r for the rearrangement r where a 1 appears in the ith and jth positions and the remaining letters $2, 2, \ldots, n$ appear in order. In terms of the usual presentation of the Specht matroid in terms of fillings, this vector corresponds to the filling with an i and a j in the first column and the remaining integers in order along the first row. The usual basis of the Specht module given by Standard Young Tableaux corresponds to the tree with edges between vertex 1 and vertex j for every $j > 1$, and declaring i to be the "smallest" letter in our filling alphabet gives a similar tree with vertex i having degree $n-1$. Of course, K_n has many other types of spanning trees, so the Specht matroid has many bases that look completely different than the standard one!

8 Matroidification

Many constructions in the representation theory of S_n involve tensor products or Hom spaces of Specht modules. Since Specht modules have a distinguished symmetric spanning set, so do their tensor products. Hence, we can extend our definitions of Specht matroids and Specht polytopes to these other contexts. In this section, we build matroids and polytopes for three famous collections of numbers arising in combinatorics and representation theory: Kronecker coefficients, Littlewood–Richardson coefficients, and plethysm coefficients.

Definition 8.1 If λ, μ, ν are partitions of n, then the *Kronecker coefficient* $g_{\lambda,\mu,\nu}$ is defined to be the dimension of the S_n-invariants of the tensor product

$$g_{\lambda,\mu,\nu} := \dim \left(\text{Specht}(\lambda) \otimes \text{Specht}(\mu) \otimes \text{Specht}(\nu) \right)^{S_n}.$$

In Definitions 8.2, 8.5, and 8.8, we denote by x_1 and x_2 a pair of complementary words of length n that correspond via Theorem 2.5 to the partition λ. Similarly, y_1 and y_2 correspond to μ, and w_1 and w_2 to ν.

Definition 8.2 The *Kronecker matrix* has rows indexed by the product

$$\{\text{rearrangements of } x_1\} \times \{\text{rearrangements of } y_1\} \times \{\text{rearrangements of } w_1\}$$

and columns indexed by the product

$$\{\text{rearrangements of } x_2\} \times \{\text{rearrangements of } y_2\} \times \{\text{rearrangements of } w_2\},$$

where the $\big((p,q,r),(s,t,u)\big)$ entry is given by the formula

$$\sum_{\sigma \in S_n} \Upsilon(\sigma s, p) \cdot \Upsilon(\sigma t, q) \cdot \Upsilon(\sigma u, r).$$

Its columns define the *Kronecker matroid* and the convex hull of its columns defines the *Kronecker polytope*.

Theorem 8.3 *The dimension of the Kronecker polytope is* $g_{\lambda,\mu,\nu}$.

Proof The tensor product Specht(λ)$\otimes$Specht(μ)$\otimes$Specht(ν) is given by the column span of the matrix with entries $\mathrm{Y}(p,s)\cdot\mathrm{Y}(q,t)\cdot\mathrm{Y}(r,u)$. The summation over S_n produces S_n-invariant vectors. □

Definition 8.4 If λ,μ,ν are partitions of l, m, and $l+m$ respectively, then the *Littlewood–Richardson coefficient* $c^{\nu}_{\lambda,\mu}$ is defined by

$$c^{\nu}_{\lambda,\mu} := \dim\Big(\,\mathrm{Specht}(\lambda)\boxtimes\mathrm{Specht}(\mu)\otimes\mathrm{Res}^{S_{l+m}}_{S_l\times S_m}\big(\mathrm{Specht}(\nu)\big)\Big)^{S_l\times S_m},$$

where $S_l\times S_m$ acts on Specht(λ) $\boxtimes$ Specht(μ) separately in the two tensor factors, and $S_l\times S_m$ acts on $\mathrm{Res}^{S_{l+m}}_{S_l\times S_m}\big(\mathrm{Specht}(\nu)\big)$ by considering $S_l\times S_m$ as a subgroup of S_{l+m} and using the S_{l+m}-action on Specht(ν). We use the notation $\boxtimes$ for the tensor product with this separated action in order to contrast with the diagonal action that we indicate by $\otimes$.

Definition 8.5 The *Littlewood–Richardson matrix* has rows indexed by the product

$$\{\text{rearrangements of } x_1\}\times\{\text{rearrangements of} y_1\}\times\{\text{rearrangements of } w_1\}$$

and columns indexed by the product

$$\{\text{rearrangements of } x_2\}\times\{\text{rearrangements of} y_2\}\times\{\text{rearrangements of } w_2\},$$

where the $((p,q,r),(s,t,u))$ entry is given by the formula

$$\sum_{\sigma\times\tau\in S_l\times S_m}\mathrm{Y}(\sigma s,p)\cdot\mathrm{Y}(\tau t,q)\cdot\mathrm{Y}\big((\sigma\times\tau)u,r\big).$$

Its columns define the *Littlewood–Richardson matroid* and the convex hull of its columns defines the *Littlewood–Richardson polytope*.

Littlewood–Richardson polytope for $\lambda=2+1$, $\mu=2+1$, $\nu=3+2+1$ appears in Fig. 4. Since $c^{\nu}_{\lambda,\mu}=2$, this polytope is actually a polygon.

Theorem 8.6 *The dimension of the Littlewood–Richardson polytope is* $c^{\nu}_{\lambda\mu}$.

Proof The proof for this theorem is analogous to that of Theorem 8.3. □

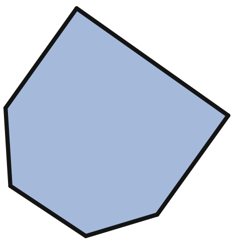


Fig. 4 Littlewood–Richardson polytope for $c^{(3,2,1)}_{(2,1),(2,1)}=2$

We now study restriction to the *wreath subgroup* $S_l \wr S_m \subseteq S_{lm}$. Thinking of lm as an $l \times m$ array of dots to be permuted, the wreath subgroup is generated by the permutations in which every dot stays in its row together with the permutations that perform the same operation in every column simultaneously. Abstractly, the wreath subgroup is isomorphic to the semidirect product $(S_l)^m \rtimes S_m$ where the second factor acts on the first by permuting coordinates.

Definition 8.7 If λ, μ, and ν are partitions of l, m, and $l \cdot m$ respectively, then the *plethysm coefficient* $p^{\nu}_{\lambda,\mu}$ is defined by

$$p^{\nu}_{\lambda,\mu} = \dim \left(\mathrm{Specht}(\lambda)^{\boxtimes m} \otimes^{\rtimes} \mathrm{Specht}(\mu) \otimes \mathrm{Res}^{S_{lm}}_{S_l \wr S_m} \left(\mathrm{Specht}(\nu) \right) \right)^{S_l \wr S_m},$$

where $(S_l)^m \rtimes S_m$ acts on $\mathrm{Specht}(\lambda)^{\boxtimes m} \otimes^{\rtimes} \mathrm{Specht}(\mu)$ by S_m on the second factor, and the normal subgroup $(S_l)^m \trianglelefteq (S_l)^m \rtimes S_m$ acts naturally on the first factor.

As before, let x_1, x_2 be a complementary pair of words of length l that correspond via Theorem 2.5 to the partition λ, and similarly suppose that y_i correspond to μ, and that w_i correspond to ν.

Definition 8.8 The *plethysm matrix* has rows indexed by the product

$$\{\text{rearrangements of } x_1\}^m \times \{\text{rearrangements of } y_1\} \times \{\text{rearrangements of } w_1\}$$

and columns indexed by the product

$$\{\text{rearrangements of } x_2\}^m \times \{\text{rearrangements of } y_2\} \times \{\text{rearrangements of } w_2\},$$

where the $\big((\hat{p}, q, r), (\hat{s}, t, u)\big)$ entry is given by the formula

$$\sum_{\hat{\sigma} \rtimes \tau \in S_l \wr S_m} \mathrm{Y}(\hat{\sigma}_1 \hat{s}_1, \hat{p}_1) \cdot \mathrm{Y}(\hat{\sigma}_2 \hat{s}_2, \hat{p}_2) \cdots \mathrm{Y}(\hat{\sigma}_m \hat{s}_m, \hat{p}_m) \cdot \mathrm{Y}(\tau t, q) \cdot \mathrm{Y}\left((\hat{\sigma} \rtimes \tau)u, r\right).$$

Its columns define the *plethysm matroid*, and the convex hull of its columns defines the *plethytope*.

Theorem 8.9 *The dimension of the plethytope is* $p^{\nu}_{\lambda\mu}$.

Proof The proof for this theorem is also analogous to that of Theorem 8.3. □

9 Computer Calculations

The following *Sage* [17] code generates the Specht matrix given a row word and a column word.

```
def distinctColumns(w1, w2):
    if len(w2) != len(w2): return False
```

```
    seen = set()
    for i in range(len(w1)):
        t = (w1[i], w2[i])
        if t in seen: return False
        seen.add(t)
    return True

def YoungCharacter(w1, w2):
    assert distinctColumns(w1, w2)
    wp = [(w1[i], w2[i]) for i in range(len(w1))]
    def ycfunc(r1, r2):
        if not distinctColumns(r1, r2):
            return 0
        rp = [(r1[i], r2[i]) for i in range(len(w1))]
        po = [wp.index(rx) + 1 for rx in rp]
        return Permutation(po).sign()
    return ycfunc

def SpechtMatrix(w1, w2):
    yc = YoungCharacter(w1, w2)
    mat = []
    for r1 in Permutations(w1):
        row = []
        for r2 in Permutations(w2):
            row = row + [yc(r1, r2)]
        mat = mat + [row]
    return matrix(QQ, mat)

sm22 = SpechtMatrix([1,1,2,2], [1,2,1,2])
print sm22
```

The output of the code is

```
[ 0  1 -1 -1  1  0]
[-1  0  1  1  0 -1]
[ 1 -1  0  0 -1  1]
[ 1 -1  0  0 -1  1]
[-1  0  1  1  0 -1]
[ 0  1 -1 -1  1  0]
```

Having a Specht matrix, we can use *Macaulay2* [11] package *Polyhedra* [5] to obtain some information about Specht polytopes.

```
loadPackage "Polyhedra";
V = matrix{{ 0, 1,-1,-1, 1, 0}, {-1, 0, 1, 1, 0,-1},
           { 1,-1, 0, 0,-1, 1}, { 1,-1, 0, 0,-1, 1},
           {-1, 0, 1, 1, 0,-1}, { 0, 1,-1,-1, 1, 0}}
P = convexHull V
fVector P
```

The output of the above code, line by line, is

```
| 0   1   -1  -1  1   0  |
| -1  0   1   1   0   -1 |
| 1   -1  0   0   -1  1  |
```

```
| 1   -1 0   0   -1 1   |
| -1 0   1   1   0   -1 |
| 0   1   -1 -1 1   0   |

              6         6
Matrix ZZ  <--- ZZ

{ambient dimension => 6            }
 dimension of lineality space => 0
 dimension of polyhedron => 2
 number of facets => 3
 number of rays => 0
 number of vertices => 3

 {3, 3, 1}
```

The following commands give us a description of the faces of codimension i and the vertices on each face of a polytope P:

```
F_i = faces(i,P)
apply(F_i,vertices)
```

For $i = 1$, the output is

```
{{ambient dimension => 6           },
 dimension of lineality space => 0
 dimension of polyhedron => 1
 number of facets => 2
 number of rays => 0
 number of vertices => 2

 {{ambient dimension => 6           },
 dimension of lineality space => 0
 dimension of polyhedron => 1
 number of facets => 2
 number of rays => 0
 number of vertices => 2

 {{ambient dimension => 6           }}
 dimension of lineality space => 0
 dimension of polyhedron => 1
 number of facets => 2
 number of rays => 0
 number of vertices => 2

{| -1 1   | , | 0   -1 | , | 0   1   |}
 | 1   0  |   | -1 1   |   | -1 0   |
 | 0   -1 |   | 1   0  |   | 1   -1 |
 | 0   -1 |   | 1   0  |   | 1   -1 |
 | 1   0  |   | -1 1   |   | -1 0   |
 | -1 1   |   | 0   -1 |   | 0   1  |
```

The following *Sage* code computes the Hilbert series of the Chow ring for a given matroid. The code computing the Chow ring was contributed to the *Sage* system by Travis Scrimshaw. In the example, we use the Specht matrix `sm22` computed above.

```
def chow_ring_dimensions(mm, R=None):
    # Setup
    if R is None:
        R = ZZ
    # We only want proper flats
    flats = [X for i in range(1, mm.rank())
             for X in mm.flats(i)]
    E = list(mm.groundset())
    flats_containing = {x: [] for x in E}
    for i,F in enumerate(flats):
        for x in F:
            flats_containing[x].append(i)

    # Create the ambient polynomial ring
    from sage.rings.polynomial\
        .polynomial_ring_constructor
    import PolynomialRing
    try:
        names = ['A{}'.format(''.join(str(x)
                 for x in sorted(F)))
                 for F in flats]
        P = PolynomialRing(R, names)
    except ValueError: # variables have
                       # improper names
        P = PolynomialRing(R, 'A', len(flats))
        names = P.variable_names()
    gens = P.gens()
    # Create the ideal of quadratic relations
    Q = [gens[i] * gens[i+j+1]
         for i,F in enumerate(flats)
         for j,G in enumerate(flats[i+1:])
         if not (F < G or G < F)]
    # Create the ideal of linear relations
    L = [sum(gens[i] for i in flats_containing[x])
         - sum(gens[i] for i in flats_containing[y])
         for j,x in enumerate(E) for y in E[j+1:]]
    # Compute Hilbert series using Macaulay2
    macaulay2.eval("restart")
    macaulay2.eval("R=QQ[" + str(gens)[1:-1] + "]")
    macaulay2.eval("I=ideal(" + str(Q)[1:-1] + ",
    " + str(L)[1:-1] + ")")
    hs = macaulay2.eval("toString hilbertSeries I")
    T = PolynomialRing(RationalField(),"T").gen()
    return sage_eval(hs, locals={'T':T})

chow_ring_dimensions(Matroid(sm22))
```

The output of the code for our example is

```
T+1
```

We now give the code for Examples 5.2 and 5.4. The following *Sage* commands compute the matroid corresponding to a given matrix, the lattice of flats of a matroid, a list of flats of a given rank, and the characteristic polynomial of a matroid.

```
X = matrix([[1, 0, 0, 1, 1, 1], [0, 1, 0, 2, 3, 4],
                         [0, 0, 1, 0, 0, 1]])
M = Matroid(X)
M
M.lattice_of_flats()
sorted([sorted(F) for F in M.lattice_of_flats()])
F1 = M.flats(1)
sorted([sorted(F) for F in F1])
rank = M.rank()
Tutte_polynomial = M.tutte_polynomial()
Tutte_polynomial
var('t')
char_poly = (-1)^rank * expand(Tutte_polynomial(1-t,0))
char_poly
```

The output of the above code is

```
Linear matroid of rank 3 on 6 elements represented over
the Rational Field
Finite lattice containing 18 elements
[[], [0], [0, 1, 2, 3, 4, 5], [0, 1, 3, 4], [0, 2],
[0, 5], [1], [1, 2], [1, 5], [2], [2, 3], [2, 4],
[2, 5], [3], [3, 5], [4], [4, 5], [5]]
[[0], [1], [2], [3], [4], [5]]
x^3 + x*y^2 + y^3 + 3*x^2 + 2*x*y + 2*y^2 + 3*x + 3*y
t
t^3 - 6*t^2 + 12*t - 7
```

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The Degree of $\mathbf{SO}(n, \mathbb{C})$

Madeline Brandt, Juliette Bruce, Taylor Brysiewicz, Robert Krone, and Elina Robeva

Abstract We provide a closed formula for the degree of $\mathrm{SO}(n, \mathbb{C})$. In addition, we test symbolic and numerical techniques for computing the degree of $\mathrm{SO}(n, \mathbb{C})$. As an application of our results, we give a formula for the number of critical points of a low-rank semidefinite programming problem. Finally, we provide evidence for a conjecture regarding the real locus of $\mathrm{SO}(n, \mathbb{C})$.

MSC 2010 codes: 14L35, 20G20, 15N30

1 Introduction

The *special orthogonal group* $\mathrm{SO}(n, \mathbb{R})$ is the group of automorphisms of $\mathbb{R}^n$ which preserve the standard inner product and have determinant equal to one. The complex special orthogonal group is the complexification of $\mathrm{SO}(n, \mathbb{R})$ or, more explicitly, the

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group of matrices $\mathrm{SO}(n, \mathbb{C}) := \{M \in \mathbb{C}^{n\times n} : \det(M) = 1 \text{ and } M^{\mathsf{T}}M = I\}$. Since the defining conditions are polynomials in the entries of the matrix M, the group $\mathrm{SO}(n, \mathbb{C})$ is also a complex variety.

The degree of a complex variety $X \subset \mathbb{C}^n$ is the generic number of points in the intersection of X with a linear space of complementary dimension. Problem 4 on Grassmannians in [19] seeks a formula for the degree of $\mathrm{SO}(n, \mathbb{C})$. Our main result provides this.

Theorem 1.1 *The degree of* $\mathrm{SO}(n, \mathbb{C})$ *equals* $2^{n-1} \det\left[\binom{2n-2i-2j}{n-2i}\right]_{1\leq i,j\leq \lfloor n/2\rfloor}$.

Our proof of Theorem 1.1 uses a formula of Kazarnovskij [14] for the degree of the image of a representation of a connected reductive algebraic group over an algebraically closed field; see Theorem 2.4 for more information. By applying this formula to the case of the standard representation of $\mathrm{SO}(n, \mathbb{C})$, we are able to express the degree in terms of its root data and other invariants. As an added feature, Theorem 4.2 provides a combinatorial interpretation of this degree in terms of non-intersecting lattice paths. In contrast with Theorem 1.1, the combinatorial statement has the benefit of being obviously non-negative.

In order to verify Theorem 1.1, as well as explore the structure of $\mathrm{SO}(n, \mathbb{C})$ in further depth, it is useful to compute this degree explicitly. We were able to do this, for small n, using symbolic and numerical computations. A comparison of the success of these approaches is illustrated in Table 1.

Remark 1.2 Let $\Bbbk$ be a field of characteristic zero. We can define $\mathrm{SO}(n, \Bbbk)$ using the same system of equations because they are defined over the prime field $\mathbb{Q}$. For a field $\Bbbk$ that is not algebraically closed, the degree of a variety can be defined in terms of the Hilbert series of its coordinate ring. Since the Hilbert series does not depend on the choice of $\Bbbk$, the degree does not either. We choose to work over $\mathbb{C}$ not only for simplicity, but also so that we may use the above definition of degree.

Remark 1.3 Our methods are not restricted to $\mathrm{SO}(n, \mathbb{C})$ and can be used to compute the degree of other algebraic groups. For example, we provide a similar closed formula for the degree of the symplectic group in Sect. 3 and a combinatorial reinterpretation in Sect. 4.

Table 1 Degree of $\mathrm{SO}(n, \mathbb{C})$ computed in various ways

n	Symbolic	Numerical	Formula
2	2	2	2
3	8	8	8
4	40	40	40
5	384	384	384
6	-	4768	4768
7	-	111616	111616
8	-	-	3433600
9	-	-	196968448

This project started in the spring of 2014, when Benjamin Recht asked the fifth author to describe the geometry of a low-rank optimization problem; see Sect. 5. In particular, Benjamin asked why the augmented Lagrangian algorithm for solving this problem [5] almost always recovers the correct optimum despite the existence of multiple local minima. It quickly became clear that to even compute the number of local extrema, one needs to know the degree of the orthogonal group. In Sect. 5, we find a formula for the number of critical points of the low-rank semidefinite programming problem; see Theorem 5.3.

The rest of this article is organized as follows. In Sect. 2, we give the reader a brief introduction to algebraic groups and state the Kazarnovskij Theorem. Section 3 proves Theorem 1.1 by applying the Kazarnovskij Theorem and simplifying the resulting expressions. After simplification, we are left with a determinant of binomial coefficients that can be interpreted combinatorially using the celebrated Gessel–Viennot Lemma; see Sect. 4. The relationship between the degree of SO($n, \mathbb{C}$) and the degree of the low-rank optimization programming problem is elaborated upon in Sect. 5. Section 6 contains descriptions of the symbolic and numerical techniques involved in the explicit computation of deg SO($n, \mathbb{C}$). Finally, in Sect. 7, we explore questions involving the real points on SO($n, \mathbb{C}$).

2 Background

In this section, we provide the reader with the language to understand the Kazarnovskij Theorem, our main tool for determining the degree of SO($n, \mathbb{C}$). We invite those who already are familiar with Lie theory to skip to the statement of Theorem 2.4. Aside from applying Theorem 2.4, no understanding of the material in this section is necessary for understanding the remainder of the proof of Theorem 1.1. A more thorough treatment of the theory of algebraic groups can be found in [6, 8, 13].

An *algebraic group* G is a variety equipped with a group structure such that multiplication and inversion are both regular maps on G. When the unipotent radical of G is trivial and G is over an algebraically closed field, we say that G is a *reductive group*. Throughout this section, we let G denote a connected reductive algebraic group over an algebraically closed field $\mathbb{k}$. Let $\mathbb{G}_{\mathrm{m}}$ denote the multiplicative group of $\mathbb{k}$, so as a set $\mathbb{G}_{\mathrm{m}} = \mathbb{k} \setminus \{0\}$. Let T denote a fixed maximal torus of G, that is a subgroup of G isomorphic to $\mathbb{G}_{\mathrm{m}}^r$ and which is maximal with respect to inclusion. The number $r \in \mathbb{N}$ is well-defined and is called the *rank* of G. After fixing T, we define the *Weyl group* of G, denoted $W(G)$, to be the quotient of the normalizer of T by its centralizer: $W(G) := N_G(T)/Z_G(T)$. Like the rank, the group $W(G)$ does not depend on the choice of T up to isomorphism.

Example 2.1 The map $\mathrm{R}\colon \mathbb{G}_\mathrm{m} \to \mathrm{SO}(2,\Bbbk)$, given by $\mathrm{R}(t) := \frac{1}{2}\begin{bmatrix} t+t^{-1} & -i(t-t^{-1}) \\ i(t-t^{-1}) & t+t^{-1}\end{bmatrix}$, parametrizes $\mathrm{SO}(2,\Bbbk)$ and is a group isomorphism. If $\Bbbk = \mathbb{C}$, then the rotation by an angle θ corresponds to the matrix $\mathrm{R}(e^{i\theta})$. Therefore, the algebraic group $\mathrm{SO}(2,\Bbbk)$ has rank 1.

If $r \geq 1$, then the maximal tori of rank r in their respective algebraic groups are

$$T_{2r} := \left\{ \begin{bmatrix} \mathrm{R}(t_1) & 0 & 0 \cdots & 0 \\ 0 & \mathrm{R}(t_2) & 0 \cdots & 0 \\ \vdots & \vdots & \vdots \ddots & \vdots \\ 0 & 0 & 0 \cdots & \mathrm{R}(t_r) \end{bmatrix} : t_i \in \mathbb{G}_\mathrm{m} \right\} \cong \mathrm{SO}(2,\Bbbk)^r \subset \mathrm{SO}(2r,\Bbbk)\,,$$

$$T_{2r+1} := \left\{ \begin{bmatrix} \mathrm{R}(t_1) & 0 & 0 \cdots & 0 & 0 \\ 0 & \mathrm{R}(t_2) & 0 \cdots & 0 & 0 \\ \vdots & \vdots & \vdots \ddots & \vdots & \vdots \\ 0 & 0 & 0 \cdots & \mathrm{R}(t_r) & 0 \\ 0 & 0 & 0 \cdots & 0 & 1 \end{bmatrix} : t_i \in \mathbb{G}_\mathrm{m} \right\} \cong \mathrm{SO}(2,\Bbbk)^r \subset \mathrm{SO}(2r+1,\Bbbk)\,.$$

Therefore, we have $\operatorname{rank} \mathrm{SO}(2r,\Bbbk) = \operatorname{rank} \mathrm{SO}(2r+1,\Bbbk) = r$ and see that the rank of $\mathrm{SO}(n,\Bbbk)$ depends on the parity of n.

The *character group* $M(T)$ is the set of algebraic group homomorphisms from T to $\mathbb{G}_\mathrm{m}$. In other words, $M(T) := \mathrm{Hom}_{\mathrm{AlgGrp}}(T, \mathbb{G}_\mathrm{m})$ consists of the group homomorphisms defined by polynomial maps. Since T is isomorphic to $\mathbb{G}_\mathrm{m}^r$, all such homomorphisms must be of the form $(t_1, t_2, \ldots, t_r) \mapsto t_1^{a_1} t_2^{a_2} \cdots t_r^{a_r}$ for some integers $a_1, a_2, \ldots, a_r$. Hence, the character group $M(T)$ is isomorphic to $\mathbb{Z}^r$ and, for this reason, it is often called the character lattice. The *group of 1-parameter subgroups* $N(T) := \mathrm{Hom}_{\mathrm{AlgGrp}}(\mathbb{G}_\mathrm{m}, T)$ is dual to $M(T)$ and is also isomorphic to $\mathbb{Z}^r$. Indeed, each 1-parameter subgroup is of the form $t \mapsto (t^{b_1}, t^{b_2}, \ldots, t^{b_r})$ for some integers $b_1, b_2, \ldots, b_r$. Moreover, there exists a natural bilinear pairing $M(T) \times N(T) \to \mathrm{Hom}_{\mathrm{AlgGrp}}(\mathbb{G}_\mathrm{m}, \mathbb{G}_\mathrm{m}) \cong \mathbb{Z}$ given by $\langle \chi, \sigma \rangle \mapsto \chi \circ \sigma$.

Now, if $\rho\colon G \to \mathrm{GL}(V)$ is a representation of G, then we attach to it special characters called weights. A *weight* of the representation ρ is a character $\chi \in M(T)$ such that the set

$$V_\chi := \bigcap_{s \in T} \ker\big(\rho(s) - \chi(s)\,\mathrm{Id}_V\big)$$

is non-trivial. This condition is equivalent to saying that all of the matrices in $\{\rho(s) : s \in T\}$ have a simultaneous eigenvector $v \in V$ such that the associated eigenvalue for $\rho(s)$ is $\chi(s)$. We write C_V for the convex hull of the weights of the representation ρ.

Example 2.2 An important example for us comes from the defining representation $\rho\colon \mathrm{SO}(n,\mathbb{C}) \hookrightarrow \mathrm{GL}(n,\mathbb{C})$. Let $e_1, e_2, \dots, e_n$ denote the standard basis for $\mathbb{C}^n$. For any $t \in \mathbb{G}_{\mathrm{m}}$, the matrix $\mathrm{R}(t) \in \mathrm{SO}(2,\mathbb{C})$ has eigenvectors $e_1 + ie_2$ and $e_1 - ie_2$ with eigenvalues t and t^{-1} respectively. From the explicit description of the maximal torus T in Example 2.1, it follows that the eigenvectors of ρ are all vectors of the form $e_{2j-1} \pm ie_{2j}$ with $1 \le j \le r$ and the corresponding eigenvalues are $t_1^{\pm1}, t_2^{\pm1}, \dots, t_r^{\pm1}$. These eigenvalues, viewed as characters, are the weights of ρ. Additionally, when $n = 2r + 1$, we see that e_{2r+1} is an eigenvector with eigenvalue 1, corresponding to the trivial character.

Another representation of a matrix group $G \subseteq \mathrm{End}(V)$ is the *adjoint representation* $\mathrm{Ad}\colon G \to \mathrm{GL}\big(\mathrm{End}(V)\big)$, where $\mathrm{Ad}(g)$ the linear map defined by $A \mapsto gAg^{-1}$. The *roots* of G are the nonzero weights of the adjoint representation. Given a linear functional ℓ on $M(T)$, we define the *positive roots* of G with respect to ℓ to be the roots χ such that $\ell(\chi) > 0$. We denote the positive roots of G by $\alpha_1, \alpha_2, \dots, \alpha_l$. For the algebraic groups in this paper, we can choose ℓ to be the inner product with the vector $(r, r-1, \dots, 1)$, so that a root of the form $e_j - e_k$ is positive if and only if $j < k$. To each root α, we associate a *coroot* $\check{\alpha}$, defined to be the linear function $\check{\alpha}(\mathbf{x}) := 2\langle \mathbf{x}, \alpha\rangle / \langle \alpha, \alpha\rangle$ where the pairing is $W(G)$-invariant. Throughout this paper, we fix the pairing to be the standard inner product.

Example 2.3 We now describe the roots of $\mathrm{SO}(n,\mathbb{C})$, starting with $n = 2r$. The simultaneous eigenvectors of $\mathrm{Ad}(s)$ over all $s \in T$ are matrices A with the following structure. These matrices are zero outside a (2×2)-block B in rows $2j-1, 2j$ and columns $2k-1, 2k$ for some $1 \le j,k \le r$. Furthermore, $B = v_1 v_2^{\mathsf{T}}$ with each vector v_k, for $1 \le k \le 2$, equals one of the eigenvectors of $\mathrm{R}(t)$, namely $e_1 \pm ie_2$. If $s \in T$ has blocks along the diagonal $\mathrm{R}(t_j)$ with $t_1, t_2, \dots, t_r \in \mathbb{G}_{\mathrm{m}}$, then the matrix $\mathrm{Ad}(s)(A)$ will also be zero except in the same (2×2)-block, and that block will be $\mathrm{R}(t_j) B\, \mathrm{R}(t_k)^{\mathsf{T}} = t_j^{\pm1} t_k^{\pm1} B$, where the signs depend on the choices of v_1 and v_2. Taking the exponent vectors of these eigenvalues, we see that the roots of $\mathrm{SO}(2r,\mathbb{C})$ are the characters of the form $\pm(e_j \pm e_k)$ for $1 \le j,k \le r$.

When $n = 2r + 1$, the matrix A has an extra row and column. If the matrix A has support only in the last column, then we have $\mathrm{Ad}(s)(A) = sAs^{-1}$. But s^{-1} acts trivially on the left, while s acts on the last column as an element of $\mathrm{GL}(n,\mathbb{C})$ as in the standard representation. As in Example 2.2, the eigenvalues are $t_1^{\pm1}, t_2^{\pm1}, \dots, t_r^{\pm1}, 1$. The same weights appear for A with support in the last row. Hence, the roots of $\mathrm{SO}(2r+1,\mathbb{C})$ are $\pm(e_j \pm e_k)$ for $1 \le j,k \le r$ and $\pm e_i$ for $1 \le i \le r$.

Associated to the algebraic group G is a Lie algebra $\mathfrak{g}$ that comes equipped with a Lie bracket $[\cdot,\cdot]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$. A *Cartan subalgebra* $\mathfrak{h}$ is a nilpotent subalgebra of $\mathfrak{g}$ that is self-normalizing; if $[x,y] \in \mathfrak{h}$ for all $x \in \mathfrak{h}$, then we have $y \in \mathfrak{h}$. Let $S(\mathfrak{h}^*)$ be the ring of polynomial functions on $\mathfrak{h}$. The Weyl group $W(G)$ acts on $\mathfrak{h}$, and this extends to an action of $W(G)$ on $S(\mathfrak{h}^*)$. The space $S(\mathfrak{h}^*)^{W(G)}$ of polynomials which are invariant up to the action of $W(G)$ is generated by r homogeneous polynomials whose degrees, $c_1 + 1, c_2 + 1, \dots, c_r + 1$, are uniquely determined. The values $c_1, c_1, \dots, c_r$ are called *Coxeter exponents*.

Table 2 Data required to apply the Kazarnovskij Theorem

Group	Dimension	Rank	Positive roots	Weights	$\lvert W(G)\rvert$	Coxeter exponents
$\mathrm{SO}(2r+1,\mathbb{C})$	$\binom{2r+1}{2}$	r	$\{e_i \pm e_j\}_{i<j} \cup \{e_i\}$	$\{\pm e_i\}$	$r!2^r$	$1, 3, 5, \ldots, 2r-1$
$\mathrm{Sp}(2r,\mathbb{C})$	$\binom{2r+1}{2}$	r	$\{e_i \pm e_j\}_{i<j} \cup \{2e_i\}$	$\{\pm e_i\}$	$r!2^r$	$1, 3, 5, \ldots, 2r-1$
$\mathrm{SO}(2r,\mathbb{C})$	$\binom{2r}{2}$	r	$\{e_i \pm e_j\}_{i<j}$	$\{\pm e_i\}$	$r!2^{r-1}$	$1, 3, 5, \ldots, 2r-3, r-1$

We are now prepared to state the Kazarnovskij Theorem.

Theorem 2.4 (Kazarnovskij Theorem, [6, Proposition 4.7.18]) *Let G be a connected reductive algebraic group of dimension m and rank r over an algebraically closed field. If $\rho\colon G \to \mathrm{GL}(V)$ is a representation with finite kernel, then we have*

$$\deg\left(\overline{\rho(G)}\right) = \frac{m!}{|W(G)|\,(c_1!c_2!\cdots c_r!)^2\,|\ker(\rho)|} \int_{C_V} (\check{\alpha}_1\check{\alpha}_2\cdots\check{\alpha}_r)^2\, dv\,,$$

where $W(G)$ is the Weyl group, the c_i are Coxeter exponents, C_V is the convex hull of the weights, and the $\check{\alpha}_i$ are the coroots.

If ρ is the standard representation for an algebraic group G, then it follows that $\deg \overline{\rho(G)} = \deg G$. Thus, in order to compute $\deg \mathrm{SO}(n,\mathbb{C})$, all we must do is apply this theorem for the standard representation of $\mathrm{SO}(n,\mathbb{C})$. The relevant data for this theorem is given in Table 2 for $\mathrm{SO}(n,\mathbb{C})$ and $\mathrm{Sp}(n,\mathbb{C})$.

3 Main Result: The Degree of $\mathrm{SO}(n,\mathbb{C})$

We now prove our main result, Theorem 1.1. At the end of this section, we also use the same method to obtain a formula for the degree of the symplectic group.

We begin by applying Theorem 2.4 to $\mathrm{SO}(2r,\mathbb{C})$ and $\mathrm{SO}(2r+1,\mathbb{C})$ to obtain

$$\deg \mathrm{SO}(2r,\mathbb{C}) = \frac{\binom{2r}{2}!}{r!2^{r-1}\big((r-1)!\big)^2 \prod_{k=1}^{r-1}\big((2k-1)!\big)^2} \int_{C_V} \prod_{1\le i<j\le r} (x_i^2 - x_j^2)^2\, dv\,,$$

$$\deg \mathrm{SO}(2r+1,\mathbb{C}) = \frac{\binom{2r+1}{2}!}{r!2^{r} \prod_{k=1}^{r}\big((2k-1)!\big)^2} \int_{C_V} \prod_{1\le i<j\le r} (x_i^2 - x_j^2)^2 \prod_{i=1}^{r}(2x_i)^2\, dv\,.$$

To compute the degree of $\mathrm{SO}(n,\mathbb{C})$, it suffices to find formulas for these integrals. We do this by expanding the integrand into monomials and integrating the result. We first use the well-known expression for the determinant of the Vandermonde matrix:

$$\prod_{1\le i<j\le r} (y_j - y_i) = \sum_{\sigma\in\mathfrak{S}_r} \mathrm{sgn}(\sigma) \prod_{i=1}^{r} y_i^{\sigma(i)-1}\,,$$

where $\mathfrak{S}_r$ denotes the symmetric group on $\{1, 2, \dots, r\}$. Substituting $y_i = x_i^2$ and squaring the entire expression yields

$$\prod_{1\le i<j\le r} (x_i^2 - x_j^2)^2 = \sum_{\sigma,\tau\in\mathfrak{S}_r} \operatorname{sgn}(\sigma\tau) \prod_{i=1}^{r} x_i^{2\sigma(i)+2\tau(i)-4} . \tag{1}$$

Every variable in the integrand is being raised to an even power, and C_V is the convex hull of weights $\{\pm e_i\}$. Because of this symmetry, the integrals over C_V are 2^r times the same integrals over the r-simplex $\Delta_r := \operatorname{conv}(0, e_1, e_2, \dots, e_r) \subset \mathbb{R}^r$. Hence, we have reduced the computation to understanding the integral of any monomial over the simplex Δ_r. The following lemma provides the required formula.

Lemma 3.1 ([15, Lemma 4.23]) *Consider the r-simplex $\Delta_r := \operatorname{conv}(0, e_1, e_2, \dots, e_r)$ in $\mathbb{R}^r$. If $\mathbf{a} = (a_1, a_2, \dots, a_r) \in \mathbb{Z}^r_{>0}$, then we have*

$$\int_{\Delta_r} \mathbf{x}^{\mathbf{a}}\, d\mathbf{x} = \int_{\Delta_r} x_1^{a_1} x_2^{a_2} \cdots x_r^{a_r}\, dx_1 dx_2 \cdots dx_r = \frac{1}{(r + \sum_i a_i)!} \prod_i a_i! .$$

With these preliminaries, we can now prove the key technical result in this section.

Proposition 3.2 *We have*

$$\int_{C_V} \prod_{1\le i<j\le r} (x_i^2 - x_j^2)^2\, dv = \frac{r!2^r}{\binom{2r}{2}!} \det\big[(2i + 2j - 4)!\big]_{1\le i,j\le r} ,$$

$$\int_{C_V} \prod_{1\le i<j\le r} (x_i^2 - x_j^2)^2 \prod_{i=1}^{r} (2x_i)^2\, dv = \frac{r!2^{3r}}{\binom{2r+1}{2}!} \det\big[(2i + 2j - 2)!\big]_{1\le i,j\le r} .$$

Proof Exploiting the symmetry of C_v along with equation (1) gives

$$I_{\text{odd}}(r) := \int_{C_V} \prod_{1\le i<j\le r} (x_i^2 - x_j^2)^2 \prod_{i=1}^{r} (2x_i)^2\, dv = 2^r \int_{\Delta_r} \prod_{1\le i<j\le r} (x_i^2 - x_j^2)^2 \prod_{i=1}^{r} (2x_i)^2\, dv$$

$$= 2^{3r} \sum_{\sigma,\tau\in\mathfrak{S}_r} \operatorname{sgn}(\sigma\tau) \int_{\Delta_r} \prod_{i=1}^{r} x_i^{2\sigma(i)+2\tau(i)-2}\, dv .$$

As the integrand is homogeneous of degree $4\binom{r}{2} + 2r$, Lemma 3.1 yields

$$I_{\text{odd}}(r) = \frac{2^{3r}}{\big(4\binom{r}{2} + 3r\big)!} \sum_{\sigma,\tau\in\mathfrak{S}_r} \operatorname{sgn}(\sigma\tau) \prod_{i=1}^{r} \big(2\sigma(i) + 2\tau(i) - 2\big)! .$$

Since $\sigma \in \mathfrak{S}_r$, we may reindex the product by $\sigma^{-1}(i)$ rather than i to obtain $\prod_{i=1}^{r}\big(2\sigma(i)+2\tau(i)-2\big)! = \prod_{i=1}^{r}\big(2i+2\tau\sigma^{-1}(i)-2\big)!$. Ranging over all $\sigma, \tau \in \mathfrak{S}_r$, each permutation in $\mathfrak{S}_r$ appears exactly $r!$ times as the composition $\upsilon := \tau\sigma^{-1}$ and $\operatorname{sgn}(\sigma\tau) = \operatorname{sgn}(\upsilon)$. Therefore, we have

$$\begin{aligned} I_{\text{odd}}(r) &= \frac{r!2^{3r}}{\big(4\binom{r}{2}+3r\big)!}\sum_{\upsilon\in\mathfrak{S}_r}\operatorname{sgn}(\upsilon)\prod_{i=1}^{r}(2i+2\upsilon(i)-2)! \\ &= \frac{r!2^{3r}}{\binom{2r+1}{2}!}\det\big[(2i+2j-2)!\big]_{1\le i,j\le r}. \end{aligned}$$

The calculation for $I_{\text{even}}(r) := \int_{C_V}\prod_{1\le i<j\le r}(x_i^2 - x_j^2)^2 \; dv$ follows the same steps. □

Proof of Theorem 1.1 Combining Theorem 2.4, the data in Table 2, and Proposition 3.2, we have

$$\begin{aligned} \deg \mathrm{SO}(2r+1,\mathbb{C}) &= \frac{\binom{2r+1}{2}!}{r!2^r\prod_{k=1}^{r}\big((2k-1)!\big)^2}\int_{C_V}\prod_{1\le i<j\le r}(x_i^2-x_j^2)^2\prod_{i=1}^{r}(2x_i)^2\; dv \\ &= \frac{2^{2r}}{\prod_{k=1}^{r}\big((2k-1)!\big)^2}\det\big[(2i+2j-2)!\big]_{1\le i,j\le r}. \end{aligned}$$

Since the determinant is linear in each row and column, we obtain

$$\deg \mathrm{SO}(2r+1,\mathbb{C}) = 2^{2r}\det\left[\frac{(2i+2j-2)!}{(2i-1)!(2j-1)!}\right] = 2^{2r}\det\left[\binom{2i+2j-2}{2i-1}\right]_{1\le i,j\le r}.$$

Reversing the order of the rows and columns of the final matrix and reindexing produces the required formula. Similarly, for the even case, we have

$$\begin{aligned} \deg \mathrm{SO}(2r,\mathbb{C}) &= \frac{\binom{2r}{2}!}{r!2^{r-1}\big((r-1)!\big)^2\prod_{k=1}^{r-1}\big((2k-1)!\big)^2}\int_{C_V}\prod_{1\le i<j\le r}(x_i^2-x_j^2)^2\; dv \\ &= \frac{2}{\big((r-1)!\big)^2\prod_{k=1}^{r-1}\big((2k-1)!\big)^2}\det\big[(2i+2j-4)!\big]_{1\le i,j\le r} \\ &= \frac{2(2^{r-1})^2}{\prod_{k=1}^{r-1}(2k)^2\prod_{k=1}^{r-1}\big((2k-1)!\big)^2}\det\big[(2i+2j-4)!\big]_{1\le i,j\le r} \end{aligned}$$

$$= \frac{2^{2r-1}}{\prod_{k=1}^{r}\big((2k-2)!\big)^2} \det\big[(2i+2j-4)!\big]_{1\le i,j\le r}$$

$$= 2^{2r-1} \det\left[\frac{(2i+2j-4)!}{(2i-2)!(2j-2)!}\right]_{1\le i,j\le r}$$

$$= 2^{2r-1} \det\left[\binom{2i+2j-4}{2i-2}\right] = 2^{2r-1} \det\left[\binom{4r-2i-2j}{2r-2i}\right]_{1\le i,j\le r}.$$

□

Since the orthogonal group $\mathrm{O}(n,\mathbb{C}) := \{M \in \mathbb{C}^{n\times n} : M^{\mathsf{T}}M = MM^{\mathsf{T}} = I\}$ has two connected components that are isomorphic to $\mathrm{SO}(n,\mathbb{C})$, we immediately get a formula for the degree of $\mathrm{O}(n,\mathbb{C})$.

Corollary 3.3 *The degree of* $\mathrm{O}(n,\mathbb{C})$ *equals* $2^n \det\big[\binom{2n-2i-2j}{n-2i}\big]_{1\le i,j\le \lfloor \frac{n}{2}\rfloor}$.

We also easily obtain the degree of the symplectic group $\mathrm{Sp}(2r,\mathbb{C})$. By definition, we have $\mathrm{Sp}(2r,\mathbb{C}) := \{M \in \mathbb{C}^{2r\times 2r} : M^{\mathsf{T}}\Omega M = \Omega\}$ where

$$\Omega := \begin{bmatrix} 0 & I_r \\ -I_r & 0 \end{bmatrix} \in \mathbb{C}^{2r\times 2r}.$$

Corollary 3.4 *We have* $\deg \mathrm{SO}(2r+1,\mathbb{C}) = 2^{2r} \deg \mathrm{Sp}(2r,\mathbb{C})$ *and*

$$\deg \mathrm{Sp}(2r,\mathbb{C}) = \det\left[\binom{2(2r+1)-2i-2j}{(2r+1)-2i-1}\right]_{1\le i,j\le r}.$$

Proof Comparing the first two rows in Table 2, we see that the Weyl groups for $\mathrm{SO}(2r+1,\mathbb{C})$ and $\mathrm{Sp}(2r,\mathbb{C})$ have the same cardinality, the Coxeter exponents are equal, the convex hull of the weights are equal, and there is a natural bijection between the coroots. In fact, among the r^2 coroots for $\mathrm{SO}(2r+1,\mathbb{C})$ and $\mathrm{Sp}(2r,\mathbb{C})$, $r(r-1)$ are equal and r differ by a factor of 2 with the coroots for $\mathrm{Sp}(2r,\mathbb{C})$ being larger. Hence, Theorem 2.4 implies that $\deg \mathrm{SO}(2r+1,\mathbb{C}) = 2^{2r} \deg \mathrm{Sp}(2r,\mathbb{C})$ and Theorem 1.1 shows that $\deg \mathrm{Sp}(2r,\mathbb{C}) = \det\big[\binom{2(2r+1)-2i-2j}{(2r+1)-2i-1}\big]_{1\le i,j\le r}$. □

4 Non-intersecting Lattice Paths

This section gives a combinatorial interpretation for the determinant appearing in our formulas for the degree of $\mathrm{SO}(n,\mathbb{C})$. In particular, we show that this determinant counts appropriate collections of non-intersecting lattice paths by using the celebrated Lindström–Gessel–Viennot Lemma; see [1, Chap. 29] or [10, Theorem 1].

To sketch this approach, let Q be a locally-finite directed acyclic graph. Since there are no directed cycles in Q and every vertex in Q is the tail of only finitely many arrows, it follows that there are only finitely many directed paths (connected sequences of distinct arrows all oriented in the same direction) between any two vertices. For pair a, b of vertices in Q, let $m_{a,b} \in \mathbb{N}$ be number of directed paths from a to b. Given two finite lists $A := \{a_1, a_2, \dots, a_r\}$ and $B := \{b_1, b_2, \dots, b_r\}$ of vertices, the associated *path matrix* is $M := [m_{a_i,b_j}]_{1 \le i,j \le r} \in \mathbb{N}^{r \times r}$. A *path system* P from A to B consists of a permutation $\sigma \in \mathfrak{S}_r$ together with r directed paths from a_i to $b_{\sigma(i)}$. For $\sigma \in \mathfrak{S}_r$, set $\operatorname{sgn}(\sigma) := (-1)^k$ where k is the number of inversions in σ. If the paths in P are pairwise vertex-disjoint, then P is a *non-intersecting* path system. The following "lemma" relates $\det M$ with non-intersecting path systems.

Lemma 4.1 (Lindström–Gessel–Viennot) *If A and B are finite lists, having the same cardinality and consisting of vertices from a locally-finite directed acyclic graph, then the determinant of the associated path matrix M equals the signed sum of the non-intersecting path systems from A to B:* $\det M = \sum_P \operatorname{sgn}(\sigma)$. □

For our application, consider the directed grid graph whose vertices are the lattice points in $\mathbb{Z}^2$ and whose arrows are unit steps in either the north or east direction. In other words, the vertex $(i, j) \in \mathbb{Z}^2$ is the tail of exactly two arrows: one with head $(i, j+1)$ and the other with head $(i+1, j)$. The next result provides our combinatorial reinterpretation for the degree of $\mathrm{SO}(n, \mathbb{C})$.

Proposition 4.2 *Let $n \in \mathbb{N}$. If $N(n)$ is the number of non-intersecting path systems in the directed grid graph from $A := \{(2-n, 0), (4-n, 0), \dots, (2\lfloor n/2 \rfloor - n, 0)\}$ to $B := \{(0, n-2), (0, n-4), \dots, (0, n - 2\lfloor n/2 \rfloor)\}$, then we have*

$$\deg \mathrm{SO}(n, \mathbb{C}) = 2^{n-1} N(n)\,.$$

Proof By construction, the only non-intersecting path systems in our directed grid graph have direct paths from $(2i-n, 0)$ to $(0, n-2i)$ for $0 \le i \le \lfloor n/2 \rfloor$. Hence, the associated element in $\mathfrak{S}_{\lceil n/2 \rceil}$ is the identity permutation and the determinant of the associated path matrix counts the total number of non-intersecting path systems.

The number of directed paths from $(0, 0)$ to (i, j) in our directed grid graph is $\binom{i+j}{i}$; simply choose which i arrows in the connected sequence are oriented east. Since the grid graph is invariant under translation, it follows that the number of direct paths from the vertex $(2i-n, 0)$ to $(0, n-2j)$ equals $\binom{2n-2i-2j}{n-2i}$. Therefore, the path matrix associated to A and B is $M = \left[\binom{2n-2i-2j}{n-2i}\right]_{1 \le i,j \le \lfloor n/2 \rfloor}$. Combining Theorem 1.1 and Lemma 4.1, we conclude that $\deg \mathrm{SO}(n, \mathbb{C}) = 2^{n-1} N(n)$. □

Remark 4.3 From Corollaries 3.3–3.4, we also see that $\deg \mathrm{O}(n, \mathbb{C}) = 2^n N(n)$ and $\deg \mathrm{Sp}(2r, \mathbb{C}) = N(2r+1)$.

Example 4.4 For $n = 5$, the 24 non-intersecting path systems are illustrated in Fig. 1. It follows that $\deg \mathrm{SO}(5, \mathbb{C}) = 2^4(24) = 384$.

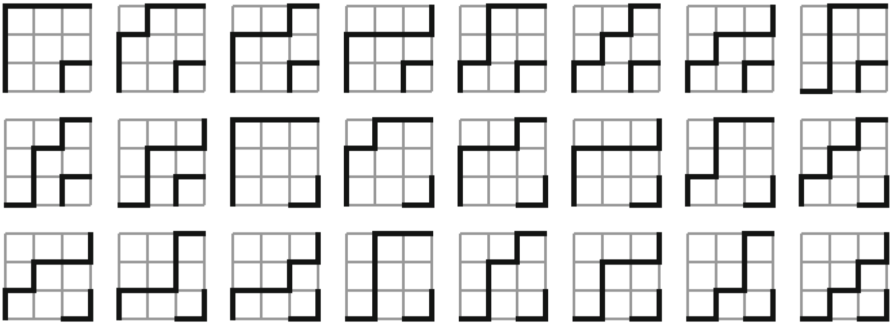

Fig. 1 The non-intersecting path systems from $\{(-3,0),(-1,0)\}$ to $\{(0,1),(0,3)\}$

Theorem 4.2 suggests that there might be a deeper relationship between the degree of SO$(n, \mathbb{C})$ and lattice paths. It would be interesting to find a direct connection. Since the degree of Sp$(2r, \mathbb{C})$ does not have a coefficient involving a power of 2, it may be the natural place to look for a combinatorial proof.

5 The Degree of a Low-Rank Optimization Problem

In this section, we show how the degree of SO$(n, \mathbb{C})$ arises in counting the number of critical points for a particular optimization problem.

To motivate our particular problem, we first consider a more general framework. The *trace* tr(A) of a square matrix C is the sum of the entries on the main diagonal, and a real symmetric matrix X is *positive semidefinite*, written $X \succeq 0$, if all of its eigenvalues are nonnegative. A semidefinite programming problem has the form:

For real symmetric matrices $C, A_1, A_2, \ldots, A_m \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^m$, minimize $\operatorname{tr}(CX)$, for all real symmetric matrices $X \in \mathbb{R}^{n\times n}$, subject to the constraints that $X \succeq 0$ and $\operatorname{tr}(A_iX) = b_i$ for all $1 \leq i \leq m$. (SDP)

Many practical problems can be modeled as, and many NP-hard problems can be approximated by, semidefinite programming problems; see [3, 11]. Although semidefinite programming problems can often be efficiently solved by interior point methods, this invariably becomes computationally prohibitive for large n. Since the rank of an optimal solution is often much smaller than n, Burer and Monteiro [5] study the hierarchy of relaxations in which X is replaced by the low-rank positive semidefinite matrix RR^{T}. Specifically, the new optimization problem is:

For real symmetric matrices $C, A_1, A_2, \ldots, A_m \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^m$, minimize $\operatorname{tr}(CRR^{\mathsf{T}})$, for all $R \in \mathbb{R}^{n\times r}$, subject to the constraints that $\operatorname{tr}(A_iRR^{\mathsf{T}}) = b_i$ for all $1 \leq i \leq m$. (NOP)

When $r < (n+1)/2$, this alternative formulation has the advantage of reducing the number of unknowns from $\binom{n+1}{2}$ to nr. However, the objective function and the contraints are no longer linear—they are quadratic and the feasible set is non-convex.

Burer and Monteiro [5] propose a fast algorithm for solving (NOP). Despite the existence of multiple local minima, this algorithm quickly finds the global minimum in practice. To help understand this phenomenon, we examine the *critical points*, those points where the partial derivatives of the associated Lagrangian function vanish, of (NOP). Before giving our formula for the number of critical points of the new optimization problem, we need the following notation.

Definition 5.1 For positive integers i and j, let $\psi_i := 2^{i-1}$, let $\psi_{0,j} := \psi_j$, and let $\psi_{i,j} := \sum_{k=i}^{j-1} \binom{i+j-2}{k}$. For $r > 2$, set

$$\psi_{i_1,i_2,\ldots,i_r} := \begin{cases} \operatorname{pf}\big[\psi_{i_k,i_\ell}\big]_{1\le k<\ell\le r} & \text{if } r \text{ is even} \\ \operatorname{pf}\big[\psi_{i_k,i_\ell}\big]_{0\le k<\ell\le r} & \text{if } r \text{ is odd,} \end{cases}$$

where pf denotes the Pfaffian of a skew-symmetric matrix. For positive integer m and n, we define $\delta(m,n,r) := \sum_I \psi_I \psi_{I'}$, where the sum runs over all strictly increasing subsequences $I := \{i_1, i_2, \ldots, i_{n-r}\}$ of $\{1, 2, \ldots, n\}$ with $i_1 + i_2 + \cdots + i_{n-r} = m$ and $I' := \{1, 2, \ldots, n\} \setminus I$ denotes the complement.

Remark 5.2 Originally defined in [16] as the number of critical points for (SDP) in which the matrix X has rank r, the number $\delta(m,n,r)$ is called the *algebraic degree* of the semidefinite programming problem. Our defining formula for $\delta(m,n,r)$ was subsequently computed in [2].

Theorem 5.3 *The number of critical points for* (NOP) *is* $2\delta(m,n,r)\deg \mathrm{SO}(r,\mathbb{C})$.

Proof Given new variables $y_1, y_2, \ldots, y_m$, the Lagrangian function associated to (NOP) is $L(R,y) := \operatorname{tr}(CRR^{\mathsf{T}}) - \sum_{i=1}^m y_i\big(\operatorname{tr}(A_iRR^{\mathsf{T}}) - b_i\big)$. Taking the partial derivatives of $L(R,y)$ yields the equations

$$\left(C - \sum_{i=1}^m y_iA_i\right)RR^{\mathsf{T}} = 0 \qquad \text{and} \qquad \operatorname{tr}(A_iRR^{\mathsf{T}}) = b_i\,, \quad \text{for } 1 \le i \le m,$$

which define the set of critical points. Analogously, the critical points for (SDP) are determined by the equations

$$\left(C - \sum_{i=1}^m y_iA_i\right)X = 0 \qquad \text{and} \qquad \operatorname{tr}(A_iX) = b_i\,, \quad \text{for } 1 \le i \le m.$$

Nie, Ranestad, and Sturmfels [16] show that the number of critical points for (SDP), for which the rank of X equals r, is $\delta(m,n,r)$. Comparing the defining systems of equations for the critical points of (NOP) and (SDP), we see that the fibre of the map $(R,y) \mapsto (RR^{\mathsf{T}}, y)$ over each point (X,y) consists of all points (R,y') for which

$X = RR^{\mathsf{T}}$ and $y' = y$. Given X and R such that $X = RR^{\mathsf{T}}$, all other matrices S such that (S, y) lies in the fibre over (X, y) have the form $S = RU$ where U is an orthogonal $(r \times r)$-matrix. In other words, the fibre is isomorphic to a copy of the orthogonal group. Therefore, the number of critical points for (NOP) equals $2\delta(m, n, r) \deg \mathrm{SO}(r, \mathbb{C})$. □

Since the number of critical points for (NOP) grows rapidly with the rank r, the appealing behaviour of the algorithm in [5] still needs to be explained.

Remark 5.4 For applications, the most important critical points for (NOP) are real and satisfy the equation $\left(C - \sum_{i=1}^{m} y_i A_i\right) \succeq 0$.

6 Computational Methods

Since Theorem 1.1 provides a formula for the degree of $\mathrm{SO}(n, \mathbb{C})$, this family of examples becomes an interesting testing ground for various symbolic and numerical methods for computing degrees. In this section, we outline three algorithmic techniques for calculating the degree of a variety. The first is based on Gröbner bases, the second uses polynomial homotopy continuation, and the third involves numerical monodromy. Table 1 summarizes the results of our computations, and the related *Macaulay2* code appears in the Appendix. Beyond contrasting these algorithms, we hope that the different routines and auxiliary data, such as Gröbner bases or witness sets, will lead to new insights into the degrees of varieties.

The standard symbolic algorithm for determining the degree of a variety first finds a Gröbner basis of the defining ideal and then uses combinatorial properties of the initial ideal to return the Hilbert polynomial; the degree can be easily extracted from the highest degree term of the Hilbert polynomial. As this method is independent of the ground field, one can speed up the calculation by working over a small finite field. With this algorithm, we were able to compute the degree of $\mathrm{SO}(n, \mathbb{C})$ for all $2 \leq n \leq 5$, but it was the slowest among the methods we compared.

The basic numerical strategy for computing the degree of $\mathrm{SO}(n, \mathbb{C})$ randomly chooses a linear subspace L of complementary dimension and counts the number of complex solutions S to the zero-dimensional system of polynomial equations corresponding to $\mathrm{SO}(n, \mathbb{C}) \cap L$. The triple $\left(\mathrm{SO}(n, \mathbb{C}), L, S\right)$ is called a *witness set* for $\mathrm{SO}(n, \mathbb{C})$. This triple is a fundamental data type in numerical algebraic geometry: the computation of a witness set is often a necessary input to other numerical algorithms, including sampling points on the variety, studying its asymptotic behaviour, computing its monodromy group, or even studying its real locus; see Sect. 7. Both numerical algorithms presented below produce a witness set for $\mathrm{SO}(n, \mathbb{C})$.

Polynomial homotopy continuation computes a witness set by finding numerical approximations for the complex solutions S. First, one constructs a polynomial system that has a similar structure to the target system and has a simple solution set. This start system is embedded in a homotopy relating it to the target system and the numerical solutions of the start system are traced towards solutions of the target system. Start systems correspond to root counts. For dense systems, one typically uses the Bézout bound whereas, for sparse systems, one uses the mixed volume of the appropriate Newton polytopes. However, for $\mathrm{SO}(n, \mathbb{C})$, both of these bounds are equal $2^{n(n+1)/2}$, which grows quickly (for $n = 6$, it is already 2 097 152). Because of the number of paths that must be tracked, we were only able to compute the degree of $\mathrm{SO}(n, \mathbb{C})$ for all $2 \leq n \leq 5$ using this method.

Our third technique takes advantage of monodromy; see [7]. Suppose L and L' are two linear subspaces of complementary dimension to $\mathrm{SO}(n, \mathbb{C})$. Given a point on the linear slice $W := \mathrm{SO}(n, \mathbb{C}) \cap L$, we can numerically track this solution along some path γ to a point in another slice $W' := \mathrm{SO}(n, \mathbb{C}) \cap L'$. Tracking the second point along a different path γ' back to W yields another point in W and induces a permutation $\sigma_{\gamma,\gamma'}$ on the points in W. Iterating this process, one expects to populate the witness set associated to W. Although there are algorithms [17] which certify that a witness set is complete, one frequently uses heuristic stopping criteria because they are much faster. This monodromy method is implemented in the *MonodromySolver* package for *Macaulay2* [9]. With the naive stopping criterion that no new points were found after ten consecutive iterations, we were able to calculate with this method the degree $\mathrm{SO}(n, \mathbb{C})$ for all $6 \leq n \leq 7$.

7 Real Points on $\mathrm{SO}(n, \mathbb{C})$

Motivated by the applications to optimization, this section investigates the structure of the real points in $\mathrm{SO}(n, \mathbb{C})$. Taking advantage of the numerical monodromy algorithm, we collect experimental data counting the number of real points in witness sets for $\mathrm{SO}(3, \mathbb{C})$, $\mathrm{SO}(4, \mathbb{C})$, and $\mathrm{SO}(5, \mathbb{C})$.

More precisely, we use the `random` function in *Macaulay2* [9] to generate a sample of linear slices of $\mathrm{SO}(n, \mathbb{C})$. Homotopy continuation allows us to track solutions from a precomputed witness set to those lying on each randomly chosen linear slice. We determine how many solutions in the random slice are real by checking whether each coordinate is within a 0.001 numerical tolerance of being real. One can actually certify reality using *alphaCertify* [12], which implements Smale's α-theory. However, for the sake of speed, we limited these formal checks to at least one witness set achieving the maximum observed number of real points. The results of computing 1,398,000, 1,004,100, and 48,200 witness sets for $\mathrm{SO}(3, \mathbb{C})$, $\mathrm{SO}(4, \mathbb{C})$, and $\mathrm{SO}(5, \mathbb{C})$ are displayed in Figs. 2 and 3.

The raw data and actually code can be found at [4]. In rare examples, the process failed to return a witness set on the randomly chosen linear slice, because the homotopy continuation was ill-conditioned. In particular, we observed 2, 51, and 81 such failures for $\mathrm{SO}(3, \mathbb{C})$, $\mathrm{SO}(4, \mathbb{C})$, and $\mathrm{SO}(5, \mathbb{C})$ respectively. Despite the fact that all witness sets computed for $\mathrm{SO}(4, \mathbb{C})$ and $\mathrm{SO}(5, \mathbb{C})$ had fewer than 40 and 384 solutions, we are not convinced that there exists a non-trivial upper bound for the number of real solutions on a witness set of $\mathrm{SO}(n, \mathbb{C})$ exists. In fact, we conjecture that, for all $n \geq 2$, $\mathrm{SO}(n, \mathbb{C})$ admits a real witness set.

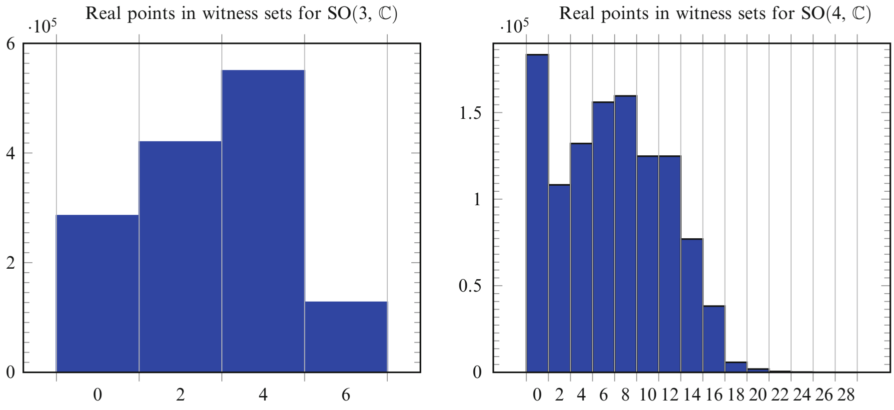

Fig. 2 Some histograms for the number of real solutions found in each witness set

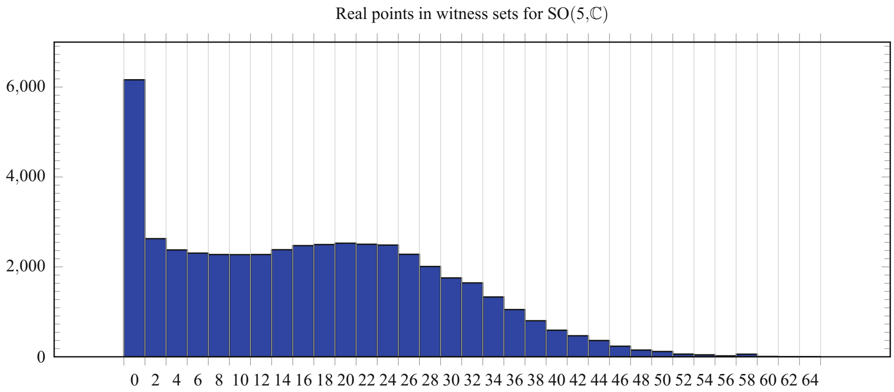

Fig. 3 Another histogram for the number of real solutions found in each witness set

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Appendix: *Macaulay2* Code

This section contains *Macaulay2* [9] code for computing the degree of $\mathrm{SO}(n, \mathbb{C})$. We typically compute the degree of $\mathrm{O}(n, \mathbb{C})$, and divide by to 2 to obtain the degree of $\mathrm{SO}(n, \mathbb{C})$, because this approach eliminates the polynomial of highest degree, the condition that the determinant equal 1.

First, we compute the degree of SO(5) using Gröbner bases. The computation is done over the finite field $\mathbb{Z}/2\mathbb{Z}$ for $\mathrm{O}(5, \mathbb{C})$ and the result is halved to give the degree of $\mathrm{SO}(5, \mathbb{C})$.

```
deg1SO = n -> (
    R := ZZ/2[x_(1,1)..x_(n,n)];
    M := genericMatrix(R,n,n);
    J := minors(1, M * transpose(M) - id_(R^n));
    (degree J) // 2)
```

Our second function uses the package *NumericalAlgebraicGeometry* to solve the zero-dimensional system arising from a linear slice of the variety $\mathrm{O}(3, \mathbb{C})$. The command `solveSystem` employs the standard method of polynomial homotopy continuation.

```
needsPackage ''NumericalAlgebraicGeometry'';
deg2SO = n -> (
    R := CC[x_(1,1)..x_(n,n)];
    M := genericMatrix(R,n,n);
    B := M * transpose(M) - id_(R^n);
    polys := unique flatten entries B;
    linearSlice := apply(binomial(n,2),
        i -> random(1,R) - random(CC));
    S := solveSystem(polys | linearSlice);
    #S // 2)
```

We next provide code that computes the degree of $\mathrm{SO}(n, \mathbb{C})$ using the package *MonodromySolver*. Again we do not include the determinant condition, but this time we do *not* need to halve the result. This is because our starting point, the identity matrix, lies on $\mathrm{SO}(n, \mathbb{C})$ and this method only discovers points on the irreducible component corresponding to our starting point. The linear slices are parametrized by the t and c variables which are varied within the function `monodromySolve` to create monodromy loops. The method stops when ten consecutive loops provide no new points. Although it is possible that this stopping criterion is satisfied prematurely, in our case the program stopped at the correct number.

```
needsPackage ''MonodromySolver'';
deg3SO = n -> (
    d := binomial(n,2);
    R := CC[c_1..c_d,
        t_(1,1,1)..t_(d,n,n)][x_(1,1)..x_(n,n)];
    M := genericMatrix(R,n,n);
```

```
    B := M * transpose(M) - id_(R^n);
    polys := unique flatten entries B;
    linearSlice := for i from 1 to d list (
        c_i + sum flatten for j from 1 to n list (
            for k from 1 to N list t_(i,j,k)*x_(j,k)));
    G := polySystem( polys | linearSlice);
    setRandomSeed 0;
    (p0, x0) := createSeedPair(G,
        flatten entries id_(CC^n));
    (V, npaths) = monodromySolve(G, p0, {x_0},
        NumberOfNodes => 2, NumberOfEdges => 4);
    # flatten points V.PartialSols)
```

Finally, we may use Theorem 1.1 to compute the degree of SO($n,\mathbb{C}$).

```
deg4SO = n -> (
    r := n // 2;
    M := matrix table(toList(1..r), toList(1..r),
        (i,j) -> binomial(2*n-2*i-2*j, n-2*i));
    2^(n-1) * det(M))
```

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Computing Toric Degenerations of Flag Varieties

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Abstract We compute toric degenerations arising from the tropicalization of the full flag varieties Fl_4 and Fl_5 embedded in a product of Grassmannians. For Fl_4 and Fl_5 we compare toric degenerations arising from string polytopes and the FFLV polytope with those obtained from the tropicalization of the flag varieties. We also present a general procedure to find toric degenerations in the cases where the initial ideal arising from a cone of the tropicalization of a variety is not prime.

MSC 2010 codes: 14T05, 13P10, 14M25, 14M15, 17B10

1 Introduction

Consider the variety Fl_n of full flags $\{0\} = V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = \mathbb{C}^n$ of vector subspaces of $\mathbb{C}^n$ with $\dim_{\mathbb{C}}(V_i) = i$. The flag variety Fl_n is naturally embedded in a product of Grassmannians using the Plücker coordinates. We denote by I_n the defining ideal of Fl_n with respect to this embedding. We produce toric degenerations of Fl_n as Gröbner degenerations coming from the initial ideals associated to the maximal cones of $\mathrm{trop}(\mathrm{Fl}_n)$. Moreover, we compare these with certain toric degenerations arising from representation theory.

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We consider 1-parameter toric degenerations of Fl_n. These are flat families $\varphi\colon \mathscr{F} \to \mathbb{A}^1$, where the fibre over zero (also called *special* fibre) is a toric variety and all other fibres are isomorphic to Fl_n. Once we have such a degeneration, some of the algebraic invariants of Fl_n will be the same for all fibres, hence the computation can be done on the toric fibre. In the case of a toric variety such invariants are easier to compute than in the case of a general variety. In fact, they have a nice combinatorial description. Moreover, toric degenerations connect different areas of mathematics, such as symplectic geometry, representation theory, and algebraic geometry.

Let $X = \mathrm{V}(I)$ be a projective variety and $\mathrm{trop}(X)$ be its tropicalization. The initial ideals associated to the top-dimensional cones of $\mathrm{trop}(X)$ are good candidates to give toric degenerations, see Lemma 2.7 and [27, Proposition 1.1] for a more general statement. For example, in the case of Grassmannians $\mathrm{Gr}(2, \mathbb{C}^n)$ each maximal cone of $\mathrm{trop}(\mathrm{Gr}(2, \mathbb{C}^n))$ gives a toric degeneration, see [5, 29, 32]. However, this is not true for the Grassmannians $\mathrm{Gr}(3, \mathbb{C}^n)$. In [27], Mohammadi and Shaw identify which maximal cones of $\mathrm{trop}(\mathrm{Gr}(3, \mathbb{C}^n))$ produce such degenerations.

The following are our main results. More detailed formulations can be found in Theorem 3.3, Theorem 3.5, and Proposition 6.4. A maximal cone C of $\mathrm{trop}(X)$ is *prime* if $\mathrm{in}_C(I) := \mathrm{in}_{\mathbf{w}}(I)$ is prime, with $\mathbf{w}$ a vector in the relative interior of C.

Theorem 1.1 *The tropical variety* $\mathrm{trop}(\mathrm{Fl}_4) \subset \mathbb{R}^{14}/\mathbb{R}^3$ *is a six-dimensional fan with* 78 *maximal cones. From prime cones we obtain four non-isomorphic toric degenerations. After applying Procedure 6.1, we obtain at least two additional non-isomorphic toric degenerations from non-prime cones.*

Theorem 1.2 *The tropical variety* $\mathrm{trop}(\mathrm{Fl}_5) \subset \mathbb{R}^{30}/\mathbb{R}^4$ *is a ten-dimensional fan with 69,780 maximal cones. From prime cones we obtain 180 non-isomorphic toric degenerations.*

Toric degenerations of flag varieties and Schubert varieties have been studied intensively in representation theory over the last two decades. We refer the reader to [13] for a nice overview on this topic and to the references therein.

The main motivation of this paper is to study the flat degenerations of flag varieties into toric varieties arising from the tropicalization and to compare these degenerations to those associated to *string polytopes* and the *Feigin–Fourier–Littelmann–Vinberg polytope (FFLV polytope).*

Theorem 1.3 *For* Fl_4*, there is at least one new toric degeneration arising from prime cones of* $\mathrm{trop}(\mathrm{Fl}_4)$ *in comparison to those obtained from string polytopes and the FFLV polytope. For* Fl_5*, there are at least 168 new toric degenerations arising from prime cones of* $\mathrm{trop}(\mathrm{Fl}_5)$ *in comparison to those obtained from string polytopes and the FFLV polytope.*

Our work is closely related to the theory of Newton–Okounkov bodies. Let $\Bbbk$ be a not necessarily algebraically closed field and X a projective variety. It is possible to associate to every prime cone in $\mathrm{trop}(X)$ a valuation with a finite *Khovanskii basis* B on the homogeneous coordinate ring $\Bbbk[X]$, see [23, Lemma 5.7]. This is a set of elements of $\Bbbk[X]$, such that their valuations generate the value semigroup $S(\Bbbk[X], \mathrm{val})$. The convex hull of $S(\Bbbk[X], \mathrm{val}) \cup \{0\}$ is referred to as the *Newton–*

Okounkov cone. After intersecting this cone with a certain hyperplane one obtains a convex body, called the *Newton–Okounkov body*. When a finite Khovanskii basis exists, [2, Theorem 1.1] states that there is a flat degeneration of the variety X into a toric variety whose normalization has as associated polytope the Newton–Okounkov body. In this case the Newton–Okounkov body is a polytope. The toric polytopes obtained in Theorem 3.3, Theorem 3.5, and Proposition 6.4 can be seen as Newton–Okounkov bodies for the valuations defined in Sect. 6.

The paper is structured as follows. In Sect. 2 we provide the necessary background. We study the tropicalization of the flag varieties Fl_n for $n = 4, 5$ and the induced toric degenerations in Sect. 3. The solutions to [30, Problem 5 on Grassmannians] and [30, Problem 6 on Grassmannians] can be found in Theorem 3.3.

In Sect. 4 we recall the definition of string cones, string polytopes, and the FFLV polytope for regular dominant integral weights. We compute for Fl_4 and Fl_5 all string polytopes for the weight ρ, which is the sum of all fundamental weights. Moreover, in Sect. 5 for every string cone we construct a weight vector $\mathbf{w}_{\underline{w}_0}$ contained in the tropicalization of the flag variety in order to further explore the connection between these two different approaches. The construction is inspired by Caldero [7].

In Sect. 6 we give an algorithmic approach to solving [23, Problem 1] for a subvariety X of a toric variety Y when each cone in $\mathrm{trop}(X)$ has multiplicity one. Procedure 6.1 aims at computing a new embedding X' of X in case $\mathrm{trop}(X)$ has some non-prime cones. Once we have such an embedding, we explain how to get new toric degenerations of X. We apply the procedure to Fl_4. Furthermore, we explain how to interpret the procedure in terms of finding valuations with finite Khovanskii basis on the algebra given by the homogeneous coordinate ring of X.

2 Preliminary Notions

In this section, we recall the definition of a flag variety and we introduce the necessary background in tropical geometry. In fact, the key ingredient in the study of Gröbner toric degenerations of Fl_n is the subfan of the Gröbner fan of I_n given by the *tropicalization* of Fl_n. We mostly refer to the approach described in [25] and we encourage the reader to look there for a more thorough introduction.

Let $\mathbb{k}$ be a field with $\mathrm{char}(\mathbb{k}) = 0$ and consider on it the trivial valuation. We are mainly interested in the case when $\mathbb{k} = \mathbb{C}$.

Definition 2.1 A *complete flag* in the vector space $\mathbb{k}^n$ is a chain

$$\mathscr{V} : \{0\} = V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = \mathbb{k}^n$$

of vector subspaces of $\mathbb{k}^n$ with $\dim_{\mathbb{k}}(V_i) = i$.

The set of all complete flags in $\mathbb{k}^n$ is denoted by Fl_n and it has an algebraic variety structure. More precisely, it is a subvariety of the product of Grassmannians $\mathrm{Gr}(1, \mathbb{k}^n) \times \mathrm{Gr}(2, \mathbb{k}^n) \times \cdots \times \mathrm{Gr}(n-1, \mathbb{k}^n)$.

Composing with the Plücker embeddings of the Grassmannians, Fl_n becomes a subvariety of $\mathbb{P}^{\binom{n}{1}-1} \times \mathbb{P}^{\binom{n}{2}-1} \times \cdots \times \mathbb{P}^{\binom{n}{n-1}-1}$ and so we can ask for its defining ideal I_n. Each point in the flag variety can be represented by an $(n-1)\times n$-matrix $M = [x_{i,j}]$ whose first d rows generate V_d. Each V_d corresponds to a point in a Grassmannian. Moreover, they satisfy the condition $V_d \subset V_{d+1}$ for $d = 0, \dots, n-1$. In order to compute the ideal I_n defining Fl_n in $\mathbb{P}^{\binom{n}{1}-1} \times \mathbb{P}^{\binom{n}{2}-1} \times \cdots \times \mathbb{P}^{\binom{n}{n-1}-1}$ we have to translate the inclusions $V_d \subset V_{d+1}$ into polynomial equations. We define the map $\varphi_n : \mathbb{k}[p_J : \varnothing \neq J \subsetneq \{1, 2, \dots, n\}] \to \mathbb{k}[x_{i,j} : 1 \leq i \leq n-1,\ 1 \leq j \leq n]$ sending each Plücker variable p_J to the determinant of the submatrix of M with row indices $1, 2, \dots, |J|$ and column indices in J. The ideal I_n of Fl_n is the kernel of φ_n. There is an action of $S_n \times \mathbb{Z}_2$ on Fl_n. The symmetric group acts by permuting the columns of M. The action of $\mathbb{Z}_2$ maps a complete flag $\mathcal{V}$ to its complement, which is defined to be $\mathcal{V}^{\perp} : \{0\} = V_n^{\perp} \subset V_{n-1}^{\perp} \subset \cdots \subset V_1^{\perp} \subset V_0^{\perp} = \mathbb{k}^n$.

Hence, we do computations up to $S_n \times \mathbb{Z}_2$-symmetry. We are interested in finding toric degenerations. These are degenerations whose special fibre is defined by a *toric* ideal, i.e. a binomial prime ideal not containing monomials. This toric ideal arises as the *initial ideal* of I_n.

Definition 2.2 Let $f = \sum a_{\mathbf{u}} x^{\mathbf{u}}$ with $\mathbf{u} \in \mathbb{Z}_{\geq 0}^{n+1}$ be a polynomial in $S = \mathbb{k}[x_0, x_1, \dots, x_n]$. For each $\mathbf{w} \in \mathbb{R}^{n+1}$ we define its *initial form* to be

$$\mathrm{in}_{\mathbf{w}}(f) = \sum_{\mathbf{w}\cdot\mathbf{u} \text{ is minimal}} a_{\mathbf{u}} x^{\mathbf{u}}.$$

If I is an ideal in S, then its *initial ideal* with respect to $\mathbf{w}$ is $\mathrm{in}_{\mathbf{w}}(I) = \langle \mathrm{in}_{\mathbf{w}}(f) : f \in I \rangle$.

An important geometric property of initial ideals is that there exists a flat family over $\mathbb{A}^1$ for which the fibre over 0 is isomorphic to $\mathrm{V}\big(\mathrm{in}_{\mathbf{w}}(I)\big)$ and all the other fibres are isomorphic to the variety $\mathrm{V}(I)$. Here, if J is a homogeneous ideal of S then we define $V(J) := \mathrm{Proj}(S/J)$ where the grading on S and hence on S/J comes from the ambient space which has S as homogeneous coordinate ring.

Let t be the coordinate in $\mathbb{A}^1$, then the flat family is given by the ideal

$$\tilde{I}_t = \langle t^{-\min_{\mathbf{u}}\{\mathbf{w}\cdot\mathbf{u}\}} f(t^{w_0}x_0, t^{w_1}x_1, \dots, t^{w_n}x_n) : f = \sum a_{\mathbf{u}} x^{\mathbf{u}} \in I \rangle \subset \mathbb{k}[t, x_0, x_1, \dots, x_n].$$

This family gives a flat degeneration of the variety $\mathrm{V}(I)$ into the variety $\mathrm{V}\left(\mathrm{in}_{\mathbf{w}}(I)\right)$ called the *Gröbner degeneration*. In order to look for toric degenerations, we study the *tropicalization* of $V(I)$.

Definition 2.3 For any $f := \sum a_{\mathbf{u}} x^{\mathbf{u}} \in S$, the *tropicalization* of f is the function $\mathrm{trop}(f) : \mathbb{R}^{n+1} \to \mathbb{R}$ given by $\mathrm{trop}(f)(\mathbf{w}) = \min\{\mathbf{w}\cdot\mathbf{u} : \mathbf{u} \in \mathbb{Z}_{\geq 0}^{n+1} \text{ and } a_{\mathbf{u}} \neq 0\}$.

Let $f := \sum a_{\mathbf{u}} x^{\mathbf{u}}$ be a homogeneous polynomial in S. If $\mathbf{w} - \mathbf{v} = m \cdot \mathbf{1}$, for some $\mathbf{v}, \mathbf{w} \in \mathbb{R}^{n+1}$, $\mathbf{1} := (1, 1, \dots, 1) \in \mathbb{R}^{n+1}$, and $m \in \mathbb{R}$, then we have that the minimum in $\mathrm{trop}(f)(\mathbf{w})$ and $\mathrm{trop}(f)(\mathbf{v})$ is achieved for the same $\mathbf{u} \in \mathbb{Z}_{\geq 0}^{n+1}$ such that $a_{\mathbf{u}} \neq 0$.

Definition 2.4 Let f be a homogeneous polynomial in S and $V(f)$ the associated hypersurface in $\mathbb{P}^n$. Then the *tropical hypersurface* of f is defined to be

$$\operatorname{trop}(V(f)) = \left\{ \mathbf{w} \in \mathbb{R}^{n+1}/\mathbb{R}\mathbf{1} \cong \mathbb{R}^n : \begin{array}{c} \text{the minimum in } \operatorname{trop}(f)(\mathbf{w}) \\ \text{is achieved at least twice} \end{array} \right\}.$$

Let I be a homogeneous ideal in S. The *tropicalization* of the variety $\mathrm{V}(I) \subset \mathbb{P}^n$ is defined to be

$$\operatorname{trop}\left(\mathrm{V}(I)\right) = \bigcap_{f \in I} \operatorname{trop}\left(\mathrm{V}(f)\right).$$

For every $\mathbf{w} \in \operatorname{trop}\left(\mathrm{V}(I)\right)$, $\mathrm{in}_{\mathbf{w}}(I)$ does not contain any monomial (see proof of [25, Theorem 3.2.3]). If $\mathrm{V}(I)$ is a $(d-1)$-dimensional irreducible projective variety, then $\operatorname{trop}(\mathrm{V}(I))$ is the support of a rational fan given by the quotient by $\mathbb{R}\mathbf{1}$ of a subfan F of the Gröbner fan of I ([25, Theorem 3.3.5]). The fan F has dimension d, which is the Krull dimension of S/I. It is possible to quotient by $\mathbb{R}\mathbf{1}$ because I is homogeneous and hence $\mathrm{in}_{\mathbf{w}}(I) = \mathrm{in}_{\mathbf{v}}(I)$ for every $\mathbf{w} - \mathbf{v} = m \cdot \mathbf{1}$ with $\mathbf{v}, \mathbf{w} \in \mathbb{R}^{n+1}$ and $m \in \mathbb{R}$. If we consider this fan structure on $\operatorname{trop}(\mathrm{V}(I))$ we have that vectors in the relative interior of a cone give rise to the same initial ideal and vectors in distinct relative cone interiors induce distinct initial ideals. For this reason, we denote by $\mathrm{in}_C(I)$ the initial ideal of I with respect to any $\mathbf{w}$ in the relative interior of C. The tropicalization of a variety X is non-empty only if X intersects the torus $T^n = (\mathbb{k}^*)^{n+1}/\mathbb{k}^*$ non-trivially. In fact, $\operatorname{trop}(X)$ is technically the tropicalization of $X \cap T^n$.

In the same way the tropicalization can be defined when S is the *total coordinate ring* (see [9, p. 207] for a definition) of $\mathbb{P}^{k_1} \times \cdots \times \mathbb{P}^{k_s}$. The ring S has a $\mathbb{Z}^s$-grading given by $\deg : \mathbb{Z}^{n+1} \to \mathbb{Z}^s$. An ideal I defining an irreducible subvariety $\mathrm{V}(I)$ of $\mathbb{P}^{k_1} \times \cdots \times \mathbb{P}^{k_s}$ is homogeneous with respect to this grading. The tropicalization of $\mathrm{V}(I)$ is contained in $\mathbb{R}^{k_1+\ldots+k_s+s}/H$, where H is an s-dimensional linear space spanned by the rows of the matrix D associated to deg. Similarly to the projective case, if $\mathrm{V}(I)$ is a d-dimensional irreducible subvariety of $\mathbb{P}^{k_1} \times \cdots \times \mathbb{P}^{k_s}$, then $\operatorname{trop}(\mathrm{V}(I))$ is the support of a fan which is the quotient by H of a rational $(d+s)$-dimensional subfan F of the Gröbner fan of I. Here the Krull dimension of S/I is $d+s$.

In the following, we always consider $\operatorname{trop}\left(\mathrm{V}(I)\right)$ with this fan structure.

Remark 2.5 A detailed definition of the tropicalization of a general toric variety X_Σ and of its subvarieties can be found in [25, Chap. 6]. Note that we only consider the tropicalization of the intersection of $\mathrm{V}(I)$ with the torus of X_Σ while in [25, Chap. 6] they introduce a generalized version of $\operatorname{trop}(\mathrm{V}(I))$ which includes the tropicalization of the intersection of $\mathrm{V}(I)$ with each orbit of X_Σ.

Another property of $\operatorname{trop}\left(\mathrm{V}(I)\right)$ is that any fan structure on it can be balanced by assigning a positive integer weight to every maximal cell. We do not explain the notion of balancing in detail and we consider an adapted version of the multiplicity defined in [25, Definition 3.4.3].

Definition 2.6 Let $I \subset S$ be a homogeneous ideal and Σ be a fan with support $|\Sigma| = |\operatorname{trop}\left(\mathrm{V}(I)\right)|$ such that, for each cone C of Σ, the ideal $\mathrm{in}_{\mathbf{w}}(I)$ is constant for $\mathbf{w}$ in the relative interior of C. For a maximal dimensional cone $C \in \Sigma$, we define the *multiplicity* as $\mathrm{mult}(C) = \sum_P \mathrm{mult}(P, \mathrm{in}_C(I))$, where the sum is taken over the minimal associated primes P of $\mathrm{in}_C(I)$ that do not contain monomials; see [11, Sect. 3] or [8, Sect. 4.7].

As we have seen, each cone of $\operatorname{trop}\left(\mathrm{V}(I)\right)$ corresponds to an initial ideal which contains no monomials. Moreover, we will see that the good candidates for toric degenerations are the initial ideals corresponding to the relative interior of the maximal cones. A maximal cone is *prime* if the corresponding initial ideal is prime.

Lemma 2.7 *Let $I \subset S$ be a homogeneous ideal and C a maximal cone of* $\operatorname{trop}(\mathrm{V}(I))$. *If* $\mathrm{in}_C(I)$ *is a toric ideal, i.e. binomial and prime, then C has multiplicity one. If C has multiplicity one, then* $\mathrm{in}_C(I)$ *has a unique toric ideal in its primary decomposition.*

Proof We first prove the lemma for S the homogeneous coordinate ring of $\mathbb{P}^n$. Let $I' = \mathrm{in}_C(I)\Bbbk[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ and consider the subvariety $\mathrm{V}(I')$ of the torus T^n. Then by [25, Remark 3.4.4] the multiplicity of a maximal cone C is the number of d-dimensional torus orbits whose union is $\mathrm{V}(I')$. If $\mathrm{in}_C(I)$ is toric, then $\mathrm{V}(I')$ is an irreducible toric variety having a unique d-dimensional torus orbit. Hence, C has multiplicity one.

Suppose now C has multiplicity one. This implies that $\mathrm{in}_C(I)$ contains one associated prime J, which does not contain monomials. The ideal J has to be binomial since it is the ideal of the unique d-dimensional torus orbit contained in $\mathrm{V}(I')$.

When S is the total coordinate ring of the product $\mathbb{P}^{k_1} \times \cdots \times \mathbb{P}^{k_s}$, the torus is given by $T^{k_1} \times \cdots \times T^{k_s} \cong T^{k_1+\cdots+k_s}$. We may assume that for each i,

$$T^{k_i} = \{[1 : a_1 : \ldots : a_{k_i}] \in \mathbb{P}^{k_i} : \ a_j \neq 0 \text{ for all } j\}.$$

The variables for $\mathbb{P}^{k_i}$ are denoted by $x_{i,0}, \ldots, x_{i,k_i}$ for each i. We fix the Laurent polynomial ring $S' = \Bbbk[x_{1,1}^{\pm 1}, \ldots, x_{1,k_1}^{\pm 1}, x_{2,1}^{\pm 1}, \ldots, x_{2,k_2}^{\pm 1}, \ldots, x_{s,1}^{\pm 1}, \ldots, x_{s,k_s}^{\pm 1}]$. We consider the ideal $I' = \mathrm{in}_C(I)S'$ in S' and the variety $\mathrm{V}(I')$ as a subvariety of $T^{k_1+\ldots+k_s}$. Then the proof proceeds as before. □

Remark 2.8 From Lemma 2.7 we conclude the multiplicity being one is a necessary but not sufficient condition for toric initial ideals. A cone can have multiplicity one but its associated initial ideal might be neither prime nor binomial. There may be associated primes that contain monomials in the decomposition of $\mathrm{in}_{\mathbf{w}}(I)$ and these do not contribute to the multiplicity. There are examples of such cones in $\operatorname{trop}(\mathrm{Fl}_5)$ as we will see in Theorem 3.5.

Let I be a homogeneous ideal in S such that the Krull dimension of S/I is d. Consider $\operatorname{trop}\left(\mathrm{V}(I)\right) \subset \mathbb{R}^{n+1}/H$ and the d-dimensional subfan $F \subset \mathbb{R}^{n+1}$ of the

Gröbner fan of I with $F/H \cong \operatorname{trop}\big(\mathrm{V}(I)\big)$. When $\mathrm{V}(I) \subset \mathbb{P}^{k_1} \times \mathbb{P}^{k_2} \times \cdots \times \mathbb{P}^{k_s}$ the linear space H is spanned by the rows of the matrix D. In particular, when $\mathrm{V}(I) \subset \mathbb{P}^n$ we have that H is equal to the span of $(1, 1, \ldots, 1)$. We now describe some properties of the toric initial ideals corresponding to maximal cones of $\operatorname{trop}\big(\mathrm{V}(I)\big)$. Let C be a cone in $\operatorname{trop}(\mathrm{V}(I))$ and $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_d\}$ be d linearly independent vectors in F generating the maximal cone C', such that $C'/H \cong C$. We can assume that the $\mathbf{w}_i$'s have integer entries since F is a rational fan. The matrix associated to C is

$$W_C = \begin{bmatrix}\mathbf{w}_1 \ \mathbf{w}_2 \cdots \mathbf{w}_d\end{bmatrix}^{\mathsf{T}}. \tag{1}$$

Consider a sublattice L of $\mathbb{Z}^{n+1}$ and the standard basis $e_1, e_2, \ldots, e_{n+1}$ of $\mathbb{Z}^{n+1}$. Given $\ell = (\ell_1, \ell_2, \ldots, \ell_{n+1}) \in L$, we set $\ell^+ = \sum_{\ell_i > 0} \ell_i e_i$ and $\ell^- = -\sum_{\ell_j < 0} \ell_j e_j$, so that $\ell = \ell^+ - \ell^-$ and $\ell^+, \ell^- \in \mathbb{N}^{n+1}$; see [9, p. 15].

Lemma 2.9 *Let I be a homogeneous ideal in S and C a maximal cone in* $\operatorname{trop}(\mathrm{V}(I))$. *If* $\operatorname{in}_C(I)$ *is toric, then there exists a sublattice L of $\mathbb{Z}^{n+1}$ and constants* $0 \neq c_\ell \in \Bbbk$ *with $\ell \in L$ such that* $\operatorname{in}_C(I) = I(W_C) := \langle \mathbf{x}^{\ell^+} - c_\ell \mathbf{x}^{\ell^-} : \ell \in L\rangle$. *In particular, L is the kernel of the map $f : \mathbb{Z}^{n+1} \to \mathbb{Z}^d$ defined by the matrix W_C. If C has multiplicity one and* $\operatorname{in}_C(I)$ *is not toric, then the unique toric ideal in the primary decomposition of* $\operatorname{in}_C(I)$ *is of the form $I(W_C)$.*

Proof Let $\operatorname{in}_C(I) \subset S$ be a toric initial ideal and let C' be the corresponding cone in F. The fan structure is defined on $\operatorname{trop}\big(\mathrm{V}(I)\big)$ so that for every $\mathbf{w}', \mathbf{w}$ in the relative interior of C' we have $\operatorname{in}_{\mathbf{w}'}(I) = \operatorname{in}_C(I) = \operatorname{in}_{\mathbf{w}}(I)$. This implies that $\operatorname{in}_C(I)$ is W_C-homogeneous, that is homogeneous with respect to the $\mathbb{Z}^d$-grading on S given by the matrix W_C. By [31, Lemma 10.12] there exists an automorphism ϕ of S sending x_i to $\lambda_i x_i$ for some $\lambda_i \in \Bbbk$, such that the ideal $\operatorname{in}_C(I)$ is isomorphic to an ideal of the form $I_L := \langle \mathbf{x}^{\ell^+} - \mathbf{x}^{\ell^-} : \ell \in L\rangle$, where L is the sublattice of $\mathbb{Z}^{n+1}$ given by the kernel of the map $f : \mathbb{Z}^{n+1} \to \mathbb{Z}^d$. Applying ϕ^{-1} to $\operatorname{in}_C(I)$ we can write each toric initial ideal as $\langle \mathbf{x}^{\ell^+} - c_\ell \mathbf{x}^{\ell^-} : \ell \in L\rangle = I(W_C)$, for some $0 \neq c_\ell \in \Bbbk$, L and W_C defined above.

Let C be a cone of multiplicity one and suppose $\operatorname{in}_C(I)$ is not prime. By Lemma 2.7, there exists a unique toric ideal J in the primary decomposition of $\operatorname{in}_C(I)$. This toric ideal J contains $\operatorname{in}_C(I)$ and we show that it can be expressed as $I(W_C)$. The variety $\mathrm{V}(I)$ is considered as a subvariety of $\mathbb{P}^n$. As in Lemma 2.7, the case in which $\mathrm{V}(I) \subset \mathbb{P}^{k_1} \times \mathbb{P}^{k_2} \times \cdots \times \mathbb{P}^{k_s}$ has an analogous proof. The tropical variety depends only on the intersection of $\mathrm{V}(I)$ with the torus, and $\operatorname{in}_C(I)\,\Bbbk[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$ is equal to J. Hence, J is a prime ideal that is homogeneous with respect to W_C so we proceed as above to establish that we have $J = \langle \mathbf{x}^{\ell^+} - c_\ell \mathbf{x}^{\ell^-} : \ell \in L\rangle = I(W_C)$. □

Remark 2.10 The lattice L and the ideal $I(W_C)$ only depend on the linear space spanned by the rays of the cone C'. Hence, they are the same for every set of d independent vectors in C' chosen to define W_C.

3 Tropicalization and Toric Degenerations

In this section, we study the tropicalization of Fl_4 and Fl_5. We analyze the Gröbner toric degenerations arising from $\mathrm{trop}(\mathrm{Fl}_4)$ and $\mathrm{trop}(\mathrm{Fl}_5)$, and we compute the polytopes associated to their normalizations. In Proposition 3.4 we describe the *tropical configurations* arising from the maximal cones of $\mathrm{trop}(\mathrm{Fl}_4)$. These are configurations of a point on a tropical line in a tropical plane corresponding to the points in the relative interior of a maximal cone.

Before stating our main results, we recall the following definition.

Definition 3.1 There exists a *unimodular equivalence* between two lattice polytopes P and Q (resp. two fans $\mathscr{F}$ and $\mathscr{G}$) if there exists an affine lattice isomorphism ϕ of the ambient lattices sending the vertices (resp. the rays) of one polytope (resp. fan) to the vertices (resp. rays) of the other. Moreover, if σ is a face of P (resp. of $\mathscr{F}$) then $\phi(\sigma)$ is a face of Q (resp. $\mathscr{G}$) and the adjacency of faces is respected.

Remark 3.2 We are interested in finding distinct fans up to unimodular equivalence as they give rise to non-isomorphic toric varieties. Often it will be possible only to determine combinatorial equivalence (see [9, Sect. 2.2]). This is a weaker condition but when it does not hold it allows us to rule out unimodular equivalence.

Theorem 3.3 *The tropical variety* $\mathrm{trop}(\mathrm{Fl}_4)$ *is a six-dimensional rational fan in* $\mathbb{R}^{14}/\mathbb{R}^3$ *with a three-dimensional lineality space. It consists of* 78 *maximal cones,* 72 *of which are prime. They are organized in five* $S_4 \times \mathbb{Z}_2$*-orbits, four of which contain prime cones. The prime cones give rise to four non-isomorphic toric degenerations.*

Proof The theorem is proved by explicit computations. We developed a *Macaulay2* package called `ToricDegenerations` containing all the functions we use. The package and the data needed for this proof are available at https://github.com/ToricDegenerations. The flag variety Fl_4 is a six-dimensional subvariety of $\mathbb{P}^3 \times \mathbb{P}^5 \times \mathbb{P}^3$. The ideal I_4 defined in the previous section is contained in the total coordinate ring R of $\mathbb{P}^3 \times \mathbb{P}^5 \times \mathbb{P}^3$ which is the polynomial ring over $\mathbb{C}$ in the variables $p_1, p_2, p_3, p_4, p_{1,2}, p_{1,3}, p_{1,4}, p_{2,3}, p_{2,4}, p_{3,4}, p_{1,2,3}, p_{1,2,4}, p_{1,3,4}, p_{2,3,4}$. The grading on R is given by the matrix

$$D = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}. \tag{2}$$

The explicit form of I_4 can be found in [26, p. 276]. As we have seen in Sect. 2 the tropicalization of Fl_4 is contained in $\mathbb{R}^{14}/H$. In this case, H is the vector space spanned by the rows of D.

We use the *Macaulay2* [19] interface to *Gfan* [21] to compute $\mathrm{trop}(\mathrm{Fl}_4)$. The given input is the ideal I_4 and the $S_4 \times \mathbb{Z}_2$-action (see [22, Sect. 3.1.1]). The output is a subfan F of the Gröbner fan of dimension 9. We quotient it by H to get $\mathrm{trop}(\mathrm{Fl}_4)$ as a six-dimensional fan contained in $\mathbb{R}^{14}/H \cong \mathbb{R}^{14}/\mathbb{R}^3$. Firstly, the function `computeWeightVectors` computes a list of vectors. There is one for every maximal cone of trop(Fl_4) and it is contained in the relative interior of the corresponding cone. Then `groebnerToricDegenerations` computes all the initial ideals and checks if they are binomial and prime over $\mathbb{Q}$. These are organized in a hash table, which is the output of the function. All 78 initial ideals are binomial and all maximal cones have multiplicity one. In order to check primeness over $\mathbb{C}$, we have to check if $\mathrm{in}_C(I_4) = I(W_C)$. This can be done by computing the degrees of $\mathrm{V}\left(\mathrm{in}_C(I_4)\right)$ and $\mathrm{V}\left(I(W_C)\right)$ seen as subvarieties of $\mathbb{P}^{13}$. If these are equal, then there are no non-toric ideals in the primary decomposition of $\mathrm{in}_C(I_4)$. The degree of $\mathrm{V}\left(I(W_C)\right)$ equals the degree of $\mathrm{V}(I_L)$, where L and I_L can be computed from W_C as in the proof of Lemma 2.9.

We consider the orbits of the $S_4 \ltimes \mathbb{Z}_2$-action on the set of initial ideals. These correspond to the orbits of maximal cones of F and trop(Fl_4). There is one orbit of non-prime initial ideals and four orbits of prime initial ideals. The varieties corresponding to initial ideals contained in the same orbit are isomorphic. Thus, for each orbit, we choose a representative of the form $\mathrm{in}_C(I_4) = I(W_C)$ for some cone C.

We now compute for each of the four prime orbits, the polytope of the normalization of the associated toric varieties. We use the *Macaulay2* package *Polyhedra* [4] for the following computations. The lattice M associated to $S/I(W_C)$ is generated over $\mathbb{Z}$ by the columns of W_C. To use *Polyhedra* we want to have a lattice with index 1 in $\mathbb{Z}^9$. Hence, in case the index of M in $\mathbb{Z}^9$ is different from 1, we consider M as the lattice generated by the columns of the matrix $(\ker((\ker(W_C))^{\mathsf{T}})^{\mathsf{T}}$. Here, for a matrix A we consider $\ker(A)$ to be the matrix whose columns minimally generate the kernel of the map $\mathbb{Z}^{14} \to \mathbb{Z}^9$ defined by A. We denote the set of generators of M by $\mathscr{B}_C = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_{14}\}$ so that $M = \mathbb{Z}\mathscr{B}_C$.

The toric variety $\mathbb{P}^3 \times \mathbb{P}^5 \times \mathbb{P}^3$ can be seen as $\mathrm{Proj}(\oplus_\ell R_{\ell(1,1,1)})$ and $I(W_C)$ as an ideal in $\oplus_\ell R_{\ell(1,1,1)}$ (see [26, Chap. 10]). The associated toric variety is $\mathrm{Proj}(\oplus_\ell \mathbb{C}[\mathbb{N}\mathscr{B}_C]_{\ell(1,1,1)})$. The polytope P of the normalization is given as the convex hull of those lattice points in $\mathbb{N}\mathscr{B}_C$ corresponding to degree $(1, 1, 1)$-monomials in $\mathbb{C}[\mathbb{N}\mathscr{B}_C]$. These can be found in the following way. We order the rows of the matrix $[\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_{14}]$ associated to $\mathscr{B}_C$ so that the first three rows give the matrix D from (2). Now the matrix $[\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_{14}]$ represents a map $\mathbb{Z}^{14} \to \mathbb{Z}^3 \oplus \mathbb{Z}^6$, where $\mathbb{Z}^3 \oplus \mathbb{Z}^6$ is the lattice M and the $\mathbb{Z}^3$ part gives the degree of the monomials associated to each lattice point on M. The lattice points, whose convex hull gives the polytope P, are those ones with the first three coordinates being 1. In other words, we have obtained P by applying the reverse procedure of constructing a toric variety from a polytope (see [9, Sect. 2.1–2.2]). The difference from the procedure given in [9, Sect. 2.1–2.2] is the $\mathbb{Z}^3$-grading and because of that we do not consider the convex hull of $\mathscr{B}_C$, but the intersection of $\mathbb{N}\mathscr{B}_C$ with these hyperplanes.

In Table 1, we list the numerical invariants of the initial ideals and their corresponding polytopes. The tropical variety trop(Fl_4) has 78 maximal cones organized in five $S_4 \times \mathbb{Z}_2$-orbits. Using *polymake* [17], we first obtain that there is no combinatorial equivalence between each pair of polytopes. This means that there is no unimodular equivalence between the corresponding normal fans, hence

Table 1 The algebraic invariants of the initial ideals associated to maximal cones in trop(Fl_4)

Orbit	Size	Cohen–Macaulay	Prime	# Generators	f-vector of associated polytope
1	24	Yes	Yes	10	(42, 141, 202, 153, 63, 13)
2	12	Yes	Yes	10	(40, 132, 186, 139, 57, 12)
3	12	Yes	Yes	10	(42, 141, 202, 153, 63, 13)
4	24	Yes	Yes	10	(43, 146, 212, 163, 68, 14)
5	6	Yes	No	10	Not applicable

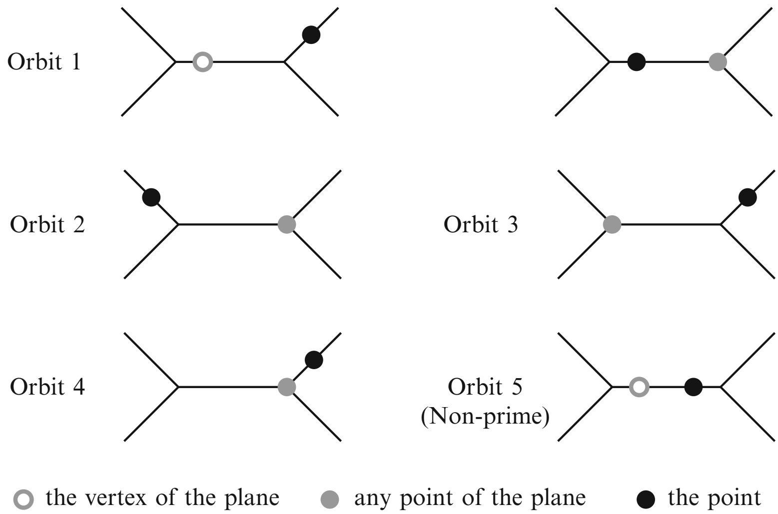


Fig. 1 The list of all tropical configurations up to symmetry that arise in Fl_4. The hollow and the full gray dot denote whether that vertex of the line is the vertex of the plane or it is contained in a ray of the plane. The black dot is the position of the point on the line

the normalization of the toric varieties associated to these toric degenerations are not isomorphic. This implies that we obtain four non-isomorphic toric degenerations. □

Proposition 3.4 *There are six tropical configurations up to symmetry, depicted in Fig. 1, arising from the maximal cones of* trop(Fl_4). *They are further organized in five* $S_4 \times \mathbb{Z}_2$*-orbits.*

Proof The tropical variety trop(Fl_4) is contained in

$$\operatorname{trop}\left(\mathrm{Gr}(1,\mathbb{C}^4)\right) \times \operatorname{trop}\left(\mathrm{Gr}(2,\mathbb{C}^4)\right) \times \operatorname{trop}\left(\mathrm{Gr}(3,\mathbb{C}^4)\right).$$

Each tropical Grassmannian parametrizes tropicalized linear spaces; see [25, Theorem 4.3.17]. This implies that every point p in trop(Fl_4) corresponds to a chain of tropical linear subspaces given by a point on a tropical line contained in a tropical plane. All tropical chains are *realizable*, meaning that they are the tropicalization of

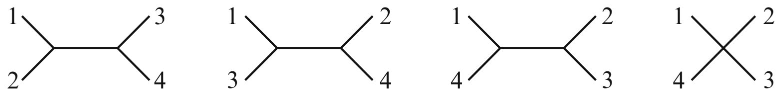


Fig. 2 Combinatorial types of tropical lines in $\mathbb{R}^3/\mathbb{R}\mathbf{1}$

the classical chains of linear spaces of $\Bbbk^4$ corresponding to a point q in Fl_4 such that $\mathrm{val}(q) = p$, where $\Bbbk = \mathbb{C}\{\{t\}\}$ and val is the natural valuation on this field; see [25, Part (3) of Theorem 3.2.3].

In this case, there is only one combinatorial type for the tropical plane and four possible types for the lines up to symmetry; see [25, Example 4.4.9]. The plane consists of six two-dimensional cones positively spanned by all possible pairs of vectors $(1, 0, 0)^\mathsf{T}, (0, 1, 0)^\mathsf{T}, (0, 0, 1)^\mathsf{T}$, and $(-1, -1, -1)^\mathsf{T}$. The combinatorial types of the tropical lines are shown in Fig. 2. The leaves of these graphs represent the rays of the tropical line labelled 1 up to 4 corresponding to the positive hull of each of the vectors $(1, 0, 0)^\mathsf{T}, (0, 1, 0)^\mathsf{T}, (0, 0, 1)^\mathsf{T}$, and $(-1, -1, -1)^\mathsf{T}$. Consider the $S_4 \times \mathbb{Z}_2$-orbits of maximal cones of $\mathrm{trop}(\mathrm{Fl}_4)$. If we compute the chain of tropical linear spaces corresponding to an element in each orbit, we get the configurations in Fig. 1. We do not include the labelling since up to symmetry we can get all possibilities. The point on the line is the black dot. In case the intersection of the line with the rays of the plane is the vertex of the plane then we denote this with a hollow dot. A vertex of the line is colored in gray if it lies on a ray of the plane. For example in orbit 2, label the rays 1 to 4 anti-clockwise starting from the top left edge. We have rays 1 and 2 in the two-dimensional positive hull of $(1, 0, 0)^\mathsf{T}$ and $(0, 1, 0)^\mathsf{T}$. The vector associated to the internal edge is $(1, 1, 0)^\mathsf{T}$. The gray point is the origin and the black point has coordinates $(a, 1, 0)^\mathsf{T}$ for $a > 1$. Orbits 1 and 4 in Fig. 1 have size 24, orbits 2 and 3 have size 12 and orbit 5 has size 6. Orbit 5 corresponds to non-prime initial ideals. Orbit 1 contains two combinatorial types of tropical configurations and one is sent to the other by the $\mathbb{Z}_2$-action on the tropical variety. The orbits 2 and 3 differ from the fact that for each combinatorial type of line the gray dot can lie on one of the four rays of the tropical plane. These possibilities are grouped in two pairs, one is in orbit 2 and the other in orbit 3. □

Theorem 3.5 *The tropical variety* $\mathrm{trop}(\mathrm{Fl}_5)$ *is a ten-dimensional fan in* $\mathbb{R}^{30}/\mathbb{R}^4$ *with a four-dimensional lineality space. It consists of 69,780 maximal cones which are grouped in* 536 $S_5 \times \mathbb{Z}_2$*-orbits. These give rise to* 531 *orbits of binomial initial ideals and among these* 180 *are prime. They correspond to* 180 *non-isomorphic toric degenerations.*

Proof The flag variety Fl_5 is a ten-dimensional variety defined by 66 quadratic polynomials in the total coordinate ring of $\mathbb{P}^4 \times \mathbb{P}^9 \times \mathbb{P}^9 \times \mathbb{P}^4$. These are of the form $\sum_{j \in J \setminus I} (-1)^{l_j} p_{I \cup \{j\}} p_{J \setminus \{j\}}$, where $J, I \subset \{1, 2, \ldots, 5\}$ and

$$l_j = \#\{k \in J : j < k\} + \#\{i \in I : i < j\} .$$

The proof is similar to the proof of Theorem 3.3. The only difference is that the action of $S_5 \times \mathbb{Z}_2$ on Fl_5 is crucial for the computations. In fact, without exploiting the symmetries the calculations to get the tropicalization would not terminate. Moreover, we only verify primeness of the initial ideals over $\mathbb{Q}$ using the *primdec* library [28] in *Singular* [10]. We compute the polytopes associated to the normalization of the 180 toric varieties in the same way as Theorem 3.3, but we change the matrix of the grading. This is now given by

$$\begin{bmatrix} 1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1 \end{bmatrix}. \tag{3}$$

Since there are no combinatorial equivalences among the normal fans to these polytopes, we deduce that the obtained toric degenerations are pairwise non-isomorphic. More information on the non-prime initial ideals is available in Table 4 in the appendix. □

4 String Polytopes and the FFLV-Polytope

This section provides an introduction to string cones, string polytopes, and the FFLV polytope with explicit computations for Fl_4 and Fl_5. String polytopes are described by Littelmann in [24], and by Berenstein and Zelevinsky in [3]. FFLV stands for Feigin, Fourier, and Littelmann, who defined this polytope in [15], and Vinberg who conjectured its existence in a special case. Both, the string polytopes and the FFLV polytope, can be used to obtain toric degenerations of the flag variety.

Let $W = S_n$ be the symmetric group, which is the Weyl group corresponding to $G = \mathrm{SL}_n$ over $\mathbb{C}$ with the longest word w_0 given in the alphabet of simple transpositions $s_i = (i, i+1) \in S_n$. We choose the Borel subgroup $B \subset SL_n$ of upper triangular matrices and the maximal torus $T \subset B$ of diagonal matrices. Further, let $U^- \subset B^-$ be the unipotent radical in the opposite Borel subgroup, i.e. the set of lower triangular matrices with 1's on the diagonal. Let $\mathrm{Lie}(G) = \mathfrak{g} = \mathfrak{sl}_n$ be the corresponding Lie algebra, i.e. $n \times n$-matrices with trace zero. Let $\mathfrak{h} = \mathrm{Lie}(T) \subset \mathfrak{g}$ be diagonal matrices. We fix a Cartan decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{b}$ with $\mathrm{Lie}(B) = \mathfrak{b}$ and $\mathrm{Lie}(U^-) = \mathfrak{n}^-$. We also have $SL_n/B = \mathrm{Fl}_n$. By R we denote the root system of $\mathfrak{g}$, see [20, Sect. 9.2] for the definition. Here R is of type $\mathtt{A}_{n-1}$. Let R^+ be the set of positive roots with respect to the given choice of $\mathfrak{b}$. We denote the simple roots generating the root lattice by $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$, and their coroots generating the dual lattice by $\alpha_i^\vee$. For positive roots $\alpha_i + \alpha_{i+1} + \cdots + \alpha_j$ with $j \geq i$ we use the short notation $\alpha_{i,j}$. Using this notation we have $\alpha_{i,i} = \alpha_i$. The number of positive roots is N, which is also the length of w_0 as reduced expression in the s_i. For a positive root $\beta \in R^+, f_\beta$ is a non-zero root vector in $\mathfrak{n}^-$ of weight $-\beta$. Let P denote

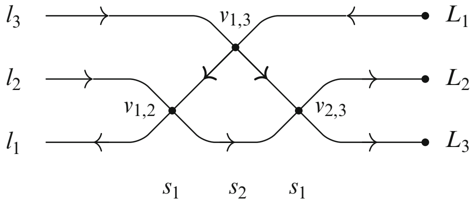


Fig. 3 Pseudoline arrangement corresponding to $\underline{w}_0 = s_1s_2s_1$ for Fl_3 with orientation induced by l_1

Fig. 4 The two local orientations with thick arrows forbidden in rigorous paths

the weight lattice of T generated by the fundamental weights $\omega_1, \omega_2, \ldots, \omega_{n-1}$. The definition can be found in [20, Sect. 13.1]. A weight $\lambda \in P$ is *regular dominant*, if $\lambda = \sum_{i=1}^{n-1} a_i\omega_i$ with $a_i \in \mathbb{Z}_{>0}$ for all i. The subset of regular dominant weights is denoted by P^{++}.

For a fixed weight $\lambda \in P^{++}$ and a reduced expression $\underline{w}_0$ of w_0 we construct the string polytope $Q_{\underline{w}_0}(\lambda)$. This description can be found in [18] and [24]. To $\underline{w}_0$ one associates a *pseudoline arrangement*. It consists of n horizontal *pseudolines* (or in short *lines*) labelled 1 to n on the left from bottom to top. Pairwise, they cross exactly once and the order of crossings depends on $\underline{w}_0$. More precisely, a simple reflection s_i induces a crossing on level i, see Fig. 3. The diagram has vertices $v_{i,j}$ for every crossing of lines l_i and l_j, as well as vertices $L_1, L_2, \ldots, L_n$ from top to bottom at the right ends of the lines. Every line l_i with $1 \leq i < n$ induces an orientation of the diagram obtained by orienting l_j for $j > i$ from left to right and l_k for $k \leq i$ from right to left.

Fix an oriented path $v_0 \to \cdots \to v_s$ in an (oriented) pseudoline arrangement and assume three adjacent vertices $v_{k-1} \to v_k \to v_{k+1}$ on the path belong to the same pseudoline l_i. Whenever a path does not change the line at a crossing, we are in this situation. Let v_k be the intersection of l_i and l_j. The path is *rigorous*, if it avoids the following two situations:

- $i < j$ and both lines are oriented to the left or
- $i > j$ and both lines are oriented to the right.

The first situation is visualized on the left of Fig. 4 and the second on the right. The thick arrow is the part of line l_i that must not be contained in a rigorous path. We denote by $\mathscr{P}_{\underline{w}_0}$ the set of all possible rigorous paths for all orientations induced by the lines l_i with $1 \leq i < n$.

Example 4.1 Consider Fl_4 with reduced expression $\underline{w}_0 = s_1s_2s_3s_2s_1s_2$. We draw the corresponding pseudoline arrangement in Fig. 5 with orientation induced by l_1. The rigorous paths for this orientation have source L_1 and sink L_2. An example of a *rigorous* path is $\mathbf{p} = L_1 \to v_{1,4} \to v_{1,3} \to v_{3,4} \to v_{2,3} \to L_2$. An example for a

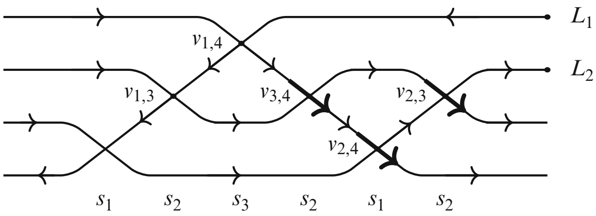


Fig. 5 A pseudoline arrangement for Fl_4 with $\underline{w}_0 = s_1s_2s_3s_2s_1s_2$ and orientation induced by l_1; thicker arrows denote forbidden line segments for rigorous paths

non-rigorous path is one that passes through a thick arrow, for example

$$\mathbf{p}' = L_1 \to v_{1,4} \to v_{3,4} \to v_{2,4} \to v_{2,3} \to L_2.$$

Back to the general case, we fix an orientation induced by $l_i, 1 \le i < n$ and consider all rigorous paths from L_i to L_{i+1}. We associate the *weight* $c_{\mathbf{p}}$ to each such path $\mathbf{p}$ as follows. Denote by $\{c_{i,j}\}_{1 \le i,j \le n}$ the standard basis of $\mathbb{R}^N$, where we set $c_{i,j} = -c_{j,i}$ if $i > j$ and $c_{j,j} = 0$. The integer N is the number of crossings in a pseudoline arrangement and hence we can associate the basis vector $c_{i,j}$ to the crossing of l_i and l_j for $1 \le i,j \le n$. Consider a rigorous path $\mathbf{p} = L_i \to v_{r_1} \to \cdots \to v_{r_m} \to L_{i+1}$. Every vertex v_{r_s} corresponds to the crossing of two lines l_k and l_j. If $\mathbf{p}$ changes from line l_k to line l_j at v_{r_s} we associate the vector $c_{k,j} \in \mathbb{R}^N$. We set $c_{\mathbf{p}}$ to be the sum of all such $c_{k,j}$ in $\mathbf{p}$ and denote it by $c_{\mathbf{p}}$.

Definition 4.2 For a fixed reduced expression $\underline{w}_0$, we define the *string cone* to be

$$C_{\underline{w}_0} = \left\{(y_{i,j}) \in \mathbb{R}^N : (c_{\mathbf{p}})^{\mathsf{T}}(y_{i,j}) \ge 0, \text{ for all } \mathbf{p} \in \mathscr{P}_{\underline{w}_0}\right\}.$$

This is not the original definition of a string cone; see [18, Corollary 5.8]. It can be extended to describe string cones for Schubert varieties, see [6].

Example 4.3 There are two rigorous paths in Fig. 3, $L_1 \to v_{1,3} \to v_{2,3} \to L_2$ and $L_1 \to v_{1,3} \to v_{1,2} \to v_{2,3} \to L_2$. The corresponding weights are $c_{1,3} - c_{2,3}$ and $c_{1,2}$ inducing the inequalities $y_{1,3} - y_{2,3} \ge 0$ and $y_{1,2} \ge 0$. Considering the orientation induced by l_2, there is a rigorous path $L_2 \to v_{2,3} \to L_3$ which gives the inequality $y_{2,3} \ge 0$. The string cone corresponding to the underlying non-oriented pseudoline arrangement in Fig. 3 is then given by $C_{s_1s_2s_1} = \{y_{1,2} \ge 0,\ y_{1,3} \ge y_{2,3} \ge 0\}$.

Each crossing of lines l_k and l_m corresponds to an index i_j associated to a simple reflection s_{i_j} in $\underline{w}_0$; see Fig. 3. Therefore, we write $c_{k,m} = c_j$. Let $1 \le i \le n-1$ and $r_1, r_2, \ldots, r_{n_i}$ be the indices such that $s_{i_{r_p}} = s_i$ in $\underline{w}_0$ for $1 \le p \le n_i$. Further, let $k_1, k_2, \ldots, k_t$ be the positions where $s_{i_{k_m}} \in \{s_{i-1}, s_{i+1}\}$ for $1 \le m \le t$. In particular, $r_1, r_2, \ldots, r_{n_i}$ are those positions inducing a crossing at level i in the corresponding pseudoline arrangement. The following appears in [24].

Definition 4.4 The *weighted string cone* $\mathscr{C}_{\underline{w}_0} \subset \mathbb{R}^N \times \mathbb{R}^{n-1}_{\geq 0}$ is obtained from $C_{\underline{w}_0}$ by adding variables $m_1, m_2, \dots, m_{n-1}$, and for every $1 \leq i \leq n-1$ and $j \in \{r_1, r_2, \dots, r_{n_i}\}$ the inequality $m_i - y_j - 2\sum_{r_p > j} y_{r_p} + \sum_{k_p > j} y_{k_p} \geq 0$, where $(y_k, m_l) \in \mathbb{R}^N \times \mathbb{R}^{n-1}_{\geq 0}$. For a weight $\lambda = \sum_{i=1}^{n-1} a_i \omega_i \in P$, the *string polytope* is $Q_{\underline{w}_0}(\lambda) := Q_{\underline{w}_0} \cap H_\lambda$, where H_λ is the intersection of the hyperplanes defined by $m_i = a_i$ for all $1 \leq i < n$.

The additional *weight inequalities* can also be obtained combinatorially as described in [6]. We will consider for all computations the weight $\rho = \sum_{i=1}^{n-1} \omega_i$. This is the weight in P^{++} with minimal choice of coefficients of fundamental weights in $\mathbb{Z}_{>0}$, namely all are 1. All string polytopes are cut out from the weighted string cone, but for different weights they are different polytopes.

The following result is a simplified version of Theorem 1 proven by Caldero [7] for flag varieties. A more general statement is given by Alexeev and Brion in [1, Theorem 3.2].

Theorem 4.5 *There exists a flat family $\mathscr{X} \to \mathbb{A}^1$ for a normal variety $\mathscr{X}$ such that for $t \neq 0$ the fibre over t is isomorphic to Fl_n and for $t = 0$ it is isomorphic to a projective toric variety X_0 with polytope $Q_{\underline{w}_0}(\lambda)$ for $\lambda \in P^{++}$.*

The proof of Theorem 4.5 uses the embedding $\mathrm{Fl}_n \hookrightarrow \mathbb{P}(\mathrm{V}(\lambda))$ and the dual canonical basis, where $V(\lambda)$ is the irreducible representation of $\mathfrak{sl}_n$ with highest weight λ.

For $A, B \subset \mathbb{R}^l$, the *Minkowski sum* is $A + B := \{a + b : a \in A, b \in B\}$. Consider the weight ρ. The string polytope $Q_{\underline{w}_0}(\rho)$ is in general *not* the Minkowski sum of string polytopes $Q_{\underline{w}_0}(\omega_1), \dots, Q_{\underline{w}_0}(\omega_{n-1})$, which motivates the following definition.

Definition 4.6 A string cone has the *weak Minkowski property* (MP), if for every lattice point $p \in Q_{\underline{w}_0}(\rho)$ there exist lattice points $p_{\omega_i} \in Q_{\underline{w}_0}(\omega_i)$ such that

$$p = p_{\omega_1} + p_{\omega_2} + \cdots + p_{\omega_{n-1}}.$$

Remark 4.7 The (non-weak) Minkowski property would require the above condition on lattice points to be true for arbitrary weights λ. Further, note that if $Q_{\underline{w}_0}(\rho)$ is the Minkowski sum of the fundamental string polytopes $Q_{\underline{w}_0}(\omega_i)$, then MP is satisfied.

Proposition 4.8 *For Fl_4 there are four string polytopes in $\mathbb{R}^{10}$ up to unimodular equivalence and three of them satisfy MP. For Fl_5 there are 28 string polytopes in $\mathbb{R}^{14}$ up to unimodular equivalence and 14 of them satisfy MP.*

Proof We first consider Fl_4. There are 16 reduced expressions for w_0. Simple transpositions s_i and s_j with $1 \leq i < i + 1 < j < n$ commute and are also called *orthogonal*. We consider reduced expressions up to changing those, i.e. there are eight symmetry classes. We fix the weight in P^{++} to be $\rho = \omega_1 + \omega_2 + \omega_3$. The string polytopes are organized in four classes up to unimodular equivalence. Table 2 summarizes the weight vectors $\mathbf{w}_{\underline{w}_0}$ constructed in Sect. 5, primeness of the binomial

Table 2 Isomorphism classes of string polytopes for $n = 4$

$\underline{w}_0$	Weight vector $\mathbf{w}_{\underline{w}_0}$	Tropical cone
String 1		
121321	(0, 32, 24, 7, 0, 16, 6, 48, 38, 30, 0, 4, 20, 52)	Rays 10, 18, 19, cone 71
212321	(0, 16, 48, 7, 0, 32, 6, 24, 22, 54, 0, 4, 36, 28)	Rays 6, 10, 19, cone 44
232123	(0, 4, 36, 28, 0, 32, 24, 6, 22, 54, 0, 16, 48, 7)	Rays 0, 3, 6, cone 3
323123	(0, 4, 20, 52, 0, 16, 48, 6, 38, 30, 0, 32, 24, 7)	Rays 0, 1, 3, cone 1
String 2		
123212	(0, 32, 18, 14, 0, 16, 12, 48, 44, 27, 0, 8, 24, 56)	Rays 2, 10, 18, cone 36
321232	(0, 8, 24, 56, 0, 16, 48, 12, 44, 27, 0, 32, 18, 14)	Rays 0, 1, 2, cone 0
String 3		
213231	(0, 16, 48, 13, 0, 32, 12, 20, 28, 60, 0, 8, 40, 22)	Rays 3, 6, 19, cone 24
String 4		
132312	(0, 16, 12, 44, 0, 8, 40, 24, 56, 15, 0, 32, 10, 26)	Rays 1, 2, 17, cone 17
FFLV		
	$w^{min} = (0, 2, 2, 1, 0, 1, 1, 2, 1, 2, 0, 1, 1, 1)$	Rays 9, 11, 12, cone 56
	$w^{reg} = (0, 3, 4, 3, 0, 2, 2, 4, 3, 5, 0, 1, 2, 3)$	Rays 9, 11, 12, cone 56

initial ideals $\text{in}_{\mathbf{w}_{\underline{w}_0}}(I_4)$, and the corresponding tropical cones with their spanning rays as they appear in the file `Flag4.txt` at github.com/ToricDegenerations; the word 121321 denotes the reduced expression $\underline{w}_0 = s_1s_2s_1s_3s_2s_1$. All of these polytopes are normal and, except for those in the class String 4, these polytopes satisfy the weak Minkowski property and the binomial initial ideals are prime. The four classes give four different toric degenerations for the embedding $\text{Fl}_4 \hookrightarrow \mathbb{P}\big(\mathrm{V}(\rho)\big)$. In order to verify whether the weak Minkowski property holds or not, we proceed as follows. We fix $\underline{w}_0$ to compute the string polytope $Q_{\underline{w}_0}(\rho)$ using *polymake*. The number of lattice points in $Q_{\underline{w}_0}(\rho)$ is $\dim \mathrm{V}(\rho) = 64$. Then we compute the polytopes $Q_{\underline{w}_0}(\omega_1), Q_{\underline{w}_0}(\omega_2), sQ_{\underline{w}_0}(\omega_3)$ and set $P = Q_{\underline{w}_0}(\omega_1) + Q_{\underline{w}_0}(\omega_2) + Q_{\underline{w}_0}(\omega_3)$. Now let $LP(P)$ be the set of lattice points in P. If $|LP(P)| < 64$, then there exists a lattice point in $Q_{\underline{w}_0}(\rho)$ that cannot be expressed as $p_1 + p_2 + p_3$ for $p_i \in Q_{\underline{w}_0}(\omega_i)$. For $\underline{w}_0 = s_1s_3s_2s_3s_1s_2$, we observe that $\big|LP(Q_{\underline{w}_0}(\omega_1)+Q_{\underline{w}_0}(\omega_2)+Q_{\underline{w}_0}(\omega_3))\big| = 62 < 64$. Hence, the class String 4 does not satisfy MP. For the classes String 1, 2, and 3 equality holds and MP is satisfied.

Now consider Fl_5. There are 62 reduced expressions $\underline{w}_0$ up to changing orthogonal transpositions. The map $L : S_5 \to S_5$ given on simple reflections by $L(s_i) = s_{4-i+1}$ induces a symmetry among the string polytopes. Namely, for a fixed $\lambda \in P^{++}$, there is a unimodular equivalence between $Q_{\underline{w}_0}(\lambda)$ and $Q_{L(\underline{w}_0)}(\lambda)$. Exploiting this symmetry, we compute 31 string polytopes for ρ. These are organized in 28 unimodular equivalence classes, that arise from further symmetries of the underlying pseudoline arrangements. Table 6 shows which reduced expressions belong to string polytopes within one class of unimodular equivalence, and which string cones satisfy MP. Proceeding as for Fl_4, we observe that 14 out of 28 classes satisfy MP. □

We will now turn to the FFLV polytope. It is defined in [15] by Feigin, Fourier, and Littelmann to describe bases of irreducible highest weight representations $V(\lambda)$. In [16], they give a construction of a flat degeneration of the flag variety into the toric variety associated to the FFLV polytope. It is also an example of the more general setup presented in [12]. We give the general definition here and compute the FFLV polytopes for Fl_4 and Fl_5 for ρ. Recall, that α_i for $1 \leq i < n$ are the simple roots of $\mathfrak{sl}_n$, and $\alpha_{p,q}$ is the positive root $\alpha_p + \alpha_{p+1} + \cdots + \alpha_q$ for $1 \leq p \leq q < n$.

Definition 4.9 A *Dyck path* is a sequence of positive roots $\mathbf{d} = (\beta_0, \beta_1, \ldots, \beta_k)$ with $k \geq 0$ satisfying the following conditions:

- if $k = 0$ then $\mathbf{d} = (\alpha_i)$ for $1 \leq i \leq n-1$,
- if $k \geq 1$ then the first and the last roots are simple, that is $\beta_0 = \alpha_i$, $\beta_k = \alpha_j$ for $1 \leq i < j \leq n-1$. Moreover, if $\beta_s = \alpha_{p,q}$ then β_{s+1} is either $\alpha_{p,q+1}$ or $\alpha_{p+1,q}$.

Denote by $\mathscr{D}$ the set of all Dyck paths. We choose the positive roots $\alpha > 0$ as an indexing set for a basis of $\mathbb{R}^N$.

Definition 4.10 The *FFLV polytope* $P(\lambda) \subset \mathbb{R}^N_{\geq 0}$ for a weight $\lambda = \sum_{i=1}^{n-1} m_i\omega_i \in P^{++}$ is defined as

$$P(\lambda) = \left\{ (r_\alpha)_{\alpha>0} \in \mathbb{R}^N_{\geq 0} : \begin{array}{c} \text{for all } \mathbf{d} \in \mathscr{D}, \text{ if } \beta_0 = \alpha_i \text{ and } \beta_k = \alpha_j \text{ then} \\ r_{\beta_0} + r_{\beta_1} + \cdots + r_{\beta_k} \leq m_i + m_{i+1} + \cdots + m_j \end{array} \right\}.$$

Example 4.11 For Fl_4, the seven Dyck paths are (α_1), (α_2), (α_3), $(\alpha_1, \alpha_{1,2}, \alpha_2)$, $(\alpha_2, \alpha_{2,3}, \alpha_3)$, $(\alpha_1, \alpha_{1,2}, \alpha_2, \alpha_{2,3}, \alpha_3)$, and $(\alpha_1, \alpha_{1,2}, \alpha_{1,3}, \alpha_{2,3}, \alpha_3)$. For our favorite choice of weight $\lambda = \rho = \omega_1 + \omega_2 + \omega_3$, we obtain the FFLV polytope

$$P(\rho) = \left\{ (r_\alpha)_{\alpha>0} : \begin{array}{c} r_{\alpha_1} \leq 1, r_{\alpha_2} \leq 1, r_{\alpha_3} \leq 1, r_{\alpha_1} + r_{\alpha_{1,2}} + r_{\alpha_2} \leq 2, \\ r_{\alpha_2} + r_{\alpha_{2,3}} + r_{\alpha_3} \leq 2, r_{\alpha_1} + r_{\alpha_{1,2}} + r_{\alpha_2} + r_{\alpha_{2,3}} + r_{\alpha_3} \leq 3, \\ r_{\alpha_1} + r_{\alpha_{1,2}} + r_{\alpha_{1,3}} + r_{\alpha_{2,3}} + r_{\alpha_3} \leq 3 \end{array} \right\} \subset \mathbb{R}^6_{\geq 0}.$$

The following is a corollary of [15, Proposition 11.6], which says that a strong version of the Minkowski property is satisfied by the FFLV polytope for Fl_n. It can alternatively be shown for $n = 4, 5$ using the methods in the proof of Proposition 4.8.

Corollary 4.12 *The FFLV polytope $P(\rho)$ satisfies the weak Minkowski property.*

Remark 4.13 The FFLV polytope is in general not a string polytope. A computation in *polymake* shows that $P(\rho)$ for Fl_5 is not combinatorially equivalent to any string polytope for ρ.

5 String Cones and the Tropicalized Flag Variety

We have seen in Sect. 2 how to obtain toric degenerations from maximal prime cones of the tropicalization of the flag varieties. We compare the different toric degenerations that arise from the different approaches. Moreover, applying [7, Lemma 3.2] we construct a weight vector from a string cone. Computational evidence for Fl_4 and Fl_5 shows that each constructed weight vector lies in the relative interior of a maximal cone in $\mathrm{trop}(\mathrm{Fl}_n)$. A similar idea for a more general case is carried out in [23, Sect. 7]. For the FFLV polytope we compute weight vectors for Fl_n with $n = 4, 5$ following a construction given in [14]; see Example 5.8.

We will now prove the result in Theorem 1.3 by analyzing the polytopes associated to the different toric degenerations of Fl_n for $n = 4, 5$.

Proof of Theorem 1.3 In order to distinguish the different toric degenerations, we consider the toric varieties associated to their special fibres. In case of the degenerations induced by the string polytopes and FFLV polytope, these toric varieties are normal. This might not be true for the degenerations found in Theorem 3.3 and Theorem 3.5. Hence, we consider two toric degenerations to be different if the normalization of their special fibres are not isomorphic.

Two toric varieties are isomorphic if their corresponding fans are unimodular equivalent. In our case the fans are the normal fans of the polytopes. For this reason we first look for combinatorial equivalences between those. If they are not combinatorially equivalent then their normal fans cannot be unimodular equivalent. We use *polymake* [17] for computations with polytopes.

From Table 3 one can see that for Fl_4 there is one toric degeneration, whose associated polytope is not combinatorially equivalent to any string polytope or the FFLV polytope for ρ. Hence, its corresponding normal toric variety is not isomorphic to any toric variety associated to these polytopes. For the toric varieties associated to the other polytopes we cannot exclude isomorphism since there might be a unimodular equivalence between pairs of normal fans.

For Fl_5, Table 5 shows that there are 168 polytopes obtained from prime cones of $\mathrm{trop}(\mathrm{Fl}_5)$ that are not combinatorially equivalent to any string polytope or the FFLV polytope for ρ. □

Remark 5.1 There are also string polytopes, which are not combinatorially equivalent to any polytope from prime cones in $\mathrm{trop}(\mathrm{Fl}_n)$ for $n = 4, 5$. These are exactly those not satisfying MP: one string polytope for Fl_4 and 14 for Fl_5; see Table 6.

Table 3 Equivalent polytopes obtained from $\mathrm{trop}(\mathrm{Fl}_4)$

Orbit	Combinatorially equivalent polytopes
1	String 2
2	String 1 (Gelfand-Tsetlin)
3	String 3 and FFLV
4	–

From now on, we fix a reduced expression $\underline{w}_0 = s_{i_1}s_{i_2}\cdots s_{i_N}$ and we consider the sequence of simple roots $S = (\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_N})$. For a positive root α, we denote by f_α the root vector in $\mathfrak{n}^- \subset \mathfrak{sl}_n$ of weight $-\alpha$. By [12, Lemma 2], the following holds.

Proposition 5.2 *The universal enveloping algebra* $U(\mathfrak{n}^-)$ *is linearly generated by monomials of the form* $\mathbf{f}^{\mathbf{m}} = f_{\alpha_{i_1}}^{m_1} f_{\alpha_{i_2}}^{m_2} \cdots f_{\alpha_{i_N}}^{m_N}$ *for* $m_i \in \mathbb{N}$.

The proposition may be interpreted as a definition of the universal enveloping algebra. Given a weight λ, the irreducible highest weight representation $V = V(\lambda)$ is cyclically generated by a highest weight vector $v_\lambda \in V(\lambda)$, i.e. $\mathrm{V}(\lambda) = U(\mathfrak{n}^-).v_\lambda$.

Example 5.3 For Fl_4, three root vectors in $\mathfrak{n}^-$ are

$$f_{\alpha_1} = \begin{bmatrix} 0&0&0&0\\ 1&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{bmatrix}, \qquad f_{\alpha_2} = \begin{bmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&1&0&0\\ 0&0&0&0 \end{bmatrix}, \qquad f_{\alpha_3} = \begin{bmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&1&0 \end{bmatrix}.$$

The action of $\mathfrak{n}^-$ on $\mathbb{C}^4$ is defined by $f_{\alpha_i}(e_i) = e_{i+1}$ and $f_{\alpha_i}(e_j) = 0$ for $j \neq i$. On $V = \bigwedge^2 \mathbb{C}^4$, the $\mathfrak{n}^-$-action is given by $f_{\alpha_i}(e_j \wedge e_k) = f_{\alpha_i}(e_j) \wedge e_k + e_j \wedge f_{\alpha_i}(e_k)$. For $e_1 \wedge e_3 \in V$, we have $f_{\alpha_2}(e_1 \wedge e_2) = e_1 \wedge e_3$. Since $V = U(\mathfrak{n}^-).(e_1 \wedge e_2)$, it follows that $v_{\omega_2} := e_1 \wedge e_2$ is a highest weight vector. Fixing $\underline{w}_0 = s_1s_2s_1s_3s_2s_1$, we obtain $U(\mathfrak{n}^-) = \langle f_{\alpha_1}^{m_1} f_{\alpha_2}^{m_2} f_{\alpha_1}^{m_3} f_{\alpha_3}^{m_4} f_{\alpha_2}^{m_5} f_{\alpha_1}^{m_6} : m_i \in \mathbb{N} \rangle$. Hence, we deduce that

$$\mathbf{f}^{(0,1,0,0,0,0)}(e_1 \wedge e_2) = \mathbf{f}^{(0,0,0,0,1,0)}(e_1 \wedge e_2).$$

As seen in Example 5.3, the monomial $\mathbf{f}^{\mathbf{m}}$ for a given weight vector $v \in V$ with $\mathbf{f}^{\mathbf{m}}(v_\lambda) = v$ is not unique. To fix this, we define a term order on the monomials $\mathbf{f}^{\mathbf{m}}$ generating $U(\mathfrak{n}^-)$ and pick the minimal monomial with this property. We fix for $\mathbf{m}, \mathbf{n} \in \mathbb{N}^N$ the order $\mathbf{f}^{\mathbf{m}} \succ \mathbf{f}^{\mathbf{n}}$ if $\deg(\mathbf{f}^{\mathbf{m}}) > \deg(\mathbf{f}^{\mathbf{n}})$ or $\deg(\mathbf{f}^{\mathbf{m}}) = \deg(\mathbf{f}^{\mathbf{n}})$ and $\mathbf{m} <_{lex} \mathbf{n}$. The connection to $\mathrm{trop}(\mathrm{Fl}_n)$ is established through Plücker coordinates. For $J := \{j_1, j_2, \ldots, j_k\} \subset [n]$, the Plücker coordinate p_J is given by

$$(e_{j_1} \wedge e_{j_1} \wedge \cdots \wedge e_{j_k})^* \in (\textstyle\bigwedge^k \mathbb{C}^n)^*.$$

Now, $\bigwedge^k \mathbb{C}^n$ is the fundamental representation $V(\omega_k) = U(\mathfrak{n}^-).(e_1 \wedge e_2 \wedge \cdots \wedge e_k)$; see Example 5.3. Denote by $\mathbf{m}_J$, the unique multiexponent such that $\mathbf{f}^{\mathbf{m}_J}$ is $\prec$-minimal satisfying $\mathbf{f}^{\mathbf{m}}(e_1 \wedge e_2 \wedge \cdots \wedge e_k) = e_{j_1} \wedge \ldots \wedge e_{j_k}$.

Following a construction given in [7, Proof of Lemma 3.2], we define the linear form $e\colon \mathbb{N}^N \to \mathbb{N}$ as $e(\mathbf{m}) = 2^{N-1}m_1 + 2^{N-2}m_2 + \cdots + 2m_{N-1} + m_N$. This is a particular choice satisfying $\mathbf{m} \succ \mathbf{n} \Rightarrow e(\mathbf{m}) > e(\mathbf{n})$ for $\mathbf{m}, \mathbf{n} \in \mathbb{N}^N$.

Definition 5.4 For a fixed reduced expression $\underline{w}_0$ the *weight* of the Plücker variable p_J is $e(\mathbf{m}_J)$. We fix the *weight vector* $\mathbf{w}_{\underline{w}_0}$ in $\mathbb{R}^{\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n}-1}$ to be

$$\mathbf{w}_{\underline{w}_0} = (e(\mathbf{m}_1), e(\mathbf{m}_2), \ldots, e(\mathbf{m}_{2,3,\ldots,n})).$$

Example 5.5 We continue as in Example 5.3 with the fixed reduced expression $\underline{w}_0 = s_1s_2s_1s_3s_2s_1$ for Fl_4. The Plücker coordinate p_{13} in Fl_4 is $(e_1 \wedge e_3)^*$. The corresponding minimal monomial among those satisfying $\mathbf{f}^{\mathbf{m}}(e_1 \wedge e_2) = e_1 \wedge e_3$ is $\mathbf{f}^{(0,1,0,0,0,0)}$. Hence, the weight of p_{13} is $e(0, 1, 0, 0, 0, 0) = 1 \cdot 2^4 = 16$. Similarly, we obtain the weights of all Plücker coordinates and

$$\mathbf{w}_{\underline{w}_0} = (0, 32, 24, 7, 0, 16, 6, 48, 38, 30, 0, 4, 20, 52).$$

Table 2 contains all weight vectors for Fl_4 constructed in the way just described.

Proposition 5.6 *Consider* Fl_n *with* $n = 4$ *or* 5*. The above construction produces a weight vector* $\mathbf{w}_{\underline{w}_0}$ *for every string cone. This weight vector lies in the relative interior of a maximal cone of* $\mathrm{trop}(\mathrm{Fl}_n)$*. If the string cone satisfies MP, then* $\mathbf{w}_{\underline{w}_0}$ *lies in the relative interior of a prime cone whose associated polytope is combinatorially equivalent to* $Q_{\underline{w}_0}(\rho)$*.*

Proof The constructed weight vectors $\mathbf{w}_{\underline{w}_0}$ can be found in Table 2 for Fl_4 and Table 6 in the appendix for Fl_5. A computation in *Macaulay2* shows that all initial ideals $\mathrm{in}_{\mathbf{w}_{\underline{w}_0}}(I_n)$ for $n = 4, 5$ are binomial, hence in the relative interiors of maximal cones of $\mathrm{trop}(\mathrm{Fl}_n)$.

Moreover, if MP is satisfied we check using *polymake* that the polytope constructed from the maximal prime cone $C_{\underline{w}_0}$ with $\mathbf{w}_{\underline{w}_0}$ in its relative interior is combinatorially equivalent to the string polytope $Q_{\underline{w}_0}(\rho)$; see Table 2 and Table 6. □

Computational evidence leads us to the following conjecture.

Conjecture 5.7 Let $n \geq 3$ be an arbitrary integer. For every reduced expression $\underline{w}_0$, the weight vector $\mathbf{w}_{\underline{w}_0}$ lies in the relative interior of a maximal cone in $\mathrm{trop}(\mathrm{Fl}_n)$. In particular, if the string cone satisfies MP this vector lies in the relative interior of the prime cone C, whose associated polytope is combinatorially equivalent to the string polytope $Q_{\underline{w}_0}(\rho)$.

The following example discusses a similar construction of weight vectors for the FFLV polytope.

Example 5.8 Consider for Fl_4 the sequence of positive roots

$$S = (\alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1, \alpha_2, \alpha_3).$$

By [12, Example 1], Proposition 5.2 is also true for this choice of S. More generally speaking, Proposition 5.2 holds for every sequence containing all positive roots ordered by height. The *height* of a positive root is the number of simple summands. Such sequences are called *PBW-sequences* with *good ordering* in [12].

With this choice of S we apply the aformentioned procedure to obtain a unique multi-exponent $\mathbf{m}_J$ for each Plücker variable p_J. Taking the convex hull of all multi-exponents $\mathbf{m}_J$ for $J \subset \{1, \ldots, 4\}$ yields the FFLV polytope from Definition 4.10 with respect to the embedding $\mathrm{Fl}_4 \hookrightarrow \mathbb{P}(\mathrm{V}(\rho))$. Then we define linear forms

$$e^{\min}(\mathbf{m}_J) = m_1 + 2m_2 + m_3 + 2m_4 + m_5 + m_6,$$
$$e^{\mathrm{reg}}(\mathbf{m}_J) = 3m_1 + 4m_2 + 2m_3 + 3m_4 + 2m_5 + m_6,$$

according to the degrees defined in [14]. We obtain in analogy to Definition 5.4 the corresponding weight vectors

$$\mathbf{w}^{\min} = (0, 2, 2, 1, 0, 1, 1, 2, 1, 2, 0, 1, 1, 1),$$
$$\mathbf{w}^{\mathrm{reg}} = (0, 3, 4, 3, 0, 2, 2, 4, 3, 5, 0, 1, 2, 3).$$

A computation in *Macaulay2* shows that $\mathrm{in}_{\mathbf{w}^{\min}}(I_4) = \mathrm{in}_{\mathbf{w}^{\mathrm{reg}}}(I_4)$ is a binomial prime ideal. Hence, $\mathbf{w}^{\min}$ and $\mathbf{w}^{\mathrm{reg}}$ lie in the relative interior of the same prime cone $C \subset \mathrm{trop}(\mathrm{Fl}_4)$. Using *polymake* [17], we verify that the polytope associated to C is combinatorially equivalent to the FFLV polytope $P(\rho)$. We did the analogue of this computation for Fl_5 and the outcome is the same, $\mathrm{in}_{\mathbf{w}^{\min}}(I_5) = \mathrm{in}_{\mathbf{w}^{\mathrm{reg}}}(I_5) = \mathrm{in}_C(I_5)$ with the polytope associated to C being combinatorially equivalent to $P(\rho)$. The weight vectors $\mathbf{w}^{\min}$ and $\mathbf{w}^{\mathrm{reg}}$ for Fl_5 can be found in Table 6 in the appendix.

6 Toric Degenerations from non-prime Cones

As we have seen in Sect. 3, not all maximal cones in the tropicalization of a variety give rise to prime initial ideals and hence to toric degenerations. In fact, there may also be tropicalizations without prime cones (see Example 6.3). Let X be a subvariety of a toric variety Y. In this section, we give a recursive procedure in Procedure 6.1 to compute a new embedding X' of X in case $\mathrm{trop}(X)$ has non-prime cones. Let C be a non-prime cone. If the procedure terminates, the new variety X' has more prime cones than $\mathrm{trop}(X)$ and at least one of them is projecting onto C. We apply this procedure to Fl_4 and compare the new toric degenerations with those obtained so far (see Proposition 6.4). The procedure terminates for Fl_4, but we are still investigating the conditions for which this is true in general.

Procedure 6.1 New embeddings of X when $\mathrm{trop}(X)$ contains non-prime cones

Input: $A = \mathbb{C}[x_0, x_1, \ldots, x_n]/I$ where $\mathbb{C}[x_0, x_1, \ldots, x_n]$ is the total coordinate ring of the toric variety Y and I defines the subvariety $\mathrm{V}(I) \subset Y$, and C is a non-prime cone of $\mathrm{trop}\big(\mathrm{V}(I)\big)$ with multiplicity 1.
Output: The algebra A', the ideal I' of a new embedding of X, and the ideal $\mathrm{in}_{C'}(I')$ of a toric degeneration of X.
Compute the primary decomposition of $\mathrm{in}_C(I)$;
Set $I(W_C)$ to be the unique prime toric component in the decomposition;
Set G to be the minimal generating set of $I(W_C)$;
Compute a list of binomials $L_C = \{f_1, f_2, \ldots, f_s\}$ in G which are not in $\mathrm{in}_C(I)$;
Set $A' := \mathbb{C}[x_0, x_1, \ldots, x_n, y_1, y_2, \ldots, y_s]/I'$ where the ideal I' is

$I' = I + \langle y_1 - f_1, y_2 - f_2, \dots, y_s - f_s \rangle$;
Compute trop $\left(\mathrm{V}(I')\right)$;
For all prime cones $C' \in \mathrm{trop}(V(I'))$ do
If $\pi(C')$ is contained in the relative interior of C then
Return the algebra A' and the ideal $\mathrm{in}_C(I')$
Else
Apply the procedure again to A' and C'.

We now explain Procedure 6.1. Consider a toric variety Y whose total coordinate ring is $\mathbb{C}[x_0, x_1, \dots, x_n]$ with associated $\mathbb{Z}^k$-degree $\deg\colon \mathbb{Z}^{n+1} \to \mathbb{Z}^k$. Let X be the subvariety of Y associated to an ideal $I \subset \mathbb{C}[x_0, x_1, \dots, x_n]$, where the Krull dimension of $A = \mathbb{C}[x_0, x_1, \dots, x_n]/I$ is d. Denote by $\mathrm{trop}(\mathrm{V}(I))$ the tropicalization of X intersected with the torus of Y. Suppose there is a non-prime cone $C \subset \mathrm{trop}(\mathrm{V}(I))$ with multiplicity one. By Lemma 2.9, we have that $I(W_C)$ is the unique toric ideal in the primary decomposition of $\mathrm{in}_C(I)$, hence $\mathrm{in}_C(I) \subset I(W_C)$. We can compute $I(W_C)$ using the function `primaryDecomposition` in *Macaulay2*. Fix a minimal binomial generating set G of $I(W_C)$, and let $L_C = \{f_1, f_2, \dots, f_s\}$ be the set consisting of binomials in G, which are not in $\mathrm{in}_C(I)$. By Hilbert's Basis Theorem, s is a finite number. The absence of these binomials in $\mathrm{in}_C(I)$ is the reason why the initial ideal is not equal to $I(W_C)$. We introduce new variables $\{y_1, y_2, \dots, y_s\}$, where $\deg(y_i) = \deg(f_i)$ for $1 \le i \le s$, and consider the algebra $A' = \mathbb{C}[x_0, x_1, \dots, x_n, y_1, y_2, \dots, y_s]/I'$, where I' is the homogeneous ideal $I + \langle y_1 - f_1, y_1 - f_1, \dots, y_s - f_s \rangle$. The new variety $\mathrm{V}(I')$ is a subvariety of a toric variety Y', which has total coordinate $\mathbb{C}[Y'] := \mathbb{C}[x_0, x_1, \dots, x_n, y_1, y_2, \dots, y_s]$. For example, if $\mathrm{V}(I)$ is a subvariety of a projective space then $\mathrm{V}(I')$ is contained in a weighted projective space.

Since the algebras A and A' are isomorphic as graded algebras, the varieties $\mathrm{V}(I)$ and $\mathrm{V}(I')$ are isomorphic. We have a monomial map

$$\pi\colon \mathbb{C}[x_0, x_1, \dots, x_n]/I \to \mathbb{C}[x_0, x_1, \dots, x_n, y_1, y_2, \dots, y_s]/I'$$

inducing a surjective map $\mathrm{trop}(\pi)\colon \mathrm{trop}\left(\mathrm{V}(I')\right) \to \mathrm{trop}\left(\mathrm{V}(I)\right)$; see [25, Corollary 3.2.13]. The map $\mathrm{trop}(\pi)$ is the projection onto the first n coordinates. Suppose there exists a prime cone $C' \subset \mathrm{trop}(\mathrm{V}(I'))$, whose projection has a non-empty intersection with the relative interior of C. Then by construction we have $\mathrm{in}_C(I) \subset \mathrm{in}_{C'}(I') \cap \mathbb{C}[x_0, x_1, \dots, x_n]$ and the procedure terminates. In this way we obtain a new initial ideal $\mathrm{in}_{C'}(I')$ which is toric and hence gives a new toric degeneration of the variety $\mathrm{V}(I') \cong \mathrm{V}(I)$. If only non-prime cones are projecting to C then run this procedure again with A' and C', where the latter is a maximal cone of trop $\left(\mathrm{V}(I')\right)$, which projects to C. We can then repeat the procedure starting from a different non-prime cone.

The function to apply Procedure 6.1 is `findNewToricDegenerations` and it is part of the package `ToricDegenerations`. This will compute only one reembedding for each non-prime cone. It is possible to use `mapMaximalCones` to obtain the image of $\mathrm{trop}(\mathrm{V}(I'))$ under the map π.

Remark 6.2 If f_i is a polynomial in $k[x_0, x_1, \ldots, x_n]$ with the standard grading and $deg(f_i) > 1$, then we need to homogenize the ideal I' before computing the tropicalization with *Gfan*. This is done by adding a new variable h. The homogenization of I' with respect to h is denoted by $I'' \subseteq \mathbb{k}[x_0, x_1, \ldots, x_n, y_1, y_2, \ldots, y_s, h]$. Then by [25, Proposition 2.6.1] for every $\mathbf{w}$ in $\mathbb{R}^{n+s+2}$ the ideal $\text{in}_{\mathbf{w}}(I')$ is obtained from $\text{in}_{(\mathbf{w},0)}(I'')$ by setting $h = 1$.

If the cone C is prime, we can construct a valuation val_C on $\mathbb{k}[x_0, x_1, \ldots, x_n]/I$ as follows. Consider the matrix W_C in Equation (1). For $m_i = c\mathbf{x}^{\boldsymbol{\alpha}_i} \in \mathbb{k}[x_0, x_1, \ldots, x_n]$ define $\text{val}(m_i) = W_C \boldsymbol{\alpha}_i$ and $\text{val}(\sum_i m_i) = \min_i\{\text{val}(m_i)\}$, where the minimum on the right side is taken with respect to the lexicographic order on $(\mathbb{Z}^d, +)$. This is a valuation on $\mathbb{k}[x_0, x_1, \ldots, x_n]$ of rank equal to the Krull dimension of A for every cone C. Composing val with the map $p\colon \mathbb{k}[x_0, x_1, \ldots, x_n] \to \mathbb{k}[x_0, x_1, \ldots, x_n]/I$, we obtain a map val_C, which is a valuation if and only if the cone C is prime. Moreover, [23] proves that a cone C in $\text{trop}\left(\text{V}(I)\right)$ is prime if and only if $A = \mathbb{k}[x_0, x_1, \ldots, x_n]/I$ has a finite Khovanskii basis for the valuation val_C constructed from the cone C. A Khovanskii basis for an algebra A with valuation val_C is a subset B of A such that $\text{val}_C(B)$ generates the value semigroup

$$S(A, \text{val}_C) = \left\{\text{val}_C(f) : f \in A \setminus \{0\}\right\}.$$

Procedure 6.1 can be interpreted as finding an extension $\text{val}_{C'}$ of val_C so that A' has finite Khovanskii basis with respect to $\text{val}_{C'}$. The Khovanskii basis is given by the images of $x_0, x_1, \ldots, x_n, y_1, y_2, \ldots, y_s$ in A'. We illustrate the procedure in the following example.

Example 6.3 Consider the algebra $A = \mathbb{C}[x, y, z]/\langle xy+xz+yz\rangle$. The tropicalization of $\text{V}(\langle xy+xz+yz\rangle) \subset \mathbb{P}^2$ has three maximal cones. The corresponding initial ideals are $\langle xz + yz\rangle$, $\langle xy + yz\rangle$ and $\langle xy + xz\rangle$, none of which is prime. Hence, they do not give rise to toric degenerations. The matrices associated to each cone are

$$W_{C_1} = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}, \qquad W_{C_2} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \qquad W_{C_3} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

We now apply Procedure 6.1 to the cone C_1. The initial ideal associated to C_1 is generated by $xz + yz$. In this case $\text{in}_{C_1}(I) = \langle z\rangle \cdot \langle x + y\rangle$ hence for the missing binomial $x+y$ we adjoin a new variable u to $\mathbb{C}[x, y, z]$ and the new relation $u-x-y$ to I. We have $I' = \langle xy + xz + yz, u - x - y\rangle$ and $A' = \mathbb{C}[x, y, z, u]/I'$ with $\text{V}(I')$ a subvariety of $\mathbb{P}^3$. After computing the tropicalization of $\text{V}(I')$ we see that there exists a prime cone C' such that $\pi(C') = C$. The matrix $W_{C'}$ associated to the cone C' is

$$W' = \begin{bmatrix} 0 & 0 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

The initial ideal $\text{in}_{C'}(I')$ gives a toric degeneration of $V(I')$. The image of the set $\{x, y, z, u\}$ in A' is a Khovanskii basis for $S(A', \text{val}_{C'})$. Repeating this process for the cones C_2 and C_3 of $\text{trop}(V(xy + xz + yz))$, we get prime cones C'_2 and C'_3 whose projections are C_2 and C_3 respectively. Hence, there is a valuation with finite Khovanskii basis and a corresponding toric degeneration for every maximal cone.

We now apply Procedure 6.1 to $\text{trop}(\text{Fl}_4)$.

Proposition 6.4 *Each of the non-prime cones of* $\text{trop}(\text{Fl}_4)$ *gives rise to three toric degenerations, which are not isomorphic to any degeneration coming from the prime cones of* $\text{trop}(\text{Fl}_4)$. *Moreover, two of the three new polytopes are combinatorially equivalent to the previously missing string polytopes for* ρ *in the class String 4.*

Proof By Theorem 3.3, we know that $\text{trop}(\text{Fl}_4)$ has six non-prime cones forming one $S_4 \times \mathbb{Z}_2$-orbit. Hence, we apply Procedure 6.1 to only one non-prime cone. The result for the other non-prime cones will be the same up to symmetry. In particular, the obtained toric degenerations from one cone will be isomorphic to those coming from another cone. We describe the results for the maximal cone C associated to the initial ideal $\text{in}_C(I_4)$ defined by the following binomials:

$$
\begin{array}{ccc}
p_4p_{1,2,3} - p_3p_{1,2,4}\,, & p_{2,4}p_{1,3,4} - p_{1,4}p_{2,3,4}\,, & p_{2,3}p_{1,3,4} - p_{1,3}p_{2,3,4}\,, \\
p_2p_{1,4} - p_1p_{2,4}\,, & p_2p_{1,3} - p_1p_{2,3}\,, & p_{2,4}p_{1,2,3} - p_{2,3}p_{1,2,4}\,, \\
p_{1,4}p_{1,2,3} - p_{1,3}p_{1,2,4}\,, & p_4p_{2,3} - p_3p_{2,4} & p_4p_{1,3} - p_3p_{1,4}\,, \\
 & p_{1,4}p_{2,3} - p_{1,3}p_{2,4}\,. &
\end{array}
$$

We define the ideal $I' = I_4 + \langle w - p_2p_{1,3,4} + p_1p_{2,3,4}\rangle$. The grading on the variables $p_1, p_2, \ldots, p_{2,3,4}$ and w is given by the matrix

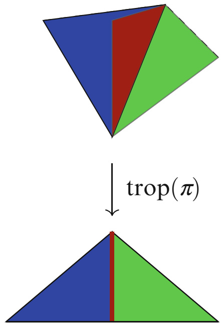
$$
\begin{bmatrix}
1&1&1&1&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&1&1&1&1&1&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&1&1&1&1
\end{bmatrix}.
$$

It extends the grading on the variables $p_1, p_2, \ldots, p_{2,3,4}$ given by the matrix D in (2). The tropical variety $\text{trop}\big(V(I')\big)$ has 105 maximal cones, 99 of which are prime. Among them we can find three maximal prime cones, which are mapped to C by $\text{trop}(\pi)$; see Fig. 6. We compute the polytopes associated to the normalizations of these three toric degenerations by applying the same methods as in Theorem 3.3. Using *polymake*, we check that two of them are combinatorially equivalent to the string polytopes for ρ in the class String 4. Moreover, none of them is combinatorially equivalent to any polytope coming from prime cones of $\text{trop}(\text{Fl}_4)$, hence they define different toric degenerations. □

Remark 6.5 Procedure 6.1 could be applied also to Fl_5, but we have not been able to do so. In fact, the tropicalization for $\text{trop}\big(V(I'_5)\big)$ did not terminate since the computation cannot be simplified by symmetries.

Fig. 6 Cones in $\mathrm{trop}\left(\mathrm{V}(I')\right)$ which project down to the non-prime cone C in $\mathrm{trop}(\mathrm{Fl}_4)$

$$\mathrm{trop}\left(\mathrm{V}(I')\right) \supset C_1, C_2, C_3$$



$$\mathrm{trop}\left(\mathrm{FI}_4\right) \supset C$$

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Appendix

In this Appendix, we provide numerical evidence of our computations.

Algebraic and Combinatorial Invariants of trop($\mathbf{Fl_5}$) Table 4 contains data on the non-prime maximal cones of $\mathrm{trop}(\mathrm{Fl}_5)$.

In Table 5, there is information on the polytopes obtained from maximal prime cones of $\mathrm{trop}(\mathrm{Fl}_5)$. It shows the f-vectors of the polytopes associated to maximal prime cones of $\mathrm{trop}(\mathrm{Fl}_5)$ for one representative in each orbit. The last column contains information on the existence of a combinatorial equivalence between these polytopes and the string polytopes resp. FFLV polytope for ρ. The initial ideals are all Cohen–Macaulay.

Algebraic Invariants of the $\mathbf{Fl_5}$ String Polytopes Table 6 contains information on the string polytopes and FFLV polytope for Fl_5, such as the weight vectors constructed in Sect. 5, primeness of the initial ideals with respect to these vectors, and the MP property. The last column contains information on unimodular equivalences among these polytopes. If there is no information in this column, then there is no unimodular equivalence between this polytope and any other polytope in the table.

Table 4 Data for non-prime initial ideals of Fl_5

# Orbits	# Generators
30	69
267	66
37	68
11	70
10	71
2	73

Table 5 Information on the polytopes obtained from maximal prime cones of trop(Fl_5)

Orbit	f-vector	Equivalences
0	475 2956 8417 14241 15690 11643 5820 1899 374 37	
1	456 2799 7843 13023 14038 10159 4938 1565 301 30	
2	425 2573 7108 11626 12333 8779 4201 1316 253 26	
3	393 2313 6200 9833 10125 7021 3297 1027 201 22	
4	433 2621 7230 11796 12473 8847 4219 1318 253 26	
5	435 2630 7246 11810 12479 8848 4219 1318 253 26	
6	425 2553 6988 11317 11888 8388 3987 1245 240 25	
7	450 2751 7677 12699 13648 9863 4800 1529 297 30	
8	435 2630 7246 11810 12479 8848 4219 1318 253 20	
9	419 2522 6922 11243 11842 8373 3985 1245 240 20	
10	453 2785 7817 12999 14027 10157 4938 1565 301 30	
11	463 2885 8237 13987 15474 11532 5788 1895 374 30	
12	463 2852 8020 13365 14459 10501 5121 1627 313 30	
13	457 2840 8078 13638 14954 10996 5413 1726 330 30	
14	454 2819 8016 13540 14870 10968 5427 1744 337 30	
15	445 2748 7770 13050 14254 10464 5161 1658 322 30	
16	441 2681 7438 12228 13056 9369 4525 1430 276 20	
17	440 2704 7602 12684 13752 10014 4897 1560 301 30	
18	471 2923 8298 13995 15369 11369 5667 1845 363 30	
19	464 2883 8200 13861 15258 11313 5651 1843 363 30	
20	467 2911 8309 14097 15574 11586 5804 1897 374 30	
21	461 2876 8225 13993 15509 11575 5814 1903 375 30	
22	397 2363 6416 10313 10755 7536 3561 1109 215 20	
23	437 2669 7447 12319 13236 9556 4642 1475 286 20	
24	425 2553 6988 11317 11888 8388 3987 1245 240 20	
25	415 2498 6861 11158 11772 8339 3976 1244 240 20	
26	470 2942 8436 14377 15944 11889 5955 1939 379 30	
27	460 2856 8109 13656 14929 10944 5374 1712 328 30	
28	449 2741 7634 12594 13487 9702 4695 1486 287 20	
29	427 2592 7181 11778 12523 8926 4270 1334 255 20	
30	425 2573 7108 11626 12333 8779 4201 1316 253 20	FFLV

(continued)

Table 5 (continued)

Orbit	f-vector	Equivalences
31	443 2708 7557 12495 13411 9667 4686 1485 287 20	
32	397 2363 6416 10313 10755 7536 3561 1109 215 20	S22
33	425 2553 6988 11317 11888 8388 3987 1245 240 20	
34	419 2522 6922 11243 11842 8373 3985 1245 240 20	
35	405 2407 6518 10442 10851 7578 3571 1110 215 20	
36	401 2387 6477 10398 10825 7570 3570 1110 215 20	
37	368 2154 5755 9111 9373 6497 3052 953 188 20	S21
38	379 2214 5892 9280 9494 6547 3063 954 188 20	S27, S28
39	393 2313 6200 9833 10125 7021 3297 1027 201 20	
40	358 2069 5453 8516 8653 5941 2778 870 174 20	S1, S18, S26, S29
41	459 2851 8111 13720 15118 11223 5614 1834 362 30	
42	467 2913 8322 14133 15629 11636 5831 1905 375 30	
43	423 2562 7083 11596 12313 8772 4200 1316 253 20	
44	425 2573 7108 11626 12333 8779 4201 1316 253 20	S24
45	397 2363 6416 10313 10755 7536 3561 1109 215 20	S23
46	461 2876 8225 13993 15509 11575 5814 1903 375 30	
47	400 2366 6377 10175 10546 7363 3480 1089 213 20	
48	393 2313 6200 9833 10125 7021 3297 1027 201 20	
49	393 2313 6200 9833 10125 7021 3297 1027 201 20	
50	379 2214 5892 9280 9494 6547 3063 954 188 20	S2, S19
51	426 2599 7257 12034 12981 9420 4602 1470 286 20	
52	428 2594 7176 11761 12514 8947 4307 1359 263 20	
53	419 2522 6922 11243 11842 8373 3985 1245 240 20	
54	466 2917 8371 14288 15879 11870 5960 1944 380 30	
55	443 2729 7692 12867 13982 10197 4987 1585 304 30	
56	453 2787 7826 13011 14021 10122 4895 1539 293 20	
57	469 2926 8358 14188 15679 11663 5839 1906 375 30	
58	458 2825 7958 13286 14398 10472 5113 1626 313 30	
59	472 2949 8435 14335 15854 11796 5902 1923 377 30	
60	440 2704 7602 12684 13752 10014 4897 1560 301 30	
61	472 2967 8561 14720 16525 12526 6410 2144 432 40	
62	457 2842 8099 13726 15153 11266 5640 1842 363 30	
63	465 2902 8296 14096 15588 11594 5795 1884 368 30	
64	459 2851 8111 13720 15118 11223 5614 1834 362 30	
65	428 2608 7269 12028 12946 9377 4576 1462 285 20	
66	441 2681 7438 12228 13056 9369 4525 1430 276 20	
67	418 2510 6876 11157 11753 8321 3969 1243 240 20	
68	406 2442 6713 10943 11587 8245 3950 1241 240 20	
69	373 2199 5926 9474 9849 6897 3267 1024 201 20	
70	427 2586 7144 11681 12383 8806 4209 1317 253 20	
71	451 2781 7840 13111 14243 10390 5089 1623 313 30	

(continued)

Table 5 (continued)

Orbit	f-vector	Equivalences
72	440 2704 7602 12684 13752 10014 4897 1560 301 30	
73	406 2442 6713 10943 11587 8245 3950 1241 240 20	
74	448 2764 7800 13061 14208 10377 5087 1623 313 30	
75	462 2873 8181 13846 15258 11321 5656 1844 363 30	
76	457 2842 8099 13726 15153 11266 5640 1842 363 30	
77	469 2927 8364 14203 15699 11678 5845 1907 375 30	
78	454 2802 7903 13216 14348 10453 5110 1626 313 30	
79	451 2787 7879 13221 14419 10565 5200 1667 323 30	
80	441 2705 7584 12611 13622 9885 4823 1537 298 30	
81	454 2803 7914 13263 14455 10598 5231 1687 330 30	
82	441 2697 7532 12465 13391 9660 4685 1485 287 20	
83	445 2721 7593 12550 13461 9694 4694 1486 287 20	
84	441 2697 7532 12465 13391 9660 4685 1485 287 20	
85	445 2725 7617 12611 13546 9764 4728 1495 288 20	
86	397 2363 6416 10313 10755 7536 3561 1109 215 20	
87	368 2154 5755 9111 9373 6497 3052 953 188 20	S5, S31
88	452 2801 7946 13385 14654 10771 5309 1699 327 30	
89	430 2624 7318 12097 12974 9329 4497 1411 269 20	
90	456 2834 8071 13670 15083 11210 5612 1834 362 30	
91	432 2633 7332 12104 12975 9341 4521 1430 276 20	
92	467 2919 8359 14230 15769 11756 5892 1922 377 30	
93	456 2834 8071 13670 15083 11210 5612 1834 362 30	
94	426 2597 7244 11998 12926 9370 4575 1462 285 20	
95	440 2708 7630 12769 13898 10169 5001 1603 311 30	
96	432 2633 7332 12104 12975 9341 4521 1430 276 20	
97	412 2479 6810 11083 11707 8306 3967 1243 240 20	
98	415 2511 6945 11391 12133 8679 4174 1313 253 20	
99	458 2845 8092 13676 15042 11132 5543 1800 353 30	
100	437 2669 7447 12319 13236 9556 4642 1475 286 20	
101	441 2703 7569 12562 13531 9780 4746 1502 289 20	
102	427 2586 7144 11681 12383 8806 4209 1317 253 20	
103	419 2522 6922 11243 11842 8373 3985 1245 240 20	
104	437 2669 7447 12319 13236 9556 4642 1475 286 20	
105	411 2470 6776 11012 11617 8235 3933 1234 239 20	
106	413 2483 6808 11043 11606 8177 3871 1201 230 20	
107	425 2553 6988 11317 11888 8388 3987 1245 240 20	
108	405 2407 6518 10442 10851 7578 3571 1110 215 20	
109	405 2427 6638 10751 11296 7969 3785 1181 228 20	S30

(continued)

Table 5 (continued)

Orbit	f-vector	Equivalences
110	465 2904 8312 14152 15700 11734 5907 1940 384 30	
111	464 2902 8323 14204 15795 11828 5960 1956 386 30	
112	438 2690 7559 12608 13667 9952 4868 1552 300 30	
113	445 2725 7617 12611 13546 9764 4728 1495 288 20	
114	437 2669 7447 12319 13236 9556 4642 1475 286 20	
115	411 2470 6776 11012 11617 8235 3933 1234 239 20	
116	424 2574 7139 11737 12529 8983 4332 1367 264 20	
117	419 2522 6922 11243 11842 8373 3985 1245 240 20	
118	401 2387 6477 10398 10825 7570 3570 1110 215 20	
119	405 2427 6638 10751 11296 7969 3785 1181 228 20	S6
120	464 2893 8261 14019 15483 11503 5746 1869 366 30	
121	454 2806 7928 13283 14448 10543 5159 1641 315 30	
122	451 2794 7928 13370 14676 10840 5387 1746 342 30	
123	444 2736 7715 12915 14053 10273 5044 1613 312 30	
124	466 2909 8318 14138 15644 11650 5837 1906 375 30	
125	456 2815 7939 13271 14398 10480 5118 1627 313 30	
126	423 2561 7078 11586 12303 8767 4199 1316 253 20	
127	429 2580 7064 11429 11972 8402 3959 1221 232 20	
128	431 2626 7309 12058 12915 9290 4494 1422 275 20	
129	428 2602 7224 11883 12684 9087 4375 1377 265 20	
130	443 2727 7679 12831 13927 10147 4960 1577 303 30	
131	432 2637 7354 12152 13024 9356 4505 1412 269 20	
132	451 2793 7920 13342 14620 10770 5331 1718 334 30	
133	434 2632 7273 11879 12557 8883 4210 1301 246 20	
134	452 2781 7813 13004 14042 10171 4944 1566 301 30	
135	453 2808 7969 13433 14725 10847 5366 1727 335 30	
136	451 2794 7928 13370 14676 10840 5387 1746 342 30	
137	433 2646 7390 12236 13150 9482 4589 1448 278 20	
138	442 2715 7629 12727 13808 10076 4948 1587 309 30	
139	432 2633 7332 12104 12975 9341 4521 1430 276 20	
140	423 2564 7096 11632 12368 8822 4227 1324 254 20	
141	413 2483 6808 11043 11606 8177 3871 1201 230 20	
142	427 2594 7196 11827 12614 9031 4347 1369 264 20	
143	431 2622 7281 11973 12769 9135 4390 1379 265 20	
144	431 2626 7309 12058 12915 9290 4494 1422 275 20	
145	410 2459 6725 10881 11411 8029 3802 1183 228 20	
146	428 2594 7176 11761 12514 8947 4307 1359 263 20	
147	419 2522 6922 11243 11842 8373 3985 1245 240 20	
148	451 2781 7840 13111 14243 10390 5089 1623 313 30	
149	464 2900 8310 14168 15740 11778 5933 1948 385 30	
150	446 2750 7757 12985 14123 10315 5058 1615 312 30	
151	420 2541 7021 11496 12218 8719 4184 1314 253 20	
152	441 2705 7584 12611 13622 9885 4823 1537 298 30	

(continued)

Table 5 (continued)

Orbit	f-vector	Equivalences
153	425 2575 7119 11651 12363 8799 4208 1317 253 20	
154	448 2764 7801 13067 14223 10397 5102 1629 314 30	
155	444 2737 7724 12949 14124 10363 5115 1647 321 30	
156	452 2772 7753 12830 13755 9876 4750 1486 282 20	
157	442 2706 7565 12529 13460 9696 4684 1473 281 20	
158	441 2708 7602 12655 13676 9915 4821 1525 292 20	
159	427 2596 7207 11850 12633 9026 4324 1350 257 20	
160	452 2781 7813 13004 14042 10171 4944 1566 301 30	
161	427 2586 7144 11681 12383 8806 4209 1317 253 20	
162	400 2382 6467 10388 10820 7569 3570 1110 215 20	
163	448 2764 7800 13061 14208 10377 5087 1623 313 30	
164	470 2943 8444 14405 16000 11959 6011 1967 387 30	
165	460 2857 8117 13684 14985 11014 5430 1740 336 30	
166	418 2530 6996 11466 12198 8712 4183 1314 253 20	
167	434 2640 7325 12025 12788 9108 4348 1353 257 20	
168	425 2577 7132 11687 12418 8849 4235 1325 254 20	
169	425 2581 7160 11772 12564 9004 4339 1368 264 20	
170	430 2614 7255 11928 12724 9109 4382 1378 265 20	
171	422 2557 7075 11597 12333 8801 4220 1323 254 20	
172	411 2470 6772 10988 11556 8150 3863 1200 230 20	S7
173	427 2586 7144 11681 12383 8806 4209 1317 253 20	
174	400 2382 6467 10388 10820 7569 3570 1110 215 20	
175	464 2898 8295 14119 15649 11673 5856 1913 376 30	

Table 6 Polytopes for Fl_5

	$\underline{w}_0$	MP	Weight vector	Prime	Equivalences
S1	1213214321	Yes	(0, 512, 384, 112, 0, 256, 96, 768, 608, 480, 0, 64, 320, 832, 15, 14, 526, 398, 126, 12, 268, 108, 780, 620, 492, 0, 8, 72, 328, 840)	Yes	S18, S26, S29
S2	1213243212	Yes	(0, 512, 384, 98, 0, 256, 96, 768, 608, 480, 0, 64, 320, 832, 30, 28, 540, 412, 123, 24, 280, 120, 792, 632, 504, 0, 16, 80, 336, 848)	Yes	
S3	1213432312	No	(0, 512, 384, 74, 0, 256, 72, 768, 584, 456, 0, 64, 320, 832, 58, 56, 568, 440, 111, 48, 304, 108, 816, 620, 492, 0, 32, 96, 352, 864)	No	
S4	1214321432	No	(0, 512, 384, 56, 0, 256, 48, 768, 560, 432, 0, 32, 288, 800, 120, 112, 624, 496, 63, 96, 352, 54, 864, 566, 438, 0, 64, 36, 292, 804)	No	
S5	1232124321	Yes	(0, 512, 288, 224, 0, 256, 192, 768, 704, 432, 0, 128, 384, 896, 15, 14, 526, 302, 238, 12, 268, 204, 780, 716, 444, 0, 8, 136, 392, 904)	Yes	
S6	1232143213	Yes	(0, 512, 288, 224, 0, 256, 192, 768, 704, 420, 0, 128, 384, 896, 30, 28, 540, 316, 252, 24, 280, 216, 792, 728, 437, 0, 16, 144, 400, 912)	Yes	
S7	1232432123	Yes	(0, 512, 260, 196, 0, 256, 192, 768, 704, 390, 0, 128, 384, 896, 60, 56, 568, 310, 246, 48, 304, 240, 816, 752, 423, 0, 32, 160, 416, 928)	Yes	
S8	1234321232	No	(0, 512, 264, 152, 0, 256, 144, 768, 656, 396, 0, 128, 384, 896, 120, 112, 624, 364, 219, 96, 352, 210, 864, 722, 462, 0, 64, 192, 448, 960)	No	
S9	1234321323	No	(0, 512, 264, 152, 0, 256, 144, 768, 656, 394, 0, 128, 384, 896, 120, 112, 624, 362, 222, 96, 352, 212, 864, 724, 459, 0, 64, 192, 448, 960)	No	
S10	1243212432	No	(0, 512, 272, 112, 0, 256, 96, 768, 608, 344, 0, 64, 320, 832, 240, 224, 736, 472, 119, 192, 448, 102, 960, 614, 350, 0, 128, 68, 324, 836)	No	
S11	1243214323	No	(0, 512, 272, 112, 0, 256, 96, 768, 608, 338, 0, 64, 320, 832, 240, 224, 736, 466, 126, 192, 448, 108, 960, 620, 347, 0, 128, 72, 328, 840)	No	

(continued)

Table 6 (continued)

	$\underline{w}_0$	MP	Weight vector	Prime	Equivalences
S12	1321324321	No	(0, 512, 192, 448, 0, 128, 384, 640, 896, 240, 0, 256, 160, 672, 15, 14, 526, 206, 462, 12, 140, 396, 652, 908, 252, 0, 8, 264, 168, 680)	No	
S13	1321343231	No	(0, 512, 192, 448, 0, 128, 384, 640, 896, 228, 0, 256, 160, 672, 29, 28, 540, 220, 476, 24, 152, 408, 664, 920, 246, 0, 16, 272, 176, 688)	No	
S14	1321432143	No	(0, 512, 192, 448, 0, 128, 384, 640, 896, 216, 0, 256, 144, 656, 60, 56, 568, 248, 504, 48, 176, 432, 688, 944, 219, 0, 32, 288, 146, 658)	No	
S15	1323432123	No	(0, 512, 132, 388, 0, 128, 384, 640, 896, 198, 0, 256, 192, 704, 60, 56, 568, 182, 438, 48, 176, 432, 688, 944, 231, 0, 32, 288, 224, 736)	No	
S16	1324321243	No	(0, 512, 136, 392, 0, 128, 384, 640, 896, 172, 0, 256, 160, 672, 120, 112, 624, 236, 492, 96, 224, 480, 736, 992, 175, 0, 64, 320, 162, 674)	No	
S17	1343231243	No	(0, 512, 48, 304, 0, 32, 288, 544, 800, 60, 0, 256, 40, 552, 240, 224, 736, 188, 444, 192, 168, 424, 680, 936, 63, 0, 128, 384, 42, 554)	No	
S18	2123214321	Yes	(0, 256, 768, 112, 0, 512, 96, 384, 352, 864, 0, 64, 576, 448, 15, 14, 270, 782, 126, 12, 524, 108, 396, 364, 876, 0, 8, 72, 584, 456)	Yes	S1, S26, S29, Gelfand-Tsetlin
S19	2123243212	Yes	(0, 256, 768, 98, 0, 512, 96, 384, 352, 864, 0, 64, 576, 448, 30, 28, 284, 796, 123, 24, 536, 120, 408, 376, 888, 0, 16, 80, 592, 464)	Yes	
S20	2123432132	No	(0, 256, 768, 76, 0, 512, 72, 384, 328, 840, 0, 64, 576, 448, 60, 56, 312, 824, 111, 48, 560, 106, 432, 362, 874, 0, 32, 96, 608, 480)	No	
S21	2132134321	Yes	(0, 256, 768, 224, 0, 512, 192, 320, 448, 960, 0, 128, 640, 336, 15, 14, 270, 782, 238, 12, 524, 204, 332, 460, 972, 0, 8, 136, 648, 344)	Yes	

(continued)

Table 6 (continued)

	$\underline{w}_0$	MP	Weight Vector	Prime	Equivalences
S22	2132143214	Yes	(0, 256, 768, 224, 0, 512, 192, 320, 448, 960, 0, 128, 640, 328, 30, 28, 284, 796, 252, 24, 536, 216, 344, 472, 984, 0, 16, 144, 656, 329)	Yes	
S23	2132343212	Yes	(0, 256, 768, 194, 0, 512, 192, 320, 448, 960, 0, 128, 640, 352, 30, 28, 284, 796, 219, 24, 536, 216, 344, 472, 984, 0, 16, 144, 656, 368)	Yes	
S24	2132432124	Yes	(0, 256, 768, 196, 0, 512, 192, 320, 448, 960, 0, 128, 640, 336, 60, 56, 312, 824, 246, 48, 560, 240, 368, 496, 1008, 0, 32, 160, 672, 337)	Yes	
S25	2134321324	No	(0, 256, 768, 152, 0, 512, 144, 272, 400, 912, 0, 128, 640, 276, 120, 112, 368, 880, 222, 96, 608, 212, 340, 468, 980, 0, 64, 192, 704, 277)	No	
S26	2321234321	Yes	(0, 64, 576, 448, 0, 512, 384, 96, 352, 864, 0, 256, 768, 112, 15, 14, 78, 590, 462, 12, 524, 396, 108, 364, 876, 0, 8, 264, 776, 120)	Yes	S1, S118, S29, Gelfand-Tsetlin
S27	2321243214	Yes	(0, 64, 576, 448, 0, 512, 384, 96, 352, 864, 0, 256, 768, 104, 30, 28, 92, 604, 476, 24, 536, 408, 120, 376, 888, 0, 16, 272, 784, 105)	Yes	
S28	2321432134	Yes	(0, 64, 576, 448, 0, 512, 384, 72, 328, 840, 0, 256, 768, 74, 60, 56, 120, 632, 504, 48, 560, 432, 106, 362, 874, 0, 32, 288, 800, 75)	Yes	
S29	2324321234	Yes	(0, 8, 520, 392, 0, 512, 384, 12, 268, 780, 0, 256, 768, 14, 120, 112, 108, 620, 492, 96, 608, 480, 78, 334, 846, 0, 64, 320, 832, 15)	Yes	S1, S18, S26, Gelfand-Tsetlin
S30	2343212324	Yes	(0, 16, 528, 304, 0, 512, 288, 24, 280, 792, 0, 256, 768, 28, 240, 224, 216, 728, 438, 192, 704, 420, 156, 412, 924, 0, 128, 384, 896, 29)	Yes	
S31	2343213234	Yes	(0, 16, 528, 304, 0, 512, 288, 20, 276, 788, 0, 256, 768, 22, 240, 224, 212, 724, 444, 192, 704, 424, 150, 406, 918, 0, 128, 384, 896, 23)	Yes	
FFLV	reg	Yes	(0, 4, 6, 6, 0, 3, 4, 6, 6, 9, 0, 2, 4, 6, 4, 3, 4, 7, 8, 2, 3, 5, 4, 6, 8, 0, 1, 2, 3, 4)	Yes	
FFLV	min	Yes	(0, 3, 4, 3, 0, 2, 2, 4, 3, 5, 0, 1, 2, 3, 1, 1, 1, 3, 3, 1, 1, 2, 1, 2, 3, 0, 1, 1, 1, 1)	Yes	

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The Multidegree of the Multi-Image Variety

Laura Escobar and Allen Knutson

Abstract The multi-image variety is a subvariety of $\mathrm{Gr}(1, \mathbb{P}^3)^n$ that parametrizes all of the possible images that can be taken by n fixed cameras. We compute its cohomology class in the cohomology ring of $\mathrm{Gr}(1, \mathbb{P}^3)^n$ and its multidegree as a subvariety of $(\mathbb{P}^5)^n$ under the Plücker embedding.

MSC 2010 codes: 14M15, 14Nxx

1 Introduction

Multi-view geometry studies the constraints imposed on a three-dimensional scene by various two-dimensional images of the scene. Each image is produced by a camera. Algebraic vision is a recent field of mathematics that uses techniques from algebraic geometry and optimization to formulate and solve problems in computer vision. One of the main objects studied in this field is a multi-view variety. Roughly speaking, a multi-view variety parametrizes all of the possible images that can be taken by a fixed collection of cameras; see [1, 11, 14] for more on multi-view varieties and [13] for various camera models.

The paper [11] presents a new viewpoint on multi-view varieties. A photographic camera maps a point in the scene to a point in the image. In contrast, a geometric camera, defined in [11], maps a point in the scene to a viewing ray, not a point. More precisely, a photographic camera corresponds to a rational map of the form $\mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ or $\mathbb{P}^3 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ whereas a geometric camera is a rational map of the form $\mathbb{P}^3 \dashrightarrow \mathrm{Gr}(1, \mathbb{P}^3)$ such that the image of each point is a line containing the point. The viewing ray corresponding to a point $p \in \mathbb{P}^3$ is the line in $\mathrm{Gr}(1, \mathbb{P}^3)$ to

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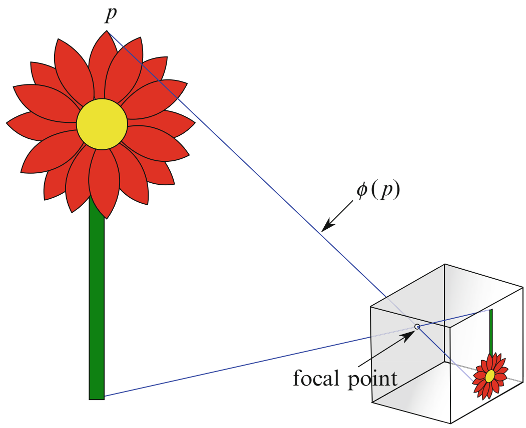
Fig. 1 A pinhole camera



which this point is mapped. Since light is assumed to travel along rays, the image of a geometric camera is two-dimensional.

We now illustrate these definitions using the example of pinhole cameras or cameræ obscuræ; see Fig. 1. As a geometric camera, a pinhole camera maps a point $p \in \mathbb{P}^3$ to the line $\phi(p)$ connecting p to the focal point. This map is undefined at the focal point. As a photographic camera, a pinhole camera maps a point $p \in \mathbb{P}^3$ to the intersection of $\phi(p)$ and the plane at the back of the camera. By varying the back plane, we see that there are many pinhole cameras associated to a given focal point. All of these cameras are equivalent up to a projective transformation. The essential part of a pinhole camera is the mapping of the scene points to viewing rays—its model as a geometric camera.

A multi-image variety is the Zariski closure of the image of a rational map $\phi\colon \mathbb{P}^3 \dashrightarrow \mathrm{Gr}(1, \mathbb{P}^3)^n$ defined by $p \mapsto \big(\phi_1(p), \phi_2(p), \ldots, \phi_n(p)\big)$ where each ϕ_i is a geometric camera. The multi-view variety is a multi-image variety in which each ϕ_i is a pinhole camera. If C_i is the Zariski closure of the image of ith camera ϕ_i inside the ith $\mathrm{Gr}(1, \mathbb{P}^3)$, then, under some assumptions, Theorem 5.1 in [11] shows that $(C_1 \times C_2 \times \cdots \times C_n) \cap V_n = \phi(\mathbb{P}^3)$, where V_n is the concurrent lines variety consisting of ordered n-tuples of lines in $\mathbb{P}^3$ that meet in a point x. The concurrent lines and multi-image varieties are embedded into $(\mathbb{P}^5)^n$ via the Plücker embedding.

The multidegree of a variety embedded into a product of projective spaces is the polynomial whose coefficients give the numbers (when finite) of intersection points in the variety intersected with a product of general linear subspaces. This article verifies the conjectured formula [11, Eq. (11)] for the multidegree of V_n and computes the multidegree of the multi-image variety. To do so, we describe V_n as the projection of a partial flag variety and use Schubert calculus to compute the cohomology classes of V_n and $(C_1 \times C_2 \times \cdots \times C_n) \cap V_n$ in the cohomology ring of $\mathrm{Gr}(1, \mathbb{P}^3)^n$. We then push forward these formulæ into $(\mathbb{P}^5)^n$ to obtain the multidegrees.

This article is organized as follows. In Sect. 2, we define the main objects of study: the multi-image variety and the concurrent lines variety. We present the main theorem which computes the multidegrees of these objects. The primary tool to prove this theorem is Schubert calculus. In Sect. 3, we give a brief introduction to Schubert calculus for $\mathrm{Gr}(1, \mathbb{P}^3)$. Section 4 calculates the cohomology class of the multi-image variety and the concurrent lines variety in terms of the Schubert cycles in $\mathrm{Gr}(1, \mathbb{P}^3)$. We prove the main theorem by taking the pushforward of these equations to the cohomology ring of $\mathbb{P}^5$. In Sect. 5, we refine these results to a computation of the K-polynomial for the concurrent lines variety.

2 The Multi-Image Variety

The Grassmannian $\mathrm{Gr}(k, \mathbb{P}^3)$ consists of all k-dimensional planes inside $\mathbb{P}^3$. A *congruence* is a two-dimensional family of lines in $\mathbb{P}^3$. The *bidegree* (α, β) of a congruence C is a pair of nonnegative integers such that the cohomology class of C in $\mathrm{Gr}(1, \mathbb{P}^3)$ has the form

$$[C] = \alpha\,[L :\ L \text{ contains a fixed point}] + \beta[L :\ L \text{ lies in a fixed plane}]\,.$$

The first integer coefficient α, called the *order*, counts the number of lines in C that pass through a general point of $\mathbb{P}^3$, and the second β, called the *class*, counts the number of lines in C that lie in a general plane of $\mathbb{P}^3$. The *focal locus* of C consists of the points in $\mathbb{P}^3$ that do not belong to α distinct lines of C.

Example 2.1 A congruence with bidegree $(1, 0)$ consists of all lines in $\mathrm{Gr}(1, \mathbb{P}^3)$ that contain a fixed point. A geometric camera for such a congruence represents a pinhole camera where the fixed point is the focal point.

Example 2.2 A two-slit camera assigns to a point $p \in \mathbb{P}^3$ the unique line passing through p and intersecting two fixed lines $L_1, L_2 \in \mathrm{Gr}(1, \mathbb{P}^3)$, see Fig. 2. Its focal locus is $\{L_1, L_2\}$. These cameras correspond to the congruences with bidegrees $(1, 1)$.

Remark 2.3 The study of congruences started with [7], which classified congruences of order 1. They were subsequently studied by many mathematicians during

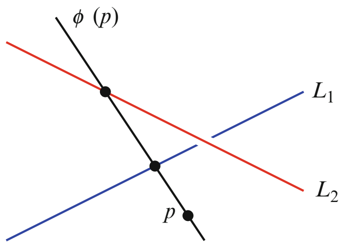


Fig. 2 A two-slit camera

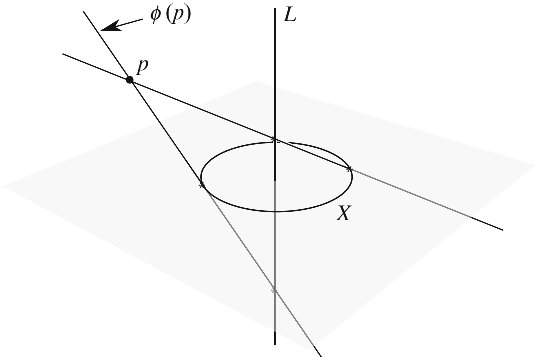


Fig. 3 A non-central panoramic camera

the second half of the nineteenth century; see [5]. Moreover, [6, Sect. 2] discusses congruences and their bidegrees, and [10] discusses the bidegrees of curves in $\mathbb{P}^1 \times \mathbb{P}^1$.

For congruences $C_1, C_2, \ldots, C_n \subset \mathrm{Gr}(1, \mathbb{P}^3)$, consider a rational map

$$\phi \colon \mathbb{P}^3 \dashrightarrow C_1 \times C_2 \times \cdots \times C_n$$

defined by $x \mapsto \big(\phi_1(x), \phi_2(x), \ldots, \phi_n(x)\big)$, where the Zariski closure of $\phi_i(\mathbb{P}^3)$ equals C_i and $x \in \phi_i(x)$ for all $1 \le i \le n$. Each map ϕ_i is defined everywhere except on the focal locus of C_i. The map ϕ corresponds to taking pictures with n rational cameras where each C_i is the ith image plane. When the congruence C_i has bidegree $(1, \beta_i)$, it follows that $\phi_i(x)$ is the unique line in C_i passing through x. In this situation, the *multi-image variety* of $(C_1, C_2, \ldots, C_n)$ is the Zariski closure of $\phi(\mathbb{P}^3)$. The *concurrent lines variety* V_n consists of ordered n-tuples of lines in $\mathbb{P}^3$ that meet in a point x. If the focal loci of the congruences are pairwise disjoint, then Theorem 5.1 in [11] shows that the multi-image equals the intersection $(C_1 \times C_2 \times \cdots \times C_n) \cap V_n$ in $\mathrm{Gr}(1, \mathbb{P}^3)^n$. The concurrent lines and multi-image varieties embed into $(\mathbb{P}^5)^n$ via the Plücker embedding $\mathrm{Gr}(1, \mathbb{P}^3) \hookrightarrow \mathbb{P}^5$.

Example 2.4 Although most cameras studied in computer vision correspond to congruences of order 1, cameras of higher order also appear; see [11, Sect. 7] and [13]. As an example of a camera associated to a congruence of bidegree $(2, 2)$, we describe a non-central panoramic camera; see Fig. 3. Consider a circle X obtained by rotating a point about a vertical axis L. There are two lines passing through a general point $p \in \mathbb{P}^3$ and intersecting both L and X. The congruence C consisting of all lines intersecting both X and L has bidegree $(2, 2)$. A physical realization of a non-central panoramic camera consists of a sensor on the circle taking measurements pointing outwards. This orientation of the sensor yields a map $\phi \colon \mathbb{P}^3 \dashrightarrow \mathrm{Gr}(1, \mathbb{P}^3)$; it assigns only one line $\phi(p)$ to a point $p \in \mathbb{P}^3$.

The *multidegree* of a variety X embedded into a product of projective spaces $\mathbb{P}^{a_1} \times \mathbb{P}^{a_2} \times \cdots \times \mathbb{P}^{a_n}$ is a homogeneous polynomial in $\mathbb{Z}[z_1, z_2, \ldots, z_n]$ whose term $q\, z_1^{r_1} z_2^{r_2} \cdots z_n^{r_n}$ indicates that there are q intersection points when X meets a product of general linear subspaces $H_1 \times H_2 \times \cdots \times H_n \subset \mathbb{P}^{a_1} \times \mathbb{P}^{a_2} \times \cdots \times \mathbb{P}^{a_n}$ such that $\dim H_i = r_i$ for all $1 \leq i \leq n$. The degree of this homogeneous polynomial equals the codimension of X in $\mathbb{P}^{a_1} \times \mathbb{P}^{a_2} \times \cdots \times \mathbb{P}^{a_n}$. Equivalently, this homogeneous polynomial represents the class of S in the cohomology ring for $\mathbb{P}^{a_1} \times \mathbb{P}^{a_2} \times \cdots \times \mathbb{P}^{a_n}$. The command `multidegree` in the *Macaulay2* [4] software system computes the multidegree of X from its defining ideal; see [9, Sect. 8.5] for more on multidegrees.

Our main result determines the multidegree of the multi-image variety.

Theorem 2.5 *The multidegree of the concurrent lines variety V_n in $(\mathbb{P}^5)^n$ equals*

$$(z_1 z_2 \cdots z_n)^3 \left(4 \sum_{\substack{(i,j) \\ i \neq j}} z_i^{-2} z_j^{-1} + 8 \sum_{\{i,j,k\}} z_i^{-1} z_j^{-1} z_k^{-1} \right).$$

If the congruence C_i has bidegree (α_i, β_i) for all $1 \leq i \leq n$, the multidegree of $(C_1 \times C_2 \times \cdots \times C_n) \cap V_n$ in $(\mathbb{P}^5)^n$ equals

$$(\alpha_1 \alpha_2 \cdots \alpha_n)(z_1 z_2 \cdots z_n)^5 \times \left(\sum_{\substack{(i,j) \\ i \neq j}} \frac{(\alpha_i+\beta_i)(\alpha_j+\beta_j)}{\alpha_i \alpha_j} z_i^{-2} z_j^{-1} + \sum_{\{i,j,k\}} \frac{(\alpha_i+\beta_i)(\alpha_j+\beta_j)(\alpha_k+\beta_k)}{\alpha_i \alpha_j \alpha_k} z_i^{-1} z_j^{-1} z_k^{-1} \right),$$

where we distribute appropriately whenever $\alpha_i = 0$ for some i. In particular, the multidegree of the multi-image variety for $(C_1, C_2, \ldots, C_n)$ is obtained by setting $\alpha_i = 1$ for all $1 \leq i \leq n$.

Remark 2.6 Ponce–Sturmfels–Trager [11, Eq. (11)] had conjectured this formula for the multidegree of V_n based on experimental evidence from *Macaulay2*. If the congruences all have bidegree $(1, 0)$, then Theorem 2.5 specializes to the equation for the multidegree appearing in Aholt–Sturmfels–Thomas [1, Corollary 3.5].

3 Schubert Varieties

In this section, we review the basic properties of Schubert varieties in $\mathrm{Gr}(k, \mathbb{P}^d)$; for more information, we recommend [3]. Given our interest in multi-image varieties and concurrent lines varieties, we highlight the case in which $d = 3$; also see [6, Sect. 6]. Fix a coordinate system for $\mathbb{P}^d$, let $\mathbb{P}^{(i,i+1,\ldots,j)}$ denote the coordinate subspace of $\mathbb{P}^n$ spanned by the coordinates $(i, i+1, \ldots, j)$, and consider the *standard flag*

$$\mathsf{E}_\bullet := \mathbb{P}^{(0)} \subset \mathbb{P}^{(0,1)} \subset \cdots \subset \mathbb{P}^{(0,1,\cdots,d)} = \mathbb{P}^d .$$

For $J := \{j_1, j_2, \ldots, j_{k+1}\} \subset [d+1] := \{1, 2, \ldots, d+1\}$, the corresponding *Schubert variety* is

$$X_J = X_{\{j_1 j_2, \cdots j_{k+1}\}} := \left\{ p \in \mathrm{Gr}(k, \mathbb{P}^d) : \begin{array}{l} \dim(p \cap \mathbb{P}^{(0,1,\ldots,j_\ell - 1)}) \geq \ell - 1 \\ \text{for all } 1 \leq \ell \leq k+1 \end{array} \right\}. \tag{1}$$

Example 3.1 There are $6 = \binom{4}{2} = \binom{d+1}{k+1}$ Schubert varieties in $\mathrm{Gr}(1, \mathbb{P}^3)$, namely

$$X_{\{1,2\}} := \{\mathbb{P}^{(0,1)}\}, \qquad X_{\{2,3\}} := \{L : L \subset \mathbb{P}^{(0,1,2)}\},$$

$$X_{\{1,3\}} := \{L : \mathbb{P}^{(0)} \subset L \subset \mathbb{P}^{(0,1,2)}\}, \qquad X_{\{2,4\}} := \{L : \dim(L \cap \mathbb{P}^{(0,1)}) \geq 1\},$$

$$X_{\{1,4\}} := \{L : \mathbb{P}^{(0)} \subset L\}, \qquad X_{\{3,4\}} := \mathrm{Gr}(1, \mathbb{P}^3).$$

The Schubert cells in $\mathrm{Gr}(1, \mathbb{P}^d)$ are defined by replacing the inequality in (1) with an equality. Since $\mathrm{Gr}(1, \mathbb{P}^d)$ is a disjoint union of the Schubert cells and each cell is contractible, the classes $[X_J]$, for all $J \subset [d+1]$, form a basis for the cohomology ring of $\mathrm{Gr}(k, \mathbb{P}^3)$. Multiplication in the cohomology ring is given by the cup product $[X_I] \smile [X_J] := [X_I(\mathsf{E}_\bullet^{\mathrm{op}}) \cap X_J]$, where $X_I(\mathsf{E}_\bullet^{\mathrm{op}})$ is defined by using $\mathbb{P}^{(d-j_\ell+1, d-j_\ell+2, \ldots, d)}$ instead of $\mathbb{P}^{(0,1,\ldots,j_\ell-1)}$ in (1). Unlike X_I, the variety $X_I(\mathsf{E}_\bullet^{\mathrm{op}})$ intersects X_J transversely, while having the same cohomology class as X_I. One obtains a basis for the cohomology ring of products of Grassmannians via the Künneth isomorphism. Expressing the product $[X_I] \smile [X_J]$ as a linear combination of this basis can be accomplished by using the Pieri rule for special classes and Littlewood–Richardson rule more generally; see the discussion before Example 3.3 and [8] respectively.

Example 3.2 In the cohomology ring of $\mathrm{Gr}(1, \mathbb{P}^3)$, we have $[X_J] \smile [X_{\{1,2\}}] = 0$ for all $J \neq \{3, 4\}$ because $\mathbb{P}^{(2,3)} \notin X_J$. Any line L containing $\mathbb{P}^{(0)}$ is not contained in $\mathbb{P}^{(1,2,3)}$, so we also have $[X_{\{1,4\}}] \smile [X_{\{2,3\}}] = 0$. Finally, we have $[X_{\{1,4\}}] \smile [X_{\{1,4\}}] = [X_{\{1,2\}}]$ because there is a unique line containing the points $\mathbb{P}^{(0)}$ and $\mathbb{P}^{(3)}$.

Schubert varieties stratify $\mathrm{Gr}(k, \mathbb{P}^d)$. The poset of their inclusions is most easily described when these strata are indexed by partitions. A *partition* of the integer n into $k+1$ parts is a list $\lambda := (\lambda_1, \lambda_2, \ldots, \lambda_{k+1})$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{k+1} > 0$ and $n = \lambda_1 + \lambda_2 + \cdots + \lambda_{k+1}$. There is a bijection between the $(k+1)$-subsets of $[d+1]$ and partitions λ with at most $k+1$ parts such that $\lambda_1 \leq d - k$ given by

$$J = \{j_1, j_2, \ldots, j_{k+1}\} \leftrightarrow \lambda = (d-k+1-j_1, d-k+2-j_2, \ldots, d-j_k, d+1-j_{k+1}).$$

Partitions can be visualized. Given a partition $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{k+1}$, the corresponding *Young diagram* is the left-justified shape of $k+1$ rows of boxes of length $\lambda_1, \lambda_2, \ldots, \lambda_{k+1}$. Figure 4 illustrates the Young diagram for the partition $(5, 3, 3, 2)$. If the Young diagram of λ fits inside the Young diagram of μ, then we simply write $\lambda \subset \mu$; by construction, this is equivalent to saying that $\lambda_i \leq \mu_i$ for all i.

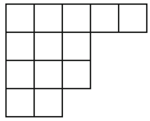


Fig. 4 The Young diagram for the partition $(5, 3, 3, 2)$

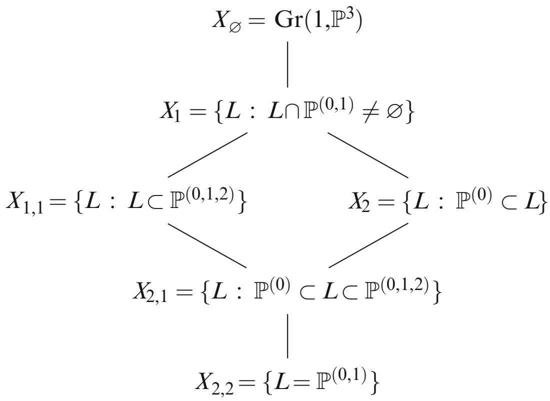


Fig. 5 The containment poset of the Schubert varieties in $\mathrm{Gr}(1, \mathbb{P}^3)$

Using our bijection between subsets and partitions, we may index Schubert cells and varieties by partitions. With this new indexing set, we have the following:

- $\dim(X_\lambda) = (k+1)(d-k) - \sum_i \lambda_i$,
- $X_\lambda \supset X_\mu$ if and only if $\mu \supset \lambda$, and
- if $\lambda = (\lambda_1, 0, 0, \ldots, 0)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_{k+1})$ are partitions with at most $k+1$ parts, then we have $[X_\lambda] \smile [X_\mu] = \sum_\nu [X_\nu]$, where the sum runs over all partitions $\nu = (\nu_1, \nu_2, \ldots, \nu_{k+1})$ such that $\nu_1 \leq d-k$, $\mu_i \leq \nu_i \leq \mu_{i-1}$ for $2 \leq i \leq k+1$ and $\sum_i \nu_i = \sum_i (\lambda_i + \mu_i)$.

The third item is known as the Pieri rule.

Example 3.3 The Schubert varieties in $\mathrm{Gr}(1, \mathbb{P}^3)$ are ordered by containment in the poset in Fig. 5. This poset is ranked by the dimensions of the Schubert varieties. By the Pieri rule, we have that $[X_1] \smile [X_2] = [X_{2,1}]$ and $[X_1] \smile [X_{1,1}] = [X_{2,1}]$.

Remark 3.4 In terms of Schubert varieties, the cohomology class of a congruence $C \subset \mathbb{P}^3$ is $[C] = \alpha[X_2] + \beta[X_{1,1}]$.

4 Computing the Multidegrees

In this section, we use Schubert calculus to obtain the cohomology classes of V_n and the multi-image variety in the cohomology ring of $\mathrm{Gr}(1, \mathbb{P}^3)^n$. Using these formulæ we then describe the multidegrees of these varieties in $\mathbb{P}^n$.

Let $M \subset \mathrm{Gr}(0,\mathbb{P}^3) \times \mathrm{Gr}(1,\mathbb{P}^3)$ be the partial flag manifold consisting of pairs (P, L), where P is a point in $\mathbb{P}^3$ and L is a line through P, and let M^n_Δ be the subvariety of M^n consisting of lists of n flags such that the point P is the same in all of them. In other words, M^n_Δ is the preimage of the small diagonal, where all points are coincident, under the projection $M^n \to (\mathbb{P}^3)^n$ which sends $\big((P_1,L_1),(P_2,L_2),\ldots,(P_n,L_n)\big)$ to $(P_1,P_2,\ldots,P_n)$. For instance, the subvariety $M^2_\Delta \subseteq M^2$ consists of pairs of flags of the form $((P,L_1),(P,L_2))$. With this notation, we see that the concurrent lines variety V_n is the image of subvariety M^n_Δ under the map $p\colon M^n \to \mathrm{Gr}(1,\mathbb{P}^3)^n$ which is defined on each factor by the composition $M \hookrightarrow \mathrm{Gr}(0,\mathbb{P}^3) \times \mathrm{Gr}(1,\mathbb{P}^3) \to \mathrm{Gr}(1,\mathbb{P}^3)$.

Using this description, we deduce the following formulæ for the cohomology classes of the concurrent lines variety V_n and the multi-image variety.

Theorem 4.1 *The class* $[V_n]$ *of the concurrent lines variety in the cohomology ring of* $\mathrm{Gr}(1,\mathbb{P}^3)^n$ *is*

$$[V_n] = \sum_{\substack{0=v_0\le v_1\le\cdots\le v_n=3\\ v_{j+1}-v_j<3}} \bigotimes_{i=0}^{n-1} \big[L : L\cap \mathbb{P}^{(v_i,v_i+1,\ldots,v_{i+1})} \neq \varnothing\big]$$

$$= \sum_{\substack{0=v_0\le v_1\le\cdots\le v_n=3\\ v_{j+1}-v_j<3}} \bigotimes_{i=1}^{n} \begin{cases} [X_2] & \text{if } v_i - v_{i-1} = 0,\\ [X_1] & \text{if } v_i - v_{i-1} = 1,\\ [\mathrm{Gr}(1,\mathbb{P}^3)] & \text{if } v_i - v_{i-1} = 2.\end{cases}$$

The first equation is more natural, as it holds in T-equivariant cohomology, whereas the second equation is only valid in ordinary cohomology.

Example 4.2 Theorem 4.1 implies that

$$\begin{aligned}
[V_3] = {}& [L : L\cap\mathbb{P}^{(0)} \neq \varnothing] \otimes [L : L\cap\mathbb{P}^{(0,1)} \neq \varnothing] \otimes [L : L\cap\mathbb{P}^{(1,2,3)} \neq \varnothing] +\\
& [L : L\cap\mathbb{P}^{(0)} \neq \varnothing] \otimes [L : L\cap\mathbb{P}^{(0,1,2)} \neq \varnothing] \otimes [L : L\cap\mathbb{P}^{(2,3)} \neq \varnothing] +\\
& [L : L\cap\mathbb{P}^{(0,1)} \neq \varnothing] \otimes [L : L\cap\mathbb{P}^{(1)} \neq \varnothing] \otimes [L : L\cap\mathbb{P}^{(1,2,3)} \neq \varnothing] +\\
& \cdots + [L : L\cap\mathbb{P}^{(0,1,2)} \neq \varnothing] \otimes [L : L\cap\mathbb{P}^{(2,3)} \neq \varnothing] \otimes [L : L\cap\mathbb{P}^{(3)} \neq \varnothing]\\
= {}& [X_2]\otimes[X_1]\otimes[\mathrm{Gr}(1,\mathbb{P}^3)] + [X_2]\otimes[\mathrm{Gr}(1,\mathbb{P}^3)]\otimes[X_1] +\\
& [X_1]\otimes[X_2]\otimes[\mathrm{Gr}(1,\mathbb{P}^3)] + [X_1]\otimes[X_1]\otimes[X_1] + [X_1]\otimes[\mathrm{Gr}(1,\mathbb{P}^3)]\otimes[X_2] +\\
& [\mathrm{Gr}(1,\mathbb{P}^3)]\otimes[X_2]\otimes[X_1] + [\mathrm{Gr}(1,\mathbb{P}^3)]\otimes[X_1]\otimes[X_2]\,.
\end{aligned}$$

Proof The transverse intersection $(M^{i-1} \times M^2_\Delta \times M^{n-i-1}) \cap (M^i \times M^2_\Delta \times M^{n-i-2})$ consists of $\big((P_1,L_1),(P_2,L_2),\ldots,(P_n,L_n)\big)$ such that $P_i = P_{i+1} = P_{i+2}$. Hence, the subvariety M^n_Δ is the transverse intersection $M^n_\Delta = \bigcap_{i=1}^{n-1}(M^{i-1} \times M^2_\Delta \times M^{n-i-1})$. It follows that

$$[M^n_\Delta] = \prod_{i=1}^{n-1} \big(1 \otimes \cdots \otimes 1 \otimes [M^2_\Delta] \otimes 1 \otimes \cdots \otimes 1\big)$$

in the cohomology of $\big(\mathrm{Gr}(0,\mathbb{P}^3) \times \mathrm{Gr}(1,\mathbb{P}^3)\big)^n$. Using Künneth isomorphism to identify the cohomology ring $H^*\big((\mathbb{P}^3)^2,\mathbb{Z}\big)$ with $H^*(\mathbb{P}^3,\mathbb{Z}) \otimes H^*(\mathbb{P}^3,\mathbb{Z})$, the class of the diagonal in $(\mathbb{P}^3)^2$ is

$$\big[(P_1,P_2) \in (\mathbb{P}^3)^2 : P_1 = P_2\big] = \sum_{0\le v\le 3} [\mathbb{P}^{(0,1,\dots,v)}] \otimes [\mathbb{P}^{(v,v+1,\dots,3)}]\,,$$

so the class of the small diagonal in $(\mathbb{P}^3)^n$ is

$$\big[(P,P,\dots,P) \in (\mathbb{P}^3)^n : P \in \mathbb{P}^3\big] = \sum_{0=v_0\le v_1\le\cdots\le v_n=3} \bigotimes_{i=0}^{n-1} [\mathbb{P}^{(v_i,v_i+1,\dots,v_{i+1})}]\,.$$

Pulling this back to M^n, we get

$$[M^n_\Delta] = \sum_{0=v_0\le v_1\le\cdots\le v_n=3} \bigotimes_{i=0}^{n-1} \big[(P,L) : P \in \mathbb{P}^{(v_i,v_i+1,\dots,v_{i+1})}\big]\,.$$

Since V_n is the image of the subvariety M^n_Δ under the map $p\colon M^n \to \mathrm{Gr}(1,\mathbb{P}^3)^n$, we see that $p_*[M^n_\Delta] = [V_n]$. The image of the subvariety $\{(P,L) : P \in \mathbb{P}^{(0,1,2,3)}\}$ under the component map from M to $\mathrm{Gr}(1,\mathbb{P}^3)$ equals $\mathrm{Gr}(1,\mathbb{P}^3)$, so the push forward of the class $[(P,L) : P \in \mathbb{P}^{(v_j,v_j+1,\dots,v_{j+1})}]$ with $v_{j+1}-v_j = 3$ is zero. When $v_{j+1}-v_j < 3$, the push forward of the class $[(P,L) : P \in \mathbb{P}^{(v_j,v_j+1,\dots,v_{j+1})}]$ is $[L : L \cap \mathbb{P}^{(v_j,v_j+1,\dots,v_{j+1})} \neq \varnothing]$. Therefore, we conclude that

$$[V_n] = \sum_{\substack{0=v_0\le v_1\le\cdots\le v_n=3\\ v_{j+1}-v_j<3}} \bigotimes_{i=0}^{n-1} [L : L \cap \mathbb{P}^{(v_i,v_i+1,\dots,v_{i+1})} \neq \varnothing]\,.$$

For any $L \in \mathrm{Gr}(1,\mathbb{P}^3)$, we have that $L \cap \mathbb{P}^{(0,1,2)} \neq \varnothing$ and $L \cap \mathbb{P}^{(1,2,3)} \neq \varnothing$, so the classes $[\mathbb{P}^{(0,1,2)}]$ and $[\mathbb{P}^{(1,2,3)}]$ push down to $[\mathrm{Gr}(1,\mathbb{P}^3)]$. □

Theorem 4.3 *If $C_i \subset \mathrm{Gr}(1,\mathbb{P}^3)$ is a general congruence, for $1 \le i \le n$, with class $\alpha_i[X_2] + \beta_i[X_{1,1}] \in H^4\big(\mathrm{Gr}(1,\mathbb{P}^3),\mathbb{Z}\big)$, then we have*

$$\begin{aligned}&[(C_1 \times C_2 \times \cdots \times C_n) \cap V_n]\\ &\quad= \sum_{\substack{0=v_0\le v_1\le\cdots\le v_n=3\\ v_{j+1}-v_j<3}} \bigotimes_{i=1}^{n} \begin{cases} \alpha_i[X_{2,2}] & \text{if } v_i - v_{i-1} = 0,\\ (\alpha_i+\beta_i)[X_{2,1}] & \text{if } v_i - v_{i-1} = 1,\\ \alpha_i[X_2] + \beta_i[X_{1,1}] & \text{if } v_i - v_{i-1} = 2.\end{cases}\end{aligned}$$

In particular, if $\alpha_i = 0$ for some $1 \le i \le n$, then the terms with $v_i - v_{i+1} = 0$ vanish.

Example 4.4 If $C_i \subset \mathrm{Gr}(1, \mathbb{P}^3)$ is congruences with bidegree (α_i, β_i) for $1 \le i \le 3$, then Theorem 4.3 implies that

$$\begin{aligned}\left[(C_1 \times C_2 \times C_3) \cap V_3\right] &= \alpha_1[X_{2,2}] \otimes (\alpha_2 + \beta_2)[X_{2,1}] \otimes (\alpha_3[X_2] + \beta_3[X_{1,1}]) \\ &\quad + \alpha_1[X_{2,1}] \otimes (\alpha_2[X_2] + \beta_2[X_{1,1}]) \otimes (\alpha_3 + \beta_3)[X_{2,1}] \\ &\quad + (\alpha_1 + \beta_1)[X_{2,1}] \otimes \alpha_2[X_{2,2}] \otimes (\alpha_3[X_2] + \beta_3[X_{1,1}]) \\ &\quad + \cdots + (\alpha_1[X_2] + \beta_1[X_{1,1}]) \otimes (\alpha_2 + \beta_2)[X_{2,1}] \otimes \alpha_3[X_{2,2}].\end{aligned}$$

Proof If $C_i \subset \mathrm{Gr}(1, \mathbb{P}^3)$ is a surface with class $\alpha_i[X_{1,1}] + \beta_i[X_2] \in H^4\left(\mathrm{Gr}(1, \mathbb{P}^3), \mathbb{Z}\right)$ for $1 \le i \le n$, then Theorem 4.1 gives

$$\begin{aligned}&[(C_1 \times C_2 \times \cdots \times C_n) \cap V_n] \\ &= [V_n] \smile ([C_1] \otimes [C_2] \otimes \cdots \otimes [C_n]) \\ &= \sum_{\substack{0=v_0 \le v_1 \le \cdots \le v_n = 3 \\ v_{j+1} - v_j < 3}} \bigotimes_{i=1}^{n} \left([L : L \cap \mathbb{P}^{(v_{i-1}, v_{i-1}+1, \ldots, v_i)} \ne \varnothing] \smile (\alpha_i[X_2] + \beta_i[X_{1,1}])\right).\end{aligned}$$

Using Examples 3.2–3.3, we analyze the three distinct cases: when $v_i - v_{i-1} = 0$, we have $[L : L \cap \mathbb{P}^{(v_i)} \ne \varnothing] = [X_2]$ and $[X_2] \smile (\alpha_i[X_2] + \beta_i[X_{1,1}]) = \alpha_i[X_{2,2}]$; when $v_i - v_{i-1} = 1$, we have $[L : L \cap \mathbb{P}^{(v_i, v_i+1)} \ne \varnothing] = [X_1]$ and

$$[X_1] \smile (\alpha_i[X_2] + \beta_i[X_{1,1}]) = (\alpha_i + \beta_i)[X_{2,1}];$$

and, when $v_i - v_{i-1} = 2$, we have $[L : L \cap \mathbb{P}^{(v_i, v_i+1, v_i+1)} \ne \varnothing] = [\mathrm{Gr}(1, \mathbb{P}^3)] = 1$ and $1 \smile (\alpha_i[X_2] + \beta_i[X_{1,1}]) = (\alpha_i[X_2] + \beta_i[X_{1,1}])$. Combining these calculations completes the proof. □

We now describe the push forward of the Schubert classes to $H^*(\mathbb{P}^5, \mathbb{Z})$ under the Plücker embedding $\iota\colon \mathrm{Gr}(1, \mathbb{P}^3) \hookrightarrow \mathbb{P}^5$. To accomplish this, we first give their equations inside $\mathbb{P}^5$. The degrees of general Schubert varieties were computed by Schubert [12]. A line $L \in \mathrm{Gr}(1, \mathbb{P}^3)$ passing through the points $(x_0 : x_1 : x_2 : x_3)$ and $(y_0 : y_1 : y_2 : y_3) \in \mathbb{P}^3$ is uniquely determined by the (2×2)-minors of the (2×4)-matrix $\left[\begin{smallmatrix} x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \end{smallmatrix}\right]$. Let $p_{i,j}$ denote the minor given by the ith and jth columns. The Plücker embedding associates the point $(p_{1,2} : p_{1,3} : \cdots : p_{3,4}) \in \mathbb{P}^5$ to each line in $\mathrm{Gr}(1, \mathbb{P}^3)$. The classes $\iota_*[X_\lambda] \in H^*(\mathbb{P}^5, \mathbb{Z})$ can then be computed as follows.

$\mathrm{Gr}(1, \mathbb{P}^3)$: The image $\iota\left(\mathrm{Gr}(1, \mathbb{P}^3)\right) = \mathrm{V}(p_{1,2}p_{3,4} - p_{1,3}p_{2,4} + p_{2,3}p_{1,4}) \subset \mathbb{P}^5$ is the quadratic hypersurface defined by the Plücker relation. Hence, it has degree 2 and codimension 1, so $\iota_*[\mathrm{Gr}(1, \mathbb{P}^3)] = 2z^1$.

X_1: Since the condition that $L \cap \mathbb{P}^{(0,1)} \ne \varnothing$ is equivalent to $p_{3,4} = 0$, the image $\iota(X_1)$ is the complete intersection $\mathrm{V}(p_{3,4}, p_{1,2}p_{3,4} - p_{1,3}p_{2,4} + p_{2,3}p_{1,4})$ in $\mathbb{P}^5$. Hence, it has degree $(1)(2) = 2$ and codimension 2, so $\iota_*[X_1] = 2z^2$.

$X_{1,1}$: Since the condition that $L \cap \mathbb{P}^{(0,1)} \subset \mathbb{P}^{(0,1,2)}$ is equivalent to $p_{1,4} = 0$, $p_{2,4} = 0$, and $p_{3,4} = 0$, the image $\iota(X_{1,1})$ is the complete intersection $\mathrm{V}(p_{1,4}, p_{2,4}, p_{3,4})$ in $\mathbb{P}^5$. Hence, it has degree $(1)(1)(1) = 1$ and codimension 3, so $\iota_*[X_{1,1}] = z^3$.

X_2: The image $\iota(X_2)$ is the complete intersection $\mathrm{V}(p_{2,3}, p_{2,4}, p_{3,4})$ in $\mathbb{P}^5$. Hence, it has degree $(1)(1)(1) = 1$ and codimension 3, so $\iota_*[X_2] = z^3$.

$X_{2,1}$: The image $\iota(X_2)$ is the complete intersection $\mathrm{V}(p_{1,4}, p_{2,3}, p_{2,4}, p_{3,4})$ in $\mathbb{P}^5$. Hence, it has degree $(1)(1)(1)(1) = 1$ and codimension 4, so $\iota_*[X_{2,1}] = z^4$.

$X_{2,2}$: The image $\iota(X_{2,2})$ is the complete intersection $\mathrm{V}(p_{1,3}, p_{1,4}, p_{2,3}, p_{2,4}, p_{3,4})$ in $\mathbb{P}^5$. Hence, it has degree $(1)(1)(1)(1)(1) = 1$ and codimension 5, so $\iota_*[X_{2,2}] = z^5$.

Remark 4.5 The small case $\mathrm{Gr}(1, \mathbb{P}^3)$ relevant for this paper is convenient, but misleading. In $\mathrm{Gr}(1, \mathbb{P}^4)$, one already encounters Schubert varieties that are not complete intersections in the Plücker embedding.

Proof of Theorem 2.5 For $1 \leq i \leq n$, the Schubert classes in the ith component of $\big(\mathrm{Gr}(1, \mathbb{P}^3)\big)^n$ push forward to $(\mathbb{P}^5)^n$ as follows:

$$[X_{2,2}] \mapsto z_i^5, \qquad [X_{2,1}] \mapsto z_i^4, \qquad [X_{1,1}] \mapsto z_i^3,$$
$$[X_2] \mapsto z_i^3, \qquad [X_1] \mapsto 2z_i^2, \qquad [\mathrm{Gr}(1, \mathbb{P}^3)] \mapsto 2z_i.$$

Hence, Theorem 4.1 implies that

$$\iota_*[V_n] = \sum_{\substack{0=v_0 \leq v_1 \leq \cdots \leq v_n = 3 \\ v_{j+1}-v_j<3}} \prod_{i=1}^{n} \begin{cases} z_i^3 & \text{if } v_i - v_{i-1} = 0, \\ 2z_i^2 & \text{if } v_i - v_{i-1} = 1, \\ 2z_i & \text{if } v_i - v_{i-1} = 2. \end{cases}$$

Since $0 = v_0 \leq v_1 \leq \cdots \leq v_{n-1} \leq v_n = 3$ and $0 \leq v_i - v_{i-1} \leq 2$ for all $1 \leq i \leq n$, there are exactly two different possibilities: either $v_i - v_{i-1} = 0$ for all but three indices at which $v_i - v_{i-1} = 1$, or $v_i - v_{i-1} = 0$ for all but two indices j and k at which $v_j - v_{j-1} = 2$ and $v_k - v_{k-1} = 1$ respectively. In the first case, we have a term of the form $8(z_1 z_2 \cdots z_n)^3 z_i^{-1} z_j^{-1} z_k^{-1}$ and, tn the second case, we have a term of the form $4(z_1 z_2 \cdots z_n)^3 z_j^{-2} z_k^{-1}$.

For $(C_1 \times C_2 \times \cdots \times C_n) \cap V_n$, Theorem 4.3 gives

$$\iota_*[(C_1 \times C_2 \times \cdots \times C_n) \cap V_n] = \sum_{\substack{0=v_0 \leq v_1 \leq \cdots \leq v_n = 3 \\ v_{j+1}-v_j<3}} \prod_{i=1}^{n} z_i^{5-(v_i - v_{i-1})} \begin{cases} \alpha_i & \text{if } v_i = v_{i-1}, \\ \alpha_i + \beta_i & \text{if } v_i > v_{i-1}. \end{cases}$$

Analyzing the two different possibilities produces the desired formula. □

5 *K*-Theory

We conclude this paper by computing the K-polynomial of the concurrent lines variety V_n. Our reference for K-polynomials is [9, Sect. 8.5], but we include some words of motivation here. Equivariant K-theory refines homology by incorporating lower-dimensional objects and the K-polynomial of a subvariety is the class represented by its structure sheaf in equivariant K-theory. More precisely, there is a filtration on equivariant K-theory of a variety such that the associated graded module over $\mathbb{Q}$ is isomorphic to the rational homology of the variety.

For our application, we need only define the K-polynomial for a subvariety of $(\mathbb{P}^m)^n$. The Cox ring of $(\mathbb{P}^m)^n$ is the polynomial ring $\mathbb{C}[x_{j,i} : 0 \le j \le m, 1 \le i \le n]$. If $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ is the standard basis of $\mathbb{Z}^n$, then the Cox ring is equipped with the $\mathbb{Z}^n$-grading induced by setting $\deg(x_{j,i}) := \mathbf{e}_i \in \mathbb{Z}^n$. Consider the abelian group generated by the (isomorphism classes of) finitely-generated $\mathbb{Z}^n$-graded modules over the Cox ring. We impose two types of relations on this group. First, any module annihilated by the ideal $\langle x_{0,i}, x_{1,i}, \dots, x_{m,i}\rangle$ for some $1 \le i \le n$ is equivalent to zero. Second, alternating sum of modules in an exact sequence is zero. The resulting quotient group is denoted by $K_0\big((\mathbb{P}^m)^n\big)$. The first type of relation ensures that choosing any module over the Cox ring to represent a coherent sheaf on $(\mathbb{P}^m)^n$ yields the same class in $K_0\big((\mathbb{P}^m)^n\big)$. For a subvariety $X \subset (\mathbb{P}^m)^n$, we write $[\mathscr{O}_X]$ for its class in $K_0\big((\mathbb{P}^m)^n\big)$, which is also known as its *K-polynomial*.

The abelian group $K_0\big((\mathbb{P}^m)^n\big)$ has a non-obvious product. However, we will need only a special case. For two subvarieties X and Y that intersect transversally, we have $[\mathscr{O}_X][\mathscr{O}_Y] = [\mathscr{O}_{X\cap Y}]$. For instance, this formula applies when X and Y are smooth and $\mathrm{codim}(X\cap Y) = \mathrm{codim}(X)+\mathrm{codim}(Y)$. Since any two hyperplanes defined the same class H in $K_0(\mathbb{P}^m)$, we have $H^{m+1} = 0$. In fact, one has $K_0(\mathbb{P}^m) \cong \mathbb{Z}[H]/\langle H^{m+1}\rangle$ and $K_0\big((\mathbb{P}^m)^n\big) \cong \mathbb{Z}[H_1, H_2, \dots, H_n]/\langle H_1^{m+1}, H_2^{m+1}, \dots, H_n^{m+1}\rangle$.

Since the diagonal in $(\mathbb{P}^3)^2$ flatly degenerates to the union of four linear subspace: $\mathrm{V}(x_{1,1}, x_{2,1}, x_{3,1}) \cup \mathrm{V}(x_{2,1}, x_{3,1}, x_{0,2}) \cup \mathrm{V}(x_{3,1}, x_{0,2}, x_{1,2}) \cup \mathrm{V}(x_{0,2}, x_{1,2}, x_{2,2})$, the K-polynomial for this diagonal is

$$
\begin{aligned}
&[\mathscr{O}_{\mathbb{P}(0)\times\mathbb{P}(0,1,2,3)}] + [\mathscr{O}_{\mathbb{P}(0,1)\times\mathbb{P}(1,2,3)}] + [\mathscr{O}_{\mathbb{P}(0,1,2)\times\mathbb{P}(2,3)}] + [\mathscr{O}_{\mathbb{P}(0,1,2,3)\times\mathbb{P}(3)}]\\
&\quad - [\mathscr{O}_{\mathbb{P}(0)\times\mathbb{P}(1,2,3)}] - [\mathscr{O}_{\mathbb{P}(0)\times\mathbb{P}(2,3)}] - [\mathscr{O}_{\mathbb{P}(0)\times\mathbb{P}(3)}] - [\mathscr{O}_{\mathbb{P}(0,1)\times\mathbb{P}(2,3)}] - [\mathscr{O}_{\mathbb{P}(0,1)\times\mathbb{P}(3)}]\\
&\quad - [\mathscr{O}_{\mathbb{P}(0,1,2)\times\mathbb{P}(3)}] + [\mathscr{O}_{\mathbb{P}(0)\times\mathbb{P}(2,3)}] + [\mathscr{O}_{\mathbb{P}(0)\times\mathbb{P}(3)}] + [\mathscr{O}_{\mathbb{P}(0)\times\mathbb{P}(3)}] + [\mathscr{O}_{\mathbb{P}(0,1)\times\mathbb{P}(3)}]\\
&\quad - [\mathscr{O}_{\mathbb{P}(0)\times\mathbb{P}(3)}]\\
&= \sum_{v=0}^{3} [\mathscr{O}_{\mathbb{P}(0,1,\dots,v)}]\Big([\mathscr{O}_{\mathbb{P}(v,v+1,\dots,3)}] - [\mathscr{O}_{\mathbb{P}(v+1,v+2,\dots,3)}]\Big)
\end{aligned}
$$

From this formula, we can follow arguments similar to those of Sect. 4 to compute the K-class of V_n in $K_0\big((\mathbb{P}^5)^n\big)$. If H_i denotes the hyperplane class in the ith factor of $(\mathbb{P}^5)^n$, then we have

$$[\mathscr{O}_{V_n}] = \sum_{\substack{0=v_0\le v_1\le\cdots\le v_n=3\\ v_{j+1}-v_j<3}} \prod_{i=1}^{n} \begin{cases} -H_i^3(2H_i - H_i^2) & \text{if } v_i - v_{i-1} = 0,\\ (2H_i^2 - H_i^3)(2H_i - 3H_i^2 + H_i^3) & \text{if } v_i - v_{i-1} = 1,\\ (2H_i - H_i^2)(2H_i^2 - 2H_i^3) & \text{if } v_i - v_{i-1} = 2.\end{cases}$$

This is an analogue of the Hilbert polynomial calculation of [1, Theorem 3.6]. However, there are two important differences. First, theirs concerns a rational map $\mathbb{P}^3 \dashrightarrow (\mathbb{P}^2)^n$ whereas ours concerns a rational map $\mathbb{P}^3 \dashrightarrow \mathrm{Gr}(1,\mathbb{P}^3)^n \to (\mathbb{P}^5)^n$. Second, their class in $H^*\big((\mathbb{P}^2)^n,\mathbb{Z}\big)$ is multiplicity-free in the sense of [2], which is what shows that every degeneration of their variety will be reduced and Cohen–Macaulay. In particular, they can have a universal Gröbner basis with squarefree initial terms; see [1, Sect. 2]. In contrast, our class in $H^*\big((\mathbb{P}^5)^n,\mathbb{Z}\big)$ is not multiplicity-free thanks to those coefficients 2 appearing in Theorem 4.1. Nevertheless, the second equation in Theorem 4.1 shows that V_n is multiplicity-free in the sense of [2] when considered as a subvariety of $\mathrm{Gr}(1,\mathbb{P}^3)^n$. It follows that every degeneration of V_n inside $\mathrm{Gr}(1,\mathbb{P}^3)^n$, but not inside $(\mathbb{P}^5)^n$, is reduced and Cohen–Macaulay.

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The Convex Hull of Two Circles in $\mathbb{R}^3$

Evan D. Nash, Ata Firat Pir, Frank Sottile, and Li Ying

Abstract We describe convex hulls of the simplest compact space curves, reducible quartics consisting of two circles. When the circles do not meet in complex projective space, their algebraic boundary contains an irrational ruled surface of degree eight whose ruling forms a genus one curve. We classify which curves arise, classify the face lattices of the convex hulls, and determine which are spectrahedra. We also discuss an approach to these convex hulls using projective duality.

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1 Introduction

Convex algebraic geometry studies convex hulls of semialgebraic sets [15]. The convex hull of finitely many points, a zero-dimensional variety, is a polytope [8, 24]. Polytopes have finitely many faces, which are themselves polytopes. The boundary of the convex hull of a higher-dimensional algebraic set typically has infinitely many faces which lie in algebraic families. Ranestad and Sturmfels [13] described this boundary using projective duality and secant varieties. For a general space curve, the boundary consists of finitely many two-dimensional faces supported on tritangent planes and a scroll of line segments, called the edge surface. These segments are stationary bisecants, which join two points of the curve whose tangents meet.

We study convex hulls of the simplest nontrivial compact space curves, those which are the union of two circles lying in distinct planes. Zero-dimensional faces of such a convex hull are extreme points on the circles. One-dimensional faces are stationary bisecants. It may have two-dimensional faces coming from the planes of

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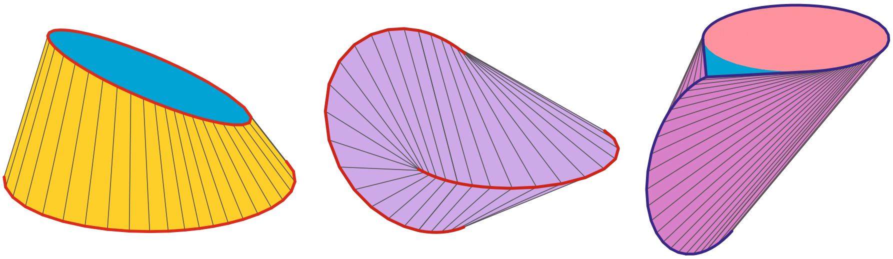


Fig. 1 Some convex hulls of two circles

the circles. It may have finitely many nonexposed faces, either points of one circle whose tangent meets the other circle, or certain tangent stationary bisecants. Figure 1 shows some of this diversity.

In the convex hull on the left, the discs of both circles are faces, and every face is exposed. In the oloid in the middle, the discs lie in the interior, an arc of each circle is extreme, and the endpoints of the arcs are nonexposed. In the convex hull on the right, there are two nonexposed stationary bisecants lying on its two-dimensional face, which is the convex hull of one circle and the point where the other circle is tangent to the plane of the first.

These objects have been studied before. Paul Schatz discovered and patented the oloid in 1929 [16]; this is the convex hull of two congruent circles in orthogonal planes, each passing through the centre of the other. It has found industrial uses [2], and is a well-known toy. A curve in $\mathbb{R}^3$ may roll along its edge surface. When rolling, the oloid develops its entire surface and has area equal to that of the sphere [3] with equator one of the circles of the oloid. Other special cases of the convex hull of two circles have been studied from these perspectives [5, 10].

This paper had its origins in Subsection 4.1 of [13], which claimed that the edge surface for a general pair of circles is composed of cylinders. We show that this is only the case when the two circles either meet in two points or are mutually tangent—in all other cases, the edge surface has higher degree and it is an irrational surface of degree eight when the circles are disjoint in $\mathbb{CP}^3$. This is related to Problem 3 on Convexity in [21], on the convex hull of three ellipsoids in $\mathbb{R}^3$. An algorithm was presented in [6] (see the video [7]), using projective duality. We sketch this in Sect. 5, and also apply duality to the convex hull of two circles.

In Sect. 2, we recall some aspects of convexity and convex algebraic geometry, and see that the convex hull of two circles is the projection of a spectrahedron. We study the edge surface and the edge curve of stationary bisecants of complex conics $C_1, C_2 \subset \mathbb{CP}^3$ in Sect. 3. We show that the edge curve is a reduced curve of bidegree $(2, 2)$ in $C_1 \times C_2$ and, if $C_1 \cap C_2 = \varnothing$ and neither circle is tangent to the plane of the other, then the edge surface has degree eight. We also classify which curves of bidegree $(2, 2)$ arise as edge curves to two conics. All possibilities occur, except a rational curve with a cusp singularity and a maximally reducible curve.

In Sect. 4, we classify the possible arrangements of two circles lying in different planes in terms that are relevant for their convex hulls. We determine the face lattice and the real edge curve of each type, and show that these convex hulls are spectrahedra only when the circles lie on a quadratic cone.

2 Convex Algebraic Geometry

We review some fundamental aspects of convexity and convex algebraic geometry, summarize our results about convex hulls of pairs of circles and their edge curves, and show that any such convex hull is the projection of a spectrahedron.

The convex hull of a subset $S \subset \mathbb{R}^d$ is

$$\operatorname{conv}(S) := \left\{ \sum_{i=1}^{n} \lambda_i s_i : s_1, s_2, \ldots, s_n \in S, 0 \leq \lambda_i, \text{ and } 1 = \sum_{i=1}^{n} \lambda_i \right\}.$$

A set K is *convex* if it equals its convex hull, and a point $p \in K$ is *extreme* if $K \neq \operatorname{conv}(K \smallsetminus \{p\})$. A compact convex set is the convex hull of its extreme points.

A convex subset F of a convex set K is a *face* if F contains the endpoints of any line segment in K whose interior meets F. A *supporting hyperplane* Π is one that meets K with K lying in one of the half-spaces of $\mathbb{R}^d$ defined by Π. A supporting hyperplane Π supports a face F of K if $F \subset K \cap \Pi$ and it *exposes* F if $F = K \cap \Pi$.

Not all faces of a convex set are exposed. The boundary of the convex hull of two coplanar circles in Fig. 2 consists of one arc on each circle and two bitangent segments. An endpoint p of an arc is not exposed. The only line supporting p is the tangent to the circle at p, and this line also supports the adjoining bitangent.

A fundamental problem from convex optimization is to describe the faces of a convex set, identify those that are exposed, and determine their lattice of inclusions (the *face lattice*). For more on convex geometry, see [1].

Convex algebraic geometry is the marriage of classical convexity with real algebraic geometry. A real algebraic variety X is an algebraic variety defined over $\mathbb{R}$. If X is irreducible and contains a smooth real point, then its real points are Zariski-dense in X, so it is often no loss to consider only the real points. Conversely, many aspects of a real algebraic variety are best understood in terms of its complex points. Studying the complex algebraic geometry aspects of a question from real algebraic

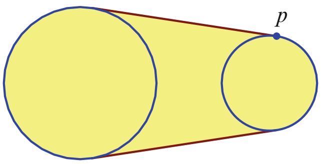


Fig. 2 Convex hull of coplanar circles

geometry is its *algebraic relaxation*. This relaxation enables the use of powerful techniques from complex algebraic geometry to address the original question.

As the real numbers are ordered, we also consider *semialgebraic sets* that are defined by polynomial inequalities. By the Tarski–Seidenberg Theorem on quantifier elimination [19, 22], the class of semialgebraic sets is closed under projections and under images of polynomial maps. A closed semialgebraic set is *basic* if it is a finite intersection of sets of the form $\{x : f(x) \geq 0\}$ for some polynomial f.

Motivating questions about convex algebraic geometry were raised in [15]. A fundamental convex semialgebraic set is the cone of positive semidefinite matrices (the PSD *cone*). These are symmetric matrices with nonnegative eigenvalues. The boundary of the PSD cone is (a connected component of) the determinant hypersurface and every face is exposed. A *spectrahedron* is an affine section $L \cap$PSD of this cone. Write $A \succeq 0$ to indicate that $A \in$ PSD. Parameterizing L shows that a spectrahedron is defined by a *linear matrix inequality*,

$$\{x \in \mathbb{R}^m : A_0 + x_1 A_1 + \cdots + x_m A_m \succeq 0\},$$

where $A_0, A_1, \ldots, A_m$ are real symmetric matrices.

Images of spectrahedra under linear maps are *spectrahedral shadows*. Semidefinite programming provides efficient methods to optimize linear objective functions over spectrahedra and their shadows, and a fundamental question is to determine if a given convex semialgebraic set may be realized as a spectrahedron or as a spectrahedral shadow, and to give such a realization. Scheiderer showed that the convex hull of a curve is a spectrahedral shadow [17], and recently proved that there are many convex semialgebraic sets which are not spectrahedra or their shadows [18].

Since the optimizer of a linear objective function lies in the boundary, convex algebraic geometry also seeks to understand the boundary of a convex semialgebraic set. This includes determining its faces and their inclusions, as well as the Zariski closure of the boundary, called the *algebraic boundary*. This was studied for rational curves [20, 23] and for curves in $\mathbb{R}^3$ by Ranestad and Sturmfels [14]. They showed that the algebraic boundary of a space curve C consists of finitely many tritangent planes and a ruled *edge surface* composed of stationary bisecant lines. A *stationary bisecant* is a secant $\overline{x, y}$ to C ($x, y \in C$) such that the tangent lines T_xC and T_yC to C at x and y meet. For a general irreducible space curve of degree d and genus g, the edge surface has degree $2(d-3)(d+g-1)$.

For example, suppose that C is a general space quartic (see [11, Rem. 5.5] or [14, Ex. 2.3]). This is the complete intersection of two real quadrics P and Q, and has genus one by the adjunction formula [9, Ex. V.1.5.2]. Its edge surface has degree $2(4-3)(4+1-1) = 8$ and is the union of four cones. In the pencil of quadrics that contain C, $sP + tQ$ for $[s, t] \in \mathbb{P}^1$, four are singular and are given by the roots of $\det(sP + tQ)$. Here, the quadratic forms P, Q are expressed as symmetric matrices. Each singular quadric is a cone and each line on that cone is a stationary bisecant of C. A general point of C lies on four stationary bisecants, one for each cone.

The union of two circles in different planes is also a space quartic, but it is not in general a complete intersection (the complex points of a complete intersection are connected). We therefore expect a different answer than for general space quartics. We give a taste of that which is to come.

Theorem 2.1 *Let C_1 and C_2 be circles in $\mathbb{R}^3$ lying in different planes. Their convex hull is a spectrahedron if and only if the scheme $C_1 \cap C_2$ has length 2. When the complex points of the circles are disjoint and neither is tangent to the plane of the other, the edge surface is irreducible and has degree eight. Its rulings are parametrized by a smooth curve of genus one in $C_1 \times C_2$. A general point of $C_1 \cup C_2$ lies on two stationary bisecants.*

Proof This is proven in Lemma 3.1, and in Theorems 3.3, 3.5 and 4.8. □

3 Stationary Bisecants to Two Complex Conics

We study stationary bisecants and edge surfaces in the algebraic relaxation of our problem of two circles, replacing circles in $\mathbb{R}^3$ by smooth conics in $\mathbb{P}^3 = \mathbb{CP}^3$.

A conic C in $\mathbb{P}^3$ spans a plane. Let C_1 and C_2 be conics spanning different planes, Π_1 and Π_2, respectively. A stationary bisecant is spanned by points $p \in C_1$ and $q \in C_2$ with $p \neq q$ whose tangent lines T_pC_1 and T_qC_2 meet. Set $\ell := \Pi_1 \cap \Pi_2$.

Lemma 3.1 *A point $p \in C_1$ lies on two stationary bisecants unless the tangent line T_pC_1 meets C_2. If the tangent line meets C_2, then it is the unique stationary bisecant through p unless $p \in C_2$ or T_pC_1 lies in the plane Π_2 of C_2. When $T_pC_1 \subset \Pi_2$, the pencil of lines in Π_2 through p are all stationary bisecants.*

Proof Consider the tangent line T_pC_1 for $p \in C_1$ and see Fig. 3 for reference. Either

$$(i) \quad T_pC_1 \not\subset \Pi_2 \qquad \text{or} \qquad (ii) \quad T_pC_1 \subset \Pi_2\,.$$

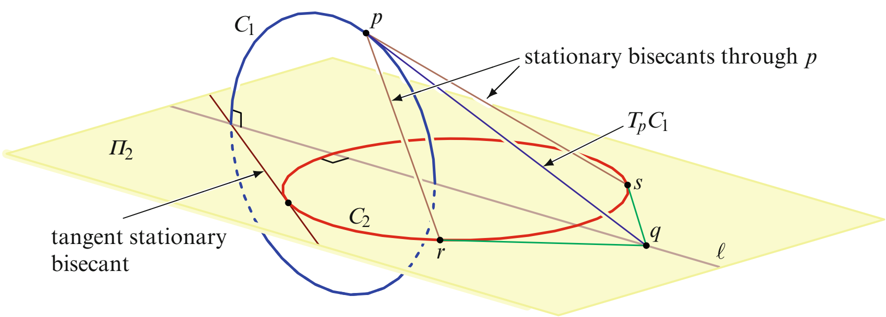


Fig. 3 Stationary bisecants

In case (i), let q be the point where T_pC_1 meets Π_2. There are further cases. When $q \notin C_2$, there are two tangents to C_2 that meet q, and the lines through p and each point of tangency (r, s in Fig. 3) give two stationary bisecants through p. If $q \in C_2$ and $p \neq q$, then the tangent line T_pC_1 is the only stationary bisecant through p.

In case (ii), the tangent line T_pC_1 meets every tangent to C_2, and every line in Π_2 through p (except T_pC_2 if $p \in C_2$) meets C_2 and is therefore a stationary bisecant. If $p \in C_2$, then the tangent line T_pC_2 is a limit of such lines. □

Remark 3.2 When C_1 is tangent to the plane Π_2 at a point p, the pencil of lines in Π_2 through p are *degenerate stationary bisecants*. When $p \notin C_2$, a general line in the pencil meets C_2 twice so that the map from C_2 to this pencil has degree two.

Lines that meet C_1 and C_2 in distinct points are given by points $(p, q) \in C_1 \times C_2$ with $p \neq q$. The *edge curve* E is the Zariski closure of the set of points (p, q) such that $\overline{p, q}$ is a stationary bisecant. As a smooth conic is isomorphic to $\mathbb{P}^1$, the edge curve is a curve in $\mathbb{P}^1 \times \mathbb{P}^1$. Subvarieties of products of projective spaces have a multidegree (see [4, § 2]). For a curve C in $\mathbb{P}^1 \times \mathbb{P}^1$, this becomes its bidegree (a, b), where a is the number of points in the intersection of C with $\mathbb{P}^1 \times \{q\}$ for q general and b the number of points in the intersection of C with $\{p\} \times \mathbb{P}^1$ for p general. As $\mathbb{P}^1 \times \{q\}$ has bidegree $(0, 1)$ and $\{p\} \times \mathbb{P}^1$ has bidegree $(1, 0)$, the intersection pairing on curves in $\mathbb{P}^1 \times \mathbb{P}^1$, expressed in terms of bidegree, is

$$(a, b) \cdot (c, d) = ad + bc \in \mathbb{Z}\,. \tag{1}$$

A curve of bidegree (a, b) is defined in homogeneous coordinates $([s : t], [u : v])$ for $\mathbb{P}^1 \times \mathbb{P}^1$ by a bihomogeneous polynomial that has degree a in s, t and b in u, v.

Theorem 3.3 *The edge curve E has bidegree* $(2, 2)$.

Proof In the projection to C_1, two points of E map to a general point $p \in C_1$, by Lemma 3.1. Thus the intersection number of E with $\{p\} \times C_2$ is 2 and vice-versa for $C_1 \times \{q\}$, for general $q \in C_2$. Consequently, E has bidegree $(2, 2)$.

We compute the defining equation of E to give a second proof. This begins with a parametrization of the conics. Let $f_{i,0}, f_{i,1}, f_{i,2}, f_{i,3} \in H^0\big(\mathbb{P}^1, \mathscr{O}(2)\big)$ for $i = 1, 2$ be two quadruples of homogeneous quadrics that each span $H^0\big(\mathbb{P}^1, \mathscr{O}(2)\big)$. Each quadruple gives a map $f_i \colon \mathbb{P}^1 \to \mathbb{P}^3$ whose image is a conic C_i. The plane Π_i of C_i is defined by the linear relation among $f_{i,0}, f_{i,1}, f_{i,2}, f_{i,3}$, and we assume that $\Pi_1 \neq \Pi_2$.

In coordinates, if $[s : t] \in \mathbb{P}^1$, then the image

$$f_i[s : t] = [f_{i,0}(s, t) : f_{i,1}(s, t) : f_{i,2}(s, t) : f_{i,3}(s, t)]$$

is the corresponding point of C_i. Its tangent line is spanned by $\partial_s f_i$ and $\partial_t f_i$, where $\partial_x := \frac{\partial}{\partial x}$, as $s\partial_s f + t\partial_t f = 2f$, for a homogeneous quadric f. The points $f_1[s : t]$ and $f_2[u : v]$ span a stationary bisecant when their tangents meet. Equivalently, when

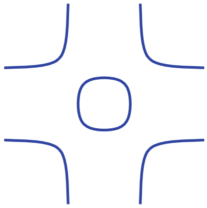


Fig. 4 Edge curve in window $|s/t|, |u/v| \le 5$

$$E(s,t,u,v) := \det \begin{bmatrix} \partial_s f_{1,0} & \partial_s f_{1,1} & \partial_s f_{1,2} & \partial_s f_{1,3} \\ \partial_t f_{1,0} & \partial_t f_{1,1} & \partial_t f_{1,2} & \partial_t f_{1,3} \\ \partial_u f_{2,0} & \partial_u f_{2,1} & \partial_u f_{2,2} & \partial_u f_{2,3} \\ \partial_v f_{2,0} & \partial_v f_{2,1} & \partial_v f_{2,2} & \partial_v f_{2,3} \end{bmatrix} = 0 . \tag{2}$$

As the first two rows have bidegree $(1,0)$ and the second two have bidegree $(0,1)$, this form $E(s,t,u,v)$ has bidegree $(2,2)$. □

Example 3.4 Suppose that C_1 and C_2 are the unlinked unit circles where C_1 is centred at the origin and lies in the xy-plane and C_2 is centred at $(3,0,0)$ and lies in the xz-plane. If we choose homogeneous coordinates $[X_0 : X_1 : X_2 : X_3]$ for $\mathbb{P}^3$ where $(x,y,z) = \frac{1}{X_0}(X_1, X_2, X_3)$, then these admit parametrizations

$$[s:t] \longmapsto [s^2{+}t^2 : s^2{-}t^2 : 2st : 0] \quad \text{and} \quad [u:v] \longmapsto [u^2{+}v^2 : 2u^2{+}4v^2 : 0 : 2uv] .$$

Dividing the determinant (2) by -16 gives the equation for the edge curve E,

$$s^2u^2 - 3s^2v^2 - 3t^2u^2 + 5t^2v^2 ,$$

which is irreducible. Figure 4 draws E in the window $|s/t|, |u/v| \le 5$ in $\mathbb{RP}^1 \times \mathbb{RP}^1$.

The Zariski closure of the union of all stationary bisecants is the ruled *edge surface* $\mathscr{E}$. By Lemma 3.1, a general point of one of the conics lies on two stationary bisecants. Therefore, each conic is a curve of self-intersections of $\mathscr{E}$, and the multiplicity of $\mathscr{E}$ at a general point of a conic is 2.

Theorem 3.5 *The edge surface $\mathscr{E}$ has degree eight when $C_1 \cap C_2 = \varnothing$ and neither is tangent to the plane of the other.*

Proof The line $\ell = \Pi_1 \cap \Pi_2$ meets each conic in two points and therefore meets $\mathscr{E}$ in at least four points. Any other point $r \in \ell \cap \mathscr{E}$ lies on a stationary bisecant m between a point p of C_1 and a point q of C_2. As $p, r \in \Pi_1$, we have $m \subset \Pi_1$, and similarly $m \subset \Pi_2$. Thus $m = \ell$, but ℓ is not a stationary bisecant, a contradiction.

Each of the four points of $\ell \cap \mathscr{E}$ has multiplicity two on $\mathscr{E}$ by Lemma 3.1 and the observation preceding the statement of the theorem. Thus, $\mathscr{E}$ has degree eight.

We give a second proof. Let m be a general line that meets $\mathscr{E}$ transversally. The points of $m \cap \mathscr{E}$ lie on stationary bisecants that meet m. We count these using

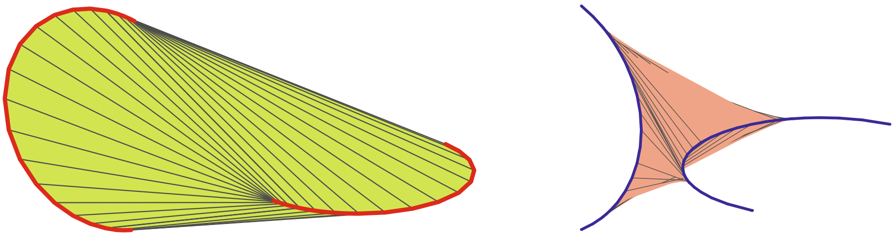


Fig. 5 Expanded view of edge surface

intersection theory. Let $M \subset C_1 \times C_2$ be the curve whose points are pairs (p, q) such that the secant line spanned by p and q meets m. Stationary bisecants that meet m are points of intersection of M and the edge curve E. We compute the bidegree of M.

Fix a point $p \in C_1$ with $p \notin \Pi_2$. Secant lines through p rule the cone over C_2 with vertex p. As this cone meets m in two points, we have $\deg(M \cap \{p\} \times C_2) = 2$. The symmetric argument with a point of C_2 shows that M has bidegree $(2, 2)$. By (1), M meets E in $(2, 2) \cdot (2, 2) = 4 + 4 = 8$ points. This proves the theorem. □

The arguments in this proof using intersection theory are similar to arguments used in the contributions [4, 12] in this volume.

Remark 3.6 Each irreducible component C of the edge curve E gives an algebraic family of stationary bisecants and an irreducible component $\mathscr{C}$ of the edge surface $\mathscr{E}$. If C has bidegree (a, b), then the corresponding component $\mathscr{C}$ of $\mathscr{E}$ has degree at most $(2, 2) \cdot (a, b) = 2(a + b)$. This is not an equality when the intersection $M \cap E$ has a basepoint or when the general point of $\mathscr{C}$ contains two stationary bisecants. This occurs when one circle is tangent to the plane of the other and there are one or more components of degenerate stationary bisecants.

Example 3.7 The real points of the edge curve appearing in Fig. 4 had two connected components (the picture showed a patch of $\mathbb{RP}^1 \times \mathbb{RP}^1$). Thus, the set of real points of the edge surface has two components. Stationary bisecants corresponding to the oval in the centre of Fig. 4 lie along the convex hull, which is shown on the left in Fig. 5. The others bound a nonconvex set that lies inside the convex hull. Figure 5 displays it in an expanded view on the right. The planes of the circles meet in the x-axis. For sufficiently small $\epsilon > 0$, the line defined by $y = z = \epsilon$ meets $\mathscr{E}$ transversally. Near each point of a circle lying on the x-axis it meets $\mathscr{E}$ in two points, one for each of the two families of stationary bisecants passing through the nearby arc of the circle. These eight points are real.

A curve of bidegree $(2, 2)$ on $\mathbb{P}^1 \times \mathbb{P}^1$ has arithmetic genus one, by the adjunction formula. If smooth, then it is an irrational genus one curve. Another way to see this is that the projection to a $\mathbb{P}^1$ factor is two-to-one, except over the branch points, of which there are four, counted with multiplicity. Indeed, writing its defining equation as a quadratic form in the variables (u, v) for the second $\mathbb{P}^1$ factor, its coefficients are quadratic forms in the variables s, t of the first $\mathbb{P}^1$. The projection to the first

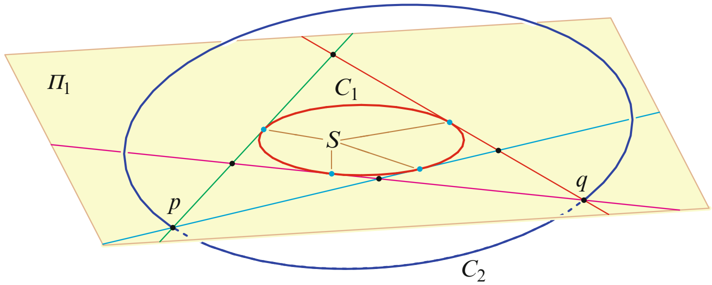


Fig. 6 Conic giving specified branch points

has branch points where the discriminant vanishes, which is a quartic form. By elementary topology, a double cover of $\mathbb{CP}^1$ with four branch points has Euler characteristic zero, again implying that it has genus one.

Lemma 3.8 *For every set S of four points of C_1, there is a conic C_2 such that the projection to C_1 of the edge curve is branched over S.*

Proof Let p be the point of intersection of two of the tangents to C_1 at points of S and q be the point of intersection of the other two tangents (see Fig. 6). Since the tangent T_sC_1 at any point $s \in S$ meets C_2 (in one of the points p or q), Lemma 3.1 implies that this is the unique stationary bisecant involving the point s. Thus, the points of S are branch points of the projection to C_1 of the edge curve. □

Remark 3.9 There are three families of conics C_2 giving an edge curve branched over S. These correspond to the three partitions of S into two parts of size two. Each partition determines two points p, q on the plane Π_1 of C_1 where the tangent lines at the points in each part meet. The corresponding family is the collection of conics C_2 that meet Π_1 transversally in p and q. If both C_1 and $S \subset C_1$ are real and we choose an affine $\mathbb{R}^3$ containing the points p and q, then we may choose C_2 to be a circle.

The isomorphism class of a complex smooth genus one curve is determined by its j-invariant [9, Sect. IV.4]. This may be computed from the branch points S of any degree two map to $\mathbb{P}^1$. Explicitly, if we choose coordinates on $\mathbb{P}^1$ so that the branch points S are $\{0, 1, \lambda, \infty\}$, then the j-invariant is

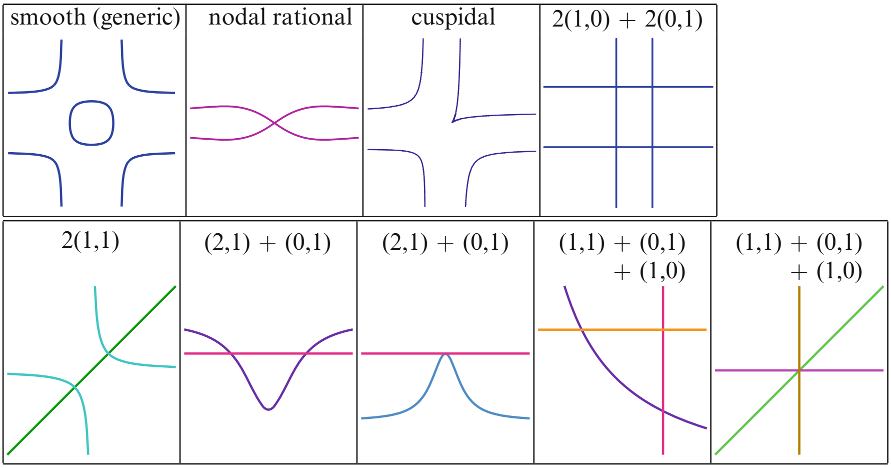
$$2^8 \cdot \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2}\,.$$

We have the following corollary of Lemma 3.8.

Theorem 3.10 *For every conic C_1 and every $J \in \mathbb{C}$, there is a conic C_2 such that the edge curve has j-invariant J. When C_1 and $S \subset C_1$ are real, C_2 may be a circle.*

We now classify the possible edge curves E to a pair of conics C_1 and C_2 lying in distinct planes Π_1 and Π_2. By Lemma 3.12, every component of E is reduced. If $E = F \cup G$ is reducible, then we have that $(2, 2) = \text{bidegree}(F) + \text{bidegree}(G)$. Thus,

Table 1 Types of $(2, 2)$-curves



the bidegrees of the components of E form a partition of $(2, 2)$. If E is irreducible, then either it is smooth of genus one or singular of arithmetic genus one and hence rational. Any curve of bidegree $(1, a)$ or $(a, 1)$ is rational. Table 1 gives the different possibilities, along with pictures of a real curve.

Theorem 3.11 *All types of* $(2, 2)$*-curves of Table 1 occur as the edge curve of a pair of conics* C_1, C_2 *lying in distinct planes except a curve with a cusp and a reducible curve* $2(1, 0) + 2(0, 1)$ *with four components.*

For existence, see Tables 2, 3, and 4, which display edge curves of two circles in all possible configurations. We rule out edge curves with a cusp and reducible edge curves of type $2(1, 0) + 2(0, 1)$. We first analyze the singularities of edge curves.

Lemma 3.12 *The edge curve E is reduced. A point* $(p, q) \in C_1 \times C_2$ *is a singular point of E only if* $p = q$ *or* $T_pC_1 \subset \Pi_2$ *or* $T_qC_2 \subset \Pi_1$*. There are five possibilities for* p, q *and the tangents, up to interchanging the conics* C_1 *and* C_2*.*

(i) $p = q$ *and the tangent to each conic at p does not lie in the plane of the other.*
(ii) $p = q$ *with* $T_pC_1 \subset \Pi_2$*, but* $T_qC_2 \not\subset \Pi_1$*.*
(iii) $p = q$ *with both* $T_pC_1 \subset \Pi_2$ *and* $T_qC_2 \subset \Pi_1$*.*
(iv) $p \neq q$ *and* $T_pC_1 \subset \Pi_2$*, but* $T_qC_2 \not\subset \Pi_1$*. Then* $p \in T_qC_2$ *is a stationary bisecant.*
(v) $p \neq q$ *and* $T_pC_1 = T_qC_2$ *is* $\Pi_1 \cap \Pi_2$*, and is a stationary bisecant.*

Proof Let $(p, q) \in C_1 \times C_2$ be a point on a curve E of bidegree $(2, 2)$. If the fibre of E in one of the projections from (p, q), say to C_2, has exactly two points, then E is smooth at (p, q). Indeed, as E is a $(2, 2)$ curve, $E \cap (C_1 \times \{q\})$ is either $C_1 \times \{q\}$ or one double or two simple points, and if two, then E is smooth at each point.

Consequently, there are three possibilities for points of E in the fibres of the projections to C_1 and C_2 containing a singular point (p, q). Either

1. (p, q) is the only point of E in both fibres,
2. (p, q) is the only point in one fibre and the other fibre is a component of E, or
3. both fibres are components of E.

In Case 2, E has at least one component with either linear bidegree $(1, 0)$ or $(0, 1)$, and in Case 3, it has at least one component with each linear bidegree.

Now let E be the edge curve, which is smooth at any point (p, q) where there is another point in one of the two fibres of projections to C_i. Lemma 3.1 implies that there are two points in E over a general point of either conic, so every component of E is smooth at a generic point and therefore E is reduced. By Lemma 3.1 and the analysis above, a point $(p, q) \in E$ is singular if and only if both tangents meet the other conic for otherwise there is a second point in one of the fibres.

If $T_pC_1 \subset \Pi_2$, then every line in Π_2 through p is a stationary bisecant, so E contains $\{p\}\times C_2$, which has bidegree $(1, 0)$. If $T_qC_2 \subset \Pi_1$, then as before E contains $C_1 \times \{q\}$, which has bidegree $(0, 1)$. If neither occurs, but E is singular at (p, q), then we are in Case (i). When $p = q$ and we are not in Case (i), then, up to interchanging the indices 1 and 2, we are in either Case (ii) or (iii). When $p \neq q$, so that one circle is tangent to the plane of the other, then we are in either Case (iv) or (v). □

Proof of Theorem 3.11 We need only to rule out that the edge curve E has type $2(1, 0) + 2(0, 1)$ or has a cusp. By Lemma 3.12, E has a component $\{p\} \times C_2$ of bidegree $(1, 0)$ exactly when C_1 is tangent to the plane Π_2 at the point p. Since $\Pi_1 \neq \Pi_2$, there is at most one such point of tangency on C_1, so E has at most one component of bidegree $(1, 0)$ and the same is true for a component of bidegree $(0, 1)$. Thus the type $2(1, 0) + 2(0, 1)$ cannot occur for an edge curve.

We show that if $(p, q) \in E$ is a singular point in Case (i) of Lemma 3.12, then E has a node at (p, q), ruling out a cusp and completing the proof.

Suppose that $p = q$ and the tangents to each conic at p do not lie in the plane of the other. Choose coordinates x, y, z, w for $\mathbb{P}^3$ so that Π_1 is the plane $z = 0$, Π_2 is the plane $x = 0$, T_pC_1 is the line $y = z = 0$, T_pC_2 is $x = y = 0$, and $p = [0 : 0 : 0 : 1]$. Then we may choose parametrizations near p for C_1 and C_2 of the form

$$\begin{aligned} C_1 &: s \longmapsto [s + as^2 : bs^2 : 0 : 1 + cs + ds^2] \\ C_2 &: u \longmapsto [0 : \beta u^2 : u + \alpha u^2 : 1 + \gamma u + \delta u^2]\,, \end{aligned} \tag{3}$$

for some $a, b, c, d, \alpha, \beta, \gamma, \delta \in \mathbb{C}$ where $b\beta \neq 0$. The edge curve is defined by

$$\det \begin{bmatrix} s + as^2 & bs^2 & 0 & 1 + cs + ds^2 \\ 1 + 2as & 2bs & 0 & c + 2ds \\ 0 & \beta u^2 & u + \alpha u^2 & 1 + \gamma u + \delta u^2 \\ 0 & 2\beta u & 1 + 2\alpha u & \gamma + 2\delta u \end{bmatrix}$$

$$= \big(\beta(ac - d) - b(\alpha\gamma - \delta)\big)s^2u^2 - 2b\alpha s^2u + 2a\beta su^2 + \beta u^2 - bs^2\,. \tag{4}$$

Indeed, the matrix has rows $f_1(s), f_1'(s), f_2(u), f_2'(u)$, where f_i is the parametrization of C_i (3). The determinant vanishes when the tangent to C_1 at $f_1(s)$ meets the tangent to C_2 at $f_2(u)$. The terms of lowest order in (4), $\beta u^2 - bs^2$, have distinct linear factors when $b\beta \neq 0$. Thus, E has a node when $s = u = 0$, which is (p, p). □

4 Convex Hull of Two Circles in $\mathbb{R}^3$

We classify the relative positions of two circles in $\mathbb{R}^3$ and show that the combinatorial type of the face lattice of their convex hull depends only upon their relative position. This relative position is determined by the combinatorial type of the face lattice and the real geometry of the edge curve. We use this classification to determine when the convex hull of two circles is a spectrahedron.

Let C_1, C_2 be circles in $\mathbb{R}^3$ lying in distinct planes Π_1 and Π_2, respectively. The intersection $C_1 \cap \Pi_2$ in $\mathbb{CP}^3$ is either two real points, two complex conjugate points, or C_1 is tangent to Π_2 at a single real point. Let m_1 be the number of real points in this intersection, and the same for m_2. Order the circles so that $m_1 \geq m_2$, and call $[m_1, m_2]$ the *intersection type* of the pair of circles.

The configuration of the circles is determined by the order of their points along the line $\ell := \Pi_1 \cap \Pi_2 \subset \mathbb{R}^3$. For example, C_1 and C_2 have *order type* $(1, 2, 1, 2)$ *along* ℓ when they have intersection type $[2, 2]$ and meet ℓ in distinct points which alternate. If $C_1 \cap C_2 \neq \varnothing$, then we write S for that shared point. For example, if C_1 meets ℓ in two real points with C_2 tangent to ℓ at one, then this pair has order type $(1, S)$. The intersection type may be recovered from the order type.

A further distinction is necessary for intersection type $[0, 0]$, when both circles meet ℓ in two complex conjugate points. In $\mathbb{CP}^3$, we have either $C_1 \cap C_2 = \varnothing$ or $C_1 \cap \ell = C_2 \cap \ell$. Write $\varnothing$ for the order type in the first case and $(2c)$ in the second. By Lemma 3.12, the edge curve is smooth in order type $\varnothing$ and singular in order type $(2c)$.

Lemma 4.1 *There are 15 possible order types of two circles in* $\mathbb{R}^3$.

Proof See Tables 2, 3, and 4 for the order types of circles and their convex hulls.

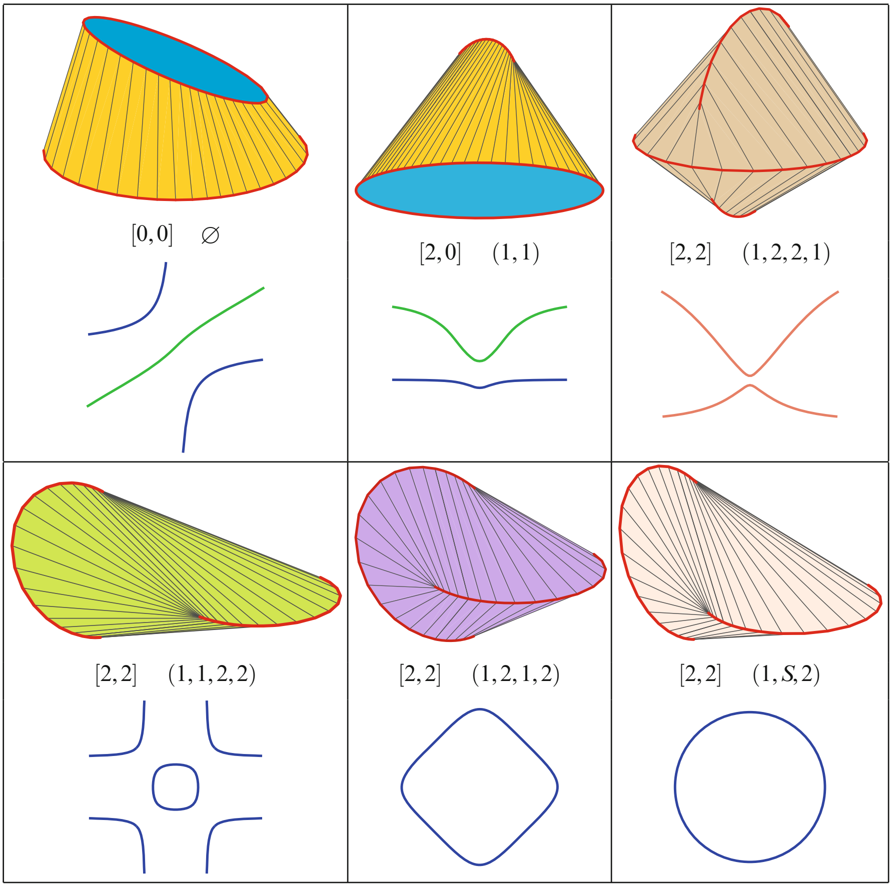
The possible intersection types are $[0, 0]$, $[1, 0]$, $[1, 1]$, $[2, 0]$, $[2, 1]$, and $[2, 2]$. For $[0, 0]$, we noted two order types, and intersection types $[1, 0]$ and $[2, 0]$ each admit one order type, namely (1) and $(1, 1)$, respectively.

For $[1, 1]$, each circle C_i is tangent to ℓ at a point p_i. Either $p_1 \neq p_2$ or $p_1 = p_2$, so there are two order types, $(1, 2)$ and (S).

For $[2, 1]$, the line ℓ is secant to circle C_1 and C_2 is tangent to ℓ at a point p_2. Either p_2 is in the exterior of C_1 or it lies on C_1 or it is interior to C_1. These give three order types, $(1, 1, 2)$, $(1, S)$, and $(1, 2, 1)$, respectively.

Finally, for $[2, 2]$ there are three order types when all four points are distinct, $(1, 1, 2, 2)$, $(1, 2, 1, 2)$, and $(1, 2, 2, 1)$. When one point is shared, we have $(1, 2, S)$ or $(1, S, 2)$. Finally, both points may be shared, giving (S, S). □

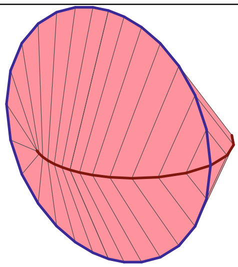
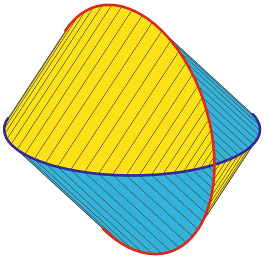
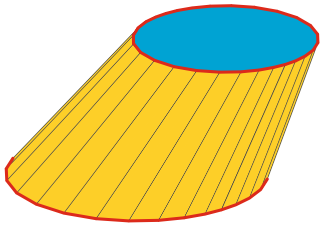
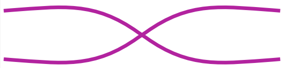
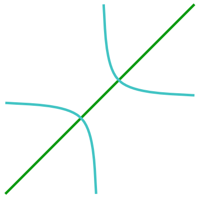
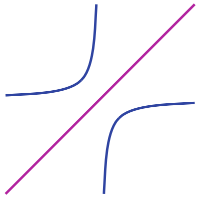
Table 2 Some convex hulls, intersection and order types, and edge curves



The order type of the circles determines the combinatorial type of the face lattice of their convex hull K. Describing the face lattice means identifying all (families of) faces of K, their incidence relations, and which are exposed/not exposed. Throughout, D_i is the disc of the circle C_i. We invite the reader to peruse our gallery in Tables 2, 3, and 4 while reading this classification. Our main result is the following.

Theorem 4.2 *The order type of C_1, C_2 determines the combinatorial type of the face lattice of K, as summarized in Table 5. There are eleven distinct combinatorial types of face lattice. The combinatorial type of the face lattice, together with the real algebraic geometry of its edge curve, determines the order type.*

Table 3 More convex hulls, intersection and order types, and edge curves

		
$[2,2]$ $(1,2,S)$	$[2,2]$ (S,S)	$[0,0]$ $(2c)$
		

We determine the face lattice for each order type. Some general statements are given in preliminary results which precede our proof of Theorem 4.2. The statements are asymmetric, with the symmetric statement obtained by interchanging 1 and 2. We first study the section $\kappa_1 := K \cap \Pi_1$ of K, which contains D_1.

Lemma 4.3 *We have* $\kappa_1 = \text{conv}(C_1, C_2 \cap \Pi_1)$.

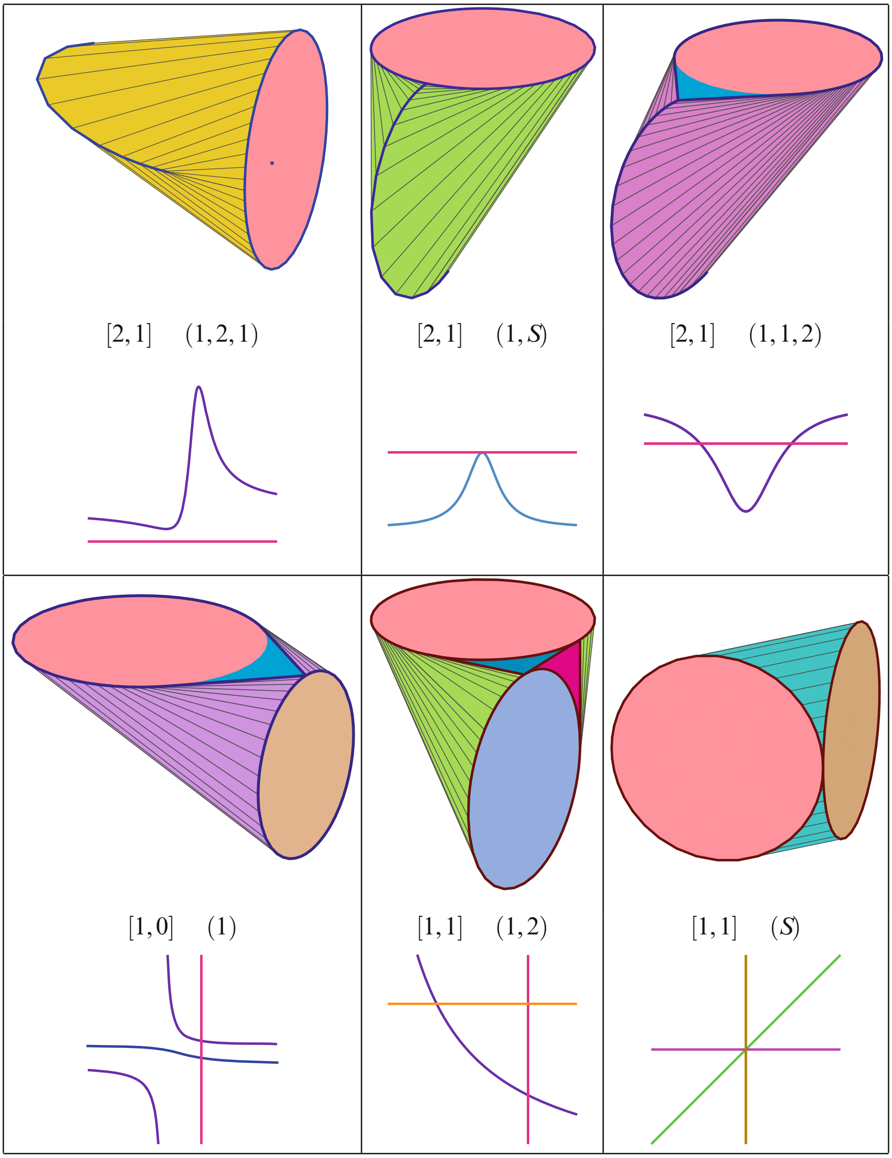
Proof As $D_i = \text{conv}(C_i)$, $K = \text{conv}(D_1, D_2)$. Therefore a point $x \in K$ is a convex combination $\lambda y + \mu z$ ($\lambda, \mu \geq 0$ with $\lambda + \mu = 1$) of points $y \in D_1$ and $z \in D_2$. If $x \in \kappa_1 \subset \Pi_1$, then as $y \in \Pi_1$, we must have that $z \in D_2 \cap \Pi_1 = \text{conv}(C_2 \cap \Pi_1)$. □

Corollary 4.4 *If $C_2 \cap \Pi_1 \subset D_1$, then we have $\kappa_1 = D_1$. Otherwise, κ_1 is the convex hull of D_1 and the one or two points of $C_2 \cap \Pi_1$ exterior to D_1. A point $p \in C_1$ is an extreme point of κ_1 if and only if C_1 and $C_2 \cap \Pi_1$ lie on the same side of T_pC_1. An extreme point $p \in C_1$ of κ_1 is not exposed if and only if T_pC_1 meets $C_2 \smallsetminus \{p\}$. Extreme points of κ_1 are extreme points of K and nonexposed points of κ_1 are nonexposed in K. Finally, κ_1 is a face of K if and only if $m_2 \leq 1$.*

Proof The first two statements follow from Lemma 4.3. The next two about extreme points p of κ_1 follow as T_pC_1 is the only possible supporting line to κ_1 at p. The next, about extreme points of K and its section κ_1, follows by Lemma 4.3, and the last is immediate as C_2 lies on one side of Π_1 if and only if $|C_2 \cap \Pi_1| < 2$. □

By Lemma 3.1, a general point $p \in C_1$ lies on two stationary bisecants. If $p \in K$ is extreme, then these may support one-dimensional *bisecant faces* of K. We determine the bisecant faces meeting most extreme points. Any plane supporting an extreme point $p \in C_1$ contains T_pC_1. If such a plane does not meet C_2, then p is exposed.

Table 4 More convex hulls, intersection and order types, and edge curves



Lemma 4.5 *Let $p \in K$ be an extreme point of K. If T_pC_1 neither meets C_2 nor lies in Π_2, then p is exposed. Such a point p lies on one bisecant face if $m_2 \leq 1$ and two if $m_2 = 2$. When there are two, one is on each side of Π_1.*

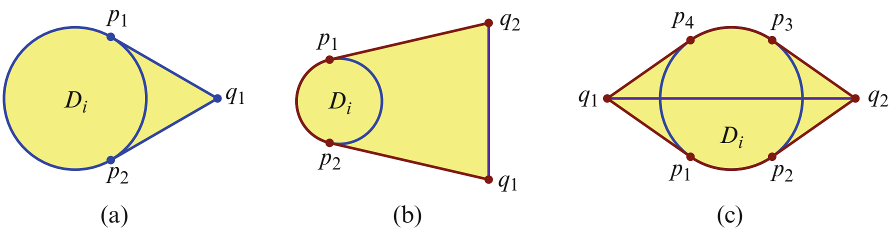


Fig. 7 Some possible slices κ_i

Proof Let $p \in C_1$ be an extreme point of K such that T_pC_1 neither meets C_2 nor lies in Π_2. By Corollary 4.4, C_1 and $C_2 \cap \Pi_1$ lie on the same side of T_pC_1 in Π_1. In the pencil $\mathbb{RP}^1$ of planes containing T_pC_1, those meeting K form an interval I containing Π_1 and an interval γ of planes meeting C_2. Each endpoint of γ is a plane containing a stationary bisecant through p. Our assumptions on p and T_pC_1 imply that $I \neq \mathbb{RP}^1$, so that p is exposed. If $m_2 \leq 1$, then Π_1 is one endpoint of I and the other is an endpoint of γ, otherwise the endpoints of I are the endpoints of γ and Π_1 is an interior point, which proves the lemma. □

Remark 4.6 Corollary 4.4 identifies the 2-faces, extreme points, and some nonexposed points of K. Lemma 4.5 identifies most exposed points and bisecant edges. The rest of the face lattice is determined in the proof of Theorem 4.2. We first understand the boundary of each section $\kappa_i = K \cap \Pi_i$. Figure 7 shows the possibilities when κ_i is not the disc D_i.

There, q_1 and q_2 are points of the other circle on the boundary of κ_i and points p_j are nonexposed points of C_i as $T_{p_j}C_i$ meets q_1 or q_2. The line segment between q_1 and q_2 is where the disc of the second circle meets Π_i.

Proof of Theorem 4.2 We give separate arguments for each order type.

Order Type $\varnothing$: By Corollary 4.4, both discs are faces of K, and every point of the circles is extreme. By Lemma 4.5, all points of the circles are exposed, and each point lies on exactly one bisecant face.

Order Type $(2c)$: As the edge curve for order type $\varnothing$ is smooth and of genus 1, while that for order type $(2c)$ is singular, the edge curve distinguishes these two order types.

Order Type $(1,1)$: Since $m_1 = 2$ and $m_2 = 0$, D_1 is the only 2-face. The section κ_2 is similar to Fig. 7b, so the extreme points on C_2 form an arc $\widehat{p_1, p_2}$ whose endpoints are not exposed, each lying on one bisecant edge. The interior points of $\widehat{p_1, p_2}$ are exposed by Lemma 4.5 and each lies on two bisecant edges. Similarly, every point of C_1 is exposed and lies on one bisecant edge.

Order Type $(1,2,1)$: Its edge curve is singular, while order type $(1,1)$ has a smooth edge curve.

Order Type $(1,2,2,1)$: Since $m_1 = m_2 = 2$, K has no 2-faces. Since C_2 meets the interior of D_1, Corollary 4.4 implies that every point of C_1 is extreme and C_2 has two intervals of extreme points. The four endpoints are not exposed and each lies on one bisecant edge. By Lemma 4.5, every point of C_1 and of the interior of the arcs on C_2 is exposed and lies on two bisecant edges.

Order Type $(1,1,2,2)$: By Corollary 4.4, K has no 2-faces and each circle has one arc of extreme points, as the sections κ_i are similar to Fig. 7a. As before, each endpoint of an arc is not exposed and lies on one bisecant edge, and each interior point of an arc is exposed and lies on two bisecant edges.

Order Types $(1,2,1,2)$ and $(1,S,2)$: The edge curve in type $(1,1,2,2)$ has two real components as seen in Example 3.7, while for type $(1,2,1,2)$ there is one real component. For type $(1,S,2)$, the edge curve is singular at the shared point.

Order Type $(1,2,S)$: Again, K has no 2-faces. All points of C_1 are extreme and C_2 has an arc of extreme points whose endpoints are not exposed and each lies on one bisecant edge. Also, all interior points of that arc and of C_1—except possibly the shared point p—are exposed and lie on two bisecant edges. The tangents T_pC_1 and T_pC_2 span a plane exposing p and p lies on no bisecant edges.

Order Type (S,S): There are no 2-faces and as in type $(1,2,S)$ every point of the circles is extreme, and the nonshared points are exposed and each lies on two bisecant edges. Each shared point is exposed by the plane spanned by the two tangents at that point and neither shared point lies on a bisecant edge.

Order Type $(1,S)$: The only 2-face is D_1. Every point of C_1 is extreme and C_2 has an arc $\widehat{p_1,p_2}$ of extreme points with one endpoint, say p_1, the shared point where C_2 is tangent to Π_1. Neither endpoint is exposed and p_2 lies on one bisecant edge (the bisecant $T_{p_1}C_2$ meets the interior of D_1). By Lemma 4.5, every point of C_1 except p_1 lies on one bisecant edge and every interior point of $\widehat{p_1,p_2}$ lies on two bisecant edges, and all of these are exposed.

Order Type $(1,1,2)$: The only 2-face is κ_1 and its shape is as in Fig 7a with the vertex q_1 where D_2 is tangent to Π_1. There is an arc $\widehat{p_1,p_2}$ of extreme points of C_1 whose endpoints are not exposed with each lying on a bisecant edge $\overline{p_i,q_1}$. The section κ_2 has the same shape and C_2 has an arc $\widehat{q_1,q_2}$ of extreme points with neither endpoint exposed. The point q_2 lies on one bisecant edge along $T_{q_2}C_2$ and q_1 lies on two bisecant edges $\overline{p_i,q_1}$. Neither of the edges $\overline{p_i,q_1}$ is exposed as Π_1 is the only supporting plane of K containing either edge. Finally, by Lemma 4.5, interior points of the arcs are exposed, with those from $\widehat{p_1,p_2}$ lying on one bisecant edge and those from $\widehat{q_1,q_2}$ lying on two.

In the order types of the last row of Table 4, the circle C_2 is tangent to Π_1 at a point q_1 and the tangent $T_{q_1}C_2$ does not meet the interior of D_1. In the pencil of planes containing $T_{q_1}C_2$, Π_1 and Π_2 are the endpoints of an interval of planes meeting $K \smallsetminus T_{q_1}C_2$ and of an interval of planes that meet K only in $T_{q_1}C_2 \cap K$. Thus, both sections κ_1 and κ_2 are 2-faces of K and the face $T_{q_1}C_2 \cap K$ is exposed.

Order Type (1): Here, $m_1 = 1$ and $m_2 = 0$. The 2-face κ_2 has the same shape as in order type $(1,1,2)$. The description of the points and bisecant edges meeting C_2 is also the same. By Lemma 4.5 and the preceding observation, every point of C_1 is exposed, and all lie on a unique bisecant edge except q_1, which lies on the two nonexposed bisecant edges $\overline{p_i,q_1}$.

Order Type $(1,2)$: This is the most complicated. Each circle is tangent to the plane of the other, sharing a tangent line, and the description is symmetric in the indices 1 and 2. The 2-faces are the sections κ_1 and κ_2, with the description

Table 5 Face lattices

Order type	0-faces	1-faces	2-faces
$\varnothing$	Points on $C_1 \cup C_2$	One family parametrized by C_1	D_1, D_2
$(1, 1)$	Points on C_1 and points on an arc of C_2	One family parametrized by C_1	D_1
$(1, 2, 2, 1)$	Points on C_1 and points on two arcs of C_2	Two families parametrized by C_1	None
$(1, 1, 2, 2)$	Points on an arc of C_1 and an arc of C_2	One family parametrized by a 2-fold branched cover of an arc	None
$(1, 2, 1, 2)$	Same as order type $(1, 1, 2, 2)$	Same as order type $(1, 1, 2, 2)$	Same as order type $(1, 1, 2, 2)$
$(1, S, 2)$	Same as order type $(1, 1, 2, 2)$	Same as order type $(1, 1, 2, 2)$	Same as order type $(1, 1, 2, 2)$
$(1, 2, S)$	Points on C_1 and an arc of C_2	Two families parametrized by $C_1 \setminus C_2$	None
(S, S)	Points on $C_1 \cup C_2$	Four families with two parametrized by each arc $C_1 \setminus C_2$	None
$(2c)$	Same as order type $\varnothing$	Same as order type $\varnothing$	Same as order type $\varnothing$
$(1, S)$	Points on C_1 and an arc of C_2	One family parametrized by $C_1 \setminus C_2$	D_1
$(1, 1, 2)$	Points on an arc of C_1 and an arc of C_2	One family parametrized by the arc of C_1	$\mathrm{conv}(D_1, p_2)$
$(1, 2, 1)$	Same as order type $(1, 1)$	Same as order type $(1, 1)$	Same as order type $(1, 1)$
(1)	Points on C_1 and an arc of C_2	One family parametrized by the arc on C_2	D_1, $\mathrm{conv}(D_2, p_1)$
$(1, 2)$	Points on an arc of C_1 and an arc of C_2	One family parametrized by either arc, and an isolated bisecant $\overline{p_1, p_2}$	$\mathrm{conv}(D_1, p_2)$, $\mathrm{conv}(D_2, p_1)$
(S)	Points on $C_1 \cup C_2$	One family parametrized by either circle except the common point	D_1, D_2

for each is nearly the same as for κ_1 in order type $(1, 1, 2)$. The exception is the bisecant edge $\overline{p_1, q_1}$ lying along the shared tangent. This is exposed, but neither endpoint is exposed. It is also isolated from the other bisecant edges, which form a continuous family.

Order Type (S): The two circles are mutually tangent at a point p. The 2-faces are D_1 and D_2, every point of either circle is extreme, including p, and each (except for p) lies on one bisecant edge. □

Table 5 summarizes the face lattices by order type. In this table, when $m_i = 1$, the point where C_i is tangent to the plane of the other circle is denoted by p_i.

By [17], the convex hull K is a spectrahedral shadow. We use our classification to describe when K is a spectrahedron.

Lemma 4.7 *Let $C_1 \in \mathbb{P}^3$ and $C_2 \in \mathbb{P}^3$ be conics in distinct planes Π_1 and Π_2. If $C_1 \cap \Pi_2 = C_2 \cap \Pi_1$, then the conics C_1 and C_2 lie on a pencil of quadrics.*

Proof Since C_1, C_2 lie on the singular quadric $\Pi_1 \cup \Pi_2$, we need only find a second quadric containing them. Choose coordinates $[x : y : z : w]$ for $\mathbb{P}^3$ so that Π_1 is defined by $w = 0$ and Π_2 by $z = 0$. Then C_1 and C_2 are given by homogeneous quadratic polynomials $f(x, y, z) = 0$ and $g(x, y, w) = 0$. Since $C_1 \cap \Pi_2 = \Pi_1 \cap C_2$, the forms $f(x, y, 0)$ and $g(x, y, 0)$ define the same scheme, so they are proportional. Scaling g if necessary, we may assume that $f(x, y, 0) = g(x, y, 0)$. Define $h(x, y, z, w)$ to be $f(x, y, z) + g(x, y, w) - f(x, y, 0)$. It follows that $h(x, y, z, 0) = f(x, y, z)$ and $h(x, y, 0, w) = g(x, y, w)$, and thus C_1 and C_2 lie on the quadric defined by h. □

Theorem 4.8 *The convex hull of two circles C_1 and C_2 lying in distinct planes in $\mathbb{R}^3$ is a spectrahedron only if they have order type (S, S) or $(2c)$ or (S).*

Proof We have that $C_1 \cap \Pi_2 = C_2 \cap \Pi_1$ in $\mathbb{P}^3$ if and only if the circles have order type (SS) or $(2c)$ or (S). By Lemma 4.7, C_1 and C_2 lie on a pencil $Q_1 + tQ_2$ of quadrics. Following Example 2.3 in [14], this pencil of quadrics contains singular quadrics given by the real roots of $\det(Q_1 + tQ_2)$. Such a singular quadric is given by the determinant of a 2×2 matrix polynomial $Ax + By + Cz + D$, and the block diagonal matrix with blocks A, B, C, and D represents $\mathrm{conv}(C)$ as a spectrahedron.

By Corollary 4.4, K has a nonexposed face when a tangent line to one circle meets the other circle in a different point. This occurs for all the remaining order types of the circles C_1 and C_2, except type $\varnothing$ where $C_1 \cap C_2 = \varnothing$ in $\mathbb{P}^3$. In this case, the edge curve is irreducible with two connected real components and the edge surface meets the interior of $\mathrm{conv}(C)$ (as there are internal stationary bisecants). Thus, $\mathrm{conv}(C)$ is not a basic semialgebraic set and thus not a spectrahedron. □

5 Convex Hulls Through Duality

We sketch an alternative approach to studying the convex hull K of two circles that uses projective duality. This is inspired by the paper [6] and accompanying video [7] that explains a solution to the problem of determining the convex hull of three ellipsoids in $\mathbb{R}^3$.

Points $\check{\Pi}$ of the dual projective space $\check{\mathbb{P}}^3$ correspond to planes Π of the primal space $\mathbb{P}^3$. A line $\check{\ell}$ represents the pencil of planes containing a fixed line $\ell \subset \mathbb{P}^3$, and a plane $\check{o}$ represents the net of planes incident on a point $o \in \mathbb{P}^3$. The dual $\check{C} \subset \check{\mathbb{P}}^3$ of a conic $C \subset \mathbb{P}^3$ is the set of planes that contain a line tangent to C.

Lemma 5.1 *The dual $\check{C}$ to a conic C is a quadratic cone in $\check{\mathbb{P}}^3$ with vertex $\check{\Pi}$ corresponding to the plane Π of C.*

Proof The pencil of planes containing the tangent line T_pC to C is a line lying on $\check{C}$ that meets $\check{\Pi}$ as $T_pC \subset \Pi$. Thus, $\check{C}$ is a cone in $\check{\mathbb{P}}^3$ with vertex $\check{\Pi}$. Let $o \in \mathbb{P}^3$ be any point that is not on Π. Then the curve $\check{o} \cap \check{C}$ is the set of planes through o that contain a tangent line T_pC to C. As there are two such planes that contain a general line ℓ through o (ℓ meets two tangents to C), the curve $\check{o} \cap \check{C}$ is a conic in $\check{o}$ and $\check{C}$ is the cone over that conic with vertex $\check{\Pi}$. □

Let C_1, C_2 be circles in $\mathbb{R}^3 \subset \mathbb{RP}^3$ lying in distinct planes Π_1, Π_2 and let K be the convex hull of $C_1 \cup C_2$. Let o be any point in the interior of K. We will consider the plane $\check{o} \subset \check{\mathbb{RP}}^3$ to be the plane at infinity and set $\check{\mathbb{R}}^3 := \check{\mathbb{RP}}^3 \smallsetminus \check{o}$. This is an affine space that contains every plane supporting K as well as all those disjoint from K, as every plane incident on o meets the interior of K. It also contains the point $\check{\infty}$ corresponding to the plane at infinity in $\mathbb{RP}^3$.

For $i = 1, 2$, let $\check{C}_i$ be the cone in $\check{\mathbb{R}}^3$ dual to the conic C_i. If $o \in \Pi_i$, then the vertex $\check{\Pi}_i$ of $\check{C}_i$ lies at infinity ($\check{\Pi}_i \in \check{o}$) and $\check{C}_i$ is a cylinder. Neither dual cone contains the point $\check{\infty}$. Let $\check{K}$ be the closure of the component of $\check{\mathbb{R}}^3 \smallsetminus \check{C}_1 \smallsetminus \check{C}_2$ containing $\check{\infty}$.

Proposition 5.2 *Points $\check{\Pi}$ in the interior of $\check{K}$ are exactly those whose corresponding plane Π is disjoint from K. Points of the boundary of $\check{K}$ correspond to supporting planes of K, and $\check{K}$ is convex and bounded.*

We present an elementary proof of this standard result about convex sets in $\mathbb{R}^d$.

Proof Choose coordinates (x, y, z) for $\mathbb{R}^3$ so that $o = (0, 0, 0)$ is the origin. An affine plane is defined by the vanishing of an affine form $\Lambda := ax + by + cz + d$, whose coefficients $[a : b : c : d]$ give homogeneous coordinates for $\check{\mathbb{RP}}^3$. In these coordinates, $\check{\infty}$ is the point $[0 : 0 : 0 : 1]$, $\check{o}$ has equation $d = 0$, and the points of the affine $\check{\mathbb{R}}^3$ have coordinates $[a : b : c : 1]$, so that $\check{\infty}$ is the origin in $\check{\mathbb{R}}^3$.

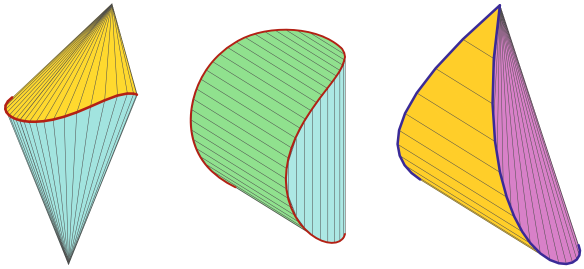
Let $v = (\alpha, \beta, \gamma) \in \mathbb{R}^3 \smallsetminus \{(0, 0, 0)\}$ and consider the linear map $\Lambda_v \colon \mathbb{R}^3 \to \mathbb{R}$ defined by $\Lambda_v(x, y, z) := \alpha x + \beta y + \gamma z$. Since $\Lambda_v^{-1}(0)$ is a plane containing the origin o, $\Lambda_v(K)$ is a closed interval $[\epsilon, \delta]$ with 0 in its interior, so that $\epsilon < 0 < \delta$. Thus, the points $\Lambda_{v,t} \colon [t\alpha : t\beta : t\gamma : 1]$ for $-\frac{1}{\delta} < t < -\frac{1}{\epsilon}$ of $\check{\mathbb{RP}}^3$ are exactly the planes in $\mathbb{RP}^3$ parallel to $\Lambda_v^{-1}(0)$ that are disjoint from K as $\Lambda_{v,t}(K) \subset (0, \infty)$ for $-\frac{1}{\delta} < t < -\frac{1}{\epsilon}$.

All other planes parallel to $\Lambda_v^{-1}(0)$ meet K, with $\Lambda_{v,-1/\delta}$ and $\Lambda_{v,-1/\epsilon}$ the planes in this family that support K. These supporting planes necessarily lie on $\check{C}_1 \cup \check{C}_2$. Hence, the interior of $\check{K}$ is exactly the set of all planes disjoint from K and its boundary is exactly the set of planes supporting K.

As o lies in the interior of K, there is a closed ball centred at o of radius $1/\rho$ contained in the interior of K. For any unit vector v, the numbers ϵ, δ defined by $\Lambda_v(K) = [\epsilon, \delta]$ satisfy $|1/\epsilon|, |1/\delta| < \rho$. Thus, the coordinates of points $[\alpha : \beta : \gamma : 1]$ in $\check{K}$ satisfy $\|(\alpha, \beta, \gamma)\| < \rho$, proving that $\check{K}$ is bounded.

Let $\Lambda = [a : b : c : 1]$ and $\Lambda' = [a' : b' : c' : 1]$ be points of $\check{K}$. It follows that $\Lambda(K), \Lambda'(K) \subset [0, \infty)$. For all $t \in [0, 1]$, set $\Lambda_t := t\Lambda + (1 - t)\Lambda'$. Since $[0, \infty)$ is convex, we see that $\Lambda_t(K) \subset [0, \infty)$ and $\Lambda_t \in \check{K}$. This proves that $\check{K}$ is convex. □

Fig. 8 Duals to convex hulls of two circles are intersections of two cylinders



Points in the boundary ∂K of $\check{K}$ are planes supporting K, and faces of $\check{K}$ correspond to exposed faces of K. For example, $\check{\Pi}_i \in \partial\check{K}$ if and only if the plane Π_i of C_i supports a two-dimensional face of K. Points of the curve in $\partial\check{K}$ where the cones $\check{C}_1$ and $\check{C}_2$ meet correspond to stationary bisecants, and line segments in the ruling of $\check{C}_i$ lying in $\partial\check{K}$ correspond to the exposed points of C_i in K. This may be seen in Fig. 8, which shows the dual bodies to the convex hulls of Fig. 1. For these, the origin o is the midpoint of the segment joining the centres of the circles.

The intersection of two cones on the left has cone points corresponding to the planes of the discs in the boundary of the convex set on the left in Fig. 1. In the centre is the dual of the oloid. The origin o is in the interior of the discs of the circles, so both cones $\check{C}_i$ are elliptical cylinders. On the right is the intersection of a cone with a horizontal cylinder meeting its vertex. The cylinder is dual to the vertical circle in the rightmost convex set in Fig. 1. The vertex is the two-dimensional face, and the two branches of the intersection curve at the vertex of the cone have limit the two nonexposed stationary bisecants.

In [6], the authors sketch an exact algorithm (beautifully explained in [7]) to compute the convex hull of three ellipsoids P, Q, and R in $\mathbb{R}^3$. Their approach inspired the previous discussion.

If the origin o lies in the interior of an ellipsoid P, then its dual $\check{P}$ is also an ellipsoid. If o lies on P, then its dual is a paraboloid and $\check{\infty}$ lies in the convex component of its complement. If o is exterior to P, then its dual is a hyperboloid of two sheets, and one of the convex components of its complement contains $\check{\infty}$.

Choosing an origin o in the interior of the convex hull K of $P \cup Q \cup R$ as in Proposition 5.2, $\check{K}$ is a bounded convex set that is the closure of the region in the complement of the duals containing the origin $\check{\infty}$. The video [7] describes the algorithm to compute K when the origin o lies in the interior of all three ellipsoids. In that case, the dual $\check{K}$ of the convex hull of the three ellipsoids is the intersection of the three dual ellipsoids $\check{P} \cap \check{Q} \cap \check{R}$. Computing $\check{K}$ requires the computation of the curves where two dual ellipsoids intersect, and points where three dual ellipsoids meet, and then decomposing the dual ellipsoids along these curves into patches.

This analysis gives three types of points in the boundary of $\check{K}$.

1. Points common to all three dual ellipsoids. These give tritangent planes in ∂K.
2. Points on curves given by the pairwise intersection of dual ellipsoids. They are bitangent planes and give bitangent edges. These form one-dimensional families of 1-faces in ∂K.
3. Points on a single dual ellipsoid. These are tangent planes to an ellipsoid at a point of K, and give a two-dimensional family of exposed points of K coming from the corresponding ellipsoid.

As we see in Fig. 8, the dual $\check{K}$ eloquently displays information about the exposed faces of K, but information about the nonexposed faces is less clear in $\check{K}$.

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The Hilbert Scheme of 11 Points in $\mathbb{A}^3$ is Irreducible

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Abstract We prove that the Hilbert scheme of 11 points on a smooth threefold is irreducible. In the course of the proof, we present several known and new techniques for producing curves on the Hilbert scheme.

MSC 2010 codes: 14C05, 14D15

1 Introduction

Let X be a smooth connected quasi-projective variety. The Hilbert scheme of d points in X is the scheme parametrizing finite subschemes of X of degree d; for an introduction see [17, 18, 20, 26, 30, 37, 38]. The Hilbert scheme of points is quasi-projective (projective if and only if X is) and connected; see [18]. Moreover, Fogarty [18] proves that, for $\dim X \leq 2$, it is smooth of dimension $d \dim X$. For higher-dimensional X, much less is known. The question of irreducibility for the Hilbert scheme of points is especially interesting because it ensures that all finite schemes are limits of reduced ones; see [4] for an application. This question is local

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and only depends on the dimension of X: the answer for n-dimensional X will be the same as for $\mathbb{A}^n$, see [1, p. 4] or [10, Lemma 2.2]. We denote the Hilbert scheme of d points in $\mathbb{A}^n$ by H_n^d. Our motivating question is the following: *For which pairs* (n, d) *is the Hilbert scheme* H_n^d *irreducible?*

By Fogarty's results, all H_2^d are irreducible. Mazzola [36] proves irreducibility of the Hilbert scheme H_n^d, for all n and $d \leq 7$. Iarrobino [27, 28] shows that, for all $n \geq 3$ and $d \geq 78$, the Hilbert scheme H_n^d is reducible. Emsalem and Iarrobino prove that H_n^d is reducible for $n \geq 4$ and $d \geq 8$; see [29, Sect. 2.2, p. 158] and [8]. Finally, Borges dos Santos, Henni, and Jardim [2] show that H_3^9 and H_3^{10} are irreducible, by comparing them with appropriate spaces of commuting matrices and using the results of Šivic [40, Theorems 26,32]. These papers do not determine the reducibility of H_n^d for $n = 3$ and $11 \leq d \leq 77$. In this article, we improve the lower bound.

Theorem 1.1 *The Hilbert scheme of* 11 *points in a smooth irreducible threefold is irreducible of dimension* 33.

We prove Theorem 1.1 in Sect. 4. We review background information in Sect. 2. In Sect. 3, we give an overview of strategy, gather general results for the proof of the Theorem 1.1, and demonstrate how to use *Macaulay2* [22] for some computations.

In Sect. 5, we discuss a special class of subschemes, coming from the first example of reducible H_3^d by Iarrobino [27]. Let $\mathfrak{m}$ be the ideal of the origin of $\mathbb{A}^3$, fix d, and consider an ideal I such that $\mathfrak{m}^{s+1} \subset I \subset \mathfrak{m}^s$ and the variety $\mathrm{V}(I)$ has degree d; the nonnegative integer s is uniquely determined. We call such ideals *very compressed* and denote by $\mathscr{H}^{\max,d}$ the subset of H_3^d corresponding to these ideals. The closure in H_3^d of the open set of smooth subschemes is the *smoothable component* R_3^d of H_3^d, and has dimension $3d$. The key result in [27] establishes that, for $d \geq 96$, we have $\dim \mathscr{H}^{\max,d} \geq 3d$, so a general very compressed ideal does not lie in the smoothable component. We prove that this result is sharp.

Proposition 1.2 *The family* $\mathscr{H}^{d,\max}$ *of very compressed ideals is contained in the smoothable component if and only if* $d \leq 95$.

The proof depends on initial ideals and a *Macaulay2* calculation; see Sect. 5.

We now outline our approach to the proof of Theorem 1.1, which builds upon the strategy in [8]. Questions about smoothability of a specified ideal I reduce to the case where I is local and has embedding dimension 3. There are 15 possible Hilbert functions of I:

$(1, 3, 1, 1, 1, 1, 1, 1, 1)$	$(1, 3, 5, 1, 1)$	$(1, 3, 2, 2, 2, 1)$
$(1, 3, 2, 1, 1, 1, 1, 1)$	$(1, 3, 3, 4)$	$(1, 3, 3, 2, 2)$
$(1, 3, 2, 2, 1, 1, 1)$	$(1, 3, 4, 3)$	$(1, 3, 3, 3, 1)$
$(1, 3, 3, 1, 1, 1, 1)$	$(1, 3, 5, 2)$	$(1, 3, 3, 2, 1, 1)$
$(1, 3, 4, 1, 1, 1)$	$(1, 3, 4, 2, 1)$	$(1, 3, 6, 1)$

For each Hilbert function $\mathbf{h}$, let $H_3^{\mathbf{h}}$ denote the scheme parametrizing local ideals with fixed Hilbert function $\mathbf{h}$, and let $\mathscr{H}_3^{\mathbf{h}}$ be the *standard graded Hilbert scheme* parametrizing homogeneous ideals with fixed Hilbert function $\mathbf{h}$. We use three different strategies to prove that, for each of these 15 functions, we have $H_3^{\mathbf{h}} \subset R_3^{11}$.

First, for some cases, knowledge about the Hilbert function of an ideal I is enough to produce a deformation (via the *ray families* in [9]), whose special fibre is I and general fibre is reducible. By Lemma 1.4, such an I is smoothable; see Sect. 4.

Second, most of the schemes $H_3^{\mathbf{h}}$ contain smooth points of the Hilbert scheme that lie in the smoothable component R_3^{11}. Such points are called *smooth and smoothable points* and examples include points corresponding to Gorenstein algebras; see [10, Corollary 2.6].

Lemma 1.3 *If $Z \subseteq H_3^{11}$ is an irreducible set that contains a smooth and smoothable point, then we have $Z \subseteq R_3^{11}$.*

Proof The locus of smooth and smoothable points is open and contained in R_3^{11}, so the intersection $Z \cap R_3^{11}$ contains an open subset of Z. Hence, the subset $Z \cap R_3^{11} \subset Z$ is dense and closed, so it is equal to Z. □

To exploit this lemma, we write $H_3^{\mathbf{h}}$ as a union of irreducible sets Z and demonstrate that each Z contains a smooth and smoothable point. To find the sets Z, we may take advantage of the morphism $\pi_{\mathbf{h}}\colon H_3^{\mathbf{h}} \to \mathscr{H}_3^{\mathbf{h}}$ sending an ideal I to its initial ideal; see [8]. We employ the following 3-step strategy:

1. Decompose $\mathscr{H}_3^{\mathbf{h}}$ into irreducible strata.
2. Using the morphism $\pi_{\mathbf{h}}\colon H_3^{\mathbf{h}} \to \mathscr{H}_3^{\mathbf{h}}$, decompose $H_3^{\mathbf{h}}$ into irreducible strata.
3. For each stratum of $H_3^{\mathbf{h}}$, find a smooth point of the Hilbert scheme that lies in the smoothable component and conclude that the whole stratum lies there.

In steps 1–2, we use inverse systems; see Sect. 2. In the simplest cases, we find that there is a bijection between irreducible strata of $\mathscr{H}_3^{\mathbf{h}}$ and $H_3^{\mathbf{h}}$, but this is not always true. For step 3, we introduce *cleavable* ideals. An ideal is *cleavable* if it can be deformed to an ideal whose support consists of at least two points.

Lemma 1.4 *A cleavable ideal $I \in H_3^{11}$ is smoothable.*

Proof Let I_t be a one-parameter flat family of ideals such that $I_0 = I$ and, for $t \neq 0$, I_t is supported at more than one point. Each irreducible component of I_t has length strictly less than 11, so it is smoothable. Hence, the ideal I is also smoothable. □

To show that an ideal I is cleavable, we construct a family over $\operatorname{Spec} \mathbb{k}[t]$, whose general fibre is reducible, and check that this family is flat.

Third, there is one case where neither of the previous methods apply, namely $\mathbf{h} = (1, 3, 6, 1)$; see Proposition 4.22. The stratum $H_3^{\mathbf{h}}$ does not seem to contain smooth points. However, the stratum is irreducible and we can describe what general points look like. We build a deformation establishing that such general points are smoothable, so irreducibility implies that the entire stratum has to be smoothable.

Throughout, we work over an algebraically closed field $\mathbb{k}$ of characteristic zero.

2 Prerequisites

Hilbert Schemes and Smoothability The *Hilbert scheme* H_n^d parametrizes subschemes of $\mathbb{A}^n$ of dimension zero and degree d. More formally, H_n^d represents the functor which assigns, to each $\Bbbk$-scheme X, the set of subschemes of $\mathbb{A}^n \times X$ that are flat and finite of degree d over X; see [26, Chapter 1]. Equivalently, letting $T := \Bbbk[\alpha_1, \alpha_2, \dots, \alpha_n]$, the scheme H_n^d parametrizes ideals I for which T/I is a vector space of dimension d. In other words, H_n^d also represents the functor which assigns, to each $\Bbbk$-algebra A, the set of ideals I in $T \otimes A$ such that the quotients $T \otimes A/I$ are locally-free A-modules of rank d.

The Zariski tangent space to H_n^d at the point representing I is the T-module $\operatorname{Hom}(I, T/I)$; see [26, Theorem 1.1]. Using *Macaulay2* [22], we can compute the dimension of this tangent space. We stress that a point is smooth if and only if the point lies on only one irreducible component of the scheme and the dimension of the tangent space at that point equals the dimension of the component of the scheme containing the point. The dimension of the tangent space increases at singular points.

On the Hilbert scheme H_n^d, there is a distinguished component corresponding to smooth schemes. Since a slightly perturbed tuple of d closed points in $\mathbb{A}^n$ is just another such tuple, the set of tuples of points is open in H_n^d. Their closure is the *smoothable component* of H_n^d and denoted by R_n^d. By construction, R_n^d is generically smooth of dimension nd. Since H_2^d is smooth, we have $R_2^d = H_2^d$.

A point of R_n^d is said to be *smoothable*. Thus, an ideal I is smoothable if and only if it can be deformed to an ideal of d distinct points. This means that one can build a one-parameter flat family of schemes over a discrete valuation ring for which the general member consists of d distinct points and the special fibre is T/I; see [6, 8] for the details. In particular, a disjoint union of smoothable schemes is smoothable and a limit of smoothable schemes is smoothable.

Hilbert Functions To analyze H_n^d, it is useful to have an invariant that refines the degree d. There are two closely-related notions of a *Hilbert function*:

- For an $\mathbb{N}$-graded T-module M, its Hilbert function $\mathbf{h}\colon \mathbb{N} \to \mathbb{N}$ is defined by $\mathbf{h}(i) := \dim M_i$. Given a homogeneous ideal $I \subset T$, we consider the Hilbert function of the quotient ring T/I.
- For a filtered T-module M with descending filtration $M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots$, the Hilbert function $\mathbf{h}\colon \mathbb{N} \to \mathbb{N}$ is defined by $\mathbf{h}(i) := \dim(M_i/M_{i+1})$. If the scheme associated to an ideal $I \subset T$ is supported at a point, then T/I is a local ring $(A, \mathfrak{m})$, and the Hilbert function $\mathbf{h}$ with respect to the filtration by powers of $\mathfrak{m}$ is defined to be $\mathbf{h}(i) := \dim(\mathfrak{m}^i/\mathfrak{m}^{i+1})$.

If I is homogeneous and T/I is local, the two notions coincide. We typically write $\mathbf{h}$ as the vector $\big(\mathbf{h}(0), \mathbf{h}(1), \dots\big)$ in which the trailing zeros are omitted.

Consider $A := T/I$ where $T = \Bbbk[\alpha_1, \alpha_2, \dots, \alpha_n]$ is a polynomial ring with its standard $\mathbb{N}$-grading and I is a homogeneous ideal. Assume that I contains no linear forms. We call such an algebra *standard graded*. The Macaulay bound is an upper bound for the growth of Hilbert functions of standard graded algebras.

More precisely, for any positive integers h and d, there exist unique integers $\delta, k_d, k_{d-1}, \dots, k_\delta \in \mathbb{N}$ such that $\delta \geq 1$, $k_d > k_{d-1} > \cdots > k_\delta \geq \delta$, and $h = \binom{k_d}{d} + \binom{k_{d-1}}{d-1} + \cdots + \binom{k_\delta}{\delta}$. This expression, denoted by $h_{(d)}$, is called the *d-binomial expansion* of h. The d-binomial expansion of h can be determined via a greedy algorithm: choose k_d to be the greatest integer such that $\binom{k_d}{d} \leq h$ and recursively compute the $(d-1)$-binomial expansion of $h - \binom{k_d}{d}$. If $h_{(d)} = \binom{k_d}{d} + \binom{k_{d-1}}{d-1} + \cdots + \binom{k_\delta}{\delta}$, then the function $h \mapsto h^{\langle d \rangle}$ is defined by

$$h^{\langle d \rangle} := \binom{k_d + 1}{d + 1} + \binom{k_{d-1} + 1}{d} + \cdots + \binom{k_\delta + 1}{\delta + 1}.$$

Example 2.1 The 2-binomial expansion of 5 and 4 are simply $5_{(2)} = \binom{3}{2} + \binom{2}{1}$ and $4_{(2)} = \binom{3}{2} + \binom{1}{1}$, so we have $5^{\langle 2 \rangle} = \binom{4}{3} + \binom{3}{2} = 7$ and $4^{\langle 2 \rangle} = \binom{4}{3} + \binom{2}{2} = 5$.

Example 2.2 If $h \leq d$, then we have $h_{(d)} = \binom{d}{d} + \binom{d-1}{d-1} + \cdots + \binom{d-h+1}{d-h+1}$ and $h^{\langle d \rangle} = h$.

Theorem 2.3 (Macaulay Bound; [34] or [3, Theorem 4.2.10]) *If* $\mathbf{h}$ *is the Hilbert function of a standard graded algebra, then we have* $\mathbf{h}(d + 1) \leq \mathbf{h}(d)^{\langle d \rangle}$ *for all* $d \in \mathbb{N}$.

Corollary 2.4 *Let* $\mathbf{h}$ *be the Hilbert function of a standard graded algebra. If* $d \in \mathbb{N}$ *satisfies* $\mathbf{h}(d) \leq d$*, then we have* $\mathbf{h}(i) \geq \mathbf{h}(i + 1)$ *for all* $i \geq d$.

Once the Macaulay bound is attained then it will also be attained for all higher degrees provided that no new generators of the ideal appear.

Theorem 2.5 (Gotzmann Persistence Theorem; [21] or [3, Theorem 4.3.3]) *Let* $\mathbf{h}$ *be the Hilbert function of a standard graded algebra* T/I*. If* $d \in \mathbb{N}$ *satisfies* $\mathbf{h}(d + 1) = \mathbf{h}(d)^{\langle d \rangle}$ *and the homogeneous ideal* I *is generated in degrees at most* d*, then we have* $\mathbf{h}(i + 1) = \mathbf{h}(i)^{\langle i \rangle}$*, for all* $i \geq d$.

Apolarity and Inverse Systems A key tool in the analysis of finite schemes is the technique of *Macaulay's inverse systems*, also known as apolarity; see [16, 19, 39] and [30, Sect. 5.1.3]. Let $S := \Bbbk[x_1, x_2, \dots, x_n]$ and $T = \Bbbk[\alpha_1, \alpha_2, \dots, \alpha_n]$ be polynomial rings with the standard grading. When $n \leq 3$, we replace the subscripted variables with x, y, z and α, β, γ respectively. For $d \in \mathbb{N}$, set $S_{\leq d} := \bigoplus_{k=0}^{d} S_k$ and $T_{\leq d} := \bigoplus_{k=0}^{d} T_k$. The polynomial ring T acts on S by letting α_i act as partial differentiation with respect to x_i. We denote this action by $\circ$, so $\alpha_i \circ F = \frac{\partial F}{\partial x_i}$ for all $F \in S$. In other words, this apolarity action gives bilinear maps $T_d \times S_e \to S_{e-d}$ for all d, e and, for each $d \in \mathbb{N}$, the pairing $T_d \times S_d \to S_0 = \Bbbk$ is perfect.

Definition 2.6 For any subset $J \subset S$, the *apolar ideal* or *annihilating ideal* $J^\perp \subset T$ is the ideal of elements $\Theta \in T$ such that $\Theta \circ F = 0$, for all $F \in J$. For a single element $F \in S$, we simply write $F^\perp$ for the apolar ideal of the singleton $\{F\}$. When J is spanned by homogeneous elements, the apolar ideal is also homogeneous. When J is spanned by a single element F, then $F^\perp$ is a *Gorenstein* ideal; see [13, Sect. 21.2].

Example 2.7 If $F = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$, then we claim $F^\perp = (\alpha_1^{a_1+1}, \alpha_2^{a_2+1}, \ldots, \alpha_n^{a_n+1})$. Indeed, we easily see that $\alpha_i^{a_i+1} \in F^\perp$, for $1 \le i \le n$. Conversely, if the polynomial $\Theta \in T$ contains a monomial $\alpha_1^{b_1} \alpha_2^{b_2} \cdots \alpha_n^{b_n}$ such that $b_i \le a_i$ for $1 \le i \le n$, then the apolar pairing of this monomial with F is a monomial whose exponent vector is $(a_1 - b_1, a_2 - b_2, \ldots, a_n - b_n)$. Since each monomial in Θ corresponds to a distinct monomial in $\Theta \circ F$, there can be no cancellation. Thus, if $\Theta \in F^\perp$, then each monomial in Θ must lie in the indicated ideal.

For a given $F \in S$, the apolar ideal $F^\perp$ is just the kernel of the linear map $T \to S$ given by $\Theta \mapsto \Theta \circ F$. Hence, we can compute $J^\perp$ by intersecting the ideals $F^\perp$, for all $F \in J$. If J is a $\Bbbk$-vector space, then it is sufficient to consider a basis for J.

Example 2.8 For $F = x^3 + yz$, we have $F^\perp = (\alpha^3 - 6\beta\gamma, \alpha\beta, \alpha\gamma, \beta^2, \gamma^2)$.

Example 2.9 For $F = x^2y + y^2z$, we have $F^\perp = (\gamma^2, \alpha\gamma, \alpha^2 - \beta\gamma, \beta^3, \alpha\beta^2)$.

Definition 2.10 A *(Macaulay) inverse system* is a T-submodule $J \subseteq S$ that is closed under the apolarity action: for $F \in J$, the derivatives $\alpha_1 \circ F, \alpha_2 \circ F, \ldots, \alpha_n \circ F$ lie in J.
The inverse system *generated by* the elements $f_1, f_2, \ldots, f_s$ of S is the vector space spanned by the f_i and all of their higher partial derivatives:

$$\langle f_1, f_2, \ldots, f_s \rangle = Tf_1 + Tf_2 + \cdots + Tf_s\,.$$

It follows that $\langle f_1, f_2, \ldots, f_s \rangle^\perp = \bigcap_{i=1}^s \langle f_i \rangle^\perp = \bigcap_{i=1}^s f_i^\perp$. An inverse system is *homogeneous* if it is generated by homogeneous elements.

Remark 2.11 As Macaulay [35] or [16, Corollaire 2] establish, the map $J \mapsto J^\perp$ defines a bijection between finite-dimensional inverse systems and local ideals supported at the origin (also known as $\mathfrak{m}$-primary ideals where $\mathfrak{m}$ is the ideal of the origin). When I is a local ideal, we write $I^\perp$ for its inverse system.

Proposition 2.12 ([19, Remark after Proposition 2.5]) *If J is a homogeneous inverse system, then J is isomorphic as a graded $\Bbbk$-vector space to $T/J^\perp$.*

Proposition 2.13 ([16, Proposition 2(a)]) *For a finite-dimensional inverse system J, we have* $\dim_\Bbbk J = \dim_\Bbbk T/J^\perp$.

Proof Choose d sufficiently large so that $J \subseteq S_{\le d}$. Hence, the map $T_{\le d} \to T/J^\perp$ is surjective, and the dimensions are equal to the codimension of $J^\perp \cap T_{\le d}$ in $T_{\le d}$. □

Remark 2.14 Given $k \in \mathbb{N}$ and an inverse system J, the vector space of polynomials of degree at most k in J is denoted by $J_{\le k}$. These vector spaces form an increasing filtration: $J_{\le k} \subseteq J_{\le k+1}$ for all $k \ge \mathbb{N}$. The inverse system J is a filtered T-module and its Hilbert function $\mathbf{h}$ is $\mathbf{h}(k) = \dim J_{\le k} - \dim J_{\le k-1}$ for all $k \in \mathbb{N}$. Moreover, we have $\sum_{k \in \mathbb{N}} \mathbf{h}(k) = \dim_\Bbbk J$ and, when J is homogeneous, $\mathbf{h}(k) = \dim J_k$.

Proposition 2.15 ([30, Lemma 2.12]) *If $F \in S$ is homogeneous, then the Hilbert function of $F^\perp$ is symmetric; if $F \in S_d$, then $\mathbf{h}(i) = \mathbf{h}(d - i)$ for $0 \le i \le d$.*

Proposition 2.16 ([7]) *Suppose that $f \in S$ is a homogeneous form of degree d. Let $\mathbf{h}$ be the Hilbert function of $\langle f \rangle$. If $\mathbf{h}(d-1) = k$, that is $\mathbf{h} = (\dots, k, 1)$, then there are independent linear functions $\ell_1, \ell_2, \dots, \ell_k \in S_1$ and a homogeneous form g such that $f = g(\ell_1, \ell_2, \dots, \ell_k)$. Equivalently, there is a linear change of coordinates so that f depends only on the variables $x_1, x_2, \dots, x_k$, not on $x_{k+1}, x_{k+2}, \dots, x_n$.*

Remark 2.17 Using the above proposition, one can show that if $\langle f \rangle$ has Hilbert function $(\dots, 1, 1)$, then $f = \ell^d$ for some linear function ℓ and $\langle f \rangle$ has Hilbert function $(1, 1, \dots, 1, 1)$. If $\mathbf{h}(d-2) = \mathbf{h}(d-1) = 2$, then either $f = \ell^d + m^d$ or $f = \ell^{d-1}m$ for some independent linear functions $\ell, m \in S_1$, and either way $\langle f \rangle$ has Hilbert function $(1, 2, 2, \dots, 2, 2, 1)$. For proof see for example [30, Theorem 1.44]: in their notation, $s = 2$, and $f^\perp$ has a quadratic generator, which up to a change of coordinates is either $\alpha\beta$ or β^2.

Dealing with nonhomogeneous inverse systems is much harder than working with homogeneous ones. Fortunately, each inverse system J has an associated homogeneous inverse system $\operatorname{lead}(J)$.

Definition 2.18 The *leading form* of a polynomial is its highest degree homogeneous part. This may not be a monomial. For an inverse system $J \subset S$, the inverse system of leading forms of J, denoted $\operatorname{lead}(J)$, is the vector subspace of S spanned by leading forms of all the elements of J.

For example, the inverse system $\langle x^3 + y^2 \rangle = \operatorname{span}\{x^3 + y^2, x^2, x, y, 1\}$ has

$$\operatorname{lead}(\langle x^3 + y^2 \rangle) = \operatorname{span}\{x^3, x^2, x, y, 1\} = \langle x^3, y \rangle.$$

There is a tight connection between a system J and $\operatorname{lead}(J)$.

Proposition 2.19 *The Hilbert functions of J and* $\operatorname{lead}(J)$ *are equal.*

Proof (Sketch) Let $f_1, f_2, \dots, f_s$ be a vector space basis for $\operatorname{lead}(J)$ consisting of homogeneous elements and let $g_1, g_2, \dots, g_s \in J$ with $\operatorname{lead}(g_i) = f_i$. One can show the g_i are a basis for J. Expressing the Hilbert functions of J and $\operatorname{lead}(J)$ in terms of the g_i and f_i gives the result. □

The *initial form* or *lowest degree form* of a polynomial g_i is its lowest degree homogeneous part. The *initial ideal* of an ideal K, denoted $\operatorname{in}(K)$, is the ideal generated by the initial forms of all elements of K.

Proposition 2.20 ([16, Proposition 3]) *Let J be a finite-dimensional inverse system with ideal $J^\perp = I$. We have* $\operatorname{lead}(J)^\perp = \operatorname{in}(I)$. *In other words, $T/\operatorname{lead}(J)^\perp$ is the associated graded algebra of $T/J^\perp$.*

Proof If $\Theta \in \operatorname{in}(I)$, then $\Theta = \operatorname{in}(\Psi)$, for some $\Psi \in I$. To see that $\Theta \in \operatorname{lead}(J)^\perp$, let $F = \operatorname{lead}(G)$ for $G \in J$. It follows that $\Theta \circ F$ is the highest degree part of $\Psi \circ G = 0$, so it is zero. This shows that $\operatorname{in}(I) \subseteq \operatorname{lead}(J)^\perp$. We have

$$\dim_{\Bbbk} J = \dim_{\Bbbk} \operatorname{lead}(J) = \dim_{\Bbbk} T/\operatorname{lead}(J)^\perp \le \dim_{\Bbbk} T/\operatorname{in}(I) = \dim_{\Bbbk} T/I = \dim_{\Bbbk} J\,,$$

where the first equality is by Proposition 2.19 and the last is by Proposition 2.13. This completes the proof. □

Remark 2.21 By Proposition 2.19 and Proposition 2.20, the Hilbert function of an inverse system J is also the Hilbert function of a standard graded algebra, namely the associated graded algebra of $T/J^{\perp}$. Hence, Macaulay's and Gotzmann's theorems apply to these functions. This enables us to classify the possible Hilbert functions $\mathbf{h}$ of local ideals $I = J^{\perp}$ in H_3^{11} with $\mathbf{h}(1) = 3$. Since $\mathbf{h}(2) \leq 6$, we need to consider every possible value for $\mathbf{h}(2)$, $1 \leq \mathbf{h}(2) \leq 6$. Also, $\sum \mathbf{h}(i) = \dim_{\Bbbk} T/I = 11$. Finally, if $\mathbf{h}(i) \leq 2$ for any $i \geq 2$, then $\mathbf{h}$ is nonincreasing from the ith step onward, by Corollary 2.4. It is then easy to list the possible Hilbert functions.

Proposition 2.22 ([16, Sect. C.2]) *Let $F(t) = \{f_1(t), f_2(t), \ldots, f_s(t)\} \subset S[[t]]$ be a collection of polynomials in $S[[t]]$, which we regard as polynomials in S whose coefficients are continuous functions of a parameter t in a neighbourhood of* 0. *The family of apolar ideals $\{F(t)^{\perp}\}$ satisfies $\lim_{t\to 0} F(t)^{\perp} \subseteq F(0)^{\perp}$. If the inverse systems $\langle F(t)\rangle$ have the same Hilbert function for all t, then we have $\lim_{t\to 0} F(t)^{\perp} = F(0)^{\perp}$ and $\{F(t)^{\perp}\}$ is a flat family.*

Proof If $\Theta \in \lim_{t\to 0} F(t)^{\perp}$, write $\Theta = \Theta(0) = \lim_{t\to 0} \Theta(t)$ where $\Theta(t) \in F(t)^{\perp}$ for $t \neq 0$. For each $t \neq 0$ we then have that $\Theta(t) \circ f_i(t) = 0$, for $i = 1, \ldots, s$. By continuity, we also have that $\Theta(0) \circ f_i(0) = 0$. This shows $\Theta \in F(0)^{\perp}$ and $\lim_{t\to 0} F(t)^{\perp} \subseteq F(0)^{\perp}$. The equality of Hilbert functions implies equality of dimensions, so the ideals are equal. □

Definition 2.23 When $J_t = \langle f_1(t), f_2(t), \ldots, f_s(t)\rangle$ is a parametrized family of inverse systems generated by polynomials f_i whose coefficients are continuous functions of t, we will say $\lim_{t\to 0} J_t = J_0$ if and only if $\lim_{t\to 0} J_t^{\perp} = J_0^{\perp}$.

Example 2.24 Consider the families $W_1 = \{\langle \ell^d, m^d\rangle : \ell, m \in S_1, \text{ independent}\}$ and $W_2 = \{\langle \ell^d, \ell^{d-1}m\rangle : \ell, m \in S_1, \text{ independent}\}$. Since the limit

$$\lim_{t\to 0} \frac{(\ell + tm)^d - \ell^d}{dt} = \ell^{d-1}m,$$

we have, by Proposition 2.22, that

$$\lim_{t\to 0} \langle \ell^d, (\ell + tm)^d\rangle = \lim_{t\to 0} \left\langle \ell^d, \frac{(\ell + tm)^d - \ell^d}{dt} \right\rangle = \langle \ell^d, \ell^{d-1}m\rangle.$$

This is because every inverse system in each family has Hilbert function $(1, 2, 2, \ldots, 2)$. This implies that W_2 is in the closure of W_1 in the Zariski topology.

3 The Hilbert Scheme of 11 Points in 3-space

In this section, we use *Macaulay2* to perform some required computations and gather some general methods.

***Macaulay2* Code** To check if an ideal I in $T = \Bbbk[a,b,c]$ is smooth, we use the following code. In particular, this example is needed for the proof of Proposition 4.16.

```
i1 : T = QQ[a,b,c];

i2 : I = ideal(b*c, a*b, a^2*c, a^3-c^2, b^5);

o2 : Ideal of T

i3 : (dim I, degree I, degree Hom(I,T/I))

o3 = (0, 11, 33)

o3 : Sequence
```

This shows that the ideal I defines a zero-dimensional scheme of degree 11 with tangent space dimension 33. If this ideal corresponds to a point in the smoothable component, then it has to be a smooth point, because the smoothable component has dimension $(3)(11) = 33$. To demonstrate that this point lies in the smoothable component, we construct a deformation. We guess a candidate ideal K and check that it satisfies the required conditions.

```
i4 : R = T[t];

i5 : K = ideal (b*c, a*b, a^2*c, a^3-c^2, b^5+t*b^4);

o5 : Ideal of R

i6 : assert (K:t == K)

i7 : minimalPrimes K

o7 = {ideal (c, a, t + b), ideal (c, b, a)}

o7 : List
```

Regarding K as a family of ideals over $\mathbb{Q}[t]$, we see that its fibre over $t = 0$ is I. Proposition III.9.7 in [25] implies that this is a flat family in a neighbourhood of 0. Since the general fibre is supported at the two points $(0, -t, 0)$ and $(0, 0, 0)$, we see that the special fibre is cleavable. Thus, Lemma 1.4 proves that I is also smoothable.

Some General Methods We now collect various results for use in Sect. 4. In our analysis of the irreducible components of some standard graded Hilbert scheme (and the fibres of $\pi_{\mathbf{h}}$), we often consider the set of quadric generators $\{q_1, q_2, \ldots, q_k\}$ of a homogeneous ideal $I \subset T$. The next lemma describes the space of cubics $\langle q_1, q_2, \ldots, q_k\rangle \cdot T_1$ in the ideal generated by these quadrics.

Lemma 3.1 *If the ideal* $I = (q_1, q_2, \ldots, q_k) \subset T = \Bbbk[\alpha_1, \alpha_2, \ldots, \alpha_n]$*, where* $2 \leq k \leq n$*, is generated by linearly independent quadrics, then we have* $\dim I_3 \geq nk - \binom{k}{2}$ *and equality holds if and only if the quadrics share a common linear factor.*

Proof Let $\mathbf{h}\colon \mathbb{N} \to \mathbb{N}$ be the Hilbert function of the quotient ring T/I. Since the 2-binomial expansion of $\mathbf{h}(2)$ is $\mathbf{h}(2) = \binom{n+1}{2} - k = \binom{n}{2} + \binom{n-k}{1}$, the Macaulay Bound on Hilbert functions establishes that $\mathbf{h}(3) \leq \mathbf{h}(2)^{\langle 2 \rangle} = \binom{n+1}{3} + \binom{n-k+1}{2}$, so we obtain $\dim I_3 = \dim T_3 - \mathbf{h}(3) \geq \binom{n+2}{3} - \binom{n+1}{3} - \binom{n-k+1}{2} = nk - \binom{k}{2}$. We have the equality $\dim I_3 = nk - \binom{k}{2}$ if and only if $\mathbf{h}(3) = \mathbf{h}(2)^{\langle 2 \rangle}$. In this situation, the Gotzmann Persistence Theorem implies that $\mathbf{h}(t+1) = \mathbf{h}(t)^{\langle t \rangle}$, for all $t \geq 2$. It follows that

$$\mathbf{h}(t) = \binom{n+t-2}{t} + \binom{n-k+t-2}{t-1} = \binom{n+t-2}{n-2} + \binom{n-k+t-2}{n-k-1} = \tfrac{1}{(n-2)!} t^{n-2} + O(t^{n-3}),$$

is the Hilbert polynomial of the variety $\mathrm{V}(I) \subset \mathbb{P}^{n-1}$, so the variety $\mathrm{V}(I)$ has codimension 1 and degree 1; see [25, Sect. I.7, p. 52]. Hence, the variety $\mathrm{V}(I)$ consists of a reduced hyperplane, with possibly some lower-dimensional components. Since each q_i vanishes on this hyperplane, all of the quadrics are divisible by the linear equation defining the hyperplane. □

The next two lemmas demonstrate that it is relatively easy to determine the irreducible components of $H_n^{\mathbf{h}}$ when $\mathbf{h}$ equals the Hilbert function of the ambient ring for all but a few values.

Lemma 3.2 *If the Hilbert function* $\mathbf{h} = \big(1, \mathbf{h}(1), \mathbf{h}(2), \ldots, \mathbf{h}(t)\big)$ *satisfies* $\mathbf{h}(i) = \dim T_i$ *for* $1 \leq i \leq t-2$*, then the Hilbert scheme* $H_n^{\mathbf{h}}$ *is a vector bundle of rank* $\mathbf{h}(t)\big(\dim S_{t-1} - \mathbf{h}(t-1)\big)$ *over* $\mathscr{H}_n^{\mathbf{h}}$*. In particular, the irreducible components of* $H_n^{\mathbf{h}}$ *are exactly the preimages of the irreducible components of* $\mathscr{H}_n^{\mathbf{h}}$*.*

Proof A direct generalization of Proposition 4.3 in [8]. □

Lemma 3.3 *If the Hilbert function* $\mathbf{h} = \big(1, \mathbf{h}(1), \mathbf{h}(2), \ldots, \mathbf{h}(t)\big)$ *satisfies* $\mathbf{h}(i) = \dim T_i$ *for* $1 \leq i \leq t-3$*, then every fibre of* $\pi_{\mathbf{h}}$ *is isomorphic to an affine space.*

Proof Let I be a homogeneous ideal in $\mathscr{H}_n^{\mathbf{h}}$. The fibre $\pi_{\mathbf{h}}^{-1}(I)$ consists of the ideals I' for which $\mathrm{in}(I') = I$. We regard I' as a perturbation of I; the generators of I' are allowed to have additional terms when compared to the corresponding generators in I. Requiring that $\mathrm{in}(I') = I$ corresponds to allowing higher degree terms in generators of I. The requirement that the Hilbert function of T/I' equal $\mathbf{h}$ imposes conditions on the coefficients of these higher degree terms. Adding terms of degree greater than t has no effect as these are already contained in I. To any generator of degree $t-2$ or $t-1$, we can freely add terms of degree t because they cannot change the Hilbert function. To any generator q_i of degree $t-2$, we can add a term a_i of degree $t-1$ if it satisfies the following condition. For any tuple of linear forms $\ell_1, \ell_2, \ldots, \ell_r \in T_1$ such that $\ell_1 q_1 + \ell_2 q_2 + \cdots + \ell_r q_r = 0$, we require that $\ell_1 a_1 + \ell_2 q_2 + \cdots + \ell_r a_r \in I'_t = I_t$. Since these are all linear conditions on the coefficients of the a_i, the solution space is an affine space. Hence, the fibre at I is isomorphic to $\mathbb{A}^k$ for some k. □

Remark 3.4 If there are only two generators q_1 and q_2 of degree $t-2$, then there is at most one linear condition on the forms a_1 and a_2. Specifically, if there are linear forms ℓ_1, ℓ_2 such that $\ell_1 q_1 + \ell_2 q_2 = 0$, then these are uniquely determined up to a scalar multiple, and the condition $\ell_1 a_1 + \ell_2 a_2 \in I_t$ implies that $\mathrm{in}(I') = I$.

Going beyond Lemma 3.2, it is possible that the fibres of $\pi_{\mathbf{h}}$ are reducible. To show that they are contained in the smoothable component of the Hilbert scheme, we would have to find a smooth and smoothable point in each component of the fibre. Unfortunately, it is generally difficult to describe the fibres of $\pi_{\mathbf{h}}$. The following statement allows us in a handful of very special cases to avoid this difficulty.

Lemma 3.5 *If $I \in \mathscr{H}_n^{\mathbf{h}}$, then I lies in every irreducible component of $\pi_h^{-1}(I)$.*

Proof If $I' \in \pi_{\mathbf{h}}^{-1}(I)$, then we have $I = \mathrm{in}(I')$ and there is a path in $\pi_{\mathbf{h}}^{-1}(I)$ from I' to I; see [13, Theorem 15.17]. Thus, the ideal I lies in the irreducible component that contains I'. □

If the homogeneous ideal I happens to be a smooth and smoothable point, then the whole fibre is contained in the smoothable component of the Hilbert scheme.

Non-linear Changes of Coordinates Following [9, 15] and [31, Sect. 2.2], we describe a useful non-linear change of coordinates. Assume $T = \mathbb{k}[\alpha, \beta, \gamma]$ and the zero-dimensional quotient $A = T/I$ is supported at the origin. The algebra A can also be viewed as a quotient of the power series ring $R = \mathbb{k}[\![\alpha, \beta, \gamma]\!]$, which has a much larger automorphism group than the polynomial ring. Let $\mathfrak{m}$ denote the maximal ideal in R. For any $\sigma_1, \sigma_2, \sigma_3 \in \mathfrak{m}$ whose images span $\mathfrak{m}/\mathfrak{m}^2$, there is an automorphism ψ of R defined by $\psi(\alpha) := \sigma_1$, $\psi(\beta) := \sigma_2$, and $\psi(\gamma) := \sigma_3$. If $J = \langle f_1, f_2, \dots, f_r \rangle$ is the inverse system of I, then the inverse system of $\psi^{-1}(I)$ is generated by $\psi^\vee(f_i)$ where $\psi^\vee$ is defined as follows; see [31, Sect. 2.2]. Setting $D_\alpha := \psi(\alpha) - \alpha$, $D_\beta := \psi(\beta) - \beta$, and $D_\gamma := \psi(\gamma) - \gamma$, we have

$$\psi^\vee(f) = \sum_{(k,m,n) \in \mathbb{Z}^3_{\geq 0}} \tfrac{x^k y^m z^n}{k!m!n!} \cdot \left(D_\alpha^k D_\beta^m D_\gamma^n \circ f \right).$$

Example 3.6 Consider $J = \langle f \rangle$ where $f = x^4 + y^4 + g$ and $\deg(g) \leq 3$. By subtracting multiples of $\alpha \circ f$ and $\alpha^2 \circ f$ from f, we may assume the monomials x^3 and x^2 do not appear in g. We perform a non-linear change of coordinates so that there are no monomials in g divisible by x^2; for an application, see the proof of Lemma 4.8. If B and C are the coefficients of x^2y and x^2z in g, then we define the automorphism ψ by $\psi(\alpha) := \alpha$, $\psi(\beta) := \beta - \frac{B}{12}\alpha^2$, and $\psi(\gamma) := \gamma - \frac{C}{12}\alpha^2$. It follows that $D_\alpha = 0$, $D_\beta = -\frac{B}{12}\alpha^2$, and $D_\gamma = -\frac{C}{12}\alpha^2$, so we obtain

$$\psi^\vee(x^4) = x^4 + yD_\beta(x^4) + zD_\gamma(x^4) + \cdots = x^4 - Bx^2y - Czx^2 + \cdots$$

where we have omitted terms of degree less than 3. Similarly, we have $\psi^\vee(y^4) = y^4$ and $\psi^\vee(g)$ is equal to g modulo terms of degree less than 3. Therefore, the image $\psi^\vee(f)$ has no terms divisible by x^2.

An Explicit Construction of Flat Families Adapting the more general results on Gorenstein schemes in [9, Sect. 5], we show that many zero-dimensional schemes R are cleavable by constructing a family whose special fibre is R and whose general fibre is reducible.

Proposition 3.7 *Let $R \subset \mathbb{A}^n$ be a finite scheme supported at the origin and let $C \subset \mathbb{A}^n$ be a smooth curve passing through the origin such that the hyperplane* $\mathrm{V}(x)$ *intersects C transversely. Furthermore, suppose that the ideal of intersection $R \cap C$ in C is (x^r) for some $r \geq 1$. If $R \subset C \cup \mathrm{V}(x^{r-1})$ as schemes, then R is cleavable.*

Proof Since $R \cap C$ is cut out of C by x^r, we can choose an $f \in I_R$ whose image in $\Bbbk[C] = \Bbbk[\mathbb{A}^n]/I_C$ is x^r, so we have $q := x^r - f \in I_C$. The image in $\Bbbk[C]$ of any $p \in I_R$ is gx^r for some $g \in \Bbbk[\mathbb{A}^n]$, so we have $p = g(x^r - q) + h$ for some $h \in \Bbbk[\mathbb{A}^n]$. It follows that $h \in I_R \cap I_C$ and $I_R = (x^r - q) + I_{R \cup C}$. In other words, the finite scheme R is cut out of $R \cup C$ by the equation $x^r - q$. Consider the deformation of $R \subset R \cup C$ defined by $\mathrm{V}(x^r - tx^{r-1} - q) \subset (R \cup C) \times \mathbb{A}^1$ where t is the local parameter on $\mathbb{A}^1$. Let $I_V := (x^r - tx^{r-1} - q) \subset \Bbbk[(R \cup C) \times \mathbb{A}^1]$ denote the defining ideal of this family. If $I_{C \times \mathbb{A}^1}$ and $I_{R \times \mathbb{A}^1}$ are the ideals in $\Bbbk[(R \cup C) \times \mathbb{A}^1]$ of $C \times \mathbb{A}^1$ and $R \times \mathbb{A}^1$ respectively, then we have $I_{C \times \mathbb{A}^1} \cap I_{R \times \mathbb{A}^1} = 0$. By assumption, we have $(x^{r-1}) \cap I_{C \times \mathbb{A}^1} \subset I_{R \times \mathbb{A}^1}$. Since the hyperplane $\mathrm{V}(x)$ is transversal to C, we also have $I_V \cap I_{C \times \mathbb{A}^1} = I_V \cdot I_{C \times \mathbb{A}^1}$. Consequently, we obtain

$$
\begin{aligned}
I_V \cap I_{C \times \mathbb{A}^1} = I_V \cdot I_{C \times \mathbb{A}^1} &= (x^r - tx^{r-1} - q) \cdot I_{C \times \mathbb{A}^1} \\
&\subset (x^r - q) \cdot I_{C \times \mathbb{A}^1} + (x^{r-1}) \cdot I_{C \times \mathbb{A}^1} \subset I_{R \times \mathbb{A}^1} \cap I_{C \times \mathbb{A}^1} = 0.
\end{aligned}
$$

To prove the flatness of the family V, it is enough to show that every polynomial $f \in \Bbbk[t]$ is not a zero-divisor in the coordinate ring of V. Suppose there exists $f \in \Bbbk[t]$ and $g \in \Bbbk[V]$ such that the product fg is zero in $\Bbbk[V]$. Restricting to C, the family $V \cap (C \times \mathbb{A}^1)$ is given by the equation $x^r - tx^{r-1}$, so it is flat. Thus, we see that g restricts to zero on $V \cap (C \times \mathbb{A}^1)$, which implies that g lies in $(I_C + I_V)/I_V \subset \Bbbk[V]$. Since $I_V \cap I_{C \times \mathbb{A}^1} = 0$, the element g belongs to

$$
(I_C + I_V)/I_V \simeq I_C/(I_V \cap I_C) = I_C \subset \Bbbk[(R \cup C) \times \mathbb{A}^1]\,,
$$

so g is an element of a flat $\Bbbk[t]$-module $\Bbbk[(R \cup C) \times \mathbb{A}^1]$. Since $fg = 0$, it follows that $g = 0$, which concludes the proof of flatness.

The fiber of this family over $t \neq 0$ is supported on at least two points: the origin and $(t, 0, \ldots, 0)$. Since these fibres are reducible, we conclude that R is cleavable. □

Corollary 3.8 *Let $R \subset \mathbb{A}^n$ be a finite scheme supported at the origin and let I be its ideal. Assume that there exist coordinates $\alpha_1, \alpha_2, \dots, \alpha_n$ on $\mathbb{A}^n$ and $c \in \mathbb{N}$ is such that $\alpha_1^c \cdot \alpha_j \in I$ for all $j \neq 1$. If $\alpha_1^c \notin I + (\alpha_2, \alpha_3, \dots, \alpha_n)$, then R is cleavable.*

Proof This follows from Proposition 3.7. If $C = \mathrm{V}(\alpha_2, \alpha_3, \dots, \alpha_n)$ and $H = \mathrm{V}(\alpha_1)$, then r is determined by the equation $R \cap C = (\alpha_1^r)$. Since $r > c$, we conclude that $R \subset C \cup H^{r-1}$. □

Corollary 3.9 *Suppose that $R \subset \mathbb{A}^3$ is a scheme of length 11. Let $I \subset \mathbb{k}[\alpha, \beta, \gamma]$ be its ideal and suppose that $\alpha\beta, \alpha\gamma \in I$. Then the ideal I is smoothable.*

Proof If R is reducible, it is smoothable because all its components are. Suppose R is irreducible supported at the origin. If any order one element lies in I, then after a non-linear coordinate change R is contained in an $\mathbb{A}^2$ and so is smoothable. If no order one element lies in I, then Corollary 3.8 applied to $c = 1$ implies that R is cleavable. Therefore, it is smoothable by Lemma 1.4. □

4 Proof of Main Theorem

In this section, we prove Theorem 1.1 by examining each possible Hilbert function. Throughout the section, we fix $n = 3$, $S = \mathbb{k}[x, y, z]$, and $T = \mathbb{k}[\alpha, \beta, \gamma]$.

Cases with Long Tails of Ones

Proposition 4.1 *If the Hilbert function* $\mathbf{h}$ *equals* $(1, 3, 3, 1, 1, 1, 1)$, $(1, 3, 5, 1, 1)$, $(1, 3, 4, 1, 1, 1)$, $(1, 3, 2, 2, 1, 1, 1)$, $(1, 3, 2, 1, 1, 1, 1, 1)$, *or* $(1, 3, 1, 1, 1, 1, 1, 1, 1)$, *then we have $H_3^{\mathbf{h}} \subset R_3^{11}$.*

Proof Fix $I \in H_3^{\mathbf{h}}$. Proposition 4.2 establishes the ideal I is cleavable and Lemma 1.4 demonstrates that the ideal I is smoothable. Therefore, we have $I \in R_3^{11}$. □

Proposition 4.2 *Let $R = \mathrm{Spec}(A) \subset \mathbb{A}^n$ be an irreducible subscheme and* $\mathbf{h}$ *be the Hilbert function of the local algebra A. If $\mathbf{h} = \big(1, \mathbf{h}(1), \mathbf{h}(2), \dots, \mathbf{h}(c), 1, 1, \dots, 1\big)$ with at least c trailing ones (that is, letting s be the greatest value such that $\mathbf{h}(s) \neq 0$, we assume that $\mathbf{h}(k) = 1$ for $c + 1 \leq k \leq s$, and $s \geq 2c$), then the scheme R is cleavable.*

The proof follows the Gorenstein case of [9, Example 5.15].

Proof Let I be the ideal of R and let J be the inverse system of I. Consider a minimal generating set of J. It has a unique generator f of degree s. As explained above, we can perform a non-linear coordinate change to assume that $f = x_1^s + g$, for some g such that $\alpha_1^c \circ g = 0$. All other generators of J are of degree at most c. By subtracting some partials of f, we may assume that they are also annihilated by α_1^c. Thus, we have $\alpha_1^c \alpha_j$ lies in I for all $j \neq 1$.

It remains to check that $\alpha_1^c \notin I + (\alpha_2, \alpha_3, \dots, \alpha_n)$. Take any $q \in (\alpha_2, \alpha_3, \dots, \alpha_n)$. Then $(\alpha_1^c - q) \circ f = \frac{s!}{(s-c)!} x_1^{s-c} - q \circ g$. We claim this is nonzero. Note that $s - c \geq c$ by assumption on the number of trailing ones. Therefore, α_1^{s-c} annihilates g. So

$$\alpha_1^{s-c} \circ \left(\frac{s!}{(s-c)!} x_1^{s-c} - q \circ g \right) = s! x_1^0 - q \circ (\alpha_1^{s-c} \circ g) = s! \neq 0.$$

This shows $(\alpha_1^c - q) \circ f \neq 0$, as claimed, so $\alpha_1^c - q \notin I$. Therefore, we conclude that $\alpha_1^c \notin I + (\alpha_2, \alpha_3, \ldots, \alpha_n)$. Thus, by Corollary 3.8 the subscheme R is cleavable. □

Cases with Short Hilbert Functions For the following three cases $\mathbf{h} = (1, 3, 3, 4)$, $\mathbf{h} = (1, 3, 4, 3)$, and $\mathbf{h} = (1, 3, 5, 2)$, the analysis of the irreducible components of their standard graded Hilbert schemes completely determines the corresponding strata in the (not graded) Hilbert scheme $H_3^{\mathbf{h}}$. In each of these cases, $H_3^{\mathbf{h}}$ is a vector bundle over $\mathscr{H}_3^{\mathbf{h}}$ by Lemma 3.2, so the irreducible components of $H_3^{\mathbf{h}}$ are exactly the preimages of the irreducible components of $\mathscr{H}_3^{\mathbf{h}}$. We cover each $\mathscr{H}_3^{\mathbf{h}}$ by a collection of irreducible sets (which are not necessarily components) and produce a smooth and smoothable ideal for each set. By Lemma 3.2 and Lemma 1.3, this is enough to guarantee that all algebras in $H_3^{\mathbf{h}}$ are smoothable.

Proposition 4.3 *For the Hilbert function* $\mathbf{h} = (1, 3, 3, 4)$, *we have* $H_3^{\mathbf{h}} \subset R_3^{11}$.

Proof Let $I \subset T$, $I \in \mathscr{H}_3^{\mathbf{h}}$ be a homogeneous ideal such that $A = T/I$ has Hilbert function $\mathbf{h}$. Then $\dim I_2 = \dim T_2 - \mathbf{h}(2) = 3$. Let $I' = (I_2)$ be the ideal generated by the quadrics in I. By Lemma 3.1, $\dim I'_3 \geq 3 \cdot 3 - \binom{3}{2} = 6$, but $\dim I'_3 \leq \dim I_3 = \dim T_3 - \mathbf{h}(3) = 6$. So $I_3 = I'_3$, equality holds in the dimension bound, and by Lemma 3.1, the quadrics in I_2 must share a common linear factor ℓ.

Then I_2 is spanned by $\ell\alpha$, $\ell\beta$, $\ell\gamma$. That is, the standard graded Hilbert scheme $\mathscr{H}_3^{\mathbf{h}}$ is parametrized by the line ℓ. It is, therefore, isomorphic to the Grassmannian $\mathrm{Gr}(1, 3) \cong \mathbb{P}^2$ and, hence, irreducible. By Lemma 3.2, $H_3^{\mathbf{h}}$ is also irreducible.

It is sufficient to find one smooth and smoothable point in $H_3^{\mathbf{h}}$. Consider the ideal $L = (\alpha\beta, \alpha\gamma, \alpha^2 + \beta^3, \beta^2\gamma^2, \beta\gamma^3, \gamma^4)$. It is smoothable by Corollary 3.9 and we check computationally that L is smooth. □

Proposition 4.4 *For the Hilbert function* $\mathbf{h} = (1, 3, 4, 3)$, *we have* $H_3^{\mathbf{h}} \subset R_3^{11}$.

Proof The standard graded Hilbert scheme $\mathscr{H}_3^{\mathbf{h}}$ is a union of two irreducible sets. We will provide a smooth and smoothable point in each of them.

Let $I \subset T$, $I \in \mathscr{H}_3^{\mathbf{h}}$ be a homogeneous ideal such that $A = T/I$ has Hilbert function $\mathbf{h}$. Then $\dim I_2 = 2$. By Lemma 3.1, the space of cubics generated by the quadrics in I_2 can have dimension either 6 or 5, and the latter occurs exactly when the quadrics share a linear factor. Let $P \subset \mathscr{H}_3^{\mathbf{h}}$ be the set of ideals I whose quadrics generate a six-dimensional space of cubics and let $Q \subset \mathscr{H}_3^{\mathbf{h}}$ be the set of ideals I whose quadrics generate a five-dimensional space of cubics. Then $\mathscr{H}_3^{\mathbf{h}} = P \cup Q$. We claim that each of P and Q is irreducible.

The subset P is parametrized by pairs of spaces (K, M), where K is a two-dimensional subspace of T_2, not of the form $\mathrm{span}\{\ell \cdot \ell_1, \ell \cdot \ell_2\}$, and M is a seven-dimensional subspace of T_3 that contains $K \cdot T_1$, equivalently a line in $T_3/K \cdot T_1$. Thus, P is realized as a projective bundle with fibre $\mathbb{P}(T_3/K \cdot T_1)$ over an open subset of $\mathrm{Gr}(2, T_2)$. In particular, P is irreducible.

In the subset Q, the quadrics q_1, q_2 that span I_2 have the form $q_1 = \ell \cdot \ell_1$ and $q_2 = \ell \cdot \ell_2$ for some lines ℓ, ℓ_1, ℓ_2. This component is parametrized by a triple (ℓ, L, N), where $\ell \in T_1$ is the common line, $L = (\ell_1, \ell_2) \subset T_1$ is the space spanned by the other two lines, and N is a seven-dimensional space of T_3 that contains the five-dimensional space $\ell \cdot L \cdot T_1$. It follows that Q is isomorphic to a Grassmannian bundle with fibre $\mathrm{Gr}(7-5, T_3/\ell \cdot L \cdot T_1)$, over a base $\mathrm{Gr}(1, T_1) \times \mathrm{Gr}(2, T_1)$; it is, therefore, irreducible.

Now $H_3^{\mathbf{h}} = \pi_{\mathbf{h}}^{-1}(P) \cup \pi_{\mathbf{h}}^{-1}(Q)$, and by Lemma 3.2 these are irreducible sets as well. To complete this case, we provide a smooth and smoothable ideal for each set. The ideal $I = (\alpha^2, \beta^2, \gamma^3, \alpha\beta\gamma^2)$ lies in P and, hence, also in $\pi_{\mathbf{h}}^{-1}(P)$. It is monomial, hence, smoothable by [8, Proposition 4.15] and it is easy to check computationally that it is a smooth point. For $\pi_{\mathbf{h}}^{-1}(Q)$ let $I = (\alpha\beta, \alpha\gamma, \alpha^3 + \gamma^3, \beta\gamma^2, \beta^3\gamma, \beta^4)$. Then I is smoothable by Corollary 3.9 and once again a smooth point. □

Proposition 4.5 *For the Hilbert function* $\mathbf{h} = (1, 3, 5, 2)$, *we have* $H_3^{\mathbf{h}} \subset R_3^{11}$.

Proof Let $I \subset T$, $I \in \mathscr{H}_3^{\mathbf{h}}$ be a homogeneous ideal such that $A = T/I$ has Hilbert function $\mathbf{h}$. Then $\dim I_2 = 1$ and $\dim I_3 = \dim T_3 - \mathbf{h}(3) = 8$. The standard graded Hilbert scheme $\mathscr{H}_{\mathbf{h}}^3$ is parametrized by pairs (L, M), where L is some one-dimensional subspace of T_2 and M is an eight-dimensional subspace of T_3 that contains the three-dimensional subspace $L \cdot T_1$. This parametrization realizes an isomorphism of $\mathscr{H}_3^{\mathbf{h}}$ to a Grassmannian bundle with base $\mathbb{P}T_2$ and fibre $\mathrm{Gr}(8-3, T_3/L \cdot T_1)$, proving that $\mathscr{H}_3^{\mathbf{h}}$ is irreducible. By Lemma 3.2, $H_3^{\mathbf{h}}$ is irreducible as well.

Now let $I = (\alpha\beta, \alpha^3, \beta^3, \gamma^3, \alpha\gamma^2, \alpha^2\gamma + \beta\gamma^2)$. One can check that $I \in H_3^{\mathbf{h}}$. Since $\alpha\beta^2, \alpha\gamma^2 \in I$ and $\alpha^2 \notin I + (\beta, \gamma)$, Corollary 3.8 with $c = 2$ implies I is smoothable. Finally one can check computationally that I is a smooth point. □

Case $\mathbf{h} = (1, 3, 4, 2, 1)$

Proposition 4.6 *For the Hilbert function* $\mathbf{h} = (1, 3, 4, 2, 1)$, *we have* $H_3^{\mathbf{h}} \subset R_3^{11}$.

Proof Let $I \in \mathscr{H}_3^{\mathbf{h}}$ be a homogeneous ideal with inverse system J. Let $f \in J_4$ and let $\mathbf{h}_f$ be the Hilbert function of $\langle f \rangle$. Since $\langle f \rangle \subset J$ we have $\mathbf{h}_f \leq \mathbf{h}$. By Proposition 2.15 and the Macaulay bound (Theorem 2.3), the Hilbert function $\mathbf{h}_f$ must be $(1, 2, 3, 2, 1)$, $(1, 2, 2, 2, 1)$, or $(1, 1, 1, 1, 1)$.

If $\mathbf{h}_f = (1, 2, 3, 2, 1)$, then see Lemma 4.7. If $\mathbf{h}_f = (1, 2, 2, 2, 1)$ then, by Remark 2.17, we can choose coordinates so that $f = x^4 + y^4$ or $f = x^3y$. For $f = x^4 + y^4$ see Lemma 4.8 and for $f = x^3y$ see Lemma 4.9. If $\mathbf{h}_f = (1, 1, 1, 1, 1)$, then see Lemma 4.10. □

Lemma 4.7 *Let* $\mathbf{h} = (1, 3, 4, 2, 1)$ *and let* $I \in \mathscr{H}_3^{\mathbf{h}}$ *be a homogeneous ideal with inverse system* J. *If the degree* 4 *generator* f *of* J *is such that the Hilbert function of* $\langle f \rangle$ *is* $(1, 2, 3, 2, 1)$, *then we have* $\pi_{\mathbf{h}}^{-1}(I) \subset R_3^{11}$.

Proof Let $I' \in \pi_{\mathbf{h}}^{-1}(I)$ with inverse system J'. Let F be the degree 4 generator of J', so that f is the leading form of F. We will construct a family J'_t so that $J'_1 = J'$ and J'_0 is $\langle x^2y^2, z^2 \rangle$, $\langle x^2y^2, zx \rangle$, or $\langle x^2y^2, z(x+y) \rangle$. First change coordinates so that

$f \in \Bbbk[x, y]$. Then $x^2, xy, y^2 \in \langle f\rangle_2 \subset J'$, so $J'_{\leq 2}$ is spanned by $\{x^2, xy, y^2, Q, S_{\leq 1}\}$ for a quadratic form $Q \in \Bbbk[x, y, z]$. Write $Q = cxz + dyz + ez^2$. If $e \neq 0$ then changing coordinates by replacing z with a suitable linear combination of x, y, z to complete the square eliminates the xz and yz terms and takes Q to z^2 modulo x^2, xy, y^2. So either $J' = \langle F, z^2\rangle$ or $J' = \langle F, z(cx + dy)\rangle$.

Write $F = f + g$, $\deg g \leq 3$. By well-known facts about binary forms (see [30, Theorem 1.43]), we have $f = \ell_1^4 + \ell_2^4 + \ell_3^4$ for some nonproportional linear forms $\ell_i \in \Bbbk[x, y]$. Observe that $18x^2y^2 = (x + y)^4 + \omega(x + \omega y)^4 + \omega^2(x + \omega^2 y)^4$ where ω is a cube root of unity. We change coordinates in $\Bbbk[x, y]$ so that $\ell_1 = x + y$ and $\ell_2 = \omega^{1/4}(x + \omega y)$. Let $f_t = \ell_1^4 + \ell_2^4 + (t\ell_3 + (1 - t)\omega^{1/2}(x + \omega^2 y))^4$, $F_t = f_t + tg$, and $J'_t = \langle F_t, Q\rangle$.

It is easy to check that $F_1 = F$, $F_0 = 18x^2y^2$, and for all but finitely many t, $\langle f_t\rangle$ has Hilbert function $(1, 2, 3, 2, 1)$ and J'_t has Hilbert function $(1, 3, 4, 2, 1)$. Then $\lim J'_t = J'_0 = \langle 18x^2y^2, Q\rangle = \langle x^2y^2, Q\rangle$, as in Definition 2.23. Rescaling x and y and interchanging if necessary, Q is one of z^2, zx, or $z(x + y)$. Now $\langle x^2y^2, z^2\rangle^\perp$ and $\langle x^2y^2, zx\rangle^\perp$ are monomial ideals, hence smoothable. The family $(\gamma^2, \alpha\gamma - \beta\gamma, \beta^2\gamma, \beta^3, \alpha^3 + t\alpha^2)$ shows that $\langle x^2y^2, z(x + y)\rangle^\perp = (\gamma^2, \alpha\gamma - \beta\gamma, \beta^2\gamma, \beta^3, \alpha^3)$ is smoothable. So all three points are smoothable and it is easy to check that each one is a smooth point. Hence, the irreducible (one-dimensional) family $\{(J'_t)^\perp\} \subset R_3^{11}$, in particular $I' = (J'_1)^\perp \in R_3^{11}$. □

Lemma 4.8 *Let* $\mathbf{h} = (1, 3, 4, 2, 1)$ *and let* $I \in \mathcal{H}_3^{\mathbf{h}}$ *be a homogeneous ideal with inverse system* J. *If the degree* 4 *generator of* J *is of the form* $\ell^4 + m^4$ *for some independent linear forms* $\ell, m \in S_1$, *then we have* $\pi_{\mathbf{h}}^{-1}(I) \subset R_3^{11}$.

Proof Assume $\ell = x$, and $m = y$. Let $I' \in \pi_{\mathbf{h}}^{-1}(I)$ with inverse system J'. We will apply Corollary 3.8. Consider the degree four generator $F = x^4 + y^4 + g \in J'$, where $\deg g \leq 3$. Since $x^2 \in J$ we can subtract the x^2 term out of g. Then the only terms of g divisible by x^2 are possibly x^3, x^2y, x^2z. After a non-linear coordinate change as in Example 3.6 we may assume that there are no such terms. Then $\alpha^2 \circ F = 12x^2$, so $\alpha^2 \notin F^\perp + (\beta, \gamma)$. Moreover $\alpha^2\beta$ and $\alpha^2\gamma$ annihilate F and so its partials lie in I'. Therefore, the assumptions of Corollary 3.8 for $c = 2$ are satisfied and I' is cleavable. By Lemma 1.4, it is smoothable. □

Lemma 4.9 *Let* $\mathbf{h} = (1, 3, 4, 2, 1)$ *and let* $I \in \mathcal{H}_3^{\mathbf{h}}$ *be a homogeneous ideal with inverse system* J. *If the degree* 4 *generator of* J *is of the form* $\ell^3 m$ *for some independent linear forms* $\ell, m \in S_1$, *then we have* $\pi_{\mathbf{h}}^{-1}(I) \subset R_3^{11}$.

Proof Assume $\ell = x$, $m = y$, so that $J = \langle x^3y, Q_1, Q_2\rangle$ for some quadratic forms Q_1, Q_2. Let $I' \in \pi_{\mathbf{h}}^{-1}(I)$ with inverse system J'. We will show I' is smoothable by writing it as a limit of smoothable points. Note, $J' = \langle x^3y + g_3 + g_2, Q_1, Q_2\rangle$ where g_i is a form of degree i for $i = 2, 3$. We introduce a parameter t and let $y_t = x + ty$. Observe that $\lim_{t\to 0}(y_t^4 - x^4)/4t = x^3y$. For general t we will define a form $g_3(t)$ so that $J'_t = \langle (y_t^4 - x^4)/4t + g_3(t) + g_2, Q_1, Q_2\rangle \to J'$ in the sense of Definition 2.23.

To define $g_3(t)$, first note that $\gamma \circ g_3 \in J_2 = \operatorname{span}\{x^2, xy, Q_1, Q_2\}$. For $i = 1, 2$ let $Q_i^\sharp = \int Q_i \, dz$ be a homogeneous form of degree 3 so that $\gamma \circ Q_i^\sharp = Q_i$. Write

$g_3 = ax^2z + bxyz + cQ_1^\sharp + dQ_2^\sharp + e(x, y)$ for some scalars a, b, c, d and a 3-form e. Now we define $g_3(t) = ax^2z + (b/2t)(y_t^2 - x^2)z + cQ_1^\sharp + dQ_2^\sharp + e(x, y)$.

Now $\gamma \circ g_3(t) \in \operatorname{span}\{x^2, y_t^2, Q_1, Q_2\}$, hence, $\operatorname{lead}(J_t') = \langle y_t^4 - x^4, Q_1, Q_2\rangle$. Since $\dim J_2 = 4$ we have $xy \notin \operatorname{span}\{x^2, Q_1, Q_2\}$. Since $xy = \lim_{t\to 0}(y_t^2 - x^2)/(2t)$ we also have $y_t^2 \notin \operatorname{span}\{x^2, Q_1, Q_2\}$ for general t. For such t the space $(\operatorname{lead}(J_t'))_2$ has dimension 4, which means that $\operatorname{lead}(J_t')$ and J_t' have Hilbert function $\mathbf{h}$. Also $\lim_{t\to 0} g_3(t) = g_3$. Therefore, $\lim_{t\to 0} J_t' = J'$, as desired. By Lemma 4.8, each $(J_t')^\perp$ with $t \neq 0$ is smoothable, which implies $I' = \lim(J_t')^\perp$ is smoothable as well. □

Lemma 4.10 *Let* $\mathbf{h} = (1, 3, 4, 2, 1)$ *and let* $I \in \mathcal{H}_3^{\mathbf{h}}$ *be a homogeneous ideal with inverse system* J*. If the degree* 4 *generator* f *of* J *is such that the Hilbert function of* $\langle f\rangle$ *is* $(1, 1, 1, 1, 1)$*, then we have* $\pi_{\mathbf{h}}^{-1}(I) \subset R_3^{11}$.

Proof By Remark 2.17, we can choose coordinates such that $f = z^4$. Let $V \subset \mathcal{H}_3^{\mathbf{h}}$ be the set of ideals I satisfying the hypothesis; $V = \{I \in \mathcal{H}_3^{\mathbf{h}} : I \subset (z^4)^\perp\}$. For $I \in V$, we have $\dim I_2 = 2$ and $\dim I_3 = 8$. Lemma 3.1 shows that $\dim T_1 \cdot I_2$ is either 5 or 6. Let $V_1 \subset V$ be the set of I such that $\dim T_1 \cdot I_2 = 6$ or, equivalently, the quadrics in I_2 have no common factor. Let $V_2 \subset V$ be the set of I such that $\dim T_1 \cdot I_2 = 5$ and $I_2 = \operatorname{span}\{\ell\ell_1, \ell\ell_2\}$ for some linear forms ℓ, ℓ_1, ℓ_2 such that

$$\operatorname{span}\{\ell_1, \ell_2\} \subseteq z^\perp = \operatorname{span}\{\alpha, \beta\}$$

(necessarily equality must hold). And let $V_3 \subset V$ be the remainder, the set of I such that $\dim T_1 \cdot I_2 = 5$ and $I_2 = \operatorname{span}\{\ell\ell_1, \ell\ell_2\}$ for some linear forms ℓ, ℓ_1, ℓ_2 such that $\operatorname{span}\{\ell_1, \ell_2\} \not\subseteq z^\perp$. We will show that each V_i and each $\pi_{\mathbf{h}}^{-1}(V_i)$ is irreducible, and give a smooth and smoothable point on each $\pi_{\mathbf{h}}^{-1}(V_i)$.

First, every ideal $I \in V$ is determined by (I_2, I_3). Suppose $I \in V_1$. The subspace $I_2 \subset (z^2)^\perp$ is parametrized by an open subset of $\operatorname{Gr}(2, (z^2)^\perp) = \operatorname{Gr}(2, 5)$. And then $I_3 \subset (z^3)^\perp$ is such that $T_1 \cdot I_2 \subset I_3$. The quotient $I_3/T_1 \cdot I_2$ is a two-dimensional subspace of $(z^3)^\perp/T_1 \cdot I_2$. So for each choice of I_2, I_3 may be chosen from $\operatorname{Gr}(8 - 6, (z^3)^\perp/T_1 \cdot I_2) = \operatorname{Gr}(2, 3)$. This shows V_1 is a Grassmannian bundle over an open subset of a Grassmannian, in particular irreducible. Let $I_2 = \operatorname{span}\{q_1, q_2\}$. Since $I \in V_1$, there are no lines ℓ_1, ℓ_2 such that $\ell_1 q_1 + \ell_2 q_2 = 0$. By Lemma 3.3 and Remark 3.4, the fibre $\pi_{\mathbf{h}}^{-1}(I)$ is a certain product of affine spaces. Explicitly it is $T_3^2 \times T_4^4$, corresponding to cubic terms that may be added to the quadric generators of I and quartic terms that may be added to the quadric and cubic generators of I. This makes $\pi_{\mathbf{h}}^{-1}(V_1)$ a (trivial!) vector bundle over V_1, hence, irreducible. A smooth and smoothable point in $\pi_{\mathbf{h}}^{-1}(V_1)$ is given by $\langle yz, x^2y, z^4\rangle^\perp = (\alpha\gamma, \beta^2, \beta\gamma^2, \alpha^3, \gamma^5)$. It is smoothable because it is a monomial ideal and we check computationally that it is a smooth point. This shows that $\pi_{\mathbf{h}}^{-1}(V_1) \subset R_3^{11}$.

If $I \in V_2$ then $I_2 = \ell\cdot\operatorname{span}\{\alpha, \beta\}$ for some linear form ℓ, so I_2 is determined by the choice of $[\ell] \in \mathbb{P}T_1$. As before, for each choice of ℓ, I_3 may be chosen from $\operatorname{Gr}(8 - 5, (z^3)^\perp/T_1 \cdot I_2) = \operatorname{Gr}(3, 4)$. Again this makes V_2 a Grassmannian bundle over an irreducible base, so V_2 is irreducible. By Remark 3.4, $\pi_{\mathbf{h}}^{-1}(V_2)$ is a trivial subbundle of a trivial vector bundle over V_2, namely $\pi_{\mathbf{h}}^{-1}(V_2) \subset V_2 \times (T_3^2 \times T_4^5)$ is defined by $\beta a_1 - \alpha a_2 \in I_4 = (z^4)^\perp$, where a_1, a_2 are the cubic terms added to the quadric generators $\ell\alpha, \ell\beta$. Hence, $\pi_{\mathbf{h}}^{-1}(V_2)$ is irreducible. A smooth and smoothable point in this set is given by the limit of the flat family $(\alpha\gamma, \beta\gamma, \beta^3 + \gamma^4, \alpha^3 - t \cdot \alpha^2, \alpha^2\beta)$.

If $I \in V_3$ then, writing $I_2 = \mathrm{span}\{\ell\ell_1, \ell\ell_2\}$, we must have $\ell \circ z = 0$, since for at least one of $i = 1, 2$ we have $\ell_i \circ z \neq 0$, but $\ell\ell_i \circ z^2 = 0$. Now ℓ may be chosen from $z^\perp$ and $\mathrm{span}\{\ell_1, \ell_2\}$ may be chosen to be any two-dimensional subspace of T_1 other than $z^\perp$. It follows that the choice of I_2 is parametrized by an open subset of $\mathbb{P}(z^\perp) \times \mathrm{Gr}(2, T_1)$. Once again, for each choice of I_2, I_3 may be chosen from the Grassmannian $\mathrm{Gr}(8 - 5, (z^3)^\perp / T_1 \cdot I_2) = \mathrm{Gr}(3, 4)$. Hence, V_3 is a Grassmannian bundle over an irreducible base, in particular irreducible. By Remark 3.4, $\pi_{\mathbf{h}}^{-1}(V_3)$ is a (nontrivial) subbundle of a trivial vector bundle over V_2, namely $\pi_{\mathbf{h}}^{-1}(V_3) \subset V_3 \times (T_3^2 \times T_4^5)$ is defined by $\ell_2 a_1 - \ell_1 a_2 \in I_4 = (z^4)^\perp$ where, as before, a_1, a_2 are the cubic terms added to the quadric generators $\ell\ell_1, \ell\ell_2$. Hence, $\pi_{\mathbf{h}}^{-1}(V_3)$ is irreducible. The ideal $(\beta^2, \beta\gamma, \alpha^3, \alpha^2\gamma, \alpha\gamma^2, \gamma^5) \in V_3$ is smoothable because it is monomial and we check computationally that it is smooth. □

Case $\mathbf{h} = (1, 3, 2, 2, 2, 1)$

Lemma 4.11 *If $\mathbf{h} = (1, \mathbf{h}(1), \dots, \mathbf{h}(k), 2, \dots, 2, 1)$ such that $\mathbf{h}(i) = \dim S_i$ for all $i \leq k$ and $\mathbf{h}$ has at least two 2s and a 1 in the last position, then the standard graded Hilbert scheme $\mathscr{H}_n^{\mathbf{h}}$ is irreducible. Each ideal $I \in \mathscr{H}_n^{\mathbf{h}}$ is the apolar ideal $J^\perp$ of an inverse system J of one of the following forms: $\langle \ell^d + m^d, S_k \rangle$, $\langle \ell^{d-1}m, S_k \rangle$, $\langle \ell^d, m^{d-1}, S_k \rangle$, $\langle \ell^d, \ell^{d-2}m, S_k \rangle$ for some linear forms ℓ, m.*

Proof Say the last 1 is in degree d, let J be a homogeneous inverse system with Hilbert function $\mathbf{h}$, and let $f \in J$ be the d-form that appears. Either $\langle f \rangle$ has Hilbert function $(\dots, 2, 2, 1)$ or $(\dots, 1, 1, 1)$. In the first case $f = \ell^d + m^d$ or $f = \ell^{d-1}m$, and J is generated by f together with S_k. The second type is a limit of the first type, similarly to Example 2.24.

In the second case $f = \ell^d$ and there is a generator g of degree $d-1$. Note g has at most 2 first derivatives since $\langle g \rangle_{d-2} \subseteq J_{d-2}$. So $\langle g \rangle$ has Hilbert function $(\dots, 2, 1, 0)$ or $(\dots, 1, 1, 0)$. If it is $(\dots, 1, 1, 0)$ then $g = m^{d-1}$ for a linear form m independent from ℓ. If the Hilbert function of g is $(\dots, 2, 1, 0)$ then $\ell^{d-2} \in \langle g \rangle_{d-2}$, so $g = \ell^{d-2}m$ for a linear form m independent from ℓ.

So either $g = \ell^{d-2}m$ or $g = m^{d-1}$. Correspondingly, either $J = \langle \ell^d, \ell^{d-2}m, S_k \rangle$ or $J = \langle \ell^d, m^{d-1}, S_k \rangle$. Both of these can be obtained as limits of inverse systems of the first two forms in appropriate ways, using Proposition 2.22. Explicitly, we have $\langle \ell^d, \ell^{d-2}m, S_k \rangle = \lim_{t\to 0} \langle \ell^{d-1}(\ell + tm), S_k \rangle$ and $\langle \ell^d, m^{d-1}, S_k \rangle = \lim_{t\to 0} \langle \ell^d + tm^d, S_k \rangle$. □

Proposition 4.12 *For the Hilbert function $\mathbf{h} = (1, 3, 2, 2, 2, 1)$, we have $H_3^{\mathbf{h}} \subset R_3^{11}$.*

Proof By Lemma 4.11, every homogeneous ideal in $\mathscr{H}_3^{\mathbf{h}}$ is the apolar ideal of an inverse system which is isomorphic to one of the following: $J_1 = \langle x^5 + y^5, z \rangle$, $J_2 = \langle x^4y, z \rangle$, $J_3 = \langle x^5, y^4, z \rangle$, or $J_4 = \langle x^5, x^3y, z \rangle$. We may dispose of the first two cases easily. We compute $I_2 = J_2^\perp = (\alpha^5, \beta^2, \alpha\gamma, \beta\gamma, \gamma^2)$. Then I_2 is smoothable because it is a monomial ideal and one can easily check computationally that it is a smooth point. By Lemma 3.5, the smooth and smoothable point I_2 lies in every component of the fibre $\pi_{\mathbf{h}}^{-1}(I_2)$, which shows that each irreducible component of the fibre is contained in R_3^{11}.

Similarly, $I_1 = J_1^{\perp} = (\alpha^5 - \beta^5, \alpha\beta, \alpha\gamma, \beta\gamma, \gamma^2)$ is smooth and it is smoothable by Corollary 3.9. Using Lemma 3.5 again, this smooth and smoothable point lies in each irreducible component of the fibre, so each irreducible component of the fibre is contained in R_3^{11}.

Now we consider the last two cases, where one finds that the homogeneous ideals $J_3^{\perp}$, $J_4^{\perp}$ are not smooth points (although they are monomial, hence, smoothable). So we need to develop a more detailed description of the fibres in these cases. In Lemma 4.13 we show that the fibre $\pi_{\mathbf{h}}^{-1}(J_3^{\perp})$ is contained in R_3^{11} and in Lemma 4.14 we do the same for J_4. □

Lemma 4.13 *If* $\mathbf{h} = (1, 3, 2, 2, 2, 1)$ *and* $J = \langle x^5, y^4, z\rangle$, $I = J^{\perp}$, *then we have* $\pi_{\mathbf{h}}^{-1}(I) \subset R_3^{11}$.

Proof First we will show that the fibre $\pi_{\mathbf{h}}^{-1}(I)$ is irreducible, then we will display a smooth and smoothable point in the fibre. To begin, I is generated by $f_1 = \alpha\beta$, $f_2 = \alpha\gamma, f_3 = \beta\gamma, f_4 = \gamma^2, \beta^5, \alpha^6$. If $I' \in \pi_{\mathbf{h}}^{-1}(I)$, then we have

$$I' = (F_1, F_2, F_3, F_4, \beta^5) + (\alpha, \beta, \gamma)^6\,, \tag{1}$$

where $F_i = f_i + g_i$ and each g_i involves monomials of degree 3 or greater that are not in I. Those monomials are $\alpha^3, \beta^3, \alpha^4, \beta^4, \alpha^5$. We can write, for each $1 \le i \le 4$, $g_i = a_i\alpha^3 + b_i\beta^3 + c_i\alpha^4 + d_i\beta^4 + e_i\alpha^5$. This embeds the fibre $\pi_{\mathbf{h}}^{-1}(I)$ into $\mathbb{A}^{20}$ with coordinates $a_1, \dots, e_4$. It remains to find its equations, that is, determine which ideals I' of the form (1) have initial ideal I. We claim that $\pi_{\mathbf{h}}^{-1}(I)$ is defined by the equations

$$b_2 = a_3 = a_4 = b_4 = a_1a_2 + c_3 = a_2^2 + c_4 = 0. \tag{2}$$

Since $\mathrm{in}(I') \supset I$ and $\dim T/\mathrm{in}(I') = \dim T/I'$ we have $\mathrm{in}(I') = I$ if and only if $\dim T/I = \dim T/I'$. Consider the elements $\tilde{g}_i = a_1\alpha^3 + b_i\beta^3 + t\cdot(c_i\alpha^4 + d_i\beta^4) + t^2\cdot e_i\alpha^5 \in T[t]$ and $\tilde{F}_i = f_i + t\tilde{g}_i$. Define the ideal

$$\tilde{I'} = (\tilde{F}_1, \tilde{F}_2, \tilde{F}_3, \tilde{F}_4, \beta^5) + (\alpha, \beta, \gamma)^6. \tag{3}$$

Clearly, the fibre of $\tilde{I'}$ over $t = 1$ is I' and over $t = 0$ is I. Also the family is flat over $\mathbb{k}[t^{\pm 1}]$ because of the torus action. Therefore, $\tilde{I'}$ is flat if and only if all fibres have the same length, if and only if $\dim T/I' = \dim T/I$. That is, $I' \in \pi_{\mathbf{h}}^{-1}(I)$ if and only if $\tilde{I'}$ is flat. Flatness of $\tilde{I'}$ is equivalent to the following condition; see [1, p. 11] or [23, Corollary 7.4.7].

$$\text{Every relation } \sum f_ir_i = 0 \text{ with } r_i \in T \text{ lifts to } \sum \tilde{F}_iR_i = 0 \text{ with } R_i \in T[t].$$

That is, there exist $R_i' \in T[t]$ such that $0 = \sum \tilde{F}_i(r_i + tR_i') = \sum r_i f_i + t\sum r_i\tilde{g}_i + t\sum R_i'\tilde{F}_i$, equivalently $\sum r_i\tilde{g}_i = -\sum R_i'\tilde{F}_i \in \tilde{I}'$. So $\tilde{I}'$ is flat if and only if the following holds.

$$\text{For every relation } \sum f_i r_i = 0 \text{ with } r_i \in T \text{ we have } \sum \tilde{g}_i r_i \in \tilde{I}'. \tag{4}$$

The relations between the f_i are the syzygies of I. They are generated by four linear syzygies, two quartic syzygies, and two quintic syzygies (direct check). It is enough to check (4) for those generators. Since $\tilde{I}' \supset (\alpha, \beta, \gamma)^6$, the property (4) is automatically satisfied for quartic and quintic syzygies. The linear generators are given by $\gamma f_1 = \beta f_2$, $\beta f_2 = \alpha f_3$, $\gamma f_2 = \alpha f_4$, and $\gamma f_3 = \beta f_4$ By (4), the fibre is cut out by the conditions $\gamma\tilde{g_1} - \beta\tilde{g_2} \in \tilde{I}'$, $\beta\tilde{g_2} - \alpha\tilde{g_3} \in \tilde{I}'$, $\gamma\tilde{g_2} - \alpha\tilde{g_4} \in \tilde{I}'$, and $\gamma\tilde{g_3} - \beta\tilde{g_4} \in \tilde{I}'$. We now check that they unfold into (2). Consider an ideal $I' \in \pi_{\mathbf{h}}^{-1}(I)$. The element $\gamma\tilde{g_1} - \beta\tilde{g_2}$ lies in $\tilde{I}'$ by (4). Since

$$\gamma\tilde{g_1} - \beta\tilde{g_2} = a_1\alpha^3\gamma + b_1\beta^3\gamma - a_2\alpha^3\beta - b_2\beta^4 \\ + t(c_1\alpha^4\gamma + d_1\beta^4\gamma - c_2\alpha^4\beta - d_2\beta^5) + t^2(e_1\alpha^5\gamma - e_2\alpha^5\beta)$$

lies in $\tilde{I}'$, its initial form lies in I, which implies $b_2 = 0$. Similarly, by considering the initial forms of $\gamma\tilde{g_1} - \alpha\tilde{g_3} \in I'$ we deduce that $a_3 = 0$; from $\gamma\tilde{g_2} - \alpha\tilde{g_4} \in I'$ we get $a_4 = 0$; from $\gamma\tilde{g_3} - \beta\tilde{g_4} \in I'$ we get $b_4 = 0$. Note the following relations:

$$\alpha^3\beta \equiv -a_1t\alpha^5, \quad \alpha^3\gamma \equiv -a_2t\alpha^5, \quad \beta^3\gamma \equiv -b_3t\beta^5 \pmod{\tilde{I}'}.$$

Using these relations, together with $b_2 = a_3 = a_4 = b_4 = 0$, we check that

$$\beta\tilde{g_2} - \alpha\tilde{g_3} \equiv -t(a_1a_2 + c_3)\alpha^5 \pmod{\tilde{I}'}.$$

This implies that $-t(a_1a_2 + c_3)\alpha^5 \in \tilde{I}'$, so by evaluating at $t = 1$ we get $(a_1a_2 + c_3)\alpha^5 \in I'$. Hence, the leading form $(a_1a_2 + c_3)\alpha^5$ is in I. Therefore, $a_1a_2 + c_3 = 0$. Similarly, $\gamma\tilde{g_2} - \alpha\tilde{g_4} \equiv -(a_2^2 + c_4)\alpha^5 \pmod{\tilde{I}'}$ which gives the condition $a_2^2 + c_4 = 0$, whereas for $\gamma\tilde{g_3} - \beta\tilde{g_4}$ and $\gamma\tilde{g_1} - \beta\tilde{g_2}$ we get trivially zero. Thus, (2) is satisfied for every I' in the fibre. Conversely, the above reasoning implies that each I' satisfying (2) lies in the fibre. This shows that the fibre is irreducible, in fact isomorphic to $\mathbb{A}^{14}$ via projection to the coordinates $a_1, a_2, b_1, b_3, c_1, c_2, d_1, \ldots, e_4$.

Finally, let $I' = (\alpha^6, \beta^5, \alpha\beta, \alpha\gamma, \beta\gamma + \alpha^5, \gamma^2)$. It is smoothable by Corollary 3.9. We verify computationally that I' is a smooth point. □

Lemma 4.14 *If* $\mathbf{h} = (1, 3, 2, 2, 2, 1)$ *and* $J = \langle x^5, x^3y, z\rangle$, $I = J^\perp$, *then we have* $\pi_{\mathbf{h}}^{-1}(I) \subset R_3^{11}$.

Proof The proof directly follows the argument of Lemma 4.13. The ideal I is generated by $f_1 = \alpha\gamma$, $f_2 = \beta^2$, $f_3 = \beta\gamma$, $f_4 = \gamma^2$, $\alpha^4\beta$, α^6. Let $I' \in \pi_{\mathbf{h}}^{-1}(I)$. Then

$$I' = (F_1, F_2, F_3, F_4, \beta^5) + (\alpha, \beta, \gamma)^6, \tag{5}$$

where $F_i = f_i + g_i$ and $g_i = a_i\alpha^3 + b_i\alpha^2\beta + c_i\alpha^4 + d_i\alpha^3\beta + e_i\alpha^5$. The syzygies among f_i's are again generated by linear, quartic, and quintic syzygies. The linear generators are $\beta f_1 - \alpha f_3$, $\gamma f_1 - \alpha f_4$, $\gamma f_2 - \beta f_3$, $\gamma f_3 - \beta f_4$. An analysis of the resulting conditions gives the following equations for $\pi_{\mathbf{h}}^{-1}(I)$:

$$a_1 - b_3 = a_3 = a_4 = b_4 = a_2 b_1 + c_3 = a_1^2 + c_4 = 0. \tag{6}$$

This shows that the fibre $\pi_{\mathbf{h}}^{-1}(I)$ is irreducible, in fact isomorphic to $\mathbb{A}^{14}$ via projection to the coordinates $a_1, a_2, b_1, b_2, c_1, c_2, d_1, \dots, e_4$. A smooth and smoothable point in the fibre is $I' = (\alpha^6, \alpha^4\beta, \alpha\gamma, \beta^2, \beta\gamma, \gamma^2 + \alpha^5)$. It is smoothable by Corollary 3.9 and is computationally verified to be a smooth point. □

Case h = (1, 3, 3, 2, 2)

Lemma 4.15 *Let $f \in \mathbb{k}[x, y]$ be a homogeneous form of degree d with $d \geq 3$. Either $f = \ell^d + m^d$ or $f = \ell^{d-1}m$ for some linear forms ℓ and m, or else f is determined up to scalar multiple by the subspace $\langle f \rangle_{d-1}$, in the sense that if $f \neq \ell^d + m^d, \ell^{d-1}m$ and $g \in \mathbb{k}[x, y]$ is a homogeneous form of degree d such that $\langle f \rangle_{d-1} = \langle g \rangle_{d-1}$, then g is a scalar multiple of f.*

Proof If $f = \ell^d$ for a linear form ℓ, then we have $\langle g \rangle_{d-1} = \langle f \rangle_{d-1} = \operatorname{span}\{\ell^{d-1}\}$, so g is a scalar multiple of ℓ^d. Otherwise let $I = f^\perp$ and $J = g^\perp$. The assumption $\langle f \rangle_{d-1} = \langle g \rangle_{d-1}$ means $I_{d-1} = J_{d-1}$. Assuming $f \neq \ell^d, \ell^d + m^d, \ell^{d-1}m$ means that $f^\perp$ has no generators of degree $\geq d$ by [5, Proposition 1.6, Theorem 1.7]. So I_d is determined by $I_{d-1} = (\langle f \rangle_{d-1})^\perp$. Since $J_{d-1} = I_{d-1}$, these generate the same degree d part, $J_d = I_d$. But J_d is perpendicular to g while I_d is perpendicular to f, so g and f are linearly dependent. □

Proposition 4.16 *For the Hilbert function* $\mathbf{h} = (1, 3, 3, 2, 2)$, *we have* $H_3^{\mathbf{h}} \subset R_3^{11}$.

Proof Let J be a graded inverse system in $S = \mathbb{k}[x, y, z]$ with Hilbert function $(1, 3, 3, 2, 2)$. Then $\dim J_4 = 2$, say $J_4 = \operatorname{span}\{f, g\}$. Each f, g has first derivatives in J_3, so each f, g involves at most two variables. If both $\langle f \rangle$ and $\langle g \rangle$ have Hilbert function $(1, *, *, 1, 1)$, then $f = \ell^4$, $g = m^4$ for independent linear forms ℓ, m, and we change coordinates so $(f, g) = (x^4, y^4)$. Otherwise at least one, say $\langle f \rangle$, has Hilbert function $(1, *, *, 2, 1)$. Then by Proposition 2.16 there is a coordinate change so that $f \in \mathbb{k}[x, y]$. We have $\langle g \rangle_3 \subseteq J_3 = \langle f \rangle_3 \subset \mathbb{k}[x, y]$. This shows that $\alpha \circ g, \beta \circ g, \gamma \circ g$ have no terms involving z. This implies g has no terms involving z. So $g \in \mathbb{k}[x, y]$ as well.

Now there are various cases, according as $f = \ell^4 + m^4$ (which we may take to be $x^4 + y^4$ after a change of coordinates), $f = \ell^3 m$ (equivalently, $x^3 y$), or something else; and $\dim\langle g \rangle_3 = 1$ or 2. In every case one checks that either $\operatorname{span}\{f, g\} = \operatorname{span}\{x^4, y^4\}$ or $\operatorname{span}\{f, g\} = \operatorname{span}\{x^4, x^3 y\}$, after a change of coordinates.

In either case, J is generated by J_4, some quadratic form Q, and possibly linear forms: J is generated, possibly redundantly, either by $\{x^4, y^4, Q, x, y, z\}$ or by

$\{x^4, x^3y, Q, x, y, z\}$, where Q is linearly independent from $\{x^2, y^2\}$ in the first case or $\{x^2, xy\}$ in the second case.

Now we claim that there is an automorphism of $S_1 = \operatorname{span}\{x, y, z\}$ that takes J to one of the following. If J_4 is generated by x^4, y^4, then we claim there is an automorphism taking J to the inverse system generated by $\{x^4, y^4, Q, x, y, z\}$ where $Q \in \{z^2, z^2 + xy, z(x + y), zx, xy\}$. If J_4 is generated by x^4, x^3y, then we claim there is an automorphism taking J to the inverse system generated by $\{x^4, x^3y, Q, x, y, z\}$ where $Q \in \{z^2, z^2 + y^2, yz, y^2 + xz, y^2, xz\}$.

First suppose J is generated by x^4, y^4, Q, x, y, z. Write $Q = axy + bxz + cyz + dz^2$, where we can eliminate x^2, y^2 terms since $x^2, y^2 \in J_2$. If $d \neq 0$ then replacing z with a suitable linear combination of z, x, y allows us to eliminate the xz, yz terms by completing the square, as well as simultaneously rescaling z to get rid of the coefficient d. Then $Q = a'xy + z^2$. If $a' = 0$ then $Q = z^2$, and if $a' \neq 0$ then rescaling x, y gives $Q = z^2 + xy$. On the other hand, if $d = 0$, then rescaling x, y, z allows us to get rid of the coefficients a, b, c, so we may assume each of them is 0 or 1. This shows $Q \in \{z^2, z^2 + xy, xy + xz + yz, xy + xz, xy + yz, xz + yz, xy, xz, yz\}$. By symmetry, interchanging x and y allows us to eliminate the cases $xy + yz, yz$ because these are isomorphic to $xy + xz, xz$. Replacing z with $z - y$ takes $xy + xz = x(y + z)$ to xz. Similarly, replacing z with $z - y$ sends $xy + xz + yz$ to $xz + yz - y^2$ and

$$\operatorname{span}\{x^2, y^2, xz + yz - y^2\} = \operatorname{span}\{x^2, y^2, xz + yz\},$$

so this case is also equivalent to $Q = xz$. This finishes the analysis of the case $J_4 = \operatorname{span}\{x^4, y^4\}$.

The case $J_4 = \operatorname{span}\{x^4, x^3y\}$ is similar. Instead of a symmetry interchanging x and y, we can replace y with $y + ax$, since $\operatorname{span}\{x^4, x^3y\} = \operatorname{span}\{x^4, x^3(y + ax)\}$. Write $Q = axz + by^2 + cyz + dz^2$, after eliminating x^2, xy terms. If $d \neq 0$ then a substitution for z eliminates xz, yz terms, yielding $Q = b'y^2 + z^2$. Rescaling y if necessary, $Q = z^2$ or $Q = y^2 + z^2$. If $d = 0$ then rescaling x, y, z to eliminate the a, b, c coefficients gives $Q \in \{xz + y^2 + yz, xz + y^2, xz + yz, y^2 + yz, xz, y^2, yz\}$. Appropriate substitutions for y and z take the cases $xz + y^2 + yz, xz + yz, y^2 + yz$ all to yz.

By Lemma 3.3, each fibre over a point in $\mathscr{H}_3^{\mathbf{h}}$ is irreducible. Thus, it suffices to find a smooth and smoothable inverse system J' such that $\operatorname{lead}(J') = J$ for each of the normal forms J. For the case that J_4 is spanned by x^4 and y^4 see Table 1. For the case that J_4 is spanned by x^4 and x^3y see Table 2. □

Case $\mathbf{h} = (1, 3, 3, 3, 1)$

We consider separately a special case, where the quadrics in the inverse systems have a most special form.

Proposition 4.17 *Let* $\mathbf{h} = (1, 3, 3, 3, 1)$*, let* $I \in H_3^{\mathbf{h}}$ *and let* J *be its inverse system. Suppose* $x^2, xy, y^2 \in J$*. Then* I *is contained in the smoothable component.*

Proof The inverse system J has a quartic generator and its leading form f is uniquely determined. Since $\mathbf{h}(2) = 3$ and $x^2, xy, y^2 \in \operatorname{lead}(J)$, we see that $f \in \Bbbk[x, y]$. We

Table 1 Smooth and smoothable inverse systems J' with Hilbert function $(1,3,3,2,2)$ and $\mathrm{lead}(J')_4$ spanned by x^4, y^4

Q	J'	Deformation of ideal of J'
z^2+xy	$\langle x^4, y^4, z^2+xy\rangle$	$(\beta\gamma, 2\alpha\beta-\gamma^2, \alpha\gamma, \alpha^5, \beta^5+t\beta^4)$
z^2	$\langle x^4+x^2y+x^2z+z^3, y^4, z^2\rangle$	$(\beta\gamma, \alpha\beta-\alpha\gamma, 3\alpha^2\gamma-\gamma^3, \alpha^3-12\alpha\gamma, \beta^5+t\beta^4)$
xy	$\langle x^4+x^2y+x^2z+xy^2, y^4, xy, z\rangle$	$(\gamma^2, \beta\gamma, \alpha\beta^2-\alpha^2\gamma, \alpha^2\beta-\alpha^2\gamma, \alpha^3-12\alpha\gamma, \beta^5+t\beta^4)$
xz	$\langle x^4+xz^2, y^4, xz\rangle$	$(\beta\gamma, \alpha\beta, \alpha^2\gamma, \alpha^3-12\gamma^2, \beta^5+t\beta^4)$
$(x+y)z$	$\langle x^4, y^4, xz+yz\rangle$	$(\gamma^2, \alpha\gamma-\beta\gamma, \alpha\beta, \beta^5, \alpha^5+t\alpha^4)$

Table 2 Smooth and smoothable inverse systems J' with Hilbert function $(1,3,3,2,2)$ and $\mathrm{lead}(J')_4$ spanned by x^4, x^3y

Q	J'	Deformation of ideal of J'
z^2	$\langle x^4, x^3y+x^2z+z^3, z^2\rangle$	$(\beta\gamma, \beta^2, 3\alpha^2\gamma-\gamma^3, \alpha^2\beta-3\alpha\gamma, \alpha^5+t\alpha^4)$
z^2+y^2	$\langle x^4, x^3y, y^2+z^2\rangle$	$(\beta\gamma, \alpha\gamma, \beta^2-\gamma^2, \alpha^4\beta, \alpha^5+t\alpha^4)$
yz	$\langle x^4, x^3y+x^2z, yz\rangle$	$(\gamma^2, \beta^2, \alpha\beta\gamma, \alpha^2\beta-3\alpha\gamma, \alpha^3\gamma, \alpha^5+t\alpha^4)$
y^2+xz	$\langle x^4+2x^2z, x^3y+xyz, y^2+xz\rangle$	$(\gamma^2+t\gamma, \beta^2\gamma, \beta^3, \alpha\beta^2, \alpha^2\beta-6\beta\gamma, \alpha^3-6\alpha\gamma+3\beta^2)$
y^2	$\langle x^4+2x^2z, x^3y+x^2z+xyz, y^2, z\rangle$	$(\gamma^2, \beta^2\gamma, \beta^3+t\beta^2, \alpha\beta^2, \alpha^2\beta-6\beta\gamma, \alpha^3-6\alpha\gamma+12\beta\gamma)$
xz	$\langle x^4, x^3y+x^2z+xz^2, xz\rangle$	$(\beta\gamma, \beta^2, \alpha^2\gamma-\alpha\gamma^2, \alpha^2\beta-3\gamma^2, \alpha^5+t\alpha^4)$

consider two cases. In each case we show that the space of possible J is irreducible and find a smooth and smoothable point there.

Suppose f is annihilated by a linear form in $\Bbbk[\alpha,\beta]$; up to coordinate change, we have $f = x^4$. Consider the family of tuples $(x^4+c+q, c_1+q_1, c_2+q_2, x, y, z)$, where c_i, c are cubics and q_i, q are quadrics, with the condition that $\gamma\circ c, \beta\circ c$ lie in $\mathrm{span}\{x^2, xy, y^2\}$ and also all derivatives of c_i lie in $\mathrm{span}\{x^2, xy, y^2\}$. The space of polynomial tuples satisfying these conditions is an affine space. Each inverse system K generated by a tuple as above has Hilbert function at most $(1,3,3,3,1)$. Thus, a *general* one has Hilbert function exactly $(1,3,3,3,1)$. Denote the irreducible family of such K's by $\mathscr{F}$. Then $\mathscr{F}$ gives a morphism to the Hilbert scheme $H_3^{\mathbf{h}}$ and the image contains J. The image contains also $J_0 = \langle x^4+x^2z, x^2y, xy^2, x, y, z\rangle$. A deformation of its ideal is given by $(\beta\gamma, \gamma^2+t\gamma, \beta^3, \alpha^3-12\alpha\gamma, \alpha^2\beta^2)$. For $t\neq 0$ this is supported at more than one point, hence, $J_0^\perp$ is smoothable. And $J_0^\perp$ is smooth as well, hence the whole image of $\mathscr{F}$ is contained in R_3^{11} by Lemma 1.3.

Suppose now f is not annihilated by a linear form in $\Bbbk[\alpha,\beta]$. Then the proof of the previous case applies with the difference we consider the family of $g+c+q$, c_1+q_1 where $g\in\Bbbk[x,y]_4$ with the condition that $\gamma\circ c$ and all derivatives of c_1 lie in $\mathrm{span}\{x^2, xy, y^2\}$. The smooth and smoothable point is given by the inverse system $\langle x^2y^2+xyz, x^3, z\rangle$ and a deformation of the corresponding ideal is given by $(\gamma^2, \beta^2\gamma, \alpha^2\gamma, \beta^3, \alpha\beta^2-4\beta\gamma, \alpha^2\beta-4\alpha\gamma, \alpha^4+\alpha^3t)$. □

Proposition 4.18 *If* $\mathbf{h} = (1,3,3,3,1)$ *and* J *is a graded inverse system with Hilbert function* $\mathbf{h}$*, then up to coordinate change the vector space* J_2 *is the span of one of the* *following sets:* $\{x^2, xy, y^2\}$*,* $\{x^2, y^2, z^2\}$*,* $\{x^2, yz, z^2\}$*,* $\{xz, yz, z^2\}$*, and* $\{x^2 + yz, xz, z^2\}$*. Setting* $A = T/(J_2^\perp)$*, we have* $\dim A_3 = 4$ *when* $J_2 = \operatorname{span}\{x^2, xy, y^2\}$*, and* $\dim A_3 = 3$ *otherwise.*

Proof Let $I := (J_2^\perp)$. By the Macaulay bound, we have $\dim A_3 \le (\dim A_2)^{\langle 2\rangle} = 3^{\langle 2\rangle} = 4$. If $\dim A_3 = 4$, then by Lemma 3.1 the quadrics in I_2 share a common linear factor. After a change of coordinates, I_2 is spanned by $\{\gamma^2, \beta\gamma, \alpha\gamma\}$, so the ideal J contains the quadrics x^2, xy, y^2.

To reduce to the case $\dim A_3 = 3$ or, equivalently, $\dim I_3 = 7$, we start with the claim that, when $I = (I_2)$ is generated by 3 quadrics and $\dim I_3 = 7$, then I is the saturated ideal of a zero-dimensional degree 3 scheme in $\mathbb{P}^2$. The space $T_1 \otimes I_2$ has dimension 9 and maps by multiplication surjectively to $T_1 \otimes I_2 \to I_3$, so the kernel has dimension 2, which means there are 2 linear syzygies among the quadrics in I_2. The minimal free resolution of $A = T/I$ is equal to

$$0 \leftarrow T \leftarrow T(-2)^{\oplus 3} \leftarrow T(-3)^{\oplus 2} \oplus T(-4)^{\oplus q} \oplus F' \leftarrow T(-4)^{\oplus p} \oplus F'' \leftarrow 0,$$

where F', F'' are sums of $T(-i)$ with $i > 4$. We will show that $p = q = 0$. First, if $p = \beta_{3,4}(I) \ne 0$ then I contains the ideal $\ell(\alpha, \beta, \gamma)$ for some linear form ℓ, by [14, discussion following Theorem 8.15, p. 162]. But this is the case $\dim A_3 = 4$ which we have already treated. Since we are now assuming $\dim A_3 = 3$, then we must have $p = 0$. Next, we compute $\dim A_4$ by considering the free resolution above:

$$\dim A_4 = \dim T_4 - 3 \dim T_2 + (2 \dim T_1 + q \dim T_0) - 0,$$

where the final 0 reflects $p = 0$. This gives $\dim A_4 = 15 - 3 \cdot 6 + 2 \cdot 3 + q = 3 + q$. At the same time, $3 + q = \dim A_4 \le (\dim A_3)^{\langle 3\rangle} = 3^{\langle 3\rangle} = 3$. So $q = 0$.

Now I is generated in degree 2 and $\dim A_4 = (\dim A_3)^{\langle 3\rangle} = 3$. By the Gotzmann Persistence Theorem, $\dim A_k = 3$ for all $k \ge 3$. This shows $Z = \operatorname{Proj} A$ has Hilbert polynomial 3, so $Z = V(I) \subset \mathbb{P}^2$ is zero-dimensional and has degree 3. To see that I is saturated, let I' be the saturation of I. Since the quadrics in I share no common linear factor, Z is not contained in any line, so I' contains no linear forms. Then $\dim(T/I')_1 = 3$. The Hilbert function of a saturated ideal is nondecreasing, so for every $k \ge 1$, $3 = \dim(T/I')_1 \le \dim(T/I')_k \le \dim(T/I)_k = 3$, which shows $I' = I$. This completes the proof of the claim that I is the saturated ideal of a degree 3 zero-dimensional scheme Z in $\mathbb{P}^2$.

Since Z is cut out by quadrics, the intersection of Z with any line has degree at most 2, so Z is one of the following.

- Z may be a disjoint union of three non-collinear reduced points. We change coordinates so that $Z = \{[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]\}$. Then $I_2 = \operatorname{span}\{\alpha\beta, \alpha\gamma, \beta\gamma\}$.
- Z may be the union of a reduced point with a zero-dimensional scheme of degree 2. We choose coordinates so that the reduced point is $[1 : 0 : 0]$ and the scheme

of degree 2 is supported at $[0 : 0 : 1]$ and is contained in the line spanned by $[0 : 0 : 1]$ and $[0 : 1 : 0]$. Then $I_2 = \operatorname{span}\{\alpha\beta, \alpha\gamma, \beta^2\}$.
- Z may be a scheme of degree 3 supported at a point which we may take to be $[0 : 0 : 1]$. Then after a change of coordinates either $I_2 = \operatorname{span}\{\alpha^2, \alpha\beta, \beta^2\}$ or $I_2 = \operatorname{span}\{\alpha^2 - \beta\gamma, \alpha\beta, \beta^2\}$.

To see the last claim, observe that if $q = \ell_1\ell_2 \in I_2$ is a reducible quadric then both components ℓ_1, ℓ_2 pass through $[0 : 0 : 1]$, because Z is not contained in any single line. If every quadric in I_2 is reducible then $I_2 = \operatorname{span}\{\alpha^2, \alpha\beta, \beta^2\}$ consists of all the quadrics that are singular at $[0 : 0 : 1]$. Otherwise, there is a smooth quadric in I_2 and we can choose coordinates so that it is $q = \alpha^2 - \beta\gamma$. A quadric $q' \in I_2$ intersects q in Z plus one more point. If the extra point is also $[0 : 0 : 1]$ then $q' = \beta^2 + \lambda q$ for some scalar λ. Otherwise $q' = \ell\beta + \lambda q$ where ℓ is the line through $[0 : 0 : 1]$ and the extra point. So I_2 is spanned by q, $\alpha\beta$, and β^2. Now, having the normal forms of Z, we calculate J_2 as $I_2^{\perp}$, obtaining the list above. □

Lemma 4.19 *Fix a three-dimensional space of quadrics Q and a subspace $A \subset \langle \alpha, \beta, \gamma \rangle$. Suppose that the derivatives of Q span $\langle x, y, z \rangle$. If the set $\mathscr{J}(Q, A)$ of inverse systems J satisfies*

1. *J has Hilbert function $(1, 3, 3, 3, 1)$,*
2. *Q equals J_2, and*
3. *A is equal to the space of linear forms annihilating the quartic in $\operatorname{lead}(J)$,*

then $\mathscr{J}(Q, A)$ is irreducible or empty.

Proof Suppose $\mathscr{J}(Q, A)$ is non-empty. Let $I = Q^{\perp}$. Let $\mathbf{h}$ be the Hilbert function of T/I. By Proposition 4.18, we have either $Q = \operatorname{span}\{x^2, xy, y^2\}$ up to coordinate change or $\mathbf{h}(3) = 3$. The equality $Q = \operatorname{span}\{x^2, xy, y^2\}$ is impossible, since derivatives of Q span x, y, z. Thus, $\mathbf{h}(3) = 3$. The remaining part of the proof resembles the proof of Proposition 4.17. Let $a = \dim A$. Consider the set $\mathscr{F}$ of tuples of polynomials

$$f + c + q,\ c_1 + q_1, \ldots, c_a + q_a \in S, \tag{7}$$

such that

1. f is homogeneous of degree four and annihilated by A (and possibly other linear forms),
2. all c_i and c are homogeneous of degree three,
3. all q_i and q are homogeneous of degree two,
4. both f and all c_i are annihilated by I,
5. the space $A \circ c$ is contained in Q.

All given conditions are linear in coefficients of polynomials, thus, $\mathscr{F}$ is an affine space. Consider an open (possibly empty) subset $\mathscr{F}_0 \subset \mathscr{F}$ consisting of tuples where f is annihilated exactly by A and such that c_i are linearly independent and span$\{c_i\}$ is disjoint from the space of partial derivatives of f. Then $\mathscr{F}_0$ is irreducible as an open set in affine space.

Consider an inverse system J generated by all linear forms and a tuple in $\mathscr{F}_0$. We now prove that its Hilbert function $\tilde{\mathbf{h}}$ is at most $(1, 3, 3, 3, 1)$ position-wise. By Proposition 2.19, it is enough to show that the Hilbert function of lead(J) is at most $(1, 3, 3, 3, 1)$. It is clear that $\tilde{\mathbf{h}}(0) = \tilde{\mathbf{h}}(4) = 1$ and $\tilde{\mathbf{h}}(1) \leq 3$. All cubic terms in lead(J) are leading forms of combinations of c_i and partials of f. Thus, they are annihilated by I. The space of cubics annihilated by I has dimension $\mathbf{h}(3) = 3$, thus, $\tilde{\mathbf{h}}(3) \leq 3$. Consider now the quadrics in lead(J). They are combinations of leading forms of partials of f, of c_i and also of $A \circ c$. All those forms lie in Q, thus, $\tilde{\mathbf{h}}(2) \leq 3$. Therefore, we have $\tilde{\mathbf{h}} \leq (1, 3, 3, 3, 1)$ position-wise.

Since $(1, 3, 3, 3, 1)$ is the maximal possible value of $\tilde{\mathbf{h}}$, the set $\mathscr{F}_{gen} \subset \mathscr{F}_0$ consisting of systems with Hilbert function $(1, 3, 3, 3, 1)$ is open and irreducible. It gives a map to the Hilbert scheme whose image is $\mathscr{J}(Q, A)$, which is irreducible as well. □

Proposition 4.20 *For the Hilbert function* $\mathbf{h} = (1, 3, 3, 3, 1)$*, we have* $H_3^{\mathbf{h}} \subset R_3^{11}$.

Proof Let J be a graded inverse system with Hilbert function $\mathbf{h} = (1, 3, 3, 3, 1)$. It has a unique degree 4 generator f. We subdivide the cases according to the Hilbert function of the inverse system K generated by f. The Hilbert function is symmetric. Using the Macaulay bound, we find that there are four different possible Hilbert functions for K: $(1, 1, 1, 1, 1)$, $(1, 2, 2, 2, 1)$, $(1, 2, 3, 2, 1)$, and $(1, 3, 3, 3, 1)$.

Case $(1, 3, 3, 3, 1)$: The polynomial f generates the entire module, so J is Gorenstein and every J' in the fibre $\pi_{\mathbf{h}}^{-1}(J)$ is also Gorenstein. By [33, Proposition 2.2] or [32, Corollary 4.3], any Gorenstein subscheme of $\mathbb{A}^3$ is smoothable, so all such points J' lie in the smoothable component of H_3^{11}.

Case $(1, 2, 3, 2, 1)$: Since f has two independent first derivatives, we know it depends on only two variables, say x, y. Since it spans a three-dimensional set of second derivatives, this will have to be $\langle x^2, xy, y^2\rangle$ and Proposition 4.17 completes the proof in this case.

Case $(1, 2, 2, 2, 1)$: Let Q be the space of quadrics inside J. By Proposition 4.17, we may assume that $Q \neq \text{span}\{x^2, xy, y^2\}$ up to coordinate change, so that derivatives of Q span x, y, z. By Proposition 4.18 and Lemma 4.19, we see that the irreducible strata are determined by Q, which has to be equal to span$\{x^2, y^2, z^2\}$, span$\{x^2, yz, z^2\}$, span$\{xz, yz, z^2\}$, or span$\{x^2 + yz, xz, z^2\}$, and by the linear forms annihilating f (up to simultaneous coordinate change). It remains to check which annihilators are possible for each Q. Let $A = T/Q^{\perp}$. By the proof of Proposition 4.18, this is a homogeneous coordinate ring of a zero-dimensional subscheme of Proj T. If a linear form σ annihilates f, then the intersection of Proj A with the projective line $(\sigma = 0)$ has degree at least two. For the four cases, we directly check that the possible annihilators are α or β or γ for $Q = \text{span}\{x^2, y^2, z^2\}$, α or β for $Q = \text{span}\{x^2, yz, z^2\}$, $\lambda_1\alpha + \lambda_2\beta$ with $\lambda_i \in \Bbbk$ arbitrary for $Q = \text{span}\{xz, yz, z^2\}$, and β for $Q = \text{span}\{x^2 + yz, xz, z^2\}$. For the first, third, and fourth case, there is a unique choice up to coordinate change.

Table 3 Smooth and smoothable points J' with Hilbert function $(1, 3, 3, 3, 1)$ such that the inverse system generated by $f \in J'_4$ has Hilbert function $(1, 2, 2, 2, 1)$

Q	$(f^{\perp})_1$	J'	Deformation of ideal of J'
$\text{span}\{x^2, y^2, z^2\}$	γ	$\langle x^4 + y^4, z^3\rangle$	$(\beta\gamma, \alpha\gamma, \alpha\beta, \gamma^4 + t\gamma^3, \alpha^4 - \beta^4)$
$\text{span}\{x^2, yz, z^2\}$	α	$\langle yz^3, x^3\rangle$	Monomial ideal, so smoothable
$\text{span}\{x^2, yz, z^2\}$	β	$\langle x^4 + z^4, yz^2\rangle$	$(\beta^2 + t\beta, \alpha\gamma, \alpha\beta, \beta\gamma^3, \alpha^4 - \gamma^4)$
$\text{span}\{xz, yz, z^2\}$	β	$\langle xz^3, yz^2\rangle$	Monomial ideal, so smoothable
$\text{span}\{x^2 + yz, xz, z^2\}$	β	$\langle xz^3, x^2z + yz^2\rangle$	$(\beta^2, \alpha\beta, \alpha^2 - \beta\gamma, \beta\gamma^3, \gamma^4 + t\gamma^3)$

Table 4 Smooth and smoothable points J' with Hilbert function $(1, 3, 3, 3, 1)$ such that the inverse system generated by $f \in J'_4$ has Hilbert function $(1, 1, 1, 1, 1)$

Q	$(f^{\perp})_1$	J'	Deformation of ideal of J'
$\text{span}\{x^2, y^2, z^2\}$	(α, β)	$\langle z^4 + xz + xy + yz, x^3, y^3\rangle$	$(\alpha\gamma - \beta\gamma, \alpha\beta - \beta\gamma, \alpha^2\beta, \alpha^4 + t\alpha^3, \beta^4, \gamma^4 - 24\alpha\beta)$
$\text{span}\{x^2, yz, z^2\}$	(α, β)	$\langle z^4, x^3 + y^2, yz^2\rangle$	$(\alpha\gamma, \alpha\beta, \beta^2\gamma, \alpha^3 + t\alpha^2 - 3\beta^2, \beta\gamma^3, \gamma^5)$
$\text{span}\{x^2, yz, z^2\}$	(β, γ)	$\langle x^4 + y^2, yz^2, z^3\rangle$	$(\alpha\gamma, \alpha\beta, \beta^2\gamma, \beta\gamma^3, \gamma^4 + t\gamma^3, \alpha\gamma^3, \alpha^4 - 12\beta^2)$
$\text{span}\{xz, yz, z^2\}$	(α, β)	$\langle y^2z + z^4, yz^2, xz^2\rangle$	$(\alpha\beta, \alpha^2 - t\alpha, \gamma^3 - 12\beta^2, \beta^3, \beta^2\gamma^2)$
$\text{span}\{x^2 + yz, xz, z^2\}$	(α, β)	$\langle z^4 + y^2, yz^2 + x^2z, xz^2\rangle$	$(\alpha\beta, \alpha^2 - \beta\gamma, \beta^3, \alpha\gamma^3, \beta\gamma^3, \gamma^4 + t\gamma^3 - 12\beta^2)$

Therefore, we have five distinct cases in total. The corresponding smooth and smoothable points are presented in Table 3.

Case $(1, 1, 1, 1, 1)$: The argument is completely analogous to the previous case up to the point where we determine possible $(f^{\perp})_1$ depending on Q. As before, let $A = T/Q^{\perp}$. An annihilator (σ_1, σ_2) is possible if and only if $l^4 \in J$, equivalently $[l] \in \operatorname{Proj} A$, where l is the linear form annihilated by (σ_1, σ_2) and $[l] \in \operatorname{Proj} T$ is its class. Hence, the possible annihilators in the four cases are (α, β) or (β, γ) or (α, γ) for $Q = \text{span}\{x^2, y^2, z^2\}$, (α, β) or (β, γ) for $Q = \text{span}\{x^2, yz, z^2\}$, (α, β) for $Q = \text{span}\{xz, yz, z^2\}$, and (α, β) for $Q = \text{span}\{x^2 + yz, xz, z^2\}$. The three possibilities for $Q = \text{span}\{x^2, y^2, z^2\}$ are equivalent, thus, we get five cases in total. The list of smooth and smoothable points is presented in Table 4. □

Case $\mathbf{h} = (1, 3, 3, 2, 1, 1)$

Proposition 4.21 *For the Hilbert function* $\mathbf{h} = (1, 3, 3, 2, 1, 1)$, *we have* $H_3^{\mathbf{h}} \subset R_3^{11}$.

Proof Consider a local ideal I with Hilbert function $\mathbf{h} = (1, 3, 3, 2, 1, 1)$, let J be the inverse system of I, choose generators of J, and let f be the generator of J of degree five. Let $\mathbf{h}_f$ be the Hilbert function of the algebra $T/f^{\perp}$ apolar to f. The case decomposes into five subcases, depending on $\mathbf{h}_f$. Since $f^{\perp} \supset I$, $T/f^{\perp}$ is a quotient of $A = T/I$, so that $\mathbf{h}_f \leq (1, 3, 3, 2, 1, 1)$. If $\mathbf{h}_f(3) = 2$ then $\mathbf{h}_f(1), \mathbf{h}_f(2) \geq 2$ by Corollary 2.4.

$\mathbf{h}_f = (1, a, b, 1, 1, 1)$: Here we argue exactly as in the proof of Proposition 4.2, so we omit some details below. After a non-linear change of coordinates, we may assume $f = x^5 + g$ with $\alpha^2 \circ g = 0$. Since J is generated by f together with elements of degree 3 or less, $\alpha^3\beta$ and $\alpha^3\gamma$ annihilate all of J. If $q \in (\beta, \gamma)$ is such that $\alpha^c - q$ annihilates f, then $c \geq 4$; hence, $\alpha^3 \notin I + (\beta, \gamma)$. Now Corollary 3.8 proves that the element is cleavable, hence, smoothable by Lemma 1.4.

$\mathbf{h}_f = (1, 3, 3, 2, 1, 1)$: In this case $I = f^{\perp}$ and so A is Gorenstein, hence, smoothable by [33, Proposition 2.2] or [32, Corollary 4.3].

$\mathbf{h}_f = (1, 2, 3, 2, 1, 1)$: After changing coordinates, we have $J = \langle f, z \rangle$, so A is smoothable by Corollary 3.9.

$\mathbf{h}_f = (1, 3, 2, 2, 1, 1)$: In this case after a non-linear change of coordinates we have $f = g + z^2$ for $g \in \Bbbk[x, y]$ with Hilbert function $(1, 2, 2, 2, 1, 1)$, see [9, Proposition 4.5, Example 4.6]. The set of those g is irreducible by a result of Iarrobino [9, Proposition 4.8]. Thus, the set of f is also irreducible. J is generated by f and a quadric q which may be chosen arbitrarily, modulo $\langle f \rangle_2$, thus, the set of pairs (f, q) is irreducible. It is now enough to find a smooth or smoothable point. Such a point is given by the ideal

$$I = \langle x^5 + y^4 + z^2, xz \rangle^{\perp} = \lim_{t \to 0} (\beta\gamma, \alpha\beta, \alpha\beta^2, \alpha^2\beta, \beta^4 + t\beta^3 - 12\gamma^2, \alpha^5 - 60\gamma^2)\,.$$

$\mathbf{h}_f = (1, 2, 2, 2, 1, 1)$: As in the previous case, after a nonlinear change of coordinates we get $f \in \Bbbk[x, y]$ and the set of $f \in \Bbbk[x, y]$ is irreducible by [9, Proposition 4.8]. J is generated by f and a quadric q with no relations. For general q after adding a multiple of q to f we get f_{new} with $\mathbf{h}_{f_{\text{new}}} = (1, 3, 2, 2, 1, 1)$ and, thus, reduce to the previous case. ☐

Case $\mathbf{h} = (1, 3, 6, 1)$

Proposition 4.22 *For the Hilbert function* $\mathbf{h} = (1, 3, 6, 1)$*, we have* $H_3^{\mathbf{h}} \subset R_3^{11}$.

This case is in contrast with previous ones. First, it is easy to check that $H_3^{\mathbf{h}} = \mathscr{H}_3^{\mathbf{h}}$, that is, every local ideal with Hilbert function $\mathbf{h}$ is homogeneous. And second, it is also easy to check that $\mathscr{H}_3^{\mathbf{h}}$ is irreducible, in fact a $\mathbb{P}^9 = \mathrm{Gr}(9, 10)$. The isomorphism is given by sending an ideal to all its cubic equations. However, in this set there seem to be no smooth points. We argue by showing that general points in $\mathscr{H}_3^{\mathbf{h}}$ are smoothable, then by irreducibility so are all points.

Proof Let $I \in H_3^{\mathbf{h}}$ with inverse system J. Necessarily I and J are homogeneous. Let $f \in J$ be the cubic generator, which is unique up to scalar. We see that $H_3^{\mathbf{h}} = \mathscr{H}_3^{\mathbf{h}}$ is parametrized by the point $[f]$ in the projective space of cubics in three variables, a $\mathbb{P}^9$, so once again $H_3^{\mathbf{h}}$ is irreducible. The cubic f is the equation of a plane cubic curve. Assume it is a smooth curve. Then we may change coordinates so that f is in Hesse normal form, that is $f = x^3 + y^3 + z^3 + 6hxyz$ for some h, see for example [11, Sect. 3.1.2]. Now $J = \langle f, S_2 \rangle$. We may directly compute

$$I = J^{\perp} = (\alpha\beta^2, \alpha^2\beta, \alpha\gamma^2, \alpha^2\gamma, \beta\gamma^2, \beta^2\gamma, \alpha^3 - \beta^3, \alpha^3 - \gamma^3, \alpha\beta\gamma - h\gamma^3)\,.$$

Corollary 3.8 with $c = 2$ implies that I is cleavable and smoothable. Thus, all I corresponding to smooth curves are smoothable, so by irreducibility $H_3^{\mathbf{h}} \subset R_3^{11}$. □

5 Smoothability of Very Compressed Algebras

In this section, we prove Proposition 1.2, in other words we show that each element of $\mathscr{H}^{\max,d}$ is smoothable. We begin by noting that the scheme is irreducible.

Lemma 5.1 *The locus $\mathscr{H}^{\max,d}$ is irreducible.*

Proof A scheme in $\mathscr{H}^{\max,d}$ is uniquely determined by choice of I inside $\mathfrak{m}^s$ and containing $\mathfrak{m}^{s+1}$. Hence, $\mathscr{H}^{\max,d}$ is a Grassmannian and, thus, irreducible. □

To check smoothability, we verify that a general point of the stratum is obtained as a $\Bbbk^*$*-limit*, a notion which we now explain. The scaling (homothety) action of $\Bbbk^*$ on $\mathbb{A}^3$ extends to an action on $\mathbb{P}^3$. Take a set Γ of d points in $\mathbb{P}^3$. For every $t \in \Bbbk^*$ we may take $t \cdot \Gamma$. The $\Bbbk^*$*-limit* of Γ is $\Gamma' = \lim_{t\to 0}(t \cdot \Gamma)$. This is a flat limit, in the sense of [25, Proposition III.9.8]. It is constructed as follows. Take the graph of the $\Bbbk^*$-action, which is a family $Z_\Gamma^\circ \subset \Bbbk^* \times \mathbb{P}^3$, whose fibre over $t \in \Bbbk^*$ is $t\Gamma$. This family is just the union of n lines in $\Bbbk^* \times \mathbb{P}^3$ through the points $(1, p)$, where $p \in \Gamma$. All its fibres are isomorphic to Γ and it is flat over $\Bbbk^*$. Let $Z_\Gamma \subset \Bbbk \times \mathbb{P}^3$ be the closure of Z_Γ°. This family is flat over $\Bbbk$, see [25, Proposition III.9.8]. Finally let $\Gamma' = Z_\Gamma \cap (t = 0)$. By construction, Γ' is smoothable (as a limit of Γ) and $\Bbbk^*$-invariant.

A general set Γ of d points imposes independent conditions on forms, hence the ideal defining the limit scheme has no small-degree generators. For example, for $d = 11$ the algebra Γ' has Hilbert function $\mathbf{h} = (1, 3, 6, 1)$. After restricting to general Γ's, the $\Bbbk^*$-limit can be made relative [8, Proof of Lemma 5.4] and we get a rational map $\varphi_d\colon R^d\mathbb{P}^3 \dashrightarrow \mathscr{H}^{\max,d}$, where $R^d\mathbb{P}^3$ is the smoothable component of the Hilbert scheme of points of $\mathbb{P}^3$.

Lemma 5.2 *The map φ_d is dominating for all* $8 \le d \le 95$.

Proof First, we prove that for every $8 \le d \le 95$ there is a smooth point $x \in R^d\mathbb{P}^3$ such that the tangent map $T_{\varphi_d}\colon \left(TR^d\mathbb{P}^3\right)_x \to \left(T\mathscr{H}^{\max,d}\right)_{\varphi_d(x)}$ is surjective. This is verified by a direct computer calculation; see *CombalggeomApprenticeship-sHilbert.m2* [12]. By [24, Theorem 17.11.1d, p. 83] the morphism f is smooth at x, thus flat, thus open, and thus the claim. □

Proposition 5.3 *For all $d \le 95$ all schemes in $\mathscr{H}^{\max,d}$ are smoothable.*

Proof By Lemma 5.1, the locus $\mathscr{H}^{\max,d}$ is irreducible. For $d < 8$ all schemes are smoothable by [8]. Assume $d \ge 8$. By Lemma 5.2, the map φ_d is dominating. Hence, a general element of $\mathscr{H}^{\max,d}$ is smoothable. But smoothability is a closed property, thus, all elements of $\mathscr{H}^{\max,d}$ are smoothable. □

Proof of Proposition 1.2 When $d \leq 95$ the claim follows from Proposition 5.3. When $d \geq 96$ the claim follows by dimension count, as in [27]. □

Remark 5.4 (Comparison with the Case of Eight Points in $\mathbb{A}^4$) From the case $d = 96$ onwards we do not get a surjective tangent map and, indeed, the dimension of the family $\mathscr{H}^{\max,96} = \mathscr{H}_3^{(1,3,6,10,20,35,21)}$ is equal to $3 \cdot 96$, thus, a general member of this family cannot be smoothable for dimensional reasons. (Points in $\mathscr{H}^{\max,96}$ define schemes supported at a single point, so $\mathscr{H}^{\max,96}$ would have to be contained in the boundary of R_3^{96}.) This was the original example of [27]. Our methods show that $\mathscr{H}^{\max,d}$ is smoothable for all $d \leq 95$, hence the bound $d = 96$ obtained in [27] is sharp for this method. Note that in [28] another, only partially related, method was used to prove that H_3^d is reducible for $d \geq 78$. It is currently unclear whether this other method can yield irreducible components for $d \leq 77$.

Even though T_{φ_d} is not surjective for $d \geq 96$, we conjecture that the maps T_{φ_d} are of maximal rank. This is no longer true for $\mathbb{A}^4$: in fact $T_{\varphi_8\,\mathbb{A}^4}$ has 20-dimensional image in the 21-dimensional Grassmannian Gr(3, 10), which accounts for the fact that there are nonsmoothable ideals of degree 8 in $\mathbb{A}^4$, as proven in [8]. An explicit example of such a scheme in $\mathbb{A}^4 = \operatorname{Spec} \Bbbk[\alpha, \beta, \gamma, \delta]$ is given by the ideal $(\alpha^2, \alpha\beta, \beta^2, \alpha\delta + \beta\gamma, \gamma^2, \gamma\delta, \delta^2) = \langle xz, xw, yz, yw, xy - zw\rangle^{\perp}$, see [8, Proposition 5.1]. This scheme gives an answer to [41, Problem 3 on Parameters and Moduli].

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Towards a Tropical Hodge Bundle

Bo Lin and Martin Ulirsch

Abstract The moduli space $\mathrm{M}_g^{\mathrm{trop}}$ of tropical curves of genus g is a generalized cone complex that parametrizes metric vertex-weighted graphs of genus g. For each such graph $\varGamma$, the associated canonical linear system $|K_\varGamma|$ has the structure of a polyhedral complex. In this article, we propose a tropical analogue of the Hodge bundle on $\mathrm{M}_g^{\mathrm{trop}}$ and study its basic combinatorial properties. Our construction is illustrated with explicit computations and examples.

MSC 2010 codes: 14T05

1 Introduction

Let $g \geq 2$ and denote by $\mathscr{M}_g$ the moduli space of smooth algebraic curves of genus g. The *Hodge bundle* $\varLambda_g$ is a vector bundle on $\mathscr{M}_g$ whose fibre over a point $[C]$ in $\mathscr{M}_g$ is the vector space $H^0(C, \omega_C)$ of holomorphic differentials on C. One can think of the total space of $\varLambda_g$ as parametrizing pairs (C, ω) consisting of a smooth algebraic curve and a differential ω on C. Since for every curve C the canonical linear system $|K_C|$ can be identified with the projectivization $\mathbb{P}\big(H^0(C, \omega_C)\big)$, the total space of the projectivization $\mathscr{H}_g := \mathbb{P}(\varLambda_g)$ of $\varLambda_g$ parametrizes pairs (C, D) consisting of a smooth algebraic curve C and a canonical divisor D on C; it is referred to as the *projective Hodge bundle*. Let $\pi\colon \mathscr{C}_g \to \mathscr{M}_g$ be the universal curve on $\mathscr{M}_g$. We may define $\varLambda_g$ formally as the pushforward $\pi_*\omega_g$ of the relative dualizing sheaf ω_g on $\mathscr{C}_g$ over $\mathscr{M}_g$.

The Hodge bundle is of fundamental importance when describing the geometry of $\mathscr{M}_g$. For example, its Chern classes, the so-called λ-*classes*, form an important

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collection of elements in the tautological ring on $\mathcal{M}_g$; see [31] for an introductory survey. The Hodge bundle admits a natural stratification by prescribing certain pole and zero orders $(m_1, m_2, \ldots, m_n)$ such that $m_1 + m_2 + \cdots + m_n = 2g - 2$ and the study of natural compactifications of these components has recently seen a surge from the perspective of algebraic geometry and Teichmüller theory; see [4].

In tropical geometry, the natural analogue of $\mathcal{M}_g$ is the moduli space $\mathrm{M}_g^{\mathrm{trop}}$ that parametrizes isomorphism classes $[\Gamma]$ of stable tropical curves Γ of genus g. In Sect. 2, we recall the construction of this moduli space. In particular, we see how this moduli space naturally admits the structure of a generalized cone complex whose cones are in a order-reversing one-to-one correspondence with the boundary strata of the Deligne-Mumford compactification $\overline{\mathcal{M}}_g$ of $\mathcal{M}_g$; see Sect. 2 or [1]. We refer the reader to [19, 20, 24, 25] for the theory in genus $g = 0$ (with marked points), to [7, 10, 13, 16, 32] for its connections to the tropical Torelli map, to [1, 11, 12, 28, 30, 33] for connections to non-Archimedean analytic geometry, and to [14, 15] for an in-depth study of the topology of $\mathrm{M}_{g,n}^{\mathrm{trop}}$. We also highlight the two survey papers [8, 9].

Let Γ be a tropical curve. We denote by K_Γ the canonical divisor on Γ and by $\mathrm{Rat}(\Gamma)$ the group of piecewise integer linear functions on Γ; see Sect. 3. In this note, we propose tropical analogues of the affine and the projective Hodge bundle, and study their basic combinatorial properties.

Definition 1.1 As a set, the *tropical Hodge bundle* $\Lambda_g^{\mathrm{trop}}$ is given as

$$\Lambda_g^{\mathrm{trop}} := \left\{([\Gamma], f) : [\Gamma] \in \mathrm{M}_g^{\mathrm{trop}} \text{ and } f \in \mathrm{Rat}(\Gamma) \text{ such that } K_\Gamma + (f) \geq 0\right\}$$

and the *projective tropical Hodge bundle* $\mathcal{H}_g^{\mathrm{trop}}$ is given as

$$\mathcal{H}_g^{\mathrm{trop}} := \left\{([\Gamma], D) : [\Gamma] \in \mathrm{M}_g^{\mathrm{trop}} \text{ and } D \in |K_\Gamma|\right\}.$$

The maps $\big([\Gamma], f\big) \mapsto [\Gamma]$ and $\big([\Gamma], D\big) \mapsto [\Gamma]$ define projections $\Lambda_g^{\mathrm{trop}} \longrightarrow \mathrm{M}_g^{\mathrm{trop}}$ and $\mathcal{H}_g^{\mathrm{trop}} \longrightarrow \mathrm{M}_g^{\mathrm{trop}}$ that are, in a slight abuse of notation, both denoted by π_g.

In [18, 22, 26], the authors describe the structure of a polyhedral complex on the linear system $|D|$ associated to a divisor D on a tropical curve Γ; we review this description in Sect. 3. Moreover, the paper [23] presents algorithms for computing this polyhedral complex. Our main result is the following.

Theorem 1.2 *Let* $g \geq 2$.

1. *The tropical Hodge bundle* $\Lambda_g^{\mathrm{trop}}$ *and the projective tropical Hodge bundle* $\mathcal{H}_g^{\mathrm{trop}}$ *carry the structure of a generalized cone complex.*
2. *The dimensions of* $\Lambda_g^{\mathrm{trop}}$ *and* $\mathcal{H}_g^{\mathrm{trop}}$ *are* $5g - 4$ *and* $5g - 5$ *respectively.*
3. *There is a proper subdivision of* $\mathrm{M}_g^{\mathrm{trop}}$ *such that, for all* $[\Gamma]$ *in the relative interior of a cone in this subdivision, the canonical linear systems* $|K_\Gamma| = \pi_g^{-1}([\Gamma])$ *have the same combinatorial type.*

We refer to the subdivision of $\mathrm{M}_g^{\mathrm{trop}}$ appearing in part 3 as the *wall-and-chamber* decomposition of $\mathrm{M}_g^{\mathrm{trop}}$. In general, the generalized cone complexes $\Lambda_g^{\mathrm{trop}}$ and $\mathcal{H}_g^{\mathrm{trop}}$

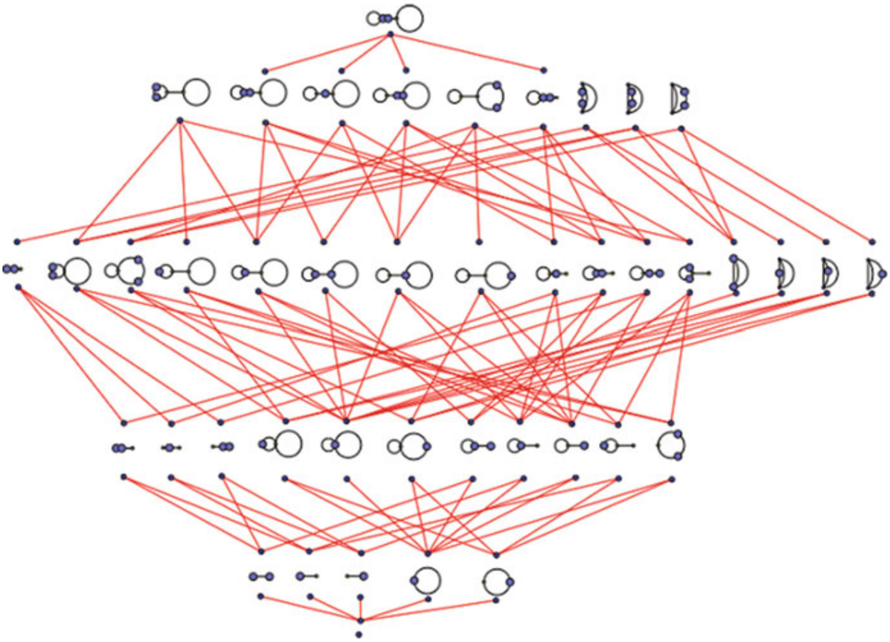


Fig. 1 The face lattice of $\mathscr{H}_2^{\mathrm{trop}}$

are not equidimensional, so part 2 simply states that the dimension of a maximal-dimensional cone in $\Lambda_g^{\mathrm{trop}}$ and $\mathscr{H}_g^{\mathrm{trop}}$ has dimension $5g-4$ and $5g-5$ respectively.

As a first example, we depict in Fig. 1 the face lattice of the tropical Hodge bundle in the case $g=2$; the numbers in purple are the positive $h(v)$ and the numbers in black denote coefficients greater than 1 in the divisors.

We outline the contents of the article. In Sect. 2–3, we review the construction of the moduli space $\mathrm{M}_g^{\mathrm{trop}}$ of stable tropical curves and the polyhedral structure of linear systems on tropical curves respectively. In Sect. 4, we prove Theorem 1.2 by simultaneously describing the polyhedral structures on both $\Lambda_g^{\mathrm{trop}}$ and $\mathscr{H}_g^{\mathrm{trop}}$. Section 5 contains a selection of explicit (sometimes partial) calculations of the polyhedral structure of $\mathscr{H}_g^{\mathrm{trop}}$ in some small genus cases. Finally, in Sect. 6, we describe a tropicalization procedure for the projective algebraic Hodge bundle via non-Archimedean analytic geometry and pose a natural *realizability problem*.

2 Moduli of Tropical Curves

A *tropical curve* is a finite metric graph Γ, with a fixed minimal model G, together with a genus function $h\colon V(G)\to\mathbb{Z}_{\geq 0}$. The *genus* of Γ or G is defined to be

$$g(\Gamma)=g(G)=b_1(G)+\sum_{v\in V(G)}h(v)\,,$$

where $b_1(G)$ denotes the Betti number of G. In the above sum, one should think of the vertex-weight terms as the contributions of $h(v)$ infinitesimally small loops at every vertex v. We say a tropical curve Γ (or the graph G) is *stable* if, for every vertex $v \in V(G)$, we have

$$2h(v) - 2 + |v| > 0\,, \tag{1}$$

where $|v|$ denotes the valence of G at v.

Definition 2.1 As a set, the *moduli space* $\mathrm{M}_g^{\mathrm{trop}}$ *of stable tropical curves of genus g* is $\mathrm{M}_g^{\mathrm{trop}} := \{$isomorphism classes $[\Gamma]$ of stable tropical curves of genus$g\}$.

Following [1], we recall the description of $\mathrm{M}_g^{\mathrm{trop}}$ as a generalized cone complex.

Proposition 2.2 ([1, Sect. 4]) *The moduli space* $\mathrm{M}_g^{\mathrm{trop}}$ *carries the structure of an equidimensional generalized rational polyhedral cone complex of dimension* $3g-3$.

A morphism $\tau \to \sigma$ between rational polyhedral cones is a *face morphism* if it induces an isomorphism onto a face of σ. In particular, the class of face morphisms includes all isomorphisms. A *generalized (rational polyhedral) cone complex* is a topological space Σ that arises as a colimit of a finite diagram of face morphisms; see [1, Sect. 2] or [29, Sect. 3.5].

In order to understand this structure on $\mathrm{M}_g^{\mathrm{trop}}$, we exploit a particular presentation as a colimit, namely $\mathrm{M}_g^{\mathrm{trop}} = \varinjlim \widetilde{M}_G$, where the rational polyhedral cones $\widetilde{M}_G$ and the underlying category J_g are defined as follows.

1. The objects in J_g are stable vertex-weighted graphs (G, h) of genus g. A *weighted edge contraction* $c \colon G \to G/e$ is an edge contraction such that, for every vertex v in G/e, we have $g\big(c^{-1}(v)\big) = h(v)$. The morphisms in J_g are generated by *weighted edge contractions* $G \to G/e$ for an edge e of G together with the automorphisms of all (G, h).
2. For every graph G, we denote by $\widetilde{M}_G = \mathbb{R}_{\geq 0}^{E(G)}$ the parameter space of all possible edge lengths on G. The assignment $G \mapsto \widetilde{M}_G$ defines a contravariant functor $J_g \to \mathbf{RPC}_{\mathbb{Z}}$ from the category J_g to the category of rational polyhedral cones. It associates to a weighted edge contraction $G \to G/e$ the embedding of the corresponding face of $\widetilde{M}_G^{\mathrm{trop}}$ and to an automorphism of G the automorphism of $\widetilde{M}_G$ that permutes the entries correspondingly.

From this colimit description, we obtain a decomposition of $\mathrm{M}_g^{\mathrm{trop}}$ into locally closed subsets

$$\mathrm{M}_g^{\mathrm{trop}} = \bigsqcup_G \mathbb{R}_{>0}^{E(G)} / \operatorname{Aut}(G)\,,$$

where the disjoint union is taken over all isomorphism classes of stable finite vertex-weighted graphs G of genus g.

Example 2.3 ([13, Theorem 2.12]) For a d-dimensional cone complex C, its f-vector is defined as $(f_0, f_1, \ldots, f_d)$, where f_i is the number of i-dimensional cones in C. The 12-dimensional moduli space $\mathrm{M}_5^{\mathrm{trop}}$ has 4555 cells and its f-vector is given by

$$f(\mathrm{M}_5^{\mathrm{trop}}) = (1, 3, 11, 34, 100, 239, 492, 784, 1002, 926, 632, 260, 71).$$

Remark 2.4 Earlier approaches, such as [7, 8, 10, 13, 16, 32], refer to the structure of a generalized cone complex as a *stacky fan*. Since there is a closely related, but not equivalent, notion with the same name in the theory of toric stacks, we prefer to follow the terminology of generalized cone complexes introduced in [1].

3 Linear Systems on Tropical Curves

Expanding on [6], a *divisor* on a tropical curve Γ is a finite formal $\mathbb{Z}$-linear sum $D = \sum_i a_i p_i$ over points p_i in Γ. In other words, D is an element in the free abelian group $\mathrm{Div}(\Gamma)$ on the points of Γ. The *degree* $\deg(D)$ of a divisor $D = \sum_i a_i p_i$ is the integer $\sum_i a_i$. We say $D = \sum_i a_i p_i$ is *effective* if $a_i \geq 0$ for all i.

A *rational function* on Γ is a continuous function $f\colon \Gamma \to \mathbb{R}$ whose restriction to every edge is a piecewise linear integral affine function. Given a rational function f on Γ and a point $p \in \Gamma$, the *order* $\mathrm{ord}_p(f)$ of f at p is the sum of the outgoing slopes of f emanating from p. Since $\mathrm{ord}_p(f)$ is equal to zero for all but finitely many points $p \in \Gamma$, we have a map from $\mathrm{Rat}(\Gamma)$ to $\mathrm{Div}(\Gamma)$ given by $f \mapsto (f) := \sum_p \mathrm{ord}_p(f) \cdot p$. Divisors of the form (f), for some function $f \in \mathrm{Rat}(\Gamma)$, are called *principal divisors* on Γ and form the subgroup $\mathrm{PDiv}(\Gamma)$ of $\mathrm{Div}(\Gamma)$. Moreover, the continuity of f implies that $\deg(f) = 0$. Two divisors D and D' on Γ are *equivalent*, written as $D \sim D'$, if there is a rational function $f \in \mathrm{Rat}(\Gamma)$ such that $D + (f) = D'$ or simply if $D - D' \in \mathrm{PDiv}(\Gamma)$.

Let us now define the main players of this game.

Definition 3.1 Let D be a divisor of degree n on a tropical curve Γ, and consider the set $R(D) := \{f \in \mathrm{Rat}(\Gamma) : D + (f) \geq 0\}$. For $f \in R(D)$, the divisor $D + (f)$ is supported on $\deg\big(D + (f)\big) = \deg(D) = n$ points, counted with multiplicity, so we define

$$S(D) := \left\{ (f, p_1, p_2, \ldots, p_n) : \begin{array}{c} f \in \mathrm{Rat}(\Gamma) \text{ and } p_1, p_2, \ldots, p_n \in \Gamma \text{ such} \\ \text{that } D + (f) = p_1 + p_2 + \cdots + p_n \geq 0 \end{array} \right\}.$$

The associated *linear system* is the set $|D| := \{D' \in \mathrm{Div}(\Gamma) : D' \geq 0 \text{ and } D' \sim D\}$.

Observe that $R(D) = S(D)/\mathfrak{S}_n$, where the symmetric group $\mathfrak{S}_n$ acts on $S(D)$ by permutation of the points $p_1, p_2, \ldots, p_n$. Moreover, the additive group $\mathbb{R} = (\mathbb{R}, +)$ operates on $R(D)$ by adding a constant function. Taking the quotient under this operation, we obtain $R(D)/\mathbb{R} = |D|$, because $(f) = 0$ if and only if f is a constant function on Γ.

The sets $S(D)$, $R(D)$, and $|D|$ are known to carry the structure of a polyhedral complex; see [18, 26]. The following proposition is a more detailed version of Lemma 1.9 in [18].

Proposition 3.2 *Given a divisor D on a tropical curve Γ, the space $S(D)$ has the structure of a polyhedral complex. More precisely, choose an orientation for each edge e of Γ and identify it with the interval $[0, l(e)]$. The cells of $S(D)$ are described by the following discrete data:*

- *a partition of $\{p_1, p_2, \ldots, p_n\}$ into disjoint subsets P_e and P_v, indexed by $v \in V(G)$ and edges $e \in E(G)$, that indicates on which edge or at which vertex every point p_i is located,*
- *a total order on each P_e, and*
- *the outgoing slope $m_e \in \mathbb{Z}$ of f at the starting point of e*

such that, for every vertex $v \in V(G)$, we have

$$\#P_v = D(v) + \sum_{\text{outward edges at } v} m_e + \sum_{\text{inward edges at } v} -(\#P_e + m_e)\,.$$

Furthermore, this polyhedral structure descends from $S(D)$ to $R(D) = S(D)/S_n$ and $|D| = R(D)/\mathbb{R}$.

Proof Set $d_v := \#P_v$ and $d_e := \#P_e$. We claim that the points in a cell of $S(D)$ can be parametrized by two types of continuous data: the value $f(v)$ at a vertex v, and the distance $d(p_i^e)$ of every point $p_i^e \in P_e$ on the edge $0 \in e = [0, l(e)]$.

The distance $d(p_i^e)$ immediately determines the point p_i. In order to reconstruct f (if it exists), we write $\sum_i p_i^e = \sum_j d_{e,j} x_j$ for points $0 < x_1 < x_2 < \cdots < x_r < l(e)$ on e, where the positive integers $d_{e,j}$ indicate the number of points p_i^e that are all located at the same point x_j. The rational function f is then determined by taking the value $f(v)$ at the origin of every edge $e = [0, l(e)]$ and continuing it piecewise linearly with slope m_e until we hit x_1, at which point we change the slope to $d_{e,1} + m_e$ until we hit x_2, where we change the slope to $d_{e,2} + d_{e,1} + m_e$, and so on until we hit the vertex v' at the end of $e = [0, l(e)]$. By continuity, for every edge, we obtain the linear condition on the parameters of a cell in $S(D)$:

$$\begin{aligned} f(v') &= f(v) + m_e x_1 + \sum_{k=1}^{r} \left(m_e + \sum_{j=1}^{k} d_{e,j} \right) (x_{k+1} - x_k) \\ &= f(v) + m_e l(e) + \sum_{i=1}^{r} d_{e,i} \big(l(e) - x_i \big)\,. \end{aligned}$$

Together with the inequalities $0 < x_1 < x_2 < \cdots < x_r < l(e)$, these linear conditions determine the polyhedral structure of a cell in $S(D)$. These parameters are still overdetermined in the sense that there may be no rational function f which satisfies these linear conditions and $D + (f) = p_1 + p_2 + \cdots + p_n \geq 0$. In this case, we obtain an empty cell.

The linear conditions on the cells of $S(D)$ are all discrete, and the points within one cell are all parametrized by the distances $d(p_i^e) \in \big(0, l(e)\big)$ and the values $f(v)$ subject to these discrete conditions. Therefore, the set $S(D)$ is a polyhedral complex that does not depend on the choice of the orientation of Γ.

The action of $\mathfrak{S}_n$ on every cell is affine linear, so the polyhedral structure descends to $R(D)$. Moreover, the additive group $\mathbb{R}$ acts on $R(D)$ by adding a constant to all $f(v)$. Therefore, the polyhedral structure also descends to $|D|$. □

We end with a technical lemma that will be used in the next section.

Lemma 3.3 *Let Γ be a tropical curve with minimal model $G = (V, E)$. If D is a divisor on Γ such that the support of D is contained in V, then the combinatorial structure of $|D|$ is independent of the length of any loop or bridge in G.*

Proof Suppose $e_b \in E$ is a bridge in G. Let Γ_1 and Γ_2 be two tropical curves with minimal model G such that

- there exists positive constants l and c such that the length of e_b in Γ_1 and Γ_2 is l and cl respectively, and
- for all $e \in E \setminus \{e_b\}$, the lengths of e in Γ_1 and Γ_2 are the same.

It suffices to show that the sets of cells in $|D|_{\Gamma_1}$ and $|D|_{\Gamma_2}$ are exactly the same.

On the first tropical curve Γ_1, we view the bridge e_b as the open interval $(0, l)$. For any cell C_1 in $|D|_{\Gamma_1}$, its data consist of an integer m_{e_b} and a partition of nonnegative integers $d_{e_b} = \sum_{j=1}^r d_{e,j}$. Suppose a divisor $D + (f_1)$ is $\sum_{j=1}^r d_{e,j} x_j$ on the bridge e_b, where $0 < x_1 < x_2 < \cdots < x_{r(e)} < l$. Since the rational function f_1 is unique up to a translation, we may assume that the value of f_1 is zero on the endpoint 0 of e_b. It follows that, for all $0 < x \leq x_1$, we have $f_1(x) = m_{e_b} x$ and, for all $1 \leq k \leq r(e)$ and all $x_k < x < x_{k+1}$, we have $f_1(x) = f_1(x_k) + \left(m_{e_b} + \sum_{j=1}^k d_{e_b,j}\right)(x - x_k)$.

Now, on the second tropical curve Γ_2, we view the bridge e_b as the open interval $(0, cl)$ with the same orientation as on Γ_1. We construct a rational function f_2 on Γ_2 as follows. Since the bridge e_b is identified with $(0, cl)$, we set $f_2(x) := m_{e_b} x$, for all $0 < x \leq cx_1$, and $f_2(x) := f_1(cx_k) + \left(m_{e_b} + \sum_{j=1}^k d_{e_b,j}\right)(x - cx_k)$, for all $1 \leq k \leq r(e)$ and all $cx_k < x < cx_{k+1}$. Since e_b is a bridge in G, the graph $G - e_b$ consists of two connected components. We denote them by G_1 and G_2, where G_1 contains the endpoint 0 of e_b and G_2 contains the endpoint cl of e_b. For notational convenience, we set

$$f_1(l) := m_{e_b} x_1 + (m_{e_b} + d_{e_b,1}) x_2 + \cdots + \left(m_{e_b} + \sum_{j=1}^r d_{e_b,j}\right)(l - x_{r(e)})$$

and $f_2(cl) := cf_1(l)$. We define f_2 on $G - e_b$ as follows:

$$f_2(x) := \begin{cases} f_1(x) & \text{if } x \in G_1, \\ f_1(x) + f_2(cl) - f_1(l) = f_1(x) + (c-1)f_1(l) & \text{if } x \in G_2. \end{cases}$$

By construction, the rational functions f_1 and f_2 correspond to the same data on G_i for $1 \le i \le 2$. In addition, on the bridge e_b, both functions correspond to the integer m_{e_b} and the partition $\sum_{j=1}^{r} d_{e_b,j}$. Hence, f_2 corresponds to a cell C_2 in $|D|_{\Gamma_2}$ that is exactly the same as C_1. Since $l = \frac{1}{c}(cl)$, we can interchange Γ_1 and Γ_2, so the cell C_1 can be obtained from C_2. Therefore, Lemma 3.3 holds for bridges.

Suppose e_l is a loop in G. In this case almost the same proof works, except that we have $f_1(l) = 0$ and $G' := G - e_l$ is connected. Since $f_2(cl) = 0$, we may define f_2 on e_l in the same way as on e_b and $f_2(x) = f_1(x)$ for all $x \in G'$. Thus, our claim also holds for loops. □

4 Structure of the Tropical Hodge Bundle

Let Γ be a tropical curve with a fixed minimal model G. As explained in Sect. 5.2 of [3], the canonical divisor on Γ is defined to be

$$K_\Gamma = K_G := \sum_{v \in V(G)} \big(2h(v) + |v| - 2\big)(v)\,,$$

where $|v|$ denotes the valence of the vertex v. It follows that $\deg(K_\Gamma) = 2g - 2$. The $h(v)$-term in the sum should be thought of as contributing $h(v)$ infinitesimally small loops at the vertex v. In fact, given a semistable curve C whose dual graph is G, the canonical divisor is the multidegree of the dualizing sheaf on C; see [2, Remark 3.1].

To understand the structure of the tropical Hodge bundle $\Lambda_g^{\mathrm{trop}}$ from Definition 1.1, we consider the pullbacks of $\Lambda_g^{\mathrm{trop}}$ and $\mathscr{H}_g^{\mathrm{trop}}$ to $\widetilde{M}_G$, namely

$$\widetilde{\Lambda}_G := \big\{([\Gamma], f) : [\Gamma] \in \widetilde{M}_G \text{ and } f \in \mathrm{Rat}(\Gamma) \text{ such that } K_\Gamma + (f) \ge 0\big\}\,,$$

$$\widetilde{\mathscr{H}}_G := \big\{([\Gamma], D) : [\Gamma] \in \widetilde{M}_G \text{ and } D \in |K_\Gamma|\big\}\,.$$

In analogy with the space $S(D)$ defined in Sect. 3, we also set

$$\widetilde{S}_G := \left\{([\Gamma], f, p_1, p_2, \ldots, p_{2g-2}) : \begin{array}{l} [\Gamma] \in \widetilde{M}_G, f \in \mathrm{Rat}(\Gamma), \text{ and } p_1, p_2, \ldots, p_{2g-2} \in \Gamma \\ \text{such that } K_\Gamma + (f) = p_1 + p_2 + \cdots + p_{2g-2} \ge 0 \end{array}\right\}.$$

Proposition 4.1 *The action of the symmetric group $\mathfrak{S}_{2g-2}$ on $\widetilde{S}_G$, which permutes the points $p_1, p_2, \ldots, p_{2g-2}$, induces a natural bijection $\widetilde{\Lambda}_G \simeq \widetilde{S}_G/\mathfrak{S}_{2g-2}$. Moreover, the action of the additive group $\mathbb{R} = (\mathbb{R}, +)$ on $\widetilde{\Lambda}_G$, given by adding constant functions to f, induces a natural bijection $\widetilde{\mathscr{H}}_G \simeq \widetilde{\Lambda}_G/\mathbb{R}$.*

Proof Since the projections $\widetilde{S}_G \to \widetilde{M}_G$ and $\widetilde{\Lambda}_G \to \widetilde{M}_G$ are invariant under the action of $\mathfrak{S}_{2g-2}$ and $\mathbb{R}$, the claims follow from the respective identities on the fibres. □

Proof of Theorem 1.2 For the first part, it suffices to show that $\widetilde{S}_G$ carries a canonical structure of a cone complex because of Proposition 4.1. Choose an orientation for each edge e of G and identify it with the closed interval $[0, l(e)]$. As in Proposition 3.2, we can describe the cells of $\widetilde{S}_G$ by the following discrete data:

- a partition of $\{p_1, p_2, \ldots, p_{2g-2}\}$ into disjoint subsets P_e and P_v, indexed by vertices $v \in V(G)$ and edges $e \in E(G)$, that indicate on which edge or at which vertex each point p_i is located,
- a total order on each P_e, and
- the integer slope m_e of f at the starting point of e

such that, for every vertex v, we have

$$d_v = 2h(v) - 2 + |v| + \sum_{\text{outward edges at } v} m_e + \sum_{\text{inward edges at } v} -(d_e + m_e)$$

where $d_v := \#P_v$ and $d_e := \#P_e$. The continuous parameters describing all elements in our cell are the values $f(v)$, the distances $d(p_i^e)$ of the point p_i^e from $0 \in [0, l(e)]$, and the lengths $l(e)$. In order to find the conditions on these parameters, we again write $\sum_i p_i^e = \sum_{j=1}^r d_{e,j} x_j$ for some $x_1 < x_2 < \cdots < x_r$. Using this notation, we have $0 < x_1 < x_2 < \cdots < x_r < l(e)$ as conditions on the $d(p_i^e) = x_i$. By the continuity of f, we also have

$$\begin{aligned}
f(x_1) - f(v) &= m_e x_1 \\
f(x_2) - f(x_1) &= (m_e + d_{e,1})(x_2 - x_1) \\
f(x_3) - f(x_2) &= (m_e + d_{e,1} + d_{e,2})(x_3 - x_2) \\
&\vdots \\
f(v') - f(x_r) &= (m_e + d_{e,1} + d_{e,2} + \cdots + d_{e,r})\big(l(e) - x_r\big)
\end{aligned}$$

Summing these equations, we obtain

$$f(v') = f(v) + (m_e + d_e)l(e) - (d_{e,1}x_1 + d_{e,2}x_2 + \cdots + d_{e,r}x_r) \tag{2}$$

Since these conditions are invariant under multiplying all parameters simultaneously by elements in $\mathbb{R}_{\geq 0}$, every non-empty cell in $\widetilde{S}_G$ has the structure of a rational polyhedral cone. Finally, the natural action of $\mathrm{Aut}(G)$ on $\widetilde{S}_G$, given by

$$\phi \cdot ([\Gamma], f, p_1, p_2, \ldots, p_{2g-2}) = \big([\phi(\Gamma)], f \circ \phi^{-1}, \phi(p_1), \phi(p_2), \ldots, \phi(p_{2g-2})\big)$$

for $\phi \in \mathrm{Aut}(G)$, is compatible with both the $\mathfrak{S}_{2g-2}$-action and the $\mathbb{R}$-action. Moreover, given a weighted edge contraction $G' = G/e$ of G, the natural map $\widetilde{S}_{G'} \hookrightarrow \widetilde{S}_G$ identifies $\widetilde{S}_{G'}$ with the subcomplex of $\widetilde{S}_G$ given by the condition $l(e) = 0$ in the above coordinates. Therefore, we can conclude that $\Lambda_g^{\mathrm{trop}} = \varinjlim \widetilde{\Lambda}_G$ and $\mathscr{H}_g^{\mathrm{trop}} = \varinjlim \widetilde{\mathscr{H}}_G$, where the limits are taken over the category J_g as in Sect. 2 above, carry the structure of a generalized cone complex.

For the second part, we need to show that the dimension of a maximal-dimensional cone in $\mathscr{H}_g$ is $5g-5$. Proposition 3.2.5 (i) in [7] demonstrates that $\dim \mathrm{M}_g^{\mathrm{trop}} = 3g-3$ and Corollary 7 in [23] establishes that the dimension of the fibre $|K_\Gamma|$ of a point $[\Gamma]$ is at most $\deg(K_\Gamma) = 2g-2$. It follows that the dimension of $\mathscr{H}_g^{\mathrm{trop}}$ is at most $(3g-3) + (2g-2) = 5g-5$. It remains to exhibit a $(5g-5)$-dimensional cone in $\mathscr{H}_g^{\mathrm{trop}}$. To accomplish this, consider the tropical curve $\Gamma_{\max}$ as indicated in Fig. 2; it has $2g-2$ vertices and $3g-3$ edges. Lemma 3.3 implies that the combinatorial structure of $|K_{\Gamma_{\max}}|$ is independent of the edge lengths, so we can choose a generic chamber. We obtain a divisor $D \in |K_{\Gamma_{\max}}|$ as indicated in Fig. 3. [22, Proposition 13] implies that the divisor D belongs to a $(2g-2)$-dimensional face in $|K_{\Gamma_{\max}}|$. Thus, there is a $(5g-5)$-dimensional cone in $\mathscr{H}_g^{\mathrm{trop}}$.

For part three, we reuse the coordinates described in first part. For every edge e of G with $d_e = \#P_e = 0$, we have an equation $m_e l(e) = f(v) - f(v')$ which is parametrized by the $l(e)$. If $d_e > 0$ and $x := \frac{1}{d_e}\sum_{j=1}^{r} d_{e,j}x_j$, then equation (2) can be rewritten as $f(v') = f(v) + (m_e + d_e)l(e) - d_e x$. Since $0 < x < l(e)$, we deduce

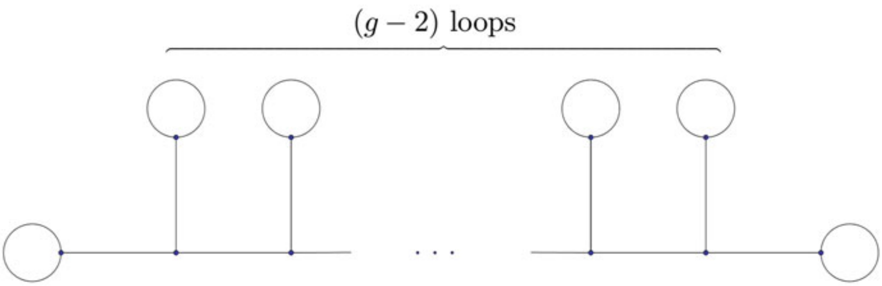


Fig. 2 The tropical curve $\Gamma_{\max}$ with $2g-2$ vertices (bold) and $3g-3$ edges

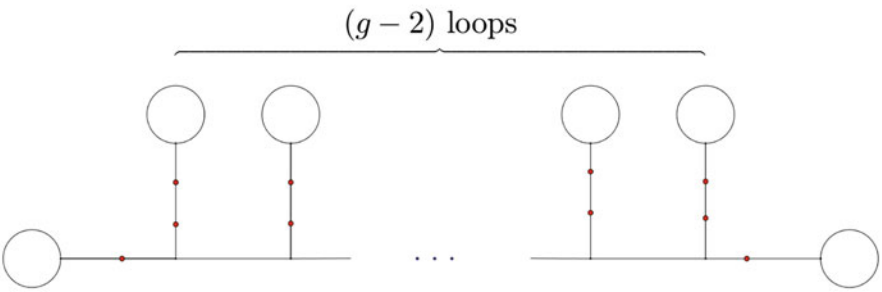


Fig. 3 The divisor D on $\Gamma_{\max}$ (red)

that $(m_e + d_e)l(e) > f(v') - f(v) > m_e l(e)$, so the cells in $\widetilde{M}_G$ are polyhedra. As the combinatorial type of $|K_\Gamma|$ is independent under scaling all edge lengths with a factor in $\mathbb{R}_{>0}$, all these polyhedra determine a subdivision of $\widetilde{M}_G$ such that on each relatively open cell of this subdivision, the corresponding $|K_\Gamma|$ has the same set of cells. In other words, the combinatorial type of $\widetilde{S}_G$ is constant. □

5 Computations in Low Genus

In this section, we present some computational results on the polyhedral structure of tropical Hodge bundles of small genus. In order to describe all cones in Λ_g^{trop}, we first list all cones in $\mathrm{M}_g^{\text{trop}}$. For each cone, we then compute its subdivision by the structure of $|K_\Gamma|$. The two cases $g = 2$ and $g = 3$ already show a surprisingly different behaviour.

Proposition 5.1 *If Γ is a tropical curve in $\mathrm{M}_2^{\text{trop}}$, then the combinatorial structure of $|K_\Gamma|$ is uniquely determined by the minimal model G of Γ. In other words, it is independent of the edge lengths in Γ.*

Proof There are seven faces in $\mathrm{M}_2^{\text{trop}}$; see in [13, Fig, 4]. For six of these faces, all edges are loops or bridges, so the claim follows from Lemma 3.3. For the "theta graph" G_θ, an explicit computation shows that the canonical linear system $|K_{G_\theta}|$ is always a one-dimensional polyhedral complex with three segments, as in Fig. 4. □

The face lattice of Λ_2^{trop} is visualized in Fig. 1.

Remark 5.2 The f-vector of Λ_2^{trop} is $(1, 5, 11, 16, 9, 1)$, which is consistent with Theorem 1.2. The unique five-dimensional face consists of the "dumbbell" graph and a triangular cell in $|K_\Gamma|$. In other words, any divisor in this cell is of the form $P+Q$, where P and Q are distinct points in the interior of the bridge in the dumbbell graph.

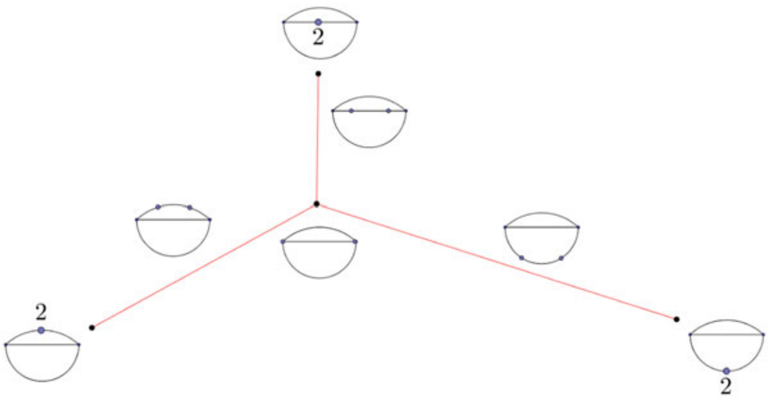


Fig. 4 The polyhedral complex $|K_{G_\theta}|$

When $g = 3$, the direct analogue of Proposition 5.1 is false. One counterexample comes from the six-dimensional cone $C \simeq \mathbb{R}^6_{>0}$ in $\mathrm{M}_3^{\mathrm{trop}}$ parametrizing tropical curves whose minimal model G is a complete metric graph K_4. The following proposition characterizes the open chambers of C regarding the structure of $|K_G|$.

Proposition 5.3 *There are* 51 *open chambers in the six-dimensional cone C in* $\mathrm{M}_3^{\mathrm{trop}}$ *parametrizing tropical curves whose minimal model G is a K_4. For all metrics in the same chamber, the canonical linear system $|K_G|$ has the same set of cells, and the polyhedral complex $|K_G|$ always has* 34 *vertices,* 60 *edges, and* 27 *two-dimensional faces (*12 *triangles and* 15 *quadrilaterals). However, there are four non-isomorphic combinatorial structures of $|K_G|$.*

Furthermore, the metric $M = (M_{12}, M_{13}, M_{14}, M_{23}, M_{24}, M_{34})$ on K_4 belongs to an open chamber if and only if, among the four subsets $\{M_{12}, M_{13}, M_{14}\}$, $\{M_{12}, M_{23}, M_{24}\}$, $\{M_{13}, M_{23}, M_{34}\}$, and $\{M_{14}, M_{24}, M_{34}\}$, the minimum is attained only once.

Proof Computations using the algorithm from [23, Sect. 2.3] give the result. □

Remark 5.4 (The Structure of $|K_G|$ for a Generic Metric) If M belongs to an open chamber, the canonical linear system $|K_G|$ always has the 13 vertices in Fig. 5. Among them, the ten labelled vertices are all connected to an extra vertex that is the divisor K_G. The remaining 20 vertices come from 4 copies of a substructure (we call a *bat*) attached at D_1, D_2, D_3, and D_4. Some edges in Fig. 5 are subdivided by other vertices in the bats. The four distinct combinatorial types of $|K_G|$ come from different ways of attaching the bats. Since M belongs to an open chamber, the minimum of M_{12}, M_{13}, M_{14} appears only once. Suppose it is M_{12}, then the bat at D_1 is attached along the edges towards D_{13} and D_{14}, as in Fig. 6. This bat appears whenever $M_{12} < \min(M_{13}, M_{14})$. Figure 7 shows the divisors D_i and D_{ij}.

The action of the symmetric group $\mathfrak{S}_4$ on the vertices of K_4 induces 4 orbits among the 51 open chambers with lengths 24, 12, 12, and 3. Each orbit corresponds to a combinatorial type of $|K_G|$. Each open chamber is an open cone in C, defined by homogeneous linear inequalities involving M_{12}, M_{13}, M_{14}, M_{23}, M_{24}, and M_{34}. The

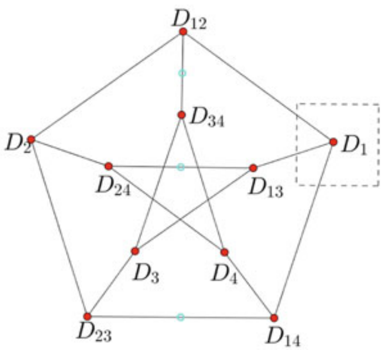


Fig. 5 The main skeleton of $|K_G|$

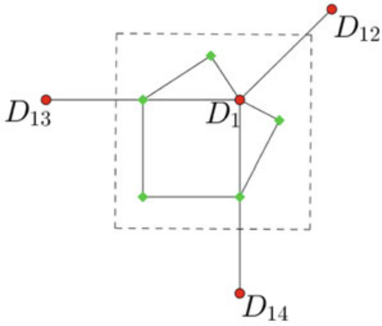


Fig. 6 Each bat has five extra (green) vertices

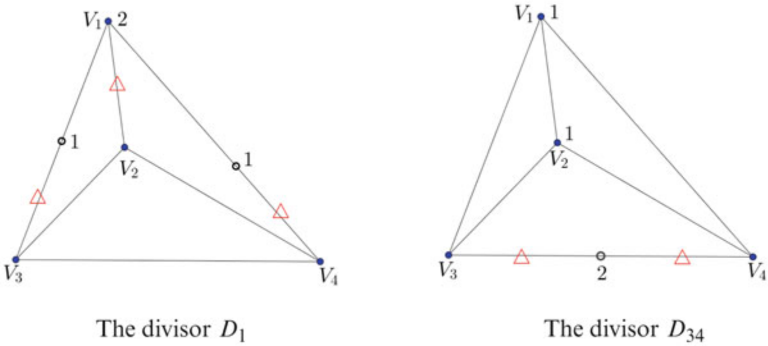
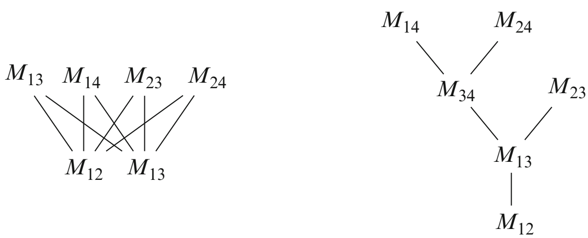


Fig. 7 Numbers are the coefficients of divisors and the triangle symbols show the equal line segments

Fig. 8 Representatives of two chamber orbits of length 12

Fig. 9 Representatives of a chamber orbit of length 3 and length 24



inequalities are displayed as the *covers* in a lattice. For example, M_{13} covering M_{12} means that the inequality $M_{13} > M_{12}$ holds. Figures 8 and 9 illustrate some of the possibilities.

6 The Realizability Problem

Let k be an algebraically closed field carrying the trivial absolute value. In [1], expanding on earlier work (see e.g. [32]), the authors have constructed a natural continuous tropicalization map $\mathrm{trop}_{\mathscr{M}_g}\colon \mathscr{M}_g^{\mathrm{an}} \to \mathrm{M}_g^{\mathrm{trop}}$ sending a point x in the non-Archimedean analytic moduli space $\mathscr{M}_g^{\mathrm{an}}$ to a tropical curve $[\Gamma_x] \in \mathrm{M}_g^{\mathrm{trop}}$. To describe this map, recall that a point $x \in \mathscr{M}_g^{\mathrm{an}}$ parametrizes an algebraic curve C over some non-Archimedean extension K of k. After a finite extension K' of K if necessary, we can extend C to a stable model $\mathscr{C} \to \operatorname{Spec} R'$ over the valuation ring R' of K'. Let G_x be the weighted dual graph of the special fibre $\mathscr{C}_s$ of $\mathscr{C}$; the vertices correspond to the components of $\mathscr{C}_s$ and we have an edge e between two vertices v and v' for every node connecting the two corresponding components C_v and $C_{v'}$. The vertex weight function is given by $h(v) = g(\widetilde{C}_v)$, where $\widetilde{C}_v$ denotes the normalization of C_v. Around every node p_e in $\mathscr{C}_s$, there exists formal coordinates x and y on $\mathscr{C}$ such that $xy = t$ for some element t in the base. The edge length of e is $l(e) = \mathrm{val}(t)$.

Let $\mathscr{H}_g^{\mathrm{an}}$ denote the non-Archimedean analytification of the total space of the algebraic Hodge bundle $\mathscr{H}_g$.

Proposition 6.1 *There is a natural tropicalization map* $\mathrm{trop}_{\mathscr{H}_g}\colon \mathscr{H}_g^{\mathrm{an}} \to \mathscr{H}_g^{\mathrm{trop}}$ *such that the following diagram commute:*

$$\begin{array}{ccc} \mathscr{H}_g^{\mathrm{an}} & \xrightarrow{\mathrm{trop}_{\mathscr{H}_g}} & \mathscr{H}_g^{\mathrm{trop}} \\ \downarrow & & \downarrow \\ \mathscr{M}_g^{\mathrm{an}} & \xrightarrow{\mathrm{trop}_{\mathrm{M}_g}} & \mathrm{M}_g^{\mathrm{trop}}. \end{array}$$

We expect that $\mathrm{trop}_{\mathscr{H}_g}$ is also continuous, but refrain from investigating this question here.

Proof An element $x \in \mathscr{H}_g^{\mathrm{an}}$ parametrizes a tuple (C, K_C) consisting of a smooth projective curve C over a non-Archimedean extension K of k together with a canonical divisor on C. We associate to (C, K_C) the point $\big([\Gamma_x], \tau_*(K_C)\big)$, where $\tau_*\colon \mathrm{Div}(C_{\overline{K}}) \longrightarrow \mathrm{Div}(\Gamma)$ denotes the specialization map constructed in [5, Sect. 2.3] given by pushing K_C forward to the non-Archimedean skeleton of $\mathscr{C}$. As shown in [5, Sect. 2.3], this is well-defined and the commutativity of the above diagram is an immediate consequence of the definition. □

It is well-known that $\mathrm{trop}_g\colon \mathscr{M}_g^{\mathrm{an}} \to \mathrm{M}_g^{\mathrm{trop}}$ is surjective. However, Theorem 1.2 shows that $\dim_{\mathbb{C}} \mathscr{H}_g = 4g - 4 < 5g - 5 = \dim \mathscr{H}_g^{\mathrm{trop}}$, so the analogous statement for $\mathscr{H}_g$ appears to be false. This gives rise to the following problem.

Problem 6.2 Characterize of the *realizability* locus $\mathrm{trop}_{\mathscr{H}_g}(\mathscr{H}_g^{\mathrm{an}})$ in $\mathscr{H}_g^{\mathrm{trop}}$.

In other words, given a stable tropical curve Γ of genus g together with a divisor D that is equivalent to K_Γ, find the algebraic and combinatorial conditions that ensure that there is an algebraic curve C over a non-Archimedean field extension K of k together with a canonical divisor $\widetilde{D}$ on C such that $\mathrm{trop}_{\mathscr{H}_g}([C], \widetilde{D}) = ([\Gamma], D)$.

Since $\mathrm{trop}_{\mathscr{M}_g}$ is surjective, we already know that every tropical curve Γ can be lifted to a smooth algebraic curve C. When the tropical curve Γ has integer edge lengths $l(e)$, we can also give a constructive approach to this problem. For a special fibre $\mathscr{C}_s$ over k whose weighted dual graph is G, use logarithmically smooth deformation theory to find a smoothing of $\mathscr{C}_s$ to a stable family $\mathscr{C} \to \mathrm{Spec}\, R$ with deformation parameters $l(e)$ at each node; see [21, Proposition 3.38]). If $l(e) = 1$ for all edges e, we may also proceed as in [5, Appendix B]. Next, let $\widetilde{D}$ be a divisor on C that specializes to the given canonical divisor D on Γ. Since we have $\deg \widetilde{D} = \deg D = 2g-2$, Clifford's theorem (or alternatively Baker's Specialization Lemma [5, Corollary 2.11]) shows that the rank of $\widetilde{D}$ is at most $g-1$. If the rank of D is smaller than $g-1$, then it cannot be a canonical divisor. If, however, the divisor $\widetilde{D}$ has rank $g-1$, then, by Riemann-Roch, it is a canonical divisor. So the realizability problem reduces to finding a lift of the divisor D of rank $g-1$.

The existence of such a divisor would follow, for example, from the smoothness of a suitable moduli space of limit linear series; see [17, 27]. Unfortunately the machinery of limit linear series is not available for nodal special fibres that are not of compact type. However, considerations undertaken from the point of view of compactifications of the moduli space of abelian differentials and its strata, such as [4], treating the special case of limits of canonical linear systems seem to provide us with a very promising approach for future investigations into this question.

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Cellular Sheaf Cohomology in *Polymake*

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Abstract This is a guide to the *polymake* extension *cellularSheaves*. We first define cellular sheaves on polyhedral complexes in Euclidean space, as well as cosheaves, and their (co)homologies. As motivation, we summarize some results from toric and tropical geometry linking cellular sheaf cohomologies to cohomologies of algebraic varieties. We then give an overview of the structure of the extension *cellularSheaves* for *polymake*. Finally, we illustrate the usage of the extension with examples from toric and tropical geometry.

MSC 2010 codes: 05-04, 14Fxx, 14T05, 52Bxx

1 Introduction

The main motivation for this *polymake* (see [11]) extension is to implement tropical homology, as introduced by Itenberg, Katzarkov, Mikhalkin, and Zharkov in [15]. Tropical homology is the homology of particular cosheaves which can be defined on any polyhedral complex. When the polyhedral complex arises as the tropicalization of a family of complex projective varieties, the tropical homology groups give information about the Hodge numbers of a generic member of the family; see Theorem 2.13. However, this is just one particular instance of cellular (co)sheaf (co)homology that our extension can handle. Cellular (co)sheaves have also appeared as a tool in recent years in the field of applied topology, notably in persistent homology, sensor networks, and network coding; see [6, 12]. With this *polymake* extension, (co)sheaves on polyhedral complexes can be constructed from

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scratch. We hope that this will allow for a range of uses of the extension beyond just the ones from combinatorial algebraic geometry that we highlight here.

Given a polyhedral complex Π in $\mathbb{R}^n$, a cellular sheaf of vector spaces on Π associates to every face of Π a vector space and to every face relation a map of the associated vector spaces. Just as with usual sheaves, we can compute the cohomology of cellular sheaves. The advantage over usual sheaves is that cellular sheaf cohomology is the cohomology of a chain complex consisting of finite-dimensional vector spaces.

We begin, in Sect. 2, by giving the definitions of cellular (co)sheaves and their (co)homologies and presenting two major classes of examples coming from toric and tropical geometry. We refer the reader to [10] and [5] for a guide to toric geometry and polytopes. For an introduction to tropical geometry, see [4] and [20]. A description of our implementation of cellular sheaves and their cohomologies in *polymake* is given in Sect. 3. In Sect. 4, we illustrate the usage of the extension in a variety of examples from tropical and algebraic geometry. Finally, Sect. 5 outlines some potential future directions and applications for our extension.

2 Cellular Sheaf Cohomology

This section defines cellular sheaves and cosheaves, as well as their cohomologies and homologies. We provide an explicit example of a sheaf and a cosheaf which have been implemented in our extension.

A *polyhedral complex* Π is a finite collection of polyhedra in $\mathbb{R}^n$ with the property that any face of a polyhedron in Π is also in Π and the intersection of any two polyhedra in Π is a face of both. Let Π^i denote the collection of polyhedra in Π of dimension i. For polyhedra $\sigma, \tau \in \Pi$, we use $\tau \leq \sigma$ to indicate that τ is a face of σ.

Definition 2.1 Given a polyhedral complex Π and a chosen orientation of each polyhedron in Π, we define the *orientation map* $\mathcal{O}\colon \Pi^{i-1} \times \Pi^i \to \{-1, 0, +1\}$, for each i, by

$$\mathcal{O}(\tau,\sigma) := \begin{cases} -1 \text{ if the orientation of } \tau \subset \partial\sigma \text{ differs from that of } \tau, \\ \;\;\,0 \text{ if } \tau \not\leq \sigma, \\ +1 \text{ if the orientation of } \tau \subset \partial\sigma \text{ coincides with that of } \tau. \end{cases}$$

A polyhedral complex Π can be considered as a category where the objects are the polyhedra and the morphisms are given by inclusions. For instance, we have $(f\colon \tau \to \sigma) \in \mathrm{Mor}(\Pi)$ if and only if $\tau \leq \sigma$. We use the notation Π^{op} to denote the category obtained from Π by using the same objects and reversing the directions of all morphisms. Viewing Π as a category, we can give a succinct definition of cellular (co)sheaves. Let $\mathrm{Vect}_{\Bbbk}$ denote the category of vector spaces over a field $\Bbbk$.

Definition 2.2 Given a polyhedral complex Π, a *cellular sheaf* $\mathscr{G}$ and a *cellular cosheaf* $\mathscr{F}$ are functors $\mathscr{G}\colon \Pi \to \mathrm{Vect}_{\Bbbk}$ and $\mathscr{F}\colon \Pi^{\mathrm{op}} \to \mathrm{Vect}_{\Bbbk}$.

Expanding on the definition, a cellular sheaf consists of the following data:

- for each polyhedron σ in Π, a vector space $\mathscr{G}(\sigma)$, and
- given $\tau, \sigma \in \Pi$ satisfying $\tau \le \sigma$, a morphism $\rho_{\tau\sigma}\colon \mathscr{G}(\tau) \to \mathscr{G}(\sigma)$.

In particular, for $\gamma \le \tau \le \sigma$, the restriction morphisms commute in the sense that $\rho_{\gamma,\sigma} = \rho_{\tau,\sigma} \circ \rho_{\gamma,\tau}$. A cellular cosheaf is similar except that the morphisms are in the opposite direction: $\iota_{\sigma,\tau}\colon \mathscr{F}(\sigma) \to \mathscr{F}(\tau)$.

A sheaf of vector spaces, in the usual sense, is a contravariant functor from the category of open sets of a topological space to $\mathrm{Vect}_{\Bbbk}$ that satisfies additional axioms. A polyhedral complex can be equipped with a finite topology known as the Alexandrov topology and our functorial definition produces a sheaf in this topology. Due to the simplicity of the cellular sheaves and the Alexandrov topology, no additional sheaf axioms are required. The reader is directed to Chapter 4 in [6] for more details.

Example 2.3 We may define a constant sheaf by setting $\mathscr{G}(\sigma)$ to be the one-dimensional vector space $\Bbbk$, for all $\sigma \in \Pi$, and setting the map $\rho_{\tau,\sigma}\colon \mathscr{G}(\tau) \to \mathscr{G}(\sigma)$ to be the identity, for all $\tau, \sigma \in \Pi$ such that $\tau \le \sigma$. A constant cosheaf can be defined in a similar fashion.

Example 2.4 Let Π be a polyhedral complex in $\mathbb{R}^n$. For $\sigma \in \Pi$, set $L(\sigma)$ to be the linear subspace of $\mathbb{R}^n$ parallel to the face σ. For $p \in \mathbb{N}$, we define the sheaf W^p by setting $W^p(\sigma) := \bigwedge^p L(\sigma)$, for all $\sigma \in \Pi$, and taking the map $\rho_{\tau,\sigma}\colon W^p(\tau) \to W^p(\sigma)$ to be the pth exterior power of the natural inclusion $L(\tau) \to L(\sigma)$, for all $\tau \le \sigma$. By convention, the sheaf W^0 is the constant sheaf from Example 2.3.

We next give an example of a cosheaf on a polyhedral complex. The homology of this particular cosheaf is the tropical homology from [15] and will come up in subsequent sections.

Example 2.5 If Π is a polyhedral complex in $\mathbb{R}^n$, then we define

$$F_p(\sigma) := \sum_{\sigma < \gamma} \bigwedge^p L(\gamma)\,.$$

If $\tau \le \sigma$, then we have $\{\gamma : \sigma < \gamma\} \subset \{\gamma : \tau < \gamma\}$, which yields the inclusion $\iota_{\sigma,\tau}\colon F_p(\sigma) \to F_p(\tau)$. As in Example 2.4, we obtain the constant cosheaf from Example 2.3 when $p = 0$.

Remark 2.6 By dualizing the vector spaces $\mathscr{G}(\sigma)$ for all $\sigma \in \Pi$, we can transform a cellular sheaf $\mathscr{G}$ into a cellular cosheaf and vice versa.

For a given (co)sheaf, we build (co)chain complexes in the following two parallel definitions. The definitions appeared in this form in Section 6.2 of [6].

Definition 2.7 Given a polyhedral complex Π and a cellular sheaf $\mathscr{G}$, the *cellular cochain groups* and *cellular cochain groups with compact support* are defined as

$$C^q(\Pi;\mathscr{G}) := \bigoplus_{\substack{\dim\sigma=q\\ \sigma \text{ compact}}} \mathscr{G}(\sigma) \qquad \text{and} \qquad C_c^q(\Pi;\mathscr{G}) := \bigoplus_{\dim\sigma=q} \mathscr{G}(\sigma)$$

respectively. The cellular cochain maps (usual and with compact support)

$$d\colon C^q(\Pi;\mathscr{G}) \to C^{q+1}(\Pi;\mathscr{G}) \qquad \text{and} \qquad d\colon C_c^q(\Pi;\mathscr{G}) \to C_c^{q+1}(\Pi;\mathscr{G})$$

are given componentwise by $d_{\tau,\sigma}\colon \mathscr{G}(\tau) \to \mathscr{G}(\sigma)$, for $\tau \in \Pi^q$ and $\sigma \in \Pi^{q+1}$, where

$$d_{\tau,\sigma} := \begin{cases} \mathscr{O}(\tau,\sigma)\cdot\rho_{\tau\sigma} & \text{if } \tau \le \sigma,\\ 0 & \text{otherwise.}\end{cases}$$

Definition 2.8 Given a polyhedral complex Π and a cellular cosheaf $\mathscr{F}$, the *cellular chain groups* and the *Borel–Moore cellular chain groups* are defined as

$$C_q(\Pi;\mathscr{F}) := \bigoplus_{\substack{\dim\sigma=q\\ \sigma \text{ compact}}} \mathscr{F}(\sigma) \qquad \text{and} \qquad C_q^{BM}(\Pi;\mathscr{F}) := \bigoplus_{\dim\sigma=q} \mathscr{F}(\sigma)$$

respectively. The cellular chain maps (usual and Borel–Moore)

$$\partial\colon C_q(\Pi;\mathscr{F}) \to C_{q-1}(\Pi;\mathscr{F}) \qquad \text{and} \qquad \partial\colon C_q^{BM}(\Pi;\mathscr{F}) \to C_{q-1}^{BM}(\Pi;\mathscr{F})$$

are given componentwise by $\partial_{\sigma,\tau}\colon \mathscr{F}(\sigma) \to \mathscr{F}(\tau)$, for $\sigma \in \Pi^q$ and $\tau \in \Pi^{q-1}$, where

$$\partial_{\sigma,\tau} := \begin{cases} \mathscr{O}(\tau,\sigma)\cdot\iota_{\sigma\tau} & \text{if } \sigma \ge \tau,\\ 0 & \text{otherwise.}\end{cases}$$

Definition 2.9 The *cellular sheaf cohomology (with compact support)* of $\mathscr{G}$ is the cohomology of the cellular cochain complex (with compact support) from Definition 2.7. The *cellular (Borel-Moore) cosheaf homology* of $\mathscr{F}$ is the homology of the cellular (Borel-Moore) chain complex from Definition 2.8.

Remark 2.10 It may seem counterintuitive that the usual cellular cochains are supported only on compact faces and cellular cochains with compact support are supported on all faces. As a sanity check, the reader is encouraged to compute the cohomology of the constant sheaf from Example 2.3 on your favourite non-compact polyhedral complex. Under reasonable hypotheses on the polyhedral complex, the cellular cohomology of the constant sheaf will be isomorphic to the ordinary singular cohomology of the polyhedral complex. The analogous statement holds for the compactly supported versions. See Example 6.2.4 in [6] for a simple example and more details.

We now present some connections between the cellular (co)homology groups of (co)sheaves and the cohomology groups of complex algebraic varieties. Demonstations of these theorems, along with the *polymake* code, appear in Sect. 4.

To a rational polytope $\Delta \subset \mathbb{R}^n$, we associate the toric variety $\mathrm{TV}(\Delta)$ corresponding to its outer-normal fan; see Sect. 1.4 in [10]. The following theorem relates the cohomology of the sheaves of p-differential forms Ω^p on $\mathrm{TV}(\Delta)$ with the cohomologies of the sheaves W^p on the polytope Δ from Example 2.4.

Theorem 2.11 ([8, Remark 12.4.1]) *If $\Delta \subset \mathbb{R}^n$ is a rational polytope and $\mathrm{TV}(\Delta)$ is the associated toric variety, then we have $H^q(\mathrm{TV}(\Delta); \Omega^p) \cong H^q(\Delta; W^p) \otimes_{\mathbb{R}} \mathbb{C}$.* When the toric variety is smooth, we see that $H^q(\mathrm{TV}(\Delta); \Omega^p) \cong H^{p,q}(\mathrm{TV}(\Delta))$ is isomorphic to the (p, q)th part in the Hodge decomposition.

The next two results involve the F-cosheaves defined in Example 2.5. Consider a hyperplane arrangement $\mathscr{A}$ in $\mathbb{P}^d_{\mathbb{C}}$. Orlik and Solomon [21, Theorem 3.65] prove that the cohomology of the complement $\mathbb{P}^d_{\mathbb{C}} \setminus \mathscr{A}$ depends only on the combinatorics of the arrangment. More precisely, the combinatorics of an arrangement is encoded by a matroid; see [18] or [22] for an introduction to matroid theory. For any matroid, there is a combinatorially described Orlik–Solomon algebra, which equals the cohomology ring of the complement $\mathbb{P}^d_{\mathbb{C}} \setminus \mathscr{A}$ when the matroid corresponds to the hyperplane arrangement.

To any matroid M, we associate a fan $B(M)$ in Euclidean space known as the Bergman fan of M; see [1]. If M is the matroid of a hyperplane arrangement $\mathscr{A}$, then the fan $B(M)$ is the tropicalization of the complement $\mathbb{P}^d_{\mathbb{C}} \setminus \mathscr{A}$ under a suitable embedding to a complex torus.

Theorem 2.12 ([27, Theorem 4]) *If M is a matroid, $B(M)$ is its Bergman fan, and F_p are the cosheaves from Example 2.5 on $B(M)$, then we have $OS^p(M) = F_p(v)^*$ where $OS^p(M)$ denotes the pth graded piece of the Orlik–Solomon algebra of M and v is the vertex of the fan.*

In Examples 4.1–4.3, we illustrate how we can compute the dimensions of the graded pieces of the Orlik–Solomon algebra of a matroid using the *polymake* applications `matroid`, `tropical` and our extension *cellularSheaves*.

Lastly, the statements relating the cohomology of the complements of arrangements and the tropical homology of matroidal fans in Theorem 2.12 can be generalized, and even refined, in the setting of tropicalization of complex projective algebraic varieties. We state a theorem and refer the reader to [15] for the precise definitions of smooth $\mathbb{Q}$-tropical projective varieties and tropical limits.

Theorem 2.13 ([15, Corollary 2]) *Consider a 1-parameter family of complex projective varieties $\pi\colon \mathscr{X} \to D^\circ$ where D° is the punctured disc. If the tropical limit $\mathrm{trop}(\mathscr{X}) = X$ is a smooth $\mathbb{Q}$-tropical projective variety, then we have*

$$\dim H^{p,q}(\mathscr{X}_t) = \dim H_q(X; F_p)$$

where $H^{p,q}(\mathscr{X}_t)$ *is the* (p,q)*th part of the Hodge decomposition of* $\mathscr{X}_t := \pi^{-1}(t)$ *for a generic point* $t \in D^{\circ}$.
As a consequence of this theorem, the homology of the F-cosheaves is also known as tropical homology.

As yet, *polymake* does not have compact tropical varieties as objects. Therefore, in the examples presented in Sect. 4, we do not produce Hodge numbers of complex projective algebraic varieties, but rather Betti numbers of limit mixed Hodge structures of non-complete varieties. Examples of this can be found in [7] for hypersurfaces and complete intersections, also in [19] from a more tropical point of view. Future plans to implement tropical homology for compact (and hence also projective) tropical varieties are outlined in Sect. 5.

3 Implementation in *Polymake*

The mathematical software package *polymake* provides a framework for computations in polyhedral geometry. It focuses on combinatorial objects such as cones, polyhedra, graphs, fans, and polyhedral complexes. Toric and tropical geometry provide many ways for using *polymake* to solve computational tasks from algebraic geometry. In particular, *polymake* provides the applications `tropical` for tropical geometry and `fulton` for toric geometry. Furthermore, *polymake* interfaces several other software packages which may be useful in our context, such as *Gfan* [17] for tropical computations and *Singular* [9] for algebraic geometry. See [14] for an overview of the most current implemented *polymake* features for tropical geometry.

The interface language for *polymake* is perl. For improved performance one can write and attach C++ code. The combinatorial objects are realized as objects with properties. For example, the object `Polytope` has the properties `VERTICES` and `F_VECTOR` among many others. Since solving certain problems can be very expensive, *polymake* adheres to the principle of lazy evaluation: properties are only computed when needed and then cached with the object, so they do not have to be recomputed.

Computation of properties is done via *polymake*'s internal rule structure. A rule takes a certain set of input properties and then computes a certain set of output properties. When asked for a certain property of an object, *polymake* creates a queue of rules to apply in order to get this property from any set of given properties, if this is possible. Take for example the following snippet of code:

```
object PolyhedralFan {

   property ORIENTATIONS : Map<Set<Set<Int> >, Int>;

   rule ORIENTATIONS: HASSE_DIAGRAM, FAN_DIM, RAYS,
      LINEALITY_SPACE{
                 ... # Code
   }
}
```

Here the object `PolyhedralFan` is equipped with a new property `ORIENTATIONS` which one needs for computing tropical homology, see Definition 2.1. A rule is created, that computes `ORIENTATIONS` from the properties `HASSE_DIAGRAM`, `FAN_DIM`, `RAYS` and `LINEALITY_SPACE` of the `PolyhedralFan`.

Internally in *polymake*, every polyhedral complex Π in $\mathbb{R}^n$ is considered as a polyhedral fan Σ in $\mathbb{R}^{n+1}$ intersected with the hyperplane defined by $x_0 = 1$. Every face of Π is indexed by a subset of the rays of Σ. The one-dimensional faces of Σ whose direction $\mathbf{v} = (v_0, v_1, \ldots, v_n)$ satisfies $v_0 = 1$ correspond to vertices of Π. The one-dimensional faces of Σ whose direction $\mathbf{v}$ satisfies $v_0 = 0$ correspond to unbounded one dimensional faces of Π.

Definition 3.1 Let Σ be a fan in $\mathbb{R}^{n+1}$ and $\Pi = \Sigma \cap \{x_0 = 1\}$. A *far vertex* of Π is a ray of Σ whose direction $\mathbf{v}$ satisfies $v_0 = 0$. A face σ of Π is

- a *far face* if its index set consists only of far vertices,
- a *non-far face* if its index set contains at least one non-far vertex,
- a *bounded face* if it contains no far vertices, and
- an *unbounded face* if it is neither a far face nor a bounded face.

Our extension *cellularSheaves* adds the properties `FAR_FACES`, `NON_FAR_FACES`, `BOUNDED_FACES`, and `UNBOUNDED_FACES` to a polyhedral complex.

Computing orientations for the polyhedral fan avoids complications caused by the different types of faces of the polyhedral complex. Since the object `PolyhedralComplex` is derived from the object `PolyhedralFan`, it will have the property `ORIENTATIONS` as well.

Installing the *cellularSheaves* Extension The extension can be installed on a Linux system with the most recent *polymake* version with the following two steps. First clone the repository with

```
git clone \
   http://www.github.com/lkastner/cellularSheaves \
   FOLDER
```

into a folder named `FOLDER`. Second start *polymake* and import the extension using

```
import_extension("FOLDER");
```

The extension introduces the new objects `Sheaf` and `CoSheaf` from Definition 2.2. A basic usage scenario looks like

```
application "fan";
$pc = new PolyhedralComplex(
   check_fan_objects(new Cone(cube(3))));
$w1 = $pc->wsheaf(1);
```

We switch to the application `fan` because this is the application our extension adds functionality to. The next line takes the three dimensional cube and turns it into a polyhedral complex. We then ask for the W^1-sheaf appearing in Example 2.4.

We implemented most methods dealing with pure linear algebra in C++. The file

```
apps/fan/include/linalg.h
```

contains the C++ code. These linear algebra methods, especially those assembling a chain complex from given block matrices, perform significantly better when implemented in C++ than in perl.

Sheaves and Cosheaves In our extension, we introduce the objects `Sheaf` and `CoSheaf`. As implemented, these objects have two properties. The first is a map from a collection of sets of integers to matrices. This property represents the vector spaces of a (co)sheaf. The second is a map from pairs of sets in this collection to matrices. These matrices represent the morphisms between these vector spaces. The vector spaces and morphisms are stored in the following two properties of a (co)sheaf:

```
property BASES : Map<Set<Int>, Matrix>;

property BLOCKS : Map<Set<Set<Int> >, Matrix >;
```

Let us rephrase this in terms of the Definitions 2.7 and 2.8. Let Π be a polyhedral complex with a sheaf $\mathscr{G}$ and let $\tau \leq \sigma$ be a face relation in Π. The faces of Π are encoded as index sets of the rays of vertices of the defining polyhedral fan Σ. As an object in *polymake*, the sheaf $\mathscr{G}$ has the property `BASES` containing the bases of the vector spaces $\mathscr{G}(\gamma)$ for all $\gamma \in \Pi$. For the sheaf $\mathscr{G}$, the property `BLOCKS` also contains a matrix representing the map $\rho_{\tau,\sigma}: \mathscr{G}(\tau) \to \mathscr{G}(\sigma)$, for each pair of faces $\tau \leq \sigma$. This matrix is written using the bases from the property `BASES`. Similarly for cosheaves, the property `BASES` of $\mathscr{F}$ contains the bases of $\mathscr{F}(\tau)$, for all $\tau \in \Pi$, and the property `BLOCKS` contains a matrix representing the map $\iota_{\sigma,\tau}: \mathscr{F}(\sigma) \to \mathscr{F}(\tau)$, for each pair of faces $\tau \leq \sigma$. For the purpose of computing sheaf cohomology only the `BLOCKS` are required; the property `BASES` may be left empty. However, in `BLOCKS`, it is also necessary to store morphisms for certain non-face relations, these consist of zero matrices of the appropriate sizes.

The main (co)sheaf constructors included in the *cellularSheaves* extension are `constant_sheaf`, `fcosheaf`, and `wsheaf`, which produce the (co)sheaves from Examples 2.3, 2.4, and 2.5. These are user methods attached to a polyhedral complex. The latter two methods take a non-negative integer p as a parameter that determines the exterior power for the F-cosheaves and W-sheaves.

Chain Complexes and Homologies The remaining new important objects are chain complexes introduced as `ChainComplex`. A chain complex comes with the properties `DIFFERENTIALS`, `BETTI_NUMBERS`, `HOMOLOGY` and `IS_WELLDEFINED`. It can be created by giving an array of matrices as the property `INPUT_DIFFERENTIALS`. It has a user method `print()` providing a human readable sequence format of the chain complex. Dually, we introduce the object `CoChainComplex`. Internally this is just a wrapper around the object `ChainComplex`.

Currently, there are four (co)homology methods in our extension for a given (co)sheaf. They differ by which faces are used in building the chain complex.

`usual_chain_complex`: This method computes $C_\bullet(\Pi; \mathscr{F})$, meaning that it only considers the bounded faces of the given polyhedral complex.

`borel_moore_complex`: This method computes $C^{BM}_\bullet(\Pi;\mathscr{F})$, meaning that it uses all non-far faces of a given polyhedral complex.

`usual_cochain_complex`: This method computes $C^\bullet(\Pi;\mathscr{G})$.

`compact_support_complex`: This method computes $C^\bullet_c(\Pi;\mathscr{G})$.

4 Examples and Usage

This section provides sample code and output for some specific examples. These examples are chosen to highlight the connections to cohomology of complex algebraic varieties described in Sect. 2.

Polytopes We consider the polyhedral complex that consists of a three-dimensional cube C and all its faces. We will compute the W-sheaves for C as well as the Betti numbers of the cohomology groups $H^q(C;W^p)$ for all p, q from 0 to 3.

```
application "fan";
$pc = new PolyhedralComplex(
   check_fan_objects(new Cone(cube(3))));
@betti = ();
for(my $i=0; $i<4; $i++){
   my $w = $pc->wsheaf($i);
   my $s = $pc->usual_cochain_complex($w);
   push @betti, $s->BETTI_NUMBERS;
}
print new Matrix(@betti);
```

The first step turns the three-dimensional cube into a polyhedral complex. We then loop over all possible W-sheaves and save the Betti numbers in a matrix, which yields the following output.

```
fan > print new Matrix(@betti);
1 0 0 0
0 3 0 0
0 0 3 0
0 0 0 1
```

We see that $\dim H^q(C;W^p) = 0$ if $p \neq q$. The diagonal $\dim H^p(C;W^p)$ is the dual h-vector of the cube. This relationship holds for any simple polytope Δ; see [3, Corollary, p. 6].

```
fan > $cube = polytope::cube(3);

fan > print $cube->DUAL_H_VECTOR;
1 3 3 1
```

The toric variety of this cube is $X = \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$. Hence,we have

$$\dim H^p(X;\Omega^p) = \begin{cases} 1 & \text{if } p = 0, 3, \\ 3 & \text{if } p = 1, 2. \end{cases}$$

and $\dim H^q(X;\Omega^p) = 0$ for $p \neq q$.

Bergman Fans and Tropical Linear Spaces We can build the Bergman fan $B(M)$ of a matroid M and compute the usual homology of the F-cosheaf. Assuming the matroid M is connected, the fan $B(M)$ has a unique bounded face, its vertex v. Therefore, the cellular chain groups $C_q\big(B(M); F_p\big)$ are trivial unless $q = 0$. In the following examples, we will see that $\dim H_0(B(M); F_p) = \dim OS^p(M)$, where $OS^p(M)$ is the pth graded part of the Orlik–Solomon algebra of M. This follows from Theorem 2.12. When v is the vertex of the Bergman fan, we have

$$H_0\big(B(M); F_p\big) = C_0\big(B(M); F_p\big) = F_p(v)$$

and $H_i\big(B(M); F_p\big) = 0$ for $i > 0$.

When M is a rank $d+1$ matroid on $n+1$ elements arising from a non-central hyperplane arrangement $\mathscr{A}$ in $\mathbb{P}^d_{\mathbb{C}}$, the Orlik–Solomon algebra is isomorphic to the cohomology ring of the complement $\mathscr{C} := \mathbb{P}^d_{\mathbb{C}} \backslash \mathscr{A}$ of the arrangement. There is a canonical embedding of $\mathscr{C} \to (\mathbb{C}^*)^n$, and its tropicalization is the Bergman fan of the matroid. Therefore, we see that the homology of the F-cosheaf on a tropicalization recovers cohomological information about the original variety.

Example 4.1 Our first example computes the tropical homology of a tropical line in $\mathbb{R}^2$. This is the tropicalization of a generic line $L \subset \mathbb{C}^2$ intersected with the torus $(\mathbb{C}^*)^2$. This space is homeomorphic to $\mathbb{P}^1_{\mathbb{C}} \setminus \{p_1, p_2, p_3\}$, so it follows that $\dim H^0\big(L \cap (\mathbb{C}^*)^2; \mathbb{C}\big) = 1$ and $\dim H^1\big(L \cap (\mathbb{C}^*)^2; \mathbb{C}\big) = 2$. The tropical line is the Bergman fan of the uniform matroid of rank two on three elements.

We start by computing the Bergman fan of the matroid in *polymake*. The algorithm *polymake* uses to compute this fan is from the program *TropLi* [25].

```
application "fan";
$m = matroid::uniform_matroid(2,3);
$berg = tropical::matroid_fan<Max>($m);
```

Next, we construct the associated F-cosheaves up to the dimension of the Bergman fan and compute their usual chain complexes.

```
$f0 = $berg->fcosheaf(0);
$f1 = $berg->fcosheaf(1);
$s0 = $berg->usual_chain_complex($f0);
$s1 = $berg->usual_chain_complex($f1);
```

We may determine the Betti numbers as follows.

```
fan > print $s0->BETTI_NUMBERS;
1 0
fan > print $s1->BETTI_NUMBERS;
2 0
```

We can also compute the Borel–Moore homology. Here every face of the Bergman fan contributes to the Borel–Moore chain groups; see Definition 2.9.

```
$bm0 = $berg->borel_moore_complex($f0);
$bm1 = $berg->borel_moore_complex($f1);
```

gives

```
fan > print $bm0->BETTI_NUMBERS;
0 2
fan > print $bm1->BETTI_NUMBERS;
0 1
```

We obtain $\dim H_q(B(M); F_p) = \dim H^{BM}_{d-q}(B(M); F_{d-p})$ for $d = 1$, which is the dimension of the Bergman fan. This is the homological version of Poincaré duality for matroidal fans and tropical manifolds from [16].

Example 4.2 In this example, we study the Bergman fan of the matroid of the complete graph on four vertices. This is also the matroid of the braid arrangement of lines in $\mathbb{P}^2_{\mathbb{C}}$ whose complement is the moduli space of 5-marked genus 0 curves $M_{0,5}$; see [1]. We use the applications `graph` and `matroid` to construct the Bergman fan.

```
application "fan";
$g = graph::complete(4);
$m = matroid::matroid_from_graph($g);
$berg = tropical::matroid_fan<Max>($m);
```

We compute the usual and the Borel-Moore homology of the F-cosheaf.

```
@betti_usual = ();
@betti_bm = ();
for(my $i=0; $i<3; $i++){
   my $f = $berg->fcosheaf($i);
   my $s = $berg->usual_chain_complex($f);
   my $bm = $berg->borel_moore_complex($f);
   push @betti_usual, $s->BETTI_NUMBERS;
   push @betti_bm, $bm->BETTI_NUMBERS;
}
```

This gives the following Betti numbers:

```
fan > print new Matrix(@betti_usual);
1 0 0
5 0 0
6 0 0

fan > print new Matrix(@betti_bm);
0 0 6
0 0 5
0 0 1
```

Again, we see that $\dim H_q\big(B(M); F_p\big) = \dim H^{BM}_{d-q}\big(B(M); F_{d-p}\big)$ for $d = 2$.

Example 4.3 A tropical linear space is not necessarily a fan. Nevertheless, the Betti numbers of the tropical homology of the tropical linear space and of its recession fan agree. In this example, we start with the Bergman fan of the uniform matroid of rank three on six elements and compare its homology with that of the tropical linear space of a valuated matroid with the aforementioned matroid as its underlying matroid. Consider the input

```
application "fan";
$m = matroid::uniform_matroid(3,6);
$berg = tropical::matroid_fan<Max>($m);
@betti_usual = ();
@betti_bm = ();
for(my $i=0; $i<3; $i++){
   my $f = $berg->fcosheaf($i);
   my $s = $berg->usual_chain_complex($f);
   my $bm = $berg->borel_moore_complex($f);
   push @betti_usual, $s->BETTI_NUMBERS;
   push @betti_bm, $bm->BETTI_NUMBERS;
}
```

The resulting output equals

```
fan > print new Matrix(@betti_usual);
1 0 0
5 0 0
10 0 0

fan > print new Matrix(@betti_bm);
0 0 10
0 0 5
0 0 1
```

We next consider a valuated matroid whose underlying matroid is uniform of rank three on six elements and construct the corresponding tropical linear space. In particular, the input

```
$v = [0,0,3,1,2,1,0,1,0,2,2,0,3,0,4,1,2,2,0,0];
$val_matroid = new matroid::ValuatedMatroid<Min>(
   BASES=>matroid::uniform_matroid(3,6)->BASES,
   VALUATION_ON_BASES=>$v,N_ELEMENTS=>6);
$tls = tropical::linear_space($val_matroid);
@betti_usual = ();
@betti_bm = ();
for(my $i=0;$i<3;$i++){
   my $fi = $tls->fcosheaf($i);
   my $si=$tls->usual_chain_complex($fi);
   my $bmi=$tls->borel_moore_complex($fi);
   push @betti_usual, $si->BETTI_NUMBERS;
   push @betti_bm, $bmi->BETTI_NUMBERS;
}
```

returns

```
fan > print new Matrix(@betti_usual);
1 0 0
5 0 0
10 0 0

fan > print new Matrix(@betti_bm);
0 0 10
0 0 5
0 0 1
```

which is the same as for the Bergman fan of the matroid above.

Example 4.4 In this example, we compute the usual cohomology and the compactly supported cohomology of the W-sheaves on a tropical linear space. These cohomologies exhibit vanishing in certain degrees. We continue with the same tropical linear space from Example 4.3. The input

```
@wbetti_usual = ();
@wbetti_cs = ();
for(my $i=0;$i<3;$i++){
   my $wi = $tls->wsheaf($i);
   my $wsi=$tls->usual_cochain_complex($wi);
   my $wcsi=$tls->compact_support_complex($wi);
   push @wbetti_usual, $wsi->BETTI_NUMBERS;
   push @wbetti_cs, $wcsi->BETTI_NUMBERS;
}
```

returns

```
fan > print new Matrix(@wbetti_usual);
1 0 0
0 4 0
0 0 1
```

```
fan > print new Matrix(@wbetti_cs);
0 0 10
0 0 32
0 0 28
```

More generally, we conjecture that the following cohomology groups vanish.

Conjecture 4.5 If $L \subset \mathbb{R}^n$ is a tropical linear space of dimension d, then we have $H^q(L; W^p) = 0$ if $p \neq q$ and $H^q_c(L; W^p) = 0$ if $q \neq d$.
This conjecture has been verified on all tropical linear spaces in $\operatorname{trop}\operatorname{Gr}(3, \mathbb{A}^6)$.

The Euler characteristics of the complexes $C^\bullet(L; W^p)$ and $C^\bullet_c(L; W^p)$ imply that

$$(-1)^p H^p(L; W^p) = \sum_{q=0}^{d} (-1)^q \binom{q}{p} f^b_q \quad \text{and} \quad (-1)^d H^d_c(L; W^p) = \sum_{q=0}^{d} (-1)^q \binom{q}{p} f_q$$

where $(f^b_0, f^b_1, \ldots, f^b_d)$ is the f^b-vector of the bounded faces of L and $(f_0, f_1, \ldots, f_d)$ is the f-vector of L. If the above conjecture holds, then understanding the f-vector of a tropical linear space comes down to understanding the possible dimensions of $H^q_\bullet(L; W^p)$. As a consequence, it would be possible to bound the f^b-vector by bounding $H^p(L; W^p)$. This would give an approach to the f-vector conjecture for tropical linear spaces, see [23], similar to the proof of the upper bound conjecture for polytopes.

Tropical Hypersurfaces Using the *a-tint* application [13], we can construct tropical hypersurfaces in *polymake* from convex piecewise integer affine functions, which are also known as tropical polynomials. The subsequent examples demonstrate how one can start directly with a given tropical polynomial and compute the tropical homology of the hypersurface.

Example 4.6 We begin with a tropical curve in $\mathbb{R}^2$ which is dual to a triangulation of a square of size one. The input

```
application "tropical";
$f = toTropicalPolynomial("max(0,x+5,y+3, x+y+9)");
$div = divisor( (projective_torus<Max>(2)),
   rational_fct_from_affine_numerator($f));
application "fan";
@betti_usual = ();
@betti_bm = ();
for(my $i=0;$i<2;$i++){
   my $fi = $div->fcosheaf($i);
   my $si=$div->usual_chain_complex($fi);
   my $bmi=$div->borel_moore_complex($fi);
   push @betti_usual, $si->BETTI_NUMBERS;
   push @betti_bm, $bmi->BETTI_NUMBERS;
}
```

gives

```
fan > print new Matrix(@betti_usual);
1 0
3 0
```

```
fan > print new Matrix(@betti_bm);
0 3
0 1
```

Example 4.7 As a final example, we calculate the homology of another tropical hypersurface. This hypersurface arises as a triangulation of the three dimensional simplex of edge length four, and is a tropical K3-surface in $\mathbb{R}^3$. From the input

```
application "tropical";
$f = toTropicalPolynomial("max(0,x,y,z, 2*x-2,
   2*y-2, 2*z-2, x+y-1, x+z-1, y+z-1, 3*x-6,
   3*y-6, 3*z-6, 2*x+y-4, 2*y+x-4, 2*x+z-4,
   2*z+x-4, 2*y+z-4, 2*z+y-4, x+y+z+1, 4*x-12,
   4*y-12, 4*z-12, 3*x+y-9, 3*y+x-9, 3*x+z-9,
   3*z+x-9, 3*y+z-9, 3*z+y-9, 2*x+2*y-8,
   2*x+2*z-8, 2*y+2*z-8, 2*x+y+z-7, x+2*z+y-7,
   2*y+z+x-7)");
$k3 = divisor((projective_torus<Max>(3)),
   rational_fct_from_affine_numerator($f));
application "fan";
@numbers = (0..2);
@cosheaves = map{$k3 -> fcosheaf($_)} @numbers;
@usualChainComplexes = map{$k3->
   usual_chain_complex($_)} @cosheaves;
@bmComplexes = map{$k3->borel_moore_complex($_)}
   @cosheaves;
@betti_usual = map{$_->BETTI_NUMBERS}
   @usualChainComplexes;
@betti_bm = map{$_->BETTI_NUMBERS} @bmComplexes;
```

we obtain the following matrices of Betti numbers

```
fan > print new Matrix(@betti_usual);
1 0 1
3 31 0
34 0 0
```

```
fan > print new Matrix(@betti_bm);
0 0 34
0 31 3
1 0 1
```

This tropical hypersurface is bigger than the polyhedral complexes we previously considered. Its f-vector is $(64, 96, 34)$. This can be seen from the usual chain complex of the F^0-cosheaf.

```
fan > $usualChainComplexes[0]->print();
 3        2        1        0        -1
k^0 --> k^34 --> k^96 --> k^64 --> k^0
```

In this example and Example 4.6, we again observe Poincaré duality for tropical homology.

5 Future Directions

Sheaves of Modules It is also possible to compute (co)homology of cellular (co)sheaves of modules. For example, given a rational polyhedral complex, there are also integral versions of the W-sheaves and F-cosheaves that are free $\mathbb{Z}$-modules. However, using the current methods `fcosheaf` and `wsheaf` can lead to incorrect cohomology groups over $\mathbb{Z}$. Still, the ranks of the torsion and the free part of the (co)homology will be correct in these cases.

The problem with using the current implementation to compute integral versions of the (co)homology of the integral versions of these (co)sheaves is that the property `BASES` does not necessarily consist of a lattice basis of the free $\mathbb{Z}$-module for each face. If one properly chooses $\mathbb{Z}$-bases for `BASES` and defines `BLOCKS` manually with the correct maps over $\mathbb{Z}$ when creating a (co)sheaf, then the current rules for computing the cellular (co)homology will compute the correct $\mathbb{Z}$-homology.

We plan to adapt `fcosheaf` and `wsheaf` to give the correct results over $\mathbb{Z}$ after switching to *polymake*'s internal chain complex object. This has recently been pushed to the *polymake* repository by Olivia Röhrig.

Tropical Compactifications and Projective Hypersurfaces How to implement compact tropical varieties is part of an ongoing discussion inside the *polymake* developer team. One possibility is to save one affine tropical variety per chart of tropical projective space. For many cases, this would result in a drastic increase of resource usage. Thus, one may want to restrict to certain classes of tropical varieties with nice compactifications.

A solution to this problem is necessary in order to use our extension to solve Problem 10 on Surfaces in [24]. Upon having an implementation of compact tropical varieties, one could for example combine our extension and the approach to tropical Enriques surfaces in [2] to determine the Hodge numbers of the surface.

Implementing Other Cellular (Co)sheaves There are many other (co)sheaves to consider on a polyhedral complex including those arising from common (co)sheaf operations, such as restrictions, pullbacks, and Verdier duals. For example, cellular sheaves on polyhedral fans have appeared in [3]. In this work, given a polyhedral fan in $\mathbb{R}^n$, Brion associates to a face σ, the quotient vector space $\mathbb{R}^n/L(\sigma)$ where $L(\sigma)$ denotes the linear span of the σ. These vector spaces come equipped with natural maps between them when there is an inclusion of faces. One can also take pth exterior powers of these vector spaces, as well as generalize this definition beyond polyhedral fans to get a collection of sheaves. The cohomology of these cellular sheaves is related to the motion spaces in discrete dynamical geometry; see [26]. Following [26], the W-sheaves come up in aspects of rigidity and the F-cosheaves of skeleta of polyhedral complexes are related to stress spaces.

Applied Topology Cellular (co)sheaves also appear frequently in applied topology and topological data analysis. Notable examples arise in the study of sensor networks, network coding, and persistent homology; see [6, 12]. The (co)sheaves appearing in these contexts are often a part of the input data of the model under consideration and do not have a simple recipe coming from the geometry of the underlying topological space like in the case of tropical homology. However, our extension allows for the construction of a sheaf from scratch. In the future, our implementation will be extended to cellular (co)sheaves on more general topological spaces using the *polymake* application *topaz*. We would also like to point out the current efforts underway by Olivia Röhrig to implement persistent homology in *polymake*.

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