Stability Results for Some Geometric Inequalities and Their Functional Versions

Umut Caglar and Elisabeth M. Werner

Abstract The Blaschke Santaló inequality and the L_p affine isoperimetric inequalities are major inequalities in convex geometry and they have a wide range of applications. Functional versions of the Blaschke Santaló inequality have been established over the years through many contributions. More recently and ongoing, such functional versions have been established for the L_p affine isoperimetric inequalities as well. These functional versions involve notions from information theory, like entropy and divergence.

We list stability versions for the geometric inequalities as well as for their functional counterparts. Both are known for the Blaschke Santaló inequality. Stability versions for the L_p affine isoperimetric inequalities in the case of convex bodies have only been known in all dimensions for $p = 1$ and for $p > 1$ only for convex bodies in the plane. Here, we prove almost optimal stability results for the L_p affine isoperimetric inequalities, for all p, for all convex bodies, for all dimensions. Moreover, we give stability versions for the corresponding functional versions of the L_p affine isoperimetric inequalities, namely the reverse log Sobolev inequality, the L_p affine isoperimetric inequalities for log concave functions, and certain divergence inequalities.

Keywords Entropy • Divergence • Affine isoperimetric inequalities • Log Sobolev inequalities

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1 Introduction and Background

We present stability results for several geometric and functional inequalities. Our main focus will be on geometric inequalities coming from affine convex geometry, namely the Blaschke Santaló inequality, e.g., [24, 55], and the L_p affine-isoperimetric and related inequalities [12, 21, 45, 51, 66] and also their functional counterparts, which includes the functional Blaschke Santaló inequality [5, 7, 22, 35] and the recently established divergence and entropy inequalities [6, 17, 20]. These inequalities are fundamental in convex geometry and geometric analysis, e.g., [10, 29, 30, 45, 46, 48, 49, 60, 64, 66, 67], and they have applications throughout mathematics. We only quote: approximation theory of convex bodies by polytopes [11, 27, 37, 54, 57, 61], affine curvature flows [3, 4, 62, 63], information theory [6, 17, 18, 20, 51, 65], valuation theory [2, 28, 29, 38, 40, 41, 42, 52, 56], and partial differential equations [43]. Therefore, it is important to know stability results of those inequalities.

Stability results answer the following question: Is the inequality that we consider sensitive to small perturbations? In other words, if a function almost attains the equality in a given inequality, is it possible to say that then this function is close to the minimizers of the inequality? For the Blaschke Santaló inequality and the functional Blaschke Santaló inequality such stability results have been established in [8] and [9], respectively. Stability results for the L_p-affine isoperimetric inequalities for convex bodies were proved in [13] for $p = 1$ and dimension $n \geq 3$. In [32, 33], stability results for the L_p-affine isoperimetric inequality were proved in dimension 2 and for $p \geq 1$.

We present here stability results for the L_p-affine isoperimetric inequalities for all p and in all dimensions. Stability results for the corresponding functional versions of these inequalities are also given.

Throughout, we will assume that K is a convex body in $\mathbb{R}^n$, i.e., a convex compact subset of $\mathbb{R}^n$ with non-empty interior $\mathrm{int}(K)$. We denote by ∂K the boundary of K and by $\mathrm{vol}(K)$ or $|K|$ its n-dimensional volume. B_2^n is the Euclidean unit ball centered at 0 and $S^{n-1} = \partial B_2^n$ its boundary. The standard inner product on $\mathbb{R}^n$ is $\langle , \rangle$. It induces the Euclidean norm, denoted by $\| \cdot \|_2$. We will use the Banach-Mazur distance $d_{BM}(K, L)$ to measure the distance between the convex bodies K and L,

$$d_{BM}(K,L) = \min\{\alpha \geq 1 : K - x \subset T(L-y) \subset \alpha(K-x), \\ \text{for } T \in GL(n), x, y \in \mathbb{R}^n\}.$$

In the case when K and L are 0-symmetric, x and y can be taken to be 0,

$$d_{BM}(K,L) = \min\{\alpha \geq 1 : K \subset T(L) \subset \alpha\, K,\ \text{for } T \in GL(n)\}.$$

2 Stability in Inequalities for Convex Bodies

2.1 The Blaschke Santaló Inequality

Let K be a convex body in $\mathbb{R}^n$ such that $0 \in \mathrm{int}(K)$. The polar K° of K is defined as

$$K^\circ = \{y \in \mathbb{R}^n \ : \ \langle x, y\rangle \le 1 \ \text{ for all } \ x \in K\}$$

and, more generally, the polar K^z with respect to $z \in \mathrm{int}(K)$ by $(K - z)^\circ$. The classical Blaschke Santaló inequality (see, e.g., [55]) states that there is a unique point $s \in \mathrm{int}(K)$, the Santaló point of K, such that the volume product $|K||K^s|$ is minimal and that

$$|K|\,|K^s| \le |B_2^n|^2$$

with equality if and only if K is an ellipsoid.

Ball and Böröczky [8] proved the following stability version of the Blaschke Santaló inequality. It will be one of the tools to prove stability versions for the L_p-affine isoperimetric inequalities.

Theorem 1 ([8]) *Let K be a convex body in $\mathbb{R}^n$, $n \ge 3$, with Santaló point at 0. If $|K||K^\circ| > (1-\varepsilon)|B_2^n|^2$, for $\varepsilon \in (0, \frac{1}{2})$, then for some $\gamma > 0$, depending only on n, we have*

$$d_{BM}(K, B_2^n) < 1 + \gamma \varepsilon^{\frac{1}{3(n+1)}} \,|\log \varepsilon|^{\frac{4}{3(n+1)}}.$$

Remark It was noted in [8] that if K is 0-symmetric, then the exponent $\frac{1}{3(n+1)}$ occurring in Theorem 1 can be replaced by $\frac{2}{3(n+1)}$. Moreover, it was also noted in [8] that taking K to be the convex body resulting from B_2^n by cutting off two opposite caps of volume ε, shows that the exponent $\frac{1}{(3(n+1)}$ cannot be replaced by anything larger than $\frac{2}{n+1}$, even for 0-symmetric convex bodies with axial rotational symmetry. Therefore the exponent of ε is of the correct order.

2.2 L_p-Affine Isoperimetric Inequalities

Now we turn to stability results for the L_p-affine isoperimetric inequalities for convex bodies. These inequalities involve the L_p-affine surface areas which are a central part of the rapidly developing L_p and Orlicz Brunn Minkowski theory and are the focus of intensive investigations (see, e.g., [19, 23, 25, 26, 30, 39, 40, 41, 42, 43, 44, 45, 46, 47, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66]).

The L_p-affine surface area $as_p(K)$ of a convex body K in $\mathbb{R}^n$ was introduced by Lutwak for all $p > 1$ in his seminal paper [45] and for all other p by Schütt and Werner [60](see also [31]). The case $p = 1$ is the classical affine surface area introduced by Blaschke in dimensions 2 and 3 [12] (see also [36, 59]).

Let $p \in \mathbb{R}$, $p \neq -n$ and assume that K is a convex body with centroid or Santaló point at the origin. Then

$$as_p(K) = \int_{\partial K} \frac{\kappa(x)^{\frac{p}{n+p}}}{\langle x, N(x)\rangle^{\frac{n(p-1)}{n+p}}} d\mu_K(x), \tag{1}$$

where $N(x)$ is the unit outer normal in $x \in \partial K$, the boundary of K, $\kappa(x)$ is the (generalized) Gaussian curvature in x and μ_K is the surface area measure on ∂K. In particular, for $p = 0$

$$as_0(K) = \int_{\partial K} \langle x, N_K(x)\rangle \, d\mu_K(x) = n|K|.$$

For $p = 1$,

$$as_1(K) = \int_{\partial K} \kappa_K(x)^{\frac{1}{n+1}} d\mu_K(x)$$

is the classical affine surface area which is independent of the position of K in space. Note also that $as_p(B_2^n) = \mathrm{vol}_{n-1}(\partial B_2^n) = n|B_2^n|$ for all $p \neq -n$. If the boundary of K is sufficiently smooth, (1) can be written as an integral over the boundary S^{n-1} of the Euclidean unit ball B_2^n,

$$as_p(K) = \int_{S^{n-1}} \frac{f_K(u)^{\frac{n}{n+p}}}{h_K(u)^{\frac{n(p-1)}{n+p}}} d\sigma(u).$$

Here, σ is the usual surface area measure on S^{n-1}, $h_K(u) = \max_{x\in K}\langle x, u\rangle$ is the support function of K in direction $u \in S^{n-1}$, and $f_K(u)$ is the curvature function, i.e. the reciprocal of the Gaussian curvature $\kappa_K(x)$ at this point $x \in \partial K$ that has u as outer normal. In particular, for $p = \pm\infty$,

$$as_{\pm\infty}(K) = \int_{S^{n-1}} \frac{1}{h_K(u)^n} d\sigma(u) = n|K^\circ|. \tag{2}$$

The L_p-affine surface area is invariant under linear transformations T with determinant 1. More precisely (see, e.g., [60]), if $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear, invertible map, then

$$as_p(T(K)) = |\det T|^{\frac{n-p}{n+p}} as_p(K). \tag{3}$$

The L_p-affine surface area is a valuation [40, 42, 58], i.e., for convex bodies K and L such that $K \cup L$ is convex,

$$as_p(K \cup L) + as_p(K \cap L) = as_p(K) + as_p(L).$$

Valuations have become a major topic in convex geometry in recent years. We refer to, e.g., [2, 28, 29, 38, 40, 41, 42, 52, 56].

We now state the L_p-affine isoperimetric inequalities for the quantities $as_p(K)$. They were proved by Lutwak for $p > 1$ [45] and for all other p by Werner and Ye [66]. The case $p = 1$ is the classical affine isoperimetric inequality [12, 21].

Theorem 2 ($p = 1$ [12, 21], $p > 1$ [45], all other p [66]) *Let K be a convex body with centroid at the origin.*

(i) If $p > 0$, then

$$\frac{as_p(K)}{as_p(B_2^n)} \leq \left(\frac{|K|}{|B_2^n|}\right)^{\frac{n-p}{n+p}},$$

with equality if and only if K is an ellipsoid. For $p = 0$, equality holds trivially for all K.

(ii) If $-n < p < 0$, then

$$\frac{as_p(K)}{as_p(B_2^n)} \geq \left(\frac{|K|}{|B_2^n|}\right)^{\frac{n-p}{n+p}},$$

with equality if and only if K is an ellipsoid.

(iii) If K is in addition in C_+^2 and if $p < -n$, then

$$c^{\frac{np}{n+p}} \left(\frac{|K|}{|B_2^n|}\right)^{\frac{n-p}{n+p}} \leq \frac{as_p(K)}{as_p(B_2^n)}.$$

The constant c in (iii) is the constant from the inverse Blaschke Santaló inequality due to Bourgain and Milman [15]. This constant has recently been improved by Kuperberg [34] (see also [50] for a different proof).

2.3 Stability for the L_p-Affine Isoperimetric Inequality for Convex Bodies

Stability results for the L_p-affine isoperimetric inequalities for convex bodies were proved by Böröczky [13] for $p = 1$ and dimension $n \geq 3$. Ivaki [32, 33] gave stability results for the L_p-affine isoperimetric inequality in dimension 2 and $p \geq 1$. We present here stability results for the L_p-affine isoperimetric inequalities for all p and in all dimensions. Before we do so, we first quote the results by Böröczky [13] and Ivaki [33].

Theorem 3 ([13]) *If K is a convex body in $\mathbb{R}^n$, $n \geq 3$, and*

$$\left(\frac{as_1(K)}{as_1(B_2^n)}\right)^{n+1} > (1-\epsilon) \left(\frac{|K|}{|B_2^n|}\right)^{n-1} \quad \textit{for } \epsilon \in \left(0, \frac{1}{2}\right), \tag{4}$$

then for some $\gamma > 0$, depending only on n, we have

$$d_{BM}(K, B_2^n) < 1 + \gamma \varepsilon^{\frac{1}{6n}} |\log \varepsilon|^{\frac{1}{6}}.$$

Later, in [8], the above approximation was improved to

$$d_{BM}(K, B_2^n) < 1 + \gamma \varepsilon^{\frac{1}{3(n+1)}} |\log \varepsilon|^{\frac{4}{3(n+1)}}.$$

Ivaki [33] gave a stability version for the Blaschke Santaló inequality from which the following stability result for the L_p-affine isoperimetric inequality in dimension 2 and $p \geq 1$ follows easily.

Theorem 4 ([33]) *Let K be an origin symmetric convex body in $\mathbb{R}^2$, and $p \geq 1$. There exists an $\epsilon_p > 0$, depending on p, such that the following holds. If for an ϵ, $0 < \epsilon < \epsilon_p$,*

$$\left(\frac{as_p(K)}{2\pi}\right)^{p+2} > (1-\epsilon)^p \left(\frac{area(K)}{\pi}\right)^{2-p}$$

then for some $\gamma > 0$, we have

$$d_{BM}(K, B_2^2) < 1 + \gamma \varepsilon^{\frac{1}{2}}. \tag{5}$$

The same author also considered the case when K is a not necessarily origin symmetric convex body in $\mathbb{R}^2$ [33]. Then the order of approximation becomes $\frac{1}{4}$ instead of $\frac{1}{2}$. Note also that there are results in dimension $n = 2$ by Böröczky and Makai [14] on stability of the Blaschke Santaló inequality, from which a stability result of the form (5) for the L_p-affine isoperimetric inequality in dimension 2 follows easily. But the order of approximation in the origin-symmetric case is $1/3$ and in the general case $1/6$.

We now present almost optimal stability results for the L_p-affine isoperimetric inequalities, for all p, for all convex bodies, for all dimensions. To do so, we use the above stability version of the Blaschke Santaló inequality by Ball and Böröczky [8], together with inequalities proved in [66].

Theorem 5 *Let K be a convex body in $\mathbb{R}^n$, $n \geq 3$, with Santaló point or centroid at 0.*

(i) Let $p > 0$. If $\left(\frac{as_p(K)}{as_p(B_2^n)}\right)^{n+p} > (1-\varepsilon)^p \left(\frac{|K|}{|B_2^n|}\right)^{n-p}$, then for some $\gamma > 0$, depending only on n, we have

$$d_{BM}(K, B_2^n) < 1 + \gamma \varepsilon^{\frac{1}{3(n+1)}} |\log \varepsilon|^{\frac{4}{3(n+1)}}.$$

(ii) Let $-n < p < 0$. If $\left(\frac{as_p(K)}{as_p(B_2^n)}\right)^{n+p} < (1-\varepsilon)^p \left(\frac{|K|}{|B_2^n|}\right)^{n-p}$, then for some $\gamma > 0$, depending only on n, we have

$$d_{BM}(K, B_2^n) < 1 + \gamma \varepsilon^{\frac{1}{3(n+1)}} |\log \varepsilon|^{\frac{4}{3(n+1)}}.$$

Remarks (i) If K is 0-symmetric, then $\varepsilon^{\frac{1}{3(n+1)}}$ can be replaced by $\varepsilon^{\frac{2}{3(n+1)}}$. This follows from [8]. See also the Remark after Theorem 1.

(ii) The example in [8] already quoted in the Remark after Theorem 1 shows that $\varepsilon^{\frac{1}{3(n+1)}}$ cannot be replaced by anything smaller than $\varepsilon^{\frac{2}{n-1}}$, even for 0-symmetric convex bodies with axial rotational symmetry. Indeed, let K be the convex body obtained from B_2^n by removing two opposite caps of volume ε each. Then

$$\left(\frac{as_p(K)}{as_p(B_2^n)}\right)^{n+p} > (1 - k\varepsilon^{\frac{n-1}{n+1}})^p \left(\frac{|K|}{|B_2^n|}\right)^{n-p} = (1-\delta)^p \left(\frac{|K|}{|B_2^n|}\right)^{n-p},$$

where we have put $\delta = k\varepsilon^{\frac{n-1}{n+1}}$ and where k is a constant that depends on n only, except for $0 < p < n$, where it also depends on p. And $d_{BM}(K, B_2^n) = 1 + \gamma \delta^{\frac{2}{n-1}}$.

Proof of Theorem 5. (i) As $as_p(B_2^n) = n|B_2^n|$, we observe that the inequality

$$\left(\frac{as_p(K)}{as_p(B_2^n)}\right)^{n+p} > (1-\varepsilon)^p \left(\frac{|K|}{|B_2^n|}\right)^{n-p}$$

is equivalent to the inequality

$$as_p(K)^{n+p} > (1-\varepsilon)^p n^{n+p} |K|^{n-p} |B_2^n|^{2p}. \tag{6}$$

It was proved in [66] that for all $p > 0$,

$$as_p(K)^{n+p} \le n^{n+p} |K|^n |K^\circ|^p.$$

Hence we get from the assumption that

$$n^{n+p} |K|^n |K^\circ|^p > (1-\varepsilon)^p n^{n+p} |K|^{n-p} |B_2^n|^{2p},$$

or equivalently, that

$$|K||K^\circ| > (1-\varepsilon)\, |B_2^n|^2,$$

and we conclude with the Ball and Böröczky stability result in Theorem 1.

(ii) The proof of (ii) is done similarly. We use the inequality

$$as_p(K)^{n+p} \ge n^{n+p} |K|^n |K^\circ|^p,$$

which holds for $-n < p < 0$ and which was also proved in [66]. □

Another stability result for the L_p-affine isoperimetric inequalities for convex bodies is obtained as a corollary to Proposition 17 below. We list it now, as we want to compare the two. Let K be a convex body in $\mathbb{R}^n$ with 0 in its interior and let the function ψ of Proposition 17 be $\psi(x) = \|x\|_K^2/2$, where $\|\cdot\|_K$ is the gauge function of the convex body K,

$$\|x\|_K = \min\{\alpha \geq 0 : x \in \alpha K\} = \max_{y \in K^\circ} \langle x, y \rangle.$$

Let

$$as_\lambda(\psi) = \int_{\mathbb{R}^n} e^{(2\lambda-1)\psi(x)-\lambda\langle \nabla\psi, x\rangle} \left(\det\left(\nabla^2\psi(x)\right)\right)^\lambda dx \tag{7}$$

be the L_λ-affine surface area of the function ψ. This quantity is discussed in detail in Section 3.3. Differentiating $\psi(x) = \|x\|_K^2/2$, we get $\langle x, \nabla\psi(x)\rangle = 2\psi(x)$. Thus, for $\psi(x) = \|x\|_K^2/2$, the expression (7) simplifies to

$$as_\lambda(\psi) = \int_{\mathbb{R}^n} \left(\det \nabla^2\psi(x)\right)^\lambda e^{-\psi(x)} dx. \tag{8}$$

Note that for the Euclidean norm $\|.\|_2$, $as_\lambda\left(\frac{\|\cdot\|_2^2}{2}\right) = (2\pi)^{\frac{n}{2}}$ and it was proved in [20] that

$$\frac{as_\lambda\left(\frac{\|\cdot\|_K^2}{2}\right)}{as_\lambda\left(\frac{\|\cdot\|_2^2}{2}\right)} = \frac{as_p(K)}{as_p(B_2^n)}, \tag{9}$$

where λ and p are related by $\lambda = \frac{p}{n+p}$. Together with Proposition 17, this immediately implies another stability result for the L_p-affine isoperimetric inequalities for convex bodies.

Corollary 6 *Let K be a convex body in $\mathbb{R}^n$ with the centroid or the Santaló point at the origin.*
(i) Let $0 < p \leq \infty$ and suppose that for some $\varepsilon \in (0, \varepsilon_0)$,

$$\frac{as_p(K)}{as_p(B_2^n)} > (1-\varepsilon)^{\frac{p}{n+p}} \left(\frac{|K|}{|B_2^n|}\right)^{\frac{n-p}{n+p}}.$$

(i) Let $-n < p < 0$ and suppose that for some $\varepsilon \in (0, \varepsilon_0)$,

$$\frac{as_p(K)}{as_p(B_2^n)} < (1-\varepsilon)^{\frac{p}{n+p}} \left(\frac{|K|}{|B_2^n|}\right)^{\frac{n-p}{n+p}}.$$

Then, in both cases (i) and (ii), there exist $c > 0$ and a positive definite matrix A such that

$$\int_{R(\varepsilon)B_2^n} \left| \|Ax\|_K^2 - \|x\|_2^2 - c \right| dx < \eta \varepsilon^{\frac{1}{129n^2}},$$

where $R(\varepsilon) = \frac{|\log \varepsilon|^{\frac{1}{2}}}{8n}$ and ε_0, η depend on n.

Proof It is easy to see (e.g., [20]) that

$$|K| = \frac{1}{2^{\frac{n}{2}} \Gamma\left(1 + \frac{n}{2}\right)} \int e^{-\frac{\|x\|_K^2}{2}} dx.$$

As $|B_2^n| = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(1+\frac{n}{2}\right)}$, we get, with $\psi(x) = \frac{\|x\|_K^2}{2}$, by (9) and the assumptions of the theorem, that for $0 < p \leq \infty$,

$$as_\lambda(\psi) > (1-\varepsilon)^\lambda (2\pi)^{n\lambda} \left(\int e^{-\psi(x)} dx \right)^{1-2\lambda}.$$

We have also used that $\lambda = \frac{p}{n+p}$. The result for $0 < p \leq \infty$ then follows immediately from Proposition 17. The case $-n < p < 0$ is treated similarly. $\square$

Remarks In general, one cannot deduce Theorem 5 from Corollary 6. However, it follows from Theorem 5 that there exists $T \in GL(n)$ and $x_0, y_0 \in \mathbb{R}^n$ such that

$$K - x_0 \subset T(B_2^n - y_0) \subset \left(1 + \gamma \varepsilon^{\frac{1}{3(n+1)}} |\log \varepsilon|^{\frac{4}{3(n+1)}}\right) (K - x_0).$$

For simplicity, assume that $x_0 = y_0 = 0$, which corresponds to the case that K is 0-symmetric. Then this means that for all $x \in \mathbb{R}^n$,

$$|\ \|x\|_K - \|T(x)\|_2\ | \leq \|T\| \left(\gamma \varepsilon^{\frac{1}{3(n+1)}} |\log \varepsilon|^{\frac{4}{3(n+1)}}\right) \|x\|_2$$

and thus

$$\begin{aligned}
&\int_{R(\varepsilon)B_2^n} \left|\ \|x\|_K^2 - \|T(x)\|_2^2\ \right| dx \\
&\leq \left(1 + \gamma \varepsilon^{\frac{1}{3(n+1)}} |\log \varepsilon|^{\frac{4}{3(n+1)}}\right) |B_2^n|\ \|T\|^2 R^{n+2}(\varepsilon) \left(\gamma \varepsilon^{\frac{1}{3(n+1)}} |\log \varepsilon|^{\frac{4}{3(n+1)}}\right) \\
&= \left(1 + \gamma \varepsilon^{\frac{1}{3(n+1)}} |\log \varepsilon|^{\frac{4}{3(n+1)}}\right) \frac{|B_2^n|}{(8n)^{n+2}} \|T\|^2 \left(\gamma \varepsilon^{\frac{1}{3(n+1)}} |\log \varepsilon|^{\frac{4}{3(n+1)} + \frac{n+2}{2}}\right).
\end{aligned}$$

Hence, allowing general T, the exponent of ε can be improved.

2.4 Stability Result for the Entropy Power Ω_K

An affine invariant quantity that is closely related to the L_p-affine surface areas is the entropy power Ω_K. It was introduced in [51] as the limit of L_p-affine surface areas,

$$\Omega_K = \lim_{p\to\infty} \left(\frac{as_p(K)}{n|K^\circ|}\right)^{n+p}. \tag{10}$$

The quantity Ω_K is related to the relative entropy of the cone measures of K and K°. We refer to [51] for the details and only mention an affine isoperimetric inequality for Ω_K proved in [51].

Theorem 7 ([51]) *If K is a convex body of volume* 1*, then*

$$\Omega_{K^\circ} \leq \Omega_{\left(\frac{B_2^n}{|B_2^n|^{\frac{1}{n}}}\right)^\circ}. \tag{11}$$

Equality holds if and only if K is a normalized ellipsoid.

We now use the previous theorems to prove stability results for inequality (11). Using the invariant property (3) and the fact that $as_p(B_2^n) = n|B_2^n|$, this inequality can be written as

$$\Omega_{K^\circ} \leq |B_2^n|^{2n}.$$

Theorem 8 *Let K be a convex body in* $\mathbb{R}^n$*,* $n \geq 3$*, of volume* 1 *and such that the Santaló point or the centroid is at* 0*. Suppose that for some* $\varepsilon \in (0, \frac{1}{2})$*,*

$$\Omega_{K^\circ} > (1-\varepsilon)|B_2^n|^{2n}. \tag{12}$$

Then for some $\gamma > 0$*, depending only on n, we have*

$$d_{BM}(K^\circ, B_2^n) < 1 + \gamma \left(\frac{2\varepsilon}{n}\right)^{\frac{1}{3(n+1)}} \left|\log \frac{2\varepsilon}{n}\right|^{\frac{4}{3(n+1)}}.$$

Remarks similar to the ones after Theorem 5 hold.

Proof It was shown in [66] that $\left(\frac{as_p(K^\circ)}{n|K|}\right)^{n+p}$ is decreasing in $p \in (0, \infty)$. By definition (7), $\lim_{p\to\infty} \left(\frac{as_p(K^\circ)}{n|K|}\right)^{n+p} = \Omega_{K^\circ}$. Therefore we get with assumption (12) that for all $p > 0$

$$\left(\frac{as_p(K^\circ)}{n|K|}\right)^{n+p} > (1-\varepsilon)|B_2^n|^{2n}.$$

Or, equivalently, as $|K| = 1$,

$$as_p(K^\circ)^{n+p} > (1-\varepsilon)n^{n+p}|K|^{n+p}|B_2^n|^{2n} = (1-\varepsilon)n^{n+p}|B_2^n|^{2p}\,|B_2^n|^{2(n-p)}$$
$$\geq (1-\varepsilon)n^{n+p}|K^\circ|^{n-p}|B_2^n|^{2p}.$$

In the last inequality we have used the Blaschke Santaló inequality $|K|\,|K^\circ| \leq |B_2^n|^2$, which we can apply as long as $n - p \geq 0$. Note that for all $\varepsilon \in (0, \frac{1}{2})$ and $p > 0$

$$1 - \varepsilon > \left(1 - \frac{2\varepsilon}{p}\right)^p.$$

Hence, using the elementary inequality above, we get for all $0 < p \leq n$ that

$$as_p(K^\circ)^{n+p} > \left(1 - \frac{2\varepsilon}{p}\right)^p n^{n+p}|K^\circ|^{n-p}|B_2^n|^{2p}.$$

Inequality (6) and the arguments used after it imply that for all $0 < p \leq n$,

$$d_{BM}(K^\circ, B_2^n) < 1 + \gamma\left(\frac{2\varepsilon}{p}\right)^{\frac{1}{3(n+1)}}\left|\log\frac{2\varepsilon}{p}\right|^{\frac{4}{3(n+1)}}.$$

Since the right-hand side of above equation is decreasing in p, minimizing over p in the interval $(0, n]$ gives the result. □

The second stability result and the corresponding comparisons (see the Remark after Corollary 6) are obtained accordingly. We skip the proof.

Theorem 9 *Let K be a convex body in $\mathbb{R}^n$, $n \geq 3$, of volume 1 and with Santaló point or centroid at 0, such that $\Omega_{K^\circ} > (1-\varepsilon)|B_2^n|^{2n}$. Then there exists $c > 0$ and a positive definite matrix A such that*

$$\int_{R(\varepsilon)B_2^n} \left|\|Ax\|_K^2 - |x|_2^2 - c\right| dx < \eta\varepsilon^{\frac{1}{129n^2}},$$

$R(\varepsilon) = \frac{|\log\varepsilon|^{\frac{1}{2}}}{8n}$ *and ε_0, η depend on n.*

3 Stability Results for Functional Inequalities

3.1 Stability for the Functional Blaschke Santaló Inequality

We will first state a functional version of the Blaschke Santaló inequality. To do so, we recall that the Legendre transform of a function $\psi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ at $z \in \mathbb{R}^n$ is defined by

$$\mathcal{L}_z\psi(y) = \sup_{x\in\mathbb{R}^n} \left(\langle x - z, y\rangle - \psi(x)\right), \quad \text{for } y \in \mathbb{R}^n. \tag{13}$$

The function $L_z\psi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is always convex and lower semicontinuous. If ψ is convex, lower semicontinuous and $\psi < +\infty$, then $L_zL_z\psi = \psi$. When $z = 0$, we write

$$\psi^*(y) = \mathcal{L}_0\psi(y) = \sup_x \left(\langle x, y\rangle - \psi(x)\right). \tag{14}$$

Work by K.M. Ball [7], S. Artstein-Avidan, B. Klartag, V.D.Milman [5], M. Fradelizi, M. Meyer [22], and J. Lehec [35] led to the functional version of the Blaschke Santaló inequality which we now state.

Theorem 10 ([5, 7, 22, 35]) *Let $\rho : \mathbb{R} \to \mathbb{R}_+$ be a log-concave non-increasing function and $\psi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be measurable. Then*

$$\inf_{z\in\mathbb{R}^n} \int_{\mathbb{R}^n} \rho(\psi(x))dx \int_{\mathbb{R}^n} \rho(\mathcal{L}_z\psi(x))dx \le \left(\int_{\mathbb{R}^n} \rho\left(\frac{\|x\|_2^2}{2}\right) dx\right)^2.$$

If ρ is decreasing, there is equality if and only if there exist a, b, c in $\mathbb{R}$, $a < 0$, $z \in \mathbb{R}^n$ and a positive definite matrix $A : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\psi(x) = \frac{\|A(x+z)\|_2^2}{2} + c, \ \textit{for} \ x \in \mathbb{R}^n$$

and moreover either $c = 0$, or $\rho(t) = e^{at+b}$, for $t > -|c|$.

Remark If $\rho(t) = e^{-t}$ and if $\varphi = e^{-\psi}$ has centroid at 0, i.e., $\int_{\mathbb{R}^n} xe^{-\psi}dx = 0$, then the inequality of the above theorem simplifies to

$$\begin{aligned}\int_{\mathbb{R}^n} \rho(\psi(x))dx \int_{\mathbb{R}^n} \rho(\mathcal{L}_z\psi(x))dx &= \left(\int_{\mathbb{R}^n} e^{-\psi(x))}dx\right)\left(\int_{\mathbb{R}^n} e^{-\psi^*(x))}dx\right) \\ &\le \left(\int_{\mathbb{R}^n} e^{-\frac{\|x\|_2^2}{2}} dx\right)^2. \end{aligned} \tag{15}$$

Barthe, Böröczky, and Fradelizi [9] established the following stability theorem for the functional Blaschke Santaló inequality.

Theorem 11 ([9]) *Let $\rho : \mathbb{R} \to \mathbb{R}_+$ be a log-concave and decreasing function with $\int_{\mathbb{R}_+} \rho < \infty$. Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a convex, measurable function. Assume that for some $\varepsilon \in (0, \varepsilon_0)$ and all $z \in \mathbb{R}^n$ the following inequality holds*

$$\int_{\mathbb{R}^n} \rho(\psi(x))dx \int_{\mathbb{R}^n} \rho(\mathcal{L}_z\psi(x))dx > (1-\varepsilon)\left(\int_{\mathbb{R}^n} \rho\left(\frac{\|x\|_2^2}{2}\right) dx\right)^2.$$

Then there exists some $z \in \mathbb{R}^n$, $c \in \mathbb{R}$ and a positive definite $n \times n$ matrix A such that

$$\int_{R(\varepsilon)B_2^n} \left| \frac{\|x\|_2^2}{2} + c - \psi(Ax+z) \right| dx < \eta \varepsilon^{\frac{1}{129n^2}},$$

where $\lim_{\varepsilon \to 0} R(\varepsilon) = \infty$ *and* $\varepsilon_0, \eta, R(\varepsilon)$ *depend on* n *and* ρ.

3.2 Stability for Divergence Inequalities

A function $\varphi : \mathbb{R}^n \to [0, \infty)$ is log concave, if it is of the form $\varphi(x) = e^{-\psi(x)}$, where $\psi : \mathbb{R}^n \to \mathbb{R}$ is a convex function. Recall that we say that $\varphi = e^{-\psi}$ has centroid at 0, respectively, the Santaló point, at 0 if,

$$\int x\varphi(x)dx = \int xe^{-\psi(x)}dx = 0, \text{ respectively } \int xe^{-\psi^*(x)}dx = 0.$$

The following entropy inequality for log concave functions was established in [17], Corollary 13.

Theorem 12 ([17]) *Let* $\varphi : \mathbb{R}^n \to [0, \infty)$ *be a log-concave function that has centroid or Santaló point at* 0. *Let* $f : (0, \infty) \to \mathbb{R}$ *be a convex, decreasing function. Then*

$$\int_{supp(\varphi)} \varphi f\left(e^{\langle \frac{\nabla\varphi}{\varphi}, x\rangle} \varphi^{-2} \left(\det\left(\nabla^2\left(-\log\varphi\right)\right)\right)\right) \geq f\left(\frac{(2\pi)^n}{\left(\int \varphi dx\right)^2}\right) \left(\int_{supp(\varphi)} \varphi dx\right). \quad (16)$$

If f *is a concave, increasing function, the inequality is reversed.*
Equality holds in both cases if and only if $\varphi(x) = ce^{-\langle Ax, x\rangle}$, *where* c *is a positive constant and* A *is an* $n \times n$ *positive definite matrix.*

Theorem 12 was proved under the assumptions that the convex or concave functions f and the log concave functions φ have enough smoothness and integrability properties so that the expressions considered in the above statement make sense. Thus, in this section, we will make the same assumptions on f and φ, i.e., we will assume that $\varphi^\circ \in L^1(\text{supp}(\varphi), dx)$, the Lebesgue integrable functions on the support of φ, that

$$\varphi \in C^2(\text{supp}(\varphi)) \cap L^1(\mathbb{R}^n, dx), \quad (17)$$

where $C^2(\text{supp}(\varphi))$ denotes the twice continuously differentiable functions on their support, and that

$$\varphi f\left(\frac{e^{\frac{\langle\nabla\varphi, x\rangle}{\varphi}}}{\varphi^2} \det\left(\nabla^2\left(-\log\varphi\right)\right)\right) \in L^1(\text{supp}(\varphi), dx). \quad (18)$$

Recall that $\varphi(x) = e^{-\psi(x)}$ and put $d\mu = e^{-\psi}dx$. Then the left-hand side of inequality (16) can be written as

$$\int_{\mathbb{R}^n} f\left(e^{2\psi-\langle\nabla\psi,x\rangle}\det\left(\nabla^2\psi\right)\right)d\mu.$$

It was shown in [17] that the left-hand side of the inequality (16) is the natural definition of f-divergence $D_f(\varphi)$ for a log concave function φ, so that (16) can be rewritten as

$$D_f(\varphi) \geq f\left(\frac{(2\pi)^n}{\left(\int\varphi dx\right)^2}\right)\left(\int_{\mathrm{supp}(\varphi)}\varphi dx\right). \tag{19}$$

In information theory, probability theory, and statistics, an f-divergence is a function that measures the difference between two (probability) distributions. We refer to, e.g., [17] for details and references about f-divergence.

Theorem 13 *Let $f : (0,\infty) \to \mathbb{R}$ be a concave, strictly increasing function. Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a convex function such that $e^{-\psi} \in C^2(\mathbb{R}^n)$ and such that $\int_{\mathbb{R}^n} xe^{-\psi(x)}dx = 0$ or $\int_{\mathbb{R}^n} xe^{-\psi^*(x)}dx = 0$. Suppose that for some $\varepsilon \in (0,\varepsilon_0)$,*

$$\int_{\mathbb{R}^n} f\left(e^{2\psi-\langle\nabla\psi,x\rangle}\det\left(\nabla^2\psi\right)\right)d\mu > \\ f\left(\frac{(2\pi)^n}{\left(\int_{\mathbb{R}^n} d\mu\right)^2}\right)\left(\int_{\mathbb{R}^n} d\mu\right) - \varepsilon f'\left(\frac{(2\pi)^n}{\left(\int_{\mathbb{R}^n} d\mu\right)^2}\right)\left(\int_{\mathbb{R}^n} d\mu\right)^{-1}.$$

Then there exist $c > 0$ and a positive definite matrix A such that

$$\int_{R(\varepsilon)B_2^n}\left|\frac{\|x\|_2^2}{2} + c - \psi(Ax)\right|dx < \eta\varepsilon^{\frac{1}{129n^2}},$$

where $\lim_{\varepsilon\to 0} R(\varepsilon) = \infty$ and $\varepsilon_0, \eta, R(\varepsilon)$ depend on n.

The analogue stability result holds, if f is convex and strictly decreasing.

Proof We treat the case when f is concave and strictly increasing. The case when f is convex and strictly decreasing is done similarly. We set $d\nu = \frac{e^{-\psi}dx}{\int e^{-\psi}dx} = \frac{\mu}{\int d\mu}$. Then ν is a probability measure and by Jensen's inequality and a change of variable,

$$\left(\int d\mu\right)\int_{\mathbb{R}^n} f\left(e^{(2\psi(x)-\langle\nabla\psi,x\rangle)}\left(\det\left(\nabla^2\psi(x)\right)\right)\right)d\nu \leq \\ \left(\int d\mu\right) f\left(\int_{\mathbb{R}^n} e^{(2\psi(x)-\langle\nabla\psi,x\rangle)}\left(\det\left(\nabla^2\psi(x)\right)\right)d\nu\right)$$

$$= f\left(\frac{1}{\int d\mu}\int_{\mathbb{R}^n} e^{-\psi^*(x)}dx\right)\left(\int d\mu\right).$$

Thus, by the assumption of the theorem, we get

$$\begin{aligned} f&\left(\frac{1}{\int d\mu}\int_{\mathbb{R}^n} e^{-\psi^*(x)}dx\right)\left(\int d\mu\right) \\ &> \left(\int d\mu\right) f\left(\frac{(2\pi)^n}{\left(\int d\mu\right)^2}\right) - \frac{\varepsilon}{\int d\mu} f'\left(\frac{(2\pi)^n}{\left(\int d\mu\right)^2}\right) \\ &\geq \left(\int d\mu\right) f\left(\frac{(2\pi)^n - \varepsilon}{\left(\int d\mu\right)^2}\right). \end{aligned}$$

The last inequality holds as by Taylor's theorem and the assumptions on f (i.e., $f'' \leq 0$), for ε small enough, there is a real number τ such that

$$\begin{aligned} f\left(\frac{(2\pi)^n - \varepsilon}{\left(\int d\mu\right)^2}\right) &= f\left(\frac{(2\pi)^n}{\left(\int d\mu\right)^2}\right) - \frac{\varepsilon}{\left(\int d\mu\right)^2} f'\left(\frac{(2\pi)^n}{\left(\int d\mu\right)^2}\right) + \frac{\varepsilon^2}{2\left(\int d\mu\right)^4} f''(\tau) \\ &\leq f\left(\frac{(2\pi)^n}{\left(\int d\mu\right)^2}\right) - \frac{\varepsilon}{\left(\int d\mu\right)^2} f'\left(\frac{(2\pi)^n}{\left(\int d\mu\right)^2}\right). \end{aligned}$$

Therefore we arrive at

$$f\left(\frac{1}{\int d\mu}\int_{\mathbb{R}^n} e^{-\psi^*(x)}dx\right) > f\left(\frac{(2\pi)^n - \varepsilon}{\left(\int d\mu\right)^2}\right).$$

Since f is strictly increasing we conclude that

$$\frac{1}{\int d\mu}\int_{\mathbb{R}^n} e^{-\psi^*(x)}dx > \frac{(2\pi)^n - \varepsilon}{\left(\int d\mu\right)^2},$$

which is equivalent to

$$\left(\int_{\mathbb{R}^n} e^{-\psi(x)}dx\right)\left(\int_{\mathbb{R}^n} e^{-\psi^*(x)}dx\right) > (2\pi)^n - \varepsilon.$$

From that we get

$$\left(\int_{\mathbb{R}^n} e^{-\psi(x)}dx\right)\left(\int_{\mathbb{R}^n} e^{-\psi^*(x)}dx\right) > (1-\varepsilon)\,(2\pi)^n.$$

As μ has its centroid at 0, we have by (15) that

$$\inf_{z\in\mathbb{R}^n}\left(\int_{\mathbb{R}^n} e^{-\psi(x)}dx\right)\left(\int_{\mathbb{R}^n} e^{-\mathcal{L}_z\psi(y)}\,dy\right) = \left(\int_{\mathbb{R}^n} e^{-\psi}\,dx\right)\left(\int_{\mathbb{R}^n} e^{-\psi^*(y)}\,dy\right)$$

and the theorem follows from the result by Barthe, Böröczky and Fradelizi [9], Theorem 11, with $\rho(t) = e^{-t}$. □

3.3 Stability for the Reverse Log Sobolev Inequality

We now prove a stability result for the reverse log Sobolev inequality. This inequality was first proved by Artstein-Avidan, Klartag, Schütt, and Werner [6] under strong smoothness assumptions. Those were subsequently removed in [20] and there, also equality characterization was achieved.

We first recall the reverse log Sobolev inequality. Let γ_n be the standard Gaussian measure on $\mathbb{R}^n$. For a log-concave probability measure μ on $\mathbb{R}^n$ with density $e^{-\psi}$, i.e., $\psi = -\log(d\mu/dx)$, let

$$S(\mu) = \int_{\mathbb{R}^n} \psi\, d\mu$$

be the Shannon entropy of μ.

Theorem 14 ([6, 20]) *Let μ be a log-concave probability measure on $\mathbb{R}^n$ with density $e^{-\psi}$ with respect to the Lebesgue measure. Then*

$$\int_{\mathbb{R}^n} \log\left(\det(\nabla^2\psi)\right)\, d\mu \le 2\ \left(S(\gamma_n) - S(\mu)\right). \tag{20}$$

Equality holds if and only if μ is Gaussian (with arbitrary mean and positive definite covariance matrix).

Inequality (20) is a reverse log Sobolev inequality as it can be shown that the log Sobolev inequality is equivalent to

$$2\left(S(\gamma_n) - S(\mu)\right) \le n\log\left(\frac{\int_{\mathbb{R}^n}\Delta\psi\, d\mu}{n}\right),$$

where Δ is the Laplacian. We refer to, e.g., [6, 20] for the details.

Note that inequality (20) follows from inequality (16) with $f(t) = \log t$. However, because of the assumptions on φ in Theorem 13, the result would only hold under those assumptions and not in the full generality stated in Theorem 14. Similarly, a stability result for Theorem 14 follows from Theorem 13 with $f(t) = \log t$. But again, because of the assumptions of Theorem 13, the result would only hold for those ψ such that $e^{-\psi}$ is in $C^2(\mathbb{R}^n)$ and has centroid at 0. We can prove a stability result for Theorem 14 without these assumptions. The proof is similar to the one of Theorem 13. We include it for completeness. But first we need to recall various items.

For a convex function $\psi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, we define D_ψ to be the convex domain of ψ, $D_\psi = \{x \in \mathbb{R}^n, \psi(x) < +\infty\}$. We always consider convex functions ψ such that int $\left(D_\psi\right) \neq \emptyset$. In the general case, when ψ is neither smooth nor strictly convex, the gradient of ψ, denoted by $\nabla\psi$, exists almost everywhere by Rademacher's theorem (e.g., [53]), and a theorem of Alexandrov [1], Busemann and Feller [16], guarantees the existence of its Hessian $\nabla^2\psi$ almost everywhere in int $\left(D_\psi\right)$. We let X_ψ be the set of points of int $\left(D_\psi\right)$ at which its Hessian $\nabla^2\psi$ in the sense of Alexandrov, Busemann, and Feller exists and is invertible. Then, by definition of the Legendre transform, for a convex function $\psi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ we have

$$\psi(x) + \psi^*(y) \geq \langle x, y \rangle$$

for every $x, y \in \mathbb{R}^n$, and with equality if and only if $x \in D_\psi$ and $y = \nabla\psi(x)$, i.e.,

$$\psi^*(\nabla\psi(x)) = \langle x, \nabla\psi(x)\rangle - \psi(x), \quad \text{a.e. in } \mathrm{D}_{\llcorner}. \tag{21}$$

Theorem 15 *Let* $\psi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *be a convex function and let* μ *be a log-concave probability measure on* $\mathbb{R}^n$ *with density* $e^{-\psi}$ *with respect to Lebesgue measure. Suppose that for some* $\varepsilon \in (0, \varepsilon_0)$*,*

$$\int_{\mathbb{R}^n} \log\bigl(\det(\nabla^2\psi)\bigr)\, d\mu > 2\left(S(\gamma_n) - S(\mu)\right) - \varepsilon.$$

Then there exist $c > 0$ *and a positive definite matrix* A *such that*

$$\int_{R(\varepsilon)B_2^n} \left|\frac{\|x\|_2^2}{2} + c - \psi(Ax)\right| dx < \eta\varepsilon^{\frac{1}{129n^2}},$$

where $\lim_{\varepsilon\to 0} R(\varepsilon) = \infty$ *and* $\varepsilon_0, \eta, R(\varepsilon)$ *depend on n.*

Proof Both terms of the inequality are invariant under translations of the measure μ, so we can assume that μ has its centroid at 0.

Put $\varepsilon = \log\beta > 0$. Since $S(\gamma_n) = \frac{n}{2}\log(2\pi e)$, the inequality of the theorem turns into

$$\int_{D_\psi} \log\bigl(\beta\, \det(\nabla^2\psi)\bigr)\, d\mu + 2\int_{D_\psi} \psi\, d\mu > \log(2\pi e)^n,$$

which in turn is equivalent to

$$\int_{D_\psi} \log\big(\beta \det(\nabla^2\psi)\big)\, d\mu + \int_{D_\psi} \log\left(e^{2\psi}\right)\, d\mu - n > \log(2\pi)^n. \tag{22}$$

We now use the divergence theorem and get

$$\int_{D_\psi} \langle x, \nabla\psi(x)\rangle\, d\mu = \int_{\mathrm{int}(D_\psi)} \mathrm{div}(x)\, d\mu - \int_{\partial D_\psi} \langle x, N_{D_\psi}(x)\rangle e^{-\psi(x)} d\sigma_{D_\psi},$$

where $N_{D_\psi}(x)$ is an exterior normal to the convex set D_ψ at the point x and σ_{D_ψ} is the surface area measure on ∂D_ψ. Since D_ψ is convex, the centroid 0 of μ is in D_ψ. Thus $\langle x, N_{D_\psi}(x)\rangle \geq 0$ for every $x \in \partial D_\psi$ and $\mathrm{div}(x) = n$ hence

$$-n \leq -\int_{D_\psi} \langle x, \nabla\psi(x)\rangle\, d\mu = \int_{D_\psi} \log\left(e^{-\langle x, \nabla\psi(x)\rangle}\right) d\mu$$

Thus we get from inequality (22),

$$\int_{D_\psi} \log\big(\beta \det(\nabla^2\psi)\, e^{2\psi(x) - \langle x, \nabla\psi(x)\rangle}\big)\, d\mu > \log(2\pi)^n.$$

With Jensen's inequality, and as $d\mu = e^{-\psi} dx$,

$$\beta \int_{D_\psi} \det(\nabla^2\psi)\, e^{\psi(x) - \langle x, \nabla\psi(x)\rangle} dx > (2\pi)^n. \tag{23}$$

By (21),

$$\int_{D_\psi} \det(\nabla^2\psi)\, e^{\psi(x) - \langle x, \nabla\psi(x)\rangle} dx = \int_{D_\psi} \det(\nabla^2\psi)\, e^{-\psi^*(\nabla\psi(x))} dx.$$

The change of variable $y = \nabla\psi(x)$ gives

$$\int_{D_\psi} e^{-\psi^*(\nabla\psi(x))} \det(\nabla^2\psi(x))\, dx = \int_{D_{\psi^*}} e^{-\psi^*(y)}\, dy, \tag{24}$$

and inequality (23) becomes

$$\int_{D_{\psi^*}} e^{-\psi^*(y)}\, dy > \frac{1}{\beta}(2\pi)^n.$$

As $\int_{D_\psi} e^{-\psi} dx = 1$ and $\beta^{-1} = e^{-\varepsilon} \geq 1 - \varepsilon$, we therefore get that

$$\left(\int_{\mathbb{R}^n} e^{-\psi} dx\right)\left(\int_{\mathbb{R}^n} e^{-\psi^*(y)} dy\right) \geq \left(\int_{D_\psi} e^{-\psi} dx\right)\left(\int_{D_{\psi^*}} e^{-\psi^*(y)} dy\right)$$
$$> (1-\varepsilon)(2\pi)^n.$$

As μ has its centroid at 0, we have by (15) that

$$\inf_{z\in\mathbb{R}^n}\left(\int_{\mathbb{R}^n} e^{-\psi(x)} dx\right)\left(\int_{\mathbb{R}^n} e^{-\mathcal{L}_z\psi(y)} dy\right) = \left(\int_{\mathbb{R}^n} e^{-\psi} dx\right)\left(\int_{\mathbb{R}^n} e^{-\psi^*(y)} dy\right).$$

The theorem now follows from Theorem 11, the stability result for the functional Blaschke Santaló inequality, due to Barthe, Böröczky, and Fradelizi [9]. □

3.4 Stability for the L_λ-Affine Isoperimetric Inequality for Log Concave Functions

The following divergence inequalities were proved in [17]. In fact, inequalities (25), (26) and consequently (16) are special cases of a more general divergence inequality proved in [17].

For $0 \leq \lambda \leq 1$, it says

$$\int\left(e^{2\psi-\langle\nabla\psi,x\rangle}\det\left(\nabla^2\psi\right)\right)^\lambda d\mu \leq \left(\frac{\int_{\mathbb{R}^n} e^{-\psi^*}dx}{\int_{\mathbb{R}^n} d\mu}\right)^\lambda \left(\int_{\mathbb{R}^n} d\mu\right) \tag{25}$$

and for $\lambda \notin [0,1]$,

$$\int\left(e^{2\psi-\langle\nabla\psi,x\rangle}\det\left(\nabla^2\psi\right)\right)^\lambda d\mu \geq \left(\frac{\int_{\mathbb{R}^n} e^{-\psi^*}dx}{\int_{\mathbb{R}^n} d\mu}\right)^\lambda \left(\int_{\mathbb{R}^n} d\mu\right). \tag{26}$$

The left-hand sides of the above inequalities are the L_λ-affine surface areas $as_\lambda(\psi)$. For a general log concave function $\varphi = e^{-\psi}$ (and not just a log concave function in $C^2(\mathbb{R}^n)$) they were introduced in [20],

$$as_\lambda(\psi) = \int_{X_\psi} e^{(2\lambda-1)\psi(x)-\lambda\langle\nabla\psi,x\rangle}\left(\det\left(\nabla^2\psi(x)\right)\right)^\lambda dx. \tag{27}$$

Since $\det\left(\nabla^2\psi(x)\right) = 0$ outside X_ψ, the integral may be taken on D_ψ for $\lambda > 0$. In particular,

$$as_0(\psi) = \int_{X_\psi} e^{-\psi(x)} dx \quad\text{and}\quad as_1(\psi) = \int_{X_{\psi^*}} e^{-\psi^*(x)} dx.$$

Assume now that $\int xe^{-\psi(x)}dx = 0$ or $\int xe^{-\psi^*(x)}dx = 0$. Then we can apply the functional Blaschke Santaló inequality (15) and get from (25) that for $\lambda \in [0, 1]$,

$$as_\lambda(\psi) \leq (2\pi)^{n\lambda} \left(\int_{\mathbb{R}^n} e^{-\psi(x)} dx \right)^{1-2\lambda}.$$

Similarly, for $\lambda \leq 0$, we get from (26)

$$as_\lambda(\psi) \geq (2\pi)^{n\lambda} \left(\int_{\mathbb{R}^n} e^{-\psi(x)} dx \right)^{1-2\lambda},$$

provided that $\varphi \in C^2(\mathbb{R}^n)$, which is the assumption on φ in inequality (16). However, these inequalities hold without such a strong smoothness assumption. This, together with characterization of equality, was proved in [20].

Theorem 16 ([20]) *Let $\psi : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ be a convex function. For $\lambda \in [0, 1]$,*

$$as_\lambda(\psi) \leq (2\pi)^{n\lambda} \left(\int_{X_\psi} e^{-\psi(x)} dx \right)^{1-2\lambda} \tag{28}$$

and for $\lambda \leq 0$,

$$as_\lambda(\psi) \geq (2\pi)^{n\lambda} \left(\int_{X_\psi} e^{-\psi(x)} dx \right)^{1-2\lambda}. \tag{29}$$

For $\lambda = 0$ equality holds trivially in these inequalities. Moreover, for $0 < \lambda \leq 1$, or $\lambda < 0$, equality holds in above inequalities if and only if $\psi(x) = \frac{1}{2}\langle Ax, x\rangle + c$, where A is a positive definite $n \times n$ matrix and c is a constant.

A stability result for these inequalities is again an immediate consequence of Theorem 13. But again, we would then get the stability result for log concave functions $\varphi \in C^2(\mathbb{R}^n)$ only, so we include the proof for general functions.

Proposition 17 *Let $\psi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a convex function such that $\int xe^{-\psi(x)}dx = 0$ or $\int xe^{-\psi^*(x)}dx = 0$.*

(i) Let $0 < \lambda \leq 1$ and suppose that for some $\varepsilon \in (0, \varepsilon_0)$,

$$as_\lambda(\psi) > (1-\varepsilon)^\lambda \, (2\pi)^{n\lambda} \left(\int_{X_\psi} e^{-\psi(x)} dx \right)^{1-2\lambda}.$$

(ii) Let $\lambda < 0$ and suppose that for some $\varepsilon \in (0, \varepsilon_0)$,

$$as_\lambda(\psi) < (1-\varepsilon)^\lambda \, (2\pi)^{n\lambda} \left(\int_{X_\psi} e^{-\psi(x)} dx \right)^{1-2\lambda}.$$

Then, in both cases (i) and (ii), there exists $c > 0$ *and a positive definite matrix* A *such that*

$$\int_{R(\varepsilon)B_2^n} \left| \frac{\|x\|_2^2}{2} + c - \psi(Ax) \right| dx < \eta \varepsilon^{\frac{1}{129n^2}},$$

where $\lim_{\varepsilon\to 0} R(\varepsilon) = \infty$ *and* $\varepsilon_0, \eta, R(\varepsilon)$ *depend on n.*

Proof (i) The case $\lambda = 1$ is the stability case for the functional Blaschke Santaló inequality of Theorem 11. Therefore we can assume that $0 < \lambda < 1$. We put $d\mu = e^{-\psi} dx$. By Hölder's inequality with $p = 1/\lambda$ and $q = 1/(1-\lambda)$,

$$\begin{aligned} as_\lambda(\psi) &= \int_{X_\psi} e^{\lambda(2\psi(x)-\langle\nabla\psi,x\rangle)} \left(\det\left(\nabla^2\psi(x)\right)\right)^\lambda d\mu \\ &\le \left(\int_{X_\psi} e^{2\psi(x)-\langle\nabla\psi,x\rangle} \det\left(\nabla^2\psi(x)\right) d\mu\right)^\lambda \left(\int_{X_\psi} d\mu\right)^{1-\lambda} \\ &= \left(\int_{D_\psi} e^{\psi(x)-\langle\nabla\psi,x\rangle} \det\left(\nabla^2\psi(x)\right) dx\right)^\lambda \left(\int_{X_\psi} e^{-\psi(x)} dx\right)^{1-\lambda} \\ &\le \left(\int_{\mathbb{R}^n} e^{-\psi^*(x)} dx\right)^\lambda \left(\int_{X_\psi} e^{-\psi(x)} dx\right)^{1-\lambda}, \end{aligned}$$

where, in the last equality, we have used (21) and (24). Therefore, by the assumption (i) of the proposition

$$\left(\int_{\mathbb{R}^n} e^{-\psi^*(x)} dx\right)^\lambda \left(\int_{X_\psi} e^{-\psi(x)} dx\right)^{1-\lambda} > (1-\varepsilon)^\lambda (2\pi)^{n\lambda} \left(\int_{X_\psi} e^{-\psi(x)} dx\right)^{1-2\lambda},$$

which is equivalent to

$$\begin{aligned} \left(\int_{\mathbb{R}^n} e^{-\psi^*(x)} dx\right) \left(\int_{\mathbb{R}^n} e^{-\psi(x)} dx\right) &> \left(\int_{\mathbb{R}^n} e^{-\psi^*(x)} dx\right) \left(\int_{X_\psi} e^{-\psi(x)} dx\right) \\ &> (1-\varepsilon)(2\pi)^n, \end{aligned}$$

and the result is again a consequence of Theorem 11 by Barthe, Böröczky, and Fradelizi [9].

Similarly, in the case (ii) the proposition follows by applying the reverse Hölder inequality. □

The following Blaschke Santaló type inequality follows directly from inequality (28). It was also proved, together with its equality characterization in [20].

Corollary 18 ([20]) *Let* $\lambda \in [0, \frac{1}{2}]$ *and let* $\psi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *be a convex function such that* $\int xe^{-\psi(x)}dx = 0$ *or* $\int xe^{-\psi^*(x)}dx = 0$. *Then*

$$as_\lambda(\psi)\; as_\lambda((\psi^*)) \le (2\pi)^n.$$

Equality holds if and only if there exist $a \in \mathbb{R}$ *and a positive definite matrix A such that* $\psi(x) = \frac{1}{2}\langle Ax, x\rangle + a$, *for every* $x \in \mathbb{R}^n$.

We have the following stability result as a direct consequence of Theorem 11.

Proposition 19 *Let* $\psi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *be a convex function such that* $\int xe^{-\psi(x)}dx = 0$ *or* $\int xe^{-\psi^*(x)}dx = 0$. *Let* $0 \le \lambda \le \frac{1}{2}$ *and suppose that for some* $\varepsilon \in (0, \varepsilon_0)$,

$$as_\lambda(\psi)\; as_\lambda((\psi^*)) \ge (1-\epsilon)(2\pi)^n.$$

Then, there exist $c > 0$ *and a positive definite matrix A such that*

$$\int_{R(\varepsilon)B_2^n} \left| \frac{\|x\|_2^2}{2} + c - \psi(Ax) \right| dx < \eta \varepsilon^{\frac{1}{129n^2}},$$

where $\lim_{\varepsilon\to 0} R(\varepsilon) = \infty$ *and* $\varepsilon_0, \eta, R(\varepsilon)$ *depend on n.*

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