Identities for Logarithmic Means: A Survey

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Abstract We provide a survey of identities for sums of the type $\sum_{n\le x} a(n)\log(x/n)$. In each case, $a(n)$ is an arithmetical function generated by a Dirichlet series satisfying a functional equation involving the gamma function. Moreover, all of the identities given in this paper feature infinite series of Bessel functions.

1 Introduction

Let $a(n)$ be an arithmetical function generated by a Dirichlet series satisfying a functional equation involving the gamma function $\Gamma(s)$. For example, let $r_k(n)$ denote the number of representations of the positive integer n as a sum of k squares. Then its generating function

$$\zeta_k(s) := \sum_{n=1}^{\infty} r_k(n) n^{-s}, \quad \sigma = \operatorname{Re} s > \tfrac{1}{2}k, \tag{1}$$

satisfies the functional equation

$$\pi^{-s}\Gamma(s)\zeta_k(s) = \pi^{s-k/2}\Gamma(\tfrac{1}{2}k - s)\zeta_k(\tfrac{1}{2}k - s).$$

Second, let $d(n)$ denote the number of positive divisors of the positive integer n. If $\zeta(s)$ denotes the Riemann zeta function, it is easily seen that

$$\zeta^2(s) = \sum_{n=1}^{\infty} d(n) n^{-s}, \quad \sigma > 1,$$

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which satisfies the functional equation

$$\pi^{-s}\Gamma^2(\tfrac{1}{2}s)\zeta^2(s) = \pi^{-1+s}\Gamma^2(\tfrac{1}{2}-\tfrac{1}{2}s)\zeta^2(1-s).$$

Third, let $\tau(n)$ denote the Ramanujan tau-function. Then Ramanujan's Dirichlet series

$$f(s) := \sum_{n=1}^{\infty} \tau(n)n^{-s}, \quad \sigma > \tfrac{13}{2},$$

satisfies the functional equation

$$(2\pi)^{-s}\Gamma(s)f(s) = (2\pi)^{-(12-s)}\Gamma(12-s)f(12-s).$$

In each case, the functional equation can be used to analytically continue the Dirichlet series to the entire complex s-plane.

For an arithmetical function $a(n)$, we often desire an asymptotic formula for $\sum_{n\leq x} a(n)$, or, if we divide by x, we ask for the *average order* of $a(n)$. If $a(n)$ is generated by a Dirichlet series satisfying a functional equation involving $\Gamma(s)$, then $\sum_{n\leq x} a(n)$ often satisfies an identity containing an infinite series of Bessel functions [10].

For example,

$$\begin{aligned}\sum_{0\leq n\leq x}{}^{\prime} r_2(n) &= \pi x + \sum_{n=1}^{\infty} r_2(n)\left(\frac{x}{n}\right)^{1/2} J_1(2\pi\sqrt{nx}) \\ &=: \pi x + P(x), \end{aligned} \tag{2}$$

where $P(x)$ is the "error term," and where $J_\nu(x)$ is the ordinary Bessel function of order ν defined by

$$J_\nu(z) := \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(\nu+n+1)}\left(\frac{z}{2}\right)^{\nu+2n}, \qquad 0 < |z| < \infty, \qquad \nu \in \mathbb{C}. \tag{3}$$

The prime $'$ on the summation sign on the left side of (2) indicates that if x is an integer, then we count only $\frac{1}{2}r_2(x)$. One of the most famous unsolved problems in analytic number theory is the *circle problem*: find the precise order of magnitude of $P(x)$ as $x \to \infty$. It is conjectured that, for every $\epsilon > 0$, $P(x) = O(x^{1/4+\epsilon})$ as x tends to ∞. A history and survey of the *circle problem* can be found in [7].

For a second example, we return to $d(n)$. First define the Bessel function $Y_\nu(z)$ of the second kind of order ν by [21, p. 64, Eq. (1)]

$$Y_\nu(z) := \frac{J_\nu(z)\cos(\nu\pi) - J_{-\nu}(z)}{\sin(\nu\pi)}, \tag{4}$$

and the modified Bessel function $K_\nu(z)$ of order ν by [21, p. 78, Eq. (6)]

$$K_\nu(z) := \frac{\pi}{2}\,\frac{e^{\pi i\nu/2}J_{-\nu}(iz) - e^{-\pi i\nu/2}J_\nu(iz)}{\sin(\nu\pi)}. \tag{5}$$

If ν is an integer n, then it is understood that we define the latter two functions by taking the limits as $\nu \to n$ in (4) and (5). Let

$$I_\nu(z) := -Y_\nu(z) - \frac{2}{\pi}K_\nu(z). \tag{6}$$

Then a famous identity of G.F. Voronoï [20] asserts that

$$\begin{aligned}\sum_{n\le x}{}' d(n) &= x\,(\log x + 2\gamma - 1) + \frac{1}{4} + \sum_{n=1}^{\infty} d(n)\left(\frac{x}{n}\right)^{1/2} I_1(4\pi\sqrt{nx}) \\ &=: x\,(\log x + 2\gamma - 1) + \frac{1}{4} + \Delta(x),\end{aligned} \tag{7}$$

where γ denotes Euler's constant and $\Delta(x)$ is the "error term." The *Dirichlet divisor problem* asks for the exact order of magnitude of $\Delta(x)$ as $x \to \infty$. Voronoï employed (7) to prove that $\Delta(x) = O(x^{1/3}\log x)$ as $x \to \infty$. It is conjectured that $\Delta(x) = O(x^{1/4+\epsilon})$, for each $\epsilon > 0$, as $x \to \infty$, and a survey of this difficult classical problem can also be found in [7].

In many cases an identity may not exist for $\sum_{n\le x} a(n)$, but it may exist for $\sum_{n\le x} a(n)(x-n)^\rho$ for sufficiently large ρ. Moreover, the weighted identity generally converges absolutely and uniformly on compact subintervals of $(0,\infty)$ making it more convenient to use than an identity for $\sum_{n\le x} a(n)$, which is discontinuous at positive integers n when $a(n) \neq 0$. Then one can apply a method of finite differences, originally due to E. Landau, to the sum $\sum_{n\le x} a(n)(x-n)^\rho$ to gain information about $\sum_{n\le x} a(n)$ [11, Theorem 4.1, pp. 106–111]. The sums $\sum_{n\le x} a(n)(x-n)^\rho$ are sometimes called Riesz means. Note that for "small" n, the contribution of $(x-n)^\rho$ is "large," while for large n, the contribution of $(x-n)^\rho$ is "small." Very roughly, arithmetic functions are "small" for "small" n and "large" for "large" n, and so $(x-n)^\rho$ acts as a "smoothing factor." In fact, W. Sierpinski [18] used an identity for $\sum_{n\le x} r_2(n)(x-n)$ to show that $P(x) = O(x^{1/3})$, as x tends to infinity.

We emphasize that the logarithmic sums $\sum_{n\le x} a(n)\log^\rho(x/n)$ can also be used in the study of the average order of certain arithmetic functions, since $\log^\rho(x/n)$ has a "smoothing" effect similar to that of $(x-n)^\rho$, for in each case, when n is "small," the contributions of these factors are "large," while when n is "large", the contributions of these factors are "small." Generally, for "small" ρ, the simplicities of the identities for $\sum_{n\le x} a(n)(x-n)^\rho$ and $\sum_{n\le x} a(n)\log^\rho(x/n)$ are comparable, but for "large" ρ, the identities for the former sum are usually more elegant than those for the latter sum. It is likely for this reason that Riesz means have been employed instead of logarithmic means in the study of the average order of arithmetic functions.

The goal of this paper is to provide a survey of identities for $\sum_{n\le x} a(n)\log(x/n)$, which, because of their intrinsic beauty, deserve to be better known than they are. We are confining our attention only to the case $\rho = 1$, because with increasing ρ, the logarithmic mean identities diminish in elegance; see, for example, [3].

2 History and Examples

To the best of our knowledge, the first logarithmic mean identity is due to A. Oppenheim [14] in 1927. Let $d(n)$ denote the number of divisors of n. Then [14, p. 342], for $x > 0$,

$$\sum_{n\le x} d(n)\log\frac{x}{n} = x\log x - x + (2\gamma-1)x + \frac{1}{4}\log x + \frac{1}{2}\log(2\pi) - \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{d(n)}{n}I_0(4\pi\sqrt{nx}), \tag{8}$$

where γ denotes Euler's constant, I_0 is defined by (6), and the series on the right-hand side of (8) converges absolutely and uniformly on compact subsets of $(0,\infty)$.

Oppenheim's proof is interesting. Let $\sigma_k(n) = \sum_{d|n} d^k$. He derives a general identity for $\sum_{n\le x}\sigma_k(n)(x-n)^{\rho}$ [14, p. 340], which we relate only for $|k| < \frac{1}{2}$ and $\rho = 0$. Let

$$F_{\nu}(z) := \cos(\tfrac{1}{2}\nu\pi)J_{\nu}(z) + \sin(\tfrac{1}{2}\nu\pi)I_{\nu}(z),$$

where $J_{\nu}(z)$ and $I_{\nu}(z)$ are defined in (3) and (6), respectively. Then

$$\sum_{n\le x}{}^{'}\sigma_k(n) = \Phi_k(x) + \sum_{n=1}^{\infty}\sigma_k(n)\left(\frac{x}{n}\right)^{(k+1)/2}F_{k+1}(4\pi\sqrt{nx}), \tag{9}$$

where $\Phi_k(x)$ is the sum of the residues of

$$\frac{\zeta(z)\zeta(z-k)}{z}x^z.$$

Using (9), Oppenheim forms an identity for

$$L(k) := \frac{1}{k}\left\{x^k\sum_{n\le x}\sigma_{-k}(n) - \sum_{n\le x}\sigma_k(n)\right\}. \tag{10}$$

He then takes the limit of his identity for $L(k)$ as $k \to 0$. Oppenheim did not provide any details, and we will not as well, except that we will evaluate, by (10) and L'Hospital's rule,

$$\begin{aligned}\lim_{k\to 0} L(k) &= \lim_{k\to 0}\left\{x^k \log x \sum_{n\le x}\sigma_{-k}(n) - x^k \sum_{n\le x}\sum_{d|n} d^{-k}\log d - \sum_{n\le x}\sum_{d|n} d^k \log d\right\}\\ &= \log x \sum_{n\le x} d(n) - 2\sum_{n\le x}\sum_{d|n}\log d\\ &= \log x \sum_{n\le x} d(n) - \sum_{n\le x}\sum_{d|n}\log n\\ &= \sum_{n\le x} d(n)\log\frac{x}{n},\end{aligned}$$

because

$$\sum_{d|n}\log d = \sum_{d|n}\log\frac{n}{d} = \log n\, d(n) - \sum_{d|n}\log d.$$

The only other proof of (8) of which we are aware is due to the first author [3, p. 371], who deduced (8) from a general identity, established by him for $\sum_{n\le x} a(n)\log^{\rho}(x/n)$ by a method completely different from that of Oppenheim.

Oppenheim [14, p. 312] further remarks, "Similarly we can obtain identities for ... and

$$\sum_{n\le x} r_2(n)\log\frac{x}{n}."$$

However, Oppenheim provides no details. In 1954, C. Müller [13] proved that

$$\sum_{n\le x} r_2(n)\log\frac{x}{n} = \pi x - \log x - \frac{1}{4}\log\frac{\Gamma^4(1/4)}{4\pi} + O(x^{-1/4}), \tag{11}$$

as $x \to \infty$. He did not provide an exact formula for the error term on the right side of (11). L. Carlitz [9] gave a simpler proof of (11), but his method did not yield a value for the constant term in closed form on the right-hand side of (11). The first author [3, p. 372] established an identity for the first time, showing that

$$\sum_{n\le x} r_2(n)\log\frac{x}{n} = \pi x - \log x + \zeta_2'(0) - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{r_2(n)}{n}J_0(2\pi\sqrt{nx}), \tag{12}$$

where $\zeta_2'(s)$ is defined in (1). The error term in (11) follows readily from the well-known asymptotic formula [21, p. 199]

$$J_\nu(x) = \left(\frac{2}{\pi x}\right)^{1/2} \cos\left(x - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi\right) + O\left(\frac{1}{x^{3/2}}\right), \tag{13}$$

as $x \to \infty$. R. Ayoub and S. Chowla [2], [12, pp. 1189–1191], unaware of [3], established (11), including showing that

$$\zeta_2'(0) = -\frac{1}{4}\log\frac{\Gamma^4(1/4)}{4\pi}.$$

D. Redmond [16] generalized the work of Ayoub and Chowla by replacing $r_2(n)$ by

$$d_{\chi_1,\chi_2}(n) = \sum_{d|n} \chi_1(d)\chi_2(n/d), \tag{14}$$

where χ_1 and χ_2 are characters modulo q_1 and q_2, respectively. Thus, when $\chi_1(n) \equiv 1$ and $\chi_2(n)$ is the primitive, non-principal character modulo 4, then $d_{\chi_1,\chi_2}(n) = \frac{1}{4}r_2(n)$. If $\chi_1(n) = \chi_2(n) \equiv 1$, then $d_{\chi_1,\chi_2}(n) = d(n)$. Redmond did not prove (8) and (12), but instead established results in which the infinite series of Bessel functions are replaced by error terms. In these two examples, Redmond claimed error terms improving that of Müller and of Ayoub and Chowla in (11), and also that obtained by approximating the Bessel functions in (8) by (13). However, in [17], Redmond acknowledged that his claimed error terms are incorrect and that his methods only yield error terms that are obtained by using (13).

C. Calderón and M.J. Zárate [8] generalized Redmond's work by proving an asymptotic formula for $\sum_{n\le x} a(n)\log^k(x/n)$, where $a(n)$ is an arithmetical function (too complicated to state here) considerably generalizing (14).

U.M.A. Vorhauer [19, p. 60, Theorem 2], apparently unfamiliar with the earlier work of Müller, Carlitz, Ayoub and Chowla, Redmond, and Calderón and Zárate, also established (11). Her result was actually a special case of a general theorem in which she established an "identity" for $\sum_{n\le x} a(n)\log^k(x/n)$, where $k \ge 0$ and $a(n)$ is generated by a Dirichlet series satisfying a functional equation involving a very general product of gamma factors. Her theorem [19, p. 59, Theorem 1] is more general than the one proved in [3], where the arithmetical functions are generated by Dirichlet series satisfying functional equations involving $\Gamma^m(s)$, where m is a positive integer. However, Vorhauer's general "identity" contains an "error term," for which estimates are given, while the identity in [3] is exact and written in terms of Bessel functions. The primary purpose of Vorhauer's paper is to provide precise estimates for the "error terms."

Let us return to Ramanujan's tau-function, one of the three examples discussed in the Introduction. From [3, p. 372], we record the identity

$$\sum_{n\leq x}\tau(n)\log\frac{x}{n}=\frac{2}{(4\pi)^{12}}\sum_{n=1}^{\infty}\frac{\tau(n)}{n^{12}}\int_0^{4\pi\sqrt{nx}}u^{11}J_{12}(u)du. \tag{15}$$

The identity (15) illustrates a common roadblock in establishing elegant logarithmic mean identities—the identities often involve integrals that are not readily evaluated in closed form, as is the case with the integrals on the right side of (15).

The present authors and their colleague, A. Zaharescu, have devoted considerable efforts in recent years to establishing two intriguing identities found in Ramanujan's lost notebook [15] that are connected, respectively, with the classical, unsolved *circle* and *divisor* problems. For example, see their paper [6], their survey paper [7], and an account of much of their work in the book [1] by Andrews and Berndt. We record only one of the two identities. First define

$$F(x)=\begin{cases}[x], & \text{if } x \text{ is not an integer,}\\ x-\frac{1}{2}, & \text{if } x \text{ is an integer.}\end{cases} \tag{16}$$

We offer now the first entry.

Entry 1 (p. 335). *Let* $F(x)$ *be defined by* (16). *If* $0<\theta<1$ *and* $x>0$, *then*

$$\sum_{n=1}^{\infty}F\left(\frac{x}{n}\right)\sin(2\pi n\theta)=\pi x\left(\frac{1}{2}-\theta\right)-\frac{1}{4}\cot(\pi\theta)$$
$$+\frac{1}{2}\sqrt{x}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\left\{\frac{J_1\left(4\pi\sqrt{m(n+\theta)x}\right)}{\sqrt{m(n+\theta)}}-\frac{J_1\left(4\pi\sqrt{m(n+1-\theta)x}\right)}{\sqrt{m(n+1-\theta)}}\right\}.$$

It was natural for us to ask if there exist logarithmic mean identities corresponding to Ramanujan's two entries on page 335 of his lost notebook. Indeed, we found such analogues. However, to establish these logarithmic analogues, we first need to establish analogues of (8) for weighted divisor functions, which we now define.

For a character χ, define the weighted divisor sum

$$d_\chi(n):=\sum_{d|n}\chi(d).$$

If χ is a character modulo q, the Gauss sum $\tau(\chi)$ is defined by

$$\tau(\chi):=\sum_{h=1}^{q-1}\chi(h)e^{2\pi ih/q}.$$

Lastly, for any character χ modulo q and $\sigma > 1$, recall that the Dirichlet L-series is defined by

$$L(s,\chi) := \sum_{n=1}^{\infty} \chi(n) n^{-s}.$$

It is well-known that $L(s,\chi)$ can be analytically continued into the entire complex s-plane.

We are now ready to state two new logarithmic mean identities [5].

Theorem 2. *If χ denotes an odd primitive character modulo q, then*

$$\sum_{n\le x} d_\chi(n) \log\frac{x}{n} = xL(1,\chi) + \frac{i\tau(\chi)}{2\pi}\log(2\pi x)L(1,\overline{\chi}) - \frac{1}{2}L'(0,\chi) + \frac{i\tau(\chi)}{2\pi}\sum_{n=1}^{\infty}\frac{d_{\overline{\chi}}(n)}{n}J_0(4\pi\sqrt{nx/q}). \tag{17}$$

Theorem 3. *If χ is an even, non-principal, primitive character modulo q, then*

$$\sum_{n\le x} d_\chi(n) \log\frac{x}{n} = xL(1,\chi) - \frac{1}{2}L'(0,\chi) - \frac{\tau(\chi)}{2\pi}\sum_{n=1}^{\infty}\frac{d_{\overline{\chi}}(n)}{n}I_0(4\pi\sqrt{nx/q}). \tag{18}$$

Theorems 2 and 3 are special cases of theorems in [3], but the identities and the details of their proofs were not previously worked out until the authors did so in [5].

We are now ready to offer the logarithmic mean identity motivated by Entry 1, established with the key aid of Theorem 2, and proved by the authors in [5]. (A corresponding entry motivated by Ramanujan's second entry on page 335 of [15] and established with the help of Theorem 3 will not be given here.)

Theorem 4. *Let $x > 0$ and $0 < \theta < 1$. Then*

$$\sum_{n\le x}\log\frac{x}{n}\sum_{r|n}\sin(2\pi r\theta) = -\frac{\log(4\pi^2 x)+\gamma}{4}\cot(\pi\theta) + \pi x\left(\frac{1}{2}-\theta\right) + \frac{1}{4\pi}\big(\gamma_1(\theta)-\gamma_1(1-\theta)\big) - \frac{1}{4\pi}\sum_{\substack{m\ge 1\\ n\ge 0}}\left\{\frac{J_0\left(4\pi\sqrt{m(n+\theta)x}\right)}{(m(n+\theta))} - \frac{J_0\left(4\pi\sqrt{m(n+1-\theta)x}\right)}{(m(n+1-\theta))}\right\}, \tag{19}$$

where $\gamma_1(\theta)$ and $\gamma_1(1-\theta)$ are the Laurent series coefficients of the Hurwitz zeta function $\zeta(s,a)$, also called generalized Stieltjes constants, defined by Berndt [4, p. 152]

$$\zeta(s,a) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \gamma_n(a)(s-1)^n. \tag{20}$$

Lastly, suppose that we consider the logarithmic mean identity associated with the arithmetic function $a(n) \equiv 1$, which, of course, is generated by $\zeta(s)$. Then one can show that [3, p. 370]

$$\begin{aligned}\sum_{n\le x} \log\frac{x}{n} &= x - \frac{1}{2}\log x - \frac{1}{2}\log(2\pi) - \frac{1}{\pi}\sum_{n=1}^{\infty}\int_{2\pi nx}^{\infty}\frac{\sin u}{u}du \\ &\sim x - \frac{1}{2}\log x - \frac{1}{2}\log(2\pi) - \sum_{n=2}^{\infty}\frac{B_n(x-[x])}{(n-1)!x^{n-1}},\end{aligned} \tag{21}$$

as x tends to infinity, upon successive integrations by parts, where $B_n(x)$, $n \ge 2$, denotes the nth Bernoulli polynomial. (Complete details may be found in [3, p. 371].) The asymptotic expansion (21) is equivalent to Stirling's asymptotic series expansion for the gamma function.

We have not recorded all known logarithmic mean identities that can be found in the literature. For example, let $F(n)$ denote the number of nonzero integral ideals of norm n in either an imaginary quadratic number field or a real quadratic number field. Then in each of these cases, identities for $\sum_{n\le x} F(n)\log(x/n)$ are derived in [3]. Redmond [16] also considered the case of an imaginary quadratic number field, but with the infinite series of Bessel functions replaced by an error term.

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