A Prime Analogue of Roth's Theorem in Function Fields

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Abstract Let $\mathbb{F}_q[t]$ denote the polynomial ring over the finite field $\mathbb{F}_q$, and let $\mathscr{P}_R$ denote the subset of $\mathbb{F}_q[t]$ containing all monic irreducible polynomials of degree R. For non-zero elements $\mathbf{r} = (r_1, r_2, r_3)$ of $\mathbb{F}_q$ satisfying $r_1 + r_2 + r_3 = 0$, let $D(\mathscr{P}_R) = D_{\mathbf{r}}(\mathscr{P}_R)$ denote the maximal cardinality of a set $A_R \subseteq \mathscr{P}_R$ which contains no non-trivial solution of $r_1x_1 + r_2x_2 + r_3x_3 = 0$ with $x_i \in A_R$ $(1 \leq i \leq 3)$. By applying the polynomial Hardy-Littlewood circle method, we prove that $D(\mathscr{P}_R) \ll_q |\mathscr{P}_R|/(\log\log\log\log |\mathscr{P}_R|)$.

1 Introduction

For $n \in \mathbb{N} = \{1, 2, \cdots\}$, let $D_3([1, n])$ denote the maximal cardinality of an integer subset of $[1, n]$ containing no non-trivial 3-term arithmetic progressions. In a fundamental paper, Roth [20] proved that $D_3([1, n]) \ll n/\log\log n$. His result was later improved by Heath-Brown [8], Szemerédi [24], Bourgain [3, 4] and Sanders [21, 22]. In 2014, Bloom [2] showed that $D_3([1, n]) \ll n(\log\log n)^4/\log n$, which gives the best upper bound up to date. Szemerédi [23] proved that subsets of the natural numbers with positive upper density contain arbitrarily long arithmetic progressions, and in 2001, Gowers [5] proved a quantitative version of Szemerédi's theorem.

One can consider analogous questions with $[1, n]$ replaced by $P[1, n]$, the set of positive primes up to n. Let $D_3(P[1, n])$ denote the maximal cardinality of an integer subset of $P[1, n]$ containing no non-trivial 3-term arithmetic progression, and let $\pi(n)$ denote the cardinality of $P[1, n]$. In [6], Green proved that

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$$D_3(P[1,n]) \ll \pi(n)\left(\frac{\log\log\log\log\log \pi(n)}{\log\log\log\log \pi(n)}\right)^{1/2}.$$

In [7], Green and Tao proved that subsets of the primes with positive upper density contain arbitrarily long arithmetic progressions.

Let $\mathbb{F}_q[t]$ denote the ring of polynomials over the finite field $\mathbb{F}_q$. For $R \in \mathbb{N} = \{1, 2, \ldots\}$, let $\mathscr{P}_R$ be the subset of $\mathbb{F}_q[t]$ containing all monic irreducible polynomials of degree R. Let $\mathbf{r} = (r_1, r_2, r_3)$ be non-zero elements of $\mathbb{F}_q$ satisfying $r_1 + r_2 + r_3 = 0$. Let $(x_1, x_2, x_3) \in \mathbb{F}_q[t]^3$ be a solution of $r_1x_1 + r_2x_2 + r_3x_3 = 0$. We say that (x_1, x_2, x_3) is a *trivial* solution if $x_1 = x_2 = x_3$. Otherwise, we say that (x_1, x_2, x_3) is a *non-trivial* solution. Let $D(\mathscr{P}_R) = D_{\mathbf{r}}(\mathscr{P}_R)$ denote the maximal cardinality of a set $A_R \subseteq \mathscr{P}_R$ for which there is no non-trivial solution of $r_1x_1 + r_2x_2 + r_3x_3 = 0$ with $x_i \in A_R$ $(1 \le i \le 3)$, and let $|\mathscr{P}_R|$ denote the cardinality of $\mathscr{P}_R$. In this paper, we prove the following theorem.

Theorem 1. *For $R \in \mathbb{N}$,*

$$D(\mathscr{P}_R) \ll_q \frac{|\mathscr{P}_R|}{\log\log\log\log |\mathscr{P}_R|}.$$

Here the implicit constant depends only on q.

In the special case that $\mathbf{r} = (1, -2, 1)$ and $\gcd(2, q) = 1$, the number $D(\mathscr{P}_R)$ denotes the maximal cardinality of a set $A_R \subseteq \mathscr{P}_R$ which contains no non-trivial 3-term arithmetic progression. In large part, this paper will follow the approach of Green. Our improvement over the analogous bound for $\mathbb{Z}$ stems from nice properties of Bohr sets in $\mathbb{F}_q[t]$ and the availability of a stronger bound for Roth's theorem in $\mathbb{F}_q[t]$ (see [14]) than in $\mathbb{Z}$. It is worth noting that when studying equations of the form $r_1x_1 + \cdots + r_sx_s = 0$ where $r_1 + \cdots + r_s = 0$ and $s \ge 4$, in [14], the authors proved that

$$D(\mathscr{P}_R) \ll_q \frac{|\mathscr{P}_R|}{(\log |\mathscr{P}_R|)^{s-3}},$$

which provides a strong bound compared to Theorem 1. Also, Lê has proved a function field analogue of Green and Tao's theorem on arithmetic progressions of primes (see [11]). While his method provides results about more general configurations in the irreducible polynomials of $\mathbb{F}_q[t]$, the approach of this paper produces stronger quantitative bounds on $D(\mathscr{P}_R)$. In addition, several estimates of exponential sums in this paper are essential to various additive combinatorial problems in function fields, including the results in [12].

In 2011, the above mentioned bound of Green was improved by Helfgott and de Roton [9] to

$$|\tilde{A}_R| \ll |\tilde{\mathscr{P}}_R| \frac{\log\log\log |\tilde{\mathscr{P}}_R|}{(\log\log |\tilde{\mathscr{P}}_R|)^{1/3}}.$$

Recently, Naslund [16] showed that for any $\epsilon > 0$,

$$|\tilde{A}_R| \ll |\tilde{\mathscr{P}}_R| \left(\frac{1}{\log \log |\tilde{\mathscr{P}}_R|} \right)^{1-\epsilon}.$$

In future work, we will show how their methods can be implemented over $\mathbb{F}_q[t]$ to improve Theorem 1.

2 Basic Setup

We start this section by introducing the Fourier analysis of $\mathbb{F}_q[t]$. Let $\mathbb{K} = \mathbb{F}_q(t)$ be the field of fractions of $\mathbb{F}_q[t]$, and let $\mathbb{K}_\infty = \mathbb{F}_q((1/t))$ be the completion of $\mathbb{K}$ at ∞. We may write each element $\alpha \in \mathbb{K}_\infty$ in the shape $\alpha = \sum_{i \le r} a_i t^i$ for some $r \in \mathbb{Z}$ and $a_i = a_i(\alpha) \in \mathbb{F}_q$ $(i \le r)$. If $a_r \ne 0$, we define $\operatorname{ord} \alpha = r$ and we write $\langle \alpha \rangle$ for $q^{\operatorname{ord} \alpha}$. We adopt the conventions that $\operatorname{ord} 0 = -\infty$ and $\langle 0 \rangle = 0$. Also, it is often convenient to refer to $a_{-1}(\alpha)$ as being the residue of α, an element of $\mathbb{F}_q$ that we denote by $\operatorname{res} \alpha$. For a real number R, we let $\hat{R}$ denote q^R. Hence, for $x \in \mathbb{F}_q[t]$, $\langle x \rangle < \hat{N}$ if and only if $\operatorname{ord} x < N$. Furthermore, we let $\mathbb{T}$ denote the compact additive subgroup of $\mathbb{K}_\infty$ defined by $\mathbb{T} = \{\alpha \in \mathbb{K}_\infty \colon \langle \alpha \rangle < 1\}$. Given any Haar measure $d\alpha$ on $\mathbb{K}_\infty$, we normalize it in such a manner that $\int_{\mathbb{T}} 1 \, d\alpha = 1$. Thus if $\mathfrak{N}$ is the subset of $\mathbb{K}_\infty$ defined by $\mathfrak{N} = \{\alpha \in \mathbb{K}_\infty \colon \operatorname{ord} \alpha < -N\}$, then the measure of $\mathfrak{N}$, $\operatorname{mes}(\mathfrak{N})$, is equal to $\hat{N}^{-1}$.

We are now equipped to define the exponential function on $\mathbb{F}_q[t]$. Suppose that the characteristic of $\mathbb{F}_q$ is p. Let $e(z)$ denote $e^{2\pi i z}$ and let $\operatorname{tr} : \mathbb{F}_q \to \mathbb{F}_p$ denote the familiar trace map. There is a non-trivial additive character $e_q : \mathbb{F}_q \to \mathbb{C}^\times$ defined for each $a \in \mathbb{F}_q$ by taking $e_q(a) = e(\operatorname{tr}(a)/p)$. This character induces a map $e : \mathbb{K}_\infty \to \mathbb{C}^\times$ by defining, for each element $\alpha \in \mathbb{K}_\infty$, the value of $e(\alpha)$ to be $e_q(\operatorname{res} \alpha)$. The orthogonality relation underlying the Fourier analysis of $\mathbb{F}_q[t]$, established in [10, Lemma 1], takes the shape

$$\int_{\mathbb{T}} e(h\alpha)\, d\alpha = \begin{cases} 1, & \text{when } h = 0, \\ 0, & \text{when } h \in \mathbb{F}_q[t] \setminus \{0\}. \end{cases}$$

For $N \in \mathbb{N}$, let $\mathscr{S}_N$ denote the subset of $\mathbb{F}_q[t]$ containing all monic polynomials of degree N. For $b, m \in \mathbb{F}_q[t]$ with m monic, $\langle b \rangle < \langle m \rangle \le N$ and $(b, m) = 1$, define a set

$$X = \Lambda_{b,m,N} = \{n \in \mathscr{S}_N \mid mn + b \text{ is irreducible}\} \tag{1}$$
$$\cong \{n' \in S_{N+\operatorname{ord} m} \mid n' \text{ is irreducible and } n' \equiv b \pmod{m}\}.$$

Thus by the prime number theorem in arithmetic progression in $\mathbb{F}_q[t]$ [19, Theorem 4.8],

$$|X| = \frac{\hat{N}\langle m\rangle}{(N + \operatorname{ord} m)\phi(m)} + O\left(\frac{\hat{N}^{1/2}\langle m\rangle^{1/2}}{N + \operatorname{ord} m}\right), \tag{2}$$

where $\phi(m) = \left|\left\{n \in \mathbb{F}_q[t] \,|\, \operatorname{ord} n < \operatorname{ord} m \text{ and } (n,m) = 1\right)\right\}\big|$. Define a function $\lambda_{b,m,N} : \mathscr{S}_N \to \mathbb{C}$ supported on X by setting

$$\lambda_{b,m,N}(n) = \begin{cases} \frac{(N+\operatorname{ord} m)\phi(m)}{\hat{N}\langle m\rangle}, & \text{when } n \in X, \\ 0, & \text{otherwise.} \end{cases}$$

In the following, we will abuse our notation and view $\lambda_{b,m,N}$ as a measure on X. By (2), we have

$$\lambda_{b,m,N}(X) = \sum_{n\in X} \lambda_{b,m,N}(n) = 1 + o(1).$$

For functions $h_1, h_2 : \mathscr{S}_N \to \mathbb{C}$, we define an inner product

$$\langle h_1, h_2\rangle_X = \sum_{n\in\mathscr{S}_N} h_1(n)\overline{h_2(n)}\lambda_{b,m,N}(n).$$

We will use the wedge symbol to denote the Fourier transforms on both $\mathbb{T}$ and $\mathscr{S}_N$. More precisely, for $f : \mathbb{T} \to \mathbb{C}$ and $h : \mathscr{S}_N \to \mathbb{C}$, the functions $f^\wedge : \mathscr{S}_N \to \mathbb{C}$ and $h^\wedge : \mathbb{T} \to \mathbb{C}$ are defined by

$$f^\wedge(n) = \int_{\mathbb{T}} f(\theta)e(-n\theta)\,d\theta \qquad \text{and} \qquad h^\wedge(\theta) = \sum_{n\in\mathscr{S}_N} h(n)e(n\theta).$$

Also, we define the convolution of two functions $f : \mathbb{T} \to \mathbb{C}$ and $g : \mathbb{T} \to \mathbb{C}$ to be

$$(f * g)(\rho) = \int_{\mathbb{T}} f(\theta)g(\rho - \theta)\,d\theta.$$

For any measure space Y, let $B(Y)$ denote the space of continuous functions on Y and define an operator $T : B(X) \to B(\mathbb{T})$ by

$$T : h \longmapsto (h\lambda_{b,m,N})^\wedge.$$

A dual operator $T^* : B(\mathbb{T}) \to B(X)$ of T is defined by

$$T^* : f \longmapsto f^\wedge|_X.$$

We have

$$\langle Th, f\rangle_{\mathbb{T}} = \langle h, T^* f\rangle_X.$$

Also, the map $TT^* : B(\mathbb{T}) \to B(\mathbb{T})$ is given by

$$TT^* : f \longmapsto f * \lambda^{\wedge}_{b,m,N}.$$

Furthermore, for an operator T and positive numbers a and b, we define

$$\|T\|_{a\to b} = \sup_f \frac{\|Tf\|_b}{\|f\|_a},$$

where $\|\cdot\|_a$ denotes the L^a norm and f ranges over continuous functions that map to $\mathbb{C}$. A main step in proving Theorem 1 will be deriving a restriction theorem for monic irreducible polynomials. Namely, we will prove the following theorem.

Theorem 2. *Suppose that $\delta > 2$ is a real number. Then there exists a constant $C(q,\delta)$, depending only on q and δ, such that*

$$\|T\|_{2\to\delta} \leq C(q,\delta)\hat{N}^{-1/\delta}.$$

As an application of Theorem 2, we are able to derive the Hardy-Littlewood majorant property for function fields. Namely, we will establish the following theorem.

Theorem 3. *Let $(a_x)_{x\in\mathscr{P}_R}$ be any sequence of complex numbers with $|a_x| \leq 1$ for all $x \in \mathscr{P}_R$. For a real number $\delta \geq 2$, we have*

$$\Big\| \sum_{x\in\mathscr{P}_R} a_x e(x\theta)\Big\|_\delta \leq C'(q,\delta)\Big\| \sum_{x\in\mathscr{P}_R} e(x\theta)\Big\|_\delta,$$

where $C'(q,\delta)$ is a constant depending only on q and δ.

Note that in the special case when δ is an even integer, by considering the underlying Diophantine equation, one can show that Theorem 3 holds with $C'(q,\delta) = 1$.

For a real number $\delta > 1$, let δ' denote the unique real number satisfying $1/\delta + 1/\delta' = 1$. Since

$$\begin{aligned}\|Tf\|_\delta &= \sup_{\|g\|_{\delta'}=1} \langle Tf, g\rangle = \sup_{\|g\|_{\delta'}=1} \langle f, T^* g\rangle \leq \|f\|_2 \sup_{\|g\|_{\delta'}=1} \|T^* g\|_2 \\ &= \|f\|_2 \sup_{\|g\|_{\delta'}=1} \langle g, TT^* g\rangle^{1/2} \leq \|f\|_2 \|TT^*\|^{1/2}_{\delta'\to\delta},\end{aligned} \tag{3}$$

to prove Theorem 2, it suffices to bound the quantity

$$\|TT^*\|_{\delta'\to\delta} = \sup_{\|f\|_{\delta'}=1} \|f * \lambda^{\wedge}_{b,m,N}\|_\delta. \tag{4}$$

In this paper, ϖ will be used to denote a monic irreducible polynomial. For a polynomial $x \in \mathbb{F}_q[t]$, we say that x is *$\hat{Q}$-rough* if for all monic irreducible polynomials ϖ with $\varpi | x$, we have $\langle \varpi \rangle > \hat{Q}$. For $Q \in \mathbb{N}$, define

$$\lambda_{b,m,N}^{(Q)}(n) = \begin{cases} \hat{N}^{-1} \prod_{\substack{\langle \varpi \rangle \le \hat{Q} \\ \varpi \nmid m}} \left(1 - 1/\langle \varpi \rangle\right)^{-1}, & \text{if } n \in \mathscr{S}_N \text{ and } mn + b \text{ is } \hat{Q}\text{-rough}, \\ 0, & \text{otherwise.} \end{cases}$$

By a sieve argument, one can show that

$$\sum_{n \in \mathscr{S}_N} \lambda_{b,m,N}^{(Q)}(n) = 1 + o(1).$$

Also, we define $\lambda_{b,m,N}^{(0)}(n) = 0$ for all $n \in \mathscr{S}_N$. Let $A = 4/(\delta - 2)$. For a positive integer $K = [A \log_q N]$ and $1 \le Q \le K$, let

$$\psi_Q = \lambda_{b,m,N}^{(Q)} - \lambda_{b,m,N}^{(Q-1)} \quad (1 \le Q \le K) \qquad \text{and} \qquad \psi_{K+1} = \lambda_{b,m,N} - \lambda_{b,m,N}^{(K)}.$$

Since $\sum_{i=1}^{K+1} \psi_i = \lambda_{b,m,N}$, by the triangle inequality, to bound $\|TT^*\|_{\delta' \to \delta}$, it suffices to consider

$$\sup_{\|f\|_{\delta'}=1} \|f * \psi_j^\wedge\|_\delta \quad (1 \le j \le K+1).$$

To obtain the above bound, we will apply the Riesz-Thorin interpolation theorem [17, 25] with the following bounds which we will prove in the next two sections:

$$\|f * \psi_Q^\wedge\|_\infty \ll_{q,\delta} \hat{Q}^{-1} \|f\|_1 \qquad \text{and} \qquad \|f * \psi_Q^\wedge\|_2 \ll_{q,\delta} N\hat{N}^{-1} \|f\|_2.$$

Notation For $k \in \mathbb{N}$, let $f(k)$ and $g(k)$ be functions of k. If $g(k)$ is positive and there exists a constant $c > 0$ such that $|f(k)| \le cg(k)$, we write $f(k) \ll g(k)$. In the following, all implicit constants depend at most on q and δ. In Sect. 6, while δ is fixed, all implicit constant depends at most on q. Throughout, the letter ϵ will denote a sufficiently small positive number. We adopt the convention that whenever ϵ appears in a statement, then we are implicitly asserting that for each $\epsilon > 0$, the statement holds for sufficiently large values of the main parameter. Note that the "value" of ϵ may consequently change from statement.

3 An L^2-L^2 Estimate

We first state Merten's theorem for $\mathbb{F}_q[t]$.

Lemma 4 ([13, Lemma 2]). *For $Q \in \mathbb{N}$, we have*

$$\prod_{\langle \varpi \rangle \leq \hat{Q}} \left(1 - 1/\langle \varpi \rangle\right)^{-1} \ll Q.$$

Lemma 5. *For a function $f : \mathbb{T} \to \mathbb{C}$ and $1 \leq Q \leq K$,*

$$\|f * \psi_Q^{\wedge}\|_2 \ll Q\hat{N}^{-1} \|f\|_2.$$

Also, one has

$$\|f * \psi_{K+1}^{\wedge}\|_2 \ll N\hat{N}^{-1} \|f\|_2.$$

Proof. Note that for $1 \leq Q \leq K + 1$,

$$\|f * \psi_Q^{\wedge}\|_2 = \|f^{\wedge}\psi_Q\|_2 \leq \|\psi_Q\|_\infty \|f^{\wedge}\|_2 = \|\psi_Q\|_\infty \|f\|_2.$$

For $1 \leq Q \leq K$, by Lemma 4,

$$\begin{aligned}\|\psi_Q\|_\infty &\leq \|\lambda_{b,m,N}^{(Q)}\|_\infty + \|\lambda_{b,m,N}^{(Q-1)}\|_\infty \\ &= \hat{N}^{-1} \prod_{\substack{\langle \varpi \rangle \leq \hat{Q} \\ \varpi \nmid m}} \left(1 - 1/\langle \varpi \rangle\right)^{-1} + \hat{N}^{-1} \prod_{\substack{\langle \varpi \rangle \leq \widehat{Q-1} \\ \varpi \nmid m}} \left(1 - 1/\langle \varpi \rangle\right)^{-1} \\ &\ll Q\hat{N}^{-1} + (Q-1)\hat{N}^{-1} \ll Q\hat{N}^{-1}.\end{aligned}$$

Similarly,

$$\|\psi_{K+1}\|_\infty \leq \|\lambda_{b,m,N}\|_\infty + \|\lambda_{b,m,N}^{(K)}\|_\infty \ll \frac{\phi(m)(N + \operatorname{ord} m)}{\hat{N}\langle m \rangle} + K\hat{N}^{-1} \ll N\hat{N}^{-1}.$$

Thus the lemma follows.

4 An L^1-L^∞ Estimate

For a function $f : \mathbb{T} \to \mathbb{C}$ and $1 \leq Q \leq K + 1$, we have

$$\|f * \psi_Q^{\wedge}\|_\infty \leq \|\psi_Q^{\wedge}\|_\infty \|f\|_1.$$

The goal of this section is to apply the Hardy-Littlewood circle method to establish the following proposition.

Proposition 6. *For* $1 \leq Q \leq K$*, we have*

$$\|\lambda_{b,m,N}^{\wedge} - \lambda_{b,m,N}^{(Q)\wedge}\|_{\infty} \ll \hat{Q}^{-1}.$$

Note that

$$\|\lambda_{b,m,N}^{\wedge} - \lambda_{b,m,N}^{(0)\wedge}\|_{\infty} = \|\lambda_{b,m,N}^{\wedge}\|_{\infty} \ll 1.$$

Thus by combining Proposition 6 with the triangle inequality, we obtain the following lemma.

Lemma 7. *For a function* $f : \mathbb{T} \to \mathbb{C}$ *and* $1 \leq Q \leq K+1$*,*

$$\|f * \psi_Q^{\wedge}\|_{\infty} \ll \hat{Q}^{-1}\|f\|_1.$$

Let $B = 2A + 12$. Note that for all $\alpha \in \mathbb{T}$, by Dirichlet's theorem for $\mathbb{F}_q[t]$ [10, Lemma 3], there exist $a, g \in \mathbb{F}_q[t]$ with g monic, $(a, g) = 1$, $\langle \alpha - a/g \rangle < N^B/(\langle g \rangle \hat{N})$ and $\langle g \rangle \leq \hat{N}/N^B$. We define the major arcs $\mathfrak{M}$ and the minor arcs $\mathfrak{m}$ as follow:

$$\mathfrak{M} = \bigcup_{\substack{\langle g \rangle \leq N^B \\ (a,g)=1 \\ g \text{ monic}}} \mathfrak{M}_{a,g} \qquad \text{and} \qquad \mathfrak{m} = \mathbb{T} \setminus \mathfrak{M},$$

where

$$\mathfrak{M}_{a,g} = \left\{\alpha \in \mathbb{T} \mid \langle \alpha - a/g \rangle < N^B/\langle g \rangle \hat{N}\right\}.$$

In order to prove Proposition 6, we will separate our analysis into major arc contributions and minor arc contributions.

4.1 Major Arc Estimates

In the following, we consider a function $h : \mathscr{S}_N \to \mathbb{C}$ which satisfies the following condition:

- **Condition*** Let $r, g \in \mathbb{F}_q[t]$ with g monic, $\langle r \rangle < \langle g \rangle$ and $\langle g \rangle \leq N^B$. Let $L = N - \lceil B \log_q N \rceil$. For $r' \in \mathscr{S}_N$ with $r' \equiv r \pmod{g}$, let

  $$Y = \left\{r' + lg \mid \langle l \rangle < \hat{L}\right\} \subseteq \mathscr{S}_N.$$

  Then

  $$\sum_{n \in Y} h(n) = \frac{\hat{L}}{\hat{N}}\left(\gamma_{r,g}(h) + O(E(h))\right),$$

  where $\gamma_{r,g}(h)$ is a constant depending on h and $E(h)$ is an error term of size $o(1)$.

Let

$$\varrho(\beta) = \hat{N}^{-1} \sum_{n \in \mathscr{S}_N} e(\beta n).$$

Lemma 8. *Suppose that $\langle \beta \rangle < N^B/\langle g \rangle \hat{N}$ and that $r, g \in \mathbb{F}_q[t]$ with g monic, $\langle r \rangle < \langle g \rangle$ and $\langle g \rangle \leq N^B$. For $h : \mathscr{S}_N \to \mathbb{C}$ satisfying Condition*, we have*

$$\sum_{\substack{n \in \mathscr{S}_N \\ n \equiv r (\mathrm{mod}\ g)}} h(n) e(\beta n) = \langle g \rangle^{-1} \gamma_{r,g}(h) \varrho(\beta) + O\big(\langle g \rangle^{-1} E(h)\big).$$

Proof. For $n \in \mathscr{S}_N$ with $n \equiv r \pmod g$, we can write $n = g(yt^L + l) + r$ with y monic, $\langle y \rangle = \hat{N}/\langle g \rangle \hat{L}$ and $\langle l \rangle < \hat{L}$. Moreover, for $\langle l \rangle < \hat{L}$, we have

$$\langle \beta(gl + r) \rangle < \frac{N^B}{\langle g \rangle \hat{N}} \cdot \langle g \rangle \cdot \frac{\hat{N}}{q^{1+\lceil B \log_q N \rceil}} \leq \frac{1}{q},$$

which implies that $e(\beta(gl + r)) = 1$. Thus by applying Condition* with $r' = gyt^L + r$,

$$\begin{aligned}
\sum_{\substack{n \in \mathscr{S}_N \\ n \equiv r (\mathrm{mod}\ g)}} h(n) e(\beta n) &= \sum_{\substack{\langle y \rangle = \hat{N}/\langle g \rangle \hat{L} \\ y \text{ monic}}} \sum_{\langle l \rangle < \hat{L}} h\big(g(yt^L + l) + r\big)\, e\big(\beta(g(yt^L + l) + r)\big) \\
&= \sum_{\substack{\langle y \rangle = \hat{N}/\langle g \rangle \hat{L} \\ y \text{ monic}}} e(\beta g y t^L) \sum_{\langle l \rangle < \hat{L}} h\big(gyt^L + lg + r\big) \\
&= \frac{\hat{L}}{\hat{N}} \gamma_{r,g}(h) \sum_{\substack{\langle y \rangle = \hat{N}/\langle g \rangle \hat{L} \\ y \text{ monic}}} e(\beta g y t^L) + O\big(\langle g \rangle^{-1} E(h)\big).
\end{aligned}$$

In addition, for $\langle z \rangle < \langle g t^L \rangle = \langle g \rangle \hat{L}$, we have

$$\langle \beta z \rangle < \frac{N^B}{\langle g \rangle \hat{N}} \cdot \langle z \rangle \leq \frac{N^B}{\langle g \rangle \hat{N}} \cdot \frac{\langle g \rangle \hat{N}}{q^{1+\lceil B \log_q N \rceil}} \leq \frac{1}{q},$$

which implies that $e(\beta z) = 1$. Thus

$$\begin{aligned}
\sum_{\substack{\langle y \rangle = \hat{N}/\langle g \rangle \hat{L} \\ y \text{ monic}}} e(\beta g y t^L) &= \frac{1}{\langle g \rangle \hat{L}} \sum_{\langle z \rangle < \langle g t^L \rangle} \sum_{\substack{\langle y \rangle = \hat{N}/\langle g \rangle \hat{L} \\ y \text{ monic}}} e\big(\beta(gyt^L + z)\big) \\
&= \frac{1}{\langle g \rangle \hat{L}} \sum_{n \in \mathscr{S}_N} e(\beta n) = \frac{\hat{N}}{\langle g \rangle \hat{L}} \varrho(\beta).
\end{aligned}$$

By combining the above estimates, we have

$$\sum_{\substack{n\in\mathscr{S}_N\\ n\equiv r(\text{mod } g)}} h(n)e(\beta n) = \langle g\rangle^{-1}\gamma_{r,g}(h)\varrho(\beta) + O\big(\langle g\rangle^{-1}E(h)\big).$$

This completes the proof of the lemma.

Lemma 9. *Let $h : \mathscr{S}_N \to \mathbb{C}$ satisfy Condition*. For $a, g \in \mathbb{F}_q[t]$ with g monic, $(a, g) = 1$ and $\langle g\rangle \le N^B$, define*

$$\sigma_{a,g}(h) = \sum_{\langle r\rangle<\langle g\rangle} e\left(\frac{ar}{g}\right)\gamma_{r,g}(h).$$

Then for $\alpha \in \mathfrak{M}_{a,g}$,

$$h^{\wedge}(\alpha) = \langle g\rangle^{-1}\sigma_{a,g}(h)\varrho\left(\alpha - \frac{a}{g}\right) + O\big(E(h)\big).$$

Proof. Write $\alpha = a/g + \beta$ with $\langle\beta\rangle < N^B\langle g\rangle^{-1}\hat{N}^{-1}$. Then by Lemma 8,

$$\begin{aligned} h^{\wedge}(\alpha) &= \sum_{n\in\mathscr{S}_N} h(n)e(n\alpha)\\ &= \sum_{\langle r\rangle<\langle g\rangle} e\left(\frac{ra}{g}\right) \sum_{\substack{n\in\mathscr{S}_N\\ n\equiv r(\text{mod } g)}} h(n)e(\beta n)\\ &= \langle g\rangle^{-1}\varrho(\beta)\sum_{\langle r\rangle<\langle g\rangle} e\left(\frac{ra}{g}\right)\gamma_{r,g}(h) + O\big(\langle g\rangle\langle g\rangle^{-1}E(h)\big)\\ &= \langle g\rangle^{-1}\varrho(\beta)\sigma_{a,g}(h) + O\big(E(h)\big). \end{aligned}$$

Thus the lemma follows.

In the following, we will show that the functions $\lambda_{b,m,N}$ and $\lambda^{(Q)}_{b,m,N}$ $(1 \le Q \le K)$ satisfy Condition*. We first recall a result of Rhin.

Lemma 10 (Rhin [18, Theorem 4]). *Let $c, d \in \mathbb{F}_q[t]$ with c monic and $(c, d) = 1$. For $D, M \in \mathbb{N}$, we denote by $N(c, d; M, D)$ the number of monic irreducible polynomials ϖ of order M satisfying $\varpi \equiv c\,(\text{mod}\, d)$ and $\text{ord}\,(\varpi t^{\text{ord}\, c} - ct^{\text{ord}\, m}) < -D + \text{ord}\,\varpi + \text{ord}\, c$. Then*

$$N(c, d; M, D) = \frac{\hat{M}}{M\phi(d)\hat{D}} + O\big((\text{ord}\, d + D + 1)\hat{M}^{1/2}\big).$$

Lemma 11. *Let $r, g \in \mathbb{F}_q[t]$ with g monic, $\langle r\rangle < \langle g\rangle$ and $\langle g\rangle \le N^B$. Then $\lambda_{b,m,N}$ satisfies Condition* with*

$$\gamma_{r,g}(\lambda_{b,m,N}) = \begin{cases} \frac{\phi(m)\langle g\rangle}{\phi(mg)}, & \text{if } (mr+b, mg) = 1, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$E(\lambda_{b,m,N}) = N^{B+1+\epsilon}\langle m\rangle^{1/2+\epsilon}\hat{N}^{-1/2}.$$

Proof. Recall the definition of X in (1). Let $r' \in \mathscr{S}_N$ with $r' \equiv r \pmod g$ and $Y = \left\{r' + lg \mid \langle l\rangle < \hat{L}\right\} \subseteq \mathscr{S}_N$. For $n = r' + lg \in Y$, $\lambda_{b,m,N}(n) = 0$ if and only if $mn + b \notin X$.

(1) Suppose that $(mr + b, mg) \neq 1$. We assume that $N^B < \hat{N}$. Then there exists a monic irreducible polynomial ϖ such that $\varpi | (mr + b, mg)$. Write $n = r + l'g + lg$ for some $l' \in \mathbb{F}_q[t]$. Then the polynomial

$$mn + b = m(r + l'g + lg) + b = (mr + b) + mg(l + l')$$

has a factor ϖ. If $mn + b \in X$, then $\varpi = mn + b$. Since

$$\langle \varpi\rangle \le \langle mg\rangle \le \langle m\rangle N^B < \langle m\rangle\hat{N} = \langle mn + b\rangle,$$

we have $\varpi \neq mn + b$. Thus we have $mn + b \notin X$. It follows that

$$\sum_{n\in Y} \lambda_{b,m,N}(n) = 0.$$

Thus the lemma follows in this case.

(2) Suppose that $(mr + b, mg) = 1$. Consider

$$N_{r'} = N_{r'}(m, g, L) = \#\left\{n = r' + lg \mid \langle l\rangle < \hat{L} \text{ and } mn + b \in X\right\},$$

which is equal to the number of monic irreducible polynomials ϖ with $\operatorname{ord}\varpi = N + \operatorname{ord} m$, $\varpi \equiv mr' + b \pmod{mg}$ and $\langle \varpi - (mr' + b)\rangle < \hat{L}\langle mg\rangle$. We now apply Lemma 10 with $c = mr' + b$, $d = mg$, $M = N + \operatorname{ord} m = \operatorname{ord} c$ and $D = N - L - \operatorname{ord} g$. Since $L = N - \lceil \log_q N^B\rceil$, we have

$$\begin{aligned} N_{r'} &= \frac{\hat{N}\langle m\rangle\hat{L}\langle g\rangle}{(N + \operatorname{ord} m)\phi(mg)\hat{N}} + O\Big(\big((\operatorname{ord} g + \operatorname{ord} m) + (N - L - \operatorname{ord} g) + 1\big)\hat{N}^{1/2}\langle m\rangle^{1/2}\Big) \\ &= \frac{\hat{L}\langle mg\rangle}{(N + \operatorname{ord} m)\phi(mg)} + O\big((\operatorname{ord} m + \lceil B\log_q N\rceil)\hat{N}^{1/2}\langle m\rangle^{1/2}\big). \end{aligned}$$

It follows that

$$\begin{aligned}
&\sum_{n\in Y}\lambda_{b,m,N}(n)\\
&=\frac{\phi(m)(N+\operatorname{ord}m)}{\hat{N}\langle m\rangle}\left(\frac{\hat{L}\langle mg\rangle}{(N+\operatorname{ord}m)\phi(mg)}+O\big((\operatorname{ord}m+\lceil B\log_q N\rceil)\hat{N}^{1/2}\langle m\rangle^{1/2}\big)\right)\\
&=\frac{\hat{L}}{\hat{N}}\left(\frac{\phi(m)\langle g\rangle}{\phi(mg)}+O\left(\frac{\hat{N}\phi(m)(N+\operatorname{ord}m)}{\hat{L}\hat{N}\langle m\rangle}(\operatorname{ord}m+B\log_q N)\hat{N}^{1/2}\langle m\rangle^{1/2}\right)\right)\\
&=\frac{\hat{L}}{\hat{N}}\left(\frac{\phi(m)\langle g\rangle}{\phi(mg)}+O\left(\frac{N^{B+1+\epsilon}\langle m\rangle^{1/2+\epsilon}}{\hat{N}^{1/2}}\right)\right).
\end{aligned}$$

Thus the lemma also follows in this case.

Lemma 12. *Suppose that $a, g \in \mathbb{F}_q[t]$ with g monic, $(a, g) = 1$ and $\langle g\rangle \le N^B$. For σ defined as in Lemma 9, one has*

$$\sigma_{a,g}(\lambda_{b,m,N}) = \begin{cases} \frac{\langle g\rangle\mu(g)}{\phi(g)}e\left(\frac{-ab\bar{m}}{g}\right), & \text{if } (m,g)=1,\\ 0, & \text{otherwise.}\end{cases}$$

Here, we write $\bar{m}$ for the multiplicative inverse of m modulo g and $\mu(\cdot)$ the Möbius function on $\mathbb{F}_q[t]$.

Proof. By Lemma 11, we have

$$\begin{aligned}
\sigma_{a,g}(\lambda_{b,m,N}) = \sum_{\langle r\rangle<\langle g\rangle} e\left(\frac{ar}{g}\right)\gamma_{r,g}(\lambda_{b,m,N}) &= \frac{\phi(m)\langle g\rangle}{\phi(mg)}\sum_{\substack{\langle r\rangle<\langle g\rangle\\(mr+b,mg)=1}} e\left(\frac{ar}{g}\right)\\
&= \frac{\phi(m)\langle g\rangle}{\phi(mg)}\sum_{\substack{\langle r\rangle<\langle g\rangle\\(mr+b,g)=1}} e\left(\frac{ar}{g}\right).
\end{aligned}$$

For $z\in\mathbb{Z}$ with $z\ge 0$, if $\varpi^z|g$ and $\varpi^{z+1}\nmid g$, we write that $\varpi^z\|g$. Let

$$g_0=\prod_{\substack{\varpi\\ \varpi^z\|g,\ \varpi\nmid m}}\varpi^z,$$

and $g_1 = g/g_0$. If $\varpi|m$, then $\varpi\nmid(mr+b)$. Thus $(mr+b, mg) = (mr+b, g_0)$, and

$$\sum_{\substack{\langle r\rangle<\langle g\rangle\\(mr+b,g)=1}} e\left(\frac{ar}{g}\right)=\sum_{\substack{\langle r\rangle<\langle g\rangle\\(mr+b,g_0)=1}} e\left(\frac{ar}{g}\right).$$

By writing $r = ug_0 + v$ with $\langle u\rangle < \langle g_1\rangle$ and $\langle v\rangle < \langle g_0\rangle$, we have

$$\sum_{\substack{\langle r\rangle<\langle g\rangle\\(mr+b,g_0)=1}} e\left(\frac{ar}{g}\right) = \sum_{\substack{\langle v\rangle<\langle g_0\rangle\\(mv+b,g_0)=1}} e\left(\frac{av}{g}\right) \sum_{\langle u\rangle<\langle g_1\rangle} e\left(\frac{au}{g_1}\right).$$

Since

$$\sum_{\langle u\rangle<\langle g_1\rangle} e\left(\frac{au}{g_1}\right) = \begin{cases} 1, & \text{if } \langle g_1\rangle = 1,\\ 0, & \text{otherwise,}\end{cases}$$

it follows that

$$\sum_{\substack{\langle v\rangle<\langle g_0\rangle\\(mv+b,g_0)=1}} e\left(\frac{av}{g}\right) \sum_{\langle u\rangle<\langle g_1\rangle} e\left(\frac{au}{g_1}\right) = \begin{cases} \displaystyle\sum_{\substack{\langle v\rangle<\langle g\rangle\\(mv+b,g)=1}} e\left(\frac{av}{g}\right), & \text{if } g_1 = 1,\\ 0, & \text{otherwise.}\end{cases}$$

One has that $(g,m) = 1$ if and only if $g_1 = 1$. When $(g,m) = 1$, we have $\frac{\phi(m)\langle g\rangle}{\phi(mg)} = \frac{\langle g\rangle}{\phi(g)}$. Therefore, to prove the lemma, it is enough to show that when $(g,m) = 1$, we have

$$\sum_{\substack{\langle v\rangle<\langle g\rangle\\(mv+b,g)=1}} e\left(\frac{av}{g}\right) = \mu(g) e\left(\frac{-ab\bar{m}}{g}\right).$$

Suppose that $(g,m) = 1$. Let $w = mv+b$. Then $(w-b)\bar{m} \equiv v \pmod g$. By checking that $\displaystyle\sum_{\substack{\langle w\rangle<\langle g\rangle\\(w,g)=1}} e\left(\frac{aw\bar{m}}{g}\right)$ is a multiplicative function in g, one can verify that

$$\sum_{\substack{\langle w\rangle<\langle g\rangle\\(w,g)=1}} e\left(\frac{aw\bar{m}}{g}\right) = \mu(g).$$

Thus

$$\begin{aligned}\sum_{\substack{\langle v\rangle<\langle g\rangle\\(mv+b,g)=1}} e\left(\frac{av}{g}\right) &= \sum_{\substack{\langle w\rangle<\langle g\rangle\\(w,g)=1}} e\left(\frac{a(w-b)\bar{m}}{g}\right) = e\left(\frac{-ab\bar{m}}{g}\right) \sum_{\substack{\langle w\rangle<\langle g\rangle\\(w,g)=1}} e\left(\frac{aw\bar{m}}{g}\right)\\ &= \mu(g) e\left(\frac{-ab\bar{m}}{g}\right).\end{aligned}$$

This completes the proof of the lemma.

Lemma 13. *Let $r, g \in \mathbb{F}_q[t]$ with g monic, $\langle r\rangle < \langle g\rangle$ and $\langle g\rangle \le N^B$. For $1 \le Q \le K$, the function $\lambda_{b,m,N}^{(Q)}$ satisfies Condition* with*

$$\gamma_{r,g}(\lambda_{b,m,N}^{(Q)}) = \begin{cases} \prod_{\substack{\langle\varpi\rangle\le\hat{Q} \\ \varpi\nmid m}} \left(1-1/\langle\varpi\rangle\right)^{-1} \prod_{\substack{\langle\varpi\rangle\le\hat{Q} \\ \varpi\nmid mg}} \left(1-1/\langle\varpi\rangle\right), & \text{if } (mr+b, mg) \text{ is } \hat{Q}\text{-rough}, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$E(\lambda_{b,m,N}^{(Q)}) = \hat{N}^{-1/(2A)+\epsilon} + \hat{N}^{-1/2+\epsilon},$$

where $A = 4/(\delta-2)$ is defined as in Sect. 2.

Proof. Let $r' \in \mathscr{S}_N$ with $r' \equiv r \pmod{g}$ and $Y = \{r' + lg \mid \langle l\rangle < \hat{L}\} \subseteq \mathscr{S}_N$. Since $(b,m) = 1$, if $\varpi \in \mathbb{F}_q[t]$ is a monic irreducible polynomial with $\varpi | m$, then $\varpi \nmid (mn+b)$. Thus it suffices to consider ϖ with $\varpi \nmid m$. Let $\varpi_1, \ldots, \varpi_R \in \mathbb{F}_q[t]$ denote the monic irreducible polynomials with $\langle\varpi_i\rangle \le \hat{Q}$ and $\varpi_i \nmid m$ $(1 \le i \le R)$. For $n = r' + lg \in Y$, $\lambda_{b,m,N}^{(Q)}(n) = 0$ if and only if $\varpi_i | (mn+b)$ for some $1 \le i \le R$.

(1) Suppose that $(mr+b, mg)$ is not $\hat{Q}$-rough. Then there exists some ϖ_i such that $\varpi_i | (mr+b, mg)$. Write $n = r + l'g + lg$ for some $l' \in \mathbb{F}_q[t]$. Thus the polynomial $mn + b = (mr+b) + mg(l+l')$ has a factor ϖ_i. Hence, $\lambda_{b,m,N}^{(Q)}(n) = 0$ and the lemma follows in this case.
(2) Suppose that $(mr+b, mg)$ is $\hat{Q}$-rough, i.e., $\varpi_i \nmid (mr+b, mg)$ $(1 \le i \le R)$. Let X_i denote the event that $\varpi_i | (mn+b)$ for $n \in Y$, and let $\mathbb{P}(X_i) = |X_i|/\hat{L}$ be the probability of X_i occurring. We denote by X_i^c the complement of X_i. Note that

$$\sum_{n\in Y} \lambda_{b,m,N}^{(Q)}(n) = \frac{1}{\hat{N}} \prod_{\substack{\langle\varpi\rangle\le\hat{Q} \\ \varpi\nmid m}} \left(1-1/\langle\varpi\rangle\right)^{-1} \cdot \left| \bigcap_{i=1}^{R} X_i^c \right| = \frac{\hat{L}}{\hat{N}} \prod_{\substack{\langle\varpi\rangle\le\hat{Q} \\ \varpi\nmid m}} \left(1-1/\langle\varpi\rangle\right)^{-1} \cdot \mathscr{V},$$

where

$$\mathscr{V} = \mathbb{P}\left(\bigcap_{i=1}^{R} X_i^c\right).$$

It remains to estimate $\mathscr{V}$.

(2.1) If $\varpi_i | g$, then $mn + b \equiv mr + b \not\equiv 0 \pmod{\varpi_i}$, i.e., $\varpi_i \nmid (mn+b)$. Thus $\mathbb{P}(X_i) = 0$.
(2.2) Suppose that $\varpi_i \nmid g$. Since $\varpi_i \nmid m$, we have $(\varpi_i, mg) = 1$. If $\hat{L} \ge \langle\varpi\rangle$, as l varies with $\langle l\rangle < \hat{L}$, then $mn + b = (mr' + b) + lmg$ runs through all

residue classes modulo ϖ_i. Thus we have $\mathbb{P}(X_i) = 1/\langle \varpi_i \rangle$. On the other hand, if $\hat{L} < \langle \varpi \rangle$, then either 0 or 1 choices of l will give $\varpi_i | (mn + b)$. Thus $\mathbb{P}(X_i) = O(\hat{L}^{-1})$. From the above estimates, we have

$$\mathbb{P}(X_i) = \frac{\epsilon_i}{\langle \varpi_i \rangle} + O(\hat{L}^{-1}),$$

where

$$\epsilon_i = \begin{cases} 0, & \text{if } \varpi_i | g, \\ 1, & \text{otherwise.} \end{cases}$$

By the inclusion-exclusion formula, we have

$$\mathscr{V} = \sum_{s=0}^{R} (-1)^s \sum_{1 \le i_1 < \cdots < i_s \le R} \prod_{j=1}^{s} \frac{\epsilon_{i_1} \cdots \epsilon_{i_s}}{\langle \varpi_{i_1} \rangle \cdots \langle \varpi_{i_s} \rangle} + O\left(\hat{L}^{-1} \sum_{s=1}^{R} \binom{R}{s} \right).$$

Note that for any $K' \in \mathbb{N}$, by considering the even terms of the above alternating sum, we have

$$\begin{aligned} \mathscr{V} &\le \sum_{s=0}^{2K'} (-1)^s \sum_{1 \le i_1 < \cdots < i_s \le R} \prod_{j=1}^{s} \frac{\epsilon_{i_1} \cdots \epsilon_{i_s}}{\langle \varpi_{i_1} \rangle \cdots \langle \varpi_{i_s} \rangle} + O\left(\hat{L}^{-1} \sum_{s=1}^{2K'} \binom{R}{s} \right) \\ &= \prod_{i=1}^{R} \left(1 - \frac{\epsilon_i}{\langle \varpi_i \rangle} \right) + O\left(\sum_{s=2K'+1}^{R} \sum_{1 \le i_1 < \cdots < i_s \le R} \prod_{j=1}^{s} \frac{\epsilon_{i_1} \cdots \epsilon_{i_s}}{\langle \varpi_{i_1} \rangle \cdots \langle \varpi_{i_s} \rangle} \right) \\ &\quad + O\left(\hat{L}^{-1} \sum_{s=1}^{2K'} \binom{R}{s} \right). \end{aligned}$$

Similarly, by considering the odd terms of the alternating sum, we have

$$\begin{aligned} \mathscr{V} &\ge \sum_{s=0}^{2K'-1} (-1)^s \sum_{1 \le i_1 < \cdots < i_s \le R} \prod_{j=1}^{s} \frac{\epsilon_{i_1} \cdots \epsilon_{i_s}}{\langle \varpi_{i_1} \rangle \cdots \langle \varpi_{i_s} \rangle} + O\left(\hat{L}^{-1} \sum_{s=1}^{2K'-1} \binom{R}{s} \right) \\ &= \prod_{i=1}^{R} \left(1 - \frac{\epsilon_i}{\langle \varpi_i \rangle} \right) + O\left(\sum_{s=2K'}^{R} \sum_{1 \le i_1 < \cdots < i_s \le R} \prod_{j=1}^{s} \frac{\epsilon_{i_1} \cdots \epsilon_{i_s}}{\langle \varpi_{i_1} \rangle \cdots \langle \varpi_{i_s} \rangle} \right) \\ &\quad + O\left(\hat{L}^{-1} \sum_{s=1}^{2K'-1} \binom{R}{s} \right). \end{aligned}$$

Thus for any $J \in \mathbb{N}$, we have

$$\mathscr{V} = \prod_{\substack{\langle \varpi \rangle \le \hat{Q} \\ \varpi \nmid mg}} \left(1 - 1/\langle \varpi \rangle\right) + O\left(\sum_{s=J}^{R} \sum_{1 \le i_1 < \cdots < i_s \le R} \prod_{j=1}^{s} \frac{\epsilon_{i_1} \cdots \epsilon_{i_s}}{\langle \varpi_{i_1} \rangle \cdots \langle \varpi_{i_s} \rangle} \right)$$
$$+ O\left(\hat{L}^{-1} \sum_{s=1}^{J} \binom{R}{s} \right).$$

To estimate the error terms, note that

$$\hat{L}^{-1} \sum_{s=1}^{J} \binom{R}{s} \ll \hat{L}^{-1} R^{J+1}.$$

Also, for $J \le s \le R$,

$$\sum_{1 \le i_1 < \cdots < i_s \le R} \prod_{j=1}^{s} \frac{\epsilon_{i_1} \cdots \epsilon_{i_s}}{\langle \varpi_{i_1} \rangle \cdots \langle \varpi_{i_s} \rangle} \le \frac{1}{s!} \left(\sum_{i=1}^{R} \frac{1}{\langle \varpi_i \rangle} \right)^s \le \frac{1}{s!} \left(\sum_{\langle \varpi \rangle \le \hat{Q}} \frac{1}{\langle \varpi \rangle} \right)^s$$
$$\le \frac{1}{s!} \left(\ln Q + c \right)^s.$$

The last inequality follows from Lemma 4 with c some fixed constant. It follows that for $J > 3(\ln Q + c)$,

$$\sum_{s=J}^{R} \sum_{1 \le i_1 < \cdots < i_s \le R} \prod_{j=1}^{s} \frac{\epsilon_{i_1} \cdots \epsilon_{i_s}}{\langle \varpi_{i_1} \rangle \cdots \langle \varpi_{i_s} \rangle}$$
$$\le \sum_{s=J}^{R} \frac{1}{s!} (\ln Q + c)^s$$
$$\le \frac{(\ln Q + c)^J}{J!} \left(1 + \frac{\ln Q + c}{J+1} + \frac{(\ln Q + c)^2}{(J+1)(J+2)} + \cdots \right)$$
$$\le \frac{(\ln Q + c)^J}{J!} (1 + 1/3 + 1/3^2 + \cdots)$$
$$\ll \left(\frac{e \ln(e^c Q)}{J} \right)^J.$$

The last inequality follows from Stirling's formula, namely that $J! = \sqrt{2\pi J}(J/e)^J\left(1 + O(1/J)\right)$. Thus we have

$$\mathscr{V} = \prod_{\substack{\langle \varpi \rangle \le \hat{Q} \\ \varpi \nmid mg}} \left(1 - 1/\langle \varpi \rangle\right) + O\left(\left(\frac{e \ln(e^c Q)}{J} \right)^J + \hat{L}^{-1} R^{J+1} \right).$$

Since $R \ll \hat{Q} \ll N^A$, by choosing $J = N/(2A \log_q N)$, we have

$$\left(\frac{e \ln(e^c Q)}{J}\right)^J \ll \hat{N}^{-1/(2A)+\epsilon} \qquad \text{and} \qquad \hat{L}^{-1} R^{J+1} \ll \hat{N}^{-1/2+\epsilon}.$$

Thus

$$\mathscr{V} = \prod_{\substack{\langle \varpi \rangle \le \hat{Q} \\ \varpi \nmid mg}} \left(1 - 1/\langle \varpi \rangle\right) + O\left(\hat{N}^{-1/(2A)+\epsilon} + \hat{N}^{-1/2+\epsilon}\right).$$

By Lemma 4, we have

$$\prod_{\substack{\langle \varpi \rangle \le \hat{Q} \\ \varpi \nmid m}} \left(1 - 1/\langle \varpi \rangle\right)^{-1} \le \prod_{\langle \varpi \rangle \le \hat{Q}} \left(1 - 1/\langle \varpi \rangle\right)^{-1} \ll Q \ll \log_q N.$$

It follows that

$$\sum_{n \in Y} \lambda_{b,m,N}^{(Q)}(n)$$
$$= \frac{\hat{L}}{\hat{N}} \left(\prod_{\substack{\langle \varpi \rangle \le \hat{Q} \\ \varpi \nmid m}} \left(1 - 1/\langle \varpi \rangle\right)^{-1} \prod_{\substack{\langle \varpi \rangle \le \hat{Q} \\ \varpi \nmid mg}} \left(1 - 1/\langle \varpi \rangle\right) + O\left(\hat{N}^{-1/(2A)+\epsilon} + \hat{N}^{-1/2+\epsilon}\right)\right).$$

This completes the proof of the lemma.

For a polynomial $x \in \mathbb{F}_q[t]$, we say that x is $\hat{Q}$*-smooth* if for all monic irreducible polynomials ϖ with $\varpi | x$, we have $\langle \varpi \rangle \le \hat{Q}$.

Lemma 14. *Suppose that $a, g \in \mathbb{F}_q[t]$ with g monic, $(a, g) = 1$ and $\langle g \rangle \le N^B$. Also, suppose that $1 \le Q \le K$. For σ defined as in Lemma 9, one has*

$$\sigma_{a,g}(\lambda_{b,m,N}^{(Q)}) = \begin{cases} \frac{\langle g \rangle \mu(g)}{\phi(g)} e\left(\frac{-ab\bar{m}}{g}\right), & \text{if } (m, g) = 1 \text{ and } g \text{ is } \hat{Q}\text{-smooth}, \\ 0, & \text{otherwise}, \end{cases}$$

where $\bar{m}$ is the multiplicative inverse of m modulo g.

Proof. By Lemma 13, we have

$$\sigma_{a,g}(\lambda_{b,m,N}^{(Q)}) = \sum_{\langle r \rangle < \langle g \rangle} e\left(\frac{ar}{g}\right) \gamma_{r,g}(\lambda_{b,m,N}^{(Q)})$$
$$= \prod_{\substack{\langle \varpi \rangle \le \hat{Q} \\ \varpi \nmid m}} \left(1 - 1/\langle \varpi \rangle\right)^{-1} \prod_{\substack{\langle \varpi \rangle \le \hat{Q} \\ \varpi \nmid mg}} \left(1 - 1/\langle \varpi \rangle\right) \sum_{\substack{\langle r \rangle < \langle g \rangle \\ (mr+b, mg) \text{ is } \hat{Q}\text{-rough}}} e\left(\frac{ar}{g}\right).$$

Note that if $(m,g)=1$ and g is $\hat{Q}$-smooth, then

$$\prod_{\substack{\langle\varpi\rangle\le\hat{Q}\\ \varpi\nmid m}}\bigl(1-1/\langle\varpi\rangle\bigr)^{-1}\prod_{\substack{\langle\varpi\rangle\le\hat{Q}\\ \varpi\nmid mg}}\bigl(1-1/\langle\varpi\rangle\bigr)=\langle g\rangle/\phi(g).$$

Thus to prove the lemma, it is enough to show that

$$\sum_{\substack{\langle r\rangle<\langle g\rangle\\ (mr+b,mg)\text{ is }\hat{Q}\text{-rough}}} e\left(\frac{ar}{g}\right)=\begin{cases}\mu(g)e\left(\frac{-ab\bar{m}}{g}\right), & \text{if } (m,g)=1 \text{ and } g \text{ is } \hat{Q}\text{-smooth},\\ 0, & \text{otherwise.}\end{cases}$$

Let

$$g_2=\prod_{\substack{\langle\varpi\rangle\le\hat{Q}\\ \varpi^z\|g,\ \varpi\nmid m}}\varpi^z,$$

and $g_3=g/g_2$. If $\varpi|m$, then $\varpi\nmid(mr+b)$. Thus $(mr+b,mg)=(mr+b,g)$, and

$$\sum_{\substack{\langle r\rangle<\langle g\rangle\\ (mr+b,mg)\text{ is }\hat{Q}\text{-rough}}} e\left(\frac{ar}{g}\right)=\sum_{\substack{\langle r\rangle<\langle g\rangle\\ (mr+b,g_2)=1}} e\left(\frac{ar}{g}\right).$$

Note that $(m,g)=1$ and that g is $\hat{Q}$-smooth if and only if $g_3=1$. Then using a similar argument as the one in the proof of Lemma 12 (with g_0 replaced by g_2 and g_1 replaced by g_3), we can show that

$$\sum_{\substack{\langle r\rangle<\langle g\rangle\\ (mr+b,g_2)=1}} e\left(\frac{ar}{g}\right)=\begin{cases}\mu(g)e\left(\frac{-ab\bar{m}}{g}\right), & \text{if } (m,g)=1 \text{ and } g \text{ is } \hat{Q}\text{-smooth},\\ 0, & \text{otherwise.}\end{cases}$$

This completes the proof of the lemma.

We now summarize the major arc contribution to Proposition 6.

Lemma 15. *For* $1\le Q\le K$*, we have*

$$\sup_{\alpha\in\mathfrak{M}}|\lambda^{\wedge}_{b,m,N}(\alpha)-\lambda^{(Q)\wedge}_{b,m,N}(\alpha)|\ll\hat{Q}^{-1}.$$

Proof. Let $\alpha\in\mathfrak{M}$. Then there exists $a,g\in\mathbb{F}_q[t]$ with g monic, $(a,g)=1$, $\langle\alpha-a/g\rangle<N^B/\bigl(\langle g\rangle\hat{N}\bigr)$ and $\langle g\rangle\le\hat{N}/N^B$. By combining Lemmas 9, 11, 12, 13 and 14, if g is $\hat{Q}$-smooth, we have

$$|\lambda^{\wedge}_{b,m,N}(\alpha)-\lambda^{(Q)\wedge}_{b,m,N}(\alpha)|\ll N^{B+1+\epsilon}\langle m\rangle^{1/2+\epsilon}\hat{N}^{-1/2}+\hat{N}^{-1/(2A)+\epsilon}+\hat{N}^{-1/2+\epsilon}\ll\hat{Q}^{-1}.$$

If g is not $\hat{Q}$-smooth, then there exists an irreducible polynomial ϖ with $\langle \varpi \rangle > \hat{Q}$ and $\varpi | g$. It follows that $\phi(g) \geq \phi(\varpi) = \langle \varpi \rangle - 1 \gg \hat{Q}$. Thus we have

$$\begin{aligned} |\lambda^{\wedge}_{b,m,N}(\alpha) - \lambda^{(Q)\wedge}_{b,m,N}(\alpha)| &\leq |\lambda^{\wedge}_{b,m,N}(\alpha)| + |\lambda^{(Q)\wedge}_{b,m,N}(\alpha)| \\ &\ll 1/\phi(g) + N^{B+1+\epsilon}\langle m \rangle^{1/2+\epsilon}\hat{N}^{-1/2} + \hat{N}^{-1/(2A)+\epsilon} + \hat{N}^{-1/2+\epsilon} \\ &\ll \hat{Q}^{-1}. \end{aligned}$$

This completes the proof of the lemma.

4.2 *Minor Arc Estimates*

We will now turn our attention to obtaining a minor arc estimate for $\lambda_{b,m,N}(\alpha)$. We will obtain the following result.

Lemma 16. *Suppose that $\langle m \rangle \leq N$. One has*

$$\sup_{\alpha \in \mathfrak{m}} |\lambda^{\wedge}_{b,m,N}(\alpha)| \ll N^{6-B/2} = N^{-A},$$

where $A = 4/(\delta - 2)$ and $B = 2A + 12$ are defined as in Sects. 2 and 3.

In order to prove this lemma, we need to establish more notation. Whenever a sum has a superscript $+$, which will look like $\sum^{+}$, the sum will be restricted to monic polynomials. Let $R \in \mathbb{N}$, and let U be a parameter with $1 \leq U < R/2$. Define τ_x by

$$\tau_x = \sum_{\substack{d|x \\ \langle d \rangle \leq \hat{U}}}^{+} \mu(d). \tag{5}$$

Let

$$\Lambda(y) = \begin{cases} \operatorname{ord} \varpi, & \text{when } y = \varpi^l \text{ for some monic, irreducible polynomial } \varpi \text{ and } l \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

We now will present a sequence of lemmas concerning the weighted exponential sum

$$\sum_{\substack{\langle y \rangle \leq \hat{R} \\ y \equiv b \ (\mathrm{mod}\ m)}}^{+} \Lambda(y) e(\alpha y);$$

from these lemmas, we will be able to extract Lemma 16. Due to the underlying shape of Dirichlet series in $\mathbb{F}_q[t]$, we are unable to take an approach similar to that in [1]. Instead, we will follow the ideas of [26, Chap. 3].

Lemma 17. *Let $\upsilon(x, y)$ denote a function on $\mathbb{F}_q[t]^2$. Then we have*

$$\sideset{}{^+}\sum_{\hat{U}<\langle y\rangle\le\hat{R}} \upsilon(1,y) + \sideset{}{^+}\sum_{\hat{U}<\langle x\rangle\le\hat{R}} \sideset{}{^+}\sum_{\hat{U}<\langle y\rangle\le\hat{R}/\langle x\rangle} \tau_x \upsilon(x,y) = \sideset{}{^+}\sum_{\langle d\rangle\le\hat{U}} \sideset{}{^+}\sum_{\hat{U}<\langle y\rangle\le\hat{R}/\langle d\rangle} \sideset{}{^+}\sum_{\langle z\rangle\le\hat{R}/\langle yd\rangle} \mu(d)\upsilon(dz,y).$$

Proof. By writing $x = dz$, we have

$$\begin{aligned}\sideset{}{^+}\sum_{\langle d\rangle\le\hat{U}} \sideset{}{^+}\sum_{\hat{U}<\langle y\rangle\le\hat{R}/\langle d\rangle} \sideset{}{^+}\sum_{\langle z\rangle\le\hat{R}/\langle yd\rangle} \mu(d)\upsilon(dz,y) &= \sideset{}{^+}\sum_{\hat{U}<\langle x\rangle\le\hat{R}} \sideset{}{^+}\sum_{\hat{U}<\langle y\rangle\le\hat{R}/\langle x\rangle} \upsilon(x,y) \sideset{}{^+}\sum_{\substack{d\mid x\\ \langle d\rangle\le\hat{U}}} \mu(d) \\ &\quad + \sideset{}{^+}\sum_{\langle x\rangle\le\hat{U}} \sideset{}{^+}\sum_{\hat{U}<\langle y\rangle\le\hat{R}/\langle x\rangle} \upsilon(x,y) \sideset{}{^+}\sum_{\substack{d\mid x\\ \langle d\rangle\le\hat{U}}} \mu(d).\end{aligned} \tag{6}$$

For $\langle x\rangle \le \hat{U}$, we have

$$\sideset{}{^+}\sum_{\substack{d\mid x\\ \langle d\rangle\le\hat{U}}} \mu(d) = \begin{cases} 1, & \text{when } x = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Thus

$$\sideset{}{^+}\sum_{\langle x\rangle\le\hat{U}} \sideset{}{^+}\sum_{\hat{U}<\langle y\rangle\le\hat{R}/\langle x\rangle} \upsilon(x,y) \sideset{}{^+}\sum_{\substack{d\mid x\\ \langle d\rangle\le\hat{U}}} \mu(d) = \sideset{}{^+}\sum_{\hat{U}<\langle y\rangle\le\hat{R}} \upsilon(1,y). \tag{7}$$

The lemma now follows from (5), (6) and (7).

Let

$$S_1(\alpha) = \sideset{}{^+}\sum_{\substack{\langle y\rangle\le\hat{U}\\ y\equiv b\,(\mathrm{mod}\,m)}} \Lambda(y)e(\alpha y), \qquad S_2(\alpha) = \sideset{}{^+}\sum_{\substack{\langle xy\rangle\le\hat{R}\\ \langle x\rangle\le\hat{U}\\ xy\equiv b\,(\mathrm{mod}\,m)}} \mu(x)(\mathrm{ord}\,y)e(\alpha xy),$$

$$S_3(\alpha) = \sideset{}{^+}\sum_{\substack{\langle xy\rangle\le\hat{R}\\ \langle x\rangle\le\hat{U}^2\\ xy\equiv b\,(\mathrm{mod}\,m)}} \sideset{}{^+}\sum_{\substack{x=uv\\ \langle u\rangle,\langle v\rangle\le\hat{U}}} \mu(u)\Lambda(v)e(\alpha xy),$$

and

$$S_4(\alpha) = \sideset{}{^+}\sum_{\substack{\langle xy\rangle\le\hat{R}\\ \langle x\rangle,\langle y\rangle>\hat{U}\\ xy\equiv b\,(\mathrm{mod}\,m)}} \tau_x \Lambda(y)e(\alpha xy).$$

Lemma 18. *One has*

$$\sum_{\substack{\langle y\rangle\le\hat{R}\\ y\equiv b\,(\mathrm{mod}\,m)}}^{+} \Lambda(y)e(\alpha y) = S_1(\alpha) + S_2(\alpha) - S_3(\alpha) - S_4(\alpha).$$

Proof. Let

$$\upsilon(x,y) = \begin{cases} \Lambda(y)e(\alpha xy), & \text{when } xy \equiv b\,(\mathrm{mod}\,m),\\ 0, & \text{otherwise.}\end{cases}$$

We first notice that

$$\sum_{\substack{\langle y\rangle\le\hat{R}\\ y\equiv b\,(\mathrm{mod}\,m)}}^{+} \Lambda(y)e(\alpha y) = S_1(\alpha) + \sum_{\hat{U}<\langle y\rangle\le\hat{R}}^{+} \upsilon(1,y).$$

Thus we are left to show that

$$\sum_{\hat{U}<\langle y\rangle\le\hat{R}}^{+} \upsilon(1,y) + S_4(\alpha) = S_2(\alpha) - S_3(\alpha).$$

Applying Lemma 17, we have

$$\sum_{\hat{U}<\langle y\rangle\le\hat{R}}^{+} \upsilon(1,y) + S_4(\alpha) = \sum_{\langle d\rangle\le\hat{U}}^{+} \sum_{\hat{U}<\langle y\rangle\le\hat{R}/\langle d\rangle}^{+} \sum_{\langle z\rangle\le\hat{R}/\langle yd\rangle}^{+} \mu(d)\upsilon(dz,y). \tag{8}$$

Since

$$S_3(\alpha) = \sum_{\langle d\rangle\le\hat{U}}^{+} \sum_{\langle y\rangle\le\hat{U}}^{+} \sum_{\langle z\rangle\le\hat{R}/\langle yd\rangle}^{+} \mu(d)\upsilon(dz,y),$$

by combining this with (8), we find that

$$\begin{aligned}\sum_{\hat{U}\le\langle y\rangle\le\hat{R}}^{+} \upsilon(1,y) + S_4(\alpha) &= \sum_{\langle d\rangle\le\hat{U}}^{+} \sum_{\langle y\rangle\le\hat{R}/\langle d\rangle}^{+} \sum_{\langle z\rangle\le\hat{R}/\langle yd\rangle}^{+} \mu(d)\upsilon(dz,y) - S_3(\alpha)\\ &= \sum_{\langle d\rangle\le\hat{U}}^{+} \sum_{\langle y\rangle\le\hat{R}/\langle d\rangle}^{+} \sum_{\substack{\langle z\rangle\le\hat{R}/\langle yd\rangle\\ dyz\equiv b\,(\mathrm{mod}\,m)}}^{+} \mu(d)\Lambda(y)e(\alpha dyz) - S_3(\alpha)\\ &= \sum_{\langle d\rangle\le\hat{U}}^{+} \sum_{\substack{\langle w\rangle\le\hat{R}/\langle d\rangle\\ dw\equiv b\,(\mathrm{mod}\,m)}}^{+} \mu(d)e(\alpha dw) \sum_{v\mid w}^{+} \Lambda(v) - S_3(\alpha)\end{aligned}$$

$$= \sideset{}{^+}\sum_{\langle d\rangle \le \hat{U}} \sideset{}{^+}\sum_{\substack{\langle w\rangle \le \hat{R}/\langle d\rangle \\ dw \equiv b \,(\mathrm{mod}\, m)}} \mu(d)(\mathrm{ord}\, w)e(\alpha dw) - S_3(\alpha)$$

$$= S_2(\alpha) - S_3(\alpha).$$

The lemma now follows.

We will now obtain upper bounds for the sums $S_1(\alpha)$, $S_2(\alpha)$, $S_3(\alpha)$ and $S_4(\alpha)$.

Lemma 19. *One has*

$$S_1(\alpha) \ll \hat{U}U.$$

Proof. By applying the triangle inequality and the trivial bound, we have

$$S_1(\alpha) \ll \sideset{}{^+}\sum_{\substack{\langle y\rangle \le \hat{U} \\ y \equiv b \,(\mathrm{mod}\, m)}} \Lambda(y) \ll \hat{U}U.$$

Lemma 20. *Suppose that* $\langle \alpha - a/g\rangle < \langle g\rangle^{-2}$ *with* $(a, g) = 1$. *Assume that* $S, R \in \mathbb{N}$ *with* $S \le R$. *Then for any real number* T *with* $T \le \hat{R}/\hat{S}$, *we have*

$$\sideset{}{^+}\sum_{\langle x\rangle \le \hat{S}} \left| \sideset{}{^+}\sum_{T < \langle y\rangle \le \hat{R}/\langle x\rangle} e(\alpha xy) \right| \ll \hat{R}S\langle g\rangle^{-1} + \hat{S}R + \langle g\rangle(RS + \mathrm{ord}\, g).$$

Proof. By the triangle inequality, we have

$$\sideset{}{^+}\sum_{\langle x\rangle \le \hat{S}} \left| \sideset{}{^+}\sum_{T < \langle y\rangle \le \hat{R}/\langle x\rangle} e(\alpha xy) \right| \le \sideset{}{^+}\sum_{\langle x\rangle \le \hat{S}} \sum_{W=0}^{R - \mathrm{ord}\, x} \left| \sideset{}{^+}\sum_{\langle y\rangle = \hat{W}} e(\alpha xy) \right|. \tag{9}$$

Also, it was proved in [10, Lemma 7] that

$$\left| \sideset{}{^+}\sum_{\langle y\rangle = \hat{W}} e(\alpha xy) \right| = \begin{cases} \hat{W}, & \text{when } \langle \|\alpha x\|\rangle < \hat{W}^{-1}, \\ 0, & \text{otherwise.} \end{cases}$$

Thus we have

$$\sideset{}{^+}\sum_{\langle x\rangle \le \hat{S}} \sum_{W=0}^{R - \mathrm{ord}\, x} \left| \sideset{}{^+}\sum_{\langle y\rangle = \hat{W}} e(\alpha xy) \right|$$

$$= \sideset{}{^+}\sum_{\langle x\rangle \le \hat{S}} \sum_{W=0}^{\min(R - \mathrm{ord}\, x,\, -\mathrm{ord}\, \|\alpha x\| - 1)} \hat{W}$$

$$\begin{aligned}
&\ll \sum_{\langle x\rangle \le \hat{S}}^{+} \min\left(\hat{R}/\langle x\rangle, \langle\|\alpha x\|\rangle^{-1}\right) \\
&\ll \sum_{\langle x\rangle < \langle g\rangle}^{+} \langle\|\alpha x\|\rangle^{-1} + \sum_{W=\operatorname{ord} g}^{S} \sum_{\langle x\rangle = \hat{W}}^{+} \min\left(\hat{R}/\langle x\rangle, \langle\|\alpha x\|\rangle^{-1}\right) \\
&= E_1 + E_2, \qquad (10)
\end{aligned}$$

where

$$E_1 = \sum_{\langle x\rangle < \langle g\rangle}^{+} \langle\|\alpha x\|\rangle^{-1} \qquad \text{and} \qquad E_2 = \sum_{W=\operatorname{ord} g}^{S} \sum_{\langle x\rangle = \hat{W}}^{+} \min\left(\hat{R}/\langle x\rangle, \langle\|\alpha x\|\rangle^{-1}\right).$$

We first bound E_1. For $\langle x\rangle < \langle g\rangle$, since $\langle \alpha - a/g\rangle < \langle g\rangle^{-2}$, we have

$$\left\langle \alpha x - \frac{ax}{g}\right\rangle < \frac{\langle x\rangle}{\langle g\rangle^2} < \frac{1}{\langle g\rangle}.$$

Since $\langle\|ax/g\|\rangle \ge \langle g\rangle^{-1}$, we deduce that

$$\langle\|\alpha x\|\rangle = \left\langle \left\| \frac{ax}{g} + \left(\alpha - \frac{a}{g}\right)x \right\| \right\rangle = \left\langle \left\| \frac{ax}{g} \right\| + \left(\alpha - \frac{a}{g}\right)x \right\rangle = \left\langle \left\| \frac{ax}{g} \right\| \right\rangle.$$

Since $(a, g) = 1$, we have

$$\begin{aligned}
E_1 &\le \sum_{\langle x\rangle < \langle g\rangle} \langle\|\alpha x\|\rangle^{-1} = \sum_{\langle x\rangle < \langle g\rangle} \left\langle \left\| \frac{ax}{g} \right\| \right\rangle^{-1} = \sum_{\langle y\rangle < \langle g\rangle} \left\langle \left\| \frac{y}{g} \right\| \right\rangle^{-1} \\
&\ll \sum_{W=0}^{\operatorname{ord} g - 1} \hat{W}\left(\frac{\langle g\rangle}{\hat{W}}\right) \ll \langle g\rangle(\operatorname{ord} g). \qquad (11)
\end{aligned}$$

We are now left to bound E_2. Note that

$$\begin{aligned}
E_2 &= \sum_{W=\operatorname{ord} g}^{S} \Bigg(\sum_{\substack{\langle x\rangle = \hat{W} \\ \langle\|\alpha x\|\rangle^{-1} \ge \hat{R}/\hat{W}}}^{+} \frac{\hat{R}}{\hat{W}} + \sum_{V=0}^{R-W-1} \sum_{\substack{\langle x\rangle = \hat{W} \\ \langle\|\alpha x\|\rangle^{-1} = \hat{V}}}^{+} \hat{V} \Bigg) \\
&\le \sum_{W=\operatorname{ord} g}^{S} \sum_{V=0}^{R-W} \sum_{\substack{\langle x\rangle = \hat{W} \\ \langle\|\alpha x\|\rangle^{-1} \ge \hat{V}}}^{+} \hat{V} \le \sum_{W=\operatorname{ord} g}^{S} \sum_{V=0}^{R-W} \sum_{\substack{\langle x\rangle < q\hat{W} \\ \langle\|\alpha x\|\rangle < q\hat{V}^{-1}}} \hat{V}.
\end{aligned}$$

By [10, Lemma 7], we deduce that

$$E_2 \ll \sum_{W=\operatorname{ord} g}^{S} \sum_{V=0}^{R-W} \sum_{\langle x\rangle < q\hat{W}} \Big| \sum_{\langle y\rangle < \hat{V}q^{-1}} e(\alpha xy)\Big|.$$

We now apply [15, Lemma 11.1] to get

$$\sum_{\langle x\rangle < q\hat{W}} \Big| \sum_{\langle y\rangle < \hat{V}q^{-1}} e(\alpha xy)\Big| \ll \hat{W}\hat{V}\Big(\langle g\rangle^{-1} + \hat{W}^{-1} + \hat{V}^{-1} + \langle g\rangle \hat{W}^{-1}\hat{V}^{-1}\Big).$$

Using this bound, we see that

$$\begin{aligned} E_2 &\ll \sum_{W=\operatorname{ord} g}^{S} \sum_{V=0}^{R-W} \hat{W}\hat{V}\Big(\langle g\rangle^{-1} + \hat{W}^{-1} + \hat{V}^{-1} + \langle g\rangle \hat{W}^{-1}\hat{V}^{-1}\Big) \\ &\ll \sum_{W=\operatorname{ord} g}^{S} \Big(\hat{R}\langle g\rangle^{-1} + \hat{R}\hat{W}^{-1} + \hat{W}R + \langle g\rangle R\Big) \qquad (12) \\ &\ll \hat{R}S\langle g\rangle^{-1} + \hat{S}R + \langle g\rangle RS. \end{aligned}$$

The lemma now follows by combining (9)–(12).

Lemma 21. *Suppose that* $\langle \alpha - a/g\rangle < \langle g\rangle^{-2}$ *with* $(a, g) = 1$ *and* $\operatorname{ord} m < U$. *Then one has*

$$S_2(\alpha) \ll \hat{U}\langle m\rangle R^2 + \hat{R}R^2\langle g\rangle^{-1} + \langle g\rangle R(R^2 + \operatorname{ord} g).$$

Proof. Note that

$$\begin{aligned} S_2(\alpha) &= \sideset{}{^+}\sum_{\substack{\langle xy\rangle \le \hat{R} \\ \langle x\rangle \le \hat{U} \\ xy \equiv b \,(\mathrm{mod}\, m)}} \mu(x)(\operatorname{ord} y)e(\alpha xy) \\ &= \sideset{}{^+}\sum_{\langle x\rangle \le \hat{U}} \mu(x) \sideset{}{^+}\sum_{\substack{\langle y\rangle \le \hat{R}/\langle x\rangle \\ xy \equiv b \,(\mathrm{mod}\, m)}} e(\alpha xy) \int_1^{\langle y\rangle} \frac{dt}{t \log q} \\ &= \sideset{}{^+}\sum_{\langle x\rangle \le \hat{U}} \mu(x) \int_1^{\hat{R}/\langle x\rangle} \Bigg(\sideset{}{^+}\sum_{\substack{t < \langle y\rangle \le \hat{R}/\langle x\rangle \\ xy \equiv b \,(\mathrm{mod}\, m)}} e(\alpha xy)\Bigg) \frac{dt}{t \log q}. \end{aligned}$$

By two applications of the triangle inequality, we get

$$S_2(\alpha) \ll \sideset{}{^+}\sum_{\langle x\rangle \le \hat{U}} \int_1^{\hat{R}/\langle x\rangle} \Bigg| \sideset{}{^+}\sum_{\substack{t < \langle y\rangle \le \hat{R}/\langle x\rangle \\ xy \equiv b \,(\mathrm{mod}\, m)}} e(\alpha xy)\Bigg| \frac{dt}{t}.$$

Switching the leftmost sum with the integral in the last expression, we obtain

$$S_2(\alpha) \ll \int_1^{\hat{R}} \sum_{\langle x\rangle \le \min(\hat{U},\hat{R}/t)}^{+} \Bigg| \sum_{\substack{t<\langle y\rangle \le \hat{R}/\langle x\rangle \\ xy\equiv b \,(\mathrm{mod}\, m)}}^{+} e(\alpha xy)\Bigg| \frac{dt}{t} \ll \int_1^{\hat{R}} \sum_{\substack{\langle x\rangle \le \hat{U} \\ (x,m)=1}}^{+} \Bigg| \sum_{\substack{t<\langle y\rangle \le \hat{R}/\langle x\rangle \\ y\equiv \bar{x}b \,(\mathrm{mod}\, m)}}^{+} e(\alpha xy)\Bigg| \frac{dt}{t},$$

where $\bar{x}$ is the multiplicative inverse of x modulo m. We now split the sum over y into two sums depending on whether or not $\langle y\rangle \le \langle m\rangle$. Write $y = \bar{x}b + my'$ and $x' = mx$. Then by the triangle inequality, we have

$$\begin{aligned} S_2(\alpha) &\ll \int_1^{\hat{R}} \hat{U}\langle m\rangle \frac{dt}{t} + \int_1^{\hat{R}} \sum_{\substack{\langle x\rangle \le \hat{U} \\ (x,m)=1}}^{+} \Bigg| \sum_{\substack{\max(t,\langle m\rangle)<\langle y\rangle \le \hat{R}/\langle x\rangle \\ y\equiv \bar{x}b \,(\mathrm{mod}\, m)}}^{+} e(\alpha xy)\Bigg| \frac{dt}{t} \\ &\ll \hat{U}\langle m\rangle R + \int_1^{\hat{R}} \sum_{\substack{\langle x\rangle \le \hat{U} \\ (x,m)=1}}^{+} \Bigg| \sum_{\max(t/\langle m\rangle,1)<\langle y'\rangle \le \hat{R}/\langle mx\rangle}^{+} e(\alpha mxy')\Bigg| \big|e(\alpha b)\big| \frac{dt}{t} \\ &\ll \hat{U}\langle m\rangle R + \int_1^{\hat{R}} \sum_{\langle x'\rangle \le \hat{U}\langle m\rangle}^{+} \Bigg| \sum_{\max(t/\langle m\rangle,1)<\langle y'\rangle \le \hat{R}/\langle x'\rangle}^{+} e(\alpha x'y')\Bigg| \frac{dt}{t}. \end{aligned}$$

Since $\operatorname{ord} m < U < R$, by Lemma 20, we deduce that

$$\begin{aligned} S_2(\alpha) \ll \hat{U}\langle m\rangle R + \int_1^{\hat{R}} \Big(&\hat{R}(U + \operatorname{ord} m)\langle g\rangle^{-1} + \hat{U}\langle m\rangle R \\ &+\langle g\rangle((U + \operatorname{ord} m)R + \operatorname{ord} g)\Big) \frac{dt}{t} \\ \ll \hat{U}\langle m\rangle R^2 &+ \hat{R}R^2\langle g\rangle^{-1} + \langle g\rangle R(R^2 + \operatorname{ord} g). \end{aligned}$$

This completes the proof of the lemma.

Lemma 22. *Suppose that* $\langle \alpha - a/g\rangle < \langle g\rangle^{-2}$ *with* $(a, g) = 1$ *and* $\operatorname{ord} m < U$. *Then one has*

$$S_3(\alpha) \ll \hat{R}R^2\langle g\rangle^{-1} + \hat{U}^2\langle m\rangle R^2 + \langle g\rangle R(R^2 + \operatorname{ord} g).$$

Proof. For any $\langle x\rangle \le \hat{U}^2$, we have

$$\sum_{\substack{x=uv \\ \langle u\rangle,\langle v\rangle \le \hat{U}}}^{+} \mu(u)\Lambda(v) \ll \sum_{v|x}^{+} \Lambda(v) = \operatorname{ord} x \ll R.$$

Write $y = \bar{x}b + my'$ and $x' = mx$, where $\bar{x}$ is the multiplicative inverse of x modulo m. Then from the above inequality, we deduce that

$$\begin{aligned}
S_3(\alpha) &= \sideset{}{^+}\sum_{\substack{\langle xy\rangle \le \hat{R} \\ \langle x\rangle \le \hat{U}^2 \\ xy \equiv b \,(\mathrm{mod}\, m)}} \sideset{}{^+}\sum_{\substack{x=uv \\ \langle u\rangle, \langle v\rangle \le \hat{U}}} \mu(u)\Lambda(v)e(\alpha xy) \\
&\ll \sideset{}{^+}\sum_{\langle x\rangle \le \hat{U}^2} \Bigg| \sideset{}{^+}\sum_{\substack{x=uv \\ \langle u\rangle, \langle v\rangle \le \hat{U}}} \mu(u)\Lambda(v)\Bigg| \cdot \Bigg| \sideset{}{^+}\sum_{\substack{\langle y\rangle \le \hat{R}/\langle x\rangle \\ xy \equiv b \,(\mathrm{mod}\, m)}} e(\alpha xy)\Bigg| \\
&\ll R \sideset{}{^+}\sum_{\substack{\langle x\rangle \le \hat{U}^2 \\ (x,m)=1}} \Bigg| \sideset{}{^+}\sum_{\substack{\langle y\rangle \le \hat{R}/\langle x\rangle \\ y \equiv \bar{x}b \,(\mathrm{mod}\, m)}} e(\alpha xy)\Bigg| \\
&= R \sideset{}{^+}\sum_{\substack{\langle x\rangle \le \hat{U}^2 \\ (x,m)=1}} \Bigg| \sideset{}{^+}\sum_{\langle y'\rangle \le \hat{R}/\langle mx\rangle} e(\alpha mxy')\Bigg| \\
&\ll R \sideset{}{^+}\sum_{\langle x'\rangle \le \hat{U}^2\langle m\rangle} \Bigg| \sideset{}{^+}\sum_{\langle y'\rangle \le \hat{R}/\langle x'\rangle} e(\alpha x'y')\Bigg|.
\end{aligned}$$

Since $\mathrm{ord}\, m < U < R$, by Lemma 20, we obtain that

$$\begin{aligned}
S_3(\alpha) &\ll R\big(\hat{R}(2U + \mathrm{ord}\, m)\langle g\rangle^{-1} + \hat{U}^2\langle m\rangle R + \langle g\rangle((2U + \mathrm{ord}\, m)R + \mathrm{ord}\, g)\big) \\
&\ll \hat{R}R^2\langle g\rangle^{-1} + \hat{U}^2\langle m\rangle R^2 + \langle g\rangle R(R^2 + \mathrm{ord}\, g).
\end{aligned}$$

This completes the proof of the lemma.

Lemma 23. *Suppose that* $\langle \alpha - a/g\rangle < \langle g\rangle^{-2}$ *with* $(a, g) = 1$ *and* $\mathrm{ord}\, m \le U$. *Then one has*

$$S_4(\alpha) \ll \hat{R}R^{9/2}\langle m\rangle^{1/2}\langle g\rangle^{-1/2} + \hat{R}R^{9/2}\langle m\rangle\hat{U}^{-1/2} + \hat{R}^{1/2}R^{9/2}\langle m\rangle^{1/2}\langle g\rangle^{1/2}.$$

Proof. By writing $x = y_1 z$, $z = rs$ and $y_1 = uv$, we have

$$\begin{aligned}
\sideset{}{^+}\sum_{\substack{\langle x\rangle = \hat{V} \\ x \equiv \tilde{x} \,(\mathrm{mod}\, m)}} |\tau_x|^2 &\le \sideset{}{^+}\sum_{\langle x\rangle = \hat{V}} \tau_x^2 \le \sideset{}{^+}\sum_{\langle x\rangle = \hat{V}} \Big(\sideset{}{^+}\sum_{y|x} 1\Big)^2 = \sideset{}{^+}\sum_{\langle y_1\rangle \le \hat{V}} \sideset{}{^+}\sum_{\substack{\langle x\rangle = \hat{V} \\ y_1|x}} \sideset{}{^+}\sum_{y_2|x} 1 \\
&= \sideset{}{^+}\sum_{\langle y_1\rangle \le \hat{V}} \sideset{}{^+}\sum_{\langle z\rangle = \hat{V}/\langle y_1\rangle} \sideset{}{^+}\sum_{y_2|y_1 z} 1 \\
&\le \sideset{}{^+}\sum_{\langle y_1\rangle \le \hat{V}} \sideset{}{^+}\sum_{d_1|y_1} \sideset{}{^+}\sum_{\langle z\rangle = \hat{V}/\langle y_1\rangle} \sideset{}{^+}\sum_{d_2|z} 1 = \sideset{}{^+}\sum_{\langle y_1\rangle \le \hat{V}} \sideset{}{^+}\sum_{d_1|y_1} \sideset{}{^+}\sum_{\substack{r,s \\ \langle rs\rangle = \hat{V}/\langle y_1\rangle}} 1 \qquad (13) \\
&\ll \hat{V}V \sideset{}{^+}\sum_{\langle y_1\rangle \le \hat{V}} \sideset{}{^+}\sum_{d_1|y_1} \langle y_1\rangle^{-1} = \hat{V}V \sideset{}{^+}\sum_{\substack{u,v \\ \langle uv\rangle \le \hat{V}}} \langle uv\rangle^{-1} \ll \hat{V}V^3.
\end{aligned}$$

Note that

$$S_4(\alpha) = \sum^{+}_{\substack{\langle xy\rangle \le \hat{R} \\ \langle x\rangle,\langle y\rangle > \hat{U} \\ xy\equiv b \,(\mathrm{mod}\, m)}} \tau_x \Lambda(y) e(\alpha xy)$$

$$= \sum_{\substack{U<V,W<R-U \\ V+W\le R}} \sum_{\substack{\tilde{x},\tilde{y} \\ \tilde{x}\tilde{y}\equiv b \,(\mathrm{mod}\, m)}} \sum^{+}_{\substack{\langle x\rangle=\hat{V} \\ x\equiv\tilde{x} \,(\mathrm{mod}\, m)}} \tau_x \sum^{+}_{\substack{\langle y\rangle=\hat{W} \\ y\equiv\tilde{y} \,(\mathrm{mod}\, m)}} \Lambda(y) e(\alpha xy).$$

Applying the Cauchy-Schwarz inequality and (13), we obtain that

$$S_4(\alpha) \ll \sum_{\substack{U<V,W<R-U \\ V+W\le R}} \sum_{\substack{\tilde{x},\tilde{y} \\ \tilde{x}\tilde{y}\equiv b \,(\mathrm{mod}\, m)}} \Bigg(\sum^{+}_{\substack{\langle x\rangle=\hat{V} \\ x\equiv\tilde{x} \,(\mathrm{mod}\, m)}} |\tau_x|^2 \Bigg)^{1/2}$$

$$\times \Bigg(\sum^{+}_{\substack{\langle x\rangle=\hat{V} \\ x\equiv\tilde{x} \,(\mathrm{mod}\, m)}} \Bigg| \sum^{+}_{\substack{\langle y\rangle=\hat{W} \\ y\equiv\tilde{y} \,(\mathrm{mod}\, m)}} \Lambda(y) e(\alpha xy) \Bigg|^2 \Bigg)^{1/2}$$

$$\ll \sum_{\substack{U<V,W<R-U \\ V+W\le R}} \sum_{\substack{\tilde{x},\tilde{y} \\ \tilde{x}\tilde{y}\equiv b \,(\mathrm{mod}\, m)}} \hat{V}^{1/2} V^{3/2} \Bigg(\sum^{+}_{\substack{\langle x\rangle=\hat{V} \\ x\equiv\tilde{x} \,(\mathrm{mod}\, m)}} \Bigg| \sum^{+}_{\substack{\langle y\rangle=\hat{W} \\ y\equiv\tilde{y} \,(\mathrm{mod}\, m)}} \Lambda(y) e(\alpha xy) \Bigg|^2 \Bigg)^{1/2}. \tag{14}$$

One has

$$\sum^{+}_{\substack{\langle x\rangle=\hat{V} \\ x\equiv\tilde{x} \,(\mathrm{mod}\, m)}} \Bigg| \sum^{+}_{\substack{\langle y\rangle=\hat{W} \\ y\equiv\tilde{y} \,(\mathrm{mod}\, m)}} \Lambda(y) e(\alpha xy) \Bigg|^2$$

$$= \sum^{+}_{\substack{\langle x\rangle=\hat{V} \\ x\equiv\tilde{x} \,(\mathrm{mod}\, m)}} \sum^{+}_{\substack{\langle y_1\rangle=\langle y_2\rangle=\hat{W} \\ y_1\equiv y_2\equiv\tilde{y} \,(\mathrm{mod}\, m)}} \Lambda(y_1)\Lambda(y_2) e(\alpha x(y_1-y_2))$$

$$= \sum^{+}_{\substack{\langle y\rangle=\hat{W} \\ y\equiv\tilde{y} \,(\mathrm{mod}\, m)}} \sum_{\substack{\langle h\rangle<\hat{W} \\ h\equiv 0 \,(\mathrm{mod}\, m)}} \Lambda(y)\Lambda(y+h) \sum^{+}_{\substack{\langle x\rangle=\hat{V} \\ x\equiv\tilde{x} \,(\mathrm{mod}\, m)}} e(\alpha xh). \tag{15}$$

For $(\tilde{x}, m) = (\tilde{y}, m) = 1$, $V + W \le R$ and $\langle m\rangle \le \min(\hat{V}, \hat{W})$, since $|\Lambda(z)| \le \operatorname{ord} z$, by writing $h = mh'$, $x = \tilde{x} + mx'$ and $h'' = m^2h'$, we have

$$\begin{aligned}
&\sideset{}{^+}\sum_{\substack{\langle y\rangle=\hat{W}\\ y\equiv\tilde{y}\,(\mathrm{mod}\,m)}}\ \sum_{\substack{\langle h\rangle<\hat{W}\\ h\equiv 0\,(\mathrm{mod}\,m)}} \Lambda(y)\Lambda(y+h) \sideset{}{^+}\sum_{\substack{\langle x\rangle=\hat{V}\\ x\equiv\tilde{x}\,(\mathrm{mod}\,m)}} e(\alpha xh)\\
&\quad\ll W^2 \sideset{}{^+}\sum_{\substack{\langle y\rangle=\hat{W}\\ y\equiv\tilde{y}\,(\mathrm{mod}\,m)}}\ \sum_{\substack{\langle h\rangle<\hat{W}\\ h\equiv 0\,(\mathrm{mod}\,m)}} \Big| \sideset{}{^+}\sum_{\substack{\langle x\rangle=\hat{V}\\ x\equiv\tilde{x}\,(\mathrm{mod}\,m)}} e(\alpha xh)\Big|\\
&\quad=\frac{\hat{W}W^2}{\langle m\rangle}\sum_{\langle h'\rangle<\hat{W}/\langle m\rangle}\Big|\sideset{}{^+}\sum_{\langle x'\rangle=\hat{V}/\langle m\rangle} e(\alpha m^2x'h')\Big|\\
&\quad\ll\frac{\hat{W}W^2}{\langle m\rangle}\sum_{\langle h''\rangle<\hat{W}\langle m\rangle}\Big|\sideset{}{^+}\sum_{\langle x'\rangle=\hat{V}/\langle m\rangle} e(\alpha x'h'')\Big|.
\end{aligned}\tag{16}$$

When $V+W\le R$ and $U\le\min(V,W)$, it follows from [15, Lemma 11.1] that

$$\begin{aligned}
\sum_{\langle h''\rangle<\hat{W}\langle m\rangle}\Big|\sideset{}{^+}\sum_{\langle x'\rangle=\hat{V}/\langle m\rangle} e(\alpha x'h'')\Big| &\le \sum_{\langle h''\rangle<\hat{W}\langle m\rangle}\Big|\sum_{\langle x'\rangle<q\hat{V}/\langle m\rangle} e(\alpha x'h'')\Big|\\
&\ll \hat{W}\hat{V}\big(\langle g\rangle^{-1}+\hat{W}^{-1}\langle m\rangle^{-1}+\hat{V}^{-1}\langle m\rangle+\langle g\rangle(\hat{W}\hat{V})^{-1}\big)\\
&\ll \hat{R}\langle g\rangle^{-1}+\hat{R}\langle m\rangle\hat{U}^{-1}+\langle g\rangle.
\end{aligned}\tag{17}$$

Upon combining (14)–(17), we have

$$\begin{aligned}
S_4(\alpha)&\ll\sum_{\substack{U<V,W<R-U\\ V+W\le R}}\ \sum_{\substack{\tilde{x},\tilde{y}\\ \tilde{x}\tilde{y}\equiv b\,(\mathrm{mod}\,m)}}\hat{V}^{1/2}V^{3/2}\hat{W}^{1/2}W\langle m\rangle^{-1/2}\big(\hat{R}\langle g\rangle^{-1}\\
&\qquad+\hat{R}\langle m\rangle\hat{U}^{-1}+\langle g\rangle\big)^{1/2}\\
&\ll\sum_{\substack{U<V,W<R-U\\ V+W\le R}}\hat{R}^{1/2}R^{5/2}\langle m\rangle^{1/2}\big(\hat{R}\langle g\rangle^{-1}+\hat{R}\langle m\rangle\hat{U}^{-1}+\langle g\rangle\big)^{1/2}\\
&\ll\hat{R}R^{9/2}\langle m\rangle^{1/2}\langle g\rangle^{-1/2}+\hat{R}R^{9/2}\langle m\rangle\hat{U}^{-1/2}+\hat{R}^{1/2}R^{9/2}\langle m\rangle^{1/2}\langle g\rangle^{1/2}.
\end{aligned}$$

Lemma 24. *Suppose that* $\langle m\rangle\le\hat{R}^{2/5}R$, $\langle g\rangle<\hat{R}\langle m\rangle$ *and* $\langle\alpha-a/g\rangle<\langle g\rangle^{-2}$ *with* $(a,g)=1$. *Then one has*

$$\begin{aligned}
\sideset{}{^+}\sum_{\substack{\langle y\rangle\le\hat{R}\\ y\equiv b\,(\mathrm{mod}\,m)}}\Lambda(y)e(\alpha y)&\ll\hat{R}^{4/5}\langle m\rangle R^4+\langle g\rangle R^3+\hat{R}R^{9/2}\langle m\rangle^{1/2}\langle g\rangle^{-1/2}\\
&\quad+\hat{R}^{1/2}R^{9/2}\langle m\rangle^{1/2}\langle g\rangle^{1/2}.
\end{aligned}$$

Proof. We deduce from Lemmas 18, 19, 21, 22 and 23 that when $\operatorname{ord} m \leq U$, we have

$$\sum_{\substack{\langle y\rangle \leq \hat{R} \\ y \equiv b \,(\operatorname{mod} m)}}^{+} \Lambda(y)e(\alpha y) \ll \hat{U}^2\langle m\rangle R^2 + \langle g\rangle R(R^2 + \operatorname{ord} g) + \hat{R}R^{9/2}\langle m\rangle^{1/2}\langle g\rangle^{-1/2}$$
$$+ \hat{R}R^{9/2}\langle m\rangle \hat{U}^{-1/2} + \hat{R}^{1/2}R^{9/2}\langle m\rangle^{1/2}\langle g\rangle^{1/2}.$$

The lemma now follows by setting $\hat{U} = \hat{R}^{2/5}R$.

We will now derive Lemma 16 from Lemma 24.

Proof (of Lemma 16). Note that

$$\lambda^{\wedge}_{b,m,N}(\alpha) = \frac{(N + \operatorname{ord} m)\phi(m)}{\hat{N}\langle m\rangle} \sum_{\substack{\langle n\rangle = \hat{N} \\ mn+b \text{ irred}}}^{+} e(\alpha n)$$
$$= \frac{\phi(m)}{\hat{N}\langle m\rangle} \sum_{\langle n\rangle = \hat{N}}^{+} \Lambda(mn+b)e(\alpha n)$$
$$+ O\Big(\frac{N + \operatorname{ord} m}{\hat{N}}\Big(\sum_{\substack{\langle \varpi\rangle = (\hat{N}\langle m\rangle)^{1/2} \\ \varpi \text{ irred}}}^{+} \frac{1}{2} + \sum_{\substack{\langle \varpi\rangle = (\hat{N}\langle m\rangle)^{1/3} \\ \varpi \text{ irred}}}^{+} \frac{1}{3} + \cdots\Big)\Big)$$
$$= \frac{\phi(m)}{\hat{N}\langle m\rangle} \sum_{\langle n\rangle = \hat{N}}^{+} \Lambda(mn+b)e(\alpha n) + O\big(\hat{N}^{-1/2}\langle m\rangle^{1/2}\big).$$

By writing $x = mn + b$, we have

$$\lambda^{\wedge}_{b,m,N}(\alpha) = \frac{\phi(m)e(-\alpha b/m)}{\hat{N}\langle m\rangle} \sum_{\substack{\langle x\rangle = \hat{N}\langle m\rangle \\ x \equiv b \,(\operatorname{mod} m)}}^{+} \Lambda(x)e(\alpha x/m) + O\big(\hat{N}^{-1/2}\langle m\rangle^{1/2}\big).$$

By the triangle inequality, we deduce that

$$\lambda^{\wedge}_{b,m,N}(\alpha) \ll \hat{N}^{-1}\Bigg(\Bigg| \sum_{\substack{\langle x\rangle \leq \hat{N}\langle m\rangle \\ x \equiv b \,(\operatorname{mod} m)}}^{+} \Lambda(x)e(\alpha x/m)\Bigg| + \Bigg| \sum_{\substack{\langle x\rangle \leq q^{-1}\hat{N}\langle m\rangle \\ x \equiv b \,(\operatorname{mod} m)}}^{+} \Lambda(x)e(\alpha x/m)\Bigg|\Bigg)$$
$$+ \hat{N}^{-1/2}\langle m\rangle^{1/2}. \tag{18}$$

Let $\alpha \in \mathfrak{m}$. By Dirichlet's approximation theorem, there exist $a, g \in \mathbb{F}_q[t]$ with g monic, $\langle g\rangle \leq \hat{N}\langle m\rangle/N^B$, $(a, g) = 1$ and $\langle \alpha/m - a/g\rangle < N^B/(\langle mg\rangle\hat{N}) \leq \langle g\rangle^{-2}$. Let $d = (g, m)$. Then

$$\left\langle \alpha - \frac{am/d}{g/d} \right\rangle < \frac{N^B}{\langle g \rangle \hat{N}} \le \frac{N^B}{\langle g/d \rangle \hat{N}}.$$

Since $\alpha \in \mathfrak{m}$, we must have $\langle g/d \rangle > N^B$, which implies that $\langle g \rangle > N^B \langle d \rangle \ge N^B$. By Lemma 24 and (18), we have

$$\begin{aligned}
&\lambda^{\wedge}_{b,m,N}(\alpha) \\
&\ll \hat{N}^{-1}\big(\hat{N}^{4/5}\langle m \rangle^{9/5} N^4 + \langle g \rangle N^3 + \hat{N} N^{9/2} \langle m \rangle^{3/2} \langle g \rangle^{-1/2} + \hat{N}^{1/2} N^{9/2} \langle m \rangle \langle g \rangle^{1/2}\big) \\
&\qquad + \hat{N}^{-1/2} \langle m \rangle^{1/2} \\
&\ll \hat{N}^{-1/5} N^{29/5} + N^{4-B} + N^{6-B/2} + \hat{N}^{-1/2} N^{1/2} \ll N^{6-B/2} = N^{-A}.
\end{aligned}$$

This completes the proof of the lemma.

We will next prove a minor arc estimate for $\lambda^{(Q)\wedge}_{b,m,N}(\alpha)$.

Lemma 25. *Let* $1 \le Q \le K$ *and* $\langle m \rangle \le N$. *Suppose that* $\langle \alpha - a/g \rangle < \langle g \rangle^{-2}$ *with* $(a, g) = 1$. *Then one has*

$$|\lambda^{(Q)\wedge}_{b,m,N}(\alpha)| \ll \log_q N \Big(N \langle g \rangle^{-1} + \hat{N}^{-1} \langle g \rangle (N^2 + \operatorname{ord} g) + \hat{N}^{-1/(3A)} N \Big).$$

Proof. Let $\{\varpi_1, \ldots, \varpi_R\}$ denote the set of monic, irreducible polynomials ϖ with $\langle \varpi \rangle \le \hat{Q}$ and $\varpi \nmid m$. By the inclusion-exclusion principle, we have

$$\lambda^{(Q)\wedge}_{b,m,N}(\alpha) = \sum_{n \in \mathscr{S}_N} \lambda^{(Q)}_{b,m,N}(n) e(\alpha n) = \hat{N}^{-1} \prod_{i=1}^{R} (1 - 1/\langle \varpi_i \rangle)^{-1} h(\alpha), \tag{19}$$

where

$$h(\alpha) = \sum_{s=0}^{R} (-1)^s \sum_{1 \le i_1 < \cdots < i_s \le R} \sideset{}{^+}\sum_{\substack{\langle y \rangle = \hat{N}\langle m \rangle / \langle \varpi_1 \cdots \varpi_s \rangle \\ \varpi_{i_1} \cdots \varpi_{i_s} y \equiv b \,(\mathrm{mod}\, m)}} e\left(\alpha \left(\frac{\varpi_{i_1} \cdots \varpi_{i_s} y - b}{m} \right) \right). \tag{20}$$

By Lemma 4, we have

$$\prod_{i=1}^{R} \big(1 - 1/\langle \varpi_i \rangle\big)^{-1} \ll Q \ll \log_q N. \tag{21}$$

Let $J = N/(2A \log_q N)$. If $0 \le s \le J$, since $\langle \varpi_i \rangle \le \hat{Q} \le N^A$, we have

$$\prod_{j=1}^{s} \langle \varpi_{i_j} \rangle \le N^{AN/(2A \log_q N)} = N^{N/(2 \log_q N)} = \hat{N}^{1/2}.$$

Therefore, by writing $y = \bar{x}b + my'$, where $\bar{x}$ is the multiplicative inverse of x modulo m, it follow from Lemma 20 that

$$\begin{aligned}
&\sum_{0\le s\le J}(-1)^s \sum_{1\le i_1<\cdots<i_s\le R} \sideset{}{^+}\sum_{\substack{\langle y\rangle=\hat{N}\langle m\rangle/\langle \varpi_1\cdots\varpi_s\rangle\\ \varpi_{i_1}\cdots\varpi_{i_s}y\equiv b\,(\mathrm{mod}\,m)}} e\left(\alpha\left(\frac{\varpi_{i_1}\cdots\varpi_{i_s}y-b}{m}\right)\right)\\
&\quad\ll \sideset{}{^+}\sum_{\substack{\langle x\rangle\le\hat{N}^{1/2}\\ (x,m)=1}}\left|\sideset{}{^+}\sum_{\substack{\langle y\rangle=\hat{N}\langle m\rangle/\langle x\rangle\\ xy\equiv b\,(\mathrm{mod}\,m)}} e\left(\frac{\alpha xy}{m}\right)\right| \ll \sideset{}{^+}\sum_{\langle x\rangle\le\hat{N}^{1/2}}\left|\sideset{}{^+}\sum_{\langle y'\rangle=\hat{N}/\langle x\rangle} e(\alpha xy')\right| \qquad (22)\\
&\quad\ll \hat{N}N\langle g\rangle^{-1}+\hat{N}^{1/2}N+\langle g\rangle(N^2+\mathrm{ord}\,g).
\end{aligned}$$

For $s > J$, we have

$$\begin{aligned}
&\sum_{J<s\le R}(-1)^s \sum_{1\le i_1<\cdots<i_s\le R} \sideset{}{^+}\sum_{\substack{\langle y\rangle=\hat{N}\langle m\rangle/\langle \varpi_1\cdots\varpi_s\rangle\\ \varpi_{i_1}\cdots\varpi_{i_s}y\equiv b\,(\mathrm{mod}\,m)}} e\left(\alpha\left(\frac{\varpi_{i_1}\cdots\varpi_{i_s}y-b}{m}\right)\right)\\
&\ll \sum_{J<s\le R}\sum_{1\le i_1<\cdots<i_s\le R}\hat{N}\langle m\rangle/\langle\varpi_{i_1}\cdots\varpi_{i_s}\rangle \qquad (23)\\
&\ll \hat{N}\langle m\rangle\sum_{J<s\le R}(s!)^{-1}\left(\langle\varpi_1\rangle^{-1}+\cdots+\langle\varpi_R\rangle^{-1}\right)^s\\
&\ll \hat{N}\langle m\rangle\sum_{J<s\le R}(s!)^{-1}(C_1\log_q\log_q N)^s,
\end{aligned}$$

where the last inequality follows from Lemma 4. By Stirling's formula, we have $s! = \sqrt{2\pi s}\left(\frac{s}{e}\right)^s\left(1+O\left(\frac{1}{s}\right)\right)$. Thus for $s > J = N/2A\log_q N$, we have

$$\begin{aligned}
\sum_{J<s\le R}(s!)^{-1}(C_1\log\log N)^s &\ll \sum_{J<s\le R}s^{-1/2}\left(\frac{C_1e\log_q\log_q N}{s}\right)^s\\
&\ll \sum_{J<s\le R}\left(\frac{2A\log_q N}{N}\right)^{1/2}\left(\frac{2C_1Ae\log_q N\log_q\log_q N}{N}\right)^s\\
&\ll \left(\frac{\log_q N}{N}\right)^{1/2}\sum_{J<s\le R}N^{s\left(-1+o(1)\right)}\\
&\ll \left(\frac{\log_q N}{N}\right)^{1/2}N^{\left(-1+o(1)\right)N/(2A\log_q N)}\\
&\ll \left(\frac{\log_q N}{N}\right)^{1/2}\hat{N}^{-1/(2A)+o(1)} \ll \hat{N}^{-1/(3A)}. \qquad (24)
\end{aligned}$$

By combining (19)–(24), we deduce that

$$|\lambda_{b,m,N}^{(Q)\wedge}(\alpha)| \ll \hat{N}^{-1}\log_q N\big(\hat{N}N\langle g\rangle^{-1} + \hat{N}^{1/2}N + \langle g\rangle(N^2 + \operatorname{ord} g) + \hat{N}^{1-1/(3A)}\langle m\rangle\big)$$
$$\ll \log_q N\big(N\langle g\rangle^{-1} + \hat{N}^{-1}\langle g\rangle(N^2 + \operatorname{ord} g) + \hat{N}^{-1/(3A)}N\big).$$

Lemma 26. *Let* $1 \le Q \le K$ *and* $\langle m\rangle \le N$. *One has*

$$\sup_{\alpha\in\mathfrak{m}} |\lambda_{b,m,N}^{(Q)\wedge}(\alpha)| \ll N^{2-B}\log_q N \ll N^{-A}.$$

Proof. Let $\alpha \in \mathfrak{m}$. By Dirichlet's approximation theorem, there exist $a, g \in \mathbb{F}_q[t]$ with g monic, $\langle g\rangle \le \hat{N}/N^B$, $(a, g) = 1$ and $\langle \alpha - a/g\rangle < N^B/(\langle g\rangle\hat{N}) \le \langle g\rangle^{-2}$. Since $\alpha \in \mathfrak{m}$, we have $\langle g\rangle > N^B$. By Lemma 25,

$$|\lambda_{b,m,N}^{(Q)\wedge}(\alpha)| \ll \log_q N\big(N\langle g\rangle^{-1} + \hat{N}^{-1}\langle g\rangle(N^2 + \operatorname{ord} g) + \hat{N}^{-1/(3A)}N\big)$$
$$\ll N^{2-B}\log_q N \ll N^{-A}.$$

We now summarize the minor arc contribution in Proposition 6.

Lemma 27. *For* $1 \le Q \le K$, *we have*

$$\sup_{\alpha\in\mathfrak{m}} \big|\lambda_{b,m,N}^{\wedge}(\alpha) - \lambda_{b,m,N}^{(Q)\wedge}(\alpha)\big| \ll N^{-A} \ll \hat{Q}^{-1}.$$

Proof. The lemma follows by combining Lemmas 16 and 26 and noting that

$$N^{-A} \ll \hat{K}^{-1} \ll \hat{Q}^{-1}.$$

Note that by combining Lemmas 15 and 27, we obtain Proposition 6.

5 Proofs of Theorems 2 and 3

We will first prove Theorem 2.

Proof (of Theorem 2). By Lemmas 5 and 7, for $1 \le Q \le K$, we have

$$\|f * \psi_Q^{\wedge}\|_\infty \ll \hat{Q}^{-1}\|f\|_1 \qquad \text{and} \qquad \|f * \psi_Q^{\wedge}\|_2 \ll Q\hat{N}^{-1}\|f\|_2.$$

By the Riesz-Thorin interpolation theorem [17, 25], we interpolate between these two bounds to find that for $\delta \ge 2$, we have

$$\|f * \psi_Q^{\wedge}\|_\delta \ll \hat{Q}^{-1+2/\delta}Q^{2/\delta}\hat{N}^{-2/\delta}\|f\|_{\delta'}.$$

Similarly, since

$$\|f * \psi_{K+1}^{\wedge}\|_\infty \ll \widehat{(K+1)}^{-1}\|f\|_1 \ll N^{-A}\|f\|_1 \quad \text{and} \quad \|f * \psi_{K+1}^{\wedge}\|_2 \ll N\hat{N}^{-1}\,\|f\|_2,$$

for $\delta > 2$, we have

$$\|f * \psi_{K+1}^{\wedge}\|_\delta \ll N^{A(-1+2/\delta)+2/\delta}\hat{N}^{-2/\delta}\|f\|_{\delta'}.$$

Upon recalling that $A = 4/(\delta - 2)$, we have

$$\|f * \psi_{K+1}^{\wedge}\|_\delta \ll N^{-2/\delta}\hat{N}^{-2/\delta}\|f\|_{\delta'}.$$

By the triangle inequality,

$$\|f * \lambda_{b,m,N}^{\wedge}\|_\delta \ll \sum_{Q=1}^{K+1}\|f * \psi_Q^{\wedge}\|_\delta \ll \hat{N}^{-2/\delta}\|f\|_{\delta'}.$$

Therefore, by (3) and (4), we have

$$\|T\|_{2\to\delta} \le \sup_{\|f\|_{\delta'}=1}\|f * \lambda_{b,m,N}^{\wedge}\|_\delta^{1/2} \ll \hat{N}^{-1/\delta}.$$

This completes the proof of the theorem.

We will now deduce Theorem 3 from Theorem 2.

Proof (of Theorem 3). When $\delta = 2$, the theorem follows from Parseval's inequality. Hence, we assume that $\delta > 2$. Let $(a_x)_{x\in\mathscr{P}_R}$ be a sequence of complex numbers with $|a_x| \le 1$ for $x \in \mathscr{P}_R$. Let

$$f(x) = \begin{cases} a_x, & \text{if } x \in \mathscr{P}_R,\\ 0, & \text{otherwise.}\end{cases}$$

Then, by setting $\lambda_{b,m,N} = \lambda_{0,1,R}$, it follows from Theorem 2 that

$$\left(\int_{\mathbb{T}}\Big|\sum_{x\in\mathscr{P}_R} a_x\frac{R}{\hat{R}}e(\alpha x)\Big|^\delta d\alpha\right)^{1/\delta} = \|Tf\|_\delta \ll \hat{R}^{-1/\delta}\|f\|_2 = \hat{R}^{-1/\delta}\Big(\sum_{x\in\mathscr{P}_R}|a_x|^2\frac{R}{\hat{R}}\Big)^{1/2}$$
$$\ll \hat{R}^{-1/\delta}.$$

Thus

$$\Big\|\sum_{x\in\mathscr{P}_R} a_x e(x\theta)\Big\|_\delta = \left(\int_{\mathbb{T}}\Big|\sum_{x\in\mathscr{P}_R} a_x e(\alpha x)\Big|^\delta d\alpha\right)^{1/\delta} \ll \hat{R}^{1-1/\delta}R^{-1}. \tag{25}$$

Also, for $\langle\beta\rangle < q^{-1}\hat{R}^{-1}$ and $x \in \mathscr{P}_R$, we have $\langle\beta x\rangle < q^{-1}$, implying that

$$\Big\| \sum_{x\in\mathscr{P}_R} e(x\theta)\Big\|_\delta = \left(\int_{\mathbb{T}} \Big| \sum_{x\in\mathscr{P}_R} e(\alpha x)\Big|^\delta d\alpha\right)^{1/\delta}$$
$$\geq \left(\int_{\langle\beta\rangle<q^{-1}\hat{R}^{-1}} \Big| \sum_{x\in\mathscr{P}_R} e(\beta x)\Big|^\delta d\beta\right)^{1/\delta} \tag{26}$$
$$\gg \left(\int_{\langle\beta\rangle<q^{-1}\hat{R}^{-1}} \hat{R}^\delta R^{-\delta} d\beta\right)^{1/\delta} \gg \hat{R}^{1-1/\delta} R^{-1}.$$

The theorem now follows by combining (25) and (26).

6 Proof of Theorem 1

To prove Theorem 1, we will employ the W-trick (see [7] for a discussion of the method). Namely, we will pass to an arithmetic progression with common difference equal to a product of small irreducible polynomials and this will allow us to avoid some obstacles modulo small irreducible polynomials. It is worth noting that if one is able to avoid using the W-trick, the resulting bound in Theorem 1 could be improved to $D_{\mathbf{r}}(\mathscr{P}_R) \ll |\mathscr{P}_R|/\log_q|\mathscr{P}_R|$.

Lemma 28. *Let $r_1, r_2, r_3 \in \mathbb{F}_q$ with $r_1 + r_2 + r_3 = 0$. Suppose that $A_R \subseteq \mathscr{P}_R$ and that there is no non-trivial solution to $r_1x_1 + r_2x_2 + r_3x_3 = 0$ with $x_1, x_2, x_3 \in A_R$. Suppose also that $|A_R| > \eta\hat{R}/R$ for some $\eta \in \mathbb{R}$ with $\eta > 0$. Let*

$$W = \left[\log_q\left(\frac{\log_q R}{4}\right)\right] \qquad \text{and} \qquad m = \prod_{\langle\varpi\rangle\leq\hat{W}} \varpi.$$

Set $\hat{N} = \hat{R}/\langle m\rangle$. Then for N sufficiently large, there exists $\mathscr{A} \subseteq \mathscr{S}_N$ such that

- *There is no non-trivial solution to $r_1x_1 + r_2x_2 + r_3x_3 = 0$ with $x_1, x_2, x_3 \in \mathscr{A}$,*
- *There exists some $b \in \mathbb{F}_q[t]$ with $(b, m) = 1$ and $\lambda_{b,m,N}(\mathscr{A}) \geq \eta$.*

Proof. Let 1_{A_R} denote the characteristic function of the set A_R. We have

$$\sum_{\substack{\langle b\rangle<\langle m\rangle \\ (b,m)=1}} \sum_{\substack{x\in S_R \\ x\equiv b \,(\mathrm{mod}\, m)}} 1_{A_R}(x) \geq \eta\hat{R}/R.$$

By the pigeonhole principle, there exists $b \in \mathbb{F}_q[t]$ with $\langle b\rangle < \langle m\rangle$ and $(b, m) = 1$ such that

$$\sum_{\substack{x\in S_R\\ x\equiv b\,(\mathrm{mod}\,m)}} 1_{A_R}(x) \geq \frac{\eta\hat{R}}{\phi(m)R}.$$

Let $\mathscr{A} = \{n \in \mathscr{S}_N \mid mn + b \in A_R\}$. Thus

$$\lambda_{b,m,N}(\mathscr{A}) = \frac{(N + \mathrm{ord}\, m)\phi(m)}{\hat{N}\langle m\rangle} \sum_{n\in\mathscr{S}_N} 1_{A_R}(mn + b) \geq \eta.$$

Since $r_1 + r_2 + r_3 = 0$ and there is no non-trivial solution to $r_1x_1 + r_2x_2 + r_3x_3 = 0$ with $x_1, x_2, x_3 \in A_R$, it follows that there is no non-trivial solutions to $r_1x_1 + r_2x_2 + r_3x_3 = 0$ with $x_1, x_2, x_3 \in \mathscr{A}$. This completes the proof of the lemma.

In order to apply Lemma 28 with the earlier work in this paper, we need to bound $\langle m\rangle$ in terms of N. Note that

$$\mathrm{ord}\, m = \sum_{\langle\varpi\rangle\leq\hat{W}} \mathrm{ord}\,\varpi = \sum_{K=1}^{W} K\big(\hat{K}/K + O(\hat{K}^{1/2}/K)\big) = q(q-1)^{-1}\hat{W} + O(\hat{W}^{1/2}).$$

Since $W = \big[\log_q\big(\frac{\log_q R}{4}\big)\big]$, for R sufficiently large in terms of q, we have

$$\mathrm{ord}\, m \in \Big[\frac{\log_q R}{4.1q}, \frac{\log_q R}{1.9}\Big];$$

from which we derive that

$$\hat{N} = \hat{R}/\langle m\rangle \in \big[\hat{R}R^{-1/1.9}, \hat{R}R^{-1/(4.1q)}\big].$$

In addition, we have $\langle m\rangle \leq R^{1/1.9} \leq N$ and $W \ll \log_q \log_q N$.

For a set $\mathscr{A} \subseteq \mathscr{S}_N$ and a monic irreducible polynomial ϖ of degree N, we embed $\mathscr{A}$ into $\mathbb{F}_q[t]/\varpi\mathbb{F}_q[t]$ via the bijection $x \to x\,(\mathrm{mod}\,\varpi)$. Also, we define Fourier analysis for $\mathbb{F}_q[t]/\varpi\mathbb{F}_q[t]$: if $f, g : \mathbb{F}_q[t]/\varpi\mathbb{F}_q[t] \to \mathbb{C}$ and $r \in \mathbb{F}_q[t]/\varpi\mathbb{F}_q[t]$, we write

$$\tilde{f}(r) = \sum_{\langle x\rangle<\langle\varpi\rangle} f(x)e(rx/\varpi) \qquad \text{and} \qquad (f * g)(r) = \sum_{\langle x\rangle<\langle\varpi\rangle} f(x)g(r-x).$$

We define functions $\kappa, \lambda : \mathbb{F}_q[t]/\varpi\mathbb{F}_q[t] \to \mathbb{C}$ by

$$\kappa(x) = \begin{cases} 1, & \text{if there exists } y \in \mathscr{A} \text{ such that } x \equiv y\,(\mathrm{mod}\,\varpi),\\ 0, & \text{otherwise,}\end{cases}$$

and $\lambda(x) = \lambda_{b,m,N}(y)$, where y is the unique element of $\mathscr{S}_N$ with $x \equiv y \pmod{\varpi}$. We also define a function $a : \mathbb{F}_q[t]/\varpi\mathbb{F}_q[t] \to \mathbb{C}$ by $a(x) = \kappa(x)\lambda(x)$. First, we estimate the function $\tilde{\lambda}$.

In what follows, we will fix $\delta = 5/2$. Thus all implicit constants below depend at most on q.

Lemma 29. *We have*

$$\sup_{z \not\equiv 0 \pmod{\varpi}} |\tilde{\lambda}(z)| \ll (\log_q N)^{-1}.$$

Proof. Note that $\tilde{\lambda}(z) = \lambda^{\wedge}_{b,m,N}(z/\varpi)$. For $z/\varpi \in \mathfrak{m}$, by Lemma 16, we have

$$\tilde{\lambda}(z) \ll N^{-A} \ll (\log_q N)^{-1}.$$

Thus we are left to prove the lemma for the case that $z/\varpi \in \mathfrak{M}_{a,g} \subseteq \mathfrak{M}$. By Lemmas 9, 11 and 12, we have

$$\tilde{\lambda}(z) = \begin{cases} \frac{\mu(g)}{\phi(g)} e\left(\frac{-ab\bar{m}}{g}\right) \varrho\left(\frac{z}{\varpi} - \frac{a}{g}\right) + O\left(\frac{N^{B+1+\epsilon}\langle m\rangle^{1/2+\epsilon}}{\hat{N}^{1/2}}\right), & \text{if } (g,m) = 1, \\ O\left(\frac{N^{B+1+\epsilon}\langle m\rangle^{1/2+\epsilon}}{\hat{N}^{1/2}}\right), & \text{otherwise.} \end{cases}$$

Because

$$\frac{N^{B+1+\epsilon}\langle m\rangle^{1/2+\epsilon}}{\hat{N}^{1/2}} \ll (\log_q N)^{-1},$$

it is enough to show that when $(g,m) = 1$, we have

$$\phi(g)^{-1} \varrho\left(\frac{z}{\varpi} - \frac{a}{g}\right) \ll (\log_q N)^{-1}.$$

For $\langle g\rangle = 1$, since $z \not\equiv 0 \pmod{\varpi}$,

$$\phi(g)^{-1} \varrho\left(\frac{z}{\varpi} - \frac{a}{g}\right) = \varrho(z/\varpi) = \hat{N}^{-1} \sum_{x \in \mathscr{S}_N} e(zx/\varpi) = 0.$$

For $\langle g\rangle > 1$, note that $|\varrho(\alpha)| \le 1$ for all $\alpha \in \mathbb{T}$. When $\langle g\rangle > 1$ and $(g,m) = 1$, by the definition of m, there exists a monic irreducible polynomial ϖ' with $\varpi' | g$ and $\langle \varpi'\rangle > \hat{W}$. Thus

$$\phi(g)^{-1} \le \phi(\varpi')^{-1} \ll \hat{W}^{-1} \ll (\log_q N)^{-1}.$$

This completes the proof of the lemma.

We now prove a discrete version of the majorant property with $\delta = 5/2$. Note that the proof below can be adapted to give a discrete majorant property for any $\delta > 2$.

Lemma 30. *There exists an absolute constant $C''(q)$ such that*

$$\sum_{\langle z\rangle < \langle \varpi\rangle} |\tilde{a}(z)|^{5/2} \le C''(q).$$

Proof. For $\langle \varpi\rangle = \hat{N} > 1$, $x \in \mathscr{S}_N$ and $\langle\theta\rangle < 1$, we have $e\big(\frac{x(z+\theta)}{\varpi}\big) = e\big(\frac{xz}{\varpi}\big)e\big(\frac{t^N\theta}{\varpi}\big)$. Thus for all $\langle\alpha\rangle < \hat{N}$, by writing $\alpha = z + \theta$ with $z \in S_{N-1}$ and $\theta \in \mathbb{T}$, we have

$$\begin{aligned}\sum_{\langle z\rangle < \langle \varpi\rangle} |\tilde{a}(z)|^{5/2} &= \sum_{\langle z\rangle < \langle \varpi\rangle} \Big|\sum_{x\in\mathscr{S}_N} \kappa(x)\lambda(x)e(zx/\varpi)\Big|^{5/2} \\ &= \int_{\langle\alpha\rangle<\hat{N}} \Big|\sum_{x\in\mathscr{S}_N} \kappa(x)\lambda(x)e(\alpha x/\varpi)\Big|^{5/2} d\alpha. \end{aligned} \tag{27}$$

By writing $\alpha = \varpi\gamma$, we deduce that

$$\int_{\langle\alpha\rangle<\hat{N}} \Big|\sum_{x\in\mathscr{S}_N} \kappa(x)\lambda(x)e(\alpha x/\varpi)\Big|^{5/2} d\alpha = \hat{N}\int_{\mathbb{T}} \Big|\sum_{x\in\mathscr{S}_N} \kappa(x)\lambda_{b,m,N}(x)e(\gamma x)\Big|^{5/2} d\gamma. \tag{28}$$

By Theorem 2,

$$\begin{aligned}&\left(\int_{\mathbb{T}} \Big|\sum_{x\in\mathscr{S}_N} \kappa(x)\lambda_{b,m,N}(x)e(\gamma x)\Big|^{5/2} d\gamma\right)^{2/5} \\ &= \|T\kappa\|_{5/2} \ll \hat{N}^{-2/5}\|\kappa\|_2 \\ &= \hat{N}^{-5/2}\Big(\sum_{x\in\mathscr{S}_N} |\kappa(x)|^2\lambda_{b,m,N}(x)\Big)^{1/2} \ll \hat{N}^{-5/2}. \end{aligned} \tag{29}$$

By combining (27)–(29), we find that

$$\sum_{\langle z\rangle < \langle \varpi\rangle} |\tilde{a}(z)|^{5/2} \ll 1.$$

This completes the proof of the lemma.

Let ς be a real parameter satisfying $0 \le \varsigma \le 1$ and define

$$\mathscr{Z} = \mathscr{Z}(\varsigma) = \big\{z \in \mathbb{F}_q[t]/\varpi\mathbb{F}_q[t] \,\big|\, |\tilde{a}(z)| \ge \varsigma\big\}.$$

Let $k = |\mathscr{Z}|$ and write $\mathscr{Z} = \{z_1, \ldots, z_k\}$. We now are able to define a Bohr set

$$\mathscr{B} = \mathscr{B}(\mathscr{Z}) = \left\{x \in \mathbb{F}_q[t]/\varpi\mathbb{F}_q[t] \,\middle|\, \left\langle \left\| \frac{xz_i}{\varpi} \right\| \right\rangle < q^{-1}\ (1 \le i \le k)\right\}.$$

Define a function $\beta : \mathbb{F}_q[t]/\varpi\mathbb{F}_q[t] \to \mathbb{C}$ by

$$\beta(x) = \begin{cases} |\mathscr{B}|^{-1}, & \text{if } x \in \mathscr{B}, \\ 0, & \text{otherwise.} \end{cases}$$

We define a function $a_1 : \mathbb{F}_q[t]/\varpi\mathbb{F}_q[t] \to \mathbb{C}$ by $a_1(x) = (a * \beta * \beta)(x)$.

Lemma 31. *There exists a positive constant $C_2(q)$ such that whenever $k \le \log_q \log_q N$, we have $\|a_1\|_\infty \le C_2(q)\hat{N}^{-1}$.*

Proof. From the definition of a_1 and Lemma 29, we have

$$\begin{aligned} a_1(x) &= \big(a * \beta * \beta\big)(x) \le \big(\lambda * \beta * \beta\big)(x) = \hat{N}^{-1} \sum_{\langle y\rangle < \langle \varpi\rangle} \tilde{\lambda}(y)\tilde{\beta}(y)^2 e\big(-xy/\varpi\big) \\ &\le \hat{N}^{-1}\tilde{\lambda}(0)\tilde{\beta}(0)^2 + \hat{N}^{-1} \sum_{\substack{\langle y\rangle < \langle \varpi\rangle \\ y \ne 0}} \tilde{\lambda}(y)\tilde{\beta}(y)^2 e\big(-xy/\varpi\big) \\ &\ll \hat{N}^{-1} + \hat{N}^{-1} \sup_{y \not\equiv 0 \,(\mathrm{mod}\, \varpi)} |\tilde{\lambda}(y)| \sum_{\langle y\rangle < \langle \varpi\rangle} |\tilde{\beta}(y)|^2 \\ &\ll \hat{N}^{-1} + (\log_q N)^{-1}|B|^{-1}. \end{aligned}$$

Recall that $\mathscr{Z} = \{z_1, \ldots, z_k\}$. Consider the mapping $\Gamma : \mathbb{F}_q[t]/\varpi\mathbb{F}_q[t] \to \mathbb{T}^k$ defined by

$$\Gamma(x) = \big(\|xz_1/\varpi\|, \ldots, \|xz_k/\varpi\|\big).$$

Let

$$\mathscr{G} = \big\{(\alpha_1, \ldots, \alpha_k) \in \mathbb{T}^k \mid \langle \alpha_i \rangle < q^{-1}\ (1 \le i \le k)\big\}.$$

By the pigeonhole principle, there exists an element $(v_1, \ldots, v_k) \in \mathbb{F}_q^k$ where

$$\mathscr{H} = \{x\ (\mathrm{mod}\ \varpi) \mid \Gamma(x) - (v_1, \ldots, v_k) \in \mathscr{G}\}$$

contains at least $\hat{N}q^{-k}$ elements. Let $y \in \mathscr{H}$. Then for any $y' \in \mathscr{H}$, we have $\Gamma(y - y') \in \mathscr{G}$. Hence, $|B| \ge \hat{N}q^{-k}$, implying that

$$|a_1(x)| \ll \hat{N}^{-1} + (\log_q N)^{-1}\hat{N}^{-1}q^k \ll \hat{N}^{-1}.$$

We will now prove upper and lower bounds for the sum

$$\hat{N}^{-1} \sum_{\langle z\rangle<\langle\varpi\rangle} \tilde{a}_1(r_1 z)\tilde{a}_1(r_2 z)\tilde{a}_1(r_3 z),$$

and we will then deduce Theorem 1 by comparing these upper and lower bounds.

Lemma 32. *Suppose that there is no non-trivial solution to* $r_1x_1 + r_2x_2 + r_3x_3 = 0$ *with* $x_i \in \mathscr{A}$ $(1 \le i \le 3)$. *Then*

$$\hat{N}^{-1} \sum_{\langle z\rangle<\langle\varpi\rangle} \tilde{a}_1(r_1 z)\tilde{a}_1(r_2 z)\tilde{a}_1(r_3 z) \ll \hat{N}^{-2}N^2 + \hat{N}^{-1}\varsigma^{1/2}.$$

Proof. Since there is no non-trivial solution to $r_1x_1 + r_2x_2 + r_3x_3 = 0$ with $x_i \in \mathscr{A}$ $(1 \le i \le 3)$, we have

$$\begin{aligned}\hat{N}^{-1} \sum_{\langle z\rangle<\langle\varpi\rangle} \tilde{a}(r_1 z)\tilde{a}(r_2 z)\tilde{a}(r_3 z) &= \sum_{\langle x_1\rangle<\langle\varpi\rangle}\sum_{\langle x_2\rangle<\langle\varpi\rangle} a(x_1)a(x_2)a(-r_1r_3^{-1}x_1 - r_2r_3^{-1}x_2)\\ &= \sum_{\langle x\rangle<\langle\varpi\rangle} a(x)^3 \le \sum_{y\in\mathscr{S}_N} \lambda_{b,m,N}(y)^3\\ &\ll \frac{(N+\operatorname{ord} m)^2\phi(m)^2}{\hat{N}^2\langle m\rangle^2} \ll N^2\hat{N}^{-2}.\end{aligned}$$

Since $\tilde{a}_1 = \tilde{a}\tilde{\beta}^2$, it follows that

$$\begin{aligned}&\hat{N}^{-1} \sum_{\langle z\rangle<\langle\varpi\rangle} \tilde{a}_1(r_1 z)\tilde{a}_1(r_2 z)\tilde{a}_1(r_3 z)\\ &= \hat{N}^{-1} \sum_{\langle z\rangle<\langle\varpi\rangle} \Big(\tilde{a}_1(r_1 z)\tilde{a}_1(r_2 z)\tilde{a}_1(r_3 z) - \tilde{a}(r_1 z)\tilde{a}(r_2 z)\tilde{a}(r_3 z)\Big)\\ &\quad + O(N^2\hat{N}^{-2})\\ &= \hat{N}^{-1} \sum_{\langle z\rangle<\langle\varpi\rangle} \tilde{a}(r_1 z)\tilde{a}(r_2 z)\tilde{a}(r_3 z)\Big(\tilde{\beta}(r_1)^2\tilde{\beta}(r_2)^2\tilde{\beta}(r_3)^2 - 1\Big)\\ &\quad + O(N^2\hat{N}^{-2}).\end{aligned} \tag{30}$$

Note that when $z \in \mathscr{Z}$ and $r \in \mathbb{F}_q$, since $\langle \| rzx/\varpi \| \rangle < q^{-1}$ for all $x \in \mathscr{B}$, we have

$$\tilde{\beta}(rz) = |\mathscr{B}|^{-1} \sum_{x\in\mathscr{B}} e(rzx/\varpi) = 1.$$

Thus

$$\sum_{z\in\mathscr{Z}} \tilde{a}(r_1 z)\tilde{a}(r_2 z)\tilde{a}(r_3 z)\Big(\tilde{\beta}(r_1 z)^2\tilde{\beta}(r_2 z)^2\tilde{\beta}(r_3 z)^2 - 1\Big) = 0. \tag{31}$$

Note that for all $z \pmod{\varpi}$,

$$\left|\tilde{\beta}(r_1 z)^2 \tilde{\beta}(r_2 z)^2 \tilde{\beta}(r_3 z)^2 - 1\right| \leq 2.$$

By combining Hölder's inequality with Lemma 30, we have

$$\begin{aligned}\sum_{\substack{\langle z\rangle < \langle \varpi\rangle \\ z \notin \mathscr{Z}}} &\tilde{a}(r_1 z)\tilde{a}(r_2 z)\tilde{a}(r_3 z)\left(\tilde{\beta}(r_1 z)^2 \tilde{\beta}(r_2 z)^2 \tilde{\beta}(r_3 z)^2 - 1\right) \\ &\ll \sup_{\substack{\langle z\rangle < \langle \varpi\rangle \\ z \notin \mathscr{Z}}} |\tilde{a}(z)|^{1/2} \sum_{\langle z\rangle < \langle \varpi\rangle} |\tilde{a}(z)|^{5/2} \ll \varsigma^{1/2}.\end{aligned} \tag{32}$$

The lemma now follows by combining (30)–(32).

Lemma 33. *Suppose that* $k \leq \log_q \log_q N$. *Then there exists a positive constant* $C_5 = C_5(q)$ *such that*

$$\hat{N}^{-1} \sum_{\langle z\rangle < \langle \varpi\rangle} \tilde{a}_1(r_1 z)\tilde{a}_1(r_2 z)\tilde{a}_1(r_3 z) \gg \eta^4 \hat{N}^{-1} q^{-C_5/\eta}.$$

Proof. Let

$$\mathscr{A}' = \left\{x \in \mathbb{F}_q[t]/\varpi\mathbb{F}_q[t] \,\middle|\, a_1(x) \geq \frac{\eta}{2\hat{N}}\right\}.$$

By Lemma 31, there exists a constant $C_2 = C_2(q) > 1$ such that $\|a_1\|_\infty \leq C_2 \hat{N}^{-1}$. Thus by Lemma 28,

$$\begin{aligned}|\mathscr{A}'|\frac{C_2}{\hat{N}} + (\hat{N} - |\mathscr{A}'|)\frac{\eta}{2\hat{N}} &\geq \sum_{\langle x\rangle < \langle \varpi\rangle} a_1(x) \\ &= \sum_{\langle x\rangle < \langle \varpi\rangle} (a * \beta * \beta)(x) \\ &= \sum_{\langle y\rangle < \langle \varpi\rangle} \beta(y) \sum_{\langle z\rangle < \langle \varpi\rangle} \beta(z - y) \sum_{\langle x\rangle < \langle \varpi\rangle} a(x - z) \\ &\geq \eta \sum_{\langle y\rangle < \langle \varpi\rangle} \beta(y) \sum_{\langle z\rangle < \langle \varpi\rangle} \beta(z - y) = \eta.\end{aligned}$$

Hence, we have

$$|\mathscr{A}'| \geq \eta\hat{N}/(2C_2 - \eta) \geq C_3 \eta \hat{N},$$

where $C_3 = 1/(2C_2) \in (0, 1)$. Let S denote the number of non-trivial solutions to $r_1x_1 + r_2x_2 + r_3x_3 = 0$ with $x_i \in \mathscr{A}'$ $(1 \le i \le 3)$. Then one has

$$\hat{N}^{-1} \sum_{\langle z\rangle<\langle\varpi\rangle} \tilde{a}_1(r_1z)\tilde{a}_1(r_2z)\tilde{a}_1(r_3z) \ge \frac{C_3^3\eta^3 S}{\hat{N}^3}. \tag{33}$$

Let $M \in \mathbb{N}$. By [14, Theorem 1], there exists a positive constant $C_4 = C_4(q)$ such that if $M \ge C_4/\eta$, then any subset of S_M of density at least $C_3\eta/2$ contains a non-trivial solution to $r_1x_1 + r_2x_2 + r_3x_2 = 0$. Furthermore, since $r_i \in \mathbb{F}_q$ $(1 \le i \le 3)$, the same is true for any space isomorphic to S_M as a vector space over $\mathbb{F}_q$. Now, let $M < N$. There are $\hat{N}(\hat{N} - 1)$ choices of (u, v) where $u \in \mathscr{S}_N$ and $0 < \langle v\rangle < \hat{N}$. Consider arithmetic progressions of the form $W_{u,v} = \{u + vl \mid \langle l\rangle < \hat{M}\} \subset \mathbb{F}_q[t]/\varpi\mathbb{F}_q[t]$. Let $\mathscr{U} = \{(u, v) \bigm| |W_{u,v} \cap \mathscr{A}'| > C_3\eta\hat{M}/2\}$. Note that $|W_{u,v} \cap \mathscr{A}'| \le \hat{M}$ for all u and v. Upon noting that every element $x \in \mathscr{A}'$ lies inside exactly $(\hat{N} - 1)\hat{M}$ sets $W_{u,v}$, we have

$$|\mathscr{U}|\hat{M} + \big(\hat{N}(\hat{N} - 1) - |\mathscr{U}|\big)C_3\eta\hat{M}/2 \ge (\hat{N} - 1)\hat{M}|\mathscr{A}'| \ge C_3\eta\hat{N}(\hat{N} - 1)\hat{M}.$$

It follows that

$$|\mathscr{U}| \ge C_3\eta\hat{N}(\hat{N} - 1)/(2 - C_3\eta) \ge C_3\eta\hat{N}(\hat{N} - 1)/2.$$

Thus there are at least $C_3\eta\hat{N}(\hat{N} - 1)/2$ sets $W_{u,v}$ for which $\mathscr{A}' \cap W_{u,v}$ has density at least $C_3\eta/2$. Provided that $C_4/\eta \le M < N$, each set $W_{u,v}$ with $(u, v) \in \mathscr{U}$ contains a non-trivial solution to $r_1x_1 + r_2x_2 + r_3x_3 = 0$. Note that for any non-trivial solution $r_1x_1 + r_2x_2 + r_3x_3 = 0$ with $x_i \in \mathscr{A}'$ $(1 \le i \le 3)$, there are at most $\hat{M}^2$ choices of (u, v) so that $(x_1, x_2, x_3) \in W_{u,v}^3$. Therefore, provided that $\lceil C_4/\eta\rceil < N$, by setting $M = \lceil C_4/\eta\rceil$, we have

$$S \ge \frac{C_3\eta\hat{N}(\hat{N} - 1)}{2\hat{M}^2} \gg \eta\hat{N}^2 q^{-2C_4/\eta}. \tag{34}$$

The lemma follows by combining (33) and (34) and setting $C_5 = 2C_4$.

We are now in a position to prove Theorem 1.

Proof (of Theorem 1). Let η, A_R, $\mathscr{A}$, N and R be defined as in Lemma 28, where R is sufficiently large in terms of q. Suppose that there is no non-trivial solutions to $r_1x_1 + r_2x_2 + r_3x_3 = 0$ with $x_i \in \mathscr{A}$ $(1 \le i \le 3)$. Recall that $k = |\mathscr{Z}| = \big|\{\langle z\rangle < \langle\varpi\rangle \bigm| |\tilde{a}(z)| \ge \varsigma\}\big|$. By Lemma 30,

$$k\varsigma^{5/2} \le \sum_{\langle x\rangle<\langle\varpi\rangle} |\tilde{a}(x)|^{5/2} \ll 1.$$

Since $k \ll \varsigma^{-5/2}$, there exists a positive constant $C_6 = C_6(q)$ such that, upon setting $\varsigma = C_6(\log_q \log_q N)^{-2/5}$, we have $k \le \log_q \log_q N$. By Lemmas 32 and 33,

$$\begin{aligned}\eta^4 \hat{N}^{-1} q^{-C_5/\eta} &\ll \hat{N}^{-1} \sum_{\langle z\rangle < \langle \varpi \rangle} \tilde{a}_1(r_1 z)\tilde{a}_1(r_2 z)\tilde{a}_1(r_3 z) \\ &\ll \hat{N}^{-2} N^2 + \hat{N}^{-1} \varsigma^{1/2} \\ &\ll \hat{N}^{-2} N^2 + \hat{N}^{-1} (\log_q \log_q N)^{-1/5} \\ &\ll \hat{N}^{-1} (\log_q \log_q N)^{-1/5}.\end{aligned}$$

Thus $\eta^4 q^{-C_5/\eta} \ll (\log_q \log_q N)^{-1/5}$, which implies that

$$\log_q \log_q \log_q N \ll -\log_q \eta + \frac{1}{\eta} \ll \frac{1}{\eta}.$$

From the above inequality, we can deduce that $\eta \ll (\log_q \log_q \log_q N)^{-1}$. Therefore, we have

$$\frac{|A_R|}{|\mathscr{P}_R|} \ll \frac{1}{\log_q \log_q \log_q N} \ll \frac{1}{\log_q \log_q \log_q R} \ll \frac{1}{\log_q \log_q \log_q \log_q |\mathscr{P}_R|}.$$

Theorem 1 now follows.

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