Chapter 2
Submanifolds of Real Space Forms

In this chapter, we review the basic theory of submanifolds of real space forms needed for our in-depth treatment of isoparametric and Dupin hypersurfaces in later chapters. In Sections 2.1–2.4, we find the formulas for the shape operators of parallel hypersurfaces and tubes over submanifolds, and we discuss the focal submanifolds of a given submanifold.

We then define curvature surfaces and Dupin hypersurfaces in Section 2.5, and prove Pinkall's [446] result (Theorem 2.25) that given any positive integer g, and any positive integers $m_1, \ldots, m_g$ with $m_1 + \cdots + m_g = n - 1$, there exists a proper Dupin hypersurface M^{n-1} in $\mathbf{R}^n$ with g distinct principal curvatures having respective multiplicities $m_1, \ldots, m_g$.

In the next two sections, we define the notions of tight and taut immersions of manifolds into real space forms and develop the basic properties of these types of immersions. These concepts are important in themselves, and they are needed in the theory of isoparametric and Dupin hypersurfaces. In Section 2.8, we study the close relationship between the concepts of taut and Dupin submanifolds in detail.

Finally, in Section 2.9, we describe the standard embeddings of projective spaces into Euclidean spaces. These examples have many remarkable properties, and they are important in the theories of tight, taut, and isoparametric hypersurfaces.

2.1 Real Space Forms

We let $\mathbf{R}^n$ denote n-dimensional Euclidean space endowed with the standard Euclidean metric of constant sectional curvature zero. The theory of isoparametric and Dupin hypersurfaces in the sphere $S^n(c)$ of constant sectional curvature $c > 0$ is essentially the same for all values of $c > 0$, and so we restrict our attention to the sphere S^n of constant sectional curvature 1, that is, the unit sphere in $\mathbf{R}^{n+1}$ with the Riemannian metric induced from the Euclidean metric in $\mathbf{R}^{n+1}$.

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Similarly, for ambient spaces of constant negative sectional curvature, we restrict our attention to the hyperbolic space H^n of constant sectional curvature -1. To get a model for H^n, we consider the Lorentz space $\mathbf{R}^{n+1}_1$ endowed with the Lorentz metric of signature $(1, n)$,

$$\langle x, y\rangle = x_1 y_1 + \cdots + x_n y_n - x_{n+1} y_{n+1}, \tag{2.1}$$

for $x = (x_1, \ldots, x_{n+1})$ and $y = (y_1, \ldots, y_{n+1})$ in $\mathbf{R}^{n+1}_1$. Then real hyperbolic space of constant sectional curvature -1 is the hypersurface in $\mathbf{R}^{n+1}_1$ given by

$$H^n = \{x \in \mathbf{R}^{n+1}_1 \mid \langle x, x\rangle = -1,\ x_{n+1} \geq 1\}, \tag{2.2}$$

on which the Lorentz metric $\langle\ ,\ \rangle$ restricts to a Riemannian metric of constant sectional curvature -1 (see Kobayashi–Nomizu [283, Vol. II, pp. 268–271] for more detail).

By a real *space form* of dimension n, we mean a complete, connected, simply connected manifold $\tilde{M}^n$ with constant sectional curvature c. If $c = 0$, then $\tilde{M}^n = \mathbf{R}^n$; if $c = 1$, then $\tilde{M}^n = S^n$, and if $c = -1$, then $\tilde{M}^n = H^n$ (see, for example, [283, Vol. I, pp. 204–209]).

Let $f : M^n \to \tilde{M}^{n+k}$ for $k \geq 1$ be an immersion with codimension k of an n-dimensional manifold M into one of the three space forms $\tilde{M}^{n+k}$ mentioned above. For $x \in M$, let T_xM denote the tangent space to M at x, and let $T_x^{\perp}M$ denote the normal space to $f(M)$ at the point $f(x) \in \tilde{M}$. Let

$$NM = \{(x, \xi) \mid x \in M,\ \xi \in T_x^{\perp}M\}, \tag{2.3}$$

be the normal bundle of $f(M)$ with natural bundle projection $\pi : NM \to M$ defined by $\pi(x, \xi) = x$. Let η be a local cross section of NM. For any vector X in the tangent space T_xM, we have the fundamental equation

$$\tilde{\nabla}_{f_*(X)}\eta = -f_*(A_\eta X) + \nabla^{\perp}_{f_*(X)}\eta, \tag{2.4}$$

where $\tilde{\nabla}$ is the Levi-Civita connection in $\tilde{M}$, f_* is the differential of f, A_η is the *shape operator* determined by the normal vector $\eta(x)$, and $\nabla^{\perp}$ is the connection in the normal bundle.

The shape operator defines smooth map $(x, \xi) \mapsto A_\xi$ from the normal bundle NM into the space of symmetric tensors of type $(1, 1)$ on M. An eigenvalue λ of A_ξ is called a *principal curvature* of A_ξ, and its corresponding eigenvector is called a *principal vector*. Since $A_{t\xi} = tA_\xi$, for $t \in \mathbf{R}$, it is sufficient to know the principal curvatures on the bundle BM of unit normal vectors to M, i.e., the *unit normal bundle* of M.

2.2 Focal Points

Let $f : M \to \tilde{M}$ be an embedded submanifold of a real space form. Let $T\tilde{M}$ denote the tangent bundle of $\tilde{M}$, and let exp:$T\tilde{M} \to \tilde{M}$ be the exponential map of $\tilde{M}$. The *normal exponential map* or *end-point map* $E : NM \to \tilde{M}$ is the restriction of the exponential map of $\tilde{M}$ to the normal bundle NM of the submanifold M. Thus, if ξ is a nonzero normal vector to $f(M)$ at $f(x)$, then $E(x, \xi)$ is the point of $\tilde{M}$ reached by traversing a distance $|\xi|$ along the geodesic in $\tilde{M}$ with initial point $f(x)$ and initial tangent vector ξ. If ξ is the zero vector in the tangent space to $\tilde{M}$ at $f(x)$, then $E(x, \xi)$ is the point $f(x)$. It is well known (see, for example, [283, Vol. I, p. 147]) that exp is smooth in a neighborhood of the 0-section in $T\tilde{M}$, and so E is also smooth in a neighborhood of the 0-section in NM. It is easy to show that the differential E_* is nonsingular at points on the zero section, so we restrict our attention to points in NM that are not in the 0-section in trying to locate the critical values of E.

The focal points of M are the critical values of the normal exponential map E. Specifically, a point $p \in \tilde{M}$ is called a *focal point of* (M, x) *of multiplicity* m if $p = E(x, \xi)$ and the differential E_* at the point (x, ξ) has nullity $m > 0$. The *focal set* of M is the set of all focal points of (M, x) for all $x \in M$. Since NM and $\tilde{M}$ have the same dimension, it follows from Sard's Theorem (see, for example, [359, p. 33]) that the focal set of M has measure zero in $\tilde{M}$.

We now assume that ξ is a unit length normal vector to $f(M)$ at a point $x \in M$. The following theorem gives the location of the focal points of (M, x) along the geodesic $E(x, t\xi)$, for $t \in \mathbf{R}$, in terms of the eigenvalues of the shape operator A_ξ at x. We will give a proof for part (a) of the theorem, the case $\tilde{M}^{n+k} = \mathbf{R}^{n+k}$. (See also Milnor [359, pp. 32–35] for a proof in the Euclidean case, and Cecil [70] for a proof in the hyperbolic case. The proof in the spherical case is similar to that in the hyperbolic case.)

Theorem 2.1. *Let $f : M^n \to \tilde{M}^{n+k}$ be a submanifold of a real space form $\tilde{M}^{n+k}$, and let ξ be a unit normal vector to $f(M^n)$ at $f(x)$. Then $p = E(x, t\xi)$ is a focal point of (M^n, x) of multiplicity $m > 0$ if and only if there is an eigenvalue λ of the shape operator A_ξ of multiplicity m such that*

(a) $\lambda = 1/t$, *if* $\tilde{M}^{n+k} = \mathbf{R}^{n+k}$,
(b) $\lambda = \cot t$, *if* $\tilde{M}^{n+k} = S^{n+k}$,
(c) $\lambda = \coth t$, *if* $\tilde{M}^{n+k} = H^{n+k}$.

Proof. (a) In the following local calculation, we consider $M^n \subset \mathbf{R}^{n+k}$ as an embedded submanifold and do not mention the embedding f explicitly. We also consider the tangent space T_xM to be a subspace of $T_x\mathbf{R}^{n+k}$. We first recall some standard terminology and equations of submanifold theory. We will denote the Levi-Civita connection on $\mathbf{R}^{n+k}$ by D rather than $\tilde{\nabla}$. For locally defined smooth vector fields X and Y defined on M, we have the decomposition of D_XY into tangential and normal components,

$$D_XY = \nabla_XY + \sigma(X, Y), \tag{2.5}$$

which defines the *Levi-Civita connection* ∇ of the induced Riemannian metric on M and the *second fundamental form* σ. For a local field of unit normal vectors ξ on M, we have the decomposition of $D_X\xi$ into tangential and normal components,

$$D_X\xi = -A_\xi X + \nabla^\perp_X \xi, \tag{2.6}$$

which defines the shape operator A_ξ and normal connection $\nabla^\perp$, as in equation (2.4) above.

Since we know that there are no focal points on the 0-section, we will compute E_* at a point of NM that is not on the 0-section. We can consider this point to have the form $(x, t\xi)$, where $|\xi| = 1$ and $t > 0$. Let $\xi_1, \dots, \xi_k$ be an orthonormal frame of normal vectors to M at x with $\xi_1 = \xi$. Let U be a normal coordinate neighborhood of x in M as defined in [283, Vol. I, p. 148]. In order to simplify the calculations below, we extend $\xi_1, \dots, \xi_k$ to orthonormal normal vector fields on U by parallel translation with respect to the normal connection $\nabla^\perp$ along geodesics in U through x.

Let $\{\epsilon_1, \dots, \epsilon_k\}$ be the standard orthonormal basis of $\mathbf{R}^k$. Let S^{k-1} be the unit sphere in $\mathbf{R}^k$ given by

$$S^{k-1} = \{a = \sum_{j=1}^k a_j \epsilon_j \mid a_1^2 + \cdots + a_k^2 = 1\}. \tag{2.7}$$

We parametrize the normal bundle NM locally in a neighborhood of the point $(x, t\xi)$ by defining

$$\Psi : (0, \infty) \times S^{k-1} \times U \to NM \tag{2.8}$$

by

$$\Psi(\mu, a, y) = \mu \sum_{j=1}^k a_j \xi_j(y), \tag{2.9}$$

where the vector $\Psi(\mu, a, y)$ is normal to M at the point $y \in U$.

Then $(E \circ \Psi)(\mu, a, y)$ is the point in $\mathbf{R}^{n+k}$ reached by traversing a distance μ along the geodesic in $\mathbf{R}^{n+k}$ beginning at y and having initial direction

$$\sum_{j=1}^k a_j \xi_j(y). \tag{2.10}$$

That is,

$$(E \circ \Psi)(\mu, a, y) = y + \mu \sum_{j=1}^k a_j \xi_j(y). \tag{2.11}$$

In this local parametrization, the point $(x, t\xi)$ is equal to $\Psi(t, \epsilon_1, x)$. Evaluating E_* at $(x, t\xi)$ is equivalent to evaluating $(E \circ \Psi)_*$ at the point (t, ϵ_1, x). We now want to express $(E \circ \Psi)_*$ in terms of a basis consisting of $\partial/\partial\mu$ for $(0, \infty)$, $\{\epsilon_j\}$, $2 \leq j \leq k$, for $T_{\epsilon_1}S^{k-1}$, and an orthonormal basis of T_xM consisting of eigenvectors X of A_ξ with corresponding eigenvalues denoted by λ.

We first evaluate $(E \circ \Psi)_*(\partial/\partial\mu)$ at the point (t, ϵ_1, x). We have

$$(E \circ \Psi)_*(\partial/\partial\mu) = \overrightarrow{\beta}(\mu)|_{\mu=t}, \text{ where } \beta(\mu) = x + \mu\, \xi_1(x), \tag{2.12}$$

where $\overrightarrow{\beta}(\mu)$ is the velocity vector (tangent vector) of the curve $\beta(\mu)$.

Thus, we get

$$(E \circ \Psi)_*(\partial/\partial\mu) = \xi_1(x) = \xi. \tag{2.13}$$

Next, the tangent space $T_{\epsilon_1}S^{k-1}$ has an orthonormal basis $\{\epsilon_2, \ldots, \epsilon_k\}$. We want to compute $(E \circ \Psi)_*\epsilon_j$ for $2 \leq j \leq k$. In S^{k-1}, the curve

$$\gamma(s) = \cos s\, \epsilon_1 + \sin s\, \epsilon_j \tag{2.14}$$

has initial point ϵ_1 and initial velocity vector ϵ_j. Thus by equation (2.11), we see that $(E \circ \Psi)_*\epsilon_j$ is the initial velocity vector to the curve

$$\beta(s) = x + t\,(\cos s\, \xi_1(x) + \sin s\, \xi_j(x)). \tag{2.15}$$

Differentiating with respect to s and substituting $s = 0$, we get

$$(E \circ \Psi)_*\epsilon_j = t\xi_j(x). \tag{2.16}$$

Equations (2.13) and (2.16) show that if

$$V = c_1 \left(\frac{\partial}{\partial\mu}\right) + \sum_{j=2}^{k} c_j\epsilon_j, \tag{2.17}$$

then $(E \circ \Psi)_*V = 0$ only if $V = 0$.

Next we compute $(E \circ \Psi)_*X$ for $X \in T_xM$. If $\delta(s)$ is a curve in U with initial point x and initial velocity vector X, then $(E \circ \Psi)_*X$ is the initial velocity vector to the curve

$$\zeta(s) = (E \circ \Psi)\delta(s) = \delta(s) + t\, \xi_1(\delta(s)). \tag{2.18}$$

Differentiating with respect to s and using $\delta(0) = x$ and $\overrightarrow{\delta}(0) = X$, we get

$$\overrightarrow{\zeta}(0) = X + t\, D_X\xi_1. \tag{2.19}$$

We know that $D_X \xi_1 = -A_{\xi_1} X + \nabla^{\perp}_X \xi_1$, and we have constructed ξ_1 so that $\xi_1(x) = \xi$ and $\nabla^{\perp}_X \xi_1 = 0$. Hence, we have

$$(E \circ \Psi)_* X = X - tA_\xi X = (I - tA_\xi)X, \tag{2.20}$$

where we are identifying X with its Euclidean parallel translate at the point $p = E(x, t\xi)$.

From equations (2.13), (2.16), and (2.20), we see that for V as in equation (2.17) and a nonzero $X \in T_x M$, we have $(E \circ \Psi)_*(X + V) = 0$ if and only if $V = 0$ and $A_\xi X = (1/t)X$, i.e., $1/t$ is an eigenvalue of A_ξ with eigenvector X. Furthermore, if $\lambda = 1/t$ is an eigenvalue of A_ξ, then the nullity of E_* at $(x, t\xi)$ is equal to the dimension of the eigenspace T_λ, i.e., the multiplicity m of λ. This completes the proof of the theorem. □

2.3 Tubes and Parallel Hypersurfaces

As above, let $f : M^n \to \tilde{M}^{n+k}$ be an immersion into a real space form, and let BM denote the bundle of unit normal vectors to $f(M)$ in $\tilde{M}$. If the codimension k is greater than one, then we define the *tube of radius* $t > 0$ *over* M by the map $f_t : BM \to \tilde{M}$,

$$f_t(x, \xi) = E(x, t\xi). \tag{2.21}$$

If $(x, t\xi)$ is not a critical point of E, then f_t is an immersion in a neighborhood of (x, ξ) in BM. It follows from Theorem 2.1 that given any point $x \in M$, there is a neighborhood U of x in M such that for all $t > 0$ sufficiently small, the restriction of f_t to the unit normal bundle BU over U is an immersion onto an $(n + k - 1)$-dimensional manifold, which is geometrically a tube of radius t over U.

In the case where M is a hypersurface, i.e., the codimension $k = 1$, then BM is a double covering of M. In that case, for local calculations, we can assume that M is orientable with a local field of unit normal vectors ξ. Then we consider the *parallel hypersurface* $f_t : M \to \tilde{M}$ given by

$$f_t(x) = E(x, t\xi), \tag{2.22}$$

for $t \in \mathbf{R}$, rather than defining f_t on the double covering BM. Note that t can take any real value in this case. For a negative value of t, the parallel hypersurface lies locally on the side of M in the direction of the unit normal field $-\xi$, instead of on the side of M in the direction of ξ. For $t = 0$, we have $f_0 = f$, the original hypersurface.

In this section, we will compute the principal curvatures of the tube f_t in terms of the principal curvatures of the original submanifold M. We will treat the case of codimension $k > 1$ here. The case of codimension $k = 1$ is similar and is actually easier, and we omit it here. The formulas in Theorem 2.2 below work for the case of codimension 1 also, except that there are no ϵ_j in that case. We will handle the case $\tilde{M} = \mathbf{R}^{n+k}$ here. The calculations for the other space forms are similar and are left to the reader.

As in the preceding section, in the following local calculation we consider $M^n \subset \mathbf{R}^{n+k}$ as an embedded submanifold and do not mention the embedding f explicitly. We also consider the tangent space T_xM to be a subspace of $T_x\mathbf{R}^{n+k}$. Let (x, ξ) be a point in BM such that f_t is an immersion at (x, ξ), i.e., $(x, t\xi)$ is not a critical point of E. Let $\xi_1, \ldots, \xi_k$ be an orthonormal frame of normal vectors to M at x with $\xi_1 = \xi$. Let U be a normal coordinate neighborhood of x in M. We extend $\xi_1, \ldots, \xi_k$ to orthonormal normal vector fields on U by parallel translation with respect to the normal connection $\nabla^\perp$ along geodesics in U through x. Thus, we have the same setup as for the calculations in the proof of Theorem 2.1.

As in the proof of Theorem 2.1, let $\{\epsilon_1, \ldots, \epsilon_k\}$ be the standard orthonormal basis of $\mathbf{R}^k$. Let S^{k-1} be the unit sphere in $\mathbf{R}^k$ given by

$$S^{k-1} = \{a = \sum_{j=1}^{k} a_j \epsilon_j \mid a_1^2 + \cdots + a_k^2 = 1\}. \tag{2.23}$$

We parametrize the unit normal bundle BM locally in a neighborhood of the point (x, ξ) by defining

$$\Psi : S^{k-1} \times U \to BM \tag{2.24}$$

by

$$\Psi(a, y) = \sum_{j=1}^{k} a_j \xi_j(y), \tag{2.25}$$

where the vector $\Psi(a, y)$ is a unit normal vector to M at the point $y \in U$.

In this local parametrization, the point (x, ξ) in BM is equal to $\Psi(\epsilon_1, x)$. Evaluating $(f_t)_*$ at (x, ξ) is equivalent to evaluating $(f_t \circ \Psi)_*$ at the point (ϵ_1, x). We now want to express $(f_t \circ \Psi)_*$ at (ϵ_1, x) in terms of a basis consisting of $\{\epsilon_j\}$, $2 \leq j \leq k$, for $T_{\epsilon_1} S^{k-1}$, and an orthonormal basis of T_xM consisting of eigenvectors X of A_ξ with corresponding eigenvalues denoted by λ.

The calculations of $(f_t \circ \Psi)_*$ are exactly the same as the calculations of $(E \circ \Psi)_*$ in the proof of Theorem 2.1, except that there is no $\partial/\partial\mu$ term. Specifically, as in equation (2.16), we get

$$(f_t \circ \Psi)_* \epsilon_j = t\xi_j(x). \tag{2.26}$$

Then for $X \in T_x M$, we get as in equation (2.20),

$$(f_t \circ \Psi)_* X = X - tA_\xi X = (I - tA_\xi)X, \tag{2.27}$$

where we are identifying X with its Euclidean parallel translate at the point $p = f_t(x, \xi)$.

Since f_t is an immersion at (x, ξ), there is a neighborhood W of the point (x, ξ) in the unit normal bundle BU such that the restriction of f_t to W is an embedded hypersurface in $\mathbf{R}^{n+k}$. To find the shape operator of $f_t W$, we need to find a local field of unit normals to $f_t W$, and then compute its covariant derivative.

If (u, η) is an arbitrary point of W, then the Euclidean parallel translate of η is a unit normal to the hypersurface $f_t W$ at the point $f_t(u, \eta)$. So we now let η denote a field of unit normals to the hypersurface $f_t W$ on the neighborhood W. We denote the corresponding shape operator of the oriented hypersurface $f_t W$ by A_t.

We use the same local parametrization of BM given above. We can identify the tangent space $T_{(x,\xi)} BM$ with $T_{\epsilon_1} S^{k-1} \times T_x M$ via the parametrization Ψ, and we can consider the shape operator A_t to be defined on $T_{\epsilon_1} S^{k-1} \times T_x M$. In particular, A_t is defined by,

$$(f_t \circ \Psi)_* (A_t Z) = -D_{(f_t \circ \Psi)_* Z}\, \eta, \tag{2.28}$$

for $Z \in T_{\epsilon_1} S^{k-1} \times T_x M$. Note that there is no term involving the normal connection $\nabla^\perp$, since the codimension of $f_t W$ is one.

We first compute $A_t \epsilon_j$ for $2 \leq j \leq k$. As in equation (2.15), we have that $(f_t \circ \Psi)_* \epsilon_j$ is the initial velocity vector to the curve

$$\beta(s) = x + t\,(\cos s\, \xi_1(x) + \sin s\, \xi_j(x)). \tag{2.29}$$

Hence, $D_{(f_t \circ \Psi)_* \epsilon_j}\, \eta$ is the initial velocity vector $\overrightarrow{\eta}(0)$ to the curve

$$\eta(\beta(s)) = \cos s\, \xi_1(x) + \sin s\, \xi_j(x). \tag{2.30}$$

Therefore, we have

$$(f_t \circ \Psi)_* (A_t \epsilon_j) = -\overrightarrow{\eta}(0) = -\xi_j(x). \tag{2.31}$$

Since we have $(f_t \circ \Psi)_* \epsilon_j = t \xi_j(x)$ by equation (2.26), we get

$$A_t \epsilon_j = -\frac{1}{t} \epsilon_j. \tag{2.32}$$

Thus, ϵ_j is a principal vector of A_t with corresponding principal curvature $-1/t$, where t is the radius of the tube.

Next we find A_tX for a vector $X \in T_xM$. Let $\delta(s)$ be a curve in M with initial point $\delta(0) = x$ and initial velocity vector $\overrightarrow{\delta}(0) = X$. Then $(f_t \circ \Psi)_*X$ is the initial velocity vector to the curve

$$\zeta(s) = \delta(s) + t\,\xi_1(\delta(s)). \tag{2.33}$$

Along this curve $\zeta(s)$, the unit normal field η to the tube is given by

$$\eta(\delta(s)) = \xi_1(\delta(s)). \tag{2.34}$$

Then $D_{(f_t \circ \Psi)_*X}\eta$ is the initial velocity vector to this curve $\xi_1(\delta(s))$, which is just $D_X\xi_1$, where again we are identifying parallel vectors in $\mathbf{R}^{n+k}$. Then using the fact that $\nabla^\perp_X \xi_1 = 0$, we get from equation (2.4)

$$D_{(f_t \circ \Psi)_*X}\eta = D_X\xi_1 = -A_\xi X, \tag{2.35}$$

since $\xi_1(x) = \xi$. Thus we have from equation (2.28) that $(f_t \circ \Psi)_*(A_tX) = A_\xi X$. Then it follows from equation (2.27) for $(f_t \circ \Psi)_*X$ that

$$A_tX = (I - tA_\xi)^{-1}A_\xi X. \tag{2.36}$$

In the case of a principal vector X such that $A_\xi X = \lambda X$, this reduces to

$$A_tX = \frac{\lambda}{1 - t\lambda}X. \tag{2.37}$$

Therefore, X is a principal vector of A_t with corresponding principal curvature $\lambda/(1 - t\lambda)$.

Principal curvatures of a tube

In summary, we have the following theorem for the shape operators of a tube over a submanifold of Euclidean space $\mathbf{R}^{n+k}$. Similar computations to those above yield the results for submanifolds of S^{n+k} and H^{n+k}, which are also stated in the theorem. In the case $k = 1$, the theorem gives the formula for the shape operator of a parallel hypersurface f_tM. In that case, there are no terms $A_t\epsilon_j$.

Theorem 2.2. *Let M^n be a submanifold of a real space form $\tilde{M}^{n+k}$ and ξ a unit normal vector to M at x such that $f_t : BM \to \tilde{M}^{n+k}$ is an immersion at the point $(x, \xi) \in BM$. Let $\{X_1, \ldots, X_n\}$ be a basis of T_xM consisting of principal vectors of A_ξ with $A_\xi X_i = \lambda_i X_i$ for $1 \leq i \leq n$. In terms of the local parametrization of BM given in this section, the shape operator A_t of the tube f_t of radius t over M at the point (x, ξ) is given in terms of its principal vectors as follows:*

For submanifolds of $\mathbf{R}^{n+k}$,

(1) *For* $2 \leq j \leq k$, $A_t \epsilon_j = -\frac{1}{t}\, \epsilon_j$,
(2) *For* $1 \leq i \leq n$, $A_t X_i = \frac{\lambda_i}{1-t\lambda_i}\, X_i$.

For submanifolds of S^{n+k},

(1) *For* $2 \leq j \leq k$, $A_t \epsilon_j = -\cot t\, \epsilon_j$,
(2) *For* $1 \leq i \leq n$, $A_t X_i = \cot(\theta_i - t)\, X_i$, *if* $\lambda_i = \cot \theta_i$, $0 < \theta_i < \pi$.

For submanifolds of H^{n+k},

(1) *For* $2 \leq j \leq k$, $A_t \epsilon_j = -\coth t\, \epsilon_j$,
(2) *For* $1 \leq i \leq n$,

   (a) $A_t X_i = \coth(\theta_i - t)\, X_i$, *if* $|\lambda_i| > 1$, *and* $\lambda_i = \coth \theta_i$,
   (b) $A_t X_i = \pm X_i$, *if* $\lambda_i = \pm 1$,
   (c) $A_t X_i = \tanh(\theta_i - t)\, X_i$, *if* $|\lambda_i| < 1$, *and* $\lambda_i = \tanh \theta_i$.

As a consequence of Theorems 2.1 and 2.2, we obtain the following useful result. In the case where M has codimension $k > 1$, the points of M are focal points of the tube $f_t M$ corresponding to the principal curvature $\mu = -1/t$ of A_t in the case $\tilde{M}^{n+k} = \mathbf{R}^{n+k}$, $\mu = -\cot t$ in the case $\tilde{M}^{n+k} = S^{n+k}$, and $\mu = -\coth t$ in the case $\tilde{M}^{n+k} = H^{n+k}$.

Theorem 2.3. *Let* M^n *be a submanifold of a real space form* $\tilde{M}^{n+k}$ *and* t *a real number such that* $f_t M$ *is a hypersurface.*

(a) *If* M *is a hypersurface, then the focal set of the parallel hypersurface* $f_t M$ *is the focal set of* M.
(b) *If* M *has codimension greater than one, then the focal set of the tube* $f_t M$ *consists of the union of the focal set of* M *with* M *itself.*

2.4 Focal Submanifolds

In this section, we find a natural manifold structure for the sheet of the focal set of a hypersurface of a real space form corresponding to a principal curvature of constant multiplicity. By considering tubes and using Theorems 2.2 and 2.3, this also enables us to give a manifold structure to a sheet of the focal set of a submanifold of codimension greater than one. These results were originally obtained in the paper of Cecil and Ryan [88], and they were suggested by the work of Nomizu [403], who obtained similar results for the sheets of the focal set of an isoparametric hypersurface. See also the related work of Reckziegel [457–459].

Let $f : M^n \to \tilde{M}^{n+1}$ be an immersed hypersurface in a real space form $\tilde{M}$. For the following local considerations, we assume that $f(M)$ is orientable with a global field of unit normals ξ and corresponding shape operator $A = A_\xi$. If the principal curvature functions on M are ordered as

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n, \tag{2.38}$$

then each λ_i is a continuous function (see Ryan [468, p. 371]). Furthermore, if a continuous principal curvature function λ has constant multiplicity m on M, then λ is a smooth function, and its m-dimensional distribution T_λ of principal vectors is also smooth on M (see, for example, Nomizu [402], Reckziegel [457, 458], or Singley [486]). We will show in this section that T_λ is also integrable, and so it is an m-dimensional foliation on M called the *principal foliation* corresponding to the principal curvature λ. Using this fact, we will then show that if λ is constant along each leaf of T_λ, then the sheet of the focal set of M corresponding to λ is a smooth $(n-m)$-dimensional submanifold of $\tilde{M}$.

Remark 2.4 (An example of principal curvature functions that are not smooth). If a continuous principal curvature function does not have constant multiplicity, then it is not necessarily a smooth function. Consider, for example, the behavior of the principal curvature functions of the monkey saddle in $\mathbf{R}^3$ given as the graph of the function

$$z = \frac{x^3 - 3xy^2}{3}. \tag{2.39}$$

This surface has two distinct principal curvatures at each point except at the umbilic point at the origin. In terms of polar coordinates (r, θ) on $\mathbf{R}^2$, the principal curvatures are given by the formula,

$$(1 + r^4)^{3/2}\,\lambda = -r^5 \cos 3\theta \pm 2r \left(1 + r^4 + \frac{r^8}{4}\cos^2 3\theta\right)^{1/2}. \tag{2.40}$$

As r approaches zero, the two principal curvature functions are asymptotically equal to $\pm 2r$, so these functions are continuous, but not smooth at the origin. (See [96] or [95, pp. 134–135] for more detail.)

If a principal curvature function λ has constant multiplicity m on M, then we can define a smooth *focal map* f_λ from an open subset $U \subset M$ (defined below) onto the sheet of the focal set of M determined by λ. Using Theorem 2.1 for the location of the focal points, we define the map f_λ by the formulas,

$$\begin{aligned} f_\lambda(x) &= f(x) + \frac{1}{\lambda}\,\xi(x), \\ f_\lambda(x) &= \cos\theta\, f(x) + \sin\theta\,\xi(x), \text{ where } \cot\theta = \lambda, \\ f_\lambda(x) &= \cosh\theta\, f(x) + \sinh\theta\,\xi(x), \text{ where } \coth\theta = \lambda, \end{aligned} \tag{2.41}$$

for $\tilde{M}$ equal to $\mathbf{R}^{n+1}$, S^{n+1}, and H^{n+1}, respectively.

In the case of $\mathbf{R}^{n+1}$, the domain U of f_λ is the set of points in M where $\lambda \neq 0$. In hyperbolic space, the domain U of f_λ is the set of points where $|\lambda| > 1$. In the case of S^{n+1}, at each point $x \in M$ the principal curvature λ gives rise to two antipodal focal points in S^{n+1} determined by substituting $\theta = \cot^{-1}\lambda$ and $\theta = \cot^{-1}\lambda + \pi$ into equation (2.41). Thus, λ gives rise to two antipodal focal maps into S^{n+1}.

For a point x in the domain U of f_λ, the hypersphere $K_\lambda(x)$ in $\tilde{M}$ through x and centered at the focal point $f_\lambda(x)$ is called the *curvature sphere* determined by λ at x. This curvature sphere is tangent to $f(M)$ at the point $f(x)$. These curvature spheres play a crucial role in the study of Dupin hypersurfaces in the context of Lie sphere geometry.

Remark 2.5 (On the definition of a curvature sphere). Note that the definition of a curvature sphere does not require that λ have constant multiplicity or be a smooth function. It can be defined pointwise. If x is in the domain U of the map f_λ defined in equation (2.41), then the curvature sphere at x corresponding to the principal curvature λ is the hypersphere $K_\lambda(x)$ in $\tilde{M}$ through x and centered at the focal point $f_\lambda(x)$.

Conformal transformations of the ambient space

The condition that a principal curvature function λ has constant multiplicity on M is important in the study of Dupin hypersurfaces. This consideration is preserved by conformal transformations of the ambient space, as the following considerations show.

Let $(\tilde{M}, g)$ and $(\tilde{M}', g')$ be two Riemannian manifolds, and suppose that $\psi : \tilde{M} \to \tilde{M}'$ is a conformal diffeomorphism such that

$$g'(\psi_* X, \psi_* Y) = e^{2h(x)} g(X, Y), \tag{2.42}$$

for all X, Y tangent to $\tilde{M}$ at x, where h is a smooth function on $\tilde{M}$. Let M be a submanifold of $\tilde{M}$, and let ξ be a local field of unit normals to M in a neighborhood of x. Then $\xi' = \psi_*(e^{-h}\xi)$ is a field of unit normals to $\psi(M)$ near $\psi(x)$ and the corresponding shape operators are related by the equation,

$$B_{\xi'} = e^{-h}(A_\xi - g(\operatorname{grad} h, \xi) I). \tag{2.43}$$

A direct calculation then yields the following relationship between the principal curvatures of M in $\tilde{M}$ and those of $\psi(M)$ in $\tilde{M}'$.

Theorem 2.6. *Let $\psi : (\tilde{M}, g) \to (\tilde{M}', g')$ be a conformal diffeomorphism of Riemannian manifolds with $g'(\psi_* X, \psi_* Y) = e^{2h(x)} g(X, Y)$ for all X, Y tangent to $\tilde{M}$ at x. Let M be an oriented hypersurface in $\tilde{M}$, and let λ be a smooth principal curvature function of constant multiplicity m on M. Then*

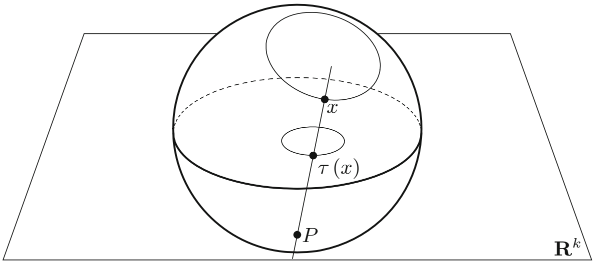

Fig. 2.1 Stereographic projection

$$\mu = e^{-h}(\lambda - g(\text{grad}\, h, \xi))$$

is a smooth principal curvature function of multiplicity m *on* $\psi(M)$*, and the respective principal distributions of* λ *and* μ *coincide on* M*.*

Remark 2.7 (Stereographic projection and inversions in spheres). We want to apply Theorem 2.6 to the case of hypersurfaces in real space forms by considering the conformal transformation given by stereographic projection from S^k or H^k into $\mathbf{R}^k$, for any positive integer k. In the spherical case, let P be an arbitrary point of the unit sphere $S^k \subset \mathbf{R}^{k+1}$, and let

$$\mathbf{R}^k = \{x \in \mathbf{R}^{k+1} \mid \langle x, P \rangle = 0\}, \tag{2.44}$$

where $\langle\ ,\ \rangle$ is the Euclidean inner product on $\mathbf{R}^{k+1}$. Then stereographic projection with pole P is the map $\tau : S^k - \{P\} \to \mathbf{R}^k$ defined geometrically as follows. For $x \in S^k - \{P\}$, the ray from x through P intersects $\mathbf{R}^k$ in exactly one point which is $\tau(x)$ (see Figure 2.1). Analytically, this is given by

$$\tau(x) = P + \frac{1}{1 - \langle x, P \rangle}\,(x - P). \tag{2.45}$$

In terms of our conformal geometric considerations, this can be written as

$$\tau(x) = P + e^{h(x)}(x - P), \tag{2.46}$$

where $e^{-h(x)} = 1 - \langle x, P \rangle$. It is easily shown that τ is a conformal diffeomorphism with $\langle \tau_* X, \tau_* Y \rangle = e^{2h(x)} \langle X, Y \rangle$, for all X, Y tangent to S^k at x.

Recall from equation (2.2) that our model of k-dimensional hyperbolic space is given by

$$H^k = \{x \in \mathbf{R}^{k+1}_1 \mid \langle x, x \rangle = -1,\ x_{k+1} \geq 1\},$$

where $\langle\ ,\ \rangle$ is the Lorentz metric,

$$\langle x, y\rangle = x_1y_1 + \cdots + x_ky_k - x_{k+1}y_{k+1},$$

on $\mathbf{R}^{k+1}_1$. Let P be a point in $\mathbf{R}^{k+1}_1$ such that $-P \in H^k$. Let D^k be the k-dimensional disk

$$D^k = \{x \in \mathbf{R}^{k+1}_1 \mid \langle x, P\rangle = 0,\ \langle x, x\rangle < 1\}, \tag{2.47}$$

on which the metric $\langle\ ,\ \rangle$ restricts to a Euclidean metric, which we denote by g. Then we define stereographic projection $\tau : H^k \to D^k$ with pole P as follows. For $x \in H^k$, the ray from P through x intersects D^k in exactly one point which is $\tau(x)$. Analytically, this is given by

$$\tau(x) = P + \frac{1}{1 + \langle x, P\rangle}\,(x - P). \tag{2.48}$$

In terms of our conformal geometry, this can be written as,

$$\tau(x) = P + e^{h(x)}(x - P), \tag{2.49}$$

where $e^{-h(x)} = 1 + \langle x, P\rangle$. One can easily show that τ is a conformal diffeomorphism with $g(\tau_*X, \tau_*Y) = e^{2h(x)}\langle X, Y\rangle$, for all X, Y tangent to H^k at the point x.

Another important type of conformal transformation is inversion,

$$\sigma : \mathbf{R}^{n+1} - \{p\} \to \mathbf{R}^{n+1} - \{p\}, \tag{2.50}$$

in a sphere centered at $p \in \mathbf{R}^{n+1}$ with radius $r > 0$. The map σ takes a point $q \in \mathbf{R}^{n+1} - \{p\}$ to the point $\sigma(q)$ on the ray from p through q such that $|q - p|\,|\sigma(q) - p| = r^2$.

We now return to the case of a hypersurface $f : M^n \to \tilde{M}^{n+1}$ in a real space form and consider the question of when the image of a focal map f_λ is a submanifold of the ambient space $\tilde{M}$, where λ is a principal curvature of constant multiplicity m on M. Here we will make the calculations only for $\tilde{M} = \mathbf{R}^{n+1}$. The proofs for the other space forms are similar.

Theorem 2.8. *Let $f : M^n \to \tilde{M}^{n+1}$ be an oriented hypersurface of a real space form. Suppose that λ is a smooth principal curvature function of constant multiplicity $m \geq 1$ in a neighborhood of a point x in the domain of f_λ. Then the rank of the focal map f_λ at x equals $n - m + 1$ if there exists $X \in T_\lambda(x)$ such that $X\lambda \neq 0$, and it equals $n - m$ otherwise.*

Proof. Here we consider the case $f : M^n \to \mathbf{R}^{n+1}$. Let ξ be the field of unit normals on M. On a neighborhood W of x on which λ is nonzero and has constant multiplicity m, we have from equation (2.41) that

$$f_\lambda(y) = f(y) + \rho(y)\xi(y),$$

for $y \in W$, where $\rho = 1/\lambda$. For $X \in T_xM$, we compute the differential of f_λ applied to X,

$$(f_\lambda)_*(X) = X + (X\rho)\,\xi + \rho D_X\xi, \tag{2.51}$$

where again we are identifying vectors that are Euclidean parallel. If $X \in T_\mu(x)$ for a principal curvature $\mu \neq \lambda$, then since $D_X\xi = -\mu X$, equation (2.51) yields

$$(f_\lambda)_*(X) = (1 - \frac{\mu}{\lambda})X + (X\rho)\,\xi, \text{ for } X \in T_\mu, \tag{2.52}$$

and thus $(f_\lambda)_*$ is injective on T_μ. This is true for all principal curvatures μ not equal to λ, and so $(f_\lambda)_*$ is injective on $T_\lambda^\perp(x)$, which is the direct sum of the principal spaces corresponding to the other principal curvatures. On the other hand, if $X \in T_\lambda(x)$, then equation (2.51) yields

$$(f_\lambda)_*(X) = (X\rho)\xi = \frac{-X\lambda}{\lambda^2}\,\xi, \text{ for } X \in T_\lambda. \tag{2.53}$$

Thus, if $X\lambda \neq 0$ for some $X \in T_\lambda(x)$, then the range of $(f_\lambda)_*$ is the $(n-m+1)$-dimensional space spanned by $T_\lambda^\perp(x)$ and ξ, while if $X\lambda = 0$ for all $X \in T_\lambda(x)$, then the range of $(f_\lambda)_*$ is the $(n-m)$-dimensional space $(f_\lambda)_*(T_\lambda^\perp(x))$. □

This proof shows that at a point x where the focal map f_λ has rank equal to $n-m+1$, a vector parallel to the normal vector $\xi(x)$ is tangent to the image of f_λ at the point $f_\lambda(x)$. Thus, it generalizes the classical result that the normal to a surface M in $\mathbf{R}^3$ is tangent to the evolute surface (focal set) when f_λ has rank two (see, for example, Goetz [175]). In the classical case, if a principal curvature λ has constant multiplicity one, and $X\lambda \neq 0$ on M for a corresponding nonzero principal vector field X, then the sheet of the focal submanifold $f_\lambda(M)$ is also an immersed surface. More generally, if λ has constant multiplicity one, then $f_\lambda(M)$ is a surface with singularities at the images under f_λ of points where $X\lambda = 0$. For example, the evolute of an ellipse in a plane has singularities at the images of the four vertices.

Another consequence of the proof of Theorem 2.8 is the following corollary. Recall that for x in the domain U of f_λ, the curvature sphere $K_\lambda(x)$ in $\tilde{M}$ is the hypersphere through x centered at the focal point $f_\lambda(x)$. Thus, the curvature sphere map K_λ is constant along a leaf of T_λ in U if and only if the focal map f_λ is constant along that leaf.

Corollary 2.9. *Let $f : M^n \to \tilde{M}^{n+1}$ be an oriented hypersurface of a real space form. Suppose that λ is a smooth principal curvature function of constant multiplicity $m \geq 1$ on M, and let U be the domain of the focal map f_λ. Then the following conditions are equivalent on U:*

(1) *λ is constant along each leaf of its principal foliation T_λ.*
(2) *The focal map f_λ is constant along each leaf of T_λ in U.*
(3) *The curvature sphere map K_λ is constant along each leaf of T_λ in U.*

Returning to the general situation of a hypersurface $f : M^n \to \tilde{M}^{n+1}$, it follows from the "constant rank theorem" (see, for example, Conlon [120, p. 39]) that the sheet $f_\lambda(U)$ of the focal set will be a submanifold of $\tilde{M}$ locally if f_λ has constant rank on U. From Theorem 2.8, we see that this is contingent on the value of $X\lambda$ for principal vectors X corresponding to the principal curvature λ. The following theorem shows that in the case where λ has constant multiplicity $m > 1$ on M, the derivative $X\lambda$ is always zero for every principal vector X corresponding to λ at every point of M, and thus f_λ has constant rank $n - m$ on U. However, this is not the case if λ has constant multiplicity $m = 1$ on M, and so we will handle the cases $m > 1$ and $m = 1$ separately, beginning with the case $m > 1$.

Integrability of the principal distribution when m > 1

Theorem 2.10. *Let $f : M^n \to \tilde{M}^{n+1}$ be an oriented hypersurface of a real space form. Suppose that λ is a smooth principal curvature function of constant multiplicity $m > 1$ on M. Then the principal distribution T_λ is integrable, and $X\lambda = 0$ for every $X \in T_\lambda$ at every point of M.*

Proof. We use the Codazzi equation, which for an oriented hypersurface in a real space form takes the form $(\nabla_X A)Y = (\nabla_Y A)X$ (see [283, Vol. II, p. 26]), that is,

$$\nabla_X(AY) - A(\nabla_X Y) = \nabla_Y(AX) - A(\nabla_Y X), \tag{2.54}$$

for vector fields X and Y tangent to M. If take X and Y to be linearly independent (local) vector fields in the principal distribution T_λ, then the Codazzi equation (2.54) becomes

$$(X\lambda)Y + \lambda\nabla_X Y - A(\nabla_X Y) = (Y\lambda)X + \lambda\nabla_Y X - A(\nabla_Y X). \tag{2.55}$$

Since the Levi-Civita connection has zero torsion, the Lie bracket $[X, Y] = \nabla_X Y - \nabla_Y X$, and equation (2.55) reduces to

$$(X\lambda)Y - (Y\lambda)X = (A - \lambda I)\,[X, Y]. \tag{2.56}$$

Since the left side of this equation is in T_λ, while the right side is $T_\lambda^\perp$, both sides are equal to zero. Thus, T_λ is integrable by the Frobenius Theorem (see, for example, [283, Vol I., p. 10]), since $[X, Y]$ is in T_λ. Furthermore, $X\lambda$ and $Y\lambda$ are both zero on M, since X and Y are linearly independent. □

Thus, in the case where λ has constant multiplicity $m > 1$ on M, the distribution T_λ is a foliation on M, which we call the *principal foliation* corresponding to λ. We next prove that the leaves of a principal foliation are m-dimensional totally umbilic submanifolds of $\tilde{M}$, where a submanifold V of a space form $\tilde{M}$ is said to be *totally umbilic* if for each $x \in V$, there is a real-valued linear function ω on $T_x^\perp V$ such that the shape operator B_η of V satisfies $B_\eta = \omega(\eta)I$ for every $\eta \in T_x^\perp V$.

In all three space forms $\tilde{M}^{n+1}$, a totally umbilic m-dimensional submanifold always lies in a totally geodesic $(m+1)$-dimensional submanifold of $\tilde{M}^{n+1}$. Thus it suffices to describe the totally umbilic hypersurfaces in each of the three space forms. In $\mathbf{R}^{m+1}$, a connected totally umbilic hypersurface is an open subset of an m-plane or an m-dimensional metric sphere. In S^{m+1}, a connected totally umbilic hypersurface is an open subset of a great or small hypersphere in S^{m+1}. Finally, in hyperbolic space H^{m+1}, a connected totally umbilic hypersurface is an open subset of a totally geodesic hyperplane, an equidistant hypersurface from a hyperplane, a horosphere, or a metric sphere (see, for example, [283, Vol. II, pp. 30–32] or Spivak [495, Vol. 4, pp. 110–114]).

Theorem 2.11. *Let $f : M^n \to \tilde{M}^{n+1}$ be an oriented hypersurface of a real space form. Suppose that λ is a smooth principal curvature function of constant multiplicity $m > 1$ on M. Then the leaves of the principal foliation T_λ are m-dimensional totally umbilic submanifolds of $\tilde{M}$.*

Proof. Let V be a leaf of the principal foliation T_λ. The normal space $T_x^\perp V$ to V in $\tilde{M}$ at a point $x \in V$ can be decomposed as

$$T_x^\perp V = T_x^\perp M \oplus T_\lambda^\perp(x),$$

where $T_\lambda^\perp(x)$ is the orthogonal complement to $T_\lambda(x)$ in T_xM. For a unit vector $\eta \in T_x^\perp V$, let B_η denote the shape operator of V corresponding to η. If η is the normal vector ξ to M at x with associated shape operator A, then $B_\eta X = AX = \lambda X$, for $X \in T_\lambda(x)$, and thus $B_\eta = \lambda I$.

Next let $\eta \in T_\lambda^\perp(x)$ be a unit length principal vector of A with corresponding principal curvature μ, so that $A\eta = \mu\eta$ for $\mu \neq \lambda$. Extend η to a vector field $Y \in T_\lambda^\perp$ on a neighborhood W of x. Then there exists a unique vector field $Z \in T_\lambda^\perp$ such that $\langle Z, Y\rangle = 0$ and

$$AY = \mu Y + Z, \tag{2.57}$$

for some smooth function μ on W. This is possible since $T_\lambda^\perp$ is invariant under A, even though the eigenvalues of A need not be smooth.

We now find the shape operator B_η. Let X be a vector field in T_λ on the neighborhood W. Since the vector field $Z = 0$ at x, one can easily show that $\nabla_X Z \in T_\lambda^\perp$ at x. Using equation (2.57), we see that the Codazzi equation (2.54) becomes

$$(X\mu)Y - (Y\lambda)X + \nabla_X Z = (A - \mu I)\nabla_X Y - (A - \lambda I)\nabla_Y X. \tag{2.58}$$

If we consider the T_λ-component of both sides of this equation, we see that the T_λ-component of $\nabla_X Y$ at x is

$$\frac{-(Y\lambda)X}{\lambda - \mu}. \tag{2.59}$$

If $\tilde{\nabla}$ is the Levi-Civita connection on $\tilde{M}^{n+1}$, we have the basic equation,

$$\tilde{\nabla}_X Y = \nabla_X Y + \langle AX, Y\rangle \xi.$$

Since $\langle AX, Y\rangle = 0$, the T_λ-component of $\tilde{\nabla}_X Y$ at x is equal to the T_λ-component of $\nabla_X Y$ at x, which is given by equation (2.59). Since $\eta = Y$ at x, the vector $-B_\eta X$ is by definition equal to the T_λ-component of $\tilde{\nabla}_X Y$, and so we have

$$B_\eta X = \frac{(\eta\lambda)X}{\lambda - \mu}. \tag{2.60}$$

This completes the proof of the theorem. □

A manifold structure for the focal set

As we saw in Theorem 2.1, the domain U of the focal map f_λ is the set where $\lambda \neq 0$ in the case $\tilde{M} = \mathbf{R}^{n+1}$, and the set where $|\lambda| > 1$ for $\tilde{M} = H^{n+1}$. At all such points, the leaf of the principal foliation T_λ through the point is an open subset of an m-dimensional metric sphere in $\tilde{M}$.

By Theorems 2.8 and 2.10, we know that in this case of multiplicity $m > 1$, the focal map f_λ is constant along each leaf of T_λ, and so it factors through a map of the space U/T_λ of leaves of T_λ, where U is the domain of f_λ. This enables us to place a manifold structure on the sheet of the focal set $f_\lambda(U)$ as follows.

Theorem 2.12. *Let $f : M^n \to \tilde{M}^{n+1}$ be an oriented hypersurface of a real space form. Suppose that λ is a smooth principal curvature function of constant multiplicity $m > 1$ on M^n. Then the focal map $f_\lambda : U \to \tilde{M}^{n+1}$ factors through an immersion of the (possibly non-Hausdorff) $(n-m)$-dimensional manifold U/T_λ into $\tilde{M}^{n+1}$. If M^n is complete with respect to the induced metric, then the manifold U/T_λ is Hausdorff.*

Proof. Since the leaves of T_λ are totally umbilic submanifolds of $\tilde{M}^{n+1}$, the foliation T_λ is regular as defined by Palais [425, p. 13], that is, every point has a coordinate chart distinguished by the foliation such that each leaf intersects the chart in at most one m-dimensional slice. This implies that the space of leaves U/T_λ is an $(n-m)$-dimensional manifold in the sense of Palais, which may not be Hausdorff. By Theorems 2.8 and 2.10, the focal map f_λ factors through a map $g_\lambda : U/T_\lambda \to \tilde{M}^{n+1}$, and the map g_λ is an immersion, since the rank of g_λ equals the rank f_λ, which is $n-m$ at each point. Finally, the regularity of the foliation T_λ implies that each leaf is a closed subset of M (see Palais [425, p. 18]). Thus, if M is complete, then each leaf is also complete (see, for example, [283, Vol. I, p. 179]). Therefore, each leaf of T_λ that intersects the domain U of f_λ is an m-dimensional metric sphere in $\tilde{M}^{n+1}$ and is thus compact. This implies that the leaf space U/T_λ is Hausdorff [425, p. 16]. □

Remark 2.13 (An example in which $f_\lambda(M)$ is not a Hausdorff manifold). The following example ([88, p. 34] or [95, p. 143]) shows that the leaf space of a regular foliation is not necessarily Hausdorff and that the image of a focal map with constant rank is not necessarily a Hausdorff manifold. Let $\phi(t)$ be the real-valued function on $\mathbf{R}$ defined by $\phi(t) = e^{-1/t}$ if $t > 0$ and $\phi(t) = 0$ if $t \leq 0$. Let K be a tube of constant radius one in $\mathbf{R}^3$ over the curve,

$$\gamma(t) = (t, 0, \phi(t)), \ t \in (-1, 1).$$

Then the curve γ itself is the sheet of the focal set of K corresponding to the principal curvature $\lambda = 1$ with appropriate choice of unit normal field.

Let N be the intersection of K with the closed upper half-space given by $z \geq 0$, with the points satisfying $z = 0, \ x \geq 0$ removed. Let M be the union of N with its mirror image in the plane $z = 0$. Then $\lambda = 1$ is still a constant principal curvature on all of M. However, the leaf space M/T_λ is not Hausdorff, since the two open semi-circular leaves L_1 and L_2 in the plane $x = 0$ cannot be separated by disjoint neighborhoods in the quotient topology. The corresponding sheet of the focal set $f_\lambda(M)$ consists of the union of the curve γ with its mirror image in the plane $z = 0$, and it is not a Hausdorff 1-dimensional manifold in a neighborhood of the origin. Nevertheless, the rank of the focal map f_λ is one on all of M.

In this example, λ has constant multiplicity $m = 1$. One can produce similar examples where $m > 1$ by imitating the construction above in $\mathbf{R}^n$ for $n > 3$.

The case of a principal curvature of multiplicity m = 1

We now consider the case where a principal curvature λ has constant multiplicity $m = 1$ on an oriented hypersurface $f : M^n \rightarrow \tilde{M}^{n+1}$. This case differs greatly from the case of multiplicity greater than one, since λ is not necessarily constant along the leaves of its principal foliation T_λ, i.e., along its lines of curvature. In fact, by Theorem 2.8, the rank of the focal map f_λ is n at points x where $X\lambda \neq 0$ for a nonzero vector $X \in T_\lambda(x)$, and it is $n-1$ at points x where $X\lambda = 0$ for all $X \in T_\lambda(x)$. Thus, in general, the sheet of the focal set $f_\lambda(U)$, where U is the domain of f_λ, is a hypersurface with singularities at points where $X\lambda = 0$ for all $X \in T_\lambda(x)$.

We are interested in the case where the sheet $f_\lambda(U)$ of the focal set is a submanifold of dimension $n-1$ in $\tilde{M}^{n+1}$. As noted above, this is not always the case, as it was in the case of higher multiplicity, and so the main result is formulated in terms of conditions that are equivalent to the condition that $f_\lambda(U)$ is a submanifold of dimension $n-1$.

In this case of multiplicity one, it is significant that for hypersurfaces in hyperbolic space H^{n+1}, the domain U of f_λ does not include those points $x \in M$ where $|\lambda(x)| \leq 1$. In fact, conditions (1–3) of Theorem 2.14 below are equivalent on U, but not on all of M. Specifically, conditions (1) and (2) are equivalent on M and they imply (3). However, one can construct a surface M in H^3 such that focal set $f_\lambda(U)$ is a curve (and so condition (3) holds), and yet not all of the lines of curvature corresponding to λ are plane curves of constant curvature. This is done by beginning with a surface $N \subset H^3$ on which all three conditions are satisfied, and then modifying N on the set where $|\lambda| < 1$ (which is disjoint from U) so as to destroy condition (2), but introduce no new focal points and thereby preserve condition (3).

Theorem 2.14. *Let $f : M^n \to \tilde{M}^{n+1}$ be an oriented hypersurface of a real space form. Suppose that λ is a smooth principal curvature function of constant multiplicity $m = 1$ on M^n. Then the following conditions are equivalent on $\tilde{M}^{n+1}$ if $\tilde{M}^{n+1} = \mathbf{R}^{n+1}$ or S^{n+1}, and on the domain U of f_λ if $\tilde{M}^{n+1} = H^{n+1}$.*

(1) *λ is constant along each leaf of its principal foliation T_λ.*
(2) *The leaves of T_λ are plane curves of constant curvature.*
(3) *The rank of the focal map f_λ is identically equal to $n-1$ on its domain U, and f_λ factors through an immersion of the $(n-1)$-dimensional space of leaves U/T_λ into $\tilde{M}^{n+1}$.*

We first give the proof in the Euclidean case and then handle the other cases via stereographic projection.

Proof (Euclidean case). (1) $\Leftrightarrow$ (3) This follows immediately from Theorem 2.8 concerning the rank of the focal map f_λ and from the connectedness of the leaves of the foliation T_λ.

(2) $\Rightarrow$ (1) This follows easily from the Frenet equations for plane curves.

(1) $\Rightarrow$ (2) Since T_λ is a 1-dimensional foliation on M, in a neighborhood of any point of M, we can find a local coordinates (t, v) given by the coordinate chart $\phi : (-\varepsilon, \varepsilon) \times V \to M$, where V is an open subset in $\mathbf{R}^{n-1}$, such that the leaves of T_λ that intersect the image $W \subset M$ of ϕ are precisely the images under ϕ of the curves $v =$ constant in $(-\varepsilon, \varepsilon) \times V$.

We first consider the case where λ is nonzero on W. By an appropriate choice of the unit normal field ξ, we may arrange that $\lambda > 0$ on W. Thus, λ is a positive constant on each leaf of T_λ that passes through W. By condition (1) and Theorem 2.8, the focal map f_λ on W is constant on each leaf of T_λ, and so the functions $g_\lambda = f_\lambda \circ \phi$ and $\rho = (1/\lambda) \circ \phi$ are functions of the coordinate $v \in V$ alone, and g_λ is an immersion on V, since it has rank $n-1$.

The point $q = f(\phi(t, v))$ lies on the hypersphere in $\mathbf{R}^{n+1}$ determined by v given by the equation

$$|z - g_\lambda(v)| = \rho(v), \; z \in \mathbf{R}^{n+1}. \tag{2.61}$$

Since the normal line to $f(M)$ at q is the same as the normal line to the sphere given in equation (2.61) at q (i.e., $f(M)$ is the envelope of the $(n-1)$-parameter family of spheres parametrized by v), one can show that for any Y tangent to V at the point v, the point q also lies on the hyperplane in $\mathbf{R}^{n+1}$ given by the equation:

$$\langle z - g_\lambda(v), (g_\lambda)_*(Y)\rangle = -\rho(v)Y(\rho), \; z \in \mathbf{R}^{n+1}. \tag{2.62}$$

Thus, for any given value of v, the leaf of T_λ in W determined by v lies on the circle obtained by intersecting the hypersphere in equation (2.61) with the 2-plane determined by equation (2.62), as Y ranges over the $(n-1)$-dimensional tangent space $T_v V$. Hence, each leaf of T_λ on which λ is nonzero lies locally on a circle, and by connectedness, it is an arc of a circle.

Finally, suppose that γ is a leaf of T_λ on which λ is identically zero. By Theorem 2.6, there exists an inversion σ of $\mathbf{R}^{n+1}$ in a hypersphere such that $(\sigma \circ f)(\gamma)$ is a leaf of the principal foliation T_μ on which the associated principal principal curvature μ, as in Theorem 2.6, is a nonzero constant. By the argument above, $(\sigma \circ f)(\gamma)$ lies on a circle, and so $f(\gamma)$ itself lies on a circle or a straight line. This completes the proof of the theorem in the Euclidean case. □

We now discuss the proof of Theorem 2.14 in the non-Euclidean cases. As in the Euclidean case, (1) ⇔ (3) follows immediately from Theorem 2.8.

To prove (1) ⇔ (2), we use stereographic projection τ, as defined in Remark 2.7. Let $f : M^n \to \tilde{M}^{n+1}$ be an oriented hypersurface with field of unit normals ξ in a real space form $\tilde{M}^{n+1} = S^{n+1}$ or H^{n+1}, and let τ be the appropriate stereographic projection for $\tilde{M}^{n+1}$. If λ is a principal curvature of $f(M)$ of multiplicity one, then $\mu = e^{-h}(\lambda - g(\text{grad } h, \xi))$ is a principal curvature of multiplicity one of the hypersurface $(\tau \circ f)(M)$ in $\mathbf{R}^{n+1}$ (or $D^{n+1} \subset \mathbf{R}^{n+1}$ in the hyperbolic case) by Theorem 2.6. By a direct calculation, one can show that the leaves of T_λ are plane curves of constant curvature in $\tilde{M}^{n+1}$ if and only if the leaves of T_μ are plane curves of constant curvature in $\mathbf{R}^{n+1}$ (or D^{n+1}). Thus, the equivalence of conditions (1) and (2) follows from the equivalence of (1) and (2) in the Euclidean case and the following important lemma.

Lemma 2.15. *Let $f : M^n \to S^{n+1}$ (respectively, H^{n+1}) be an oriented hypersurface with field of unit normals ξ. Suppose that λ is a smooth principal curvature function of constant multiplicity $m = 1$ on M^n, and let X denote a field of unit principal vectors of λ on M. Let*

$$\mu = e^{-h}(\lambda - \langle \text{grad } h, \xi \rangle)$$

be the corresponding principal curvature of multiplicity $m = 1$ of the hypersurface $(\tau \circ f) : M \to \mathbf{R}^{n+1}$ (respectively, D^{n+1}), where τ is stereographic projection. Then $X\lambda = 0$ at $x \in M$ if and only if $X\mu = 0$ at x.

Proof. We will do the proof for a hypersurface in S^{n+1}, and the proof in H^{n+1} is quite similar. This is a local calculation, so we will consider M as an embedded hypersurface in S^{n+1} and suppress the mention of the embedding f. We use stereographic projection $\tau : S^{n+1} - \{P\} \to \mathbf{R}^{n+1}$ with pole P as given in equation (2.46), that is

$$\tau(x) = P + e^{h(x)}(x - P),$$

where $e^{-h(x)} = 1 - \langle x, P\rangle$. Then, a direct calculation yields,

$$\text{grad } h = e^h(P - \langle x, P\rangle\, x).$$

Using the fact that $\langle x, \xi\rangle = 0$ for $x \in M$ and ξ the local field of unit normals to M, we get

$$\mu = e^{-h}(\lambda - \langle e^h P, \xi\rangle) = e^{-h}\lambda - \langle P, \xi\rangle,$$

and

$$X\mu = -e^{-h}(Xh)\lambda + e^{-h}(X\lambda) - \langle P, D_X\xi\rangle, \tag{2.63}$$

where D is the Euclidean covariant differentiation on $\mathbf{R}^{n+2}$. Since $\langle X, \xi\rangle = 0$, it follows that $D_X\xi = \tilde{\nabla}_X\xi$, where $\tilde{\nabla}$ is the Levi-Civita connection on S^{n+1}. Then, we know that $\tilde{\nabla}_X\xi = -AX = -\lambda X$, so $D_X\xi = -\lambda X$. This and the fact that

$$Xh = \langle \text{grad } h, X\rangle = e^h\langle P, X\rangle,$$

enable us to rewrite the expression for $X\mu$ in equation (2.63) as,

$$X\mu = -\langle P, X\rangle\lambda + e^{-h}(X\lambda) + \langle P, X\rangle\lambda = e^{-h}(X\lambda).$$

From this it is clear that $X\mu = 0$ if and only if $X\lambda = 0$.

For a hypersurface M in H^{n+1}, we have from equation (2.49) that stereographic projection $\tau : H^{n+1} \to D^{n+1}$ with pole P is given by

$$\tau(x) = P + e^{h(x)}(x - P),$$

where $e^{-h(x)} = 1 + \langle x, P\rangle$. From this we can compute that

$$\text{grad } h = -e^h(P + \langle x, P\rangle\, x),$$

where $\langle\ ,\ \rangle$ is the Lorentz metric, and the rest of the proof follows in a way similar to the spherical case. □

Remark 2.16. In the context of Lie sphere geometry (see, for example, Pinkall [446] or the book [77, p. 67]), it is easy to prove that the property that a curvature sphere map is constant along each leaf of its principal foliation is invariant under Möbius (conformal) transformations, and more generally, under Lie sphere transformations. From this, Lemma 2.15 follows easily.

In the case where the hypersurface M is complete with respect to the metric induced from $\tilde{M}$, Theorem 2.14 enables us to give a manifold structure to the sheet $f_\lambda(U)$ of the focal set, where U is the domain of f_λ, similar to that obtained in Theorem 2.12 for the case of higher multiplicity.

Theorem 2.17. *Let $f : M^n \to \tilde{M}^{n+1}$ be an oriented hypersurface of a real space form which is complete with respect to the induced metric. Suppose that λ is a smooth principal curvature function of constant multiplicity $m = 1$ on M^n such that the equivalent conditions* (1)–(3) *of Theorem 2.14 are satisfied on the domain U of the focal map f_λ. Then f_λ factors through an immersion of the* $(n-1)$*-dimensional manifold U/T_λ into $\tilde{M}^{n+1}$.*

Proof. The proof is almost identical to the proof of Theorem 2.12 for the case of higher multiplicity. As in that case, the completeness of M implies that each leaf of T_λ is complete with respect to the induced metric. This implies that each leaf of T_λ in M is a covering space of the metric circle in $\tilde{M}^{n+1}$ on which it lies (see, for example, [283, Vol. I, p. 176]). However, since the circle is not simply connected, this does not guarantee that the leaf itself is compact, as in the multiplicity $m > 1$ case. Even so, using the fact that each leaf of T_λ is a covering of the circle on which it lies, one can produce a direct argument (which we omit here) that the leaf space U/T_λ is Hausdorff. □

We close this section with three results that have proven to be valuable in the study of isoparametric and Dupin hypersurfaces. Theorems 2.11 and 2.14 show that if a principal curvature λ of a hypersurface $f : M^n \to \tilde{M}^{n+1}$ has constant multiplicity $m \geq 1$ on M and is constant along the leaves of its principal foliation, then the leaves of T_λ are totally umbilic submanifolds in $\tilde{M}^{n+1}$. The next result shows that the case where λ assumes a critical value along a certain leaf of T_λ has even more geometric significance.

Theorem 2.18. *Let $f : M^n \to \tilde{M}^{n+1}$ be an oriented hypersurface of a real space form. Suppose that λ is a smooth principal curvature function of constant multiplicity $m \geq 1$ on M^n which is constant along each leaf of its principal foliation T_λ. Then λ assumes a critical value along a leaf γ of T_λ if and only if γ is totally geodesic in M.*

Proof. Let x be a point on the leaf γ. Then the normal space to γ in M at x is $T_\lambda^\perp(x)$. We know that $T_\lambda^\perp(x)$ is the direct sum of the principal spaces $T_\mu(x)$, where μ ranges over the principal curvatures of M at x that are not equal to λ. Let $\eta \in T_\mu(x)$, for $\mu \neq \lambda$. By the same calculation used to obtain equation (2.60), we get that the shape operator B_η of γ in M at x has the form

$$B_\eta X = \frac{(\eta\lambda)X}{\lambda - \mu},$$

for $X \in T_\lambda(x)$. The leaf γ is totally geodesic in M if and only if $B_\eta = 0$ for each $\eta \in T_\mu(x)$ for each $\mu \neq \lambda$ for all $x \in \gamma$. This occurs precisely when $\eta\lambda = 0$ for all $\eta \in T_\lambda^\perp(x)$ for all $x \in \gamma$. Since λ is assumed constant along γ, this happens precisely when λ assumes a critical value along γ. □

Since the principal curvatures are constant on an isoparametric hypersurface, the following corollary follows immediately from Theorems 2.11, 2.14, and 2.18, as was first shown by Nomizu [403].

Corollary 2.19. *Let $f : M^n \to \tilde{M}^{n+1}$ be an isoparametric hypersurface of a real space form. Then for each principal curvature λ, the leaves of the principal foliation T_λ are totally umbilic in $\tilde{M}^{n+1}$ and totally geodesic in M^n.*

The following result is similar to Theorem 2.18, and it is useful in the study of Dupin hypersurfaces. Recall from Corollary 2.9 that if a principal curvature λ has constant multiplicity $m \geq 1$, then λ is constant along each leaf of T_λ in the domain U of f_λ if and only if f_λ itself is constant along each leaf of T_λ in U. In that case, the curvature sphere map K_λ is also constant along each leaf of T_λ in U. As in Theorem 2.18, the case where λ has a critical value along a leaf of T_λ has special geometric significance.

Theorem 2.20. *Let $f : M^n \to \tilde{M}^{n+1}$ be an oriented hypersurface of a real space form. Suppose that λ is a smooth principal curvature function of constant multiplicity $m \geq 1$ on M^n which is constant along each leaf of its principal foliation T_λ. Then λ assumes a critical value along a leaf γ of T_λ in the domain U of f_λ if and only if γ is totally geodesic in the curvature sphere K_λ determined by γ.*

Proof. A leaf γ of T_λ in U is a submanifold of the curvature sphere K_λ, and its normal space in K_λ is $T_\lambda^\perp$, since K_λ is tangent to M along γ. The rest of the proof is then exactly the same as the proof of Theorem 2.18. □

2.5 Curvature Surfaces and Dupin Hypersurfaces

Let $f : M^n \to \tilde{M}^{n+1}$ be an oriented hypersurface of a real space form. A connected submanifold S of M^n is called a *curvature surface* if at each $x \in S$, the tangent space T_xS is equal to some principal space $T_\lambda(x)$. In that case, the corresponding principal curvature $\lambda : S \to \mathbf{R}$ is a smooth function on S.

For example, if dim T_λ is constant on an open subset U of M^n, then each leaf of the principal foliation T_λ is a curvature surface on U. Curvature surfaces are plentiful, since the results of Reckziegel [458] and Singley [486] imply that there is an open dense (possibly not connected) subset Ω of M^n on which the multiplicities of the principal curvatures are locally constant. On Ω, each leaf of each principal foliation is a curvature surface.

Remark 2.21 (Curvature surfaces of submanifolds of codimension $k > 1$). Reckziegel [458] generalized the notion of a curvature surface to the case of an immersed submanifold $f : M^n \to \tilde{M}^{n+k}$ of a space form $\tilde{M}^{n+k}$ with codimension $k > 1$. In that case, Reckziegel defines a curvature surface to be a connected submanifold $S \subset M^n$ for which there is a parallel (with respect to the normal connection) section $\eta : S \to B^{n+k-1}$ of the unit normal bundle B^{n+k-1} such that for each $x \in S$, the tangent space T_xS is equal to some eigenspace of $A_{\eta(x)}$. In that case, the corresponding principal curvature $\lambda : S \to \mathbf{R}$ is a smooth function on S.

It is also possible to have a curvature surface S that is not a leaf of a principal foliation, because the multiplicity of the corresponding principal curvature is not constant on a neighborhood of S, as the following example due to Pinkall [447] shows.

Example 2.22 (A curvature surface that is not a leaf of a principal foliation). Let T^2 be a torus of revolution in $\mathbf{R}^3$, and embed $\mathbf{R}^3$ into $\mathbf{R}^4 = \mathbf{R}^3 \times \mathbf{R}$. Let η be a field of unit normals to T^2 in $\mathbf{R}^3$. Let M^3 be a tube of sufficiently small radius $\varepsilon > 0$ around T^2 in $\mathbf{R}^4$, so that M^3 is a compact smooth embedded hypersurface in $\mathbf{R}^4$. The normal space to T^2 in $\mathbf{R}^4$ at a point $x \in T^2$ is spanned by $\eta(x)$ and $e_4 = (0, 0, 0, 1)$. The shape operator A_η of T^2 has two distinct principal curvatures at each point of T^2, while the shape operator A_{e_4} of T^2 is identically zero. Thus the shape operator A_ζ for the normal

$$\zeta = \cos\theta\ \eta(x) + \sin\theta\ e_4,$$

at a point $x \in T^2$, is given by

$$A_\zeta = \cos\theta\ A_{\eta(x)}.$$

From the formulas for the principal curvatures of a tube in Theorem 2.2, we see that at all points of M^3 where $x_4 \neq \pm\varepsilon$, there are three distinct principal curvatures of multiplicity one, which are constant along their corresponding lines of curvature (curvature surfaces of dimension one). However, on the two tori, $T^2 \times \{\pm\varepsilon\}$, the principal curvature $\kappa = 0$ has multiplicity two. These two tori are curvature surfaces for this principal curvature, since the principal space corresponding to κ is tangent to each torus at every point.

Theorem 2.10 has the following generalization to curvature surfaces of submanifolds of arbitrary codimension. The proof is the same as that of Theorem 2.10 with obvious minor modifications.

Theorem 2.23. *Suppose that S is a curvature surface of dimension $m > 1$ of a submanifold $f : M^n \to \tilde{M}^{n+k}$ for $k \geq 1$. Then the corresponding principal curvature is constant along S.*

An oriented hypersurface $f : M^n \to \tilde{M}^{n+1}$ is called a *Dupin hypersurface* if:

(a) along each curvature surface, the corresponding principal curvature is constant.
Furthermore, a Dupin hypersurface M is called *proper Dupin* if, in addition to Condition (a), the following condition is satisfied:
(b) the number g of distinct principal curvatures is constant on M.

By Theorem 2.23, Condition (a) is automatically satisfied along a curvature surface of dimension $m > 1$, and thus the key case is when the dimension of the curvature surface equals one.

Condition (b) is equivalent to requiring that each continuous principal curvature function has constant multiplicity on M^n. The torus T^2 in Example 2.22 above is a proper Dupin hypersurface of $\mathbf{R}^3$, but the tube M^3 over T^2 in $\mathbf{R}^4$ is Dupin, but not proper Dupin, since the number of distinct principal curvatures is not constant on M^3.

Remark 2.24 (On the terms: Dupin and proper Dupin). In some early papers on the subject (see, for example, Thorbergsson [533], and Grove–Halperin [184]) and in the book [95, p. 166], a hypersurface which satisfies Conditions (a) and (b) was called "Dupin" instead of "proper Dupin." Pinkall introduced the term "proper Dupin" in his paper [446], and that has become the standard terminology in the subject. In the book [95, p. 189], hypersurfaces such as the tube M^3 over T^2 in $\mathbf{R}^4$ were called "semi-Dupin."

Pinkall's local construction of proper Dupin hypersurfaces

The following local construction due to Pinkall [446] shows that proper Dupin hypersurfaces are very plentiful.

Theorem 2.25. *Given positive integers $m_1, \ldots, m_g$ with*

$$m_1 + \cdots + m_g = n - 1,$$

there exists a proper Dupin hypersurface in $\mathbf{R}^n$ with g distinct principal curvatures having respective multiplicities $m_1, \ldots, m_g$.

Proof. The proof is by an inductive local construction which will be clear once the first few steps are done. The proof uses the fact that the proper Dupin property is preserved by inversion of $\mathbf{R}^n$ in a hypersphere $S \subset \mathbf{R}^n$ (see Remark 2.7). This follows from Theorems 2.6 and 2.10, and an argument similar to the proof of Lemma 2.15 for stereographic projection. This construction does not, in general, result in a compact proper Dupin hypersurface.

Let $M_1 \subset \mathbf{R}^{m_1+1}$ be a sphere of radius one centered at the origin. Construct a cylinder

$$M_1 \times \mathbf{R}^{m_2} \subset \mathbf{R}^{m_1+1} \times \mathbf{R}^{m_2} = \mathbf{R}^{m_1+m_2+1}$$

over the submanifold $M_1 \subset \mathbf{R}^{m_1+1} \subset \mathbf{R}^{m_1+m_2+1}$. This cylinder has two distinct principal curvatures at each point, $\lambda_1 = 1$ of multiplicity m_1, and $\lambda_2 = 0$ of multiplicity m_2. The next step is to invert the cylinder $M_1 \times \mathbf{R}^{m_2}$ in a hypersphere $S_1 \subset \mathbf{R}^{m_1+m_2+1}$ chosen so that the image of the cylinder under the inversion has an open subset M_2 on which neither of its two principal curvatures equals zero at any point. Then M_2 is a proper Dupin hypersurface in $\mathbf{R}^{m_1+m_2+1}$ having two distinct nonzero principal curvatures with multiplicities m_1 and m_2.

Next construct a cylinder

$$M_2 \times \mathbf{R}^{m_3} \subset \mathbf{R}^{m_1+m_2+1} \times \mathbf{R}^{m_3} = \mathbf{R}^{m_1+m_2+m_3+1}$$

over the submanifold $M_2 \subset \mathbf{R}^{m_1+m_2+1}$. This cylinder has three distinct principal curvatures at each point with respective multiplicities m_1, m_2, m_3, where m_3 is the multiplicity of the principal curvature that is identically zero. As above, invert the cylinder $M_2 \times \mathbf{R}^{m_3}$ in a hypersphere $S_2 \subset \mathbf{R}^{m_1+m_2+m_3+1}$ chosen so that the image of the cylinder under the inversion has an open subset M_3 on which none of its three principal curvatures equals zero at any point. One continues the process by constructing the cylinder

$$M_3 \times \mathbf{R}^{m_4} \subset \mathbf{R}^{m_1+m_2+m_3+1} \times \mathbf{R}^{m_4} = \mathbf{R}^{m_1+m_2+m_3+m_4+1}$$

and so on, until one finally obtains the desired proper Dupin hypersurface $M_g \subset \mathbf{R}^{m_1+\cdots+m_g+1}$ with g distinct principal curvatures having respective multiplicities $m_1, \ldots, m_g$. □

As noted above, the proper Dupin hypersurfaces constructed in Theorem 2.25 are not compact, in general, and compact proper Dupin hypersurfaces are much more rare.

An important class of compact proper Dupin hypersurfaces consists of the isoparametric hypersurfaces in spheres S^n and those hypersurfaces in $\mathbf{R}^n$ obtained from isoparametric hypersurfaces in S^n via stereographic projection. For example, the well-known cyclides of Dupin in $\mathbf{R}^3$ are obtained from a standard product torus $S^1(r) \times S^1(s) \subset S^3$, $r^2 + s^2 = 1$, in this way. These examples will be discussed in more detail in later chapters.

In fact, Thorbergsson [533] proved that the number g of distinct principal curvatures of a compact proper Dupin hypersurface M embedded in S^n (or $\mathbf{R}^n$) can only be $1, 2, 3, 4$, or 6, the same restriction as for an isoparametric hypersurface in S^n. There are also restrictions on the multiplicities of the principal curvatures due to Stolz [502] and Grove and Halperin [184] (see Sections 3.7 and 5.8 for more detail).

We will see in Chapter 4 that both the Dupin and proper Dupin conditions are invariant under Lie sphere transformations. Because of this, Lie sphere geometry has proven to be a useful setting for the study of Dupin hypersurfaces, and we will use Lie sphere geometry extensively in Chapter 5 on Dupin hypersurfaces.

Remark 2.26 (Dupin submanifolds of higher codimension). In the case of an immersed submanifold $f : M^n \to \tilde{M}^{n+k}$ of a space form $\tilde{M}^{n+k}$ with codimension $k > 1$, Pinkall defined $f(M^n)$ to be Dupin if along each curvature surface (in the sense of Remark 2.21), the corresponding principal curvature is constant. In that case, $f(M^n)$ is called proper Dupin if the number of distinct principal curvatures is constant on the unit normal bundle B^{n+k-1}. One can show that Pinkall's definition is equivalent to the definition of a Dupin submanifold given in Section 4.4 in the context of Lie sphere geometry (see Remark 4.10 on page 217).

2.6 Height Functions and Tight Submanifolds

In this section, we give a brief review of the aspects of the theory of tight submanifolds that will be needed later in the book. For more complete coverage of the topic, the reader is referred to Chapter 1 of the book [95] or the survey articles of Kuiper [302, 303], or Banchoff and Kühnel [24]. Our treatment here is based on the book [95, pp. 6–33].

We begin with a review of the critical point theory needed in the theory of tight and taut submanifolds (see Milnor [359, pp. 4–6] for more detail). Let $f : M_1 \to M_2$ be a smooth function between manifolds M_1 and M_2. A point $x \in M_1$ is called a *critical point* of f if the derivative map

$$f_* : T_x M_1 \to T_{f(x)} M_2$$

at x is not surjective. If $y \in M_2$ is the image of a critical point x under f, then y is called a *critical value* of f. All other points in the image of f are called *regular values* of f. Note that if M_1 and M_2 have the same dimension, then x is a critical point of f if and only if f_* is singular at x.

Suppose $\phi : M \to \mathbf{R}$ is a smooth function on a manifold M; then $x \in M$ is a critical point of ϕ if and only if $\phi_* = 0$ at x. If $(x_1, \ldots, x_n)$ are local coordinates on M in a neighborhood of x, then x is a critical point of ϕ if and only if

$$\frac{\partial \phi}{\partial x_1}(x) = \cdots = \frac{\partial \phi}{\partial x_n}(x) = 0. \tag{2.64}$$

If x is a critical point of ϕ, then the behavior of ϕ near x is determined by the *Hessian* H_x of ϕ at x, which is given in local coordinates by the symmetric matrix

$$H_x = \left[\frac{\partial^2 \phi}{\partial x_i \partial x_j} \right]. \tag{2.65}$$

A critical point x of ϕ is said to be *degenerate* if the rank of the Hessian H_x is less than $n = \dim M$. If rank $H_x = n$, then x is called a *nondegenerate* critical point. The *index* of a nondegenerate critical point x is the number of negative eigenvalues of the symmetric matrix H_x. The behavior of ϕ in a neighborhood of a nondegenerate critical point is determined by the index according to the following lemma (see, for example, Milnor [359, p. 6]).

Lemma 2.27 (Lemma of Morse). *Let p be a nondegenerate critical point of index k of a function $\phi : M \to \mathbf{R}$. Then there is a local coordinate system $(x_1, \ldots, x_n)$ in a neighborhood U with origin at p such that the identity*

$$\phi = \phi(p) - x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_n^2 \tag{2.66}$$

holds throughout U.

From the lemma we see that a critical point of index n is a local maximum of ϕ, and a critical point of index 0 is a local minimum of ϕ. All other nondegenerate critical points are various types of saddle points.

A real-valued function ϕ is called a *Morse function* or *nondegenerate function* if all of its critical points are nondegenerate. From the lemma, we see that if M is compact, then a Morse function ϕ on M can only have a finite number of critical points, since the critical points are isolated.

Let $\phi : M \to \mathbf{R}$ be a Morse function such that the sublevel set,

$$M_r(\phi) = \{x \in M \mid \phi(x) \leq r\}, \tag{2.67}$$

is compact for all $r \in \mathbf{R}$. Of course, this is always true if M itself is compact. Let $\mu_k(\phi, r)$ be the number of critical points of ϕ of index k in $M_r(\phi)$. For compact M, let $\mu_k(\phi)$ be the number of critical points of ϕ of index k in M, and let $\mu(\phi)$ be the total number of critical points of ϕ on M. For a field $\mathbf{F}$, let

$$\beta_k(\phi, r, \mathbf{F}) = \dim_{\mathbf{F}} H_k(M_r(\phi), \mathbf{F}), \tag{2.68}$$

where $H_k(M_r(\phi), \mathbf{F})$ is the k-th homology group of $M_r(\phi)$ over the field $\mathbf{F}$. That is, $\beta_k(\phi, r, \mathbf{F})$ is the k-th $\mathbf{F}$-Betti number of $M_r(\phi)$. Further, let

$$\beta_k(M, \mathbf{F}) = \dim_{\mathbf{F}} H_k(M, \mathbf{F}) \tag{2.69}$$

be the k-th $\mathbf{F}$-Betti number of M. The *Morse inequalities* (see, for example, Morse–Cairns [379, p. 270]) state that

$$\mu_k(\phi, r) \geq \beta_k(\phi, r, \mathbf{F}), \tag{2.70}$$

for all $\mathbf{F}, r, k$. For a compact M, the *Morse number* $\gamma(M)$ of M is defined by

$$\gamma(M) = \min\{\mu(\phi) \mid \phi \text{ is a Morse function on } M\}. \tag{2.71}$$

The Morse inequalities imply that

$$\gamma(M) \geq \beta(M, \mathbf{F}) = \sum_{k=0}^{n} \beta_k(M, \mathbf{F}), \tag{2.72}$$

for any field $\mathbf{F}$. If there exists a field $\mathbf{F}$ such that $\mu(\phi) = \beta(M, \mathbf{F})$, then ϕ is called a *perfect* Morse function. In that case, ϕ has the minimum number of critical points possible in view of the Morse inequalities.

Kuiper [301] noted the following reformulation of the condition that the Morse inequalities are actually equalities, and he used it very effectively in his papers on tight and taut immersions.

Theorem 2.28. *Let ϕ be a Morse function on a compact manifold M. For a given field $\mathbf{F}$, the equality $\mu_k(\phi, r) = \beta_k(\phi, r, \mathbf{F})$ holds for all k, r if and only if the map on homology*

$$H_*(M_r(\phi), \mathbf{F}) \to H_*(M, \mathbf{F})$$

induced by the inclusion $M_r(\phi) \subset M$ is injective for all r.

This theorem follows immediately from Theorem 29.2 of Morse–Cairns [379, p. 260], and we will omit it here, although we will make a few comments about some of the key ideas in the proof.

Suppose that p is a nondegenerate critical point of index k of a Morse function ϕ on M and $\phi(p) = r$. For the sake of simplicity, assume that p is the only critical point at the critical level r. A fundamental result in critical point theory (see, for example, Milnor [359, pp. 12–24] or Morse–Cairns [379, pp. 184–202]) states that $M_r(\phi)$ has the homotopy type of $M_r^-(\phi)$ with a k-cell attached, where $M_r^-(\phi)$ consists of all points in M for which $\phi < r$. Morse and Cairns [379, pp. 258–261] characterize the effect of attaching this k-cell as follows. Let

$$\triangle\beta_i(r) = \beta_i(M_r(\phi)) - \beta_i(M_r^-(\phi)).$$

Then the $\triangle\beta_i(r)$ are 0 for all i, except that $\triangle\beta_k(r) = 1$ if the critical point is of "linking type," and $\triangle\beta_{k-1}(r) = -1$ if the critical point p if of "non-linking type." From this, it is clear that the two conditions in Theorem 2.28 are equivalent, and they hold precisely when every critical point of ϕ is of linking type.

Let $f : M^n \to \mathbf{R}^m$ be a smooth immersion, and let S^{m-1} denote the unit sphere in $\mathbf{R}^m$. For $p \in S^{m-1}$, the *linear height function* $l_p : \mathbf{R}^m \to \mathbf{R}$ is defined by the formula

$$l_p(q) = \langle p, q \rangle, \tag{2.73}$$

where $\langle\ ,\ \rangle$ is the usual Euclidean inner product on $\mathbf{R}^m$. This induces a smooth function l_p defined on M by $l_p(x) = l_p(f(x))$.

Critical points of height functions

The critical point behavior of linear height functions is related to the shape operator of $f(M^n)$ according to the following well-known theorem.

Theorem 2.29. *Let $f : M \to \mathbf{R}^m$ be a smooth immersion of an n-dimensional manifold M into $\mathbf{R}^m$, and let $p \in S^{m-1}$.*

(a) *A point $x \in M$ is a critical point of l_p if and only if p is orthogonal to T_xM.*
(b) *Suppose l_p has a critical point at x. Then for $X, Y \in T_xM$, the Hessian H_x of l_p at x satisfies $H_x(X, Y) = \langle A_pX, Y \rangle$.*

Proof. (a) Let $x \in M$, and let U be a neighborhood of x on which f is an embedding. We omit the mention of f in the following local calculation. Let $X \in T_xM$, and let $\gamma(t)$ be a curve in U with initial point $\gamma(0) = x$ and initial tangent vector $\overrightarrow{\gamma}(0) = X$. By definition we have

$$Xl_p(x) = \frac{d}{dt} l_p(\gamma(t))|_{t=0} = \langle\, p, \overrightarrow{\gamma}(t)\,\rangle|_{t=0} = \langle p, X \rangle. \tag{2.74}$$

Thus, $Xl_p(x) = 0$ if and only if $\langle p, X\rangle = 0$, and so x is a critical point of l_p if and only if p is orthogonal to T_xM.

(b) To compute the Hessian, let X and Y be tangent to M at a critical point x of l_p, and extend Y to a vector field tangent to M on the neighborhood U of x. It is easy to show that the Hessian of l_p at the critical point x is given by $H_x(X, Y) = X(Yl_p)$ at the point x. Using part (a), we compute

$$H_x(X, Y) = X(Yl_p) = X\langle Y, p\rangle = \langle D_X Y, p\rangle. \tag{2.75}$$

Let ξ be a field of unit normals on U with $\xi(x) = p$. Then $\langle Y, \xi\rangle = 0$ on U, and thus

$$\begin{aligned} 0 &= D_X\langle Y, \xi\rangle = \langle D_X Y, \xi\rangle + \langle Y, D_X\xi\rangle \\ &= \langle D_X Y, \xi\rangle + \langle Y, -A_\xi X\rangle = \langle D_X Y, p\rangle + \langle Y, -A_p X\rangle. \end{aligned} \tag{2.76}$$

From equations (2.75) and (2.76), we get $H_x(X, Y) = \langle A_p X, Y\rangle$. □

As an immediate consequence, we get the following corollary, which is an "Index Theorem" for height functions.

Corollary 2.30. *Let $f : M \to \mathbf{R}^m$ be a smooth immersion of an n-dimensional manifold M into $\mathbf{R}^m$, and suppose that p is a unit vector orthogonal to T_xM.*

(a) *The function l_p has a degenerate critical point at x if and only if the shape operator A_p is singular.*
(b) *If l_p has a nondegenerate critical point at x, then the index of l_p at x is equal to the number of negative eigenvalues of A_p.*

We next consider the *Gauss map* $\nu : BM \to S^{m-1}$, where BM is the unit normal bundle of M, defined by $\nu(x, \xi) = \xi$. The following well-known theorem is obtained by a direct calculation using coordinates on the unit normal bundle BM similar to those used in the proof of Theorem 2.1 on page 11, and we omit the proof here.

Theorem 2.31. *The nullity of the Gauss map ν at a point $\xi \in BM$ is equal to the nullity of the shape operator A_ξ. In particular, ξ is a critical point of ν if and only if A_ξ is singular.*

From Theorem 2.31 and Corollary 2.30, we immediately obtain the following theorem.

Theorem 2.32. *For $p \in S^{m-1}$, the height function l_p is a Morse function on M if and only if p is a regular value of the Gauss map ν.*

Since BM and S^{m-1} are manifolds of the same dimension, Sard's Theorem (see, for example, Milnor [360, p. 10]) implies the following corollary.

Corollary 2.33. (a) *For almost all $p \in S^{m-1}$, the function l_p is a Morse function.*
(b) *Suppose l_p has a nondegenerate critical point of index k at $x \in M$. Then there is a Morse function l_q having a critical point $y \in M$ of index k (q and y can be chosen as close to p and x, respectively, as desired).*

Proof. (a) This follows from Theorem 2.32 and Sard's Theorem.

(b) By Theorem 2.29 (a), we know that $p = \nu(\xi)$, where ξ is a unit normal vector to M at x. Since l_p has a nondegenerate critical point of index k at x, the derivative map ν_* is nonsingular at (x, ξ), and A_ξ has k negative eigenvalues and $n-k$ positive eigenvalues. Thus, there is a neighborhood V of (x, ξ) in BM such that ν_* is nonsingular on V, and the restriction of ν to V is a diffeomorphism of V onto a neighborhood U of p in S^{m-1}. Let $q \in U$ be a regular value of ν. Then $q = \nu(y, \eta)$ for some (y, η) in V, and l_q is a Morse function having a critical point at $y \in M$. Furthermore, since ν_* is nonsingular on V, the number of negative eigenvalues of A_η equals the number of negative eigenvalues of A_ξ, so the index of l_q at y is also k. By Sard's Theorem, the points q and y can be chosen to be as close to p and x, respectively, as desired. □

Tight immersions

Suppose now that M is compact. An immersion $f : M \to \mathbf{R}^m$ is said to be a *tight immersion* if there exists a field $\mathbf{F}$ such that every nondegenerate linear height function l_p has $\beta(M, \mathbf{F})$ critical points on M, i.e., every nondegenerate height function is a perfect Morse function. By Theorem 2.28 above, we see that f is tight if and only if for every nondegenerate linear height function l_p, the map on homology

$$H_*(M_r(l_p), \mathbf{F}) \to H_*(M, \mathbf{F}) \tag{2.77}$$

induced by the inclusion $M_r(l_p) \subset M$ is injective for all r. Note that

$$M_r(l_p) = \{x \in M \mid \langle p, f(x) \rangle \leq r\}. \tag{2.78}$$

Thus, $M_r(l_p)$ is the inverse image under f of the half-space in $\mathbf{R}^m$ determined by the inequality $l_p(q) \leq r$. In this formulation, one requires the map on homology in equation (2.77) to be injective for all half-spaces determined by nondegenerate height functions.

Remark 2.34 (Immersions of minimal total absolute curvature). For smooth immersions of manifolds into Euclidean space, tightness is closely related to the property that the immersion has minimal total absolute curvature in the sense of Chern and Lashof [103, 104] (see [95, pp. 9–17] for more detail).

A celebrated result in the theory of tight immersions is the Chern–Lashof Theorem [103, 104] which states that a tight immersion of a sphere S^n is a convex hypersurface $M^n \subset \mathbf{R}^{n+1} \subset \mathbf{R}^m$. This was generalized to tight topological immersions by Kuiper [302], and so we will state the theorem in that generality. (See also [95, p. 86] for a proof.)

Recall that a map f of a topological space X into $\mathbf{R}^m$ is said to be *substantial* if the image $f(X)$ is not contained in any hyperplane in $\mathbf{R}^m$.

Theorem 2.35. *(Chern–Lashof Theorem) Let $f : S^n \to \mathbf{R}^m$ be a substantial topological immersion such that almost all linear height functions have exactly two critical points. Then $m = n+1$, and f embeds S^n as a convex hypersurface in $\mathbf{R}^{n+1}$.*

Remark 2.36. Kuiper [301] first used the term "convex immersions" for tight immersions, because of the Chern–Lashof Theorem. Banchoff [19] was the first to use the term "tight" for such immersions, in conjunction with his introduction of the two-piece property.

An important advance in the theory due to Kuiper [303] was to remove the restriction mentioned above that the half-space be determined by a nondegenerate linear height function, so that one can use all half-spaces. Kuiper accomplished this by using Čech homology and its "continuity property," as we will now describe.

Kuiper's formulation of tightness then generalizes to continuous maps on compact topological spaces, and so we will define it in that context. For the sake of definiteness, we will use the field $\mathbf{F} = \mathbf{Z}_2$, which has been satisfactory in almost all known applications of the theory of tight immersions thus far.

A map f of a compact topological space X into $\mathbf{R}^m$ is called a *tight map* if for every closed half-space h in $\mathbf{R}^m$, the induced homomorphism

$$H_*(f^{-1}h) \to H_*(X) \tag{2.79}$$

in Čech homology with $\mathbf{Z}_2$ coefficients is injective. A subset of $\mathbf{R}^m$ is called a *tight set* if the inclusion map $f : X \to \mathbf{R}^m$ given by $f(x) = x$ is a tight map.

Remark 2.37 (On the use of Čech homology). Kuiper used Čech homology instead of singular homology because of its continuity property. In particular, this property is used in Kuiper's proof (see Theorem 2.41) that for a tight immersion of smooth manifold, one can use all half-spaces instead of only those that are determined by nondegenerate height functions. This fact simplifies many arguments in the theory of tight immersions and maps. Of course, for triangulable spaces (and thus for smooth manifolds), Čech homology agrees with singular homology.

Remark 2.38 (Tightness is a projective property). In the definition of a tight map above, one does not use the Euclidean metric on $\mathbf{R}^m$, but only the underlying affine space A^m. We can consider A^m as the complement of a hyperplane in real projective space $\mathbf{RP}^m$. If $f : X \to A^m$ is a tight map, and $\sigma : \mathbf{RP}^m \to \mathbf{RP}^m$ is a projective transformation such that the image $\sigma(f(X))$ lies in A^m, then $\sigma f : X \to A^m$ is also a tight map. This follows immediately from the definition, since for every half-space h in A^m, the set $(\sigma f)^{-1}h = f^{-1}(\sigma^{-1}h) = f^{-1}h'$, for the appropriate half-space h'.

Remark 2.39 (Orthogonal projections of tight maps). Suppose that $f : X \to \mathbf{R}^m$ is a tight map, and $\phi : \mathbf{R}^m \to \mathbf{R}^k$ is orthogonal projection onto a Euclidean subspace of $\mathbf{R}^m$. Then $\phi \circ f : X \to \mathbf{R}^k$ is also tight. To see this, let h be the closed half-space in $\mathbf{R}^k$ given by the inequality $l_p \leq r$, for $p \in \mathbf{R}^k$ and $r \in \mathbf{R}$. Then the same inequality in $\mathbf{R}^m$ gives a half-space h' in $\mathbf{R}^m$ such that $\phi^{-1}h = h'$. Thus, $(\phi \circ f)^{-1}h = f^{-1}h'$, and the tightness of $\phi \circ f$ follows from the tightness of f. Similarly, if $f : X \to \mathbf{R}^m$ is tight and $i : \mathbf{R}^m \to \mathbf{R}^{m+j}$ is inclusion of $\mathbf{R}^m$ into a higher dimensional Euclidean space, then $i \circ f : X \to \mathbf{R}^{m+j}$ is also tight. In that case, if h is a half-space in $\mathbf{R}^{m+j}$ and $h' = h \cap \mathbf{R}^m$, then $(i \circ f)^{-1}h = f^{-1}h'$.

We will show in Theorem 2.41 that if an immersion $f : M \to \mathbf{R}^m$ is a tight immersion in the sense that every nondegenerate height function is a perfect Morse function on M, then f is a tight map as defined above. The important point here is to show that the injectivity condition on homology in equation (2.77) holds for half-spaces determined by degenerate height functions, as well as those determined by nondegenerate height functions.

The main ingredients of the proof are the continuity property of Čech homology and the following lemma due to Kuiper [303] (see also [95, pp. 24–26]).

Lemma 2.40. *Let $f : M \to \mathbf{R}^m$ be an immersion of a compact manifold. Suppose U is an open subset of M containing $M_r(l_p)$ for some $p \in S^{m-1}$ and real number r. Then there exists a nondegenerate height function l_q and a real number s such that*

$$M_r(l_p) \subset M_s^-(l_q) \subset M_s(l_q) \subset U. \tag{2.80}$$

Proof. Since $M_r(l_p)$ is compact and U is open, one can easily show that there exists $\varepsilon > 0$ such that $M_{r+\varepsilon}(l_p) \subset U$. Let K be the maximum absolute value that any linear height function assumes on M.

We will use the spherical metric $d(p,z) = \cos^{-1}\langle p, z\rangle$ on S^{m-1}. If $d(p,z) = \alpha$, then there is a unit vector p' is orthogonal to p such that

$$z = \cos\alpha\, p + \sin\alpha\, p'. \tag{2.81}$$

Then

$$p = \sec\alpha\, z - \tan\alpha\, p'. \tag{2.82}$$

For any $x \in M$, we have

$$|l_p(x) - l_z(x)| = |\langle p - z, f(x)\rangle| = |\langle(\sec\alpha - 1)z - \tan\alpha\, p', f(x)\rangle|$$
$$\leq |\sec\alpha - 1|\, |l_z(x)| + |\tan\alpha|\, |l_{p'}(x)| \leq (|\sec\alpha - 1| + |\tan\alpha|)K. \tag{2.83}$$

Choose the positive number α sufficiently small so that

$$|\sec\alpha - 1| < \varepsilon/4K \text{ and } |\tan\alpha| < \varepsilon/4K. \tag{2.84}$$

Then

$$|l_p(x) - l_z(x)| < \varepsilon/2 \tag{2.85}$$

for any $x \in M$, and therefore

$$M_r(l_p) \subset M^-_{r+\varepsilon/2}(l_z) \subset M_{r+\varepsilon/2}(l_z) \subset M_{r+\varepsilon}(l_p) \subset U. \tag{2.86}$$

Let W be the open disk in S^{m-1} centered at p of radius α. If $q \in W$, then we can write

$$q = \cos\theta\, p + \sin\theta\, p', \tag{2.87}$$

for some unit vector p' orthogonal to p, where $0 \leq \theta < \alpha$. We can replace z by q and α by θ in equation (2.83), and get

$$|l_p(x) - l_q(x)| \leq (|\sec\theta - 1| + |\tan\theta|)K. \tag{2.88}$$

Since $0 \leq \theta < \alpha$, we have

$$|\sec\theta - 1| < |\sec\alpha - 1| < \varepsilon/4K \text{ and } |\tan\theta| < |\tan\alpha| < \varepsilon/4K, \tag{2.89}$$

and we still get $|l_p(x) - l_q(x)| < \varepsilon/2$ for all $x \in M$. Thus we have

$$M_r(l_p) \subset M^-_{r+\varepsilon/2}(l_q) \subset M_{r+\varepsilon/2}(l_q) \subset M_{r+\varepsilon}(l_p) \subset U. \tag{2.90}$$

This holds for any point q in the open neighborhood W of p in S^{m-1}. Since the set of regular values of the Gauss map is dense in S^{m-1}, there exists a point q in W such that l_q is nondegenerate, and equation (2.80) holds for that q and $s = r + \varepsilon/2$ by equation (2.90). □

With this lemma, we can prove the following important result due to Kuiper [303]. The proof given here is similar to the proof of Theorem 5.4 of [95, pp. 25–26].

Theorem 2.41. *Let $f : M \to \mathbf{R}^m$ be an immersion of a compact, connected manifold. Suppose that every nondegenerate linear height function l_p has $\beta(M, \mathbf{Z}_2)$ critical points on M. Then for every closed half-space h in $\mathbf{R}^m$, the induced homomorphism*

$$H_*(f^{-1}h) \to H_*(M) \tag{2.91}$$

in Čech homology with $\mathbf{Z}_2$ coefficients is injective.

Proof. For a given half-space h, we have $f^{-1}h = M_r(l_p)$ for some $p \in S^{m-1}$, $r \in \mathbf{R}$. If l_p is nondegenerate, then the map in equation (2.91) is injective by Theorem 2.28, since l_p has $\beta(M, \mathbf{Z}_2)$ critical points on M.

Suppose now that $f^{-1}h = M_r(l_p)$ for a degenerate height function l_p, and some $r \in \mathbf{R}$. We need to show that the map in equation (2.91) is injective in that case also. Here we use the continuity property of Čech homology. We will produce a nested sequence of half-spaces h_i, $i = 1, 2, 3, \ldots$ satisfying

$$f^{-1}(h_i) \supset f^{-1}(h_{i+1}) \supset \cdots \supset \bigcap_{j=1}^{\infty} f^{-1}(h_j) = M_r(l_p), \;\; i = 1, 2, 3, \ldots \tag{2.92}$$

such that the homomorphism in $\mathbf{Z}_2$-homology

$$H_*(f^{-1}(h_i)) \to H_* M \text{ is injective}, \;\; i = 1, 2, 3, \ldots \tag{2.93}$$

If equations (2.92) and (2.93) are satisfied, then the map

$$H_*(f^{-1}(h_i)) \to H_*(f^{-1}(h_j)) \text{ is injective for all } i > j. \tag{2.94}$$

The continuity property of Čech homology (see Eilenberg–Steenrod [145, p. 261]) says that

$$H_*(M_r(l_p)) = \overleftarrow{\lim_{i\to\infty}} H_*(f^{-1}(h_i)). \tag{2.95}$$

Equations (2.94) and Theorem 3.4 of Eilenberg–Steenrod [145, p. 216] on inverse limits imply that the map

$$H_*(M_r(l_p)) \to H_*(f^{-1}(h_i)) \tag{2.96}$$

is injective for each i. Thus, the map

$$H_*(M_r(l_p)) \to H_*(M) \tag{2.97}$$

is injective, as needed.

It remains to construct the sequence $\{h_i\}$. This is done by an inductive procedure using Lemma 2.40 to find at each step a nondegenerate height function l_q and a real number s such that

$$M_r(l_p) \subset M_s^-(l_q) \subset M_s(l_q) \subset M_{r+(1/i)}^-(l_p). \tag{2.98}$$

At each step, the set U in Lemma 2.40 should be taken to be the previous $M_s^-(l_q)$, except for $i = 1$ when $U = M_{r+1}^-(l_p)$. We take h_i to be the half-space $l_q \leq s$ constructed at the i-th step. Note that equation (2.93) is satisfied since each l_q is nondegenerate, and the half-spaces h_i are nested as in equation (2.92). Finally, since

$$f^{-1}(h_{i+1}) \subset M_{r+(1/i)}(l_p), \tag{2.99}$$

we get

$$\bigcap_{j=1}^{\infty} f^{-1}(h_j) = M_r(l_p), \tag{2.100}$$

and the theorem is proven. □

The two-piece property

Another important idea in the theory of tight immersions is the two-piece property due to Banchoff [21]. A continuous map $f : X \to \mathbf{R}^m$ of a compact, connected topological space is said to have the *two-piece property* (TPP) if $f^{-1}h$ is connected for every closed half-space h in $\mathbf{R}^m$. A compact, connected space $X \subset \mathbf{R}^m$ embedded in $\mathbf{R}^m$ is said to have the TPP if the inclusion map $f : X \to \mathbf{R}^m$ has the TPP. In that case, the TPP means that every hyperplane in $\mathbf{R}^m$ cuts X into at most two pieces, whence the name "two-piece property." The following result is immediate.

Theorem 2.42. *Let $f : X \to \mathbf{R}^m$ be continuous map of a compact, connected space X into $\mathbf{R}^m$. If f is tight, then f has the TPP.*

Proof. If the map $f : X \to \mathbf{R}^m$ is tight, then f has the TPP, since tightness implies that $\beta_0(f^{-1}h, \mathbf{Z}_2)$ is less than or equal to one, and $\beta_0(f^{-1}h, \mathbf{Z}_2)$ is equal to the number of connected components of $f^{-1}h$. □

More generally, a map f of a compact connected topological space X into $\mathbf{R}^m$ is said to be *k-tight* if for every closed half-space h in $\mathbf{R}^m$ and for every integer $0 \leq i \leq k$, the induced homomorphism $H_i(f^{-1}h) \to H_i(X)$ in Čech homology with $\mathbf{Z}_2$ coefficients is injective. Thus, 0-tightness is just the two-piece property. If $f : M \to \mathbf{R}^m$ is a smooth immersion of a compact, connected manifold, then f is k-tight if and only if every nondegenerate height function l_p has exactly $\beta_i(M, \mathbf{Z}_2)$ critical points of index i for every integer i such that $0 \leq i \leq k$.

In the setting of smooth immersions of compact manifolds into $\mathbf{R}^m$, we have the following theorem due to Banchoff.

Theorem 2.43. *Let $f : M \to \mathbf{R}^m$ be an immersion of a smooth compact, connected manifold. Then f has the TPP if and only if every nondegenerate linear height function l_p has exactly one minimum and one maximum on M.*

The basic idea here is that if a hyperplane determined by a height function l_p cuts $f(M)$ into more than two pieces, then l_p must have either more than one maximum or more than one minimum, since l_p has a maximum or a minimum on each piece. Conversely, if l_p has more than one maximum or more than one minimum, then there exists a hyperplane determined by l_p that cuts $f(M)$ into more than two pieces (see [95, pp. 29–31] for a complete proof).

Banchoff [21] also noted the following corollary in the case where M is a 2-dimensional surface.

Corollary 2.44. *A TPP immersion $f : M^2 \to \mathbf{R}^m$ of a smooth compact, connected manifold 2-dimensional surface M^2 is tight.*

Proof. Let l_p be a nondegenerate linear height function on M^2. Let $\mu_k(l_p)$ be the number of critical points of l_p of index k. Since f has the TPP, we know that $\mu_0(l_p) = \beta_0(M^2, \mathbf{Z}_2) = 1$ and $\mu_2(l_p) = \beta_2(M^2, \mathbf{Z}_2) = 1$, i.e., l_p has one minimum and one maximum on M^2. Then the Morse relation involving the Euler characteristic $\chi(M^2)$ (see, for example, Milnor [359, p. 29]),

$$\sum_{k=0}^{2}(-1)^k \mu_k(l_p) = \sum_{k=0}^{2}(-1)^k \beta_k(M^2, \mathbf{Z}_2) = \chi(M^2), \tag{2.101}$$

implies that $\mu_1(l_p) = \beta_1(M^2, \mathbf{Z}_2)$ as well, and thus f is a tight immersion. □

Bound on the codimension of a substantial TPP immersion

Another important result in the theory of tight immersions concerns the upper bound on the codimension of a substantial smooth TPP immersion. Kuiper [300] proved part (a) of Theorem 2.46 below, and we will give the proof as in [95, pp. 33–34]. The proof of part (b) of Theorem 2.46 is much more difficult. It is due to Kuiper [301] for $n = 2$, and to Little and Pohl [333] for higher dimensions, and we refer the reader to [95, p. 105] for a complete proof.

The standard embeddings of projective spaces mentioned in part (b) are described in detail in Section 2.9 (see also Tai [505] or [95, pp. 87–98]). The standard embedding of $\mathbf{RP}^2$ into $S^4 \subset \mathbf{R}^5$ is the well-known Veronese surface, which we will now describe.

Remark 2.45 (Veronese surface). Let S^2 be the unit sphere in $\mathbf{R}^3$ given by the equation

$$x^2 + y^2 + z^2 = 1. \tag{2.102}$$

Define a map from S^2 into $\mathbf{R}^6$ by

$$(x, y, z) \mapsto (x^2, y^2, z^2, \sqrt{2}yz, \sqrt{2}zx, \sqrt{2}xy). \tag{2.103}$$

One can easily check that this map takes the same value on antipodal points of S^2, and so it induces a map $\phi : \mathbf{RP}^2 \to \mathbf{R}^6$. An elementary calculation then proves that ϕ is a smooth embedding of $\mathbf{RP}^2$. Furthermore, if $(u_1, \ldots, u_6)$ are the standard coordinates on $\mathbf{R}^6$, then the image of ϕ lies in the Euclidean hyperplane $\mathbf{R}^5 \subset \mathbf{R}^6$ given by the equation:

$$u_1 + u_2 + u_3 = 1, \tag{2.104}$$

since $x^2 + y^2 + z^2 = 1$. One can easily show further that ϕ is a substantial embedding into $\mathbf{R}^5$, and that the image of ϕ is contained in the unit sphere $S^5 \subset \mathbf{R}^6$ given by the equation,

$$u_1^2 + \cdots + u_6^2 = 1. \tag{2.105}$$

Thus, ϕ is a substantial embedding of $\mathbf{RP}^2$ into the 4-sphere $S^4 = S^5 \cap \mathbf{R}^5$ (see Section 2.9 for more detail).

To see that ϕ has the TPP, note that a hyperplane in $\mathbf{R}^5$ given by an equation

$$a_1u_1 + \cdots + a_6u_6 = c, \tag{2.106}$$

for $c \in \mathbf{R}$, cuts $\mathbf{RP}^2$ in a conic. Such a conic does not separate $\mathbf{RP}^2$ into more than two pieces, and so ϕ has the TPP. Since $\mathbf{RP}^2$ has dimension two, ϕ is also tight by Corollary 2.44. Finally, since ϕ is tight and spherical, it is a taut embedding of $\mathbf{RP}^2$ into $S^4 \subset \mathbf{R}^5$ by Theorem 2.69, which will be proven in the next section.

From the Veronese embedding ϕ, we can obtain a tight substantial embedding of $\mathbf{RP}^2$ into a 4-dimensional Euclidean space $\mathbf{R}^4$ in two different ways. First let $\tau : S^4 - \{P\} \to \mathbf{R}^4$ be stereographic projection with pole P not in the image of ϕ (see Remark 2.7). Then by Theorem 2.70 (see page 61), $\tau \circ \phi$ is a taut (and hence tight by Theorem 2.55 on page 55) embedding of $\mathbf{RP}^2$ into $\mathbf{R}^4$. Secondly, we can compose ϕ with orthogonal projection of $\mathbf{R}^6$ onto the 4-space $\mathbf{R}^4$ spanned by the vectors $\{(e_1 - e_2)/\sqrt{2}, e_4, e_5, e_6\}$, where $\{e_1, \ldots, e_6\}$ is the standard basis of $\mathbf{R}^6$. This gives a parametrization

$$(x, y, z) \mapsto \left(\frac{x^2 - y^2}{\sqrt{2}}, \sqrt{2}yz, \sqrt{2}zx, \sqrt{2}xy\right), \tag{2.107}$$

which induces an embedding $f : \mathbf{RP}^2 \to \mathbf{R}^4$. Since tightness is preserved by orthogonal projections (see Remark 2.39), f is a tight embedding of $\mathbf{RP}^2$ into $\mathbf{R}^4$.

TPP immersions with maximal codimension

In the following theorem, the term "up to a projective transformation of $\mathbf{R}^m$" means in the sense defined in Remark 2.38 on page 42.

Theorem 2.46. *Let $f : M^n \to \mathbf{R}^m$ be a substantial smooth immersion of a compact, connected n-dimensional manifold.*

(a) *If f has the TPP, then $m \leq n(n+3)/2$.*
(b) *If f has the TPP and $m = n(n+3)/2$ for $n \geq 2$, then f is a standard embedding $f : \mathbf{RP}^n \to \mathbf{R}^m$ of a projective space, up to a projective transformation of $\mathbf{R}^m$.*

Proof. (a) Let l_p be a nondegenerate linear height function on M with an absolute maximum at a point $x \in M$. After a translation, we can assume that $f(x)$ is the origin of our coordinate system on $\mathbf{R}^m$, so that $l_p(x) = 0$. Since l_p has a maximum at $f(x)$, we know by Theorem 2.29 that the vector p is normal to $f(M)$ at $f(x)$, and the Hessian $H(X,Y) = \langle A_p X, Y\rangle$ of l_p at x is negative-definite. Let $T_x^{\perp}M$ denote the normal space to $f(M)$ at $f(x)$, and let V be the vector space of symmetric bilinear forms on T_xM. Define a linear map $\phi : T_x^{\perp}M \to V$ by $\phi(q) = A_q$, i.e.,

$$\phi(q)(X,Y) = \langle A_q X, Y\rangle, \quad X, Y \in T_xM. \tag{2.108}$$

The dimension of $T_x^{\perp}M$ is $m-n$, and the dimension of V is $n(n+1)/2$. Thus, if $m - n > n(n+1)/2$, i.e., $m > n(n+3)/2$, then the kernel of ϕ contains a nonzero vector.

We now complete the proof by showing that if f has the TPP, then the kernel of ϕ contains only the zero vector, and thus $m \leq n(n+3)/2$. Suppose there exists a vector $q \neq 0$ in $T_x^{\perp}M$ with $A_q = 0$. Let $z(t) = p + tq$. Then $z(t) \in T_x^{\perp}M$ for all t, and

$$A_{z(t)} = A_p + tA_q = A_p,$$

for all t. Thus, $l_{z(t)}$ has a nondegenerate maximum at x for all t. Note that $l_{z(t)}(x) = 0$ for all t, since $f(x)$ is at the origin of the coordinate system.

On the other hand, since f is substantial, there exists a point $y \in M$ such that $l_q(y) \neq 0$. Then we have

$$l_{z(t)}(y) = l_p(y) + t l_q(y),$$

and thus $l_{z(t)}(y) > 0$ for a suitable choice of t. For that value of t, the function $l_{z(t)}$ does not assume its absolute maximum at x. Thus, f does not have the TPP, since if h is the half-space determined by the inequality $l_{z(t)}(u) \geq 0$, for $u \in \mathbf{R}^m$, then $f^{-1}h$ has at least two components, the single point $\{x\}$ and a component containing y.
(b) For a proof of part (b), see Kuiper [301] for $n = 2$, and Little and Pohl [333] for higher dimensions (see also [95, pp. 98–105] for a complete proof). □

Remark 2.47 (The case of dimension $n = 1$). For $n = 1$, part (a) of Theorem 2.46 states that if $f : S^1 \to \mathbf{R}^m$ is a substantially immersed closed curve with the TPP, then $m \leq 2$, and hence the curve is a plane curve. In fact, much more can be said. For $n = 1$, the TPP is equivalent to requiring that the closed curve have total absolute curvature equal to 2π, the minimum value possible. Fenchel [155] proved that a closed curve $f : S^1 \to \mathbf{R}^3$ with total absolute curvature equal to 2π is an embedded convex plane curve, and Borsuk [48] obtained the same conclusion for curves in $\mathbf{R}^m$ with $m > 3$. (See also Chern [102] for a proof of Fenchel's Theorem.) If the curve f is knotted, then the total absolute curvature is greater than 4π (see Fary [153] and Milnor [357, 358]). For a related result regarding the total curvature of a knotted torus, see the paper of Kuiper and Meeks [306].

Kuiper [300] also proved the following generalization of part (a) of Theorem 2.46 which is useful in determining the possible codimensions of tight immersions of projective planes (see Theorem 2.95 on page 81). For this theorem, we need the following notation. Let $(\beta_0, \ldots, \beta_n)$ be an $(n + 1)$-tuple of nonnegative integers. Let $c(\beta_0, \ldots, \beta_n)$ be the maximal dimension of a linear family of symmetric bilinear forms in n variables which contains a positive definite form and such that no form in the family has index k if $\beta_k = 0$. Note that $c(\beta_0, \ldots, \beta_n) \leq n(n + 1)/2$, the dimension of the space of all symmetric bilinear forms in n variables.

In our applications, of course, the $(n + 1)$-tuple $(\beta_0, \ldots, \beta_n)$ will be the $\mathbf{Z}_2$-Betti numbers of a compact manifold M. In the case where M is $\mathbf{FP}^2$ for $\mathbf{F} = \mathbf{R}, \mathbf{C}, \mathbf{H}$, or $\mathbf{O}$ (Cayley numbers), the number $c(\beta_0, \ldots, \beta_n)$ can be computed, and so the following theorem can be used to give bounds on the codimension of a tight immersion of these projective planes into Euclidean spaces (see Theorem 2.95 on page 81).

Theorem 2.48. *Let $f : M^n \to \mathbf{R}^m$ be a substantial tight immersion of a compact, connected n-dimensional manifold, and let β_k denote the k-th $\mathbf{Z}_2$-Betti number of M. Then*

$$m - n \leq c(\beta_0, \ldots, \beta_n) \leq n(n + 1)/2.$$

Proof. We will use the notation of the proof of Theorem 2.46. Let V be the vector space of all symmetric bilinear forms on T_xM. Consider the linear map $\phi : T_x^{\perp}M \to V$ defined by $\phi(q) = A_q$, that is,

$$\phi(q)(X, Y) = \langle A_qX, Y\rangle, \quad X, Y \in T_xM. \tag{2.109}$$

Since f is tight, it has the TPP, and so the proof of Theorem 2.46 shows that ϕ is injective on $T_x^{\perp}M$, which has dimension $m - n$. Thus, we have

$$m - n = \dim\,(\text{Image }\phi). \tag{2.110}$$

The image of ϕ is a vector space that contains a positive definite bilinear form. Furthermore, if $\beta_k = 0$, then no bilinear form in Image ϕ can have index k, for if

$\phi(q)$ has index k, then l_q has a nondegenerate critical point of index k at x. Then by Corollary 2.33, there exists a nondegenerate linear height function l_z having a critical point y of index k, contradicting tightness, since $\beta_k = 0$.

Thus, the space Image ϕ contains a positive definite form and no form of index k if $\beta_k = 0$. Then by definition, the dimension of Image ϕ is less than or equal to $c(\beta_0, \ldots, \beta_n)$, which is less than or equal to $n(n+1)/2$, the dimension of the space V of all symmetric bilinear forms in n variables. So we have

$$m - n = \dim\,(\text{Image}\ \phi) \leq c(\beta_0, \ldots, \beta_n) \leq n(n+1)/2,$$

as needed. □

The product of two tight immersions is tight

We close this section with a proof of the fact that a product of two tight immersions is tight. This was first noted by Kuiper [300], and we follow the presentation given in [95, pp. 43–46].

Let $f : M \to \mathbf{R}^m$ and $g : M' \to \mathbf{R}^{m'}$ be immersions of compact, connected manifolds, each having dimension greater than or equal to 1. The product immersion,

$$f \times g : M \times M' \to \mathbf{R}^m \times \mathbf{R}^{m'} = \mathbf{R}^{m+m'}, \tag{2.111}$$

is defined by

$$(f \times g)(x, y) = (f(x), g(y)). \tag{2.112}$$

Let p be a unit vector in $\mathbf{R}^{m+m'}$. We can decompose p in a unique way as,

$$p = \cos\theta\ q + \sin\theta\ q', \tag{2.113}$$

for $q \in \mathbf{R}^m$, $q' \in \mathbf{R}^{m'}$, and $0 \leq \theta \leq \pi/2$.

Lemma 2.49. *Let $p = \cos\theta\ q + \sin\theta\ q'$ for $0 \leq \theta \leq \pi/2$.*

(a) *l_p is nondegenerate on $M \times M'$ if and only if $0 < \theta < \pi/2$, and l_q, $l_{q'}$ are nondegenerate on M, M', respectively.*
(b) *If l_p is nondegenerate, then the number $\mu(l_p)$ of critical points of l_p on $M \times M'$ is given by $\mu(l_p) = \mu(l_q)\mu(l_{q'})$*

Proof. Let $(x, y) \in M \times M'$, and let $X \in T_xM$, $Y \in T_yM'$. Then a straightforward calculation yields

$$(l_p)_*(X, Y) = \cos\theta\ (l_q)_*X + \sin\theta\ (l_{q'})_*Y. \tag{2.114}$$

If $\theta = 0$, the set of critical points of l_p is the set of all points of the form (x, y), where x is a critical point of l_q, and y is any point in M'. Similarly, if $\theta = \pi/2$, the set of critical points of l_p is the set of all points of the form (x, y), where x any point in M, and y is a critical point of $l_{q'}$. In either case, the critical points of l_p are not isolated, so they are degenerate critical points, and l_p is not a Morse function.

If $0 < \theta < \pi/2$, then we see from equation (2.114) that l_p has a critical point at (x, y) if and only if l_q has a critical point at x and $l_{q'}$ has a critical point at y. We now compute the Hessian of l_p at such a critical point (x, y). Let $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_{n'})$ be local coordinates in neighborhoods of x in M, and y in M', respectively. Clearly, with respect to the local coordinates $(u_1, \ldots, u_n, v_1, \ldots, v_{n'})$ in a neighborhood of (x, y) in $M \times M'$, the Hessian of l_p at the critical point (x, y) has the form

$$H_{(x,y)}(l_p) = \begin{bmatrix} \cos\theta\, H_x(l_q) & 0 \\ 0 & \sin\theta\, H_y(l_{q'}) \end{bmatrix}. \tag{2.115}$$

Thus, $H_{(x,y)}(l_p)$ is nonsingular if and only if $H_x(l_q)$ and $H_y(l_{q'})$ are nonsingular. Hence, l_p is a nondegenerate function if and only if l_q and $l_{q'}$ are nondegenerate functions. Furthermore, from equation (2.115), we see that the index of l_p at a nondegenerate critical point (x, y) is equal to the sum of the indices of l_q at x and $l_{q'}$ at y. Thus, for $0 \leq k \leq n + n'$, we have

$$\mu_k(l_p) = \sum_{i+j=k} \mu_i(l_q)\mu_j(l_{q'}). \tag{2.116}$$

From this, we compute

$$\begin{aligned} \mu(l_p) &= \sum_{k=0}^{n+n'} \mu_k(l_p) \\ &= \sum_{k=0}^{n+n'} \sum_{i+j=k} \mu_i(l_q)\mu_j(l_{q'}) \\ &= \mu(l_q)\, \mu(l_{q'}). \end{aligned}$$

□

Theorem 2.50. *Suppose $f : M \to \mathbf{R}^m$ and $g : M' \to \mathbf{R}^{m'}$ are tight immersions of compact manifolds. Then $f \times g$ is a tight immersion of $M \times M'$ into $\mathbf{R}^{m+m'}$.*

Proof. In the notation of Lemma 2.49, we have for any nondegenerate height function l_p on $M \times M'$,

$$\mu(l_p) = \mu(l_q)\mu(l_{q'}),$$

for appropriate nondegenerate functions l_q and $l_{q'}$. Since f and g are tight, we know that

$$\mu(l_q) = \beta(M, \mathbf{Z}_2) \text{ and } \mu(l_{q'}) = \beta(M', \mathbf{Z}_2),$$

for every such pair of nondegenerate functions l_q and $l_{q'}$. Thus,

$$\mu(l_p) = \beta(M, \mathbf{Z}_2)\, \beta(M', \mathbf{Z}_2) = \beta(M \times M', \mathbf{Z}_2),$$

where the last equality is due to the Künneth formula (see, for example, Greenberg [183, p. 198]). Hence, $f \times g$ is tight. □

2.7 Distance Functions and Taut Submanifolds

In this section, we give a brief review of the theory of distance functions and taut submanifolds in Euclidean space. Various aspects of this theory will be treated in more detail later in this book. For more complete coverage of the topic, the reader is referred to Chapter 2 of the book [95] or the survey article [76]. Our treatment here is based on [95, pp. 113–127].

Let $f : M \to \mathbf{R}^m$ be a smooth immersion of an n-dimensional manifold M into Euclidean space $\mathbf{R}^m$. For $p \in \mathbf{R}^m$, the *Euclidean distance function* L_p is defined on $\mathbf{R}^m$ by

$$L_p(q) = |p - q|^2. \tag{2.117}$$

The restriction of L_p to M gives a real-valued function $L_p : M \to \mathbf{R}$ defined by $L_p(x) = |p - f(x)|^2$. As with linear height functions, Sard's Theorem implies that for almost all $p \in \mathbf{R}^m$, the function L_p is a Morse function on M.

We recall the normal exponential map $E : NM \to \mathbf{R}^m$ defined in Section 2.2 by

$$E(x, \zeta) = f(x) + \zeta, \tag{2.118}$$

where ζ is a normal vector to $f(M)$ at $f(x)$. As in Section 2.2, the focal set of M is the set of critical values of the map E. Hence, by Sard's Theorem, the focal set of M has measure zero in $\mathbf{R}^m$. As noted in Theorem 2.1, if $p = E(x, t\xi)$, where $|\xi| = 1$, then p is a focal point of (M, x) of multiplicity $\nu > 0$ if and only if $1/t$ is a principal curvature of multiplicity ν of the shape operator A_ξ.

Index Theorem for distance functions

The critical point behavior of the L_p functions is described by the following well-known *Index Theorem* (see Milnor [359, pp. 32–38] for a proof).

Theorem 2.51. *Let $f : M \to \mathbf{R}^m$ be a smooth immersion of an n-dimensional manifold M into Euclidean space $\mathbf{R}^m$, and let $p \in \mathbf{R}^m$.*

(a) *A point $x \in M$ is a critical point of L_p if and only if $p = E(x, \zeta)$ for some $\zeta \in T_x^{\perp}M$.*
(b) *L_p has a degenerate critical point at x if and only if p is a focal point of (M, x).*
(c) *If L_p has a nondegenerate critical point at x, then the index of L_p at x is equal to the number of focal points of (M, x) (counting multiplicities) on the segment from $f(x)$ to p.*

The following corollary, due to Nomizu and Rodriguez [405, p. 199], follows from Theorem 2.51 in the same way that Corollary 2.33 follows from Theorem 2.29 for height functions. In the proof of Corollary 2.52, one uses the normal exponential map E instead of the Gauss map ν. In particular, part (a) of the corollary follows from part (b) of Theorem 2.51, since the focal set of M has measure zero in $\mathbf{R}^m$.

Corollary 2.52. *Let $f : M \to \mathbf{R}^m$ be a smooth immersion of an n-dimensional manifold M into Euclidean space $\mathbf{R}^m$.*

(a) *For almost all $p \in \mathbf{R}^m$, L_p is a Morse function on M.*
(b) *Suppose L_p has a nondegenerate critical point of index k at $x \in M$. Then there is a Morse function L_q having a critical point $y \in M$ of index k (q and y may be chosen as close to p and x, respectively, as desired).*

Taut immersions

We can now define taut immersions in a similar way to how we defined tight immersions in the previous section. Suppose first that M is a compact n-dimensional manifold. An immersion $f : M \to \mathbf{R}^m$ is said to be a *taut immersion* if there exists a field $\mathbf{F}$ such that every nondegenerate Euclidean distance function L_p has $\beta(M, \mathbf{F})$ critical points on M, i.e., every nondegenerate distance function is a perfect Morse function. By Theorem 2.28 in the previous section, we see that f is taut if and only if for every nondegenerate Euclidean distance function L_p, the map on homology

$$H_*(M_r(L_p), \mathbf{F}) \to H_*(M, \mathbf{F}) \tag{2.119}$$

induced by the inclusion $M_r(L_p) \subset M$ is injective for all r. Note that

$$M_r(L_p) = \{x \in M \mid |p - f(x)|^2 \leq r\}. \tag{2.120}$$

Thus, $M_r(L_p)$ is the inverse image under f of the closed ball in $\mathbf{R}^m$ with center p and radius $\sqrt{r}$. In this formulation of tautness, one requires the map on homology in equation (2.119) to be injective for all closed balls determined by nondegenerate distance functions, that is, the map

$$H_*(f^{-1}B, \mathbf{F}) \to H_*(M, \mathbf{F}) \tag{2.121}$$

is injective for every closed ball B centered at a point $p \in \mathbf{R}^m$ such that L_p is a nondegenerate function.

Remark 2.53 ($\mathbf{F}$-taut implies $\mathbf{Z}_2$-taut). Grove and Halperin [185]), and independently Terng and Thorbergsson [531], extended the notion of tautness to submanifolds of complete Riemannian manifolds. Their definition agrees with the definition of tautness above for submanifolds of Euclidean space. Recently Wiesendorf [554] showed that if a compact submanifold of a complete Riemannian manifold is taut with respect to some field $\mathbf{F}$, then it is also $\mathbf{Z}_2$-taut. Thus, we will use $\mathbf{Z}_2$-tautness at all times.

As with tight immersions, if we use $\mathbf{Z}_2$–Čech homology and its continuity property, we can prove results similar to Lemma 2.40 and Theorem 2.41 which imply that we can use all closed balls B in $\mathbf{R}^m$ in equation (2.121) and not just those determined by nondegenerate distance functions.

Next one shows that if the injectivity condition in equation (2.121) holds for all closed balls B, then it also holds for all closed half-spaces h and for all complements of open balls in $\mathbf{R}^m$. For half-spaces, this comes from approximating $f^{-1}h$ by $f^{-1}B$, for an appropriate large closed ball B, in a manner similar to the proof of Lemma 2.40. For complements of open balls, one uses the fact that $L_p(x) \geq r$ if and only if $-L_p(x) \leq -r$, and L_p is a perfect Morse function on M if and only if $-L_p$ is a perfect Morse function on M. Using these ideas and techniques similar to those in the proofs of Lemma 2.40 and Theorem 2.41, one can prove the following theorem, and we omit the proof here.

Theorem 2.54. *Let $f : M \to \mathbf{R}^m$ be an immersion of a compact, connected manifold. Suppose that every nondegenerate Euclidean distance function L_p has $\beta(M, \mathbf{Z}_2)$ critical points on M. Then for every closed ball, complement of an open ball, and closed half-space Ω in $\mathbf{R}^m$, the induced homomorphism,*

$$H_*(f^{-1}\Omega) \to H_*(M) \tag{2.122}$$

in Čech homology with $\mathbf{Z}_2$ coefficients is injective.

If S^{m-1} is the metric hypersphere in $\mathbf{R}^m$ with center p and radius r, then S^{m-1} is a taut subset of $\mathbf{R}^m$. To see this, note that if q is any point in $\mathbf{R}^m$ other than p, then L_q is a nondegenerate function having exactly two critical points on S^{m-1} at the two points where the line determined by p and q intersects the sphere S^{m-1}. We will show later (see Theorems 2.73 and 2.74 on page 63) that every taut immersion of an $(m-1)$-sphere into $\mathbf{R}^m$ is a metric hypersphere. This result was proven for $m = 2$ and $m = 3$ by Banchoff [20], and for higher dimensions by Carter and West [61], and independently by Nomizu and Rodriguez [405] using a different proof.

As in the case of tightness, one can generalize the notion of tautness to continuous maps of compact spaces as follows. A map f of a compact topological space X into

$\mathbf{R}^m$ is called a *taut map* if for every closed ball, complement of an open ball, and closed half-space Ω in $\mathbf{R}^m$, the induced homomorphism

$$H_*(f^{-1}\Omega) \to H_*(X) \tag{2.123}$$

in Čech homology with $\mathbf{Z}_2$ coefficients is injective. A subset of $\mathbf{R}^m$ is called a *taut set* if the inclusion map $f : X \to \mathbf{R}^m$ is a taut map.

An obvious consequence of this definition and the definition of a tight map is the following theorem, since tightness only requires that the map on homology in equation (2.123) be injective when Ω is a closed half-space.

Theorem 2.55. *Let $f : X \to \mathbf{R}^m$ be continuous map of a compact, connected space X into $\mathbf{R}^m$. If f is taut, then f is tight.*

Of course, Theorem 2.54 shows that if an immersion $f : M \to \mathbf{R}^m$ of a compact, connected manifold M is taut in the sense that every nondegenerate L_p-function is a perfect Morse function, then f is a taut map as defined above.

The spherical two-piece property

As is the case with tightness, there is a two-piece property associated to tautness due to Banchoff [20]. A continuous map $f : X \to \mathbf{R}^m$ of a compact, connected topological space is said to have the *spherical two-piece property* (STPP) if $f^{-1}\Omega$ is connected for every closed ball, complement of an open ball, and closed half-space Ω in $\mathbf{R}^m$. A compact, connected space $X \subset \mathbf{R}^m$ embedded in $\mathbf{R}^m$ is said to have the STPP if the inclusion map $f : X \to \mathbf{R}^m$ has the STPP. In that case, the STPP means that every hypersphere and hyperplane in $\mathbf{R}^m$ cuts X into at most two pieces, whence the name "spherical two-piece property."

The following results can be proven in a way to very similar Theorem 2.42, Theorem 2.43, and Corollary 2.44 for the two-piece property in the last section, and we omit the proofs here. These are due to Banchoff [20].

Theorem 2.56. *Let $f : X \to \mathbf{R}^m$ be continuous map of a compact, connected space X into $\mathbf{R}^m$. If f is taut, then f has the STPP.*

Theorem 2.57. *Let $f : M \to \mathbf{R}^m$ be an immersion of a smooth compact, connected manifold. Then f has the STPP if and only if every nondegenerate distance function L_p has exactly one minimum and one maximum on M.*

Corollary 2.58. *An STPP immersion $f : M^2 \to \mathbf{R}^m$ of a smooth compact, connected manifold 2-dimensional surface M^2 is taut.*

Carter and West [61] introduced the term "taut immersion" in a paper published in 1972. They also noted that tautness can be defined for proper immersions of non-compact manifolds as follows. Recall that a map $f : X \to \mathbf{R}^m$ of a topological space X is called *proper* if $f^{-1}K$ is compact for every compact subset K of $\mathbf{R}^m$.

If $f : M \to \mathbf{R}^m$ is a proper immersion of a smooth manifold, then $f^{-1}B$ is compact for every closed ball B in $\mathbf{R}^m$, and so the Morse inequalities (2.68) for a nondegenerate distance function L_p can be applied.

We say that such a proper immersion of a non-compact manifold is *taut* if for every closed ball B in $\mathbf{R}^m$, the map $H_*(f^{-1}B) \to H_*M$ (in $\mathbf{Z}_2$–Čech homology) is injective. This definition agrees with the definition of a taut immersion of a compact manifold by Theorem 2.54, since in that case if the map $H_*(f^{-1}B) \to H_*M$ is injective for all closed balls B, then the map $H_*(f^{-1}\Omega) \to H_*M$ is also injective for all closed half-spaces and all complements of open balls Ω.

From Theorem 2.28 on page 38, we see that a proper immersion of a non-compact manifold is taut if and only if for every nondegenerate L_p, the equation

$$\mu_k(L_p, r) = \beta_k(L_p, r, \mathbf{Z}_2) \tag{2.124}$$

holds for all $r \in \mathbf{R}$, $k \in \mathbf{Z}$.

More generally, a proper immersion $f : M \to \mathbf{R}^m$ of a smooth connected manifold is said to be *k-taut* if for every closed ball B in $\mathbf{R}^m$ and for every integer $i \leq k$, the induced homomorphism $H_i(f^{-1}B) \to H_i(M)$ in Čech homology with $\mathbf{Z}_2$ coefficients is injective. If M is compact and connected, then 0-tautness is equivalent to the STPP, since in that case if the map $H_0(f^{-1}B) \to H_*M$ is injective for all closed balls B, then the map $H_0(f^{-1}\Omega) \to H_*M$ is also injective for all closed half-spaces and all complements of open balls Ω, and so f has the STPP.

Next we have the following important consequence of the 0-tautness for smooth immersions due to Banchoff [20], and Carter and West [61].

Theorem 2.59. *Let $f : M \to \mathbf{R}^m$ be a proper immersion of a smooth connected manifold. If f is 0-taut, then f is an embedding.*

Proof. Suppose $f(x_1) = f(x_2) = p$ for two distinct points x_1, x_2 in M. Since f is an immersion, there exist neighborhoods U_1 and U_2 of x_1 and x_2, respectively, on which f is an embedding. Let B be the closed ball of radius 0 centered at p. Then $f^{-1}B$ has at least two connected components, $\{x_1\} \subset U_1$ and $\{x_2\} \subset U_2$, contradicting the assumption that f is 0-taut. □

Constructions preserving tautness

In the following three remarks, we discuss three important constructions which preserve tautness: cylinders over taut submanifolds, tubes over taut submanifolds, and hypersurfaces of revolution with a taut submanifold as the profile submanifold.

Remark 2.60 (Cylinders over taut submanifolds). An example of a taut embedding of a non-compact manifold is a circular cylinder in $\mathbf{R}^3$, or more generally a *spherical cylinder* defined by the product embedding,

$$f \times g : S^k \times \mathbf{R}^{n-k} \to \mathbf{R}^{k+1} \times \mathbf{R}^{n-k} = \mathbf{R}^{n+1}, \tag{2.125}$$

where f embeds S^k as a metric sphere, and g is the identity map on $\mathbf{R}^{n-k}$. Every nondegenerate L_p function has two critical points on $S^k \times \mathbf{R}^{n-k}$, one of index 0 and one of index k. More generally, suppose that $f : M \to \mathbf{R}^k$ is a taut immersion of a compact, connected manifold M, and g is the identity map on $\mathbf{R}^{m-k}$ then

$$f \times g : M \times \mathbf{R}^{m-k} \to \mathbf{R}^k \times \mathbf{R}^{m-k} = \mathbf{R}^m, \tag{2.126}$$

is taut. To see this, note that if $p = (p_1, p_2)$ and $x = (x_1, x_2)$ are points in $\mathbf{R}^k \times \mathbf{R}^{m-k}$, then

$$L_p(x) = L_{p_1}(x_1) + L_{p_2}(x_2). \tag{2.127}$$

Thus, L_p has a critical point at x if and only if L_{p_1} has a critical point at x_1 and $p_2 = x_2$. Such a critical point is nondegenerate if and only if the critical point of L_{p_1} at x_1 is nondegenerate, and in that case, these two critical points have the same index. Since $M \times \mathbf{R}^{m-k}$ and M have the same Betti numbers, f is taut if and only if $f \times g$ is taut.

Remark 2.61 (Parallel hypersurfaces and tubes over taut submanifolds). Carter and West [61] and Pinkall [447, p. 83] pointed out that constructing parallel hypersurfaces or tubes over taut submanifolds preserves tautness.

First suppose that $f : M \to \mathbf{R}^{n+1}$ is an embedded compact, connected oriented hypersurface with global field of unit normals ξ. Suppose that t is a real number such that the parallel map $f_t : M \to \mathbf{R}^{n+1}$ given by

$$f_t(x) = f(x) + t\xi(x) \tag{2.128}$$

is an embedding. Then f_t is a parallel hypersurface of the original embedding $f_0 = f$. By Theorem 2.3 on page 18, the parallel hypersurfaces f_t and f have the same focal set. Suppose that $p \in \mathbf{R}^{n+1}$ is not a focal point of these hypersurfaces. Let L_p denote the restriction of the distance function determined by p to the original embedding f, and let $\tilde{L}_p$ denote its restriction to the parallel hypersurface f_t. By Theorem 2.51, L_p has a critical point at $x \in M$ if and only if $\tilde{L}_p$ has a critical point at x, since the normal line to $f(M)$ at $f(x)$ is the same as the normal line to $f_t(M)$ at $f_t(x)$. So the functions L_p and $\tilde{L}_p$ have the same number of critical points on M. Thus, f_t is taut if and only if f is taut.

Next consider the case where $f : M \to \mathbf{R}^{n+k}$ is a tautly embedded compact, connected submanifold of codimension $k > 1$ in $\mathbf{R}^{n+k}$. Again consider $t > 0$ sufficiently small so that the tube $f_t : BM \to \mathbf{R}^{n+k}$ is an embedded hypersurface, where BM is the unit normal bundle of $f(M)$ in $\mathbf{R}^{n+k}$.

By Theorem 2.3, the focal set of the tube f_t is the union of the focal set of $f(M)$ with $f(M)$ itself. Let $p \in \mathbf{R}^{n+k}$ be a point that is not a focal point of the tube f_t. Let L_p denote the restriction of the distance function determined by p to the original embedding f, and let $\tilde{L}_p$ denote its restriction to the tube f_t. Each critical point $x \in M$ of L_p corresponds to two critical points of $\tilde{L}_p$ on the tube f_t at points where the line from p to $f(x)$ intersects the tube. These critical points are

$$z_1 = f(x) + t\xi, \text{ and } z_2 = f(x) - t\xi, \tag{2.129}$$

where $\xi = (p - f(x))/|p - f(x)|$. Thus, the number of critical points of $\tilde{L}_p$ is equal to twice the number of critical points of L_p. (Note that if $p \in f(M)$, then $\tilde{L}_p$ is a degenerate function, whereas L_p may be nondegenerate. This will not affect tautness since $f(M)$ has measure zero in $\mathbf{R}^{n+k}$.)

Therefore, f is taut if and only if f_t is taut, since the sum of the $\mathbf{Z}_2$-Betti numbers of the unit normal bundle BM (the domain of the tube f_t) is equal to twice the sum of the $\mathbf{Z}_2$-Betti numbers of M. This last fact follows from the Gysin sequence of the unit normal bundle BM of M (see, for example, [418, Lemma 4.7, p. 264]), as was pointed out by Pinkall [447, p. 83]). Thus we have the following theorem due to Pinkall.

Theorem 2.62. *Let $f : M \to \mathbf{R}^n$ be a compact, connected embedded submanifold of $\mathbf{R}^n$ of codimension greater than one, and let $t > 0$ be sufficiently small so that the tube $f_t : BM \to \mathbf{R}^n$ is a compact, connected embedded hypersurface in $\mathbf{R}^n$. Then $f(M)$ is taut with respect to $\mathbf{Z}_2$ coefficients if and only if the tube $f_t(M)$ is taut with respect to $\mathbf{Z}_2$ coefficients.*

Remark 2.63 (Taut hypersurfaces of revolution). Suppose M is a taut compact, connected hypersurface embedded in $\mathbf{R}^{k+1}$ which is disjoint from a hyperplane $\mathbf{R}^k \subset \mathbf{R}^{k+1}$ through the origin. Let e_{k+1} be a unit normal to the hyperplane $\mathbf{R}^k$ in $\mathbf{R}^{k+1}$. Embed $\mathbf{R}^{k+1}$ in $\mathbf{R}^{n+1}$, and let $\mathbf{R}^{n-k+1}$ be the orthogonal complement of $\mathbf{R}^k$ in $\mathbf{R}^{n+1}$. Let $SO(n-k+1)$ denote the group of isometries in $SO(n+1)$ that keep $\mathbf{R}^k$ pointwise fixed. If we consider $\mathbf{R}^{n+1}$ as $\mathbf{R}^k \times \mathbf{R}^{n-k+1}$, then each point of $M \subset \mathbf{R}^{k+1}$ has the form (x, y), where $y = ce_{k+1}$ for some $c > 0$. Let

$$W = \{(x, Ay) \mid (x, y) \in M,\ A \in SO(n-k+1)\} \tag{2.130}$$

be the hypersurface in $\mathbf{R}^{n+1}$ obtained by rotating M about the axis $\mathbf{R}^k$. Then W is diffeomorphic to $M \times S^{n-k}$, and the sum of the $\mathbf{Z}_2$-Betti numbers of W satisfies $\beta(W) = 2\beta(M)$.

We now show that W is taut in $\mathbf{R}^{n+1}$. First, if $p \in \mathbf{R}^k \subset \mathbf{R}^{n+1}$, then L_p has an absolute minimum on M at some point $z \in M$. Hence, L_p has critical points at all the points of W in the orbit of z under the action of $SO(n-k+1)$. Since these critical points are not isolated, they are degenerate critical points. Thus, every point $p \in \mathbf{R}^k$ is a focal point of W. Next consider any $(k+1)$-plane of the form

$$V = \mathbf{R}^k \oplus \text{ Span } \{Ae_{k+1}\}, \tag{2.131}$$

for a fixed $A \in SO(n-k+1)$. Then $W \cap V$ consists of two disjoint congruent copies of M. If $z \in W \cap V$, then the normal line to W through z lies in V. Now suppose that L_p, for $p \in \mathbf{R}^{n+1}$, is a nondegenerate function on W. Then p does not lie in the axis $\mathbf{R}^k$, so p lies in the space V spanned by $\mathbf{R}^k$ and p itself. All of the critical points of L_p on W lie in $W \cap V$. Since M is taut, L_p has exactly $\beta(M)$ critical points on each of the two copies of M in $W \cap V$. Thus, L_p has $\beta(W) = 2\beta(M)$ critical points on W. This is true for all Morse functions of the form L_p on W, and so W is tautly embedded in $\mathbf{R}^{n+1}$.

Basic results on taut embeddings

We now follow the development of the theory in Carter and West [61]. The first theorem is essentially Banchoff's [20] observation that for an STPP embedding, every local support sphere is a global support sphere.

Theorem 2.64. (a) *Let $f : M \rightarrow \mathbf{R}^m$ be a 0-taut embedding of a connected manifold M. Suppose p is the first focal point of (M,x) on a normal ray to $f(M)$ at $f(x)$. If q is any point except p on the closed segment from $f(x)$ to p, then L_q has a strict absolute minimum on M at x. Further, the function L_p itself has an absolute minimum at x.*

(b) *Let $f : M \rightarrow \mathbf{R}^m$ be an STPP embedding of a compact, connected n-dimensional manifold. Suppose that p is a focal point of (M,x) such that the sum of the multiplicities of the focal points of (M,x) on the closed segment from $f(x)$ to p is n. If q is any point beyond p on the normal ray from $f(x)$ through p, then L_q has a strict absolute maximum at p. Further, the function L_p itself has an absolute maximum at x.*

Proof. (a) For any point $q \neq p$ on the closed segment from $f(x)$ to p, the function L_q has a strict local minimum at x by the Index Theorem (Theorem 2.51). Since the intersection of $f(M)$ with the closed ball centered at q through $f(x)$ is connected by 0-tautness, this intersection consists of the point $f(x)$ alone. Therefore, $f(M)$ lies outside the corresponding open ball centered at q with radius equal to the length of the segment from q to $f(x)$. Thus $f(M)$ lies outside the union of these open balls as q varies from $f(x)$ to p, and so $f(M)$ lies outside the open ball centered at p through $f(x)$. Therefore, L_p has an absolute minimum at x.
(b) This is proven in a way similar to (a) using maxima rather than minima. □

This theorem has the following three useful corollaries. Here l_ξ denotes the linear height function in the direction ξ.

Corollary 2.65. *Let $f : M \rightarrow \mathbf{R}^m$ be a 0-taut embedding of a connected manifold M. Suppose there are no focal points of (M,x) on the normal ray to $f(M)$ in the direction ξ at $f(x)$. Then $f(M)$ lies in the closed half-space determined by the inequality $l_\xi \leq l_\xi(x)$.*

Proof. For all q on the normal ray in question, part (a) of Theorem 2.64 above implies that the set $f(M)$ is disjoint from the open sphere centered at q of radius $|q - f(x)|$. Hence, $f(M)$ is disjoint from the union of such open balls, which is the open half-space determined by the inequality $l_\xi(u) > l_\xi(x)$, for $u \in \mathbf{R}^m$. □

From Theorem 2.64 and Corollary 2.65, we see that the existence of a normal vector ξ such that $A_\xi = \lambda I$ has strong implications for an STPP embedding, as the next two corollaries show.

Corollary 2.66. *Let $f : M \to \mathbf{R}^m$ be a 0-taut embedding of a connected manifold M. If $A_\xi = 0$ for some unit normal ξ to $f(M)$ at a point $f(x)$, then $f(M)$ lies in the hyperplane in $\mathbf{R}^m$ determined by the condition $l_\xi = l_\xi(x)$.*

Proof. Since $A_\xi = 0$, there do not exist any focal points on the normal line determined by ξ. If we apply Corollary 2.65 to each of the normal rays determined by ξ, we get that $f(M)$ lies in the intersection of the two closed half spaces determined by the hyperplane with equation $l_\xi = l_\xi(x)$, and so $f(M)$ lies in that hyperplane. □

Corollary 2.67. *Let $f : M \to \mathbf{R}^m$ be an STPP embedding of a compact, connected manifold M. If $A_\xi = \lambda I, \lambda \neq 0$, for some unit normal ξ to $f(M)$ at a point $f(x)$, then $f(M)$ lies in the hypersphere in $\mathbf{R}^m$ centered at the focal point $p = f(x) + (1/\lambda)\xi$ with radius $1/|\lambda|$.*

Proof. Let q be a point on the open segment from $f(x)$ to p. By Theorem 2.64 (a), the set $f(M)$ does not intersect the open ball centered at q of radius $|q - f(x)|$. Hence $f(M)$ is disjoint from the union of such open balls, i.e., the open ball centered at p of radius $1/|\lambda|$. Similarly, by part (b) of Theorem 2.64, the set $f(M)$ is disjoint from the complement of the closed ball centered at p with radius $1/|\lambda|$. Thus $f(M)$ lies in the hypersphere centered at p with radius $1/|\lambda|$. □

This has the following immediate corollary in the case where M is a hypersurface. This was first proven by Banchoff [20] in the case of where M is a 2-dimensional surface.

Corollary 2.68. *Let $f : M^n \to \mathbf{R}^{n+1}$ be a codimension one STPP embedding of a compact, connected manifold M . If $f(M)$ has one umbilic point, then f embeds M as a metric sphere in $\mathbf{R}^{n+1}$.*

Proof. By Corollaries 2.66 and 2.67, if $f(M)$ has one umbilic point, then $f(M)$ is contained in a hyperplane in $\mathbf{R}^{n+1}$ or in a metric hypersphere $S^n \subset \mathbf{R}^{n+1}$. The image of the compact n-dimensional manifold M cannot be contained in a hyperplane, and so $f(M)$ is a compact, connected n-dimensional submanifold of S^n. Thus, $f(M)$ is S^n itself. □

The relationship between tight and taut maps

We now wish to explore the relationship between tightness and tautness further. The first result is that a tight, spherical map is taut. Here $f : X \to \mathbf{R}^m$ is *spherical* if the image of f lies in a metric hypersphere in $\mathbf{R}^m$.

Theorem 2.69. *Let $f : M \to S^m \subset \mathbf{R}^{m+1}$ be a tight spherical map of a compact topological space X. Then f is a taut map into $\mathbf{R}^{m+1}$.*

Proof. Let Ω be a closed ball or the complement of an open ball in $\mathbf{R}^{m+1}$. Then $\Omega \cap S^m = h \cap S^m$ for some closed half-space in $\mathbf{R}^{m+1}$. Since $f(X)$ is contained in S^m, we have

$$f^{-1}\Omega = f^{-1}(\Omega \cap S^m) = f^{-1}(h \cap S^m) = f^{-1}h.$$

Since f is tight, the map $H_*(f^{-1}h) \to H_*(X)$ is injective, and so the map $H_*(f^{-1}\Omega) \to H_*(X)$ is injective, and f is taut. □

Let $\tau : S^m - \{P\} \to \mathbf{R}^m$ be stereographic projection with pole $P \in S^m$ as in equation (2.45). Via the map τ, the space $S^m - \{P\}$ is conformally equivalent to $\mathbf{R}^m$, or we may consider S^m as $\mathbf{R}^m \cup \{\infty\}$, the one-point compactification of $\mathbf{R}^m$. A conformal transformation of $\mathbf{R}^m \cup \{\infty\}$ takes the collection of all hyperspheres and hyperplanes onto itself. Hence, tautness and the STPP are preserved by such a conformal transformation. We formulate this conformal invariance of tautness and the STPP specifically in the following theorem. In this way, we see that tautness is equivalent to the combination of tight and spherical via stereographic projection.

Theorem 2.70. *Let X be a compact topological space.*

(a) *If $f : X \to \mathbf{R}^m$ is a taut (respectively STPP) map, and φ is a conformal transformation of $\mathbf{R}^m \cup \{\infty\}$ such that $\varphi(f(X)) \subset \mathbf{R}^m$, then $\varphi \circ f$ is a taut (respectively STPP) map of X into $\mathbf{R}^m$.*
(b) *If $f : X \to S^m \subset \mathbf{R}^{m+1}$ is taut (respectively STPP), and $\tau : S^m - \{P\} \to \mathbf{R}^m$ is stereographic projection with pole P not in $f(X)$, then $\tau \circ f$ is a taut (respectively STPP) map of X into $\mathbf{R}^m$.*
(c) *If $f : X \to \mathbf{R}^m$ is taut (respectively STPP) and $\tau^{-1} : \mathbf{R}^m \to S^m \subset \mathbf{R}^{m+1}$ is inverse stereographic projection with respect to any pole P, then $\tau^{-1} \circ f$ is a taut (respectively STPP) map of X into S^m.*

By similar considerations, we see another method to obtain taut embeddings of non-compact manifolds, as in Carter and West [61] (see also [95, pp. 120–121]).

Theorem 2.71. *Suppose that $f : M \to \mathbf{R}^m$ is a taut embedding of a compact, connected manifold M, and φ is a conformal transformation of $\mathbf{R}^m \cup \{\infty\}$ such that $\varphi(f(x)) = \infty$ for some $x \in M$. Then $\varphi \circ f$ is a taut embedding of $M - \{x\}$ into $\mathbf{R}^m$.*

Proof. If B is any closed ball in $\mathbf{R}^m$, then $\varphi^{-1}B$ is a closed ball, the complement of an open ball, or a closed half-space in $\mathbf{R}^m$. Since f is taut, the map $H_*(f^{-1}(\varphi^{-1}B)) \to$

$H_*(M)$ is injective. Since this map factors through the homomorphism $H_*(M - \{x\}) \to H_*(M)$, the map

$$H_*(f^{-1}(\varphi^{-1}B)) \to H_*(M - \{x\})$$

is injective also, as needed. □

Recall that a map f of a topological space X into $\mathbf{R}^m$ is said to be substantial if the image $f(X)$ is not contained in any hyperplane in $\mathbf{R}^m$. From the theorem above, we immediately get a way to obtain more examples of taut submanifolds in $\mathbf{R}^m$ by taking the image under stereographic projection of taut submanifolds in S^m.

Corollary 2.72. *Let M be a compact manifold. Then there exists a substantial, non-spherical taut (respectively STPP) embedding $f : M \to \mathbf{R}^m$ if and only if there exists a substantial taut (respectively STPP) spherical embedding*

$$\tilde{f} : M \to \mathbf{R}^{m+1}.$$

Proof. Let $f : M \to \mathbf{R}^m$ be a substantial, non-spherical taut embedding. Let τ^{-1} be the inverse of stereographic projection with respect to any pole $P \in S^m$. Then $\tilde{f} = \tau^{-1} \circ f$ is a taut embedding of M into $S^m \subset \mathbf{R}^{m+1}$. Furthermore, $\tau^{-1} \circ f$ is substantial in $\mathbf{R}^{m+1}$, since if the image of $\tau^{-1} \circ f$ lies in a hyperplane π in $\mathbf{R}^{m+1}$, then it lies in the hypersphere $\Sigma^{m-1} = \pi \cap S^m$. This implies that the image of f lies in the hyperplane or hypersphere $\tau(\Sigma^{m-1})$ in $\mathbf{R}^m$, contradicting the assumption that f is substantial and non-spherical in $\mathbf{R}^m$.

Conversely, suppose that $\tilde{f} : M \to S^m \subset \mathbf{R}^{m+1}$ is a substantial taut spherical embedding. Since $\tilde{f}$ is substantial in $\mathbf{R}^{m+1}$, the image of $\tilde{f}$ does not lie in a hypersphere in $S^m \subset \mathbf{R}^{m+1}$. Let P be any point in S^m that is not in the image of $\tilde{f}$, and let $\tau : S^m - \{P\} \to \mathbf{R}^m$ be stereographic projection with pole P. Then $f = \tau \circ \tilde{f}$ is a taut embedding of M into $\mathbf{R}^m$, and it is substantial and non-spherical in $\mathbf{R}^m$, since the image of $\tilde{f}$ does not lie in a hypersphere in S^m. □

This corollary is useful, because many important examples of taut submanifolds lie in a sphere S^m in $\mathbf{R}^{m+1}$. In particular, all isoparametric (constant principal curvatures) hypersurfaces and their focal submanifolds in S^m are taut [93], as we will see in Section 3.6 (see Corollary 3.56 on page 139).

Using Theorem 2.70, we can thus obtain many new taut submanifolds in $\mathbf{R}^m$ via stereographic projection. In particular, the cyclides of Dupin in $\mathbf{R}^m$ are obtained from a standard product of two spheres (which is an isoparametric hypersurface in S^m, see Section 3.8.2 on page 148),

$$S^p(r) \times S^{m-1-p}(s) \subset S^m, \quad r^2 + s^2 = 1, \tag{2.132}$$

via stereographic projection, and thus they are taut in $\mathbf{R}^m$.

Taut embeddings of spheres

Banchoff [20], and Carter and West [61] pointed out that Theorem 2.70, when combined with known results for tight immersions, yields theorems for taut immersions. In particular, in conjunction with the Chern–Lashof Theorem (Theorem 2.35), which states that if $f : S^n \to \mathbf{R}^m$ is a tight immersion, then f embeds S^n as a convex hypersurface in a Euclidean space $\mathbf{R}^{n+1} \subset \mathbf{R}^m$, one gets the following theorem.

Theorem 2.73. *Let $f : S^n \to \mathbf{R}^m$ be a substantial taut immersion. Then $m = n + 1$, and f embeds S^n as a metric hypersphere.*

Proof. Since a taut immersion is tight, the Chern–Lashof Theorem implies that $m = n + 1$ and f embeds S^n as a convex hypersurface in $\mathbf{R}^{n+1}$. If $f(S^n)$ were not a metric hypersphere, then by part (c) of Theorem 2.70, the map $\tau^{-1} \circ f$ would be a taut substantial embedding of S^n into $\mathbf{R}^{n+2}$, contradicting the Chern–Lashof Theorem. □

We next give a different proof of Theorem 2.73 due to Nomizu and Rodriguez [405]. This is an important type of proof using the properties of distance functions and the Index Theorem (Theorem 2.51), as opposed to the proof above which is based on the theory of tight immersions. Another key element in this proof is the characterization of spheres as compact, totally umbilical submanifolds in Euclidean space. A similar approach can be used to characterize totally umbilic submanifolds of hyperbolic space (see Cecil–Ryan [90]) and to characterize totally geodesic embeddings of $\mathbf{CP}^n$ and complex quadrics Q^n in complex projective space $\mathbf{CP}^m$ in terms of the critical point behavior of distance functions (see Cecil [71]).

Since the proof of the following theorem relies on the characterization of metric spheres as totally umbilic submanifolds, it only works for S^n with $n \geq 2$. For $n = 1$, one can use the approach of Theorem 2.73 above, or else use Banchoff's [20] elementary direct proof using the spherical two-piece property. The following theorem is due to Nomizu and Rodriguez [405]. Here we are following the proof in [95, p. 126].

Theorem 2.74. *Let $M^n, n \geq 2$, be a connected, complete Riemannian manifold isometrically immersed in $\mathbf{R}^m$. If every nondegenerate distance function L_p has index 0 or n at each of its critical points, then M^n is embedded as a totally geodesic n-plane or a metric n-sphere $S^n \subset \mathbf{R}^{n+1} \subset \mathbf{R}^m$.*

Proof. As noted above, the proof is accomplished by showing that the immersion $f : M^n \to \mathbf{R}^m$ is totally umbilic, that is, for every normal vector ξ to $f(M^n)$ at every point $f(x)$, the shape operator A_ξ is a multiple of the identity endomorphism on T_xM^n.

Let ξ be a unit normal to $f(M^n)$ at a point $f(x)$. If $A_\xi = 0$, then A_ξ is a multiple of the identity as needed. If not, then we may assume that A_ξ has a positive eigenvalue by considering $A_{-\xi} = -A_\xi$, if necessary. Let λ be the largest positive eigenvalue of A_ξ. Let t be a real number such that $1/\lambda < t < 1/\mu$, where μ is the next largest positive eigenvalue of A_ξ (if λ is the only positive eigenvalue, just consider $1/\lambda < t$). Then for $p = f(x) + t\xi$, the Index Theorem (Theorem 2.51) implies that the distance function L_p has a nondegenerate critical point of index k at x, where k is the multiplicity of the eigenvalue λ. While L_p may not be a nondegenerate function, Corollary 2.52 implies that there exists a nondegenerate distance function L_q having a critical point y of index k, where q and y can be chosen to be as close to p and x, respectively, as desired. By the hypothesis of the theorem, since k is greater than 0, we get $k = n$, and so $A_\xi = \lambda I$. Since this is true for any unit normal ξ at any point $x \in M^n$, we have that f is totally umbilical. The result then follows from a theorem of E. Cartan [57] which states that a complete Riemannian n-manifold isometrically and totally umbilically immersed in $\mathbf{R}^m$ is embedded as a totally geodesic n-plane or a metric n-sphere $S^n \subset \mathbf{R}^{n+1} \subset \mathbf{R}^m$. (See also B.Y. Chen [98] or M. Spivak [495, Vol. 4, p. 110] for a proof of Cartan's theorem.) □

Remark 2.75 (Another proof of Theorem 2.73). As a consequence of Theorem 2.74, we get another proof of Theorem 2.73 that a taut immersion $f : S^n \to \mathbf{R}^m$ is an embedding of S^n as a metric hypersphere in $\mathbf{R}^{n+1} \subset \mathbf{R}^m$. Specifically, if $f : S^n \to \mathbf{R}^m$ is taut, then every nondegenerate distance function L_p has exactly one maximum and one minimum. Thus all of the critical points of L_p have index 0 or n, and so f embeds S^n as a metric hypersphere in $\mathbf{R}^{n+1} \subset \mathbf{R}^m$ by Theorem 2.74.

Taut embeddings of maximal codimension

Finally, as with Theorem 2.46 for TPP immersions, there is a bound on the codimension of a substantial STPP embedding of a compact, connected n-dimensional manifold into $\mathbf{R}^m$. This follows fairly directly from Theorem 2.46, Corollary 2.72, and the fact that the STPP implies the TPP. This result is due to Banchoff [20] for $n = 2$ and to Carter and West [61] for $n \geq 3$ (see also [95, pp. 124–125]). The standard embeddings of $\mathbf{RP}^n$ into $\mathbf{R}^m$, $m = n(n+3)/2$, are described in detail in Section 2.9. The term "projectively equivalent" means up to a projective transformation in the sense defined in Remark 2.38.

Theorem 2.76. *Let $f : M^n \to \mathbf{R}^m$, $n \geq 2$, be a substantial smooth immersion of a compact, connected n-dimensional manifold.*

(a) *If f has the STPP, then $m \leq n(n+3)/2$.*
(b) *If f has the STPP and $m = n(n+3)/2$, then f is projectively equivalent to a standard embedding of $\mathbf{RP}^n$ into $\mathbf{R}^m$, and the image $f(M)$ lies in a metric sphere $S^{m-1} \subset \mathbf{R}^m$.*

Proof. Since the STPP implies the TPP, part (a) follows immediately from part (a) of Theorem 2.46.

To prove part (b), suppose that $m = n(n+3)/2$ for $n \geq 2$. If the image $f(M)$ does not lie in a metric sphere in $\mathbf{R}^m$, then by Corollary 2.72 (for the STPP), there exists a substantial spherical STPP embedding $\tilde{f} : M \to \mathbf{R}^{m+1}$. This contradicts part (a) of the theorem. Furthermore, f is a TPP embedding of an n-dimensional manifold into $\mathbf{R}^m$ with $m = n(n+3)/2$ for $n \geq 2$. Thus, by part (b) of Theorem 2.46, f is a standard embedding $f : \mathbf{RP}^n \to \mathbf{R}^m$ of a projective space, up to a projective transformation of $\mathbf{R}^m$. □

We also have the following similar result for taut embeddings of non-compact manifolds into $\mathbf{R}^{m-1}$ due to Carter and West [61] (see also [95, pp. 124–125]).

Theorem 2.77. *Let $g : M^n \to \mathbf{R}^m$, $n \geq 2$, be a substantial smooth proper immersion of a non-compact, connected n-dimensional manifold.*

(a) *If g is taut, then $m \leq \frac{n(n+3)}{2} - 1$.*
(b) *If g is taut and $m = \frac{n(n+3)}{2} - 1$, then $g = \tau \circ f$, where $f : \mathbf{RP}^n \to \mathbf{R}^{m+1}$ is projectively equivalent to a standard embedding and the image of f lies in a metric sphere $S^m \subset \mathbf{R}^{m+1}$, and $\tau : S^m - \{P\} \to \mathbf{R}^m$ is stereographic projection with pole $P \in f(M)$.*

Remark 2.78 (Tight and taut immersions into hyperbolic space). In hyperbolic space H^m there are three types of totally umbilic hypersurfaces: spheres, horospheres, and equidistant hypersurfaces (those at a fixed oriented distance from a totally geodesic hyperplane, including hyperplanes themselves). These have constant sectional curvature which is positive, zero, or negative, for spheres, horospheres, and equidistant hypersurfaces, respectively. Thus, there are three natural types of distance functions L_p, L_h, and L_π, which measure the distance from a given point p, horosphere h, or hyperplane π, respectively. Just as in Euclidean space (see Theorem 2.74), the totally umbilic hypersurfaces of H^m can be characterized in terms of the critical point behavior of these distance functions as follows (see Cecil–Ryan [90]).

Theorem 2.79. *Let $M^n, n \geq 2$, be a connected, complete Riemannian manifold isometrically immersed in H^m. Every Morse function of the form L_p or L_π has index 0 or n at all of its critical points if and only if M is embedded as a sphere, horosphere, or equidistant hypersurface in a totally geodesic $H^{n+1} \subset H^m$.*

An immersion $f : M \to H^m$ is called *taut*, *horo-tight*, or *tight*, respectively, if every nondegenerate function L_p, L_h, or L_π, has the minimum number of critical points required by the Morse inequalities. See Cecil and Ryan [90, 91], [95, pp. 233–236], and Izumiya et al. [227, 228], for more on these conditions.

2.8 The Relationship between Taut and Dupin

In this section, we examine the relationship between taut and Dupin submanifolds. We begin with a theorem of Thorbergsson [533] which states that the proper Dupin condition implies tautness for complete embedded hypersurfaces in real space forms. After stating the theorem, we will make some comments regarding

Thorbergsson's approach to proving this result, and we refer the reader to [533] for a complete proof.

Theorem 2.80. *Let $M^n \subset \tilde{M}^{n+1}$ be a complete, connected proper Dupin hypersurface embedded in a real space form $\tilde{M}^{n+1}$. Then M is taut.*

We now discuss Thorbergsson's method of proof. Let $p \in \tilde{M}^{n+1}$ and let L_p be the distance function $L_p(x) = d(p,x)^2$, where $d(p,x)$ is the distance from p to x in $\tilde{M}^{n+1}$. By Sard's Theorem, the restriction of L_p to M is a Morse function for almost all $p \in \tilde{M}^{n+1}$.

To prove that M is tautly embedded, one must show that every nondegenerate (Morse) function of the form L_p has the minimum number of critical points required by the Morse inequalities on M. Equivalently, one can show that every critical point of every nondegenerate distance function is of linking type (see, Morse–Cairns [379, p. 258] and the comments after Theorem 2.28 on page 38). That is the method used by Thorbergsson, as we now discuss.

Specifically, since M is a complete embedded hypersurface in a (simply connected) real space form, M is orientable, and we take ξ to be a field of unit normal vectors on M. Since M is proper Dupin, it has g distinct principal curvatures at each point, and thus we have g smooth principal curvature functions:

$$\lambda_1 > \lambda_2 > \cdots > \lambda_g, \tag{2.133}$$

with respective constant multiplicities $m_1, \ldots, m_g$ on M.

Let $E : NM \to \tilde{M}^{n+1}$ be the normal exponential map of M as defined in Section 2.2. By Theorem 2.1, a point $p = E(x, t\xi(x))$ is a focal point of (M,x) of multiplicity $m > 0$ if and only if there is a principal curvature λ of A_ξ of multiplicity m such that:

$$\begin{aligned} \lambda &= 1/t, \text{ if } \tilde{M}^{n+1} = \mathbf{R}^{n+1}, \\ \lambda &= \cot t, \text{ if } \tilde{M}^{n+1} = S^{n+1}, \\ \lambda &= \coth t, \text{ if } \tilde{M}^{n+1} = H^{n+1}. \end{aligned} \tag{2.134}$$

Thus, as noted earlier, if a principal curvature function λ has constant multiplicity m on M, then we can define a smooth focal map f_λ from an open subset $U \subset M$ (defined below) onto the sheet of the focal set of M determined by λ. Using equation (2.134) for the location of the focal points, we define the map f_λ by the formulas:

$$\begin{aligned} f_\lambda(x) &= f(x) + \frac{1}{\lambda}\,\xi(x), \\ f_\lambda(x) &= \cos\theta\, f(x) + \sin\theta\, \xi(x), \text{ where } \cot\theta = \lambda, \\ f_\lambda(x) &= \cosh\theta\, f(x) + \sinh\theta\, \xi(x), \text{ where } \coth\theta = \lambda, \end{aligned} \tag{2.135}$$

for $\tilde{M}$ equal to $\mathbf{R}^{n+1}$, S^{n+1}, and H^{n+1}, respectively.

In the case of $\mathbf{R}^{n+1}$, the domain U of the focal map f_λ is the set of points in M where $\lambda \neq 0$. In hyperbolic space, the domain U of f_λ is the set of points where $|\lambda| > 1$. In the case of S^{n+1}, at each point $x \in M$ the principal curvature λ gives rise to two antipodal focal points in S^{n+1} determined by substituting $\theta = \cot^{-1}\lambda$ and $\theta = \cot^{-1}\lambda + \pi$ into equation (2.135). Thus, λ gives rise to two antipodal focal maps into S^{n+1}.

Define $\rho_i(x) = 1/\lambda_i(x)$ if $\tilde{M}^{n+1} = \mathbf{R}^{n+1}$ and $\lambda_i(x) \neq 0$; $\rho_i(x) = \cot^{-1}\lambda_i(x)$ if $\tilde{M}^{n+1} = S^{n+1}$; and $\rho_i(x) = \coth^{-1}\lambda_i(x)$ if $\tilde{M}^{n+1} = H^{n+1}$ and $|\lambda_i(x)| > 1$. Then the focal point $f_i(x)$ corresponding to the principal curvature $\lambda_i(x)$ is $f_i(x) = E(x, \rho_i(x)\xi(x))$.

For $x \in M$, let $S_i(x)$ denote the leaf of the principal foliation T_i determined by λ_i through the point x. If x is in the domain U of the focal map f_i, then by Theorems 2.11 and 2.14, the Dupin condition implies that the leaf $S_i(x)$ is a compact m_i-dimensional metric sphere contained in a totally geodesic $(m_i + 1)$-dimensional submanifold of $\tilde{M}^{n+1}$ (which does not necessarily contain the focal point $f_i(x)$). The m_i-sphere $S_i(x)$ is also contained in the metric hypersphere (the curvature sphere) in $\tilde{M}^{n+1}$ with center $f_i(x)$ and radius $|\rho_i(x)|$, and $S_i(x)$ is either a great or small sphere in this curvature sphere.

Using these facts about the principal foliations T_i by m_i-spheres on the domain U of f_i, Thorbergsson gave an inductive procedure using iterated sphere bundles to construct concrete $\mathbf{Z}_2$-cycles in M to show that every critical point of every nondegenerate distance function L_p is of linking type, and thus M is taut. (See Thorbergsson's paper [533] for the detailed construction.)

As noted earlier in Theorem 2.62, using the Gysin sequence of the unit normal bundle of the submanifold M, Pinkall [447] proved the following result concerning submanifolds of codimension greater than one. We restate the theorem here for the sake of completeness.

Theorem 2.81. *Let $f : M \to \mathbf{R}^n$ be a compact, connected embedded submanifold of $\mathbf{R}^n$ of codimension greater than one, and let $t > 0$ be sufficiently small so that the tube $f_t : BM \to \mathbf{R}^n$ is a compact, connected embedded hypersurface in $\mathbf{R}^n$. Then $f(M)$ is taut with respect to $\mathbf{Z}_2$ coefficients if and only if the tube $f_t(M)$ is taut with respect to $\mathbf{Z}_2$ coefficients.*

We can use this to generalize Theorem 2.80 to submanifolds of higher codimension as follows. Recall from Remark 2.21 that if $f : M \to \mathbf{R}^n$ is an immersed submanifold of $\mathbf{R}^n$ with codimension greater than one, then a connected submanifold $S \subset M$ is called a curvature surface of $f(M)$ if there exists a parallel (with respect to the normal connection) section of the unit normal bundle $\eta : S \to B^{n-1}$ such that for each $x \in S$, the tangent space T_xS is equal to some eigenspace of $A_{\eta(x)}$. As in Remark 2.26, the submanifold $f(M)$ is called Dupin if along each curvature surface, the corresponding principal curvature is constant. In that case, $f(M)$ is called proper Dupin if the number of distinct principal curvatures is constant on the unit normal bundle B^{n-1}. We can now prove the following result due to Pinkall [447].

Theorem 2.82. *Let M be a compact, connected proper Dupin submanifold of codimension greater than one embedded in $\mathbf{R}^n$. Then M is taut with respect to $\mathbf{Z}_2$ coefficients.*

Proof. Let $f : M \to \mathbf{R}^n$ be the embedding of M as a compact, connected proper Dupin submanifold, and let $t > 0$ be sufficiently small so that the tube $f_t : BM \to \mathbf{R}^n$ is a compact, connected embedded hypersurface in $\mathbf{R}^n$. Since $f(M)$ is proper Dupin, the multiplicities of its principal curvatures are constant on its unit normal bundle B^{n-1}. Then by using Theorem 2.2 (page 17) regarding the shape operators of a tube, one easily shows that $f_t(M)$ is a proper Dupin hypersurface embedded in $\mathbf{R}^n$. By Theorem 2.80, $f_t(M)$ is a taut hypersurface, and thus by Theorem 2.81, $f(M)$ is also taut. □

More generally, tautness has been established for *Dupin submanifolds with constant multiplicities* of higher codimension by Terng [527] and [529, p. 467]. These are Dupin submanifolds $M \subset \mathbf{R}^n$ of codimension greater than one such that the multiplicities of the principal curvatures of any parallel normal field $\xi(t)$ along any piecewise smooth curve on M are constant.

Taut implies Dupin

In the opposite direction of Theorem 2.80, Pinkall [447] and Miyaoka [364] (for hypersurfaces) independently proved the following theorem, which is also valid for submanifolds of S^n. We give Pinkall's proof below, following the presentation given in [95, pp. 194–196].

Theorem 2.83. *Every taut submanifold $M \subset \mathbf{R}^n$ is Dupin (but not necessarily proper Dupin).*

Remark 2.84. Although a taut submanifold is always Dupin, it need not be proper Dupin, as we see from Example 2.22 (page 33). In that example, the tube M^3 of sufficiently small radius ϵ over a torus of revolution $T^2 \subset \mathbf{R}^3 \subset \mathbf{R}^4$ is taut (see Remark 2.61), but it is not proper Dupin, since there are only two distinct principal curvatures on the set $T^2 \times \{\pm\epsilon\}$, but three distinct principal curvatures elsewhere on M.

To begin the proof of Theorem 2.83, let $M \subset \mathbf{R}^n$ be a connected taut submanifold of arbitrary codimension. To prove that M is Dupin, we must show that along any curvature surface the corresponding principal curvature is constant. As shown in Theorem 2.23, this is always true if the dimension of the curvature surface is greater than one. Thus, the proof consists in showing that along any 1-dimensional curvature surface (line of curvature), the corresponding principal curvature is constant.

Let γ be a line of curvature in M. By definition γ is a connected 1-dimensional submanifold of M for which there is a parallel (with respect to the normal connection $\nabla^{\perp}$) unit normal field ξ defined along γ such that for each $x \in \gamma$, the tangent space $T_x\gamma$ is a principal space of the shape operator A_ξ. Assuming that $\lambda(x) \neq 0$ for some $x \in \gamma$, the curvature sphere determined by λ at x is the hypersphere in $\mathbf{R}^n$ with center at the focal point

$$f_\lambda(x) = x + \frac{1}{\lambda(x)}\xi(x), \tag{2.136}$$

and radius $1/|\lambda(x)|$.

The following lemma is a generalization of the classical result that if the curvature of a plane curve has nonvanishing derivative on a parameter interval, then the corresponding one-parameter family of osculating circles is nested one within another (see, for example, Stoker [501, p. 31]).

Lemma 2.85. *Let $\gamma(s)$ be a unit speed parametrization of a line of curvature of a submanifold $M \subset \mathbf{R}^n$ with corresponding principal curvature function λ. Suppose that λ and its derivative λ' are both nonzero along γ. Then along γ, the family of curvature spheres determined by λ is nested.*

Proof. By appropriate choice of sign of the parallel unit normal field ξ and the direction of the unit speed parametrization, we can assume that $\lambda < 0$ and $\lambda' > 0$ on γ, where the prime denotes differentiation with respect to s. Let $\xi(s)$ denote the normal vector field $\xi(\gamma(s))$. Let s_1 and s_2 be any two parameter values with $s_1 < s_2$, and let p_1 and p_2 be the λ-focal points of $x_1 = \gamma(s_1)$ and $x_2 = \gamma(s_2)$, as in equation (2.136). Let $\alpha(s)$ be the evolute curve (focal curve)

$$\alpha(s) = \gamma(s) + \frac{1}{\lambda(s)}\xi(s). \tag{2.137}$$

Using the fact that $\nabla^{\perp}\xi = 0$, we can compute that the velocity vector $\overrightarrow{\xi}(s)$ to the curve $\xi(s)$ is given by

$$\overrightarrow{\xi}(s) = -A_\xi(\overrightarrow{\gamma}(s)) = -\lambda(s)\overrightarrow{\gamma}(s). \tag{2.138}$$

Using this, we calculate that the velocity vector to the curve $\alpha(s)$ is

$$\overrightarrow{\alpha}(s) = \left(\frac{1}{\lambda(s)}\right)'\xi(s). \tag{2.139}$$

Thus, the arc-length of the evolute curve from p_1 to p_2 is

$$\frac{1}{\lambda(s_1)} - \frac{1}{\lambda(s_2)} = \sigma - \rho, \tag{2.140}$$

where ρ is the Euclidean distance $d(x_1, p_1)$ and $\sigma = d(x_2, p_2)$. The left side of equation (2.140) equals $\sigma - \rho$, since $\lambda(s_1)$ and $\lambda(s_2)$ are both negative. We know that α is not a straight line segment, since ξ is not constant along γ. Thus, $d(p_1, p_2) < \sigma - \rho$, and by the triangle inequality, the closed ball $B_\rho(p_1)$ with center p_1 and radius ρ is contained in the interior of the closed ball $B_\sigma(p_2)$. □

Proof (of Theorem 2.83). As noted earlier, to prove that M is Dupin we must show that if γ is any line of curvature on M, then the corresponding principal curvature λ is constant along γ. If λ is identically zero on γ, then λ is constant along γ as needed. Otherwise, there exists a unit speed parametrization $\gamma(s)$ on a real parameter interval (a, b) with $\lambda(s) < 0$ and $\lambda'(s) > 0$ for all $s \in (a, b)$, as in Lemma 2.85. For each s in the interval (a, b), let B_s be the closed ball of radius $1/|\lambda(s)|$ centered at the λ-focal point $\alpha(s)$ given in equation (2.137). Let

$$\beta(s) = \dim H_*(M \cap B_s, \mathbf{Z}_2). \tag{2.141}$$

By the tautness of M, the number $\beta(s)$ is a finite integer for each s in (a, b). We will obtain a contradiction by proving that the function $\beta(s)$ is strictly increasing on the parameter interval (a, b), which is clearly impossible for an integer-valued function.

To see this, let s_1 and s_2 be any two parameter values in (a, b) with $s_1 < s_2$, and let B_1 and B_2 be the corresponding closed balls centered at the λ-focal points $\alpha(s_1)$ and $\alpha(s_2)$, respectively. We will prove that the homomorphism,

$$j : H_*(M \cap B_1) \to H_*(M \cap B_2), \tag{2.142}$$

induced by the inclusion $B_1 \subset B_2$ is injective, but not surjective, and thus $\beta(s_1) < \beta(s_2)$.

The injectivity of the map j follows immediately from the tautness of M, since the injective map

$$H_*(M \cap B_1) \to H_*(M), \tag{2.143}$$

factors through the sequence

$$H_*(M \cap B_1) \xrightarrow{j} H_*(M \cap B_2) \to H_*(M). \tag{2.144}$$

To show that j is not surjective, consider any parameter value s_0 with $s_1 < s_0 < s_2$. Let $p_0 = \alpha(s_0)$ denote the λ-focal point of $\gamma(s_0)$, and let B_0 be the closed ball centered at p_0 of radius $1/|\lambda(s_0)|$. Let q be a point on the normal ray from $\gamma(s_0)$ to p_0 such that q is beyond p_0 and before the next focal point (if any exist) of $(M, \gamma(s_0))$ on the normal ray. The point q can be chosen arbitrarily close to p_0. By Corollary 2.52, there exists a point $p \in \mathbf{R}^n$ arbitrarily near to q (and hence to p_0 also) such that L_p is a Morse function having a nondegenerate critical point x arbitrarily near to $\gamma(s_0)$ and no other critical points at the same level. Let $r = d(p, x)$. By Lemma 2.85, we have

$$B_1 \subset \operatorname{int}(B_0), \quad B_0 \subset \operatorname{int}(B_2), \tag{2.145}$$

where $\text{int}(B_0)$ denotes the interior of B_0. Since p and x can be chosen arbitrarily close to p_0 and $\gamma(s_0)$, respectively, there exists a $\delta > 0$ such that

$$B_1 \subset \text{int}(B_{r-\delta}(p)), \quad B_{r+\delta}(p) \subset \text{int}(B_2). \tag{2.146}$$

Let k be the index of L_p at the critical point x. Since M is taut, the k-th Betti number increases by one as the critical point x is passed, and thus the homomorphism

$$H_k(M \cap B_{r-\delta}(p)) \to H_k(M \cap B_{r+\delta}(p)) \tag{2.147}$$

is injective but not surjective. The map j factors through the sequence of homomorphisms induced by inclusions

$$H_k(M \cap B_1) \to H_k(M \cap B_{r-\delta}(p)) \to H_k(M \cap B_{r+\delta}(p)) \to H_k(M \cap B_2).$$

By tautness, all of the maps in this sequence are injective, but the middle one is not surjective, as shown in equation (2.147). Thus, the map j is not surjective, and so $\beta(s_1) < \beta(s_2)$. This is true for all $s_1 < s_2$ in the interval (a, b), which is impossible for the integer-valued function β. This completes the proof of Theorem 2.83. □

Ozawa's Theorem

In the case where M is compact, we can use a theorem of Ozawa [421] to obtain a result which is slightly stronger than Theorem 2.83, as was noted in [76]. Note that the definition of a Dupin hypersurface in Section 2.5 does not require that given a principal space T_λ at a point $x \in M$, there exists a curvature surface S through x whose tangent space at x is T_λ. However, using the following result of Ozawa [421], we can show that tautness does imply that this property holds on M (see Corollary 2.88 below). We first state Ozawa's result and then use it to derive this corollary. Ozawa proved his result using Morse–Bott critical point theory (see [49]) and a careful analysis of the critical submanifolds, and we refer the reader to Ozawa's paper for a complete proof.

Theorem 2.86. *Let M be a taut compact, connected submanifold of $\mathbf{R}^n$, and let L_p be a Euclidean distance function on M. Let $x \in M$ be a critical point of L_p and let S be the connected component of the critical set of L_p which contains x. Then S is*

(a) *a smooth compact manifold of dimension equal to the nullity of the Hessian of L_p at the critical point x,*
(b) *nondegenerate as a critical manifold,*
(c) *taut in $\mathbf{R}^n$.*

Part (a) of the theorem implies that for each $p \in \mathbf{R}^n$, the critical set of L_p is a union of smooth, compact submanifolds of $\mathbf{R}^n$. Note that the critical set of L_p is the pre-image of p under the normal exponential map of the submanifold M. Thus, part (a) of the theorem implies that for each $p \in \mathbf{R}^n$, the pre-image of p under the normal exponential map is a union of submanifolds.

Remark 2.87 (Taut embeddings into complete Riemannian manifolds). Using different approaches, Grove and Halperin [185]), and independently, Terng and Thorbergsson [531], extended the notion of tautness to properly embedded submanifolds of complete Riemannian manifolds. Specifically, a submanifold M of a complete Riemannian manifold N is said to be taut if there exists a field $\mathbf{F}$ such that each energy functional:

$$E_p(\gamma) = \int_0^1 |\gamma'(t)|^2 dt, \tag{2.148}$$

on the space $\mathcal{P}(N, M \times p)$ of H^1-paths $\gamma : [0, 1] \to N$ from M to a fixed point $p \in N$ is a perfect Morse function with respect to $\mathbf{F}$, if p is not a focal point of M. (Here a path is H^1 if it is absolutely continuous and the length of its derivative is square integrable.)

This definition can be shown to agree with the usual definition of tautness for submanifolds of Euclidean space. Terng and Thorbergsson [531] showed that many of the important properties of taut embeddings into Euclidean space have natural analogues in this more general setting.

In a recent paper, Wiesendorf [554] proved that a compact, connected submanifold M embedded in a complete Riemannian manifold N is taut if and only if for each point p in N, the pre-image of p under the normal exponential map of M is a union of submanifolds, as in Ozawa's Theorem above.

Wiesendorf also proved that if M is taut with respect to any field $\mathbf{F}$, then M is also taut with respect to $\mathbf{Z}_2$. In addition, Wiesendorf proved several results concerning singular Riemannian foliations, all of whose leaves are taut (see also Lytchak [338, 339], Lytchak and Thorbergsson [340, 341]).

In the context of taut submanifolds of complete Riemannian manifolds, Taylor [524] gave a classification of immersions of S^{n-1} into a complete Riemannian manifold N^n which have odd order in homotopy and are taut. (See also Hebda [191, 192], Kahn [232], and Ruberman [467] for related results.)

Using Ozawa's theorem, we can prove the following corollary (as in [76, p. 154]).

Corollary 2.88. *Let M be a taut compact, connected submanifold of $\mathbf{R}^n$. Then*

(a) *M is a Dupin submanifold.*
(b) *Given a principal space T_λ of a shape operator A_ξ at a point $x \in M$, there exists a curvature surface S through x whose tangent space at x is equal to T_λ, and λ is constant along S.*

Proof. Note that part (b) implies part (a), so we will prove part (b). Let $f : M \to \mathbf{R}^n$ be a taut embedding. Let ξ be a unit normal vector at an arbitrary point $x \in M$, and let λ be a principal curvature of A_ξ. We first consider the case where $\lambda \neq 0$. Let

$p = f(x) + (1/\lambda)\xi$ be the focal point of (M,x) determined by the principal curvature λ of A_ξ. Then the distance function L_p has a degenerate critical point at x and the nullity of the Hessian of L_p at x is equal to the multiplicity m of λ as an eigenvalue of A_ξ (see [359, p. 36]). By Ozawa's theorem, the connected component S of the critical set of L_p containing x is a smooth submanifold (a critical submanifold) of dimension m. We will now show that S is the desired curvature surface and that the corresponding principal curvature is constant along S.

The function L_p has a constant value, which is $1/\lambda^2$, on the critical submanifold S. Thus, for every point $y \in S$, the vector $p - f(y)$ is normal to $f(M)$ at $f(y)$, and it has length $1/|\lambda|$. So we can extend the normal vector ξ to a unit normal vector field to $f(M)$ along S, which we also denote by ξ, by setting $\xi(y) = \lambda(p - f(y))$. Note that p is a focal point of (M,y) for every point $y \in S$, and Ozawa's theorem implies that the number λ is an eigenvalue of $A_{\xi(y)}$ of multiplicity $m = \dim S$ for every point $y \in S$. Thus, the principal curvature λ is constant along S. We next show that T_yS equals the principal space $T_\lambda(y)$ at each point $y \in S$, and that the normal field ξ is parallel along S with respect to the normal connection. Consider the focal map,

$$f_\lambda(y) = f(y) + \frac{1}{\lambda}\xi(y),$$

for $y \in S$. Then $f_\lambda(y) = p$ for all $y \in S$. Let X be any tangent vector to S at any point $y \in S$. Then $(f_\lambda)_*X = 0$, since f_λ is constant on S. On the other hand,

$$(f_\lambda)_*X = f_*X + \frac{1}{\lambda}\xi_*X,$$

and $\xi_*X = D_X\xi = f_*(-A_\xi X) + \nabla_X^\perp\xi$. Therefore,

$$(f_\lambda)_*X = f_*(X - \frac{1}{\lambda}A_\xi X) + \frac{1}{\lambda}\nabla_X^\perp\xi.$$

Since $(f_\lambda)_*X = 0$, we see that $A_\xi X = \lambda X$ and $\nabla_X^\perp\xi = 0$. Thus, ξ is parallel along S and $T_yS \subset T_\lambda(y)$. Since T_yS and $T_\lambda(y)$ have the same dimension, they are equal. So S is the curvature surface through y corresponding to the principal curvature λ, which is constant along S.

Now suppose that $\lambda = 0$ is an eigenvalue of A_ξ at x. Let $\sigma : \mathbf{R}^{n+1} - \{q\} \to \mathbf{R}^{n+1} - \{q\}$ be an inversion, as in equation (2.50), centered at a point $q \in \mathbf{R}^n$ chosen so that $q \notin f(M)$, and so that the principal curvature μ of the embedding $\sigma f : M \to \mathbf{R}^n$ corresponding to λ by Theorem 2.6 is not zero. Since σf is taut by Theorem 2.70 and $\mu \neq 0$, the argument above shows that there exists a curvature surface V of σf through x whose tangent space at x is equal to T_μ, and μ is constant along V. Applying the inversion σ again, we get a curvature surface $S = \sigma(V)$ corresponding to the principal curvature λ of $f = \sigma^2 f$, and λ is constant along S, as needed in part (b) of the theorem. This completes the proof. □

Remark 2.89 (On the relationship between taut and "semi-Dupin"). In the book [95, p. 189], a Dupin (but not necessarily proper Dupin) hypersurface which satisfies Condition (b) in Corollary 2.88 was called "semi-Dupin." Corollary 2.88 gives an affirmative answer to one direction of Conjecture 6.19 in [95, p. 189], that is, taut implies semi-Dupin for a compact, connected submanifold of $\mathbf{R}^n$. Perhaps the converse can be proved using the approach of Wiesendorf [554] .

2.9 Standard Embeddings of Projective Spaces

In this section, we consider the standard embeddings of projective spaces into Euclidean space. These are important in the theory of tight and taut submanifolds, as well as in the theory of isoparametric hypersurfaces (see Subsection 3.8.3, page 151), and we will present some of the associated results here also. This section is based on the paper of Tai [505] (see also Section 9 of Chapter 1 of [95, pp. 87–98]).

As noted in Theorem 2.46 on page 48, Kuiper [300] showed that if $f : M^n \to \mathbf{R}^m$ is a substantial TPP immersion, then $m \leq n(n+3)/2$. In a much deeper result, he also showed that a substantial TPP immersion $f : M^2 \to \mathbf{R}^5$ (so having maximal codimension) is a Veronese surface (see Remark 2.45), which is a standard embedding $f : \mathbf{RP}^2 \to \mathbf{R}^5$, up to a projective transformation (as defined in Remark 2.38). Kuiper's result was then generalized by Little and Pohl [333], who showed that a TPP immersion $f : M^n \to \mathbf{R}^m$, $m = n(n+3)/2$, is a standard embedding of $\mathbf{RP}^n$, up to projective transformation. (See also [95, pp. 98–108] for a proof of the result of Little and Pohl).

Kuiper and Pohl [307] also generalized some of these results to the topological category by proving that if $f : \mathbf{RP}^2 \to \mathbf{R}^m$, $m \geq 5$, is a substantial TPP topological embedding, then $m = 5$, and f is either a smooth standard embedding (up to projective transformation) or the TPP polyhedral embedding of Banchoff [22] (see also Example 5.21 of [95, p. 37]).

We now begin our presentation of the standard embeddings, following the approach and using the notation of Tai [505]. Let $\mathbf{F}$ be one of the division algebras, $\mathbf{R}$, $\mathbf{C}$ or $\mathbf{H}$ (quaternions). For $q \in \mathbf{H}$, we can write,

$$q = r_0 + r_1 i + r_2 j + r_3 k, \tag{2.149}$$

where r_0, r_1, r_2, r_3 are real numbers, and the *conjugate* of q is defined by

$$\bar{q} = r_0 - r_1 i - r_2 j - r_3 k. \tag{2.150}$$

The norm of q is given by $|q| = (q\bar{q})^{1/2}$. If $q \in \mathbf{C}$, then $\bar{q}$ is the usual complex conjugate, and if $q \in \mathbf{R}$, then $\bar{q} = q$. We let $d = 1, 2, 4$, respectively, for the algebras $\mathbf{R}$, $\mathbf{C}$, $\mathbf{H}$. If A is a matrix with coefficients in $\mathbf{F}$, we define $A^* = \bar{A}^T$, where A^T denotes the transpose of A. Then it is easy to check that the following two equations hold, whenever the indicated operations make sense,

$$(AB)^* = B^*A^*, \tag{2.151}$$

$$\Re\,(\text{trace}\,(AB)) = \Re\,(\text{trace}\,(BA)). \tag{2.152}$$

Here $\Re$ denotes the real part.

Let $M(n+1, \mathbf{F})$ denote the space of all $(n+1) \times (n+1)$ matrices over $\mathbf{F}$. Let

$$H(n+1, \mathbf{F}) = \{A \in M(n+1, \mathbf{F}) \mid A^* = A\} \tag{2.153}$$

be the space of *Hermitian* matrices over $\mathbf{F}$. If A is Hermitian, then the off-diagonal entries in A are in $\mathbf{F}$, while the diagonal entries are in $\mathbf{R}$. Thus, $H(n+1, \mathbf{F})$ is a real vector space with dimension given by

$$\dim H(n+1, \mathbf{F}) = \frac{n(n+1)d}{2} + n + 1. \tag{2.154}$$

Let

$$U(n+1, \mathbf{F}) = \{A \in M(n+1, \mathbf{F}) \mid AA^* = I\}. \tag{2.155}$$

Then for $\mathbf{F} = \mathbf{R}, \mathbf{C}, \mathbf{H}$, respectively, $U(n+1, \mathbf{F})$ is equal to $O(n+1)$, $U(n+1)$, $Sp(n+1)$, respectively.

The space $\mathbf{F}^{n+1}$ is a Euclidean space of real dimension $(n+1)d$. The usual Euclidean inner product on $\mathbf{F}^{n+1} = \mathbf{R}^{(n+1)d}$ is given by

$$\langle x, y \rangle = \Re\,(x^* y), \tag{2.156}$$

where x and y in $\mathbf{F}^{n+1}$ are considered as column vectors, such as,

$$x = \begin{bmatrix} x_0 \\ x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}, \tag{2.157}$$

and thus $x^* = (\bar{x}_0, \ldots, \bar{x}_n)$, a row vector. Then

$$\langle x, Ay \rangle = \langle A^* x, y \rangle, \tag{2.158}$$

for all $A \in M(n+1, \mathbf{F})$.

The space $M(n+1,\mathbf{F})$ can be considered as a Euclidean space of real dimension $(n+1)^2 d$, and the usual Euclidean inner product is given by

$$\langle A, B\rangle = \Re\ (\text{trace}\ (AB^*)), \tag{2.159}$$

for $A, B \in M(n+1,\mathbf{F})$. On the subspace $H(n+1,\mathbf{F})$, this simplifies to

$$\langle A, B\rangle = \Re\ (\text{trace}\ (AB)). \tag{2.160}$$

Let $S^{(n+1)d-1}$ be the unit sphere in $\mathbf{F}^{n+1}$, and let $\mathbf{FP}^n$ be the quotient space of $S^{(n+1)d-1}$ under the equivalence relation,

$$(x_0, \dots, x_n) \simeq (x_0\lambda, \dots, x_n\lambda), \quad \lambda \in \mathbf{F}, \quad |\lambda| = 1. \tag{2.161}$$

Consider the map from $S^{(n+1)d-1}$ into $H(n+1,\mathbf{F})$ given by

$$x \mapsto xx^* = \begin{bmatrix} |x_0|^2 & x_0\bar{x}_1 & \cdots & x_0\bar{x}_n \\ x_1\bar{x}_0 & |x_1|^2 & \cdots & x_1\bar{x}_n \\ \cdots & \cdots & \cdots & \cdots \\ x_n\bar{x}_0 & x_n\bar{x}_1 & \cdots & |x_n|^2 \end{bmatrix} \tag{2.162}$$

for x a column vector as in equation (2.157) with $|x| = 1$. Note that if $y = x\lambda$ for $\lambda \in \mathbf{F}$ with $|\lambda| = 1$, then $xx^* = yy^*$. Furthermore, if $xx^* = yy^*$, then multiplication of this equation by x on the right gives

$$x = yy^*x = y\lambda, \tag{2.163}$$

where $\lambda = y^*x$ is in $\mathbf{F}$ and $|\lambda| = 1$. Thus, the map in equation (2.162) induces a well-defined, injective map $\phi : \mathbf{FP}^n \to H(n+1,\mathbf{F})$.

The image of ϕ consists precisely of those matrices in $M(n+1,\mathbf{F})$ satisfying the equation,

$$A = A^* = A^2, \quad \text{rank}\, A = 1. \tag{2.164}$$

In fact, $\phi(x)$ is just the matrix representation of orthogonal projection of $\mathbf{F}^{n+1}$ onto the $\mathbf{F}$-line spanned by the vector x. One can verify that ϕ is a smooth immersion on $\mathbf{FP}^n$ by a direct calculation, or else deduce this fact as a consequence of the equivariance given in Theorem 2.91 below. Thus, ϕ is a smooth embedding of $\mathbf{FP}^n$ into $H(n+1,\mathbf{F})$. We can, and often will, consider $(x_0, \dots, x_n)$ to be homogeneous coordinates on $\mathbf{FP}^n$.

This embedding $\phi : \mathbf{FP}^n \to H(n+1,\mathbf{F})$ is often called the *standard embedding* of $\mathbf{FP}^n$ into the Euclidean space $H(n+1,\mathbf{F})$. In the case $\mathbf{F} = \mathbf{R}$, the formula in equation (2.162) agrees with the formula $x \mapsto xx^T$ for the Veronese embedding, but this is not true for $\mathbf{F} = \mathbf{C}$ or $\mathbf{H}$, since x^* does not equal x^T in those cases.

The condition $|x| = 1$ is equivalent to the condition trace $\phi(x) = 1$. Hence, the image of ϕ lies in the hyperplane in $H(n+1, \mathbf{F})$ given by the linear equation trace $A = 1$. We now show that the image of ϕ does not lie in any lower dimensional plane, and hence ϕ is a substantial map into the space

$$\mathbf{R}^N = \{A \in H(n+1, \mathbf{F}) \mid \text{trace}\, A = 1\}, \tag{2.165}$$

where

$$N = \frac{n(n+1)d}{2} + n. \tag{2.166}$$

For the remainder of this section, N will always have the value given in equation (2.166).

Theorem 2.90. *The standard embedding $\phi : \mathbf{FP}^n \to \mathbf{R}^N$ is substantial in $\mathbf{R}^N$, and its image lies in a metric sphere in $\mathbf{R}^N$.*

Proof. Let p be an arbitrary point in the unit sphere $S^{(n+1)d-1}$, and let X be a unit tangent vector to $S^{(n+1)d-1}$ at p. Consider the curve,

$$\alpha(t) = \cos t\, p + \sin t\, X. \tag{2.167}$$

Then

$$\phi_*(X) = \frac{d}{dt}[\alpha(t)\, \alpha^*(t)]|_{t=0} = pX^* + Xp^*. \tag{2.168}$$

Let $\{e_0, \ldots, e_n\}$ be the standard basis of $\mathbf{F}^{n+1}$ as a vector space over the field $\mathbf{F}$. If we take $p = e_i$ and $X = e_j u$, for $j \neq i$ and u a unit length element of $\mathbf{F}$, equation (2.168) implies that $\phi_*(X)$ is a matrix which is zero except for u in the (j, i) position and $\bar{u}$ in the (i, j) position. This shows that all off-diagonal elements of $H(n+1, \mathbf{F})$ occur as tangent vectors to ϕ.

If we take $p = e_0$, $X = e_j$ and evaluate at $t = \pi/4$, we get

$$\phi_*(X) = e_j e_j^* - e_0 e_0^*, \tag{2.169}$$

showing that all real diagonal matrices with trace zero also occur. Thus, ϕ embeds $\mathbf{FP}^n$ substantially into the Euclidean space $\mathbf{R}^N$ given in equation (2.165).

Finally, note that

$$\langle xx^*, xx^* \rangle = \text{trace}\, [(xx^*)^2] = \text{trace}\, [xx^*] = 1, \tag{2.170}$$

so that the image of ϕ lies in the intersection of $\mathbf{R}^N$ with the unit sphere in $M(n+1, \mathbf{F})$, which is a metric sphere in $\mathbf{R}^N$. □

We next show that the embedding $\phi : \mathbf{FP}^n \to H(n+1,\mathbf{F})$ is equivariant with respect to the linear action of $U(n+1,\mathbf{F})$ on $M(n+1,\mathbf{F})$ defined by

$$U(A) = UAU^*, \tag{2.171}$$

for $U \in U(n+1,\mathbf{F})$ and $A \in M(n+1,\mathbf{F})$. An elementary calculation shows that this group action preserves the inner product on $M(n+1,\mathbf{F})$. Further, we have

$$\phi(Ux) = (Ux)(Ux)^* = Uxx^*U^* = U(\phi(x)), \tag{2.172}$$

for $x \in \mathbf{FP}^n$ and $U \in U(n+1,\mathbf{F})$. Thus we have the following theorem.

Theorem 2.91. *The embedding $\phi : \mathbf{FP}^n \to H(n+1,\mathbf{F})$ is equivariant with respect to and invariant under the action of $U(n+1,\mathbf{F})$, i.e.,*

$$\phi(Ux) = U(\phi(x)) \in \phi(\mathbf{FP}^n), \tag{2.173}$$

for all $x \in \mathbf{FP}^n$, and $U \in U(n+1,\mathbf{F})$.

The standard embeddings are taut

As noted in Sections 2.6 and 2.7, the standard embeddings of projective spaces play a special role in the theory of tight and taut immersions of manifolds into Euclidean spaces. We now prove that these standard embeddings are taut, substantial embeddings of $\mathbf{FP}^n$ into $\mathbf{R}^N$.

Theorem 2.92. *The embedding $\phi : \mathbf{FP}^n \to H(n+1,\mathbf{F})$ is taut. Hence, the embedding $\phi : \mathbf{FP}^n \to \mathbf{R}^N$ is taut and substantial.*

Proof. We will prove that the embedding $\phi : \mathbf{FP}^n \to \mathbf{R}^N \subset H(n+1,\mathbf{F})$ is tight, and since ϕ is spherical by Theorem 2.90, ϕ is also taut by Theorem 2.69. We already know that the embedding $\phi : \mathbf{FP}^n \to \mathbf{R}^N$ is substantial by Theorem 2.90.

To establish the tightness if ϕ, we will prove that every nondegenerate linear height function l_A, for $A \in H(n+1,\mathbf{F})$, has the minimum number of critical points required by the Morse inequalities. Thus $\phi : \mathbf{FP}^n \to H(n+1,\mathbf{F})$ is tight. Since $\mathbf{R}^N$ is a Euclidean subspace of $H(n+1,\mathbf{F})$, every height function in $\mathbf{R}^N$ corresponds to a height function in $H(n+1,\mathbf{F})$, and so ϕ is also tight as an embedding into $\mathbf{R}^N$.

Let $A \in H(n+1,\mathbf{F})$, and let x be a point in the sphere $S^{(n+1)d-1}$. Then x is also a homogeneous coordinate vector of the point in $\mathbf{FP}^n$ corresponding to the $\mathbf{F}$-line in $\mathbf{F}^{n+1}$ determined by x. We compute the value of the linear height function l_A at x as,

$$l_A(x) = \langle A, \phi(x)\rangle = \langle A, xx^*\rangle = \Re \text{ trace } (Axx^*) = \Re \text{ trace } (x^*Ax) = \langle x, Ax\rangle.$$

Thus, if X is a tangent vector to the sphere at x, we have

$$Xl_A = \langle X, Ax\rangle + \langle x, AX\rangle = 2\langle X, Ax\rangle. \tag{2.174}$$

Therefore, l_A has a critical point at x if and only if $\langle X, Ax\rangle = 0$ for all X tangent to the sphere at x. This means that the vector Ax is normal to the sphere $S^{(n+1)d-1}$ at x, and so Ax is a real multiple of the vector x. Thus, the critical points of the height function l_A on the sphere correspond to real eigenvectors of the matrix A. The usual inductive process (maximizing $\langle x, Ax\rangle$) can be used to produce $n+1$ real eigenvalues (not necessarily distinct) of A, each with a d-dimensional eigenspace. In fact, if x and $\lambda \in \mathbf{R}$ are such that $Ax = \lambda x$, and u is a unit length element in $\mathbf{F}$, then $A(xu) = \lambda(xu)$, and thus xu is also an eigenvector of A corresponding to the real eigenvalue λ. However, for any given $x \in S^{(n+1)d-1}$, all the points xu in $S^{(n+1)d-1}$ determine the same point of the projective space $\mathbf{FP}^n$. Thus, l_A has precisely $n+1$ critical points on $\mathbf{FP}^n$ provided that the $n+1$ real eigenvalues of A are distinct.

We now compute the Hessian of l_A at a point x such that $Ax = \lambda x$ for $\lambda \in \mathbf{R}$. Let X and Y be tangent to the sphere at x. To get $H(X, Y)$, we differentiate equation (2.174) in the direction Y. We use the decomposition of the Euclidean covariant derivative:

$$D_Y X = \nabla_Y X - \langle X, Y\rangle x, \tag{2.175}$$

where $\nabla_Y X$ is the component tangent to the sphere $S^{(n+1)d-1}$, and the normal component is $-\langle X, Y\rangle x$. We extend X to a vector field tangent to the sphere in a neighborhood of x and then differentiate the expression $2\langle X, Ax\rangle$ in the direction Y to get

$$\begin{aligned} H(X, Y) &= Y(2\langle X, Ax\rangle) = 2(\langle D_Y X, Ax\rangle + \langle X, AY\rangle) \\ &= 2\,(\langle \nabla_Y X, Ax\rangle - \langle X, Y\rangle\langle x, Ax\rangle + \langle X, AY\rangle) \\ &= 2\,(-\langle X, Y\rangle\lambda + \langle AX, Y\rangle) = 2\,\langle (A - \lambda I)X, Y\rangle, \end{aligned} \tag{2.176}$$

since the term $\langle \nabla_Y X, Ax\rangle$ equals zero, because $\nabla_Y X$ is tangent to the sphere, while $Ax = \lambda x$ is normal to the sphere at x. Equation (2.176) shows that the Hessian is nondegenerate if and only if all of the eigenvalues of A with eigenspaces orthogonal to the $\mathbf{F}$-line determined by x are distinct from λ. In particular, l_A is a Morse function on $\mathbf{FP}^n$ if and only if all $n+1$ eigenvalues are distinct. In that case, a consideration of the Hessian shows that l_A has one critical point of index k for each of the following values,

$$k = 0, d, 2d, \ldots, nd. \tag{2.177}$$

Thus, every Morse function of the form l_A has $n+1$ critical points with indices given in equation (2.177). This shows that the embedding $\phi : \mathbf{FP}^n \to H(n+1, \mathbf{F})$ is tight. In the case of $\mathbf{F} = \mathbf{R}$, this follows from the well-known fact that the $\mathbf{Z}_2$-Betti numbers of $\mathbf{RP}^n$ are as follows:

$$\beta_i(\mathbf{RP}^n, \mathbf{Z}_2) = 1, \ \text{for } 0 \leq i \leq n. \tag{2.178}$$

In the cases of $\mathbf{F} = \mathbf{C}$ or $\mathbf{F} = \mathbf{H}$, the construction of a Morse function on $\mathbf{FP}^n$ having exactly one critical point of index k for each k in equation (2.177) and no other critical points determines the Betti numbers of these spaces as follows,

$$\beta_i(\mathbf{FP}^n, \mathbf{Z}_2) = 1, \text{ for } i = 0, d, 2d, \ldots, nd, \text{ and 0 otherwise.} \tag{2.179}$$

This follows from the *lacunary principle* in Morse theory (see, for example, Morse–Cairns [379, p. 272] or Milnor [359, p. 31]), which states if a Morse function $f : M \to \mathbf{R}$ on a compact manifold has no critical points of index $i-1$ and no critical points of index $i+1$, then for any field $\mathbf{K}$, the $\mathbf{K}$-Betti numbers of M satisfy $\beta_{i-1} = \beta_{i+1} = 0$, and $\beta_i = \mu_i$, where μ_i is the number of critical points of f of index i on M. □

Tight embeddings of projective spaces

Kuiper [300] presented a variation of the standard embeddings of projective spaces due to H. Hopf [201] which gives tight substantial embeddings of $\mathbf{FP}^n$ into lower dimensional Euclidean spaces produced by composing ϕ with orthogonal projections onto certain subspaces of $H(n+1, \mathbf{F})$.

Theorem 2.93. *There exists a tight substantial embedding of* $\mathbf{FP}^n$ *into* $\mathbf{R}^m$ *for*

$$(2n-1)d + 1 \leq m \leq N, \text{ where } N = \frac{n(n+1)d}{2} + n.$$

Proof. The embeddings are obtained by projecting the standard embedding onto an appropriate subspace $\mathbf{R}^m$ of $H(n+1, \mathbf{F})$. Define the following quadratic functions in the homogeneous coordinates $(x_0, \ldots, x_n)$ of $\mathbf{FP}^n$,

$$z_k = \sum_{\substack{i+j=k \\ i \leq j}} x_i \bar{x}_j, \ k = 0, \ldots, 2n-1. \tag{2.180}$$

The values of z_k are real for $k = 0$ and are in $\mathbf{F}$ for $k > 0$. These functions are easily shown to be linearly independent, and so the mapping $\psi : \mathbf{FP}^n \to \mathbf{R}^K$, where $K = (2n-1)d + 1$, given by

$$\psi(x) = (z_0, \ldots, z_{2n-1}) \tag{2.181}$$

is a substantial map of $\mathbf{FP}^n$ into $\mathbf{R}^K$. Furthermore, the values of all the homogeneous coordinates $(x_0, \ldots, x_n)$ can be recovered by knowing $(z_0, \ldots, z_{2n-1})$, so the mapping ψ is injective on $\mathbf{FP}^n$. Finally, one can compute that ψ is an immersion, and thus ψ is a substantial embedding of $\mathbf{FP}^n$ into $\mathbf{R}^K$.

The embedding ψ is related to the standard embedding ϕ as follows. For each k, $0 \leq k \leq 2n-1$, let M_k be the matrix having a 1 in the (i,j) position for $i+j=k$, $i \leq j$, and zero elsewhere. The M_k are mutually orthogonal, and so we can write,

$$\psi(x) = \sum_{k=0}^{2n-1} z_k M_k/|M_k|. \tag{2.182}$$

Note that $\psi = \sigma \circ \phi$, where ϕ is the standard embedding and σ is the orthogonal projection of $H(n+1, \mathbf{F})$ onto the Euclidean subspace $\mathbf{R}^K$ determined by real multiples of M_0 and $\mathbf{F}$-multiples of the other M_k. By Remark 2.39 on page 42, the map $\psi = \sigma \circ \phi$ is tight, since it is an orthogonal projection of a tight map.

To obtain a tight substantial embedding of $\mathbf{FP}^n$ into $\mathbf{R}^m$ for $K < m < N$, one needs to adjoin appropriate coordinates of the embedding ϕ which are linearly independent from the coordinates of the embedding ψ, i.e., project $\phi(\mathbf{FP}^n)$ into a subspace $\mathbf{R}^m$ of $H(n+1, \mathbf{F})$ that contains $\mathbf{R}^K$. Such an embedding is tight and substantial for the same reasons as those given for ψ. □

Remark 2.94 (Taut embeddings of Grassmann manifolds). The standard embeddings of projective spaces can be generalized to produce taut embeddings of Grassmann manifolds over $\mathbf{F} = \mathbf{R}$, $\mathbf{C}$, or $\mathbf{H}$ into $\mathbf{R}^m$ (see, for example, Kuiper [303, p. 113]).

For projective planes, one can get even sharper results. From Theorem 2.93 with $n=2$, we get the existence of substantial tight embeddings of $\mathbf{FP}^2$ into $\mathbf{R}^m$ for

$$3d+1 \leq m \leq 3d+2. \tag{2.183}$$

In fact, we can also obtain taut embeddings of $\mathbf{FP}^2$ into $\mathbf{R}^m$ for these values of m as follows. First of all, the standard embedding ϕ of $\mathbf{FP}^2$ into $\mathbf{R}^{3d+2}$ is taut and spherical, as was shown in Theorems 2.90 and 2.92. By composing ϕ with stereographic projection with respect to a pole not in the image of ϕ, we obtain a taut, non-spherical embedding of $\mathbf{FP}^2$ into $\mathbf{R}^{3d+1}$ by Corollary 2.72.

Using methods similar to those employed in the proof of Theorem 2.92, Tai [505] showed that the analogous embedding of $\mathbf{OP}^2$ (Cayley projective plane) into $\mathbf{R}^{26}$ is tight and spherical, and thus taut. Again by Corollary 2.72, we can obtain a substantial non-spherical taut embedding of $\mathbf{OP}^2$ into $\mathbf{R}^{25}$ via stereographic projection.

Kuiper [302, pp. 215–217] proved that these are the only dimensions possible for tight substantial embeddings of these projective planes as follows.

Theorem 2.95. *There exist tight substantial embeddings of the projective planes $\mathbf{FP}^2$ into $\mathbf{R}^m$ for precisely the following dimensions.*

(a) $\mathbf{RP}^2$ *into* $\mathbf{R}^4$ *or* $\mathbf{R}^5$*,*
(b) $\mathbf{CP}^2$ *into* $\mathbf{R}^7$ *or* $\mathbf{R}^8$*,*
(c) $\mathbf{HP}^2$ *into* $\mathbf{R}^{13}$ *or* $\mathbf{R}^{14}$*,*
(d) $\mathbf{OP}^2$ *into* $\mathbf{R}^{25}$ *or* $\mathbf{R}^{26}$.

Proof. The existence of tight embeddings into the spaces listed in the theorem has been noted above. There do not exist embeddings of $\mathbf{FP}^2$ into lower dimensional Euclidean spaces because the normal Stiefel–Whitney class $\bar{w}_d(\mathbf{FP}^2) \neq 0$, where $d = 1, 2, 4, 8$ for $\mathbf{F} = \mathbf{R}, \mathbf{C}, \mathbf{H}, \mathbf{O}$, respectively. (See, for example, Husemöller [212, p. 263] and Borel–Hirzebruch [47, p. 533] for the case $\mathbf{F} = \mathbf{O}$.)

The upper bound in the case $\mathbf{F} = \mathbf{R}$ is given in Theorem 2.46 (due to Kuiper) on page 48 regarding tight immersions of maximal codimension. For the other division algebras $\mathbf{C}, \mathbf{H}, \mathbf{O}$, we need to use Theorem 2.48 (also due to Kuiper) on page 49 to obtain the upper bound as follows.

The $\mathbf{Z}_2$-Betti numbers of $\mathbf{FP}^2$ are known to be as follows,

$$\beta_i(\mathbf{FP}^2, \mathbf{Z}_2) = 1 \text{ for } i = 0, d, 2d, \quad \beta_i(\mathbf{FP}^2, \mathbf{Z}_2) = 0 \text{ for } i \neq 0, d, 2d. \tag{2.184}$$

By Theorem 2.48, we know that the substantial codimension of a tight smooth immersion is less than or equal to $c(\beta_0, \ldots, \beta_{2d})$, which is the maximal dimension of a linear family of symmetric bilinear forms in $2d$ variables which contains a positive definite form and such that no form of the family has index k if $\beta_k = 0$. Thus, we will complete the proof if we show that for the β_i given in equation (2.184), we have $c(\beta_0, \ldots, \beta_{2d}) = 4, 6, 10$, for $d = 2, 4, 8$, respectively.

The result that we need is contained in Hurwitz [211] (see also Kuiper [302, pp. 232–234]). There it is shown that the desired linear family of symmetric bilinear forms with maximal dimension can be represented by the set of symmetric matrices of the form

$$\begin{bmatrix} \lambda I & B \\ B^T & \mu I \end{bmatrix}, \tag{2.185}$$

where B is the 2×2, 4×4, or 8×8 matrix in the upper left corner of the matrix in equation (2.186) below, depending on whether $\mathbf{F} = \mathbf{C}, \mathbf{H}, \mathbf{O}$, respectively. From this, we see that $c(\beta_0, \ldots, \beta_{2d})$ has the desired values.

$$\begin{bmatrix} x_1 & -x_2 & -x_3 & -x_4 & -x_5 & -x_6 & -x_7 & -x_8 \\ x_2 & x_1 & -x_4 & x_3 & -x_6 & x_5 & -x_8 & x_7 \\ x_3 & x_4 & x_1 & -x_2 & -x_7 & x_8 & x_5 & -x_6 \\ x_4 & -x_3 & x_2 & x_1 & x_8 & x_7 & -x_6 & -x_5 \\ x_5 & x_6 & x_7 & -x_8 & x_1 & -x_2 & -x_3 & x_4 \\ x_6 & -x_5 & -x_8 & -x_7 & x_2 & x_1 & x_4 & x_3 \\ x_7 & x_8 & -x_5 & x_6 & x_3 & -x_4 & x_1 & -x_2 \\ x_8 & -x_7 & x_6 & x_5 & -x_4 & -x_3 & x_2 & x_1 \end{bmatrix} \tag{2.186}$$

□

Remark 2.96 (Manifolds which are like projective planes). Recall that the Morse number of a compact manifold M is the minimum number of critical points that any Morse function has on M. A compact, connected manifold with Morse number 3 was called a *manifold which is like a projective plane* by Eells and Kuiper [144], who gave many examples of such manifolds M^{2k}, all necessarily of dimensions $2k = 2, 4, 8$, or 16. They are obtained from $\mathbf{R}^{2k}$ under compactification by a k-sphere. Of course, the projective planes $\mathbf{FP}^2$ for $\mathbf{F} = \mathbf{R}, \mathbf{C}, \mathbf{H}, \mathbf{O}$ are examples. Kuiper [303, p. 132] showed that if $f : M^{2k} \to \mathbf{R}^m$ is a tight substantial topological embedding of a manifold which is like a projective plane, then $m \leq 3k + 2$. Moreover, if $f : M^{2k} \to \mathbf{R}^{3k+2}$ is a tight smooth substantial embedding of a manifold like a projective plane, then M^{2k} is embedded as an algebraic submanifold. For $k = 1, 2$, respectively, Kuiper showed that M^{2k} is $\mathbf{RP}^2$, $\mathbf{CP}^2$, respectively, and f is a standard embedding up to a real projective transformation of $\mathbf{R}^{3k+2}$. The hypothesis of smoothness is necessary in these results, as the piecewise linear embeddings of $\mathbf{RP}^2$ into $\mathbf{R}^5$ due to Banchoff [19], and of $\mathbf{CP}^2$ into $\mathbf{R}^8$ due to Kühnel and Banchoff [299] show.

In Theorem 2.46 on page 48, we showed that the substantial codimension of a tight immersion of an n-manifold always satisfies the inequality $1 \leq k \leq n(n+1)/2$. In the following theorem, we show that every value k in this interval can be realized.

Theorem 2.97. *For every integer k satisfying $1 \leq k \leq n(n+1)/2$, there exists a tight substantial embedding of an n-dimensional manifold M into $\mathbf{R}^{n+k}$.*

Proof. For $k = 1$, we have the embedding of S^n as a metric hypersphere in $\mathbf{R}^{n+1}$. For $k = 2$, take the standard product embedding of $S^{n-1} \times S^1$ into $\mathbf{R}^{n+2}$, which is tight by Theorem 2.50 on page 51 concerning a product of tight immersions. More generally, for $2 \leq k \leq n$, we can take the standard product of S^{n-k+1} with $k-1$ copies of S^1,

$$S^{n-k+1} \times S^1 \times \cdots \times S^1 \subset \mathbf{R}^{n-k+2} \times \mathbf{R}^2 \times \cdots \times \mathbf{R}^2 = \mathbf{R}^{n+k}. \tag{2.187}$$

Finally, for codimensions $n+1 \leq k \leq n(n+1)/2$, we can use the tight embeddings of $\mathbf{RP}^n$ given in Theorem 2.93. □