Chapter 1
Introduction

A smooth real-valued function F defined on a Riemannian manifold $\tilde{M}$ is called an *isoparametric function* if both of its classical Beltrami differential parameters $\Delta_1 F = |\text{grad } F|^2$ and $\Delta_2 F = \Delta F$ (Laplacian of F) are smooth functions of F itself. That is, both of the differential parameters are constant on each level set of F. An *isoparametric family* of $\tilde{M}$ is the collection of level sets of a nonconstant isoparametric function F on $\tilde{M}$.

In the case where $\tilde{M}$ is a real space form $\mathbf{R}^n$, S^n or H^n (hyperbolic space), a necessary and sufficient condition for an oriented hypersurface $M \subset \tilde{M}$ to belong to an isoparametric family is that all of its principal curvatures are constant (see Section 3.1). Thus, an oriented hypersurface of a real space form $\tilde{M}$ is called an *isoparametric hypersurface* if it has constant principal curvatures.

For $\tilde{M}$ equal to $\mathbf{R}^n$ or H^n, the classification of isoparametric hypersurfaces is complete and relatively simple, but as Cartan [52–55] showed in a series of four papers in 1938–1940, the subject is much deeper and more complicated for hypersurfaces in the sphere S^n.

A hypersurface M^{n-1} in a real space form $\tilde{M}^n$ is *proper Dupin* if the number g of distinct principal curvatures is constant on M^{n-1}, and each principal curvature function is constant along each leaf of its corresponding principal foliation. This is an important generalization of the isoparametric property that traces back to the book of Dupin [143] published in 1822. Proper Dupin hypersurfaces have been studied effectively in the context of Lie sphere geometry.

The theories of isoparametric and Dupin hypersurfaces are beautiful and filled with well-known important examples, and they have been analyzed from several points of view: geometric, algebraic, analytic, and topological. In this book, we cover the fundamental framework of these theories, and we study the main examples in detail. We also give a comprehensive treatment of the extension of these theories to real hypersurfaces with special curvature properties in complex and quaternionic space forms. We now give a brief overview of the contents of the book.

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Chapter 2 contains important results from the theory of submanifolds of real space forms that are needed in our study of isoparametric and Dupin hypersurfaces. In Sections 2.1–2.4, we find formulas for the shape operators of parallel hypersurfaces and tubes over submanifolds, and we discuss the focal submanifolds of a given submanifold. This leads naturally to the notions of curvature surfaces and Dupin hypersurfaces in Section 2.5. There we prove Pinkall's [446] local result (Theorem 2.25) which states that given any positive integer g, and any positive integers $m_1, \ldots, m_g$ with $m_1 + \cdots + m_g = n - 1$, there exists a proper Dupin hypersurface M^{n-1} in $\mathbf{R}^n$ with g distinct principal curvatures having respective multiplicities $m_1, \ldots, m_g$.

In Sections 2.6 and 2.7, we cover some basic results concerning tight and taut immersions of manifolds into real space forms. These are fundamental ideas in themselves, and they are needed to develop certain important results in the theory of isoparametric and Dupin hypersurfaces. In Section 2.8, we study the close relationship between the concepts of taut and Dupin submanifolds. Finally, we close the chapter with a treatment of the standard embeddings of projective spaces into Euclidean spaces. These examples play a significant role in the theories of tight, taut, and isoparametric hypersurfaces.

Chapter 3 is devoted to the basic theory of isoparametric hypersurfaces in real space forms developed primarily by Cartan [53–56] and Münzner [381, 382] (first published as preprints in the early 1970s). In Section 3.1, we describe the aspects of the theory that are common to all three space forms, and then prove the classification of isoparametric hypersurfaces M^{n-1} in Euclidean space $\mathbf{R}^n$ and in hyperbolic space H^n using Cartan's formula involving the principal curvatures of M^{n-1}.

The rest of the chapter is devoted to the much more complicated theory of isoparametric hypersurfaces in the sphere S^n. In Sections 3.2–3.6, we present Münzner's theory, including the proof that an isoparametric hypersurface in $S^n \subset \mathbf{R}^{n+1}$ with g distinct principal curvatures is always contained in a level set of a homogeneous polynomial of degree g on $\mathbf{R}^{n+1}$ satisfying certain differential equations on the length of its gradient and its Laplacian. From this it can be shown that every connected isoparametric hypersurface in S^n is contained in a unique compact, connected isoparametric hypersurface in S^n.

Using Münzner's construction, it can also be shown that each compact, connected isoparametric hypersurface $M^{n-1} \subset S^n$ has two focal submanifolds of codimension greater than one. These codimensions are determined by the multiplicities of the principal curvatures of M^{n-1}. From this it follows that M^{n-1} separates S^n into two ball bundles over these two focal submanifolds. Münzner then used cohomology theory to show that this topological situation implies that the number g of distinct principal curvatures of M^{n-1} can only be $1, 2, 3, 4$, or 6. At approximately the same time as Münzner's work, Takagi and Takahashi [511] classified homogeneous isoparametric hypersurfaces and found examples having g distinct principal curvatures for each of the values $g = 1, 2, 3, 4$ or 6.

Thorbergsson [533] applied Münzner's theory to show that the number g of distinct principal curvatures of a compact proper Dupin hypersurface M^{n-1} embedded in S^n is always $1, 2, 3, 4$, or 6, since M^{n-1} separates S^n into two ball bundles over two focal submanifolds of M^{n-1}, as in the isoparametric case. Several authors then used this same topological information to find a complete list of possibilities for the multiplicities of the principal curvatures of a compact proper Dupin hypersurface in S^n. This is discussed in Section 3.7.

In Section 3.8, we describe many important examples of isoparametric hypersurfaces in S^n from various points of view, and we discuss many classification results that have been obtained. Then in Section 3.9, we give a thorough treatment of the important paper of Ferus, Karcher, and Münzner [160], who used representations of Clifford algebras to construct an infinite collection of isoparametric hypersurfaces with $g = 4$ principal curvatures, now known as isoparametric hypersurfaces of *FKM-type*. Many of the hypersurfaces of FKM-type are not homogeneous. At the end of that section (see Subsection 3.9.1), we discuss progress that has been made on the classification of isoparametric hypersurfaces with four principal curvatures.

Isoparametric hypersurfaces in spheres have also occurred in considerations of several concepts in Riemannian geometry, such as the spectrum of the Laplacian, constant scalar curvature, and Willmore submanifolds. These applications and others are discussed in Section 3.10.

Chapter 4 describes the method for studying submanifolds of Euclidean space $\mathbf{R}^n$ or the sphere S^n in the setting of Lie sphere geometry. For proper Dupin hypersurfaces this has proven to be a valuable approach, since Dupin hypersurfaces occur naturally as envelopes of families of spheres, which can be handled well in Lie sphere geometry. Since the proper Dupin condition is invariant under Lie sphere transformations, this is also a natural setting for classification theorems. In Section 4.6, we formulate the related notion of tautness in the setting of Lie sphere geometry and prove that it is also invariant under Lie sphere transformations. The material in this chapter is covered in more detail in Chapters 2–4 of the book [77].

In Chapter 5, we study proper Dupin hypersurfaces in a real space form $\tilde{M}^n$ in detail. As noted above, proper Dupin hypersurfaces can also be studied in the context of Lie sphere geometry, and many classification results have been obtained in that setting. In this chapter, we use the viewpoint of the metric geometry of $\tilde{M}^n$ and that of Lie sphere geometry to obtain results about proper Dupin hypersurfaces.

An important class of proper Dupin hypersurfaces consists of the isoparametric hypersurfaces in S^n, and those hypersurfaces in $\mathbf{R}^n$ obtained from isoparametric hypersurfaces in S^n via stereographic projection. For example, the well-known cyclides of Dupin in $\mathbf{R}^3$ are obtained from a standard product torus $S^1(r) \times S^1(s) \subset S^3$, $r^2 + s^2 = 1$, in this way. These examples are discussed in more detail in Section 5.5.

However, in contrast to the situation for isoparametric hypersurfaces, there are both local and global aspects to the theory of proper Dupin hypersurfaces with quite different results. As noted above, Thorbergsson [533] proved that the restriction $g = 1, 2, 3, 4,$ or 6 on the number of distinct principal curvatures of an isoparametric hypersurface in S^n also holds for a compact proper Dupin hypersurface M^{n-1} embedded in S^n. On the other hand, Pinkall [446] (see Theorem 2.25) showed that it is possible to construct a non-compact proper Dupin hypersurface with any number

g of distinct principal curvatures having any prescribed multiplicities. The proof involves Pinkall's standard constructions of building tubes, cylinders, and surfaces of revolution over lower dimensional Dupin submanifolds. These constructions are described in Section 5.1.

Pinkall's constructions lead naturally to proper Dupin hypersurfaces with the property that each point has a neighborhood with a local principal coordinate system, i.e., one in which the coordinate curves are principal curves. We discuss this type of hypersurface in Section 5.2. In particular, we show that if $M \subset S^n$ is an isoparametric hypersurface with $g \geq 3$ principal curvatures, then there does not exist any local principal coordinate system on M (see Pinkall [442, p. 42] and Cecil–Ryan [95, pp. 180–184]). We then give necessary and sufficient conditions for a hypersurface in a real space form with a fixed number g of distinct principal curvatures to have a local principal coordinate system in a neighborhood of each of its points.

An important notion in the local classification of proper Dupin hypersurfaces is reducibility. A proper Dupin hypersurface is called *reducible* if it is locally Lie equivalent to a proper Dupin hypersurface in $\mathbf{R}^n$ obtained as the result of one of Pinkall's standard constructions. In Section 5.3, we discuss reducible proper Dupin hypersurfaces in detail and develop Lie geometric criteria for reducibility.

In Section 5.4, we introduce the method of moving frames in Lie sphere geometry, which has been used to obtain local classifications of proper Dupin hypersurfaces with 2, 3, or 4 distinct principal curvatures. In Section 5.5, we use this method to give a complete local classification of proper Dupin hypersurfaces with $g = 2$ distinct principal curvatures, i.e., the cyclides of Dupin. This is a nineteenth century result for $n = 3$, and it was obtained in dimensions $n > 3$ by Pinkall [446] in 1985. In Sections 5.6 and 5.7, we discuss local classification results for the cases $g = 3$ and $g = 4$, respectively, that have been obtained using the moving frames approach.

As demonstrated by Thorbergsson's restriction on the number of distinct principal curvatures, compact proper Dupin hypersurfaces in S^n are relatively rare, and several important classification results have been obtained for them. These results are discussed in detail in Section 5.8 together with the important counterexamples of Pinkall-Thorbergsson [448] and Miyaoka–Ozawa [377] to the conjecture of Cecil and Ryan [95, p. 184] that every compact proper Dupin hypersurface embedded in S^n is Lie equivalent to an isoparametric hypersurface.

As noted earlier, the Dupin and taut conditions for submanifolds of real space forms are very closely related (see Section 2.8). In Sections 5.9 and 5.10, we discuss important classification results that have been obtained for taut submanifolds in Euclidean space $\mathbf{R}^n$. Many of these have been proven by using classifications of compact proper Dupin hypersurfaces.

The study of real hypersurfaces in complex projective space $\mathbf{CP}^n$ and complex hyperbolic space $\mathbf{CH}^n$ began at approximately the same time as Münzner's work on isoparametric hypersurfaces in spheres. A key early work was Takagi's [507] classification in 1973 of homogeneous real hypersurfaces in $\mathbf{CP}^n$. These hypersurfaces necessarily have constant principal curvatures, and they serve as model spaces for many subsequent classification theorems. Later Montiel [378] provided a similar list of standard examples in complex hyperbolic space $\mathbf{CH}^n$. These examples of Takagi and Montiel are presented in detail in Sections 6.3–6.5.

Let M be an oriented real hypersurface in $\mathbf{CP}^n$ or $\mathbf{CH}^n$, $n \geq 2$, with field of unit normals ξ. The hypersurface M is said to be *Hopf* if the *structure vector* $W = -J\xi$ is a principal vector at every point of M, where J is the complex structure on the ambient space. In that case, if $AW = \alpha W$, then α is called the *Hopf principal curvature* on M. A fundamental result is that the Hopf principal curvature α is always constant on a Hopf hypersurface M.

The hypersurfaces on the lists of Takagi and Montiel are Hopf hypersurfaces with constant principal curvatures. Furthermore, all tubes over complex submanifolds of $\mathbf{CP}^n$ or $\mathbf{CH}^n$ are Hopf. Conversely, if the rank of the focal map determined by the Hopf principal curvature is constant, then the image of that focal map is a complex submanifold of the ambient space. This was first shown by Cecil–Ryan [94] in $\mathbf{CP}^n$, and by Montiel [378] in $\mathbf{CH}^n$. These basic results concerning Hopf hypersurfaces are covered in Sections 6.6–6.8.

In Section 6.7, we study parallel hypersurfaces, focal sets and tubes over submanifolds of complex space forms using techniques similar to those used in Sections 2.2–2.4 for submanifolds of real space forms. This yields formulas that can be used to compute the principal curvatures of the hypersurfaces on the lists of Takagi and Montiel. In Section 6.9, we present an alternative approach to the study of parallel hypersurfaces and tubes using the method of Jacobi fields. This method has been effective in proving some important results in the field, and we will use it extensively.

Most of the examples on the lists of Takagi and Montiel are tubes over complex submanifolds. In Chapter 7, we study the basic geometry of complex submanifolds in complex space forms, and we focus on certain important examples in $\mathbf{CP}^n$ that arise in the classification of Hopf hypersurfaces with constant principal curvatures. Specifically, in Sections 7.2–7.5, we determine the behavior of the principal curvatures of the Veronese embedding of $\mathbf{CP}^m$ in $\mathbf{CP}^n$, the Segre embedding of $\mathbf{CP}^h \times \mathbf{CP}^k$ in $\mathbf{CP}^n$, the Plücker embedding of complex Grassmannians in $\mathbf{CP}^n$, and the half-spin embedding of $SO(2d)/U(d)$ in $\mathbf{CP}^n$.

In Chapter 8, we present the classification of Hopf hypersurfaces with constant principal curvatures. This is due to Kimura [270] in $\mathbf{CP}^n$ and to Berndt [27] in $\mathbf{CH}^n$. Simply stated, these theorems say that any connected Hopf hypersurface in a complex space form is an open subset of a hypersurface on Takagi's list for $\mathbf{CP}^n$, and on Montiel's list for $\mathbf{CH}^n$. These classifications are major results in the field.

In the case of $\mathbf{CH}^n$, the classification follows from a generalization to complex space forms of Cartan's formula for isoparametric hypersurfaces in real space forms (see Section 8.1). The proof of the classification theorem in the case of $\mathbf{CP}^n$ is more involved. A Hopf hypersurface M in $\mathbf{CP}^n$ with constant principal curvatures gives rise to an isoparametric hypersurface $\tilde{M} = \pi^{-1}M$ in the sphere S^{2n+1}, where $\pi : S^{2n+1} \to \mathbf{CP}^n$ is the Hopf fibration. By Münzner's results, the number $\tilde{g}$ of principal curvatures of $\tilde{M}$ can only be $1, 2, 3, 4$, or 6. By a careful analysis of the relationship between the principal curvatures of M and those of $\tilde{M}$, one can show that the number

g of principal curvatures of M is either $2, 3$ or 5, the same as for the hypersurfaces on Takagi's list. When $g = 2$ or 3, we prove that M is an open subset of a hypersurface on Takagi's list by an elementary argument involving the shape operators of the complex focal submanifold of M corresponding to the Hopf principal curvature α.

In the case $g = 5$, the classification is much more difficult. We first prove that the complex focal submanifold determined by α is a parallel submanifold, i.e., it has parallel second fundamental form. Such parallel complex submanifolds of $\mathbf{CP}^n$ were classified by Nakagawa and Takagi [391], and the list includes the special embeddings studied in Chapter 7. Using the analysis of the shape operators of these parallel submanifolds in Sections 7.2–7.5, we ultimately determine which parallel submanifolds have tubes with constant principal curvatures. From this we can deduce that every connected Hopf hypersurface with constant principal curvatures in $\mathbf{CP}^n$ is an open subset of a hypersurface on Takagi's list.

In Section 8.5, we study other characterizations of the hypersurfaces on the lists of Takagi and Montiel. In particular, a real hypersurface M in $\mathbf{CP}^n$ or $\mathbf{CH}^n$ is said to be *pseudo-Einstein* if there exist functions ρ and σ on M such that the Ricci tensor S of M satisfies the equation $SX = \rho X + \sigma \langle X, W \rangle W$, for all tangent vectors X to M, where W is the structure vector defined above.

Of course, if σ is identically zero, then M is Einstein, but there do not exist any Einstein real hypersurfaces in $\mathbf{CP}^n$ or $\mathbf{CH}^n$. For $n \geq 3$, Cecil and Ryan [94] proved in 1982 that a pseudo-Einstein hypersurface in $\mathbf{CP}^n$ is an open subset of a geodesic sphere, a tube of a certain radius over a totally geodesic $\mathbf{CP}^k$, $1 \leq k \leq n-2$, or a tube of a certain radius over a complex quadric $Q^{n-1} \subset \mathbf{CP}^n$. All of these hypersurfaces are on Takagi's list. M. Kon [289] obtained the same conclusion in 1979 under the assumption that the functions ρ and σ are constant.

In 1985, Montiel [378] showed that a pseudo-Einstein hypersurface in $\mathbf{CH}^n$, $n \geq 3$, is an open subset of a geodesic sphere, a tube over a complex hyperplane, or a horosphere. It is important to note that the classifications of pseudo-Einstein hypersurfaces in $\mathbf{CP}^2$ by H.S. Kim and Ryan [260], and in $\mathbf{CH}^2$ by Ivey and Ryan [222], are different than the classification theorems for $n \geq 3$. These classifications of pseudo-Einstein hypersurfaces will be discussed in detail in Section 8.5. There we also study several other related classifications of hypersurfaces based on conditions on the shape operator, the curvature tensor, or the Ricci tensor.

In Section 8.6, we study non-Hopf hypersurfaces in complex space forms. These include the *Berndt orbits*, which are a family of non-Hopf homogeneous hypersurfaces in $\mathbf{CH}^n$ for $n \geq 2$ having three distinct constant principal curvatures. In Section 8.7, we discuss various ways to extend the definition of "isoparametric" to hypersurfaces in complex forms. These formulations of the concept are equivalent for hypersurfaces in real space forms, but different in complex space forms. In Section 8.8, we discuss some open problems that remain in the theory of real hypersurfaces in complex space forms.

In 1986, Martinez and Pérez [353] began the study of real hypersurfaces in quaternionic space forms, and in 1991 Berndt [28] found a list of standard examples of real hypersurfaces in quaternionic space forms with constant principal curvatures, leading to further research in this area. These examples together with classification results and open problems are described in Chapter 9.

In this book, all manifolds and maps are taken to be smooth, i.e., C^∞, unless explicitly stated otherwise. Notation generally follows the book of Kobayashi and Nomizu [283].