Chapter 1
Continuous Models

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Abstract The aim of this introductory chapter is to summarize the main mechanical models which describe the physics of musical instruments and of their constitutive parts. These models derive from the general principles of the mechanics of continuous media (solids and fluids). In this framework, the phenomena are described at a scale of the so-called *particle*, or *element*, whose dimensions are infinitesimal in the sense of differential calculus. Particular emphasis is given to the bending of structures and to the equations of acoustic waves in air, because of their relevance in musical acoustics. One section is devoted to the excitation mechanisms of musical instruments. Analogies between vibrations of solids (such as strings) and fluids (in pipes) are underlined. Elementary considerations on the numerical formulation of the models are also given. This chapter should be considered as a summary which contains reference results to help in reading the rest of the book. It focuses on the origin of the equations and on their underlying assumptions, living aside the complete demonstrations.

1.1 Strings, Membranes, Bars, Plates, and Shells

1.1.1 Introduction

In this chapter, we present *linear* models. This means, in particular, that we limit ourselves to the case of *small displacements* (geometric linearity) and to materials whose constitutive stress–strain relations are linear (material linearity).

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Conversely, in Chap. 8, several examples of nonlinearity will be examined. For the time being, the dissipative phenomena in solids are ignored. Chapter 5 will be specifically devoted to damping.

We study here the class of *elastic* solids. If such a solid is deformed under the effect of a given load, the deformation disappears as the load is gradually removed until the solid returns to its initial state. For example, if we hit the bar of a vibraphone with a soft mallet, and if we touch the bar after a few seconds in order to suppress the sound, we find that the state of the bar is unchanged. However, if a xylophone wooden bar is hit with a hard mallet, irreversible *plastic* deformations may appear locally on the bar. A hard stroke might even break it!

In order to excite the structures used in musical acoustics, it is often necessary to apply a *prestress* to some of them. This is, for example, the case of strings and membranes subjected to a static tension field at rest. A piano (or a guitar) soundboard is also subjected to a strong prestress due to the tension of the strings attached to, or passing over, the bridge (see Fig. 1.1) [7, 40]. The prestress works only if the structure departs from its equilibrium. It is sometimes called *geometric stiffness*.

The dynamics of structures which are vibrating parts of musical instruments are governed by both elasticity and geometric stiffness. If only elasticity is present, we are in the extreme case of bars and plates. Geometric stiffness dominates in the case of strings and membranes. In practice, a structure with zero elasticity can never be found. Systems with geometric stiffness, such as ideal strings and membranes, where the intrinsic elasticity is ignored, should be considered as theoretical limiting cases (see Sect. 1.1.2).

The case of shells is more complex and will be considered separately. This book is limited to the study of thin shallow shells. Such structures are found both in percussion (cymbals, gongs, etc.) and string instruments (soundboard of bowed string instruments, for example). The presence of curvature has several important

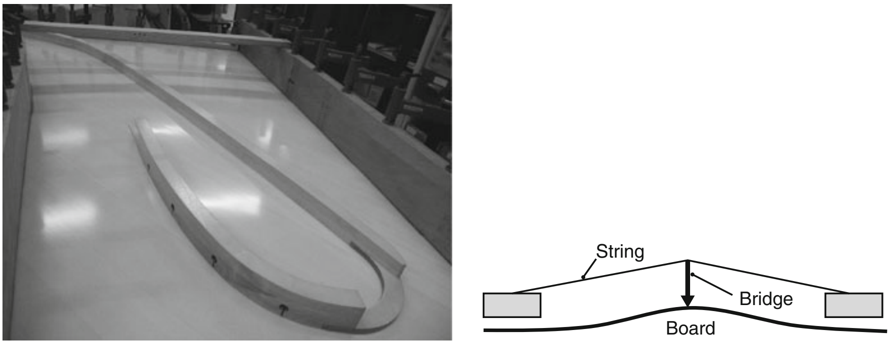


Fig. 1.1 (*Left*) A grand piano soundboard with its bridge (© Pleyel). (*Right*) Simplified diagram of the prestress supported by a piano soundboard. The soundboard is initially curved. Under the influence of string tension, the bridge presses on the soundboard. This transverse force is partially converted into longitudinal stress in the soundboard

effects: change of the radiation properties compared to flat plates (see Chap. 13), increase of the maximum static load supported by the soundboard, easier starting nonlinear behavior for large amplitude of vibration for shells with free edges such as gongs (see Chap. 8).

A common feature between membranes, plates, and thin shells follows from the fact that their models involve only two spatial dimensions. If the vibration wavelength is large compared to the thickness, it is justified to integrate the stress along this dimension and neglect the thickness strain. As a consequence, the "3D" model is reduced to a "2D" one. In addition, to obtain string and bar equations, it is assumed that these structures are *slender*, where one dimension (the length) is large compared to the other two. This leads then to a "1D" model where the integration of the stress is now made along the two dimensions of a cross-section.

1.1.2 Membranes and Strings

Preamble For a complete demonstration of the membrane equation, the reader can consult the literature devoted to the mechanics of continuous media (see [51]). The presentation is limited here to the heterogeneous membrane equation in orthonormal Cartesian coordinates.

We consider an infinitesimal element of membrane with coordinate vector $\mathbf{x}$ and density $\rho(\mathbf{x})$, for which the elastic stiffness is ignored. At equilibrium, the membrane is located in the plane $(\mathbf{e}_x, \mathbf{e}_y)$ and subjected to a tension field. This tension field is described by a symmetrical tensor of order 2

$$\underline{\underline{\tau}} = \begin{bmatrix} \tau_{11} & \tau_{12} \\ \tau_{12} & \tau_{22} \end{bmatrix}, \tag{1.1}$$

where the τ_{ij} are the components of the tensor (see Fig. 1.2). From this tensor, we can derive the tensional forces acting on a membrane elements:

$$\begin{cases} \text{on the surfaces with normal vector oriented along } \mathbf{e}_x : \boldsymbol{\tau}_x = \tau_{11}\mathbf{e}_x + \tau_{21}\mathbf{e}_y, \\ \text{on the surfaces with normal vector oriented along } \mathbf{e}_y : \boldsymbol{\tau}_y = \tau_{12}\mathbf{e}_x + \tau_{22}\mathbf{e}_y, \end{cases} \tag{1.2}$$

A tension field is measured in force per unit length, and its unit is thus in $\mathrm{N\,m^{-1}}$. Integrating this tension along the perimeter of a given surface gives the total external force which is necessary to apply at the periphery to balance the internal tension field.

The off-diagonal tension components have the symmetry property $\tau_{12} = \tau_{21}$ to ensure equilibrium of the moments on the membrane element (reciprocity principle). Each component τ_{ij} is a function of the coordinate vector $\mathbf{x}$. It is assumed that the membrane can move freely along $\mathbf{e}_z$ so that its vertical displacement $\xi(x, y, t)$ at time t is governed by the equilibrium between the inertial forces and the restoring

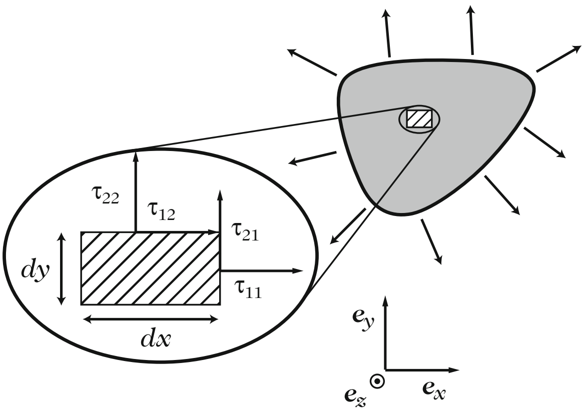


Fig. 1.2 Tension field exerted on a membrane element. $\mathbf{e}_x$, $\mathbf{e}_y$, and $\mathbf{e}_z$ are the unit vectors in Cartesian coordinates

forces due to the tension field. Gravity is ignored. With the assumptions of small displacements, the rotations θ_x and θ_y of the membrane element on both planes $(\mathbf{e}_x, \mathbf{e}_z)$ and $(\mathbf{e}_y, \mathbf{e}_z)$ are given by:

$$\begin{cases} \theta_x \approx \sin\theta_x \approx \tan\theta_x \approx \dfrac{\partial \xi}{\partial x}, \\ \theta_y \approx \sin\theta_y \approx \tan\theta_y \approx \dfrac{\partial \xi}{\partial y}\,. \end{cases} \tag{1.3}$$

Similarly, for any function $G(x,y)$ on the membrane, a first-order expansion yields

$$\begin{cases} G(x+dx,y) = G(x,y) + \dfrac{\partial G}{\partial x}dx, \\ G(x,y+dy) = G(x,y) + \dfrac{\partial G}{\partial y}dy\,. \end{cases} \tag{1.4}$$

Balancing the forces applied on each sides of the element, and projecting them on the vertical axis $\mathbf{e}_z$, we obtain the equation of transverse motion of a heterogeneous membrane:

$$\rho(\mathbf{x})h\ddot{\xi} = \frac{\partial}{\partial x}\left(\tau_{11}\frac{\partial \xi}{\partial x} + \tau_{12}\frac{\partial \xi}{\partial y}\right) + \frac{\partial}{\partial y}\left(\tau_{12}\frac{\partial \xi}{\partial x} + \tau_{22}\frac{\partial \xi}{\partial y}\right), \tag{1.5}$$

where h is the thickness. Denoting:

$$\mathbf{grad}\xi = \frac{\partial \xi}{\partial x}\mathbf{e}_x + \frac{\partial \xi}{\partial y}\mathbf{e}_y, \tag{1.6}$$

we can write this equation in a more compact form:

$$\rho(\mathbf{x})h\ddot{\xi} = \text{div}\left(\underline{\underline{\tau}}.\mathbf{grad}\xi\right) . \tag{1.7}$$

If the membrane is also subjected to external pressure forces that cannot be neglected, as for drums and timpani, then the projection of Newton's second law along the vertical axis $\mathbf{e}_z$ leads to the equation with a "source" term:

$$\rho(\mathbf{x})h\ddot{\xi} = \text{div}\left(\underline{\underline{\tau}}.\mathbf{grad}\xi\right) + p(x,y,0^-,t) - p(x,y,0^+,t) . \tag{1.8}$$

In Eq. (1.8), the source term corresponds to a *pressure jump* across the membrane. For timpani, this pressure jump is equal to the difference between the sound pressure in the cavity and the sound pressure in the external air, in the membrane plane $z = 0$. More generally, the membrane may be subjected to a distribution of external forces localized or distributed on its surface. This surface distribution of forces $f_s(x,y,t)$ (with dimension of a pressure) is due, for example, to the action of a timpani mallet or of a drum stick. In this case, the equation of motion becomes[1]

$$\rho(\mathbf{x})h\ddot{\xi} = \text{div}\left(\underline{\underline{\tau}}.\mathbf{grad}\xi\right) + p(x,y,0^-,t) - p(x,y,0^+,t) + f_s(x,y,t) . \tag{1.9}$$

1.1.2.1 1D Approximation: Transverse Motion of Strings

The string length of musical instruments are large compared to the radius of the cross-section, so that it is justified to neglect the deformation in both transverse dimensions. Rewriting Eq. (1.5) through integration of inertial and tension forces along $\mathbf{e}_y$ yields the 1D approximation of the transverse motion equation (along $\mathbf{e}_z$) for a heterogeneous string:

$$\mu(x)\ddot{\xi} = \frac{\partial}{\partial x}\left[T(x)\frac{\partial \xi}{\partial x}\right], \tag{1.10}$$

where $\mu = \rho_s S$ is the linear density of the string and T the tension at rest (in N). S is the cross-sectional area of the string.

If the string is subjected to external forces along its length (linear density of forces $f_{\text{ext}}(x,t)$ in $\text{N}\,\text{m}^{-1}$) the equation of motion, including the source term, becomes

$$\mu(x)\ddot{\xi} = \frac{\partial}{\partial x}\left[T(x)\frac{\partial \xi}{\partial x}\right] + f_{\text{ext}}(x,t) . \tag{1.11}$$

[1]Equation (1.9), written here in Cartesian coordinates, can be generalized to other coordinate systems.

Since we are dealing here with a 1D model, there is no need to consider air pressure forces here.

Comments

1. Another transverse motion $\eta(x, t)$ oriented along $\mathbf{e}_y$ exists on the string. The equation of motion for $\eta(x, t)$ is analogous to (1.10). In general, both *polarizations* are excited.
2. In the absence of coupling terms in the model, ξ and η are independent of each other. In stringed musical instruments, however, this coupling does exist: it is mainly due to motion of the bridge at the end and to the existence of nonlinear terms for large amplitude motion (see Chap. 8).
3. Strings (and membranes) are also subjected to *longitudinal* vibrations. Such vibrations arise because fluctuations in length induce stress fluctuations. As a consequence, the stress becomes a function of the amplitude (and thus a function of time). This transverse–longitudinal coupling is usually neglected under the assumption of small amplitude. Nevertheless, it can be easily observed in piano strings, for example. This point will be clarified in Chap. 8.

1.1.2.2 Homogeneous Membranes and Strings Under Uniform Tension

For a uniformly stretched membrane made of a homogeneous material, the tension tensor $\underline{\underline{\tau}}$ becomes isotropic, which can be written as $\underline{\underline{\tau}} = \tau \underline{\underline{\mathbb{1}}}$, where $\underline{\underline{\mathbb{1}}}$ denotes the unit tensor. Equation (1.7) becomes

$$\tau \operatorname{div}(\mathbf{grad}\xi) = \tau \Delta\xi = \rho h \ddot{\xi}, \tag{1.12}$$

where the Laplacian in Cartesian coordinates is

$$\Delta\xi = \frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2} . \tag{1.13}$$

For timpani and drums, the most easier way to obtain a uniform tension is to choose a circular geometry for the membrane. In this case, the use of polar coordinates (r, θ) is preferable and Eq. (1.12) is written:

$$\rho h \ddot{\xi} = \tau \left(\frac{\partial^2 \xi}{\partial r^2} + \frac{1}{r}\frac{\partial \xi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \xi}{\partial \theta^2} \right) . \tag{1.14}$$

Homogeneous String Under Uniform Tension

With a uniform tension T, Eq. (1.11) reduces to:

$$\mu \ddot{\xi} = T \frac{\partial^2 \xi}{\partial x^2} + f_{\text{ext}}(x, t) . \tag{1.15}$$

This partial differential equation of order 2 can be rewritten in the form of a system of two equations of order 1 involving force and velocity:

$$\begin{cases} \dfrac{\partial f}{\partial t} = T\dfrac{\partial v}{\partial x}, \\ \mu\dfrac{\partial v}{\partial t} = \dfrac{\partial f}{\partial x} + f_{\text{ext}}(x,t) \;\; \text{with} \;\; v = \dfrac{\partial \xi}{\partial t} \,. \end{cases} \tag{1.16}$$

The latter formulation is useful in numerical analysis and sound synthesis, where it is often easier to solve systems of equations of lower order. It also helps in highlighting formal analogies with electrical transmission lines (see Chap. 4).

1.1.3 *Stress and Strain*

Before starting to examine the deformation of elastic solids, it is necessary to briefly recall the concepts of *strain* and *stress* that form the basis of continuum mechanics. For more details the reader may refer to specialized textbooks (see, for example, [49]).

1.1.3.1 Strain

General Formulation

The concept of strain can be introduced by writing the variation of length of an elementary vector $\mathbf{ds}_1$, whose both ends are subjected to the displacements $\boldsymbol{\xi}$ and $\boldsymbol{\xi} + \mathbf{d}\boldsymbol{\xi}$, respectively (see Fig. 1.3).

Starting from the general formula giving the length of sides in a triangle, we get the length of the vector $\mathbf{ds}_2$, noting that:

$$ds_2^2 - ds_1^2 \;\simeq\; 2\,\mathbf{ds}_1.\mathbf{d}\boldsymbol{\xi} \;=\; 2\,[d\xi_1 dx_1 + d\xi_2 dx_2 + d\xi_3 dx_3], \tag{1.17}$$

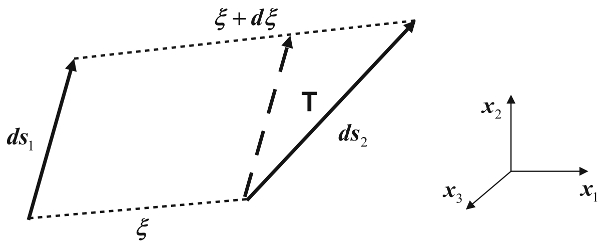


Fig. 1.3 Displacement of a vector in a deformable solid. The length of the vector $\mathbf{ds}_2$ is calculated from the length of sides in triangle T

where the second-order terms in $d\xi^2$ are neglected, and where x_i (i = {1,2,3}) denote the coordinates of the initial vector $\mathbf{ds}_1$. The displacement $\boldsymbol{\xi}(x)$ depends on the coordinates x_i, therefore the differential of each of its components is written:

$$d\xi_i = \sum_{j=1}^{3} \frac{\partial \xi_i}{\partial x_j} dx_j \; . \tag{1.18}$$

The subscript j in (1.18) is called *summation index* (or *dummy index*), since it appears in both the partial derivative and differential form. In continuum mechanics, we use the *Einstein convention* which consists in ignoring the summation sign ($\sum$) when it applies to a dummy index, in order to simplify the notation. In these conditions, Eq. (1.17) becomes

$$ds_2^2 - ds_1^2 \;\simeq\; 2\,\frac{\partial \xi_i}{\partial x_j} dx_j dx_i \; . \tag{1.19}$$

In addition, one can show the following property of symmetry:

$$\frac{\partial \xi_i}{\partial x_j} = \frac{\partial \xi_j}{\partial x_i}. \tag{1.20}$$

Therefore, each term in (1.19) can be written as follows:

$$ds_2^2 - ds_1^2 \;\simeq\; \varepsilon_{ij} dx_j dx_i \;\; \text{with} \;\; \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial \xi_i}{\partial x_j} + \frac{\partial \xi_j}{\partial x_i}\right) \; . \tag{1.21}$$

The quantities ε_{ij} form a set of six distinct components called *strain tensor*, which is written $\underline{\underline{\varepsilon}}$. It is a tensor of rank 2, since it depends on two indices. This tensor is symmetrical since $\varepsilon_{ij} = \varepsilon_{ji}$. Its six components fully characterize the strain of a continuous medium in three dimensions.

1.1.3.2 Stress

In the mechanics of rigid bodies, a general load is represented by a set of *forces* and *moments*. In fluid mechanics, it is necessary to also introduce the concept of *pressure*. However, these notions are not sufficient to represent the internal constraints acting in a deformable solid. It is observed first that the contact load exerted by an infinitesimal element on its neighbors inside the deformable medium cannot be reduced to a simple set of forces and moments. Secondly, the resulting forces are not oriented normally to each contact surface, as it is the case for perfect fluids. It is therefore necessary to introduce the concept of *stress* reflecting the fact that, on each elementary surface of contact between two particles, the *surface force*

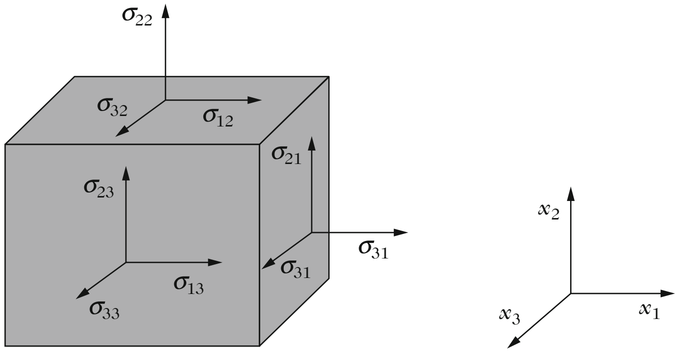


Fig. 1.4 Stress components exerted on a small elementary cubic volume

density vector is defined by three components. To clarify this, the concept of stress is illustrated on a small elementary cubic volume, with edges parallel to the axes (see Fig. 1.4).

The elementary forces applied on each surface dS_j ($j = \{1, 2, 3\}$) can be decomposed into three components dF_i ($i = \{1, 2, 3\}$). The surface density of force is defined as:

$$\sigma_{ij} = \frac{dF_i}{dS_j} . \tag{1.22}$$

The balance of moments leads to the symmetry property:

$$\sigma_{ij} = \sigma_{ji} . \tag{1.23}$$

In total, on each side of the elementary volume, we get nine components σ_{ij} which reduce to six components, due to symmetry. This set, denoted $\underline{\underline{\sigma}}$, is the *stress tensor* for the continuous medium. It is a symmetric tensor of rank 2, as for the strain tensor. In vector and tensor notation, we write the resulting force on a surface dS with normal vector $\boldsymbol{n}$:

$$\boldsymbol{dF} = \underline{\underline{\sigma}}.\boldsymbol{n}dS . \tag{1.24}$$

$\boldsymbol{T} = \underline{\underline{\sigma}}.\boldsymbol{n}$ is the *stress vector* on the surface. Finally, in the presence of a body force field $\boldsymbol{f}$, and taking further the inertial forces into account, the local equilibrium equation in a given solid element of density ρ is written:

$$\rho\ddot{\boldsymbol{\xi}} = \mathrm{div}\underline{\underline{\sigma}} + \boldsymbol{f}, \tag{1.25}$$

where $\ddot{\boldsymbol{\xi}}$ is the local acceleration.

1.1.4 Constitutive Equations of Materials: Linear Elasticity

In the dynamics of rigid bodies, the motion reduces to a set of translations and rotations, as a result of the application of forces and moments. In this case, inertial quantities such as the masses and the moments of inertia of the body make the links between load and motion. In continuum mechanics, we need a finer description of the internal properties of the deformable body to interpret static and dynamic strain and stress. The applied load results in a distribution of stress in the body. In the example discussed in the following subsection, a tensile force along the axis of a specimen leads to an almost uniaxial stress. As a result of stress, the structure will deform more or less according to its internal properties. We call *constitutive equations of materials* all properties (elasticity, viscosity, and thermal expansion) that make the link between stress and strain. We restrict ourselves to the particular class of linear elastic materials.

1.1.4.1 A One-Dimensional Traction Experiment

Let us perform a simple traction experiment on an homogeneous cylinder with cross-section S, whose length at rest L_0 is significantly larger than $\sqrt{S}$. It is observed that the relative extension $\frac{L-L_0}{L_0} \approx \varepsilon_{xx}$ linearly varies with the surface density of force $\sigma_{xx} \approx \frac{F}{S}$, as long as it does not exceed the yield strength σ_e (see Fig. 1.5).

In addition, it is observed that the experiment is reversible and that the sample recovers its initial form when the tensile force is removed. As a consequence, we write

$$\sigma_{xx} = E\,\varepsilon_{xx}, \tag{1.26}$$

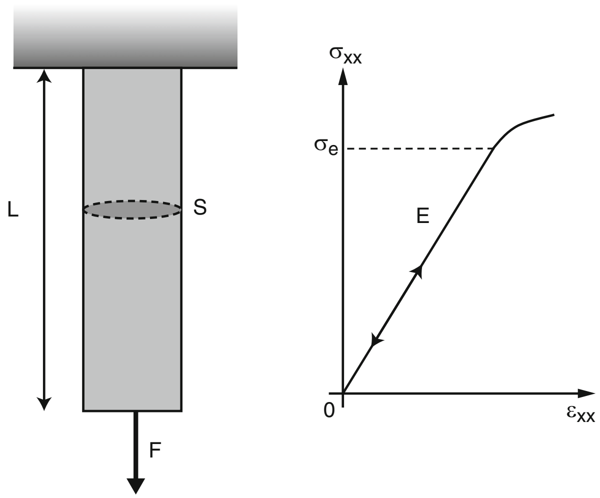


Fig. 1.5 Tension experiment. The force F is applied along the axis of the cylinder with initial cross-section S and length L_0. The relative extension $\frac{L-L_0}{L_0} \approx \varepsilon_{xx}$ is measured for increasing and decreasing values of the stress $\sigma_{xx} \approx \frac{F}{S}$. As long as σ_{xx} is less than the yield strength σ_e (which depends on the material) the curve $\sigma_{xx} = f(\varepsilon_{xx})$ is linear and reversible. The slope is the Young's modulus E

where E (in N/m^2) is the Young's modulus of the material. One important point here is that the stress is proportional to the strain: we are in the situation of a *linear* elastic behavior. This relationship, based on experimental facts, is also called *Hooke's law*.

Comment The relation (1.26) does not take the variation of the cross-section in the body consecutive to elongation (or compression) into account. This can only be done with the 3D generalization of Hooke's law (see the next section).

1.1.4.2 Elasticity Tensor

In a deformable body, one stress component σ_{ij} is likely to generate several components of strain ε_{kl}. Assume a linear constitutive law, we can generalize Eq. (1.26) to:

$$\sigma_{ij} = A_{ijkl}\, \varepsilon_{kl}, \tag{1.27}$$

where A_{ijkl} represents the *elasticity tensor* of the material, also denoted $\underline{\underline{\underline{\underline{A}}}}$. It is a fourth rank tensor. In theory, this tensor should have $3^4 = 81$ distinct components. However, if we recall that $\underline{\underline{\sigma}}$ and $\underline{\underline{\varepsilon}}$ are symmetrical, it reduces to 36, which is the maximum number of independent components for $\underline{\underline{\underline{\underline{A}}}}$. Because of additional energetic considerations, this number is reduced to 21 in the case of an anisotropic material [8]. Finally, taking also the symmetry of the material into account allows to reduce again the number of elasticity components.

Isotropic Material

In an *isotropic* material, all directions are equivalent. In this particular case, only 12 components A_{ijkl} are non-zero, and they are defined as function of two independent elasticity coefficients only, λ and μ, called *Lamé parameters* [49]. For such a material, the stress–strain relations are written:

$$\begin{pmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \sigma_{zx}\\ \sigma_{yz}\\ \sigma_{xy}\end{pmatrix} = \begin{pmatrix}\lambda+2\mu & \lambda & \lambda & 0 & 0 & 0\\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0\\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0\\ 0 & 0 & 0 & 2\mu & 0 & 0\\ 0 & 0 & 0 & 0 & 2\mu & 0\\ 0 & 0 & 0 & 0 & 0 & 2\mu\end{pmatrix}\begin{pmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\\ \varepsilon_{zx}\\ \varepsilon_{yz}\\ \varepsilon_{xy}\end{pmatrix}, \tag{1.28}$$

which can be written equivalently using the following compact tensor form:

$$\underline{\underline{\sigma}} = \lambda(\mathrm{tr}\underline{\underline{\varepsilon}})\underline{\underline{\mathbb{1}}} + 2\mu\underline{\underline{\varepsilon}}, \tag{1.29}$$

where $\text{tr}\underline{\underline{\varepsilon}}$ is nothing but the divergence of the displacement field: $\text{div}\boldsymbol{\xi}$. We also call it *dilatation*. In order to derive the strain tensor from the stress tensor, it is sufficient to invert Eq. (1.29), which leads to:

$$\underline{\underline{\varepsilon}} = \frac{1+\nu}{E}\underline{\underline{\sigma}} - \frac{\nu}{E}(\text{tr}\underline{\underline{\sigma}})\underline{\underline{1}}, \tag{1.30}$$

with

$$E = \frac{\mu\,(3\lambda + 2\mu)}{\lambda + \mu} \quad \text{et} \quad \nu = \frac{\lambda}{2(\lambda+\mu)}\,. \tag{1.31}$$

We recognize here the Young's modulus E and the Poisson's ratio ν. The latter is a dimensionless coefficient such that $-1 < \nu < 1/2$. By inverting the system (1.31), we obtain the expression of the Lamé parameters as function of E and ν:

$$\lambda = E\frac{\nu}{(1+\nu)(1-2\nu)} \quad ; \quad \mu = \frac{E}{2(1+\nu)}\,. \tag{1.32}$$

For a uniaxial stress field σ_{xx} applied (see the previous experiment of tension) to a "3D" specimen, we get from (1.29): $\varepsilon_{xx} = \sigma_{xx}/E$, $\varepsilon_{yy} = -\nu\sigma_{xx}/E$, and $\varepsilon_{zz} = -\nu\sigma_{xx}/E$ (see Fig. 1.6).

We can conclude that the Poisson's ratio gives a measure of the lateral compression of a specimen along the axes $\mathbf{e}_y$ and $\mathbf{e}_z$ under the effect of a traction along $\mathbf{e}_x$. This property is called "Poisson's effect."

Orthotropic Material

Unlike isotropic materials, anisotropic materials do not show identical elastic properties in all directions. Wood, for example, is an orthotropic material, which is a special case of anisotropy. To be convinced of such a behavior, a simple experiment can be made which consists in bending a guitar soundboard in Spruce with the hands. The experienced rigidity is higher when the bending is applied in the direction of the fibers compared to the case where the bending moment is applied in a direction perpendicular to them. More generally, for an orthotropic material, we can distinguish three orthogonal directions: longitudinal (L), radial (R), and tangential (T) (see Fig. 1.7).

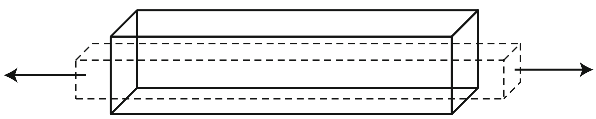


Fig. 1.6 Tension of a 3D bar and lateral compression (Poisson's effect)

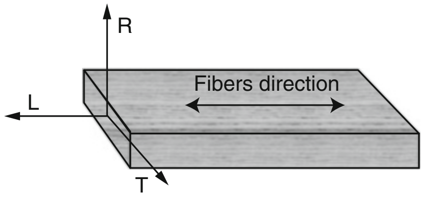


Fig. 1.7 Orthotropic material. A sample made from an orthotropic material (such as wood) has different elastic properties depending on whether the longitudinal direction (L), the radial direction (R), or the tangential direction (T) is considered

For a system of axes with coordinates (x, y, z) corresponding to these directions, the strain tensor in such a material is expressed as follows [28]:

$$\begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{zx} \\ \varepsilon_{yz} \\ \varepsilon_{xy} \end{pmatrix} = \begin{pmatrix} \frac{1}{E_L} & -\frac{\nu_{RL}}{E_R} & -\frac{\nu_{TL}}{E_T} & 0 & 0 & 0 \\ -\frac{\nu_{LR}}{E_L} & \frac{1}{E_R} & -\frac{\nu_{TR}}{E_T} & 0 & 0 & 0 \\ -\frac{\nu_{LT}}{E_L} & -\frac{\nu_{RT}}{E_R} & \frac{1}{E_T} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2G_{LT}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2G_{TR}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2G_{RL}} \end{pmatrix} \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{zx} \\ \sigma_{yz} \\ \sigma_{xy} \end{pmatrix} \tag{1.33}$$

where the Poisson's ratios ν_{ij} correspond to a contraction in direction j consecutive to an extension applied in direction i. As an example, ν_{LR} corresponds to a contraction in the radial direction consecutive to an extension in the longitudinal direction.

The symmetry properties of the material lead to the following equalities:

$$\frac{\nu_{LR}}{E_L} = \frac{\nu_{RL}}{E_R} \; ; \; \frac{\nu_{LT}}{E_L} = \frac{\nu_{TL}}{E_T} \; ; \; \frac{\nu_{RT}}{E_R} = \frac{\nu_{TR}}{E_T} \, . \tag{1.34}$$

In summary, the elastic properties of an orthotropic material are defined by nine independent coefficients:

- Three Young's moduli (or elasticity moduli): E_L, E_R, andE_T,
- Three Poisson's ratios: ν_{LR}, ν_{RT}, and ν_{TL},
- Three shear moduli: G_{LT}, G_{TR}, and G_{RL}.

Comment For a guitar soundboard made of Spruce, the ratio E_L/E_T usually lies between 10 and 20. The directions of the fibers correspond to those of the strings so that the board resists to the shear induced by the bridge. Flexibility in the tangential direction is partially compensated by stiffeners glued on the inferior face of the board. Using more recent materials, such as carbon fiber and composites, it is possible to control the elastic properties in all three directions [8]. Today, a number of soundboards of stringed instruments are made by mixing, in various proportions, wood and carbon fibers [9]. It will be seen in Chap. 13 that the choice of materials in instrument making is not only governed by static considerations but also by radiation criteria, which is fully understandable for musical instruments.

1.1.5 Bars and Plates

We are now interested in the case of elastic solids without prestress. As previously done for the membranes, dissipation phenomena are ignored. The bar model can be applied to elastic solids whose one dimension is of a higher order of magnitude than the two others (slender solid), and for which a one-dimensional model can thus be developed. Plates correspond to 2D plane solids where the order of magnitude of the thickness is lower than the length of the sides. For a bar subjected to small perturbations it is justified to decouple the different regimes of vibrations: traction, torsion, and bending [39]. For pedagogical reasons, we will examine these limiting cases in the order of increasing difficulty. Thus, traction and torsion will be presented before the bending, although this does not correspond to the relative significance of these regimes in musical acoustics. In xylophones and other mallet instruments, for example, bending vibrations are responsible for the essential part of the sound. Torsional vibrations are also present, but are usually unwanted. Finally, longitudinal motion is unsignificant.

1.1.5.1 Traction (or Compression) of a Bar

Consider an isotropic elastic bar loaded along its main axis (denoted $\mathbf{e}_x$) (see Fig. 1.8). In this case, the displacement $\boldsymbol{\xi}(x,t)$ at each point is axial (or longitudinal). In musical acoustics, the axial vibrations of piano strings play a major role, especially during the attack transient [5].

The strain in the bar is $\varepsilon_{xx} = \varepsilon = \frac{\partial \xi}{\partial x}$. The axial stress is $\sigma_{xx} = \sigma = E\varepsilon$ where E is the Young's modulus. For a bar of length L and cross-section S, the elastic potential energy is given by:

$$E_p = \frac{1}{2}\int_0^L ES\left(\frac{\partial \xi}{\partial x}\right)^2 dx, \tag{1.35}$$

and the kinetic energy is

$$E_c = \frac{1}{2}\int_0^L \rho S\left(\frac{\partial \xi}{\partial t}\right)^2 dx\,. \tag{1.36}$$

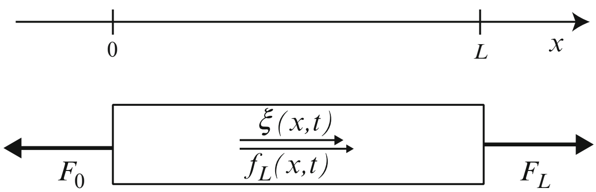


Fig. 1.8 Traction (or compression) of a bar. One-dimensional model

Application of Hamilton's principle yields the equation of motion:

$$\rho S \frac{\partial^2 \xi}{\partial t^2} - \frac{\partial}{\partial x}\left(ES \frac{\partial \xi}{\partial x} \right) = f_L(x,t) \tag{1.37}$$

where $f_L(x,t)$ is the force per unit length applied to the bar.

Comment. *Unless otherwise specified, quantities E, ρ, and S defined above are functions of the abscissa x. Thus, the present model allows to treat the case of heterogeneous bars and/or bars of variable thickness.*

Denoting F_0 and F_L the forces applied at both ends $x = 0$ and $x = L$ (see Fig. 1.8), the boundary conditions are written:

$$\begin{cases} \left(ES \frac{\partial \xi}{\partial x} \right)_{x=0} = -F_0, \\ \left(ES \frac{\partial \xi}{\partial x} \right)_{x=L} = F_L \,. \end{cases} \tag{1.38}$$

In this book, we will have to consider general boundary conditions (BC) of the type:

$$\alpha ES \frac{\partial \xi}{\partial x} + \beta \xi = F \quad \text{with} \quad (\alpha, \beta) \in \mathbb{R}, \tag{1.39}$$

These general conditions include the two following simple cases:

$$\frac{\partial \xi}{\partial x} = 0 \quad \text{(free BC)} \quad ; \quad \xi = 0 \quad \text{(fixed BC)}\,. \tag{1.40}$$

1.1.5.2 Torsion of a Bar

The bars of keyboard percussion instruments often show a torsional motion around the main axis, especially when they are struck near the edge. The amplitude of this motion can be significant if the bar is cut in its central part since, in this case, the torsional stiffness decreases notably (see below for an accurate definition of torsional stiffness). The torsional vibrations can be musically annoying, because the corresponding frequencies are generally not in harmonic correspondence with the main components of the bending vibrations (see the next section) that mainly contribute to the sound of the instrument.

Bowed strings (which can be considered as prestressed bars) are also subjected to a torsional moment induced by the bow. Woodhouse and colleagues have shown that these vibrations, particularly through their dissipative function, have an important role in the stability of the motion of the bowed string [23, 56, 57] (see Chap. 11).

To model torsional vibrations, we consider the simple case of a cylinder with a circular cross-section of radius a (see Fig. 1.9).

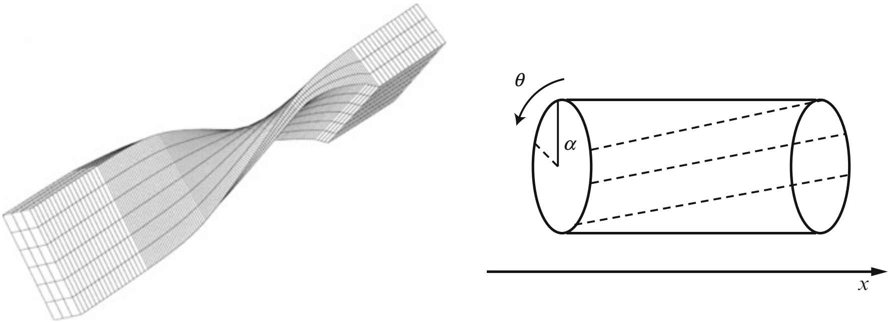


Fig. 1.9 (*Left*) Torsional motion of a marimba bar. Such a motion, which can be musically very annoying, often appears when a bar with an undercut is struck near the edge. For such a bar of complex geometry, the equations that govern the torsional motion can only be solved numerically. (*Right*) Torsion of a cylindrical bar with a circular cross-section of radius a. We assume that, under the effect of a torsional torque, each cross-section of the cylinder of abscissa x rotates with an angle $\theta(x,t)$. The *dashed lines* show the deformation of the generatrices of the cylinder, consecutive to the rotation $\theta(x,t)$. The cylinder can be viewed as a kind of "spaghetti" bundle where each generatrix remains straight in the rotation

We note $\theta(x,t)$ the angular displacement of a cross-section of abscissa x, $dS = rdrd\theta$ a surface element in this section, σ the modulus of the torsional stress, G the torsional modulus of the cylinder's material, and ϕ the angular displacement of a generatrix initially parallel to the x axis. First, we can write

$$\phi = r\frac{\partial\theta}{\partial x} . \tag{1.41}$$

According to the definition of the torsional modulus, we have $\sigma = G\phi$. In this case, the relation between the moment $M(x)$ applied to the cross-section S of the cylinder and the rotation is given by:

$$M = \int_S \sigma r dS = \int_S G r^2 \frac{\partial\theta}{\partial x} dS = GJ\frac{\partial\theta}{\partial x}, \tag{1.42}$$

where $J = \int_S r^2 dS$ is *the rotational inertia of the section*. This quantity has the dimension of a length to the fourth power. It is equal to $J = \pi a^4/2$ for a circular cross section of radius a.

Newton's second law (or law of conservation of angular momentum) applied to an element dx of the bar leads to the balance of moments:

$$I\frac{\partial^2\theta}{\partial t^2} = \frac{\partial M}{\partial x} + m_e, \tag{1.43}$$

where I is here the mass moment of inertia with respect to the axis x and per unit length of the cylinder; m_e is the density per unit length of the external momenta applied to the cylinder. Combining (1.43) and (1.42), we obtain the partial differential equation governing θ:

$$I\frac{\partial^2\theta}{\partial t^2} = \frac{\partial}{\partial x}\left(GJ\frac{\partial\theta}{\partial x}\right) + m_e, \tag{1.44}$$

which, in the case of a uniform cylinder of constant cross-section, reduces to:

$$I\frac{\partial^2\theta}{\partial t^2} = GJ\frac{\partial^2\theta}{\partial x^2} + m_e \,. \tag{1.45}$$

In the particular case of a homogeneous beam of constant circular cross-section (and in this case, only!), we also have $I = \rho J$. We formally obtain a wave equation with the same form as for the vibrating string (or for the longitudinal vibrations of a bar), but notice that the physical phenomena at the origin of these equations are fully different.

1.1.5.3 Bending of an Isotropic Bar

We now consider the bending of a slender bar in the plane $(\mathbf{e}_x, \mathbf{e}_y)$ with the following assumptions:

1. One dimension (along the main axis) is large compared to the other two.
2. The material is elastic and linear.
3. The cross-sections are symmetrical so that we make no distinction between the mean fiber (locus of the center of gravity of the cross-sections) and the neutral fiber (locus of the points which are not subjected to bending stress during the deformation).
4. The Poisson's effect (lateral contraction–extension) is ignored.
5. Each point of a cross-section at abscissa x moves vertically in the direction of the axis $\mathbf{e}_y$ with amplitude $v(x)$ small compared to the bar's thickness.
6. The cross-sections are subjected to a rotation θ_z around the axis $\mathbf{e}_z$ so that they remain straight and perpendicular to the mean fiber during the motion.
7. The rotations are small, so that we can perform a first-order approximation $\theta_z \approx \frac{\partial v}{\partial x}$. In addition, we neglect the rotational kinetic energy of the sections.

Within the framework of these *Euler–Bernoulli assumptions*, a displacement field is of the form:

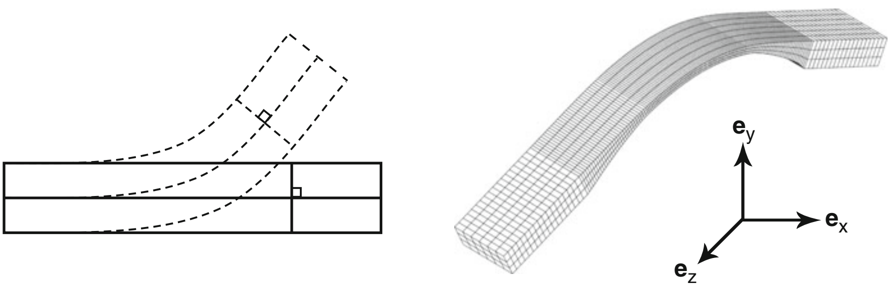


Fig. 1.10 (*Left*) Euler–Bernoulli kinematics. During the motion, the cross-sections remain straight and perpendicular to the neutral fiber. (*Right*) Bending motion of a marimba bar. The bending of such a bar is well described by a model of a bar with variable cross-section

$$\xi_x = -y\theta_z \approx -y\frac{\partial v}{\partial x} \;\; ; \;\; \xi_y = v \;\; ; \quad \xi_z = 0, \tag{1.46}$$

from which we get, from (1.21), the strain tensor:

$$\begin{cases} \varepsilon_{xx} = \dfrac{\partial \xi_x}{\partial x} = -y\dfrac{\partial^2 v}{\partial x^2}, \\ \varepsilon_{xy} = \varepsilon_{yx} = \dfrac{1}{2}\left(\dfrac{\partial \xi_x}{\partial y} + \dfrac{\partial \xi_y}{\partial x}\right) = 0, \\ \varepsilon_{yy} = \dfrac{\partial \xi_y}{\partial y} = 0\,. \end{cases} \tag{1.47}$$

The hypothesis of isotropic material yields the stress tensor. Here, we write [see Eq. (1.26)] $\sigma_{xx} = E\varepsilon_{xx}$, since the Poisson's effect is neglected.

Let us now express *the elastic potential energy*, also called *strain energy*, of the bending bar. For an elementary spring of stiffness k, the elastic energy stored under the effect of a traction (or compression) of elongation x from equilibrium is $E_p = \frac{1}{2}kx^2$. By analogy, the elementary force dF_x applied to an element of length dx of the bar with cross-section dS is, according to (1.22): $dF_x = \sigma_{xx}dS$. Therefore, the elementary elastic potential energy is: $dE_p = \frac{1}{2}\sigma_{xx}\varepsilon_{xx}\,dx\,dS$. By integrating this expression on the complete volume of the bar of length L, we get

$$E_p = \frac{1}{2}\int_S\int_0^L Ey^2\left(\frac{\partial^2 v}{\partial x^2}\right)^2 dx\,dS = \frac{1}{2}\int_0^L EI_z(x)\left(\frac{\partial^2 v}{\partial x^2}\right)^2 dx \tag{1.48}$$

where $I_z(x) = \int_S y^2 dS$ is the *principal moment of inertia along the axis 0z* of the cross-section S. The quantity $\mathscr{M} = EI_z\frac{\partial^2 v}{\partial x^2}$ is the bending moment and $\mathscr{C} = \frac{\partial^2 v}{\partial x^2}$ is the curvature.

The kinetic energy E_c of the beam of density ρ is written:

$$E_c = \frac{1}{2}\int_S\int_0^L \rho\dot{\xi}^2 dxdS = \frac{1}{2}\int_S\int_0^L \rho\left[y^2\left(\frac{\partial \dot{v}}{\partial x}\right)^2 + \dot{v}^2\right] dx\,dS\,. \tag{1.49}$$

This energy can be rewritten as:

$$E_c = \frac{1}{2}\int_0^L \rho I_z\dot{\theta}_z^2 dx + \frac{1}{2}\int_0^L \rho S\dot{v}^2 dx, \tag{1.50}$$

which shows that the kinetic energy is the sum of both a rotational and a translational energy. Under the Euler–Bernoulli framework, the rotational inertia is neglected, so that we get

$$E_c \simeq \frac{1}{2}\int_0^L \rho S\dot{v}^2 dx\,. \tag{1.51}$$

Starting from the energetic quantities, we obtain the equation of motion through application of Hamilton's principle. Denoting $f(x,t)$ the density per unit length of non-conservative external forces applied to the bar, and δv a cinematically acceptable virtual displacement (test function), we get the virtual mechanical work δW_{nc} of these forces from the expression:

$$\delta W_{nc} = \int_0^L f(x,t)\, \delta v\, dx\,. \tag{1.52}$$

The method consists in deriving the equation of motion and the boundary conditions that must be verified by the displacement field $v(x,t)$ to ensure that the following integral is equal to zero between two arbitrary moments of time t_1 and t_2 [21, 25]:

$$\int_{t_1}^{t_2} \left(\delta E_c - \delta E_p + \delta W_{nc}\right) dt = 0\,. \tag{1.53}$$

The variations of both kinetic and potential energy are given by:

$$\delta E_c = \frac{\partial E_c}{\partial v}\delta v + \frac{\partial E_c}{\partial \dot{v}}\delta \dot{v} = -\frac{d}{dt}\left(\frac{\partial E_c}{\partial \dot{v}}\right)\delta v = -\int_0^L \rho S \ddot{v}\, \delta v\, dx, \tag{1.54}$$

and

$$\delta E_p = \frac{\partial E_p}{\partial v''}\delta v'' = \left[EI_z \frac{\partial^2 v}{\partial x^2}\delta v'\right]_0^L - \left[\frac{\partial}{\partial x}\left(EI_z \frac{\partial^2 v}{\partial x^2}\right)\delta v\right]_0^L + \int_0^L \frac{\partial}{\partial x}\left(EI_z \frac{\partial^2 v}{\partial x^2}\right)\delta v\, dx, \tag{1.55}$$

where $v'' = \frac{\partial^2 v}{\partial x^2}$ and $v' = \frac{\partial v}{\partial x}$.

By inserting (1.55) and (1.54) in (1.53), we derive the bending equation of motion of the bar, within the simplified framework of Euler–Bernoulli assumptions:

$$\rho S \ddot{v} + \frac{\partial^2}{\partial x^2}\left(EI_z \frac{\partial^2 v}{\partial x^2}\right) = f, \tag{1.56}$$

with the boundary conditions:

$$\left[EI_z \frac{\partial^2 v}{\partial x^2}\delta v'\right]_0^L = 0 \quad \text{and} \quad \left[\frac{\partial}{\partial x}\left(EI_z \frac{\partial^2 v}{\partial x^2}\right)\delta v\right]_0^L = 0\,. \tag{1.57}$$

Equation (1.56) is of fourth-order in space. Therefore, four boundary conditions, two conditions at each end, are necessary to properly define the problem. From (1.57), we see that only four combinations are possible at each end:

$$\begin{cases} \text{Simply supported edge: } v = 0 \text{ and } \mathscr{M} = EI_z \dfrac{\partial^2 v}{\partial x^2} = 0, \\ \text{Clamped edge : } \dfrac{\partial v}{\partial x} = 0 \text{ and } v = 0, \\ \text{Free edge: } \mathscr{T} = \dfrac{\partial}{\partial x}\left(EI_z \dfrac{\partial^2 v}{\partial x^2}\right) = 0 \text{ and } \mathscr{M} = EI_z \dfrac{\partial^2 v}{\partial x^2} = 0, \\ \text{Guided edge: } \mathscr{T} = \dfrac{\partial}{\partial x}\left(EI_z \dfrac{\partial^2 v}{\partial x^2}\right) = 0 \text{ and } \dfrac{\partial v}{\partial x} = 0\,. \end{cases} \tag{1.58}$$

The quantity $\mathscr{T} = \dfrac{\partial}{\partial x}\left(EI_z \dfrac{\partial^2 v}{\partial x^2}\right)$ is the *shear force*.

- In musical acoustics, the Euler–Bernoulli model gives satisfactory results provided that the ratio between the length and a characteristic dimension of the cross-section (radius or side length) is greater than or equal to about 10. For keyboard percussion instruments (xylophone, vibraphone, and marimba), this model is valid for the lowest bars only. As the length of the bar decreases, it is necessary to choose a kinematic model accounting for the fact that the cross-sections do not remain perpendicular to the neutral axis during the motion. It becomes also necessary to take the rotational inertia of the sections into account (Timoshenko model) [19]. For a detailed comparison of different models of bars, the reader can refer to [27].

1.1.5.4 Bending of Thin Elastic Plates

The "thin plate" hypotheses (or Kirchhoff–Love model) generalize for plates the Euler–Bernoulli assumptions applied to the bars (see Sect. 1.1.5.3). A detailed presentation of the equation of bending plates is beyond the scope of this book. We can refer, for example, to the work by Yu [58] or Geradin and Rixen [21].

Here, only the main steps of the modeling are summarized, using the same approach as for bars in the previous paragraph. The case of orthotropic plates is selected as an illustration. It is particularly useful in musical acoustics since it can be applied to wooden plates used in lutherie [13, 55]. The problem is treated in Cartesian coordinates, and the transverse displacement is denoted $w(x, y, t)$. We assume that the coordinates coincide with the symmetry axes of the material and that $\boldsymbol{e}_z$ is the transverse direction. We therefore consider that the displacement field $\boldsymbol{\xi}$ in the plate is of the form:

$$\xi_x = -z\frac{\partial w}{\partial x} \quad ; \quad \xi_y = -z\frac{\partial w}{\partial y} \quad ; \quad \xi_z = w, \tag{1.59}$$

from which we get the strain tensor, assumed to be plane:

$$\begin{cases} \varepsilon_{xx} = \dfrac{\partial \xi_x}{\partial x} = -z\dfrac{\partial^2 w}{\partial x^2}, \\ \varepsilon_{yy} = \dfrac{\partial \xi_y}{\partial y} = -z\dfrac{\partial^2 w}{\partial y^2}, \\ \varepsilon_{xy} = \varepsilon_{yx} = \dfrac{1}{2}\left(\dfrac{\partial \xi_x}{\partial y} + \dfrac{\partial \xi_y}{\partial x}\right) = -z\dfrac{\partial^2 w}{\partial x \partial y}\ . \end{cases} \tag{1.60}$$

The orthotropy of the material leads to the following relations between plane stress and strain:

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{pmatrix} = \begin{pmatrix} \dfrac{E_x}{1-\nu_{xy}\nu_{yx}} & \dfrac{\nu_{yx}E_x}{1-\nu_{xy}\nu_{yx}} & 0 \\ \dfrac{\nu_{yx}E_x}{1-\nu_{xy}\nu_{yx}} & \dfrac{E_y}{1-\nu_{xy}\nu_{yx}} & 0 \\ 0 & 0 & 2G_{xy} \end{pmatrix} \begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{pmatrix} \tag{1.61}$$

where the coefficients ν_{ij} are such that $1 - \nu_{ij}\nu_{ji} > 1$ [13].

The bending moments are obtained by integration of the elementary moments on the plate thickness h:

$$\mathscr{M}_x = \int_{-h/2}^{h/2} z\sigma_{xx} dz\ ;\ \mathscr{M}_y = \int_{-h/2}^{h/2} z\sigma_{yy} dz\ ;\ \mathscr{M}_{xy} = \mathscr{M}_{yx} = \int_{-h/2}^{h/2} z\sigma_{xy} dz, \tag{1.62}$$

from which we derive the relations between moments and curvatures:

$$\begin{pmatrix} \mathscr{M}_x \\ \mathscr{M}_y \\ \mathscr{M}_{xy} \end{pmatrix} = -\begin{pmatrix} D_1 & D_2/2 & 0 \\ D_2/2 & D_3 & 0 \\ 0 & 0 & D_4/2 \end{pmatrix} \begin{pmatrix} \dfrac{\partial^2 w}{\partial x^2} \\ \dfrac{\partial^2 w}{\partial y^2} \\ \dfrac{\partial^2 w}{\partial x \partial y} \end{pmatrix}, \tag{1.63}$$

where

$$D_1 = \frac{E_x h^3}{12(1-\nu_{xy}\nu_{yx})}\ ; \qquad D_2 = \frac{E_x \nu_{yx} h^3}{6(1-\nu_{xy}\nu_{yx})} = \frac{E_y \nu_{xy} h^3}{6(1-\nu_{xy}\nu_{yx})}, \tag{1.64}$$

$$D_3 = \frac{E_y h^3}{12(1-\nu_{xy}\nu_{yx})}\ ; \qquad D_4 = \frac{G_{xy} h^3}{3}\ . \tag{1.65}$$

The equation of motion is again obtained from the Hamilton integral [Eq. (1.53)]. The variation of the kinetic energy resulting from a virtual displacement δw is written:

$$\delta E_c = -\int_S \rho_p h \ddot{w}\, \delta w\, dS\,. \tag{1.66}$$

The variation of the potential energy that generalizes the case of bars is

$$\delta E_p = \int_S \left[\mathscr{M}_x \delta w''_{xx} + \mathscr{M}_y \delta w''_{yy} + 2\mathscr{M}_{xy} \delta w''_{xy} \right] dS\,. \tag{1.67}$$

Finally, for a surface density of transverse force $f(x, y, t)$, the virtual mechanical work is written:

$$\delta W_{nc} = \int_S f(x, y, t)\, \delta w\, dx\,. \tag{1.68}$$

Applying Hamilton's principle to the set of Eqs. (1.66)–(1.68), we derive the bending equation of the plate:

$$\rho_p h \frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 \mathscr{M}_x}{\partial x^2} + \frac{\partial^2 \mathscr{M}_y}{\partial y^2} + 2\frac{\partial^2 \mathscr{M}_{xy}}{\partial x \partial y} + f(x, y, t)\,. \tag{1.69}$$

Equation of Motion of Plates in Terms of Displacement

We eliminate the bending moments from Eqs. (1.63) and (1.69) to get the equation describing the transverse displacement of the orthotropic plate:

$$\begin{aligned} &\rho_p h \frac{\partial^2 w}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(D_1 \frac{\partial^2 w}{\partial x^2} + \frac{D_2}{2}\frac{\partial^2 w}{\partial y^2}\right) + \frac{\partial^2}{\partial y^2}\left(D_3 \frac{\partial^2 w}{\partial y^2} + \frac{D_2}{2}\frac{\partial^2 w}{\partial x^2}\right) \\ &\quad + \frac{\partial^2}{\partial x \partial y}\left(D_4 \frac{\partial^2 w}{\partial x \partial y}\right) = f(x, y, t)\,. \end{aligned} \tag{1.70}$$

which becomes, in the particular case of a homogeneous plate:

$$\rho_p h \frac{\partial^2 w}{\partial t^2} + D_1 \frac{\partial^4 w}{\partial x^4} + D_3 \frac{\partial^4 w}{\partial y^4} + (D_2 + D_4)\frac{\partial^4 w}{\partial x^2 \partial y^2} = f(x, y, t)\,. \tag{1.71}$$

For an isotropic plate, we have $E_x = E_y = E$ and $\nu_{xy} = \nu_{yx} = \nu$, so that the rigidity constants are written:

$$\begin{aligned} D_1 &= D_3 = \frac{Eh^3}{12(1-\nu^2)} = D\,;\; D_2 = 2\nu D, \\ D_4 &= \frac{\mu h^3}{3} = \frac{Eh^3}{6(1+\nu)} = 2(1-\nu)D\,. \end{aligned} \tag{1.72}$$

In conclusion, we obtain the classical equation of homogeneous and isotropic thin plates under Kirchhoff–Love assumptions:

$$\rho_p h \frac{\partial^2 w}{\partial t^2} + D\left[\frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w}{\partial y^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2}\right] = f(x,y,t)\ . \tag{1.73}$$

More generally, we assume the following notation, which is independent of the coordinate system:

$$\rho_p h \frac{\partial^2 w}{\partial t^2} + D\nabla^4 w = f, \tag{1.74}$$

where ∇^4 represents the bi-Laplacian. The symbol Δ^2 is also used for designating this operator.

Boundary Conditions

In Sect. 1.1.5.3, Eq. (1.55) has shown that the boundary conditions are the results of integration by parts carried out to express the variation of elastic potential energy as a function of the virtual displacement (noted δv for bars). We proceed here in a similar manner for the variation of potential energy in plates written in (1.67). Here the integration is performed on a surface, and thus the results and the number of possible boundary conditions depend on the geometry of the plate. For rectangular plates, for example, Leissa lists 21 possible cases for the boundary conditions [36]. For a given edge (at $x = x_0$ for example), the most commonly encountered conditions are the following:

1. Clamped edge: displacement $w = 0$ and rotation $\frac{\partial w}{\partial x} = 0$,
2. Simply supported edge: displacement $w = 0$ and bending moment $\mathcal{M}_x = 0$,
3. Free edge: bending moment $\mathcal{M}_x = 0$ and shear force $\mathcal{T}_x = \frac{\partial \mathcal{M}_x}{\partial x} + 2\frac{\partial \mathcal{M}_{xy}}{\partial y} = 0$.

The boundary conditions for a free edge are written in Cartesian coordinates:

$$\begin{cases} \mathcal{M}_x = D_1 \dfrac{\partial^2 w}{\partial x^2} + \dfrac{D_2}{2}\dfrac{\partial^2 w}{\partial y^2} = 0, \\ \mathcal{T}_x = \dfrac{\partial}{\partial x}\left(D_1 \dfrac{\partial^2 w}{\partial x^2} + \dfrac{D_2}{2}\dfrac{\partial^2 w}{\partial y^2}\right) + \dfrac{\partial}{\partial y}\left(D_4 \dfrac{\partial^2 w}{\partial x \partial y}\right) = 0\ . \end{cases} \tag{1.75}$$

For a corner at the intersection of two free edges, we must add the condition:

$$\mathcal{M}_{xy} = 0, \quad \text{or} \quad \frac{\partial^2 w}{\partial x \partial y} = 0\ . \tag{1.76}$$

Fig. 1.11 The vibrations of bells are described by models of shells. © Australian-Dream.Fotolia.com

For an isotropic and homogeneous material, the conditions (1.75) become

$$\begin{cases} \dfrac{\partial^2 w}{\partial x^2} + \nu \dfrac{\partial^2 w}{\partial y^2} = 0 \quad \text{and} \\ \dfrac{\partial^3 w}{\partial x^3} + (2-\nu) \dfrac{\partial^3 w}{\partial x \partial y^2} = 0 \,. \end{cases} \tag{1.77}$$

1.1.6 Equation of Shells

A *shell* is a continuous medium which is completely defined by a surface and a thickness (see, as an example, the bells on Fig. 1.11). A plate corresponds to the special case of a shell with a plane surface. In musical acoustics, models of shells can be applied to a large number of percussion instruments (gongs, cymbals, bells, etc.) and to soundboards of string instruments. Here, we restrict our study to a brief presentation of the theory of *thin* elastic shells (the *thin shell theory*).[2] This theory, due to Love, is applicable when the thickness of the shell is small compared to other dimensions [51].

As for the Kirchhoff–Love model previously applied to plates, we assume that the local displacement field in the cross-sections consists in a translation and a rotation, so that each cross-section remains plane during the motion (see Figure 1.12). Translations and rotations differ from one section to another, otherwise, we would

[2]Here, we do not treat the cylindrical shells theory, which naturally applies to wind instruments, because it requires significant developments that are beyond the scope of this book. Nevertheless, we provide valuable references in Chap. 13 which deals with sound–structure interaction.

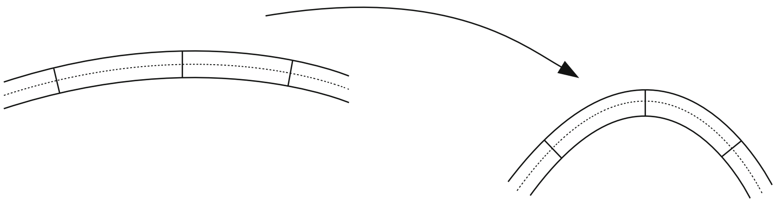


Fig. 1.12 Deformation of a thin shell. Each cross-section is subjected to a combination of translation and rotation

simply get a rigid body global displacement for the shell and, consecutively, no strain. In what follows, rotational inertia and transverse shear are ignored.

The main difference between strain tensors of plate and shell, respectively, is that, because of the non-zero curvature, the deformation induced by a transverse load is not only of the *bending* type as in the case of bars and plates, but also includes a *membrane-like* deformation. This means that a strain normal to the load exists in the thickness of the shell. The strain tensor is therefore formed by the sum of two contributions (see in the following Section the example of the spherical cap).

The Love model can be further simplified under the assumption that the shells are *slightly curved, or shallow,* and excited along their transverse dimension. The resulting model is traditionally called "Donnell–Mushtari–Vlasov model" (or DMV model) and is one of the most often used in shell theory. The fundamental assumptions of this model are

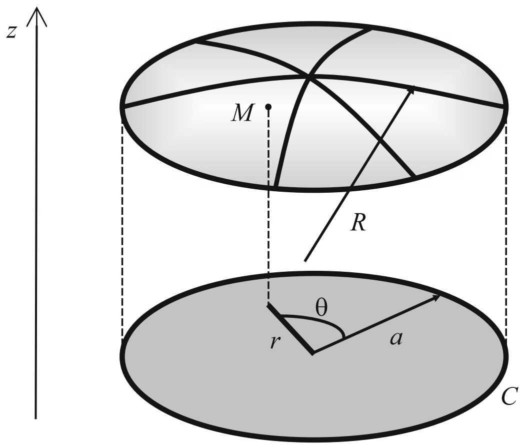
1. The membrane displacement is neglected in the bending strain.
2. Inertia of the membrane displacement is neglected.

To solve the problem it is convenient to introduce an auxiliary variable called *force function* or *Airy function*. Consequently, the motion equations take the form of a system of partial differential equations with two unknowns: the transverse displacement and the force function. For more details, the reader is invited to read the specialized literature on shells [3, 37, 51].

1.1.6.1 Thin Shallow Spherical Shells

In order to illustrate its general concepts, the main features of the DMV model are applied to the particular case of thin shallow spherical shells. This example has the advantage of being simple enough, while showing the influence of the curvature. In addition, it allows to properly explain the dynamics of cymbals and gongs.

Fig. 1.13 Geometry of a thin spherical cap



Presentation

We are interested in a thin spherical cap of constant thickness h small compared to the radius a of the circle $\mathscr{C}$ obtained by projecting the cap on the horizontal plane (see Fig. 1.13).

We assume that the cap is slightly curved (shallow shell), which is expressed by the condition $a \ll R$, where R is the radius of curvature of the shell. It is supposed that the Kirchhoff–Love conditions are fulfilled: rigid body displacement for each cross-section, shear and rotational inertia neglected. Only the case of a homogeneous and isotropic shell is considered. Because of the rotational symmetry, we use the polar coordinates (r, θ, z) where (r, θ) are the coordinates of the projection of a current point M of the shell on the disk of radius a, and z its vertical coordinate in the shell thickness, with $-h/2 \leq z \leq h/2$.

Displacement Field

With the Kirchhoff–Love assumptions, the components of the displacement field $\boldsymbol{\xi}$ in the shell are written:

$$\begin{cases} \xi_r = u(r,\theta) + z\beta_r(r,\theta), \\ \xi_\theta = v(r,\theta) + z\beta_\theta(r,\theta), \\ \xi_z = w(r,\theta), \end{cases} \tag{1.78}$$

where (u, v, w) are the components of a translation vector and where β_r and β_θ are the elementary rotations of a cross-section of the shell along r and θ. First-order expansions of these rotations are written [51]:

$$\begin{cases} \beta_r(r,\theta) = \dfrac{u}{R} - \dfrac{\partial w}{\partial r}, \\ \beta_\theta(r,\theta) = \dfrac{v}{R} - \dfrac{1}{r}\dfrac{\partial w}{\partial \theta}\,. \end{cases} \tag{1.79}$$

Strain Tensor

We obtain the strain tensor by calculating the elongation of a small vector under an elementary displacement [see Eq. (1.19)]. In the case of thin shallow shells, the strain tensor can be put on the form:

$$\underline{\underline{\varepsilon}} = (\underline{\underline{\varepsilon}})_m + z(\underline{\underline{\varepsilon}})_f \tag{1.80}$$

where $(\underline{\underline{\varepsilon}})_m$ is a *membrane* type tensor that expresses the strains in the thickness of the shell, and where $(\underline{\underline{\varepsilon}})_f$ is a *bending* type tensor with components representing the changes of curvature consecutive to the displacement. These components are written:

$$\begin{cases} (\varepsilon_{rr})_m = \dfrac{\partial u}{\partial r} + \dfrac{w}{r}, \\ (\varepsilon_{\theta\theta})_m = \dfrac{u}{r} + \dfrac{1}{r}\dfrac{\partial v}{\partial \theta} + \dfrac{w}{R}, \\ (\varepsilon_{r\theta})_m = (\varepsilon_{\theta r})_m = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta} + r\dfrac{\partial}{\partial r}\left(\dfrac{v}{r}\right), \end{cases} \tag{1.81}$$

and

$$\begin{cases} (\varepsilon_{rr})_f = -\dfrac{\partial^2 w}{\partial r^2}, \\ (\varepsilon_{\theta\theta})_f = -\dfrac{1}{r}\left(\dfrac{\partial w}{\partial r} + \dfrac{1}{r}\dfrac{\partial^2 w}{\partial \theta^2}\right), \\ (\varepsilon_{r\theta})_f = (\varepsilon_{\theta r})_f = -2\dfrac{\partial}{\partial r}\left(\dfrac{1}{r}\dfrac{\partial w}{\partial \theta}\right)\,. \end{cases} \tag{1.82}$$

The other components of the tensor $\underline{\underline{\varepsilon}}$ are equal to zero.

Stress–Strain Relation

For a homogeneous and isotropic material, characterized by its Young's modulus E and Poisson's ratio ν, the non-zero components of the stress tensor are written:

$$\begin{cases} \sigma_{rr} = \dfrac{E}{1-\nu^2}\left(\varepsilon_{rr} + \nu\varepsilon_{\theta\theta}\right), \\ \sigma_{\theta\theta} = \dfrac{E}{1-\nu^2}\left(\varepsilon_{\theta\theta} + \nu\varepsilon_{rr}\right), \\ \sigma_{r\theta} = \sigma_{\theta r} = \dfrac{E}{2(1+\nu)}\varepsilon_{r\theta} . \end{cases} \tag{1.83}$$

Resulting Forces and Moments

As for bars and plates (see Sect. 1.1.5.4), the resulting forces applied to each elementary volume of the shell are obtained through integration of the stress vector over the thickness:

$$N_r = \int_{-h/2}^{h/2} \sigma_{rr}\, dz \ ; \ N_\theta = \int_{-h/2}^{h/2} \sigma_{\theta\theta}\, dz \ ; \ N_{r\theta} = N_{\theta r} = \int_{-h/2}^{h/2} \sigma_{r\theta}\, dz . \tag{1.84}$$

Similarly, the resulting moments are obtained by calculating:

$$M_r = \int_{-h/2}^{h/2} \sigma_{rr} z\, dz \ ; \ M_\theta = \int_{-h/2}^{h/2} \sigma_{\theta\theta} z\, dz \ ; \ M_{r\theta} = M_{\theta r} = \int_{-h/2}^{h/2} \sigma_{r\theta} z\, dz . \tag{1.85}$$

As a result, we get

$$\begin{cases} N_r = \dfrac{Eh}{1-\nu^2}\left[(\varepsilon_{rr})_m + \nu(\varepsilon_{\theta\theta})_m\right], \\ N_\theta = \dfrac{Eh}{1-\nu^2}\left[(\varepsilon_{\theta\theta})_m + \nu(\varepsilon_{rr})_m\right], \\ N_{r\theta} = N_{\theta r} = \dfrac{Eh}{2(1+\nu)}(\varepsilon_{r\theta})_m . \end{cases} \tag{1.86}$$

and

$$\begin{cases} M_r = \dfrac{Eh^3}{12(1-\nu^2)}\left[(\varepsilon_{rr})_f + \nu(\varepsilon_{\theta\theta})_f\right], \\ M_\theta = \dfrac{Eh^3}{12(1-\nu^2)}\left[(\varepsilon_{\theta\theta})_f + \nu(\varepsilon_{rr})_f\right], \\ M_{r\theta} = M_{\theta r} = \dfrac{Eh^3}{24(1+\nu)}(\varepsilon_{r\theta})_f . \end{cases} \tag{1.87}$$

These expressions show that the resulting forces applied by an element of shell on its neighboring elements are entirely due to membrane deformations. The resulting moments are the consequence of changes in curvature, as in the case of plates.

Equations of Motion

The equations of motion for the shell are obtained by writing down the balance of forces and moments on a shell element, and applying Newton's second law. After some calculations, we get

$$\begin{cases} D\nabla^4 w + \dfrac{N_r + N_\theta}{R} + \rho h \dfrac{\partial^2 w}{\partial t^2} = f \quad \text{with } D = \dfrac{Eh^3}{12(1-\nu^2)}, \\ \dfrac{\partial (r N_r)}{\partial r} + \dfrac{\partial N_{r\theta}}{\partial \theta} - N_\theta = 0, \\ \dfrac{\partial (r N_{r\theta})}{\partial r} + \dfrac{\partial N_\theta}{\partial \theta} + N_{r\theta} = 0 \, . \end{cases} \tag{1.88}$$

These three spherical shells Eq. (1.88) are expressed in terms of transverse displacement w and force components N_r, N_θ, and $N_{r\theta}$. These four unknowns are not independent since the three force components depend on the coordinates u, v, and w [see Eq. (1.86)]. The first equation in (1.88) shows that the additional term $(N_r + N_\theta)/R$ is due to membrane forces and tends to zero when the radius of curvature tends to infinity: it corresponds to the case of plates (Eq. 1.74). In most cases, we are primarily interested in the vertical component w of the displacement. The variables u and v can be eliminated in Eq. (1.88) by introducing a force function (or Airy function) F such that:

$$N_r = \frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2} \; ; \; N_\theta = \frac{\partial^2 F}{\partial r^2} \; ; \; N_{r\theta} = N_{\theta r} = \frac{1}{r^2}\frac{\partial F}{\partial \theta} - \frac{1}{r}\frac{\partial F}{\partial r \partial \theta}, \tag{1.89}$$

As a result, we obtain

$$\nabla^2 F = N_r + N_\theta \, . \tag{1.90}$$

Finally, the system (1.88) is written:

$$\begin{cases} D\nabla^4 w + \dfrac{\nabla^2 F}{R} + \rho h \dfrac{\partial^2 w}{\partial t^2} = f, \\ \nabla^4 F = \dfrac{Eh}{R}\nabla^2 w \, . \end{cases} \tag{1.91}$$

If necessary, the Airy function F can be further eliminated, in order to derive an equation in terms of w. However, in most cases, and, in particular, in the context of numerical resolution, it is more appropriate to keep a formulation based on a system with two unknowns, which offers the advantage to involve differential operators of lower orders.

Boundary Conditions

The boundary conditions for the thin shallow shell are obtained from Hamilton's principle using the same method as for bars and plates (see Eq. 1.57). This yields the following possibilities at the periphery of the shell (in $r = a$):

$$\begin{cases} N_r = 0 \quad \text{or} \quad u = 0, \\ N_{r\theta} + \dfrac{M_{r\theta}}{R} = 0 \quad \text{or} \quad v = 0, \\ \dfrac{1}{r}\dfrac{\partial M_{r\theta}}{\partial \theta} = 0 \quad \text{or} \quad w = 0, \\ M_r = 0 \quad \text{or} \quad \beta_r = 0\,. \end{cases} \tag{1.92}$$

A shell clamped at its edge, for example, has the following four boundary conditions:

$$u = v = w = 0 \ \text{ and } \ \beta_r = 0, \tag{1.93}$$

while the boundary conditions for a spherical shell with free edges are written:

$$N_r = 0 \ ; \ N_{r\theta} + \frac{M_{r\theta}}{R} = 0 \ ; \ \frac{\partial M_{r\theta}}{\partial \theta} = 0 \ ; \ M_r = 0 \ . \tag{1.94}$$

1.2 3D Acoustic Waves

The equation of three-dimensional (3D) acoustic waves in a non-dissipative medium at rest, at low level, forms the basis of each book of acoustics. The knowledge of the solution allows at least deriving a first approximation of the eigenfrequencies of wind instruments, known since Bernoulli to be very close to the played frequencies, or the frequencies of cavity appearing in string and percussion instruments. Further in this book, we need to partially remove the above-mentioned restrictions, particularly concerning dissipation. Conversely, we will start by studying cases much simpler than the 3D problem. We now establish the three-dimensional equation to set the framework of many following chapters. For more details on this subject, we refer the reader to some basic textbooks on acoustics [12, 41, 44, 48], but also on fluid mechanics [6].

Under the above-defined conditions, the acoustic wave equation is the result of the elimination of two acoustic variables, the velocity $\mathbf{v}$ and the density ρ. Only the acoustic pressure, which is a scalar quantity, is kept from two conservation equations and a state equation. For a given physical quantity, the corresponding acoustic quantity is defined as the variation of the quantity around an average value, considered as time invariant. This variation is assumed to be small, which allows to use linear approximations of phenomena (see the comments in the box below). The propagation of a sound wave in a fluid mainly depends on the fact that this fluid has a mass and a compressibility. It can therefore be seen as a combination of masses and springs, or rather springs with a certain mass, that are modeled in continuum mechanics as infinitesimal objects, called "particles." The compressibility of the fluid makes an acoustic motion different from a regular, incompressible flow, with a density remaining constant. Some results presented here are summarized and detailed further in Chap. 5 where the dissipation effects are studied.

Acoustic Quantities and Decibels
The variation of the acoustic quantities is small, but its magnitude can vary considerably, by a factor 10^7, from the lowest audible sound to the sound of a taking off airplane: this is the reason why the decibel scale is used. For a given acoustic pressure amplitude (i.e., the root-mean-square pressure), p, we define $N_{\mathrm{dB}} = 20\log(p/p_0)$, where p_0 is the reference value equal to 2×10^{-5} Pa. This value is almost the lowest sound level perceived by the ear at 1000 Hz. For a musical instrument playing *piano*, the sound level is approximately 60 dB, or 0.02 Pa, and for a *fortissimo* play, the sound hardly exceeds 100 dB, or 2 Pa, which is very far from the atmospheric pressure, equal to 10^5 Pa. However, it will be seen in Part III that the pressure at the input of a wind instrument can reach 170 dB, or 6000 Pa (this value still remains well below the atmospheric pressure, although it will produce new phenomena which be discussed in Part III of the book). Finally, to take into account that hearing perceives frequencies between 1000 and 3000 Hz much better than other frequencies, a weighted decibel, the dBA [20, 59] has been defined.

1.2.1 State Equation of a Gas

At equilibrium, the gas has a density ρ_0, expressed in kg m^{-3}, a uniform temperature T, expressed in °K, and a pressure p_0, expressed in N m^{-2} or Pascals (Pa). If we consider a volume V equal to nM/ρ, nM being the mass of the fluid[3] in the volume V, these quantities are linked by a state equation, $f(P, V, T) = 0$. As there are only two independent thermodynamic variables, we can express any variations of the quantities defining the fluid in terms of two of them.

Thus for the specific heat received by a fluid element $dQ = TdS$, (where S is the entropy per unit mass, which is a state function), we can express it in terms of the variations dP and dV (or dP and $d\rho = -\rho dV/V$). For acoustic motions of

[3] n is number of moles, and M the molar mass.

a sufficiently high frequency, it is assumed that motions are isentropic ($dQ = 0$), i.e., there is no heat exchange between the fluid elements (dissipation is discussed in Chap. 5). We derive a proportionality relation between pressure variations and density variations, which gives the first of the three sought equations. It is written:

$$dP = c^2 d\rho, \tag{1.95}$$

where $c \triangleq \sqrt{(\partial P/\partial \rho)_S}$ is a coefficient which will be later identified as the speed of sound waves. We note that c is simply related to the isentropic compressibility

$$\chi_S \triangleq \frac{1}{\rho}\left(\frac{\partial \rho}{\partial P}\right)_S = \frac{1}{\rho c^2}. \tag{1.96}$$

- If we write that pressure $P = p_0 + p$ and density $\rho = \rho_0 + \rho'$ slightly vary around their equilibrium values,[4] p_0 and ρ_0, we obtain from (1.95):

$$p = \frac{1}{\rho_0 \chi_S}\rho' = c^2 \rho'. \tag{1.97}$$

We note also that for constant entropy, as for the density variation, the temperature variation is proportional to the pressure variation. So if we write $T = T_0 + \tau$, the acoustic temperature τ is proportional to the acoustic pressure p. We find the expression of the corresponding coefficient in Chap. 5, as well as the values of the compressibility and the speed of sound for a given gas law, including temperature. We will show that if $PV = n\mathrm{R}T$, or $MP = \mathrm{R}T\rho$, where R is the constant of an ideal gas, we have

$$\chi_T \triangleq \frac{1}{\rho}\left(\frac{\partial \rho}{\partial P}\right)_T = \frac{1}{p_0} \text{ which leads to } \chi_S = \frac{1}{\gamma p_0} \text{ and } c^2 = \frac{\gamma p_0}{\rho_0} = \gamma \frac{\mathrm{R}T_0}{M}, \tag{1.98}$$

where $\gamma = C_p/C_v$, the ratio of specific heats at constant pressure and volume. This allows calculating the theoretical value of the speed of sound with respect to the temperature. Numerical values of the speed of sound, density, and other constants of air are given in Chap. 5.

1.2.2 Momentum Conservation

Here we write the conservation of momentum, i.e., the Newton's second law. We use the Eulerian variables, which are best suited for this study: these are the variables that an observer sees when he is looking at the fluid evolution from a fixed point in space, **r**, instead of following the evolution of a fluid element (Lagrangian description).

[4] This difference in notation for pressure and density, although it is apparently illogical, is convenient for the following of the statement.

- Considering a given quantity f depending on space and time, its time variation depends on the infinitesimal motion with velocity $\mathbf{v} = d\mathbf{r}/dt$. We write

$$\begin{aligned} df &= f(\mathbf{r} + \mathbf{dr}, t + dt) - f(\mathbf{r}, t) \\ &= f(\mathbf{r} + \mathbf{dr}, t + dt) - f(\mathbf{r}, t + dt) + f(\mathbf{r}, t + dt) - f(\mathbf{r}, t) \\ &= \mathbf{grad}f(\mathbf{r}, t + dt).d\mathbf{r} + [\partial f(\mathbf{r}, t)/\partial t]\ dt, \end{aligned}$$

  or at the first order

$$df = \mathbf{grad}f(\mathbf{r}, t).d\mathbf{r} + [\partial f(\mathbf{r}, t)/\partial t]\ dt = \mathbf{grad}f(\mathbf{r}, t).\mathbf{v}dt + [\partial f(\mathbf{r}, t)/\partial t]\ dt. \tag{1.99}$$

  Thus, we can define in general the following operator:

$$\frac{d}{dt} = (\mathbf{v}.\mathbf{grad}) + \frac{\partial}{\partial t}. \tag{1.100}$$

- Now we write the Newton's second law for an infinitesimal volume:

$$\rho \frac{d\mathbf{v}}{dt} = -\mathbf{grad}P + \rho \mathbf{F} \tag{1.101}$$

  where $\mathbf{F}$ is an external force per unit mass. In one dimension, it is read as follows: the product of the acceleration by the mass of the fluid element is equal to the pressure difference at both sides added to an external force. The $-$ sign on the right-hand side comes from the fact that, at position $(x + dx)$, a positive pressure applies a force along the negative x-axis, in contrast to what happens at position x.

  Equation (1.101) is the equation of the momentum conservation $\rho\mathbf{v}$. For a finite volume D bounded by a surface S, it can be written in the following integral form:

$$\iiint_D \rho \frac{d\mathbf{v}}{dt} dD = -\iint_S P d\mathbf{S} + \iiint_D \rho \mathbf{F} dD. \tag{1.102}$$

  The derivative with respect to time of the momentum in the volume D is equal to the outflow, which is simply the pressure force applied on the surface, added to the effect of forces external to the fluid.
- In linear acoustics, we can now linearize Eq. (1.101) to the first order, which is the Euler equation. For a fluid at rest, the total velocity $\mathbf{v}$ is the acoustic velocity, which is small, i.e., of the first order, and we obtain

$$\rho_0 \frac{\partial \mathbf{v}}{\partial t} = -\mathbf{grad}p + \rho_0 \mathbf{F}. \tag{1.103}$$

Equation (1.101) implies that the zeroth order of **F** is zero (because ρ_0 is not zero): thus, **F** is of the first order. We obtain here the second vector equation connecting p, ρ, and **v**, this time with a source external to the fluid.[5]

Bernoulli's Law

Returning to the non-linearized Euler Equation (1.101), it is seen that if the force **F** is irrotational, this is also the case for the velocity, and they both derive from a potential, written $-V$ and φ: $\mathbf{F} = -\mathbf{grad}V$ and $\mathbf{v} = \mathbf{grad}\varphi$. If the motion is isentropic (i.e., adiabatic and reversible), the pressure P depends on ρ only, and we can write

$$\frac{1}{\rho}\mathbf{grad}P = \frac{1}{\rho}\frac{dP}{d\rho}\mathbf{grad}\rho = \mathbf{grad}\int \frac{1}{\rho}\frac{dP}{d\rho}d\rho = \mathbf{grad}\int \frac{dP}{\rho}.$$

In addition $(\mathbf{v}.\mathbf{grad})\mathbf{v} = \frac{1}{2}\mathbf{grad}v^2 - (\mathbf{v} \times \mathbf{rot}\mathbf{v}) = \frac{1}{2}\mathbf{grad}v^2$, and we obtain from (1.101):

$$\mathbf{grad}\left[\frac{\partial \boldsymbol{\varphi}}{\partial t} + \frac{1}{2}v^2 + \int \frac{dP}{\rho} + V\right] = 0.$$

We can integrate this equation in space: by calculating the scalar product of this quantity and **v**, and noting that the quantity $\mathbf{v}.\mathbf{grad} = v\partial/\partial_n$ is the derivative in the direction of **v**, i.e., along a streamline. We obtain the Bernoulli's law by integrating along such a line:

$$\frac{\partial \boldsymbol{\varphi}}{\partial t} + \frac{1}{2}v^2 + \int \frac{dP}{\rho} + V = \text{function}(t). \tag{1.104}$$

We can include the right-hand side function in the potential φ, which is defined apart from a space-independent function, and we obtain a right-hand side equal to zero. The quasi-static version of (1.104), in homogeneous medium and without external force, will be useful to describe the flow at the input of a reed instrument. It is written:

$$P + \frac{1}{2}\rho v^2 = \text{constant}. \tag{1.105}$$

Concerning the version obtained in linear acoustics for a homogeneous medium at rest, it is simply written: $p = -\rho_0 \partial\varphi/\partial t$, where φ is the velocity potential.

[5]We could also add a source to Eq. (1.97): it would be a heat source, varying in time, which does not occur in musical instruments.

1.2.3 Conservation of Mass

We need another equation to link the quantities p, ρ, and $\mathbf{v}$: it is the conservation of mass, that we first write in an integral form. The mass entering a domain D bounded by a surface S per unit of time, to which we possibly add the one produced by a density of source mass ρq, is equal to the increase in fluid mass in the domain per unit of time:

$$-\iint_S \rho \mathbf{v}.\mathbf{dS} + \iiint_D \rho q(\mathbf{r}, t) dD = \frac{\partial}{\partial t} \iiint_D \rho\, dD, \tag{1.106}$$

if $\mathbf{dS}$ is the outgoing normal of the volume. We will see examples of sources $q(\mathbf{r}, t)$, which are flow sources per unit volume, especially for reed instruments. By using the divergence theorem, we get

$$\iiint_D \left[\text{div}(\rho \mathbf{v}) + \frac{\partial \rho}{\partial t} \right] dD = \iiint_D \rho q(\mathbf{r}, t) dD.$$

This expression is valid for any domain D and can therefore be written in a differential form:

$$\text{div}(\rho \mathbf{v}) + \frac{\partial \rho}{\partial t} = \rho q(\mathbf{r}, t). \tag{1.107}$$

For the same reasons as those given for $\boldsymbol{F}$ (see Sect. 1.2.2), q is of order 1, and the linearization gives for a homogeneous medium at rest:

$$\rho_0 \text{div} \mathbf{v} + \frac{\partial \rho'}{\partial t} = \rho_0 q(\mathbf{r}, t). \tag{1.108}$$

1.2.4 Acoustic Wave Equation

- In summary, the three linearized equations (1.97), (1.103), and (1.108) *(from here, we omit the subscript 0 for the average density)*, where only two of them have an external source, are written:

$$p = \frac{1}{\rho \chi_S} \rho' = c^2 \rho',$$

$$\rho \text{div} \mathbf{v} + \frac{\partial \rho'}{\partial t} = \rho q(\mathbf{r}, t),$$

$$\rho \frac{\partial \mathbf{v}}{\partial t} = -\mathbf{grad} p + \rho \mathbf{F}.$$

- Eliminating the acoustic density ρ', we derive two equations for the two most usual quantities, acoustic pressure and particle velocity:

$$\mathrm{div}\mathbf{v} + \chi_S \frac{\partial p}{\partial t} = q(\mathbf{r}, t)\ ; \tag{1.109}$$

$$\mathbf{grad}p + \rho \frac{\partial \mathbf{v}}{\partial t} = \rho \mathbf{F}. \tag{1.110}$$

- The mass-spring fluid system is finally characterized by two main parameters, its density ρ and its compressibility χ_S. If we remove the velocity, using a cross derivation, we obtain the wave equation for the pressure:

$$\Delta p - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = \rho \left(\mathrm{div}\mathbf{F} - \frac{\partial q}{\partial t} \right) \tag{1.111}$$

  where $\Delta = \nabla^2$. The choice of the pressure as the unique acoustic variable is very common, because, on the one hand, the pressure is a scalar quantity, and, on the other hand, the pressure is, to a first approximation, the quantity to which the ear is sensitive. We see that the production of sound is due to the time variation of the flow q. If the flow is constant, there is no sound.

1.2.5 Simple Solutions: Traveling and Standing Waves

We consider the particular case of a plane wave, where all quantities vary in the x direction, only. After the change of variables $(x, t) \rightarrow (x - ct, x + ct)$, the general solution is of the form:

$$p = f^+(x - ct) + f^-(x + ct), \tag{1.112}$$

which is the sum of two traveling waves, an outgoing one and an incoming one, of any shape. The wave speed is c, the square of which is the inverse of the product of the two parameters ρ and χ_S [see Eq. (1.95)]. Notice that when there is no term in the right-hand side in the equation, the velocity potential is governed by the same equation as the pressure (provided that it has been adequately chosen). This is also the case for the acoustic velocity. Therefore, in one dimension, the general solution for both the potential and velocity has an expression similar to Eq. (1.112).

For the outgoing wave, we have $\partial f^+/\partial t = -c\partial f^+/\partial x$, and we deduce $p = \rho c v_x$. The quantity p/v is the *specific acoustic impedance*, which, for both waves, is called the *characteristic impedance*. It is equal to $Z_S = \rho c = \sqrt{\rho/\chi_S}$, while $c = 1/\sqrt{\rho\chi_S}$. The pair of parameters of our fluid-spring system, ρ and χ_S, is equivalent to another pair, which characterizes a plane traveling wave: the speed of sound c and the specific characteristic impedance, or impedance of the medium, Z_S. We can alternatively use one pair or the other, depending on the context.

- There is another simple general solution of the wave equation, which separates the space and time variables. If we search for a solution of the form $p(\mathbf{r}, t) = R(\mathbf{r})T(t)$, Eq. (1.111) without sources becomes

$$c^2 \frac{\Delta R}{R} = \frac{1}{T}\frac{d^2 T}{dt^2}. \tag{1.113}$$

The left-hand side is a function of space only, and the right-hand side a function of time only. Therefore, each side is a constant, known as the separation constant, that we denote $-\omega^2$ if it is negative.[6] The function of time depends on two constants, A and φ:

$$T(t) = A\cos(\omega t + \varphi).$$

For plane waves, we can also find the function of space, and finally write:

$$p(\mathbf{r}, t) = (a\cos kx + b\sin kx)\cos(\omega t + \varphi),$$

where a and b are two constants. $k = \omega/c$ is the wavenumber. It is related to the spatial period λ, i.e., the wavelength, by $k = 2\pi/\lambda$. The solution is a *standing wave* solution, since all points of the space are vibrating in phase (or in antiphase, depending on the sign of the spatial solution): the phase is equal to φ or $\varphi + \pi$. They are clearly distinguishable from traveling waves (1.112), which are not separable into functions of space and time.

Complex Notation: Fourier and Laplace Transforms

If a quantity varies sinusoidally, for example, $p(t) = A\cos(\omega t + \varphi)$, it is very convenient to associate a complex quantity to it:

$$p_c(t) = ae^{j\omega t}\text{, where } a = Ae^{j\varphi}. \tag{1.114}$$

The interesting quantity is the real part of this complex quantity $p(t) = \Re e\,[p_c(t)]$. a is called *complex amplitude*. This simplifies all linear calculations, such as addition, scalar multiplication, derivation, and integration. For

(continued)

[6] Exponentially time-increasing or time-decreasing solutions may also exist if the constant is positive (but this is rare). The case of complex values is treated in the following section. Furthermore we can continue this operation by separating the variables of space. This will happen several times in this book.

example,

$$p_1(t) + p_2(t) = \Re e\left[p_{c1}(t) + p_{c2}(t)\right].$$

Thus we can use this method for the wave equation (1.111). Most often, the c subscript is omitted because there is no confusion. The solution with separate variables $p(\mathbf{r}, t) = R(\mathbf{r})T(t)$ can also be written using complex quantities. It can happen that the constant ω is complex, thus the functions $R(\mathbf{r})$ and $T(t)$ are complex (a complete problem must include the boundary conditions, which are governing the possible values of the separation constant). For this case there are no standing waves, as it can be easily verified by taking the real part: the phase varies in space.

Looking for solutions with sinusoidal variation is usual. Therefore we use a complex formulation, with a time dependence in $\exp(j\omega t)$. This means that we look for particular solutions, related to the time variation of the source.

This search for particular sinusoidal solutions should be distinguished from the search for *general* solution using a Fourier transform. For the latter, we will choose the definition:

$$P(\mathbf{r}, \omega) = \int_{-\infty}^{+\infty} p(\mathbf{r}, t)e^{-j\omega t}dt, \text{ with} \tag{1.115}$$

$$p(\mathbf{r}, t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} P(\mathbf{r}, \omega)e^{j\omega t}d\omega. \tag{1.116}$$

In these expressions, $p(\mathbf{r}, t)$ is the real physical general solution, which implies $P(\mathbf{r}, -\omega) = P^*(\mathbf{r}, \omega)$. Thus, if the pressure at one given point is $p(t) = a\cos(\omega_0 t + \varphi)$, we have

$$* \; P(\omega) = (a/2)\left[\delta(\omega - \omega_0)\exp(j\varphi) + \delta(\omega + \omega_0)\exp(-j\varphi)\right], \tag{1.117}$$

* where δ is the delta function. Recall that, for any function $g(x)$, and for any interval $[a, b]$ including the origin, the delta function satisfies

$$\int_a^b g(x)\delta(x)dx = g(0).$$

For example, the wave equation (1.111) without sources becomes, in the Fourier domain, the Helmholtz equation: $\left(\Delta + k^2\right)P(\mathbf{r}, \omega) = 0$.

For initial values problems, we also use the Laplace transform, where $s = \sigma + j\omega$ has a positive real part:

(continued)

$$P(\mathbf{r}, \mathbf{s}) = \int_0^{+\infty} p(\mathbf{r}, t)e^{-st}dt \;\; \text{with} \tag{1.118}$$

$$p(\mathbf{r}, t) = \frac{1}{2\pi j}\int_{-j\infty+\varepsilon}^{j\infty+\varepsilon} P(\mathbf{r}, s)e^{st}ds. \tag{1.119}$$

As a reminder, the derivation rules for this transform are

$$\text{If } f(t) \to F(s), \;\text{ then } f'(t) \to sF(s) - f(0) \text{ and } f''(t) = s^2F(s) - sf(0) - f'(0). \tag{1.120}$$

1.3 Energy, Intensity, and Power

In this section, two simple examples are treated in parallel: the vibrating string and the acoustic waves, in order to emphasize some interesting analogies.

1.3.1 Example of the Vibrating String

We consider an ideal homogeneous vibrating string without external source. In view of the results presented in Sect. 1.1.2, the equation of motion is written:

$$\mu\frac{\partial^2\xi}{\partial t^2} - T\frac{\partial^2\xi}{\partial x^2} = 0\,. \tag{1.121}$$

An energy-based formulation of the problem is obtained by multiplying this equation by the speed $v = \frac{\partial\xi}{\partial t}$ and integrating the resulting expression over the entire length of the string. After an integration by parts, we find

$$\int_0^L \mu\frac{\partial^2\xi}{\partial t^2}\frac{\partial\xi}{\partial t}dx + \int_0^L T\frac{\partial^2\xi}{\partial x\partial t}\frac{\partial\xi}{\partial x}dx + \left[-T\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial t}\right]_0^L = 0 \tag{1.122}$$

The first two integrals in (1.122) correspond to the time variation of the total energy $E = E_c + E_p$ of the string, where

$$\begin{cases} E_c = \int_0^L e_c \, dx \quad \text{where} \quad e_c = \frac{1}{2}\mu \left(\frac{\partial \xi}{\partial t}\right)^2 , \\ E_p = \int_0^L e_p \, dx \quad \text{where} \quad e_p = \frac{1}{2} T \left(\frac{\partial \xi}{\partial x}\right)^2 . \end{cases} \tag{1.123}$$

In these expressions, e_c is the kinetic energy per unit length and e_p the elastic potential energy per unit length. For the total energy per unit length, $e = e_c + e_p$, Eq. (1.122) is simply rewritten as:

$$\int_0^L \frac{\partial e}{\partial t} dx + \left[-T \frac{\partial \xi}{\partial x} \frac{\partial \xi}{\partial t} \right]_0^L = 0, \tag{1.124}$$

which can be alternatively formulated in the form:

$$\int_0^L \left(\frac{\partial e}{\partial t} + \frac{\partial}{\partial x} \left[-T \frac{\partial \xi}{\partial x} \frac{\partial \xi}{\partial t} \right] \right) dx = 0 . \tag{1.125}$$

The quantity $\Pi = -T \frac{\partial \xi}{\partial x} \frac{\partial \xi}{\partial t}$ has the dimension of an instantaneous power at point M with abscissa x. It is the product of the force $f = -T \frac{\partial \xi}{\partial x}$ by the velocity $\frac{\partial \xi}{\partial t}$ of the string at point M. For a wave traveling to the right, this power is imparted to the points situated at the right-hand side of M (see Fig. 1.14).

Finally, the fact that the integral in (1.125) vanishes implies that we can write at any point:

$$\frac{\partial e}{\partial t} + \frac{\partial \Pi}{\partial x} = 0 . \tag{1.126}$$

Equation (1.126) is a classical example of conservation law. It links the time variation of a density (here e) to the spatial variation of a flow (here Π). This equation shows that wave propagation corresponds to a continuous energy transfer from one point to another in the medium.

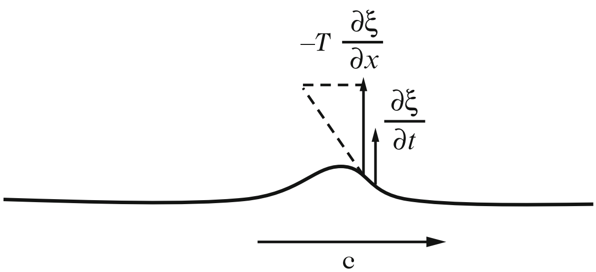


Fig. 1.14 Energy transfer on a string

1.3.2 Example of Linear Acoustic Waves

In linear acoustics, writing an equation for energy as a corollary of linearized equations is questionable, since the goal is to calculate quantities of order 2. However, it can be shown from the non-linearized equations that the following result is correct. Using (1.109) and (1.110), the quantity div($p\mathbf{v}$), can be calculated. Since div($p\mathbf{v}$) $= p$div($\mathbf{v}$) $+ \mathbf{v}.\mathbf{grad}p$, we obtain

$$\text{div}\,\mathbf{I} = -\frac{\partial}{\partial t}[E] + pq + \rho\mathbf{v}.\mathbf{F}, \tag{1.127}$$

$$\text{where } \mathbf{I} = p\mathbf{v} \;\; ; \;\; E = \frac{1}{2}\left[\chi_s p^2 + \rho\mathbf{v}.\mathbf{v}\right]. \tag{1.128}$$

The quantity E is the total energy per unit volume: it can be shown that $E_p = \frac{1}{2}\chi_s p^2$ is the potential energy density, and $E_c = \frac{1}{2}\rho\mathbf{v}.\mathbf{v}$ the kinetic energy density. The vector $\mathbf{I} = p\mathbf{v}$ is *the acoustic intensity*. It is connected to the power per unit area by $dP = \mathbf{I}.\mathbf{n}dS$, where $\mathbf{n}$ is the unit vector in the velocity's direction.

By integrating on a volume V and using the divergence theorem, Eq. (1.127) can be interpreted as follows: the power $\iint_S p\mathbf{v}.\mathbf{dS}$ going out the volume V through the surface S is equal to the total energy decrease in the volume added to the power supplied by sources. In periodic regime, averaged over a period, the term containing E vanishes (the average of the time derivative of a periodic quantity is zero), and in the absence of a source, the average outgoing power is zero. This is due to the fact that we assume the system to be conservative.

1.3.3 Power and Impedance

1.3.3.1 Instantaneous and Average Acoustic Power: Acoustic Impedance

With the previous definition of the acoustic power, we consider the *instantaneous* power through the surface of area S in harmonic regime: $\mathscr{P} = Sp(t)v(t) = p(t)u(t)$, where $u(t) = Sv(t)$ is the flow rate. We assume that the pressure and velocity are uniform on this surface, and that $p_c(t) = A \exp j(\omega t + \varphi)$. With the complex variables, we define also the *acoustic impedance* $Z = p_c/u_c$ and the *acoustic admittance* $Y = u_c/p_c$, thus and $u_c = Yp_c$. We get

$$u(t) = \Re e\left[u_c(t)\right] = \Re e\left[YAe^{j\omega t+\varphi}\right] = A\Re e(Y)\cos(\omega t+\varphi) - A\Im m(Y)\sin(\omega t+\varphi),$$

and the instantaneous acoustic power is written as follows:

$$\mathscr{P} = p(t)u(t) = \mathscr{P}_m + \frac{1}{2}A^2\left[\Re e(Y)\cos 2(\omega t+\varphi) - \Im m(Y)\sin 2(\omega t+\varphi)\right]. \tag{1.129}$$

The term $\mathscr{P}_m = \frac{1}{2}A^2\Re e(Y)$ is the power averaged over a period. The second term, of zero mean value, is called "fluctuating power," as the system restores during a half-period the energy it has received in the previous half-period.[7]

In complex notation (see Sect. 1.2.5), the calculation of quadratic quantities turns out to be rather tricky. However, in practice, as previously, we often calculate the average values of some variables over a period T, only. The general result for two variables p_1 and p_2 is

$$\frac{1}{T}\int_0^T p_1(t)p_2(t)dt = \frac{1}{2}\Re e\left[p_{c1}(t)p_{c2}^*(t)\right] = \frac{1}{4}\left[p_{c1}(t)p_{c2}^*(t) + p_{c1}^*(t)p_{c2}(t)\right],$$

where $*$ is the conjugate quantity. With these expressions, we can write the most useful formulas:

$$\mathscr{P}_m = \frac{1}{2}\Re e(p_c u_c^*) = \frac{1}{2}\left|u_c\right|^2 \Re e(Z) = \frac{1}{2}\left|p_c\right|^2 \Re e(Y) \tag{1.131}$$

1.3.3.2 Power Supplied to a Passive System

- The power is either provided to a system, or provided by this system, depending on the sign of the real part of the impedance (resp. admittance). Consider a simple example in mechanics: a force **f** is applied to a passive system, it performs the mechanical work **f**.**x**, where **x** is the displacement of the system. The generated power, i.e., the work per unit time, is therefore **f**.**v**. This scalar product is equal to the provided power, which is also the power dissipated in the passive system, and it is, by definition, positive.
- By convention we define the impedance as the ratio of a quantity providing energy to a quantity characterizing a passive system (we can see that if we choose the reaction force of the passive system on the excitation, equal to −**f**, we would have a definition leading to a negative real part for the impedance). An example of such a system is a volume of air excited by a vibrating wall: the definition of the impedance is the ratio between the force applied by the wall on the air and the velocity (either of the wall or of air, because these velocities are identical for a perfectly reflecting wall). The opposite convention would be to use the force exerted by air on the wall. If the passive system is not dissipative, i.e., conservative, the impedance is purely imaginary.

[7] The average power $\mathscr{P}_m$ is also called *active* power. A "reactive" power is also defined by

$$\mathscr{P}_r = \frac{1}{2}\Im m(pv^*) = -\frac{1}{2}\left|p\right|^2 \Im m(Y) = \frac{1}{2}\left|v\right|^2 \Im m(Z). \tag{1.130}$$

We choose arbitrarily its sign, which will be positive or negative depending on whether the system is dominated by stiffness or mass (we do not go further here, because the chosen quantities are formal).

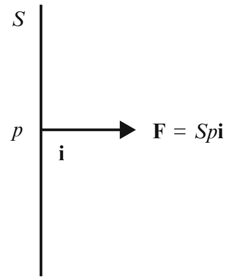


Fig. 1.15 Acoustic pressure force

- A special case is that of the force of acoustic pressure, because pressure is a scalar quantity. Let us consider a passive surface S on which the pressure p is applied (see Fig. 1.15). We choose, by convention, to define the specific impedance as p/v, where v is the projection of the velocity onto the normal to the surface in the direction of the force due to the pressure (and the acoustic impedance as $p/(Sv)$). We see that this convention is consistent with the previous choice. It is also the case for the input impedance of a wind instrument: as the resonator is passive, if a pressure is applied at its input, the input impedance of the resonator is always defined by choosing the velocity projection along the axis directed towards the exit of the pipe.

1.3.3.3 Standing Waves

In sinusoidal regime, for the complex quantity corresponding to a standing wave, we can write: $p_c = f(\mathbf{r}) \exp(j\omega t + \varphi)$, where φ does not depend on the spatial dimension. Using the Euler equation (1.110), we derive that the velocity is in phase quadrature with the pressure, since $\partial/\partial t = j\omega$. Therefore, the impedance is purely imaginary for the three velocity components, thus the admittance vector $\mathbf{Y} = \mathbf{v}/p$ is purely imaginary, and the average acoustic intensity over a period is zero in all directions. It is noticeable that standing waves do not carry any energy averaged over a period. It is (almost) the case for the oscillation of the air column of a wind instrument or, similarly, for the vibration of a string.

1.4 Sources in Musical Acoustics: Excitation Mechanisms

In the previous sections, a number of differential equations were written to describe the structures used in musical instruments. To use them, we must know the sources, which can be introduced either in the differential equation itself, such as for Eq. (1.9) or Eq. (1.111), or in the boundary and initial conditions. In this section, some type of sources found in musical acoustics are presented.

1.4.1 Generalities About Sources and Types of Oscillations

To build a model, we need to consider that one or more physical quantities are imposed in a region of space and time, and that they are insensitive to the medium in which they are imposed. These quantities play the role of sources, or generators. Thus in a linear circuit, it is often assumed that one can impose a voltage (possibly with an internal impedance, according to Thevenin and Norton's theorems). All quantities in the circuit are proportional to the magnitude of this source. The power supplied to the circuit depends on the circuit, i.e., on the impedance viewed at the source. Thus a source term not only implies a notion of imposed magnitude, but also a notion of supplied power.

In musical acoustics, we have to consider sources that produce oscillations and, in turn, sounds. Different types of oscillations can be encountered:

- The oscillations are *linear* if the result is proportional to the cause, or, in case of multiple causes, if the result is a linear combination of these causes (superposition principle). If a source is sinusoidal, a result proportional to the cause is also sinusoidal with the same frequency. Otherwise, the oscillations are nonlinear.
- Oscillations are *free* after extinction of the sources, and are *forced* during application of the sources.

Musical instruments enter into two main categories:

- Instruments with a *transient* excitation, followed by free oscillations. In this case, the sound usually lasts longer than the excitation: this is the case for percussion and string instruments, except for bowed strings. For these instruments, the free oscillations can be either linear or nonlinear (piano, timpani, cymbals,…). The excitation is produced by means of an impact (hammer, stick, and mallet), or a pluck (plectrum and finger).
- Instruments with a *continuous* excitation, which is often constant or slowly varying. Oscillations arising from a constant or slowly varying excitation are necessarily nonlinear, and are called "self-sustained oscillations." This is the case for bowed string instruments excited by a continuous bow-string friction process, and for all wind instruments, where the excitation is the result of jet–edge interaction (flute-like instruments) or air–structure interaction (vocal folds, lips, and reed).

Some constitutive parts of an instrument can be regarded as steady-state oscillating sources generating *forced* oscillations: a string fixed at the bridge of a soundboard (or soundbox), for example, acts as a source of oscillations for the board. Such a source generally has a low internal impedance, which means that the string transmits to the board the *force* developed at the point of coupling, almost entirely. Conversely, the soundboard (or soundbox) is an oscillating source that induces forced oscillations to the surrounding air. This source is generally of high internal impedance: the board transmits its *velocity* to the ambient fluid.

For wind instruments, the air in the pipe plays the same role for the external air as the soundboard. It can be assumed, especially at low frequencies, that the flow produced at the holes is not affected by radiation.

For most models expressed in the form of differential equations, such as acoustic waves, source terms appear in the right-hand side of the conservation equations. In some cases, however, the sources are included in the boundary conditions, or even in initial conditions. In a number of cases, both formulations are equivalent. A simple example is the one of a plucked string released from its initial position at the origin of time: the problem can be either treated as an initial value problem, or as a problem with a second member that contains a plucking force (see Chap. 3).

1.4.2 Acoustic Sources

1.4.2.1 Flow Source

Equations (1.109) and (1.110) show two types of acoustic sources, referred to as sources of type 1 (flow sources) and type 2 (force sources). We will reexamine these concepts in Chap. 12 devoted to radiation in free space. Let us now illustrate Eq. (1.109) with the example of a vibrating body that imposes its velocity to the fluid (at rest). One can imagine the membrane of a small loudspeaker, with displacement $\xi(t)$, acting at a given point **a** in space. This loudspeaker is assumed to be a point-source, which means in practice that its dimensions are much smaller than the wavelength (see the following Sect. 1.5). The speaker is located in an enclosure, to avoid a short-circuit between both sides of the membrane. In three dimensions, the speaker can be viewed as a "pulsating" sphere, radiating uniformly in all directions (see Fig. 1.16).

This source, which is well-known in acoustics, is obviously an idealized object, only conceivable in "thought experiments." The displacement per unit volume is $\xi(t)\delta(\mathbf{r}-\mathbf{a})$, where δ is the Dirac delta function (integrating this quantity on any volume yields $\xi(t)$). Consequently, the velocity per unit volume is $\delta(\mathbf{r}-\mathbf{a})\partial[\xi(t)]/\partial t$,

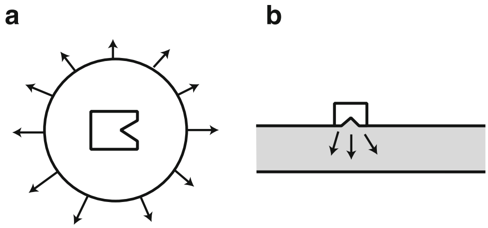


Fig. 1.16 "Punctual" speaker, located in an enclosed space, radiating: (**a**) in free space; (**b**) in a pipe

and the flow per unit volume is $q = S_m \delta(\mathbf{r}-\mathbf{a})\partial\left[\xi(t)\right]/\partial t$, where S_m is the membrane area.[8]

- Let us now turn back to wind instruments, assuming a cylindrical cross-section. The elementary acoustic solutions are plane waves. We put the little speaker on the side of the pipe (Fig. 1.16), and we study the case of a sinusoidal excitation with angular frequency ω. A flow $u_s(t) = S_m\, \partial\left[\xi(t)\right]/\partial t = S_m j\omega\xi$ is produced, at point $x = a$. For a pipe of cross-section S, the projection of the velocity vectors on the x axis yields

$$u_s = S\left[v_x(x+dx) - v_x(x)\right] = S\left[v_x(a^+) - v_x(a^-)\right]. \tag{1.132}$$

It is assumed further that the speaker does not disturb the pressure field, which remains plane and continuous at the position of the speaker (this question will be discussed in more details in Chap. 7 with regard to the effects of side holes). The law of dynamics: $dp/dx = -j\omega\rho v_x$ (1.110) applies here. Therefore, the pressure p is continuous at position $x = a$ whereas its derivative is discontinuous. This gives for the wave equation:

$$\frac{d^2p}{dx^2} + \frac{\omega^2}{c^2}p = -\frac{j\omega\rho}{S}u_s\delta(x-a). \tag{1.133}$$

(We can verify this result by integrating this equation between a^- and a^+: it is the discontinuity of the derivative that brings the Dirac delta function on the right-hand side).

- If we now return to an arbitrary dependence in time $u_s(t)$, we obtain

$$\frac{\partial^2 p}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = -\frac{\rho}{S}\frac{du_s(t)}{dt}\delta(x-a). \tag{1.134}$$

This is consistent with what was obtained in three dimensions. The presence of the cross-section S here is a consequence of the unidimensional character of the Dirac delta function which is inversely proportional to a length, and not to a volume. We can imagine a practical illustration by considering the key of an instrument with side holes that are instantaneously closed. The flow is almost a

[8]Assuming a given function of time for the displacement, the source term in Eq. (1.111) is entirely known. A "realistic" simple function is, for example, $hH(t)$, where h is the amplitude and $H(t)$ the unit step function (or Heaviside step function). In practice, this means that the membrane is suddenly moved, then blocked. In this case, the source term in (1.111) becomes $-\rho h S_m \delta(\mathbf{r}-\mathbf{a})\partial\left[\delta(t)\right]/\partial t$, since $\delta(t)$ is the derivative of $H(t)$. In the next chapters, a particular case of elementary source called Green's function, where the source is written $\delta(\mathbf{r}-\mathbf{a})\delta(t-t_0)$, will be examined in details. To achieve it in our "thought experiment," the velocity should be a step function, and therefore, the displacement should increase indefinitely, which is not realistic! Another way to obtain this Green's function is to write the acoustic wave equation in terms of velocity potential [see Eq. (1.105)]. In this latter case, the source term becomes $hS_m\delta(\mathbf{r}-\mathbf{a})\delta(t)$.

pulse, but as the closure is not instantaneous, the pulse is not perfect. However, we clearly hear a sound with definite pitch.

Since we impose a displacement, we can also consider a problem without a source, but rather with imposed boundary conditions.[9] Several problems exist with imposed flow, or rather with flow function $u_s = F(p)$ of the pressure, particularly for reed instruments. At the origin of the transients, this function is linear, and we can write $u_s = F_0 + Ap$. Since we are interested in the derivative only, the source term is written $-\frac{\rho}{S}Ap\delta(x-a)$. As the oscillation starts, growing exponentially, the source must provide energy, and the coefficient A must be negative. Otherwise, it would not be a source, but a dissipating system.

1.4.2.2 Relation Between Applied Force and Acoustic Force Strength

What happens if the membrane of the speaker, which is now supposed to be free on both sides, is set perpendicular to the pipe, thus preventing any continuous flow? It exerts a force **f** on the fluid, which has to be balanced by the pressure. By projecting this force on the x-axis, we obtain

$$f + S\left[p(a^-) - p(a^+)\right] = 0\,.$$

With an imposed force, we obtain a source of pressure difference and not the difference of its derivative. We can write an expression similar to (1.134), by interchanging the roles of pressure and velocity, i.e., by using the conservation of mass instead of Euler equation:

$$\frac{\partial^2 v_x}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 v_x}{\partial t^2} = -\frac{\chi_S}{S}\frac{df(t)}{dt}\delta(x-a).$$

Taking the derivative of both terms with respect to x, and integrating with respect to time, we get

$$\frac{\partial^2 p}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = \frac{1}{S}f(t)\frac{d}{dx}\delta(x-a). \tag{1.135}$$

In Chap. 10, it will be shown that the production of sound in flute-like instruments can be represented by such an aeroacoustical force strength.

[9] A wave equation or a boundary condition including a source is called *heterogeneous*. It can be shown that it is always possible to transform a heterogeneous boundary condition into a homogeneous one by changing the wave equation.

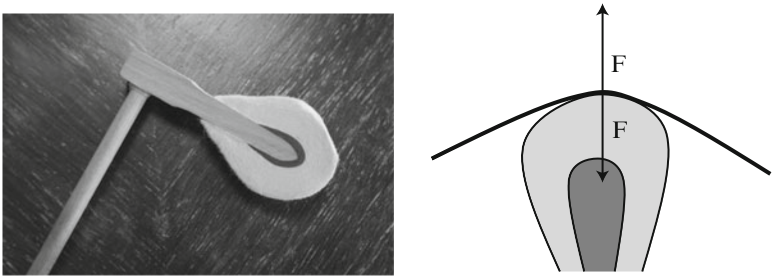


Fig. 1.17 (*Left*) A piano hammer (© Itemm). (*Right*) Interaction between an exciter and the vibrating system. Example of the piano hammer. As a result of the motion of the key pressed by the pianist, the hammer strikes the string(s) and transmits a force F that depends on time and impact velocity. According to the principle of action and reaction, the string exerts an equal and opposite force that leads to push the hammer back after an interaction time of a few milliseconds

1.4.3 Transient Mechanical Excitation

We describe here the transient vibrations of strings and percussion instruments subjected to impact or friction (plucking). During such transients, the energy of the exciter is transmitted to the vibrating system during a finite duration. In this time interval, the interaction between the exciter and the system can be rather complex and it is generally not possible to ignore the reaction of the structure on the exciter. In the case of the piano, for example, the impact force is not imposed: it is the result of the temporal evolution of the strings in contact with the hammer (see Fig. 1.17).

1.4.3.1 Friction and Plucking

Transient excitation by friction, or plucking, occurs in plucked string instruments such as guitar, lute, harp or harpsichord. In most works, the plucking action is simply viewed as an initial condition for the displacement of the string (see [45]). This model provides a first approximation of the spectral content of the free vibration of the string. However, it does not account for the interaction with the exciter or the player. This initial stage is essential since it contributes to determine the timbre of the produced sound. Auditory experiments performed with recorded sounds where the initial transients are truncated show that the listeners are not able to recognize the instruments anymore. Some elements to consider for a better physical description of the plucking are given below.

- The string is moved from its initial position by a force localized on a small portion of the string, that we can write as $F(\mathbf{x}, t) = F(t)\delta(\mathbf{x}-\mathbf{x_o})$. As long as the frictional force exerted by the finger (or plectrum) on the string remains below a given threshold F_M, it stays stuck to the exciter: this corresponds to the *stick* phase. The amplitude of F then continues to increase and is balanced by the restoring force resulting from the angle formed by the two sides of the string on both sides of the exciter. During this phase, the motion of the string might contain a torsional component.

- When the restoring force reaches the threshold F_M, then the string slides under the finger and begins to produce free oscillations. During this *slip* phase, which is relatively short compared to the stick phase, the finger (or plectrum) is likely to introduce a damping which decreases as the relative velocity between exciter and string at the contact point increases. We will find again such a succession of stick and slip phases in the mechanics of the bowed string (see Chap. 11). In the latter case, the essential difference follows from the fact that such transitions occur repeatedly with a cadence that gradually synchronizes with the oscillation of the string.

A detailed description of the excitation of a guitar string by the friction of the finger was done by Pavlidou [43]. More recently, a similar model has been developed for a plucked harp string [14, 15, 34]. We briefly recall here some principles of the Pavlidou model. The two transverse polarizations and torsional waves are taken into account on the string. We also consider the motion of the bridge at one end. The tension T of the string is assumed to be constant during the motion. The finger model first includes a muscle, represented by a nonlinear spring with a spring force $S(\xi)$ which is a hyperbolic function of its elongation ξ (see Fig. 1.18) written as:

$$S(\xi) = \begin{cases} \dfrac{\sigma_1 \xi}{\sigma_2 - \xi} & \text{for} \quad \sigma_2 > \xi \geq 0, \\[2ex] \dfrac{\sigma_1 \xi}{\sigma_2 + \xi} & \text{for} \quad -\sigma_2 < \xi < 0, \end{cases} \tag{1.136}$$

where σ_1 and σ_2 are constants derived from experimental measurements. The model also includes the upper part of the finger, considered as a lever arm with a speed imposed by the guitarist, and the nail, circular, in direct contact with the string. The interaction with the pulp of the finger is not considered here (see Fig. 1.19).

During the three phases of the motion, the model is obtained by considering:

1. **During the stick phase**: (a) the translational motion of the string element interacting with the exciter, (b) the rotational motion of the fingertip, (c) the rotational motion of the string element, and (d) the relative velocity between string and finger.
2. **During the slip phase**: the friction coefficient μ depends on the relative velocity V_{rel} between string and nail. Typically, such a function is of the form (see Fig. 1.20):

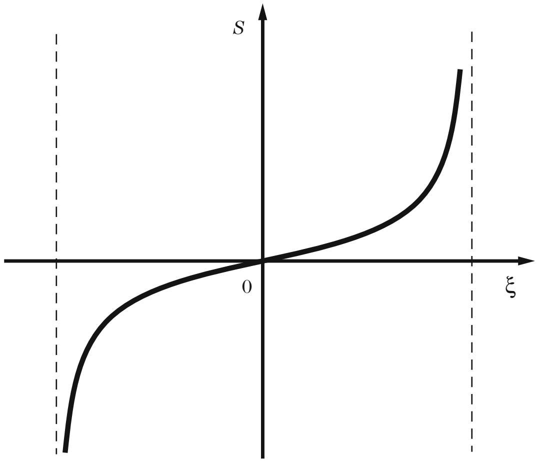


Fig. 1.18 Force-elongation diagram of a muscle (from Pavlidou [43])

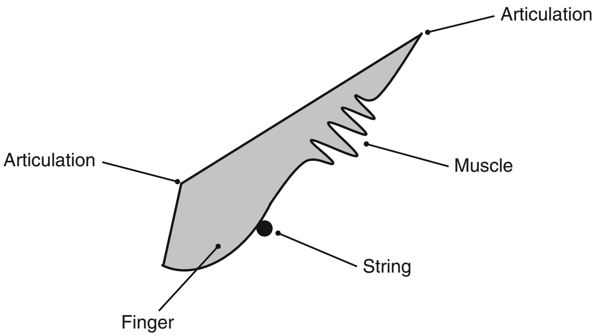


Fig. 1.19 Finger model (from Pavlidou [43]). This diagram shows the two last phalanges of the finger. The last one is in direct contact with the string, and its motion is guided by both the articulation and muscle that connect it to the upper phalanx

$$\mu = \begin{cases} \dfrac{(-V_0\mu_s - \mu_d V_{\text{rel}})}{V_{\text{rel}} + V_0} & \text{for} \quad V_{\text{rel}} > 0, \\ \dfrac{(V_0\mu_s - \mu_d V_{\text{rel}})}{V_0 - V_{\text{rel}}} & \text{for} \quad V_{\text{rel}} < 0\,. \end{cases} \tag{1.137}$$

In this equation μ_s is the static friction coefficient, μ_d the dynamic friction coefficient, and V_0 the initial velocity of the finger impacting the string. A similar model accounts for the friction of the rosin on a violin bow (see, for example, [50], and Chap. 11 of this book). For more information on friction models, the reader can refer to [1].

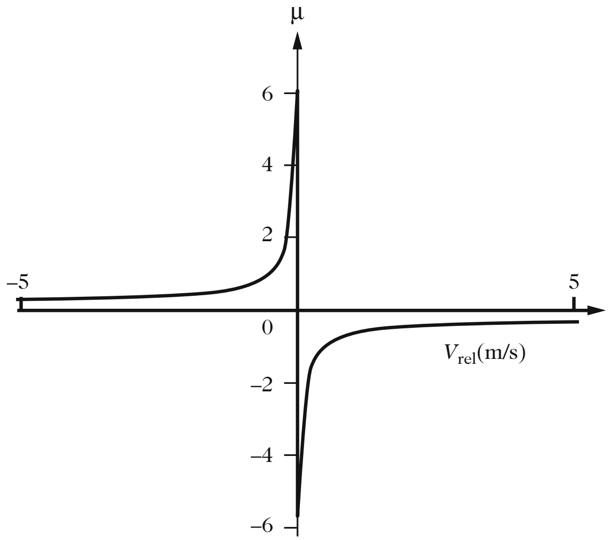
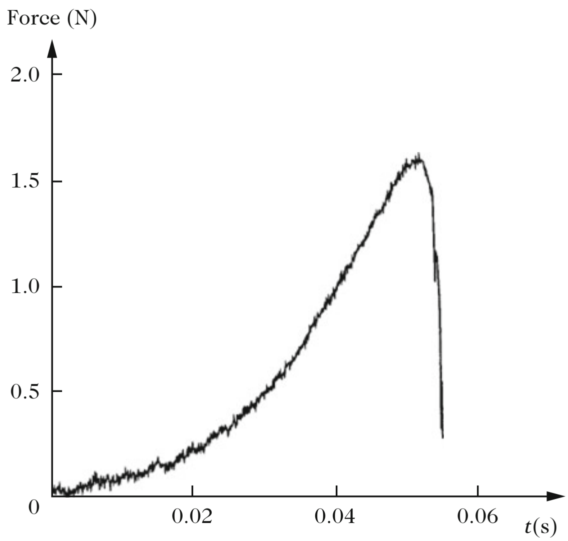


Fig. 1.20 Friction model during the slip phase

Fig. 1.21 String–plectrum interaction force of a harpsichord note



3. **During the free oscillations**: the motion of the string is completely defined by three equations (two transverse polarizations and one torsional oscillation) with initial conditions obtained from the equations of the stick phase.

Giordano and Winans measured the string–plectrum interaction force for a harpsichord string (see Fig. 1.21). They showed a gradual increase of the force during the stick phase followed by a rapid decrease during the slip phase [24].

For plucked strings, there is an average rise time of about 20 ms followed by a rapid decrease (<1 ms), after which the free oscillation starts. Systematic variations of interaction parameters show that the sound quality primarily depends on the following properties:

- The characteristics of the finger-string friction, mostly during the slip phase. This mainly affects the relaxation phase.
- The elastic properties of the finger muscle.
- The input admittance of the string at the bridge. This admittance affects the transmission rate of energy from string to soundboard. The reaction force exerted by the string on the exciter depends on this rate: the player says that he "feels" his instrument under his finger.
- The initial direction of the finger motion (angle of attack). This parameter affects the initial polarizations of the string that are coupled at the bridge [32].

Finally, the plucking velocity primarily affects the amplitude of vibration (and sound), and weakly the timbre, as long as the assumption of linear vibrations of the string is valid.

1.4.3.2 Elastic Hertzian Impact

Impact excitation concerns the piano and almost all percussion instruments. Musical experience shows that the sound produced depends, among other things, on the properties of the exciter: the thickness of the felt varies among the piano hammers, and timpani mallets show a large variety of rigidity (see Fig. 1.22). The head of xylophone mallets also differs from each other in terms of weight and stiffness. To be convinced of the relevance of exciter properties, just look at percussionists in an orchestra: they change their sticks and mallets several times during a performance.

Fig. 1.22 Examples of kettledrum mallets. They differ from each other through the stiffness of the felt and the elasticity of the stick

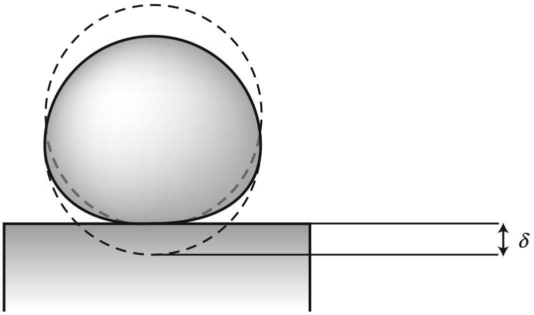


Fig. 1.23 Contact between an elastic sphere and an infinite rigid plane. The quantity δ indicates the *compression* of the sphere, which represents its change in thickness consecutive to the impact

During the impact, an interaction force is generated between the impactor and the struck structure, as a result of the deformation of both elastic solids in contact. Historically, the first theory of contact between two semi-infinite elastic solids is due to Hertz and was published in 1882 [29]. This theory predicts, in particular, the stress distribution in the contact area. One of the most famous result of this theory is the expression of the interaction force F:

$$F = K\delta^{3/2}, \tag{1.138}$$

where δ is the *compression* or, in other words, the summation of strains on both surfaces (see Fig. 1.23), and K is a constant which depends on both the curvature and elastic coefficients of the solids. This constant is given by [33]:

$$\frac{1}{K} = \frac{3}{4}\left[\frac{1-\nu_1^2}{E_1} + \frac{1-\nu_2^2}{E_2}\right]\sqrt{\frac{1}{R_1} + \frac{1}{R_2}} = \frac{3}{4}\frac{1}{E_{eq}}\frac{1}{\sqrt{R_{eq}}}, \tag{1.139}$$

where E_1 and E_2 are the Young's moduli, ν_1 and ν_2 the Poisson's ratios, R_1 and R_2 the radii of curvature of the solids at the contact point. The quantities E_{eq} and R_{eq} are equivalent Young's modulus and radius, respectively, often used to simplify the formula.

The formula (1.139) can be applied to the case of a sphere impacting an infinite rigid plane, as $R_2 \to \infty$. If, in addition, $E_1 \gg E_2$, which means that the impactor is significantly more rigid than the impacted surface. As a consequence, the coefficient K becomes

$$K = \frac{4}{3}\frac{E_2}{1-\nu_2^2}\sqrt{R_1} \quad \text{in} \quad \mathrm{N\,m^{-3/2}}\,. \tag{1.140}$$

In other words, the softer solid imposes the main properties of the impact.

- Hertz's contact theory remains valid as long as the dimensions of the contact area remain small compared to both the dimensions of the solids and radii of curvature. It does not predict good results for head materials subjected to large deformations as, for example, rubber (see [16]). Hertz's theory also ignores the effects of inertia and elastic waves in the media in contact. However, if the contact time is small compared to the period of the studied phenomena, this theory reasonably accounts for experimental observations, which explains its wide use.

Pulse Duration and Maximum Impact Force

Based on Hertz's law, an estimation for both the pulse duration and maximum of the impact force can be derived. If V_0 is the initial velocity of the impactor on the solid surface at rest, and denoting $m_r = m_1 m_2/(m_1 + m_2)$ the reduced mass of the two solids, the conservation of the total energy of the system (without dissipation) is written [33]:

$$\frac{1}{2} m_r \dot{\delta}^2 + \frac{2}{5} K \delta^{5/2} = \frac{1}{2} m_r V_0^2 \, . \tag{1.141}$$

The maximum of the compression is obtained when $\dot{\delta} = 0$, which provides

$$\delta_{\text{Max}} = \left(\frac{5 m_r}{4K} \right)^{2/5} V_0^{4/5} \, . \tag{1.142}$$

The duration τ of the force impulse is obtained by integrating (1.141):

$$\tau = 2 \int_0^{\delta_{\text{Max}}} \frac{d\delta}{\sqrt{V_0^2 - \frac{4K}{5m_r} \delta^{5/2}}} = 2 \left(\frac{25 m_r^2}{16 K^2 V_0} \right)^{1/5} \int_0^1 \frac{d\xi}{\sqrt{1 - \xi^{5/2}}}, \tag{1.143}$$

which yields finally:

$$\tau = 3.218 \left(\frac{m_r^2}{K^2 V_0} \right)^{1/5} \, . \tag{1.144}$$

Equation (1.144) shows, in particular, that the pulse duration only weakly depends on the impact velocity. This result is in agreement with measurements made on a large number of mallets [11]. From an experimental point of view, the constant can be derived K from measurements of the maximum impact force and pulse width using Eqs. (1.138), (1.142), and (1.144). We find

$$K = 35.4 \frac{1}{\tau^3} \sqrt{\frac{m_r^3}{F_{\text{Max}}}} \, . \tag{1.145}$$

1.4.3.3 Empirical Generalization of Hertz's Law

Since Ghosh in 1927 [22], several authors proposed a generalization of Hertz's law for piano hammers, on the form:

$$F = K\delta^p, \tag{1.146}$$

where the superscript p is between 2.0 and 4.0 approximately [26]. This expression has been also used for modeling the impact of timpani mallets [46]. The power law (1.146) is essentially empirical and is not based on an accurate analysis of stress and strain, in contrast with Hertz's law. This expression fairly accounts for the compression of the felt (a porous material) wrapped around the wooden tip. It offers the practical advantage to identify experimental force-compression laws with two parameters only.

1.4.3.4 Impact with Dissipation

In practice, the contact force is not purely elastic. Due to their rheological properties, hammer and mallet materials are subjected to internal dissipation. Moreover, impacts can be strong and lead to additional dissipation due to plastic deformation. We can, for example, easily observe proofs of impacts on wooden xylophone bars.

Viscous Dissipation

N.B. In this section, plastic deformation is ignored.

As a consequence of viscous dissipation, the force-deformation curve $F(\delta)$ shows a hysteresis loop (see Fig. 1.24). This loop is due to the viscoelasticity of the material, a "memory effect" which produces a relaxation (decrease) of stress over time after application of strain. The first attempt to extend Hertz's law to viscoelastic media was made by Pao [42]. His theory leads to a modified expression of the form:

$$F = F_0 \left[\delta^{3/2} - \int_0^t \Psi(\xi - t)\delta^{3/2}(\xi) d\xi \right], \tag{1.147}$$

where $\Psi(t - \xi)$ is a relaxation function which can be represented by a sum of exponentials. This expression was revisited by Stulov [53] for piano hammers:

$$F = F_0 \left[\delta^p - \frac{\varepsilon}{\tau_o} \int_0^t \exp\left(\frac{\xi - t}{\tau_o} \right) \delta^p(\xi) d\xi \right], \tag{1.148}$$

where ε is a dimensionless coefficient that reflects the hysteresis area, i.e., the energy lost per cycle, and where τ_o is the relaxation time. In the case of piano hammer felt, τ_o is approximately 1–2 ms.

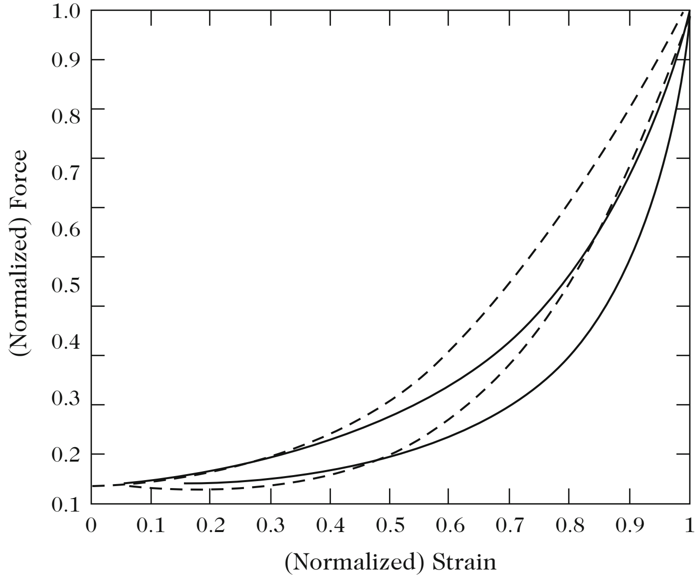


Fig. 1.24 Force-compression curve for a dissipative mallet with hysteresis loop. *Solid line*: experimental curve; *dotted line*: differential model (1.149)

For piano and percussion instruments, the contact pressure on the felt is applied relatively slowly, since the impact velocities do not exceed 5 m/s. This allows writing $F \gg \tau_o \frac{dF}{dt}$. Calculating $\frac{dF}{dt}$ from (1.148), we find

$$F_{tot} \simeq F + \tau_o \frac{dF}{dt} = K \left[\delta^p + R \frac{d\delta^p}{dt} \right], \tag{1.149}$$

where $K = F_0(1 - \varepsilon)$ is the stiffness coefficient and $R = \frac{\tau_o}{1-\varepsilon}$ the coefficient of viscous dissipation. The differential formulation (1.149) has been used in models of piano and drums [31]. An estimation for the coefficient R can be obtained by energetic considerations [17]. This differential expression is simpler to use than the integral formulation (1.148), though it remains valid for medium or low impact velocities only. Figure 1.24 shows a comparison between an experimental curve and a differential model of type (1.149) for a timpani mallet.

Plastic Strain

By definition, a plastic solid shows stable residual strains, after cessation of the excitation. This behavior does not depend explicitly on time (see, for example, [38]). For an elastic, *perfectly plastic* solid, strain ε is linear (and characterized by a Young's modulus E) below the yield stress (or threshold) of plasticity σ_s.

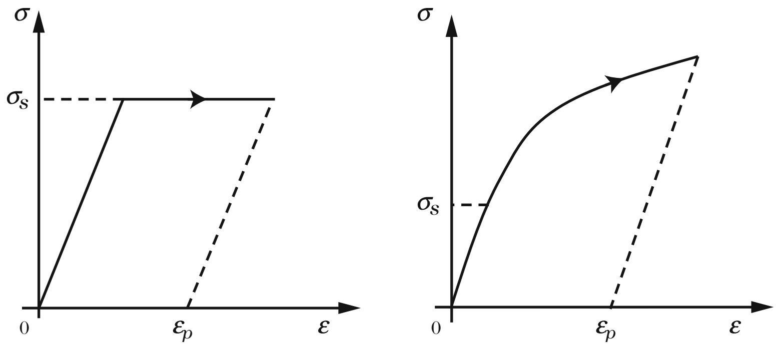


Fig. 1.25 (*Left*) Behavior law for an elastic perfectly plastic solid. (*Right*) Behavior law for an elastoplastic solid

The stress then remains constant as ε increases above σ_s/E. As the stress then decreases, another curve is drawn, distinct from the linear part, showing a permanent deformation ε_p at $\sigma = 0$ (see Fig. 1.25).

In the case of an *elastoplastic* solid, the stress continues to increase beyond the threshold of plasticity σ_s, generally in a nonlinear manner with respect to ε. Again, we note the existence of a permanent deformation after cessation of the loading (see Fig. 1.25).

In the case of contact between two spheres, Johnson showed that the impact becomes plastic when the average pressure between the two solids is about $p_m = 1.1\sigma_s$ [30]. From this property, this author deduced that the velocity of contact above which plastic deformations are likely to occur is about $V_l = 0.14\,\mathrm{m\,s^{-1}}$ for a steel of medium hardness and $V_l < 0.08\,\mathrm{m\,s^{-1}}$ for aluminum. The important point indicated by these values is that plasticity is present in most of the common impacts, even at low speed. For an impact between two solids made of the same material of density ρ, with a threshold of plasticity σ_s and for which the the relative velocity of impact is V, the following table, due to Johnson [30], provides a good order of magnitude:

- $\frac{\rho V^2}{\sigma_s} < 10^{-6}$ → elastic behavior.
- $10^{-6} < \frac{\rho V^2}{\sigma_s} < 10^{-3}$ → elastoplastic behavior
- $10^{-3} < \frac{\rho V^2}{\sigma_s} < 10^{-1}$ → perfectly plastic behavior.

One effect of plasticity is that the force-compression curve again shows a hysteresis loop (see Fig. 1.26).

This curve shows a maximum at point A (of coordinates F_c, δ_c). We see that there is a non-zero residual deformation δ_p as the interaction force equals zero, corresponding to the situation where the two solids move away from each other (point B on the curve). As a consequence, the energy restored during the decrease of the force, which corresponds to the area below the curve AB, is less than the

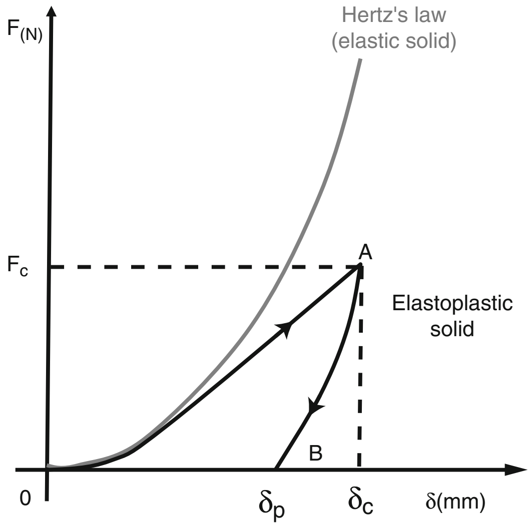


Fig. 1.26 Force-compression curve for an impact between two elastoplastic solids. From [54]

energy stored during the impact, corresponding to the area below the curve OA. The *restitution coefficient* is the ratio between these two energies. It is less than unity for a plastic impact.

Several formulations were proposed to extend Hertz's law to elastoplastic or perfectly plastic case. Stronge [52] suggests to model the loading curve OA using Hertz's law, and the unloading curve AB with an equation of the form:

$$F = \frac{4}{3} E_{\text{eq}} \sqrt{R^*_{\text{eq}}} (\delta - \delta_p)^{3/2}. \tag{1.150}$$

where R^*_{eq} is a curvature radius greater than R_{eq}, due to plastic deformation.

Vu-Quoc and Zhang developed a numerical model of impact between two elastoplastic spheres whose central idea is based on the decomposition of the contact radius into an elastic part and a plastic part [54]. The detailed presentation of this theory is beyond the scope of the present book. The results show that this model is able to predict the variations of the coefficient of restitution with the impact velocity accurately. This represents a significant advance over previous models in the sense that most of the parameters of this model can be directly related to material and geometric properties of the solids. Impact modeling still remains an open field of study, especially in the field of granular media.

1.5 Lumped Elements; Helmholtz Resonator

Having established the differential equations and discussed about the sources, we study a very particular case, usually encountered at low frequencies. Under certain conditions, the acoustic wave equation can be simplified. Leaving aside the sources in (1.111), there are conditions for which the term containing the time derivative (of order two) of the pressure is very small compared to that containing the spatial derivative. To establish these equations we make a dimensional analysis, assuming that there is a length L and a time τ by which we make dimensionless distance and time. If we note these quantities $\bar{x}$, $\bar{y}$, $\bar{z}$, $\bar{t}$, we get

$$\frac{\partial^2 p}{\partial \bar{x}^2} + \frac{\partial^2 p}{\partial \bar{y}^2} + \frac{\partial^2 p}{\partial \bar{z}^2} = (\mathrm{He})^2 \frac{\partial^2 p}{\partial \bar{t}^2},$$

where $\mathrm{He} = \frac{L}{c\tau}$ is the Helmholtz number, which is mostly written with an angular frequency $\omega = 1/\tau$:

$$\mathrm{He} = \frac{\omega L}{c} = kL = \frac{2\pi L}{\lambda}, \tag{1.151}$$

where λ is the wavelength. We see that at the limit of low frequencies, spatial variations become predominant and we can solve the Laplace equation $\Delta p = 0$, valid for an incompressible (not viscous) fluid. This occurs particularly near singularities such as the open end of a pipe, when considering only a "compact" area, i.e., an area of dimensions small compared to the wavelength c/ω: such a zone is called "lumped zone." In fact we can go a step further by examining the two first order equations involving pressure and velocity, (1.109) and (1.110).

- In Eq. (1.109) without right-hand side, the time-derivative term becomes very small under the following conditions: at low frequencies—in fact at low Helmholtz number, as it has been seen—or if the pressure is small, which also occurs near the end of a pipe, or if the compressibility is low (the fluid is close to incompressibility). Then, by integrating the term div$\mathbf{v}$ in the considered zone, we see that the flow rate is zero. Thus, in one dimension, the incoming flow rate is equal to the outgoing one, and it is the same for the velocity. The two equations can be reduced to an acoustic "Ohm's law" [10]:

$$v_x(x) = v_x(x + \delta x) = v \ \ ; \ \ p(x) - p(x + \delta x) = \rho(\partial v_x/\partial t)\, \delta x = \rho c(\partial v_x/\partial \bar{t})\mathrm{He}\ . \tag{1.152}$$

- In the second Eq. (1.110), the time-derivative term becomes very small under the following conditions: at low frequencies, if the velocity is small, or if the fluid is very light (small ρ). This occurs in particular near a rigid wall, where the velocity is zero, and, in turn, the term **grad**p is zero: the pressure is uniform in the

[10] The electroacoustic analogy called "acoustic impedance" associates acoustic pressure and velocity to electric voltage and current, respectively. Ohm's law states that between two points the difference in one of the quantities is proportional to the other quantity.

considered area, and using the divergence theorem, the total flow rate entering the zone of volume V is equal to $V\chi_S \partial p/\partial t$. In one dimension this gives the second Ohm's law[11]:

$$p(x) = p(x+\delta x) = p \ ; \ \ v_x(x) - v_x(x+\delta x) = \chi_S(\partial p/\partial t)\,\delta x = c\chi_S(\partial p/\partial \bar{t})\mathrm{He}. \tag{1.153}$$

- A physical system exists that combines these two effects: the Helmholtz resonator. It is made of a rigid cavity of volume V with a neck of length ℓ and section S (see Fig. 1.27), the neck being open onto a large space imposing a low pressure at the exit of the neck.

In order to model the resonator at wavelengths much greater than its dimensions, taking the sources into account, we have to consider the connection between the neck and the volume. Using again the divergence theorem, the flow rate u entering the cavity, where the pressure is uniform p_V, is written:

$$u = C_a \partial p_V/\partial t - u_{\text{source}}, \tag{1.154}$$

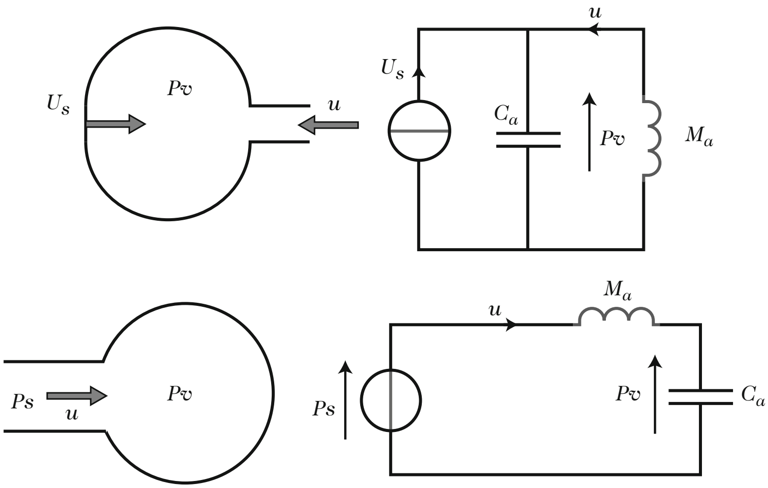


Fig. 1.27 Helmholtz resonator: its behavior is that of a system with localized constants at its resonance frequency. The volume of the cavity is V, the length and thickness of the neck are ℓ and S. We have: $C_a = \chi_S V$ and $M_a = \rho\ell/S$. The case shown on the *top* of the figure is that of a flow rate excitation [first Eq. (1.156)], the *lower one* corresponds to a pressure excitation [second Eq. (1.156)]

[11]We will encounter several times this concept of lumped-element systems. Notice that the finite difference calculation (Sect. 1.7.1) is based upon the division of a continuous system into lumped elements.

where $C_a = \chi_S V$ is the *acoustic compliance* and $u_{\text{source}} = \iiint_V q(\mathbf{r}, t)dV$ [see Eq. (1.109)]. In the neck, the flow entering from outside is equal to the flow rate u entering the cavity. As the transverse dimension of the neck is small compared to the wavelength, we assume that the pressure is uniform (i.e., plane) in a slice of fluid. We will see later (Chap. 7) how to connect the pressure at the exit of the neck with that of the cavity. For now, we assume that they are equal. Using the equation obtained above in one dimension, we derive

$$-p_V = M_a \frac{\partial u}{\partial t} - p_{\text{source}}. \tag{1.155}$$

where $M_a = \rho\ell/S$ is the *acoustic mass* and $p_{\text{source}} = p_e + \rho \int_{\text{neck}} F_x dx$. In this last expression the incoming pressure is added to the acoustic force strength.

- We will see that the correct matching leads to the same equation with a modified length ℓ (Chap. 7). We can then distinguish the two problems depending on whether the source is a source of flow in the cavity or a source of pressure in the neck, and write the second-order differential equations:

$$\begin{aligned} C_a \frac{\partial^2 p_V}{\partial t^2} + \frac{1}{M_a} p_V &= \frac{\partial u_{\text{source}}}{\partial t} \quad \text{and} \\ M_a \frac{\partial^2 u}{\partial t^2} + \frac{1}{C_a} u &= \frac{\partial p_{\text{source}}}{\partial t}. \end{aligned} \tag{1.156}$$

These equations are similar to those of electric circuits which are well known: if we choose the "acoustic impedance" analogy (pressure, flow, mass and compliance are analog of voltage, current, inductance, capacitance, respectively), the first equation is that of an antiresonant circuit (mass and compliance in parallel), while the second is that of a resonant circuit (mass and compliance in series). Of course, we expect to find in these equations a derivative of order 1, associated with damping. We will formally introduce damping in Chap. 2 to study the behavior of these equations. However in most situations, damping is frequency dependent, the modeling as a derivative of order 1 being a rough approximation.

Another way to model approximately an acoustic (or mechanical) system is to calculate a modal expansion, and to truncate the modal series to the first mode, as it is often done in this book.

1.6 Vibrating Strings-Sound Pipes Analogies

The analogy between longitudinal waves in solids and fluids is obviously very natural. It is so true that the term analogy can be discussed. However, it is not really useful in musical acoustics. As previously noticed, we are most often interested in transverse vibrations of solids, mainly in 2D, while for the fluid we are mainly concerned with 1D and 3D models. We therefore limit this section to the analogy between lumped elements ("0D") and the analogy between vibrating strings and sound pipes, considered as 1D systems. Notice that, in all cases, we choose a pair of main quantities whose product gives the power, and whose ratio gives the impedance. Thus *the mechanical impedance is the ratio force/velocity*, and the admittance (or mobility) the inverse ratio. For a fluid, several pairs may be chosen.

- Let us start by lumped elements: we consider a fluid element of surface S moving with a velocity v and subjected to a pressure force $f = Sp$, which are the projection of vectors on the same axis. If we consider these two quantities, the acoustic system appears to be a mechanical system. But this choice of basic quantities is not the most useful for a main reason: as seen in Sect. 1.5, at the junction between ducts, the conserved quantity is the flow rate $u = Sv$. This is similar to the forces at the junction of several elementary mechanical elements, such as springs, having an identical velocity. For joined ducts, we will show that, at the lowest frequencies, the pressure is uniform. An important example for wind instruments is the junction between the main pipe and the "chimney" of a tonehole. Flow conservation at a junction is a major reason for which the most used impedance for a fluid is the impedance called "acoustic impedance," based upon the equivalence between pressure p and velocity of a mechanical system v, and between acoustic flow rate $u = Sv$ and force f, respectively. We consider here scalar quantities obtained by projecting a vector on an axis. Thus the *acoustic impedance*[12] *is defined as the ratio pressure/flow rate* $Z = p/Sv$. Notice that the power through a section S of pipe is given by the product of these two quantities, whatever the chosen pair ($f = Sp, v$) or ($p, u = Sv$).
- This analogy is presented in Table 1.1 for vibrating strings and sound pipes in one dimension.[13] We call it "reverse" analogy, because it reverses potential and kinetic energies. However, it is the most useful in practice. It can be extended, in particular, to continuous sources of self-sustained oscillations: velocity for bowed strings, pressure for reed instruments. On the other hand, for flutes, because of the nature of the source of self-sustained oscillations, it is preferable to choose the direct analogy, where the pair (f, v) matches the pair (p, Sv). For this reason, Table 1.1 also mentions this analogy.

[12]Sometimes we will also use an acoustic impedance called *specific* defined by the ratio pressure/velocity (see Sect. 1.2.4). This choice is convenient for some problems of unbounded media, or for energy transmission between two media with different sound speed or density.

[13]This table has some specificity with regard to the dimensions. The quantity f_{ext}, for example, is a force per unit length, whereas the quantity F in Eq. (1.110) is a force per unit mass. In addition, the equation of vibrating strings is written in terms of velocity: this is rather unusual, but it allows to easily highlight some analogies. Finally, the wave equations are written here for a homogeneous medium, although we will have to deal with heterogeneous strings and horns, for which the analogies remain valid.

Table 1.1 Reverse and direct analogies for strings and pipes

	Strings	Pipes (reverse analogy)	Pipes (direct analogy)
Variable 1	$f(x,t) = -T\partial\xi/\partial x =$ force applied to the right of x by the tension T (towards the $x>0$)	$u =$ flow rate	$p =$ pressure
Variable 2	$v =$ velocity	$p =$ pressure	$u = Sv =$ flow rate
Equation 1	$\frac{\partial v}{\partial x} = -\frac{1}{T}\frac{\partial f}{\partial t}$	$\frac{\partial p}{\partial x} = -\frac{\rho}{S}\frac{\partial u}{\partial t}$	$\frac{\partial u}{\partial x} = -S\chi_S\frac{\partial p}{\partial t}$
Equation 2	$\frac{\partial f}{\partial x} = -\rho S\frac{\partial v}{\partial t} + f_{\text{ext}}$	$\frac{\partial u}{\partial x} = -S\chi_S\frac{\partial p}{\partial t} + q$	$\frac{\partial p}{\partial x} = -\frac{\rho}{S}\frac{\partial u}{\partial t} + \frac{1}{S}f_{\text{ext}}$
Source	$f_{\text{ext}} =$ external force per unit length	$q =$ flow rate per unit length	$f_{\text{ext}} =$ external force per unit length
Parameter 1	$T =$ tension	$S/\rho =$ cross-section/density	$1/\chi_s S =$ 1/(compressibility × section)
Parameter 2	$\rho S =$ density × cross-section	$\chi_S S =$ compressibility × cross-section	$\rho/S =$ density/cross-section
Element 1	$C =$ compliance $= 1/K$ ($K =$ stiffness)	$M_a = \rho\ell/S =$ acoustic mass	$C_a = \chi_S V =$ acoustic compliance
Element 2	$M = \rho\ell S =$ mass	$C_a = \chi_S V =$ acoustic compliance	$M_a = \rho\ell/S =$ acoustic mass
Wave speed	$c = \sqrt{T/\rho S}$	$c = 1/\sqrt{\rho\chi_s}$	$c = 1/\sqrt{\rho\chi_s}$
Ratio	$Y =$ mechanical admittance $= v/f$	$Z =$ acoustic impedance $= p/u$	$Y =$ acoustic admittance $= u/p$
Wave characteristic	$Y_c = \frac{1}{\sqrt{T\rho S}} = c/T$	$Z_c = \sqrt{\rho/\chi_S S^2} = \rho c/S$	$Y_c = \sqrt{S^2\chi_s/\rho} = S/\rho c$
Additional variable	$\xi =$ displacement $= \int v dt$	$\varphi =$ velocity potential $= -\int p/\rho dt$	volume displacement $= \int u dt$
Wave equation	$\rho S\frac{\partial^2 v}{\partial t^2} - T\frac{\partial^2 v}{\partial x^2} = \frac{\partial f_{\text{ext}}}{\partial t}$	$S\chi_S\frac{\partial^2 p}{\partial t^2} - \frac{S}{\rho}\frac{\partial^2 p}{\partial x^2} = \frac{\partial q}{\partial t}$	$\frac{\rho}{S}\frac{\partial^2 u}{\partial t^2} - \frac{1}{\chi_s S}\frac{\partial^2 u}{\partial x^2} = \frac{1}{S}\frac{\partial f_{\text{ext}}}{\partial t}$

1.6.1 Note on the Definition of Impedances for Forced Oscillations

A definition was given in Sect. 1.3.3.1 for the impedance in case of forced oscillations. This concept can be slightly extended, considering a simple situation with a point excitation of linear and sinusoidal forced oscillations, for example a source of acoustic flow, so that we can define:

- a *transfer impedance* as the ratio of the pressure response P at a point b to the flow source U provided at point a: $Z_t = P(b)/U(a)$;
- a *driving-point impedance* as the previous quantity for the particular case $a = b$: $Z = P(a)/U(a)$.

Of course, the inverse ratio is called admittance. When we use mechanical quantities, both are vectors, so that impedance and admittance become matrices. Similarly if there are multiple sources and multiple receivers, we can have an impedance matrix, and if there are continuous sources and receivers, we have an operator. These concepts are developed in Chap. 3 extensively.

Here we used the frequency domain. In the time domain, we would write equivalent equations, for example:

$$p(b) = \widehat{Z(t)} * u(a),$$

where $\widehat{Z(t)}$ is an impulse response, i.e., the inverse Fourier transform of the impedance. If the impedance is a pressure response to a unit flow, the corresponding impulse response is the pressure response to a flow pulse $\delta(t)$. In addition, there are other concepts of impedance:

- in the case of one-dimensional waves, *the local impedance*: we consider a pipe excited by several sources, and a point a downstream the sources. The part downstream of the point a is therefore passive. Then, the impedance $Z = P(a)/U(a)$ at point a is determined by the characteristics of the downstream medium, and has the same value as the impedance at the driving-point a. This is the typical case of the input impedance of a wind instrument, and, by extension, of the admittance matrix imposed by a soundboard or a sound box to the string. This concept can be generalized to one or more dimensions, but it will not be treated here.
- *the impedance of an element*: if between two points of a pipe a and b, the flow rate u is constant, the pipe element has an impedance $Z = [P(a) - P(b)]/U$ (we can define something similar if the pressure is constant, see Sect. 1.5). Thus, in the table of analogies 1.1, the impedances $Z = j\omega M$, $Z = 1/j\omega C, \ldots$, correspond to the impedances of a mass, the compliance of a spring, etc.

1.7 Numerical Methods

Analytical methods for solving the wave equation will be presented in Chaps. 3 and 4. As this equation becomes more complex through addition of extra terms (damping, stiffness, interaction with an exciter, etc.) which are needed to properly model an instrument, then the use of numerical methods becomes necessary.

A detailed description of the numerical techniques used for solving partial differential equations involved in models of musical instruments is beyond the scope of this book. However, we find it useful to say a few words about two of the most common techniques used, namely the finite difference and the finite element methods. To introduce and illustrate the foundations of these methods, the simple example of the ideal wave equation, without source terms, is selected. Emphasis is put on time-domain numerical modeling. For more information on the application of finite differences to simulations in musical acoustics, one can refer to the book by Bilbao [10].

1.7.1 Finite Difference Methods

In order to illustrate the use of finite difference methods we discuss the typical example of the wave equation where the initial conditions are given in explicit form. The objective is to solve the following system numerically:

$$\begin{cases} \dfrac{\partial^2 \xi}{\partial t^2} = c^2 \dfrac{\partial^2 \xi}{\partial x^2} \quad \forall x, \quad \forall t > 0, \\ \xi(x,0) = \xi_0(x) \quad ; \quad \dfrac{\partial \xi}{\partial t}(x,0) = \xi_1(x) \,. \end{cases} \tag{1.157}$$

Here, the variable $\xi(x,t)$ might designate, for example, the transverse oscillation of a string, the longitudinal motion of a bar, or the sound pressure in a 1D pipe. The resolution method consists in replacing this continuous variable by a discrete variable $\xi_j^n = \xi(x_j, t^n)$ which is defined at some discrete spatial points x_j, and for a discrete series of instants t^n only. Equally spaced points $x_j = j\Delta x$ are generally selected, where Δx is the spatial step. Similarly, the simplest methods use constant time steps Δt, so that we can write $t^n = n\Delta t$. Variable step methods are also used, particularly when it is needed to refine a subdomain of the meshing [2].

In what follows, we limit ourselves to the case of a uniform mesh in space and time. The basic principle of finite difference methods is to approximate the partial derivatives in time and space by linear combinations of ξ_j^n. Thus, for example, if we want to make a second-order approximation of the partial derivatives appearing in (1.157), we write

$$\begin{cases} \dfrac{\partial^2 \xi}{\partial t^2}(x_j, t^n) \approx \dfrac{\xi_j^{n+1} - 2\xi_j^n + \xi_j^{n-1}}{\Delta t^2}, \\ \dfrac{\partial^2 \xi}{\partial x^2}(x_j, t^n) \approx \dfrac{\xi_{j+1}^n - 2\xi_j^n + \xi_{j-1}^n}{\Delta x^2} \end{cases} \tag{1.158}$$

The resulting difference equation can be rewritten:

$$\xi_j^{n+1} = 2(1-\alpha^2)\xi_j^n + \alpha^2(\xi_{j-1}^n + \xi_{j+1}^n) - \xi_j^{n-1} \quad \text{with} \quad \alpha = c\frac{\Delta t}{\Delta x}. \tag{1.159}$$

In (1.159), a recurrence equation is obtained that allows to explicitly calculate the future value ξ_j^{n+1} as a function of the values taken by the same variable at earlier instants, at the same point and at neighboring points. This is called an *explicit* finite difference scheme. Equation (1.159) is initialized at both instants ($n = 0$ and $n = 1$) by means of the initial conditions in displacement and velocity defined in (1.157).

The number of spatial steps included in a recurrence equation depends on the order of the scheme. For a better accuracy, it might be necessary to use higher order approximations. As an example, the following approximation:

$$\frac{\partial^2 \xi}{\partial x^2}(x_j, t^n) \approx \frac{1}{\Delta x^2}\left(-\frac{1}{12}\xi_{j-2}^n + \frac{4}{3}\xi_{j-1}^n - \frac{5}{2}\xi_j^n + \frac{4}{3}\xi_{j+1}^n - \frac{1}{12}\xi_{j+2}^n\right) \tag{1.160}$$

is of the fourth-order in space. The explicit scheme presented in (1.159) is a special case. If ξ_j^{n+1} cannot be directly expressed as a function of the values of the variable at earlier instants, an *implicit* scheme is obtained [4].

1.7.1.1 Stability of the Discretization Scheme

With a given discrete approximation, cumulative errors might propagate during the calculation over time, causing the "explosion" of the solution because of numerical instability. Each numerical scheme thus requires a prior analysis of its stability properties. Such an analysis can be performed by energy methods or by Fourier techniques. Some guidance on this latter method are given below. The reader is invited to consult the specialized literature for more details [2, 47].

In the Fourier method, discrete solutions of the form $\xi_j^n = \hat{\xi}^n \exp(-ikx_j)$ are tested, where $i = \sqrt{-1}$ and k the wavenumber. Within the framework of our reference example (1.159), this leads to the equation:

$$\hat{\xi}^{n+1} - 2(1-2\beta)\hat{\xi}^n + \hat{\xi}^{n-1} = 0 \quad \text{where} \quad \beta = 2\alpha^2 \sin^2\frac{k\Delta x}{2}. \tag{1.161}$$

The resulting scheme will be stable provided that the solutions do not contain terms whose amplitude grows continuously with time. It is known that the general solution of (1.161) is $\hat{\xi}^n = a_1 d_1^n + a_2 d_2^n$ where d_1 and d_2 are solutions of the associated characteristic equation $d^2 - 2(1-2\beta)d + 1 = 0$. If $\beta > 2$, either the modulus of d_1, or the modulus of d_2 is greater than unity, and instability occurs. Therefore, the stability condition imposes here $\beta \leq 2$. This result must be true for any k, thus leading to the condition:

$$\alpha = c\frac{\Delta t}{\Delta x} \leq 1\ . \tag{1.162}$$

This last condition is called Courant–Friedrichs–Levy (or CFL) condition. It shows that, for stability reasons, time and space steps cannot be selected independently. A specific condition of stability corresponds to each scheme. Some implicit schemes can guarantee an unconditional stability, but at the cost of lower accuracy, for a given order of approximation [18].

1.7.1.2 Numerical Dispersion

Numerical schemes also have dispersive properties, which means that the propagation velocities are not correctly estimated. The dispersion properties of a given scheme are analyzed on the basis of the Fourier transform, which states that the solution can be represented as a superposition of plane waves of the form $\xi(x_j, t^n) = \xi_j^n \exp i(\omega t^n - kx_j)$. By introducing this form in the recurrence equation (1.159), it is found that the relation of numerical dispersion between angular frequency ω and wavenumber k, is given by:

$$D_{\text{num}}(\omega, k) = \sin^2\frac{\omega\Delta t}{2} - \frac{c^2\Delta t^2}{\Delta x^2}\sin^2\frac{k\Delta x}{2} = 0\ . \tag{1.163}$$

Equation (1.163) shows that the numerical phase velocity is equal to:

$$c_{\text{num}} = \frac{\omega}{k} = \frac{c}{\alpha K}\arcsin\left[\alpha \sin K\right] \quad \text{with} \quad K = \frac{k\Delta x}{2} = \frac{\pi\Delta x}{\lambda}\ . \tag{1.164}$$

As K tends to zero in (1.164), i.e., when the spatial step is small compared to the wavelength λ, then c_{num} tends to the continuous phase velocity c. Thus, if we want to numerically reproduce a wave propagation with good accuracy, it is necessary to discretize the equation with a large number of points per wavelength. Figure 1.28 shows the ratio $R = c_{\text{num}}/c$ as a function of the parameter K for different values of the stability parameter α. It can be seen that the dispersion properties of the scheme are degraded as α decreases. Notice that the ideal wave equation shows a remarkable result for the limiting stability value $\alpha = 1$. In this case, Eq. (1.164) shows that the numerical phase velocity is strictly equal to c, whatever the wavelength. This limiting condition of stability therefore provides the exact solution of the wave equation for the particular centered explicit scheme selected here. This result is due

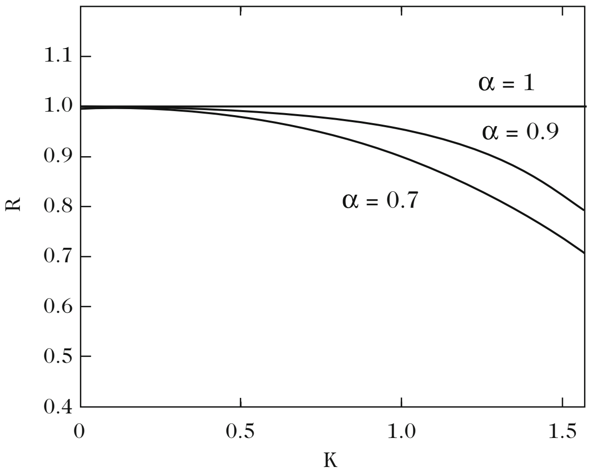


Fig. 1.28 Numerical dispersion of a second-order centered finite difference scheme applied to the wave equation, for different values of the stability parameter$\alpha = c\Delta t/\Delta x$

to the particular form of this equation. We could show, for example, that with a fluid damping term introduced in (1.157), then it is not anymore possible to find a value of α giving the exact solution of the problem.

1.7.2 Finite Element Method

The ideal wave equation with simple boundary conditions (equation of the ideal string of length L fixed at both ends) is now solved by means of the finite element method. The main idea is that the solution $\xi(x,t)$ is now approximated by a linear combination of *basic functions* $\phi_i(x)$ [35]:

$$\xi(x,t) \simeq \sum_{i=1}^{N} q_i(t)\phi_i(x), \tag{1.165}$$

where N is the number of discrete points with abscissa x_i on the string and $q_i(t)$ the unknown functions of time which are expected to provide the *best possible approximation* for $\xi(x,t)$.

The method is illustrated using *hat* functions for $\phi_i(x)$. Hat functions (or triangular functions) are equal to 1 for x_i and show linear slopes from 0 to 1 between x_i and its adjacent points x_{i-1} and x_{i+1} (see Fig. 1.29). The 2D equivalent of such piecewise linear functions are *triangle functions* and, in 3D, *tetrahedron functions*. These basic functions are the most commonly used.

Figure 1.30 shows that we can achieve a discrete approximation of the string motion at each time using a linear combination of *hat* functions.

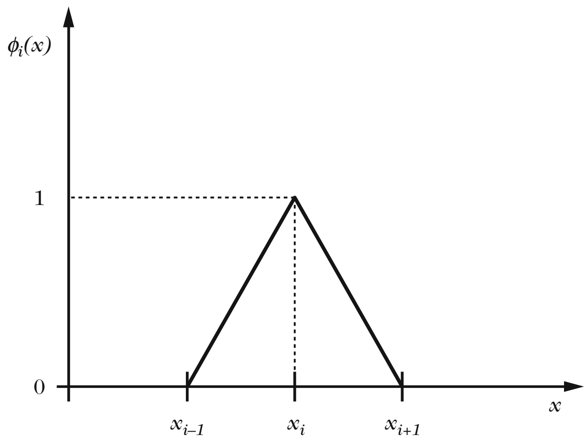


Fig. 1.29 "Hat" function (or triangular function)

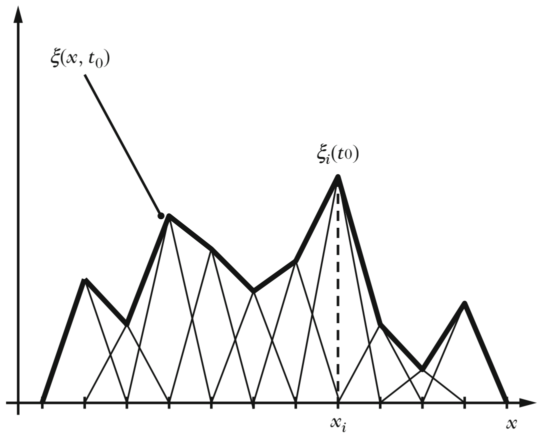


Fig. 1.30 Discrete approximation of a string motion using *hat* functions

Variational Formulation of the Wave Equation

The finite element method is based on a variational formulation (or *weak* formulation) of the motion equations. Let us note $c^{-2}\xi_{tt} - \xi_{xx} - f = 0$ the wave equation, where the subscripts are the partial derivatives with respect to time and space. Considering $v(x)$ as a continuous *test-function*, at least differentiable once, and bounded on the interval $[0, L]$, that satisfies the boundary conditions at both ends of the string. We can check that

$$\int_0^L (c^{-2}\xi_{tt} - \xi_{xx} - f)v \, dx = 0 \, . \tag{1.166}$$

After integration by parts, and taking the boundary conditions into account, (1.166) is written:

$$\int_0^L c^{-2}\xi_{tt} v \, dx + \int_0^L \xi_x v_x \, dx - \int_0^L f v \, dx = 0 \, . \tag{1.167}$$

The discrete formulation of the problem is done by replacing $\xi(x,t)$ in (1.167) by its approximation (1.165). Considering that every hat function $\phi_k(x)$ satisfies all conditions imposed to $v(x)$, the following system of differential equations is obtained

$$c^{-2}\sum_{i=1}^{N} \ddot{q}_i \int_0^L \phi_i(x)\phi_k(x)\, dx + \sum_{i=1}^{N} q_i \int_0^L \frac{d\phi_i}{dx}\frac{d\phi_k}{dx}\, dx = \int_0^L f\phi_k \, dx, \tag{1.168}$$

that can be written as

$$\mathbb{M}\ddot{\mathbf{Q}} + \mathbb{K}\mathbf{Q} = \mathbf{F} \tag{1.169}$$

where $\mathbf{Q}$ is the vector whose components are the $q_i(t)$ and where $\mathbf{F}$ is the vector whose components are the projection of the excitation f on the discrete string.

The parameter $h = L/(N+1)$ is the space step. The following results for the coefficients of mass $\mathbb{M}$ and stiffness matrices $\mathbb{K}$ can be verified, in case of *hat* functions:

$$\begin{cases} \displaystyle\int_0^L \phi_i^2 dx = \frac{2h}{3} \, ; & \displaystyle\int_0^L \left(\frac{d\phi_i}{dx}\right)^2 dx = \frac{2}{h} \quad \forall i = 1, 2, \ldots N, \\ \displaystyle\int_0^L \phi_i\phi_{i+1} dx = \frac{h}{6} \, ; & \displaystyle\int_0^L \frac{d\phi_i}{dx}\frac{d\phi_{i+1}}{dx} dx = -\frac{1}{h} \quad \forall i = 1, 2, \ldots N-1, \end{cases} \tag{1.170}$$

which leads to

$$\mathbb{M} = c^{-2}h \begin{pmatrix} 2/3 & 1/6 & 0 & \ldots & 0 & 0 & 0 \\ 1/6 & 2/3 & 1/6 & 0 & \ldots & 0 & 0 \\ 0 & 1/6 & 2/3 & 1/6 & 0 & \ldots & 0 \\ & & \ldots & \ldots & \ldots & & \\ 0 & 0 & \ldots & 0 & 1/6 & 2/3 & 1/6 \\ 0 & 0 & 0 & \ldots & 0 & 1/6 & 2/3 \end{pmatrix} \tag{1.171}$$

and

$$\mathbb{K} = \frac{1}{h} \begin{pmatrix} 2 & -1 & 0 & \ldots & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & \ldots & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & \ldots & 0 \\ & & \ldots & \ldots & \ldots & & \\ 0 & 0 & \ldots & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & \ldots & 0 & -1 & 2 \end{pmatrix} . \tag{1.172}$$

Therefore, for $i \neq 1$ and $i \neq N$, the discrete variable $q_i(t)$, which represents here the displacement $\xi_i(t)$ at point $x_i = ih$ of the string, since $\phi_i(x_i) = 1$, is governed by the equation:

$$\left(\frac{1}{6}\frac{d^2\xi_{i-1}}{dt^2} + \frac{2}{3}\frac{d^2\xi_i}{dt^2} + \frac{1}{6}\frac{d^2\xi_{i+1}}{dt^2}\right) - c^2\frac{\xi_{i+1} - 2\xi_i + \xi_{i-1}}{h^2} = F_i \,. \tag{1.173}$$

Comparing this finite element approximation (1.173) with the second-order centered finite difference method approximation (1.159), one can see that the second-order partial derivative in space is identical in both cases:

$$\frac{\partial^2\xi}{\partial x^2}(x_i) \simeq \frac{\xi_{i+1} - 2\xi_i + \xi_{i-1}}{h^2} \,. \tag{1.174}$$

However, the approximation of the second-order partial derivative in time is not punctual here, since it involves three spatial points:

$$\frac{\partial^2\xi}{\partial t^2}(x_i) \simeq \frac{1}{6}\frac{d^2\xi_{i-1}}{dt^2} + \frac{2}{3}\frac{d^2\xi_i}{dt^2} + \frac{1}{6}\frac{d^2\xi_{i+1}}{dt^2} \,. \tag{1.175}$$

Without going into details, let us mention here the existence of techniques known as "mass lumping" which consists in the simplification:

$$\frac{\partial^2\xi}{\partial t^2}(x_i) \simeq \frac{d^2\xi_i}{dt^2}, \tag{1.176}$$

which is equivalent to replacing the matrix $\mathbb{M}$ by $c^{-2}h\mathbb{I}$ where $\mathbb{I}$ is the identity matrix [18].

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