Chapter 4
Submanifolds in Lie Sphere Geometry

This chapter is an outline of the method for studying submanifolds of Euclidean space $\mathbf{R}^n$ or the sphere S^n in the context of Lie sphere geometry. For Dupin hypersurfaces this has proven to be a valuable approach, since Dupin hypersurfaces occur naturally as envelopes of families of spheres, which can be handled well in Lie sphere geometry. Since the Dupin property is invariant under Lie sphere transformations, this is also a natural setting for classification theorems.

In Section 4.5, we give a Lie geometric criterion for a Legendre submanifold to be Lie equivalent to the Legendre lift of an isoparametric hypersurface in S^n, and we develop the important invariants known as Lie curvatures of a Legendre submanifold. Finally, in Section 4.6, we formulate the notion of tautness in the setting of Lie sphere geometry and prove that it is invariant under Lie sphere transformations.

For the early development of Lie sphere geometry, see the paper of Lie [326] and the books of Lie and Scheffers [327], Klein [281], Blaschke [42] and Bol [44]. For a historical treatment of the subject, see the papers of Hawkins [190] and Rowe [466]. For a modern treatment of Möbius geometry, see the book of Hertrich-Jeromin [198]. The material in this chapter is covered in more detail in Chapters 2–4 of the book [77], and the figures in this chapter are also taken from that book.

4.1 Möbius Geometry of Unoriented Spheres

We begin with the "Möbius geometry" of unoriented hyperspheres in Euclidean space $\mathbf{R}^n$ or in the unit sphere S^n in $\mathbf{R}^{n+1}$. We always assume that $n \geq 2$.

We can go back and forth between these two ambient spaces $\mathbf{R}^n$ and S^n via stereographic projection, which we recall here. Let $\mathbf{R}^{n+1}$ have coordinates $x = (x_1, \ldots, x_{n+1})$, and denote the usual inner product in $\mathbf{R}^{n+1}$ by $x \cdot y$, where

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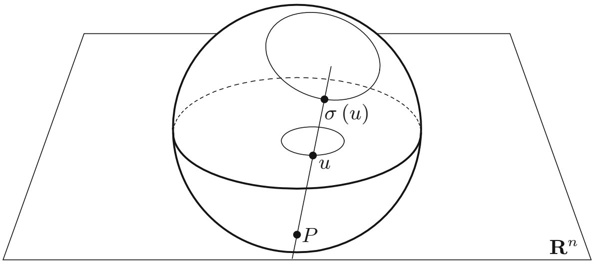


Fig. 4.1 Inverse stereographic projection

$$x \cdot y = x_1 y_1 + \cdots + x_{n+1} y_{n+1}. \tag{4.1}$$

In this chapter, we will use the notation $x \cdot y$ instead of $\langle x, y \rangle$ (as used in the preceding chapters) to denote the Euclidean inner product, because we want to use $\langle x, y \rangle$ for the Lie scalar product, which we will introduce later in this chapter.

The unit sphere S^n is the set of points $x \in \mathbf{R}^{n+1}$ such that $x \cdot x = 1$. We identify $\mathbf{R}^n$ with the hyperplane given by the equation $x_1 = 0$ in $\mathbf{R}^{n+1}$. Let $P = (-1, 0, \ldots, 0)$ be the south pole of S^n.

As in Remark 2.7 on page 21, we define stereographic projection with pole P to be the map $\tau : S^n - \{P\} \to \mathbf{R}^n$ given by the formula,

$$\tau(x_1, \ldots, x_{n+1}) = \left(0, \frac{x_2}{x_1 + 1}, \ldots, \frac{x_{n+1}}{x_1 + 1}\right). \tag{4.2}$$

To describe inverse stereographic projection $\sigma : \mathbf{R}^n \to S^n - \{P\}$ (see Figure 4.1), we write a point $u \in \mathbf{R}^n$ as $u = (u_2, \ldots, u_{n+1})$, that is, we omit the first coordinate 0. Then inverse stereographic projection is given by the formula:

$$\sigma(u) = \left(\frac{1 - u \cdot u}{1 + u \cdot u}, \frac{2u}{1 + u \cdot u}\right). \tag{4.3}$$

Later in this section we will show that stereographic projection τ maps a hypersphere S in S^n that does not contain the point P to a hypersphere $\tau(S)$ in $\mathbf{R}^n$. If S does contain P, then τ maps $S - \{P\}$ to a hyperplane in $\mathbf{R}^n$. Obviously, the inverse map σ has similar properties.

Remark 4.1. Sometimes the map σ is referred to as "stereographic projection," as in the book *Lie Sphere Geometry* [77]. However, in this book, we will call the map τ "stereographic projection," and the map σ "inverse stereographic projection."

To construct the space of unoriented hyperspheres in S^n, we need to consider the Lorentz space $\mathbf{R}^{n+2}_1$ of dimension $n + 2$ endowed with the Lorentz metric (bilinear form) of signature $(1, n+1)$ defined for $x = (x_1, \ldots, x_{n+2})$ and $y = y_1, \ldots, y_{n+2})$ by

$$(x, y) = -x_1 y_1 + x_2 y_2 + \cdots + x_{n+2} y_{n+2}. \tag{4.4}$$

This metric is also referred to as the *Lorentz scalar product*.

We borrow the terminology of relativity theory and say that vector x in $\mathbf{R}^{n+2}_1$ is *spacelike*, *timelike*, or *lightlike*, respectively, depending on whether (x, x) is positive, negative, or zero. We will use this terminology even when we are using a metric of different signature.

In the Lorentz space $\mathbf{R}^{n+2}_1$, the set of all lightlike vectors forms a cone of revolution, called the *light cone* or *isotropy cone*. Lightlike vectors are often called *isotropic* in the literature. Timelike vectors are "inside the cone" and spacelike vectors are "outside the cone."

We identify $\mathbf{R}^{n+1}$ with the spacelike subspace of $\mathbf{R}^{n+2}_1$ determined by the equation $x_1 = 0$, and we consider S^n to be the unit sphere in this space $\mathbf{R}^{n+1}$. We next embed this space $\mathbf{R}^{n+1}$ as an affine subspace of projective space $\mathbf{RP}^{n+1}$ as follows. Define projective space $\mathbf{RP}^{n+1}$ to be the space of lines through the origin in $\mathbf{R}^{n+2}$. Equivalently, $\mathbf{RP}^{n+1}$ is the set of equivalence classes $[x]$ for the equivalence relation $\simeq$ on $\mathbf{R}^{n+2} - \{0\}$ defined by $x \simeq y$ if and only if $y = tx$ for some nonzero real number t.

We embed the space $\mathbf{R}^{n+1}$ determined by the equation $x_1 = 0$ in $\mathbf{R}^{n+2}_1$ as an affine hyperplane in $\mathbf{RP}^{n+1}$ by the map $\phi : \mathbf{R}^{n+1} \to \mathbf{RP}^{n+1}$,

$$\phi(x_2, \ldots, x_{n+2}) = [(1, x_2, \ldots, x_{n+2})]. \tag{4.5}$$

If $x \in \mathbf{R}^{n+2}_1$ is a spacelike, timelike, or lightlike vector, then the corresponding point $[x]$ in $\mathbf{RP}^{n+1}$ will be referred to as spacelike, timelike, or lightlike point, respectively.

Let S^n be the unit sphere in $\mathbf{R}^{n+1}$. The image Σ of S^n under the embedding ϕ consists of all points $[(1, y)]$ for $y \in S^n$. If we compute the Lorentz scalar product on such a point $(1, y)$, we get

$$((1, y), (1, y)) = -1 \cdot 1 + y \cdot y = -1 + 1 = 0.$$

Conversely, if the Lorentz scalar product of $(1, y)$ with itself is zero, then y is in S^n. Thus the image $\Sigma = \phi(S^n)$ consists precisely of the projective classes of lightlike vectors in $\mathbf{R}^{n+2}_1$.

We identify $\mathbf{R}^n$ with the subspace of $\mathbf{R}^{n+1}$ determined by the equation $x_2 = 0$. We next consider the composition of the map ϕ above with inverse stereographic projection σ, that is, $\phi\sigma : \mathbf{R}^n \to \mathbf{RP}^{n+1}$ given by

$$\phi\sigma(u) = \left[\left(1, \frac{1 - u \cdot u}{1 + u \cdot u}, \frac{2u}{1 + u \cdot u}\right)\right] = \left[\left(\frac{1 + u \cdot u}{2}, \frac{1 - u \cdot u}{2}, u\right)\right]. \tag{4.6}$$

Let $(z_1, \ldots, z_{n+2})$ be homogeneous coordinates on $\mathbf{RP}^{n+1}$. Then $\phi\sigma(\mathbf{R}^n)$ is just the set of points in $\mathbf{RP}^{n+1}$ lying on the n-sphere Σ given by the equation $(z, z) = 0$, with the exception of the *improper point* $[(1, -1, 0, \ldots, 0)]$, that is, the image under ϕ of the south pole $P \in S^n$. We will refer to the points in Σ other than $[(1, -1, 0, \ldots, 0)]$ as *proper points*, and will call Σ the *Möbius sphere* or *Möbius space*.

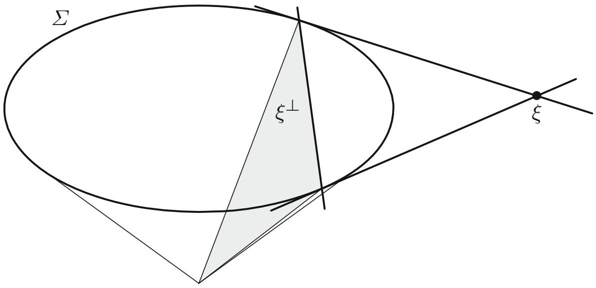


Fig. 4.2 Intersection of Σ with $\xi^\perp$

Spheres in Möbius geometry

The basic construction in the Möbius geometry of unoriented spheres is a correspondence between the set of all hyperspheres and hyperplanes in $\mathbf{R}^n$ and the manifold of all spacelike points in projective space $\mathbf{RP}^{n+1}$, and we now give a brief description of this correspondence.

Let ξ be a spacelike vector in $\mathbf{R}^{n+2}_1$. The polar hyperplane $\xi^\perp$ of $[\xi]$ in $\mathbf{RP}^{n+1}$ intersects the sphere Σ in an $(n-1)$-sphere S^{n-1} (see Figure 4.2).

This sphere S^{n-1} is the image under $\phi\sigma$ of a hypersphere in $\mathbf{R}^n$, unless it contains the improper point, in which case it is the image under $\phi\sigma$ of a hyperplane in $\mathbf{R}^n$. Thus we have a bijective correspondence between the set of all hyperspheres and hyperplanes in $\mathbf{R}^n$ and the manifold of all spacelike points $\mathbf{RP}^{n+1}$. We next derive the analytic formulas for this correspondence.

The hypersphere in $\mathbf{R}^n$ with center p and radius $r > 0$ has equation

$$(u-p)\cdot(u-p) = r^2. \tag{4.7}$$

A straightforward calculation shows that this is equivalent to the following equation in homogeneous coordinates in $\mathbf{RP}^{n+1}$,

$$(\xi, \phi\sigma(u)) = 0, \tag{4.8}$$

where ξ is the spacelike vector,

$$\xi = \left(\frac{1+p\cdot p - r^2}{2}, \frac{1-p\cdot p + r^2}{2}, p\right), \tag{4.9}$$

and $\phi\sigma(u)$ is given by equation (4.6). Thus the point $u \in \mathbf{R}^n$ lies on the sphere given by equation (4.7) if and only if $\phi\sigma(u)$ lies on the polar hyperplane to $[\xi]$. Since $(\xi,\xi) = r^2 > 0$, the point $[\xi]$ is spacelike. Note also that $\xi_1 + \xi_2 = 1$. The homogeneous coordinates of $[\xi]$ are only determined up to a nonzero scalar multiple, but we can conclude that $\xi_1 + \xi_2 \neq 0$ for any homogeneous coordinates of $[\xi]$.

Conversely, if $[z]$ is a spacelike point in $\mathbf{RP}^{n+1}$ with $z_1 + z_2 \neq 0$, then $[z]$ corresponds to a hypersphere in $\mathbf{R}^n$ as follows. Let $\xi = z/(z_1 + z_2)$ so that $[\xi] = [z]$ is a spacelike point with $\xi_1 + \xi_2 = 1$. Then $(\xi, \xi) = r^2 > 0$ for some $r > 0$, and there exists a unique $p \in \mathbf{R}^n$ such that ξ can be written in the form of equation (4.9). This $p \in \mathbf{R}^n$ and $r > 0$ determine the sphere in $\mathbf{R}^n$ corresponding to $[\xi]$ via equation (4.8).

Next consider the hyperplane in $\mathbf{R}^n$ given by the equation

$$u \cdot N = h, \quad |N| = 1. \tag{4.10}$$

A direct calculation shows that (4.10) is equivalent to the equation

$$(\eta, \phi\sigma(u)) = 0, \text{where } \eta = (h, -h, N). \tag{4.11}$$

Note that $\eta_1 + \eta_2 = 0$, and this is true for any nonzero scalar multiple of η. This condition $\eta_1 + \eta_2 = 0$ is equivalent to the equation

$$(\eta, (1, -1, 0, \ldots, 0)) = 0,$$

and thus the improper point $[(1, -1, 0, \ldots, 0)]$ lies on the hypersphere of Σ obtained by intersecting Σ with the polar hyperplane of η.

Conversely, if $[z]$ is a spacelike point in $\mathbf{RP}^{n+1}$ with $z_1 + z_2 = 0$, then $(z, z) = v \cdot v$, where $v = (z_3, \ldots, z_{n+2})$ is a nonzero vector in $\mathbf{R}^n$. If we take $\eta = z/|v|$, then η has the form $(h, -h, N)$ for some real number h and some unit vector $N \in \mathbf{R}^n$, and the polar hyperplane of $[\eta]$ intersects Σ in an $(n-1)$-sphere corresponding to the hyperplane in $\mathbf{R}^n$ given by equation (4.10).

Thus we have a correspondence between each spacelike point in $\mathbf{RP}^{n+1}$ and a unique hypersphere or hyperplane in $\mathbf{R}^n$. The set of all spacelike points in $\mathbf{RP}^{n+1}$ can be realized as an $(n+1)$-dimensional manifold in the following natural way. Let W^{n+1} be the set of vectors in $\mathbf{R}^{n+2}_1$ satisfying $(\zeta, \zeta) = 1$. This is a hyperboloid of revolution of one sheet in $\mathbf{R}^{n+2}_1$. If $[\xi]$ is a spacelike point in $\mathbf{RP}^{n+1}$, then there are precisely two vectors $\zeta = \pm\xi/\sqrt{(\xi, \xi)}$ in W^{n+1} with $[\zeta] = [\xi]$. Thus the set of all spacelike points in $\mathbf{RP}^{n+1}$ is diffeomorphic to the quotient manifold $W^{n+1}/\simeq$, where $\simeq$ is projective equivalence.

Note that this correspondence also demonstrates that inverse stereographic projection σ maps a hypersphere or hyperplane in $\mathbf{R}^n$ to a hypersphere in the sphere Σ corresponding to the intersection of Σ with the polar hyperplane of the appropriate spacelike point $[\xi]$ or $[\eta]$. Conversely, any hypersphere in Σ is obtained by intersecting Σ with the polar hyperplane of some spacelike point $[\xi]$ or $[\eta]$ in $\mathbf{RP}^{n+1}$, and stereographic projection τ maps this hypersphere in Σ to a hypersphere or hyperplane in $\mathbf{R}^n$ determined by equation (4.7) or (4.10), as the case may be.

The space of hyperspheres in the sphere S^n

Similarly, we can construct a bijective correspondence between the space of all hyperspheres in the unit sphere $S^n \subset \mathbf{R}^{n+1}$ and the manifold of all spacelike points in $\mathbf{RP}^{n+1}$ as follows. The hypersphere S in S^n with center $p \in S^n$ and (spherical) radius $\rho, 0 < \rho < \pi$, is given by the equation

$$p \cdot y = \cos \rho, \quad 0 < \rho < \pi, \tag{4.12}$$

for $y \in S^n$. If we take $[z] = \phi(y) = [(1, y)]$, then

$$p \cdot y = \frac{-(z, (0, p))}{(z, e_1)},$$

where $e_1 = (1, 0, \ldots, 0)$. Thus equation (4.12) is equivalent to the equation

$$(z, (\cos \rho, p)) = 0, \tag{4.13}$$

in homogeneous coordinates in $\mathbf{RP}^{n+1}$. Therefore, y lies on the hypersphere S given by equation (4.12) if and only if $[z] = \phi(y)$ lies on the polar hyperplane in $\mathbf{RP}^{n+1}$ of the spacelike point

$$[\xi] = [(\cos \rho, p)]. \tag{4.14}$$

Remark 4.2 (The space of hyperspheres in hyperbolic space H^n). One can also construct the space of unoriented hyperspheres in hyperbolic space H^n with constant sectional curvature -1. To do this, we let $\mathbf{R}^{n+1}_1$ denote the Lorentz subspace of $\mathbf{R}^{n+2}_1$ spanned by the orthonormal basis $\{e_1, e_3, \ldots, e_{n+2}\}$. Then H^n is the hypersurface

$$\{y \in \mathbf{R}^{n+1}_1 \mid (y, y) = -1, \ y_1 \geq 1\},$$

on which the restriction of the Lorentz metric (,) is a positive definite metric of constant sectional curvature -1 (see Kobayashi–Nomizu [283, Vol. II, p. 268–271] for more detail). The distance between two points p and q in H^n is given by

$$d(p, q) = \cosh^{-1}(-(p, q)).$$

Thus the equation for the unoriented sphere in H^n with center p and radius ρ is

$$(p, y) = -\cosh \rho. \tag{4.15}$$

As with S^n, we first embed $\mathbf{R}^{n+1}_1$ into $\mathbf{RP}^{n+1}$ as an affine space by the map

$$\psi(y) = [y + e_2].$$

Let $p \in H^n$ and let $z = y + e_2$ for $y \in H^n$. Then we have

$$(p, y) = (z, p)/(z, e_2).$$

Thus, the condition (4.15) for y to lie on sphere S with center p and radius ρ is equivalent to the condition that $[z] = [y+e_2]$ lies on the polar hyperplane in $\mathbf{RP}^{n+1}$ to

$$[\xi] = [p + \cosh\rho\, e_2], \tag{4.16}$$

and we can associate the sphere S with the point $[\xi]$.

Orthogonal spheres

Möbius geometry in $\mathbf{R}^n$ or S^n is often identified with the conformal geometry of these spaces via the following considerations. Let S_1 and S_2 denote hyperspheres in $\mathbf{R}^n$ with centers p_1 and p_2 and radii r_1 and r_2, respectively. These two spheres intersect orthogonally (see Figure 4.3) if and only if

$$|p_1 - p_2|^2 = r_1^2 + r_2^2. \tag{4.17}$$

Suppose that S_1 and S_2 correspond to the spacelike points $[\xi_1]$ and $[\xi_2]$ via equation (4.9). Then a straightforward calculation shows that equation (4.17) is equivalent to the condition

$$(\xi_1, \xi_2) = 0, \tag{4.18}$$

in homogeneous coordinates in $\mathbf{RP}^{n+1}$.

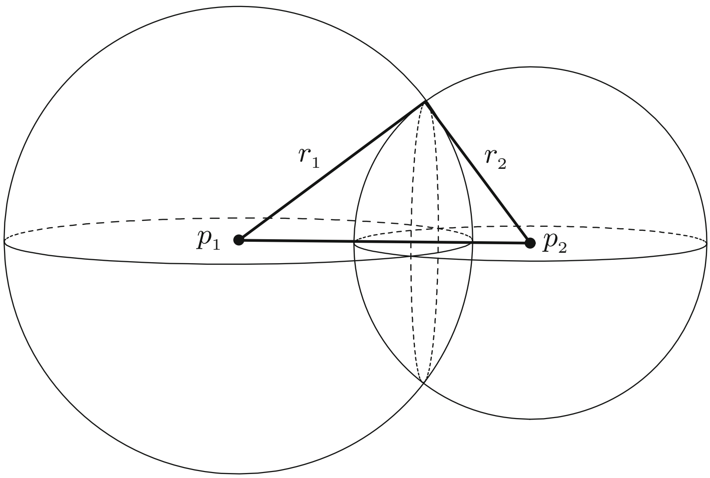


Fig. 4.3 Orthogonal spheres

Similarly, a hyperplane π in $\mathbf{R}^n$ intersects a hypersphere S in $\mathbf{R}^n$ orthogonally if and only if the center p of S lies in the hyperplane π. If π is given by equation (4.10) above, then this condition is $p \cdot N = 0$. One can easily verify that this equation is equivalent to the condition $(\xi, \eta) = 0$ in homogeneous coordinates in $\mathbf{RP}^{n+1}$, where ξ and η correspond to S and π via equations (4.8) or (4.11), respectively. Finally, two hyperplanes π_1 and π_2 in $\mathbf{R}^n$ are orthogonal if and only if their unit normals N_1 and N_2 are orthogonal. A direct calculation shows that this is equivalent to the equation $(\eta_1, \eta_2) = 0$ in homogeneous coordinates for the spacelike points $[\eta_1]$ and $[\eta_2]$ corresponding to π_1 and π_2 via equation (4.11). Thus, in all cases of hyperspheres or hyperplanes in $\mathbf{R}^n$, orthogonal intersection corresponds to a polar relationship in $\mathbf{RP}^{n+1}$ given by equations (4.8) or (4.11).

Möbius transformations

We conclude this section with a discussion of Möbius transformations. Recall that a linear transformation $A \in GL(n+2)$ induces a projective transformation $P(A)$ on $\mathbf{RP}^{n+1}$ defined by $P(A)[x] = [Ax]$. The map P is a homomorphism of $GL(n+2)$ onto the group $PGL(n+1)$ of projective transformations of $\mathbf{RP}^{n+1}$, and its kernel is the group of nonzero multiples of the identity transformation $I \in GL(n+2)$.

A *Möbius transformation* is a projective transformation α of $\mathbf{RP}^{n+1}$ that preserves the condition $(\eta, \eta) = 0$ for $[\eta] \in \mathbf{RP}^{n+1}$, that is, $\alpha = P(A)$, where $A \in GL(n+2)$ maps lightlike vectors in $\mathbf{R}^{n+2}_1$ to lightlike vectors. It can be shown (see, for example, [77, pp. 26–27]) that such a linear transformation A is a nonzero scalar multiple of a linear transformation $B \in O(n+1,1)$, the orthogonal group for the Lorentz inner product space $\mathbf{R}^{n+2}_1$. Thus, $\alpha = P(A) = P(B)$.

The Möbius transformation $\alpha = P(B)$ induced by an orthogonal transformation $B \in O(n+1,1)$ maps spacelike points to spacelike points in $\mathbf{RP}^{n+1}$, and it preserves the polarity condition $(\xi, \eta) = 0$ for any two points $[\xi]$ and $[\eta]$ in $\mathbf{RP}^{n+1}$. Therefore by the correspondence given in equations (4.8) and (4.11) above, α maps the set of hyperspheres and hyperplanes in $\mathbf{R}^n$ to itself, and it preserves orthogonality and hence angles between hyperspheres and hyperplanes. A similar statement holds for the set of all hyperspheres in S^n.

Let H denote the group of Möbius transformations and let

$$\psi : O(n+1,1) \to H \tag{4.19}$$

be the restriction of the map P to $O(n+1,1)$. The discussion above shows that ψ is onto, and the kernel of ψ is $\{\pm I\}$, the intersection of $O(n+1,1)$ with the kernel of P. Therefore, H is isomorphic to the quotient group $O(n+1,1)/\{\pm I\}$.

One can show that the group H is generated by Möbius transformations induced by inversions in spheres in $\mathbf{R}^n$. This follows from the fact that the corresponding orthogonal groups are generated by reflections in hyperplanes. In fact, every orthogonal transformation on an indefinite inner product space $\mathbf{R}^n_k$ is a product of at most n reflections, a result due to Cartan and Dieudonné. (See Cartan [58, pp. 10–12], Chapter 3 of E. Artin's book [15], or [77, pp. 30–34]).

Since a Möbius transformation $\alpha = P(B)$ for $B \in O(n+1,1)$ maps lightlike points to lightlike points in $\mathbf{RP}^{n+1}$ in a bijective way, it induces a diffeomorphism of the n-sphere Σ which is conformal by the considerations given above. It is well known that the group of conformal diffeomorphisms of the n-sphere is precisely the Möbius group.

4.2 Lie Geometry of Oriented Spheres

We now turn to Lie's construction of the space of oriented spheres which is a natural setting for the study of Dupin hypersurfaces. As noted in the previous section, each unoriented hypersphere or hyperplane in $\mathbf{R}^n$ corresponds to a spacelike point $[\xi]$ in $\mathbf{RP}^{n+1}$ via the polarity relationships in equations (4.8) and (4.11). If $[\xi]$ is a spacelike point in $\mathbf{RP}^{n+1}$, then there are precisely two unit length spacelike vectors $\pm\xi/\sqrt{(\xi,\xi)}$ that determine the same spacelike point $[\xi]$ in $\mathbf{RP}^{n+1}$. Thus, as noted earlier, the set of spacelike points in $\mathbf{RP}^{n+1}$ is diffeomorphic to the quotient manifold $W^{n+1}/\simeq$, where W^{n+1} is the set of all unit spacelike vectors in $\mathbf{R}^{n+2}_1$ and $\simeq$ is projective equivalence.

We can associate the two points $\pm\xi/\sqrt{(\xi,\xi)}$ to the two orientations of the hypersphere or hyperplane corresponding to $[\xi]$ by the following construction. We first embed $\mathbf{R}^{n+2}_1$ as an affine space in projective space $\mathbf{RP}^{n+2}$ by the embedding $z \mapsto [(z,1)]$, i.e., we introduce one more coordinate x_{n+3} to give $\mathbf{R}^{n+3}$ and then let $\mathbf{RP}^{n+2}$ be the space of lines through the origin in $\mathbf{R}^{n+3}$. If $\zeta \in W^{n+1}$ is a unit spacelike vector in $\mathbf{R}^{n+2}_1$, then

$$-\zeta_1^2 + \zeta_2^2 + \cdots + \zeta_{n+2}^2 = 1,$$

so the point $[(\zeta,1)]$ in $\mathbf{RP}^{n+2}$ lies on the quadric Q^{n+1} in $\mathbf{RP}^{n+2}$ given in homogeneous coordinates by the equation

$$\langle x,x\rangle = -x_1^2 + x_2^2 + \cdots + x_{n+2}^2 - x_{n+3}^2 = 0, \qquad (4.20)$$

which defines the indefinite scalar product $\langle\ ,\ \rangle$ of signature $(n+1,2)$ on the space $\mathbf{R}^{n+3}$, which we now denote as $\mathbf{R}^{n+3}_2$ to indicate the signature of the indefinite scalar product $\langle\ ,\ \rangle$. This scalar product is called the *Lie metric* or *Lie scalar product*, and the quadric Q^{n+1} is called the *Lie quadric*.

We now give the details of how the set of points on the Lie quadric corresponds to the set of all oriented hyperspheres, oriented hyperplanes and point spheres in $\mathbf{R}^n$, or equivalently, to the set of all oriented hyperspheres and point spheres in S^n.

First consider a point $[x] = [(x_1,\ldots,x_{n+3})]$ on Q^{n+1} with last coordinate $x_{n+3} \neq 0$. Then we can divide x by x_{n+3} and represent $[x]$ by a vector of the form

$(\zeta, 1)$ with $\zeta \in W^{n+1}$. Thus, ζ represents an unoriented hypersphere or unoriented hyperplane in $\mathbf{R}^n$ via the Möbius geometric correspondence.

Suppose first that $\zeta_1 + \zeta_2$ is nonzero. Then $[\zeta]$ represents a hypersphere in Möbius geometry via equation (4.8). Specifically, we can divide ζ by $\zeta_1 + \zeta_2$ and get a vector ξ that is projectively equivalent to ζ that satisfies $\xi_1 + \xi_2 = 1$. Then, as in Möbius geometry, $(\xi, \xi) = r^2$ for some $r > 0$, and we can take $p = (\xi_3, \ldots, \xi_{n+2})$ in $\mathbf{R}^n$ so that ξ has the form

$$\xi = \left(\frac{1 + p \cdot p - r^2}{2}, \frac{1 - p \cdot p + r^2}{2}, p \right). \tag{4.21}$$

Since $(\xi, \xi) = r^2$ and $(\zeta, \zeta) = 1$, we see that $\zeta = \pm\xi/r$. So the two unit vectors $\pm\zeta$ in $[\xi] \in \mathbf{RP}^{n+1}$ give rise to two points

$$[(\pm\zeta, 1)] = [(\pm\xi/r, 1)] = [(\xi, \pm r)]$$

in the Lie quadric. We associate these two points to the two orientations of the unoriented hypersphere S in $\mathbf{R}^n$ corresponding to $[\xi] = [\zeta]$ as follows. For $p \in \mathbf{R}^n$ and $r > 0$, and ξ given by equation (4.21), the point $[(\xi, r)]$ in Q^{n+1} corresponds to the oriented hypersphere in $\mathbf{R}^n$ with center p, radius r, and orientation given by the inner field of unit normals. The point $[(\xi, -r)]$ corresponds to the same sphere in $\mathbf{R}^n$ with the opposite orientation.

Next we handle the case where $(\zeta, \zeta) = 1$, but $\zeta_1 + \zeta_2 = 0$. In this case, $[(\zeta, 1)]$ corresponds to an oriented hyperplane in $\mathbf{R}^n$ as follows. Since $\zeta_1 + \zeta_2 = 0$, the vector ζ can be written in the form $\zeta = (h, -h, N)$, with $|N| = 1$ since $(\zeta, \zeta) = 1$. Then the two projective points on Q^{n+1} induced by ζ and $-\zeta$ are

$$[(h, -h, N, \pm 1)]. \tag{4.22}$$

These represent the two orientations of the hyperplane in $\mathbf{R}^n$ with equation $u \cdot N = h$. We adopt the convention that $[(h, -h, N, 1)]$ corresponds to the orientation given by the field of unit normals N, while $[(h, -h, N, -1)] = [(-h, h, -N, 1)]$ corresponds to the opposite orientation.

Finally, we consider the case of $[x] = [(x_1, \ldots, x_{n+3})]$ in Q^{n+1} with $x_{n+3} = 0$. Then if we take $z = (x_1, \ldots, x_{n+2})$, we have

$$0 = \langle x, x \rangle = -x_1^2 + x_2^2 + \ldots + x_{n+2}^2 = (z, z),$$

and so $[z] \in \mathbf{RP}^{n+1}$ represents a point in the Möbius sphere Σ, or equivalently a point in $\mathbf{R}^n \cup \{\infty\}$, where ∞ corresponds to the improper point $[(1, -1, 0, \ldots, 0)] \in \Sigma$. Thus, $[x]$ represents a *point sphere* or sphere with radius zero in $\mathbf{R}^n \cup \{\infty\}$. Point spheres do not have an orientation assigned to them.

Lie coordinates of oriented spheres

In summary, we have the following bijective correspondence between the set of all oriented hyperspheres, oriented hyperplanes and point spheres in $\mathbf{R}^n \cup \{\infty\}$ and the set of points on the Lie quadric Q^{n+1}.

Euclidean	**Lie**
points : $u \in \mathbf{R}^n$	$\left[\left(\frac{1+u\cdot u}{2}, \frac{1-u\cdot u}{2}, u, 0\right)\right]$
∞	$[(1,-1,0,0)]$
spheres: center p, signed radius r	$\left[\left(\frac{1+p\cdot p-r^2}{2}, \frac{1-p\cdot p+r^2}{2}, p, r\right)\right]$
planes: $u \cdot N = h$, unit normal N	$[(h,-h,N,1)]$

(4.23)

We will use the term *Lie sphere* to denote any oriented hypersphere, oriented hyperplane, or point sphere in $\mathbf{R}^n \cup \{\infty\}$, and we will refer to the coordinates on the right side of the table above as the *Lie coordinates* of the corresponding Lie sphere.

We can begin with a point $[x] = [(x_1, \ldots, x_{n+3})]$ in Q^{n+1} and find the corresponding Euclidean object as follows. If $x_1 + x_2 \neq 0$, then we can divide x by $x_1 + x_2$ to obtain a point $y = (y_1, \ldots, y_{n+3})$ with $y_1 + y_2 = 1$. Then if $y_{n+3} \neq 0$, we can take $r = y_{n+3}$, and $p = (y_3, \ldots, y_{n+2})$, and see that y is in the correct form for the Lie coordinates of the oriented hypersphere with center $p \in \mathbf{R}^n$ and signed radius r. If $y_{n+3} = 0$, then y is in the correct form for the point $u = (y_3, \ldots, y_{n+2})$ in $\mathbf{R}^n$.

Next if $x_1 + x_2 = 0$ and $x_{n+3} \neq 0$, then we can divide x by x_{n+3} to get a vector $y = (h, -h, N, 1)$, which clearly represents an oriented hyperplane in $\mathbf{R}^n$. Finally, if $x_1 + x_2 = 0$ and $x_{n+3} = 0$, then the equation $\langle x, x\rangle = 0$ forces x to have the form $(h, -h, 0, \ldots, 0) \simeq (1, -1, 0, \ldots, 0)$, and so $[x]$ is the improper point corresponding to the point ∞.

Oriented spheres in S^n and H^n

If we wish to consider oriented hyperspheres and point spheres in the unit sphere S^n in $\mathbf{R}^{n+1}$, then the table above can be simplified. First, we have shown that in Möbius geometry, the unoriented hypersphere S in S^n with center $p \in S^n$ and spherical radius ρ, $0 < \rho < \pi$, corresponds to the point $[\xi] = [(\cos\rho, p)]$ in $\mathbf{RP}^{n+1}$. To correspond the two orientations of this sphere to points on the Lie quadric, we first note that

$$(\xi, \xi) = -\cos^2\rho + 1 = \sin^2\rho.$$

Since $\sin \rho > 0$ for $0 < \rho < \pi$, we can divide ξ by $\sin \rho$ and consider the two vectors $\zeta = \pm \xi / \sin \rho$ that satisfy $(\zeta, \zeta) = 1$. We then map these two points into the Lie quadric to get the points

$$[(\zeta, 1)] = [(\xi, \pm \sin \rho)] = [(\cos \rho, p, \pm \sin \rho)].$$

in Q^{n+1}. We can incorporate the sign of the last coordinate into the radius and thereby arrange that the oriented sphere S with signed radius $\rho \neq 0$, where $-\pi < \rho < \pi$, and center p corresponds to the point

$$[x] = [(\cos \rho, p, \sin \rho)]. \tag{4.24}$$

in Q^{n+1}. This formula still makes sense if the radius $\rho = 0$, in which case it yields the point sphere $[(1, p, 0)]$.

We adopt the convention that the positive radius ρ in (4.24) corresponds to the orientation of the sphere given by the field of unit normals which are tangent vectors to geodesics from $-p$ to p, and a negative radius corresponds to the opposite orientation. Each oriented sphere can be considered in two ways, with center p and signed radius $\rho, -\pi < \rho < \pi$, or with center $-p$ and the appropriate signed radius $\rho \pm \pi$.

For a given point $[x]$ in the quadric Q^{n+1}, we can determine the corresponding oriented hypersphere or point sphere in S^n as follows. Multiplying by -1, if necessary, we can arrange that the first coordinate x_1 of x is nonnegative. If x_1 is positive, then it follows from equation (4.24) that the center p and signed radius $\rho, -\pi/2 < \rho < \pi/2$, are given by

$$\tan \rho = x_{n+3}/x_1, \quad p = (x_2, \ldots, x_{n+2})/(x_1^2 + x_{n+3}^2)^{1/2}. \tag{4.25}$$

If $x_1 = 0$, then x_{n+3} is nonzero, and we can divide by x_{n+3} to obtain a point with coordinates $(0, p, 1)$. This corresponds to the oriented hypersphere in S^n with center p and signed radius $\pi/2$, which is a great sphere in S^n.

We can also find a representation for oriented hyperspheres in hyperbolic space H^n. We know from equation (4.16) in Möbius geometry that the unoriented hypersphere S in H^n with center $p \in H^n$ and hyperbolic radius ρ corresponds to the point $[p + \cosh \rho \, e_2]$ in $\mathbf{RP}^{n+1}$. Following exactly the same procedure as in the spherical case, we find that the oriented hypersphere in H^n with center p and signed radius ρ corresponds to a point $[x] \in Q^{n+1}$ given by

$$[x] = [p + \cosh \rho \, e_2 + \sinh \rho \, e_{n+3}]. \tag{4.26}$$

Oriented contact of spheres

As we saw in the previous section, the angle between two spheres is the fundamental geometric quantity in Möbius geometry, and it is the quantity that is preserved by Möbius transformations. In Lie's geometry of oriented spheres, the corresponding fundamental notion is that of oriented contact of spheres. By definition, two oriented spheres S_1 and S_2 in $\mathbf{R}^n$ are in *oriented contact* if they are tangent to each other and they have the same orientation at the point of contact. (See Figures 4.4 and 4.5 for the two possibilities.)

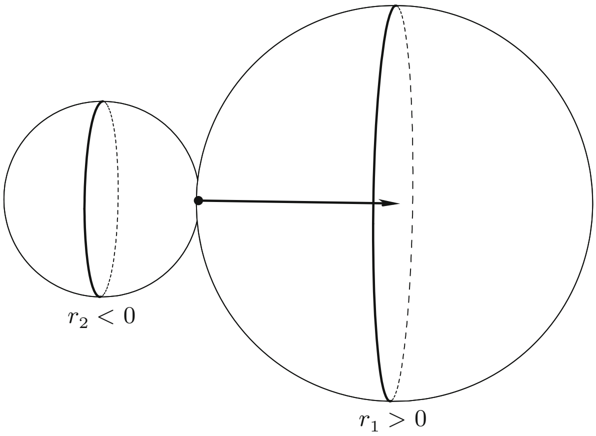
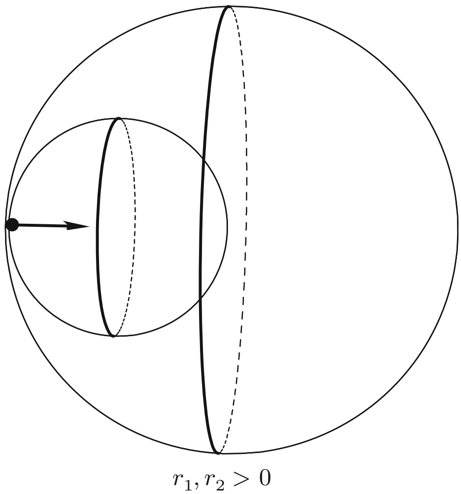


Fig. 4.4 Oriented contact of spheres, first case

Fig. 4.5 Oriented contact of spheres, second case



If p_1 and p_2 are the respective centers of S_1 and S_2, and r_1 and r_2 are their respective signed radii, then the analytic condition for oriented contact is

$$|p_1 - p_2| = |r_1 - r_2|. \tag{4.27}$$

Similarly, we say that an oriented hypersphere sphere S with center p and signed radius r and an oriented hyperplane π with unit normal N and equation $u \cdot N = h$ are in oriented contact if π is tangent to S and their orientations agree at the point of contact. This condition is given by the equation

$$p \cdot N = r + h. \tag{4.28}$$

Next we say that two oriented planes π_1 and π_2 are in oriented contact if their unit normals N_1 and N_2 are the same. These planes can be considered to be two oriented spheres in oriented contact at the improper point. Finally, a proper point u in $\mathbf{R}^n$ is in oriented contact with a sphere or a plane if it lies on the sphere or plane, and the improper point is in oriented contact with each plane, since it lies on each plane.

An important fact in Lie sphere geometry is that if S_1 and S_2 are two Lie spheres which are represented as in equation (4.23) by $[k_1]$ and $[k_2]$, then the analytic condition for oriented contact is equivalent to the equation

$$\langle k_1, k_2 \rangle = 0. \tag{4.29}$$

This can be checked easily by a direct calculation.

Parabolic pencils of spheres

By standard linear algebra in indefinite inner product spaces (see, for example, [77, p. 21]), it follows from the fact that the signature of $\mathbf{R}^{n+3}_2$ is $(n+1, 2)$ that the Lie quadric contains projective lines in $\mathbf{RP}^{n+2}$, but no linear subspaces of $\mathbf{RP}^{n+2}$ of higher dimension. These projective lines on Q^{n+1} play a crucial role in the theory of submanifolds in the context of Lie sphere geometry.

One can show further that if $[k_1]$ and $[k_2]$ are two points of Q^{n+1}, then the line $[k_1, k_2]$ in $\mathbf{RP}^{n+2}$ lies on Q^{n+1} if and only if the spheres corresponding to $[k_1]$ and $[k_2]$ are in oriented contact, i.e., $\langle k_1, k_2 \rangle = 0$. Moreover, if the line $[k_1, k_2]$ lies on Q^{n+1}, then the set of spheres in $\mathbf{R}^n$ corresponding to points on the line $[k_1, k_2]$ is precisely the set of all spheres in oriented contact with both $[k_1]$ and $[k_2]$. Such a 1-parameter family of spheres is called a *parabolic pencil* of spheres in $\mathbf{R}^n \cup \{\infty\}$.

Each parabolic pencil contains exactly one point sphere, and if that point sphere is a proper point, then the parabolic pencil contains exactly one hyperplane π in $\mathbf{R}^n$ (see Figure 4.6), and the pencil consists of all spheres in oriented contact with a certain oriented plane π at p. Thus, we can associate the parabolic pencil with the point (p, N) in the unit tangent bundle to $\mathbf{R}^n \cup \{\infty\}$, where N is the unit normal to the oriented plane π.

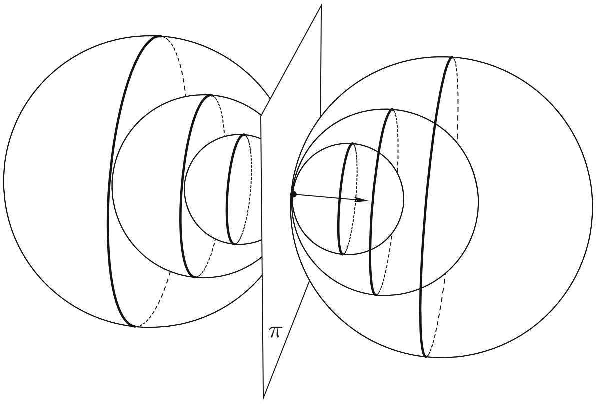


Fig. 4.6 Parabolic pencil of spheres

If the point sphere in the pencil is the improper point, then the parabolic pencil is a family of parallel hyperplanes in oriented contact at the improper point. If N is the common unit normal to all of these planes, then we can associate the pencil with the point (∞, N) in the unit tangent bundle to $\mathbf{R}^n \cup \{\infty\}$.

Similarly, we can establish a correspondence between parabolic pencils and elements of the unit tangent bundle T_1S^n that is expressed in terms of the spherical metric on S^n. If ℓ is a line on the quadric, then ℓ intersects both $e_1^\perp$ and $e_{n+3}^\perp$ at exactly one point, where $e_1 = (1, 0, \ldots, 0)$ and $e_{n+3} = (0, \ldots, 0, 1)$. So the parabolic pencil corresponding to ℓ contains exactly one point sphere (orthogonal to e_{n+3}) and one great sphere (orthogonal to e_1), given respectively by the points,

$$[k_1] = [(1, p, 0)], \quad [k_2] = [(0, \xi, 1)]. \tag{4.30}$$

Since ℓ lies on the quadric we know that $\langle k_1, k_2 \rangle = 0$, and this condition is equivalent to the condition $p \cdot \xi = 0$, i.e., ξ is tangent to S^n at p. Thus, the parabolic pencil of spheres corresponding to the line ℓ can be associated with the point (p, ξ) in T_1S^n. More specifically, the line ℓ can be parametrized as

$$[K_t] = [\cos t\, k_1 + \sin t\, k_2] = [(\cos t, \cos t\, p + \sin t\, \xi, \sin t)].$$

From equation (4.24) above, we see that $[K_t]$ corresponds to the oriented sphere in S^n with center

$$p_t = \cos t\, p + \sin t\, \xi, \tag{4.31}$$

and signed radius t. The pencil consists of all oriented spheres in S^n in oriented contact with the great sphere corresponding to $[k_2]$ at the point (p, ξ) in T_1S^n. Their centers p_t lie along the geodesic in S^n with initial point p and initial velocity vector ξ. Detailed proofs of all these facts are given in [77, pp. 21–23].

Lie sphere transformations

We conclude this section with a discussion of Lie sphere transformations. By definition, a *Lie sphere transformation* is a projective transformation of $\mathbf{RP}^{n+2}$ which maps the Lie quadric Q^{n+1} to itself. In terms of the geometry of $\mathbf{R}^n$ or S^n, a Lie sphere transformation maps Lie spheres to Lie spheres, and since it is a projective transformation, it maps lines on Q^{n+1} to lines on Q^{n+1}. Thus, it preserves oriented contact of spheres in $\mathbf{R}^n$ or S^n. Conversely, Pinkall [443] (see also [77, pp. 28–30]) proved the so-called "Fundamental Theorem of Lie sphere geometry," which states that any line preserving diffeomorphism of Q^{n+1} is the restriction to Q^{n+1} of a projective transformation, that is, a transformation of the space of oriented spheres which preserves oriented contact is a Lie sphere transformation.

By the same type of reasoning given for Möbius transformations, one can show that the group G of Lie sphere transformations is isomorphic to the group $O(n+1,2)/\{\pm I\}$, where $O(n+1,2)$ is the group of orthogonal transformations of $\mathbf{R}^{n+3}_2$. As with the Möbius group, it follows from the theorem of Cartan and Dieudonné (see [77, pp. 30–34]) that the Lie sphere group G is generated by Lie inversions, that is, projective transformations that are induced by reflections in $O(n+1,2)$.

The Möbius group H can be considered to be a subgroup of G in the following manner. Each Möbius transformation on the space of unoriented spheres, naturally induces two Lie sphere transformations on the space Q^{n+1} of oriented spheres as follows. If A is in $O(n+1,1)$, then we can extend A to a transformation B in $O(n+1,2)$ by setting $B=A$ on $\mathbf{R}^{n+2}_1$ and $B(e_{n+3})=e_{n+3}$. In terms the standard orthonormal basis in $\mathbf{R}^{n+3}_2$, the transformation B has the matrix representation,

$$B = \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix}. \tag{4.32}$$

Although A and $-A$ induce the same Möbius transformation in H, the Lie transformation $P(B)$ is not the same as the Lie transformation $P(C)$ induced by the matrix

$$C = \begin{bmatrix} -A & 0 \\ 0 & 1 \end{bmatrix} \simeq \begin{bmatrix} A & 0 \\ 0 & -1 \end{bmatrix},$$

where $\simeq$ denotes equivalence as projective transformations. Note that $P(B) = \Gamma P(C)$, where Γ is the Lie transformation represented in matrix form by

$$\Gamma = \begin{bmatrix} I & 0 \\ 0 & -1 \end{bmatrix} \simeq \begin{bmatrix} -I & 0 \\ 0 & 1 \end{bmatrix}.$$

From equation (4.23), we see that Γ has the effect of changing the orientation of every oriented sphere or plane. The transformation Γ is called the *change of orientation transformation* or "Richtungswechsel" in German. Hence, the two Lie sphere transformations induced by the Möbius transformation $P(A)$ differ by this change of orientation factor.

Thus, the group of Lie sphere transformations induced from Möbius transformations is isomorphic to $O(n+1,1)$. This group consists of those Lie transformations that map $[e_{n+3}]$ to itself, and it is a double covering of the Möbius group H. Since these transformations are induced from orthogonal transformations of $\mathbf{R}^{n+3}_2$, they also map $e^{\perp}_{n+3}$ to itself, and thereby map point spheres to point spheres. When working in the context of Lie sphere geometry, we will refer to these transformations as "Möbius transformations."

Laguerre transformations

A Lie sphere transformation that maps the improper point to itself is a *Laguerre transformation*. Since oriented contact must be preserved, Laguerre transformations can also be characterized as those Lie sphere transformations that take planes to planes. Like Möbius geometry, Laguerre geometry can be studied on its own, independent of Lie sphere geometry (see, for example, Blaschke [42]). One can show (see, for example, [77, p. 47]) that the group G of Lie sphere transformations is generated by the union of the groups of Möbius and Laguerre.

An important Laguerre transformation in the study of submanifolds is *Euclidean parallel transformation* P_t that adds t to the signed radius of every oriented sphere in $\mathbf{R}^n$ while keeping the center fixed. In terms of the standard basis of $\mathbf{R}^{n+3}_2$, the transformation P_t has the matrix representation,

$$P_t = \begin{bmatrix} 1-(t^2/2) & -t^2/2 & 0\dots0 & -t \\ t^2/2 & 1+(t^2/2) & 0\dots0 & t \\ 0 & 0 & I & 0 \\ t & t & 0\dots0 & 1 \end{bmatrix}. \tag{4.33}$$

One can check that if the column vector consisting of the Lie coordinates (see equation (4.23)) of the oriented sphere with center $p \in \mathbf{R}^n$ and signed radius r is multiplied on the left by this matrix P_t, the result is the column vector consisting of the Lie coordinates of the oriented hypersphere with center p and signed radius $r+t$.

There is also a parallel transformation that adds t to the signed radius of every oriented sphere in S^n or H^n while keeping the center fixed. In the case of S^n, using the fact that $[x] = [(\cos\rho, p, \sin\rho)]$ represents the oriented hypersphere in S^n with center $p \in S^n$ and signed radius ρ, one can check that *spherical parallel transformation* P_t is given by the following transformation in $O(n+1,2)$,

$$\begin{aligned} P_t e_1 &= \cos t\, e_1 + \sin t\, e_{n+3}, \\ P_t e_{n+3} &= -\sin t\, e_1 + \cos t\, e_{n+3}, \\ P_t e_i &= e_i, \quad 2 \le i \le n+2. \end{aligned} \tag{4.34}$$

In hyperbolic space, the sphere with center $p \in H^n$ and signed radius ρ corresponds to the point $[p + \cosh\rho\, e_2 + \sinh\rho\, e_{n+3}]$ in Q^{n+1}, and so *hyperbolic parallel transformation* is accomplished by the transformation,

$$\begin{aligned} P_t e_i &= e_i, \quad i = 1, 3, \ldots, n+2. \\ P_t e_2 &= \cosh t\, e_2 + \sinh t\, e_{n+3}, \\ P_t e_{n+3} &= \sinh t\, e_2 + \cosh t\, e_{n+3}. \end{aligned} \tag{4.35}$$

The following theorem of Cecil and Chern [79] (see also [77, p. 49]) demonstrates the important role played by parallel transformations.

Theorem 4.3. *Any Lie sphere transformation α can be written as*

$$\alpha = \phi P_t \psi,$$

where ϕ and ψ are Möbius transformations and P_t is some Euclidean, spherical or hyperbolic parallel transformation.

4.3 Contact Structure and Legendre Submanifolds

The goal of this section is to define a contact structure on the unit tangent bundle T_1S^n and on the $(2n-1)$-dimensional manifold Λ^{2n-1} of projective lines on the Lie quadric Q^{n+1}, and to describe its associated Legendre submanifolds. This will enable us to study submanifolds of $\mathbf{R}^n$ or S^n within the context of Lie sphere geometry in a natural way. This theory was first developed extensively in a modern setting by Pinkall [447] (see also Cecil–Chern [79] or [77, pp. 51–60]).

We consider T_1S^n to be the $(2n-1)$-dimensional submanifold of

$$S^n \times S^n \subset \mathbf{R}^{n+1} \times \mathbf{R}^{n+1}$$

given by

$$T_1S^n = \{(x, \xi) \mid |x| = 1,\ |\xi| = 1,\ x \cdot \xi = 0\}. \tag{4.36}$$

As shown in the previous section, the points on a line ℓ lying on Q^{n+1} correspond to the spheres in a parabolic pencil of spheres in S^n. In particular, as in equation (4.30), ℓ contains one point $[k_1] = [(1, x, 0)]$ corresponding to a point sphere in S^n, and one point $[k_2] = [(0, \xi, 1)]$ corresponding to a great sphere in S^n, where the coordinates are with respect to the standard orthonormal basis $\{e_1, \ldots, e_{n+3}\}$ of $\mathbf{R}^{n+3}_2$. Thus we get a bijective correspondence between the points (x, ξ) of T_1S^n and the space Λ^{2n-1} of lines on Q^{n+1} given by the map:

$$(x, \xi) \mapsto [Y_1(x, \xi), Y_{n+3}(x, \xi)], \tag{4.37}$$

where

$$Y_1(x, \xi) = (1, x, 0), \quad Y_{n+3}(x, \xi) = (0, \xi, 1). \tag{4.38}$$

We use this correspondence to place a natural differentiable structure on Λ^{2n-1} in such a way as to make the map in equation (4.37) a diffeomorphism.

We now show how to define a contact structure on the manifold T_1S^n. By the diffeomorphism in equation (4.37), this also determines a contact structure on Λ^{2n-1}. Recall that a $(2n-1)$-dimensional manifold V^{2n-1} is said to be a *contact manifold* if it carries a globally defined 1-form ω such that

$$\omega \wedge (d\omega)^{n-1} \neq 0 \tag{4.39}$$

at all points of V^{2n-1}. Such a form ω is called a *contact form*. A contact form ω determines a codimension one distribution (the *contact distribution*) D on V^{2n-1} defined by

$$D_p = \{Y \in T_pV^{2n-1} \mid \omega(Y) = 0\}, \tag{4.40}$$

for $p \in V^{2n-1}$. This distribution is as far from being integrable as possible, in that there exist integral submanifolds of D of dimension $n-1$ but none of higher dimension (see, for example, [77, p. 57]). The distribution D determines the corresponding contact form ω up to multiplication by a nonvanishing smooth function.

A tangent vector to T_1S^n at a point (x, ξ) can be written in the form (X, Z) where

$$X \cdot x = 0, \quad Z \cdot \xi = 0. \tag{4.41}$$

Differentiation of the condition $x \cdot \xi = 0$ implies that (X, Z) also satisfies

$$X \cdot \xi + Z \cdot x = 0. \tag{4.42}$$

We now show that the form ω defined by

$$\omega(X, Z) = X \cdot \xi, \tag{4.43}$$

is a contact form on T_1S^n. At a point (x, ξ), the distribution D is the $(2n-2)$-dimensional space of vectors (X, Z) satisfying $X \cdot \xi = 0$, as well as the equations (4.41) and (4.42). The equation $X \cdot \xi = 0$ together with equation (4.42) implies that

$$Z \cdot x = 0, \tag{4.44}$$

for vectors (X, Z) in D.

Note that if we take $Y_1(x,\xi) = (1,x,0)$, and $Y_{n+3}(x,\xi) = (0,\xi,1)$ as in equation (4.38), then

$$dY_1(X,Z) = (0,X,0), \quad dY_{n+3}(X,Z) = (0,Z,0). \tag{4.45}$$

Thus,

$$\langle dY_1(X,Z), Y_{n+3}(x,\xi)\rangle = X \cdot \xi = \omega(X,Z). \tag{4.46}$$

To prove that the form ω defined by equation (4.43) is a contact form and to study submanifolds in the context of Lie sphere geometry, we use the method of moving frames, as in Cecil–Chern [79] or the book [77]. (See also the paper of Jensen [229] and the forthcoming book of Jensen, Musso and Nicolodi [230].)

Moving frames in Lie sphere geometry

Since we want to define frames on the manifold Λ^{2n-1}, it is better to use frames for which some of the vectors are lightlike, rather than orthonormal frames. For the sake of brevity, we use the following ranges of indices in this section:

$$1 \leq a,b,c \leq n+3, \quad 3 \leq i,j,k \leq n+1. \tag{4.47}$$

A *Lie frame* is an ordered set of vectors $\{Y_1,\ldots,Y_{n+3}\}$ in $\mathbf{R}^{n+3}_2$ satisfying the relations

$$\langle Y_a, Y_b\rangle = g_{ab}, \tag{4.48}$$

for

$$[g_{ab}] = \begin{bmatrix} J & 0 & 0 \\ 0 & I_{n-1} & 0 \\ 0 & 0 & J \end{bmatrix}, \tag{4.49}$$

where I_{n-1} is the $(n-1)\times(n-1)$ identity matrix and

$$J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{4.50}$$

If $(y_1,\ldots,y_{n+3})$ are homogeneous coordinates on $\mathbf{RP}^{n+2}$ with respect to a Lie frame, then the Lie metric has the form

$$\langle y,y\rangle = 2(y_1y_2 + y_{n+2}y_{n+3}) + y_3^2 + \cdots + y_{n+1}^2. \tag{4.51}$$

The space of all Lie frames can be identified with the group $O(n+1,2)$ of which the Lie sphere group G, being isomorphic to $O(n+1,2)/\{\pm I\}$, is a quotient group. In this space, we use the *Maurer–Cartan forms* ω_a^b defined by the equation

$$dY_a = \sum \omega_a^b Y_b, \tag{4.52}$$

and we adopt the convention that the sum is always over the repeated index. Differentiating equation (4.48), we get

$$\omega_{ab} + \omega_{ba} = 0, \tag{4.53}$$

where

$$\omega_{ab} = \sum g_{bc}\omega_a^c. \tag{4.54}$$

Equation (4.53) says that the following matrix is skew-symmetric,

$$[\omega_{ab}] = \begin{bmatrix} \omega_1^2 & \omega_1^1 & \omega_1^i & \omega_1^{n+3} & \omega_1^{n+2} \\ \omega_2^2 & \omega_2^1 & \omega_2^i & \omega_2^{n+3} & \omega_2^{n+2} \\ \omega_j^2 & \omega_j^1 & \omega_j^i & \omega_j^{n+3} & \omega_j^{n+2} \\ \omega_{n+2}^2 & \omega_{n+2}^1 & \omega_{n+2}^i & \omega_{n+2}^{n+3} & \omega_{n+2}^{n+2} \\ \omega_{n+3}^2 & \omega_{n+3}^1 & \omega_{n+3}^i & \omega_{n+3}^{n+3} & \omega_{n+3}^{n+2} \end{bmatrix}. \tag{4.55}$$

Taking the exterior derivative of equation (4.52) yields the *Maurer–Cartan equations*,

$$d\omega_a^b = \sum \omega_a^c \wedge \omega_c^b. \tag{4.56}$$

To show that the form defined by equation (4.43) is a contact form on T_1S^n we want to choose a local frame $\{Y_1,\ldots,Y_{n+3}\}$ on T_1S^n with Y_1 and Y_{n+3} given by equation (4.38). When we transfer this frame to Λ^{2n-1}, it will have the property that for each point $\lambda \in \Lambda^{2n-1}$, the line $[Y_1, Y_{n+3}]$ of the frame at λ is the line on the quadric Q^{n+1} corresponding to λ.

On a sufficiently small open subset U in T_1S^n, we can find smooth mappings,

$$v_i : U \to \mathbf{R}^{n+1}, \quad 3 \leq i \leq n+1,$$

such that at each point $(x,\xi) \in U$, the vectors $v_3(x,\xi),\ldots,v_{n+1}(x,\xi)$ are unit vectors orthogonal to each other and to x and ξ. By equations (4.41) and (4.42), we see that the vectors

$$\{(v_i,0),(0,v_i),(\xi,-x)\}, \quad 3 \leq i \leq n+1, \tag{4.57}$$

form a basis to the tangent space to T_1S^n at (x, ξ). We now define a Lie frame on U as follows:

$$\begin{aligned} Y_1(x,\xi) &= (1,x,0), \\ Y_2(x,\xi) &= (-1/2,x/2,0), \\ Y_i(x,\xi) &= (0,v_i(x,\xi),0), \quad 3 \le i \le n+1, \\ Y_{n+2}(x,\xi) &= (0,\xi/2,-1/2) \\ Y_{n+3}(x,\xi) &= (0,\xi,1). \end{aligned} \tag{4.58}$$

Note that Y_1 and Y_{n+3} are defined on all of T_1S^n. We compute the derivatives dY_1 and dY_{n+3} and find

$$\begin{aligned} dY_1(v_i,0) &= (0,v_i,0) = Y_i, \\ dY_1(0,v_i) &= (0,0,0), \\ dY_1(\xi,-x) &= (0,\xi,0) = Y_{n+2} + (1/2)Y_{n+3}, \end{aligned} \tag{4.59}$$

and

$$\begin{aligned} dY_{n+3}(v_i,0) &= (0,0,0), \\ dY_{n+3}(0,v_i) &= (0,v_i,0) = Y_i, \\ dY_{n+3}(\xi,-x) &= (0,-x,0) = (-1/2)Y_1 - Y_2. \end{aligned} \tag{4.60}$$

Comparing these equations with the equation (4.52), we see that the 1-forms,

$$\{\omega_1^i, \omega_{n+3}^i, \omega_1^{n+2}\}, \quad 3 \le i \le n+1, \tag{4.61}$$

form the dual basis to the basis given in (4.57) for the tangent space to T_1S^n at (x, ξ). Furthermore,

$$\omega_1^{n+2}(X,Z) = \langle dY_1(X,Z), Y_{n+3}(x,\xi)\rangle = X \cdot \xi = \omega(X,Z), \tag{4.62}$$

so ω_1^{n+2} is the form ω in equation (4.43).

To prove that ω_1^{n+2} satisfies the condition (4.39) for a contact form, we use the Maurer–Cartan equations and the skew-symmetry of the matrix in equation (4.55) to show by a straightforward calculation that

$$\begin{aligned} \omega_1^{n+2} \wedge (d\omega_1^{n+2})^{n-1} &= \omega_1^{n+2} \wedge (\textstyle\sum \omega_1^i \wedge \omega_i^{n+2})^{n-1} \\ &= (-1)^{n-1}(n-1)! \quad \omega_1^{n+2} \wedge \omega_1^3 \wedge \omega_{n+3}^3 \wedge \cdots \wedge \omega_1^{n+1} \wedge \omega_{n+3}^{n+1} \neq 0. \end{aligned} \tag{4.63}$$

Here the last form is nonzero because the set (4.61) is a basis for the cotangent space to T_1S^n at (x, ξ). We can use the diffeomorphism given in (4.37) to transfer this contact form ω_1^{n+2} to the manifold Λ^{2n-1} of lines on the Lie quadric.

Finally, suppose that

$$Z_1 = \alpha Y_1 + \beta Y_{n+3}, \quad Z_{n+3} = \gamma Y_1 + \delta Y_{n+3}, \tag{4.64}$$

for smooth functions $\alpha, \beta, \gamma, \delta$ with $\alpha\delta - \beta\gamma \neq 0$ on T_1S^n, so that the line $[Z_1, Z_{n+3}]$ equals the line $[Y_1, Y_{n+3}]$ at all points of T_1S^n. Let θ_1^{n+2} be the 1-form defined by $\theta_1^{n+2} = \langle dZ_1, Z_{n+3}\rangle$. Then using equation (4.48), we can compute

$$\begin{aligned} \theta_1^{n+2} &= \langle dZ_1, Z_{n+3}\rangle = \langle d(\alpha Y_1 + \beta Y_{n+3}), \gamma Y_1 + \delta Y_{n+3}\rangle \\ &= \alpha\delta\langle dY_1, Y_{n+3}\rangle + \beta\gamma\langle dY_{n+3}, Y_1\rangle = (\alpha\delta - \beta\gamma)\langle dY_1, Y_{n+3}\rangle \\ &= (\alpha\delta - \beta\gamma)\omega_1^{n+2}. \end{aligned} \tag{4.65}$$

Thus, θ_1^{n+2} is also a contact form on T_1S^n.

Legendre submanifolds

Returning briefly to the general theory, let V^{2n-1} be a contact manifold with contact form ω and corresponding contact distribution D, as in equation (4.40). An immersion $\phi : W^k \to V^{2n-1}$ of a smooth k-dimensional manifold W^k into V^{2n-1} is called an *integral submanifold* of the distribution D if $\phi^*\omega = 0$ on W^k, i.e., for each tangent vector Y at each point $w \in W$, the vector $d\phi(Y)$ is in the distribution D at the point $\phi(w)$. (See Blair [41, p. 36].) It is well known (see, for example, [77, p. 57]) that the contact distribution D has integral submanifolds of dimension $n-1$, but none of higher dimension. These integral submanifolds of maximal dimension are called *Legendre submanifolds* of the contact structure.

In our specific case, we now formulate conditions for a smooth map $\mu : M^{n-1} \to T_1S^n$ to be a Legendre submanifold. We consider T_1S^n as a submanifold of $S^n \times S^n$ as in equation (4.36), and so we can write $\mu = (f, \xi)$, where f and ξ are both smooth maps from M^{n-1} to S^n. We have the following theorem (see [77, p. 58]) giving necessary and sufficient conditions for μ to be a Legendre submanifold.

Theorem 4.4. *A smooth map $\mu = (f, \xi)$ from an $(n-1)$-dimensional manifold M^{n-1} into T_1S^n is a Legendre submanifold if and only if the following three conditions are satisfied.*

(1) *Scalar product conditions:* $f \cdot f = 1, \quad \xi \cdot \xi = 1, \quad f \cdot \xi = 0.$
(2) *Immersion condition: there is no nonzero tangent vector X at any point $x \in M^{n-1}$ such that $df(X)$ and $d\xi(X)$ are both equal to zero.*
(3) *Contact condition:* $df \cdot \xi = 0.$

Note that by equation (4.36), the scalar product conditions are precisely the conditions necessary for the image of the map $\mu = (f, \xi)$ to be contained in T_1S^n. Next, since $d\mu(X) = (df(X), d\xi(X))$, Condition (2) is necessary and sufficient for μ to be an immersion. Finally, from equation (4.43), we see that $\omega(d\mu(X)) = df(X) \cdot \xi(x)$, for each $X \in T_xM^{n-1}$. Hence Condition (3) is equivalent to the requirement that $\mu^*\omega = 0$ on M^{n-1}.

We now want to translate these conditions into the projective setting, and find necessary and sufficient conditions for a smooth map $\lambda : M^{n-1} \to \Lambda^{2n-1}$ to be a Legendre submanifold. We again make use of the diffeomorphism defined in equation (4.37) between T_1S^n and Λ^{2n-1}.

For each $x \in M^{n-1}$, we know that $\lambda(x)$ is a line on the quadric Q^{n+1}. This line contains exactly one point $[Y_1(x)] = [(1, f(x), 0)]$ corresponding to a point sphere in S^n, and one point $[Y_{n+3}(x)] = [(0, \xi(x), 1)]$ corresponding to a great sphere in S^n. These two formulas define maps f and ξ from M^{n-1} to S^n which depend on the choice of orthonormal basis $\{e_1, \ldots, e_{n+2}\}$ for the orthogonal complement of e_{n+3}.

The map $[Y_1]$ from M^{n-1} to Q^{n+1} is called the *Möbius projection* or *point sphere map* of λ, and the map $[Y_{n+3}]$ from M^{n-1} to Q^{n+1} is called the *great sphere map*. The maps f and ξ are called the *spherical projection* of λ, and the *spherical field of unit normals* of λ, respectively.

In this way, λ determines a map $\mu = (f, \xi)$ from M^{n-1} to T_1S^n, and because of the diffeomorphism (4.37), λ is a Legendre submanifold if and only if μ satisfies the conditions of Theorem 4.4.

It is often useful to have conditions for when λ determines a Legendre submanifold that do not depend on the special parametrization of λ in terms of the point sphere and great sphere maps, $[Y_1]$ and $[Y_{n+3}]$. In fact, in many applications of Lie sphere geometry to submanifolds of S^n or $\mathbf{R}^n$, it is better to consider $\lambda = [Z_1, Z_{n+3}]$, where Z_1 and Z_{n+3} are not the point sphere and great sphere maps.

Legendre submanifolds in Lie sphere geometry

Pinkall [447] gave the following projective formulation of the conditions needed for a Legendre submanifold. In his paper, Pinkall referred to a Legendre submanifold as a "Lie geometric hypersurface." The proof that the three conditions of the theorem below are equivalent to the three conditions of Theorem 4.4 can be found in [77, pp. 59–60].

Theorem 4.5. *Let $\lambda : M^{n-1} \to \Lambda^{2n-1}$ be a smooth map with $\lambda = [Z_1, Z_{n+3}]$, where Z_1 and Z_{n+3} are smooth maps from M^{n-1} into $\mathbf{R}^{n+3}_2$. Then λ determines a Legendre submanifold if and only if Z_1 and Z_{n+3} satisfy the following conditions.*

(1) *Scalar product conditions: for each $x \in M^{n-1}$, the vectors $Z_1(x)$ and $Z_{n+3}(x)$ are linearly independent and*

$$\langle Z_1, Z_1 \rangle = 0, \quad \langle Z_{n+3}, Z_{n+3} \rangle = 0, \quad \langle Z_1, Z_{n+3} \rangle = 0.$$

(2) *Immersion condition: there is no nonzero tangent vector X at any point $x \in M^{n-1}$ such that $dZ_1(X)$ and $dZ_{n+3}(X)$ are both in*

$$\text{Span}\,\{Z_1(x), Z_{n+3}(x)\}.$$

(3) *Contact condition: $\langle dZ_1, Z_{n+3}\rangle = 0$.*

These conditions are invariant under a reparametrization $\lambda = [W_1, W_{n+3}]$, where $W_1 = \alpha Z_1 + \beta Z_{n+3}$ and $W_{n+3} = \gamma Z_1 + \delta Z_{n+3}$, for smooth functions $\alpha, \beta, \gamma, \delta$ on M^{n-1} with $\alpha\delta - \beta\gamma \neq 0$.

The Legendre lift of a submanifold of a real space form

Every oriented hypersurface in a real space form S^n, $\mathbf{R}^n$ or H^n naturally induces a Legendre submanifold of Λ^{2n-1}, as does every submanifold of codimension $m > 1$ in these spaces. Conversely, a Legendre submanifold naturally induces a smooth map into S^n which may have singularities. We now study the details of these maps.

Let $f : M^{n-1} \to S^n$ be an immersed oriented hypersurface with field of unit normals $\xi : M^{n-1} \to S^n$. The induced Legendre submanifold is given by the map $\lambda : M^{n-1} \to \Lambda^{2n-1}$ defined by $\lambda(x) = [Y_1(x), Y_{n+3}(x)]$, where

$$Y_1(x) = (1, f(x), 0), \quad Y_{n+3}(x) = (0, \xi(x), 1). \tag{4.66}$$

The map λ is called the *Legendre lift* of the immersion f with field of unit normals ξ.

To show that λ is a Legendre submanifold, we check the conditions of Theorem 4.5. Condition (1) is satisfied since both f and ξ are maps into S^n, and $\xi(x)$ is tangent to S^n at $f(x)$ for each x in M^{n-1}. Since f is an immersion, $dY_1(X) = (0, df(X), 0)$ is not in Span $\{Y_1(x), Y_{n+3}(x)\}$, for any nonzero vector $X \in T_xM^{n-1}$, and so Condition (2) is satisfied. Finally, Condition (3) is satisfied since

$$\langle dY_1(X), Y_{n+3}(x)\rangle = df(X) \cdot \xi(x) = 0,$$

because ξ is a field of unit normals to f.

In the case of a submanifold $\phi : V \to S^n$ of codimension $m + 1$ greater than one, the domain of the Legendre lift is be the unit normal bundle B^{n-1} of the submanifold $\phi(V)$. We consider B^{n-1} to be the submanifold of $V \times S^n$ given by

$$B^{n-1} = \{(x, \xi) | \phi(x) \cdot \xi = 0,\ d\phi(X) \cdot \xi = 0,\ \text{for all } X \in T_xV\}.$$

The *Legendre lift* $\phi(V)$ (or the *Legendre submanifold induced by* ϕ) is the map $\lambda : B^{n-1} \to \Lambda^{2n-1}$ defined by

$$\lambda(x, \xi) = [Y_1(x, \xi), Y_{n+3}(x, \xi)], \tag{4.67}$$

where

$$Y_1(x,\xi) = (1, \phi(x), 0), \quad Y_{n+3}(x,\xi) = (0, \xi, 1). \tag{4.68}$$

Geometrically, $\lambda(x,\xi)$ is the line on the quadric Q^{n+1} corresponding to the parabolic pencil of spheres in S^n in oriented contact at the contact element $(\phi(x), \xi) \in T_1S^n$. In [77, pp. 61–62], we show that λ satisfies the conditions of Theorem 4.5,

Similarly, suppose that $F : M^{n-1} \to \mathbf{R}^n$ is an oriented hypersurface with field of unit normals $\eta : M^{n-1} \to \mathbf{R}^n$, where we identify $\mathbf{R}^n$ with the subspace of $\mathbf{R}^{n+3}_2$ spanned by $\{e_3, \ldots, e_{n+2}\}$. The Legendre lift of (F, η) is the map $\lambda : M^{n-1} \to \Lambda^{2n-1}$ defined by $\lambda = [Y_1, Y_{n+3}]$, where

$$Y_1 = (1 + F \cdot F, 1 - F \cdot F, 2F, 0)/2, \quad Y_{n+3} = (F \cdot \eta, -(F \cdot \eta), \eta, 1). \tag{4.69}$$

By equation (4.23), $[Y_1(x)]$ corresponds to the point sphere and $[Y_{n+3}(x)]$ corresponds to the hyperplane in the parabolic pencil determined by the line $\lambda(x)$ for each $x \in M^{n-1}$. One can easily verify that Conditions (1)–(3) of Theorem 4.5 are satisfied in a manner similar to the spherical case. In the case of a submanifold $\psi : V \to \mathbf{R}^n$ of codimension greater than one, the Legendre lift of ψ is the map λ from the unit normal bundle B^{n-1} to Λ^{2n-1} defined by $\lambda(x,\eta) = [Y_1(x,\eta), Y_{n+3}(x,\eta)]$, where

$$\begin{aligned} Y_1(x,\eta) &= (1 + \psi(x) \cdot \psi(x), 1 - \psi(x) \cdot \psi(x), 2\psi(x), 0)/2, \\ Y_{n+3}(x,\eta) &= (\psi(x) \cdot \eta, -(\psi(x) \cdot \eta), \eta, 1). \end{aligned} \tag{4.70}$$

The verification that the pair $\{Y_1, Y_{n+3}\}$ satisfies conditions (1)–(3) of Theorem 4.5 is similar to that for submanifolds of S^n of codimension greater than one.

Finally, as in Section 4.1, we consider H^n to be the submanifold of the Lorentz space $\mathbf{R}^{n+1}_1$ spanned by $\{e_1, e_3, \ldots, e_{n+2}\}$ defined by:

$$H^n = \{y \in \mathbf{R}^{n+1}_1 | (y,y) = -1, \quad y_1 \geq 1\},$$

where (,) is the Lorentz metric on $\mathbf{R}^{n+1}_1$ obtained by restricting the Lie metric. Let $h : M^{n-1} \to H^n$ be an oriented hypersurface with field of unit normals $\zeta : M^{n-1} \to \mathbf{R}^{n+1}_1$. The Legendre lift of (h, ζ) is given by the map $\lambda = [Y_1, Y_{n+3}]$, where

$$Y_1(x) = h(x) + e_2, \quad Y_{n+3}(x) = \zeta(x) + e_{n+3}. \tag{4.71}$$

Note that $(h,h) = -1$, so $\langle Y_1, Y_1 \rangle = 0$, while $(\zeta, \zeta) = 1$, so $\langle Y_{n+3}, Y_{n+3} \rangle = 0$. One can easily check that the conditions (1)–(3) are satisfied. Finally, if $\gamma : V \to H^n$ is an immersed submanifold of codimension greater than one, then the Legendre submanifold $\lambda : B^{n-1} \to \Lambda^{2n-1}$ is again defined on the unit normal bundle B^{n-1} in the obvious way.

Conversely, suppose that $\lambda : M^{n-1} \to \Lambda^{2n-1}$ is an arbitrary Legendre submanifold. We have seen above that we can parametrize λ as $\lambda = [Y_1, Y_{n+3}]$, where

$$Y_1 = (1, f, 0), \quad Y_{n+3} = (0, \xi, 1). \tag{4.72}$$

for the spherical projection f and spherical field of unit normals ξ. Both f and ξ are smooth maps, but neither need be an immersion or even have constant rank (see Example 4.6 below). The Legendre lift of an oriented hypersurface in S^n is the special case where the spherical projection f is an immersion, i.e., f has constant rank $n-1$ on M^{n-1}. In the case of the Legendre lift of a submanifold $\phi : V^k \to S^n$, the spherical projection $f : B^{n-1} \to S^n$ defined by $f(x, \xi) = \phi(x)$ has constant rank k.

If the range of the point sphere map $[Y_1]$ does not contain the improper point $[(1, -1, 0, \ldots, 0)]$, then λ also determines a *Euclidean projection* $F : M^{n-1} \to \mathbf{R}^n$, and a *Euclidean field of unit normals*, $\eta : M^{n-1} \to \mathbf{R}^n$. These are defined by the equation $\lambda = [Z_1, Z_{n+3}]$, where

$$Z_1 = (1 + F \cdot F, 1 - F \cdot F, 2F, 0)/2, \quad Z_{n+3} = (F \cdot \eta, -(F \cdot \eta), \eta, 1). \tag{4.73}$$

Here $[Z_1(x)]$ corresponds to the unique point sphere in the parabolic pencil determined by $\lambda(x)$, and $[Z_{n+3}(x)]$ corresponds to the unique plane in this pencil. As in the spherical case, the smooth maps F and η need not have constant rank.

Finally, if the range of the Euclidean projection F lies inside some disk Ω in $\mathbf{R}^n$, then one can define a hyperbolic projection and hyperbolic field of unit normals by placing a hyperbolic metric on Ω.

There are, however, many Dupin submanifolds whose spherical (or Euclidean) projection is not an immersion and does not have constant rank. Examples of this type can be obtained by applying a parallel transformation P_t to a Dupin submanifold λ whose spherical or Euclidean projection is an immersion, where P_t is chosen in such a way that the spherical or Euclidean projection of $P_t\lambda$ contains a focal point of the original hypersurface. In particular, consider the following example from [77, pp. 63–64].

Example 4.6. A Euclidean projection F that is not an immersion.
An example where the Euclidean (or spherical) projection does not have constant rank is illustrated by the cyclide of Dupin in Figure 4.7. Here the corresponding Legendre submanifold is a map $\lambda : T^2 \to \Lambda^5$, where T^2 is a 2-dimensional torus. The Euclidean projection $F : T^2 \to \mathbf{R}^3$ maps the circle S^1 containing the points A, B, C and D to the point P. However, the map λ into the space of lines on the quadric (corresponding to contact elements) is an immersion. The four arrows in Figure 4.7 represent the contact elements corresponding under the map λ to the four points indicated on the circle S^1.

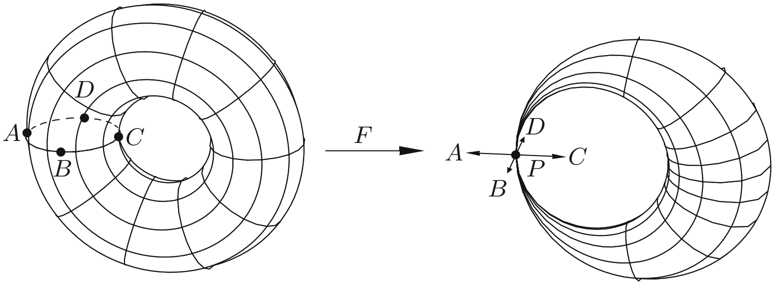


Fig. 4.7 A Euclidean projection F with a singularity

4.4 Curvature Spheres and Dupin Submanifolds

In this section, we discuss the notions of curvature spheres and Dupin hypersurfaces in the context of Lie sphere geometry, and we prove that the Dupin property is invariant under Lie sphere transformations.

We begin with the case of an oriented hypersurface $f : M^{n-1} \to S^n$ with field of unit normals $\xi : M^{n-1} \to S^n$. As we showed in Section 2.2, a point

$$f_t(x) = \cos t\, f(x) + \sin t\, \xi(x) \tag{4.74}$$

is a focal point of (M^{n-1}, x) of multiplicity $m > 0$ if and only if $\cot t$ is a principal curvature of multiplicity m at x. Note that each principal curvature $\kappa = \cot t = \cot(t + \pi)$ produces two distinct antipodal focal points on the normal geodesic to $f(M^{n-1})$ at f with parameter values t and $t + \pi$. The oriented hypersphere centered at a focal point p and in oriented contact with $f(M^{n-1})$ at $f(x)$ is called a *curvature sphere* of f at x. The two antipodal focal points determined by κ are the two centers of the corresponding curvature sphere. Thus, the correspondence between principal curvatures and curvature spheres is bijective. The multiplicity of the curvature sphere is by definition equal to the multiplicity of the corresponding principal curvature.

Curvature spheres in Lie sphere geometry

We now formulate the notion of curvature sphere in the context of Lie sphere geometry. As in equation (4.66), the Legendre lift $\lambda : M^{n-1} \to \Lambda^{2n-1}$ of the oriented hypersurface (f, ξ) is given by $\lambda = [Y_1, Y_{n+3}]$, where

$$Y_1 = (1, f, 0), \quad Y_{n+3} = (0, \xi, 1). \tag{4.75}$$

For each $x \in M^{n-1}$, the points on the line $\lambda(x)$ can be parametrized as

$$[K_t(x)] = [\cos t \, Y_1(x) + \sin t \, Y_{n+3}(x)] = [(\cos t, f_t(x), \sin t)], \tag{4.76}$$

where f_t is given in equation (4.74) above. By equation (4.24), the point $[K_t(x)]$ in Q^{n+1} corresponds to the oriented sphere in S^n with center $f_t(x)$ and signed radius t. This sphere is in oriented contact with the oriented hypersurface $f(M^{n-1})$ at $f(x)$. Given a tangent vector $X \in T_xM^{n-1}$, we have

$$dK_t(X) = (0, df_t(X), 0). \tag{4.77}$$

Thus, $dK_t(X) = (0, 0, 0)$ for a nonzero vector $X \in T_xM^{n-1}$ if and only if $df_t(X) = 0$, i.e., $p = f_t(x)$ is a focal point of f at x corresponding to the principal curvature $\cot t$. The vector X is a principal vector corresponding to the principal curvature $\cot t$, and it is also called a principal vector corresponding to the curvature sphere $[K_t]$.

This characterization of curvature spheres depends on the parametrization of $\lambda = [Y_1, Y_{n+3}]$ given by the point sphere and great sphere maps $[Y_1]$ and $[Y_{n+3}]$, and it has only been defined in the case where the spherical projection f is an immersion. We now give a projective formulation of the definition of a curvature sphere that is independent of the parametrization of λ and is valid for an arbitrary Legendre submanifold.

Let $\lambda : M^{n-1} \to \Lambda^{2n-1}$ be a Legendre submanifold parametrized by the pair $\{Z_1, Z_{n+3}\}$, as in Theorem 4.5. Let $x \in M^{n-1}$ and $r, s \in \mathbf{R}$ with at least one of r and s not equal to zero. The sphere,

$$[K] = [rZ_1(x) + sZ_{n+3}(x)],$$

is called a *curvature sphere* of λ at x if there exists a nonzero vector X in T_xM^{n-1} such that

$$r \, dZ_1(X) + s \, dZ_{n+3}(X) \in \text{Span}\,\{Z_1(x), Z_{n+3}(x)\}. \tag{4.78}$$

The vector X is called a *principal vector* corresponding to the curvature sphere $[K]$. This definition is invariant under a change of parametrization of the form considered in Theorem 4.5 on page 208. Furthermore, if we take the special parametrization $Z_1 = Y_1$, $Z_{n+3} = Y_{n+3}$ given in equation (4.75), then condition (4.78) holds if and only if $r \, dY_1(X) + s \, dY_{n+3}(X)$ actually equals $(0, 0, 0)$.

From equation (4.78), it is clear that the set of principal vectors corresponding to a given curvature sphere $[K]$ at x is a subspace of T_xM^{n-1}. This set is called the *principal space* corresponding to the curvature sphere $[K]$. Its dimension is the *multiplicity* of $[K]$.

Lie equivalent Legendre submanifolds

We next show that a Lie sphere transformation maps curvature spheres to curvature spheres. We first need to discuss the notion of Lie equivalent Legendre submanifolds. Let $\lambda : M^{n-1} \to \Lambda^{2n-1}$ be a Legendre submanifold parametrized by $\lambda = [Z_1, Z_{n+3}]$. Suppose $\beta = P(B)$ is the Lie sphere transformation induced by an orthogonal transformation B in the group $O(n+1, 2)$. Since B is orthogonal, the maps, $W_1 = BZ_1$, $W_{n+3} = BZ_{n+3}$, satisfy the Conditions (1)–(3) of Theorem 4.5, and thus $\gamma = [W_1, W_{n+3}]$ is a Legendre submanifold which we denote by $\beta\lambda : M^{n-1} \to \Lambda^{2n-1}$. We say that the Legendre submanifolds λ and $\beta\lambda$ are *Lie equivalent*. In terms of submanifolds of real space forms, we say that two immersed submanifolds of $\mathbf{R}^n$, S^n, or H^n are *Lie equivalent* if their Legendre lifts are Lie equivalent.

Theorem 4.7. *Let $\lambda : M^{n-1} \to \Lambda^{2n-1}$ be a Legendre submanifold and β a Lie sphere transformation. The point $[K]$ on the line $\lambda(x)$ is a curvature sphere of λ at x if and only if the point $\beta[K]$ is a curvature sphere of the Legendre submanifold $\beta\lambda$ at x. Furthermore, the principal spaces corresponding to $[K]$ and $\beta[K]$ are identical.*

Proof. Let $\lambda = [Z_1, Z_{n+3}]$ and $\beta\lambda = [W_1, W_{n+3}]$ as above. For a tangent vector $X \in T_x M^{n-1}$ and real numbers r and s, at least one of which is not zero, we have

$$\begin{aligned} r\, dW_1(X) + s\, dW_{n+3}(X) &= r\, d(BZ_1)(X) + s\, d(BZ_{n+3})(X) \\ &= B(r\, dZ_1(X) + s\, dZ_{n+3}(X)), \end{aligned} \tag{4.79}$$

since B is a constant linear transformation. Thus, we see that

$$r\, dW_1(X) + s\, dW_{n+3}(X) \in \text{Span}\,\{W_1(x), W_{n+3}(x)\}$$

if and only if

$$r\, dZ_1(X) + s\, dZ_{n+3}(X) \in \text{Span}\,\{Z_1(x), Z_{n+3}(x)\}.$$

□

We next consider the case when the Lie sphere transformation β is a spherical parallel transformation P_t given in equation (4.34), that is,

$$\begin{aligned} P_t e_1 &= \cos t\, e_1 + \sin t\, e_{n+3}, \\ P_t e_{n+3} &= -\sin t\, e_1 + \cos t\, e_{n+3}, \\ P_t e_i &= e_i, \quad 2 \le i \le n+2. \end{aligned} \tag{4.80}$$

Recall that P_t has the effect of adding t to the signed radius of each oriented sphere in S^n while keeping the center fixed.

If $\lambda : M^{n-1} \to \Lambda^{2n-1}$ is a Legendre submanifold parametrized by the point sphere map $Y_1 = (1, f, 0)$ and the great sphere map $Y_{n+3} = (0, \xi, 1)$, then $P_t\lambda = [W_1, W_{n+3}]$, where

$$W_1 = P_t Y_1 = (\cos t, f, \sin t), \quad W_{n+3} = P_t Y_{n+3} = (-\sin t, \xi, \cos t). \tag{4.81}$$

Note that W_1 and W_{n+3} are not the point sphere and great sphere maps for $P_t\lambda$. Solving for the point sphere map Z_1 and the great sphere map Z_{n+3} of $P_t\lambda$, we find

$$\begin{aligned} Z_1 &= \cos t\, W_1 - \sin t\, W_{n+3} = (1, \cos t\, f - \sin t\, \xi, 0), \\ Z_{n+3} &= \sin t\, W_1 + \cos t\, W_{n+3} = (0, \sin t\, f + \cos t\, \xi, 1). \end{aligned} \tag{4.82}$$

From this, we see that $P_t\lambda$ has spherical projection and spherical unit normal field given, respectively, by

$$\begin{aligned} f_{-t} &= \cos t\, f - \sin t\, \xi = \cos(-t) f + \sin(-t)\xi, \\ \xi_{-t} &= \sin t\, f + \cos t\, \xi = -\sin(-t) f + \cos(-t)\xi. \end{aligned} \tag{4.83}$$

The minus sign occurs because P_t takes a sphere with center $f_{-t}(x)$ and radius $-t$ to the point sphere $f_{-t}(x)$. We call $P_t\lambda$ a *parallel submanifold* of λ. Formula (4.83) shows the close correspondence between these parallel submanifolds and the parallel hypersurfaces f_t to f, in the case where f is an immersed hypersurface.

In the case where the spherical projection f is an immersion at a point $x \in M^{n-1}$, we know that the number of values of t in the interval $[0, \pi)$ for which f_t is not an immersion is at most $n-1$, the maximum number of distinct principal curvatures of f at x. Pinkall [446, p. 428] proved that this statement is also true for an arbitrary Legendre submanifold, even if the spherical projection f is not an immersion at x by proving the following theorem (see also [77, pp. 68–72] for a proof).

Theorem 4.8. *Let $\lambda : M^{n-1} \to \Lambda^{2n-1}$ be a Legendre submanifold with spherical projection f and spherical unit normal field ξ. Then for each $x \in M^{n-1}$, the parallel map,*

$$f_t = \cos t\, f + \sin t\, \xi,$$

fails to be an immersion at x for at most $n-1$ values of $t \in [0, \pi)$.

As a consequence of Pinkall's theorem, one can pass to a parallel submanifold to obtain the following important corollary. Note that parts (a)–(c) of the corollary are pointwise statements, while (d)–(e) hold on an open set U if they can be shown to hold in a neighborhood of each point of U.

Now let x be an arbitrary point of M^{n-1}. If the spherical projection f of λ is an immersion at x, then it is an immersion on a neighborhood of x, and the corollary holds on this neighborhood by known results concerning hypersurfaces in S^n given in Chapter 2, and by the correspondence between the curvature spheres of λ and the principal curvatures of f. If the spherical projection f is not an immersion at x, then by Theorem 4.8, there exists parallel transformation P_{-t} such that the spherical projection f_t of the Legendre submanifold $P_{-t}\lambda$ is an immersion at x, and hence on a neighborhood of x. So the corollary holds for $P_{-t}\lambda$ on this neighborhood of x, and by Theorem 4.7, the corollary also holds for λ on this neighborhood x.

Corollary 4.9. *Let $\lambda : M^{n-1} \to \Lambda^{2n-1}$ be a Legendre submanifold. Then:*

(a) *at each point $x \in M^{n-1}$, there are at most $n-1$ distinct curvature spheres $K_1, \ldots, K_g$,*
(b) *the principal vectors corresponding to a curvature sphere K_i form a subspace T_i of the tangent space $T_x M^{n-1}$,*
(c) *the tangent space $T_x M^{n-1} = T_1 \oplus \cdots \oplus T_g$,*
(d) *if the dimension of a given T_i is constant on an open subset U of M^{n-1}, then the principal distribution T_i is integrable on U,*
(e) *if $\dim T_i = m > 1$ on an open subset U of M^{n-1}, then the curvature sphere map K_i is constant along the leaves of the principal foliation T_i.*

We can also generalize the notion of a curvature surface defined in Section 2.5 (page 32) for hypersurfaces in real space forms to Legendre submanifolds. Specifically, let $\lambda : M^{n-1} \to \Lambda^{2n-1}$ be a Legendre submanifold. A connected submanifold S of M^{n-1} is called a *curvature surface* if at each $x \in S$, the tangent space $T_x S$ is equal to some principal space T_i. For example, if $\dim T_i$ is constant on an open subset U of M^{n-1}, then each leaf of the principal foliation T_i is a curvature surface on U. It is also possible to have a curvature surface S which is not a leaf of a principal foliation as in Example 2.22 on page 33.

Dupin submanifolds in Lie sphere geometry

Next we generalize the definition of a Dupin hypersurface in a real space form to the setting of Legendre submanifolds in Lie sphere geometry. We say that a Legendre submanifold $\lambda : M^{n-1} \to \Lambda^{2n-1}$ is a *Dupin submanifold* if:

(a) along each curvature surface, the corresponding curvature sphere map is constant.

The Dupin submanifold λ is called *proper Dupin* if, in addition to Condition (a), the following condition is satisfied:

(b) the number g of distinct curvature spheres is constant on M.

In the case of the Legendre lift $\lambda : M^{n-1} \to \Lambda^{2n-1}$ of an immersed Dupin hypersurface $f : M^{n-1} \to S^n$, the submanifold λ is a Dupin submanifold, since a curvature sphere map of λ is constant along a curvature surface if and only if the corresponding principal curvature map of f is constant along that curvature surface. Similarly, λ is proper Dupin if and only if f is proper Dupin, since the number of distinct curvatures spheres of λ at a point $x \in M^{n-1}$ equals the number of distinct principal curvatures of f at x. Particularly important examples of proper Dupin submanifolds are the Legendre lifts of isoparametric hypersurfaces in S^n.

Remark 4.10 (Relationship to the Euclidean definition of Dupin). Reckziegel [458] gives a definition of principal curvatures and curvature surfaces in the case of an immersed submanifold $\phi : V \to S^n$ of codimension $\nu + 1 > 1$. In that case, Reckziegel defines a curvature surface to be a connected submanifold $S \subset V$ for which there is a parallel section of the unit normal bundle $\eta : S \to B^{n-1}$ such that for each $x \in S$, the tangent space T_xS is equal to some eigenspace of $A_{\eta(x)}$. The corresponding principal curvature function $\kappa : S \to \mathbf{R}$ is then a smooth function on S. As noted in Remark 2.26 on page 35, Pinkall [447] calls a submanifold $\phi(V)$ of codimension greater than one Dupin if along each curvature surface (in the sense of Reckziegel), the corresponding principal curvature is constant. A Dupin submanifold $\phi(V)$ is proper Dupin if the number of distinct principal curvatures is constant on the unit normal bundle B^{n-1}. One can show that Pinkall's definition is equivalent to requiring that the Legendre lift $\lambda : B^{n-1} \to \Lambda^{2n-1}$ of the submanifold $\phi(V)$ is a proper Dupin submanifold in the sense of Lie sphere geometry, as defined above.

Lie invariance of the Dupin condition

By Theorem 4.7 both the Dupin and proper Dupin conditions are invariant under Lie sphere transformations (see Theorem 4.11 below), and many important classification results for Dupin submanifolds have been obtained in the setting of Lie sphere geometry, as we will see in Chapter 5.

Theorem 4.11. *Let $\lambda : M^{n-1} \to \Lambda^{2n-1}$ be a Legendre submanifold and β a Lie sphere transformation.*

(a) *If λ is Dupin, then $\beta\lambda$ is Dupin.*
(b) *If λ is proper Dupin, then $\beta\lambda$ is proper Dupin.*

Proof. By Theorem 4.7, a point $[K]$ on the line $\lambda(x)$ is a curvature sphere of λ at $x \in M$ if and only if the point $\beta[K]$ is a curvature sphere of $\beta\lambda$ at x, and the principal spaces corresponding $[K]$ and $\beta[K]$ are identical. Since these principal spaces are the same, if S is a curvature surface of λ corresponding to a curvature sphere map $[K]$, then S is also a curvature surface of $\beta\lambda$ corresponding to a curvature sphere map $\beta[K]$, and clearly $[K]$ is constant along S if and only if $\beta[K]$ is constant along S. This proves part (a) of the theorem. Part (b) also follows immediately from Theorem 4.7, since for each $x \in M$, the number g of distinct curvature spheres of λ at x equals the number of distinct curvatures spheres of $\beta\lambda$ at x. So if this number g is constant on M for λ, then it is constant on M for $\beta\lambda$. □

4.5 Lie Curvatures and Isoparametric Hypersurfaces

In this section,we introduce certain natural Lie invariants, known as Lie curvatures, due to R. Miyaoka [365], that have been important in the study of Dupin and isoparametric hypersurfaces in the context of Lie sphere geometry. We also find a criterion (Theorem 4.16) for when a Legendre submanifold is Lie equivalent to the Legendre lift of an isoparametric hypersurface in S^n. This theorem has been used in proving various classification results for Dupin hypersurfaces.

Let $\lambda : M^{n-1} \to \Lambda^{2n-1}$ be an arbitrary Legendre submanifold. As before, we can write $\lambda = [Y_1, Y_{n+3}]$, where

$$Y_1 = (1, f, 0), \quad Y_{n+3} = (0, \xi, 1), \tag{4.84}$$

where f and ξ are the spherical projection and spherical field of unit normals, respectively.

For $x \in M^{n-1}$, the points on the line $\lambda(x)$ can be written in the form,

$$\mu Y_1(x) + Y_{n+3}(x), \tag{4.85}$$

that is, we take μ as an inhomogeneous coordinate along the projective line $\lambda(x)$. Then the point sphere $[Y_1]$ corresponds to $\mu = \infty$. The next two theorems give the relationship between the coordinates of the curvature spheres of λ and the principal curvatures of f, in the case where f has constant rank. In the first theorem, we assume that the spherical projection f is an immersion on M^{n-1}. By Theorem 4.8, we know that this can always be achieved locally by passing to a parallel submanifold.

Theorem 4.12. *Let $\lambda : M^{n-1} \to \Lambda^{2n-1}$ be a Legendre submanifold whose spherical projection $f : M^{n-1} \to S^n$ is an immersion. Let Y_1 and Y_{n+3} be the point sphere and great sphere maps of λ as in equation (4.84). Then the curvature spheres of λ at a point $x \in M^{n-1}$ are*

$$[K_i] = [\kappa_i Y_1 + Y_{n+3}], \quad 1 \leq i \leq g,$$

where $\kappa_1, \ldots, \kappa_g$ are the distinct principal curvatures at x of the oriented hypersurface f with field of unit normals ξ. The multiplicity of the curvature sphere $[K_i]$ equals the multiplicity of the principal curvature κ_i.

Proof. Let X be a nonzero vector in $T_x M^{n-1}$. Then for any real number μ,

$$d(\mu Y_1 + Y_{n+3})(X) = (0, \mu\, df(X) + d\xi(X), 0).$$

This vector is in Span $\{Y_1(x), Y_{n+3}(x)\}$ if and only if

$$\mu\, df(X) + d\xi(X) = 0,$$

i.e., μ is a principal curvature of f with corresponding principal vector X. $\square$

We next consider the case where the point sphere map Y_1 is a curvature sphere of constant multiplicity m on M^{n-1}. By Corollary 4.9, the corresponding principal distribution is a foliation, and the curvature sphere map $[Y_1]$ is constant along the leaves of this foliation. Thus the map $[Y_1]$ factors through an immersion $[W_1]$ from the space of leaves V of this foliation into Q^{n+1}. We can write $[W_1] = [(1, \phi, 0)]$, where $\phi : V \to S^n$ is an immersed submanifold of codimension $m+1$. The manifold M^{n-1} is locally diffeomorphic to an open subset of the unit normal bundle B^{n-1} of the submanifold ϕ, and λ is essentially the Legendre lift of $\phi(V)$, as defined in Section 4.3. The following theorem relates the curvature spheres of λ to the principal curvatures of ϕ. Recall that the point sphere and great sphere maps for λ are given as in equation (4.68) by

$$Y_1(x, \xi) = (1, \phi(x), 0), \quad Y_{n+3}(x, \xi) = (0, \xi, 1). \tag{4.86}$$

Theorem 4.13. *Let $\lambda : B^{n-1} \to \Lambda^{2n-1}$ be the Legendre lift of an immersed submanifold $\phi(V)$ in S^n of codimension $m+1$. Let Y_1 and Y_{n+3} be the point sphere and great sphere maps of λ as in equation (4.86). Then the curvature spheres of λ at a point $(x, \xi) \in B^{n-1}$ are*

$$[K_i] = [\kappa_i Y_1 + Y_{n+3}], \quad 1 \leq i \leq g,$$

where $\kappa_1, \ldots, \kappa_{g-1}$ are the distinct principal curvatures of the shape operator A_ξ, and $\kappa_g = \infty$. For $1 \leq i \leq g-1$, the multiplicity of the curvature sphere $[K_i]$ equals the multiplicity of the principal curvature κ_i, while the multiplicity of $[K_g]$ is m.

The proof of this theorem is similar to that of Theorem 4.12, but one must introduce local coordinates on the unit normal bundle to get a complete proof (see [77, p. 74]).

Given these two theorems, we define a *principal curvature* of a Legendre submanifold $\lambda : M^{n-1} \to \Lambda^{2n-1}$ at a point $x \in M^{n-1}$ to be a value κ in the set $\mathbf{R} \cup \{\infty\}$ such that $[\kappa Y_1(x) + Y_{n+3}(x)]$ is a curvature sphere of λ at x, where Y_1 and Y_{n+3} are as in equation (4.84).

Lie curvatures and Möbius curvatures

The principal curvatures of a Legendre submanifold are not Lie invariants, and they depend on the special parametrization for λ given in equation (4.84). However, R. Miyaoka [365] pointed out that the cross-ratios of the principal curvatures are Lie invariants. This is due to the fact that a projective transformation preserves the cross-ratio of four points on a projective line.

We now formulate Miyaoka's theorem specifically. Let $\lambda : B^{n-1} \to \Lambda^{2n-1}$ be a Legendre submanifold, and let β be a Lie sphere transformation. The Legendre submanifold $\beta\lambda$ has point sphere and great sphere maps which we denote by

$$Z_1 = (1, h, 0), \quad Z_{n+3} = (0, \zeta, 1),$$

where h and ζ are the spherical projection and spherical field of unit normals of $\beta\lambda$. Let

$$[K_i] = [\kappa_i Y_1 + Y_{n+3}], \quad 1 \leq i \leq g,$$

denote the distinct curvature spheres of λ at a point $x \in M^{n-1}$. By Theorem 4.7, the points $\beta[K_i]$, $1 \leq i \leq g$, are the distinct curvature spheres of $\beta\lambda$ at x. We can write

$$\beta[K_i] = [\gamma_i Z_1 + Z_{n+3}], \quad 1 \leq i \leq g.$$

Then these γ_i are the principal curvatures of $\beta\lambda$ at x.

Next recall that the *cross-ratio* of four distinct numbers a, b, c, d in $\mathbf{R} \cup \{\infty\}$ is given by

$$[a, b; c, d] = \frac{(a-b)(d-c)}{(a-c)(d-b)}. \tag{4.87}$$

We use the usual conventions involving operations with ∞. For example, if $d = \infty$, then the expression $(d-c)/(d-b)$ evaluates to one, and the cross-ratio $[a, b; c, d]$ equals $(a-b)/(a-c)$.

Miyaoka's theorem can now be stated as follows.

Theorem 4.14. *Let $\lambda : M^{n-1} \to \Lambda^{2n-1}$ be a Legendre submanifold and β a Lie sphere transformation. Suppose that $\kappa_1, \ldots, \kappa_g, g \geq 4$, are the distinct principal curvatures of λ at a point $x \in M^{n-1}$, and $\gamma_1, \ldots, \gamma_g$ are the corresponding principal curvatures of $\beta\lambda$ at x. Then for any choice of four numbers h, i, j, k from the set $\{1, \ldots, g\}$, we have*

$$[\kappa_h, \kappa_i; \kappa_j, \kappa_k] = [\gamma_h, \gamma_i; \gamma_j, \gamma_k]. \tag{4.88}$$

Proof. The left side of equation (4.88) is the cross-ratio, in the sense of projective geometry, of the four points $[K_h], [K_i], [K_j], [K_k]$ on the projective line $\lambda(x)$. The right side of equation (4.88) is the cross-ratio of the images of these four points under β. The theorem now follows from the fact that the projective transformation β preserves the cross-ratio of four points on a line. □

The cross-ratios of the principal curvatures of λ are called the *Lie curvatures* of λ. There is also a set of similar invariants for the Möbius group defined as follows. Here we consider a Möbius transformation to be a Lie sphere transformation that takes point spheres to point spheres. Hence the transformation β in Theorem 4.14 is a Möbius transformation if and only if $\beta[Y_1] = [Z_1]$. This leads to the following corollary of Theorem 4.14.

Corollary 4.15. *Let $\lambda : M^{n-1} \to \Lambda^{2n-1}$ be a Legendre submanifold and β a Möbius transformation. Then for any three distinct principal curvatures $\kappa_h, \kappa_i, \kappa_j$ of λ at a point $x \in M^{n-1}$, none of which equals ∞, we have*

$$\Phi(\kappa_h, \kappa_i, \kappa_j) = (\kappa_h - \kappa_i)/(\kappa_h - \kappa_j) = (\gamma_h - \gamma_i)/(\gamma_h - \gamma_j), \tag{4.89}$$

where γ_h, γ_i, and γ_j are the corresponding principal curvatures of $\beta\lambda$ at the point x.

Proof. Note that we are using equation (4.89) to define the ratio Φ, which is called a *Möbius curvature* of λ. Since β is a Möbius transformation, the point $[Y_1]$, corresponding to $\mu = \infty$, is taken by β to the point Z_1 with coordinate $\gamma = \infty$. Since β preserves cross-ratios, we have

$$[\kappa_h, \kappa_i; \kappa_j, \infty] = [\gamma_h, \gamma_i; \gamma_j, \infty]. \tag{4.90}$$

Since the cross-ratio on the left in equation (4.90) equals the left side of equation (4.89), and the cross-ratio on the right in equation (4.90) equals the right side of equation (4.89), the corollary holds. □

Criterion for Lie equivalence to an isoparametric hypersurface

We close this section with a local Lie geometric characterization of Legendre submanifolds that are Lie equivalent to the Legendre lift of an isoparametric hypersurface in S^n (see Cecil [73]). Here a line in $\mathbf{RP}^{n+2}$ is called *timelike* if it contains only timelike points. This means that an orthonormal basis for the 2-plane in $\mathbf{R}^{n+3}_2$ determined by the timelike line consists of two timelike vectors. An example is the line $[e_1, e_{n+3}]$.

Theorem 4.16. *Let $\lambda : M^{n-1} \to \Lambda^{2n-1}$ be a Legendre submanifold with g distinct curvature spheres $[K_1], \ldots, [K_g]$ at each point. Then λ is Lie equivalent to the Legendre lift of an isoparametric hypersurface in S^n if and only if there exist g points $[P_1], \ldots, [P_g]$ on a timelike line in $\mathbf{RP}^{n+2}$ such that*

$$\langle K_i, P_i \rangle = 0, \quad 1 \leq i \leq g.$$

Proof. If λ is the Legendre lift of an isoparametric hypersurface in S^n, then all the spheres in a family $[K_i]$ have the same radius ρ_i, where $0 < \rho_i < \pi$. By formula (4.24), this is equivalent to the condition $\langle K_i, P_i \rangle = 0$, where

$$P_i = \sin \rho_i \, e_1 - \cos \rho_i \, e_{n+3}, \quad 1 \leq i \leq g, \tag{4.91}$$

are g points on the timelike line $[e_1, e_{n+3}]$. Since a Lie sphere transformation preserves curvature spheres, timelike lines and the polarity relationship, the same is true for any image of λ under a Lie sphere transformation.

Conversely, suppose that there exist g points $[P_1], \ldots, [P_g]$ on a timelike line ℓ such that $\langle K_i, P_i \rangle = 0$, for $1 \leq i \leq g$. Let β be a Lie sphere transformation that maps ℓ to the line $[e_1, e_{n+3}]$. Then the curvature spheres $\beta[K_i]$ of $\beta\lambda$ are orthogonal to the points $[Q_i] = \beta[P_i]$ on the line $[e_1, e_{n+3}]$. This means that the spheres corresponding to $\beta[K_i]$ have constant radius on M^{n-1}. By applying a parallel transformation P_t, if necessary, we can arrange that none of these curvature spheres has radius zero. Then $P_t\beta\lambda$ is the Legendre lift of an isoparametric hypersurface in S^n. □

Remark 4.17. In the case where λ is Lie equivalent to the Legendre lift of an isoparametric hypersurface in S^n, one can say more about the position of the points $[P_1], \ldots, [P_g]$ on the timelike line ℓ. By Theorem 3.26 (page 108) due to Münzner, the radii ρ_i of the curvature spheres of an isoparametric hypersurface satisfy the equation

$$\rho_i = \rho_1 + (i-1)\frac{\pi}{g}, \quad 1 \leq i \leq g, \tag{4.92}$$

for some $\rho_1 \in (0, \pi/g)$. Hence, after Lie sphere transformation, the $[P_i]$ have the form (4.91) for ρ_i as in equation (4.92).

On an isoparametric hypersurface, the distinct principal curvatures have the form

$$\cot \rho_i, \quad 1 \leq i \leq g, \tag{4.93}$$

for ρ_i as in equation (4.92). From this, we can determine the Lie curvatures of an isoparametric hypersurface, which are obviously constant.

For the sake of definiteness, we make the calculation as follows. First we order the principal curvatures so that

$$\kappa_1 < \cdots < \kappa_g, \tag{4.94}$$

and so the κ_i decrease as the ρ_i increase.

We first consider the case $g = 4$. Then the ordering of the principal curvatures in equation (4.94) leads to a unique Lie curvature Ψ defined by

$$\Psi = [\kappa_1, \kappa_2; \kappa_3, \kappa_4] = (\kappa_1 - \kappa_2)(\kappa_4 - \kappa_3)/(\kappa_1 - \kappa_3)(\kappa_4 - \kappa_2). \tag{4.95}$$

With this ordering of the principal curvatures, the Lie curvature Ψ satisfies the inequality $0 < \Psi < 1$. Using equations (4.93) and (4.95), one can compute that $\Psi = 1/2$ on any isoparametric hypersurface with $g = 4$ principal curvatures, i.e., the four curvature spheres form a *harmonic set* in the sense of projective geometry (see, for example, [472, p. 59]).

Computation of the Lie curvature

There is, however, a simpler way to compute Ψ by considering the focal submanifolds. By Theorem 3.44 (page 131), each isoparametric hypersurface M^{n-1} embedded in S^n has two distinct focal submanifolds, each of codimension greater than one. The hypersurface M^{n-1} is a tube of constant radius over each of these focal submanifolds. Therefore, the Legendre lift of M^{n-1} is obtained from the Legendre lift of either focal submanifold by parallel transformation. Thus, the Legendre lift of M^{n-1} has the same Lie curvature as the Legendre lift of either focal submanifold.

Let $\phi : V \to S^n$ be one of the focal submanifolds of an isoparametric hypersurface M^{n-1} with $g = 4$ principal curvatures. By Theorem 3.21 (page 105) and Theorem 3.26 (page 108), we see that if ξ is any unit normal to $\phi(V)$ at any point, then the shape operator A_ξ has three distinct principal curvatures,

$$\kappa_1 = -1, \quad \kappa_2 = 0, \quad \kappa_3 = 1.$$

By Theorem 4.13, the Legendre lift of ϕ has a fourth principal curvature $\kappa_4 = \infty$. Thus, the Lie curvature of this Legendre lift is

$$\Psi = (-1-0)(\infty-1)/(-1-1)(\infty-0) = 1/2, \tag{4.96}$$

as stated above.

We can determine the Lie curvatures of an isoparametric hypersurface M^{n-1} in S^n with $g = 6$ principal curvatures in the same way. Let $\phi(V)$ be one of the focal submanifolds of M^{n-1}. By Münzner's formula (4.92) and Theorem 3.21 (page 105), the Legendre lift of $\phi(V)$ has six constant principal curvatures,

$$\kappa_1 = -\sqrt{3},\ \kappa_2 = -1/\sqrt{3},\ \kappa_3 = 0,\ \kappa_4 = 1/\sqrt{3},\ \kappa_5 = \sqrt{3},\ \kappa_6 = \infty,$$

as in Theorem 4.13. The corresponding six curvature spheres $[K_1], \ldots, [K_6]$ are situated symmetrically on a projective line, as in Figure 4.8.

There are only three geometrically distinct configurations which can obtained by choosing four of the six curvature spheres. These give the cross-ratios:

$$[\kappa_3, \kappa_4; \kappa_5, \kappa_6] = 1/3, \quad [\kappa_2, \kappa_3; \kappa_5, \kappa_6] = 1/4, \quad [\kappa_2, \kappa_3; \kappa_4, \kappa_6] = 1/2.$$

Of course, if a certain cross-ratio has the value r, then one can obtain the values,

$$\{r,\ 1/r,\ 1-r,\ 1/(1-r),\ (r-1)/r,\ r/(r-1)\}, \tag{4.97}$$

by permuting the order of the spheres (see, for example, Samuel [472, p. 58]).

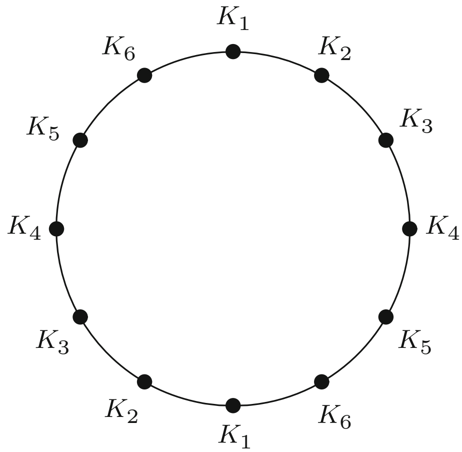


Fig. 4.8 Curvature spheres on a projective line, $g = 6$

4.6 Lie Invariance of Tautness

In this section, we discuss the notion of tautness for Legendre submanifolds in the context of Lie sphere geometry. This was introduced in a paper of Cecil and Chern [79], although the approach taken here is due to Álvarez Paiva [14], who used functions whose level sets form a parabolic pencil of spheres rather than the usual distance functions or height functions to formulate tautness. This approach leads to a natural proof of the invariance of tautness under Lie sphere transformations. In this section, we follow Section 4.6 of the book [77] closely, although we will omit some of the calculations given there. (See also another paper of Álvarez Paiva [13] that extends the notion of tautness to symplectic geometry.)

In the proof of the Lie invariance of tautness, it is more convenient to consider embeddings of compact, connected manifolds into S^n rather than $\mathbf{R}^n$. Theorem 2.70 on page 61 shows that these two theories are equivalent.

As noted in Theorem 2.28 on page 38, Kuiper [301] reformulated tightness and tautness in terms of an injectivity condition on homology which has turned out be very useful. Let f be a nondegenerate function on a manifold V. We consider the sublevel set

$$V_r(f) = \{x \in V \mid f(x) \leq r\}, \quad r \in \mathbf{R}. \tag{4.98}$$

The next theorem, which follows immediately from Theorem 29.2 of Morse–Cairns [379, p. 260] was a key to Kuiper's formulation of these conditions. (This is the same as Theorem 2.28, see page 38 for more discussion).

Theorem 4.18. *Let f be a nondegenerate function on a compact, connected manifold V. For a given field $\mathbf{F}$, the number $\mu(f)$ of critical points of f equals the sum $\beta(V, \mathbf{F})$ of the $\mathbf{F}$-Betti numbers of V if and only if the map on homology,*

$$H_*(V_r(f), \mathbf{F}) \to H_*(V, \mathbf{F}), \tag{4.99}$$

induced by the inclusion $V_r(f) \subset V$ is injective for all $r \in \mathbf{R}$.

Of course, for an embedding $\phi : V \to S^n$ and a height function ℓ_p, the set $V_r(\ell_p)$, is equal to $\phi^{-1}(B)$, where B is the closed ball in S^n obtained by intersecting S^n with the half-space in $\mathbf{R}^{n+1}$ determined by the inequality $\ell_p(q) \leq r$. Kuiper [304] used the continuity property of $\mathbf{Z}_2$-Čech homology to formulate tautness in terms of $\phi^{-1}(B)$, for all closed balls B in S^n, not just those centered at non-focal points of ϕ. Thus, Kuiper proved the following theorem (see also Theorem 2.54 on page 54 for the Euclidean version).

Theorem 4.19. *Let $\phi : V \to S^n$ be an embedding of a compact, connected manifold V into S^n. Then ϕ is taut if and only if for every closed ball B in S^n, the induced homomorphism $H_*(f^{-1}(B)) \to H_*(V)$ in $\mathbf{Z}_2$-Čech homology is injective.*

The key to the approach of Álvarez Paiva [14] is to formulate tautness of Legendre submanifolds in terms of functions whose level sets form a parabolic pencil of unoriented spheres, instead of using linear height functions. This is quite natural in the context of Lie sphere geometry, and it is equivalent to the usual formulation of tautness in the case of the Legendre lift of an embedding $\phi : V \to S^n$.

The specific construction is as follows (see [77, pp. 83–84]). Given a contact element $(p, \xi) \in T_1 S^n$, we want to define a function

$$r_{(p,\xi)} : S^n - \{p\} \to (0, \pi),$$

whose level sets are unoriented spheres in the parabolic pencil of unoriented spheres determined by (p, ξ). (We will often denote $r_{(p,\xi)}$ simply by r when the context is clear.) Every point x in $S^n - \{p\}$ lies on precisely one sphere S_x in the pencil as the spherical radius r of the spheres in the pencil varies from 0 to π. The radius $r_{(p,\xi)}(x)$ of S_x is defined implicitly by the equation

$$\cos r = x \cdot (\cos r\, p + \sin r\, \xi). \tag{4.100}$$

This equation says that x lies in the unoriented sphere S_x in the pencil with center

$$q = \cos r\, p + \sin r\, \xi, \tag{4.101}$$

and spherical radius $r \in (0, \pi)$ (see Figure 4.9).

This defines a smooth function

$$r_{(p,\xi)} : S^n - \{p\} \to (0, \pi). \tag{4.102}$$

Note that the contact element $(p, -\xi)$ determines the same pencil of unoriented spheres and the function $r_{(p,-\xi)} = \pi - r_{(p,\xi)}$. Some sample values of the function $r_{(p,\xi)}$ are

$$r_{(p,\xi)}(\xi) = \pi/4, \quad r_{(p,\xi)}(-p) = \pi/2, \quad r_{(p,\xi)}(-\xi) = 3\pi/4.$$

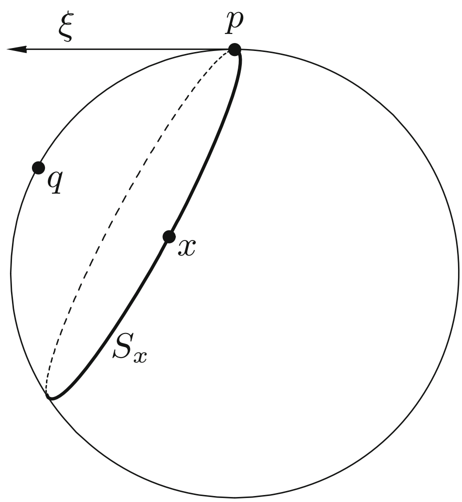


Fig. 4.9 The sphere S_x in the parabolic pencil determined by (p, ξ)

Consider an immersion $\phi : V \to S^n$, where V is a k-dimensional manifold with $k < n$. If $x \in V$, we say that the sphere S_x and $\phi(V)$ are *tangent* at $\phi(x)$ if

$$d\phi(T_x V) \subset T_{\phi(x)} S_x, \tag{4.103}$$

where $d\phi$ is the differential of ϕ.

Critical point behavior

The following lemma describes the critical point behavior of a function of the form $r_{(p,\xi)}$ on an immersed submanifold $\phi : V \to S^n$. This lemma is similar to the Index Theorem L_p functions (Theorem 2.51 on page 53), and it is proven by a direct calculation of the first and second derivatives of r. We will omit the proof here and refer the reader to [77, pp. 84–88] for a complete proof.

Lemma 4.20. *Let $\phi : V \to S^n$ be an immersion of a connected manifold V with* $\dim V < n$ *into S^n, and let $(p, \xi) \in T_1 S^n$ such that $p \notin \phi(V)$.*

(a) *A point $x_0 \in V$ is a critical point of the function $r_{(p,\xi)}$ if and only if the sphere S_{x_0} containing $\phi(x_0)$ in the parabolic pencil of unoriented spheres determined by (p, ξ) and the submanifold $\phi(V)$ are tangent at $\phi(x_0)$.*
(b) *If $r_{(p,\xi)}$ has a critical point at $x_0 \in V$, then this critical point is degenerate if and only if the sphere S_{x_0} is a curvature sphere of $\phi(V)$ at x_0.*

Next we show that except for (p, ξ) in a set of measure zero in T_1S^n, the function $r_{(p,\xi)}$ is a Morse function on $\phi(V)$. This is accomplished using Sard's Theorem in a manner similar to the proof of Corollary 2.33 on page 40.

In our particular case, from Lemma 4.20 we know that the function $r_{(p,\xi)}$, for $p \notin \phi(V)$, is a Morse function on $\phi(V)$ unless the parabolic pencil of unoriented spheres determined by (p, ξ) contains a curvature sphere of $\phi(V)$. We now show that the set of (p, ξ) in T_1S^n such that the parabolic pencil determined by (p, ξ) contains a curvature sphere of $\phi(V)$ has measure zero in T_1S^n.

Let B^{n-1} denote the unit normal bundle of the submanifold $\phi(V)$ in S^n. Note that in the case where $\phi(V)$ is a hypersurface, B^{n-1} is a two-sheeted covering of V. We first recall the normal exponential map,

$$q : B^{n-1} \times (0, \pi) \to S^n, \tag{4.104}$$

defined as follows. For a point (x, N) in B^{n-1} and $r \in (0, \pi)$, we define

$$q((x, N), r) = \cos r\, x + \sin r\, N. \tag{4.105}$$

Next we define a $(2n-1)$-dimensional manifold W^{2n-1} by

$$W^{2n-1} = \{((x, N), r, \eta) \in B^{n-1} \times (0, \pi) \times S^n \mid \eta \cdot q((x, N), r) = 0\}. \tag{4.106}$$

The manifold W^{2n-1} is a fiber bundle over $B^{n-1} \times (0, \pi)$ with fiber diffeomorphic to S^{n-1}. For each point $((x, N), r) \in B^{n-1} \times (0, \pi)$, the fiber consists of all unit vectors η in $\mathbf{R}^{n+1}$ that are tangent to S^n at the point $q((x, N), r)$.

We define a map,

$$F : W^{2n-1} \to T_1S^n, \tag{4.107}$$

by

$$F((x, N), r, \eta) = (\cos r\, q + \sin r\, \eta,\ \sin r\, q - \cos r\, \eta), \tag{4.108}$$

where $q = q((x, N), r)$ is defined in equation (4.105).

The next lemma shows that if the parabolic pencil of unoriented spheres determined by $(p, \xi) \in T_1S^n$ contains a curvature sphere of $\phi(V)$, then (p, ξ) is a critical value of F. Since the set of critical values of F has measure zero by Sard's Theorem (see, for example, Milnor [359, p. 33]), this will give the desired conclusion. The proof of this lemma is a fairly straightforward calculation of the differential of the map F, and we refer the reader to [77, pp. 89–91] for a detailed proof.

Lemma 4.21. *Let $\phi : V \to S^n$ be an immersion of a connected manifold V with* $\dim V < n$ *into S^n, and let B^{n-1} be the unit normal bundle of $\phi(V)$. Define*

$$F : W^{2n-1} \to T_1S^n,$$

as in equation (4.108). If the parabolic pencil of unoriented spheres determined by (p, ξ) *in* T_1S^n *contains a curvature sphere of* $\phi(V)$*, then* (p, ξ) *is a critical value of* F*. Thus, the set of such* (p, ξ) *has measure zero in* T_1S^n*.*

Corollary 4.22. *Let* $\phi : V \to S^n$ *be an immersion of a connected manifold* V *with* $\dim V < n$ *into* S^n*. For almost all* $(p, \xi) \in T_1S^n$*, the function* $r_{(p,\xi)}$ *is a Morse function on* V*.*

Proof. By Lemma 4.20, the function $r_{(p,\xi)}$ is a Morse function on V if and only if $p \notin \phi(V)$ and the parabolic pencil of unoriented spheres determined by (p, ξ) does not contain a curvature sphere of $\phi(V)$. The set of (p, ξ) such that $p \in \phi(V)$ has measure zero, since $\phi(V)$ is a submanifold of codimension at least one in S^n. The set of (p, ξ) such that the parabolic pencil determined by (p, ξ) contains a curvature sphere of $\phi(V)$ has measure zero by Lemma 4.21. Thus, except for (p, ξ) in the set of measure zero obtained by taking the union of these two sets, the function $r_{(p,\xi)}$ is a Morse function on V. □

Tautness in Lie sphere geometry

We will now formulate the definition of tautness for Legendre submanifolds in Lie sphere geometry. Recall the diffeomorphism from T_1S^n to the space Λ^{2n-1} of lines on the Lie quadric Q^{n+1} given by equations (4.37) and (4.38),

$$(p, \xi) \mapsto [(1, p, 0), (0, \xi, 1)] = \ell \in \Lambda^{2n-1}. \tag{4.109}$$

Under this correspondence, an oriented sphere S in S^n belongs to the parabolic pencil of oriented spheres determined by $(p, \xi) \in T_1S^n$ if and only if the point $[k]$ in Q^{n+1} corresponding to S lies on the line ℓ. Thus, the parabolic pencil of oriented spheres determined by a contact element (p, ξ) contains a curvature sphere S of a Legendre submanifold $\lambda : B^{n-1} \to \Lambda^{2n-1}$ if and only if the corresponding line ℓ contains the point $[k]$ corresponding to S.

A compact, connected Legendre submanifold $\lambda : B^{n-1} \to \Lambda^{2n-1}$ is said to be *Lie-taut* if for almost every line ℓ on the Lie quadric Q^{n+1}, the number of points $x \in B^{n-1}$ such that $\lambda(x)$ intersects ℓ is $\beta(B^{n-1}, \mathbf{Z}_2)/2$, i.e., one-half the sum of the $\mathbf{Z}_2$-Betti numbers of B^{n-1}. Here by "almost every," we mean except for a set of measure zero.

Equivalently, this definition says that for almost every contact element (p, ξ) in T_1S^n, the number of points $x \in B^{n-1}$ such that the contact element corresponding to $\lambda(x)$ is in oriented contact with some sphere in the parabolic pencil of oriented spheres determined by (p, ξ) is $\beta(B^{n-1}, \mathbf{Z}_2)/2$.

The property of Lie-tautness is clearly invariant under Lie sphere transformations, i.e., if $\lambda : B^{n-1} \to \Lambda^{2n-1}$ is Lie-taut and α is a Lie sphere transformation, then the Legendre submanifold $\alpha\lambda : B^{n-1} \to \Lambda^{2n-1}$ is also Lie-taut. This follows from the fact that the line $\lambda(x)$ intersects a line ℓ if and only if the line $\alpha(\lambda(x))$

intersects the line $\alpha(\ell)$, and α maps the complement of a set of measure zero in Λ^{2n-1} to the complement of a set of measure zero in Λ^{2n-1}.

Remark 4.23 (Comments on the definition of Lie-tautness). The factor of one-half in the definition comes from the fact that Lie sphere geometry deals with oriented contact and not just unoriented tangency, as we will see in the proof of Theorem 4.24 below. Recall that if $\phi : V \to S^n$ is an embedding of a compact, connected manifold V into S^n and B^{n-1} is the unit normal bundle of $\phi(V)$, then the Legendre lift of ϕ is defined to be the Legendre submanifold $\lambda : B^{n-1} \to \Lambda^{2n-1}$ given by

$$\lambda(x, N) = [(1, \phi(x), 0), (0, N, 1)], \tag{4.110}$$

where N is a unit normal vector to $\phi(V)$ at $\phi(x)$. If V has dimension $n-1$, then B^{n-1} is a two-sheeted covering of V. If V has dimension less than $n-1$, then B^{n-1} is diffeomorphic to a tube W^{n-1} of sufficiently small radius over $\phi(V)$ so that W^{n-1} is an embedded hypersurface in S^n. In either case,

$$\beta(B^{n-1}, \mathbf{Z}_2) = 2\beta(V, \mathbf{Z}_2).$$

This is obvious in the case where V has dimension $n-1$, and it was proved by Pinkall [447] in the case where V has dimension less than $n-1$.

Since Lie-tautness is invariant under Lie sphere transformations, the following theorem establishes that tautness is Lie invariant. Recall that a taut immersion $\phi : V \to S^n$ is in fact an embedding (see Theorem 2.59 on page 56). Here we use the proof of Theorem 4.28 of the book [77, pp. 93–95].

Theorem 4.24. *Let $\phi : V \to S^n$ be an embedding of a compact, connected manifold V with $\dim V < n$ into S^n. Then $\phi(V)$ is a taut submanifold in S^n if and only if the Legendre lift $\lambda : B^{n-1} \to \Lambda^{2n-1}$ of ϕ is Lie-taut.*

Proof. Suppose that $\phi(V)$ is a taut submanifold in S^n, and let

$$\lambda : B^{n-1} \to \Lambda^{2n-1}$$

be the Legendre lift of ϕ. Let $(p, \xi) \in T_1S^n$ such that $p \notin \phi(V)$ and such that the parabolic pencil of unoriented spheres determined by (p, ξ) does not contain a curvature sphere of $\phi(V)$. By Lemma 4.21, the set of such (p, ξ) is the complement of a set of measure zero in T_1S^n. For such (p, ξ), the function $r_{(p,\xi)}$ is a Morse function on V, and the sublevel set

$$V_s(r_{(p,\xi)}) = \{x \in V \mid r_{(p,\xi)}(x) \leq s\} = \phi(V) \cap B, \quad 0 < s < \pi, \tag{4.111}$$

is the intersection of $\phi(V)$ with a closed ball $B \subset S^n$. By tautness and Theorem 4.19, the map on $\mathbf{Z}_2$-Čech homology,

$$H_*(V_s(r_{(p,\xi)})) = H_*(\phi^{-1}(B)) \to H_*(V), \tag{4.112}$$

is injective for every $s \in \mathbf{R}$, and so by Theorem 4.18, the function $r_{(p,\xi)}$ has $\beta(V, \mathbf{Z}_2)$ critical points on V.

By Lemma 4.20, a point $x \in V$ is a critical point of $r_{(p,\xi)}$ if and only if the unoriented sphere S_x in the parabolic pencil determined by (p, ξ) containing x is tangent to $\phi(V)$ at $\phi(x)$. At each such point x, exactly one contact element $(x, N) \in B^{n-1}$ is in oriented contact with the oriented sphere $\tilde{S}_x$ through x in the parabolic pencil of oriented spheres determined by (p, ξ). Thus, the number of critical points of $r_{(p,\xi)}$ on V equals the number of points $(x, N) \in B^{n-1}$ such that (x, N) is in oriented contact with an oriented sphere in the parabolic pencil of oriented spheres determined by (p, ξ).

Thus there are

$$\beta(V, \mathbf{Z}_2) = \beta(B^{n-1}, \mathbf{Z}_2)/2$$

points $(x, N) \in B^{n-1}$ such that (x, N) is in oriented contact with an oriented sphere in the parabolic pencil of oriented spheres determined by (p, ξ). This means that there are $\beta(B^{n-1}, \mathbf{Z}_2)/2$ points $(x, N) \in B^{n-1}$ such that the line $\lambda(x, N)$ intersects the line ℓ on Q^{n+1} corresponding to the contact element (p, ξ). Since this true for almost every $(p, \xi) \in T_1S^n$, the Legendre lift λ of ϕ is Lie-taut.

To prove the converse, we use a Čech homology argument similar to that of Kuiper [303] used in the proof of Theorem 2.41 on page 44. Suppose that the Legendre lift $\lambda : B^{n-1} \to \Lambda^{2n-1}$ of ϕ is Lie-taut. Then for all $(p, \xi) \in T_1S^n$ except for a set Z of measure zero, the number of points $(x, N) \in B^{n-1}$ that are in oriented contact with some sphere in the parabolic pencil of oriented spheres determined by (p, ξ) is $\beta(B^{n-1}, \mathbf{Z}_2)/2 = \beta(V, \mathbf{Z}_2)$. This means that the corresponding function $r_{(p,\xi)}$ has $\beta(V, \mathbf{Z}_2)$ critical points on V. By Theorem 4.18, this implies that for a closed ball $B \subset S^n$ such that $\phi^{-1}(B) = V_s(r_{(p,\xi)})$ for $(p, \xi) \notin Z$ and $s \in \mathbf{R}$, the map on homology,

$$H_*(\phi^{-1}(B)) \to H_*(V), \tag{4.113}$$

is injective. On the other hand, if B is a closed ball corresponding to a sublevel set of $r_{(p,\xi)}$ for $(p, \xi) \in Z$, then since Z has measure zero, one can produce a nested sequence,

$$\{B_i\}, \quad i = 1, 2, 3, \ldots,$$

of closed balls (coming from $r_{(p,\xi)}$ for $(p, \xi) \notin Z$) satisfying

$$\phi^{-1}(B_i) \supset \phi^{-1}(B_{i+1}) \supset \cdots \supset \cap_{j=1}^{\infty} \phi^{-1}(B_j) = \phi^{-1}(B), \tag{4.114}$$

for $i = 1, 2, 3, \ldots$, such that the homomorphism in $\mathbf{Z}_2$-homology,

$$H_*(\phi^{-1}(B_i)) \to H_*(V), \text{ is injective for } i = 1, 2, 3, \ldots \tag{4.115}$$

If equations (4.114) and (4.115) are satisfied, then the map

$$H_*(\phi^{-1}(B_i)) \to H_*(\phi^{-1}(B_j)) \text{ is injective for all } i > j. \tag{4.116}$$

The continuity property of Čech homology (see Eilenberg–Steenrod [145, p. 261]) says that

$$H_*(\phi^{-1}(B)) = \overleftarrow{\lim_{i\to\infty}}\, H_*(\phi^{-1}(B_i)).$$

Equation (4.116) and Theorem 3.4 of Eilenberg–Steenrod [145, p. 216] on inverse limits imply that the map

$$H_*(\phi^{-1}(B)) \to H_*(\phi^{-1}(B_i))$$

is injective for each i. Thus, from equation (4.115), we get that the map

$$H_*(\phi^{-1}(B)) \to H_*(V)$$

is also injective. Since this holds for all closed balls B in S^n, the embedding $\phi(V)$ is taut by Theorem 4.19. □

Another formulation of the Lie invariance of tautness is the following corollary, as in [77, p. 95].

Corollary 4.25. *Let $\phi : V \to S^n$ and $\psi : V \to S^n$ be two embeddings of a compact, connected manifold V with $\dim V < n$ into S^n, such that their corresponding Legendre lifts are Lie equivalent. Then ϕ is taut if and only if ψ is taut.*

Proof. Since the Legendre lifts of ϕ and ψ are Lie equivalent, the unit normal bundles of $\phi(V)$ and $\psi(V)$ are diffeomorphic, and we will denote them both by B^{n-1}. Now let $\lambda : B^{n-1} \to \Lambda^{2n-1}$ and $\mu : B^{n-1} \to \Lambda^{2n-1}$ be the Legendre lifts of ϕ and ψ, respectively. By Theorem 4.24, ϕ is taut if and only if λ is Lie-taut, and ψ is taut if and only if μ is Lie-taut. Further, since λ and μ are Lie equivalent, λ is Lie-taut if and only if μ is Lie-taut, so it follows that ϕ is taut if and only if ψ is taut. □