Chapter 6
Free Group Factors and Freeness

The concept of freeness was actually introduced by Voiculescu in the context of operator algebras, more precisely, during his quest to understand the structure of special von Neumann algebras, related to free groups. We wish to recall here the relevant context and show how freeness shows up there very naturally and how it can provide some information about the structure of those von Neumann algebras.

Operator algebras are $*$- algebras of bounded operators on a Hilbert space which are closed in some canonical topologies. (C^*-algebras are closed in the operator norm, and von Neumann algebras are closed in the weak operator topology; the first topology is the operator version of uniform convergence, the latter of pointwise convergence.) Since the group algebra of a group can be represented on itself by bounded operators given by left multiplication (this is the regular representation of a group), one can take the closure in the appropriate topology of the group algebra and get thus C^*-algebras and von Neumann algebras corresponding to the group. The *group von Neumann algebra* arising from a group G in this way is usually denoted by $\mathcal{L}(G)$. This construction, which goes back to the foundational papers of Murray and von Neumann in the 1930s, is, for G an infinite discrete group, a source of important examples in von Neumann algebra theory, and much of the progress in von Neumann algebra theory was driven by the desire to understand the relation between groups and their von Neumann algebras better. The group algebra consists of finite sums over group elements; going over to a closure means that we allow also some infinite sums. One should note that the weak closure, in the case of infinite groups, is usually much larger than the group algebra, and it is very hard to control which infinite sums are added. Von Neumann algebras are quite large objects and their classification is notoriously difficult.

J.A. Mingo, R. Speicher, *Free Probability and Random Matrices*,
Fields Institute Monographs 35, DOI 10.1007/978-1-4939-6942-5_6

6.1 Group (von Neumann) algebras

Let G be a discrete group. We want to consider compactly supported continuous functions $a : G \to \mathbb{C}$, equipped with convolution $(a,b) \mapsto a * b$. Note that compactly supported means just finitely supported in the discrete case, and thus the set of such functions can be identified with the *group algebra* $\mathbb{C}[G]$ of formal finite linear combinations of elements in G with complex coefficients, $a = \sum_{g\in G} a(g)g$, where only finitely many $a(g) \neq 0$. Integration over such functions is with respect to the counting measure; hence, the convolution is then written as

$$a*b = \sum_{g\in G}(a*b)(g)g = \sum_{g\in G}\left(\sum_{h\in G} a(h)b(h^{-1}g)\right)g = \sum_{h\in G} a(h)h \sum_{k\in G} b(k)k = ab,$$

and is hence nothing but the multiplication in $\mathbb{C}[G]$. Note that the function $\delta_e = 1\cdot e$ is the identity element in the group algebra $\mathbb{C}[G]$, where e is the identity element in G.

Now define an inner product on $\mathbb{C}[G]$ by setting

$$\langle g,h\rangle = \begin{cases} 1, & \text{if } g = h \\ 0, & \text{if } g \neq h \end{cases} \tag{6.1}$$

on G and extending sesquilinearly to $\mathbb{C}[G]$. From this inner product, we define the 2-norm on $\mathbb{C}[G]$ by $\|a\|_2^2 = \langle a,a\rangle$. In this way $(\mathbb{C}[G], \|\cdot\|_2)$ is a normed vector space. However, it is not complete in the case of infinite G (for finite G the following is trivial). The completion of $\mathbb{C}[G]$ with respect to $\|\cdot\|_2$ consists of all functions $a : G \to \mathbb{C}$ satisfying $\sum_{g\in G} |a(g)|^2 < \infty$ and is denoted by $\ell_2(G)$ and is a Hilbert space.

Now consider the unitary group representation $\lambda : G \to \mathcal{U}(\ell_2(G))$ defined by

$$\lambda(g)\cdot \sum_{h\in G} a(h)h := \sum_{h\in G} a(h)gh. \tag{6.2}$$

This is the *left regular representation* of G on the Hilbert space $\ell_2(G)$. It is obvious from the definition that each $\lambda(g)$ is an isometry of $\ell_2(G)$, but we want to check that it is in fact a unitary operator on $\ell_2(G)$. Since clearly $\langle gh,k\rangle = \langle h, g^{-1}k\rangle$, the adjoint of the operator $\lambda(g)$ is $\lambda(g^{-1})$. But then since λ is a group homomorphism, we have $\lambda(g)\lambda(g)^* = I = \lambda(g)^*\lambda(g)$, so that $\lambda(g)$ is indeed a unitary operator on $\ell_2(G)$.

Now extend the domain of λ from G to $\mathbb{C}[G]$ in the obvious way:

$$\lambda(a) = \lambda\left(\sum_{g\in G} a(g)g\right) = \sum_{g\in G} a(g)\lambda(g).$$

This makes λ into an algebra homomorphism $\lambda : \mathbb{C}[G] \to B(\ell_2(G))$, i.e. λ is a representation of the group algebra on $\ell_2(G)$. We define two new (closed) algebras via this representation. The *reduced group C^*-algebra* $C^*_{\text{red}}(G)$ of G is the closure of $\lambda(\mathbb{C}[G]) \subset B(\ell_2(G))$ in the operator norm topology. The *group von Neumann algebra of* G, denoted $\mathcal{L}(G)$, is the closure of $\lambda(\mathbb{C}[G])$ in the strong operator topology on $B(\ell_2(G))$.

One knows that for an infinite discrete group G, $\mathcal{L}(G)$ is a type II_1 von Neumann algebra, i.e. $\mathcal{L}(G)$ is infinite dimensional, but yet there is a trace τ on $\mathcal{L}(G)$ defined by $\tau(a) := \langle ae, e\rangle$ for $a \in \mathcal{L}(G)$, where $e \in G$ is the identity element. To see the trace property of τ, it suffices to check it for group elements; this extends then to the general situation by linearity and normality. However, for $g, h \in G$, the fact that $\tau(gh) = \tau(hg)$ is just the statement that $gh = e$ is equivalent to $hg = e$; this is clearly true in a group. The existence of a trace shows that $\mathcal{L}(G)$ is a proper subalgebra of $B(\ell_2(G))$; this is the case because there does not exist a trace on all bounded operators on an infinite dimensional Hilbert space. An easy fact is that if G is an ICC group, meaning that the conjugacy class of each $g \in G$ with $g \neq e$ has infinite cardinality, then $\mathcal{L}(G)$ is a factor, i.e. has trivial centre (see [106, Theorem 6.75]). Another fact is that if G is an amenable group (e.g. the infinite permutation group $S_\infty = \cup_n S_n$), then $\mathcal{L}(G)$ is the hyperfinite II_1 factor R.

Exercise 1.

(*i*) Show that $\mathcal{L}(G)$ is a factor if and only if G is an ICC group.
(*ii*) Show that the infinite permutation group $S_\infty = \cup_n S_n$ is ICC. (Note that each element from S_∞ moves only a finite number of elements.)

6.2 Free group factors

Now consider the case where $G = \mathbb{F}_n$, the *free group on n generators*; n can here be a natural number $n \geq 1$ or $n = \infty$. Let us briefly recall the definition of $\mathbb{F}_n$ and some of its properties. Consider the set of all words, of arbitrary length, over the $2n + 1$-letter alphabet $\{a_1, a_2, \ldots, a_n, a_1^{-1}, a_2^{-1}, \ldots, a_n^{-1}\} \cup \{e\}$, where the letters of the alphabet satisfy no relations other than $ea_i = a_i e = a_i$, $ea_i^{-1} = a_i^{-1}e = a_i^{-1}$, $a_i^{-1}a_i = a_i a_i^{-1} = e$. We say that a word is *reduced* if its length cannot be reduced by applying one of the above relations. Then the set of all reduced words in this alphabet together with the binary operation of concatenating words and reducing constitutes the free group $\mathbb{F}_n$ on n generators. $\mathbb{F}_n$ is the group generated by n symbols satisfying no relations other than those required by the group axioms. Clearly $\mathbb{F}_1$ is isomorphic to the abelian group $\mathbb{Z}$, while $\mathbb{F}_n$ is non-abelian for $n > 1$ and in fact has trivial centre. The integer n is called the *rank* of the free group; it is fairly easy, though not totally trivial, to see (e.g. by reducing it via abelianization to a corresponding question about abelian free groups) that $\mathbb{F}_n$ and $\mathbb{F}_m$ are isomorphic if and only if $m = n$.

Exercise 2. Show that $\mathbb{F}_n$ is, for $n \geq 2$, an ICC group.

Since $\mathbb{F}_n$ has the infinite conjugacy class property, one knows that the group von Neumann algebra $\mathcal{L}(\mathbb{F}_n)$ is a II_1 factor, called a *free group factor*. Murray and von Neumann showed that $\mathcal{L}(\mathbb{F}_n)$ is not isomorphic to the hyperfinite factor, but otherwise nothing was known about the structure of these free group factors, when free probability was invented by Voiculescu to understand them better.

While as pointed out above we have that $\mathbb{F}_n \simeq \mathbb{F}_m$ if and only if $m = n$, the corresponding problem for the free group factors is still unknown; see however some results in this direction in section 6.12.

Free group factor isomorphism problem: Let $m, n \geq 2$ (possibly equal to ∞), $n \neq m$. Are the von Neumann algebras $\mathcal{L}(\mathbb{F}_n)$ and $\mathcal{L}(\mathbb{F}_m)$ isomorphic?

The corresponding problem for the reduced group C^*-algebras was solved by Pimsner and Voiculescu [143] in 1982: they showed that $C^*_{\mathrm{red}}(\mathbb{F}_n) \not\simeq C^*_{\mathrm{red}}(\mathbb{F}_m)$ for $m \neq n$.

6.3 Free product of groups

There is the notion of free product of groups. If G, H are groups, then their free product $G * H$ is defined to be the group whose generating set is the disjoint union of G and H and which has the property that the only relations in $G * H$ are those inherited from G and H and the identification of the neutral elements of G and H. That is, there should be no non-trivial algebraic relations between elements of G and elements of H in $G * H$. In a more abstract language, the free product is the coproduct in the category of groups. For example, in the category of groups, the n-fold direct product of n copies of $\mathbb{Z}$ is the lattice $\mathbb{Z}^n$; the n-fold coproduct (free product) of n copies of $\mathbb{Z}$ is the free group $\mathbb{F}_n$ on n generators.

In the category of groups, we can understand $\mathbb{F}_n$ via the decomposition $\mathbb{F}_n = \mathbb{Z} * \mathbb{Z} * \cdots * \mathbb{Z}$. Is there a similar *free product of von Neumann algebras* that will help us to understand the structure of $\mathcal{L}(\mathbb{F}_n)$? The notion of *freeness* or *free independence* makes this precise. In order to understand what it means for elements in $\mathcal{L}(G)$ to be *free*, we need to deal with infinite sums, so the algebraic notion of freeness will not do: we need a state.

6.4 Moments and isomorphism of von Neumann algebras

We will try to understand a von Neumann algebra with respect to a state. Let M be a von Neumann algebra and let $\varphi : M \to \mathbb{C}$ be a state defined on M, i.e. a positive linear functional. Select finitely many elements $a_1, \ldots, a_k \in M$. Let us first recall the notion of ($*$-)moments and ($*$-)distribution in such a context.

Definition 1. 1) The collection of numbers gotten by applying the state to words in the alphabet $\{a_1, \ldots, a_k\}$ is called the collection of *joint moments* of $a_1, \ldots, a_k$, or the *distribution* of $a_1, \ldots, a_k$.

2) The collection of numbers gotten by applying the state to words in the alphabet $\{a_1, \ldots, a_k, a_1^*, \ldots, a_k^*\}$ is called the collection of *joint $*$-moments* of $a_1, \ldots, a_k$, or the *$*$-distribution* of $a_1, \ldots, a_k$.

Theorem 2. *Let $M = \mathrm{vN}(a_1, \ldots, a_k)$ be generated as von Neumann algebra by elements $a_1, \ldots, a_k$ and let $N = \mathrm{vN}(b_1, \ldots, b_k)$ be generated as von Neumann algebra by elements $b_1, \ldots, b_k$. Let $\varphi : M \to \mathbb{C}$ and $\psi : N \to \mathbb{C}$ be faithful normal states. If $a_1, \ldots, a_k$ and $b_1, \ldots, b_k$ have the same $*$-distributions with respect to φ and ψ, respectively, then the map $a_i \mapsto b_i$ extends to a $*$-isomorphism of M and N.*

Exercise 3. Prove Theorem 2 by observing that the assumptions imply that the GNS-constructions with respect to φ and ψ are isomorphic.

Though the theorem is not hard to prove, it conveys the important message that all information about a von Neumann algebra is, in principle, contained in the *-moments of a generating set with respect to a faithful normal state.

In the case of the group von Neumann algebras $\mathcal{L}(G)$, the canonical state is the trace τ. This is defined as a vector state, so it is automatically normal. It is worth to notice that it is also faithful (and hence $(\mathcal{L}(G), \tau)$ is a tracial W^*-probability space).

Proposition 3. *The trace τ on $\mathcal{L}(G)$ is a faithful state.*

Proof: Suppose that $a \in \mathcal{L}(G)$ satisfies $0 = \tau(a^*a) = \langle a^*ae, e\rangle = \langle ae, ae\rangle$, thus $ae = 0$. So we have to show that $ae = 0$ implies $a = 0$. To show that $a = 0$, it suffices to show that $\langle a\xi, \eta\rangle = 0$ for any $\xi, \eta \in \ell_2(G)$. It suffices to consider vectors of the form $\xi = g, \eta = h$ for $g, h \in G$, since we can get the general case from this by linearity and continuity. Now, by using the traciality of τ, we have

$$\langle ag, h\rangle = \langle age, he\rangle = \langle h^{-1}age, e\rangle = \tau(h^{-1}ag) = \tau(gh^{-1}a) = \langle gh^{-1}ae, e\rangle = 0,$$

since the first argument to the last inner product is 0. □

6.5 Freeness in the free group factors

We now want to see that the algebraic notion of freeness of subgroups in a free product of groups translates with respect to the canonical trace τ to our notion of free independence.

Let us say that a product in an algebra $\mathcal{A}$ is *alternating* with respect to subalgebras $\mathcal{A}_1, \ldots, \mathcal{A}_s$ if adjacent factors come from different subalgebras. Recall that our definition of free independence says: the subalgebras $\mathcal{A}_1, \ldots, \mathcal{A}_s$ are free if any product in centred elements over these algebras which alternates is centred.

Proposition 4. *Let G be a group containing subgroups $G_1, \ldots, G_s$ such that $G = G_1 * \cdots * G_s$. Let τ be the state $\tau(a) = \langle ae, e\rangle$ on $\mathbb{C}[G]$. Then the subalgebras $\mathbb{C}[G_1], \ldots, \mathbb{C}[G_s] \subset \mathbb{C}[G]$ are free with respect to τ.*

Proof: Let $a_1a_2\cdots a_k$ be an element in $\mathbb{C}[G]$ which alternates with respect to the subalgebras $\mathbb{C}[G_1],\ldots,\mathbb{C}[G_s]$, and assume the factors of the product are centred with respect to τ. Since τ is the "coefficient of the identity" state, this means that if $a_j\in\mathbb{C}[G_{i_j}]$, then a_j looks like $a_j=\sum_{g\in G_{i_j}}a_j(g)g$ and $a_j(e)=0$. Thus we have

$$\tau(a_1a_2\cdots a_k)=\sum_{g_1\in G_{i_1},\ldots,g_k\in G_{i_k}}a_1(g_1)a_2(g_2)\cdots a_k(g_k)\tau(g_1g_2\cdots g_k).$$

Now, $\tau(g_1g_2\cdots g_k)\neq 0$ only if $g_1g_2\cdots g_k=e$. But $g_1g_2\cdots g_k$ is an alternating product in G with respect to the subgroups $G_1,G_2,\ldots,G_s$, and since $G=G_1*G_2*\cdots*G_s$, this can happen only when at least one of the factors, let's say g_j, is equal to e; but in this case $a_j(g_j)=a_j(e)=0$. So each summand in the sum for $\tau(a_1a_2\cdots a_k)$ vanishes, and we have $\tau(a_1a_2\cdots a_k)=0$, as required. □

Thus freeness of the subgroup algebras $\mathbb{C}[G_1],\ldots,\mathbb{C}[G_s]$ with respect to τ is just a simple reformulation of the fact that $G_1,\ldots,G_s$ are free subgroups of G. However, a non-trivial fact is that this reformulation carries over to closures of the subalgebras.

Proposition 5. (1) *Let A be a C^*-algebra, $\varphi:A\to\mathbb{C}$ a state. Let $B_1,\ldots,B_s\subset A$ be unital $*$-subalgebras which are free with respect to φ. Put $A_i:=\overline{B_i}^{\|\cdot\|}$, the norm closure of B_i. Then $A_1,\ldots,A_s$ are also free.*

(2) *Let M be a von Neumann algebra, $\varphi:M\to\mathbb{C}$ a normal state. Let $B_1,\ldots,B_s$ be unital $*$-subalgebras which are free. Put $M_i:=\mathrm{vN}(B_i)$. Then $M_1,\ldots,M_s$ are also free.*

Proof: (1) Consider $a_1,\ldots,a_k$ with $a_i\in A_{j_i}$, $\varphi(a_i)=0$, and $j_i\neq j_{i+1}$ for all i. We have to show that $\varphi(a_1\cdots a_k)=0$. Since B_i is dense in A_i, we can, for each i, approximate a_i in operator norm by a sequence $(b_i^{(n)})_{n\in\mathbb{N}}$, with $b_i^{(n)}\in B_i$, for all n. Since we can replace $b_i^{(n)}$ by $b_i^{(n)}-\varphi(b_i^{(n)})$ (note that $\varphi(b_i^{(n)})$ converges to $\varphi(a_i)=0$), we can assume, without restriction, that $\varphi(b_i^{(n)})=0$. But then we have

$$\varphi(a_1\cdots a_k)=\lim_{n\to\infty}\varphi(b_1^{(n)}\cdots b_k^{(n)})=0,$$

since, by the freeness of $B_1,\ldots,B_s$, we have $\varphi(b_1^{(n)}\cdots b_k^{(n)})=0$ for each n.

(2) Consider $a_1,\ldots,a_k$ with $a_i\in M_{j_i}$, $\varphi(a_i)=0$, and $j_i\neq j_{i+1}$ for all i. We have to show that $\varphi(a_1\cdots a_k)=0$. We approximate essentially as in the C^*-algebra case; we only have to take care that the multiplication of our k factors is still continuous in the appropriate topology. More precisely, we can now approximate, for each i, the operator a_i in the strong operator topology by a sequence (or a net, if you must) $b_i^{(n)}$. By invoking Kaplansky's density theorem, we can choose those such that we keep everything bounded, namely, $\|b_i^{(n)}\|\leq\|a_i\|$ for all n. Again we can centre the sequence, so that we can

assume that all $\varphi(b_i^{(n)}) = 0$. Since the joint multiplication is on bounded sets continuous in the strong operator topology, we have then still the convergence of $b_1^{(n)} \cdots b_k^{(n)}$ to $a_1 \cdots a_k$ and, thus, since φ is normal, also the convergence of $0 = \varphi(b_1^{(n)} \cdots b_k^{(n)})$ to $\varphi(a_1 \cdots a_k)$.

□

6.6 The structure of free group factors

What does this tell us for the free group factors? It is clear that each generator of the free group gives a Haar unitary element in $(\mathcal{L}(\mathbb{F}_n), \tau)$. By the discussion above, those elements are $*$-free. Thus the free group factor $\mathcal{L}(\mathbb{F}_n)$ is generated by n $*$-free Haar unitaries $u_1, \ldots, u_n$. Note that, by Theorem 2, we will get the free group factor $\mathcal{L}(\mathbb{F}_n)$ whenever we find somewhere n Haar unitaries which are $*$-free with respect to a faithful normal state. Furthermore, since we are working inside von Neumann algebras, we have at our disposal measurable functional calculus, which means that we can also deform the Haar unitaries into other, possibly more suitable, generators.

Theorem 6. *Let M be a von Neumann algebra and τ a faithful normal state on M. Assume that $x_1, \ldots, x_n \in M$ generate M, $\mathrm{vN}(x_1, \ldots, x_n) = M$ and that*

- *$x_1, \ldots, x_n$ are $*$-free with respect to τ,*
- *each x_i is normal, and its spectral measure with respect to τ is diffuse (i.e. has no atoms).*

Then $M \simeq \mathcal{L}(\mathbb{F}_n)$.

Proof: Let x be a normal element in M which is such that its spectral measure with respect to τ is diffuse. Let $A = \mathrm{vN}(x)$ be the von Neumann algebra generated by x. We want to show that there is a Haar unitary $u \in A$ that generates A as a von Neumann algebra. A is a commutative von Neumann algebra and the restriction of τ to A is a faithful state. A cannot have any minimal projections as that would mean that the spectral measure of x with respect to τ was not diffuse. Thus there is a normal $*$-isomorphism $\pi : A \to L^\infty[0, 1]$ where we put Lebesgue measure on $[0, 1]$. This follows from the well-known fact that any commutative von Neumann algebra is $*$-isomorphic to $L^\infty(\mu)$ for some measure μ and that all spaces $L^\infty(\mu)$ for μ without atoms are $*$-isomorphic; see, for example, [170, Chapter III, Theorem 1.22].

Under π the trace τ becomes a normal state on $L^\infty[0, 1]$. Thus there is a positive function $h \in L^1[0, 1]$ such that for all $a \in A$, $\tau(a) = \int_0^1 \pi(a)(t)h(t)\,dt$. Since τ is faithful, the set $\{t \in [0, 1] \mid h(t) = 0\}$ has Lebesgue measure 0. Thus $H(s) = \int_0^s h(t)\,dt$ is a continuous positive strictly increasing function on $[0, 1]$ with range $[0, 1]$. So by the Stone-Weierstrass theorem, the C^*-algebra generated by 1 and H is all of $C[0, 1]$. Hence the von Neumann algebra generated by 1 and H is all of $L^\infty[0, 1]$. Let $v(t) = \exp(2\pi i H(t))$. Then H is in the von Neumann algebra generated by v, so the von Neumann algebra generated by v is $L^\infty[0, 1]$. Also,

$$\int_0^1 v(t)^n h(t)\,dt = \int_0^1 \exp(2\pi i n H(t)) H'(t)\,dt = \int_0^1 e^{2\pi i n s}\,ds = \delta_{0,n}.$$

Thus v is Haar unitary with respect to h. Finally let $u \in A$ be such that $\pi(u) = v$. Then the von Neumann algebra generated by u is A and u is a Haar unitary with respect to the trace τ.

This means that for each i we can find in $\mathrm{vN}(x_i)$ a Haar unitary u_i which generates the same von Neumann algebra as x_i. By Proposition 5, freeness of the x_i goes over to freeness of the u_i. So we have found n Haar unitaries in M which are $*$-free and which generate M. Thus M is isomorphic to the free group factor $\mathcal{L}(\mathbb{F}_n)$. □

Example 7. Instead of generating $\mathcal{L}(\mathbb{F}_n)$ by n $*$-free Haar unitaries, it is also very common to use n free semi-circular elements. (Note that for self-adjoint elements $*$-freeness is of course the same as freeness.) This is of course covered by the theorem above. But let us be a bit more explicit on deforming a semi-circular element into a Haar unitary. Let $s \in M$ be a semi-circular operator. The spectral measure of s is $\sqrt{4-t^2}/(2\pi)\,dt$, i.e.

$$\tau(f(s)) = \frac{1}{2\pi}\int_{-2}^{2} f(t)\sqrt{4-t^2}\,dt.$$

If

$$H(t) = \frac{t}{4\pi}\sqrt{4-t^2} + \frac{1}{\pi}\sin^{-1}(t/2) \qquad \text{then} \qquad H'(t) = \frac{1}{2\pi}\sqrt{4-t^2},$$

and $u = \exp(2\pi i H(s))$ is a Haar unitary, i.e.

$$\tau(u^k) = \int_{-2}^{2} e^{2\pi i k H(t)} H'(t)\,dt = \int_{-1/2}^{1/2} e^{2\pi i k r}\,dr = \delta_{0,k},$$

which generates the same von Neumann subalgebra as s.

6.7 Compression of free group factors

Let M be any II_1 factor with faithful normal trace τ and e a projection in M. Let $eMe = \{exe \mid x \in M\}$; eMe is again a von Neumann algebra, actually a II_1 factor, with e being its unit, and it is called the *compression* of M by e. It is an elementary fact in von Neumann algebra theory that the isomorphism class of eMe depends only on $t = \tau(e)$, and we denote this isomorphism class by M_t. A deeper fact of Murray and von Neumann is that $(M_s)_t = M_{st}$. We can define M_t for all $t > 0$ as follows. For a positive integer n, let $M_n = M \otimes M_n(\mathbb{C})$, and for any t, let $M_t = eM_ne$ for any sufficiently large n and any projection e in M_n with trace t, where here we use the non-normalized trace $\tau \otimes \mathrm{Tr}$ on M_n. Murray and von

Neumann then defined the *fundamental group* of M, $\mathcal{F}(M)$, to be $\{t \in \mathbb{R}^+ \mid M \simeq M_t\}$ and showed that it is a multiplicative subgroup of $\mathbb{R}^+$. (See [106, Ex. 13.4.5 and 13.4.6].) It is a theorem that when G is an amenable ICC group, we have that $\mathcal{L}(G)$ is the hyperfinite II_1 factor and $\mathcal{F}(\mathcal{L}(G)) = \mathbb{R}^+$; see [170].

Rădulescu showed that $\mathcal{F}(\mathcal{L}(\mathbb{F}_\infty)) = \mathbb{R}^+$; see [144]. For finite n, $\mathcal{F}(\mathcal{L}(\mathbb{F}_n))$ is unknown; but it is known to be either $\mathbb{R}^+$ or $\{1\}$. In the rest of this chapter, we will give the key ideas about those compression results for free group factors.

The first crucial step was taken by Voiculescu who showed in 1990 in [179] that for integer m, n, k, we have $\mathcal{L}(\mathbb{F}_n)_{1/k} \simeq \mathcal{L}(\mathbb{F}_m)$, where $(m-1)/(n-1) = k^2$, or equivalently

$$\mathcal{L}(\mathbb{F}_n) \simeq M_k(\mathbb{C}) \otimes \mathcal{L}(\mathbb{F}_m), \qquad \text{where} \qquad \frac{m-1}{n-1} = k^2. \tag{6.3}$$

So if we embed $\mathcal{L}(\mathbb{F}_m)$ into $M_k(\mathbb{C}) \otimes \mathcal{L}(\mathbb{F}_m) \simeq \mathcal{L}(\mathbb{F}_n)$ as $x \mapsto 1 \otimes x$, then $\mathcal{L}(\mathbb{F}_m)$ is a subfactor of $\mathcal{L}(\mathbb{F}_n)$ of Jones index k^2; see [105, Example 2.3.1]. Thus, $(m-1)/(n-1) = [\mathcal{L}(\mathbb{F}_n) : \mathcal{L}(\mathbb{F}_m)]$. Notice the similarity to Schreier's index formula for free groups. Indeed, suppose G is a free group of rank n and H is a subgroup of G of finite index. Then H is necessarily a free group, say of rank m, and Schreier's index formula says that $(m-1)/(n-1) = [G : H]$.

Rather than proving Voiculescu's theorem, Equation (6.3), in full generality, we shall first prove a special case which illustrates the main ideas of the proof and then sketch the general case.

Theorem 8. *We have* $\mathcal{L}(\mathbb{F}_3)_{1/2} \simeq \mathcal{L}(\mathbb{F}_9)$.

To prove this theorem, we must find in $\mathcal{L}(\mathbb{F}_3)_{1/2}$ nine free normal elements with diffuse spectral measure which generate $\mathcal{L}(\mathbb{F}_3)_{1/2}$. In order to achieve this, we will start with normal elements x_1, x_2, x_3, together with a faithful normal state φ, such that

- the spectral measure of each x_i is diffuse (i.e. no atoms) and
- x_1, x_2, x_3 are $*$-free.

Let N be the von Neumann algebra generated by x_1, x_2, and x_3. Then $N \simeq \mathcal{L}(\mathbb{F}_3)$. We will then show that there is a projection p in N such that

- $\varphi(p) = 1/2$
- there are 9 free and diffuse elements in pNp which generate pNp.

Thus $\mathcal{L}(\mathbb{F}_3)_{1/2} \simeq pNp \simeq \mathcal{L}(\mathbb{F}_9)$.

The crucial issue above is that we will be able to choose our elements x_1, x_2, x_3 in such a form that we can easily recognize p and the generating elements of pNp. (Just starting abstractly with three $*$-free normal diffuse elements will not be very helpful, as we have then no idea how to get p and the required nine free elements.)

Actually, since our claim is equivalent to $\mathcal{L}(\mathbb{F}_3) \simeq M_2(\mathbb{C}) \otimes \mathcal{L}(\mathbb{F}_9)$, it will surely be a good idea to try to realize x_1, x_2, x_3 as 2×2 matrices. This will be achieved in the next section with the help of circular operators.

6.8 Circular operators and complex Gaussian random matrices

To construct the elements x_1, x_2, x_3 as required above, we need to make a digression into circular operators. Let X be an $2N \times 2N$ GUE random matrix. Let

$$P = \begin{pmatrix} I_n & 0_n \\ 0_n & 0_n \end{pmatrix} \qquad \text{and} \qquad G = \sqrt{2}\, PX(1-P).$$

Then G is a $N \times N$ matrix with independent identically distributed entries which are centred complex Gaussian random variables with complex variance $1/N$; such a matrix we call a *complex Gaussian random matrix*. We can determine the limiting $*$-moments of G as follows.

Write $Y_1 = (G+G^*)/\sqrt{2}$ and $Y_2 = -i(G-G^*)/\sqrt{2}$ then $G = (Y_1+iY_2)/\sqrt{2}$ and Y_1 and Y_2 are independent $N \times N$ GUE random matrices. Therefore by the asymptotic freeness of independent GUE (see section 1.11), Y_1 and Y_2 converge as $N \to \infty$ to free standard semi-circulars s_1 and s_2.

Definition 9. Let s_1 and s_2 be free and standard semi-circular. Then we call $c = (s_1 + is_2)/\sqrt{2}$ a *circular operator.*

Since s_1 and s_2 are free, we can easily calculate the free cumulants of c. If $\varepsilon = \pm 1$ let us adopt the following notation for $x^{(\varepsilon)}$: $x^{(-1)} = x^*$ and $x^{(1)} = x$. Recall that for a standard semi-circular operator s

$$\kappa_n(s, \dots, s) = \begin{cases} 1, & n = 2 \\ 0, & n \neq 2 \end{cases}.$$

Thus

$$\begin{aligned} \kappa_n(c^{(\varepsilon_1)}, \dots, c^{(\varepsilon_n)}) &= 2^{-n/2}\kappa_n(s_1 + \varepsilon_1 i s_2, \dots, s_1 + i\varepsilon_n s_2) \\ &= 2^{-n/2}\big(\kappa_n(s_1, \dots, s_1) + i^n \varepsilon_1 \cdots \varepsilon_n \kappa_n(s_2, \dots, s_2)\big) \end{aligned}$$

since all mixed cumulants in s_1 and s_2 are 0. Thus $\kappa_n(c^{(\varepsilon_1)}, \dots, c^{(\varepsilon_n)}) = 0$ for $n \neq 2$, and

$$\kappa_2(c^{(\varepsilon_1)}, c^{(\varepsilon_2)}) = 2^{-1}\big(\kappa_2(s_1, s_1) - \varepsilon_1\varepsilon_2\kappa_2(s_2, s_2)\big) = \frac{1 - \varepsilon_1\varepsilon_2}{2} = \begin{cases} 1 & \varepsilon_1 \neq \varepsilon_2 \\ 0 & \varepsilon_1 = \varepsilon_2 \end{cases}.$$

Hence, $\kappa_2(c, c^*) = \kappa_2(c^*, c) = 1$, $\kappa_2(c, c) = \kappa_2(c^*, c^*) = 0$, and all other $*$-cumulants are 0. Thus

$$\tau((c^*c)^n) = \sum_{\pi \in NC(2n)} \kappa_\pi(c^*, c, c^*, c, \ldots, c^*, c) = \sum_{\pi \in NC_2(2n)} \kappa_\pi(c^*, c, c^*, c, \ldots, c^*, c).$$

Now note that any $\pi \in NC_2(2n)$ connects, by parity reasons, automatically only c with c^*, hence $\kappa_\pi(c^*, c, c^*, c, \ldots, c^*, c) = 1$ for all $\pi \in NC_2(2n)$, and we have

$$\tau((c^*c)^n) = |NC_2(2n)| = \tau(s^{2n}),$$

where s is a standard semi-circular element. Since $t \mapsto \sqrt{t}$ is a uniform limit of polynomials in t, we have that the moments of $|c| = \sqrt{c^*c}$ and $|s| = \sqrt{s^2}$ are the same and $|c|$ and $|s|$ have the same distribution. The operator $|c| = |s|$ is called a *quarter-circular operator* and has moments

$$\tau(|c|^k) = \frac{1}{\pi} \int_0^2 t^k \sqrt{4 - t^2}\, dt.$$

An additional result which we will need is Voiculescu's theorem on the polar decomposition of a circular operator.

Theorem 10. *Let (M, τ) be a W^*-probability space and $c \in M$ a circular operator. If $c = u\,|c|$ is its polar decomposition in M, then*

(*i*) *u and $|c|$ are $*$-free,*
(*ii*) *u is a Haar unitary,*
(*iii*) *$|c|$ is a quarter circular operator.*

The proof of (*i*) and (*ii*) can either be done using random matrix methods (as was done by Voiculescu [180]) or by showing that if u is a Haar unitary and q is a quarter-circular operator such that u and q are $*$-free, then uq has the same $*$-moments as a circular operator (this was done by Nica and Speicher [137]). The latter can be achieved, for example, by using the formula for cumulants of products, equation (2.23). For the details of this approach, see [137, Theorem 15.14].

Theorem 11. *Let $(\mathcal{A}, \varphi)$ be a unital $*$-algebra with a state φ. Suppose $s_1, s_2, c \in \mathcal{A}$ are $*$-free and s_1 and s_2 semi-circular and c circular. Then*

$$x = \frac{1}{\sqrt{2}} \begin{pmatrix} s_1 & c \\ c^* & s_2 \end{pmatrix} \in (M_2(\mathcal{A}), \varphi_2)$$

is semi-circular.

Here we have used the standard notation $M_2(\mathcal{A}) = M_2(\mathbb{C}) \otimes \mathcal{A}$ for 2×2 matrices with entries from $\mathcal{A}$ and $\varphi_2 = \mathrm{tr} \otimes \varphi$ for the composition of the normalized trace with φ.

Proof: Let $\mathbb{C}\langle x_{11}, x_{12}, x_{21}, x_{22}\rangle$ be the polynomials in the non-commuting variables $x_{11}, x_{12}, x_{21}, x_{22}$. Let

$$p_k(x_{11}, x_{12}, x_{21}, x_{22}) = \frac{1}{2}\mathrm{Tr}\left(\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}^k\right).$$

Now let $\mathcal{A}_N = M_N(L^{\infty-}(\Omega))$ be the $N \times N$ matrices with entries in $L^{\infty-}(\Omega) := \bigcap_{p\geq 1} L^p(\Omega)$, for some classical probability space Ω. On $\mathcal{A}_N$ we have the state $\varphi_N(X) = \mathrm{E}(N^{-1}\mathrm{Tr}(X))$. Now suppose in $\mathcal{A}_N$ we have S_1, S_2, and C, with S_1 and S_2 GUE random matrices and C a complex Gaussian random matrix and with the entries of S_1, S_2, C independent. Then we know that S_1, S_2, C converge in $*$-distribution to s_1, s_2, c, i.e. for any polynomial p in four non-commuting variables, we have $\varphi_N(p(S_1, C, C^*, S_2)) \to \varphi(p(s_1, c, c^*, s_2))$. Now let

$$X = \frac{1}{\sqrt{2}}\begin{pmatrix} S_1 & C \\ C^* & S_2 \end{pmatrix}.$$

Then X is in $\mathcal{A}_{2N}$, and

$$\begin{aligned}\varphi_{2N}(X^k) = \varphi_N\big(p_k(S_1, C, C^*, S_2)\big) \to \varphi\big(p_k(s_1, c, c^*, s_2)\big) &= \varphi(\frac{1}{2}\mathrm{Tr}(x^k)) \\ &= \mathrm{tr}\otimes\varphi(x^k).\end{aligned}$$

On the other hand, X is a $2N \times 2N$ GUE random matrix; so $\varphi_{2N}(X^k)$ converges to the k^{th} moment of a semi-circular operator. Hence x in $M_2(\mathcal{A})$ is semi-circular. □

Exercise 4. Suppose s_1, s_2, c, and x are as in Theorem 11. Show that x is semi-circular by computing $\varphi(\mathrm{tr}(x^n))$ directly using the methods of Lemma 1.9.

We can now present the realization of the three generators x_1, x_2, x_3 of $\mathcal{L}(\mathbb{F}_3)$ which we need for the proof of the compression result.

Lemma 12. *Let $\mathcal{A}$ be a unital $*$-algebra and φ a state on $\mathcal{A}$. Suppose s_1, s_2, s_3, s_4, c_1, c_2, u in $\mathcal{A}$ are $*$-free, with s_1, s_2, s_3, and s_4 semi-circular, c_1 and c_2 circular, and u a Haar unitary. Let*

$$x_1 = \begin{pmatrix} s_1 & c_1 \\ c_1^* & s_2 \end{pmatrix}, \quad x_2 = \begin{pmatrix} s_3 & c_2 \\ c_2^* & s_4 \end{pmatrix}, \quad x_3 = \begin{pmatrix} u & 0 \\ 0 & 2u \end{pmatrix}.$$

Then x_1, x_2, x_3 are $*$-free in $M_2(\mathcal{A})$ with respect to the state $\mathrm{tr}\otimes\varphi$; x_1 and x_2 are semi-circular and x_3 is normal and diffuse.

Proof: We model x_1 by X_1, x_2 by X_2, and x_3 by X_3 where

$$X_1 = \begin{pmatrix} S_1 & C_1 \\ C_1^* & S_2 \end{pmatrix}, \quad X_2 = \begin{pmatrix} S_3 & C_2 \\ C_2^* & S_3 \end{pmatrix}, \quad X_3 = \begin{pmatrix} U & 0 \\ 0 & 2U \end{pmatrix}$$

and S_1, S_2, S_3, S_4 are $N \times N$ GUE random matrices, C_1 and C_2 are $N \times N$ complex Gaussian random matrices, and U is a diagonal deterministic unitary matrix, chosen so that the entries of X_1 are independent from those of X_2 and that the diagonal entries of U converge in distribution to the uniform distribution on the unit circle. Then X_1, X_2, X_3 are asymptotically $*$-free by Theorem 4.4. Thus x_1, x_2, and x_3 are $*$-free because they have the same distribution as the limiting distribution of X_1, X_2, and X_3. By the previous Theorem 11, x_1 and x_2 are semi-circular. x_3 is clearly normal, and its spectral distribution is given by the uniform distribution on the union of the circle of radius 1 and the circle of radius 2. □

6.9 Proof of $\mathcal{L}(\mathbb{F}_3)_{1/2} \simeq \mathcal{L}(\mathbb{F}_9)$

We will now present the proof of Theorem 8.

Proof: We have shown that if we take four semi-circular operators s_1 s_2, s_3, s_4, two circular operators c_1, c_2, and a Haar unitary u in a von Neumann algebra M with trace τ such that $s_1, s_2, s_3, s_4, c_1, c_2, u$ are $*$-free, then

- the elements

$$x_1 = \begin{pmatrix} s_1 & c_1 \\ c_1^* & s_2 \end{pmatrix}, \qquad x_2 = \begin{pmatrix} s_3 & c_2 \\ c_2^* & s_4 \end{pmatrix}, \qquad x_3 = \begin{pmatrix} u & 0 \\ 0 & 2u \end{pmatrix}$$

  are $*$-free in $(M_2(M), \mathrm{tr} \otimes \tau)$,
- x_1 and x_2 are semi-circular, and x_3 is normal and has diffuse spectral measure.

Let $N = \mathrm{vN}(x_1, x_2, x_3) \subseteq M_2(M)$. Then, by Theorem 6, $N \simeq \mathcal{L}(\mathbb{F}_3)$. Since

$$\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} = x_3^* x_3 \in N, \quad \text{we also have the spectral projection} \quad p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in N,$$

and thus $px_1(1-p) \in N$ and $px_2(1-p) \in N$. We have the polar decompositions

$$\begin{pmatrix} 0 & c_1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & v_1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & |c_1| \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & c_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & v_2 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & |c_2| \end{pmatrix},$$

where $c_1 = v_1|c_1|$ and $c_2 = v_2|c_2|$ are the polar decompositions of c_1 and c_2, respectively, in M.

Hence we see that $N = \mathrm{vN}(x_1, x_2, x_3)$ is generated by the ten elements

$$y_1 = \begin{pmatrix} s_1 & 0 \\ 0 & 0 \end{pmatrix} \quad y_2 = \begin{pmatrix} 0 & 0 \\ 0 & s_2 \end{pmatrix} \quad y_3 = \begin{pmatrix} 0 & v_1 \\ 0 & 0 \end{pmatrix} \quad y_4 = \begin{pmatrix} 0 & 0 \\ 0 & |c_1| \end{pmatrix} \quad y_5 = \begin{pmatrix} s_3 & 0 \\ 0 & 0 \end{pmatrix}$$

$$y_6 = \begin{pmatrix} 0 & 0 \\ 0 & s_4 \end{pmatrix} \quad y_7 = \begin{pmatrix} 0 & v_2 \\ 0 & 0 \end{pmatrix} \quad y_8 = \begin{pmatrix} 0 & 0 \\ 0 & |c_2| \end{pmatrix} \quad y_9 = \begin{pmatrix} u & 0 \\ 0 & 0 \end{pmatrix} \quad y_{10} = \begin{pmatrix} 0 & 0 \\ 0 & u \end{pmatrix}.$$

Let us put

$$v := \begin{pmatrix} 0 & v_1 \\ 0 & 0 \end{pmatrix}; \quad \text{then} \quad v^*v = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad vv^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = p = p^2.$$

Since we can write now any $py_{i_1} \cdots y_{i_n} p$ in the form $py_{i_1} 1 y_{i_2} 1 \cdots 1 y_{i_n} p$ and replace each 1 by $p^2 + v^*v$, it is clear that $\bigcup_{i=1}^{10} \{py_i p, py_i v^*, vy_i p, vy_i v^*\}$ generate pNp. This gives for pNp the generators

$$s_1, \quad v_1 s_2 v_1^*, \quad v_1 v_1^*, \quad v_1 |c_1| v_1^*, \quad s_3, \quad v_1 s_4 v_1^*, \quad v_2 v_1^*, \quad v_1 |c_2| v_1^*, \quad u, \quad v_1 u v_1^*.$$

Note that $v_1 v_1^* = 1$ can be removed from the set of generators. To check that the remaining nine elements are $*$-free and diffuse, we recall a few elementary facts about freeness.

Exercise 5. Show the following:

(*i*) if $\mathcal{A}_1$ and $\mathcal{A}_2$ are free subalgebras of $\mathcal{A}$, if $\mathcal{A}_{11}$ and $\mathcal{A}_{12}$ are free subalgebras of $\mathcal{A}_1$, and if $\mathcal{A}_{21}$ and $\mathcal{A}_{22}$ are free subalgebras of $\mathcal{A}_2$; then $\mathcal{A}_{11}, \mathcal{A}_{12}, \mathcal{A}_{21}, \mathcal{A}_{22}$ are free;
(*ii*) if u is a Haar unitary $*$-free from $\mathcal{A}$, then $\mathcal{A}$ is $*$-free from $u\mathcal{A}u^*$;
(*iii*) if u_1 and u_2 are Haar unitaries and u_2 is $*$-free from $\{u_1\} \cup \mathcal{A}$ then $u_2 u_1^*$ is a Haar unitary and is $*$-free from $u_1 \mathcal{A} u_1^*$.

By construction $s_1, s_2, s_3, s_4, |c_1|, |c_2|, v_1, v_2, u$ are $*$-free. Thus, in particular, $s_2, s_4, |c_1|, |c_2|, v_2, u$ are $*$-free. Hence, by (*ii*), $v_1 s_2 v_1^*, v_1 s_4 v_1^*, v_1 |c_1| v_1^*, v_1 |c_2| v_1^*$, $v_1 u v_1^*$ are $*$-free and, in addition, $*$-free from u, s_1, s_3, v_2. Thus

$$u, \quad s_1, \quad s_3, \quad v_1 s_2 v_1^*, \quad v_1 s_4 v_1^*, \quad v_1 |c_1| v_1^*, \quad v_1 |c_2| v_1^*, \quad v_1 u v_1^*, \quad v_2$$

are $*$-free. Let $\mathcal{A} = \text{alg}(s_2, s_4, |c_1|, |c_2|, u)$. We have that v_2 is $*$-free from $\{v_1\} \cup \mathcal{A}$, so by (*iii*), $v_2 v_1^*$ is $*$-free from $v_1 \mathcal{A} v_1^*$. Thus, $v_2 v_1^*$ is $*$-free from

$$v_1 s_2 v_1^*, \quad v_1 s_4 v_1^*, \quad v_1 |c_1| v_1^*, \quad v_1 |c_2| v_1^*, \quad v_1 u v_1^*$$

and it was already $*$-free from s_1, s_3 and u. Thus by (*i*) our nine elements

$$s_1, \quad s_3, \quad v_1 s_2 v_1^*, \quad v_1 s_4 v_1^*, \quad v_1 |c_1| v_1^*, \quad v_1 |c_2| v_1^*, \quad u, \quad v_1 u v_1^*, \quad v_2 v_1^*$$

are $*$-free. Since they are either semi-circular, quarter-circular, or Haar elements, they are all normal and diffuse; as they generate pNp, we have that pNp is generated by nine $*$-free normal and diffuse elements and thus, by Theorem 6, $pNp \simeq \mathcal{L}(\mathbb{F}_9)$. Hence $\mathcal{L}(\mathbb{F}_3)_{1/2} \simeq \mathcal{L}(\mathbb{F}_9)$. □

6.10 The general case $\mathcal{L}(\mathbb{F}_{\mathbf{n}})_{\mathbf{1/k}} \simeq \mathcal{L}(\mathbb{F}_{\mathbf{1+(n-1)k^2}})$

Sketch We sketch now the proof for the general case of Equation (6.3). We write $\mathcal{L}(\mathbb{F}_n) = \mathrm{vN}(x_1, \ldots, x_n)$ where for $1 \le i \le n-1$ each x_i is a semi-circular element of the form

$$x_i = \frac{1}{\sqrt{k}} \begin{pmatrix} s_1^{(i)} & c_{12}^{(i)} & \ldots & c_{1k}^{(i)} \\ c_{12}^{(i)*} & s_2^{(i)} & \ldots & c_{2k}^{(i)} \\ \vdots & & \ddots & \vdots \\ c_{1k}^{(i)*} & \ldots & \ldots & s_k^{(i)} \end{pmatrix} \quad \text{and where} \quad x_n = \begin{pmatrix} u & 0 & \ldots & 0 \\ 0 & 2u & \ldots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \ldots & ku \end{pmatrix},$$

with all $s_j^{(i)}$ $(j = 1, \ldots, k; i = 1, \ldots, n-1)$ semi-circular, all $c_{pq}^{(i)}$ $(1 \le p < q \le k;$ $i = 1, \ldots, n-1)$ circular, and u a Haar unitary, so that all elements are $*$-free.

So we have $(n-1)k$ semi-circular operators, $(n-1)\binom{k}{2}$ circular operators, and one Haar unitary. Each circular operator produces two free elements, so we have in total

$$(n-1)k + 2(n-1)\binom{k}{2} + 1 = (n-1)k^2 + 1$$

free and diffuse generators. Thus $\mathcal{L}(\mathbb{F}_n)_{1/k} \simeq \mathcal{L}(\mathbb{F}_{1+(n-1)k^2})$. □

6.11 Interpolating free group factors

The formula $\mathcal{L}(\mathbb{F}_n)_{1/k} \simeq \mathcal{L}(\mathbb{F}_m)$, which up to now makes sense only for integer m, n, and k, suggests that one might try to define $\mathcal{L}(\mathbb{F}_r)$ also for noninteger r by compression. A crucial issue is that, by the above formula, different compressions should give the same result. That this really works and is consistent was shown, independently, by Dykema [67] and Rădulescu [145].

Theorem 13. *Let R be the hyperfinite II_1 factor and $\mathcal{L}(\mathbb{F}_\infty) = \mathrm{vN}(s_1, s_2, \ldots)$ be a free group factor generated by countably many free semi-circular elements s_i, such that R and $\mathcal{L}(\mathbb{F}_\infty)$ are free in some W^*-probability space (M, τ). Consider orthogonal projections $p_1, p_2, \cdots \in R$ and put $r := 1 + \sum_j \tau(p_j)^2 \in [1, \infty]$. Then the von Neumann algebra*

$$\mathcal{L}(\mathbb{F}_r) := \mathrm{vN}(R, p_j s_j p_j (j \in \mathbb{N})) \tag{6.4}$$

is a factor and depends, up to isomorphism, only on r.

These $\mathcal{L}(\mathbb{F}_r)$ for $r \in \mathbb{R}$, $1 \le r \le \infty$ are the *interpolating free group factors*. Note that we do not claim to have noninteger free groups $\mathbb{F}_r$. The notation $\mathcal{L}(\mathbb{F}_r)$ cannot be split into smaller components.

Dykema and Rădulescu showed the following results.

Theorem 14. *1) For $r \in \{2, 3, 4, \dots, \infty\}$ the interpolating free group factor $\mathcal{L}(\mathbb{F}_r)$ is the usual free group factor.*
2) We have for all $r, s \geq 1$: $\mathcal{L}(\mathbb{F}_r) \star \mathcal{L}(\mathbb{F}_s) \simeq \mathcal{L}(\mathbb{F}_{r+s})$.
3) We have for all $r \geq 1$ and all $t \in (0, \infty)$ the same compression formula as in the integer case:

$$\big(\mathcal{L}(\mathbb{F}_r)\big)_t \simeq \mathcal{L}(\mathbb{F}_{1+t^{-2}(r-1)}). \tag{6.5}$$

The compression formula above is also valid in the case $r = \infty$; since then $1 + t^{-2}(r - 1) = \infty$, it yields in this case that any compression of $\mathcal{L}(\mathbb{F}_\infty)$ is isomorphic to $\mathcal{L}(\mathbb{F}_\infty)$; or in other words, we have that the fundamental group of $\mathcal{L}(\mathbb{F}_\infty)$ is equal to $\mathbb{R}^+$.

6.12 The dichotomy for the free group factor isomorphism problem

Whereas for $r = \infty$, the compression of $\mathcal{L}(\mathbb{F}_r)$ gives the same free group factor (and thus we know that the fundamental group is maximal in this case); for $r < \infty$ we get some other free group factors. Since we do not know whether these are isomorphic to the original $\mathcal{L}(\mathbb{F}_r)$, we cannot decide upon the fundamental group in this case. However, on the positive side, we can connect different free group factors by compressions; this yields that some isomorphisms among the free group factors will imply other isomorphisms. For example, if we would know that $\mathcal{L}(\mathbb{F}_2) \simeq \mathcal{L}(\mathbb{F}_3)$, then this would imply that also

$$\mathcal{L}(\mathbb{F}_5) \simeq \big(\mathcal{L}(\mathbb{F}_2)\big)_{1/2} \simeq \big(\mathcal{L}(\mathbb{F}_3)\big)_{1/2} \simeq \mathcal{L}(\mathbb{F}_9).$$

The possibility of using arbitrary $t \in (0, \infty)$ in our compression formulas allows to connect any two free group factors by compression, which gives then the following dichotomy for the free group isomorphism problem. This is again due to Dykema and Rădulescu.

Theorem 15. *We have exactly one of the following two possibilities.*

(i) All interpolating free group factors are isomorphic: $\mathcal{L}(\mathbb{F}_r) \simeq \mathcal{L}(\mathbb{F}_s)$ for all $1 < r, s \leq \infty$. In this case the fundamental group of each $\mathcal{L}(\mathbb{F}_r)$ is equal to $\mathbb{R}^+$.
(ii) The interpolating free group factors are pairwise non-isomorphic: $\mathcal{L}(\mathbb{F}_r) \not\simeq \mathcal{L}(\mathbb{F}_s)$ for all $1 < r \neq s \leq \infty$. In this case the fundamental group of each $\mathcal{L}(\mathbb{F}_r)$, for $r \neq \infty$, is equal to $\{1\}$.