International Series in
Operations Research & Management Science

Salvatore Greco
Matthias Ehrgott
José Rui Figueira *Editors*

Multiple Criteria Decision Analysis

State of the Art Surveys

Second Edition

Springer

International Series in Operations Research & Management Science

Volume 233

More information about this series at http://www.springer.com/series/6161

Salvatore Greco • Matthias Ehrgott
José Rui Figueira
Editors

Multiple Criteria Decision Analysis

State of the Art Surveys

Volume 1 and 2

Second Edition

Springer

Editors
Salvatore Greco
Department of Economics and Business
University of Catania
Catania, Italy

Portsmouth Business School
Centre of Operations Research
and Logistics (CORL)
University of Portsmouth
Portsmouth, UK

Matthias Ehrgott
Department of Management Science
Lancaster University
Lancaster, UK

José Rui Figueira
CEG-IST, Instituto Superior Técnico
Universidade de Lisboa
Lisboa, Portugal

ISSN 0884-8289 ISSN 2214-7934 (electronic)
International Series in Operations Research & Management Science
ISBN 978-1-4939-3093-7 ISBN 978-1-4939-3094-4 (eBook)
DOI 10.1007/978-1-4939-3094-4

Library of Congress Control Number: 2015957403

Springer New York Heidelberg Dordrecht London

Printed on acid-free paper

Springer Science+Business Media LLC New York is part of Springer Science+Business Media (www.springer.com)

Contents

List of Figures

List of Tables

Introduction

José Rui Figueira, Salvatore Greco, and Matthias Ehrgott

1 Ten Years of Success of Multiple Criteria Decision Analysis and Reasons for This New Edition

After 10 years we present an updated revision of the collection of state-of-the-art surveys on Multiple Criteria Decision Analysis (MCDA). This is a good occasion to briefly comment on the latest advances in the domain. We believe that in the last 10 years we have seen great progress of MCDA, from both a theoretical point of view and a real-life application point of view. We have seen the consolidation of the main "traditional" methodologies such as multiple attribute utility theory, outranking methods, interactive multiobjective optimization, as well as the growing success of new approaches such as Evolutionary Multiobjective Optimization (EMO). The spectrum of applications has been constantly expanding with particular emphasis on very complex problems such as industrial design or grid optimization. Taking into account this evolution of the domain, we partly modified the structure and the content of the book giving space to new methodologies (e.g., EMO or multi-criteria portfolio decision analysis for project selection) or splitting chapters into several new ones (e.g., the chapter on multiobjective programming of the previous edition that has now been substituted by three chapters, one on vector and set optimization, one on continuous multiobjective, and one on multiobjective combinatorial optimization). Moreover, all authors, sometimes with the help of

J.R. Figueira
CEG-IST, Instituto Superior Técnico, Universidade de Lisboa, A. Rovisco Pais, 1, 1049-001 Lisboa, Portugal

S. Greco
University of Catania, Catania, Italy

University of Portsmouth, Portsmouth, UK

M. Ehrgott
Department of Management Science, Lancaster University, Bailrigg, Lancaster LA1 4YX, UK

a new colleague, have updated the contents of their contributions incorporating the novelties of the last 10 years. Of course, many sophisticated technical details that appear in the new edition of the book will sooner or later be destined to be superseded by the incessant evolution of research and applications. We think, however, that the basic principles as stated by the experts who prepared the different chapters in the book will remain reference points for the years to come. Moreover, we believe that the spirit with which experts on MCDA are working today, in these so rich and fruitful years, will remain forever in this book. This spirit is strongly related with the spirit with which, in the late 1960s and early 1970s of the last century, the "pioneers" (many of who are among the many authors of the chapters in this book) outlined the basic principles of MCDA with the genuine aim to give a satisfactory answer to concrete real world problems for which the classical methods of operations research were not able to find adequate answers. Therefore the basic principles of the presented methodologies and their relationships with the MCDA spirit are things that we recommend the reader to look for in each chapter. After these words about the intuition that guided the revision of this book, let us enter "in medias res", coming back to the introduction of the first edition that was of course also updated.

2 Human Reflection About Decision

Decision-making has inspired reflections of many thinkers since ancient times. The great philosophers Aristotle, Plato, and Thomas Aquinas, to mention only a few, discussed the capacity of humans to decide and in some manners claimed that this possibility is what distinguishes humans from animals. To illustrate some important aspects of decision-making, let us briefly quote two important thinkers, Ignatius of Loyola (1491–1556) and Benjamin Franklin (1706–1790).

> To consider, reckoning up, how many advantages and utilities follow for me from holding the proposed office or benefice [. . .], and, to consider likewise, on the contrary, the disadvantages and dangers which there are in having it. Doing the same in the second part, that is, looking at the advantages and utilities there are in not having it, and likewise, on the contrary, the disadvantages and dangers in not having the same. [. . .] After I have thus discussed and reckoned up on all sides about the thing proposed, to look where reason more inclines: and so, according to the greater inclination of reason, [. . .], deliberation should be made on the thing proposed.

This fragment from the "Spiritual Exercises" of St. [14] has been taken from a paper by Fortemps and Slowinski [12].

> London, Sept 19, 1772
> Dear Sir,
> In the affair of so much importance to you, wherein you ask my advice, I cannot, for want of sufficient premises, advise you what to determine, but if you please I will tell you how. [. . .], my way is to divide half a sheet of paper by a line into two columns; writing over the one Pro, and over the other Con. [. . .] When I have thus got them all together in one view, I endeavor to estimate their respective weights; and where I find two, one on each side, that

> seem equal, I strike them both out. If I find a reason pro equal to some two reasons con, I strike out the three. If I judge some two reasons con, equal to three reasons pro, I strike out the five; and thus proceeding I find at length where the balance lies; and if, after a day or two of further consideration, nothing new that is of importance occurs on either side, I come to a determination accordingly. [. . .] I have found great advantage from this kind of equation, and what might be called moral or prudential algebra. Wishing sincerely that you may determine for the best, I am ever, my dear friend, yours most affectionately.
> B. Franklin

This letter from Benjamin Franklin to Joseph Prestly has been taken from a paper by MacCrimmon [17].

What is interesting in the above two quotations is the fact that decision is strongly related to the comparison of different points of view, some in favor and some against a certain decision. This means that decision is intrinsically related to a plurality of points of view, which can roughly be defined as criteria. Contrary to this very natural observation, for many years the only way to state a decision problem was considered to be the definition of a single criterion, which amalgamates the multidimensional aspects of the decision situation into a single scale of measure. For example, even today textbooks of operations research suggest to deal with a decision problem as follows: To first define an objective function, i.e., a single point of view like a comprehensive profit index (or a comprehensive cost index) representing the preferability (or dis-preferability) of the considered actions and then to maximize (minimize) this objective. This is a very reductive, and in some sense also unnatural, way to look at a decision problem. Thus, for at least 40 years, a new way to look at decision problems has more and more gained the attention of researchers and practitioners. This is the approach considered by Loyola and Franklin, i.e., the approach of explicitly taking into account the pros and the cons of a plurality of points of view, in other words the domain of multiple criteria decision analysis. Therefore, MCDA intuition is closely related to the way humans have always been making decisions. Consequently, despite the diversity of MCDA approaches, methods and techniques, the basic ingredients of MCDA are very simple: A finite or infinite set of actions (alternatives, solutions, courses of action, . . .), at least two criteria, and, obviously, at least one decision-maker (DM). Given these basic elements, MCDA is an activity which helps making decisions mainly in terms of choosing, ranking, or sorting the actions.

3 Technical Reflection About Decision: MCDA Researchers Before MCDA

Of course, not only philosophers reasoned about decision. Many important technical aspects of MCDA are linked to classic works in economics, in particular, welfare economics, utility theory, and voting-oriented social choice theory (see [27]). Aggregating the opinion or the preferences of voters or individuals of a community into collective or social preferences is quite similar a problem to devising comprehensive preferences of a decision-maker from a set of conflicting criteria in MCDA [7].

Despite the importance of Ramon Llull's (1232–1316) and Nicolaus Cusanus' (1401–1464) concerns about and interests in this very topic, the origins of voting systems are often attributed to Le Chevalier Jean-Charles de Borda (1733–1799) and Marie Jean Antoine Nicolas de Caritat (1743–1794), Le Marquis de Condorcet. However, Ramon Llull introduced the pairwise comparison concept before Condorcet [13], while Nicolaus Cusanus introduced the scoring method about three and a half centuries before Borda [26]. Furthermore, it should be noted that a letter from Pliny the Younger ($\approx$ AD 105) to Titus Aristo shows that he introduced the ternary approval voting strategy and was interested in voting systems a long time before Ramon Llull and Nicolaus Cusanus [18, Chapter 2]. Anyway, Borda's scoring method [4] has some similarities with current utility and value theories as has Condorcet's method [10] with the outranking approach of MCDA. In the same line of concerns, i.e., the aggregation of individual preferences into collective ones, Jeremy Bentham (1748–1832) introduced the utilitarian calculus to derive the total utility for the society from the aggregation of the personal interests of the individuals of a community [3]. Inspired by Bentham's works, Francis Ysidro Edgeworth (1845–1926), a utilitarian economist, was mainly concerned with the maximization of the utility of the different competing agents in an economy. Edgeworth tried to find the competitive equilibrium points for the different agents. He proposed to draw indifference curves (lines of equal utility) for each agent and then derive the contract curve, a curve that corresponds to the notion of the Pareto or efficient set [20]. Not long afterward, Vilfredo Federico Damaso Pareto (1848–1923) gave the following definition of ophelimity [utility] for the whole community [21].

> We will say that the members of a collectivity enjoy maximum ophelimity in a certain position when it is impossible to find a way of moving from that position very slightly in such a manner that the ophelimity enjoyed by each of the individuals of that collectivity increases or decreases. That is to say, any small displacement in departing from that position necessarily has the effect of increasing the ophelimity which certain individuals enjoy, of being agreeable to some, and disagreeable to others.

From this definition it is easy to derive the concept of dominance, which today is one of the fundamental concepts in MCDA.

MCDA also benefits from the birth and development of game theory. Félix Edouard Justin Emile Borel (1871–1956) and John von Neumann (1903–1957) are considered the founders of game theory [5, 6, 19, 29]. Many concepts from this discipline had a strong impact on the development of MCDA.

The concept of efficient point was first introduced in 1951 by Tjalling Koopmans (1910–1985) in his paper "Analysis of production as an efficient combination of activities" [15].

> A possible point in the commodity space is called efficient whenever an increase in one of its coordinates (the net output of one good) can be achieved only at the cost of a decrease in some other coordinate (the net output of a good).

In the same year (1951) Harold William Kuhn (born 1925) and Albert William Tucker (1905–1995) introduced the concept of vector maximum problem [16]. In the 1960s, basic MCDA concepts were explicitly considered for the first time. As two examples we mention Charnes' and Cooper's works on goal programming [8] and the proposition of ELECTRE methods by Roy [22]. The 1970s saw what is conventionally considered the "official" starting point of MCDA, the conference on "Multiple Criteria Decision Making" organized in 1972 by Cochrane and Zeleny at Columbia University in South Carolina [9]. Since then MCDA has seen a tremendous growth which continues today.

4 Reasons for This Collection of State-of-the-Art Surveys

The idea of MCDA is so natural and attractive that thousands of articles and dozens of books have been devoted to the subject, with many scientific journals regularly publishing articles about MCDA. To propose a new collection of state-of-the-art surveys of MCDA in so rich a context may seem a rash enterprise. Indeed, some objections come to mind. There are many and good handbooks and reviews on the subject (to give an idea consider [1, 11, 24, 25, 28]). The main ideas are well established for some years and one may question the contributions this volume can provide. Moreover, the field is so large and comprises developments so heterogeneous that it is almost hopeless to think that an exhaustive vision of the research and practice of MCDA can be given.

We must confess that at the end of the work of editing this volume we agree with the above remarks. However, we believe that a new and comprehensive collection of state-of-the-art surveys on MCDA can be very useful. The main reasons which, despite our original resistance, brought us to propose this book are the following:

1. Many of the existing handbooks and reviews are not too recent. Since MCDA is a field which is developing very quickly this is an important reason.
2. Even though the field of research and application of MCDA is so large, there are some main central themes around which MCDA research and applications have been developed. Therefore our approach was to try to present the—at least in our opinion—most important of these ideas.

With reference to the first point, we can say that we observed many theoretical developments which changed MCDA over the last 20 years. We tried to consider these changes as much as possible and in this perspective strong points of the book are the following:

1. It presents the most up-to-date discussions on well-established methodologies and theories such as outranking-based methods and MAUT.
2. The book also contains surveys of new, recently emerged fields such as conjoint measurement, fuzzy preferences, fuzzy integrals, rough sets, and others.

Following these points we drafted a list of topics and asked well-known researchers to present them. We encouraged the authors to cooperate with the aim to present different perspectives if topics had some overlap. We asked the authors to present a comprehensive presentation of the most important aspects of the field covered by their chapters, a simple yet concise style of exposition, and considerable space devoted to bibliography and survey of relevant literature. We also requested a sufficiently didactic presentation and a text that is useful for researchers in MCDA as well as for people interested in real-life applications.

The importance of these requirements is also related to the specific way the MCDA community looks at its research field. It can be summarized in the observation that there is a very strong and vital link between theoretical and methodological developments on the one hand and real applications on the other hand. Thus, the validity of theoretical and methodological developments can only be measured in terms of the progress given to real-world practice. Moreover, interest of MCDA to deal with concrete problems is related to the consideration of a sound theoretical basis which ensures the correct application of the methodologies taken into account.

In fact, not only the chapters of our book but rather all MCDA contributions should satisfy the requirements stated out above because they should be not too "esoteric" and therefore understandable for students, theoretically well founded, and applicable to some advantage in reality.

5 A Guided Tour of the Book

Of course, this book can be read from the first to the last page. However, we think that this is not the only possibility and it may not even be the most interesting possibility. In the following we propose a guided tour of the book suggesting some reference points that are hopefully useful for the reader.

5.1 Part I: The History and Current State of MCDA

This part is important because MCDA is not just a collection of theories, methodologies, and techniques, but a specific perspective to deal with decision problems. Losing this perspective, even the most rigorous theoretical developments and applications of the most refined methodologies are at risk of being meaningless because they miss an adequate consideration of the aims and of the role of MCDA. We share this conviction with most MCDA researchers.

From this perspective it is important to have a clear vision of the origin of the main basic concepts of the domain. For this reason, Murat Köksalan, Jyrki Wallenius, and Stanley Zionts present the early history of MCDA and related areas showing how many developments in the field were made by major contributors to operations research, management science, economics, and other areas.

Then Bernard Roy discusses "pre-theoretical" assumptions of MCDA and gives an overview of the field. Bernard Roy, besides making many important theoretical contributions, engaged himself in thorough reflections on the meaning and the value of MCDA, proposing some basic key concepts that are accepted throughout the MCDA community.

5.2 Part II: Foundations of MCDA

This part of the book is related to a fundamental problem of MCDA, the representation of preferences. Classically, for example in economics, it is supposed that preference can be represented by a utility function assigning a numerical value to each action such that the more preferable an action, the larger its numerical value. Moreover, it is very often assumed that the comprehensive evaluation of an action can be seen as the sum of its numerical values for the considered criteria. Let us call this the classical model. It is very simple but not too realistic. Indeed, there is a lot of research studying under which conditions the classical model holds. These conditions are very often quite strict and it is not reasonable to assume that they are satisfied in all real-world situations. Thus, other models relaxing the conditions underlying the classical model have been proposed. This is a very rich field of research, which is first of all important for those interested in the theoretical aspects of MCDA. However, it is also of interest to readers engaged in applications of MCDA. In fact, when we adopt a formal model it is necessary to know what conditions are supposed to be satisfied by the preferences of the DM. In the two chapters of this part, problems related to the representations of preferences are discussed.

Stefano Moretti, Meltem Öztürk, and Alexis Tsoukiàs present a very exhaustive review of preference modeling, starting from classical results but arriving at the frontier of some challenging issues of scientific activity related to fuzzy logic and non-classical logic.

Denis Bouyssou and Marc Pirlot discuss the axiomatic basis of the different models to aggregate multiple criteria preferences. We believe that this chapter is very important for the future of MCDA. Initially, the emphasis of MCDA research was on proposal of new methods. But gradually the necessity to understand the basic conditions underlying each method and its specific axiomatization became more and more apparent. This is the first book on MCDA with so much space dedicated to the subject of foundations of MCDA.

5.3 Part III: Outranking Methods

In this part of the book the class of outranking-based multiple criteria decision methods is presented. Given what is known about the decision-maker's preferences and given the quality of the performances of the actions and the nature of the problem, an outranking relation is a binary relation S defined on the set of potential actions A such that aSb if there are enough arguments to decide that a is at least as good as b, whereas there is no essential argument to refute that statement [23]. Methods which strictly apply this definition of outranking relation are the ELECTRE methods. They are very important in many respects, not least historically, since ELECTRE I was the first outranking method [2].

However, within the class of outranking methods we generally consider all methods which are based on pairwise comparison of actions. Thus, another class of very well-known multiple criteria methods, PROMETHEE methods, is considered in this part of the book. Besides ELECTRE and PROMETHEE methods, many other interesting MCDA methods are based on the pairwise comparison of actions. José Figueira, Vincent Mousseau, and Bernard Roy present the ELECTRE methods; Jean-Pierre Brans and Yves De Smet present the PROMETHEE methods; and Jean-Marc Martel and Benedetto Matarazzo review the rich literature of other outranking methods.

5.4 Part IV: Multi-attribute Utility and Value Theories

In this part of the book we consider multiple attribute utility theory (MAUT). This MCDA approach tries to assign a utility value to each action. This utility is a real number representing the preferability of the considered action. Very often the utility is the sum of the marginal utilities that each criterion assigns to the considered action. Thus, this approach very often coincides with what we called the classical approach before. As we noted in commenting Part I, this approach is very simple at first glance. It is often applied in real life, e.g., every time we aggregate some indices by means of a weighted sum, we are applying this approach. Despite its simplicity, the approach presents some technical problems. The first is related to the axiomatic basis and the construction of marginal utility functions (i.e., the utility functions relative to each single criterion), both in case of decision under certainty and uncertainty. These problems are considered by James Dyer in a comprehensive chapter about the fundamentals of this approach.

Yannis Siskos, Vangelis Grigoroudis, and Nikolaos Matsatsinis present the very well-known UTA methods, which on the basis of the philosophy of the aggregation–disaggregation approach and using linear programming build a MAUT model that is as consistent as possible with the DM's preferences expressed in actual previous decisions or on a "training sample". The philosophy of aggregation–disaggregation can be summarized as follows: How is it possible to assess the decision-maker's preference model leading to exactly the same decision as the actual one or at least the most "similar" decision?

Thomas Saaty presents a very well-known methodology to build utility functions, the AHP (Analytic Hierarchy Process), and its more recent extension, the ANP (Analytic Network Process). AHP is a theory of measurement that uses pairwise comparisons along with expert judgments to deal with the measurement of qualitative or intangible criteria. The ANP is a general theory of relative measurement used to derive composite priority ratio scales from individual ratio scales that represent relative measurements of the influence of elements that interact with respect to control criteria. The ANP captures the outcome of dependence and feedback within and between clusters of elements. Therefore AHP with its dependence assumptions on clusters and elements is a special case of the ANP.

Carlos Bana e Costa, Jean-Claude Vansnick, and Jean-Marie De Corte present another MCDA methodology based on the additive utility model. This methodology is MACBETH (Measuring Attractiveness by a Categorical Based Evaluation Technique). It is an MCDA approach that requires only qualitative judgments about differences of values of attractiveness of one action over another action to help an individual or a group to quantify the relative preferability of different actions. In simple words, the MACBETH approach tries to answer the following questions: How can we build an interval scale of preferences on a set of actions without forcing evaluators to produce direct numerical representations of their preferences? How can we coherently aggregate these qualitative evaluations using an additive utility model?

5.5 Part V: Non-classical MCDA Approaches

Many approaches have been proposed in MCDA besides outranking methods and multi-attribute utility theory. In this part of the book we try to collect information about some of the most interesting proposals. First, the question of uncertainty in MCDA is considered. Theo Stewart and Ian Durbach discuss risk and uncertainty in MCDA. It is necessary to distinguish between internal uncertainties (related to decision-maker values and judgments) and external uncertainties (related to imperfect knowledge concerning consequences of actions). The latter, corresponding to the most accepted interpretation of uncertainty in the specialized literature, has been considered in the chapter. Four broad approaches for dealing with external uncertainties are discussed. These are multi-attribute utility theory and some extensions; stochastic dominance concepts, primarily in the context of pairwise comparisons of alternatives; the use of surrogate risk measures such as additional decision criteria; and the integration of MCDA and scenario planning.

Salvatore Greco, Benedetto Matarazzo, and Roman Słowiński present the decision rule approach to MCDA. This approach represents the preferences in terms of "if ..., then ..." decision rules such as, for example, "if the maximum speed of car x is at least 175 km/h and its price is at most $12000, then car x is comprehensively at least medium". This approach is related to rough set theory and to artificial intelligence. Its main advantages are the following. The DM gives information in the form of examples of decisions, which requires relatively low cognitive effort and which is quite natural. The decision model is also expressed in a very natural way by decision rules. This permits an absolute transparency of the methodology for the

DM. Another interesting feature of the decision rule approach is its flexibility, since any decision model can be expressed in terms of decision rules and, even better, the decision rule model can be much more general than all other existing decision models used in MCDA.

Michel Grabisch and Christophe Labreuche present the fuzzy integral approach that is known in MCDA for the last two decades. In very simple words this methodology permits a flexible modeling of the importance of criteria. Indeed, fuzzy integrals are based on a capacity which assigns an importance to each subset of criteria and not only to each single criterion. Thus, the importance of a given set of criteria is not necessarily equal to the sum of the importance of the criteria from the considered subset. Consequently, if the importance of the whole subset of criteria is smaller than the sum of the importances of its individual criteria, then we observe a redundancy between criteria, which in some way represents overlapping points of view. On the other hand, if the importance of the whole subset of criteria is larger than the sum of the importances of its members, then we observe a synergy between criteria, the evaluations of which reinforce one another. On the basis of the importance of criteria measured by means of a capacity, the criteria are aggregated by means of specific fuzzy integrals, the most important of which are the Choquet integral (for cardinal evaluations) and the Sugeno integral (for ordinal evaluations).

Helen Moshkovich, Alexander Mechitov, and David Olson present the verbal decision methods MCDA. This is a class of methods originated from the work of one of the MCDA pioneers, the late Oleg Larichev. The idea of verbal decision analysis is to build a decision model using mostly qualitative information expressed in terms of a language that is natural for the DM. Moreover, measurement of criteria and preference elicitation should be psychologically valid. The methods, besides being mathematically sound, should check the DM's consistency and provide transparent recommendations.

Most real-world decision problems take place in a complex environment where conflicting systems of logic, uncertain, and imprecise knowledge, and possibly vague preferences have to be considered. To face such complexity, preference modeling requires the use of specific tools, techniques, and concepts which allow the available information to be represented with the appropriate granularity. In this perspective, fuzzy set theory has received a lot of attention in MCDA for a long time. Didier Dubois and Patrice Perny try to provide a tentative assessment of the role of fuzzy sets in decision analysis, taking a critical standpoint on the state-of-the-art, in order to highlight the actual achievements and trying to better assess what is often considered debatable by decision scientists observing the fuzzy decision analysis literature.

5.6 Part VI: Multiobjective Optimization

The classical formulation of an operations research model is based on the maximization or minimization of an objective function subject to some constraints. A very rich and powerful arsenal of methodologies and techniques has been developed and continues to be developed within operations research. However, it is very difficult to summarize all the points of view related to the desired results of the decision at hand in only one objective function. Thus, it seems natural to consider a very general formulation of decision problems where a set of objective functions representing different criteria have to be "optimized". To deal with these types of problems requires not only to generalize the methodologies developed for classical single-objective optimization problems, but also to introduce new methodologies and techniques permitting to compare different objectives according to the preferences of the DM. In this part of the book we tried to give adequate space to these two sides of multiobjective programming problems.

Gabriele Eichfelder and Johannes Jahn discuss recent developments of vector and set optimization. Based on the concept of a pre-order, optimal elements are defined. In vector optimization, properties of optimal elements and existence results are gained. Further, an introduction to vector optimization with a variable ordering structure is given. In set optimization basic concepts are summed up.

Margaret Wiecek, Matthias Ehrgott, and Alexander Engau present their view of the state-of-the-art in continuous multiobjective programming. After an introduction they formulate the multiobjective program (MOP) and define the most important solution concepts. They summarize properties of efficient and nondominated sets and review optimality conditions and solution techniques for MOPs and approximation of efficient and nondominated sets. They discuss also specially structured problems including linear, nonlinear, parametric, and bi-level MOPs, and finally they present a perspective on future research directions.

Within the general field of multiobjective programming, research on combinatorial optimization problems with multiple objectives has been particularly active. Matthias Ehrgott, Xavier Gandibleux, and Anthony Przybylski review exact methods for multiobjective combinatorial optimization problems, covering extensions of single objective algorithms to the multiobjective case, scalarization approaches, the two-phase method and branch and bound algorithms.

Masahiro Inuiguchi, Kosuke Kato, and Hideki Katagiri review fuzzy multi-criteria optimization focusing on possibilistic treatments of objective functions with fuzzy coefficients and on interactive fuzzy stochastic multiple objective programming approaches.

Dylan Jones and Mehrdad Tamiz present a review of the field of goal programming describing the current range of goal programming variants and the range of techniques that goal programming has been combined or integrated with is discussed. A range of modern applications of goal programming are also given.

Kaisa Miettinen, Jussi Hakanen, and Dmitry Podkopaev give an overview of interactive methods for solving multi-objective optimization problems. In interactive methods, the decision-maker progressively provides preference information so that her or his most satisfactory Pareto optimal solution can be found. The basic features of several methods are introduced and some theoretical results are provided. In addition, references to modifications and applications as well as to other methods are indicated. As the role of the decision-maker is very important in interactive methods, methods presented are classified according to the type of preference information that the decision-maker is assumed to provide.

Juergen Branke discusses relationships between MCDA and evolutionary multi-objective optimization (EMO). EMO promises to efficiently generate an approximate set of Pareto optimal solutions in a single optimization run. This allows the decision-maker to select the most preferred solution from the generated set, rather than having to specify preferences a priori. In recent years, there has been a growing interest in combining the ideas of evolutionary multi-objective optimization and MCDA. MCDA can be used before optimization, to specify partial user preferences, after optimization, to help select the most preferred solution from the set generated by the evolutionary algorithm, or be tightly integrated with the evolutionary algorithm to guide the optimization towards the most preferred solution. This chapter surveys the state-of-the-art of using preference information within evolutionary multi-objective optimization

5.7 Part VII: Applications

It is apparent that the validity and success of all the developments of MCDA research are measured by the number and quality of the decisions supported by MCDA methodologies. Applications in this case discriminate between results that are really interesting for MCDA and results that, even though beautiful and interesting for economics, mathematics, psychology, or other scientific fields, are not interesting for MCDA. The applications of MCDA in real-world problems are very numerous and in very different fields. Therefore it was clear from the outset that it would be impossible to cover all the fields of application of MCDA. We decided to select some of the most significant areas.

Jaap Spronk, Ralph Steuer, and Constantin Zopounidis discuss the contributions of MCDA in finance. A very valuable feature of their chapter is the focus on justification of the multidimensional character of financial decisions and the use of different MCDA methodologies to support them. The presentation of the contributions of MCDA in finance permits to structure complex evaluation problems in a scientific context and in a transparent and flexible way, with the introduction of both quantitative (i.e., financial ratios) and qualitative criteria in the evaluation process.

Carlos Henggeler Antunes, António Gomes Martins, and Carla Oliveira Henriques present applications of MCDA in energy planning problems. In modern technologically developed societies, decisions concerning energy planning must be made in complex and sometimes ill-structured contexts, characterized by technological evolution, changes in market structures, and new societal concerns. Decisions to be made by different agents (at utility companies, regulatory bodies, and governments) must take into account several aspects of evaluation such as technical, socio-economic, and environmental ones, at various levels of decision-making (ranging from the operational to the strategic level) and with different time frames. Thus, energy planning problems inherently involve multiple, conflicting, and incommensurate axes of evaluation. The chapter aims at examining to which extent the use of MCDA in energy planning applications has been influenced by those changes currently underway in the energy sector, in the overall socio-economic context, and in particular to which extent it is adapted to the new needs and structuring and modeling requirements.

João Clímaco, José Craveirinha, and Rita Girão-Silva present multiple criteria decision analysis in telecommunication network planning and design. Decision making processes in this field take place in an increasingly complex and turbulent environment involving multiple and potentially conflicting options. Telecommunication networks is not only an area where different socio-economic decisions involving communication issues have to be made, but it is also an area where technological issues are of paramount importance. This interaction between a complex socio-economic environment and the extremely fast development of new telecommunication technologies and services justifies the interest in using multiple criteria evaluation in decision-making processes. The chapter presents a review of contributions in these areas, with particular emphasis on network modernization planning and routing problems and outlines an agenda of current and future research trends and issues for MCDA in this area.

Giuseppe Munda addresses applications of MCDA in problems concerning sustainable development. Sustainable development is strongly related to environmental questions, i.e., sustainable development generalizes environmental management taking into account not only an ecological but also socio-economic, technical, and ethical perspectives. Ecological problems were among the first to be dealt with by MCDA. Therefore there is a strong tradition in this field and many interesting stimuli for MCDA research came from there. The extensive perspective of sustainable development is very significant because it improves the quality of decisions concerning the environment taking into account other criteria, which are not strictly environmental but which strongly interact with it. In making sustainability policies operational, basic questions to be answered are sustainability of what and whom? As a consequence, sustainability issues are characterized by a high degree of conflict. Therefore, in this context MCDA appears as an adequate approach.

Alec Morton, Jeff Keisler, and Ahti Salo consider multi-criteria portfolio analysis. It spans several methods which typically employ MCDA to guide the selection of a subset (i.e., portfolio) of available objects, with the aim of maximizing the performance of the resulting portfolio with regard to multiple criteria, subject to the requirement that the resources consumed by the selected portfolio do not exceed their availability, and that it satisfies other relevant constraints as well. A survey of the applications of portfolio decision analysis in several domains, such as allocation of research and development expenditure, military procurement, prioritization of healthcare projects, and environment and energy planning is also presented.

5.8 Part VIII: MCDM Software

Application of an MCDA method requires such a considerable amount of computation that even the development of many MCDA methodologies without the use of a specialized software is hardly imaginable. While software is an even more important element in the application of MCDA methodologies, this does not mean that to have a good software is sufficient to apply an MCDA methodology correctly. Clearly, software is a tool and it should be used as a tool. Before using a software, it is necessary to have a sound knowledge of the adopted methodology and of the decision problem at hand.

After these remarks about cautious use of software, the problem is: What software is available for MCDA? Heinz Roland Weistroffer and Yan Li present well-known MCDA software packages. While there is certainly some MCDA software available that is not present in the chapter, it can help the reader. He or she may not only get suggestions of well-known software, but also information about aspects to be taken into account when evaluating a software for adoption in an application.

References

1. Bana e Costa, C. (ed.): Readings in Multiple Criteria Decision Aid. Springer, Heidelberg (1990)
2. Benayoun, R., Roy, B., Sussman, B.: ELECTRE: Une méthode pour guider le choix en présence de points de vue multiples. Note de travail 49, SEMA-METRA International, Direction Scientifique (1966)
3. Bentham, J.: The Principles of Morals and Legislation. Prometheus Books, New York (1988)
4. Borda, J.C.: Mémoire sur les elections au scrutin. Histoire de l'Academie Royale des Sciences (Paris) Année MDCCLXXXI, pp. 657–665 (1784). Translated by Alfred de Garzia: mathematical derivation of an election system. Isis **44**(1–2), 42–51 (1953)
5. Borel, E.: La théorie du jeu et les équations integrales à noyau symétrique gauche. C. R. Acad. Sci. (Paris) **173**, 1304–1308 (1921)
6. Borel, E.: Traité du Calcul des Probabilités et de ses Applications. Gauthier-Villars, Paris (1938)
7. Bouyssou, D., Marchant, T., Pirlot, M., Perny, P., Tsoukiàs, A., Vincke, P.: Evaluation and Decision Model: A Critical Perspective. Kluwer Academic Publishers, Dordrecht (2000)
8. Charnes, A., Cooper, W.: Management Models and Industrial Applications of Linear Programming. Wiley, New York (1961)
9. Cochrane, J., Zeleny, M.: Multiple Criteria Decision Making. University of South Carolina Press, Columbia (1973)
10. Condorcet, M.: Essai sur l'Application de l'Analyse à la Probabilité des Décisions Rendues à la Pluralité des Voix. L'Imprimerie Royale, Paris (1785)
11. Fandel, G., Spronk, J., Matarazzo, B.: Multiple Criteria Decision Methods and Applications. Springer, Berlin (1985)
12. Fortemps, P., Slowinski, R.: A graded quadrivalent logic for preference modelling: Loyola-like approach. Fuzzy Optim. Decis. Making **1**, 93–111 (2002)
13. Hagele, G., Pukelsheim, F.: Llull's writings on electoral systems. Studia Lulliana **41**, 3–38 (2001)
14. Ignatius of Loyola St.: Spiritual Exercises, pp. 178–183. Antonio Bladio, Rome First edition (1548)

15. Koopmans, T.: Analysis of production as an efficient combination of activities. In: Koopmans, T. (ed.) Activity Analysis of Production and Allocations. Cowles Commission Monograph, vol. 13, pp. 33–97. Wiley, New York (1951)
16. Kuhn, H., Tucker, A.: Nonlinear programming. In: Proceedings of the Second Symposium on Mathematical Statistics and Probability, pp. 481–492. University of California Press, Berkeley (1951)
17. MacCrimmon, K.: An overview of multiple objective decision making. In: Cochrane, J., Zeleny, M. (eds.) Multiple Criteria Decision Making, pp. 18–43. University of South Carolina Press, Columbia (1973)
18. McLean, I., Urken, A. (eds.): Classics of Social Choice. The University of Michigan Press, Ann Arbor (1995)
19. Neumann, J.: Zur Theorie der Gesellschaftsspiele. Math. Ann. **100**, 295–320 (1928)
20. Newman, P. (ed.): F.Y. Edgeworth's "Mathematical Psychics" and Further Papers on Political Economy. Oxford University Press, Oxford (2003)
21. Pareto, V.: Manuale di Economia Politica. Società Editrice Libraria, Milano (1906). Translated into English by Ann S. Schwier, edited by Ann S. Schwier and Alfred N. Page, Manual of Political Economy. Augustus M. Kelley, New York (1971)
22. Roy, B.: Classement et choix en présence de point de vue multiples: Le méthode ELECTRE. Revue Francaise d'Informatique et de Recherche Opérationnelle **8**, 57–75 (1968)
23. Roy, B.: Critéres multiple et modélisation des préférence: l'apport des relations de surclassement. Rev. Econ. Polit. **1**, 1–44 (1974)
24. Roy, B.: Méthodologie Multicritère d'Aide à la Décision. Economica, Paris (1985)
25. Roy, B., Bouyssou, D.: Aide Multicritère à la Décision: Méthodes et Cas. Economica, Paris (1993)
26. Sigmund, P.: Nicholas of Cusa and Medieval Political Thought. Harvard University Press, Cambridge, MA (1963)
27. Stadler, W.: A survey of multicriteria optimization or the vector maximum problem, Part I: 1776–1960. J. Optim. Theory Appl. **29**(1), 1–52 (1979)
28. Vincke, P.: Multicriteria Decision-Aid. Wiley, Chichester (1992)
29. von Neumann, J., Morgenstern, O.: Theory of Games and Economic Behavior. Princeton University Press, Princeton (1943)

Part I
The History and Current State of MCDA

Chapter 1
An Early History of Multiple Criteria Decision Making

Murat Köksalan, Jyrki Wallenius, and Stanley Zionts

Abstract This historical note is based on a plenary talk 'A History of Early Developments in Multiple Criteria Decision Making', presented by Stanley Zionts at the 21st International Conference on Multiple Criteria Decision Making held in Jyväskylä, Finland, June 2011. It draws heavily on our book, Multiple Criteria Decision Making: From Early History to the 21st Century, published by World Scientific, Singapore, 2011 (Copyright © 2012 John Wiley & Sons, Ltd.) The note summarizes major early developments and contributors of multiple criteria decision making and closely related fields.

Keywords Decision making • History • Multiple criteria • Pareto-optimality • Utility theory

1.1 Introduction

There are different ways in which a historical note could be organized. We have chosen to organize our paper around prominent individuals, who have all contributed significantly to the development of our field.

This chapter is reprinted with permission from 'An Early History of Multiple Criteria Decision Making', Murat Köksalan, Jyrki Wallenius and Stanley Zionts, Journal of Multi-Criteria Decision Analysis, Volume 20, Issue 1–2, 87–94, Copyright © 2013, John Wiley & Sons. DOI: 10.1002/mcda.1481

M. Köksalan
Industrial Engineering Department, Middle East Technical University, Ankara, Turkey
e-mail: koksalan@metu.edu.tr

J. Wallenius
Aalto School of Business, Aalto University, Helsinki, Finland
e-mail: jyrki.walennius@aalto.fi

S. Zionts (✉)
School of Management, SUNY at Buffalo, Buffalo, NY, USA
e-mail: szionts@buffalo.edu

S. Greco et al. (eds.), *Multiple Criteria Decision Analysis*, International Series in Operations Research & Management Science 233,
DOI 10.1007/978-1-4939-3094-4_1

The practice of decision making is as old as man. A story in the life of King Solomon (1011–931 BC) was probably the first recorded instance of an approach to a problem that might be considered a Multiple Criteria Decision Making (MCDM) problem, actually a mediation or negotiation problem.[1] Solomon was purportedly an extremely wise man. His wisdom allegedly helped his kingdom of Israel become extremely wealthy. Solomon reigned for 40 years. Among his many accomplishments, King Solomon has been credited with constructing many buildings, including the famous temple in Jerusalem.

A famous example of Solomon's wisdom, reported in the bible, supposedly occurred when two women came before Solomon to resolve a quarrel about which was the true mother of a baby. One mother had her baby die during the night after rolling over on the baby in her sleep and crushing it. Each claimed the surviving child as her own. When Solomon suggested dividing the living child in two with a sword, the true mother was revealed to him because she was willing to give up her child to the lying woman, as heartbreaking a decision as it was. Solomon then declared the woman who showed compassion to be the true mother and ruled to give the baby to her. Although there is skepticism about the truth of the story, it is an interesting example of negotiation and mediation/arbitration, a close relative of MCDM.

1.2 Early Developments

Although we do not know when formal study of decision making began, it is possible to trace the origins of decision analysis/utility theory and multiple objective mathematical programming. The roots of the former are older than of the latter. The roots of modern decision analysis/utility theory go back to the subjective expected utility model because of Ramsey and de Finetti (early 1930s), the formal treatment of utility by von Neumann and Morgenstern (1940s), and the early work on the indifference contours by Edgeworth (1880s) and Samuelson (1940s), although many other important early contributions can be identified. The work of Howard Raiffa, Robert Schlaifer and Ron Howard in the 1950s was important to the development of decision analysis in its present form. The developments of multiple objective mathematical programming were rather independent of decision analysis/utility theory. Often the demarcating feature between decision analysis/utility theory and multiple objective mathematical programming was that the former addressed problems under uncertainty, the latter deterministic problems. An impetus to the development of multiple objective mathematical programming was provided by linear programming and goal programming in particular. Multiple objective mathematical programming

[1] Saul Gass, private communication.

developed mainly during the 1970s. Characteristic of this development was certain skepticism of explicit assessment of value or utility functions. The main focus was on finding most preferred solutions, or generating an approximation to the entire efficient frontier.

According to our book, the first known recorded work on MCDM (although not using that name) was carried out by the famous American statesman Benjamin Franklin. Even before Franklin's times, Aristotle (384–322 BC), a famous Greek philosopher and polymath, in Nicomachean Ethics defines 'preferences' as 'rational desires'. This might have been the first time where someone made the connection between rational decision making and human desires (preferences).

1.2.1 Moral Algebra

Benjamin Franklin allegedly had a simple paper system for deciding what to do regarding important issues, which he called 'Moral Algebra',[2] Franklin explained his procedure in a letter to a friend, Joseph Priestly. When trying to decide his position on an important issue, he would write on one side of a sheet of a paper the arguments in favour of the issue, and on the other side, the arguments against it. He would then cross out arguments on each side of the paper of relatively equal importance. When all the arguments on one side were crossed out, the side with any arguments not crossed out was the position on the argument that he felt he should support. Supposedly, Franklin used this in making important decisions. He also talked about using weights in making decisions. His approach was clearly an early MCDM approach.

Interestingly, Benjamin Franklin's 'moral algebra' can be used for formal argumentation theory in Artificial Intelligence (Stephen Toulmin, The Uses of Argument, 1958). Moreover, it resurfaced formally as the 'Even Swaps' method for making tradeoffs because of Hammond, Keeney and Raiffa (Harvard Business Review, 1998).

1.2.2 Some Early Voting Results

The Marquis de Condorcet (whose name was Marie Jean Antoine Nicolas Caritat 1743–1794) produced several interesting results regarding holding fair elections. One of them is known as Condorcet's paradox, stating that majority preferences may

[2]Franklin began an autobiography in 1771 and continued in several spurts until 1788. His last writings were of his activities as late as 1757. Interestingly enough, his autobiography was published in French a year after his death as 'Memoires De La Vie Privee'. It was later translated into English as 'The Private Life of the Late Benjamin Franklin, LL.D'.

be intransitive, even though individual preferences are perfectly transitive. Another was what became known as the Condorcet voting method, which he proposed in 1785, for holding fair elections with more than two candidates. According to Condorcet, if one person is to be elected from n candidates, then the person elected would have to win in a head-to-head contest each of the other n-1 candidates. Jean-Charles de Borda (1733–1799), a French mathematician and political scientist, disagreed with Condorcet and simply advocated the use of summed rankings.[3] De Borda felt that Condorcet's proposal was not working in practice, as it might not generate a winner. De Borda proposed a system of ranking candidates by allocating them points on the basis of their rank. Both Condorcet and de Borda have had a considerable influence on the 'outranking methods' school of thought, which began to develop during 1960s in France.

1.2.3 Pareto-Optimality

The economist Vilfredo Pareto (1848–1923) was probably the first researcher whose work might formally be classified as MCDM.[4] His main publications are Cours d'économie politique (1896–1897) and Manual of Political Economy (1906).

In his 1906 publication, he made the famous observation that 20 % of the population of Italy owned 80 % of the property. This was later generalized by Joseph M. Juran as the Pareto principle (also termed the 80–20 rule). Pareto was the first (or at least one of the first) to mathematically study the aggregation of conflicting criteria into a single composite index. He was also the first to introduce the concept of efficiency (which became known as Pareto-optimality), one of the key concepts of economics, negotiation science and modern MCDM theory, irrespective of the 'school of thought'. A Pareto-optimal allocation of resources is achieved when it is not possible to make anyone better off without making at least one other person worse off. The original suggestion was for the multiple person (or bargaining) context, but it easily generalizes to single person, multiple criteria problems.

[3] De Borda spent most of his life as a military engineer and military marine officer. Five French ships were named Borda in his honor. He also made great contributions to various measurement systems.

[4] Pareto was born in Paris to an Italian father and French mother. He studied civil engineering and worked as a civil engineer, first for the Italian railways and later for private industry. He began to study economics and sociology in his mid forties. In 1923, shortly before his death, Pareto was nominated to Mussolini's government, but did not wish to serve.

1.2.4 Indifference Curves and Edgeworth Box

Francis Edgeworth (1845–1926) developed the foundations of utility theory, introducing the notion of an indifference curve.[5] Another major contribution, the Edgeworth box, is a way of representing various distributions of resources. Edgeworth described 'the box' in his famous book (although not in its currently presented form): Mathematical Psychics: An essay on the application of mathematics to the moral sciences (1881). Interestingly enough, in 1906, Pareto reworked Edgeworth's original presentation into the now-familiar box representation. Given some endowment in an Edgeworth box, the contract curve is the set of Pareto efficient allocations in a two-agent economy. Both Pareto's and Edgeworth's work has had a profound effect on economics, negotiation science and modern MCDM.

Because of the importance of both Edgeworth and Pareto, the MCDM Society began giving Edgeworth-Pareto awards in 1992 and has continued giving this award since.

1.2.5 Set Theory, Number Theory

Georg Cantor (1845–1918) is known to be the inventor of set theory; he also developed the concept of one-to-one correspondences or isomorphisms between members of sets.[6] He made substantial contributions to number theory, including the concepts of different categories of infinity. Some of Cantor's contributions are the foundations of the mathematical concepts used in MCDM.

The MCDM Society, to acknowledge Cantor's contributions, has been giving Georg Cantor awards to leading MCDM scholars beginning in 1992. The authors wish to thank Professor Theo Stewart, University of Cape Town, South Africa, for his comments and suggestions on our paper.

[5]Edgeworth, born in Ireland, was a philosopher, politician, economist, statistician and barrister, although he never practiced law or earned his living as a barrister. He was educated by private tutors until he entered a university. He studied ancient and modern languages, and later economics and mathematics. He became an influential person in the development of neoclassical economics. He was the first to apply certain formal mathematical concepts to decision making.

[6]Cantor was a German mathematician who was born in St. Petersburg, Russia, and moved with his family to Germany when he was 11 years old. He became an outstanding violinist. His PhD thesis was on number theory. He took a position at the University of Halle, where he spent his entire career, except for a brief early period teaching at a Berlin girl's school. Although he became a full professor at a young age of 34 years, his work was extremely controversial at the time. He later became very highly regarded, and in 1904, the Royal Society of London awarded Cantor its Sylvester medal, its highest award for work in mathematics for improving natural knowledge.

1.3 Origins of Decision Analysis, Utility Theory

There have been many early developments in utility theory. We mention these, together with the main contributors under several headings.

1.3.1 Economics as a Modern Science

In 1926, Ragnar Frisch (1895–1973), a Norwegian economist, published a seminal article, 'Sur un Problème D'économie Pure', which proposed both ordinal and cardinal utility. He also wrote an article proposing that economics use the mathematical tools used by sciences such as physics.[7]

Perhaps Frisch's most relevant publication in MCDM was a paper published in a rather obscure journal titled 'Numerical Determination of a Quadratic Preference Function for Use in Macroeconomic Programming', Giornale Degli Economisti e Annali Di Economica, 1961. In that paper and in a sequel one, Frisch developed an interview technique to elicit a person's utility (value) function. Ralph Keeney and Howard Raiffa may have been unaware of Frisch's paper while working on their 1976 book. Frisch very much wanted to have his utility function elicitation technique used by the Norwegian Parliament. Although he was quite close to a senior Norwegian governmental minister, he was unable to assess the Parliament's utility function and have the government use his approach. Members of Parliament did not wish to make their utility function explicit! A very current problem!

1.3.2 Expected Subjective Utility

Frank Ramsey (1903–1930) presented the first set of axioms for choices between alternatives with uncertain outcomes, leading to an expected (subjective) utility model in 1926.[8] This work was published after his death as an essay 'Truth and Probability' in 1931 (in The Foundations of Mathematics and Other Logical Essays, Routledge and Kegan, London, 156–198.). At the same time, Bruno de Finetti published his famous articles about the notion of subjective probability (B. de

---

[7]Frisch was a pioneer of economics as a modern science. Frisch's research is very creative. As told by former colleagues, when pursuing a new topic, he would seldom read about what others had carried out. He felt that this would constrain him. Instead, being a religious man, when not being able to solve a problem, he would isolate himself and pray. Frisch was awarded the first Nobel Prize in Economic Sciences in 1969.

[8]Ramsey was born in Cambridge. He was a brilliant mathematician and philosopher, with an interest in logic. It was John Maynard Keynes who urged him to work on problems of economics. Ramsey died from a liver disease at the age of 26 years.

Finetti: 'Probabilism: A Critical Essay on the Theory of Probability and on the Value of Science', translation of 1931 article in Erkenntnis, Volume 31, September 1989). It seems that the two men never met, but essentially presented the same idea at the same time. We cannot quote one without quoting the other. Both Ramsey's and de Finetti's work are very important to the 'multiattribute utility theory' school, spearheaded by Howard Raiffa and Ralph Keeney, and more broadly to decision analysis.

The Decision Analysis Society of INFORMS (The Institute for Operations Research and the Management Sciences) awards a medal in Ramsey's name each year.

1.3.3 Theory of Games

John von Neumann (1903–1957) and Oskar Morgenstern (1902–1977) developed the mini-max theorem of game theory and the foundations of utility theory in their monumental book, Theory of Games and Economic Behavior published originally by Princeton University Press in 1944.[9]

John Nash (1928–2015) was responsible for many further developments in game theory.[10] His papers (for example, 'Equilibrium Points in n-Person Games', (Proceedings of the National Academy of Sciences, 36, 48–49, 1950) and 'The Bargaining Problem', Econometrica, 18, 155–162, 1950) have greatly influenced modern economics and negotiation science.

1.3.4 Revealed Preferences

In 1938, in a paper titled 'A note on the pure theory of consumer's behavior' published in Economica, Paul Samuelson (1915–2009) described the concept he later called 'revealed preference'. In this paper, Samuelson stated what has since

[9]Von Neumann was a Hungarian-American mathematician and is generally regarded as one of the greatest mathematicians in modern history. He has contributed to many fields, including mathematics, computer science and nuclear physics. He joined Princeton University in 1930, and, with Albert Einstein and Kurt Gödel, formed the Institute for Advanced Studies there in 1933.

Morgenstern was born in Germany and grew up in Austria. His PhD was in political science, but he later became a professor of economics. He later joined the faculty at Princeton, worked with John von Neumann, carried out joint research and published their famous book on game theory.

[10]Nash received two degrees from Carnegie Institute of Technology and a PhD from Princeton. He had a letter of recommendation from his advisor at Carnegie, Richard Duffin that consisted of one sentence, 'This man is a genius'. He was a schizophrenic prodigy. An Oscar winning film was made about him, on the basis of the best-selling book by Sylvia Nassar 'A Beautiful Mind' (2001). He was awarded the Nobel Prize in economics in 1994. Until his death in May 2015 in a taxicab accident, Nash remained active in research in game theory.

become known as the Weak Axiom of Revealed Preference by writing '... if an individual selects batch one over batch two, he does not at the same time select two over one'. Ten years later, Samuelson described how one could use the revealed preference relation to construct a set of indifference curves. The proof was for two goods only and was largely graphical. Samuelson recognized that a general proof for multiple goods was necessary, but never pursued it. The revealed preference theory has over the years had a considerable impact on the theory of consumer behaviour. In later years, it was subjected to numerous empirical tests, among others by Anthony Koo in an Econometrica paper in 1963. The influence of the revealed preference theory on modern MCDM may not be obvious; perhaps because many MCDM scholars did not have a background in economics. However, it certainly did influence single dimensional utility assessment theory and practice.[11]

1.3.5 Bounded Rationality

Against the mainstream economics, Herbert Simon (1916–2001) claimed that decision making does not obey the postulates of the 'rational man'. In a series of articles and books starting in the 1940s, Simon wrote about decision making.[12] Among other things, he developed a behavioral theory on the basis of limited or bounded rationality (H. Simon: 'A Behavioral Model of Rational Choice', Quarterly Journal of Economics, 1955.)

Simon claimed that humans do not solve problems by maximizing utility, but are 'satisficers', who set aspiration levels (that a solution must satisfy) when solving problems. If humans are able to find a solution to a problem (or a small number of solutions) that satisfies the stated aspiration levels, they accept the solution. If not, they must relax (or weaken) their aspiration levels. On the other hand, if a set of aspiration levels admits too many solutions, then the aspiration levels are tightened. The process of setting aspiration levels, determining whether there are solution(s) that satisfy them and resetting aspiration levels is repeated until a solution is found. It has been suggested that the theory has normative as well as descriptive value. Aspiration levels play a major role in modern MCDM techniques, for example in goal programming and the methods based on the use of Wierzbicki's Achievement

[11]Samuelson wrote a column for Newsweek magazine. One article included Samuelson's most quoted remark, and a favorite economics joke: 'To prove that Wall Street is an early omen of movements still to come in GNP, commentators quote economic studies alleging that market downturns predicted four out of the last five recessions. That is an understatement. Wall Street indexes predicted nine out of the last five recessions! And its mistakes were beauties'. http://online.wsj.com/article/SB126072304261489561.html. Samuelson received the Nobel Prize in Economics in 1970 for his many contributions to Neoclassical Economics.

[12]Allen Newell (1927–1992) and Herbert Simon are widely regarded as the fathers of Artificial Intelligence, a field that has also influenced modern MCDM.

Scalarizing Function (Lecture Notes in Economics and Mathematical Systems 177, Springer, Berlin, 468–486, 1980). Simon was awarded the Nobel Prize in economics in 1978.[13]

1.3.6 Social Choice and Individual Values

Kenneth Arrow's impossibility theorem or paradox demonstrates that no voting system can convert the preferences of individuals into a community-wide ranking, while also meeting certain reasonable criteria with three or more discrete options to choose from. These criteria are called unrestricted domain, nondictatorship, Pareto efficiency and independence of irrelevant alternatives. Arrow presented the theorem in his PhD thesis and published it in his 1951 book Social Choice and Individual Values, Wiley & Sons. Arrow's theorem has generated much research on how to 'circumvent' the original difficulty, by weakening one of the assumptions. Arrow received the 1972 Nobel Prize in economics. The work of Amartya Sen is also relevant for generalizing social choice theory. In fact, research in social choice theory has inspired research in MCDM (for example, the book by Arrow, K. J. and Raynaud, H., Social Choice and Multicriterion Decision-Making, The MIT Press, Cambridge, MA, 1986) and in Artificial Intelligence (Computational Social Choice). See also approval voting; a scheme devised by Steven Brams and Peter Fishburn in 1978.

1.3.7 Theory of Value

Gerard Debreu published his classic book 'Theory of Value: An Axiomatic Analysis of Economic Equilibrium' in 1959, and an influential paper 'Topological Methods in Cardinal Utility Theory' in 1960. He was awarded the Nobel Prize in Economics in 1983.

1.3.8 Games and Decisions

R. Duncan Luce (born 1925) and Howard Raiffa (born 1924) published a book, Games and Decisions: Introduction and Critical Survey, Wiley & Sons, in 1957,

[13]Herbert Simon's autobiography Models of My Life, Massachusetts Institute of Technology Press, 1996, contains an interesting discussion about how Bill Cooper and Herbert Simon (with Lee Bach and Provost Smith) laid the foundations for the Graduate School of Industrial Administration at Carnegie Mellon University. They regarded business education (at the time) as a 'wasteland of vocationalism that needed to be transformed into science-based professionalism, as medicine and engineering ... ' As a curiosity, as a professor, Simon did not allow students to take notes; if they did, they did not give him their full attention.

which was a predecessor of modern decision theory. Shortly thereafter, Ron Howard wrote a paper ‘Sequential Decision Processes’, with G. E. Kimball, in Notes on Operations Research, in 1959. Jointly with James E. Matheson, Howard also wrote ‘Decision Analysis: Applied Decision Theory’, published in the Proceedings of the Fourth International Conference on Operational Research, in 1966. In that article, they supposedly used the term ‘decision analysis’ for the first time. Howard Raiffa published two important books on decision analysis during the 1960s, the first with Robert Schlaifer in 1961 (Applied Statistical Decision Theory), and the second in 1968 (Decision Analysis, Introductory Lectures on Choices under Uncertainty). The latter extensively develops the decision tree approach. We must also mention the unpublished RAND Memorandum from 1968 by Raiffa (H. Raiffa: ‘Preference for multi-attributed alternatives’, RAND Memorandum, RM-5868-DOT/RC, December 1968), which elegantly connects axioms, preferences/values and the practice of decision making. Raiffa’s memorandum spurred much research in multiattribute utility (R. Keeney and H. Raiffa, Decisions with Multiple Objectives: Preferences and Value Tradeoffs, Wiley, 1976).

R. D. Luce and J.W. Tukey’s paper (Simultaneous Conjoint Measurement: A New Scale Type of Fundamental Measurement, Journal of Mathematical Psychology, 1, 1–27, 1964) has become a popular ‘utility measurement’ technique, used in particular in various marketing contexts.

1.3.9 Behavioral Decision Theory

Ward Edwards (1927–2005) is generally regarded as the father of behavioral decision research. He published two seminal articles, one in 1954 and the other in 1961, establishing behavioral decision research as a new field. In his 1954 article ‘The Theory of Decision Making’, Psychological Bulletin, Ward Edwards introduced the expected utility model to psychologists and posed the sensible question: do people actually behave as if they have a utility function? However, Edwards’ later publication, ‘Behavioral decision theory’ in the Annual Review of Psychology, 1961, formally named the field. The paper discussed issues such as how people make decisions and how people could improve decisions. For decades, behavioral decision theory and normative research in decision analysis/MCDM would develop separately. However, in recent years, it has increasingly been recognized that we ought to have a better understanding as to how humans make decisions, to provide better support to decision makers.

A book commemorating his contributions (J. W. Weiss and D. J. Weiss, 2009, A Science of Decision Making: The Legacy of Ward Edwards, Oxford University Press) includes papers by many of his former colleagues, as well as the two papers described previously. For 41 years, Edwards hosted the annual Bayesian Research Conference in California. Beginning in 2006, the conferences became the Edwards Bayesian Research Conference.

When addressing the critique of the 'utility theory', we should also include the contributions of M. Allais. He presented the famous Allais paradox in 1953, as a criticism to the 'American school of thought', where he showed that it is not uncommon for humans to violate the axioms of rational choice. And these could not be judged as errors. Allais paradox has spurred much research in behavioral decision theory, among others by Amos Tversky and Daniel Kahneman, also a Nobel Prize recipient in economics (2002).

1.3.10 Utility Theory

Another prolific contributor to utility theory was Peter Fishburn (born 1936). Peter Fishburn made many fundamental contributions to the theory of social choice and utility during his career. He wrote two well-known books in the 1960s: Decision and Value Theory in 1964 and Utility Theory for Decision Making in 1970, summarizing many of his earlier thoughts. Both books further advanced utility theory and helped pave the way for multi-attribute utility theory.

1.3.11 The 'Outranking Methods'

The ELECTRE methods comprise a family of MCDM methods that originated in France during the mid-1960s. The acronym ELECTRE stands for ELimination Et Choix Traduisant la REalité (ELimination and Choice Translating REality). The method was first proposed by Bernard Roy (born 1934) and his colleagues at SEMA (Société d'Economie et de Mathématiques Appliquées), a consulting company. A team at SEMA was working on the multiple criteria problem of how firms chose among possible new activities. They had encountered problems using weighted sums. Bernard Roy, who had a background in graph theory, was called in as a consultant, and the group developed the original ELECTRE method. As it was first applied in 1965, the ELECTRE method was to help choose the best action(s) from a given set of actions, but it was soon applied to problems of ranking and sorting as well. The method became more widely known when a paper titled 'La méthode ELECTRE' by Bernard Roy appeared in a French Operations Research journal Revue d'Informatique et de Recherche Opérationelle, in 1968. It evolved into ELECTRE I, and future versions of it are as follows: ELECTRE II, ELECTRE III, ELECTRE IV, ELECTRE IS and ELECTRE TRI. Bernard Roy is widely recognized as the father of the ELECTRE method, which was one of the earliest so called 'outranking' approaches. Many authors have since developed and used outranking based approaches, both in collaboration with Bernard Roy and independently.

1.4 Origins of Multiple Objective Mathematical Programming

Multiple objective mathematical programming owes much to linear programming. George Dantzig's (1914–2005) contributions to linear programming are monumental. In 1947, George Dantzig proposed the simplex algorithm as an efficient method for solving linear programming problems. He did this in the SCOOP program (Scientific Computation of Optimum Programs), a US government research program. The goal of the program was to make war-time operations more efficient. Leonid Kantorovich (1912–1986), in the Soviet Union, had earlier proposed a similar method for economic planning, but his contributions were unknown in the Western World (Mathematical Methods of Organizing and Planning Production, 1939) until after Dantzig published his first work. Kantorovich was awarded the Nobel Prize in Economics in 1975; he was the only Soviet or Russian to be awarded the prize. The development of the digital computer at roughly the same time made Dantzig's and Kantorovich's contributions extremely important: it became possible to use the simplex algorithm to solve real-world problems. Linear programming quickly became popular in industry. Saul Gass's popular textbook on Linear Programming, first published by McGraw Hill in 1958, helped applications to become more widespread. Important follow-up work was conducted by many researchers, for example, Harold W. Kuhn (born 1925) and Albert W. Tucker (1905–1995) on nonlinear programming. The Karush–Kuhn–Tucker conditions for optimality have found widespread use in multiple objective mathematical programming.

1.4.1 Efficient Vectors

In the early 1950s, Tjalling C. Koopmans (1910–1985) extended Pareto's work introducing the notion of 'efficient vectors', in the context of a resource allocation problem: 'Analysis of Production as an Efficient Combination of Activities', in Activity Analysis of Production and Allocation, published by Physica Verlag in 1951. This was certainly a precursor of multiple objective mathematical programming. He was awarded the Nobel Prize in Economics in 1975.

Arthur M. Geoffrion (born 1937) published two important articles in the late 1960s, laying the foundation for multiple objective mathematical programming and the theory of vector maximization. The first was about solving bi-criteria mathematical programs and was published in Operations Research in 1967. The second paper developed the notion of 'proper efficiency'. It was published in the Journal of Mathematical Analysis and Applications in 1968.

1.4.2 Goal Programming

In 1955, Abraham Charnes (1917–1992), William Cooper (1914–2012) and R. O. Ferguson published an article 'Optimal Estimation of Executive Compensation by Linear Programming' in Management Science that contained the essence of goal programming, even though the name goal programming was not used in print until the publication of Charnes and Cooper's book: Management Models and Industrial Applications of Linear Programming, in 1961. The idea of goal programming was simple; it is related to Simon's level of aspiration concept. Ask the decision maker to specify target values for goals and formulate the problem as one of minimizing the weighted deviations from those target values. Alternatively, instead of weights, one could use a lexicographic (pre-emptive) model for the goals. In the original version, all constraint and goal functions were assumed linear. Hence, goal programming could be regarded as a generalization of linear programming. Much of the early work in goal programming was practice driven. Goal programming has had a great impact on research in multiple objective mathematical programming. Although goal programming has been extended in various directions, in the 1970s, it also served as an impetus to scholars to develop interactive man–machine, multiobjective mathematical programming techniques.

At a conference organized by Mihajlo D. Mesarovic at then Case Institute of Technology (now Case Western Reserve University) in 1963, Mesarovic presented Abraham Charnes and William W. Cooper with a poem that he wrote:

Programming sticks upon the shoals
Of incommensurate multiple goals,
And where the tops are no one knows
When all our peaks become plateaus
The top is anything we think
When measuring makes the mountain shrink.
The upshot is, we cannot tailor
Policy by a single scalar,
Unless we know the priceless price
Of honor, justice, pride, and vice.
This means a crisis is arising
For simple-minded maximizing.

The poem was published in M. D. Mesarovic, Views on General Systems Theory, John Wiley and Sons, 1964, p. 61.

1.4.3 Parametric Programming

Thomas L. Saaty (born 1926), published two papers with Saul Gass (born 1926) in Operations Research dealing with a parametric objective function, one in 1954 and the other in 1955. Their algorithm could be used for generating efficient solutions by varying the weights of a composite (aggregate) value function, a popular early technique of MCDM. In the Saaty–Gass spirit, it was not uncommon in the following decades to work with an additive, linear value function. It was, however, soon realized that this technique could not be used to generate unsupported efficient solutions. A quarter century later, Andrzej Wierzbicki presented his Achievement Scalarizing Function, to remedy this problem.

Demonstrating the breadth of his interests, Saaty (with George Dantzig) published a book Compact City, A Plan for a Liveable Urban Environment, in 1973. He is the inventor, architect and primary theoretician of the analytic hierarchy process, a widely used method of MCDM. The first publications on analytic hierarchy process occurred in 1977.

1.4.4 Automatic Control

Several researchers addressed vector-valued criteria in automatic (or optimal) control. One of the first was Lotfi Zadeh (born 1921). He wrote a short paper 'Optimality and Non-Scalar-Valued Performance Criteria', which he published in IEEE Transactions on Automatic Control in 1963. Solutions in the Pareto optimal set could be identified by solving a series of standard optimal control problems, using a weighted sum of individual criteria of the controllers. Several of the early MCDM scholars had a background in automatic control, clearly influencing their thinking.

1.4.5 Restricted Bargaining

Stanley Zionts (born 1937) and Bruno Contini (born 1936) worked on a bargaining model involving multiple criteria in mid-1960s. They published an article on their work in Econometrica in 1968. After completing his doctorate at Carnegie Institute of Technology, Zionts took a position with the Ford Foundation in India, where he applied the bargaining model to a problem of the steel industry of India, representing one of the early real-world applications of formal bargaining models.

1.5 Conclusion

In this paper, we have presented what we know and were able to glean from our research about the early history of MCDM and related areas. Many developments in the field were made by major contributors to operations research, management science, economics and other areas. Quite a few of the early contributors were awarded the Nobel Prize in Economic Sciences. The history of the field is important because it identifies the 'roots' of our field. A saying attributed to Reverend Henry

Ward Beecher, a nineteenth century clergyman, urges people to give their children roots and wings. We have identified the roots of our field. The future will show where the wings take us. As indicated previously, we have drawn material from our book, Multiple Criteria Decision Making: From Early History to the 21st Century, published by World Scientific, Singapore, 2011. We also wish to thank those who commented on our book, this paper or the talk presented at the 21st MCDM conference in Jyväskylä, Finland in June 2011. Our presentation here continues through the 1960s. For more information, we urge readers to read our book mentioned previously.

Chapter 2
Paradigms and Challenges

Bernard Roy

Abstract The purpose of this introductory part is to present an overall view of what MCDA is today. In Sect. 2.1, I will attempt to bring answers to questions such as: what is it reasonable to expect from MCDA? Why decision aiding is more often multicriteria than monocriterion? What are the main limitations to objectivity? Sect. 2.2 will be devoted to a presentation of the conceptual architecture that constitutes the main keys for analyzing and structuring problem situations. Decision aiding cannot and must not be envisaged jointly with a hypothesis of perfect knowledge. Different ways for apprehending the various sources of imperfect knowledge will be introduced in Sect. 2.3. A robustness analysis is necessary in most cases. The crucial question of how can we take into account all criteria comprehensively in order to compare potential actions between them will be tackled in Sect. 2.4. In this introductory part, I will only present a general framework for positioning the main operational approaches that exist today. In Sect. 2.5, I will discuss some more philosophical aspects of MCDA. For providing some aid in a decision context, we have to choose among different paths which one seems to be the most appropriate, or how to combine some of them: the path of realism which leads to the quest for a discussion for discovering, the axiomatic path which is often associated with the quest of norms for prescribing, or the path of constructivism which goes hand in hand with the quest of working hypothesis for recommending.

Keywords Multiple criteria decision aiding • Imperfect knowledge • Aggregation procedures

B. Roy (✉)
LAMSADE, Université Paris-Dauphine, Place du Maréchal De Lattre de Tassigny, 75775 Paris Cedex 16, France
e-mail: roy@lamsade.dauphine.fr

S. Greco et al. (eds.), *Multiple Criteria Decision Analysis*, International Series in Operations Research & Management Science 233,
DOI 10.1007/978-1-4939-3094-4_2

2.1 What Are the Expectations That Multicriteria Decision Aiding (MCDA) Responds To?

The purpose of this introductory chapter is to present an overview of what MCDA is today. Since the 60s, this discipline has produced, and it still produces, a great number of theoretical as well as applied papers and books. The major part of them will be presented in the following chapters of this book. It is important at the outset to understand their specific contributions are in terms of enlarging the operations research field and, more generally, to bringing light to decision making contexts. That is why I shall begin this chapter by considering the three following questions: what is reasonable to expect from MCDA? Why is decision aiding is more often multicriteria than monocriterion? What are the main limitations to objectivity which must be taken into account? The next section will be devoted to a brief presentation of three basic concepts which can be viewed as initial and fundamental keys for analyzing and structuring problem situations. In practice, it is very important to draw attention to questions such as: what is the quality of the information which can be obtained? What is the meaning of the data which are available or can be elaborated? In Sect. 2.3, I shall examine how the existing models and procedures take into account various types of answers to such questions which refer to a given problem's real world context.

Another difficulty in an MCDA context comes from the fact that comparisons between potential actions must be made comprehensively, with respect to all criteria. Various aggregation techniques which will be described in detail throughout the successive chapters of this book have been proposed and used in order to overcome this kind of difficulty. In Sect. 2.4, I shall present a general framework for positioning the main operational approaches in which these aggregation techniques come into play. Some more general philosophical considerations will complete this introductory chapter.

2.1.1 What Is Reasonable to Expect from Decision Aiding (DA)?

Decision aiding can be defined (see [63]) as follows: Decision aiding is the activity of people using models (not necessarily completely formalized ones) to help to obtain elements of responses to the questions asked by a stakeholder in a decision process. These elements work towards clarifying the decision and usually towards recommending, or simply favoring, a behavior that will increase the consistency between the evolution of the process and this stakeholder's objectives and value system. In this definition, the word "recommending" is used to draw attention to the fact that both analyst and decision maker are aware that the decision maker is completely free to behave as he or she sees fit after the recommendation is made.

This term is increasingly used in DA to replace "prescription". The latter is, in many cases, inappropriate (see [36, 60]) for designating what a team of analysts accompanying a decision making process might achieve.

Thus defined, DA aims at establishing, on recognized scientific bases, with reference to working hypotheses, formulations of propositions (elements of responses to questions, a presentation of satisfying solutions or possible compromises, ...) which are then submitted to the judgment of a decision maker and/or the various actors involved in the decision making process. According to the case, DA can thus reasonably contribute to:

- Analyzing the decision making context by identifying the actors, the various possibilities of action, their consequences, and the stakes;
- Organizing and/or structuring how the decision making process will unfold, to increase consistency between the values underlying the objectives and goals and the final decision;
- Getting the actors to cooperate, by proposing keys to better mutual understanding and a framework conducive to debate;
- Drawing up recommendations based on results from models and computational procedures designed within the framework of a working hypothesis;
- Participating in the process to legitimate the final decision.

For a deeper understanding of the bases reviewed above, the reader can refer to [13, 14, 20, 21, 42, 50, 61, 72, 74].

2.1.2 Why Is DA More Often Multicriteria than Monocriterion?

Even when DA is provided for a single decision maker, it is rare for her or him to have in mind a single clear criterion. Thus, when DA takes place in a multi-actor decision making process, it is even rarer for there to be a priori a single, well-defined criterion deemed acceptable by all actors to guide the process. This process is often not very rational. Each actor plays a more or less well defined role which gives priority to her or his own objectives and value system.

In both cases, it is necessary to take into consideration various points of view dealing with, for example, finance, human resources, environmental aspects, delays, security, quality, ethics, ... By considering each pertinent point of view separately, independently from the others, it is generally possible to arrive at a clear and common elicitation of preferences regarding the single point of view considered. This naturally leads to associating a specific criterion to each pertinent point of view. Each of these criteria is used to evaluate any potential action on an appropriate qualitative or quantitative scale. In most cases, there is no obvious and acceptable arithmetic rule which can keep account of these heterogeneous scales by substituting a single scale based on a common unit for each of them (see Sect. 2.4 below).

Such a scale, bringing a common unit into play, must be introduced a priori when we want to avoid a multicriteria approach, i.e., when we prefer to choose what is called a *monocriterion* approach. This choice, in many decision making contexts, might:

- lead to wrongly neglecting certain aspects of realism;
- facilitate the setting up of equivalencies, the fictitious nature of which remains invisible;
- tend to present features of one particular value system as objective.

On the contrary, a multicriteria approach contributes to avoiding such dangers by:

- delimiting a broad spectrum of points of view liable to structure the decision process with regard to the actors involved;
- constructing a family of criteria which preserves, for each of them, without any fictitious conversion, the original concrete meaning of the corresponding evaluations;
- facilitating debate on the respective role (weight, veto, aspiration level, rejection level, ...) that each criterion might be called upon to play during the decision aiding process.

Additional considerations about relative advantages on monocriterion and multicriteria approaches can be found in [12, 15, 22, 58, 63].

2.1.3 Can MCDA Be Always Totally Objective?

In many cases, those who claim to shed light objectively on a decision in fact take a stand—consciously or unconsciously—for an a priori position or for a prevailing hypothesis which they then seek to justify. Arguments for making a decision are thus put forward more in the spirit of advocacy than in that of an objective search (see [4, 34]).

In what follows, we will only consider situations in which DA is motivated by a strong desire for objectivity. Even in such situations, it is important to be sensitive to the existence of some fundamental limitations on objectivity. Their origins lie in the following facts:

(a) The borderline between what is and what is not feasible is often fuzzy in real decision making contexts. Moreover, this borderline is frequently modified in the light of what is found through the study itself.

(b) Even in cases for which DA is provided for a well-defined decision maker, his or her preferences very seldom seem well-shaped. In and among areas of firm convictions lie hazy zones of uncertainty, half held belief, or indeed conflicts and contradictions. Such sources of ambiguity or arbitrariness concerning preferences which are to be elicited and modeled are even more present when the decision maker (entity for whom or in the name of whom decision aiding is provided for) is a mythical person, or when decision aiding is provided in a multicriteria context. We have to admit, therefore, that the study itself contributes to eliminating questioning, solving conflicts, transforming contradictions and destabilizing certain convictions. Any interaction and questioning between the analyst and the decision maker, or any actors involved into the decision making process, may have some an unpredictable or imperceptible effect.

(c) Much of the data is imprecise, uncertain or ill defined. Making data say more than it actually means is a risk. Moreover, some data only reflects the value system of a given individual.

(d) It is impossible to say that a decision is good or bad only by referring to a mathematical model. The organizational, pedagogical and cultural aspects of the entire decision-making process leading to a particular decision also contribute to its quality and success.

Rather than dismissing or canceling the subjectivity which results from the limitations of objectivity described above, decision aiding must make an objective place for it. (For a pedagogical overview of MCDA approaches, see [60, 61, 66, 70, 72].)

2.2 Three Basic Concepts

From the beginning to the end of work in MCDA, three concepts usually play a fundamental role for analyzing and structuring the decision aiding process in close connection with the decision process itself. The presentation of these concepts in the three following sub-sections is obviously succinct. It nevertheless aims to draw attention to some important features.

2.2.1 Alternative, and More Generally, Potential Action

By *potential action*, we usually designate that which constitutes the object of the decision, or that which decision aiding is directed towards. The concept of *action* does not a priori incorporate any notion of feasibility, or possible implementation. An action is qualified as potential when it is deemed possible to implement it, or simply when it deserves some interest within the decision aiding process.

The concept of *alternative* corresponds to the particular case in which modeling is such that two distinct potential actions can in no way be conjointly put into operation. This mutual exclusion comes from a way of modeling which in a comprehensive way tackles that which is the object of the decision, or that towards which DA is directed. Many authors implicitly suppose that potential actions are, by definition, mutually exclusive. Nevertheless, such an hypothesis is in no way compulsory. In many real world decision aiding contexts, it can be more appropriate to adopt another way of modeling such that several potential actions can be implemented conjointly (see examples in [57, 63]).

In all cases, A will denote the set of potential actions considered at a given stage of the DA process. This set is not necessarily stable, i.e., it can evolve throughout the decision aiding process. Such an evolution may come from the study's environment, but also from the study itself. The study may shed light on some aspects of the problem, which could lead to revising some of the data and then, possibly, to modifying the borderline between what is and what is not feasible.

By a, we will denote any potential action or alternative. When the number of actions is finite ($|A| = m$) we shall let:

$$A = \{a_1, a_2, \ldots, a_m\}.$$

When modeling of actions can be done by referring to some variables $x_1, x_2, \ldots$ it is possible to write:

$$a = (x_1, x_2, \ldots).$$

In such cases, A is generally defined by a set of analytic constraints which characterize the borderline between what is feasible and is not feasible. Multiobjective mathematical programming constitutes an important particular case of this type of modeling (see [26] and Part VI).

In another type of modeling, the value of each variable x_i ($i = 1, 2, \ldots, n$) designates a possible score on an appropriate scale X_i built for evaluating actions according to a specified point of view or criterion. In such cases, A can be viewed as a subset of the Cartesian product $X = \prod_{i=1}^{n} X_i$. This type of modeling is commonly used in multiattribute utility theory (MAUT) (see Part IV). Let us observe that this type of modeling necessitates some precautions: since each potential action is identified with the n components of its evaluation, it loses all concrete identity; in particular, two actions having the same evaluations $x_1, \ldots, x_n$ are no longer distinguishable.

More details and illustrations of the concepts and ways of modeling presented above could be found in [63, Chap. 5], [90, Chap. 1], and [38].

2.2.2 *Criterion and Family of Criteria*

The reader, who is not yet familiar with some of the terms used in this subsection, will find more precise definitions in [66, Chap. 1, Appendix 1, Glossary].

Let us remember that a criterion g is a tool constructed for evaluating and comparing potential actions according to a point of view which must be (as far as it is possible) well-defined. This evaluation must take into account, for each action a, all the pertinent effects or attributes linked to the point of view considered. It is denoted by $g(a)$ and called the *performance* of a according to this criterion.

Frequently, $g(a)$ is a real number, but in all cases, it is necessary to define explicitly the set X_g of all the possible evaluations to which this criterion can lead.

For allowing comparisons, it should be possible to define a complete order $<_g$ on X_g: $(<_g, X_g)$ is called the *scale* of criterion g. To be accepted by all stakeholders, a criterion should not bring into play, in a way which might be determinant, any aspects reflecting a value system that some of these stakeholders would find necessary to reject. This implies in particular that the direction to which the preferences increase along the scale (and more generally the complete order $<_g$) is not open to contest.

Elements $x \in X_g$ are called *degrees* or *scores* of the scale. Each degree can be characterized by a number, a verbal statement or a pictogram. When in order to compare two actions according to criterion g we compare the two degrees used for evaluating their respective performances, it is important to analyze the concrete meaning in terms of preferences covered by such degrees. This leads to distinguishing various types of scales:

(a) *Purely ordinal scale*: Scale such that the gap between two degrees does not have a clear meaning in terms of difference preferences; this is the case with:

- a *verbal scale* when nothing allows us to state that the pairs of consecutive degrees reflect equal preference differences all along the scale;
- a *numerical scale* when nothing allows us to state that a given difference y between two degrees reflects an invariant preference difference when we move the pair of degrees considered along the scale.

This type of scale is often called a qualitative scale.

(b) *Quantitative scale*: Numerical scale whose degrees are defined by referring to a clear, concrete defined quantity in a way that it gives meaning, on the one hand, to the absence of quantity (degree 0), and on the other hand, to the existence of a unit allowing us to interpret each degree as the addition of a given number (integer or fractional) of such units. In such conditions, the ratio between two degrees can receive a meaning which does not depend on the two particular degrees considered; this is another way of defining quantitative scales which are also called *cardinal* or *ratio scales*.

(c) *Other types*: In MCDA, we do not always work with scales belonging to one of the above two extreme types (especially interval scales). The most interesting intermediate types are presented in [43, Sect. 2] and [66].

In MCDA, it is essential to know which type of scale we are working with to be sure of using its degrees in a meaningful way. According to the type of scales considered, certain kinds of reasoning and arithmetical operations are significant in terms of preference (see Chap. 3).

Moreover, the use of the degrees in a significant way must take the following fact into account: the difference between two degrees that are sufficiently close together may be non significant for justifying an indisputable preference in favor of one of the two actions. This stems from the procedure used to position the two actions on the scale considered. This procedure can appear insufficiently precise (with regard to the complexity of the reality in question), or insufficiently reliable (with regard to uncertainty concerning the future) for founding such an indisputable preference on such a small difference. I will come back to this subject in the next section.

In most cases, the first step of DA consists of building n criteria with $n > 1$ (see Sect. 2.1.2 above). They constitute what we call the *family F of criteria*. In order to be sure that F is able to play its role in the DA process correctly, i.e., in laying the foundations for convictions, communicating concerning the latter, debating and orienting the process towards the decision, and in contributing in some cases to legitimating this decision, it is necessary to verify that:

- what is apprehended by each criterion is sufficiently intelligible for each stakeholder;
- each criterion is perceived as a relevant instrument for comparing potential actions on the scale associated with this criterion, without prejudging its relative importance (importance that may vary considerably from one stakeholder to the next);
- the n criteria considered all together satisfy some logical requirements (exhaustiveness, cohesiveness, and non redundancy) which insure coherence of the family (for more details, see [66, Chap. 1, Appendix 1, Glossary], [73]).

It is important to observe that none of the above requirements implies that the n criteria of F must be independent. The concept of independence is very complex, and if dependence is desirable, it is necessary to specify what type of independence is needed. Multicriteria analysis has led to important distinctions between structural independence, preferential independence, and utility independence (see [39, 56], [63, Chap. 10], and [73, Chap. 2]).

Additional developments concerning this basic concept of criterion can be found in [2, 3, 5, 6, 8, 69].

2.2.3 Problematic as a Way in Which DA May Be Envisaged

The word “problematic” is used here in the sense indicated by the heading. Other expressions such as “statement”, “problem formulation” or “problem type” have been used as substitutes, but in my view, they are inappropriate and may lead to misunderstanding.

Let us underline first that DA must not be envisaged solely in the perspective of solving a problem of choice. In some cases, DA consists only of elaborating an appropriate set A of potential actions, building a suitable family F of criteria, and determining, for all or some $a \in A$, their performances sometimes completed by additional information (possible values for discriminating thresholds, aspiration and/or rejection levels, weights, ...). For designating this manner of conceiving of DA’s aim without seeking to elaborate any prescription, or recommendation, we use the term *description problematic* often coded $P.\delta$.

In MCDA, the word *problematic* refers to the way in which DA is envisaged. This means that the problematic deals with answers to questions such as the following: in what terms should we pose the problem?, what type of results should we try to obtain?, how does the analyst see himself fitting into the decision process to aid in arriving at these results?, what kind of procedure seems the most appropriate for guiding his investigation? In addition to $P.\delta$, three other reference problematics are currently used in practice. They can be briefly described as follows (for more details, see [54], [63, Chap. 6]):

- The *choice problematic* ($P.\alpha$): The aid is oriented towards and relies on a *selection* of a small number (as small as possible) of "good" actions in such a way that a single alternative may finally be chosen; this does not mean that the selection is necessarily oriented towards the determination of one or all the actions of A which can be regarded as optimum; the selection procedure can also, more modestly, be based on comparisons between actions so as to justify the elimination of the greatest number of them, the subset N of those actions which are selected (which can be viewed as a first choice) containing all the most satisfying actions, which remain non comparable between one another.
- The *sorting problematic* ($P.\beta$): The aid is oriented towards, and relies on an *assignment* of each action to the one category deemed the most appropriate for receiving it among a family of predefined categories. Categories are designed a priori to receive actions which will be or might be processed in the same way during the following step. For instance, a family of four categories can be based on a comprehensive appreciation leading to distinguishing between: actions for which implementation (1) is fully justified, (2) could be advised after only minor modifications, (3) can only be advised after major modifications, (4) is unadvisable. Let us observe that categories are not necessarily ordered as it is the case in the above example.
- The *ranking problematic* ($P.\gamma$): The aid is oriented towards and relies on a *partial* or *complete* ordering (pre-order) on A that may be considered as an appropriate instrument for comparing actions pairwise. This pre-order is the result of a *classifying* procedure allowing us to put together in classes actions which can be judged as indifferent, and to rank these classes (some of them may remain non-comparable).

The four problematics described above are not the only possible ones (see [10, 12]). Whatever the problematic adopted, the result arrived at by exploiting a given set of data through a single procedure is (except under unusual conditions) not sufficient for founding a prescription or a recommendation (see Sect. 2.4 below).

2.3 How to Take into Account Imperfect Knowledge and Ill-Determination?

DA cannot be correctly provided without trying to analyze and to take into account reasons and factors which can be responsible for contingency, arbitrariness, and ignorance in the way the problem is addressed and procedures implemented. In addition to their subjective characteristics, these reasons and factors may take on various forms whose presence and/or importance greatly depends on the decision making context considered. Their presence comes essentially from three sources (for more details, see [17, 59, 70, 71]):

- *Source* α ($S.\alpha$): The imprecise, uncertain and, more generally, poorly understood or ill-defined nature of certain specific features or factual quantities or qualities present in the problem.
- *Source* β ($S.\beta$): the conditions for implementing the decision taken; these will be influenced by:
  - The state of the context at the time the decision is implemented if it is a once-and-for-all decision;
  - The successive states of the context if the decision is sequential.
- *Source* γ ($S.\gamma$): the fuzzy or incomplete, sometimes unstable and easily influenced character of the system or systems of values to be taken into account; these values involve, in particular, the system and most often the systems of preferences which should prevail in order to evaluate the feasibility and relative interest of diverse potential actions, by considering the conditions envisaged for implementing these actions.

Once the problem is formulated, and during the entire decision aiding process, special attention should be paid to the three sources. Their examination must highlight what is imprecise, uncertain, unstable or ill determined. This can leads, for instance:

- starting from $S.\alpha$, to delimiting a domain of reasonable instantiation values for various data and parameters;
- from $S.\beta$, to building a set of scenarios describing different possible future contexts;
- from $S.\gamma$, to eliciting a set of weight vectors; for this purpose, it is important to remember that it makes no sense and is theoretically incorrect to specify measures of relative importance for the criteria without considering the nature of the overall evaluation model which will be used, i.e., without having defined the type of mathematical aggregation rules (see next section) which allow us to derive comprehensive preferences.

The DA process must clearly take into account all the results of this study. To do so, many approaches (formalisms, models, methods, ...) have been proposed. A panorama of such approaches can be found in Chap. 12, and in [41, 82]. These approaches rest upon various concepts, tools and theories; the main ones are:

- probability theory mainly used in MAUT (see Chap. 9), but also used in many other approaches, particularly for building criteria when uncertainty can be characterized by a probabilistic distribution;
- possibility theory [23, 24];
- multi-valued logic (see Chap. 4, [86]);

- concept of discriminating thresholds and quasi or pseudo-criterion (see Chap. 3, and for more details, [37, 76–78, 93]) mainly used in outranking methods (see Chap. 5).
- concept of fuzzy binary relations [19, 25, 28, 29, 46, 55];
- rough sets theory (see Chap. 13);
- ordinal regression for MCDA (see [33]).

Regardless of the tools used for taking into account imperfect knowledge and ill determination, the analyst must seek for obtaining robust solutions and/or robust conclusions.

Solutions and conclusions are qualified as robust if their design takes account of the existence of vague approximations and zones of ignorance, thus enabling the DM to forestall any impacts that he or she may deem regrettable, for instance when the goal of at is effectively reached is far short of expectations, or when the properties the DM wanted to maintain are deteriorated.

For decision aiding to be truly useful, it must provide the decision maker with the kinds of answers that will provide enough information enabling the decision maker (according to his/her subjectivity) to reach a decision in a situation where there are two conflicting risks:

- being poorly protected in the event of very poor performance relative to the impacts that he/she deems regrettable;
- being in a position that leads the DM to abandon any hope of good, or even very good, performance.

For more details, regarding tools that the analyst can use for obtaining robust solutions and/or robust conclusions, and regarding also more generally robustness concerns, the reader can see [1, 27, 35, 40, 52, 53, 64, 65, 68–71, 84, 91, 92].

Let us note that the analyst can make use of sensitivity analysis in the framework of robustness concern. In [64, 65], the reader could find some comments on links and differences between robustness analysis and sensitivity analysis (see also [84]).

2.4 An Operational Point of View

As soon as more than one criterion comes into play, a crucial question arises: how can we take into account all criteria comprehensively in order to compare potential actions between them? Let us consider two potential actions a and b characterized by their respective performances on the n criteria considered. More often, a will be better than b for some of the criteria, and b better than a for others. In such cases, in comparing a and b, on what basis can we found a *comprehensive judgment*, i.e., taking into account, in a comprehensive way, the n performances of a and the n performances of b. This problem is usually called the *aggregation problem*. In many of the chapters in this book, the reader will find a wide variety of solutions to this fundamental problem. In the present introductory chapter, I shall present only a general framework for positioning the main operational approaches provided for DA today (for more details on what the operational approach concept covers, see, [63, Chap. 11] and [72]).

2.4.1 About Multicriteria Aggregation Procedures

Let us suppose that two potential actions characterized by their *n* performances are shown to a respondent (for instance the DM) to elicit her or his preference, so that he or she can say:

- "I prefer the first to the second or vice versa";
- "I am indifferent about the two";
- "I am unable to compare these two actions".

What we call the preference system for the respondent refers to the results of such comparisons, as they could be stated (for all pairs of actions) expressed in terms of:

$$\textit{Preference},\quad \textit{indifference}\quad \text{or}\quad \textit{incomparability}$$

In certain cases, there may be information qualifying the intensity of preference.

Overall, assuming there is a rational, rigorous and stable (implicit or explicit) procedure, capable of defining the totality of the respondent's preference system in the respondent's mind, even before the outset of the decision-aiding process, is not very realistic.

What is usually called *preference model* is, on the contrary, a well-defined model, which produces results allowing the comparison of any pair of potential actions in crisp or fuzzy terms denoting preference, indifference, or incomparability.

Above all, a preference model in DA must be a tool for delving deeper into, exploring, interpreting, debating, and even arguing the subject.

It is not necessary (I even think it unrealistic) for the analyst to attempt to design the model so that it becomes a representation, albeit simplified, of what is actually happening in the respondent's mind when he/she compares two actions.

The most frequently used decision aiding operational approaches make use of a preference model that is based on a mathematical procedure called the *multicriteria aggregation procedure* (MCAP). Let:

$$g_1(a), \ldots, g_n(a) \quad \text{and} \quad g_1(b), \ldots, g_n(b)$$

denote the performances by which two potential actions *a* and *b* are characterized.

A MCAP is a procedure that produces a result allowing for the comparison of both actions in a comprehensive way.

A MCAP brings into play:

- *A logic of aggregation* that takes account of the conditions under which compensation between bad and good performances are accepted or refused.

- *Various inter-criterion and technical parameters* (e.g., weights, scaling constants, vetoes, aspiration levels, rejection levels). The specific role that each criterion can play with respect to the others is defined by the numerical values assigned to these parameters

Outside of the logic on which the MCAP is based, the inter-criterion and technical parameters have no meaning: they have no real existence in the mind of the decision maker. Thus, no true value that has to be approximated as well as possible exists. Rather, the analyst must strive to assign an appropriate value to each parameter so that the resulting preference model can become a useful tool for decision aiding.

For more details on the above considerations, see [16, 44, 62, 73, 75, 85].

Methods which are based on a mathematically explicit MCAP come under one of two types of operational approaches usually designated by the expressions approach based on a *synthesizing criterion* and approach based on a *synthesizing preference relational system.*

2.4.2 Approach Based on a Synthesizing Criterion

This approach is the most traditional. It can be characterized as follows: Formal rules taking account of the n performances $g_1(a), \ldots, g_n(a)$ are defined to assign a well-defined degree (most often a numerical value $v(a)$) to each $a \in A$ on an appropriate scale. The comparison between two actions is determined by their respective positions on this scale.

The way the aggregation problem is addressed in this approach leads to defining a complete pre-order on the set A. Most often, the formal rules consist in mathematical formulas that lead to an explicit definition of a unique criterion synthesizing the n criteria:

$$v(a) = V[g_n(a), \ldots, g_n(a)].$$

This is the case with MAVT, MAUT, SMART, TOPSIS, MACBETH, AHP, and so on (see Chaps. 8–10). The complete preorder on A can also be obtained by the use of a set of formal rules without any mathematically explicit expression of the synthesizing criterion, which remains implicit (see [7, 8]). In any case, this approach does not allow any incomparability.

Building a synthesizing criterion using such a multicriteria approach is not equivalent to a monocriterion approach. The dangers of the monocriterion approach have been presented above (see Sect. 2.1.2). Nevertheless, even if a multicriteria approach based on a synthesizing criterion contributes to reducing these dangers, it forces us to introduce a common scale (monetary scale, utility scale, ...) on which performances of each of the n criteria have to be evaluated. Moreover, with this approach, imperfect knowledge and ill-determination (*cf.* Sect. 2.3 above) can be taken into account essentially through probability distribution, fuzzy numbers, or in some cases through rough set theory but never through preference or indifference thresholds.

2.4.3 The Operational Approach Based on a Synthesizing Preference Relational System

As is the first, this second operational approach is based on a mathematically explicit MCAP. A major difference with the preceding approach comes from the fact that here the MCAP does not work on each potential action a separately from the others, but it successively compares a to each of the other $b \in A$.

In other words, the aggregation problem is no longer addressed in terms of defining a complete preorder on A, it is now addressed in terms of pairwise comparisons so as to design a synthesizing preference relational system. Taking into account the n performances of a and the n performances of b, the role of the MCAP is to give an answer to the question: what is the preference relation which can be validated between a and b, and in some cases with what degree of credibility? Mathematical rules, which lead to answering this question, are based on:

- various inter-criteria parameters, as in the first approach; but also, unlike the first approach, on discriminating thresholds (see Sect. 2.3 above) and veto thresholds;
- a logic of aggregation which easily allows us to take into account (and this is much more difficult with the approach based on a synthesizing criterion), on one hand, some limitations to compensation, and on the other, no quantitative performances.

The synthesizing preference relational system can be reduced to a single binary relation, which can be crisp or fuzzy. But it can also bring into play more than one binary relation. In all instances, the advantages of this second type of MCAP relative to the first cause certain difficulties to arise when we consider the operational approach based on such an MCAP. These difficulties stem from the fact that:

- pairwise comparisons can cause some intransitivities to appear;
- incomparability can be the most appropriate conclusion for comparing certain pairs (a, b);
- consequently, a synthesizing relational preference system is not a tool which is immediately usable for elaborating a recommendation.

For these reasons, this second operational approach necessitates completing the MCAP by a second procedure called *exploitation procedure*. This procedure is conditioned by the problematic considered (see above Sect. 2.3).

This second operational approach has led to various methods, most of which are covered by the label of *outranking methods* (mainly PROMETHEE and ELECTRE methods). The second part of this book is devoted to them. Other works related to this approach are presented in Part IV.

2.4.4 About Other Operational Approaches

All the operational approaches which are based on a mathematically explicit MCAP are not exactly in accordance with one of the two preceding approaches. Regarding this subject, the reader can refer to [9, 18, 32, 45, 81].

Finally, let us mention the existence of operational approaches which are not based on a mathematically explicit MCAP. In such approaches the analyst takes account of the DM's preference system by interacting with him or her.

This type of operational approaches (called interactive) is based on a formal procedure organizing a dynamic sequence of questions that the DM (or his/her representative) must answer.

The respondent is asked questions involving actions characterized by their performance in the criteria space. Each question only refers to a small area in the criteria space. The answers are given in terms of preference, indifference or incomparability.

The procedure is designed so that, through trial-and-error, the respondent progresses to the point of reaching one or several satisfying actions.

For more details on this kind of approach, see Chap. 22, [30, 47, 49], [73, Chap. 7], [83, 87–90, 94, 96].

In any case, whatever the operational approach considered, there is a possible confusion which should be avoided. Except under very unusual conditions, the results arrived at by treating a set of data through any appropriate procedure should not be confused with a well founded scientific recommendation. Repeated calculations using different but equally realistic versions of the DA problem (sets of data, scenarios, ...) are generally necessary to elaborate a recommendation on the basis of robust conclusions stemming from the multiple results thus obtained. The statement of the proposals which make up the recommendation should be submitted to the assessment and discernment of the decision maker and/or the actors involved in the DA process (see [36, 48, 65–68, 84]).

2.5 Conclusion

The final objective of MCDA is, of course, to help managers to make "better" decisions. But what is the meaning of better? This meaning depends, in part, on the process by which the decision is made and implemented. This, combined with limitations on objectivity described above (see Sect. 2.1.3), shows that we cannot hope to prove scientifically, in a decision making context, that a given decision is the best one. So, one cannot consider that, in every situation, the right selection, the right assignment, the right ranking—all of which decision aiding strives to discover, or to approach as closely as possible—exists somewhere. This implies that the concepts, models and procedures presented in this book must not be viewed as being conceived from the perspective of discovering, with a better or a worst good approximation, a pre-existing truth which could be universally accepted. They have to be seen as keys capable of opening doors giving access to answers and/or expectations as described in Sect. 2.1.1.

Designed and implemented in this manner, decision aiding based on appropriate concepts, models and procedures can play a significant and beneficial role in guiding a decision making process.

Above all, the purpose of MCDA is to enable us to enhance the degree of conformity and consistency between the evolving decision-making process and the value system and the objectives of the people involved in the process. For this purpose, concepts, tools and procedures must be designed to help us make our way on a road fraught with ambiguity, uncertainty and countless crossroads.

To achieve this goal, three non exclusive paths can be envisaged:

- the path of realism which leads to the quest for a description for discovering;
- the axiomatic path which is often associated with the quest for norms for prescribing;
- the path of constructivism which goes hand in hand with the quest for a working hypothesis for recommending.

(for more details on each of these paths, see [60]). In a DA process, it is important, when following one or a combination of such paths, to shed light on:

- those aspects of reality which give meaning, value and order to facts;
- the influence exerted upon this reality by observing it, organizing it, provoking within it certain forms of debate, or even having certain tools placed there.

Personally, I consider that the path of realism can only play a role in producing certain descriptions of physical, institutional, socio-economic, financial or psychological systems which form the decision making context. Insofar as such descriptions are produced by other disciplines than DA strictly speaking, the contribution of DA comes essentially, in my opinion, from the constructivism path taken in conjunction with (observing certain precautions) the axiomatic path. Interesting developments and other points of view can be found in [11, 13, 31, 33, 38, 42, 51, 79, 80, 95, 97–99].

Acknowledgements I would like to express my most heartfelt thanks to Evelyn Perry and Dominique François for their invaluable help in putting together the final version of this chapter. Blurred nuances, obscure passages, possible misinterpretations—these are the risks we take when presenting a synthesis of rich and complex concepts, results and points of view in a language that is not our own. Such risks have been greatly reduced through a number of informative discussions with Evelyn Perry, discussions including a sentence-by-sentence examination of my first draft, typed from dictation by Dominique François, who with all her customary care and intelligence has followed the process resulting in the present version from beginning to end. The questions asked and suggestions given by Dominique François and Evelyn Perry have also proved extremely useful. Once again, I would like to thank them both for the quality of their collaboration.

References

1. Aissi, H., Roy, B.: Robustness in multi-criteria decision aiding. In: Ehrgott, M., Figueira, J.R., Greco, S. (eds.) Trends in Multiple Criteria Decision Analysis, pp. 101–139. Springer Science + Business Media Inc., New York (2010)
2. Aït Younes, A.: Problèmes liés à la construction d'un pseudo-critère: développements théoriques et implémentation informatique. Ph.D. thesis, Université Paris-Dauphine (2001)
3. Aït Younes, A., Roy, B.: Prise en compte d'une connaissance imparfaite á l'aide d'un pseudocrit 'ere: procédure interactive de construction. J. Decis. Syst. **13**(1), 113–142 (2004)
4. Armstrong, J.: Advocacy and objectivity in science. Manag. Sci. **25**(5), 423–428 (1979)
5. Azibi, R.: Construction des critères en aide à la décision: aspects méthodologiques, techniques et pratiques. Ph.D. thesis, Université Paris-Dauphine (2003)
6. Azibi, R., Vanderpooten, D.: Elaboration de critères agrégeant des conséquences dispersées: deux exemples concrets. In: Colorni, A., Paruccini, M., Roy, R. (eds.) A-MCD-A: Multiple Criteria Decision Aiding, pp. 13–30. Joint Research Center, The European Commission, Luxembourg (2001)
7. Azibi, R., Vanderpooten, D.: Construction of rule-based assignment models. Eur. J. Oper. Res. **138**(2), 274–293 (2002)
8. Azibi, R., Vanderpooten, D.: Aggregation of dispersed consequences for constructing criteria: the evaluation of flood risk reduction strategies. Eur. J. Oper. Res. **44**, 397–411 (2003)
9. Bana e Costa, C. (ed.): Readings in Multiple Criteria Decision Aid. Springer, Heidelberg (1990)
10. Bana e Costa, C.A.: Structuration, construction et exploitation d'un modèle multicritère d'aide à la décision. Ph.D. thesis, Instituto Superior Técnico, Lisboa (1992)
11. Bana e Costa, C., Pirlot, M.: Thoughts on the future of multicriteria field: basic convictions and outline for general methodology. In: Clímaco, J. (ed.) Multicriteria Analysis, pp. 562–568. Springer, Berlin (1997)
12. Bana e Costa, C., Stewart, T., Vansnick, J.C.: Multicriteria decision analysis: some thoughts based on the tutorial and discussion sessions of ESIGMA meetings. Eur. J. Oper. Res. **99**, 28–37 (1997)
13. Banville, M., Landry, M., Martel, J.-M., Boulaire, C.: A stakeholder approach to MCDA. Syst. Res. Behav. Sci. **15**, 15–32 (1998)
14. Belton, V., Pictet, J.: Talking about the practice of MCDA. In: Bouyssou, D., Jacquet-Lagrèze, E., Perny, P., Slowinski, R., Vanderpooten, D., Vincke, P. (eds.) Aiding Decisions with Multiple Criteria: Essays in Honour of Bernard Roy, pp. 71–88. Kluwer Academic Publishers, Boston (2001)
15. Bertier, P., de Montgolfier, J.: Comment choisir en tenant compte de points de vue non commensurables? Anal. Prév. **XI**, 521–548 (1971)
16. Bouyssou, D.: Some remarks on the notion of compensation in MCDM. Eur. J. Oper. Res. **26**, 150–160 (1986)
17. Bouyssou, D.: Modelling inaccurate determination, uncertainty, imprecision using multiple criteria. In: Lockett, A., Islei, G. (eds.) Improving Decision Making in Organisations. Lecture Notes in Economics and Mathematical Systems, vol. 335, pp. 78–87. Springer, Berlin (1989)
18. Chipperfield, A., Whidborne, J., Fleming, P.: Evolutionary algorithms and simulated annealing for MCDM. In: Gal, T., Stewart, T., Hanne, T. (eds.) Multicriteria Decision Making: Advances in MCDM Models, Algorithms, Theory and Applications, pp. 16.1–16.32. Kluwer Academic Publishers, Boston (1999)
19. Czyzak, P., Slowinski, R.: Possibilistic construction of fuzzy outranking relation for multiple-criteria ranking. Fuzzy Sets Syst. **81**(1), 123–131 (1996)
20. Damart, S.: A note on the concept of knowledge management in decision aid activity. In: DSI-Age 2002 (Decision Making and Decision Support in the Internet Age 2002), Cork, Ireland, 4–7 July 2002

21. David, A.: Decision-aid between tools and organisations. In: Bouyssou, D., Jacquet-Lagrèze, E., Perny, P., Slowinski, R., Vanderpooten, D., Vincke, P. (eds.) Aiding Decisions with Multiple Criteria: Essays in Honour of Bernard Roy, pp. 45–69. Kluwer Academic Publishers, Boston (2001)
22. de Montgolfier, J., Bertier, P.: Approche Multicritère des Problèmes de Décision. Editions Hommes et Techniques, Paris (1978)
23. Dubois, D.: Modèles mathématiques de l'imprécis et de l'incertain en vue d'applications aux techniques d'aide à la décision. Mémoire d'habilitation. Université Paul Sabatier, Toulous (1983)
24. Dubois, D., Prade, H.: Théorie des Possibilités: application à la Représentation des Connaissances en Informatique. Masson, Paris (1985)
25. Dubois, D., Marichal, J., Prade, H., Roubens, M.: The use of the discrete Sugeno integral in decision-making: a survey. In: Colorni, A., Paruccini, M., Roy, B. (eds.) A-MCD-A: Multiple Criteria Decision Aiding, pp. 81–98. Joint Research Center, The European Commission, Luxembourg (2001)
26. Ehrgott, M., Gandibleux, X. (eds.): Multiple Criteria Optimization. State of the Art Annotated Bibliographic Surveys. Kluwer Academic Publishers, Dordrecht (2002)
27. Escudero, L.: Robust decision making as a decision making aid under uncertainty. In: Rios, S. (ed.) Decision Theory and Decision Analysis: Trends and Challenges, pp. 127–138. Kluwer Academic Publishers, Dordrecht (1994)
28. Fargier, H., Perny, P.: Modélisation des préférences par une règle de concordance généralisée. In: Colorni, A., Paruccini, M., Roy, B. (eds.) A-MCD-A: Multiple Criteria Decision Aiding, pp. 99–116. Joint Research Center, The European Commission, Luxembourg (2001)
29. Fodor, J., Roubens, M.: Fuzzy Preference Modeling and Multicriteria Decision Support. Kluwer Academic Publishers, Dordrecht (1994)
30. Gardiner, L., Vanderpooten, D.: Interactive multiple criteria procedures: some reflections. In: Clímaco, J. (ed.) Multicriteria Analysis, pp. 290–301. Springer, Berlin (1997)
31. Genard, J., Pirlot, M.: Multi-criteria decision-aid in a philosophical perspective. In: Bouyssou, D., Jacquet-Lagrèze, E., Perny, P., Slowinski, R., Vanderpooten, D., Vincke, P. (eds.) Aiding Decisions with Multiple Criteria: Essays in Honour of Bernard Roy, pp. 89–117. Kluwer Academic Publishers, Dordrecht (2001)
32. Greco, S., Matarazzo, B., Slowinski, R.: The use of rough sets and fuzzy sets in MCDM. In: Gal, T., Stewart, T., Hanne, T. (eds.) Multicriteria Decision Making: Advances in MCDM Models, Algorithms, Theory, and Applications, pp. 14.1–14.59. Kluwer Academic Publishers, Dordrecht (1999)
33. Greco, S., Słowiński, R., Figueira, J.R., Mousseau, V.: Robust ordinal regression. In: Ehrgott, M., Figueira, J.R., Greco, S. (eds.) Trends in Multiple Criteria Decision Analysis, pp. 273–320. Springer Science + Business Media Inc., New York (2010)
34. Groupe de Reflexion sur l'Economie des Transports Urbains (GRETU): Une étude Économique a Montré. . . Mythes et Réalités sur les études de Transports. Editions Cujas, Paris (1980)
35. Gupta, S., Rosenhead, J.: Robustness in sequential investment decisions. Manag. Sci. **15**(2), 18–29 (1972)
36. Hatchuel, A.: Coopération et conception collective: variété et crises des rapports de prescription. In: De Terssac, G., Friedberg, E. (eds.) Coopération et Conception, pp. 101–121. Octares Editions, Toulouse (1996)
37. Jacquet-Lagrèze, E.: Modélisation des préférences: préordres, quasi-ordres et relations floues. Thèse de doctorat de 3e Cycle, Université Paris V René Decartes (1975)
38. Keeney, R.: Creativity in decision-making with value-focused thinking. Sloan Manage. Rev. **35**(4), 33–41 (1994)
39. Keeney, R., Raiffa, H.: Decisions with Multiple Objectives: Preferences and Value Tradeoffs. Wiley, New York (1976)
40. Kouvelis, P., Yu, G.: Robust Discrete Optimization and Its Applications. Kluwer Academic Publishers, Dordrecht (1997)

41. Kruse, R., Schwecke, E., Heinsohn, J. (eds.): Uncertainty and Vagueness in Knowledge Based Systems. Springer, Berlin (1991)
42. Landry, M.: A note on the concept of problem. Organ. Stud. **16**, 315–343 (1995)
43. Martel, J., Roy, B.: Analyse de la Signifiance de Diverses Procédures d'agrégation Multicritère. Annales du LAMSADE 1. Université Paris-Dauphine, Paris (2002)
44. Mousseau, V.: Problèmes liés à l'évaluation de l'importance relative des critères en aide multi-critère à la décision: réflexions théoriques, expérimentations et implémentations informatiques. Ph.D. thesis, Université Paris-Dauphine (1993)
45. Perny, P., Pomerol, J.C.: Use of artificial intelligence in MCDM. In: Gal, T., Stewart, T., Hanne, T. (eds.) Multicriteria Decision Making: Advances in MCDM Models, Algorithms, Theory, and Applications, pp. 15:1–15:43. Kluwer Academic Publishers, Dordrecht (1999)
46. Perny, P., Roy, B.: The use of fuzzy outranking relations in preference modelling. Fuzzy Sets Syst. **49**, 33–53 (1992)
47. Perny, P., Vanderpooten, D.: An interactive multiobjective procedure for selecting medium-term countermeasures after nuclear accidents. J. Multi-Criteria Decis. Anal. **7**(1), 48–60 (1998)
48. Pomerol, J.-C.: La décision humaine: reconnaissance plus raisonnement. In: Bouyssou, D., Dubois, D., Pirlot, M., Prade, H. (eds.) Concepts et Méthodes pour l'Aide à la Décision, Traité IC2, Information, Commande, Communication. Hermès, Paris (2006)
49. Pomerol, J., Barba-Romero, S.: Multicriterion Decision Making in Management. Kluwer Academic Publishers, Dordrecht (2000)
50. Rauschmayer, F.: How to consider ethics in MCDA? In: Colorni, A., Paruccini, M., Roy, B. (eds.) A-MCD-A: Multiple Criteria Decision Aiding, pp. 273–280. Joint Research Center, The European Commission, Luxembourg (2001)
51. Rosenhead, J.: Rational Analysis of a Problematic World. Wiley, New York (1989). Second edition revised in (2001)
52. Rosenhead, J.: Robustness analysis: keeping your options open. In: Rosenhead, J., Mingers, J. (eds.) Rational Analysis for a Problematic World Revisited: Problem Structuring Methods for Complexity, Uncertainty and Conflict, pp. 181–207. Wiley, Chichester (2001)
53. Rosenhead, J.: Robustness to the first degree. In: Rosenhead, J., Mingers, J. (eds.) Rational Analysis for a Problematic World Revisited: Problem Structuring Methods for Complexity, Uncertainty and Conflict, pp. 209–223. Wiley, Chichester (2001)
54. Roy, B.: Vers une méthodologie générale d'aide à la décision. Revue METRA **14**(3), 459–497 (1975)
55. Roy, B.: Partial preference analysis and decision-aid: the fuzzy outranking relation concept. In: Bell, D., Keeney, R., Raiffa, H. (eds.) Conflicting Objectives in Decisions, pp. 40–75. Wiley, New York (1977)
56. Roy, B.: Quelques remarques sur le concept d'indépendance dans l'aide à la décision multicritère. Found. Control Eng. **8**(3–4), 183–191 (1983)
57. Roy, B.: Méthodologie Multicritère d'aide à la Décision. Economica, Paris (1985)
58. Roy, B.: Des critères multiples en recherche opérationnelle: pourquoi? In: Rand, G. (ed.) Operational Research '87, pp. 829–842. North-Holland, Amsterdam (1988)
59. Roy, B.: Main sources of inaccurate determination, uncertainty and imprecision. Math. Comput. Model. **12**(10–11), 1245–1254 (1989)
60. Roy, B.: Decision science or decision aid science? Eur. J. Oper. Res. **66**, 184–203 (1993)
61. Roy, B.: On operational research and decision aid. Eur. J. Oper. Res. **73**, 23–26 (1994)
62. Roy, B.: Les Logiques Compensatoires et les Autres. Note de Recherche du LAMSADE 16. Université Paris-Dauphine, Paris (1996)
63. Roy, B.: Multicriteria Methodology for Decision Aiding. Nonconvex Optimization and its Applications, vol. 12. Kluwer Academic Publishers, Dordrecht (1996)
64. Roy, B.: Une Lacune en RO-AD: Les Conclusions Robustes. Cahier du LAMSADE 144. Université Paris-Dauphine, Paris (1997)
65. Roy, B.: A missing link in OR-DA, robustness analysis. Found. Comput. Decis. Sci. **23**(3), 141–160 (1998)

66. Roy, B.: Decision-aiding today: what should we expect? In: Gal, T., Stewart, T., Hanne, T. (eds.) Multicriteria Decision Making: Advances in MCDM Models, Algorithms, Theory, and Applications, pp. 1.1–1.35. Kluwer Academic Publishers, Boston (1999)
67. Roy, B.: A propos de robustesse en recherche opérationnelle et aide à la décision. In: Billaut, J.C., Moukrim, A., Sanlaville, E. (eds.) Flexibilité et Robustesse en Ordonnancement, pp. 35–50. Hermès, Paris (2005) http://basepub.dauphine.fr/handle/123456789/1211
68. Roy, B.: Robustesse de quoi et vis-à-vis de quoi mais aussi robustesse pourquoi en aide à la décision? In: Henggeler-Antunes, C., Figueira, J., Clímaco, J. (eds.) Proceedings of 56th Meeting of the European Working Group Multiple Criteria Decision Aiding, Coimbra, Portugal, 3–5 October 2002. CCRC, Coimbra (2004)
69. Roy, B.: A propos de la signification des dépendances entre critères: quelle place et quels modes de prise en compte en aide à la décision? RAIRO Oper. Res. **43**(3), 255–275 (2009)
70. Roy, B.: Robustness in operational research and decision aiding: a multi-faceted issue. Eur. J. Oper. Res. **200**(3), 629–638 (2010)
71. Roy, B.: To better respond to the robustness concern in decision aiding: four proposals based on a twofold observation. In: Zopounidis, C., Pardalos, P. (eds.) Handbook of Multicriteria Analysis, pp. 3–24. Springer Science + Business Media Inc., New York (2010)
72. Roy, B.: Two conceptions of decision aiding. Int. J. Multicriteria Decis. Mak. **1**, 74–79 (2010)
73. Roy, B., Bouyssou, D.: Aide Multicritère à la Décision: Méthodes et Cas. Economica, Paris (1993)
74. Roy, B., Damart, S.: L'analyse coûts-avantages, outil de concertation et de légitimation? METROPOLIS **108–109**, 7–16 (2002)
75. Roy, B., Mousseau, V.: A theoretical framework for analysing the notion of relative importance of criteria. J. Multi-Criteria Decis. Anal. **5**, 145–159 (1996)
76. Roy, B., Vincke, P.: Systèmes relationnels de préférences en présence de critères multiples avec seuils. Cahiers du CERO **22**(1), 23–38 (1980)
77. Roy, B., Vincke, P.: Relational systems of preference with one or more pseudo-criteria: some new concepts and results. Manag. Sci. **30**(11), 1323–1335 (1984)
78. Roy, B., Vincke, P.: Pseudo-orders: definition, properties and numerical representation. Math. Soc. Sci. **14**(3), 263–274 (1987)
79. Roy, B., Vincke, P.: The case of the vanishing optimum revisited again. J. Multi-Criteria Decis. Anal. **7**, 351 (1998)
80. Schärlig, A.: The case of the vanishing optimum. J. Multi-Criteria Decis. Anal. **5**, 160–164 (1996)
81. Slowinski, R. (ed.): Intelligent Decision Support: Handbook of Applications and Advances of the Rough Sets Theory. Kluwer Academic Publishers, Dordrecht (1992)
82. Slowinski, R., Teghem, J. (eds.): Stochastic Versus Fuzzy Approaches to Multiobjective Mathematical Programming under Uncertainty. Kluwer Academic Publishers, Dordrecht (1990)
83. Stewart, T.: Concepts of interactive programming. In: Gal, T., Stewart, T., Hanne, T. (eds.) Multicriteria Decision Making: Advances in MCDM Models, Algorithms, Theory, and Applications, pp. 10.1–10.28. Kluwer Academic Publishers, Boston (1999)
84. Tanino, T.: Sensitivity analysis in multicriteria decision making. In: Gal, T., Stewart, T., Hanne, T. (eds.) Multicriteria Decision Making: Advances in MCDM Models, Algorithms, Theory, and Applications, pp. 7.1–7.29. Kluwer Academic Publishers, Boston (1999)
85. Tsoukiàs, A.: Sur la généralisation des concepts de concordance et discordance en aide multicritr̀e à la décision. Mémoire d'habilitation. Université Paris Dauphine, Paris (1997). Appeared also as Document du LAMSADE 117
86. Tsoukiàs, A., Vincke, P.: A new axiomatic foundation of partial comparability. Theor. Decis. **39**, 79–114 (1995)
87. Vanderpooten, D.: The interactive approach in MCDA: a technical framework and some basic conceptions. Math. Comput. Model. **12**, 1213–1220 (1989)
88. Vanderpooten, D.: The use of preference information in multiple criteria interactive procedures. In: Lockett, A.G., Islei, G. (eds.) Improving Decision Making in Organisations, pp. 390–399. Springer, Berlin (1989)

89. Vanderpooten, D.: Modelling in decision aiding. In: Bouyssou, D., Jacquet-Lagrèze, E., Perny, P., Slowinski, R., Vanderpooten, D., Vincke, P. (eds.) Aiding Decisions with Multiple Criteria: Essays in Honour of Bernard Roy, pp. 195–210. Kluwer Academic Publishers, Dordrecht (2001)
90. Vincke, P.: Multicriteria Decision-Aid. Wiley, Chichester (1992)
91. Vincke, P.: Robust and neutral methods for aggregating preferences into an outranking relation. Eur. J. Oper. Res. **112**(2), 405–412 (1999)
92. Vincke, P.: Robust solutions and methods in decision-aid. J. Multi-Criteria Decis. Anal. **8**(3), 181–187 (1999)
93. Vincke, P.: Preferences and numbers. In: Colorni, A., Paruccini, M., Roy, B. (eds.) A-MCD-A: Multiple Criteria Decision Aiding, pp. 343–354. Joint Research Center, The European Commission, Luxembourg (2001)
94. Wierzbicki, A.P.: The use of reference objectives in multiobjective optimisation. In: Fandel, G., Gal, T. (eds.) MCDM Theory and Application. Lecture Notes in Economics and Mathematical Systems, vol. 177, pp. 468–486. Springer, Berlin (1980)
95. Wierzbicki, A.: On the role of intuition in decision making and some ways of multicriteria aid of intuition. J. Multi-Criteria Decis. Anal. **6**(2), 65–76 (1997)
96. Wierzbicki, A.: Reference point approaches. In: Gal, T., Stewart, T., Hanne, T. (eds.) Multicriteria Decision Making: Advances in MCDM Models, Algorithms, Theory, and Applications, pp. 9.1–9.39. Kluwer Academic Publishers, Boston (1999)
97. Zeleny, M.: Multiple Criteria Decision Making. McGraw Hill, New York (1982)
98. Zeleny, M. (ed.): MCDM: Past Decade and Future Trends, A Source Book of Multiple Criteria Decision Making. JAI Press, London (1984)
99. Zionts, S.: The case of the vanishing optimum revisited. J. Multi-Criteria Decis. Anal. **6**, 247 (1997)

Part II
Foundations of MCDA

Chapter 3
Preference Modelling

Stefano Moretti, Meltem Öztürk, and Alexis Tsoukiàs

Abstract This chapter provides the reader with a presentation of preference modelling fundamental notions as well as some recent results in this field. Preference modelling is an inevitable step in a variety of fields: economy, sociology, psychology, mathematical programming, even medicine, archaeology, and obviously decision analysis. Our notation and some basic definitions, such as those of binary relation, properties and ordered sets, are presented at the beginning of the chapter. We start by discussing different reasons for constructing a preference model. We then go through a number of issues that influence the construction of preference models. Different formalisations besides classical logic such as fuzzy sets and non-classical logics become necessary. We then present different types of preference structures reflecting the behavior of a decision-maker: classical, extended and valued ones. It is relevant to have a numerical representation of preferences: functional representations, value functions. The concepts of thresholds and minimal representation are also introduced in this section. We also deal with the problem of how to extend a preference relation over a set A of "objects" to the set of all subsets of A. In Sect. 3.9, we briefly explore the concept of deontic logic (logic of preference) and other formalisms associated with "compact representation of preferences" introduced for special purposes. We end the chapter with some concluding remarks.

Keywords Preference modelling • Decision aiding • Uncertainty • Fuzzy sets • Ordered relations • Binary relations • Preference extensions

---

S. Moretti (✉) • M. Öztürk • A. Tsoukiàs
LAMSADE-CNRS, Université Paris Dauphine, 75775 Paris Cedex 16, France
e-mail: moretti@lamsade.dauphine.fr; ozturk@lamsade.dauphine.fr; tsoukias@lamsade.dauphine.fr

S. Greco et al. (eds.), *Multiple Criteria Decision Analysis*, International Series in Operations Research & Management Science 233,
DOI 10.1007/978-1-4939-3094-4_3

3.1 Introduction

The purpose of this chapter is to present fundamental notions of preference modelling as well as some recent results in this field. Basic references on this issue can be considered: [5, 109, 112, 117, 157, 168, 229, 235, 239, 260, 265].

The chapter is organised as follows: The purpose for which formal models of preference and more generally of objects comparison are studied, is introduced in Sect. 3.2. In Sect. 3.3, we analyse the information used when such models are established and introduce different sources and types of uncertainty. Our notation and some basic definitions, such as those of binary relation, properties and ordered sets, are presented in Sect. 3.4. Besides classical logic, different formalisms can be used in order to establish a preference model, such as fuzzy sets and non-classical logics. These are discussed in Sect. 3.5. In Sect. 3.6, we then present different types of preference structures reflecting the behavior of a decision-maker: classical, extended and valued ones. It appears relevant to have a numerical representation of preferences: functional representations, value functions and intervals. These are discussed in Sect. 3.7. The concepts of thresholds and minimal representation are also introduced in this section. In Sect. 3.8, we present different approaches to the analysis of the problem of how to extend a preference relation over a set A of "objects" to the set of all subsets of A. Finally, after briefly exploring the concept of deontic logic (logic of preference) and other related issued in Sect. 3.9, we end the chapter with some concluding remarks.

3.2 Purpose

Preference modelling is an inevitable step in a variety of fields. Scientists build models in order to better understand and to better represent a given situation; such models may also be used for more or less operational purposes (see [51]). It is often the case that it is necessary to compare objects in such models, basically in order to either establish if there is an order between the objects or to establish whether such objects are "near". Objects can be everything, from candidates to time intervals, from computer codes to medical patterns, from prospects (lotteries) to production systems. This is the reason why preference modelling is used in a great variety of fields such as economy [12–14, 80], sociology, psychology [60, 68, 71, 158, 159], political science [22, 254], artificial intelligence [97], computer science [117, 252, 265], temporal logic (see [6]) and the interval satisfiability problem [131, 217] mathematical programming [224, 225], electronic business, medicine and biology [32, 62, 154, 162, 195], archaeology [148], and obviously decision analysis.

In this chapter, we are going to focus on preference modelling for decision aiding purposes, although the results have a much wider validity.

Throughout this chapter, we consider the case of somebody (possibly a decision-maker) who tries to compare objects taking into account different points of view.

We denote the set of alternatives A,[1] to be labelled $a, b, c, \ldots$ and the set of points of view J, labelled $j = 1, 2, \ldots, m$. In this framework, a data $g_j(a)$ corresponds to the evaluation of the alternative a from the point of view $j \in J$.

As already mentioned, comparing two objects can be seen as looking for one of the two following possible situations:

– object a is "before" object b, where "before" implies some kind of order between a and b, such an order referring either to a direct preference (a is preferred to b) or being induced from a measurement and its associated scale (a occurs before b, a is longer, bigger, more reliable, than b);
– object a is "near" object b, where "near" can be considered either as indifference (object a or object b will do equally well for some purpose), or as a similarity, or again could be induced by a measurement (a occurs simultaneously with b, they have the same length, weight, reliability).

The two above-mentioned "attitudes" (see [198]) are not exclusive. They just stand to show what type of problems we focus on. From a decision aiding point of view we traditionally focus on the first situation. Ordering relations is the natural basis for solving ranking or choice problems. The second situation is traditionally associated with problems where the aim is to be able to put together objects sharing a common feature in order to form "homogeneous" classes or categories (a classification problem).

The first case we focus on is the ordering relation: given the set A, establishing how each element of A compares to each other element of A from a "preference" point of view enables to obtain an order which might be used to make either a choice on the set A (identify the best) or to rank the set A. Of course, we have to consider whether it is possible to establish such an ordering relation and of what type (certain, uncertain, strong, weak etc.) for all pairs of elements of A. We also have to establish what "not preference" represents (indifference, incomparability etc.). In the following sections (namely in Sect. 3.6), we are going to see that different options are available, leading to different, so called, preference structures.

In the second case we focus on the "nearness" relation since the issue here is to put together objects which ultimately are expected to be "near" (whatever the concept of "near" might represent). In such a case, there is also the problem how to consider objects which are "not near". Typical situations in this case include the problems of grouping, discriminating and assigning [143]. A further distinction in such problems concerns the fact that the categories within which the objects might be associated could already exist or not and the fact that such categories might be ordered or not. Putting objects into non pre-existing non ordered categories is the typical classification problem, conversely, assigning objects to pre-existing ordered categories is known as the "sorting" problem [216, 221, 303].

It should be noted that although preference relations have been naturally associated to ranking and choice problem statements, such a separation can be argued.

[1] We can use the word *action* instead of alternative.

For instance, there are sorting procedures (which can be seen as classification problems) that use preference relations instead of "nearness" ones [181, 193, 297]. The reason is the following: in order to establish that two objects belong to the same category we usually either try to check whether the two objects are "near" or whether they are near a "typical" object of the category (see for instance [221]). If, however, a category is described, not through its typical objects, but through its boundaries, then, in order to establish if an object belongs to such a category it might make sense to check whether such an object performs "better" than the "minimum", or "least" boundary of the category and that will introduce the use of a preference relation.

In [198] it is claimed that decision aiding should not exclusively focus on preference relations, but also on "nearness relations", since quite often the problem statement to work with in a problem formulation is that of classification (on the existence of different problem statements and their meaning the reader is referred to [83, 242, 245, 284]).

3.3 Nature of Information

As already mentioned, the purpose of our analysis is to present the literature associated with objects comparison for either a preference or a nearness relation. Nevertheless, such an operation is not always as intuitive as it might appear. Building up a model from reality is always an abstraction (see [45]). This can always be affected by the presence of uncertainty due to our imperfect knowledge of the world, our limited capability of observation and/or discrimination, the inevitable errors occurring in any human activity etc. [243]. We call such an uncertainty exogenous. Besides, such an activity might generate uncertainty since it creates an approximation of reality, thus concealing some features of reality. We call this an endogenous uncertainty (see [267]).

As pointed out in [285], preference modelling can be seen as either the result of direct comparison (asking a decision-maker to compare two objects and to establish the relation between them) from which it might be possible to infer a numerical representation, or as the result of the induction of a preference relation from the knowledge of some "measures" associated to the compared objects.

In the first case, uncertainty can arise from the fact that the decision-maker might not be able to clearly state a preference relation for any pair of actions. We do not care why this may happen, we just consider the fact that the decision-maker may reply when asked if "x is preferred to y": yes, no, I do not know, yes and no, I am not sure, it might be, it is more preference than indifference, but … etc. The problem in such cases is how to take such replies into account when defining a model of preferences.

In the second case, we may have different situations such as: incomplete information (missing values for some objects), uncertain information (the value of an object lies within an interval to which an uncertainty distribution might be associated, but the precise value is unknown), ambiguous information (contradictory statements about the present state of an object). The problem here is how to establish a preference model on the basis of such information and to what extent the uncertainty associated with the original information will be propagated to the model and how.

Such uncertainties can be handled through the use of various formalisms (see Sect. 3.5 of this chapter). Two basic approaches can be distinguished (see also [103]).

1. Handling uncertain information and statements. In such a case, we consider that the concepts used in order to model preferences are well-known and that we could possibly be able to establish a preference relation without any uncertainty, but we consider this difficult to do in the present situation with the available information. A typical example is the following: we know that x is preferred to y if the price of x is lower than the price of y, but we know very little about the prices of x and y. In such cases we might use an uncertainty distribution (classical probability, ill-known probabilities, possibility distributions, see [69, 102, 109, 153]) in order to associate a numerical uncertainty with each statement.
2. Handling ambiguous concepts and linguistic variables. With such a perspective we consider that sentences such as "x is preferred to y" are ill-defined, since the concept of preference itself is ill-defined, independently from the available information. A typical example is a sentence of the type: "the largest the difference of price between x and y is, the strongest the preference is". Here we might know the prices of x and y perfectly, but the concept of preference is defined through a continuous valuation. In such cases, we might use a multi-valued logic such that any preferential sentence obtains a truth value representing the "intensity of truth" of such a sentence. This should not be confused with the concept of "preference intensity", since such a concept is based on the idea of "measuring" preferences (as we do with temperature or with weight) and there is no "truth" dimension (see [165, 168, 234, 235]). On the other hand such a subtle theoretical distinction can be transparent in most practical cases since often happens that similar techniques are used under different approaches.

3.4 Notation and Basic Definitions

The notion of binary relation appears for the first time in De Morgan's study [77] and is defined as a set of ordered pairs in Peirce's works [218–220]. Some of the first work dedicated to the study of preference relations can be found in [104, 253] (more in general the concept of models of arbitrary relations will be introduced in [261, 262]). Throughout this chapter, we adopt Roubens' and Vincke's notation [239].

Definition 4.1 (Binary Relation). Let A be a finite set of elements $(a, b, c, \dots, n)$, a binary relation R on the set A is a subset of the cartesian product $A \times A$, that is, a set of ordered pairs (a, b) such that a and b are in $A : R \subseteq A \times A$.

For an ordered pair (a, b) which belongs to R, we indifferently use the notations:

$$(a, b) \in R \text{ or } aRb \text{ or } R(a, b)$$

Let R and T be two binary relations on the same set A. Some set operations are:

The Inclusion :	$R \subseteq T$ if $aRb \longrightarrow aTb$
The Union :	$a(R \cup T)b$ iff aRb or(inclusive) aTb
The Intersection :	$a(R \cap T)b$ iff aRb and aTb
The Relative product :	$a(R.T)b$ iff $\exists c \in A$: aRc and cTb
	$(aR^2b$ iff $aR.Rb)$

When such concepts apply we respectively denote (R^a), (R^s), $(\hat{R})$ the asymmetric, the symmetric and the complementary part of binary relation R:

$$aR^ab \text{ iff } aRb \text{ and } not(bRa)$$

$$aR^sb \text{ iff } aRb \text{ and } bRa$$

$$a\hat{R}b \text{ iff } not(aRb) \text{ and } not(bRa)$$

The complement (R^c), the converse (the dual)$(\overline{R})$ and the co-dual (R^{cd}) of R are respectively defined as follows:

$$aR^cb \text{ iff } not(aRb)$$

$$a\overline{R}b \text{ iff } bRa$$

$$aR^{cd}b \text{ iff } not(bRa)$$

The relation R is called

reflexive,	if	$aRa, \quad \forall a \in A$
irreflexive,	if	$aR^ca, \quad \forall a \in A$
symmetric,	if	$aRb \longrightarrow bRa, \quad \forall a, b \in A$
antisymmetric,	if	$(aRb, bRa) \longrightarrow a = b, \quad \forall a, b \in A$
asymmetric,	if	$aRb \longrightarrow bR^ca, \quad \forall a, b \in A$
complete,	if	$(aRb \text{ or } bRa), \quad \forall a \neq b \in A$
strongly complete,	if	$aRb \text{ or } bRa, \quad \forall a, b \in A$
transitive,	if	$(aRb, bRc) \longrightarrow aRc, \quad \forall a, b, c \in A$
negatively transitive,	if	$(aR^cb, bR^cc) \longrightarrow aR^cc, \quad \forall a, b, c \in A$
negatively transitive,	if	$aRb \longrightarrow (aRc \text{ or } cRb), \quad \forall a, b, c \in A$
semitransitive,	if	$(aRb, bRc) \longrightarrow (aRd \text{ or } dRc), \quad \forall a, b, c, d \in A$
Ferrers relation,	if	$(aRb, cRd) \longrightarrow (aRd \text{ or } cRb), \quad \forall a, b, c, d \in A$

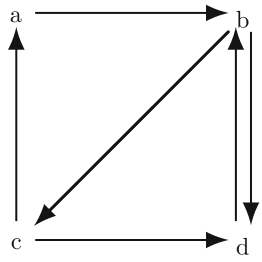


Fig. 3.1 Graphical representation of *R*

Fig. 3.2 Matrix representation of *R*

	a	b	c	d
a	0	1	0	0
b	0	0	1	1
c	1	0	0	1
d	0	1	0	0

The equivalence relation E associated with the relation R is a reflexive, symmetric and transitive relation, defined by:

$$aEb \text{ iff } \quad \forall a \in A \begin{cases} aRc \Longleftrightarrow bRc \\ cRa \Longleftrightarrow cRb \end{cases}$$

A binary relation R may be represented by a direct graph (A, R) where the nodes represent the elements of A, and the arcs, the relation R. Another way to represent a binary relation is to use a matrix M^R; the element M^R_{ab} of the matrix (the intersection of the line associated to a and the column associated to b) is 1 if aRb and 0 if $not(aRb)$.

Example 4.1. Let R be a binary relation defined on a set A, such that the set A and the relation R are defined as follows: $A = \{a, b, c, d\}$ and $R = \{(a, b), (b, d), (b, c), (c, a), (c, d), (d, b)\}$.

The graphical and matrix representation of R are given in Figs. 3.1 and 3.2.

3.5 Languages

Preference models are formal representations of comparisons of objects. As such they have to be established through the use of a formal and abstract language capturing both the structure of the world being described and the manipulations of it. It seems natural to consider formal logic as such a language. However, as already mentioned in the previous sections, the real world might be such that classical formal logic might appear too rigid to allow the definition of useful and expressive models. For this purpose, in this section, we introduce some further formalisms which extend the expressiveness of classical logic, while keeping most of its calculus properties.

3.5.1 Classic Logic

The interested reader can use two references: [183, 279] as introductory books to the use and the semantics of classical logic. All classic books mentioned in this chapter, implicitly or explicitly use classical logic, since binary relations are just sets and the calculus of sets is algebraically equivalent to truth calculus. Indeed the semantics of logical formulas as established by Tarski [261, 262], show the equivalence between membership of an element to a set and truth of the associate sentence.

Building a binary preference relation, a valuation of any proposition takes the values $\{0, 1\}$:

$$\mu(aRb) = 1 \text{ iff } aRb \text{ is true}$$

$$\mu(aRb) = 0 \text{ iff } aRb \text{ is false.}$$

The reader will note that all notations introduced in the previous section are based on the above concept. He/she should also note that when we write "a preference relation P is a subset of $A \times A$", we introduce a formal structure where the universe of discourse is $A \times A$ and P is the model of the sentence "x in relation P with y", that is, P is the set of all elements of $A \times A$ (ordered pairs of x and y) for which the sentence is true.

The above semantic can be in sharp contrast with decision analysis experience. For this purpose we will briefly introduce two more semantics: fuzzy sets and four-valued logic.

3.5.2 Fuzzy Sets

In this section, we provide a survey of basic notions of fuzzy set theory. We present definitions of connectives and several valued binary relation properties in order to be able to use this theory in the field of decision analysis. Basic references for this section include [102, 122, 258, 301].

Fuzzy sets were first introduced by Zadeh [298, 299]. The concept and the associated logics were further developed by other researchers: [99, 133, 163, 164, 185, 186, 196, 201].

Fuzzy measures can be introduced for two different uses: either they can represent a concept imprecisely known (although well defined) or a concept which is vaguely perceived such as in the case of a linguistic variable. In the first case they represent possible values, while in the second they are better understood as a continuous truth valuation (in the interval [0, 1]). To be more precise:

– in the first case we associate a possibility distribution (an ordinal distribution of uncertainty) to classical logic formulas;
– in the second case we have a multi-valued logic where the semantics allow values in the entire interval [0, 1].

A fuzzy set can be associated either with the set of alternatives considered in a decision aiding model (consider the case where objects are represented by fuzzy numbers) or with the preference relations. In decision analysis we may consider four possibilities[2]:

- Alternatives with crisp values and crisp preference relations
- Alternatives with crisp values and fuzzy preference relations
- Alternatives with fuzzy values and crisp preference relations (defuzzification, [178] with gravity center, [295] with means interval)
- Alternatives with fuzzy values and fuzzy preference relations (possibility graphs, [101]; four fuzzy dominance index, [240]); in this chapter we are going to focus on fuzzy preference relations

In the following we introduce the definitions required for the rest of the chapter.

Definition 5.1 (Fuzzy Set). A fuzzy set (or a fuzzy subset) F on a set Ω is defined by the result of an application:

$$\mu_F : \Omega \longrightarrow [0, 1]$$

where $\forall x \in \Omega$, $\mu(x)$ is the membership degree of x to F.

Definition 5.2 (Negation). A function $n : [0, 1] \longrightarrow [0, 1]$ is a negation if and only if it is non-increasing and:

[2]Lets take an example: Imagine that we have to choose one car between two. We have to know the performance of each car in order to establish the relation of preference:

- in the first case, the performance of each car is known and noted between 1 and 10 ($p(car1) = 8$ and $p(car2) = 5$); the relation of preference is known too (car1 is preferred to car2: $car1Pcar2$ ($\mu(car1Pcar2) = 1$))
- in the second case, the performance of each car is known and noted between 1 and 10 ($p(car1) = 8$ and $p(car2) = 7$); we are not sur about the preference relation that is why the relation of preference is fuzzy ($\mu(car1Pcar2) = 0.75$)
- in the third case, the performance of each car is fuzzy (in this case the performances of each car will be defined by fuzzy numbers; in this case we can use triangular or trapezoidal fuzzy number to represent the performance); the relation of preference is crisp (car1 is preferred to car2: $car1Pcar2$ ($\mu(car1Pcar2) = 1$))
- in the fourth case, the performance of each car is fuzzy (in this case the performances of each car will be defined by fuzzy numbers); the preference relation is also fuzzy (($\mu(car1Pcar2) = 0.75$))

$n(0) = 1$ and $n(1) = 0$

If the negation n is strictly decreasing and continuous then it is called *strict*.

In the following we investigate the two basic classes of operators, the operators for the intersection (triangular norms called t-norms) and the union (triangular conorms called t-conorms or s-norms) of fuzzy sets:

Definition 5.3 (t-norm). A function $T : [0,1]^2 \longrightarrow [0,1]$ is a triangular norm (t-norm), if and only if it satisfies the four conditions:

Equivalence Condition: $T(1,x) = x \ \forall x \in [0,1]$
T is commutative: $T(x,y) = T(y,x) \ \forall x,y \in [0,1]$
T is nondecreasing in both elements: $T(x,y) \leq T(u,v)$ for all $0 \leq x \leq u \leq 1$ and $0 \leq y \leq v \leq 1$
T is associative: $T(x,T(y,z)) = T(T(x,y),z) \ \forall x,y,z \in [0,1]$

The function T defines a general class of intersection operators for fuzzy sets.

Definition 5.4 (t-conorm). A function $S : [0,1]^2 \longrightarrow [0,1]$ is a (t-conorm), if and only if it satisfies the four conditions:

Equivalence Condition: $S(0,x) = x \ \forall x \in [0,1]$
S is commutative: $S(x,y) = S(y,x) \ \forall x,y \in [0,1]$
S is nondecreasing in both elements: $S(x,y) \leq S(u,v)$ for all $0 \leq x \leq u \leq 1$ and $0 \leq y \leq v \leq 1$
S is associative: $S(x,S(y,z)) = S(S(x,y),z) \ \forall x,y,z \in [0,1]$

T-norms and t-conorms are related by duality. For suitable negation operators[3] pairs of t-norms and t-conorms satisfy the generalisation of the De Morgan law:

Definition 5.5 (De Morgan Triplets). Suppose that T is a t-norm, S is a t-conorm and n is a strict negation. $\langle T,S,n\rangle$ is a De Morgan triple if and only if:

$$n(S(x,y)) = T(n(x),n(y))$$

Such a definition extends De Morgan's law to the case of fuzzy sets. There exist different proposed De Morgan triplets: [91, 100, 125, 251, 290, 294, 296].

The more frequent t-norms and t-conorms and negations are presented in Table 3.1.

We make use of De Morgan's triplet $\langle T,S,n\rangle$ in order to extend the definitions of the operators and properties introduced above in crisp cases. First, we give the definitions of the operators of implication I_T and equivalence E_T:

$$I_T(x,y) = sup\{z \in [0,1] : T(x,z) \leq y\}$$
$$E_T(x,y) = T(I_T(x,y), I_T(y,x)$$

[3] A suitable one can be the complement operator defined: $n(\mu(x)) = 1 - \mu(x)$.

Table 3.1 Principal t-norms, t-conorms and negations

Names	t-Norms	t-Conorms
Zadeh	$min(x,y)$	$max(x,y)$
Probabilistic	$x * y$	$x + y - xy$
Lukasiewicz	$max(x + y - 1, 0)$	$min(x + y, 1)$
Hamacher ($\gamma > 0$)	$(xy)/$ $(\gamma + (1-\gamma)(x + y - xy))$	$(x + y + xy - (1-\gamma)xy)/$ $(1 - (1-\gamma)xy)$
Yager ($p > 0$)	$max(1 - ((1-x)^p + (1-y)^p)^{1/p}, 0)$	$min((x^p + y^p)^{1/p}, 1)$
Weber ($\lambda > -1$)	$max((x + y - 1 + \lambda xy)/(1 + \lambda), 0))$	$min(x + y + \lambda xy, 1)$
Drastic	x if $y = 1$	x if $y = 0$
	y if $x = 1$	y if $x = 0$
	0 if not	1 if not

Since preference modelling makes use of binary relations, we extend the definitions of binary relation properties to the valued case. For the sake of simplicity $\mu(R(x,y))$ will be denoted $R(x,y)$: a valued binary relation $R(x,y)$ is ($\forall a, b, c, d \in A$)

reflexive,	if	$R(a,a) = 1$
irreflexive,	if	$R(a,a) = 0$
symmetric,	if	$R(a,b) = R(b,a)$
T-antisymmetric,	if	$a \neq b \longrightarrow T(R(a,b), R(b,a)) = 0$
T-asymmetric,	if	$T(R(a,b), R(b,a)) = 0$
S-complete,	if	$a \neq b \quad \longrightarrow \quad S(R(a,b), R(b,a)) = 1$
S-strongly complete,	if	$S(R(a,b), R(b,a)) = 1$
T-transitive,	if	$T(R(a,c), R(c,b)) \leq R(a,b)$
negatively S-transitive,	if	$R(a,b) \leq S(R(a,c), R(c,b))$
T-S-semitransitive,	if	$T(R(a,d), R(d,b)) \leq S(R(a,c), R(c,b))$
T-S-Ferrers relation,	if	$T(R(a,b), R(c,d)) \leq S(R(a,d), R(c,b))$

Different instances of De Morgan triplets will provide different definitions for each property.

The equivalence relation is one of the most-used relations in decision analysis and is defined in fuzzy set theory as follows:

Definition 5.6 (Equivalence Relation). A function $E : [0,1]^2 \longrightarrow [0,1]$ is an equivalence if and only if it satisfies:

$$E(x,y) = E(y,x) \ \forall \ x,y \in [0,1]$$
$$E(0,1) = E(1,0) = 0$$
$$E(x,x) = 1 \ \forall \ x \in [0,1]$$
$$x \leq x' \leq y' \leq y \Longrightarrow E(x,y) \leq E(x',y')$$

In Sect. 3.6.3 and Chap. 10, some results obtained by the use of fuzzy set theory are represented.

3.5.3 Four-Valued Logics

When we compare objects, it might be the case that it is not possible to establish precisely whether a certain relation holds or not. The problem is that such a hesitation can be due either to incomplete information (missing values, unknown replies, unwillingness to reply etc.) or to contradictory information (conflicting evaluation dimensions, conflicting reasons for and against the relation, inconsistent replies etc.). For instance, consider the query "is Anaxagoras intelligent?" If you know who Anaxagoras is you may reply "yes" (you came to know that he is a Greek philosopher) or "no" (you discover he is a dog). But if you know nothing you will reply "I do not know" due to your ignorance (on this particular issue). If on the other hand you came to know both that Anaxagoras is a philosopher and a dog you might again reply "I do not know", not due to ignorance, but to inconsistent information. Such different reasons for hesitation can be captured through four-valued logics allowing for different truth values for four above-mentioned cases. Such logics were first studied by Dubarle in 1963 (appeared in [98]) and introduced in the literature in [26, 27]. Further literature on such logics can be found in [9, 11, 33, 106, 119, 123, 160, 264, 268].

In the case of preference modelling, the use of such logics was first suggested in [85, 266]. Such logics extend the semantics of classical logic through two hypotheses:

- the complement of a first order formula does not necessarily coincide with its negation;
- truth values are only partially ordered (in a bilattice), thus allowing the definition of a boolean algebra on the set of truth values.

The result is that using such logics, it is possible to formally characterise different states of hesitation when preferences are modelled (see [270, 271]). Further more, using such a formalism, it becomes possible to generalise the concordance/discordance principle (used in several decision aiding methods) as shown in [274] and several characterisation problems can be solved (see for instance [272]). More recently (see [10, 81, 124, 210, 211, 226, 275]) it has been suggested to use the extension of such logics for continuous valuations.

3.6 Preference Structures

Definition 6.1 (Preference Structure). *A* preference structure is a collection of binary relations defined on the set *A* and such that:

– for each couple a, b in A; at least one relation is satisfied
– for each couple a, b in A; if one relation is satisfied, another one cannot be satisfied.

In other terms a preference structure defines a partition[4] of the set $A \times A$. In general it is recommended to have two other hypotheses with this definition (also denoted as fundamental relational system of preferences):

- Each preference relation in a preference structure is uniquely characterised by its properties (symmetry, transitivity, etc.).
- For each preference structure, there exists a unique relation from which the different relations composing the preference structure can be deduced. Any preference structure on the set A can thus be characterised by a unique binary relation R in the sense that the collection of the binary relations are defined through the combinations of the epistemic states of this characteristic relation.[5] For instance aPb if and only if aRb and not bRa.

3.6.1 $\langle P, I \rangle$ Structures

The most traditional preference model considers that the decision-maker confronted with a pair of distinct elements of a set A, either:

– clearly prefers one element to the other, or
– does not express a preference among them.

The subset of ordered pairs (a, b) belonging to $A \times A$ such that the statement "a is preferred to b" is true, is called *preference relation* and is denoted by P.

The subset of pairs (a, b) belonging to $A \times A$ such that the statement "a and b are not preferred" is true, is called (in this case) *indifference relation* and is denoted by I (I being considered the complement of $P \cup P^{-1}$ with respect to $A \times A$). We will see later on in Sect. 3.6.2.2 that this relation can be further decomposed in indifference and incomparability.

In the literature, there are two different ways of defining a specific preference structure:

– the first defines it by the properties of the binary relations of the relation set;
– the second uses the properties of the characteristic relation. In the rest of the section, we give definitions in both ways.

Definition 6.2 ($\langle P, I \rangle$ Structure). A $\langle P, I \rangle$ structure on the set A is a pair $\langle P, I \rangle$ of relations on A such that:

[4]To have a partition of the set $A \times A$, the inverse of the asymmetric relation must be considered too.

[5]While several authors prefer using both of them, there are others for which one is sufficient. For example Fishburn does not require the use of preference structures with a characteristic relation.

- P is asymmetric,
- I is reflexive, symmetric.

The characteristic relation R of a $\langle P, I\rangle$ structure can be defined as a combination of the relations P and I as:

$$aRb \text{ iff } a(P \cup I)b \tag{3.1}$$

In this case P and I can be defined from R as follows:

$$aPb \text{ iff } aRb \text{ and } bR^c a \tag{3.2}$$

$$aIb \text{ iff } aRb \text{ and } bRa \tag{3.3}$$

The construction of *orders* is of a particular interest, especially in decision analysis since they allow an easy operational use of such preference structures. We begin by representing the most elementary orders (weak order, complete order). In order to define such structures we add properties to the relations P and I (namely different forms of transitivity).

Definition 6.3 (Total Order). Let R be a binary relation on the set A, R being a characteristic relation of $\langle P, I \rangle$, the following definitions are equivalent:

(i) R is a total order.
(ii) R is reflexive, antisymmetric, complete and transitive
(iii) $\begin{cases} I = \{(a,a), \forall a \in A\} \\ P \text{ is transitive} \\ P \cup I \text{ is reflexive and complete} \end{cases}$
(iv) $\begin{cases} P \text{ is transitive} \\ P.I \subset P (\text{or equivalently } IP \subset P) \\ P \cup I \text{ is reflexive and complete} \end{cases}$

With this relation, we have an indifference between any two objects only if they are identical. The total order structure consists of an arrangement of objects from the best one to the worst one without any ex aequo.

In the literature, one can find different terms associated with this structure: total order, complete order, simple order or linear order.

Definition 6.4 (Weak Order). Let R be a binary relation on the set A, R being a characteristic relation of $\langle P, I\rangle$, the following definitions are equivalent:

(i) R is a weak order
(ii) R is reflexive, complete and transitive
(iii) $\begin{cases} I \text{ is transitive} \\ P \text{ is transitive} \\ P \cup I \text{ is reflexive and complete} \end{cases}$

This structure is also called complete preorder or total preorder. In this structure, indifference is an equivalence relation. The associated order is indeed a total order of the equivalence (indifference) classes of A.

These first two structures consider indifference (or absence of preference) as a transitive relation. This is empirically falsifiable. Literature studies on the intransitivity of indifference show this; undoubtedly the most famous is that of [179], which gives the example of a cup of sweetened tea.[6] Before him, [12, 107, 129, 142] and [230] already suggested this phenomenon. For historical commentary on the subject, see [118]. Relaxing the property of transitivity of indifference results in two well-known structures: semi-orders and interval orders.

Definition 6.5 (Semiorder). Let R be a binary relation on the set A, R being a characteristic relation of $\langle P, I \rangle$, the following definitions are equivalent:

(i) R is a semiorder
(ii) R is reflexive, complete, Ferrers relation and semitransitive
(iii) $\begin{cases} P.I.P \subset P \\ P^2 \cap I^2 = \emptyset \\ P \cup I \text{ is reflexive and complete} \end{cases}$
(iv) $\begin{cases} P.I.P \subset P \\ P^2 I \subset P \text{ (or equivalently } IP^2 \subset P) \\ P \cup I \text{ is reflexive and complete} \end{cases}$

Definition 6.6 (Interval Order (IO)). Let R be a binary relation on the set A, R being a characteristic relation of $\langle P, I \rangle$, the following definitions are equivalent:

(i) R is an interval order
(ii) R is reflexive, complete and Ferrers relation
(iii) $\begin{cases} P.I.P \subset P \\ P \cup I \text{ is reflexive and complete} \end{cases}$

A detailed study of this structure can be found in [112, 187, 229]. It is easy to see that this structure generalises all the structures previously introduced.

Can we relax transitivity of preference? Although it might appear counterintuitive there is empirical evidence that such a situation can occur: [182, 276]. Similar work can be found in: [3, 46, 48–50, 111, 113, 114, 286].

[6] One can be indifferent between a cup of tea with n milligrams of sugar and one with $n+1$ milligrams of sugar, if one admits the transitivity of the indifference, after a certain step of transitivity, one will have the indifference between a cup of tea with n milligram of sugar and that with $n+N$ milligram of sugar with N large enough, even if there is a very great difference of taste between the two; which is contradictory with the concept of indifference.

3.6.2 Extended Structures

The $\langle P, I\rangle$ structures presented in the previous section neither take into account all the decision-maker's attitudes, nor all possible situations. In the literature, there are several non exclusive ways to extend such structures:

– Using sophisticated numerical representations (such as n ordered points, triangles, trapezoids, etc.);
– Introduction of several distinct preference relations representing (one or more) hesitation(s) between preference and indifference or preference intensities;
– Introduction of one or more situations of incomparability.

3.6.2.1 Preference Relations on *n* Ordered Points

As we showed by the end of the previous section intervals may be used in order to represent sophisticated preferences (for instance where the indifference is not necessarily transitive). The use of intervals in order to take into account imprecision and vagueness in handling preferences is well known in the literature, but a general theory on how such models behave was lacking until recently. Öztürk et al. [212] have generalized the concept of interval by introducing the notion of n-point intervals where each object is represented by n ordered points (for more details see also [208]). They provided an exhaustive study of two-point and three-point intervals comparison and show the way to generalize such results to n-point intervals. Their results may be interpreted in two ways:

- What are the all preference structures that can be defined using n-point interval representations and satisfying some axioms?
- How to define all different ways to compare two objects represented by n-point intervals in order to obtain a $\langle P, I\rangle$-preference structure?

Their approach is based on two notions that they called a *relative position* (intuitively showing how "far" is the actual position of the two intervals w.r.t. to complete disjunction: one interval completely to the right of the other) and a *component set* associated with each relative position (where all redundant information is discarded and where the coding is done in a compact way).

Concerning the first point it turns out that the comparison of two-point intervals allows to establish three different preference structures: two types of weak orders, bi-weak order and interval order. The use of three-point intervals allows to establish seven types of preference structures: three types of weak orders, three types of bi-weak orders, three types of interval orders, one three-weak order, one split-interval order, one triangle order and two types of intransitive preference structures. In their paper they showed also the equivalence between the usual definitions of such preferences structures, their numerical representation and the properties that characterize them. Such results confirm the descriptive power of the framework which allows to provide a complete characterization for preference structures that have never been studied before, as well as other structures well known in the literature (for instance it is possible to interpret within the same framework triangle orders and weak orders).

Concerning the second point they were also interested to the relation between n-point intervals and fuzzy numbers. In order to interpret a fuzzy number as a n-point interval one may alternatively consider ordinal fuzzy intervals as a family of α-cuts of ordinary (i.e. with continuous membership function) fuzzy numbers or intervals; the family of cuts correspond to a finite number of different values of threshold α. Using such an approach they showed how to make use of their comparison rules in order to compare fuzzy intervals and analyzed the link between their framework and the four comparison indices introduced by Dubois and Prade [101] for fuzzy intervals. Three of these correspond to strict preference relations obtained for two-point intervals while the fourth is associated with a non-strict preference relation that is an interval order. In a similar way, they investigated special fuzzy numbers having only two non-zero levels of membership. Their comparison by means of Dubois and Prade comparison indices corresponds to preference structures met in the comparison of three-point intervals, namely three types of interval orders and one type of weak order.

3.6.2.2 Several Preference Relations

One can wish to give more freedom to the decision-maker and allow more detailed preference models, introducing one or more intermediate relations between indifference and preference. Such relations might represent one or more zones of ambiguity and/or uncertainty where it is difficult to make a distinction between preference and indifference. Another way to interpret such "intermediate" relations is to consider them as different "degrees of preference intensity". From a technical point of view these structures are similar and we are not going to further discuss such semantics. We distinguish two cases: one where only one such intermediate relation is introduced (usually called weak preference and denoted by Q), and another where several such intermediate relations are introduced.

1. $\langle P, Q, I \rangle$ preference structures. In such structures we introduce one more preference relation, denoted by Q which is an asymmetric and irreflexive binary relation. The usual properties of preference structures hold. Usually such structures arise from the use of thresholds when objects with numerical values are compared or, equivalently, when objects whose values are intervals are compared. The reader who wants to have more information on thresholds can go to Sect. 3.7.1, where all definitions and representation theorems are given.

   $\langle P, Q, I \rangle$ preference structures have been generally discussed in [283]. Two cases are studied in the literature:

   – *PQI* interval orders and semi-orders (for their characterisation see [273]). The detection of such structures has been shown to be a polynomial problem (see [200]).

– double threshold orders (for their characterisation see [272, 283]) and more precisely pseudo-orders (see [246, 247]).

One of the difficulties of such structures is that it is impossible to define P, Q and I from a single characteristic relation R as is the case for other conventional preference structures.

2. $\langle P_1, \cdots, P_n \rangle$ preference structures. Practically, such structures generalise the previous situation where just one intermediate relation was considered. Again, such structures arise when multiple thresholds are used in order to compare numerical values of objects. The problem was first introduced in [74] and then extensively studied in [89, 90, 238], see also [2, 88, 190, 278]. Typically such structures concern the coherent representation of multiple interval orders. The particular case of multiple semi-orders was studied in [86].

3.6.2.3 Incomparability

In the classical preference structures presented in the previous section, the decision-maker is supposed to be able to compare all alternatives, the absence of preference being considered indifference (we can have aPb, bPa or aIb). But certain situations, such as lack of information, uncertainty, ambiguity, multi-dimensional and conflicting preferences, can create incomparability between alternatives. Within this framework, the partial structures use a third symmetric and irreflexive relation J ($aJb \iff not(aPb), not(bPa), not(aIb), not(aQb), not(bQa)$), called incomparability, to deal with this kind of situation. To have a partial structure $\langle P, I, J \rangle$ or $\langle P, Q, I, J \rangle$, we add to the definitions of the preceding structures (total order, weak order, semi-order, interval order and pseudo-order), the relation of incomparability ($J \neq \emptyset$); and we obtain respectively partial order, partial preorder (quasi-order), partial semi-order, partial interval order and partial pseudo-order [239].

Definition 6.7 (Partial Order). Let R be a binary relation ($R = P \cup I$) on the set A, R being a characteristic relation of $\langle P, I, J \rangle$, the following definitions are equivalent:

(i) R is a partial order.
(ii) R is reflexive, antisymmetric, transitive
(iii) $\begin{cases} P \text{ is asymmetric, transitive} \\ I \text{ is reflexive, symmetric} \\ J \text{ is irreflexive and symmetric} \\ I = \{(a,a), \forall a \in A\} \end{cases}$

Definition 6.8 (Quasi-Order). Let R be a binary relation ($R = P \cup I$) on the set A, R being a characteristic relation of $\langle P, I, J \rangle$, the following definitions are equivalent:

(i) R is a quasi-order.
(ii) R is reflexive, transitive

$$\text{(iii)} \quad \begin{cases} P \text{ is asymmetric, transitive} \\ I \text{ is reflexive, symmetric and transitive} \\ J \text{ is irreflexive and symmetric} \\ (P.I \cup I.P) \subset P \end{cases}$$

A fundamental result [104, 112] shows that every partial order (resp. partial preorder) on a finite set can be obtained as an intersection of a finite number of total orders (resp. total preorders, see [40]).

A further analysis of the concept of incomparability can be found in [270, 271]. In these papers it is shown that the number of preference relations that can be introduced in a preference structure, so that it can be represented through a characteristic binary relation, depends on the semantics of the language used for modelling. In other terms, when classical logic is used in order to model preferences, no more than three different relations can be established (if one characteristic relation is used). The introduction of a four-valued logic allows to extend the number of independently defined relations to 10, thus introducing different types of incomparability (and hesitation) due to the different combination of positive and negative reasons (see [274]). It is therefore possible, with such a language, to consider an incomparability due to ignorance separately from one due to conflicting information.

3.6.3 Valued Structures

In this section, we present situations where preferences between objets are defined by a valued preference relation such that $\mu(R(a,b))$ represents either the intensity or the credibility of the preference of a over b[7] or the proportion of people who prefer a to b or the number of times that a is preferred to b. In this section, we make use of results cited in [122, 222]. To simplify the notation, the valued relation $\mu(R(a,b))$ is denoted $R(a,b)$ in the rest of this section. We begin by giving a definition of a valued relation:

Definition 6.9 (Valued Relation). A valued relation R on the set A is a mapping from the cartesian product $A \times A$ onto a bounded subset of $\mathbb{R}$, often the interval [0,1].

Remark 6.1. A valued relation can be interpreted as a family of crisp nested relations. With such an interpretation, each α-cut level of a fuzzy relation corresponds to a different crisp nested relation.

[7]This value can be given directly by the decision-maker or calculated by using different concepts, such values (indices) are widely used in many MCDA methods such as ELECTRE, PROMETHEE [56, 244].

In this section, we show some results obtained by the use of fuzzy set theory as a language which is capable to deal with uncertainty. The seminal paper by Orlovsky [206] can be considered as the first attempt to use fuzzy set theory in preference modelling. Roy in [241] will also make use of the concept of fuzzy relations in trying to establish the nature of a pseudo-order. In his paper Orlovsky defines the strict preference relation and the indifference relation with the use of Lukasiewicz and min t-norms. After him, a number of researchers were interested in the use of fuzzy sets in decision aiding, most of these works are published in the journal Fuzzy Sets and Systems.

In the following we give some definitions of fuzzy ordered sets. We derive the following definitions from the properties listed in Sect. 3.5.2:

Definition 6.10 (Fuzzy Total Order). A binary relation R on the set A, is a fuzzy total order iff:

- R is antisymmetric, strongly complete and T-transitive

Definition 6.11 (Fuzzy Weak Order). A binary relation R on the set A is a fuzzy weak order iff:

- R is strongly complete and transitive

Definition 6.12 (Fuzzy Semi-order). A binary relation R on the set A is a fuzzy semi-order iff:

- R is strongly complete, a Ferrers relation and semitransitive

Definition 6.13 (Fuzzy Interval Order (IO)). A binary relation R on the set A is a fuzzy interval order iff:

- R is a strongly complete Ferrers relation

Definition 6.14 (Fuzzy Partial Order). A binary relation R on the set A is a fuzzy partial order iff:

- R is antisymmetric reflexive and T-transitive

Definition 6.15 (Fuzzy Partial Preorder). A binary relation R on the set A is a fuzzy partial preorder iff:

- R is reflexive and T-transitive

All the above definitions are given in terms of the characteristic relation R. The second step is to define valued preference relations (valued strict preference, valued indifference and valued incomparability) in terms of the characteristic relation [120–122, 207, 223]. For this, Eqs. (3.1)–(3.3) are interpreted in terms of fuzzy logical operations:

$$P(a,b) = T[R(a,b), nR(b,a)] \tag{3.4}$$

$$I(a,b) = T[R(a,b), R(b,a)] \tag{3.5}$$

$$R(a,b) = S[P(a,b), I(a,b)] \tag{3.6}$$

However, it is impossible to obtain a result satisfying these three equations using a De Morgan triplet. Alsina [7] and Fodor and Roubens [122] present this result as an impossibility theorem that proves the non-existence of a single, consistent many-valued logic as a logic of preference. A way to deal with this contradiction is to consider some axioms to define $\langle P, I, J \rangle$. Fodor, Ovchinnikov, Roubens propose to define three general axioms that they call Independence of Irrelevant Alternatives (IA), Positive Association (PA), Symmetry (SY). With their axioms, the following propositions hold:

Proposition 6.1 (Fuzzy Weak Order). *If $\langle P, I \rangle$ is a fuzzy weak order then:*

- *P is a fuzzy strict partial order*
- *I is a fuzzy similarity relation (reflexive, symmetric, transitive)*

Proposition 6.2 (Fuzzy Semi-order). *if $\langle P, I \rangle$ is a fuzzy semi-order then:*

- *P is a fuzzy strict partial order*
- *I is not transitive*

Proposition 6.3 (Fuzzy Interval Order (IO)). *if $\langle P, I \rangle$ is a fuzzy interval order then:*

- *P is a fuzzy strict partial order*
- *I is not transitive*

De Baets, Van de Walle and Kerre [75, 280, 281] define the valued preference relations without considering a characteristic relation:

P is T-asymmetric ($P \cap_T P^{-1}$)= $\emptyset$

I is reflexive and J is irreflexive ($I(a,a) = 1, J(a,a) = 0 \ \forall a \in A$)

I and J are symmetric ($I = I^{-1}, J = J^{-1}$)

$P \cap_T I = \emptyset, P \cap_T J = \emptyset, I \cap_T J = \emptyset$

$P \cup_T P^{-1} \cup_T I \cup_T = A \times A$

With a continuous t-norm and without zero divisors, these properties are satisfied only in crisp case. To deal with this problem, we have to consider a continuous t-norm with zero divisor.

In multiple criteria decision aiding, we can make use of fuzzy sets in different ways. One of these helps to construct a valued preference relation from the crisp values of alternatives on each criteria. As an example we cite the work of Perny and Roy [223]. They define a fuzzy outranking relation R from a real valued function θ defined on $\mathbb{R} \times \mathbb{R}$, such that $R(a,b) = \theta(g(a), g(b))$ verifies the following conditions for all a, b in A:

$$\forall y \in X, \quad \theta(x,y) \quad \text{is a nondecreasing function of } x \tag{3.7}$$

$$\forall x \in X, \quad \theta(x,y) \quad \text{is a nonincreasing function of } y \tag{3.8}$$

$$\forall z \in X, \quad \theta(z,z) = 1 \tag{3.9}$$

The resulting relation R is a fuzzy semi-order (i.e. reflexive, complete, semi-transitive and Ferrers fuzzy relation). Roy [241] proposed in Electre III to define the outranking relation R characterized by a function θ for each criterion as follows:

$$\theta(x,y) = \frac{p(x) - min\{y-x, p(x)\}}{p(x) - min\{y-x, q(x)\}}$$

where $p(x)$ and $q(x)$ are thresholds of the selected criteria.

We may work with alternatives representing some imprecision or ambiguity for a criterion. In this case, we make use of fuzzy sets to define the evaluation of the alternative related to the criterion. In the ordered pair $\{x, \mu_j^a\}$, μ_j^a represents the grade of membership of x for alternative a related to the criterion j. The fuzzy set μ is supposed to be normal ($sup_x(\mu_j^a) = 1$) and convex ($\forall x, y, z \in \mathbb{R}$, $y \in [x,z]$, $\mu_j^a(y) \leq min\{\mu_j^a(x), \mu_j^a(z)\}$). The credibility of the preference of a over b is obtained from the comparison of the fuzzy intervals (normal, convex fuzzy sets) of a and b with some conditions:

- The method used should be sensitive to the specific range and shape of the grades of membership.
- The method should be independent of the irrelevant alternatives.
- The method should satisfy transitivity.

Fodor and Roubens [122] propose the use of two procedures.

In the first one, the credibility of the preference of a over b for j is defined as the possibility that $a \geq b$:

$$\Pi_j(a \geq b) = \bigvee_{x \geq y} [\mu_j^a(x) \wedge \mu_j^b(y)] = sup_{x \geq y}[min(\mu_j^a(x), \mu_j^b(y))] \tag{3.10}$$

The credibility as defined by (3.10) is a fuzzy interval order (Π_j is reflexive, complete and a Ferrers relation) and

$$min(\Pi_j(a,b), \Pi_j(b,a)) = sup_x min(\mu_j^a(x), \mu_j^b(x))$$

In the case of a symmetrical fuzzy interval (μ^a), the parameters of the fuzzy interval can be defined in terms of the valuation $g_j(a)$ and thresholds $p(g_j(a))$ and $q(g_j(a))$. Some examples using trapezoidal fuzzy numbers can be found in the work of Fodor and Roubens.

The second procedure proposed by Fodor and Roubens makes use of the shapes of membership functions, satisfies the three axioms cited at the beginning of the section (PA, SY and SY) and gives the credibility of preference and indifference as follows:

$$P_j(a,b) = R_j^d(a,b) = 1 - \Pi_j(b \geq a) = N_j(a > b) \tag{3.11}$$

$$I_j(a,b) = min[\Pi_j(a \geq b), \Pi_j(b \geq a)] \tag{3.12}$$

where Π (the possibility degree) and N (the necessity degree) are two dual distributions of the possibility theory that are related to each other with the equality: $\Pi(A) = 1 - N(A)$ (see [103] for an axiomatic definition of the theory of possibility).

3.7 Domains and Numerical Representations

In this section we present several results concerning the numerical representation of the preference structures introduced in the previous section (see also [5]). This is an important operational problem. Given a set A and a set of preference relations holding between the elements of A, it is important to know whether such preferences fit a precise preference structure admitting a numerical representation. If this is the case, it is possible to replace the elements of A with their numerical values and then work with these. Otherwise, when to the set A is already associated a numerical representation (for instance a measure), it is important to test which preference structure should be applied in order to faithfully interpret the decision-maker's preferences [285].

3.7.1 Representation Theorems

Theorem 7.1 (Total Order). *Let $R = \langle P, I \rangle$ be a reflexive relation on a finite set A, the following definitions are equivalent:*

(i) R is a total order structure (see Definition 6.3)

(ii) $\exists\, g: A \mapsto \mathbb{R}^+$ *satisfying for* $\forall a, b \in A$: $\begin{cases} aPb & \textit{iff} \quad g(a) > g(b) \\ a \neq b & \Longrightarrow \quad g(a) \neq g(b) \end{cases}$

(iii) $\exists\, g: A \mapsto \mathbb{R}^+$ *satisfying for* $\forall a, b \in A$: $\begin{cases} aRb & \textit{iff} \quad g(a) > g(b) \\ a \neq b & \Longrightarrow \quad g(a) \neq g(b) \end{cases}$

In the infinite not enumerable case, it can be impossible to find a numerical representation of a total order. For a detailed discussion on the subject, see [25]. The necessary and sufficient conditions to have a numerical representation for a total order are present in many works: [59, 79, 109, 168].

Theorem 7.2 (Weak Order). *Let $R = \langle P, I \rangle$ be a reflexive relation on a finite set A, the following definitions are equivalent:*

(i) *R is a weak order structure (see Definition 6.4)*

(ii) $\exists\, g\colon A \mapsto \mathbb{R}^+$ *satisfying for* $\forall a,b \in A$: $\begin{cases} aPb \text{ iff } g(a) > g(b) \\ aIb \text{ iff } g(a) = g(b) \end{cases}$

(iii) $\exists\, g\colon A \mapsto \mathbb{R}^+$ *satisfying for* $\forall a,b \in A$: $aRb \quad \text{iff} \quad g(a) \geq g(b)$

Remark 7.1. Numerical representations of preference structures are not unique. All monotonic strictly increasing transformations of the function g can be interpreted as equivalent numerical representations.[8]

Intransitivity of indifference or the appearance of intermediate hesitation relations is due to the use of thresholds that can be constant or dependent on the value of the objects under comparison (in this case values of the threshold might obey further coherence conditions).

Theorem 7.3 (Semi-order). *Let* $R = \langle P, I\rangle$ *be a binary relation on a finite set A, the following definitions are equivalent:*

(i) *R is a semi-order structure (see Definition 6.5)*

(ii) $\exists\, g\colon A \mapsto \mathbb{R}^+$ *and a constant* $q \geq 0$ *satisfying* $\forall a,b \in A$:
$\begin{cases} aPb \text{ iff } g(a) > g(b) + q \\ aIb \text{ iff } |g(a) - g(b| \leq q \end{cases}$

(iii) $\exists\, g\colon A \mapsto \mathbb{R}^+$ *and a constant* $q \geq 0$ *satisfying* $\forall a,b \in A$:
$aRb \text{ iff } g(a) \geq g(b) - q$

(iv) $\exists\, g\colon A \mapsto \mathbb{R}^+$ *and* $\exists q : \mathbb{R} \mapsto \mathbb{R}^+$ *satisfying* $\forall a,b \in A$:
$\begin{cases} aRb \quad \text{iff} \quad g(a) \geq g(b) - q(g(b)) \\ (g(a) > g(b)) \longrightarrow (g(a) + q(g(a)) \geq g(b) + q(g(b))) \end{cases}$

For the proofs of these theorems see [112, 169, 229, 253].

The threshold represents a quantity for which any difference smaller than this one is not significant for the preference relation. As we can see, the threshold is not necessarily constant, but if it is not, it must satisfy the inequality which defines a coherence condition.

Here too, the representation of a semi-order is not unique and all monotonic increasing transformations of g appear as admissible representations provided the condition that the function q also obeys the same transformation.[9]

Theorem 7.4 (PI Interval Order). *Let* $R = \langle P, I\rangle$ *be a binary relation on a finite set A, the following definitions are equivalent:*

(i) *R is an interval order structure (see Definition 6.6)*

(ii) $\exists\, g\colon A \mapsto \mathbb{R}^+$ *satisfying* $\forall a,b \in A$: $\begin{cases} aPb \text{ iff } g(a) > g(b) + q(b) \\ aIb \text{ iff } \begin{array}{l} g(a) \leq g(b) + q(b) \\ g(b) \leq g(a) + q(a) \end{array} \end{cases}$

[8] The function g defines an ordinal scale for both structures.

[9] But in this case the scale defined by g is more complex than an ordinal scale.

It should be noted that the main difference between an interval order and a semiorder is the existence of a coherence condition on the value of the threshold. One can further generalise the structure of interval order, by defining a threshold depending on both of the two alternatives. As a result, the asymmetric part appears without circuit: [1, 2, 4, 5, 84, 259]. For extensions on the use of thresholds see [116, 144, 189]. The special case where a "frontier" has to be explicitly considered instead of threshold is discussed in [47]. For the extension of the numerical representation of interval orders in the case A is infinite not denumerable see [35, 59, 64, 110, 197, 205].

We can now see the representation theorems concerning preference structures allowing an intermediate preference relation (Q). Before that, let us mention that numerical representations with thresholds are equivalent to numerical representations of intervals. It is sufficient to note that associating a value $g(x)$ and a strictly positive value $q(g(x))$ to each element x of A is equivalent to associating two values: $l(x) = g(x)$ (representing the left extreme of an interval) and $r(x) = g(x) + q(g(x))$ (representing the right extreme of the interval to each x; obviously: $r(x) > l(x)$ always holds).

Theorem 7.5 (PQI Interval Orders). *Let $R = \langle P, Q, I \rangle$ be a relation on a finite set A, the following definitions are equivalent:*

(i) R is a PQI interval Order
(ii) There exists a partial order L such that:

1) $I = L \cup R \cup I_d$ where $I_d = \{(x,x), x \in A\}$ and $R = L^{-1}$;
2) $(P \cup Q \cup L).P \subset P$; *3)* $P.(P \cup Q \cup R) \subset P$;
4) $(P \cup Q \cup L).Q \subset P \cup Q \cup L$; *5)* $Q.(P \cup Q \cup R) \subset P \cup Q \cup R$.

$$(iii)\ \exists\, l,r\colon A \mapsto \mathbb{R}^+ \text{ satisfying: } \begin{cases} r(a) \geq l(a) \\ aPb \quad \text{iff } l(a) > r(b) \\ aQb \quad \text{iff } r(a) > r(b) \geq l(a) \geq l(b) \\ aIb \quad \text{iff } r(a) \geq r(b) \geq l(a) \quad \text{or} \\ \qquad\quad r(b) \geq r(a) \geq l(a) \geq l(b) \end{cases}$$

For proofs, further theory on the numerical representation and algorithmic issues associated with such a structure see [199, 200, 273].

Theorem 7.6 (Double Threshold Order). *Let $R = \langle P, Q, I \rangle$ be a relation on a finite set A, the following definitions are equivalent:*

(i) R is a double Threshold Order (see [283])

$$(ii)\ \begin{cases} Q.I.Q \subset Q \cup P \\ P.I.P \subset P \\ Q.I.P \subset P \\ P.Q^{-1}.P \subset P \end{cases}$$

(iii) $\exists$ *g,q,p: A* $\mapsto \mathbb{R}^+$ *satisfying:* $\begin{cases} aPb \text{ iff } g(a) > g(b) + p(b)) \\ aQb \text{ iff } g(b) + p(b) \geq g(a) > g(b) + q(b) \\ aIb \text{ iff } g(b) + q(b) > g(a) > g(b) - q(a) \end{cases}$

Theorem 7.7 (Pseudo-order). *Let* $R = \langle P, Q, I \rangle$ *be a relation on a finite set A, the following definitions are equivalent:*

(i) *R is a pseudo-order*

$$(ii) \begin{cases} \text{is a double threshold order} \\ \langle (P \cup Q), I \rangle \text{ is a semi-order} \\ \langle P, (Q \cup I \cup Q^{-1}) \rangle \text{is a semi-order} \\ P.I.Q \subset P \end{cases}$$

$$(iii) \begin{cases} \text{is a double threshold order} \\ g(a) > g(b) \Longleftrightarrow \quad g(a) + q(a) > g(b) + q(b) \\ \qquad\qquad\qquad\qquad g(a) + p(a) > g(b) + p(b) \end{cases}$$

A pseudo-order is a particular case of double threshold order, such that the thresholds fulfil a coherence condition. It should be noted however, that such a coherence is not sufficient in order to obtain two constant thresholds. This is due to different ways in which the two functions can be defined (see [90]). For the existence of multiple constant thresholds see [86].

For partial structures of preference, the functional representations admit the same formulas, but equivalences are replaced by implications. In the following, we present a numerical representation of a partial order and a quasi-order examples:

Theorem 7.8 (Partial Order). *If* $\langle P, I, J \rangle$ *presents a partial order structure,then* $\exists$ *g: A* $\mapsto \mathbb{R}^+$ *such that:*

$$\{ \; aPb \implies g(a) > g(b)$$

Theorem 7.9 (Partial Weak Order). *If* $\langle P, I, J \rangle$ *presents a partial weak order structure, then* $\exists$ *g: A* $\mapsto \mathbb{R}^+$ *such that:*

$$\begin{cases} aPb \implies g(a) > g(b) \\ aIb \implies g(a) = g(b) \end{cases}$$

The detection of the dimension of a partial order[10] is a NP hard problem [89, 112].

Remark 7.2. In the preference modelling used in decision aiding, there exist two different approaches: In the first one, the evaluations of alternatives are known (they can be crisp or fuzzy) and we try to reach conclusions about the preferences between

[10]When it is a partial order of dimension 2, the detection can be made in a polynomial time.

the alternatives. For the second one, the preferences between alternatives (pairwise comparison) are given by an expert (or by a group of experts), and we try to define an evaluation of the alternatives that can be useful. The first approach uses the inverse implication of the equivalences presented above (for example for a total order we have $g(a) > g(b) \longrightarrow aPb$); and the second one the other implication of it (for the same example, we have $aPb \longrightarrow g(a) > g(b)$).

Remark 7.3. There is a body of research on the approximation of a preference structure by another one; here we cite some studies on the research of a total order with a minimum distance to a tournament (complete and antisymmetric relation): [21, 23, 34, 63, 152, 188, 257].

3.7.2 Minimal Representation

In some decision aiding situations, the only available preferential information can be the kind of preference relation holding between each pair of alternatives. In such a case we can try to build a numerical representation of each alternative by choosing a particular functional representation of the ordered set in question and associating this with the known qualitative relations.

This section aims at studying some minimal or parsimonious representations of ordered sets, which can be helpful for this kind of situation. Particularly, given a countable set A and a preference relation $R \subseteq A \times A$, we are interested to find a numerical representation $\hat{f} \in \mathcal{F} = \{f : A \mapsto \mathbb{R},\ f \text{ homomorph to } R\}$, such that for all $x \in A$, $\hat{f}$ is minimal.

3.7.2.1 Total Order, Weak Order

The way to build a minimal representation for a total order or a weak order is obvious since the preference and the indifference relations are transitive: The idea is to minimise the value of the difference $g(a) - g(b)$ for all a, b in A. To do this we can define a unit $k = \min_{a,b \in A}(g(a) - g(b))$ and the minimal evaluation $m = \min_{a \in A}(g(a))$. The algorithm will be:

- Choose any value for k and m , e.g. $k = 1, m = 0$;
- Find the alternative i which is dominated by all the other alternatives j in A and evaluate it by $g(i) = m$
- For all the alternatives l for which we have lIi, note $g(l) = g(i)$
- Find the alternative i' which is dominated by all the alternatives j' in $A - \{i\}$ and evaluate it by $g(i') = m + k$
- For all the alternatives l' for which we have $l'Ii'$, note $g(l') = g(i')$
- Stop when all the alternatives are evaluated

3.7.2.2 Semi-order

The first study on the minimal representation of semi-orders was done in [228] who proved its existence and proposed an algorithm to build it. One can find more information about this in [87, 184, 198, 229]. varPirlot uses an equivalent definition of the semi-order which uses a second positive constant: *Total Semi-order*: A reflexive relation $R = (P, I)$ on a finite set A is a semi-order iff there exists a real function g, defined on A, a non negative constant q and a positive constant ε such that $\forall a, b \in A$

$$\begin{cases} aPb \text{ iff } g(a) > g(b) + q + \varepsilon \\ aIb \text{ iff } |g(a) - g(b)| \leq q \end{cases}$$

Such a triple (g, q, ε) is called an $\varepsilon -$ *representation* of (P, I). Any representation (g, q), as in the definition of semi-order given in Section 3.6.1, yields an ε-representation where

$$\varepsilon = min_{(a,b)\in P}(g(a) - g(b) - q)$$

Let (A, R) be an associated to the semi-order $R = (P, I)$, we denote $G(q, \varepsilon)$ the valued graph obtained by giving the value $(q+\varepsilon)$ to the arcs P and $(-q)$ to the arcs I.

Theorem 7.10. *If $R = (P, I)$ is a semi-order on the finite set A, there exists an ε-representation with threshold q iff:*

$$\frac{q}{\varepsilon} \geq \alpha = max_C\{\frac{|C \cap P|}{|C \cap I| - |C \cap P|},\ C \text{ circuit of } (A,\ R)\}$$

where $|C \cap P|$ (resp. $|C \cap I|$), represents the number of arcs P (resp. I) in the circuit C of the graph (A, R).

An algorithm to find a numerical representation of a semi-order is as follows:

- Choose any value for εk, e.g. $\varepsilon = 1$;
- Choose a large enough value of $\frac{q}{\varepsilon}$(e.g. $\frac{q}{\varepsilon} = |P|$);
- Solve the maximal value path problem in the graph $G(q, \varepsilon)$ (e.g. by using the Bellman algorithm, see [176]).

Denote by $g_{q,\varepsilon}$, the solution of the maximal path problem in $G(q, \varepsilon)$; we have:

$$g_{q,\varepsilon} \leq g(a) \forall a \in A$$

Example 7.1. We consider the example given by varPirlot and Vincke (see [229]):
Let $S = (P, I)$ be a semiorder on $A = \{a, b, c\}$ defined by $P = \{(a, c)\}$.

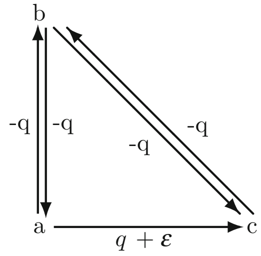


Fig. 3.3 Graphical representation of the semiorder

Table 3.2 Various ε-representations with $\varepsilon = 1$

		a	b	c
$q = 1$	$g_1 = 1$	2	1	0
$q = 1$	$g_2 = 1$	9.5	8.5	7.5
$q = 2.5$	$g_3 = 1$	3.5	1	0
$q = 2.5$	$g_4 = 1$	10.5	8.5	7
$q = 2.5$	$g_5 = 1$	3.5	2.5	0

The first inequality of Sect. 3.7.2.2 gives the following equations:

$$g(a) \geq g(c) + q + \varepsilon$$
$$g(a) \geq g(b) - q$$
$$g(b) \geq g(a) - q$$
$$g(b) \geq g(c) - q$$
$$g(c) \geq g(b) - q$$

Figure 3.3 shows the graphical representation of this semiorder.

As the non-trivial circuit $C = \{(a,c),(c,b),(b,a)\}$ is $-q + \varepsilon$ $(-q + \varepsilon = (q + \varepsilon) + (-q) + (-q))$, necessary and sufficient conditions for the existence of an ε-representation is $q \geq \varepsilon$.

The Table 3.2 provides an example of possible numerical representation of this semiorder:

Definition 7.1. A representation (g^*, q^*, ε) is minimal in the set of all non-negative ε-representations (g, q, ε) of a semiorder iff $\forall a \in A\ g^*(a) \leq g(a)$.

Theorem 7.11. *The representation $(g_{q^*,\varepsilon}, q^*, \varepsilon)$ is minimal in the set of all ε-representations of a semiorder R.*

3.7.2.3 Interval Order

An interval can be represented by two real functions l and r on the finite set A which satisfy:

$$(\forall a \in A,\ l(a) \leq r(a))^{11}$$

Definition 7.2. A reflexive relation $R = (P \cup I)$ on a finite set A is an interval order iff there exists a pair of functions $l, r : A \longrightarrow R^+$ and a positive constant ε such that $\forall a, b \in A$

$$\begin{cases} aPb & \text{iff } l(a) > r(b) + q + \varepsilon \\ aIb & \text{iff } l(a) \geq r(b) \quad \text{and} \quad l(b) \geq r(a) \end{cases}$$

Such a triplet (l, r, ε) is called an ε-representation of the interval order $P \cup I$.

Definition 7.3. The ε-representation (l^*, r^*, ε) of the interval order $P \cup I$ is minimal iff for any other ε-representation (l, r, ε) we have, $\forall a \in A$,

$$l^*(a) \leq l(a)$$
$$r^*(a) \leq r(a)$$

Theorem 7.12. *For any interval order $P \cup I$, there exists a minimal ε-representation (l^*, r^*, ε); the values of l^* and r^* are integral multiples of ε.*

3.7.2.4 *PQI* Interval Order

Ngo The and Tsoukiàs [199] have extended the results concerning the minimal representation of interval orders to the case of *PQI* interval orders. After presenting some real life examples which showed that it does not make sense to have a minimal representation of a *PQI* interval orders, they studied the problem through an instance of a *PQI* interval orders which is a separated *PQI* interval orders (it corresponds to the presentation of the condition (ii) of Theorem 7.2 where the indifference is separated into three relations, the identity, a partial order and its inverse). They obtained a result enabling to order the endpoints of intervals using an ε-representation like in the case of interval orders and they proposed two algorithms: the first one determining a general numerical representation (in $O(n^2)$) and the second one minimising the first one (in $O(n)$).

[11] One can imagine that $l(a)$ represents the evaluation of the alternative a ($g(a)$) which is the left limit of the interval and $r(a)$ represents the value of ($g(a) + q(a)$) which is the right limit of the interval. One can remark that a semi-order is an interval order with a constant length.

3.8 Extending Preferences to Sets

The problem of how to extend a preference relation over a set A of "objects" (e.g., alternatives, opportunities, candidates, etc.) to the set of all subsets of A is a very general problem inspired to many individual and collective decision making situations. Consider, for instance, the comparison of the stability of groups in coalition formation theory, or the ranking of likely sets of events in the axiomatic analysis of subjective probability, or the evaluation of equity of sets of rights inside a society, or the comparison of assets in portfolio analysis. In those situations, and in many others, a ranking of the single elements of a (finite) universal set A is not sufficient to compare the subsets of A. On the other hand, for many practical problems, only the information about preferences among single objects is available. Consequently, a central question is: how to derive a ranking over the set of all subsets of A in a way that is "compatible" with the primitive ranking over the single elements of A?

This question has been carried out in the tradition of the literature on extending an order on a set A to its *power set* (denoted by 2^A) with the objective to axiomatically characterise families of ordinal preferences over subsets (see, for instance, [19, 20, 36, 38, 115, 128, 161, 171]). In this context, an order $\succcurlyeq$ on the power set 2^A is required to be an *extension* of a primitive order R on A. This means that the relative ranking of any two singleton sets according to $\succcurlyeq$ must be the same as the relative ranking of the corresponding alternatives according to R (i.e., for each $a, b \in A$, $\{a\} \succcurlyeq \{b\} \Leftrightarrow aRb$).

The interpretation of the properties used to characterise extensions is deeply interconnected to the meaning that is attributed to sets. According to the survey of [20], the main contributions from the literature on ranking sets of objects may be grouped in three main classes of problems: (1) *complete uncertainty*, where a decision-maker is asked to rank sets which are considered as formed by mutually exclusive objects (i.e., only one object from a set will materialise), and taking into account that he cannot influence the selection of an object from a set (see, for instance, [19, 161, 203]); (2) *opportunity sets*, where sets contain again mutually exclusive objects but, in this case, a decision maker compares sets taking into account that he will be able to select a single element from a set (see, for example, [38, 171, 231, 232]); (3) *sets as final outcomes*, where each set contains objects that are assumed to materialise simultaneously (if that set is selected; for instance, see [36, 115, 237]).

In order to better clarify the differences between these three classes of problems, and to stress the importance of the nature of problems in the selection of intuitive axioms, consider the following example. Let $A = \{a, b\}$ be a universal set of two alternatives. Suppose that a decision-maker prefers a over b. Then under the *complete uncertainty* interpretation, it is reasonable to expect that the decision-maker will prefer set $\{a\}$ to $\{a, b\}$, since the possibility that alternative b materialises does exist if set $\{a, b\}$ is selected. But under the interpretation of *opportunity sets*, the two sets $\{a\}$ and $\{a, b\}$ could be simply considered indifferent. Finally, under the interpretation of *sets as final outcomes*, if objects are goods, one could guess that to have $\{a, b\}$ is better, because the decision-maker will receive both b and a. But the judgement depends on the nature of a and b and on possible effects of incompatibility between the two objects.

Let R be a binary relation on the set A, being R the characteristic relation of a preference structure $\langle P, I \rangle$. In the following, in order to rank the elements of 2^A, we use a binary relation $\succcurlyeq$ on the set 2^A, being $\succcurlyeq$ the characteristic relation of a preference structure $\langle \succ, \sim \rangle$. For example, assume that a linear order R on the set A is given. For each $S \in 2^A \setminus \{\emptyset\}$, we denote by $\max(S, R)$ the *best element* of S with respect to R such that $\max(S)Rb$ for each $b \in S$, and by $\min(S, R)$ the *worst element* of S with respect to R such that $bR\min(S)$ for each $b \in S$. Perhaps the two simplest extensions of R are the *MAX extension* and the *MIN extension*, which are defined, respectively, as a binary relation $\succcurlyeq^{\max}$ on 2^A such that $(S \succcurlyeq^{\max} T) \Leftrightarrow (\max(S)R\max(T))$, and as a binary relation $\succcurlyeq^{\min}$ on 2^A such that $(S \succcurlyeq^{\min} T) \Leftrightarrow (\min(S)R\min(T))$, for each $S, T \in 2^A \setminus \{\emptyset\}$.

3.8.1 Complete Uncertainty

In this section we introduce some axioms used in the literature in order to characterise extensions under complete uncertainty. In this context, a decision-maker is assumed to face a decision problem of establishing a ranking over all possible sets of outcomes, provided that the objects of a set are interpreted as mutually exclusive outcomes, and a final outcome is selected at a later stage according to a random procedure. As an example in this class, consider the problem faced by a policy maker that must compare different public policies, where a public policy may bring, after a certain period of time, to alternative (mutually exclusive) outcomes, whose realisation may be influenced by unforeseen contingencies.

Historically, one of the most studied axioms for extensions in this class of problems is the *dominance* property, that is referred to as the *Gärdenfors principle* in [161], in recognition of the use of this axiom in [127]. This property requires that adding an element which is better (worse) than all elements in a given set $S \in 2^A$ according to a preference relation R on the universal set A, leads to a set that is better (worse) than the original set according to preference relation $\succcurlyeq$ over 2^A.

Definition 8.1 (Dominance, DOM). Let R be a binary relation on A. A binary relation $\succcurlyeq$ on 2^A satisfies the *dominance* property (with respect to R) iff for all $S \in 2^A$ and for all $a \in A$,

$$\text{(i)}\ [aPb \text{ for all } b \in S] \Rightarrow S \cup \{a\} \succ S;$$
$$\text{(ii)}\ [bPa \text{ for all } b \in S] \Rightarrow S \succ S \cup \{a\}.$$

It is important to note that, if R on A is reflexive and antisymmetric and $\succcurlyeq$ on 2^A is reflexive and transitive, than the property of dominance for $\succcurlyeq$ (w.r.t. R) implies that $\succcurlyeq$ is an extension of R (i.e., if aPb, then the DOM property implies that $\{a\} \succ \{a, b\}$ and also $\{a, b\} \succ \{b\}$; so, by transitivity, $\{a\} \succ \{b\}$).

Another important axiom which has extensively been used in the literature is the *independence* property (introduced by Kannai and Peleg [161] with the name of *monotonicity axiom*). It requires that if there exists a strict preference between two sets $S, T \in 2^A$ and the same alternative $a \in A$ is added to both sets, then the ranking between the two formed sets must exist (according to $\succcurlyeq$) and cannot be reversed.

Definition 8.2 (Independence, IND). Let R be a binary relation on A. A binary relation $\succcurlyeq$ on 2^A satisfies the *independence* property (with respect to R) iff for all $S, T \in 2^A$, for all $a \in A \setminus (S \cup T)$,

$$S \succ T \Rightarrow (S \cup \{a\}) \succcurlyeq (T \cup \{a\}).$$

The following theorem [161], says that if a reflexive and transitive relation $\succcurlyeq$ on 2^A satisfies DOM (i.e. $\succcurlyeq$ is an extension of R on A) and IND, then any set $A \in 2^A \setminus \{\emptyset\}$ is indifferent (with respect to $\succcurlyeq$) to the set consisting of the best element and the worst element in A (according to the primitive linear order R).

Theorem 8.1. *Let R a linear order on A and let $\succcurlyeq$ be a reflexive and transitive relation on 2^A. If $\succcurlyeq$ satisfies DOM and IND (w.r.t. R), then*

$$S \simeq \{\max(S, R), \min(S, R)\}$$

for all $S \in 2^A \setminus \{\emptyset\}$.

For a proof of this theorem see [20, 161]. Both DOM and IND are quite intuitive properties for extensions when objects are mutually exclusive. Surprisingly, the following proposition shows that DOM and IND properties are incompatible when completeness of the ranking on the 2^A is assumed (and $|A| \geq 6$).

Theorem 8.2. *Let R be a linear order on A, with $|A| \geq 6$. There exists no total preorder $\succcurlyeq$ on 2^A which satisfies DOM and IND.*

For a proof of this theorem see [20, 161]. Other (possibility or impossibility) results can be obtained by modifying axioms IND and DOM [18, 39, 128], or weakening the assumption that $\succcurlyeq$ is a total preorder on 2^A [203, 213]. Many other extensions have been proposed and axiomatically studied in the literature for problems under complete uncertainty [20]. In particular, we refer to the *lexi-max* and *lexi-min* extensions [36, 213], which are obtained, respectively, as the lexicographical generalizations of the MAX and the MIN extensions, and the *median-based* extensions [203], where the relative ranking of the median alternatives is used as the criterion for comparing two sets.

3.8.2 Opportunity Sets

For this family of problems, the elements in 2^A are interpreted as sets of opportunities from which a decision-maker is allowed to select precisely one element. Note that the substantial difference from the context of complete uncertainty is that for opportunity sets the choice of an outcome from a set is left to the decision-maker, whereas in the context of complete uncertainty the selection procedure is based on a random device that cannot be influenced by the decision-maker. An example of opportunity set is the set of consumption bundles that a consumer may afford given his budget and the market price of goods in the bundle. Another example could be the sets of candidates (e.g., corresponding to different parties) that are available to a voter in a particular election [136].

In [171], a characterisation of the MAX extension $\succcurlyeq^{\max}$ for opportunity sets[12] is provided. The axiom of *extension robustness* used in [171] requires that adding to a set $A \in 2^A$ a set $B \in 2^A$ that is at most as good as A determines a set that is indifferent to A.

Definition 8.3 (Extension Robustness, EXT ROB). A binary relation $\succcurlyeq$ on 2^A satisfies the *extension robustness* property if and only if for all $S, T \in 2^A$,

$$S \succcurlyeq T \Rightarrow S \sim (S \cup T).$$

One of the main results in [171] is that a binary relation $\succcurlyeq$ on 2^A satisfies the EXT ROB property if and only if there exists a linear order R on A such that $\succcurlyeq$ coincides with $\succcurlyeq^{\max}$, the MAX extension on R.

The MAX extension has been subject to some criticism when used to compare sets of opportunities. A certain branch of the economic literature, illustrated by the contributions of [16, 37, 38, 105, 134, 135, 137, 167, 214] have attempted to define rankings of opportunity sets without explicitly refer to the future choice behavior of a decision-maker. The problem of ranking opportunity sets in this context amounts to define what it means for a set of opportunities to offer more *freedom of choice* than another. We do not enter here in the philosophical debate on the concept of "freedom" (see, for instance, [136]) and how its definition may be related to the nature of different constraints (physical, economical, legal, etc. [136, 147]). Moreover, there is no unity in the opportunity sets literature about the notion of freedom to be used for ranking opportunity sets. According to [24], different notions of freedom have been proposed: freedom of choice per se, introduced by the seminal article of Pattanaik and Xu [214], where the absence of preference information means that a measure of freedom can only reflect quantitative aspects of opportunity sets; freedom as *autonomy*, which keeps into account the autonomy of the decision-makers in making choices and where the autonomy is defined according to the

[12]The MAX extension is also known, in the context of opportunity sets, as the *indirect-utility criterion*, i.e. the criterion to rank sets is the best possible choice that can be made.

independence of the choices of a decision-maker of his conditioning or of the will of other decision-makers [156]; freedom as the valuation of *exercise of choice*, where the significance of the choices is evaluated according to some notion of diversity or similarity among alternatives (e.g., see [215, 236]); *negative* freedom, where ranking is aimed to represent the measure of absence of coercion or oppression imposed by other decision-makers on individual choices rather than any other constraints [282].

3.8.3 Sets as Final Outcomes

In this section, the problem of how to rank sets of elements that materialise simultaneously is considered. For instance, consider the formation of coalitions that should work jointly for a common goal, or the election of new members to join an organisation, or many situations where matching problems arise. A standard application of this kind of problems is the *college admissions problem* [126, 237], where colleges need to rank sets of students based on their ranking of individual applicants.

We start with the introduction of the *fixed cardinality ranking* approach [237], where the number of elements in ranked sets is fixed a priori. For instance, in the college admission problem, where the objective is to evaluate individual students for the admissibility to the first class, colleges are assumed to have a fixed quota $q \in \mathbb{N}$ specifying the maximum number of students they can admit. Therefore, matching analysis concentrates on the preferences of colleges over sets of students of size q. In order to analyse this kind of problems, Roth [237] introduced the property of *responsiveness*, which requires that if one element a in a set A is replaced by another element b, then the ranking between the new set $A \setminus \{a\} \cup \{b\}$ and the original set A is determined by the ranking between a and b according to R. In the following, we denote by $\mathcal{A}_q$ the set of all subsets of A of cardinality $q \in \{1, \ldots, |A|\}$, that is $\mathcal{A}_q = \{S \in 2^A \text{ s.t. } |S| = q\}$.

Definition 8.4 (Responsiveness, RESP). Let R be a binary relation on the set A. A binary relation $\succcurlyeq_q$ on $\mathcal{A}_q$ satisfies the *responsiveness* property on $\mathcal{A}_q$ (and with respect to R) iff for all $S \in \mathcal{A}_q$, for all $a \in A$ and for all $b \in A \setminus S$ we have that

$$\big[[S \succcurlyeq_q (S \setminus \{a\}) \cup \{b\}] \Leftrightarrow aRb\big] \text{ and } \big[[(S \setminus \{a\}) \cup \{b\} \succcurlyeq_q S] \Leftrightarrow bRa\big].$$

Clearly, the RESP property is aimed at preventing complementarity effects. As shown in [36], the RESP property was used to characterize[13] the family of *lexicographic rank-ordered extensions*, which generalise the idea of lexi-min and lexi-max orderings.

[13]Together with another property called *fixed-Cardinality neutrality*, saying that the labelling of the alternatives is irrelevant in establishing the ranking among sets of fixed cardinality q.

Another simple way to generate rankings of sets as final results is to look for a utility representation of the ranking over sets [108, 109, 235]. In particular, it is interesting to study under which conditions an extension $\succcurlyeq$ of R is additively representable [76, 109].

A still different approach was introduced by Fishburn [115], where the information available to establish an extension is not only a primitive ranking on the universal set A, but also a *signed ordering* on the "complements" of the alternatives in A is available. Looking at A as a set of possible candidates for a committee, for instance, the model based on signed ordering allows for the consideration of comparisons like "it is more important to prevent a candidate a from being in the committee than having candidate b in the committee", or "leaving candidate a off the committee is preferred to leaving b off the committee", etc. Properties of signed orderings and conditions for their extensions in this richer informational content are presented in [115].

Recently, Moretti and Tsoukiàs [192] introduced a new class of orderings of sets as final results, and they called the elements of this class *Shapley extensions*, for their attitude to preserve the ranking provided by the *Shapley value* [191, 255] of associated coalitional games. In general, Shapley extensions do not need to satisfy the RESP property (even if an axiomatic characterization using this property on the class of monotonic total preorders is provided in [192]) and therefore they can be used to keep into account possible complementarity effects among objects.

3.8.4 An Overview to Related Theories

The problem of electing a committee is also well-studied in *voting theory* [54, 55, 166]. In such situations, voters face the problem to choose from a finite set A of candidates a nonempty subset K of committee members. For a general discussion on voting methods see, for example, [66, 175]. Here we focus on the aspects of the problem which are directly related to the extension of preference of voters over single candidates to subsets of candidates. Following an example illustrated by Uckelman [277], suppose that a group of voters are invited to elect a committee of three persons from the set of five candidates $A = \{1, 2, 3, 4, 5\}$. Now, suppose that a voter believes that 1 and 2 are the best candidates: we may represent this fact with a preference structure $\langle P, I\rangle$ on A such that $1\ I\ 2\ P\ 3\ I\ 4\ I\ 5$. Consequently, it could be reasonable to assume that any committee containing one of them is better than any committee with neither. In addition, suppose also that such a voter also believes that 1 and 2 will fight if they are on the committee together (so, any committee with both of them is worse than any committee with neither). Thus, the voter would rank the committees in the following way:

$$\begin{aligned}&\{1,3,4\} \sim \{1,3,5\} \sim \{1,4,5\} \sim \{2,3,4\} \sim \{2,3,5\} \sim \{2,4,5\}\\ &\quad \succ \{3,4,5\} \succ \{1,2,3\} \sim \{1,2,4\} \sim \{1,2,5\}. \qquad (3.13)\end{aligned}$$

Which criterion can be adopted to extend a characteristic relation R of the preference structure $\langle P, I\rangle$ on single candidates, in order to end up in a characteristic relation $\succcurlyeq$ of the preference structure $\langle \succ, \sim\rangle$ on committees? Put in a more general way, how to consider the fact that committee membership for one candidate is not necessarily independent of the question of committee membership for some other candidate? Several approaches have been proposed in literature to extend preference of voters. In [55, 166], voters are assumed to rank committees according to their *Hamming distance* from their top preferences, where the top preference of a voter is the committee it most prefers. Let n be the number of voters and $k \leq n$ be the number of seats in the committee. A *ballot* is a binary k-vector, $(p_1, p_2, \ldots, p_k)$, where p_i equals 0 or 1, for each $i \in \{1, \ldots, k\}$. These binary vectors indicate the approval or disapproval of each candidate by a voter. For instance, ordering candidates increasingly, the committee $\{1, 3, 4\}$ corresponds to the ballot $(1, 0, 1, 1, 0)$ (shortly, 10110). The Hamming distance $d(p, q)$ between two binary k-vectors p and q is the number of components on which they differ. Note that the Hamming distance between $\{1, 3, 4\}$ and $\{2, 4, 5\}$ is $d(10110, 01011) = 4$, whereas the distance between $\{1, 3, 4\}$ and $\{1, 2, 3\}$ is $d(10110, 11100) = 2$. Consequently, the ordering induced by the Hamming distance from $\{1, 3, 4\}$ is $\{1, 3, 4\} \succ \{1, 2, 3\} \succ \{2, 3, 4\}$, thus putting an optimal committee last and one least favoured committee in second place, which does not represent the true ranking $\langle \succ, \sim\rangle$ introduced in (3.13).

In *financial theory*, the goal of *portfolio management* is to allocate resources and budgets to a group of *assets* (e.g., stocks, projects, initiatives etc.) that maximise the return and minimise the risk. Typically, the answer to the investment problem is not the selection of the most preferred assets: a diversified portfolio will likely have less risk than the weighted average risk of its constituent assets (see, for instance, [250]). Therefore, the problem to extend preferences over single assets to a preference over portfolios of assets is very important in practical investment problems. Since the pioneering paper of Markowitz [180], where the classical model of *Mean-Variance optimisation* has been developed, many different techniques for portfolio management have been proposed in the area of multi-criteria analysis [249, 302].

We conclude this section with a short introduction to some applications in *artificial intelligence* which require the specification of preferences over sets of information items that a computer should be able to process [82]. For instance, web search engines are designed to retrieve information relative to a particular query on the World Wide Web, presenting the retrieved information as a list of *hits* (e.g., web pages, images, media files, etc.). Since search engines operate according to predefined algorithms or procedures, efficient methods to specify and compute the relevance of sets of hits to a specified query are demanded.

In order to specify preferences of decision-makers on sets of items, one possibility is to assume that preferences are numerical and to use compact representation of such valuations as, for instance, the *bidding languages* for *combinatorial auctions* [202, 277]. An alternative approach is provided by *ordinal preferences* and methods that have been introduced in literature for elicitation and compact representation of ordinal preferences over combinatorial domains. A well-known language for eliciting and representing ordinal preferences over combinatorial domains is known under the name of (*Ceteris Paribus*) *CP-nets* [43], which is tailored for representing preference relations on the domain of each variable conditioned by the values of the variables it depends on.

More recently, richer (and more sophisticated) approaches have been introduced: TCP-nets [53], which extend CP-nets by allowing statements of *conditional importance* between single variables; *conditional preference theories* [293], which further extend TCP-nets; *conditional importance* networks (CI-nets) [44], that also generalise TCP nets, with the further simplification that CI-nets do not include any conditional preference statements on the values of the variables.

Specific solution for information retrieval problems have been also introduced. A new language has been developed in [82], namely (*Depth and Diversity*) *DD-pref*, that allows for set-based preference learning starting from the specification of few examples in a numerical form and keeping into account possible effects of interaction among single items. Effects of complementarity have been modelled in [300] as a trading off "relevance" against "novelty" of information; or by measuring the "marginal relevance" [61], with the objective to minimise redundancy in a set of information items corresponding to a certain query.

3.9 Logic of Preferences

The increasing importance of preference modelling immediately interested people from other disciplines, particularly logicians and philosophers. The strict relation with deontic logic (see [8]) raised some questions such as:

- does a general logic exist where any preferences can be represented and used?
- if yes, what is the language and what are the axioms?
- is it possible, via this formalisation, to give a definition of bad or good as absolute values?

It is clear that this attempt had a clear positivist and normative objective: to define the one well-formed logic that people should follow when expressing preferences. The first work on the subject is the one by Halldén [140], but it is Von Wright's book [288] that tries to give the first axiomatisation of a logic of preferences. Inspired by this work some important contributions have been made [67, 68, 145, 146, 149, 233]. Influence of this idea can also be found in [155, 234], but in related fields (statistics and value theory respectively). The discussion apparently was concluded by von Wright [289], but Huber [150, 151] continued on later on Halldin [141] and Widmeyer [291, 292] also worked on this.

The general idea can be presented as follows. At least two questions should be clarified: preferences among what? How should preferences be understood? Von Wright [288] argues that preferences can be distinguished as extrinsic and intrinsic. The first ones are derived as *a reason from a specific purpose*, while the second ones are *self-referential* to an actor expressing the preferences. In this sense intrinsic preferences are the expression of the actor's system of values of the actor. Moreover, preferences can be expressed for different things, the most general being (following Von Wright) *"states of affairs"*. That is, the expression "a is preferred to b" should be understood as the preference of a state (a world) where a occurs (whatever a represents: sentences, objects, relations etc.) over a state where b occurs. On this basis Von Wright expressed a theory based on five axioms:

A^W1. $\forall x, y \ \ p(x,y) \ \rightarrow \ \neg p(y,x)$
A^W2. $\forall x, y, z \ \ p(x,y) \ \wedge \ p(y,z) \ \rightarrow \ p(x,z)$
A^W3. $p(a,b) \ \equiv \ p(a \wedge \neg b, \neg a \wedge b)$
A^W4. $p(a \vee b, c) \ \equiv \ p(a \wedge b \wedge \neg c, \neg a \wedge \neg b \wedge c) \ \wedge \ p(a \wedge \neg b \wedge \neg c, \neg a \wedge \neg b \wedge c) \ \wedge \ p(\neg a \wedge b \wedge \neg c, \neg a \wedge \neg b \wedge c)$
A^W5. $p(a,b) \ \equiv \ p(a \wedge c, b \wedge c) \ \wedge \ p(a \wedge \neg c, b \wedge \neg c)$

The first two axioms are asymmetry and transitivity of the preference relation, while the following three axioms face the problem of combinations of states of affairs. The use of specific elements instead of the variables and quantifiers reflects the fact that von Wright considered the axioms not as logical ones, but as "reasoning principles". This distinction has important consequences on the calculus level. In the first two axioms, preference is considered as a binary relation (therefore the use of a predicate), in the three "principles", preference is a proposition. Von Wright does not make this distinction directly, considering the expression aPb ($p(a,b)$ in our notation) as a well-formed formulation of his logic. However, this does not change the problem since the first two axioms are referred to the binary relation and the others are not. The difference appears if one tries to introduce quantifications; in this case the three principles appear to be weak. The problem with this axiomatisation is that empirical observation of human behavior provides counterexamples of these axioms. Moreover, from a philosophical point of view (following the normative objective that this approach assumed), a logic of intrinsic preferences about general states of affairs should allow to define what is good (the always preferred?) and what is bad (the always not preferred?). But this axiomatisation fails to enable such a definition.

Chisholm and Sosa [68] rejected axioms A^W3 to A^W5 and built an alternative axiomatisation based on the concepts of *"good"* and *"intrinsically better"*. Their idea is to postulate the concept of good and to axiomatise preferences consequently. So a *good* state of affairs is one that is always preferred to its negation ($p(a, \neg a)$); Chisholm and Sosa, use this definition only for its operational potential as they argue that it does not capture the whole concept of "good"). In this case we have:

A^S1. $\forall x, y \ \ p(x,y) \ \rightarrow \ \neg p(y,x)$
A^S2. $\forall x, y, z \ \ \neg p(x,y) \ \wedge \ \neg p(y,z) \ \rightarrow \ \neg p(x,z)$
A^S3. $\forall x, y \ \ \neg p(x, \neg x) \ \wedge \ \neg p(\neg x, x) \ \wedge \ \neg p(y, \neg y) \ \wedge \ \neg p(\neg y, y) \ \rightarrow \ \neg p(y,x) \ \wedge \ \neg p(x,y)$
A^S4. $\forall x, y \ \ p(x,y) \ \wedge \ \neg p(y, \neg y) \ \wedge \ \neg p(\neg y, y) \ \rightarrow \ p(x, \neg x)$
A^S5. $\forall x, y \ \ p(y, \neg x) \ \wedge \ \neg p(y, \neg y) \ \wedge \ \neg p(\neg y, y) \ \rightarrow \ p(x, \neg x)$

Again in this axiomatisation there are counterexamples of the axioms. The assumption of the concept of good can be argued as it allows circularities in the definitions of preferences between combinations of states of affairs. This criticism leaded Hansson [146] to consider only two fundamental, universally recognised axioms:

$\mathrm{A}^H 1.\ \forall x, y, z\ \ s(x, y)\ \wedge\ s(y, z)\ \rightarrow\ s(x, z)$
$\mathrm{A}^H 2.\ \forall x, y\ \ s(x, y)\ \vee\ s(y, x)$

where s is a "large preference relation" and two specific preference relations are defined, p (strict preference) and i (indifference):

$\mathrm{D}^H 1.\ \forall x, y\ \ p(x, y)\ \equiv\ s(x, y)\ \wedge\ \neg s(y, x)$
$\mathrm{D}^H 2.\ \forall x, y\ \ i(x, y)\ \equiv\ s(x, y)\ \wedge\ s(y, x)$

He also introduces two more axioms, although he recognises their controversial nature:

$\mathrm{A}^H 3.\ \forall x, y, z\ \ s(x, y)\ \wedge\ s(x, z)\ \rightarrow\ s(x, y \vee z)$
$\mathrm{A}^H 4.\ \forall x, y, z\ \ s(x, z)\ \wedge\ s(y, z)\ \rightarrow\ s(x \vee y, z)$

Von Wright in his reply [289], trying to argue for his theory, introduced a more general frame to define intrinsic *"holistic"* preferences or as he called them *"ceteris paribus"* preferences. In this approach he considers a set S of states where the elements are the ones of A (n elements) and all the 2^n combinations of these elements. Given two states s and t (elementary or combinations of m states of S) you have i ($i = 2^{n-m}$) combinations C_i of the other states. You call an s-world any state that holds when s holds. A combination C_i of states is also a state so you can define it in the same way a C_i-world. Von Wright gives two definitions (strong and weak) of preference:

1. (strong): s is preferred to t under the circumstances C_i **iff** every C_i-world that is also an s-world and not a t-world is preferred to every C_i-world that is also t-world and not s-world.
2. (weak): s is preferred to t under the circumstances C_i **iff** some C_i-world that is an s-world is preferred to a C_i-world that is a t-world, but a C_i-world that is a t-world that is preferred to a C_i-world that is an s-world does not exist.

Now s is *"ceteris paribus"* preferred to t **iff** it is preferred under all C_i. We leave the discussion to the interested reader, but we point out that, with these definitions, it is difficult to axiomatise both transitivity and complete comparability unless they are assumed as necessary truths for "coherence" and "rationality" (see [289]).

It can be concluded that the philosophical discussion about preferences failed the objective to give a unifying frame of generalised preference relations that could hold for any kind of states, based on a well-defined axiomatisation (for an interesting discussion see [194]). It is still difficult (if not impossible) to give a definition of good or bad in absolute terms based on reasoning about preferences and the properties of these relations are not unanimously accepted as axioms of preference modelling. For more recent advances in deontic logic see [204].

More recently, Von Wright's ideas and the discussion about "logical representation of preferences" attracted attention again. This is due to problems found in the field of Artificial Intelligence field due to essentially two reasons:

– the necessity to introduce some "preferential reasoning" (see [41, 42, 52, 94–96, 170, 177, 256]);
– the large dimension of the sets to which such a reasoning might apply, thus demanding a compact representation of preferences (see [28, 30, 31, 93, 172])

Even if these motivations may appear different, the link between them is surprisingly strong as they use related languages. In fact, in both of these cases, the idea is to propose a language allowing a succinct representation of the problem without enumerating a prohibitive number of alternatives and being as close as possible to the way that a decision maker expresses his preferences in a natural language. The two common approaches consist on the use of the propositional logic or a graphical language for the representation of preferences which may be given as an ordinal data (generally a preorder) or as an utilitarian preferences.

Concerning the propositional logic, a survey may be found in [174]. In this field some authors have been interested on the use of penalties or weighted bases with propositional formulae (see [29, 65, 72, 78, 139, 209, 227, 248], among others) others have proposed the use of distance between logical worlds (see [172, 173]).

Graphical languages have been proposed for qualitative and quantitative preferences specially when the set of alternatives is defined as the cartesian product of finite domains and when there are some interactions between criteria. *Generalized additive decomposable* (*GAI*) utility functions have been introduced by Fishburn [109] in order to represent interaction between criteria by preserving some decompasability of the model. One of the earliest studies to exploit separable preferences in a graphical model is the extension of influence diagrams (see [263]), then Bacchus and Grove [15] have introduced the GAI-nets, the first graphical model based on conditional independence structure. The elicitation in GAI-nets have been addressed in [57, 58, 132]. Another important research line is about *CP-nets* which propose a qualitative graphical representation of preferences interpreting conditional independence of preference statements under a *ceteris paribus* (all else being equal) principle. The idea of using *ceteris paribus* principle is due to Von Wright [288] and have became to be used by AI researches for 20 years, firstly by Doyle [94, 95], and then others have been interested in different aspects such as elicitation, consistency, computation of a result, ... (for more details see [43, 92, 130]).

3.10 Conclusion

We hope that this chapter on preference modelling, gave the non-specialist reader a general idea of the field by providing a list of the most important references of a very vast and technical literature. In this chapter, we have tried to present the necessary technical support for the reader to understand the following chapters. One can note that our survey does not interpret all the questions related to preference modelling. Let us mention some of them:

– How to get and validate preference information [17, 287]
– The relation between preference modelling and the problem of meaningfulness in measurement theory [235]
– Statistical analysis of preferential data [70, 138]
– Interrogations on the relations between preferences and the value system, and the nature of these values [60, 73, 269, 288].

References

1. Abbas, M.: Any complete preference structure without circuit admits an interval representation. Theor. Decis. **39**, 115–126 (1995)
2. Abbas, M., Vincke, P.: Preference structures and threshold models. J. Multi-Criteria Decis. Anal. **2**, 171–178 (1993)
3. Abbas, M., Pirlot, M., Vincke, P.: Tangent circle graphs and orders. Discret. Appl. Math. **155**, 429–441 (2007)
4. Agaev, R., Aleskerov, F.: Interval choice: classic and general cases. Math. Soc. Sci. **26**, 249–272 (1993)
5. Aleskerov, F., Bouyssou, D., Monjardet, B.: Utility Maximization, Choice and Preference, 2nd edn. Springer, Berlin (2007)
6. Allen, J.F.: Maintaining knowledge about temporal intervals. J. ACM **26**, 832–843 (1983)
7. Alsina, C.: On a family of connectives for fuzzy sets. Fuzzy Sets Syst. **16**, 231–235 (1985)
8. Åqvist, L.: Deontic logic. In: Gabbay, D., Guenther, F. (eds.) Handbook of Philosophical Logic, vol. II, pp. 605–714. D. Reidel, Dordrecht (1986)
9. Arieli, O., Avron, A.: The value of the four values. Artif. Intell. **102**, 97–141 (1998)
10. Arieli, O., Cornelis, C., Deschrijver, G.: Preference modeling by rectangular bilattices. In: Proceedings of the 3rd International Conference on Modeling Decisions for Artificial Intelligence (MDAI'06). Lecture Notes in Artificial Intelligence, vol. 3885, pp. 22–33. Springer, Berlin (2006)
11. Arieli, O., Avron, A., Zamansky, A.: Ideal paraconsistent logics. Stud. Logica **99**, 31–60 (2011)
12. Armstrong, W.E.: The determinateness of the utility function. Econ. J. **49**, 453–467 (1939)
13. Armstrong, W.E.: Uncertainty and utility function. Econ. J. **58**, 1–10 (1948)
14. Armstrong, W.E.: A note on the theory of consumer's behavior. Oxf. Econ. **2**, 119–122 (1950)
15. Bacchus, F., Grove, A.J.: Graphical models for preference and utility. In: UAI, pp. 3–10 (1995)
16. Baharad, E., Nitzan, S.: Extended preferences and freedom of choice. Soc. Choice Welf. **17**, 629–637 (2000)
17. Bana e Costa, C., Vansnick, J.: Preference relation and MCDM. In: Gal, T., Stewart, T., Hanne, T. (eds.) Multicriteria Decision Making: Advances in MCDM Models, Algorithms, Theory, and Applications. International Series in Operations Research & Management Science, pp. 4.1–4.23. Kluwer Academic, Dordrecht (1999)
18. Barberà, S., Pattanaik, P.K.: Extending an order on a set to the power set: some remarks on kannai and peleg's approach* 1. J. Econ. Theory **32**(1), 185–191 (1984)
19. Barberà, S., Barrett, C.R., Pattanaik, P.K.: On some axioms for ranking sets of alternatives. J. Econ. Theory **33**, 301–308 (1984)
20. Barberà, S., Bossert, W., Pattanaik, P.K.: Ranking sets of objects. In: Barberà, S., Hammond, P., Seidl, C. (eds.) Handbook of Utility Theory, vol. II, pp. 893–977. Kluwer Academic Publishers, Dordrecht (2004)

21. Barthélémy, J.-P., Monjardet, B.: The median procedure in cluster analysis and social choice theory. Math. Soc. Sci. **1**(3), 235–267 (1981)
22. Barthélemy, J.-P., Flament, C., Monjardet, B.: Ordered sets and social sciences. In: Rankin, I. (ed.) Ordered Sets. **42**(3), 721–787. D. Reidel, Dordrecht (1982)
23. Barthélémy, J.-P., Guénoche, A., Hudry, O.: Median linear orders: heuristics and a branch and bound algorithm. Eur. J. Oper. Res. **42**(3), 313–325 (1989)
24. Baujard, A.: Conceptions of freedom and ranking opportunity sets: a typology. Technical Report, CREM-CNRS (2006)
25. Beardon, A.F., Candeal, J.C., Herden, G., Induráin, E., Mehta, G.B.: The non-existence of a utility function and a structure of non-representable preference relations. J. Math. Econ. **37**, 17–38 (2002)
26. Belnap, N.D.: How a computer should think. In: Proceedings of the Oxford International Symposium on Contemporary Aspects of Philosophy, pp. 30–56. Oxford, England (1976)
27. Belnap, N.D.: A useful four-valued logic. In: Epstein, G., Dunn, J. (eds.) Modern Uses of Multiple Valued Logics, pp. 8–37. D. Reidel, Dordrecht (1977)
28. Benferhat, S., Kaci, S.: A possibilistic logic handling of strong preferences. In: Proceedings 9th IFSA World Congress and 20th NAFIPS International Conference, vol. 2, pp. 962–967. IEEE Press, Piscataway (2001)
29. Benferhat, S., Dubois, D., Prade, H.: Towards a possibilistic logic handling of preferences. Appl. Intell. **14**, 403–417 (2001)
30. Benferhat, S., Dubois, D., Kaci, S., Prade, H.: Bipolar representation and fusion of preferences in the possibilistic logic framework. In: Proceedings of the 8th International Conference on Knowledge Representation and Reasoning, KR'02, pp. 421–432. Morgan Kauffamn, San Francisco (2002)
31. Benferhat, S., Dubois, D., Kaci, S., Prade, H.: Possibilistic logic representation of preferences: relating prioritized goals and satisfaction levels expressions. In: Proceedings of the 15th European Conference on Artificial Intelligence, ECAI 2002, pp. 685–689. ISO Press, Amsterdam (2002)
32. Benzer, S.: The fine structure of the gene. Sci. Am. **206**(1), 70–84 (1962)
33. Bergstra, J.A., Bethke, I., Rodenburg, P.: A propositional logic with four values: true, false, divergent and meaningless. J. Appl. Non-Classical Log. **5**, 199–217 (1995)
34. Bermond, J.C.: Ordres à distance minimum d'un tournoi et graphes partiels sans circuits maximaux. Math. Sci. Humaines **37**, 5–25 (1972)
35. Bosi, G., Candeal, J.C., Induráin, E.: Continuous representability of interval orders and biorders. J. Math. Psychol. **51**, 122–125 (2007)
36. Bossert, W.: Preference extension rules for ranking sets of alternatives with a fixed cardinality. Theor. Decis. **39**, 301–317 (1995)
37. Bossert, W.: Opportunity sets and individual well-being. Soc. Choice Welf. **14**, 97–112 (1996)
38. Bossert, W., Pattanaik, P.K., Xu, Y.: Ranking opportunity sets: an axiomatic approach. J. Econ. Theory **63**, 326–345 (1994)
39. Bossert, W., Pattanaik, P.K., Xu, Y.: Choice under complete uncertainty: axiomatic characterizations of some decision rules. Econ. Theory **16**, 295–312 (2000)
40. Bossert, W., Sprumont, Y., Suzumura, K.: Upper semicontinuous extensions of binary relations. J. Math. Econ. **37**, 231–246 (2002)
41. Boutilier, C.: Toward a logic for qualitative decision theory. In: Proceedings of the 4th International Conference on Knowledge Representation and Reasoning, KR'94, pp. 75–86. Morgan Kaufmann, San Francisco (1994)
42. Boutilier, C., Brafman, R.I., Hoos, H.H., Poole, D.: Reasoning with conditional ceteris paribus preference statements. In: Proceedings of the 15th Conference on Uncertainty in Artificial Intelligence, UAI'99, pp. 71–80. Morgan Kaufmann, San Francisco (1999)
43. Boutilier, C., Brafman, R., Domshlak, C., Hoos, H.H., Poole, D.: Cp-nets: a tool for representing and reasoning with conditional ceteris paribus preference statements. J. Artif. Intell. Res. **21**(1), 135–191 (2004)

44. Bouveret, S., Endriss, U., Lang, J.: Conditional importance networks: a graphical language for representing ordinal, monotonic preferences over sets of goods. In: Proceedings of IJCAI 2009, pp. 67–72 (2009)
45. Bouyssou, D.: Modelling inaccurate determination, uncertainty, imprecision using multiple criteria. In: Lockett, A., Islei, G. (eds.) Improving Decision Making in Organisations. Lecture Notes in Economics and Mathematical Systems, vol. 335, pp. 78–87. Springer, Berlin (1989)
46. Bouyssou, D.: Outranking relations: do they have special properties? J. Multi-Criteria Decis. Anal. **5**, 99–111 (1996)
47. Bouyssou, D., Marchant, T.: Biorders with frontier. Order **28**, 53–87 (2011)
48. Bouyssou, D., Pirlot, M.: Conjoint measurement without additivity and transitivity. In: Meskens, N., Roubens, M. (eds.) Advances in Decision Analysis, pp. 13–29. Kluwer Academic, Dordrecht (1999)
49. Bouyssou, D., Pirlot, M.: A characterization of strict concordance relations. In: Bouyssou, D., Jacquet-Lagrèze, E., Perny, P., Słowiński, R., Vanderpooten, D., Vincke, P. (eds.) Aiding Decisions with Multiple Criteria: Essays in Honour of Bernard Roy, pp. 121–145. Kluwer, Dordrecht (2002)
50. Bouyssou, D., Pirlot, M.: Non transitive decomposable conjoint measurement. J. Math. Psychol. **46**, 677–703 (2002)
51. Bouyssou, D., Marchant, T., Pirlot, M., Perny, P., Tsoukiàs, A., Vincke, P.: Evaluation and Decision Models: A Critical Perspective. Kluwer Academic, Dordrecht (2000)
52. Brafman, R.I., Friedman, N.: On decision-theoretic foundations for defaults. Artif. Intell. **133**, 1–33 (2001)
53. Brafman, R.I., Domshlak, C., Shimony, S.E.: On graphical modeling of preference and importance. J. Artif. Intell. Res. **25**(1), 389–424 (2006)
54. Brams, S.J., Fishburn, P.C.: Voting procedures. In: Arrow, K., Sen, A.K., Suzumura, K. (eds.) Handbook of Social Choice and Welfare, vol. 1, pp. 173–236. Elsevier, Amsterdam (2002)
55. Brams, S.J., Kilgour, D.M., Sanver, M.R.: A minimax procedure for electing committees. Public Choice **132**(3), 401–420 (2007)
56. Brans, J.P., Mareschal, B., Vincke, P.: PROMETHEE: a new family of outranking methods in multicriteria analysis. In: Brans, J.P. (ed.) Operational Research. IFORS, vol. 84, pp. 477–490. North Holland, Amsterdam (1984)
57. Braziunas, D., Boutilier, C.: Local utility elicitation in gai models. In: UAI, pp. 42–49 (2005)
58. Braziunas, D., Boutilier, C.: Minimax regret based elicitation of generalized additive utilities. In: UAI, pp. 25–32 (2007)
59. Briges, D.S., Mehta, G.B.: Representations of Preference Orderings. Springer, Berlin (1995)
60. Broome, J.: Weighting Goods. Basil Blackwell, London (1991)
61. Carbonell, J., Goldstein, J.: The use of mmr, diversity-based reranking for reordering documents and producing summaries. In: Proceedings of the 21st Annual International ACM SIGIR Conference on Research and Development in Information Retrieval, pp. 335–336. ACM, New York (1998)
62. Carrano, A.V.: Establishing the order to human chromosome-specific DNA fragments. In: Woodhead, A., Barnhart, B. (eds.) Biotechnology and the Human Genome, pp. 37–50. Plenum Press, New York (1988)
63. Charon-Fournier, I., Germa, A., Hudry, O.: Utilisation des scores dans des méthodes exactes déterminant les ordres médians des tournois. Math. Inf. Sci. Humaines **119**, 53–74 (1992)
64. Chateauneuf, A.: Continuous representation of a preference relation on a connected topological space. J. Math. Econ. **16**, 139–146 (1987)
65. Chevaleyre, Y., Endriss, U., Lang, J.: Expressive power of weighted propositional formulas for cardinal preference modeling. In: Proceedings of KR'06, pp. 145–152 (2006)
66. Chevaleyre, Y., Endriss, U., Lang, J., Maudet, N.: Preference handling in combinatorial domains: from ai to social choice. AI Mag. **29**(4), 37 (2009)
67. Chisholm, R.M., Sosa, E.: Intrinsic preferability and the problem of supererogation. Synthese **16**, 321–331 (1966)

68. Chisholm, R.M., Sosa, E.: On the logic of intrinsically better. Am. Philos. Q. **3**, 244–249 (1966)
69. Cohen, M., Jaffray, J.-Y.: Rational behavior under complete ignorance. Econometrica **48**, 1281–1299 (1980)
70. Coombs, C.H.: A Theory of Data. Wiley, New York (1964)
71. Coombs, C.H., Smith, J.E.K.: On the detection of structures in attitudes and developmental processes. Psychol. Rev. **80**, 337–351 (1973)
72. Coste-Marquis, S., Lang, J., Liberatore, P., Marquis, P.: Expressive power and succinctness of propositional languages for preference representation. In: Proceedings of KR'04, pp. 203–212 (2004)
73. Cowan, T.A., Fishburn, P.C.: Foundations of preferences. In: Eberlein, G., Berghel, H. (eds.) Essays in Honor of Werner Leinfellner, pp. 261–271. D. Reidel, Dordrecht (1988)
74. Cozzens, M.B., Roberts, F.S.: Double semiorders and double indifference graphs. SIAM J. Algebraic Discret. Methods **3**, 566–583 (1982)
75. De Baets, B., Kerre, E., Van De Walle, B.: Fuzzy preference structures and their characterization. J. Fuzzy Math. **3**, 373 (1995)
76. De Finetti, B.: Sul significato soggettivo della probabilità. Fundam. Math. **17**, 298–329 (1931)
77. De Morgan, A.: On the symbols of logic i. Trans. Camb. Philos. Soc. **9**, 78–127 (1864)
78. De Saint-Cyr, F.D., Lang, J., Schiex, T.: Penalty logic and its link with Dempster-Shafer theory. In: Proceedings of the Tenth International Conference on Uncertainty in Artificial Intelligence, pp. 204–211. Morgan Kaufmann, Los Altos (1994)
79. Debreu, G.: Representation of a preference ordering by a numerical function. In: Thrall, R., Coombs, C.H., Davies, R. (eds.) Decision Processes, pp. 159–175. Wiley, New York (1954)
80. Debreu, G.: Theory of Value: An Axiomatic Analysis of Economic Equilibrium. Wiley, New York (1959)
81. Deschrijver, G., Arieli, O., Cornelis, C., Kerre, E.: A bilattice-based framework for handling graded truth and imprecision. Int. J. Uncertainty Fuzziness Knowledge Based Syst. **15**, 13–41 (2007)
82. desJardins, M., Eaton, E., Wagstaff, K.L.: Learning user preferences for sets of objects. In: Proceedings of the 23rd International Conference on Machine Learning, pp. 273–280. ACM, New York (2006)
83. Dias, L.C, Tsoukiàs, A.: On the constructive and other approaches in decision aiding. In: Hengeller Antunes, C.A., Figueira, J. (eds.) Proceedings of the 57th Meeting of the EURO MCDA Working Group, pp. 13–28. CCDRC, Coimbra (2004)
84. Diaye, M.A.: Variable interval orders. Math. Soc. Sci. **38**, 21–33 (1999)
85. Doherty, P., Driankov, D., Tsoukiàs, A.: Partial logics and partial preferences. In: Proceedings of the CEMIT 92 International Conference, Tokyo, pp. 525–528 (1992)
86. Doignon, J.P.: Thresholds representation of multiple semiorders. SIAM J. Algebraic Discret Methods **8**, 77–84 (1987)
87. Doignon, J.P.: Sur les représentations minimales des semiordres et des ordres d'intervalles. Math. Sci. Humaines **101**, 49–59 (1988)
88. Doignon, J.P., Falmagne, J.: Well graded families of relations. Discret. Math. **173**(1–3), 35–44 (1997)
89. Doignon, J.P., Ducamp, A., Falmagne, J.C.: On realizable biorders and the biorder dimension of a relation. J. Math. Psychol. **28**, 73–109 (1984)
90. Doignon, J.P., Monjardet, B., Roubens, M., Vincke, P.: Biorder families, valued relations and preference modelling. J. Math. Psychol. **30**, 435–480 (1986)
91. Dombi, J.: A general class of fuzzy operators, the de morgan class of fuzzy operators and fuzziness measures induced by fuzzy operators. Fuzzy Sets Syst. **8**(2), 149–163 (1982)
92. Domshlak, C.: Modeling and reasoning about preferences with CP-nets. Ph.D. thesis, Ben-Gurion University (2002)
93. Domshlak, C., Brafman, R.I.: CP-nets - reasoning and consistency testing. In: Proceedings of the 8th International Conference on Knowledge Representation and Reasoning, KR'02, pp. 121–132. Morgan Kaufmann, San Francisco (2002)

94. Doyle, J.: Constructive belief and rational representation. Comput. Intell. **5**, 1–11 (1989)
95. Doyle, J.: Rationality and its roles in reasoning. In: Proceedings of the 8th National Conference on Artificial Intelligence, AAAI 90, pp. 1093–1100. MIT Press, Cambridge (1990)
96. Doyle, J.: Reasoned assumptions and rational psychology. Fundam. Inform. **20**, 35–73 (1994)
97. Doyle, J., Wellman, M.P.: Modular utility representation for decision-theoretic planning. In: Proceeding of the 1st International Conference on Artificial Intelligence Planning Systems, pp. 236–242 (1992)
98. Dubarle, D.: Essai sur la généralisation naturelle de la logique usuelle. Math. Inf. Sci. Humaines **107**, 17–73 (1989). 1963 Manuscript, Published Posthumously
99. Dubois, D., Prade, H.: Operations on fuzzy numbers. Int. J. Syst. Sci. **9**, 613–626 (1978)
100. Dubois, D., Prade, H.: Fuzzy Sets System - Theory and Applications. Academic, New York (1980)
101. Dubois, D., Prade, H.: Ranking fuzzy numbers in the setting of possibility theory. Inf. Sci. **30**, 183–224 (1983)
102. Dubois, D., Prade, H.: Possibility Theory. Plenum Press, New York (1988)
103. Dubois, D., Prade, H.: Possibility theory, probability theory and multiple-valued logics: a clarification. Ann. Math. Artif. Intell. **32**, 35–66 (2001)
104. Dushnik, B., Miller, E.W.: Partially ordered sets. Am. J. Math. **63**, 600–610 (1941)
105. Dutta, B., Sen, A.K.: Ranking opportunity sets and arrow impossibility theorems: correspondence results. J. Econ. Theory **71**, 90–101 (1996)
106. Fages, F., Ruet, P.: Combining explicit negation and negation by failure via belnap's logic. Theor. Comput. Sci. **171**, 61–75 (1997)
107. Fechner, G.T.: Elemente der Psychophysik. Breitkof und Hartel, Leipzig (1860)
108. Fishburn, P.C.: Methods of estimating additive utilities. Manag. Sci. **13**, 435–453 (1967)
109. Fishburn, P.C.: Utility Theory for Decision Making. Wiley, New York (1970)
110. Fishburn, P.C.: Interval representations for interval orders and semiorders. J. Math. Psychol. **10**, 91–105 (1973)
111. Fishburn, P.C.: Nontransitive measurable utility. J. Math. Psychol. **26**, 31–67 (1982)
112. Fishburn, P.C.: Interval Orders and Interval Graphs. J. Wiley, New York (1985)
113. Fishburn, P.C.: Nontransitive additive conjoint measurement. J. Math. Psychol. **35**, 1–40 (1991)
114. Fishburn, P.C. Nontransitive preferences in decision theory. J. Risk Uncertain. **4**, 113–134 (1991)
115. Fishburn, P.C.: Signed orders and power set extensions. J. Econ. Theory **56**, 1–19 (1992)
116. Fishburn, P.C.: Generalisations of semiorders: a review note. J. Math. Psychol. **41**, 357–366 (1997)
117. Fishburn, P.C.: Preference structures and their numerical presentations. Theor. Comput. Sci. **217**, 359–389 (1999)
118. Fishburn, P.C., Monjardet, B.: Norbert wiener on the theory of measurement (1914, 1915, 1921). J. Math. Psychol. **36**, 165–184 (1992)
119. Fitting, M.C.: Bilattices and the semantics of logic programming. J. Log. Program. **11**, 91–116 (1991)
120. Fodor, J.C.: An axiomatic approach to fuzzy preference modelling. Fuzzy Sets Syst. **52**, 47–52 (1992)
121. Fodor, J.C.: Valued preference structures. Eur. J. Oper. Res. **79**, 277–286 (1994)
122. Fodor, J.C., Roubens, M.: Fuzzy Preference Modelling and Multicriteria Decision Support. Kluwer Academic Publishers, Boston (1994)
123. Font, J.M., Moussavi, M.: Note on a six valued extension of three valued logics. J. Appl. Non-Classical Log. **3**, 173–187 (1993)
124. Fortemps, P., Słowiński, R.: A graded quadrivalent logic for ordinal preference modelling: loyola-like approach. Fuzzy Optim. Decis. Making **1**, 93–111 (2002)
125. Frank, M.: On the simultaneous associativity of $f(x,y)$ and $x+y-f(x,y)$. Aequationes Math. **19**, 194–226 (1979)

126. Gale, D., Shapley, L.S.: College admissions and the stability of marriage. Am. Math. Mon. **69**, 9–15 (1962)
127. Gärdenfors, P.: Manipulation of social choice functions. J. Econ. Theory **13**, 217–228 (1976)
128. Geist, C., Endriss, U.: Automated search for impossibility theorems in social choice theory: ranking sets of objects. J. Artif. Intell. Res. **40**, 143–174 (2011)
129. Georgescu-Roegen, N.: The pure theory of consumer's behavior. Q. J. Econ. **50**, 545–593 (1936)
130. Goldsmith, J., Lang, J., Truszczynski, M., Wilson, N.: The computational complexity of dominance and consistency in cp-nets. In: IJCAI, pp. 144–149 (2005)
131. Golumbic, M.C., Shamir, R.: Complexity and algorithms for reasoning about time: a graph theoretic approach. J. ACM **40**, 1108–1133 (1993)
132. Gonzales, C., Perny, P.: Gai networks for utility elicitation. In: KR, pp. 224–234 (2004)
133. Gougen, J.A.: The logic of inexact concepts. Synthese **19**, 325–373 (1969)
134. Gravel, N.: Can a ranking of opportunity sets attach an intrinsic importance to freedom of choice? Am. Econ. Rev. **84**, 454–458 (1994)
135. Gravel, N.: Ranking opportunity sets on the basis of their freedom of choice and their ability to satisfy preferences: a difficulty. Soc. Choice Welf. **15**, 371–382 (1998)
136. Gravel, N.: What is freedom. In: Peil, J., Van Staveren, I. (eds.) Handbook of Economics and Ethics, pp. 166–174. Edward Edgar Publishing, London (2008)
137. Gravel, N., Laslier, J.-F., Trannoy, A.: Individual freedom of choice in a social setting. In: Laslier, J.-F., Fleurbaey, M., Gravel, N., Trannoy, A. (eds.) Freedom in Economics: New Perspectives in Normative Analysis, pp. 76–92. Routledge, London (1997)
138. Green, P.E., Tull, D.S., Albaum, G.: Research for Marketing Decisions, 2nd edn. Prentice Hall, Englewood Cliffs (1988)
139. Haddawy, P., Hanks S.: Representations for decision-theoretic planning: utility functions for deadline goals. In: Proceedings of the 3rd International Conference on Principles of Knowledge Representation and Reasoning (KR'92). Cambridge, pp. 71–82, 25-29 October 1992
140. Halldén, S.: On the Logic of Better. Library of Theoria, Lund (1957)
141. Halldin, C.: Preference and the cost of preferential choice. Theor. Decis. **21**, 35–63 (1986)
142. Halphen, E.: La notion de vraisemblance. Technical Report 4(1), Publication de l'I.S.U.P (1955)
143. Hand, D.J.: Discrimination and Classification. Wiley, New York (1981)
144. Hansen, E.: Global Optimization Using Interval Analysis. M. Dekker, New York (1992)
145. Hansson, B.: Choice structures and preference relations. Synthese **18**, 443–458 (1966)
146. Hansson, B.: Fundamental axioms for preference relations. Synthese **18**, 423–442 (1966)
147. Hayek, F.A.: The Constitution of Liberty. University of Chicago Press, Chicago (1960)
148. Hodson, F.R., Kendall, D.G., Tautu, P.: Mathematics in the Archaeological and Historical Sciences. Edinburgh University Press, Edinburgh (1971)
149. Houthakker, H.S.: The logic of preference and choice. In: Tymieniecka, A. (ed.) Contributions to Logic and Methodology in Honour of J.M. Bochenski, pp. 193–207. North Holland, Amsterdam (1965)
150. Huber, O.: An axiomatic system for multidimensional preferences. Theor. Decis. **5**, 161–184 (1974)
151. Huber, O.: Non transitive multidimensional preferences. Theor. Decis. **10**, 147–165 (1979)
152. Hudry, O., Woirgard, F.: Ordres médians et ordres de Slater des tournois. Math. Inf. Sci. Humaines **133**, 23–56 (1996)
153. Jaffray, J.-Y., Wakker, P.P.: Decision making with belief functions: compatibility and incompatibility with the sure-thing principle. J. Risk Uncertain. **7**, 255–271 (1993)
154. Janowitz, M., LaPointe, F.-J., McMorris, F.R., Mirkin, B., Roberts, F.S. (eds.): Bioconsensus. DIMACS Series, vol. 61. American Mathematical Society, Providence (2003)
155. Jeffrey, R.C.: The Logic of Decision. McGraw Hill, New York (1965)
156. Jones, P., Sugden, R.: Evaluating choice. Int. Rev. Law Econ. **2**, 47–65 (1982)

157. Kacprzyk, J., Roubens, M.: Non Conventional Preference Relations in Decision Making. LNMES, vol. 301. Springer, Berlin (1988)
158. Kahneman, D., Tversky, A.: Prospect theory: an analysis of decision under risk. Econometrica **47**, 263–291 (1979)
159. Kahneman, D., Slovic, P., Tversky, A.: Judgement Under Uncertainty: Heuristics and Biases. Cambridge University Press, Cambridge (1981)
160. Kaluzhny, Y., Muravitsky, A.Y.: A knowledge representation based on the Belnap's four valued logic. J. Appl. Non-Classical Log. **3**, 189–203 (1993)
161. Kannai, Y., Peleg, B.: A note on the extension of an order on a set to the power set. J. Econ. Theory **32**, 172–175 (1984)
162. Karp, R.M.: Mapping the genome: some combinatorial problems arising in molecular biology. In: Proceeding 25th STOC, pp. 278–285. ACM Press, New York (1993)
163. Kaufmann, A.: Theory of Fuzzy Sets. Academic, New York (1975)
164. Kaufmann, A., Gupta, M.M.: Introduction to Fuzzy Arithmetic-Theory and Application. Van Nostrand Reinhold, New York (1985)
165. Keeney, R.L., Raiffa, H.: Decisions with Multiple Objectives: Preferences and Value Trade-offs. Wiley, New York (1976)
166. Kilgour, D.M., Brams, S.J., Sanver, M.R.: How to elect a representative committee using approval balloting. In: Simeone, B., Pukelsheim, F. (eds.) Mathematics and Democracy, pp. 83–95. Springer, Berlin (2006)
167. Klemisch-Ahlert, M.: Freedom of choice: a comparison of different rankings of opportunity sets. Soc. Choice Welf. **10**, 189–207 (1993)
168. Krantz, D.H., Luce, R.D., Suppes, P., Tversky, A.: Foundations of Measurement, Vol. 1: Additive and Polynomial Representations. Academic, New York (1971)
169. Krantz, D.H., Luce, R.D., Suppes, P., Tversky, A.: Foundations of Measurement, Vol. 2: Geometrical, Threshold and Probabilistic Representations. Academic, New York (1989)
170. Kraus, S., Lehmann, D., Magidor, M.: Nonmonotonic reasoning, preferential models and cumulative logics. Artif. Intell. **44**, 167–207 (1990)
171. Kreps, D.M.: A representation theorem for preference for flexibility. Econometrica: Journal of the Econometric Society, pp. 565–577 (1979)
172. Lafage, C., Lang, J.: Logical representation of preferences for group decision making. In: Proceedings of the 7th International Conference on Knowledge Representation and Reasoning, KR'2000, pp. 457–468 (2000)
173. Lafage, C., Lang, J.: Propositional distances and preference representation. In: Proceedings of the ECSQARU'2001, pp. 48–59 (2001)
174. Lang, J.: Logical representation of preferences. In: Bouyssou, D., Dubois, D., Pirlot , M., Prade, H. (eds.) Decision Making Process, pp. 321–355. Wiley, New York (2006)
175. Lang, J., Xia, L.: Sequential composition of voting rules in multi-issue domains. Math. Soc. Sci. **57**(3), 304–324 (2009)
176. Lawler, E.: Combinatorial Optimization: Networks and Matroids. Holt, Rinehart & Winston, New York (1976)
177. Lehmann, D.: Nonmonotonic logics and semantics. J. Log. Comput. **11**, 229–256 (2001)
178. Li, E., Lee, R.J.: Comparison of fuzzy numbers based on the probabiliy measure of fuzzy events. Comput. Math. Appl. **15**, 887–896 (1988)
179. Luce, R.D.: Semiorders and a theory of utility discrimination. Econometrica **24**, 178–191 (1956)
180. Markowitz, H.M.: Portfolio Selection: Efficient Diversification of Investments. Yale University Press, New Haven (1968)
181. Massaglia, R., Ostanello, A.: N-tomic: a support system for multicriteria segmentation problems. In: Korhonen, P., Lewandowski, A., Wallenius, J. (eds.) Multiple Criteria Decision Support. LNEMS, vol. 356, pp. 167–174. Springer, Berlin (1991)
182. May, K.O.: Intransitivity, utility and the aggregation of preference patterns. Econometrica **22**, 1–13 (1954)
183. Mendelson, E.: Introduction to Mathematical Logic. Van Nostrand, Princeton (1964)

184. Mitas, J.: Minimal representation of semiorders with intervals of same length. In: Bouchitté, V., Morvan, M. (eds.) Orders, Algorithms and Applications. LNCS, vol. 831, pp. 162–175. Springer, Berlin (1994)
185. Mizumoto, M., Tanaka, K.: The four operations of arithmetic on fuzzy numbers. Syst. Comput. Controls **7**(5), 73–81 (1976)
186. Mizumoto, M., Tanaka, K.: Some properties of fuzzy sets of type 2. Inf. Control **31**, 312–340 (1976)
187. Monjardet, B.: Axiomatiques et propriétés des quasi ordres. Math. Sci. Humaines **63**, 51–82 (1978)
188. Monjardet, B.: Relations à éloignement minimum des relations binaires: note bibliographique. Math. Inf. Sci. Humaines **67**, 115–122 (1979)
189. Moore, R.: Interval Analysis. Prentice-Hall, Englewood Cliffs (1966)
190. Moreno, J.A.: A transitivity approach to preference relational systems. Eur. J. Oper. Res. **60**(1) 68–75 (1992)
191. Moretti, S., Patrone, F.: Transversality of the shapley value. Top **16**(1), 1–41 (2008)
192. Moretti, S., Tsoukiàs, A.: Ranking sets of possibly interacting objects using Shapley extensions. In: Proceedings of the 13th International Conference on Principles of Knowledge Representation and Reasoning (KR 2012), pp. 199–209. AAAI Publications (2012)
193. Moscarola, J., Roy, B.: Procédure automatique d'examen de dossiers fondée sur une segmentation trichotomique en présence de critères multiples. RAIRO Rech. Opér. **11**(2), 145–173 (1977)
194. Mullen, J.D.: Does the logic of preference rest on a mistake? Metaphilosophy **10**, 247–255 (1979)
195. Nagaraja, R.: Current approaches to long-range physical mapping of the human genome. In: Anand, R. (ed.) Techniques for the Analysis of Complex Genomes, pp. 1–18. Academic Publishers, New York (1992)
196. Nahmias, S.: Fuzzy variables. Fuzzy Sets Syst. **1**, 97–110 (1976)
197. Nakamura, Y.: Real interval representations. J. Math. Psychol. **46**, 140–177 (2002)
198. Ngo The, A.: Structures de préférence non conventionnelles en aide à la décision. Ph.D. thesis, Université Paris Dauphine, Paris (2002)
199. Ngo The, A., Tsoukiàs, A.: Numerical representation of PQI interval orders. Discret. Appl. Math. **147**, 125–146 (2005)
200. Ngo The A., Tsoukiàs, A., Vincke, P..: A polynomial time algorithm to detect PQI interval orders. Int. Trans. Oper. Res. **7**, 609–623 (2000)
201. Nguyen, T.H., Kreinovitch, V., Nesterov, V., Nakamura, M.: On hardware support for interval computations and for soft computing: theorems. IEEE Trans. Fuzzy Syst. **5**, 108–127 (1978)
202. Nisan, N.: Bidding and allocation in combinatorial auctions. In: Proceedings of the 2nd ACM Conference on Electronic Commerce, pp. 1–12. ACM, New York (2000)
203. Nitzan, S.I., Pattanaik, P.K.: Median-based extensions of an ordering over a set to the power set: An axiomatic characterization* 1. J. Econ. Theory **34**(2), 252–261 (1984)
204. Nute, D.: Defeasible Deontic Logic. Kluwer Academic, Dordrecht (1997)
205. Oloriz, E., Candeal, J.C., Induráin, E.: Representability of interval orders. J. Econ. Theory **78**, 219–227 (1998)
206. Orlovsky, S.A.: Decision making with a fuzzy preference relation. Fuzzy Sets Syst. **1**, 155–167 (1978)
207. Ovchinnikov, S.N., Roubens, M.: On strict preference relations. Fuzzy Sets Syst. **43**, 319–326 (1991)
208. Öztürk, M.: Logical and mathematical structures for interval comparisons. Ph.D. thesis, Université Paris Dauphine, Paris (2005)
209. Öztürk, M., Marquis, P.: Representing interval orders by weighted bases: some complexity results. Math. Soc. Sci. **57**, 367–388 (2009)
210. Öztürk, M., Tsoukiàs, A.: Modelling uncertain positive and negative reasons in decision aiding. Decis. Support. Syst. **43**(4), 1512–1526 (2007)

211. Öztürk, M., Tsoukiàs, A.: Bipolar preference modelling and aggregation in decision support. Int. J. Intell. Syst. **23**, 970–984 (2008)
212. Öztürk, M., Pirlot, M., Tsoukiàs, A.: Representing preferences using intervals. Artif. Intell. **175**(7–8), 1194–1222 (2011)
213. Pattanaik, P.K., Peleg, B.: An axiomatic characterization of the lexicographic maximin extension of an ordering over a set to the power set. Soc. Choice Welf. **1**(2), 113–122 (1984)
214. Pattanaik, P.K., Xu, Y.: On ranking opportunity sets in terms of freedom of choice. Rech. Écon. Louvain **56**(3–4), 383–390 (1990)
215. Pattanaik, P.K., Xu, Y.: On diversity and freedom of choice. Math. Soc. Sci. **40**(2), 123–130 (2000)
216. Pawlak, Z., Słowiński, R.: Decision analysis using rough sets. Int. Trans. Oper. Res. **1**, 107–114 (1994)
217. Pe'er, I., Shamir, R.: Satisfiability problems on intervals and unit intervals. Theor. Comput. Sci. **175**, 349–372 (1997)
218. Peirce, C.S.: On the algebra of logic. Am. J. Math. **3**, 15–57 (1880)
219. Peirce, C.S.: On the logic of number. Am. J. Math. **4**, 85–95 (1881)
220. Peirce, C.S.: A Theory of Probable Inference, pp. 187–203. John Hopkins University, Baltimore (1883)
221. Perny, P.: Multicriteria filtering methods based on concordance/non-discordance principles. Ann. Oper. Res. **80**, 137–167 (1998)
222. Perny, P., Roubens, M.: Fuzzy preference modelling. In: Słowiński, R. (ed.) Fuzzy Sets in Decision Analysis, Operations Research and Statistics, pp. 3–30. Kluwer Academic, Dordrecht (1998)
223. Perny, P., Roy, B.: The use of fuzzy outranking relations in preference modelling. Fuzzy Sets Syst. **49**, 33–53 (1992)
224. Perny, P., Spanjaard, O.: Preference-based search in state space graphs. In: Proceedings of the AAAI'2002 Conference, Edmonton, pp. 751–756 (2002)
225. Perny, P., Spanjaard, O.: A preference-based approach to spanning trees and shortest paths problems. Eur. J. Oper. Res. **162**(3), 584–601 (2005)
226. Perny, P., Tsoukiàs, A.: On the continuous extension of a four valued logic for preference modelling. In: Proceedings of the IPMU 1998 Conference, Paris, pp. 302–309 (1998)
227. Pinkas, G.: Reasoning, nonmonotonicity and learning in connectionist networks that capture propositional knowledge. Artif. Intell. **77**, 203–247 (1995)
228. Pirlot, M.: Minimal representation of a semiorder. Theor. Decis. **28**, 109–141 (1990)
229. Pirlot, M., Vincke, P.: Semi Orders. Kluwer Academic, Dordrecht (1997)
230. Poincaré, H.: La Valeur de la Science. Flammarion, Paris (1905)
231. Puppe, C.: An axiomatic approach to preference for freedom of choice. J. Econ. Theory **68**(1), 174–199 (1996)
232. Puppe, C.: Individual freedom and social choice. In: Fleurbaey, M., Laslier, J.-F., Gravel, N., Trannoy, A. (eds.) Freedom in Economics: New Perspectives in Normative Analysis, pp. 49–68. Routledge, London (1997)
233. Rescher, N.: Semantic foundations for the logic of preference. In: Rescher, N. (ed.) The Logic of Decision and Action, pp. 37–62. University of Pittsburgh, Pittsburgh (1967)
234. Rescher, N.: Introduction to Value Theory. Prentice Hall, Englewood Cliffs (1969)
235. Roberts, F.S.: Measurement Theory, with Applications to Decision Making, Utility and the Social Sciences. Addison-Wesley, Boston (1979)
236. Rosenbaum, E.F.: On measuring freedom. J. Theor. Polit. **12**(2), 205 (2000)
237. Roth, A.E.: The college admissions problem is not equivalent to the marriage problem. J. Econ. Theory **36**(2), 277–288 (1985)
238. Roubens, M., Vincke, P.: On families of semiorders and interval orders imbedded in a valued structure of preference: a survey. Inf. Sci. **34**, 187–198 (1984)
239. Roubens, M., Vincke, P.: Preference Modelling. LNEMS, vol. 250. Springer, Berlin (1985)

240. Roubens, M., Vincke, P.: Fuzzy possibility graphs and their application to ranking fuzzy numbers. In: Kacprzyk, J., Roubens, M. (eds.) Non-conventional Preference Relations in Decision Making. LNEMS, vol. 301, pp. 119–128. Springer, Berlin (1988)
241. Roy, B.: Partial preference analysis and decision aid: the fuzzy outranking relation concept. In: Bell, D.E., Keeney, R.L., Raiffa, H. (eds.) Conflicting Objectives in Decisions, pp. 40–75. Wiley, New York (1977)
242. Roy, B.: Multicriteria Methodology for Decision Aiding. Kluwer Academic, Dordrecht (1996). First edition in French. Economica, Paris (1985)
243. Roy, B.: Main sources of inaccurate determination, uncertainty and imprecision in decision models. In: Munier, B., Shakun, M. (eds.) Compromise, Negotiation and Group Decision, pp. 43–67. Reidel, Dordrecht (1988)
244. Roy, B.: The outranking approach and the foundations of ELECTRE methods. Theor. Decis. **31**, 49–73 (1991)
245. Roy, B., Bouyssou, D.: Aide Multicritère à la Décision: Méthodes et Cas. Economica, Paris (1993)
246. Roy, B., Vincke, P.: Relational systems of preference with one or more pseudo-criteria: some new concepts and results. Manag. Sci. **30**, 1323–1335 (1984)
247. Roy, B., Vincke, P.: Pseudo-orders: definition, properties and numerical representation. Math. Soc. Sci. **14**, 263–274 (1987)
248. Saint-Cyr, D., Dupin, F., Lang, Jérôme, Schiex, Thomas.: Penalty logic and its link with Dempster-Shafer theory. Proceedings of the Tenth international conference on Uncertainty in artificial intelligence, pp. 204–211. Morgan Kaufmann San Francisco (1994)
249. Salo, A., Keisler, J., Morton, A.: Portfolio Management. Springer, Berlin (2011)
250. Samuelson, P.A.: General proof that diversification pays. J. Financ. Quant. Anal. **2**(1), 1–13 (1967)
251. Schweizer, B., Sklar, A.: Probabilistic Metric Spaces. Elsevier Science, New York (1983)
252. Scott, D.: Some ordered sets in computer science. In: Rival, I. (ed.) Ordered Sets, pp. 677–718. D. Reidel, Dordrecht (1982)
253. Scott, D., Suppes, P.: Foundational aspects of theories of measurement. J. Symb. Log. **23**, 113–128 (1958)
254. Sen, A.K.: Social choice theory. In: Arrow, K.J., Intriligator, M.D. (eds.) Handbook of Mathematical Economics, vol. 3, pp. 1073–1181. North-Holland, Amsterdam (1986)
255. Shapley, L.: A value for n-person games. In: Kuhn, H., Tucker, A.W. (eds.) Contributions to the Theory of Games II, pp. 307–317. Princeton University Press, Princeton (1953)
256. Shoham, Y.: Nonmonotonic logic: meaning and utility. In: Proceedings of the 10th International Joint Conference in Artificial Intelligence, IJCAI87, pp. 388–393. Morgan Kaufmann, San Francisco (1987)
257. Slater, P.: Inconsistencies in a schedule of paired comparisons. Biometrika **48**, 303–312 (1961)
258. Słowiński, R.: Fuzzy Sets in Decision Analysis, Operations Research and Statistics. Kluwer Academic, Dordrecht (1998)
259. Subiza, B.: Numerical representation of acyclic preferences. J. Math. Psychol. **38**, 467–476 (1994)
260. Tanguiane, A.S.: Aggregation and Representation of Preferences. Springer, Berlin (1991)
261. Tarski, A.: Contributions to the theory of models i, ii. Indag. Math. **16**, 572–588 (1954)
262. Tarski, A.: Contributions to the theory of models iii. Indag. Math. **17**, 56–64 (1955)
263. Tatman, J.A., Shachter, R.D.: Dynamic programming and influence diagrams. IEEE Trans. Syst. Man Cybern. **20**(2), 365–379 (1990)
264. Thomason, R., Horty, J.: Logics for inheritance theory. In: Reinfrank, M., de Kleer, J., Ginsberg, M.L., Sandewall, E. (eds.) Non-monotonic Reasoning. Lecture Notes in Artificial Intelligence, vol. 346, pp. 220–237. Springer, Berlin (1987)
265. Trotter, W.T.: Combinatorics and Partially Ordered Sets. John Hopkins University Press, Baltimore (1992)

266. Tsoukiàs, A.: Preference modelling as a reasoning process: a new way to face uncertainty in multiple criteria decision support systems. Eur. J. Oper. Res. **55**, 309–318 (1991)
267. Tsoukiàs, A.: Sur la généralisation des concepts de concordance et discordance en aide multicritr̀e à la décision. In: Mémoire présenté pour l'obtention de l'habilitation à diriger des recherches, Université Paris Dauphine, Paris (1997). Appeared also as Document du LAMSADE no. 117
268. Tsoukiàs, A.: A first-order, four valued, weakly paraconsistent logic and its relation to rough sets semantics. Found. Comput. Decis. Sci. **12**, 85–108 (2002)
269. Tsoukiàs, A., Vincke, P.: A survey on non conventional preference modelling. Ricerca Operativa **61**, 5–49 (1992)
270. Tsoukiàs A., Vincke, P.: A new axiomatic foundation of partial comparability. Theor. Decis. **39**, 79–114 (1995)
271. Tsoukiàs, A., Vincke, P.: Extended preference structures in MCDA. In: Climaco, J. (ed.) Multicriteria Analysis, pp. 37–50. Springer, Berlin (1997)
272. Tsoukiàs, A., Vincke, P.: Double threshold orders: a new axiomatization. J. Multi-Criteria Decis. Anal. **7**, 285–301 (1998)
273. Tsoukiàs, A., Vincke, P.: A characterization of PQI interval orders. Discret. Appl. Math. **127**, 387–397 (2003)
274. Tsoukiàs, A., Perny, P., Vincke, P.: From concordance/discordance to the modelling of positive and negative reasons in decision aiding. In: Bouyssou, D., Jacquet-Lagrèze, E., Perny, P., Słowiński, R., Vanderpooten, D., Vincke, P. (eds.) Aiding Decisions with Multiple Criteria: Essays in Honour of Bernard Roy, pp. 147–174. Kluwer Academic, Dordrecht (2002)
275. Turunen, E., Öztürk, M., Tsoukiàs, A.: Paraconsistent semantics for pavelka style fuzzy-sentential logic. Fuzzy Sets Syst. **161**, 1926–1940 (2010)
276. Tversky, A.: Intransitivity of preferences. Psychol. Rev. **76**, 31–48 (1969)
277. Uckelman, J.: More than the sum of its parts: compact preference representation over combinatorial domains. Ph.D. thesis, University of Amsterdam (2009). ILLC Publication DS-2009-12.
278. Valadares Tavares, L.: Generalized transitivity and preferences modelling: the concept of hyper-order. Eur. J. Oper. Res. **36**, 14–26 (1988)
279. van Dalen, D.: Logic and Structure. Springer, Berlin (1983)
280. Van De Walle, B., De Baets, B., Kerre, E.: Recent advances in fuzzy preference modelling. In: Ruan, D., D'hondt, P., Govaerts, P., Kerre, E.E. (eds.) Intelligent Systems and Soft Computing for Nuclear Science and Industry pp. 98–104. World Scientific, Publishing, Singapore Singapore, (1996)
281. Van De Walle, B., De Baets, B., Kerre, E.: Charaterizable fuzzy preference structures. Ann. Oper. Res. **80**, 105–136 (1998)
282. Van Hees, M.: On the analysis of negative freedom. Theor. Decis. **45**(2), 175–197 (1998)
283. Vincke, P.: P,Q,I preference structures. In: Kacprzyk, J., Roubens, M. (eds.) Non conventional Preference Relations in Decision Making. LNEMS, vol. 301, pp. 72–81. Springer, Berlin (1988)
284. Vincke, P.: Multicriteria Decision-Aid. Wiley, New York (1992)
285. Vincke, P.: Preferences and numbers. In: Colorni, A., Paruccini, M., Roy, B. (eds.) A-MCD-A - Aide Multi Critère à la Décision: Multiple Criteria Decision Aiding, pp. 343–354. Joint Research Center, The European Commission, Luxembourg (2001)
286. Vind, K.: Independent preferences. J. Math. Econ. **20**, 119–135 (1991)
287. von Winterfeldt, D., Edwards, W.: Decision Analysis and Behavioral Research. Cambridge University Press, Cambridge (1986)
288. von Wright, G.: The Logic of Preference. Edinburgh University Press, Edinburgh (1963)
289. von Wright, G.: The logic of preference reconsidered. Theor. Decis. **3**, 140–169 (1972)
290. Weber, S.: A general concept of fuzzy connectives, negations and implications based on t-norms and t-conorms. Fuzzy Sets Syst. **11**(2), 115–134 (1983)
291. Widmeyer, G.R.: Logic modeling with partially ordered preferences. Decis. Support Syst. **4**, 87–95 (1988)

292. Widmeyer, G.R.: Reasoning with preferences and values. Decis. Support Syst. **6**, 183–191 (1990)
293. Wilson, N.: Extending CP-nets with stronger conditional preference statements. In: Proceedings of the 19th National Conference on Artificial Intelligence (AAAI-04), San Jose. AAAI Press, San Jose (2004)
294. Yager, R.R.: On the general class of fuzzy connectives. Fuzzy Sets Syst. **4**(3), 235–242 (1980)
295. Yager, R.R.: A procedure for ordering fuzzy subsets of the unit interval. Inf. Sci. **24**, 143–161 (1981)
296. Yu, Y.D.: Triangular norms and tnf-sigma-algebras. Fuzzy Sets Syst. **16**(3), 251–264 (1985)
297. Yu, W.: Aide multicritère à la décision dans le cadre de la problématique du tri: méthodes et applications. Ph.D. thesis, LAMSADE, Université Paris Dauphine, Paris (1992)
298. Zadeh, L.A.: Fuzzy sets. Inf. Control **8**, 338–353 (1965)
299. Zadeh, L.A.: The concept of a linguistic variable and its application to approximate reasoning. Inf. Sci. **1**, 119–249 (1975)
300. Zhai, C.X., Cohen, W.W., Lafferty, J.: Beyond independent relevance: methods and evaluation metrics for subtopic retrieval. In: Proceedings of the 26th Annual International ACM SIGIR Conference on Research and Development in Information Retrieval, pp. 10–17. ACM, New York (2003)
301. Zimmermann, H.J.: Fuzzy Set Theory and Its Applications. Kluwer Academic, Dordrecht (1985)
302. Zopounidis, C.: Multicriteria decision aid in financial management. Eur. J. Oper. Res. **119**(2), 404–415 (1999)
303. Zopounidis, C., Doumpos, M.: Multicriteria classification and sorting methods: a literature review. Eur. J. Oper. Res. **138**, 229–246 (2002)

Chapter 4
Conjoint Measurement Tools for MCDM

A Brief Introduction

Denis Bouyssou and Marc Pirlot

Abstract This paper offers a brief and nontechnical introduction to the use of conjoint measurement in multiple criteria decision making. The emphasis is on the, central, additive value function model. We outline its axiomatic foundations and present various possible assessment techniques to implement it. Some extensions of this model, e.g., nonadditive models or models tolerating intransitive preferences are then briefly reviewed.

Keywords Conjoint measurement • Additive value function • Preference modelling

4.1 Introduction and Motivation

Conjoint measurement is a set of tools and results first developed in Economics [63] and Psychology [179] in the beginning of the 1960s. Its, ambitious, aim is to provide measurement techniques that would be adapted to the needs of the Social Sciences in which, most often, multiple dimensions have to be taken into account.

Soon after its development, people working in decision analysis realized that the techniques of conjoint measurement could also be used as tools to structure preferences [72, 209]. This is the subject of this paper which offers a brief and nontechnical introduction to conjoint measurement models and their use in multiple criteria decision making. More detailed treatments may be found in [87, 103, 155, 173, 259]. Advanced references include [81, 164, 261]. This text is a slightly updated version of [42].

D. Bouyssou (✉)
CNRS LAMSADE UMR 7243 and Université Paris Dauphine, 75775 Paris Cedex 16, France
e-mail: bouyssou@lamsade.dauphine.fr

M. Pirlot
Faculté Polytechnique, Université de Mons, 9, rue de Houdain, 7000 Mons, Belgium
e-mail: marc.pirlot@umons.ac.be

S. Greco et al. (eds.), *Multiple Criteria Decision Analysis*, International Series in Operations Research & Management Science 233,
DOI 10.1007/978-1-4939-3094-4_4

4.1.1 Conjoint Measurement Models in Decision Theory

The starting point of most works in decision theory is a binary relation $\succsim$ on a set A of objects. This binary relation is usually interpreted as an "at least as good as" relation between alternative courses of action gathered in A.

Manipulating a binary relation can be quite cumbersome as soon as the set of objects is large. Therefore, it is not surprising that many works have looked for a *numerical representation* of the binary relation $\succsim$. The most obvious numerical representation amounts to associate a real number $V(a)$ to each object $a \in A$ in such a way that the comparison between these numbers faithfully reflects the original relation $\succsim$. This leads to defining a real-valued function V on A, such that:

$$a \succsim b \Leftrightarrow V(a) \geq V(b), \tag{4.1}$$

for all $a, b \in A$. When such a numerical representation is possible, one can use V instead of $\succsim$ and, e.g., apply classical optimization techniques to find the most preferred elements in A given $\succsim$. We shall call such a function V a *value function*.

It should be clear that not all binary relations $\succsim$ may be represented by a value function. Condition (4.1) imposes that $\succsim$ is complete (i.e., $a \succsim b$ or $b \succsim a$, for all $a, b \in A$) and transitive (i.e., $a \succsim b$ and $b \succsim c$ imply $a \succsim c$, for all $a, b, c \in A$). When A is finite or countably infinite, it is well-known [81, 164] that these two conditions are, in fact, not only necessary but also sufficient to build a value function satisfying (4.1).

Remark 1. The general case is more complex since (4.1) implies, for instance, that there must be "enough" real numbers to distinguish objects that have to be distinguished. The necessary and sufficient conditions for (4.1) can be found in [81, 164]. An advanced treatment is [20]. Sufficient conditions that are well-adapted to cases frequently encountered in Economics can be found in [61, 64]; see [53] for a synthesis. •

It is vital to note that, when a value function satisfying (4.1) exists, it is by no means unique. Taking any increasing function ϕ on $\mathbb{R}$, it is clear that $\phi \circ V$ gives another acceptable value function. A moment of reflection will convince the reader that only such transformations are acceptable and that if V and U are two real-valued functions on A satisfying (4.1), they must be related by an increasing transformation. In other words, a value function in the sense of (4.1) defines an *ordinal scale*.

Ordinal scales, although useful, do not allow the use of sophisticated assessment procedures, i.e., of procedures that allow an analyst to assess the relation $\succsim$ through a structured dialogue with the decision-maker. This is because the knowledge that $V(a) \geq V(b)$ is strictly equivalent to the knowledge of $a \succsim b$ and no inference can be drawn from this assertion besides the use of transitivity.

It is therefore not surprising that much attention has been devoted to numerical representations leading to more constrained scales. Many possible avenues have been explored to do so. Among the most well-known, let us mention:

- the possibility to compare *probability distributions* on the set A [81, 257]. If it is required that, not only (4.1) holds but that the numbers attached to the objects should be such that their expected values reflect the comparison of probability distributions on the set of objects, a much more constrained numerical representation clearly obtains,
- the introduction of "preference difference" comparisons of the type: the difference between a and b is larger than the difference between c and d, see [63, 105, 159, 164, 202, 224, 247]. If it is required that, not only (4.1) holds, but that the differences between numbers also reflect the comparisons of preference differences, a more constrained numerical representation obtains.

When objects are evaluated according to several dimensions, i.e., when $\succsim$ is defined on a product set, new possibilities emerge to obtain numerical representations that would specialize (4.1). The purpose of conjoint measurement is to study such kinds of models.

There are many situations in decision theory which call for the study of binary relations defined on product sets. Among them let us mention, following [261]:

- *Multiple criteria decision making* using a preference relation comparing alternatives evaluated on several attributes [22, 49, 51, 155, 206, 217, 259],
- *Decision under uncertainty* using a preference relation comparing alternatives evaluated on several states of nature [92, 139, 221, 228, 260, 261],
- *Consumer theory* manipulating preference relations for bundles of several goods [62],
- *Intertemporal decision making* using a preference relation between alternatives evaluated at several moments in time [155, 160, 161],
- *Inequality measurement* comparing distributions of wealth across several individuals [5, 24, 25, 267].

The purpose of this paper is to give an introduction to the main models of conjoint measurement useful in multiple criteria decision making. The results and concepts that are presented may however be of interest in all of the afore-mentioned areas of research.

Remark 2. Restricting ourselves to applications in multiple criteria decision making will not allow us to cover every aspect of conjoint measurement. Among the most important topics left aside, let us mention: the introduction of statistical elements in conjoint measurement models [75, 142] and the test of conjoint measurement models in experiments [149, 173, 178]. •

Given a binary relation $\succsim$ on a product set $X = X_1 \times X_2 \times \cdots \times X_n$, the theory of conjoint measurement consists in finding conditions under which it is possible to build a convenient numerical representation of $\succsim$ and to study the uniqueness of this representation. The central model is the *additive value function* model in which:

$$x \succsim y \Leftrightarrow \sum_{i=1}^{n} v_i(x_i) \geq \sum_{i=1}^{n} v_i(y_i) \tag{4.2}$$

where v_i are real-valued functions, called *partial value functions*, on the sets X_i and it is understood that $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$. Clearly if $\succsim$ has a representation in model (4.2), taking any common increasing transformation of the v_i will *not* lead to another representation in model (4.2).

Specializations of this model in the above-mentioned areas give several central models in decision theory:

- The Subjective Expected Utility model, in the case of decision-making under uncertainty,
- The discounted utility model for dynamic decision making,
- Inequality measures *à la* Atkinson/Sen in the area of social welfare.

The axiomatic analysis of this model is now quite firmly established [63, 164, 261]; this model forms the basis of many decision analysis techniques [103, 155, 259, 261]. This is studied in Sects. 4.3 and 4.4 after we introduce our main notation and definitions in Sect. 4.2.

Remark 3. One possible objection to the study of model (4.2) is that the choice of an *additive* model seems arbitrary and restrictive. It should be observed here that the functions v_i will precisely be assessed so that additivity holds. Furthermore, the use of a simple model may be seen as an advantage in view of the limitations of the cognitive abilities of most human beings.

It is also useful to notice that this model can be reformulated so as to make addition disappear. Indeed if there are partial value functions v_i such that (4.2) holds, it is clear that $V = \sum_{i=1}^{n} v_i$ is a value function satisfying (4.1). Since V defines an ordinal scale, taking the exponential of V leads to another valid value function W. Clearly W has now a multiplicative form:

$$x \succsim y \Leftrightarrow W(x) = \prod_{i=1}^{n} w_i(x_i) \geq W(y) = \prod_{i=1}^{n} w_i(y_i).$$

where $w_i(x_i) = \mathrm{e}^{v_i(x_i)}$.

The reader is referred to [70] for the study of situations in which V defines a scale that is more constrained than an ordinal scale, e.g., because it is supposed to reflect preference differences or because it allows to compute expected utilities. In such cases, the additive form (4.2) is no more equivalent to the multiplicative form considered above. •

In Sect. 4.5 we present a number of extensions of this model going from nonadditive representations of transitive relations to model tolerating intransitive indifference and, finally, nonadditive representations of nontransitive relations.

Remark 4. In this paper, we shall restrict our attention to the case in which alternatives may be evaluated on the various attributes without risk or uncertainty. Excellent overviews of these cases may be found in [155, 259]; recent references include [181, 192]. •

Before starting our study of conjoint measurement oriented towards MCDM, it is worth recalling that conjoint measurement aims at establishing measurement models in the Social Sciences. To many, the very notion of "measurement in the Social Sciences" may appear contradictory. It may therefore be useful to briefly consider how the notion of measurement can be modelled in Physics, an area in which the notion of "measurement" seems to arise quite naturally, and to explain how a "measurement model" may indeed be useful in order to structure preferences.

4.1.2 An Aside: Measuring Length

Physicists usually take measurement for granted and are not particularly concerned with the technical and philosophical issues it raises (at least when they work within the realm of Newtonian Physics). However, for a Social Scientist, these questions are of utmost importance. It may thus help to have an idea of how things appear to work in Physics before tackling more delicate cases.

Suppose that you are on a desert island and that you want to "measure" the length of a collection of rigid straight rods. Note that we do not discuss here the "pre-theoretical" intuition that "length" is a property of these rods that can be measured, as opposed, say, to their softness or their beauty.

A first simple step in the construction of a measure of length is to place the two rods side by side in such a way that one of their extremities is at the same level (see Fig. 4.1). Two things may happen: either the upper extremities of the two rods coincide or not. This seems to be the simplest way to devise an experimental procedure leading to the discovery of which rod "has more length" than the other.

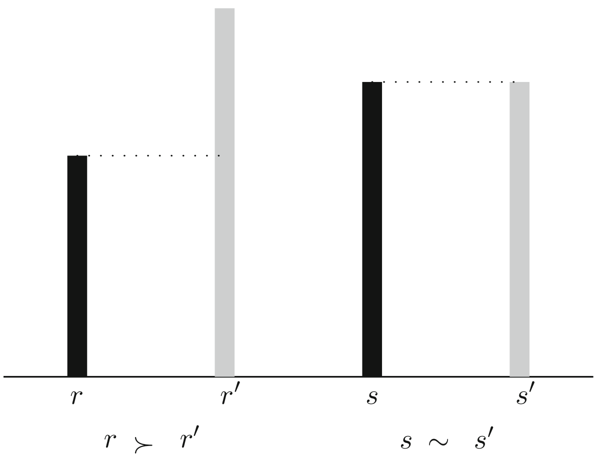


Fig. 4.1 Comparing the length of two rods

Technically, this leads to defining two binary relations $\succ$ and $\sim$ on the set of rods in the following way:

$$r \succ r' \text{ when the extremity of } r \text{ is higher than the extremity of } r',$$
$$r \sim r' \text{ when the extremities of } r \text{ and } r' \text{ are at the same level.}$$

Clearly, if length is a quality of the rods that can be measured, it is expected that these pairwise comparisons are somehow consistent, e.g.,

- if $r \succ r'$ and $r' \succ r''$, it should follow that $r \succ r''$,
- if $r \sim r'$ and $r' \sim r''$, it should follow that $r \sim r''$,
- if $r \sim r'$ and $r' \succ r''$, it should follow that $r \succ r''$.

Although quite obvious, these consistency requirements are stringent. For instance, the second and the third conditions are likely to be violated if the experimental procedure involves some imprecision, e.g if two rods that slightly differ in length are nevertheless judged "equally long". They represent a form of *idealization* of what could be a perfect experimental procedure.

With the binary relations $\succ$ and $\sim$ at hand, we are still rather far from a full-blown measure of length. It is nevertheless possible to assign numbers to each of the rods in such a way that the comparison of these numbers reflects what has been obtained experimentally. When the consistency requirements mentioned above are satisfied, it is indeed generally possible to build a real-valued function Φ on the set of rods that would satisfy:

$$r \succ r' \Leftrightarrow \Phi(r) > \Phi(r') \text{ and}$$
$$r \sim r' \Leftrightarrow \Phi(r) = \Phi(r').$$

If the experiment is costly or difficult to perform, such a numerical assignment may indeed be useful because it summarizes, once for all, what has been obtained in experiments. Clearly there are many possible ways to assign numbers to rods in this way. Up to this point, they are equally good for our purposes. The reader will easily check that defining $\succsim$ as $\succ$ or $\sim$, the function Φ is noting else than a "value function" for length: any increasing transformation may therefore be applied to Φ.

The next major step towards the construction of a measure of length is the realization that it is possible to form new rods by simply placing two or more rods "in a row", i.e., you may *concatenate* rods. From the point of view of length, it seems obvious to expect this concatenation operation $\circ$ to be "commutative" ($r \circ s$ has the same length as $s \circ r$) and associative ($(r \circ s) \circ t$ has the same length as $r \circ (s \circ t)$).

You clearly want to be able to measure the length of these composite objects and you can always include them in our experimental procedure outlined above (see Fig. 4.2). Ideally, you would like your numerical assignment Φ to be somehow compatible with the concatenation operation: knowing the numbers assigned to two rods, you want to be able to deduce the number assigned to their concatenation. The most obvious way to achieve that is to require that the numerical assignment of

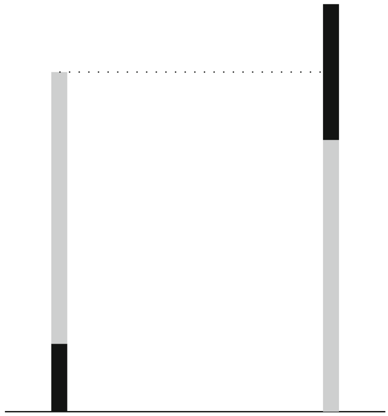


Fig. 4.2 Comparing the length of composite rods

a composite object can be deduced by addition from the numerical assignments of the objects composing it, i.e., that

$$\Phi(r \circ r') = \Phi(r) + \Phi(r').$$

This clearly places many additional constraints on the results of your experiment. An obvious one is that $\succ$ and $\sim$ should be compatible with the concatenation operation $\circ$, e.g.,

$$r \succ r' \text{ and } t \sim t' \text{ should lead to } r \circ t \succ r' \circ t'.$$

These new constraints may or may not be satisfied. When they are, the usefulness of the numerical assignment Φ is even more apparent: a simple arithmetic operation will allow us to infer the result of an experiment involving composite objects.

Let us take a simple example. Suppose that you have five rods $r_1, r_2, \ldots, r_5$ and that, because space is limited, you can only concatenate at most two rods and that not all concatenations are possible. Let us suppose, for the moment, that you do not have much technology available so that you may only experiment using *different* rods. You may well collect the following information, using obvious notation exploiting the transitivity of $\succ$ which holds in this experiment,

$$r_1 \circ r_5 \succ r_3 \circ r_4 \succ r_1 \circ r_2 \succ r_5 \succ r_4 \succ r_3 \succ r_2 \succ r_1.$$

Your problem is then to find a numerical assignment Φ to rods such that using an addition operation, you can infer the numerical assignment of composite objects consistently with your observations. Let us consider the following three assignments:

	Φ	Φ'	Φ''
r_1	14	10	14
r_2	15	91	16
r_3	20	92	17
r_4	21	93	18
r_5	28	100	29

These three assignments are equally valid to reflect the comparisons of single rods. Only the first and the third allow to capture the comparisons of composite objects that were performed. Note that, going from Φ to Φ'' does not involve just changing the "unit of measurement": since $\Phi(r_1) = \Phi''(r_1)$ this would imply that $\Phi = \Phi''$, which is clearly false.

Such numerical assignments have limited usefulness. Indeed, it is tempting to use them to predict the result of comparisons that we have not been able to perform. But this turns out to be quite disappointing: using Φ you would conclude that $r_2 \circ r_3 \sim r_1 \circ r_4$ since $\Phi(r_2) + \Phi(r_3) = 15 + 20 = 35 = \Phi(r_1) + \Phi(r_4)$, but, using Φ'', you would conclude that $r_2 \circ r_3 \succ r_1 \circ r_4$ since $\Phi''(r_2) + \Phi''(r_3) = 16 + 17 = 33$ while $\Phi''(r_1) + \Phi''(r_4) = 14 + 18 = 32$.

Intuitively, "measuring" calls for some kind of a *standard* (e.g., the "Mètre-étalon" that can be found in the Bureau International des Poids et Mesures in Sèvres, near Paris). This implies choosing an appropriate "standard" rod *and* being able to prepare perfect copies of this standard rod (we say here "appropriate" because the choice of a standard should be made in accordance with the lengths of the objects to be measured: a tiny or a huge standard will not facilitate experiments). Let us call s_0 the standard rod. Let us suppose that you have been able to prepare a large number of perfect copies $s_1, s_2, \ldots$ of s_0. We therefore have:

$$s_0 \sim s_1, s_0 \sim s_2, s_0 \sim s_3, \ldots$$

Let us also agree that the length of s_0 is 1. This is your, arbitrary, unit of length. How can you use s_0 and its perfect copies so as to determine unambiguously the length of any other (simple or composite) object? Quite simply, you may prepare a "standard sequence of length n", $S(n) = s_1 \circ s_2 \circ \ldots \circ s_{n-1} \circ s_n$, i.e., a composite object that is made by concatenating n perfect copies of our standard rod s_0. The length of a standard sequence of length n is exactly n since we have concatenated n objects that are perfect copies of the standard rod of length 1. Take any rod r and let us compare r with several standard sequences of increasing length: $S(1), S(2), \ldots$.

Two cases may arise. There may be a standard sequence $S(k)$ such that $r \sim S(k)$. In that case, we know that the number $\Phi(r)$ assigned to r must be exactly k. This is unlikely however. The most common situation is that we will find two consecutive standard sequences $S(k-1)$ and $S(k)$ such that $r \succ S(k-1)$ and $S(k) \succ r$ (see Fig. 4.3). This means that $\Phi(r)$ must be such that $k - 1 < \Phi(r) < k$. We seem to be in trouble here since, as before, $\Phi(r)$ is not exactly determined. How can you proceed? This depends on your technology for preparing perfect copies.

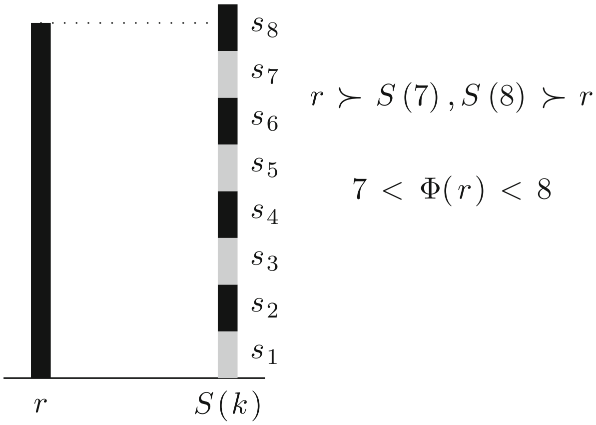


Fig. 4.3 Using standard sequences

Imagine that you are able to prepare perfect copies not only of the standard rod but also of any object. You may then prepare several copies $(r_1, r_2, \ldots)$ of the rod r. You can now compare a composite object made out of two perfect copies of r with your standard sequences $S(1), S(2), \ldots$. As before, you shall eventually arrive at locating $\Phi(r_1 \circ r_2) = 2\Phi(r)$ within an interval of width 1. This means that the interval of imprecision surrounding $\Phi(r)$ has been divided by two. Continuing this process, considering longer and longer sequences of perfect copies of r, you will keep on reducing the width of the interval containing $\Phi(r)$. This means that you can approximate $\Phi(r)$ with any given level of precision. Mathematically, a unique value for $\Phi(r)$ will be obtained using a simple argument.

Supposing that you are in position to prepare perfect copies of any object is a strong technological requirement. When this is not possible, there still exists a way out. Instead of preparing a perfect copy of r you may also try to increase the granularity of your standard sequence. This means building an object t that you would be able to replicate perfectly and such that concatenating t with one of its perfect replicas gives an object that has exactly the length of the standard object s_0, i.e., $\Phi(t) = 1/2$. Considering standard sequences based on t, you will be able to increase by a factor 2 the precision with which we measure the length of r. Repeating the process, i.e., subdividing t, will lead, as before, to a unique limiting value for $\Phi(r)$.

The mathematical machinery underlying the measurement process informally described above (called "extensive measurement") rests on the theory of ordered groups. It is beautifully described and illustrated in [164]. Although the underlying principles are simple, we may expect complications to occur, e.g., when not all concatenations are feasible, when there is some level (say the velocity of light if we were to measure speed) that cannot be exceeded or when it comes to relate different measures. See [164, 177, 212] for a detailed treatment.

Clearly, this was an overly detailed and unnecessary complicated description of how length could be measured. Since our aim is to eventually deal with "measurement" in the Social Sciences, it may however be useful to keep the above process in mind. Its basic ingredients are the following:

- well-behaved relations $\succ$ and $\sim$ allowing to compare objects,
- a concatenation operation $\circ$ allowing to consider composite objects,
- consistency requirements linking $\succ$, $\sim$ and $\circ$,
- the ability to prepare perfect copies of some objects in order to build standard sequences.

Basically, conjoint measurement is a quite ingenious way to perform related measurement operations when no concatenation operation is available. This will however require that objects can be evaluated along several dimensions. Before explaining how this might work, it is worth explaining the context in which such measurement might prove useful.

Remark 5. It is often asserted that "measurement is impossible in the Social Sciences" precisely because the Social Scientist has no way to define a concatenation operation. Indeed, it would seem hazardous to try to concatenate the intelligence of two subjects or the pain of two patients (see [77, 138]). Under certain conditions, the power of conjoint measurement will precisely be to provide a means to bypass this absence of readily available concatenation operation when the objects are evaluated on several dimensions.

Let us remark that, even when there seems to be a concatenation operation readily available, it does not always fit the purposes of extensive measurement [211]. Consider for instance an individual expressing preferences for the quantity of the two goods he consumes. The objects therefore take the well structured form of points in the positive orthant of $\mathbb{R}^2$. There seems to be an obvious concatenation operation here: $(x, y) \circ (z, w)$ might simply be taken to be $(x + y, z + w)$. However a fairly rational person, consuming pants and jackets, may indeed prefer $(3, 0)$ (three pants and no jacket) to $(0, 3)$ (no pants and three jackets) but at the same time prefer $(3, 3)$ to $(6, 0)$. This implies that these preferences cannot be explained by a measure that would be additive with respect to the concatenation operation consisting in adding the quantities of the two goods consumed. Indeed $(3, 0) \succ (0, 3)$ implies $\Phi(3, 0) > \Phi(0, 3)$, which implies $\Phi(3, 0) + \Phi(3, 0) > \Phi(0, 3) + \Phi(3, 0)$. Additivity with respect to concatenation should then imply that $(3, 0) \circ (3, 0) \succ (0, 3) \circ (3, 0)$, that is $(6, 0) \succ (3, 3)$. •

4.1.3 An Example: Even Swaps

The even swaps technique described and advocated in [155, 157, 209] is a simple way to deal with decision problems involving several attributes that does not have recourse to a formal representation of preferences, which will be the subject of

Table 4.1 Evaluation of the five offices on the five attributes

	a	*b*	*c*	*d*	*e*
Commute	45	25	20	25	30
Clients	50	80	70	85	75
Services	A	B	C	A	C
Size	800	700	500	950	700
Cost	1850	1700	1500	1900	1750

conjoint measurement. Because this technique is simple and may be quite useful, we describe it below using the same example as in [157]. This will also allow us to illustrate the type of problems that are dealt with in decision analysis applications of conjoint measurement.

Example 6 (Even Swaps Technique). A consultant considers renting a new office. Five different locations have been identified after a careful consideration of many possibilities, rejecting all those that do not meet a number of requirements.

His feeling is that five distinct characteristics, we shall say five attributes, of the possible locations should enter into his decision: his daily commute time (expressed in minutes), the ease of access for his clients (expressed as the percentage of his present clients living close to the office), the level of services offered by the new office (expressed on an ad hoc scale with three levels: *A* (all facilities available), *B* (telephone and fax), *C* (no facilities)), the size of the office expressed in square feet, and the monthly cost expressed in dollars.

The evaluation of the five offices is given in Table 4.1. The consultant has well-defined preferences on each of these attributes, independently of what is happening on the other attributes. His preference increases with the level of access for his clients, the level of services of the office and its size. It decreases with commute time and cost. This gives a first easy way to compare alternatives through the use of *dominance*.

An alternative y is dominated by an alternative x if x is at least as good as y on *all* attributes while being strictly better for at least one attribute. Clearly dominated alternatives are not candidate for the final choice and may, thus, be dropped from consideration. The reader will easily check that, on this example, alternative b dominates alternative e: e and b have similar size but b is less expensive, involves a shorter commute time, an easier access to clients and a better level of services. We may therefore forget about alternative e. This is the only case of "pure dominance" in our table. It is however easy to see that d is "close" to dominating a, the only difference in favor of a being on the cost attribute ($50 per month). This is felt more than compensated by the differences in favor of d on all other attributes: commute time (20 min), client access (35 %) and size (150 ft^2).

Dropping all alternatives that are not candidate for choice, this initial investigation allows us to reduce the problem to:

	b	c	d
Commute	25	20	25
Clients	80	70	85
Services	B	C	A
Size	700	500	950
Cost	1700	1500	1900

A natural way to proceed is then to assess tradeoffs. Observe that all alternatives but c have a common evaluation on commute time. We may therefore ask the consultant, starting with office c, what gain on client access would compensate a loss of 5 min on commute time. We are looking for an alternative c' that would be evaluated as follows:

	c	c'
Commute	20	**25**
Clients	70	$\mathbf{70 + \delta}$
Services	C	C
Size	500	500
Cost	1500	1500

and judged indifferent to c. Although this is not an easy question, it is clearly crucial in order to structure preferences.

Remark 7. In this paper, we do not consider the possibility of lexicographic preferences, in which such tradeoffs do not occur, see [83, 84, 204]. Lexicographic preferences may also be combined with the possibility of "local" tradeoffs, see [30, 88, 174]. •

Remark 8. Since tradeoffs questions may be difficult, it is wise to start with an attribute requiring few assessments (in the example, all alternatives but one have a common evaluation on commute time). Clearly this attribute should be traded against one with an underlying "continuous" structure (cost, in the example). •

Suppose that the answer is that for $\delta = 8$, it is reasonable to assume that c and c' would be indifferent. This means that the decision table can be reformulated as follows:

	b	c'	d
Commute	25	25	25
Clients	80	78	85
Services	B	C	A
Size	700	500	950
Cost	1700	1500	1900

It is then apparent that all alternatives have a similar evaluation on the first attribute which, therefore, is not useful to discriminate between alternatives and may be forgotten. The reduced decision table is as follows:

	b	c'	d
Clients	80	78	85
Services	*B*	*C*	*A*
Size	700	500	950
Cost	1700	1500	1900

There is no case of dominance in this reduced table. Therefore further simplification calls for the assessment of new tradeoffs. Using cost as the reference attribute, we then proceed to "neutralize" the service attribute. Starting with office c', this means asking for the increase in monthly cost that the consultant would just be prepared to pay to go from level "*C*" of service to level "*B*". Suppose that this increase is roughly \$250. This defines alternative c''. Similarly, starting with office d we ask for the reduction of cost that would exactly compensate a reduction of services from "*A*" to "*B*". Suppose that the answer is \$100 a month, which defines alternative d'. The decision table is reshaped as:

	b	c''	d'
Clients	80	78	85
Services	*B*	**B**	**B**
Size	700	500	950
Cost	1700	**1750**	**1800**

We may forget about the second attribute which does not discriminate any more between alternatives. When this is done, it is apparent that c'' is dominated by b and can be suppressed. Therefore, the decision table at this stage looks like the following:

	b	d'
Clients	80	85
Size	700	950
Cost	1700	1800

Unfortunately, this table reveals no case of dominance. New tradeoffs have to be assessed. We may now ask, starting with office b, what additional cost the consultant would be ready to incur to increase its size by 250 ft^2. Suppose that the rough answer is \$250 a month, which defines b'. We are now facing the following table:

	b'	d'
Clients	80	85
Size	**950**	950
Cost	**1950**	1800

Attribute size may now be dropped from consideration. But, when this is done, it is clear that d' dominates b'. Hence it seems obvious to recommend office d as the final choice.

The above process is simple and looks quite obvious. If this works, why be interested at all in "measurement" if the idea is to help someone to come up with a decision?

First observe that in the above example, the set of alternatives was relatively small. In many practical situations, the set of objects to compare is much larger than the set of alternatives in our example. Using the even swaps technique could then require a considerable number of difficult tradeoff questions. Furthermore, as the output of the technique is not a preference model but just the recommendation of an alternative in a given set, the appearance of new alternatives (e.g., because a new office is for rent) would require starting a new round of questions. This is likely to be highly frustrating. Finally, the informal even swaps technique may not be well adapted to the, many, situations, in which the decision under study takes place in a complex organizational environment. In such situations, having a formal model to be able to communicate and to convince is an invaluable asset. Such a model will furthermore allow to conduct extensive sensitivity analysis and, hence, to deal with imprecision both in the evaluations of the objects to compare and in the answers to difficult questions concerning tradeoffs.

This clearly leaves room for a more formal approach to structure preferences. But where can "measurement" be involved in the process? It should be observed that, beyond surface, there are many analogies between the even swaps process and the measurement of length considered above.

First, note that, in both cases, objects are compared using binary relations. In the measurement of length, the binary relation $\succ$ reads "is longer than". Here it reads "is preferred to". Similarly, the relation $\sim$ reading before "has equal length" now reads "is indifferent to". We supposed in the measurement of length process that $\succ$ and $\sim$ would nicely combine in experiments: if $r \succ r'$ and $r' \sim r''$ then we should observe that $r \succ r''$. Implicitly, a similar hypothesis was made in the even swaps technique. To realize that this is the case, it is worth summarizing the main steps of the argument.

We started with Table 4.1. Our overall recommendation was to rent office d. This means that we have reasons to believe that d is preferred to all other potential locations, i.e., $d \succ a$, $d \succ b$, $d \succ c$, and $d \succ e$. How did we arrive logically at such a conclusion?

Based on the initial table, using dominance and quasi-dominance, we concluded that b was preferable to e and that d was preferable to a. Using symbols, we have $b \succ e$ and $d \succ a$. After assessing some tradeoffs, we concluded, using dominance, that $b \succ c''$. But remember, c'' was built so as to be indifferent to c' and, in turn, c' was built so as to be indifferent to c. That is, we have $c'' \sim c'$ and $c' \sim c$.

Later, we built an alternative d' that is indifferent to d ($d \sim d'$) and an alternative b' that is indifferent to b ($b \sim b'$). We then concluded, using dominance, that d' was preferable to b' ($d' \succ b'$). Hence, we know that:

$$d \succ a, b \succ e,$$

$$c'' \sim c', c' \sim c, b \succ c'',$$

$$d \sim d', b \sim b', d' \succ b'.$$

Using the consistency rules linking $\succ$ and $\sim$ that we considered for the measurement of length, it is easy to see that the last line implies $d \succ b$. Since $b \succ e$, this implies $d \succ e$. It remains to show that $d \succ c$. But the second line leads to, combining $\succ$ and $\sim$, $b \succ c$. Therefore $d \succ b$ leads to $d \succ c$ and we are home. Hence, we have used the same properties for preference and indifference as the properties of "is longer than" and "has equal length" that we hypothesized in the measurement of length.

Second it should be observed that expressing tradeoffs leads, indirectly, to equating the "length" of "preference intervals" on different attributes. Indeed, remember how c' was constructed above: saying that c and c' are indifferent more or less amounts to saying that the interval [25, 20] on commute time has exactly the same "length" as the interval [70, 78] on client access. Consider an alternative f that would be identical to c except that it has a client access at 78 %. We may again ask which increase in client access would compensate a loss of 5 min on commute time. In a tabular form we are now comparing the following two alternatives:

	f	f'
Commute	20	25
Clients	78	$78 + \delta$
Services	C	C
Size	500	500
Cost	1500	1500

Suppose that the answer is that for $\delta = 10$, f and f' would be indifferent. This means that the interval [25, 20] on commute time has exactly the same length as the interval [78, 88] on client access. Now, we know that the preference intervals [70, 78] and [78, 88] have the same "length". Hence, tradeoffs provide a means to equate two preference intervals on the same attribute. This brings us quite close to the construction of standard sequences. This, we shall shortly do.

How does this information about the "length" of preference intervals relate to judgements of preference or indifference? Exactly as in the even swaps technique. You can use this measure of "length" modifying alternatives in such a way that they only differ on a single attribute and then use a simple dominance argument.

Conjoint measurement techniques may roughly be seen as a formalization of the even swaps technique that leads to building a numerical model of preferences much in the same way that we built a numerical model for length. This will require assessment procedures that will rest on the same principles as the standard sequence technique used for length. This process of "measuring preferences" is not an easy one. It will however lead to a numerical model of preference that will not only allow us to make a choice within a limited number of alternatives but that can serve as an input of computerized optimization algorithms that will be able to deal with much more complex cases.

4.2 Definitions and Notation

Before entering into the details of how conjoint measurement may work, a few definitions and notation will be needed.

4.2.1 Binary Relations

A *binary relation* $\succsim$ on a set A is a subset of $A \times A$. We write $a \succsim b$ instead of $(a, b) \in \succsim$. A binary relation $\succsim$ on A is said to be:

- *reflexive* if $[a \succsim a]$,
- *complete* if $[a \succsim b$ or $b \succsim a]$,
- *symmetric* if $[a \succsim b] \Rightarrow [b \succsim a]$,
- *asymmetric* if $[a \succsim b] \Rightarrow [Not[b \succsim a]]$,
- *transitive* if $[a \succsim b$ and $b \succsim c] \Rightarrow [a \succsim c]$,
- *negatively transitive* if $[Not[a \succsim b]$ and $Not[b \succsim c]] \Rightarrow Not[a \succsim c]$,

for all $a, b, c \in A$.

The *asymmetric* (resp. *symmetric*) part of $\succsim$ is the binary relation $\succ$ (resp. $\sim$) on A defined letting, for all $a, b \in A$, $a \succ b \Leftrightarrow [a \succsim b$ and $Not(b \succsim a)]$ (resp. $a \sim b \Leftrightarrow [a \succsim b$ and $b \succsim a]$). A similar convention will hold when $\succsim$ is subscripted and/or superscripted.

A *weak order* (resp. an *equivalence relation*) is a complete and transitive (resp. reflexive, symmetric and transitive) binary relation. For a detailed analysis of the use of binary relation as tools for preference modelling we refer to [4, 81, 90, 205, 211, 213]. The weak order model underlies the examples that were presented in the introduction. Indeed, the reader will easily prove the following.

Proposition 9. *Let $\succsim$ be a weak order on A. Then:*

- *$\succ$ is transitive,*
- *$\succ$ is negatively transitive,*
- *$\sim$ is transitive,*
- *$[a \succ b$ and $b \sim c] \Rightarrow a \succ c$,*
- *$[a \sim b$ and $b \succ c] \Rightarrow a \succ c$,*

for all $a, b, c \in A$.

4.2.2 Binary Relations on Product Sets

In the sequel, we consider a set $X = \prod_{i=1}^{n} X_i$ with $n \geq 2$. Elements $x, y, z, \ldots$ of X will be interpreted as alternatives evaluated on a set $N = \{1, 2, \ldots, n\}$ of attributes. A typical binary relation on X is still denoted as $\succsim$, interpreted as an "at least as good as" preference relation between multi-attributed alternatives with $\sim$ interpreted as indifference and $\succ$ as strict preference.

For any nonempty subset J of the set of attributes N, we denote by X_J (resp. X_{-J}) the set $\prod_{i \in J} X_i$ (resp. $\prod_{i \notin J} X_i$). With customary abuse of notation, (x_J, y_{-J}) will denote the element $w \in X$ such that $w_i = x_i$ if $i \in J$ and $w_i = y_i$ otherwise. When $J = \{i\}$ we shall simply write X_{-i} and (x_i, y_{-i}).

Remark 10. Throughout this paper, we shall work with a binary relation defined on a product set. This setup conceals the important work that has to be done in practice to make it useful:

- the structuring of objectives [3, 22, 23, 152–154, 200, 207],
- the definition of adequate attributes to measure the attainment of objectives [104, 122, 151, 156, 217, 258, 266],
- the definition of an adequate family of attributes [32, 155, 217, 218, 259],
- the modelling of uncertainty, imprecision and inaccurate determination [31, 49, 155, 215].

The importance of this "preliminary" work should not be forgotten in what follows. •

4.2.3 Independence and Marginal Preferences

In conjoint measurement, one starts with a preference relation $\succsim$ on X. It is then of vital importance to investigate how this information makes it possible to define preference relations on attributes or subsets of attributes.

Let $J \subseteq N$ be a nonempty set of attributes. We define the *marginal relation* $\succsim_J$ induced on X_J by $\succsim$ letting, for all $x_J, y_J \in X_J$:

$$x_J \succsim_J y_J \Leftrightarrow (x_J, z_{-J}) \succsim (y_J, z_{-J}), \text{ for all } z_{-J} \in X_{-J},$$

with asymmetric (resp. symmetric) part $\succ_J$ (resp. $\sim_J$). When $J = \{i\}$, we often abuse notation and write $\succsim_i$ instead of $\succsim_{\{i\}}$. Note that if $\succsim$ is reflexive (resp. transitive), the same will be true for $\succsim_J$. This is clearly not true for completeness however.

Definition 11 (Independence). Consider a binary relation $\succsim$ on a set $X = \prod_{i=1}^{n} X_i$ and let $J \subseteq N$ be a nonempty subset of attributes. We say that $\succsim$ is independent for J if, for all $x_J, y_J \in X_J$,

$$[(x_J, z_{-J}) \succsim (y_J, z_{-J}), \text{ for some } z_{-J} \in X_{-J}] \Rightarrow x_J \succsim_J y_J.$$

If $\succsim$ is independent for all nonempty subsets of N, we say that $\succsim$ is *independent*. If $\succsim$ is independent for all subsets containing a single attribute, we say that $\succsim$ is *weakly independent*.

In view of (4.2), it is clear that the additive value model will require that $\succsim$ is independent. This crucial condition says that common evaluations on some attributes do not influence preference. Whereas independence implies weak independence, it is well-know that the converse is not true [261].

Remark 12. Under certain conditions, e.g., when X is adequately "rich", it is not necessary to test that a weak order $\succsim$ is independent for J, for *all* $J \subseteq N$ in order to know that $\succsim$ is independent, see [28, 113, 155]. This is often useful in practice. •

Remark 13. Weak independence is referred to as "weak separability" in [261]; in Sect. 4.5, we use "weak separability" (and "separability") with a different meaning. •

Remark 14. Independence, or at least weak independence, is an almost universally accepted hypothesis in multiple criteria decision making. It cannot be overemphasized that it is easy to find examples in which it is inadequate.

If a meal is described by the two attributes, main course and wine, it is highly likely that most gourmets will violate independence, preferring red wine with beef and white wine with fish. Similarly, in a dynamic decision problem, a preference for variety will often lead to violating independence: you may prefer Pizza to Steak, but your preference for meals today (first attribute) and tomorrow (second attribute) may well be such that (Pizza, Steak) preferred to (Pizza, Pizza), while (Steak, Pizza) is preferred to (Steak, Steak).

Many authors [154, 217, 259] have argued that such failures of independence were almost always due to a poor structuring of attributes (e.g., in our choice of meal example above, preference for variety should be explicitly modelled). •

When $\succsim$ is a weakly independent weak order, marginal preferences are well-behaved and combine so as to give meaning to the idea of dominance that we already encountered. The proof of the following is left to the reader as an easy exercise.

Proposition 15. *Let $\succsim$ be a weakly independent weak order on $X = \prod_{i=1}^{n} X_i$. Then:*

- *$\succsim_i$ is a weak order on X_i,*
- *$[x_i \succsim_i y_i$, for all $i \in N] \Rightarrow x \succsim y$,*
- *$[x_i \succsim_i y_i$, for all $i \in N$ and $x_j \succ_j y_j$ for some $j \in N] \Rightarrow x \succ y$,*

for all $x, y \in X$.

4.3 The Additive Value Model in the "Rich" Case

The purpose of this section and the following is to present the conditions under which a preference relation on a product set may be represented by the additive value function model (4.2) and how such a model can be assessed. We begin here with the case that most closely resembles the measurement of length described in Sect. 4.1.2.

4.3.1 Outline of Theory

When the structure of X is supposed to be "adequately rich", conjoint measurement is a quite clever adaptation of the process that we described in Sect. 4.1.2 for the measurement of length. What will be measured here are the "length" of preference intervals on an attribute using a preference interval on another attribute as a standard.

4.3.1.1 The Case of Two Attributes

Consider first the two attribute case. Hence the relation $\succsim$ is defined on a set $X = X_1 \times X_2$. Clearly, in view of (4.2), we need to suppose that $\succsim$ is an *independent weak order*. Consider two levels $x_1^0, x_1^1 \in X_1$ on the first attribute such that $x_1^1 \succ_1 x_1^0$, i.e., x_1^1 is preferable to x_1^0. This makes sense because, we supposed that $\succsim$ is *independent*. Note also that we shall have to exclude the case in which all levels on the first attribute would be indifferent in order to be able to find such levels.

Choose any $x_2^0 \in X_2$. The, arbitrarily chosen, element $(x_1^0, x_2^0) \in X$ will be our "reference point". The basic idea is to use this reference point and the "unit" on the first attribute given by the reference preference interval $[x_1^0, x_1^1]$ to build a standard sequence on the preference intervals on the second attribute. Hence, we are looking for an element $x_2^1 \in X_2$ that would be such that:

$$(x_1^0, x_2^1) \sim (x_1^1, x_2^0). \tag{4.3}$$

Clearly this will require the structure of X_2 to be adequately "rich" so as to find the level $x_2^1 \in X_2$ such that the reference preference interval on the first attribute $[x_1^0, x_1^1]$ is exactly matched by a preference interval of the same "length" on the second attribute $[x_2^0, x_2^1]$. Technically, this calls for a solvability assumption or, more restrictively, for the supposition that X_2 has a (topological) structure that is close to that of an interval of $\mathbb{R}$ and that $\succsim$ is "somehow" continuous.

If such a level x_2^1 can be found, model (4.2) implies:

$$\begin{aligned} v_1(x_1^0) + v_2(x_2^1) &= v_1(x_1^1) + v_2(x_2^0) \text{ so that} \\ v_2(x_2^1) - v_2(x_2^0) &= v_1(x_1^1) - v_1(x_1^0). \end{aligned} \tag{4.4}$$

Let us fix the origin of measurement letting:

$$v_1(x_1^0) = v_2(x_2^0) = 0,$$

and our unit of measurement letting:

$$v_1(x_1^1) = 1 \text{ so that } v_1(x_1^1) - v_1(x_1^0) = 1.$$

Using (4.4), we therefore obtain $v_2(x_2^1) = 1$. We have therefore found an interval between levels on the second attribute ($[x_2^0, x_2^1]$) that exactly matches our reference interval on the first attribute ($[x_1^0, x_1^1]$). We may proceed to build our standard sequence on the second attribute (see Fig. 4.4) asking for levels $x_2^2, x_2^3, \ldots$ such that:

$$\begin{aligned} (x_1^0, x_2^2) &\sim (x_1^1, x_2^1), \\ (x_1^0, x_2^3) &\sim (x_1^1, x_2^2), \\ &\ldots \\ (x_1^0, x_2^k) &\sim (x_1^1, x_2^{k-1}). \end{aligned}$$

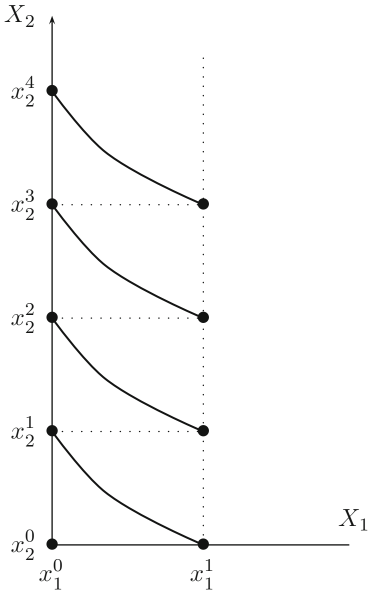


Fig. 4.4 Building a standard sequence on X_2

As above, using (4.2) leads to:

$$v_2(x_2^2) - v_2(x_2^1) = v_1(x_1^1) - v_1(x_1^0),$$
$$v_2(x_2^3) - v_2(x_2^2) = v_1(x_1^1) - v_1(x_1^0),$$
$$\dots$$
$$v_2(x_2^k) - v_2(x_2^{k-1}) = v_1(x_1^1) - v_1(x_1^0),$$

so that:

$$v_2(x_2^2) = 2, v_2(x_2^3) = 3, \dots, v_2(x_2^k) = k.$$

This process of building a standard sequence of the second attribute therefore leads to defining v_2 on a number of, carefully, selected elements of X_2.

Remember the standard sequence that we built for length in Sect. 4.1.2. An implicit hypothesis was that the length of any rod could be exceeded by the length of a composite object obtained by concatenating a sufficient number of perfect copies of a standard rod. Such an hypothesis is called "Archimedean" since it mimics the property of the real numbers saying that for any positive real numbers x, y it is true that $nx > y$ for some integer n, i.e., y, no matter how large, may always be exceeded by taking any x, no matter how small, and adding it with itself and repeating the operation a sufficient number of times. Clearly, we will need a similar hypothesis here. Failing it, there might exist a level $y_2 \in X_2$ that will never be "reached" by our standard sequence, i.e., such that $y_2 \succ_2 x_2^k$, for $k = 1, 2, \dots$. For measurement models in which this Archimedean condition is omitted, see [197, 242].

Remark 16. At this point a good exercise for the reader is to figure out how we may extend the standard sequence to cover levels of X_2 that are "below" the reference level x_2^0. This should not be difficult.

Now that a standard sequence is built on the second attribute, we may use any part of it to build a standard sequence on the first attribute. This will require finding levels $x_1^2, x_1^3, \dots \in X_1$ such that (see Fig. 4.5):

$$(x_1^2, x_2^0) \sim (x_1^1, x_2^1),$$
$$(x_1^3, x_2^0) \sim (x_1^2, x_2^1),$$
$$\dots$$
$$(x_1^k, x_2^0) \sim (x_1^{k-1}, x_2^1).$$

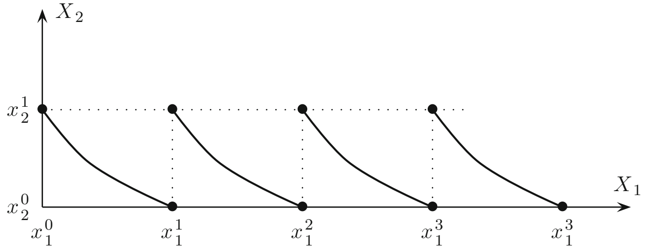


Fig. 4.5 Building a standard sequence on X_1

Using (4.2) leads to:

$$v_1(x_1^2) - v_1(x_1^1) = v_2(x_2^1) - v_2(x_2^0),$$
$$v_1(x_1^3) - v_1(x_1^2) = v_2(x_2^1) - v_2(x_2^0),$$
$$\dots$$
$$v_1(x_1^k) - v_1(x_1^{k-1}) = v_2(x_2^1) - v_2(x_2^0),$$

so that:

$$v_1(x_1^2) = 2, v_1(x_1^3) = 3, \dots, v_1(x_1^k) = k.$$

As was the case for the second attribute, the construction of such a sequence will require the structure of X_1 to be adequately rich, which calls for a solvability assumption. An Archimedean condition will also be needed in order to be sure that all levels of X_1 can be reached by the sequence.

We have defined a "grid" in X (see Fig. 4.6) and we have $v_1(x_1^k) = k$ and $v_2(x_2^k) = k$ for all elements of this grid. Intuitively such numerical assignments seem to define an adequate additive value function on the grid. We have to prove that this intuition is correct. Let us first verify that, for all integers $\alpha, \beta, \gamma, \delta$:

$$\alpha + \beta = \gamma + \delta = \epsilon \Rightarrow (x_1^\alpha, x_2^\beta) \sim (x_1^\gamma, x_2^\delta). \tag{4.5}$$

When $\epsilon = 1$, (4.5) holds by construction because we have: $(x_1^0, x_2^1) \sim (x_1^1, x_2^0)$. When $\epsilon = 2$, we know that $(x_1^0, x_2^2) \sim (x_1^1, x_2^1)$ and $(x_1^2, x_2^0) \sim (x_1^1, x_2^1)$ and the claim is proved using the transitivity of $\sim$.

Consider the $\epsilon = 3$ case. We have $(x_1^0, x_2^3) \sim (x_1^1, x_2^2)$ and $(x_1^0, x_2^3) \sim (x_1^1, x_2^2)$. It remains to be shown that $(x_1^2, x_2^1) \sim (x_1^1, x_2^2)$ (see the dotted arc in Fig. 4.6). This does not seem to follow from the previous conditions that we more or less explicitly used: transitivity, independence, "richness", Archimedean. Indeed, it does not. Hence, we have to suppose that: $(x_1^2, x_2^0) \sim (x_1^0, x_2^2)$ and $(x_1^0, x_2^1) \sim (x_1^1, x_2^0)$ imply $(x_1^2, x_2^1) \sim (x_1^1, x_2^2)$. This condition, called the Thomsen condition, is clearly necessary for (4.2). The above reasoning easily extends to all points on the grid, using weak ordering, independence and the Thomsen condition. Hence, (4.5) holds on the grid.

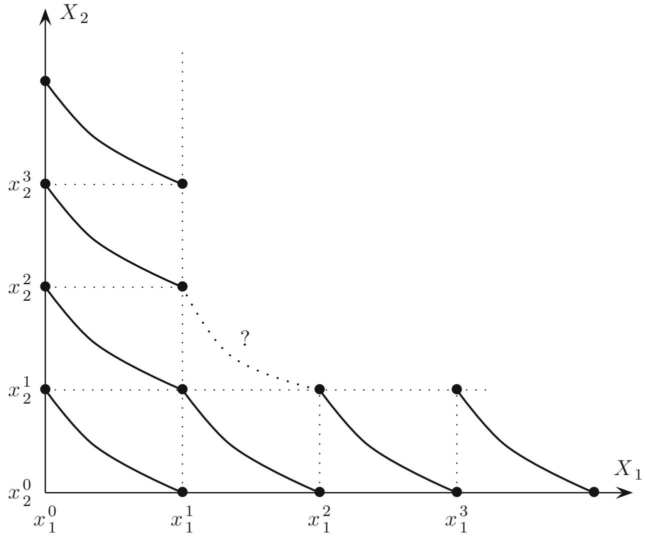


Fig. 4.6 The grid

It remains to show that:

$$\epsilon = \alpha + \beta > \epsilon' = \gamma + \delta \Rightarrow (x_1^{\alpha}, x_2^{\beta}) \succ (x_1^{\gamma}, x_2^{\delta}). \tag{4.6}$$

Using transitivity, it is sufficient to show that (4.6) holds when $\epsilon = \epsilon' + 1$. By construction, we know that $(x_1^1, x_2^0) \succ (x_1^0, x_2^0)$. Using independence this implies that $(x_1^1, x_2^k) \succ (x_1^0, x_2^k)$. Using (4.5) we have $(x_1^1, x_2^k) \sim (x_1^{k+1}, x_2^0)$ and $(x_1^0, x_2^k) \sim (x_1^k, x_2^0)$. Therefore we have $(x_1^{k+1}, x_2^0) \succ (x_1^k, x_2^0)$, the desired conclusion.

Hence, we have built an additive value function of a suitably chosen grid (see Fig. 4.7). The logic of the assessment procedure is then to assess more and more points somehow considering more finely grained standard sequences. The two techniques evoked for length may be used here depending on the underlying structure of X. Going to the limit then unambiguously defines the functions v_1 and v_2. Clearly such v_1 and v_2 are intimately related. Once we have chosen an arbitrary reference point (x_1^0, x_2^0) and a level x_1^1 defining the unit of measurement, the process just described entirely defines v_1 and v_2. It follows that the only possible transformations that can be applied to v_1 and v_2 is to multiply both by the same positive number α and to add to both a, possibly different, constant. This is usually summarized saying that v_1 and v_2 define interval scales with a common unit.

The above reasoning is a rough sketch of the proof of the existence of an additive value function when $n = 2$, as well as a sketch of how it could be assessed. Careful readers will want to refer to [81, 164, 261].

Remark 17. The measurement of length through standard sequences described above leads to a scale that is unique once the unit of measurement is chosen. At this

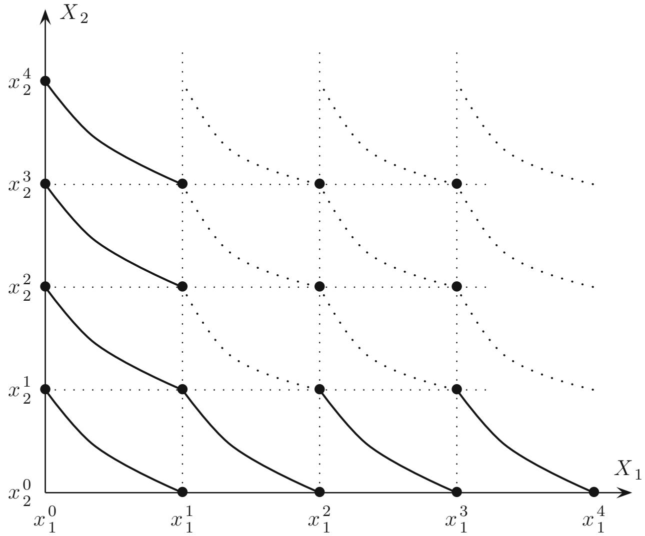


Fig. 4.7 The entire grid

point, a good exercise for the reader is to find an intuitive explanation to the fact that, when measuring the "length" of preference intervals, the origin of measurement becomes arbitrary. The analogy with the measurement of duration on the one hand and dates, as given in a calendar, on the other hand should help. •

Remark 18. As was already the case with the even swaps technique, it is worth emphasizing that this assessment technique makes no use of the vague notion of the "importance" of the various attributes. The "importance" is captured here in the lengths of the preference intervals on the various attributes.

A common but critical mistake is to confuse the additive value function model (4.2) with a weighted average and to try to assess weights asking whether an attribute is "more important" than another. This makes no sense. •

4.3.1.2 The Case of More Than Two Attributes

The good news is that the process is exactly the same when there are more than two attributes. With one surprise: the Thomsen condition is no more needed to prove that the standard sequences defined on each attribute lead to an adequate value function on the grid. A heuristic explanation of this strange result is that, when $n = 2$, there is no difference between independence and weak independence. This is no more true when $n \geq 3$ and assuming independence is much stronger than just assuming weak independence.

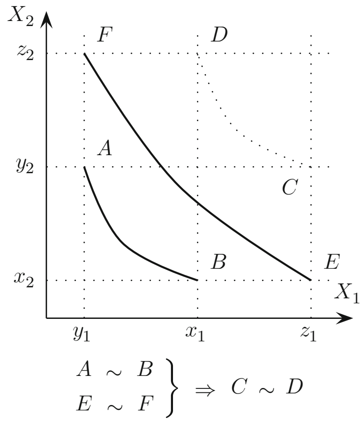


Fig. 4.8 The Thomsen condition

4.3.2 *Statement of Results*

We use below the "algebraic approach" [162, 164, 179]. A more restrictive approach using a topological structure on X is given in [63, 81, 261]. We formalize below the conditions informally introduced in the preceding section. The reader not interested in the precise statement of the results or, better, having already written down his own statement, may skip this section.

Definition 19 (Thomsen Condition). Let $\succsim$ be a binary relation on a set $X = X_1 \times X_2$. It is said to satisfy the Thomsen condition if

$$(x_1, x_2) \sim (y_1, y_2) \text{ and } (y_1, z_2) \sim (z_1, x_2) \Rightarrow (x_1, z_2) \sim (z_1, y_2),$$

for all $x_1, y_1, z_1 \in X_1$ and all $x_2, y_2, z_2 \in X_2$.

Figure 4.8 shows how the Thomsen condition uses two "indifference curves" (i.e., curves linking points that are indifferent) to place a constraint on a third one. This was needed above to prove that an additive value function existed on our grid. Remember that the Thomsen condition is only needed when $n = 2$; hence, we only stated it in this case.

Definition 20 (Standard Sequences). A standard sequence on attribute $i \in N$ is a set $\{a_i^k : a_i^k \in X_i, k \in K\}$ where K is a set of consecutive integers (positive or negative, finite or infinite) such that there are $x_{-i}, y_{-i} \in X_{-i}$ satisfying *Not*$[x_{-i} \sim_{-i} y_{-i}]$ and $(a_i^k, x_{-i}) \sim (a_i^{k+1}, y_{-i})$, for all $k \in K$.

A standard sequence on attribute $i \in N$ is said to be *strictly bounded* if there are $b_i, c_i \in X_i$ such that $b_i \succ_2 a_i^k \succ_2 c_i$, for all $k \in K$. It is then clear that, when model (4.2) holds, any strictly bounded standard sequence must be finite.

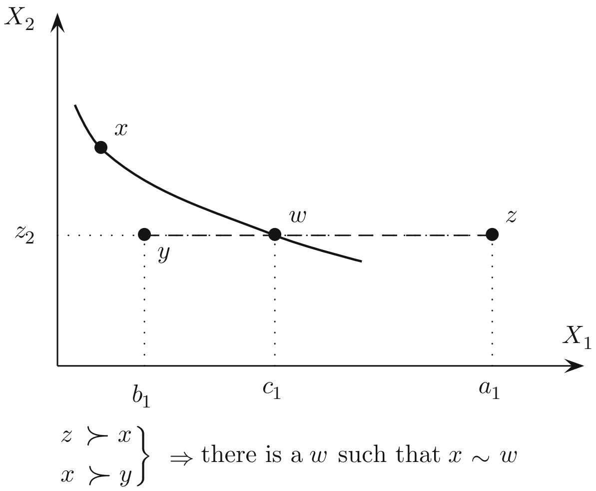


Fig. 4.9 Restricted Solvability on X_1

Definition 21 (Archimedean). For all $i \in N$, any strictly bounded standard sequence on $i \in N$ is finite.

The following condition rules out the case in which a standard sequence cannot be built because all levels are indifferent.

Definition 22 (Essentiality). Let $\succsim$ be a binary relation on a set $X = X_1 \times X_2 \times \cdots \times X_n$. Attribute $i \in N$ is said to be essential if $(x_i, a_{-i}) \succ (y_i, a_{-i})$, for some $x_i, y_i \in X_i$ and some $a_{-i} \in X_{-i}$.

Definition 23 (Restricted Solvability). Let $\succsim$ be a binary relation on a set $X = X_1 \times X_2 \times \cdots \times X_n$. Restricted solvability is said to hold with respect to attribute $i \in N$ if, for all $x \in X$, all $z_{-i} \in X_{-i}$ and all $a_i, b_i \in X_i$, $[(a_i, z_{-i}) \succsim x \succsim (b_i, z_{-i})] \Rightarrow [x \sim (c_i, z_{-i})$, for some $c_i \in X_i]$.

Remark 24. Restricted solvability is illustrated in Fig. 4.9 in the case where $n = 2$. It says that, given any $x \in X$, if it is possible find two levels $a_i, b_i \in X_i$ such that when combined with a certain level $z_{-i} \in X_{-i}$ on the other attributes, (a_i, z_{-i}) is preferred to x and x is preferred to (b_i, z_{-i}), it should be possible to find a level c_i, "in between" a_i and b_i, such that (c_i, z_{-i}) is exactly indifferent to x.

A much stronger hypothesis is unrestricted solvability asserting that for all $x \in X$ and all $z_{-i} \in X_{-i}$, $x \sim (c_i, z_{-i})$, for some $c_i \in X_i$. Its use leads however to much simpler proofs [81, 110].

It is easy to imagine situations in which restricted solvability might hold while unrestricted solvability would fail. Suppose, e.g., that a firm has to choose between several investment projects, two attributes being the Net Present Value (NPV) of the projects and their impact on the image of the firm in the public. Consider a project consisting in investing in the software market. It has a reasonable NPV and no adverse consequences on the image of the firm. Consider another project that could have dramatic consequences on the image of the firm, because it leads to investing the market of cocaine. Unrestricted solvability would require that by sufficiently increasing the NPV of the second project it would become indifferent to the more standard project of investing in the software market. This is not required by restricted solvability. •

We are now in position to state the central results concerning model (4.2). Proofs may be found in [164, 263].

Theorem 25 (Additive Value Function When $n = 2$). *Let $\succsim$ be a binary relation on a set $X = X_1 \times X_2$. If restricted solvability holds on all attributes and each attribute is essential then $\succsim$ has a representation in model (4.2) if and only if $\succsim$ is an independent weak order satisfying the Thomsen and the Archimedean conditions.*

Furthermore in this representation, v_1 and v_2 are interval scales with a common unit, i.e., if v_1, v_2 and w_1, w_2 are two pairs of functions satisfying (4.2), there are real numbers α, β_1, β_2 with $\alpha > 0$ such that, for all $x_1 \in X_1$ and all $x_2 \in X_2$

$$v_1(x_1) = \alpha w_1(x_1) + \beta_1 \text{ and } v_2(x_2) = \alpha w_2(x_2) + \beta_2.$$

When $n \geq 3$ and at least three attributes are essential, the above result simplifies in that the Thomsen condition can now be omitted.

Theorem 26 (Additive Value Function When $n \geq 3$). *Let $\succsim$ be a binary relation on a set $X = X_1 \times X_2 \times \ldots \times X_n$ with $n \geq 3$. If restricted solvability holds on all attributes and at least three attributes are essential then $\succsim$ has a representation in model (4.2) if and only if $\succsim$ is an independent weak order satisfying the Archimedean condition.*

Furthermore in this representation $v_1, v_2, \ldots, v_n$ are interval scales with a common unit.

Remark 27. As mentioned in introduction, the additive value model is central to several fields in decision theory. It is therefore not surprising that much energy has been devoted to analyze variants and refinements of the above results. Among the most significant ones, let us mention:

- the study of cases in which solvability holds only on some or none of the attributes [99, 109–112, 146, 147, 196],
- the study of the relation between the "algebraic approach" introduced above and the topological one used in [63], see e.g., [150, 158, 261, 263].

The above results are only valid when X is the entire Cartesian product of the sets X_i. Results in which X is a subset of the whole Cartesian product $X_1 \times X_2 \times \ldots \times X_n$ are not easy to obtain, see [56, 225] (the situation is "easier" in the special case of homogeneous product sets, see [262, 265]).

In [34, 35, 255, 256] additive representations are obtained on the basis of twofold ordered partitions of X. These primitives are less rich than a preference relation of X. •

4.3.3 Implementation: Standard Sequences and Beyond

We have already shown above how additive value functions can be assessed using the standard sequence technique. It is worth recalling here some of the characteristics of this assessment procedure:

- It requires the set X_i to be *rich* so that it is possible to find a preference interval on X_i that will exactly match a preference interval on another attribute. This excludes using such an assessment procedure when some of the sets X_i are discrete.
- It relies on *indifference* judgements which, a priori, are less firmly established than preference judgements.
- It relies on judgements concerning fictitious alternatives which, a priori, are harder to conceive than judgements concerning real alternatives.
- The various assessments are thoroughly intertwined and, e.g., an imprecision on the assessment of x_2^1, i.e., the endpoint of the first interval in the standard sequence on X_2 (see Fig. 4.4) will propagate to many assessed values.
- The assessment of tradeoffs may be plagued with cognitive biases, see [65, 246].

The assessment procedure based on standard sequences is therefore rather demanding; this should be no surprise given the proximity between this form of measurement and extensive measurement illustrated above on the case of length. Hence, the assessment procedure based on standard sequences seems to be seldom used in the practice of decision analysis [155, 259]. The literature on the experimental assessment of additive value functions, see e.g., [246, 258, 266], suggests that this assessment is a difficult task that may be affected by several cognitive biases.

Many other simplified assessment procedures have been proposed that are less firmly grounded in theory. In many of them, the assessment of the partial value functions v_i relies on *direct* comparison of preference differences without recourse to an interval on another attribute used as a "meter stick". We refer to [71] for a theoretical analysis of these techniques. They are also studied in detail in [70].

These procedures include:

- *direct rating* techniques in which values of v_i are directly assessed with reference to two arbitrarily chosen points [73, 74],
- procedures based on *bisection*, the decision-maker being asked to assess a point that is "half way" in terms of preference two reference points [259],
- procedures trying to build *standard sequences* on each attribute in terms of "preference differences" [164, Chap. 4].

An excellent overview of these techniques may be found in [259].

4.4 The Additive Value Model in the "Finite" Case

4.4.1 Outline of Theory

In this section, we suppose that $\succsim$ is a binary relation on a finite set $X \subseteq X_1 \times X_2 \times \cdots \times X_n$ (contrary to the preceding section, dealing with subsets of product sets will raise no difficulty here). The finiteness hypothesis clearly invalidates the standard sequence mechanism used till now. On each attribute there will only be finitely many "preference intervals" and exact matches between preference intervals will only happen exceptionally, see [264].

Clearly, independence remains a necessary condition for model (4.2) as before. Given the absence of structure of the set X, it is unlikely that this condition is sufficient to ensure (4.2). The following example shows that this intuition is indeed correct.

Example 28. Let $X = X_1 \times X_2$ with $X_1 = \{a, b, c\}$ and $X_2 = \{d, e, f\}$. Consider the weak order on X such that, abusing notation in an obvious way,

$$ad \succ bd \succ ae \succ af \succ be \succ cd \succ ce \succ bf \succ cf.$$

It is easy to check that $\succsim$ is independent. Indeed, we may for instance check that:

$$ad \succ bd \text{ and } ae \succ be \text{ and } af \succ bf,$$
$$ad \succ ae \text{ and } bd \succ be \text{ and } cd \succ ce.$$

This relation cannot however be represented in model (4.2) since:

$$af \succ be \Rightarrow v_1(a) + v_2(f) > v_1(b) + v_2(e),$$
$$be \succ cd \Rightarrow v_1(b) + v_2(e) > v_1(c) + v_2(d),$$
$$ce \succ bf \Rightarrow v_1(c) + v_2(e) > v_1(b) + v_2(f),$$
$$bd \succ ae \Rightarrow v_1(b) + v_2(d) > v_1(a) + v_2(e).$$

Summing the first two inequalities leads to:

$$v_1(a) + v_2(f) > v_1(c) + v_2(d).$$

Summing the last two inequalities leads to:

$$v_1(c) + v_2(d) > v_1(a) + v_2(f),$$

a contradiction.

Note that, since no indifference is involved, the Thomsen condition is trivially satisfied. Although it is clearly necessary for model (4.2), adding it to independence will therefore not solve the problem.

The conditions allowing to build an additive value model in the finite case were investigated in [1, 2, 223]. Although the resulting conditions turn out to be complex, the underlying idea is quite simple. It amounts to finding conditions under which a system of linear inequalities has a solution.

Suppose that $x \succ y$. If model (4.2) holds, this implies that:

$$\sum_{i=1}^{n} v_i(x_i) > \sum_{i=1}^{n} v_i(y_i). \tag{4.7}$$

Similarly if $x \sim y$, we obtain:

$$\sum_{i=1}^{n} v_i(x_i) = \sum_{i=1}^{n} v_i(y_i). \tag{4.8}$$

The problem is then to find conditions on $\succsim$ such that the system of finitely many equalities and inequalities (4.7) and (4.8) has a solution. This is a classical problem in Linear Algebra [107].

Definition 29 (Relation E^m). Let m be an integer ≥ 2. Let $x^1, x^2, \ldots, x^m, y^1, y^2, \ldots, y^m \in X$. We say that

$$(x^1, x^2, \ldots, x^m) E^m (y^1, y^2, \ldots, y^m)$$

if, for all $i \in N$, $(x_i^1, x_i^2, \ldots, x_i^m)$ is a permutation of $(y_i^1, y_i^2, \ldots, y_i^m)$.

Suppose that $(x^1, x^2, \ldots, x^m) E^m (y^1, y^2, \ldots, y^m)$ then model (4.2) implies that

$$\sum_{j=1}^{m} \sum_{i=1}^{n} v_i(x_i^j) = \sum_{j=1}^{m} \sum_{i=1}^{n} v_i(y_i^j).$$

Therefore if $x^j \succsim y^j$ for $j = 1, 2, \ldots, m-1$, it cannot be true that $x^m \succ y^m$. This condition must hold for all $m = 2, 3, \ldots$.

Definition 30 (Condition C^m). Let m be an integer ≥ 2. We say that condition C^m holds if

$$[x^j \succsim y^j \text{ for } j = 1, 2, \ldots, m-1] \Rightarrow Not[x^m \succ y^m]$$

for all $x^1, x^2, \ldots, x^m, y^1, y^2, \ldots, y^m \in X$ such that

$$(x^1, x^2, \ldots, x^m) E^m (y^1, y^2, \ldots, y^m).$$

Remark 31. It is not difficult to check that:

- $C^{m+1} \Rightarrow C^m$,
- $C^2 \Rightarrow \succsim$ is independent,
- $C^3 \Rightarrow \succsim$ is transitive.

•

We already observed that C^m was implied by the existence of an additive representation. The main result for the finite case states that requiring that $\succsim$ is complete and that C^m holds for $m = 2, 3, \ldots$ is also sufficient. Proofs can be found in [81, 164].

Theorem 32. *Let $\succsim$ be a binary relation on a finite set $X \subseteq X_1 \times X_2 \times \cdots \times X_n$. There are real-valued functions v_i on X_i such that (4.2) holds if and only if $\succsim$ is complete and satisfies C^m for $m = 2, 3, \ldots$.*

Remark 33. Contrary to the "rich" case considered in the preceding section, we have here necessary and sufficient conditions for the additive value model (4.2). However, it is important to notice that the above result uses a denumerable scheme of conditions. It is shown in [224] that this denumerable scheme cannot be truncated: for all $m \geq 2$, there is a relation $\succsim$ on a finite set X such that C^m holds but violating C^{m+1}. This is studied in more detail in [180, 250, 268]. Therefore, no finite scheme of axioms is sufficient to characterize model (4.2) for all finite sets X.

Given a finite set X *of given cardinality*, it is well-known that the denumerable scheme of condition can be truncated. The precise relation between the cardinality of X and the number of conditions needed raises difficult combinatorial questions that are studied in [101, 102]. •

Remark 34. It is clear that, if a relation $\succsim$ has a representation in model (4.2) with functions v_i, it also has a representation using functions $v_i' = \alpha v_i + \beta_i$ with $\alpha > 0$. Contrary to the rich case, the uniqueness of the functions v_i is more complex as shown by the following example.

Example 35. Let $X = X_1 \times X_2$ with $X_1 = \{a, b, c\}$ and $X_2 = \{d, e\}$. Consider the weak order on X such that, abusing notation in an obvious way,

$$ad \succ bd \succ ae \succ cd \succ be \succ ce.$$

This relation has a representation in model (4.2) with

$$v_1(a) = 3, v_1(b) = 1, v_1(c) = 0, v_2(d) = 3, v_2(e) = 0.5.$$

An equally valid representation would be given taking $v_1(b) = 2$. Clearly this new representation cannot be deduced from the original one applying a positive affine transformation. •

Remark 36. Theorem 32 has been extended to the case of an arbitrary set X in [146, 147], see also [99, 105]. The resulting conditions are however quite complex. This explains why we spent time on this "rich" case in the preceding section. •

Remark 37. The use of a denumerable scheme of conditions in Theorem 32 does not facilitate the interpretation and the test of conditions. However it should be noticed that, on a given set X, the test of the C^m conditions amounts to finding if a system of finitely many linear inequalities has a solution. It is well-known that Linear Programming techniques are quite efficient for such a task. •

4.4.2 Implementation: LP-Based Assessment

We show how to use LP techniques in order to assess an additive value model (4.2), without supposing that the sets X_i are rich. For practical purposes, it is not restrictive to assume that we are only interested in assessing a model for a limited range on each X_i. We therefore assume that the sets X_i are bounded so that, using independence, there is a worst value x_{i*} and a most preferable value x_i^*. Using the uniqueness properties of model (4.2), we may always suppose, after an appropriate normalization, that:

$$v_1(x_{1*}) = v_2(x_{2*}) = \ldots = v_n(x_{n*}) = 0 \text{ and} \tag{4.9}$$

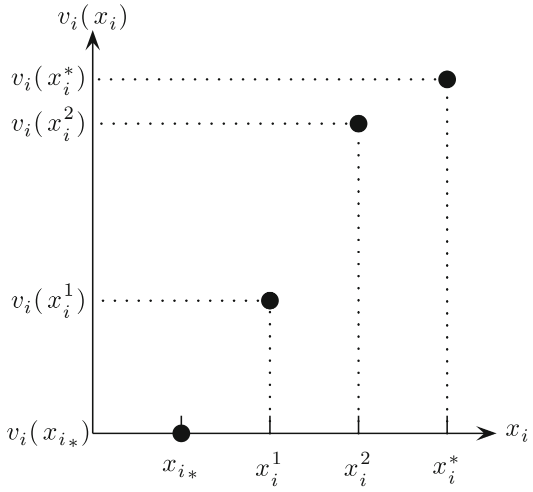
$$\sum_{i=1}^{n} v_i(x_i^*) = 1. \tag{4.10}$$

Two main cases arise (see Figs. 4.10 and 4.11):

- attribute $i \in N$ is discrete so that the evaluation of any conceivable alternative on this attribute belongs to a finite set. We suppose that $X_i = \{x_{i*}, x_i^1, x_i^2, \ldots, x_i^{r_i}, x_i^*\}$. We therefore have to assess $r_i + 1$ values of v_i,
- the attribute $i \in N$ has an underlying continuous structure. It is hardly restrictive in practice to suppose that $X_i \subset \mathbb{R}$, so that the evaluation of an alternative on this attribute may take any value between x_{i*} and x_i^*. In this case, we may opt for the assessment of a piecewise linear approximation of v_i partitioning the set X_i in $r_i + 1$ intervals and supposing that v_i is linear on each of these intervals. Note that the approximation of v_i can be made more precise simply by increasing the number of these intervals.

With these conventions, the assessment of the model (4.2) amounts to giving a value to $\sum_{i=1}^{n}(r_i + 1)$ unknowns. Clearly any judgment of preference linking x and y translate into a *linear inequality* between these unknowns. Similarly any judgment of indifference linking x and y translate into a *linear equality*. Linear

Fig. 4.10 Value function when X_i is discrete



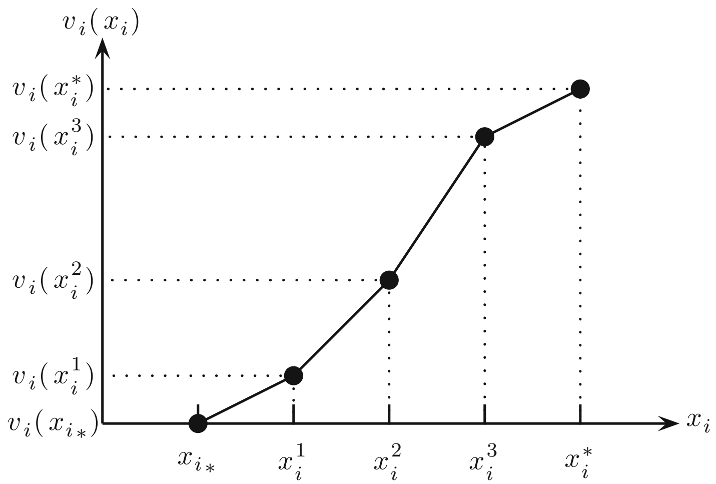


Fig. 4.11 Value function when X_i is continuous

Programming (LP) offers a powerful tool for testing whether such a system has solutions. Therefore, an assessment procedure can be conceived on the following basis:

- obtain judgments in terms of preference or indifference linking several alternatives in X,
- convert these judgments into linear (in)equalities,
- test, using LP, whether this system has a solution.

If the system has no solution then one may either propose a solution that will be "as close as possible" from the information obtained, e.g., violating the minimum number of (in)equalities or suggest the reconsideration of certain judgements. If the system has a solution, one may explore the set of all solutions to this system since they are all candidates for the establishment of model (4.2). These various techniques depend on:

- the choice of the alternatives in X that are compared: they may be real or fictitious, they may differ on a different number of attributes,
- the way to deal with the inconsistency of the system and to eventually propose some judgments to be reconsidered,
- the way to explore the set of solutions of the system and to use this set as the basis for deriving a prescription.

Linear programming offers of simple and versatile technique to assess additive value functions. All restrictions generating linear constraints of the coefficient of the value function can easily be accommodated. This idea has been often exploited, see [22, 51]. We present below two techniques using it. It should be noticed that rather different techniques have been proposed in the literature on Marketing [54, 133–135, 140, 141, 148, 169, 170, 199].

4.4.2.1 UTA [145]

UTA ("UTilité Additive", i.e., additive utility in French) is one of the oldest techniques belonging to this family. It is supposed in UTA that there is a subset $Ref \subset X$ of reference alternatives that the decision-maker knows well either because he/she has experienced them or because they have received particular attention. The technique amounts to asking the DM to provide a weak order on Ref. Each preference or indifference relation contained in this weak order is then translated into a linear constraint:

- $x \sim y$ gives an equality $v(x) - v(y) = 0$ and
- $x \succ y$ gives an inequality $v(x) - v(y) > 0$,

where $v(x)$ and $v(y)$ can be expressed as a linear combination of the unknowns as remarked earlier. Strict inequalities are then translated into large inequalities as is usual in Linear Programming, i.e., $v(x) - v(y) > 0$ becomes $v(x) - v(y) \geq \epsilon$ where $\epsilon > 0$ is a very small positive number that should be chosen according to the precision of the arithmetics used by the LP package.

The test of the existence of a solution to the system of linear constraints is done via standard Goal Programming techniques [55] adding appropriate deviation variables. In UTA, each equation $v(x) - v(y) = 0$ is translated into an equation $v(x) - v(y) + \sigma_x^+ - \sigma_x^- + \sigma_y^+ - \sigma_y^- = 0$, where $\sigma_x^+, \sigma_x^-, \sigma_y^+$ and σ_y^- are nonnegative deviation variables. Similarly each inequality $v(x) - v(y) \geq \epsilon$ is written as $v(x) - v(y) + \sigma_x^+ - \sigma_x^- + \sigma_y^+ - \sigma_y^- \geq \epsilon$. It is clear that there will exist a solution to the original system of linear constraints if there is a solution of the LP in which all deviation variables are zero. This can easily be tested using the objective function

$$\text{Minimize } Z = \sum_{x \in Ref} \sigma_x^+ + \sigma_x^- \tag{4.11}$$

Two cases arise. If the optimal value of Z is 0, there is an additive value function that represents the preference information. It should be observed that, except in exceptional cases (e.g., if the preference information collected is identical to the preference information collected with the standard sequence technique), there are infinitely many such additive value functions [that are not related via a simple change of origin and of unit, since we already fixed them through normalization (4.9) and (4.10)]. The one given as the "optimal" one by the LP does not have a special status since it is highly dependent upon the arbitrary choice of the objective function; instead of minimizing the sum of the deviation variables, we could have as well, and still preserving linearity, minimized the largest of these variables. The whole polyhedron of feasible solutions of the original (in)equalities corresponds to adequate additive value functions: we have a whole set $\mathcal{V}$ of additive value functions representing the information collected on the set of reference alternatives Ref.

The size of $\mathcal{V}$ is clearly dependent upon the choice of the alternatives in Ref. Using standard techniques in LP, several functions in $\mathcal{V}$ may be obtained, e.g., the ones maximizing or minimizing, within $\mathcal{V}$, $v_i(x_i^*)$ for each attribute [145]. It is often interesting to present them to the decision-maker in the pictorial form of Figs. 4.10 and 4.11.

If the optimal value of Z is strictly greater than 0, there is no additive value function representing the preference information available. The solution given as optimal (note that it is not guaranteed that this solution leads to the minimum possible number of violations w.r.t. the information provided—this would require solving an integer linear programme) is, in general, highly dependent upon the choice of the objective function.

This absence of solution to the system might be due to several factors:

- The piecewise linear approximation of the v_i for the "continuous" attributes may be too rough. It is easy to test whether an increase in the number of linear pieces on some of these attributes may lead to a nonempty set of additive value functions.
- The information provided by the decision-maker may be of poor quality. It might then be interesting to present to the decision-maker one additive value function (e.g., one may present an average function after some post-optimality analysis) in the pictorial form of Figs. 4.10 and 4.11 and to let him react to this information either by modifying his/her initial judgments or even by letting him/her react directly on the shape of the value functions. This is the solution implemented in the well-known PREFCALC system [143].
- The preference provided by the decision-maker might be inconsistent with the conditions implied by an additive value function. The system should then help

locate these inconsistencies and allow the DM to think about them. Alternatively, since many alternative attribute descriptions are possible, it may be worth investigating whether a different definition of the various attributes may lead to a preference model consistent with model (4.2). Several examples of such analysis may be found in [154, 155, 259]

When the above techniques fail, the optimal solution of the LP, even if not compatible with the information provided, may still be considered as an adequate model. Again, since the objective function introduced above is somewhat arbitrary and it is recommended in [145] to perform a post-optimality analysis, e.g., considering additive value functions that are "close" to the optimal solution through the introduction of a linear constraint:

$$Z \leq Z^* + \delta,$$

where Z^* is the optimal value of the objective function of the original LP and δ is a "small" positive number. As above, the result of the analysis is a set $\mathcal{V}$ of additive value functions defined by a set of linear constraints. A representative sample of additive value functions within $\mathcal{V}$ may be obtained as above.

It should be noted that many possible variants of UTA can be conceived building on the following comments. They include:

- the addition of monotonicity properties of the v_i with respect to the underlying continuous attributes,
- the addition of constraints on the shape of the marginal value functions v_i, e.g., requiring them to be concave, convex or S-shaped,
- the addition of constraints linked to a possible indication of preference intensity for the elements of *Ref* given by the DM, e.g., the difference between x and y is larger than the difference between z and w.

For applications of UTA-like techniques, we refer to [19, 57, 66, 68, 137, 144, 186, 190, 230–232, 235–238, 240, 244, 245, 269–273]. Variants of the method are considered in [26, 27, 29, 79, 131, 132, 136, 137, 229, 234, 239]. This method is reviewed in [67, 233, 241].

4.4.2.2 MACBETH [10]

It is easy to see that (4.9) and (4.10) may equivalently be written as:

$$x \succsim y \Leftrightarrow \sum_{i=1}^{n} k_i u_i(x_i) \geq \sum_{i=1}^{n} k_i u_i(y_i), \tag{4.12}$$

where

$$u_1(x_{1*}) = u_2(x_{2*}) = \ldots u_n(x_{n*}) = 0, \tag{4.13}$$

$$u_1(x_1^*) = u_2(x_2^*) = \dots u_n(x_n^*) = 1 \text{ and} \tag{4.14}$$

$$\sum_{i=1}^{n} k_i = 1. \tag{4.15}$$

With such an expression of an additive value function, it is tempting to break down the assessment into two distinct parts: a value function u_i is assessed on each attribute and, then, scaling constants k_i are assessed taking the shape of the value functions u_i as given. This is the path followed in MACBETH.

Remark 38. Again, note that we are speaking here of k_i as *scaling constants* and not as *weights*. As already mentioned weights that would reflect the "importance" of attributes are irrelevant to assess the additive value function model. Notice that, under (4.12)–(4.15) the ordering of the scaling constant k_i is dependent upon the choice of x_{i*} and x_i^*. Increasing the width of the interval $[x_{i*}, x_i^*]$ will lead to increasing the value of the scaling constant k_i. The value k_i has, therefore, nothing to do with the "importance" of attribute i. This point is unfortunately too often forgotten when using a weighted average of some numerical attributes. In the latter model, changing the units in which the attributes are measured should imply changing the "weights" accordingly. •

The assessment procedure of the u_i is conceived in such a way as to avoid comparing alternatives differing on more than one attribute. In view of what was said before concerning the standard sequence technique, this is clearly an advantage of the technique. But can it be done? The trick here is that MACBETH asks for judgments related to the difference between the desirability of alternatives and not only judgments in terms of preference or indifference. Partial value functions u_i are approximated in a similar way than in UTA: for discrete attributes, each point on the function is assessed, for continuous ones, a piecewise linear approximation is used.

MACBETH asks the DM to compare pairs of levels on each attribute. If no difference is felt between these levels, they receive an identical partial value level. If a difference is felt between x_i^k and x_i^r, MACBETH asks for a judgment qualifying the strength of this difference. The method and the associated software propose three different semantical categories:

Categories	Description
$C1$	Weak
$C2$	Strong
$C3$	Extreme

with the possibility of using intermediate categories, i.e., between null and weak, weak and strong, strong and extreme (giving a total of six distinct categories). This information is then converted into linear inequations using the natural interpretation that if the "difference" between the levels x_i^k and x_i^r has been judged larger than the "difference" between $x_i^{k'}$ and $x_i^{r'}$ then it should follow that $u_i(x_i^k) - u_i(x_i^r) > u_i(x_i^{k'}) - u_i(x_i^{r'})$. Technically the six distinct categories are delimited by thresholds that are used in the establishment of the constraints of the LP. The software associated to MACBETH offers the possibility to compare all pairs of levels on each attribute for a total of $(r_i+1)r_i/2$ comparisons. Using standard Goal Programming techniques, as in UTA, the test of the compatibility of a partial value function with this information is performed via the solution of a LP. If there is a partial value function compatible with the information, a "central" function is proposed to the DM who has the possibility to modify it. If not, the results of the LP are exploited in such a way to propose modifications of the information that would make it consistent.

The assessment of the scaling constant k_i is done using similar principles. The DM is asked to compare the following $(n+2)$ alternatives by pairs:

$$
\begin{gathered}
(x_{1*}, x_{2*}, \ldots, x_{n*}), \\
(x_1^*, x_{2*}, \ldots, x_{n*}), \\
(x_{1*}, x_2^*, \ldots, x_{n*}), \\
\ldots \\
(x_{1*}, x_{2*}, \ldots, x_n^*) \text{ and} \\
(x_1^*, x_2^*, \ldots, x_n^*),
\end{gathered}
$$

placing each pair in a category of difference. This information immediately translates into a set of linear constraints on the k_i. These constraints are processed as before. It should be noticed that, once the partial value functions u_i are assessed, it is not necessary to use the levels x_{i*} and x_i^* to assess the k_i since they may well lead to alternatives that are too unrealistic. The authors of MACBETH suggest to replace x_{i*} by a "neutral" level which appears neither desirable nor undesirable and x_i^* by a "desirable" level that is judged satisfactory. Although this clearly impacts the quality of the dialogue with the DM, this has no consequence on the underlying technique used to process information.

We refer to [6–9, 11–15, 17, 18, 203] for applications of the MACBETH technique. This method is also studied in detail in [16].

4.5 Extensions

The additive value model (4.2) is the central model for the application of conjoint measurement techniques to decision analysis. In this section, we consider various extensions to this model.

4.5.1 Transitive Decomposable Models

The transitive decomposable model has been introduced in [164] as a natural generalization of model (4.2). It amounts to replacing the addition operation by a general function that is increasing in each of its arguments.

Definition 39 (Transitive Decomposable Model). Let $\succsim$ be a binary relation on a set $X = \prod_{i=1}^{n} X_i$. The transitive decomposable model holds if, for all $i \in N$, there is a real-valued function v_i on X_i and a real-valued function g on $\prod_{i=1}^{n} v_i(X_i)$ that is increasing in all its arguments such that:

$$x \succsim y \Leftrightarrow g(v_1(x_1), \ldots, v_n(x_n)) \geq g(v_1(y_1), \ldots, v_n(y_n)), \tag{4.16}$$

for all $x, y \in X$.

An interesting point with this model is that it admits an intuitively appealing simple characterization. The basic axiom for characterizing the above transitive decomposable model is weak independence, which is clearly implied by (4.16). The following theorem is proved in [164, Chap. 7].

Theorem 40. *A preference relation $\succsim$ on a finite or countably infinite set X has a representation in the transitive decomposable model iff $\succsim$ is a weakly independent weak order.*

Remark 41. This result can be extended to sets of arbitrary cardinality adding a, necessary, condition implying that the weak order $\succsim$ has a numerical representation, see [61, 64]. •

The weak point of such a model is that the function g is left unspecified so that the model will be difficult to assess. Furthermore, the uniqueness results for v_i and g are clearly much less powerful than what we obtained with model (4.2), see [164, Chap. 7]. Therefore, practical applications of this model generally imply specifying the type of function g, possibly by verifying further conditions on the preference relation that impose that g belongs to some parameterized family of functions, e.g., some polynomial function of the v_i. This is studied in detail in [164, Chap. 7] and [21, 106, 176, 180, 198, 210, 251]. Since such models have, to the best of our knowledge, never been used in decision analysis, we do not analyze them further.

The structure of the decomposable model however suggests that assessment techniques for this model could well come from Artificial Intelligence with its "rule induction" machinery. Indeed the function g in model (4.16) may also be seen as a set of "rules". We refer to [123–126, 130] for a thorough study of the potentiality of such an approach.

Remark 42. A simple extension of the decomposable model consists in simply asking for a function g that would be nondecreasing in each of its arguments. The following result is proved in [39] (see also [126]) (it can easily be extended to cover the case of an arbitrary set X, adding a, necessary, condition implying that $\succsim$ has a numerical representation).

We say that $\succsim$ is weakly separable if, for all $i \in N$ and all $x_i, y_i \in X_i$, it is never true that $(x_i, z_{-i}) \succ (y_i, z_{-i})$ and $(y_i, w_{-i}) \succ (x_i, w_{-i})$, for some $z_{-i}, w_{-i} \in X_{-i}$. Clearly this is a weakening of weak independence since it tolerates to have at the same time $(x_i, z_{-i}) \succ (y_i, z_{-i})$ and $(x_i, w_{-i}) \sim (y_i, w_{-i})$.

Theorem 43. *A preference relation $\succsim$ on a finite or countably infinite set X has a representation in the weak decomposable model:*

$$x \succsim y \Leftrightarrow g(u_1(x_1), \ldots, u_n(x_n)) \geq g(u_1(y_1), \ldots, u_n(y_n))$$

with g nondecreasing in all its arguments iff $\succsim$ is a weakly separable weak order.

A recent trend of research has tried to characterize special functional forms for g in the weakly decomposable model, such as max, min or some more complex forms. The main references include [50, 126, 128, 226, 243]. •

Remark 44. The use of "fuzzy integrals" as tools for aggregating criteria has recently attracted much attention [69, 114–120, 182–185], the Choquet Integral and the Sugeno integral being among the most popular. It should be strongly emphasized that the very definition of these integrals requires to have at hand a weak order on $\cup_{i=1}^{n} X_i$, supposing w.l.o.g. that the sets X_i are disjoint. This is usually called a "commensurability hypothesis". Whereas this hypothesis is quite natural when dealing with an homogeneous Cartesian product, as in decision under uncertainty (see e.g., [261]), it is far less so in the area of multiple criteria decision making. A neat conjoint measurement analysis of such models and their associated assessment procedures is an open research question, see [121]. It has recently been solved for the case of the Sugeno integral [52, 129]. •

4.5.2 Intransitive Indifference

Decomposable models form a large family of preferences though not large enough to encompass all cases that may be encountered when asking subjects to express preferences. A major restriction is that not all preferences may be assumed to be weak orders. The example of the sequence of cups of coffee, each differing from the previous one by an imperceptible quantity of sugar added [171], is famous; it leads to the notions of semiorder and interval order [4, 82, 90, 171, 205], in which indifference is not transitive, while strict preference is.

Ideally, taking intransitive indifference into account, we would want to arrive at a generalization of (4.2) in which:

$$x \sim y \Leftrightarrow |V(x) - V(y)| \leq \epsilon,$$

$$x \succ y \Leftrightarrow V(x) > V(y) + \epsilon,$$

where $\epsilon \geq 0$ and $V(x) = \sum_{i=1}^{n} v_i(x_i)$.

In the finite case, it is not difficult to extend the conditions presented in Sect. 4.4 to cover such a case. Indeed, we are still looking here for the solution to a system of linear constraints. Although this seems to have never been done, it would not be difficult to adapt the LP-based assessment techniques to this case.

On the contrary, extending the standard sequence technique of Sect. 4.3 is a formidable challenge. Indeed, remember that these techniques crucially rest on indifference judgments which lead to the determination of "perfect copies" of a given preference interval. As soon as indifference is not supposed to be transitive, "perfect copies" are not so perfect and much trouble is expected. We refer to [108, 163, 167, 168, 172, 188, 189, 205, 248] for a study of these models.

Remark 45. Even if the analysis of such models proves difficult, it should be noted that the semi-ordered version of the additive value model may be interpreted as having a "built-in" sensitivity analysis via the introduction of the threshold ϵ. Therefore, in practice, we may usefully view ϵ not as a parameter to be assessed but as a simple trick to avoid undue discrimination, because of the imprecision inevitably involved in our assessment procedures, between close alternatives •

Remark 46. Clearly the above model can be generalized to cope with a possibly non-constant threshold. The literature on the subject remains minimal however, see [205]. •

4.5.3 Nontransitive Preferences

Many authors [187, 252] have argued that the reasonableness of supposing that strict preference is transitive is not so strong when it comes to comparing objects evaluated on several attributes. As soon as it is supposed that subjects may use an "ordinal" strategy for comparing objects, examples inspired from the well-known Condorcet paradox [220, 227] show that intransitivities will be difficult to avoid. Indeed it is possible to observe predictable intransitivities of strict preference in carefully controlled experiments [252]. There may therefore be a descriptive interest to studying such models. When it comes to decision analysis, intransitive preferences are often dismissed on two grounds:

- On a practical level, it is not easy to build a recommendation on the basis of a binary relation in which $\succ$ would not be transitive. Indeed, social choice theorists, facing a similar problem, have devoted much effort to devising what could be called reasonable procedures to deal with such preferences [60, 86, 165, 166, 191, 201, 222]. This literature does not lead, as was expected, to the emergence of a single suitable procedure in all situations.

- On a more conceptual level, many others have questioned the very rationality of such preferences using some version of the famous "money pump" argument [175, 208].

P.C. Fishburn has forcefully argued [97] that these arguments might not be as decisive as they appear at first sight. Furthermore some MCDM techniques make use of such intransitive models, most notably the so-called outranking methods [33, 78, 80, 216, 253, 254]. Besides the intellectual challenge, there might therefore be a real interest in studying such models.

A. Tversky [252] was one of the first to propose such a model generalizing (4.2), known as the *additive difference model*, in which:

$$x \succsim y \Leftrightarrow \sum_{i=1}^{n} \Phi_i(u_i(x_i) - u_i(y_i)) \geq 0 \tag{4.17}$$

where Φ_i are increasing and odd functions.

It is clear that (4.17) allows for intransitive $\succsim$ but implies its completeness. Clearly, (4.17) implies that $\succsim$ is independent. This allows us to unambiguously define marginal preferences $\succsim_i$. Although model (4.17) can accommodate intransitive $\succsim$, a consequence of the increasingness of the Φ_i is that the marginal preference relations $\succsim_i$ are weak orders. This, in particular, excludes the possibility of any perception threshold on each attribute which would lead to an intransitive indifference relation on each attribute. Imposing that Φ_i are nondecreasing instead of being increasing allows for such a possibility. This gives rise to what is called the "weak additive difference model" in [30].

As suggested in [30, 93, 95, 96, 255], the subtractivity requirement in (4.17) can be relaxed. This leads to *nontransitive additive* conjoint measurement models in which:

$$x \succsim y \Leftrightarrow \sum_{i=1}^{n} p_i(x_i, y_i) \geq 0 \tag{4.18}$$

where the p_i are real-valued functions on X_i^2 and may have several additional properties (e.g., $p_i(x_i, x_i) = 0$, for all $i \in \{1, 2, \ldots, n\}$ and all $x_i \in X_i$).

This model is an obvious generalization of the (weak) additive difference model. It allows for intransitive and incomplete preference relations $\succsim$ as well as for intransitive and incomplete marginal preferences $\succsim_i$. An interesting specialization of (4.18) obtains when p_i are required to be *skew symmetric* i.e., such that $p_i(x_i, y_i) = -p_i(y_i, x_i)$. This skew symmetric nontransitive additive conjoint measurement model implies that $\succsim$ is complete and independent.

An excellent overview of these nontransitive models is [97]. Several axiom systems have been proposed to characterize them. P.C. Fishburn gave [93, 95, 96] axioms for the skew symmetric version of (4.18) both in the finite and the infinite case. Necessary and sufficient conditions for a nonstandard version of (4.18) are presented in [100]. Vind [255, 256] gives axioms for (4.18) with $p_i(x_i, x_i) = 0$ when $n \geq 4$. Bouyssou [30] gives necessary and sufficient conditions for (4.18) with and without skew symmetry in the denumerable case when $n = 2$.

The additive difference model (4.17) was axiomatized in [98] in the infinite case when $n \geq 3$ and [30] gives necessary and sufficient conditions for the weak additive difference model in the finite case when $n = 2$. Related studies of nontransitive models include [58, 88, 174, 195]. The implications of these models for decision-making under uncertainty were explored in [94] (for a different path to nontransitive models for decision making under risk and/or uncertainty, see [89, 91]).

It should be noticed that even the weakest form of these models, i.e., (4.18) without skew symmetry, involves an addition operation. Therefore it is unsurprising that the axiomatic analysis of these models share some common features with the additive value function model (4.2). Indeed, except in the special case in which $n = 2$, this case relating more to ordinal than to conjoint measurement (see [36, 96]), the various axiom systems that have been proposed involve either:

- a denumerable set of cancellation conditions in the finite case or,
- a finite number of cancellation conditions together with unnecessary structural assumptions in the general case [these structural assumptions generally allow us to obtain nice uniqueness results for (4.18): the functions p_i are unique up to the multiplication by a common positive constant].

A different path to the analysis of nontransitive conjoint measurement models has recently been proposed in [37, 39, 40] and surveyed in [43]. In order to get a feeling for these various models, it is useful to consider the various strategies that are likely to be implemented when comparing objects differing on several dimensions [59, 193, 194, 219, 249, 252].

Consider two alternatives x and y evaluated on a family of n attributes so that $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$.

A first strategy that can be used in order to decide whether or not it can be said that "x is at least as good as y" consists in trying to measure the "worth" of each alternative on each attribute and then to combine these evaluations adequately. Giving up all idea of transitivity and completeness, this suggests a model in which:

$$x \succsim y \Leftrightarrow F(u_1(x_1), \ldots, u_n(x_n), u_1(y_1), \ldots, u_n(y_n)) \geq 0 \qquad (4.19)$$

where u_i are real-valued functions on the X_i and F is a real-valued function on $\prod_{i=1}^{n} u_i(X_i)^2$. Additional properties on F, e.g., its nondecreasingness (resp. nonincreasingness) in its first (resp. last) n arguments, will give rise to a variety of models implementing this first strategy.

A second strategy relies on the idea of measuring "preference differences" separately on each attribute and then combining these (positive or negative) differences in order to know whether the aggregation of these differences leads to an advantage for x over y. More formally, this suggests a model in which:

$$x \succsim y \Leftrightarrow G(p_1(x_1, y_1), p_2(x_2, y_2), \ldots, p_n(x_n, y_n)) \geq 0 \qquad (4.20)$$

where p_i are real-valued functions on X_i^2 and G is a real-valued function on $\prod_{i=1}^n p_i(X_i^2)$. Additional properties on G (e.g., its oddness or its nondecreasingness in each of its arguments) or on p_i (e.g., $p_i(x_i, x_i) = 0$ or $p_i(x_i, y_i) = -p_i(y_i, x_i)$) will give rise to a variety of models in line with the above strategy.

Of course these two strategies are not incompatible and one may well consider using the "worth" of each alternative on each attribute to measure "preference differences". This suggests a model in which:

$$x \succsim y \Leftrightarrow H(\phi_1(u_1(x_1), u_1(y_1)), \ldots, \phi_n(u_n(x_n), u_n(y_n))) \geq 0 \tag{4.21}$$

where u_i are real-valued functions on X_i, ϕ_i are real-valued functions on $u_i(X_i)^2$ and H is a real-valued function on $\prod_{i=1}^n \phi_i(u_i(X_i)^2)$.

The use of general functional forms, instead of additive ones, greatly facilitate the axiomatic analysis of these models. It mainly relies on the study of various kinds of *traces* induced by the preference relation on coordinates and does not require a detailed analysis of tradeoffs between attributes.

The price to pay for such an extension of the scope of conjoint measurement is that the number of parameters that would be needed to assess such models is quite high. Furthermore, none of them is likely to possess any remarkable uniqueness properties. Therefore, although proofs are constructive, these results will not give direct hints on how to devise assessment procedures. The general idea here is to use numerical representations as guidelines to understand the consequences of a limited number of cancellation conditions, without imposing any transitivity or completeness requirement on the preference relation and any structural assumptions on the set of objects. Such models have proved useful to:

- understand the ordinal character of some aggregation models proposed in the literature [214, 216], known as the "outranking methods" (see [78, 80, 216] for surveys) as shown in [38, 42, 44–46, 48, 127],
- understand the links between aggregation models aiming at enriching a dominance relation and more traditional conjoint measurement approaches [39],
- to include in a classical conjoint measurement framework, noncompensatory preferences in the sense of [30, 47, 76, 84, 85] as shown in [38, 41, 42, 44, 46, 127].

Acknowledgements We thank two anonymous referees for their very helpful comments and suggestions. Part of this work was accomplished while Denis Bouyssou was visiting the *Service de Mathématique de la Gestion* at the *Université Libre de Bruxelles* (Brussels, Belgium). The warm hospitality of the *Service de Mathématique de la Gestion*, the support of the Belgian *Fonds National de la Recherche Scientistic* and the Brussels-Capital Region through a "Research in Brussels" action grant are gratefully acknowledged. Many Figures in this text were inspired by the ones drawn by Christophe Gonzales in [110].

References

1. Adams, E.W.: Elements of a theory of inexact measurement. Philos. Sci. **32**, 205–228 (1965)
2. Adams, E.W., Fagot, R.F.: A model of riskless choice. Behav. Sci. **4**, 1–10 (1959)
3. Adelman, L., Sticha, P.J., Donnell, M.L.: An experimental investigation of the relative effectiveness of two techniques for structuring multiattributed hierarchies. Organ. Behav. Hum. Decis. Process. **37**, 188–196 (1986)
4. Aleskerov, F., Bouyssou, D., Monjardet, B.: Utility Maximization, Choice and Preference, 2nd edn. Studies in Economic Theory, vol. 16. Springer, Berlin (2007) [ISBN 978-3-540-34182-6]
5. Atkinson, A.B.: On the measurement of inequality. J. Econ. Theory **2**, 244–263 (1970)
6. Bana e Costa, C.A.: The use of multi-criteria decision analysis to support the search for less conflicting policy options in a multi-actor context: case study. J. Multicrit. Decis. Anal. **10**(2), 111–125 (2001)
7. Bana e Costa, C.A., Chagas, M.P.: A career choice problem: an example of how to use macbeth to build a quantitative value model based on qualitative value judgments. Eur. J. Oper. Res. **153**(2), 323–331 (2004)
8. Bana e Costa, C.A., Oliveira, M.D.: A multicriteria decision analysis model for faculty evaluation. Omega **40**(4), 424–436 (2012)
9. Bana e Costa, C.A., Oliveira, R.C.: Assigning priorities for maintenance and repair and refurbishment in managing a municipal housing stock. Eur. J. Oper. Res. **138**, 380–91 (2002)
10. Bana e Costa, C.A., Vansnick, J.-C.: MACBETH – an interactive path towards the construction of cardinal value functions. Int. Trans. Oper. Res. **1**, 489–500 (1994)
11. Bana e Costa, C.A., Ensslin, L., Corrêa, É.C., Vansnick, J.-C.: Decision support systems in action: integrated application in a multicriteria decission aid process. Eur. J. Oper. Res. **113**(2), 315–335 (1999)
12. Bana e Costa, C.A., Nunes da Silva, F., Vansnick, J.-C.: Conflict dissolution in the public sector: a case-study. Eur. J. Oper. Res. **130**(2), 388–401 (2001)
13. Bana e Costa, C.A., Corrêa, É.C., De Corte, J.-M., Vansnick, J.-C.: Facilitating bid evaluation in public call for tenders: a socio-technical approach. Omega **30**(3), 227–242 (2002)
14. Bana e Costa, C.A., Costa-Lobo, M.L., Ramos, I.A.J., Vansnick, J.C.: Multicriteria approach for strategic town planning: the case of Barcelos. In: Bouyssou, D., Jacquet-Lagrèze, É., Perny, P., Słowiński, R., Vanderpooten, D., Vincke, Ph. (eds.) Aiding Decisions with Multiple Criteria: Essays in Honour of Bernard Roy, pp. 429–456. Kluwer, Boston (2002)
15. Bana e Costa, C.A., Antão da Silva, P., Correia, F.N.: Multicriteria evaluation of flood control measures: the case of Ribeira do Livramento. Water Resour. Manag. **18**(3), 263–283 (2004)
16. Bana e Costa, C.A., De Corte, J.M., Vansnick, J.-C.: On the mathematical foundation of MACBETH. In: Figueira, J., Greco, S., Ehrgott, M. (eds.) Multiple Criteria Decision Analysis. State of the Art Surveys. International Series in Operations Research & Management Science, vol. 142, pp. 409–442. Springer, Berlin (2005)
17. Bana e Costa, C.A., Fernandes, T.G., Correia, P.V.D.: Prioritisation of public investments in social infrastructures using multicriteria value analysis and decision conferencing: a case study. Int. Trans. Oper. Res. **13**(4), 279–297 (2006)
18. Bana e Costa, C.A., Oliveira, C.S., Vieira, V.: Prioritization of bridges and tunnels in earthquake risk mitigation using multicriteria decision analysis: application to lisbon. Omega **36**(3), 442–450 (2008)
19. Baourakis, G., Matsatsinis, N.F., Siskos, Y.: Agricultural product development using multidimensional and multicriteria analyses: the case of wine. Eur. J. Oper. Res. **94**(2), 321–334 (1996)
20. Beardon, A.F., Candeal, J.C., Herden, G., Induráin, E., Mehta, G.B.: The non-existence of a utility function and the structure of non-representatble preference relations. J. Math. Econ. **37**, 17–38 (2002)

21. Bell, D.E.: Multilinear representations for ordinal utility functions. J. Math. Psychol. **31**, 44–59 (1987)
22. Belton, V., Stewart, T.: Multiple Criteria Decision Analysis: An Integrated Approach. Kluwer, Boston (2001)
23. Belton, V., Ackermann, F., Shepherd, I.: Integrated support from problem structuring through alternative evaluation using COPE and V•I•S•A. J. Multicrit. Decis. Anal. **6**, 115–130 (1997)
24. Ben-Porath, E., Gilboa, I.: Linear measures, the Gini index and the income-equality tradeoff. J. Econ. Theory **64**, 443–467 (1994)
25. Ben-Porath, E., Gilboa, I., Schmeidler, D.: On the measurement of inequality under uncertainty. J. Econ. Theory **75**, 194–204 (1997)
26. Beuthe, M., Scannella, G.: Applications comparées des méthodes d'analyse multicritère UTA. RAIRO Recherche Opérationnelle/Oper. Res. **30**(3), 293–315 (1996)
27. Beuthe, M., Scannella, G.: Comparative analysis of UTA multicriteria methods. Eur. J. Oper. Res. **130**(2), 246–262 (2001)
28. Blackorby, C., Primont, D., Russell, R.: Duality, Separability, and Functional Structure: Theory and Economic Applications. North-Holland, New York (1978)
29. Bous, G., Fortemps, P., Glineur, F., Pirlot, M.: ACUTA: a novel method for eliciting additive value functions on the basis of holistic preference statements. Eur. J. Oper. Res. **206**(2), 435–444 (2010)
30. Bouyssou, D.: Some remarks on the notion of compensation in MCDM. Eur. J. Oper. Res. **26**, 150–160 (1986)
31. Bouyssou, D.: Modelling inaccurate determination, uncertainty, imprecision using multiple criteria. In: Lockett, A.G., Islei, G. (eds.) Improving Decision Making in Organisations, pp. 78–87. Springer, Heidelberg (1989)
32. Bouyssou, D.: Builing criteria: a prerequisite for MCDA. In: Bana e Costa, C.A. (ed.) Readings in Multiple Criteria Decision Aid, pp. 58–81. Springer, Berlin (1990)
33. Bouyssou, D.: Outranking methods. In: Floudas, C.A., Pardalos, P.M. (eds.) Encyclopedia of Optimization, vol. 4, pp. 249–255. Kluwer, Boston (2001)
34. Bouyssou, D., Marchant, T.: Ordered categories and additive conjoint measurement on connected sets. J. Math. Psychol. **53**(2), 92–105 (2009)
35. Bouyssou, D., Marchant, T.: Additive conjoint measurement with ordered categories. Eur. J. Oper. Res. **203**(1), 195–204 (2010)
36. Bouyssou, D., Marchant, T.: Biorders with frontier. Order **28**(1), 53–87 (2011). doi:10.1007/s11083-010-9153-5
37. Bouyssou, D., Pirlot, M.: Nontransitive decomposable conjoint measurement. J. Math. Psychol. **46**, 677–703 (2002)
38. Bouyssou, D., Pirlot, M.: A characterization of strict concordance relations. In: Bouyssou, D., Jacquet-Lagrèze, É., Perny, P., Słowiński, R., Vanderpooten, D., Vincke, Ph. (eds.) Aiding Decisions with Multiple Criteria: Essays in Honour of Bernard Roy, pp. 121–145. Kluwer, Boston (2002)
39. Bouyssou, D., Pirlot, M.: Preferences for multiattributed alternatives: traces, dominance, and numerical representations. J. Math. Psychol. **48**(3), 167–185 (2004)
40. Bouyssou, D., Pirlot, M.: 'Additive difference' models without additivity and subtractivity. J. Math. Psychol. **48**(4), 263–291 (2004)
41. Bouyssou, D., Pirlot, M.: A characterization of concordance relations. Eur. J. Oper. Res. **167**(2), 427–443 (2005)
42. Bouyssou, D., Pirlot, M.: Conjoint measurement tools for MCDM. A brief introduction. In: Figueira, J., Greco, S., Ehrgott, M. (eds.) Multiple Criteria Decision Analysis. State of the Art Surveys. International Series in Operations Research & Management Science, vol. 142, pp. 73–130. Springer, Berlin (2005)
43. Bouyssou, D., Pirlot, M.: Following the traces: an introduction to conjoint measurement without transitivity and additivity. Eur. J. Oper. Res. **163**(2), 287–337 (2005)
44. Bouyssou, D., Pirlot, M.: Further results on concordance relations. Eur. J. Oper. Res. **181**(1), 505–514 (2007)

45. Bouyssou, D., Pirlot, M.: On some ordinal models for decision making under uncertainty. Ann. Oper. Res. **163**(1), 19–48 (2008)
46. Bouyssou, D., Pirlot, M.: An axiomatic analysis of concordance-discordance relations. Eur. J. Oper. Res. **199**(2), 468–477 (2009)
47. Bouyssou, D., Vansnick, J.-C.: Noncompensatory and generalized noncompensatory preference structures. Theor. Decis. **21**, 251–266 (1986)
48. Bouyssou, D., Pirlot, M., Vincke, Ph.: A general model of preference aggregation. In: Karwan, M.H., Spronk, J., Wallenius, J. (eds.) Essays in Decision Making, pp. 120–134. Springer, Berlin (1997)
49. Bouyssou, D., Marchant, Th., Pirlot, M., Perny, P., Tsoukiàs, A., Vincke, Ph.: Evaluation and Decision models: A Critical Perspective, 1st edn. International Series in Operations Research and Management Science, vol. 32. Kluwer, Dordrecht (2000) [ISBN 0-7923-7250-6]
50. Bouyssou, D., Greco, S., Matarazzo, B., Pirlot, M., Słowiński, R.: Characterization of 'max', 'min' and 'order statistics' multicriteria aggregation functions. In: Communication to *IFORS'2002*, Edinburgh, 8–12 July 2002
51. Bouyssou, D., Marchant, Th., Pirlot, M., Tsoukiàs, A., Vincke, Ph.: Evaluation and Decision Models with Multiple Criteria: Stepping Stones for the Analyst, 1st edn. International Series in Operations Research and Management Science, vol. 86. Springer, Boston (2006) [ISBN 0-387-31098-3]
52. Bouyssou, D., Marchant, T., Pirlot, M.: A conjoint measurement approach to the discrete Sugeno integral. In: Brams, S., Gehrlein, W.V., Roberts, F.S. (eds.) The Mathematics of Preference, Choice and Order. Essays in Honor of Peter C. Fishburn, pp. 85–109. Springer, Boston (2009) [ISBN 978-3-540-79127-0]
53. Briges, D.S., Mehta, G.B.: Representations of Preference Orderings. Springer, Berlin (1995)
54. Carroll, J.D., Green, P.E.: Psychometric methods in marketing research. Part 1. Conjoint analysis. J. Mark. Res. **32**, 385–391 (1995)
55. Charnes, A., Cooper, W.W.: Management Models and Industrial Applications of Linear Programming. Wiley, New York (1961)
56. Chateauneuf, A., Wakker, P.P.: From local to global additive representation. J. Math. Econ. **22**(6), 523–545 (1993)
57. Cosset, J.-C., Siskos, Y., Zopounidis, C.: The evaluation of country risk: a decision support approach. Glob. Finance J. **3**, 79–95 (1992)
58. Croon, M.A.: The axiomatization of additive difference models for preference judgements. In: Degreef, E., van Buggenhaut, G. (eds.) Trends in Mathematical Psychology, pp. 193–227. North-Holland, Amsterdam (1984)
59. Dahlstrand, V., Montgomery, H.: Information search and evaluation processes in decision-making: a computer-based process tracking study. Acta Psychol. **56**, 113–123 (1984)
60. de Donder, P., Le Breton, M., Truchon, M.: Choosing from a weighted tournament. Math. Soc. Sci. **40**, 85–109 (2000)
61. Debreu, G.: Representation of a preference ordering by a numerical function. In: Thrall, R., Coombs, C.H., Davies, R. (eds.) Decision Processes, pp. 159–175. Wiley, New York (1954)
62. Debreu, G.: Theory of Value: An Axiomatic Analysis of Economic Equilibrium. Wiley, New York (1959)
63. Debreu, G.: Topological methods in cardinal utility theory. In: Arrow, K.J., Karlin, S., Suppes, P. (eds.) Mathematical Methods in the Social Sciences, pp. 16–26. Stanford University Press, Stanford (1960)
64. Debreu, G.: Continuity properties of Paretian utility. Int. Econ. Rev. **5**(3), 285–293 (1964)
65. Delquié, Ph.: Inconsistent trade-offs between attributes: new evidence in preference assessment biases. Manag. Sci. **39**(11), 1382–1395 (1993)
66. Dimitras, A.I., Zopounidis, C., Hurson, C.: Assessing financial risks using a multicriteria sorting procedure: the case of country risk assessment. Found. Comput. Decis. Sci. **29**, 97–109 (2001)
67. Doumpos, M., Zopounidis, C.: Preference disaggregation and statistical learning for multicriteria decision support: a review. Eur. J. Oper. Res. **209**(3), 203–214 (2011)

68. Doumpos, M., Zanakis, S.H., Zopounidis, C.: Multicriteria preference disaggregation for classification problems with an application to global investing risk. Decis. Sci. **32**(2), 333–385 (2001)
69. Dubois, D., Marichal, J.-L., Prade, H., Roubens, M., Sabbadin, R.: The use of the discrete Sugeno integral in decision-making: a survey. Int. J. Uncertainty Fuzziness Knowledge Based Syst. **9**(5), 539–561 (2001)
70. Dyer, J.S.: MAUT – multiattribute utility theory. In: Figueira, J., Greco, S., Ehrgott, M. (eds.) Multiple Criteria Decision Analysis. State of the Art Surveys. International Series in Operations Research & Management Science, vol. 142, pp. 265–296. Springer, Berlin (2005)
71. Dyer, J.S., Sarin, R.K.: Measurable multiattribute value functions. Oper. Res. **27**, 810–822 (1979)
72. Edwards, W.: Social utilities. Eng. Econ. **6**, 119–129 (1971)
73. Edwards, W.: How to use multiattribute utility measurement for social decision making. IEEE Trans. Syst. Man Cybern. **7**(5), 326–340 (1977)
74. Edwards, W., Hutton Barron, F.: SMART and SMARTER: improved simple methods for multiattribute utility measurement. Organ. Behav. Hum. Decis. Process. **60**, 306–325 (1994)
75. Falmagne, J.-C.: On a class of probabilistic conjoint measurement models: some diagnostic properties. J. Math. Psychol. **19**(2), 73–88 (1979)
76. Fargier, H., Perny, P.: Modélisation des préférences par une règle de concordance généralisée. In: Colorni, A., Paruccini, M., Roy, B. (eds.) A-MCD-A, Aide Multicritère à la Décision/Multiple Criteria Decision Aid, pp. 99–115. European Commission, Joint Research Centre, Luxembourg (2001)
77. Ferguson, A., Myers, C.S., Bartlett, R.J., Banister, H., Bartlett, F.C., Brown, W., Campbell, N.R., Craik, K.J.W., Drever, J., Guild, J., Houstoun, R.A., Irwin, J.O., Kaye, G.W.C., Philpott, S.J.F., Richardson, L.F., Shaxby, J.H., Smith, T., Thouless, R.H., Tucker, W.: Quantitative Estimates of Sensory Events, vol. 2, pp. 331–349. British Association for the Advancement of Science, London (1940)
78. Figueira, J.R., Mousseau, V., Roy, B.: ELECTRE methods. In: Figueira, J., Greco, S., Ehrgott, M. (eds.) Multiple Criteria Decision Analysis. State of the Art Surveys. International Series in Operations Research & Management Science, vol. 142, pp. 133–162. Springer, Berlin (2005)
79. Figueira, J.R., Greco, S., Słowiński, R.: Building a set of additive value functions representing a reference preorder and intensities of preference: GRIP method. Eur. J. Oper. Res. **195**(2), 460–486 (2009)
80. Figueira, J.R., Greco, S., Roy, B., Słowiński, R.: ELECTRE Methods: Main features and recent developments. In: Zopounidis, C., Pardalos, P.M. (eds.) Handbook of Multicriteria Analysis. Applied Optimization, vol. 103, pp. 51–90. Springer, Berlin (2010) [ISBN 978-3-540-92827-0]
81. Fishburn, P.C.: Utility Theory for Decision-Making. Wiley, New York (1970)
82. Fishburn, P.C.: Intransitive indifference in preference theory: a survey. Oper. Res. **18**(2), 207–228 (1970)
83. Fishburn, P.C.: Lexicographic orders, utilities and decision rules: a survey. Manag. Sci. **20**(11), 1442–1471 (1974)
84. Fishburn, P.C.: Axioms for lexicographic preferences. Rev. Econ. Stud. **42**, 415–419 (1975)
85. Fishburn, P.C.: Noncompensatory preferences. Synthese **33**, 393–403 (1976)
86. Fishburn, P.C.: Condorcet social choice functions. SIAM J. Appl. Math. **33**, 469–489 (1977)
87. Fishburn, P.C.: A survey of multiattribute/multicriteria evaluation theories. In: Zionts, S. (ed.) Multicriteria Problem Solving, pp. 181–224. Springer, Berlin (1978)
88. Fishburn, P.C.: Lexicographic additive differences. J. Math. Psychol. **21**, 191–218 (1980)
89. Fishburn, P.C.: Nontransitive measurable utility. J. Math. Psychol. **26**, 31–67 (1982)
90. Fishburn, P.C.: Interval Orders and Intervals Graphs. Wiley, New York (1985)
91. Fishburn, P.C.: Nonlinear Preference and Utility Theory. Johns Hopkins University Press, Baltimore (1988)

92. Fishburn, P.C.: Normative theories of decision making under risk and under uncertainty. In: Non-conventional Preference Relations in Decision Making, pp. 1–21. Springer, Berlin (1988)
93. Fishburn, P.C.: Continuous nontransitive additive conjoint measurement. Math. Soc. Sci. **20**, 165–193 (1990)
94. Fishburn, P.C.: Skew symmetric additive utility with finite states. Math. Soc. Sci. **19**, 103–115 (1990)
95. Fishburn, P.C.: Additive non-transitive preferences. Econ. Lett. **34**, 317–321 (1990)
96. Fishburn, P.C.: Nontransitive additive conjoint measurement. J. Math. Psychol. **35**, 1–40 (1991)
97. Fishburn, P.C.: Nontransitive preferences in decision theory. J. Risk Uncertain. **4**, 113–134 (1991)
98. Fishburn, P.C.: Additive differences and simple preference comparisons. J. Math. Psychol. **36**, 21–31 (1992)
99. Fishburn, P.C.: A general axiomatization of additive measurement with applications. Nav. Res. Logist. **39**(6), 741–755 (1992)
100. Fishburn, P.C.: On nonstandard nontransitive additive utility. J. Econ. Theory **56**, 426–433 (1992)
101. Fishburn, P.C.: Finite linear qualitative probability. J. Math. Psychol. **40**, 21–31 (1996)
102. Fishburn, P.C.: Cancellation conditions for multiattribute preferences on finite sets. In: Karwan, M.H., Spronk, J., Wallenius, J. (eds.) Essays in Decision Making, pp. 157–167. Springer, Berlin (1997)
103. French, S.: Decision Teory – An Introduction to the Mathematics of Rationality. Ellis Horwood, London (1993)
104. Fryback, D.G., Keeney, R.L.: Constructing a complex judgmental model: an index of trauma severity. Manag. Sci. **29**, 869–883 (1983)
105. Furkhen, G., Richter, M.K.: Additive utility. Econ. Theory **1**, 83–105 (1991)
106. Furkhen, G., Richter, M.K.: Polynomial utility. Econ. Theory **1**, 231–249 (1991)
107. Gale, D.: The Theory of Linear Economic Models. McGraw-Hill, New York (1960)
108. Gilboa, I., Lapson, R.: Aggregation of semiorders: intransitive indifference makes a difference. Econ. Theory **5**, 109–126 (1995)
109. Gonzales, Ch.: Additive utilities when some components are solvable and others not. J. Math. Psychol. **40**, 141–151 (1996)
110. Gonzales, Ch.: Utilités additives: existence et construction. Thèse de doctorat, Université Paris 6 (1996)
111. Gonzales, Ch.: Two factor additive conjoint measurement with one solvable component. J. Math. Psychol. **44**, 285–309 (2000)
112. Gonzales, Ch.: Additive utility without restricted solvability on every component. J. Math. Psychol. **47**(1), 47–65 (2003)
113. Gorman, W.M.: The structure of utility functions. Rev. Econ. Stud. **35**, 367–390 (1968)
114. Grabisch, M.: The application of fuzzy integrals to multicriteria decision making. Eur. J. Oper. Res. **89**, 445–456 (1996)
115. Grabisch, M.: Fuzzy integral as a flexible and interpretable tool of aggregation. In: Aggregation and Fusion of Imperfect Information, pp. 51–72. Physica, Heidelberg (1998)
116. Grabisch, M., Labreuche, C.: Fuzzy measures and integrals in MCDA. In: Figueira, J., Greco, S., Ehrgott, M. (eds.) Multiple Criteria Decision Analysis. State of the Art Surveys. International Series in Operations Research & Management Science, vol. 142, pp. 563–608. Springer, Berlin (2005)
117. Grabisch, M., Labreuche, C.: A decade of application of the Choquet and Sugeno integrals in multi-criteria decision aid. 4OR Q. J. Oper. Res. **6**(1), 1–44 (2008)
118. Grabisch, M., Perny, P.: Agrégation multicritère. In: Bouchon-Meunier, B., Marsala, C. (eds.) Logique Floue, Principes, Aide à la Décision, IC^2, pp. 82–120. Hermès, Paris (2002)

119. Grabisch, M., Roubens, M.: Application of the Choquet integral in multicriteria decision making. In: Grabisch, M., Murofushi, T., Sugeno, M. (eds.) Fuzzy Measures and Integrals, pp. 348–374. Physica, Heidelberg (2000)
120. Grabisch, M., Orlovski, S.A., Yager, R.R.: Fuzzy aggregation of numerical preferences. In: Słowiński, R. (ed.) Fuzzy Sets in Decision Analysis, Operations Research and Statistics, pp. 31–68. Kluwer, Boston (1998)
121. Grabisch, M., Labreuche, Ch., Vansnick, J.-C.: On the extension of pseudo-boolean functions for the aggregation of interacting criteria. Eur. J. Oper. Res. **148**(1), 28–47 (2003)
122. Grassin, N.: Constructing criteria population for the comparison of different options of high voltage line routes. Eur. J. Oper. Res. **26**, 42–47 (1886)
123. Greco, S., Matarazzo, B., Słowiński, R.: The use of rough sets and fuzzy sets in MCDM. In: Gal, T., Hanne, T., Stewart, T. (eds.) Multicriteria decision making, Advances in MCDM Models, Algorithms, Theory and Applications, pp. 14.1–14.59. Kluwer, Dordrecht (1999)
124. Greco, S., Matarazzo, B., Słowiński, R.: Rough approximation of a preference relation by dominance relations. Eur. J. Oper. Res. **117**, 63–83 (1999)
125. Greco, S., Matarazzo, B., Słowiński, R.: Rough sets theory for multicriteria decision analysis. Eur. J. Oper. Res. **129**, 1–47 (2001)
126. Greco, S., Matarazzo, B., Słowiński, R.: Conjoint measurement and rough set approach for multicriteria sorting problems in presence of ordinal criteria. In: Colorni, A., Paruccini, M., Roy, B. (eds.) A-MCD-A, Aide Multicritère à la Décision/Multiple. Criteria Decision Aid, pp. 117–144. European Commission, Joint Research Centre, Luxembourg (2001)
127. Greco, S., Matarazzo, B., Słowiński, R.: Axiomatic basis of noncompensatoty preferences. In: Communication to *FUR X* (Foundations of Utility and Risk Theory), Torino, 30 May–2 June 2001
128. Greco, S., Matarazzo, B., Słowiński, R.: Preference representation by means of conjoint measurement and decision rule model. In: Bouyssou, D., Jacquet-Lagrèze, É., Perny, P., Słowiński, R., Vanderpooten, D., Vincke, Ph. (eds.) Aiding Decisions with Multiple Criteria: Essays in Honour of Bernard Roy, pp. 263–313. Kluwer, Boston (2002)
129. Greco, S., Matarazzo, B., Słowiński, R.: Axiomatic characterization of a general utility function and its particular cases in terms of conjoint measurement and rough-set decision rules. Eur. J. Oper. Res. **158**(2), 271–292 (2004)
130. Greco, S., Matarazzo, B., Słowiński, R.: Decision rule approach. In: Figueira, J., Greco, S., Ehrgott, M. (eds.) Multiple Criteria Decision Analysis. State of the Art Surveys. International Series in Operations Research & Management Science, vol. 142, pp. 507–562. Springer, Berlin (2005)
131. Greco, S., Mousseau, V., Słowiński, R.: Ordinal regression revisited: multiple criteria ranking using a set of additive value functions. Eur. J. Oper. Res. **191**(2), 416–436 (2008)
132. Greco, S., Słowiński, R., Figueira, J.R., Mousseau, V.: Robust ordinal regression. In: Ehrgott, M., Figueira, J., Greco, S. (eds.) Trends in Multiple Criteria Decision Analysis. International Series in Operations Research & Management Science, vol. 142, pp. 241–283. Springer, Berlin (2010)
133. Green, P.E., Krieger, A.M.: Conjoint analysis with product-positioning applications. In: Eliashberg, J., Lilien, G.L. (eds.) Marketing. Handbooks in Operations Research and Management Science, vol. 5, pp. 467–515. North-Holland, Amsterdam (1993) [ISBN 0-444-88957-4]
134. Green, P.E., Srinivasan, V.: Conjoint analysis in consumer research: issues and outlook. J. Consum. Res. **5**, 103–152 (1978)
135. Green, P.E., Tull, D.S., Albaum, G.: Research for Marketing Decisions. Prentice Hall, Englewood Cliffs (1988)
136. Grigoroudis, E., Siskos, Y.: Preference disaggregation for measuring and analysing customer satisfaction: the MUSA method. Eur. J. Oper. Res. **143**(1), 148–170 (2002)
137. Grigoroudis, E., Siskos, Y., Saurais, O.: TELOS: a customer satisfaction evaluation software. Comput. Oper. Res. **27**(7–8), 799–817 (2000)
138. Guild, J.: Part III of Quantitative estimation of sensory events. Interim report, British Academy for the Advancement of Science, pp. 296–328 (1936)

139. Gul, F.: Savage's theorem with a finite number of states. J. Econ. Theory **57**, 99–110 (1992)
140. Gustafsson, A., Herrmann, A., Huber, F. (eds.): Conjoint Measurement: Methods and Applications, 3rd edn. Springer, New York (2007) [ISBN 978-3-540-40479-8]
141. Halme, M., Kallio, M.: Estimation methods for choice-based conjoint analysis of consumer preferences. Eur. J. Oper. Res. **214**(1), 160–167 (2011)
142. Iverson, G., Falmagne, J.-C.: Statistical issues in measurement. Math. Soc. Sci. **10**, 131–153 (1985)
143. Jacquet-Lagrèze, E.: Interactive assessment of preferences using holistic judgments. The PREFCALC system. In: Bana e Costa, C.A. (ed.) Readings in Multiple Criteria Decision Aid, pp. 335–350. Springer, Berlin (1990)
144. Jacquet-Lagrèze, É.: An application of the UTA discriminant model for the evaluation of R&D projects. In: Pardalos, P., Siskos, Y., Zopounidis, C. (eds.) Advances in Multicriteria Analysis, pp. 203–211. Kluwer, Dordrecht (1995)
145. Jacquet-Lagrèze, É., Siskos, Y.: Assessing a set of additive utility functions for multicriteria decision making: the UTA method. Eur. J. Oper. Res. **10**, 151–164 (1982)
146. Jaffray, J.-Y.: On the extension of additive utilities to infinite sets. J. Math. Psychol. **11**, 431–452 (1974)
147. Jaffray, J.-Y.: Existence, propriétés de continuité, additivité de fonctions d'utilité sur un espace partiellement ou totalement ordonné. Thèse de doctorat d'état, Université Paris 6 (1974)
148. Johnson, R.M.: Trade-off analysis of consumer values. J. Mark. Res. **11**, 121–127 (1974)
149. Karabatsos, G., Ullrich, J.R.: Enumerating and testing conjoint measurement models. Math. Soc. Sci. **43**(3), 485–504 (2002)
150. Karni, E., Safra, Z.: The hexagon condition and additive representation for two dimensions: an algebraic approach. J. Math. Psychol. **42**, 393–399 (1998)
151. Keeney, R.L.: Measurement scales for quantifying attributes. Behav. Sci. **26**, 29–36 (1981)
152. Keeney, R.L.: Structuring objectives for problems of public interest. Oper. Res. **36**, 396–405 (1988)
153. Keeney, R.L.: Building models of values. Eur. J. Oper. Res. **37**(2), 149–157 (1988)
154. Keeney, R.L.: Value-Focused Thinking. Harvard University Press, Cambridge (1992)
155. Keeney, R.L., Raiffa, H.: Decisions with Multiple Objectives: Preferences and Value Trade-offs. Wiley, New York (1976)
156. Keeney, R.L., Robillard, G.A.: Assessing and evaluating environ-mental impacts at proposed nuclear power plant sites. J. Environ. Econ. Manag. **4**, 153–166 (1977)
157. Keeney, R.L., Hammond, J.S., Raiffa, H.: Smart Choices: A Guide to Making Better Decisions. Harvard University Press, Boston (1999)
158. Köbberling, V.: Comment on: Edi Karni & Zvi Safra (1998) The hexagon condition and additive representation for two dimensions: an algebraic approach. J. Math. Psychol. **47**(3), 370 (2003)
159. Köbberling, V.: Preference foundations for difference representations. Econ. Theory **27**, 375–391 (2006)
160. Koopmans, T.C.: Stationary ordinal utility and impatience. Econometrica **28**, 287–309 (1960)
161. Koopmans, T.C.: Representation of prefernce orderings over time. In: McGuire, C.B., Radner, R. (eds.) Decision and Organization, pp. 57–100. Noth-Holland, Amsterdam (1972)
162. Krantz, D.H.: Conjoint measurement: the Luce-Tukey axiomatization and some extensions. J. Math. Psychol. **1**, 248–277 (1964)
163. Krantz, D.H.: Extensive measurement in semiorders. Philos. Sci. **34**, 348–362 (1967)
164. Krantz, D.H., Luce, R.D., Suppes, P., Tversky, A.: Foundations of Measurement. Additive and Polynomial Representations, vol. 1. Academic, New York (1971)
165. Laffond, G., Laslier, J.-F., Le Breton, M.: Condorcet choice correspondences: a set-theoretical comparison. Math. Soc. Sci. **30**, 23–36 (1995)
166. Laslier, J.-F.: Tournament Solutions and Majority Voting. Springer, Berlin (1997)
167. Lemaire, B., Le Menestrel, M.: Homothetic interval orders. Discrete Math. **306**(15), 1669–1683 (2006)

168. Lemaire, B., Le Menestrel, M.: Generalized homothetic biorders. Discret. Math. **309**(12), 3793–3810 (2009)
169. Louviere, J.J.: Analyzing Decision Making: Metric Conjoint Analysis. Sage, Park (1988)
170. Louviere, J.J., Islam, T.: A comparison of importance weights and willingness-to-pay measures derived from choice-based conjoint, constant sum scales and best–worst scaling. J. Bus. Res. **61**(9), 903–911 (2008)
171. Luce, R.D.: Semiorders and a theory of utility discrimination. Econometrica **24**, 178–191 (1956)
172. Luce, R.D.: Three axiom systems for additive semiordered structures. SIAM J. Appl. Math. **25**, 41–53 (1973)
173. Luce, R.D.: Conjoint measurement: a brief survey. In: Bell, D.E., Keeney, R.L., Raiffa, H. (eds.) Conflicting Objectives in Decisions, pp. 148–171. Wiley, New York (1977)
174. Luce, R.D.: Lexicographic tradeoff structures. Theor. Decis. **9**, 187–193 (1978)
175. Luce, R.D.: Utility of Gains and Losses: Measurement-Theoretical and Experimental Approaches. Lawrence Erlbaum, Mahwah (2000)
176. Luce, R.D., Cohen, M.: Factorizable automorphisms in solvable conjoint structures, I. J. Pure Appl. Algebra **27**(3), 225–261 (1983)
177. Luce, R.D., Marley, A.A.J.: Extensive measurement when concatenation is restricted. In: Morgenbessser, S., Suppes, P., White, M.G. (eds.) Philosophy, Science and Method: Essays in Honor of Ernest Nagel, pp. 235–249. St. Martin's Press, New York (1969)
178. Luce, R.D., Steingrimsson, R.: Theory and tests of the conjoint commutativity axiom for additive conjoint measurement. J. Math. Psychol. **55**(5), 379–385 (2011)
179. Luce, R.D., Tukey, J.W.: Simultaneous conjoint measurement: a new type of fundamental measurement. J. Math. Psychol. **1**, 1–27 (1964)
180. Luce, R.D., Krantz, D.H., Suppes, P., Tversky, A.: Foundations of Measurement. Representation, Axiomatisation and Invariance, vol. 3. Academic, New York (1990)
181. Maas, A., Wakker, P.P.: Additive conjoint measurement for multiattribute utility. J. Math. Psychol. **38**, 86–101 (1994)
182. Marichal, J.-L.: An axiomatic approach of the discrete Choquet integral as a tool to aggregate interacting criteria. IEEE Trans. Fuzzy Syst. **8**, 800–807 (2000)
183. Marichal, J.-L.: On Sugeno integrals as an aggregation function. Fuzzy Sets Syst. **114**, 347–365 (2000)
184. Marichal, J.-L.: On Choquet and Sugeno integrals as aggregation functions. In: Grabisch, M., Murofushi, T., Sugeno, M. (eds.) Fuzzy Measures and Integrals, pp. 247–272. Physica, Heidelberg (2000)
185. Marichal, J.-L., Roubens, M.: Determination of weights of interacting criteria from a reference set. Eur. J. Oper. Res. **124**, 641–50 (2000)
186. Matsatsinis, N.F., Siskos, Y.: MARKEX: an intelligent decision support system for product development decisions. Eur. J. Oper. Res. **113**(2), 336–354 (1999)
187. May, K.O.: Intransitivity, utility and the aggregation of preference patterns. Econometrica **22**, 1–13 (1954)
188. Le Menestrel, M., Lemaire, B.: Biased extensive measurement: the homogeneous case. J. Math. Psychol. **48**(1), 9–14 (2004)
189. Le Menestrel, M., Lemaire, B.: Biased extensive measurement: the general case. J. Math. Psychol. **50**(6), 570–581 (2006)
190. Mihelis, G., Grigoroudis, E., Siskos, Y., Politis, Y., Malandrakis, Y.: Customer satisfaction measurement in the private bank sector. Eur. J. Oper. Res. **130**(2), 347–360 (2001)
191. Miller, N.R.: Graph theoretical approaches to the theory of voting. Am. J. Polit. Sci. **21**, 769–803 (1977)
192. Miyamoto, J., Wakker, P.P.: Multiattribure utility theory without expected utility foundations. Oper. Res. **44**(2), 313–326 (1996)
193. Montgomery, H.: A study of intransitive preferences using a think aloud procedure. In: Jungerman, H., de Zeeuw, G. (eds.) Decision-Making and Change in Human Affairs, pp. 347–362. D. Reidel, Dordrecht (1977)

194. Montgomery, H., Svenson, O.: On decision rules and information processing strategies for choice among multiattribute alternatives. Scand. J. Psychol. **17**, 283–291 (1976)
195. Nakamura, Y.: Lexicographic additivity for multi-attribute preferences on finite sets. Theor. Decis. **42**, 1–19 (1997)
196. Nakamura, Y.: Additive utility on densely ordered sets. J. Math. Psychol. **46**, 515–530 (2002)
197. Narens, L.: Abstract Measurement Theory. MIT press, Cambridge (1985)
198. Narens, L., Luce, R.D.: The algebra of measurement. J. Pure Appl. Algebra **8**(2), 197–233 (1976)
199. Netzer, O., Toubia, O., Bradlow, E.T., Dahan, E., Evgeniou, T., Feinberg, F.M., Feit, E.M., Hui, S.K., Johnson, J., Liechty, J.C., Orlin, J.B., Rao, V.R.: Beyond conjoint analysis: advances in preference measurement. Mark. Lett. **19**(3–4), 337–354 (2008)
200. Ostanello, A.: Action evaluation and action structuring – different decision aid situations reviewed through two actual cases. In: Bana e Costa, C.A. (ed.) Readings in Multiple Criteria Decision Aid, pp. 36–57. Springer, Berlin (1990)
201. Peris, J.E., Subiza, B.: Condorcet choice correspondences for weak tournaments. Soc. Choice Welf. **16**, 217–231 (1999)
202. Pfanzagl, J.: Theory of Measurement, 2nd edn. Physica, Würzburg (1971)
203. Phillips, L.D., Bana e Costa, C.A.: Transparent prioritisation, budgeting and resource allocation with multi-criteria decision analysis and decision conferencing. Ann. Oper. Res. **154**(1), 51–68 (2008)
204. Pirlot, M., Vincke, Ph.: Lexicographic aggregation of semiorders. J. Multicrit. Decis. Anal. **1**, 47–58 (1992)
205. Pirlot, M., Vincke, Ph.: Semiorders. Properties, Representations, Applications. Kluwer, Dordrecht (1997)
206. Pomerol, J.Ch., Barba-Romero, S.: Multicriterion Decision in Management, Principles and Practice. Kluwer, Dordrecht (2000)
207. Pöyhönen, M., Vrolijk, H., Hämäläinen, R.P.: Behavioral and procedural consequences of structural variations in value trees. Eur. J. Oper. Res. **134**(1), 216–227 (2001)
208. Raiffa, H.: Decision Analysis – Introductory Lectures on Choices Under Uncertainty. Addison-Wesley, Reading (1968)
209. Raiffa, H.: Preference for multi-attributed alternatives. RAND Memorandum RM-5868-DOT/RC, RAND, Santa Monica (1968)
210. Richter, M.K.: Rational choice and polynomial measurement theory. J. Math. Psychol. **12**, 99–113 (1975)
211. Roberts, F.S.: Measurement Theory with Applications to Decision Making, Utility and the Social Sciences. Addison-Wesley, Reading (1979)
212. Roberts, F.S., Luce, R.D.: Axiomatic thermodynamics and extensive measurement. Synthese **18**, 311–326 (1968)
213. Roubens, M., Vincke, Ph.: Preference Modelling. Springer, Berlin (1985)
214. Roy, B.: Classement et choix en présence de points de vue multiples (la méthode ELECTRE). RIRO **2**, 57–75 (1968)
215. Roy, B.: Main sources of inaccurate determination, uncertainty and imprecision in decision models. Math. Comput. Model. **12**(10–11), 1245–1254 (1989)
216. Roy, B.: The outranking approach and the foundations of ELECTRE methods. Theor. Decis. **31**, 49–73 (1991)
217. Roy, B.: Multicriteria Methodology for Decision Aiding. Kluwer, Dordrecht (1996) [Original version in French, Méthodologie multicritère d'aide à la décision. Economica, Paris (1985)]
218. Roy, B., Bouyssou, D.: Aide multicritère à la décision: méthodes et cas. Economica, Paris (1993)
219. Russo, J.E., Dosher, B.A.: Strategies for multiattribute binary choice. J. Exp. Psychol. Learn. Mem. Cogn. **9**, 676–696 (1983)
220. Saari, D.G.: Connecting and resolving Sen's and Arrow's theorems. Soc. Choice Welf. **15**, 239–261 (1998)
221. Savage, L.J.: The Foundations of Statistics. Wiley, New York (1954)

222. Schwartz, T.: The Logic of Collectice Choice. Columbia University Press, New York (1986)
223. Scott, D.: Measurement structures and linear inequalities. J. Math. Psychol. **1**, 233–247 (1964)
224. Scott, D., Suppes, P.: Foundational aspects of theories of measurement. J. Symb. Log. **23**, 113–128 (1958)
225. Segal, U.: A sufficient condition for additively separable functions. J. Math. Econ. **23**, 295–303 (1994)
226. Segal, U., Sobel, J.: Min, max and sum. J. Econ. Theory **106**, 126–150 (2002)
227. Sen, A.K.: Social choice theory. In: Arrow, K.J., Intriligator, M.D. (eds.) Handbook of Mathematical Economics, vol. 3, pp. 1073–1181. North-Holland, Amsterdam (1986)
228. Shapiro, L.: Conditions for expected uility maximization. Ann. Stat. **7**, 1288–1302 (1979)
229. Siskos, Y.: A way to deal with fuzzy preferences in multi-criteria decision problems. Eur. J. Oper. Res. **10**(3), 314–324 (1982)
230. Siskos, Y.: Evaluating a system of furniture retail outlets using an interactive ordinal regression model. Eur. J. Oper. Res. **23**, 179–193 (1986)
231. Siskos, Y., Assimakopoulos, N.: Multicriteria highway planning: a case study. Math. Comput. Model. **12**(10–11), 1401–1410 (1989)
232. Siskos, Y., Despotis, D.K.: A DSS oriented method for multiobjective linear programming problems. Decis. Support Syst. **5**(1), 47–56 (1989)
233. Siskos, Y., Grigoroudis, E.: New trends in aggregation-disaggregation approaches. In: Zopounidis, C., Pardalos, P.M. (eds.) Handbook of Multicriteria Analysis. Applied Optimization, vol. 103, pp. 189–214. Springer, Berlin (2010) [ISBN 978-3-540-92827-0]
234. Siskos, Y., Yannacopoulos, D.: UTASTAR, an ordinal regression method for building additive value functions. Investigaçao Operacional **5**(1), 39–53 (1985)
235. Siskos, Y., Zopounidis, C.: The evaluation of venture capital investment activity: an interactive assessment. Eur. J. Oper. Res. **31**(3), 304–313 (1987)
236. Siskos, Y., Despotis, D.K., Ghediri, M.: Multiobjective modelling for regional agricultural planning: case study in Tunisia. Eur. J. Oper. Res. **77**(3), 375–391 (1994)
237. Siskos, Y., Zopounidis, C., Pouliezos, A.: An integrated DSS for financing firms by an industrial development bank in Greece. Decis. Support Syst. **12**(2), 151–168 (1994)
238. Siskos, Y., Grigoroudis, E., Zopounidis, C., Saurais, O.: Measuring customer satisfaction using a collective preference disaggregation model. J. Glob. Optim. **12**(2), 175–195 (1998)
239. Siskos, Y., Spyridakos, A., Yannacopoulos, D.: Using artificial intelligence and visual techniques into preference disaggregation analysis: the MIIDAS system. Eur. J. Oper. Res. **113**(2), 281–299 (1999)
240. Siskos, Y., Matsatsinis, N.F., Baourakis, G.: Multicriteria analysis in agricultural marketing: the case of French olive oil market. Eur. J. Oper. Res. **130**(2), 315–331 (2001)
241. Siskos, Y., Grigoroudis, E., Matsatsinis, N.F.: UTA methods. In: Figueira, J., Greco, S., Ehrgott, M. (eds.) Multiple Criteria Decision Analysis. State of the Art Surveys. International Series in Operations Research & Management Science, vol. 142, pp. 265–296. Springer, Berlin (2005)
242. Skala, H.J.: Non-Archimedean Utility Theory. Kluwer, Dordrecht (1975)
243. Sounderpandian, J.: Value functions when decision criteria are not totally substitutable. Oper. Res. **39**, 592–600 (1991)
244. Spyridakos, A., Siskos, Y., Yannacopoulos, D., Skouris, A.: Multicriteria job evaluation for large organizations. Eur. J. Oper. Res. **130**(2), 375–387 (2001)
245. Stewart, T.: An interactive multiple objective linear programming method based on piecewise linear additive value functions. IEEE Trans. Syst. Man Cybern. **SMC-17**(5), 799–805 (1987)
246. Stillwell, W.G., von Winterfeldt, D., John, R.S.: Comparing hierarchical and nonhierarchical weighting methods for eliciting multiattribute value models. Manag. Sci. **33**, 442–50 (1987)
247. Suppes, P., Winet, M.: An axiomatization of utility based on the notion of utility difference. Manag. Sci. **1**, 259–270 (1955)
248. Suppes, P., Krantz, D.H., Luce, R.D., Tversky, A.: Foundations of Measurement. Geometrical, Threshold, and Probabilistic Representations, vol. 2. Academic, New York (1989)

249. Svenson, O.: Process description of decision making. Organ. Behav. Hum. Perform. **23**, 86–112 (1979)
250. Titiev, R.J.: Measurement structures in classes that are not universally axiomatizable. J. Math. Psychol. **9**, 200–205 (1972)
251. Tversky, A.: A general theory of polynomial conjoint measurement. J. Math. Psychol. **4**, 1–20 (1967)
252. Tversky, A.: Intransitivity of preferences. Psychol. Rev. **76**, 31–48 (1969)
253. Vincke, Ph.: Multi-criteria Decision Aid. Wiley, New York (1992) [Original version in French, L'Aide Multicritère à la Décision", éditions de l'Université de Bruxelles-éditions. Ellipses, Brussels (1989)]
254. Vincke, Ph.: Outranking approach. In: Gal, T., Stewart, T., Hanne, T. (eds.) Multicriteria Decision Making. Advances in MCDM Models, Algorithms, Theory and Applications, pp. 11.1–11.29. Kluwer, Dordrecht (1999)
255. Vind, K.: Independent preferences. J. Math. Econ. **20**, 119–135 (1991)
256. Vind, K.: Independence, Additivity, Uncertainty. Springer, Berlin (2003)
257. von Neumann, J., Morgenstern, O.: Theory of Games and Economic Behavior, 2nd edn. Wiley, New York (1947)
258. von Nitzsch, R., Weber, M.: The effect of attribute ranges on weights in multiattribute utility measurements. Manag. Sci. **39**(8), 937–943 (1993)
259. von Winterfeldt, D., Edwards, W.: Decision Analysis and Behavioral Research. Cambridge University Press, Cambridge (1986)
260. Wakker, P.P.: Cardinal coordinate independence for expected utility. J. Math. Psychol. **28**(1), 110–117 (1984)
261. Wakker, P.P.: Additive Representations of Preferences: A New Foundation of Decision Analysis. Kluwer, Dordrecht (1989)
262. Wakker, P.P.: Additive representations on rank-ordered sets, I. The algebraic approach. J. Math. Psychol. **35**(4), 501–531 (1991)
263. Wakker, P.P.: Additive representations of preferences, a new foundation of decision analysis; the algebraic approach. In: Doignon, J.-P., Falmagne, J.-C. (eds.) Mathematical Psychology: Current Developments, pp. 71–87. Springer, Berlin (1991)
264. Wakker, P.P.: Additive representation for equally spaced structures. J. Math. Psychol. **35**, 260–266 (1991)
265. Wakker, P.P.: Additive representations on rank-ordered sets, II. The topological approach. J. Math. Econ. **22**(1), 1–26 (1993)
266. Weber, M., Eisenfuhr, F., von Winterfeld, D.: The effects of splitting attributes on weights in multiattribute utility measurement. Manag. Sci. **34**(4), 431–445 (1988)
267. Weymark, J.A.: Generalized Gini inequality indices. Math. Soc. Sci. **1**, 409–430 (1981)
268. Wille, U.: Linear measurement models: axiomatizations and axiomatizability. J. Math. Psychol. **44**, 617–650 (2000)
269. Zopounidis, C., Doumpos, M.: Stock evaluation using a preference disaggregation methodology. Decis. Sci. **30**, 313–336 (1999)
270. Zopounidis, C., Doumpos, M.: PREFDIS: a multicriteria decision support system for sorting decision problems. Comput. Oper. Res. **27**(7–8), 779–797 (2000)
271. Zopounidis, C., Doumpos, M.: Building additive utilities for multi-group hierarchical discrimination: the MHDIS method. Optim. Methods Softw. **14**(3), 219–240 (2000)
272. Zopounidis, C., Doumpos, M.: A preference disaggregation decision support system for financial classification problems. Eur. J. Oper. Res. **130**(2), 402–413 (2001)
273. Zopounidis, C., Doumpos, M.: Multicriteria preference disaggregation for classification problems with an application to global investing risk. Decis. Sci. **32**(2), 333–385 (2001)

Part III
Outranking Methods

Chapter 5
ELECTRE Methods

José Rui Figueira, Vincent Mousseau, and Bernard Roy

Abstract Over the last three decades a large body of research in the field of ELECTRE family methods appeared. This research has been conducted by several researchers mainly in Europe. The purpose of this chapter is to present a survey of the ELECTRE methods since their first appearance in mid-60s, when ELECTRE I was proposed by Bernard Roy and his colleagues at SEMA consultancy company. The chapter is organized in five sections. The first section presents a brief history of ELECTRE methods. The second section is devoted to the main features of ELECTRE methods. The third section describes the different ELECTRE methods existing in the literature according to the three main problematics: choosing, ranking and sorting. The fourth section presents the recent developments and future issues on ELECTRE methods. Finally, the fifth section is devoted to the software and applications. An extensive and up-to-date bibliography is also provided in the end of this chapter.

Keywords Multiple criteria decision aiding • Outranking approaches • ELECTRE methods

J.R. Figueira (✉)
CEG-IST, Instituto Superior Técnico, Universidade de Lisboa, A. Rovisco Pais, 1, 1049-001 Lisboa, Portugal
e-mail: figueira@tecnico.ulisboa.pt

V. Mousseau
Laboratoire de Génie Industriel, Ecole Centrale de Paris, Grande Voie des Vignes, 92 295 Châtenay-Malabry Cedex, France
e-mail: vincent.mousseau@ecp.fr

B. Roy
LAMSADE, Université Paris-Dauphine, Place du Maréchal De Lattre de Tassigny, 75 775 Paris Cedex 16, France
e-mail: roy@lamsade.dauphine.fr

S. Greco et al. (eds.), *Multiple Criteria Decision Analysis*, International Series in Operations Research & Management Science 233,
DOI 10.1007/978-1-4939-3094-4_5

5.1 Introduction: A Brief History

How far back in history should we go to discover the origins of ELECTRE methods? Some years ago B. Roy and D. Vanderpooten [133] published an article ("The European School of MCDA: Emergence, Basic Features and Current Works", *Journal of Multi-Criteria Decision Analysis*) on this very topic. This introduction is largely based on their paper, but additional material has been included to define the origins more precisely and to look more deeply into the history of ELECTRE methods. We have also benefited from an old, but nonetheless excellent, bibliography containing a lot of references collated by Y. Siskos et al. [145]. The latter only covers the period 1966–1982, but contains many valuable references.

The origins of ELECTRE methods go back to 1965 at the European consultancy company SEMA, which is still active today. At that time, a research team from SEMA worked on a concrete, multiple criteria, real-world problem regarding decisions dealing with the development of new activities in firms. For "solving" this problem a general multiple criteria method, MARSAN (*Méthode d'Analyse, de Recherche, et de Sélection d'Activités Nouvelles*) was built. The analysts used a weighted-sum based technique included in the MARSAN method for the selection of the new activities [67]. When using the method the engineers from SEMA noticed serious drawbacks in the application of such a technique. B. Roy was thus consulted and soon tried to find a new method to overcome the limitations of MARSAN. The ELECTRE method for choosing the best action(s) from a given set of actions was thus devised in 1965, and was later referred to as ELECTRE I (electre one). In that same year (July, 1965) the new multiple criteria outranking method was presented for the first time at a conference (*les journées d'études sur les méthodes de calcul dans les sciences de l'homme*), in Rome (Italy). Nevertheless, the original ideas of ELECTRE methods were first merely published as a research report in 1966, the notorious *Note de Travail 49 de la SEMA* [14]. Shortly after its appearance, ELECTRE I was found to be successful when applied to a vast range of fields [22], but the method did not become widely known until 1968 when it was published in *RIRO, la Revue d'Informatique et de Recherche Opérationnelle* [102]. This article presents a comprehensive description of ELECTRE and the foundations of the outranking approach; the reader may also consult the graph theory book by B. Roy [103]. The method has since evolved and given rise to an "unofficial" version, ELECTRE Iv (electre one vee). This version took into account the notion of a veto threshold. A further version known as ELECTRE IS (electre one esse) appeared subsequently (see [130]) and was used for modeling situations in which the data was imperfect (see below). This is the current version of ELECTRE methods for choice problematic.

The acronym ELECTRE stands for [14, 108]: *ELimination Et Choix Traduisant la REalité* (ELimination and Choice Expressing the REality), and was cited for commercial reasons. At the time it seemed adequate and served well to promote the new tool. Nevertheless, the developments in ELECTRE methods over the last three decades, the way in which we consider the tool today and the methodological foundations of multiple criteria decision aiding have made the meaning of the acronym unsatisfactory.

An atypical ELECTRE method was also created to deal with the problem of highway layout in the *Ile de France* region; it was called the meaningful compensation method [15, 16, 29, 104, 124]. This approach was based on substitution rates. These rates were ill-defined (stakeholders views about their values strongly differed), it was only possible to fix a minimum and maximum value for each one. On such basis a set of embedded fuzzy relations has been defined.

In the late 60s, a different real-world decision making situation arose in media planning, concerning the definition of an advertising plan. For such a purpose the question was: how to establish an adequate system of ranking for periodicals (magazines, newspapers, ...)? This led to the birth of ELECTRE II (electre two): a method for dealing with the problem of ranking actions from the best option to the worst [1, 51, 121, 122]. However, in a world where perfect knowledge is rare, imperfect knowledge only could be taken into account in ELECTRE methods through the use of probabilistic distributions and expected utility criterion. Clearly more work needed to be done. Research in this area was still in its initial stages. Another way to cope with uncertain, imprecision and ill-determination has been introduced, the threshold approach [23, 57, 58, 134]. For more details and a comprehensive treatment of this issue see [18, 109, 110]. Just a few years later a new method for ranking actions was devised: ELECTRE III (electre three), [106, 135]. The main new ideas introduced by this method were the use of pseudo-criteria (see [105]) and fuzzy binary outranking relations. Another ELECTRE method, known as ELECTRE IV (electre four), arose from a new real-world problem related to the Paris subway network [46, 53, 125, 126, 128]. It now became possible to rank actions without using the relative criteria importance coefficients; this is the only ELECTRE method which does not make use of such coefficients. In addition, the new method was equipped with an embedded outranking relations framework.

Methods created up to this point were particularly designed to help decision making in choosing and ranking actions. However, in the late 70s a new technique of sorting actions into predefined and ordered categories was proposed i.e. the trichotomy procedure [78, 79, 107]. This is a decision tree based approach. Several years later, in order to help decision making in a large banking company which faced to the problem of accepting or refusing credits requested by firms, a specific method, ELECTRE A, was devised and applied in ten sectors of activity. This should have remained confidential. One of the most recent sorting method, ELECTRE TRI (electre tree), was greatly inspired by these earlier works. It removed everything they had of specific given their context of application. Indeed, this new method is, at the same time, both simpler and more general [160, 161]. Very recently, a new ELECTRE sorting method was proposed, ELECTRE TRI-C [4], where categories are defined by characteristic typical reference actions instead of boundary actions as in the previous methods. Another extension allowing to define each category through the definition of several characteristics reference actions *per* category can be found in [5].

ELECTRE methods are still evolving (see [42]). Section 5.4 presents recent developments on the topic and avenues for future research.

5.2 Main Features of ELECTRE Methods

This section presents a set of key issues concerning ELECTRE methods: the context in which they are relevant, modeling with an outranking relation, their structure, the role of criteria, and how to account for imperfect knowledge.

5.2.1 In What Context Are ELECTRE Methods Relevant?

ELECTRE methods are relevant when facing decision situations with the following characteristics (see, [112, 124, 138]).

1. The decision-maker (DM) wants to include in the model at least three criteria. However, aggregation procedures are particularly adequate in situations when decision models include more than five criteria (up to 12 or 13). Let g_j, $j = 1, \ldots, n$ denote a coherent family of criteria and let A denote the set of potential actions; $g_j(a)$ represents the performance of action a on criterion g_j.

   And, at least one of the following situations must be verified.

2. Actions are evaluated (for at least one criterion) on an ordinal scale (see [96]) or on a weakly interval scale (see [72]). These scales are not suitable for the comparison of differences. Hence, it is difficult and/or artificial to define a coding that makes sense in terms of preference differences of the ratios $\frac{g_j(a)-g_j(b)}{g_j(c)-g_j(d)}$, where a, b, c, and d are four different actions.
3. A strong heterogeneity related with the nature of the scales associated with the criteria exists (e.g., environment impact, cost, aesthetics, duration, noise, distance, security, ...). This makes it difficult to define a unique and common scale that could be used to substitute the original ones.
4. Compensation of the loss on a given criterion by a gain on another one may not be acceptable for the DM. Therefore, such situations require the use of noncompensatory aggregation procedures (see Chap. 2).
5. For at least one criterion the following holds true: small differences of preferences must not be considered as significant. This requires the introduction of discriminating (indifference and preference) thresholds (see Chap. 4).

5.2.2 Modeling Preferences Using an Outranking Relation

Preferences in ELECTRE methods are modeled by using binary outranking relations, S, whose meaning is *at least as good as*. Considering two actions a and b, four situations may occur:

- aSb and not bSa, i.e., $a \succ b$ (a is preferred to b).
- bSa and not aSb, i.e., $b \succ a$ (b is preferred to a).
- aSb and bSa, i.e., aIb (a is indifferent to b).
- Not aSb and not bSa, i.e., aRb (a is incomparable to b).

ELECTRE methods build one or several (crispy, fuzzy or embedded) outranking relations.

Note that using outranking relations to model preferences introduces a new preference relation, R, (incomparability). This relation is useful to account for situations in which the DM and/or the analyst are not able to compare two actions.

The construction of an outranking relation is based on two major concepts:

1. *Concordance*. For an outranking aSb to be validated, a *sufficient* majority of criteria should be in favor of this assertion.
2. *Non-discordance*. When the concordance condition holds, none of the criteria in the minority should oppose too strongly to the assertion aSb.

These two conditions must be fulfilled for validating the assertion aSb.

Given a binary relation on set A, from the first procedure, is extremely helpful to build a graph $G = (V, U)$, where V is the set of vertices and U the set of arcs. For each action $a \in A$ we associate a vertex $i \in V$ and for each pair of actions $(a, b) \in A$ the arc (i, l) exists if aSb. If there is no arc between vertices i and l, it means that a and b are incomparable; if two reversal arcs exist, there is an indifference between both a and b.

The outranking relation(s) is(are) not necessarily transitive. This requires an exploitation procedure to derive from such relation(s) results that fit the problematic (see Chap. 2).

Outranking relations may lead to preference intransitivities stemming from two different situations: Condorcet effect and incomparabilities between actions.

5.2.3 Structure of ELECTRE Methods

ELECTRE methods comprise two main procedures:

- A multiple criteria aggregation procedure, allowing for the construction of one or several outranking relation(s) aiming to compare in a comprehensive way each pair of actions.
- An exploitation procedure that lead to produce results according to the nature of problematic (choosing, ranking or sorting)

Hence, each method is characterized by its construction and its exploitation procedures. For more details the reader may consult the following references: [81, 111, 112, 124, 154, 157].

5.2.4 About the Relative Importance of Criteria

The relative role attached to criteria in ELECTRE methods is defined by two distinct sets of parameters: the relative importance coefficients and the veto thresholds.

The importance coefficients in ELECTRE methods refer to intrinsic "weights". For a given criterion the weight, w_j, reflects its voting power when it contributes to the majority which is in favor of an outranking. The weights do not depend neither on the ranges nor the encoding of the scales. Let us point out that these parameters cannot be interpreted as substitution rates as in compensatory aggregation procedures AHP [136], MACBETH [11] and MAUT [64].

Veto thresholds express the power attributed to a given criterion to be against the assertion "a outranks b", when the difference of the performances between $g(b)$ and $g(a)$ is greater than this threshold. Such a threshold can be constant along a scale or it can also vary.

A large amount of works have been published on the topic of relative importance of criteria. The following list is not exhaustive: [39, 74, 80, 98, 99, 129, 135, 142, 155].

It should be noticed that there are no true values for weights and veto thresholds.

5.2.5 Discriminating Thresholds

To take into account the imperfect character of the evaluation of actions and the arbitrariness when building a family of criteria (see Chap. 2), ELECTRE methods make use of discriminating (indifference and preference) thresholds. This leads to the construction of a pseudo-criterion model for each criterion (see Chap. 3).

Discriminating thresholds account for the arbitrariness and the imperfect nature of the performances, and are used for modeling situations in which the difference between performances associated with two different actions on a given criterion may either:

- Justify the preference in favor of one of the two actions (*preference threshold*, p_j).
- Be compatible with indifference between the two actions (*indifference thresholds*, q_j).
- Be interpreted as an hesitation between opting for a preference or an indifference between the two actions.

These thresholds ($p_j \geqslant q_j \geqslant 0$) can be constant or vary along the scale. When they are variable we must distinguish between *direct* (the performance of the worst action is taken into account) and *inverse* (when they are computed by using the best performance).

How to assign values to such thresholds? There are several techniques which can be used, some of them comes directly from the definition of threshold and other ask for the concept of *dispersion threshold* (see Sect. 5.4.2).

A dispersion threshold allow us to take into account the concept of probable value and the notion of optimistic and pessimistic values. It translates the plausible difference, due to over or under-estimations, which affect the evaluation of a *consequence* or of a performance level.

It should be noticed that there are no true values for thresholds. Therefore, the values chosen to assign to the thresholds are the most convenient (the best adapted) for expressing the imperfect character of the knowledge.

For more details about thresholds see, [3, 19, 108, 113, 115–117, 124].

5.3 A Short Description of ELECTRE Methods

A comprehensive treatment of ELECTRE methods may be found in the books by B. Roy and D. Bouyssou [124] and Ph. Vincke [158]. Much of the theory developed on this field is presented in these books. This theory, however, was foreshadowed in earlier papers namely by B. Roy and his colleagues at SEMA and later at LAMSADE (some of these papers were cited in the introduction). The books [74, 108, 113, 138, 139] are also good references in the area. ELECTRE software manuals also contain much material both on theoretical and pedagogical issues [3, 51, 84, 130, 153, 161]. Finally, several other works deserve to be mentioned because they include information concerning ELECTRE methods: [9, 20, 21, 24, 45, 60, 91, 99, 142].

In what follows we will only summarize the elementary concepts underlying ELECTRE methods; details will be omitted. More sophisticated presentations can, however, be found in the references cited above.

Description of methods is presented in problematic and chronological order.

5.3.1 Choice Problematic

Let us remind the purpose of choice problematic before presenting methods. The objective of this problematic consists of aiding DMs in selecting a subset, as small as possible of actions, in such a way that a single action may finally be chosen.

The order in which methods will be presented permit us to understand the historical introduction of the two fundamental concepts in multiple criteria decision aiding, *veto thresholds* and *pseudo-criteria*.

5.3.1.1 ELECTRE I

The purpose underlying the description of this method is rather theoretical and pedagogical. The method does not have a significant practical interest, given the very nature of real-world applications, having usually a vast spectrum of quantitative and qualitative elementary consequences, leading to the construction of a contradictory and very heterogeneous set of criteria with both numerical and ordinal scales associated with them. In addition, a certain degree of imprecision, uncertainty or ill-determination is always attached to the knowledge collected from real-world problems.

The method is very simple and it should be applied only when all the criteria have been coded in numerical scales with identical ranges. This is a simplified version of the original method. In such a situation we can assert that an action "*a* outranks *b*" (that is, "*a is at least as good as b*") denoted by *aSb*, only when two conditions hold.

Coding scales of criteria to get a new one with identical ranges is not a simple activity; it cannot be easy to get this common scale. It is necessary that this coding justifies the use of the max operator, introduced hereafter to model the discordance. Otherwise, the use of this operator will not be completely justified. Thus, it is very difficult to built identical numerical scales to compare actions and give a meaning to the max operator. This justifies the introduction of ELECTRE Iv.

On the one hand, the *strength of the concordant coalition* must be powerful enough to support the above assertion. By strength of the concordant coalition, we mean the sum of the weights associated with the criteria forming such a coalition. It can be defined by the following *concordance index* (assuming, for the sake of formulae simplicity, that $\sum_{j \in \mathcal{J}} w_j = 1$):

$$c(aSb) = \sum_{\{j \,:\, g_j(a) \geqslant g_j(b)\}} w_j$$

(where $\{j \;:\; g_j(a) \geqslant g_j(b)\}$ is the set of indices for all the criteria belonging to the concordant coalition with the outranking relation *aSb*).

In other words, the value of the concordance index must be greater than or equal to a given *concordance level*, s, whose value generally falls within the range $[0.5, 1 - \min_{j \in \mathcal{J}} w_j]$, i.e., $c(aSb) \geqslant s$.

On the other hand, no *discordance* against the assertion "*a* is at least as good as *b*" may occur. The discordance is measured by a *discordance level* defined as follows:

$$d(aSb) = \max_{\{j \,:\, g_j(a) < g_j(b)\}} \Big\{ g_j(b) - g_j(a) \Big\}$$

This level measures in some way the power of the discordant coalition, meaning that if its value surpasses a given level, v, the assertion is no longer valid. Discordant coalition exerts no power whenever $d(aSb) \leqslant v$.

Both concordance and discordance indices have to be computed for every pair of actions (a, b) in the set A, where $a \neq b$.

It is easy to see that such a computing procedure leads to a binary relation in comprehensive terms (taking into account the whole set of criteria) on the set A. Hence for each pair of actions (a, b), only one of the preference situations mentioned in Sect. 5.4.2 may occur.

This preference-indifference framework with the possibility to resort to incomparability, says nothing about how to select the best compromise action, or a subset of actions the DM will focus his attention on. In the construction procedure of ELECTRE I method only one outranking relation S is matter of fact.

The second procedure consists of exploiting this outranking relation in order to identify a small as possible subset of actions, from which the best compromise action could be selected. The identification of this small number justifies the removal of the actions that do not belong to the kernel. The basic idea of the kernel concept in ELECTRE methods is that all the actions that do not belong to the kernel are outranked for at least one action in the kernel. Such a subset, $\hat{A}$, may be determined with the help of the *graph kernel* concept, K_G. The justification of the use of this concept can be found in [124]. When the graph contains no direct cycles, there exists always a single kernel; otherwise, the graph contains no kernels or several. But, let us point out that a graph G may contain direct cycles. If that is the case, a preprocessing step must take place where maximal direct cycles are reduced to singleton elements, forming thus a partition on A. Let $\bar{A}$ denote that partition. Each class on $\bar{A} = \{\bar{A}_1, \bar{A}_2, \ldots\}$ is now composed of a set of (considered) equivalent actions. When no cycle exists there is a unique kernel in the graph. It should be noticed that a new preference relation, $\succ$, is defined on $\bar{A}$:

$$\bar{A}_p \succ \bar{A}_q \;\Leftrightarrow\; \exists a \in \bar{A}_p \text{ and } \exists b \in \bar{A}_q \text{ such that } aSb \;\text{ for }\; \bar{A}_p \neq \bar{A}_q$$

In ELECTRE I all the actions which form a cycle are considered indifferent, which may be, criticized. ELECTRE IS was designed to mitigate this inconvenient (see Sect. 5.3.1.3). In addition, the way of comparing this subsets is also criticized.

5.3.1.2 ELECTRE Iv

The name ELECTRE Iv was an unofficial name created for designating ELECTRE I with veto threshold [74]. This method is equipped with a different but extremely useful tool. The new tool made possible for analysts and DMs to overcome the difficulties related to the heterogeneity of scales. It is a generalization of the first method. The discordance is taken in account through the concept of veto threshold.

The new tool introduced was the *veto thresholds*, v_j, that can be attributed to certain criteria g_j belonging to F. The concept of veto threshold is related in some way, to the definition of an upper bound beyond which the discordance about the assertion "*a* outranks *b*" cannot surpass and allow an outranking (see Sect. 4.4). In practice, the idea of threshold is, however, quite different from the idea of discordance level like in ELECTRE I. Indeed, while discordance level is related to the scale of criterion g_j in absolute terms for an action a from A, threshold veto is related to the performance differences between $g_j(a)$ and $g_j(b)$.

In presence of qualitative criteria and if do not introduce a numerical coding of the scale it is necessary to identify all minimal ordered pairs leading to a discordance of the assertion aSb. Let us notice that this remark remains valid for all the ELECTRE method presented hereafter. However, it is always possible to define a way of coding the minimal ordered pairs that renders possible the application of the formulae presented in this chapter.

In terms of structure and formulae, little changes occur when moving from ELECTRE I to ELECTRE Iv. The only difference being the discordance condition, now called *no veto condition*, which may be stated as follows:

$$g_j(a) + v_j(g_j(a)) \geqslant g_j(b), \quad \forall \; j \in \mathcal{J}$$

where, $v_j(g_j(a))$ is a variable threshold.

To validate the assertion "a outranks b" it is necessary that, among the minority of criteria that are opposed to this assertion, none of them puts its veto.

ELECTRE Iv uses the same exploitation procedure as ELECTRE I, i.e., the way of selecting the kernel and exploiting the graph is the identical in both ELECTRE I and ELECTRE Iv.

But, this method is by no means complete; the problem of imperfect knowledge remains.

ELECTRE Iv as well as ELECTRE I are obsolete.

5.3.1.3 ELECTRE IS

How general an ELECTRE method can be when applied to choice decision-making problems? Is it possible to take into account simultaneously the heterogeneity of criteria scales, and imperfect knowledge about real-world decision-making situations? Previous theoretical research done on thresholds and semi-orders may, however, illuminate the issue of inaccurate data and permit to build a more general procedure, the so-called ELECTRE IS method.

One of the two majors novelties of ELECTRE IS is the use of pseudo-criteria instead of true-criteria. This method is an extension of the previous one aiming at taking into account a double objective: primarily the use of possible no nil indifference and preference thresholds for certain criteria belonging to F and, correlatively, a backing up (reinforcement) of the veto effect when the importance of the concordant coalition decreases. Both concordance and no veto conditions change. Let us present separately the formulae for each one of theses conditions.

- *Concordance condition* Let us start by building the following two indices sets:

  1. concerning the coalition of criteria in which aSb

$$\mathcal{J}^S = \Big\{ j \in \mathcal{J} \; : \; g_j(a) + q_j\big(g_j(a)\big) \geqslant g_j(b) \Big\}$$

2. concerning the coalition of criteria in which bQa

$$\mathcal{J}^Q = \Big\{ j \in \mathcal{J} \ : \ g_j(a) + q_j\big(g_j(a)\big) < g_j(a) \leqslant g_j(b) + p_j\big(g_j(b)\big) \Big\}$$

The concordance condition will be:

$$c(aSb) = \sum_{j \in \mathcal{J}^S} w_j + \sum_{j \in \mathcal{J}^Q} \varphi_j w_j \geqslant s$$

where,

$$\varphi_j = \frac{g_j(a) + p_j\big(g_j(a)\big) - g_j(b)}{p_j\big(g_j(a)\big) - q_j\big(g_j(a)\big)}$$

the coefficient φ_j decreases linearly from 1 to 0, when g_j describes the range $[g_j(a) + q_j\big(g_j(a)\big),\ g_j(a) + p_j\big(g_j(a)\big)]$.

- *no veto condition* The no veto condition can be stated as follows:

$$g_j(a) + v_j\big(g_j(a)\big) \geqslant g_j(b) + q_j\big(g_j(b)\big)\eta_j$$

where,

$$\eta_j = \frac{1 - c(aSb) - w_j}{1 - s - w_j}, \quad \text{with} \ \ 1 - s - w_j \neq 0$$

The second major novelty is related to the exploitation procedure, actions belonging to a cycle are no longer considered as indifferent as in the previous versions of ELECTRE for choice problems. Now, we take into account the concept of degree of robustness of “a outranks b”. It is a reinforcement of veto effect and allow us to build true classes of *ex æquo* (ties) and thus define an acycle graph over these classes. In such conditions there is always a single kernel. For more details see [3, 124].

5.3.2 Ranking Problematic

In ranking problematic we are concerned with the ranking of all the actions belonging to a given set of actions from the best to the worst, possibly with *ex æquo* (ties). There are three different ELECTRE methods to deal with this problematic.

5.3.2.1 ELECTRE II

From an historical and pedagogical point of view it is interesting to present ELECTRE II. This method was the first of ELECTRE methods especially designed to deal with ranking problems.

Without going into further detail, it is important to point out that ELECTRE II was the first method, to use a technique based on the construction of an embedded outranking relations sequence.

The construction procedure is very closed to ELECTRE Iv, in the sense that it is also a true-criteria based procedure. Hence, it is not surprising that no veto condition remains the same, but it can be the defined (there are two possible values for the veto thresholds according to the two different relations). However, concordance condition is modified in order to take into account the notion of embedded outranking relations. There are two embedded relations: a *strong outranking* relation followed by a *weak outranking* relation. Both the strong and weak relations are built thanks to the definition of two concordance levels, $s^1 > s^2$, where $s^1, s^2 \in [0.5,\ 1 - min_{j \in \mathcal{J}} w_j]$. Now, the concordance condition with the assertion "a outranks b" can be defined as follows:

$$c(aSb) \geqslant s^r \ \text{ and } \ c(aSb) \geqslant c(bSa), \quad \text{for } r = 1, 2$$

The exploitation procedure is a four-step algorithm:

1. *Partitioning the set* A. First, let us consider the relation S^1 over A. In a similar way like in ELECTRE I, this relation may define on A one or several cycles. If all the actions belonging to each maximal cycle are grouped together into a single class, a partition on A will be obtained. Let $\bar{A}$ denote this partition. When each class of $\bar{A}$ is not a singleton, the actions belonging to that class will be considered as *ex æquo*. For the purpose of comparison between elements of $\bar{A}$ a preference relation $\succ^1$ will be used. This relation has the same meaning as the relation $\succ$ for ELECTRE I.
2. *Building a complete pre-order* Z_1 *on* $\bar{A}$. After obtaining $\bar{A}$, the procedure identifies a subset B^1 of classes of $\bar{A}$ following the rule "no one else does not prefer them" according to the relation $\succ^1$. After removing B^1 from $\bar{A}$ and applying the same rule to $\bar{A} \backslash B^1$, a subset B^2 will be found. The procedure iterates in the same way till define the final partition on $\bar{A}$, $\{B^1, B^2, \ldots\}$.

   Now, on the basis of S^1, we may define a rough version of the complete pre-order Z_1, while placing in the head of this pre-order and in an *ex æquo* position all classes of B^1, then those of B^2 and so forth. In order to define Z_1 in a more accurate way, we examine if it is possible to refine this pre-order on the basis of the relation S^2. This refinement consists of using the information that brings this less believable outranking to decide between the various classes of a subset B^p when it contains several classes. This refinement of the rough version is obtained while using S^2 to define over B^p a complete pre-order that takes place between B^{p-1} and B^{p+1}.

3. *Determining a complete pre-order* Z_2 *on* $\bar{A}$. The procedure to obtain this pre-order is quite similar to the above one; only two modifications are needed:
   - apply the rule "they are not preferred to any other" instead of "no other is preferred to them"; let $\{B^{1'}, B^{2'}, \ldots\}$ denote the partition thus obtained;
   - define the rough version of the complete pre-order Z_2 by putting it in the queue of this pre-order, and in an *ex æquo* position all classes of $B^{1'}$, then those of $B^{2'}$ and so forth.
4. *Defining the partial pre-order* Z. The partial pre-order Z is an intersection of Z_1 and Z_2, $Z = Z_1 \cap Z_2$, is defined in the following way:

$$aZb \quad \Leftrightarrow \quad aZ_1b \text{ and } aZ_2b.$$

5.3.2.2 ELECTRE III

ELECTRE III was designed to improve ELECTRE II and thus deal with inaccurate, imprecise, uncertain or ill-determination of data. This purpose was actually achieved, and ELECTRE III was applied with success during the last two decades on a broad area of real-life applications.

In the current description of ELECTRE III we will omit several formulae details. The novelty of this method is the introduction of pseudo-criteria instead of true-criteria.

The construction of an outranking relation in ELECTRE III requires the definition of a *credibility index* for the outranking relation aSb; let $\sigma(aSb)$ denote this index. It is defined by using both the concordance index (as determined in ELECTRE IS), $c(aSb)$, and a discordance index for each criterion g_j in F, that is, $d_j(aSb)$.

The discordance of a criterion g_j aims at taking into account the fact that this criterion is more or less discordant with the assertion aSb. The discordance index reaches its maximal value when criterion g_j puts its veto to the outranking relation; it is minimal when the criterion g_j is not discordant with that relation. To define the value of the discordance index on the intermediate zone, we simply admitted that this value grows in proportion to the difference $g_j(b) - g_j(a)$. This index can now be presented as follows:

$$d_j(aSb) = \begin{cases} 1 \quad \text{if} \quad g_j(b) > g_j(a) + v_j(g_j(a)) \\ 0 \quad \text{if} \quad g_j(b) \leqslant g_j(a) + p_j(g_j(a)) \\ \frac{g_j(b) - g_j(a) - p_j(g_j(a))}{v_j(g_j(a)) - p_j(g_j(a))}, \quad \text{otherwise} \end{cases}$$

The credibility index is defined as follows,

$$\sigma(aSb) = c(aSb) \prod_{j \in j=1}^{n} T_j(aSb),$$

where $T_j(aSb) = \frac{1-d_j(aSb)}{1-c(aSb}$ if and only if $d_j(aSb) > c(aSb)$, and $T_j(aSb) = 1$ otherwise.

Notice that, when $d_j(aSb) = 1$, it implies that $\sigma(aSb) = 0$, since $c(aSb) < 1$.

The definition of $\sigma(aSb)$ is thus based on the following main ideas:

(a) When there is no discordant criterion, the credibility of the outranking relation is equal to the comprehensive concordance index.
(b) When a discordant criterion activates its veto power, the assertion is not credible at all, thus the index is null.
(c) For the situations in which the comprehensive concordance index is strictly lower than the discordance index on the discordant criterion, the credibility index becomes lower than the comprehensive concordance index, because of the opposition effect on this criterion.
(d) For the situations where the comprehensive credibility index is strictly greater than the discordance index on the criterion, the credibility remains equal to the comprehensive concordance index.

The index $\sigma(aSb)$ corresponds to the index $c(aSb)$ weakened by possible veto effects.

A fist modification of the valued credibility outranking relation used in the ELECTRE III and ELECTRE TRI was proposed in [82]. The modification requires the implementation of the discordance concept. Such a modification is shown to preserve the original discordance concept; the new outranking relation makes it easier to solve inference programs (see Sect. 5.4).

Another modification was proposed by Roy and Słowiński [131] so as to take into account two new effects, called reinforced preference and veto effects.

In ELECTRE III as well as in ELECTRE II, the partial pre-order Z is built as the intersection of two complete pre-orders, Z_1 and Z_2, which are obtained according to two variants of the same principle, both acting in an antagonistic way on the floating actions. The partial pre-order Z_1 is defined as a partition on the set A into q ordered classes, $\bar{B}_1, \ldots, \bar{B}_h, \ldots, \bar{B}_q$, where $\bar{B}_1$ is the head-class in Z_1. Each class $\bar{B}_h$ is composed of *ex æquo* elements according to Z_1. The complete pre-order Z_2 is determined in a similar way, where A is partitioned into u ordered classes, $\underline{B}_1, \ldots,$ $\underline{B}_h, \ldots, \underline{B}_u$, $\underline{B}_u$ being the head-class. Each one of these classes is obtained as a final distilled of a distillation procedure.

The procedure designed to compute Z_1 starts (first distillation) by defining an initial set $D_0 = A$; it leads to the first final distilled $\bar{B}_1$. After getting $\bar{B}_h$, in the distillation $h + 1$, the procedure sets $D_0 = A \backslash (\bar{B}_1 \cup \ldots \cup \bar{B}_h)$. According to Z_1, the actions in class $\bar{B}_h$ are, preferable to those of class $\bar{B}_{h+1}$; for this reason, distillations that lead to these classes will be called as descending (top-down).

The procedure leading to Z_2 is quite identic, but now the actions in $\bar{B}_{h+1}$ are preferred to those in class $\bar{B}_h$; these distillations will be called ascending (bottom-up).

The partial pre-order Z will be computed as the intersection of Z_1 and Z_2.

A complete pre-order is finally suggested taking into account the partial pre-orders and some additional considerations. The way the incomparabilities which remain in the pre-order are treated is nevertheless subject to criticism.

5.3.2.3 ELECTRE IV

In Sect. 5.2.4 we pointed out the difficulty to define the relative importance coefficients of criteria. However, in several circumstances we are not able, we do not want, or we do not know how to assign a value to those coefficients. It does not mean that we would be satisfied with the pre-order obtained, when applying ELECTRE III with the same value for all the coefficients w_j. Another approach we could take would be determining a pre-order, which takes into account all the pre-orders obtained from the application of several combinations of the weights. Obviously, this situation will be unmanageable.

ELECTRE IV is also a procedure based on the construction of a set of embedded outranking relations. There are five different relations, $S^1, \ldots, S^5$. The S^{r+1} relation ($r = 1, 2, 3, 4$) accepts an outranking in a less credible circumstances than the relation S^r. It means (while remaining on a merely ordinal basis) the assignment of a value σ_r for the credibility index $\sigma(aSb)$ to the assertion aSb. The chosen values must be such that $\sigma_r > \sigma_{r+1}$. Furthermore, the movement from one credibility value σ_r to another σ_{r+1} must be perceived as a considerable loss.

The ELECTRE IV exploiting procedure is the same as in ELECTRE III.

5.3.3 Sorting Problematic

A set of categories must be a priori defined. The definition of a category is based on the fact that it should be conceived a priori to receive actions, which will be or might be processed in the same way (in the step that follows the assignment). In sorting problematic, each action is considered independently from the others in order to determine the categories to which it seems justified to assign it, by means of comparisons to profiles (bounds, limits), norms or references. Results are expressed using the absolute notion of "assigned" or "not assigned" to a category, "similar" or "not similar" to a reference profile, "adequate" or "not adequate" to some norms. The sorting problematic refers to absolute judgements. It consists of assigning each action to one of the pre-defined categories which are defined by norms or typical elements of the category. The assignment of an action a results from the intrinsic evaluation of a on all criteria. From the norms defining the categories (the assignment of a to a specific category does not influence the category, to which another action b should be assigned).

5.3.3.1 ELECTRE TRI

In ELECTRE TRI categories are ordered from the worst (C_1) to the best (C_k). Each category must be characterized by a lower and an upper profile. Let $C = \{C_1, \ldots, C_h, \ldots, C_k\}$ denote the set of categories. The assignment of a given action a to a certain category C_h results from the comparison of a to the profiles defining the lower and upper limits of the categories; b_h being the upper limit of category C_h and the lower limit of category C_{h+1}, for all $h = 1, \ldots, k$. for a given category limit b_h this comparison rely on the credibility of the assertions aSb_h and b_hSa. This credibility index is defined as in ELECTRE III. In what follows, we will assume, without any loss of generality, that preferences increase with the value on each criterion.

After determine the credibility index, we should defining a λ-cutting level of the fuzzy relation in order to obtain a crisp outranking relation. The conversion of a fuzzy relation into a crisp one, is done by the definition of a cutting level, called *λ-cutting level*. It can be defined as the credibility index smallest value compatible with the assertion aSb_h.

Now, the objective of the second procedure is to exploit the four binary relations defined in Sect. 5.4.2. The role of this exploitation is to propose an assignment. This assignment can be grounded on two well-known logics. The conjunctive logic in which an action can be assigned to a category when its evaluations is, on each criterion, at least as good as the lower limit of the category. The action is hence assigned to the highest category fulfilling this condition. In the disjunctive logic, if the action has, on at least on one criterion, an evaluation at least as good as the lower limit of the category. The action is hence assigned to the highest category fulfilling this condition. With this disjunctive rule, the assignment of an action is generally higher than with the conjunctive rule. This is why the conjunctive rule is usually interpreted as pessimistic while the disjunctive rule is interpreted as optimistic. This interpretation (optimistic-pessimistic) can be permuted according to the semantic attached to the outranking relation. When no incomparability occurs in the comparison of an action a to the limits of categories, a is assigned to the same category by the conjunctive and disjunctive rules. When a is assigned to different categories by the conjunctive and disjunctive rules, a is incomparable to all "intermediary" limits within the highest and lowest assignment categories.

ELECTRE TRI is a generalization of the two above mentioned rules. The generalization is the following:

- in the conjunctive rule: replace, in the condition "*on each criterion*" by "*on a sufficient majority of criteria and in absence of veto*"
- in the disjunctive rule: replace, the condition "on at least on one criterion" by "*on a sufficient minority of criteria and in absence of veto*"

The two procedures can be stated as follows,

1. *Pseudo-conjunctive rule*. An action a will be assigned to the highest category C_h such that aSb_{h-1}.

(a) Compare a successively with b_r, $r = k-1, k-2, \ldots, 0$.
(b) The limit b_h is the first encountered profile such that aSb_h. Assign a to category C_{h+1}.

2. *Pseudo-disjunctive rule*. An action a will be assigned to the lowest category C_h such that $b_h \succ a$.

(a) Compare a successively with b_r, $r = 1, 2, \ldots, k-1$.
(b) The limit b_h is the first encountered profile such that $b_h \succ a$. Assign a to category C_h.

5.3.3.2 ELECTRE TRI C and ELECTRE TRI nC

ELECTRE TRI C [4] is a new method for sorting problems designed for dealing with decision aiding situations where each category from a completely ordered set is defined by a single characteristic reference action. The characteristic reference actions are co-constructed through an interactive process involving the analyst and the decision maker. ELECTRE TRI C has been also conceived to verify a set of natural structural requirements (conformity, homogeneity, monotonicity, and stability). The method makes use of two joint assignment rules, where the result is a range of categories for each action to be assigned. The two joint rules, called descending rule and ascending rule, can be presented as follows:
Descending rule Choose a credibility level $\lambda \in [0.5, 1]$. Decrease h from $(q+1)$ until the first value t, such that $\sigma(a, b_t) \geqslant \lambda$:

(a) For $t = q$, select C_q as a possible category to assign action a.
(b) For $0 < t < q$, if $\rho(a, b_t) > \rho(a, b_{t+1})$, then select C_t as a possible category to assign a; otherwise, select C_{t+1}.
(c) For $t = 0$, select C_1 as a possible category to assign a.

Ascending rule Choose a credibility level $\lambda \in [0.5, 1]$. Increase h from 0 until the first value k, such that $\sigma(b_k, a) \geqslant \lambda$:

(a) For $k = 1$, select C_1 as a possible category to assign action a.
(b) For $1 < k < (q+1)$, if $\rho(a, b_k) > \rho(a, b_{k-1})$, then select C_k as a possible category to assign a; otherwise, select C_{k-1}.
(c) For $k = (q+1)$, select C_q as a possible category to assign a.

Each one of the two joint rules requires the selecting function $\rho(a, b_h)$, which allows to choose between the two consecutive categories where an action a can be assigned to. The results appear in one of the following forms, and the decision maker may choose:

1. A single category, when the two selected categories are the same;
2. One of the two selected categories, when such categories are consecutive;
3. One of the two selected categories or one of the intermediate categories, when such categories are not consecutive.

In [5], ELECTRE TRI C method was generalized to ELECTRE TRI nC method where each category is defined by a set of several reference characteristic actions, rather than one. This aspect is enriching the definition of each category and allows to obtain more narrow ranges of categories to which an action can be assigned to, than the ELECTRE TRI nC method. The joint assignments rules are similar to the previous ones.

5.4 Recent Developments

Although, several decades past since the birth of the first ELECTRE method, research on ELECTRE family method stills active today. Some of the recent developments are shortly described in this section.

5.4.1 Robustness Concerns

When dealing with real-world decision problems, DMs and analysts are often facing with several sources of imperfect knowledge regarding the available data. This leads to the assignment of arbitrary values to certain "variables". In addition, modeling activity frequently requires to choose between some technical options, introducing thus an additional source of arbitrariness to the problem. For these reasons, analysts hesitate when assigning values to the *preference parameters* (weights, thresholds, categories lower and upper limits, ...), and the *technical parameters* (discordance and concordance indices, λ-cutting level, ...) of ELECTRE methods.

In practice, it is frequent to define a *reference system* built from the assignment of *central values* to these two types of parameters. Then, an exploitation procedure should be applied in order to obtain outputs which are used to elaborate recommendations. But, what about the meaningfulness of such recommendations? They strongly depend on the set of central values attributed to the parameters. Should the analyst analyze the influence of a variation of each parameter, considered separately, on the results? And, then enumerate those parameters which lead to a strong impact on the results when their values vary from the central positions. This is a frequent way to proceed in classical operations research methods and it is called *sensitivity analysis* [36, 62, 91, 94]. But, this kind of analyzes has rather a theoretical interest than a practical one. Analysts are most often interested in building recommendations which remain acceptable for a large range of the parameters values. Such recommendations should be elaborated from what we call the *robust conclusions* (Chap. 2, [114, 118, 119, 124]).

Definition 1. A conclusion, C^r, is said to be robust with respect to a domain, Ω, of possible values for the preference and technical parameters, if there is no a particular set of parameters, $\bar{\omega} \in \Omega$, which clearly invalidates the conclusion C^r.

A *robustness concern* consists of all the possible ways that contribute to build synthetic recommendations based on the robust conclusions.

Possible ways to deal with robustness concerns in ELECTRE methods are illustrated, for example, in [2, 30, 31, 33, 135], Chaps. 8, 9, and 10 in [124].

5.4.2 *Elicitation of Parameter Values*

Implementing ELECTRE methods requires to determine values (or intervals of variation) for the preference parameters.

Definition 2. A preference elicitation process proceeds through an interaction between DMs and analysts in which DMs express information about their preferences within a specific aggregation procedure.

It is possible to distinguish among direct and indirect elicitation techniques.

The objective is not to find the true values of the parameters, but the most adequate values to take into account the preferences, the wishes, or the willing of the DM.

5.4.2.1 Direct Elicitation Techniques

In direct elicitation procedures DMs should provide information directly on the values of the preference parameters. A major drawback of such techniques is that it is difficult to understand the precise meaning of the assertions of the DMs. This is why ELECTRE methods are usually implemented by using indirect elicitation procedures [120].

5.4.2.2 Indirect Elicitation Techniques

Indirect elicitation techniques do not require from DMs to provide answers to questions related to the values of the preference parameters. Two modes of questioning can be considered:

1. The first consists of asking the DM as follows: for you, what are the most important . . . , the least important . . . criteria? Can you make a ranking of them (with possible ties) in a decreasing order of importance? According to your ranking do you think that the variations of importance between two consecutive criteria are approximately the same, whatever the rank of the two relevant criteria? If not, can you compare the variations of importance? In order to facilitate the dialogue with the DM, the criteria can be shown as a "pack of cards" as it is the case of the SRF procedure [39].

2. In the second mode of questioning several (realistic or fictitious) actions have to be defined. They will serve as references so that the DM may express his/her preferences. A set of reference actions will be shown to the decision maker. Each one is characterized by its performance on each criterion. Then, we ask the DM as follows: if the set does not suit you, you may alter it. Given a pair of actions a and b, which one do you prefer? Once such a pairwise comparison has been done, do you think that the intensity of preferences of a over b is stronger that of c over d? In the case of ordered categories is defined, can you assign each reference action to a category that you deem is the most important? DIVAPIME [81] is a procedure that was designed for this second type of questioning.

Other techniques, designed from the second type of questioning are frequently called aggregation-disaggregation procedures. Such techniques make use of the disaggregation paradigm [59, 70]. They make use of optimization procedures to determine (infer) the parameters values.

Recent developments concerning elicitation techniques have been proposed for the ELECTRE TRI method. Inference procedures have been developed to elicit the parameters values from assignment examples, i.e., an assignment that is imposed by DMs on specific actions. It is possible to infer all the preference parameters simultaneously [83]; we will refer to such a case by complete inference. The induced mathematical programming model to be solved is, however, non-linear. Thus, its resolution is computationally difficult for real-world problems. In such cases, it is possible to infer a subset of parameters only (see Fig. 5.1):

- Concordant coalition parameters: weights and λ-cutting level [86];
- Discordance related parameters: veto thresholds [32];
- Category limits [87].

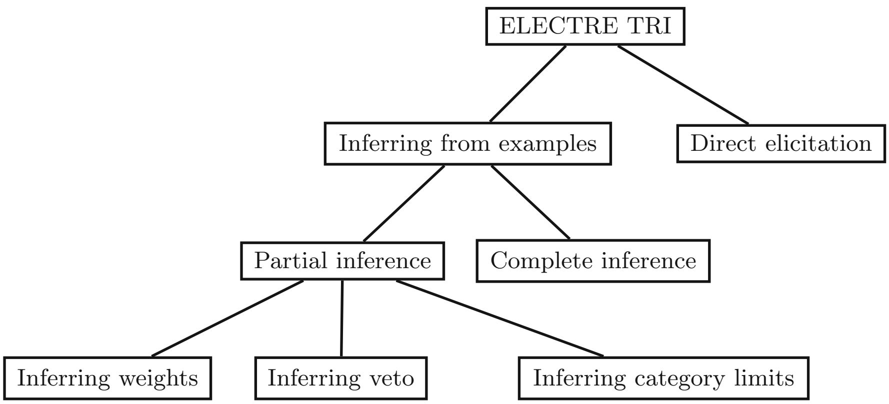


Fig. 5.1 Inferring parameter values for ELECTRE TRI

5.5 Software and Applications

This section is devoted to software, the Decision Deck project, and applications of ELECTRE methods.

5.5.1 ELECTRE Software

The implementation of ELECTRE methods in real-world decision problems involving real DMs requires software packages. Some of them are widely used in large firms and universities, in particular ELECTRE IS, ELECTRE III–IV, ELECTRE TRI and IRIS. Among the software available at LAMSADE are (http://www.lamsade.dauphine.fr/english/software.html):

1. *ELECTRE IS* is a generalization of ELECTRE I. It is an implementation of ELECTRE IS described in Sect. 5.3.1. This software runs on a IBM-compatible computer on Windows 98 and higher.
2. *ELECTRE III–IV* is a software which implements ELECTRE III and ELECTRE IV methods described in Sect. 5.3.2. It runs on Windows 3.1, 95, 98, 2000, Millennium and XP.
3. *ELECTRE TRI* is a multiple criteria decision aiding tool designed to deal with sorting problems. This software implements ELECTRE TRI method described in Sect. 5.3.3. The ELECTRE TRI software versions 2.x were developed with the C++ programming language and runs on Microsoft Windows 3.1, 95, 98, Me, 2000, XP and NT. This software integrates, *ELECTRE TRI Assistant* which enables the user to define the weights indirectly, i.e., fixing the model parameters by giving some assignment examples (corresponding to desired assignments or past decisions). The weights are thus inferred through a certain form of regression. Hence, ELECTRE TRI Assistant reduces the cognitive effort required from the DM to elicit the preference parameters.
4. *IRIS*. Interactive Robustness analysis and parameters' Inference for multiple criteria sorting problems. This DSS has been built to support the assignment of actions described by their evaluation on multiple criteria to a set of predefined ordered categories, using a variant of ELECTRE TRI. Rather than demanding precise values for the model's parameters, IRIS allows to enter constraints on these values, namely assignment examples that it tries to restore. When the constraints are compatible with multiple assignments for the actions, IRIS infers parameter values and allows to draw robust conclusions by indicating the range of assignments (for each action) that do not contradict any constraint. If it is not possible to fulfill all of the constraints, IRIS tells the user where is the source of inconsistency. It was developed with Delphi Borland and runs on Windows 98, Me, 2000, NT and XP.
5. *SRF* was designed to determine the relative importance coefficients for ELECTRE family methods. It is based on a very simple procedure (the pack of cards

technique created by J. Simos) and try to assess these coefficients by questioning the DM in an indirect way (see Sect. 5.4.2.2). It was developed with the Delphi Borland 3.0 and runs on Windows 98, Me, 2000 and XP.

The software ELECTRE IS, III–IV, TRI and TRI Assistant were developed under a collaborative project between researchers from the Institute of Computing Science of the Technical University of Poznan (Poland) and LAMSADE, Université Paris-Dauphine (France), while IRIS and SRF result from a collaborative project between researchers from LAMSADE and the Faculty of Economics of the University of Coimbra/INESC-Coimbra (Portugal).

5.5.2 The Decision Deck Project

The Decision Deck project (www.decision-deck.org) aims at collaboratively developing Open Source software tools implementing Multiple Criteria Decision Aid (MCDA). The Decision Deck software include the implementation of ELECTRE methods. Its purpose is to provide effective tools for three types of users:

- practitioners who use MCDA tools to support actual decision makers involved in real world decision problems;
- teachers who present MCDA methods in courses, for didactic purposes;
- researchers who want to test and compare methods or to develop new ones.

From a practical point of view, the Decision Deck project works on developing multiple software resources that are able to interact. Consequently, several complementary efforts focusing on different aspects contribute to Decision Deck's various goals.

The project continues and expands the series of activities that have been pursued by the Decision Deck Community, including:

- d2: a rich open source Java client offering several MCDA methods, namely ELECTRE methods
- XMCDA: a standardized XML recommendation to represent objects and data structures issued from the field of MCDA. Its main objective is to allow different MCDA algorithms to interact and be easily callable;
- XMCDA web services: distributed open source computational MCDA resources, namely ELECTRE methods
- d3: an open source rich internet application for XMCDA web services management;
- diviz: an open source Java client and server for XMCDA web services composition, work flow management and deployment.

All these efforts involve developments on at least one of the following research topics of the Decision Deck project:

- global architecture of MCDA systems;
- implementations and developments of MCDA algorithms;
- data models and management of MCDA objects;
- decision process modeling and management;
- graphical user interface.

5.5.3 Applications

Since their first appearance ELECTRE methods were successful applied in many areas.

1. *Agriculture and Forest Management*: [7, 35, 71, 132, 146, 148, 149]
2. *Energy*: [12, 13, 23, 47, 48, 63, 123, 144]
3. *Environment and Water Management*: [16, 48, 49, 52, 68, 69, 89, 90, 92, 97, 98, 100, 132, 137, 141, 142, 149, 150]
4. *Finance*: [6, 34, 54–56, 65, 73, 162–165]
5. *Military*: [10, 44, 159]
6. *Project selection (call for tenders)*: [17, 25, 28, 76, 122, 156].
7. *Transportation*: [15, 16, 27, 46, 53, 85, 125–127, 134, 135]
8. *Varia*: [37, 38, 93, 95, 122, 147].

New applications of ELECTRE methods can be found for sorting cropping systems [7], land-use suitability assessment [61], greenhouse gases emission reduction [48], risk zoning of an area subjected to mining-induced hazards [75], participatory decision-making on the localization of waste-treatment plants [88], material selection of bipolar plates for polymer electrolyte membrane fuel cell [140], assisted reproductive technology [43], promotion of social and economic development [8], sustainable demolition waste management strategy [101], assessing the risk of nanomaterials [151], public transportation [66], and rationalising photovoltaic energy investments [143].

5.6 Conclusion

Since their first appearance, in 1965 (see [14]), ELECTRE methods, on one side, had a strong impact on the Operational Research community, mainly in Europe, and led to the development of other outranking methods (see, for example, Chaps. 6 and 7), as well as other complementary multiple criteria methodologies. Most importantly, the development of ELECTRE methods is strongly connected with the birth of the European Working Group on Multiple Criteria Decision Aiding (www.cs.put.poznan.pl/ewgmcda/). On the other side, ELECTRE methods experienced a widespread and large use in real-world situations.

Despite their almost four decades of existence, research stills active in this field. We can also mention some of recent developments and avenues for future research in ELECTRE methods: generalization of the concordance and non-discordance methods [152]; robustness analysis [30, 31, 33]; the concepts of possible and necessary outranking [50], parameters elicitation techniques [83]; interaction between criteria [41, 77], multiple DMs and social interaction [26]; strong features and weaknesses of ELECTRE methods [40, 42].

References

1. Abgueguen, R.: La Sélection des Supports de Presse. Robert Lafont, Paris (1971)
2. Aissi, H., Roy, B.: Robustness in multi-criteria decision aiding. In: Ehrgott, M., Figueira, J.R., Greco, S. (eds.) Trends in Multiple Criteria Decision Analysis, pp. 101–139. Springer, New York (2010)
3. Aït Younes, A., Azibi, R., Roy, B.: ELECTRE IS: manuel d'utilisation. Document du LAMSADE 118 and 118bis, Université Paris-Dauphin (2000)
4. Almeida-Dias, J., Figueira, J.R., Roy, B.: ELECTRE TRI-C: a multiple criteria sorting method based on characteristic reference actions. Eur. J. Oper. Res. **204**, 565–580 (2010)
5. Almeida-Dias, J., Figueira, J.R., Roy, B.: A multiple criteria sorting method defining each category by several characteristic reference actions: the ELECTRE TRI-nC method. Cahier du LAMSADE 294, Université Paris-Dauphine (2010)
6. Andenmatten, A.: Evaluation du risque de défaillance des émetteurs d'obligations, une approche par l'aide multicritère à la décision. Presses Polytechniques et Universitaires Romandes, Lausanne (1995)
7. Arondel, C., Girardin, P.: Sorting cropping systems on the basis of their impact on groundwater quality. Eur. J. Oper. Res. **127**, 467–482 (2000)
8. Autran-Gomes, L., Rangel, L., Moreira, R.: Using ELECTRE IV in the promotion of social and economic development: a case study in Rio de Janeiro. Found. Comput. Decis. Sci. **34**, 155–172 (2009)
9. Bana e Costa, C. (ed.): Readings in Multiple Criteria Decision Aid. Springer, Heidelberg (1990)
10. Bana e Costa, C., Dinis das Neves, C.: Describing and formalizing the evaluation process of Portuguese navy officiers. In: Lockett, A., Islei, G. (eds.) Improving Decision Making in Organisations. Lecture Notes in Economics and Mathematical Systems, vol. 335, pp. 355–369. Springer, Berlin (1989)
11. Bana e Costa, C., Vansnick, J.: MACBETH – an interactive path towards the construction of cardinal value functions. Int. Trans. Oper. Res. **1**, 489–500 (1994)
12. Barda, O., Dupuis, J., Lencioni, P.: Multicriteria location of thermal power plants. Eur. J. Oper. Res. **45**, 332–346 (1990)
13. Beccali, M., Cellura, M., Ardente, D.: Decision making in energy planning: the ELECTRE multicriteria analysis approach compared to a fuzzy-sets methodology. Energy Convers. Manag. **39**(16–18), 1869–1881 (1998)
14. Benayoun, R., Roy, B., Sussman, B.: ELECTRE: Une méthode pour guider le choix en présence de points de vue multiples. Note de travail 49, SEMA-METRA International, Direction Scientifique (1966)
15. Bertier, P., Gagey, G., de Montgolfier, J., Roy, B.: Choix de tracés autoroutiers en milieu suburbain: Faut-il vraiment endommager des sites urbains et/ou forestiers, lesquels? Communication aux journées de formation "transports et agglomérations", Nice (1972)

16. Bétolaud, Y., Février, R.: Conservation des forêts suburbaines et passage des autoroutes. In: Revue Forestière Française, pp. 197–200 (1973)
17. Blondeau, P., Spérandio, M., Allard, F.: Multicriteria analysis of ventilation in summer period. Build. Environ. 37(2), 165–176 (2002)
18. Bouyssou, D.: Modelling inaccurate determination, uncertainty, imprecision using multiple criteria. In: Lockett, A., Islei, G. (eds.) Improving Decision Making in Organisations. Lecture Notes in Economics and Mathematical Systems, vol. 335, pp. 78–87. Springer, Berlin (1989)
19. Bouyssou, D., Roy, B.: La notion de seuils de discrimination en analyse multicritère. INFOR **25**(4), 302–313 (1987)
20. Bouyssou, D., Marchant, T., Pirlot, M., Perny, P., Tsoukiàs, A., Vincke, Ph.: Evaluation and Decision Model. A Critical Perspective. Kluwer Academic, Dordrecht (2000)
21. Bouyssou, D., Jacquet-Lagrèze, E., Perny, P., Słowiński, R., Vanderpooten, D., Vincke, Ph. (eds.): Aiding Decisions with Multiple Criteria: Essays in Honour of Bernard Roy. Kluwer Academic, Dordrecht (2001)
22. Buffet, P., Grémy, J., Marc, M., Sussmann, B.: Peut-on choisir en tenant compte de critères multiples? Une méthode ELECTRE et trois applications. Revue METRA **VI**, 283–316 (1967)
23. Charpentier, A., Jacquet-Lagrèze, E.: La promotion de l'électricité dans l'entreprise et l'utilisation des méthodes multicritères. In: Thiriez, H., Zionts, S. (eds.) Multiple Criteria Decision Making. Lecture Notes in Economics and Mathematical Systems, vol. 130, pp. 364–377. Springer, Berlin (1976)
24. Colorni, A., Paruccini, M., Roy, B. (eds.): A-MCD-A – Aide Multi-Critère à la Décision – Multiple Criteria Decision Aiding. The European Commission Joint Research Center, Luxemburg (2001)
25. Colson, G.: The OR's prize winner and the software ARGOS: how a multijudge and multicriteria ranking GDSS helps a jury to attribute a scientific award. Comput. Oper. Res. **27**(7–8), 741–755 (2000)
26. Damart, S.: Aide à la décision et processus de décision collectifs: La contribution des outils d'aide à la décision aux démarches de concertation. Ph.D. thesis, Université Paris-Dauphine (2003)
27. Damart, S., Mousseau, V., Sommerlatt, I.: Du mode d'implication d'acteurs multiples dans le cadre de l'utilisation d'un modèle d'affectation multicritère: analyse au regard d'une application à la tarification des transports publics. INFOR **40**(3), 199–222 (2002)
28. Danila, N.: Méthodologie d'aide à la décision dans le cas d'investissements fort dépendants. Ph.D. thesis, Université Paris-Dauphine (1980)
29. de Montgolfier, J., Bertier, P.: Approche Multicritère des Problèmes de Décision. Editions Hommes et Techniques, Paris (1978)
30. Dias, L., Clímaco, J.: On computing ELECTRE's credibility indices under partial information. J. Multicrit. Decis. Anal. **8**(2), 74–92 (1999)
31. Dias, L., Clímaco, J.: ELECTRE TRI for groups with imprecise information on parameter values. Group Decis. Negot. **9**(5), 355–377 (2000)
32. Dias, L., Mousseau, V.: Inferring ELECTRE's veto-related parameters from outranking examples. Eur. J. Oper. Res. **170**, 172–191 (2006)
33. Dias, L., Mousseau, V., Figueira, J., Clímaco, J.: An aggregation/disaggregation approach to obtain robust conclusions with ELECTRE TRI. Eur. J. Oper. Res. **138**, 332–348 (2002)
34. Dimitras, A.I., Zopounidis, C., Hurson, C.: Assessing financial risks using a multicriteria sorting procedure: the case of country risk asessment. Found. Comput. Decis. Sci. **29**, 97–109 (2001)
35. Duckstein, L., Gershon, M.: Multicriterion analysis of a vegetation management problem using ELECTRE II. Appl. Math. Model. **7**(4), 254–261 (1983)
36. Evans, J.: Sensitivity analysis in decision theory. Decis. Sci. **15**, 239–247 (1984)
37. Ferrand, D., Martel, J.: Le choix multicritère des items d'une échelle de mesure. Math. Sci. Humaines **89**, 35–59 (1985)
38. Fichefet, J., Leclercq, J.: A multicriteria microcomputer-assisted system for the identification of bacteria. Eur. J. Oper. Res. **24**, 12–22 (1986)

39. Figueira, J., Roy, B.: Determining the weights of criteria in the ELECTRE type methods with a revised Simos' procedure. Eur. J. Oper. Res. **139**, 317–326 (2002)
40. Figueira, J.R., Roy, B.: A note on the paper, "ranking irregularities when evaluating alternatives by using some ELECTRE methods, by Wang and Triantaphyllou, omega (2008)". OMEGA **37**, 731–733 (2009)
41. Figueira, J.R., Greco, S., Roy, B.: ELECTRE methods with interaction between criteria: an extension of the concordance index. Eur. J. Oper. Res. **199**(2), 496–505 (2009)
42. Figueira, J.R., Greco, S., Słowiński, R., Roy, B.: ELECTRE methods: main features and recent developments. In: Zopounidis, C., Pardalos, P. (eds.) Handbook of Multicriteria Analysis, pp. 51–89. Springer, New York (2010)
43. Figueira, J.R., Almeida-Dias, J., Matias, S., Roy, R., Carvalho, M.J., Plancha, C.E.: ELECTRE TRI-C, a multiple criteria decision aiding sorting model applied to assisted reproduction. Int. J. Med. Inform. **80**(4), 262–273 (2011)
44. Gabrel, V.: Méthodologie pour la planification de production de système d'observation de la terre par satellites: aspects algorithmiques et multicritères. Ph.D. thesis, Université Paris-Dauphine (1994)
45. Gal, T., Stewart, T., Hanne, T. (eds.): Advances in MultCriteria Decision Making – MCDM Models, Algorithms, Theory, and Applications. Kluwer Academic, Boston (1999)
46. Gargaillo, L.: Réponse à l'article "Le plan d'extension du métro en banlieue parisienne, un cas type de l'analyse multicritère". Les Cahiers Scientifiques de la Revue Transports **7**, 52–57 (1982)
47. Georgopoulou, E., Lalas, D., Papagiannakis, L.: A multicriteria decision aid approach for energy planning problems: the case of renewable energy option. Eur. J. Oper. Res. **103**(1), 38–54 (1997)
48. Georgopoulou, E., Sarafidis, Y., Mirasgedis, S., Zaimi, S., Lalas, D.: A multiple criteria decision-aid approach in defining national priorities for greenhouse gases emissions reduction in the energy sector. Eur. J. Oper. Res. **146**, 199–215 (2003)
49. Gershon, M., Duckstein, L., McAniff, R.: Multiobjective river basin planning with qualitative criteria. Water Resour. Res. **18**, 193–202 (1982)
50. Greco, S., Słowiński, R., Figueira, J.R., Mousseau, V.: Robust ordinal regression. In: Ehrgott, M., Figueira, J.R., Greco, S. (eds.) Trends in Multiple Criteria Decision Analysis, pp. 273–320. Springer, New York (2010)
51. Grolleau, J., Tergny, J.: Manuel de référence du programme ELECTRE II. Document de travail 24, SEMA-METRA International, Direction Scientifique (1971)
52. Hokkanen, J., Salminen, P.: Choosing a solid waste management system using multicriteria decision analysis. Eur. J. Oper. Res. **98**(1), 19–36 (1997)
53. Hugonnard, J., Roy, B.: Le plan d'extension du métro en banlieue parisienne, un cas type d'application de l'analyse multicritère. Les Cahiers Scientifiques de la Revue Transports **6**, 77–108 (1982)
54. Hurson, C., Zopounidis, C.: On the use of multicriteria decision aid methods to portfolio selection. J. Euro-Asian Manag. **1**(2), 69–94 (1995)
55. Hurson, C., Zopounidis, C.: Gestion de portefeuille et analyse multicritère. Economica, Paris (1997)
56. Hurson, C., Zopounidis, C.: On the use of multicriteria decision aid methods to portfolio selection. In: Clímaco, J. (ed.) Multicriteria Analysis, pp. 496–507. Springer, Berlin (1997)
57. Jacquet-Lagrèze, E.: Le problème de l'agrégation des préférences: Une classe de procédures à seuil. Math. Sci. Humaines **43**, 29–37 (1973)
58. Jacquet-Lagrèze, E.: How we can use the notion of semi-orders to build outranking relations in multicriteria decision making. In: Wendt, E., Vleck, C. (eds.) Utility, Subjective Probability and Human Decision Making, pp. 87–112. Reidel, Dordrecht (1975)
59. Jacquet-Lagrèze, E., Siskos, Y.: Assessing a set of additive utility functions for multicriteria decision making: the UTA method. Eur. J. Oper. Res. **10**, 151–164 (1982)
60. Jacquet-Lagrèze, E., Siskos, Y. (eds.) Méthode de Décision Multicritère. Editions Hommes et Techniques, Paris (1983)

61. Joerin, F., Thériault, M., Musy, A.: Using GIS and outranking multicriteria analysis for land-use suitability assessment. Int. J. Geogr. Inf. Sci. **15**(2), 153–174 (2001)
62. Kampke, R.: Sensitivity analysis for assessing preferentially independent order relations. Comput. Oper. Res. **23**(12), 1119–1130 (1996)
63. Karagiannidis, A., Moussiopoulos, N.: Application of ELECTRE III for the integrated management of municipal solid wastes in the greater Athens area. Eur. J. Oper. Res. **97**(3), 439–449 (1997)
64. Keeney, R., Raiffa, H.: Decisions with Multiple Objectives: Preferences and Value Tradeoffs. Wiley, New York (1976)
65. Khoury, N., Martel, J., Yourougou, P.: A multicriterion approach to country selection for global index founds. Document de travail FSA 91-13, Faculté des Sciences de l'Administration, Université Laval, Québec (1991)
66. Labbouz, S., Roy, B., Diab, Y., Christen, M.: Implementing a public transport line: multi-criteria decision-making methods that facilitate concertation. Oper. Res. Int. J. **8**, 5–31 (2008)
67. Laffy, R.: La méthode MARSAN pour la recherche de produits nouveaux. Presented at the ESOMAR congress, Copenhagen (1966)
68. Lahdelma, R., Salminen, P., Hokkanen, J.: Locating a waste treatment facility by using stochastic multicriteria acceptability analysis with ordinal criteria. Eur. J. Oper. Res. **142**, 345–356 (2002)
69. Mahmoud, M., Garcia, L.: Comparison of different multicriteria evaluation methods for the Red Bluff diversion dam. Environ. Model. Softw. **15**(5), 471–478 (2000)
70. March, J.: Ambiguity and the engineering of choice. Bell J. Econ. **9**, 587–608 (1978)
71. Martel, J., Rousseau, A.: Cadre de référence d'une démarche multicritère de gestion intégrée des ressources en milieu forestier. Document technique 93-11, Ministère de l'environment et de la faune, Québec (1993)
72. Martel, J., Roy, B.: Analyse de la signifiance de diverses procédures d'agrégation multicritère. Annales du LAMSADE 1, Université Paris-Dauphine (2002)
73. Martel, J., Khoury, N., Bergeron, M.: An application of a multicriteria approach to portfolio comparisons. J. Oper. Res. Soc. **39**, 617–628 (1988)
74. Maystre, L., Pictet, J., Simos, J.: Les Méthodes Multicritères ELECTRE. Presses Polytechniques et Universitaires Romandes, Lausanne (1994)
75. Merad, M.M., Verdel, T., Roy, B., Kouniali, S.: Use of multi-criteria decision-aids for risk zoning and management of large area subjected to mining-induced hazards. Tunn. Undergr. Space Technol. **19**, 165–178 (2004)
76. Michel, J.: La sélection des projects du programme architecture nouvelle. In: Actes du congrès Association Française pour la Cybernétique Economique et Technique (AFCET): Aide à la Décision, Paris, pp. 111–121, 16–17 May 1974
77. Montano-Guzmán, L.: Fuzzy measures and integrals in the MCDA sorting problematic: methodology and application to the diagnosis of firms. Ph.D. thesis, Université Libre de Bruxelles (2002)
78. Moscarola, J.: Aide à la décision en présence de critères multiples fondée sur une procédure trichotomique: méthodologie et application. Thèse de doctorat de 3e cycle, Université Paris V René Decartes (1977)
79. Moscarola, J., Roy, B.: Procédure automatique d'examen de dossiers fondée sur une segmentation trichotomique en présence de critères multiples. RAIRO Recherche Opérationnelle **11**(2), 145–173 (1977)
80. Mousseau, V.: Problèmes liés à l'évaluation de l'importance relative des critères en aide multicritère à la décision: réflexions théoriques, expérimentations et implémentations informatiques. Ph.D. thesis, Université Paris-Dauphine (1993)
81. Mousseau, V.: Eliciting information concerning the relative importance of criteria. In: Pardalos, P., Siskos, Y., Zopounidis, C. (eds.) Advances in Multicriteria Analysis. Nonconvex Optimization and Its Applications, pp. 17–43. Kluwer Academic, Dordrecht (1995)
82. Mousseau, V., Dias, L.: Valued outranking relations in ELECTRE providing manageable disaggregation procedures. Eur. J. Oper. Res. **156**, 417–482 (2004)

83. Mousseau, V., Słowiński, R.: Inferring an ELECTRE TRI model from assignment examples. J. Glob. Optim. **12**(2), 157–174 (1998)
84. Mousseau, V., Słowiński, R., Zielniewicz, P.: ELECTRE TRI 2.0a: methodological guide and user's documentation. Document du LAMSADE 111, Université Paris-Dauphine (1999)
85. Mousseau, V., Roy, B., Sommerlatt, I.: Elaboration d'un outil d'aide à la décision en vue de l'évolution de la tarification des transports publics en Ile de France. J. Decis. Syst. **9**(2), 289–315 (2000)
86. Mousseau, V., Figueira, J., Naux, J.: Unsing assignment examples to infer weights for ELECTRE TRI method: some experimental results. Eur. J. Oper. Res. **130**(2), 263–275 (2001)
87. Ngo The, A., Mousseau, V.: Using assignment examples to infer category limits for the ELECTRE TRI method. J. Multicrit. Decis. Anal. **11**(1), 29–43 (2002)
88. Norese, M.F.: ELECTRE III as a support for participatory decision-making on the localisation of waste-treatment plants. Land Use Policy **23**, 76–85 (2006)
89. Norese, M.F., Viale, S.: A multi-profile sorting procedure in the public administration. Eur. J. Oper. Res. **138**, 365–379 (2002)
90. Parent, E., Schnabele, P.: Le choix d'un aménagement aquacole: Exemple d'utilisation de la méthode ELECTRE III et comparaison avec d'autres méthodes multicritères d'aide à la décision. Cahier du LAMSADE 47, Université Paris-Dauphine (1988)
91. Pomerol, J., Barba-Romero, S.: Multicriterion Decision Making in Management. Kluwer Academic, Dordrecht (2000)
92. Raju, K., Pillai, C.: Multicriterion decision making in river basin planning and development. Eur. J. Oper. Res. **112**(2), 249–257 (1999)
93. Richard, J.: Aide à la décision stratégique en PME. In: Jacquet-Lagrèze, E., Siskos, Y. (eds.) Méthode de décision multicritère, pp. 119–143. Editions Hommes et Techniques, Paris (1983)
94. Rios Insua, D., French, S.: A framework for sensitivity analysis in discrete multi-objective decision making. Eur. J. Oper. Res. **54**, 176–190 (1991)
95. Roba, E., Sussmann, B., Theys, M.: Les Méthodes de Choix Multicritère Appliquées à la Sélection du Personnel: Models of Manpower. The English University Press, London (1970)
96. Roberts, F.: Measurement Theory, with Applications to Decision Making, Utility and the Social Sciences. Addison-Wesley, New York (1979)
97. Rogers, M., Bruen, M.: Choosing realistic values of indifference, preference and veto thresholds for use with environmental criteria within ELECTRE. Eur. J. Oper. Res. **107**(3), 542–551 (1998)
98. Rogers, M., Bruen, M.: A new system for weighting environmental criteria for use within ELECTRE III. Eur. J. Oper. Res. **107**(3), 552–563 (1998)
99. Rogers, M., Bruen, M., Maystre, L.: ELECTRE and Decision Support. Kluwer Academic, London (2000)
100. Rohmer, J., Veret, J.: Choix d'une stratégie commerciale dans la distribution succursaliste. In: Jacquet-Lagrèze, E., Siskos, Y. (eds.) Méthode de décision multicritère, pp. 91–118. Editions Hommes et Techniques, Paris (1983)
101. Roussat, N., Dujet, C., Méhui, J.: Choosing a sustainable demolition waste management strategy using multicriteria decision analysis. Waste Manag. **29**, 12–20 (2009)
102. Roy, B.: Classement et choix en présence de points de vue multiples (la méthode ELECTRE). RIRO **8**, 57–75 (1968)
103. Roy, B.: Algèbre Moderne et Théorie des Graphes Orientées vers les Sciences Economiques et Sociales, vol. 1/vol. 2. Dunod, Paris (1969/1970)
104. Roy, B.: Critères multiples et modélisation des préférences: l'apport des relations de surclassement. Revue d'Économie Politique **1**, 1–44 (1974)
105. Roy, B.: From optimization to multicriteria decision aid: three main operational attitudes. In: Thiriez, H., Zionts, S. (eds.) Multiple Criteria Decision Making. Lecture Notes in Economics and Mathematical Systems, vol. 130, pp. 1–34. Springer, Berlin (1976)
106. Roy, B.: ELECTRE III: Un algorithme de classements fondé sur une représentation floue des préférences en présence de critères multiples. Cahiers du CERO **20**(1), 3–24 (1978)

107. Roy, B.: A multicriteria analysis for trichotomic segmentation problems. In: Nijkamp, P., Spronk, J. (eds.) Multiple Criteria Analysis: Operational Methods, pp. 245–257. Gower Press, Farnborough (1981)
108. Roy, B.: Méthodologie Multicritère d'aide à la Décision. Economica, Paris (1985)
109. Roy, B.: Main sources of inaccurate determination, uncertainty and imprecision in decision models. In: Munier, B., Shakun, M. (eds.) Compromise, Negotiation and Group Decision, pp. 43–62. Reidel Publishing, Dordrecht (1988)
110. Roy, B.: Main sources of inaccurate determination, uncertainty and imprecision. Math. Comput. Model. **12**(10/11), 1245–1254 (1989)
111. Roy, B.: The outranking approach and the foundations of ELECTRE methods. In: Bana e Costa, C.A. (ed.) Readings in Multiple Criteria Decision Aid, pp. 155–183. Springer, Berlin (1990)
112. Roy, B.: The outranking approach and the foundations of ELECTRE methods. Theor. Decis. **31**, 49–73 (1991)
113. Roy, B.: Multicriteria Methodology for Decision Aiding. Nonconvex Optimization and Its Applications, vol. 12. Kluwer Academic, Dordrecht (1996)
114. Roy, B.: A missing link in OR-AD: robustness analysis. Found. Comput. Decis. Sci. **23**(3), 141–160 (1998)
115. Roy, B.: Decision-aiding today: What should we expect? In: Gal, T., Stewart, T., Hanne, T. (eds.) Advances in MultCriteria Decision Making – MCDM Models, Algorithms, Theory, and Applications, pp. 1.1–1.35. Kluwer Academic, Boston (1999)
116. Roy, B.: A French-English decision aiding glossary. In: Newsletter of the EWG on MCDA (Supplement 1). Series, 3(1). Faculty of Economics, INESC-Coimbra, Portugal (2000)
117. Roy, B.: Optimisation et analyse multicritère. In: Jessua, C., Labrousse, C., Vitry, D. (eds.) Dictionnaire des Sciences Economiques, pp. 640–643. Presses Universitaires de Frances, Paris (2001)
118. Roy, B.: Robustesse de quoi et vis-à-vis de quoi mais aussi robustesse pourquoi en aide à la décision? In: Henggeler-Antunes, C., Figueira, J., Clímaco, J. (eds.) Proceedings of the 56th Meeting of the European Working Group Multiple Criteria Decision Aiding, Coimbra, 3–5 October 2002. Commissão de Coordenação da Região Centro, Coimbra (2004)
119. Roy, B.: To better respond to the robustness concern in decision aiding: four proposals based on a twofold observation. In: Zopounidis, C., Pardalos, P. (eds.) Handbook of Multicriteria Analysis, pp. 3–24. Springer, New York (2010)
120. Roy, B.: Two conceptions of decision aiding. Int. J. Multicrit. Decis. Mak. **1**, 74–79 (2010)
121. Roy, B., Bertier, P.: La méthode ELECTRE II. Note de travail 142, SEMA-METRA Metra International (1971)
122. Roy, B., Bertier, P.: La méthode ELECTRE II – Une application au média-planning. In: Ross, M. (ed.) OR'72, pp. 291–302. North-Holland, Amsterdam (1973)
123. Roy, B., Bouyssou, D.: Comparison of two decision-aid models applied to a nuclear plant siting example. Eur. J. Oper. Res. **25**, 200–215 (1986)
124. Roy, B., Bouyssou, D.: Aide Multicritère à la Décision: Méthodes et Cas. Economica, Paris (1993)
125. Roy, B., Hugonnard, J.: Classement des prolongements de lignes de métro en banlieue parisienne (présentation d'une méthode multicritère originale). Cahiers du CERO **24**(2–4), 153–171 (1982)
126. Roy, B., Hugonnard, J.: Le plan d'extension du métro en banlieue parisienne, un cas type de l'analyse multicritère. Les Cahiers Scientifiques de la Revue Transports **6**, 77–108 (1982)
127. Roy, B., Hugonnard, J.: Ranking of suburban line extension projects on the Paris metro system by a multicriteria method. Transp. Res. **16A**(4), 301–312 (1982)
128. Roy, B., Hugonnard, J.: Réponse à Monsieur Gargaillo. Les Cahiers Scientifiques de la Revue Transports **7**, 58–64 (1982)
129. Roy, B., Mousseau, V.: A theoretical framework for analysing the notion of relative importance of criteria. J. Multicrit. Decis. Anal. **5**, 145–159 (1996)

130. Roy, B., Skalka, J.: ELECTRE IS: aspects méthodologiques et guide d'utilisation. Document du LAMSADE 30, Université Paris Dauphine (1984)
131. Roy, B., Słowiński, R.: Handling effects of reinforced preference and counter-veto in credibility of outranking. Eur. J. Oper. Res. **188**(1), 185–190 (2008)
132. Roy, B., Słowiński, R., Treichel, W.: Multicriteria programming of water resources supply systems for rural areas. Water Resour. Bull. **28**(1), 13–31 (1992)
133. Roy, B., Vanderpooten, D.: The European school of MCDA: emergence, basic features and current works. J. Multicrit. Decis. Anal. **5**(1), 22–37 (1996)
134. Roy, B., Jacquet-Lagrèze, E., Blancher, M.: Elaboration de critères permettant une intégration des divers aspects liés au temps dans l'aide à la décision en matière de transports. 1ère phase 1975, 2e phase 1977, Délégation Générale à la Recherche Scientifique et Technique DGRST-SAEI. Action Thématique Programmée ATP Socio-Economie des Transports (1977)
135. Roy, B., Présent, M., Silhol, D.: A programming method for determining which Paris metro stations should be renovated. Eur. J. Oper. Res. **24**, 318–334 (1986)
136. Saaty, T.: The Analytical Hierarchy Process. McGraw-Hill, New York (1980)
137. Salminen, P., Hokkanen, J., Lahdelma, R.: Comparing multicriteria methods in the context of environmental problems. Eur. J. Oper. Res. **104**(3), 485–496 (1998)
138. Scharlig, A.: Décider sur Plusieurs Critères, Panorama de l'Aide à la Décision Multicritère. Presses Polytechniques et Universitaires Romandes, Lausanne (1985)
139. Scharlig, A.: Pratiquer ELECTRE et PROMÉTHÉE: Un complément à décider sur plusieurs critères. Presses Polytechniques et Universitaires Romandes, Lausanne (1996)
140. Shanian, A., Savadogo, O.: A non-compensatory compromised solution for material selection of bipolar plates for polymer electrolyte membrane fuel cell (PEMFC) using ELECTRE IV. Electrochim. Acta **51**, 5307–5315 (2006)
141. Silva Lopez, R.: Une technique systématique de modélisation multicritère pour l'évaluation d'alternatives de lutte contre la pollution industrielle: Technologies propres ou curatives. Document du LAMSADE 60, Université Paris-Dauphine (1990)
142. Simos, J.: Gestion des Déchets Solides Urbains Genevois: Les Faits, le Traitement, l'Analyse. Presses Polytechniques et Universitaires Romandes, Lausanne (1990)
143. Siskos, P., Houridis, S.: Rationalising photovoltaic energy investments with multicriteria decision analysis: a Greek case study. Int. J. Multicrit. Decis. Mak. **1**(2), 205–229 (2011)
144. Siskos, Y., Hubert, Ph.: Multi-criteria analysis of the impacts of energy alternatives: a survey and a new comparative approach. Eur. J. Oper. Res. **13**, 278–299 (1983)
145. Siskos, Y., Wascher, G., Winkels, H.: A bibliography on outranking approaches (1966–1982). Cahier du LAMSADE 31, Université Paris Dauphine (1983)
146. Srinivasa, R., Duckstein, L., Arondel, C.: Multicriterion analysis for sustainable water resources planning: a case study in Spain. Water Resour. Manag. **14**, 435–456 (2000)
147. Tanguy, Y., Leclere, N.: Une application des méthodes de choix multi-critères à l'étude de l'adéquation des personnels aux postes de travail. In: Actes du congrès Association Française pour la Cybernétique Economique et Technique (AFCET): Aide à la Décision, Paris, pp. 139–147, 16–17 May 1974
148. Tecle, A., Duckstein, L.: A procedure for selecting MCDM techniques for forest resources management. In: Goicoechea, A., Duckstein, L., Zionts, S. (eds.) Multiple Criteria Decision Making and Support at the Interface of Industry, Business and Finance, pp. 19–32. Springer, New York (1992)
149. Tecle, A., Fogel, M., Duckstein, L.: Multicriterion analysis of forest watershed management alternatives. Water Resour. Bull. **24**, 1169–1178 (1988)
150. Teng, J., Tzeng, G.: Multicriteria evaluation for strategies of improving and controlling air quality in the super city: a case study of Taipei city. J. Environ. Manag. **40**(3), 213–229 (1994)
151. Tervonen, T., Linkov, I., Figueira, J.R., Steevens, J., Chappel, M., Merad, M.M.: Risk-based classification system of nanomaterials. J. Nanoparticle Res. **11**(4), 757–766 (2009)
152. Tsoukiàs, A., Perny, P., Vincke, Ph.: From concordance/discordance to the modelling of positive and negative reasons in decision aiding. In: Bouyssou, D., Jacquet-Lagrèze, E.,

Perny, P., Słowiński, R., Vanderpooten, D., Vincke, Ph. (eds.) Aiding Decisions with Multiple Criteria: Essays in Honour of Bernard Roy, pp. 147–174. Kluwer Academic, Dordrecht (2002)
153. Vallée, D., Zielniewicz, P.: ELECTRE III-IV, version 3.x, Aspects Méthodologiques (tome 1), Guide d'utilisation (tome 2). Document du LAMSADE 85 et 85bis, Université Paris Dauphine (1994)
154. Vanderpooten, D.: The construction of prescriptions in outranking methods. In: Bana e Costa, C. (ed.) Readings in Multiple Criteria Decision Aid, pp. 184–216. Springer, Heidelberg (1990)
155. Vansnick, J.: On the problem of weight in multiple criteria decision making (the non-compensatory approach). Eur. J. Oper. Res. **24**, 288–294 (1986)
156. Ventura, M.: Evry 1: Concours d'aménagement urbain. Cahier 31, Institut d'Amenagement et d'Urbanisme de la Région Parisienne (IAURP), Paris (1973)
157. Vincke, Ph.: Exploitation of a crisp relation in a ranking problem. Theor. Decis. **32**(3), 221–240 (1992)
158. Vincke, Ph.: Multicriteria Decision-Aid. Wiley, New York (1992)
159. Wijnmalen, D.: A case-study in military decision-making with mission-oriented multiple objectives. In: Clímaco, J. (ed.) Multicriteria Analysis, pp. 549–560. Springer, Berlin (1997)
160. Yu, W.: Aide multicritère à la décision dans le cadre de la problématique du tri: concepts, méthodes et applications. Ph.D. thesis, Université Paris-Dauphine (1992)
161. Yu, W.: ELECTRE TRI: Aspects méthodologiques et manuel d'utilisation. Document du LAMSADE 74, Université Paris-Dauphine (1992)
162. Zopounidis, C.: Evaluation du risque de défaillance de l'entreprise: Méthodes et cas d'application. Economica, Paris (1995)
163. Zopounidis, C.: Operational Tools in the Management of Financial Risks. Kluwer Academic, Dordrecht (1998)
164. Zopounidis, C., Dimitras, A.: Multicriteria Decision Aid Methods for the Prediction of Business Failure. Kluwer Academic, Dordrecht (1998)
165. Zopounidis, C., Hurson, C.: Portfolio selection and multicriteria analysis. In: Floudas, C., Pardalos, P. (eds.) Encyclopedia of Optimization, vol. IV, pp. 310–322. Kluwer Academic, Dordrecht (2001)

Chapter 6
PROMETHEE Methods

Jean-Pierre Brans and Yves De Smet

Abstract This paper gives an overview of the PROMETHEE-GAIA methodology for MCDA. It starts with general comments on multicriteria problems, stressing that a multicriteria problem cannot be treated without additional information related to the preferences and the priorities of the decision-makers. The information requested by PROMETHEE and GAIA is particularly clear and easy to define for both decision-makers and analysts. It consists in a preference function associated to each criterion as well as weights describing their relative importance. The PROMETHEE I, the PROMETHEE II ranking, as well as the GAIA visual interactive module are then presented. Additionally, comments about potential rank reversal occurrences are provided. The two next sections are devoted to the PROMETHEE VI sensitivity analysis procedure (human brain) and to the PROMETHEE V procedure for multiple selection of alternatives under constraints. A sorting method based on the PROMETHEE flow scores, called FlowSort, is described. An overview of the PROMETHEE GDSS procedure for group decision making is then given. Finally the D-Sight implementation of the PROMETHEE-GAIA methodology is presented.

Keywords MCDA • Outranking methods • PROMETHEE-GAIA • D-Sight

J.-P. Brans (✉)
Centrum voor Statistiek en Operationeel Onderzoek, Vrije Universiteit Brussel, Pleinlaan 2, 1050 Brussels, Belgium
e-mail: brans.jp@skynet.be

Y. De Smet
Computer and Decision Engineering Department - CoDE-SMG, Université Libre de Bruxelles, Boulevard du Triomphe CP 210-01, 1050 Brussels, Belgium
e-mail: yves.de.smet@ulb.ac.be

S. Greco et al. (eds.), *Multiple Criteria Decision Analysis*, International Series in Operations Research & Management Science 233,
DOI 10.1007/978-1-4939-3094-4_6

6.1 Preamble

This chapter is an updated version of [15]. Since 2005, a number of works have been focused on the PROMETHEE and GAIA methods. We decided to include in this paper some of these contributions and more specifically those regarding the following papers:

- In 1996, W. De Keyser and P. Peeters [19] initially pointed out rank reversal occurrences in the PROMETHEE I ranking. Recently, several authors analyzed conditions under which these phenomena could potentially happen. Their main results will be presented in Sect. 6.6;
- In his Ph.D. thesis [36], P. Nemery de Bellevaux proposed a sorting method based on the PROMETHEE flow scores. This approach will be summarized in Sect. 6.10;
- A new PROMETHEE and GAIA based software, called D-Sight, is now available. Section 6.12 will be dedicated to its description.

Of course, we cannot address all the contributions that have been proposed since 2005 [more than 40 new articles have been published in scientific journals since 2005 with one of their keywords corresponding to PROMETHEE (source: Science Direct)]. Far from being exhaustive, we can cite applications to portfolio and stock selection problems [1, 32, 46], to environmental issues [24, 26, 39, 44, 49], to energy management [22, 31], to chemometrics [18, 38, 41, 50], to statistical distribution selection [27] ... Recent methodological extensions include the use of the Choquet integral to model interactions between criteria [20], an extension of the Promethee II method based on generalized fuzzy numbers [28], the use of PROMETHEE in new classification methods [25, 40] ... Finally, we would like to give prominence to the latest comprehensive literature review realized by Behzadian et al. [4]. The authors have listed more than 200 papers published in 100 different journals. The applications fields cover finance, health care, logistics and transportation, hydrology and water management, manufacturing and assembly ...

B. Mareschal decided, for personal reasons, not to be a co-author of this revised chapter. We respect his decision and thank him, once again, for his continuous involvement in the development of the PROMETHEE and GAIA methodology.

6.2 History

The PROMETHEE I (partial ranking) and PROMETHEE II (complete ranking) were developed by J.P. Brans and presented for the first time in 1982 at a conference organized by R. Nadeau and M. Landry at the Université Laval, Québec, Canada (L'Ingénierie de la Décision. Elaboration d'instruments d'Aide à la Décision). The same year several applications using this methodology were already treated by G. Davignon in the field of health care.

A few years later J.P. Brans and B. Mareschal developed PROMETHEE III (ranking based on intervals) and PROMETHEE IV (continuous case). The same authors proposed in 1988 the visual interactive module GAIA which is providing a marvellous graphical representation supporting the PROMETHEE methodology.

In 1992 and 1994, J.P. Brans and B. Mareschal further suggested two nice extensions: PROMETHEE V (MCDA including segmentation constraints) and PROMETHEE VI (representation of the human brain).

A considerable number of successful applications has been treated by the PROMETHEE methodology in various fields such as Banking, Industrial Location, Manpower planning, Water resources, Investments, Medicine, Chemistry, Health care, Tourism, Ethics in OR, Dynamic management, ... The success of the methodology is basically due to its mathematical properties and to its particular friendliness of use.

6.3 Multicriteria Problems

Let us consider the following multicriteria problem:

$$\max\{g_1(a), g_2(a), \ldots, g_j(a), \ldots, g_k(a) | a \in A\}, \tag{6.1}$$

where A is a finite set of possible alternatives $\{a_1, a_2, \ldots a_i, \ldots, a_n\}$ and $\{g_1(\cdot), g_2(\cdot), \ldots, g_j(\cdot), \ldots g_k(\cdot)\}$ a set of evaluation criteria. There is no objection to consider some criteria to be maximized and the others to be minimized. The expectation of the decision-maker is to identify an alternative optimizing all the criteria.

Usually this is an *ill-posed mathematical* problem as there exists no alternative optimizing all the criteria at the same time. However most (nearly all) human problems have a multicriteria nature. According to our various human aspirations, it makes no sense, and it is often not fair, to select a decision based on one evaluation criterion only. In most of cases at least technological, economical, environmental, social and educational criteria should always be taken into account. Multicriteria problems are therefore extremely important and request an appropriate treatment.

If A is finite, the basic data of a multicriteria problem (6.1) consist of an evaluation table (Table 6.1).

Let us consider as an example the problem of an individual purchasing a car. Of course the price is important and it should be minimized. However it is clear that in general individuals are not considering only the price. Not everybody is driving the cheapest car! Most people would like to drive a luxury or sports car at the price of an economy car. Indeed they consider many criteria such as price, reputation, comfort, speed, reliability, consumption, ... As there is no car optimizing all the criteria at the same time, a *compromise* solution should be selected. Most decision problems have such a multicriteria nature.

Table 6.1 Evaluation table

a	$g_1(\cdot)$	$g_2(\cdot)$	$\dots$	$g_j(\cdot)$	$\dots$	$g_k(\cdot)$
a_1	$g_1(a_1)$	$g_2(a_1)$	$\dots$	$g_j(a_1)$	$\dots$	$g_k(a_1)$
a_2	$g_1(a_2)$	$g_2(a_2)$	$\dots$	$g_j(a_2)$	$\dots$	$g_k(a_2)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$
a_i	$g_1(a_i)$	$g_2(a_i)$	$\dots$	$g_j(a_i)$	$\dots$	$g_k(a_i)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$
a_n	$g_1(a_n)$	$g_2(a_n)$	$\dots$	$g_j(a_n)$	$\dots$	$g_k(a_n)$

The solution of a multicriteria problem depends not only on the basic data included in the evaluation table but also on the decision-maker himself. All individuals do not purchase the same car. There is no absolute best solution! The best compromise solution also depends on the individual *preferences* of each decision-maker, on the *"brain"* of each decision-maker.

Consequently, *additional information* representing these preferences is required to provide the decision maker with useful decision aid.

The natural dominance relation associated to a multicriteria problem of type (6.1) is defined as follows:

For each $(a, b) \in A$:

$$\begin{cases} \forall j : g_j(a) \geq g_j(b) \\ \exists k : g_k(a) > g_k(b) \end{cases} \iff aPb,$$
$$\forall j : g_j(a) = g_j(b) \quad \iff aIb, \tag{6.2}$$
$$\begin{cases} \exists s : g_s(a) > g_s(b) \\ \exists r : g_r(a) < g_r(b) \end{cases} \iff aRb,$$

where P, I, and R respectively stand for *preference*, *indifference* and *incomparability*. This definition is quite obvious. An alternative is better than another if it is at least as good as the other on all criteria. If an alternative is better on a criterion s and the other one better on criterion r, it is impossible to decide which is the best one without additional information. Both alternatives are therefore incomparable!

Alternatives which are not dominated by any other are called *efficient solutions*. Given an evaluation table for a particular multicriteria problem, most of the alternatives (often all of them) are usually efficient. The dominance relation is very poor on P and I. When an alternative is better on one criterion, the other is often better on another criterion. Consequently incomparability holds for most pairwise comparisons, so that it is impossible to decide without additional information. This information can for example include:

- Trade-offs between the criteria;
- A value function aggregating all the criteria in a single function (utility function) in order to obtain a single criterion problem for which an optimal solution exists;

- Weights giving the relative importance of the criteria;
- Preferences associated to each pairwise comparison within each criterion;
- Thresholds fixing preference limits;
- ...

Many multicriteria decision aid methods have been proposed. All these methods start from the same evaluation table, but they vary according to the additional information they request. The PROMETHEE methods require very clear additional information, that is easily obtained and understood by both decision-makers and analysts.

The purpose of all multicriteria methods is to enrich the dominance graph, i.e. to reduce the number of incomparabilities (R). When a utility function is built, the multicriteria problem is reduced to a single criterion problem for which an optimal solution exists. This seems exaggerated because it relies on quite strong assumptions (do we really make all our decisions based on a utility function defined somewhere in our brains?) and it completely transforms the structure of the decision problem. For this reason B. Roy proposed to build outranking relations including only realistic enrichments of the dominance relation (see [42, 43]). In that case, not all the incomparabilities are withdrawn but the information is reliable. The PROMETHEE methods belong to the class of outranking methods.

In order to build an appropriate multicriteria method some requisites could be considered:

Requisite 1: The amplitude of the deviations between the evaluations of the alternatives within each criterion should be taken into account:

$$d_j(a,b) = g_j(a) - g_j(b). \tag{6.3}$$

This information can easily be calculated, but is not considered in the efficiency theory. When these deviations are negligible the dominance relation can possibly be enriched.

Requisite 2: As the evaluations $g_j(a)$ of each criterion are expressed in their own units, *the scaling effects* should be completely eliminated. It is not acceptable to obtain conclusions depending on the scales in which the evaluations are expressed. Unfortunately not all multicriteria procedures are respecting this requisite!

Requisite 3: In the case of pairwise comparisons, an appropriate multicriteria method should provide the following information:

a is preferred to b;
a and b are indifferent;
a and b are incomparable.

The purpose is of course to reduce as much as possible the number of incomparabilities, but not when it is not realistic. Then the procedure may be considered as fair. When, for a particular procedure, all the incomparabilities are systematically withdrawn the provided information can be more disputable.

Requisite 4: Different multicriteria methods request different additional information and operate different calculation procedures so that the solutions they propose can be different. It is therefore important to develop methods being *understandable* by the decision-makers. "Black box" procedures should be avoided.

Requisite 5: An appropriate procedure should not include technical parameters having no significance for the decision-maker. Such parameters would again induce "Black box" effects.

Requisite 6: An appropriate method should provide information on the *conflicting nature* of the criteria.

Requisite 7: Most of the multicriteria methods are allocating weights of relative importance of the criteria. These weights reflects a major part of the *"brain"* of the decision-maker. It is not easy to fix them. Usually the decision-makers strongly hesitate. An appropriate method should offer *sensitivity tools* to test easily different sets of weights.

The PROMETHEE methods and the associated GAIA visual interactive module are taking all these requisites into account. On the other hand some mathematical properties that multicriteria problems possibly enjoy can also be considered. See for instance [47]. Such properties related to the PROMETHEE methods have been analyzed by [6] in a particularly interesting paper.

The next sections describe the PROMETHEE I and II rankings, the GAIA methods, as well as the PROMETHEE V and VI extensions of the methodology. The PROMETHEE III and IV extensions are not discussed here. Additional information can be found in [16]. Several actual applications of the PROMETHEE methodology are also mentioned in the list of references.

6.4 The PROMETHEE Preference Modelling Information

The PROMETHEE methods were designed to treat multicriteria problems of type (6.1) and their associated evaluation table.

The additional information requested to run PROMETHEE is particularly clear and understandable by both the analysts and the decision-makers. It consists of:

- Information between the criteria;
- Information within each criterion.

6.4.1 Information Between the Criteria

Table 6.2 should be completed, with the understanding that the set $\{w_j, j = 1, 2, \ldots, k\}$ represents weights of relative importance of the different criteria. These weights are non-negative numbers, independent from the measurement units of

Table 6.2 Weights of relative importance

$g_1(\cdot)$	$g_2(\cdot)$	...	$g_j(\cdot)$	...	$g_k(\cdot)$
w_1	w_2	...	w_j	...	w_k

the criteria. The higher the weight, the more important the criterion. There is no objection to consider normalized weights, so that:

$$\sum_{j=1}^{k} w_j = 1. \tag{6.4}$$

In the PROMETHEE software PROMCALC, DECISION LAB or D-Sight , the user is allowed to introduce arbitrary numbers for the weights, making it easier to express the relative importance of the criteria. These numbers are then divided by their sum so that the weights are normalized automatically.

Assessing weights to the criteria is not straightforward. It involves the priorities and perceptions of the decision-maker. The selection of the weights is his *space of freedom*. PROMCALC, DECISION LAB and D-Sight include several sensitivity tools to experience different set of weights in order to help to fix them.

6.4.2 *Information Within the Criteria*

PROMETHEE is not allocating an intrinsic absolute utility to each alternative, neither globally, nor on each criterion. We strongly believe that the decision-makers are not proceeding that way. The preference structure of PROMETHEE is based on *pairwise comparisons*. In this case the deviation between the evaluations of two alternatives on a particular criterion is considered. For small deviations, the decision-maker will allocate a small preference to the best alternative and even possibly no preference if he considers that this deviation is negligible. The larger the deviation, the larger the preference. There is no objection to consider that these preferences are real numbers varying between 0 and 1. This means that for each criterion the decision-maker has in mind a function

$$P_j(a,b) = F_j\left[d_j(a,b)\right] \quad \forall a,b \in A, \tag{6.5}$$

where:

$$d_j(a,b) = g_j(a) - g_j(b) \tag{6.6}$$

and for which:

$$0 \leq P_j(a,b) \leq 1. \tag{6.7}$$

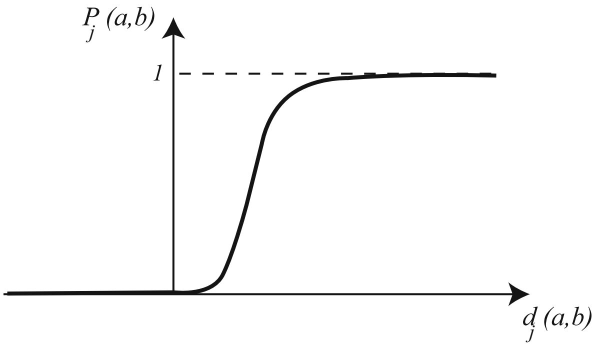


Fig. 6.1 Preference function

In case of a criterion to be maximized, this function is giving the preference of a over b for observed deviations between their evaluations on criterion $g_j(\cdot)$. It should have the following shape (see Fig. 6.1). The preferences equals 0 when the deviations are negative.

The following property holds:

$$P_j(a,b) > 0 \;\Rightarrow\; P_j(b,a) = 0. \tag{6.8}$$

For criteria to be minimized, the preference function should be reversed or alternatively given by:

$$P_j(a,b) = F_j\left[-d_j(a,b)\right]. \tag{6.9}$$

We have called the pair $\{g_j(\cdot), P_j(a,b)\}$ the *generalized criterion* associated to criterion $g_j(\cdot)$. Such a generalized criterion has to be defined for each criterion. In order to facilitate the identification six types of particular preference functions have been proposed (see Table 6.3). In each case 0, 1 or 2 parameters have to be defined, their significance is clear:

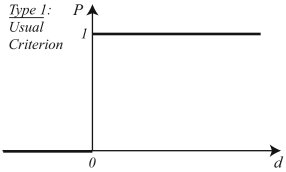
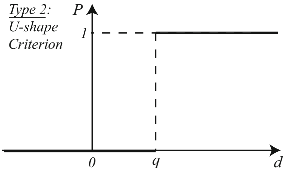
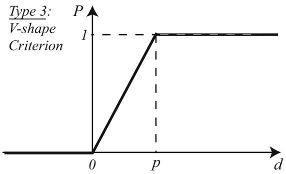
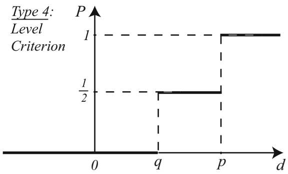
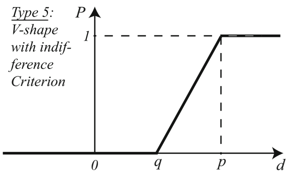
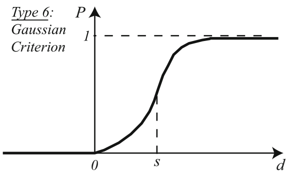
q is a threshold of indifference;
p is a threshold of strict preference ($P_j(a,b) = 1$);
s is an intermediate value between q and p.

The q indifference threshold is the largest deviation which is considered as negligible by the decision maker, while the p preference threshold is the smallest deviation which is considered as sufficient to generate a full preference.

The identification of a generalized criterion is then limited to the selection of the appropriate parameters. It is an easy task.

The PROMCALC, DECISION LAB and D-Sight software are proposing these six shapes only. As far as we know they have been satisfactory in most real-world applications. However there is no objection to consider additional generalized criteria.

Table 6.3 Types of generalized criteria ($P(d)$: preference function)

Generalized criterion	Definition	Parameters to fix
 Type 1: Usual Criterion	$P(d) = \begin{cases} 0 & d \leq 0 \\ 1 & d > 0 \end{cases}$	–
 Type 2: U-shape Criterion	$P(d) = \begin{cases} 0 & d \leq q \\ 1 & d > q \end{cases}$	q
 Type 3: V-shape Criterion	$P(d) = \begin{cases} 0 & d \leq 0 \\ \frac{d}{p} & 0 \leq d \leq p \\ 1 & d > p \end{cases}$	p
 Type 4: Level Criterion	$P(d) = \begin{cases} 0 & d \leq q \\ \frac{1}{2} & q < d \leq p \\ 1 & d > p \end{cases}$	p, q
 Type 5: V-shape with indifference Criterion	$P(d) = \begin{cases} 0 & d \leq q \\ \frac{d-q}{p-q} & q < d \leq p \\ 1 & d > p \end{cases}$	p, q
 Type 6: Gaussian Criterion	$P(d) = \begin{cases} 0 & d \leq 0 \\ 1 - e^{-\frac{d^2}{2s^2}} & d > 0 \end{cases}$	s

In case of type 5 a threshold of indifference q and a threshold of strict preference p have to be selected.

In case of a Gaussian criterion (type 6) the preference function remains increasing for all deviations and has no discontinuities, neither in its shape, nor in its derivatives. A parameter s has to be selected, it defines the inflection point of the preference function. We then recommend to determine first a q and a p and to fix s in between. If s is close to q the preferences will be reinforced for small deviations, while close to p they will be softened.

As soon as the evaluation table $\{g_j(\cdot)\}$ is given, and the weights w_j and the generalized criteria $\{g_j(\cdot), P_j(a,b)\}$ are defined for $i = 1, 2, \ldots, n; j = 1, 2, \ldots, k$, the PROMETHEE procedure can be applied.

6.5 The PROMETHEE I and II Rankings

The PROMETHEE procedure is based on pairwise comparisons (cf. [7–14, 17, 33, 34]). Let us first define aggregated preference indices and outranking flows.

6.5.1 Aggregated Preference Indices

Let $a, b \in A$, and let:

$$\begin{cases} \pi(a,b) = \sum_{j=1}^{k} P_j(a,b) w_j, \\ \pi(b,a) = \sum_{j=1}^{k} P_j(b,a) w_j. \end{cases} \tag{6.10}$$

$\pi(a,b)$ is expressing with which degree a is preferred to b over all the criteria and $\pi(b,a)$ how b is preferred to a. In most of the cases there are criteria for which a is better than b, and criteria for which b is better than a, consequently $\pi(a,b)$ and $\pi(b,a)$ are usually positive. The following properties hold for all $(a,b) \in A$.

$$\begin{cases} \pi(a,a) = 0, \\ 0 \leq \pi(a,b) \leq 1, \\ 0 \leq \pi(b,a) \leq 1, \\ 0 \leq \pi(a,b) + \pi(b,a) \leq 1. \end{cases} \tag{6.11}$$

It is clear that:

$$\begin{cases} \pi(a,b) \sim 0 \text{ implies a weak global preference of } a \text{ over } b, \\ \pi(a,b) \sim 1 \text{ implies a strong global preference of } a \text{ over } b. \end{cases} \tag{6.12}$$

In addition, it is obvious that $P_j(a,b), P_j(b,a), \pi(a,b)$ and $\pi(b,a)$ are real numbers (without units) completely independent of the scales of the criteria $g_j(.)$.

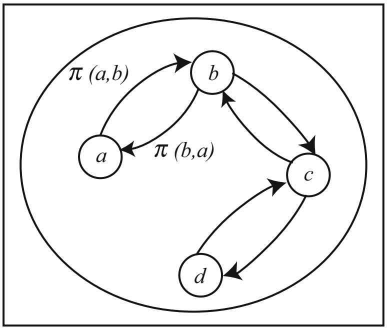


Fig. 6.2 Valued outranking graph

As soon as $\pi(a, b)$ and $\pi(b, a)$ are computed for each pair of alternatives of A, a complete valued outranking graph, including two arcs between each pair of nodes, is obtained (see Fig. 6.2).

6.5.2 Outranking Flows

Each alternative a is facing $(n - 1)$ other alternatives in A. Let us define the two following outranking flows:

- *the positive outranking flow:*

$$\phi^{+}(a) = \frac{1}{n-1} \sum_{x \in A} \pi(a, x), \tag{6.13}$$

- *the negative outranking flow:*

$$\phi^{-}(a) = \frac{1}{n-1} \sum_{x \in A} \pi(x, a). \tag{6.14}$$

The positive outranking flow expresses how an alternative a is *outranking* all the others. It is its *power, its outranking character*. The higher $\phi^{+}(a)$, the better the alternative (see Fig. 6.3a).

The negative outranking flow expresses how an alternative a is *outranked* by all the others. It is its *weakness, its outranked character*. The lower $\phi^{-}(a)$ the better the alternative (see Fig. 6.3b).

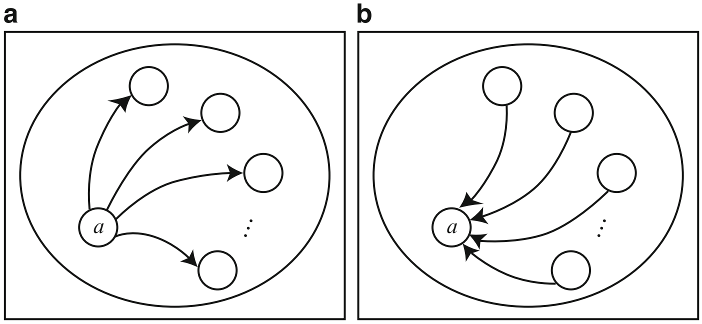


Fig. 6.3 The PROMETHEE outranking flows. (**a**) The $\phi^+(a)$ outranking flow. (**b**) The $\phi^-(a)$ outranking flow

6.5.3 The PROMETHEE I Partial Ranking

The PROMETHEE I partial ranking (P^I, I^I, R^I) is obtained from the positive and the negative outranking flows. Both flows do not usually induce the same rankings. PROMETHEE I is their intersection.

$$\begin{cases} aP^Ib & \text{iff} \quad \begin{cases} \phi^+(a) > \phi^+(b) \text{ and } \phi^-(a) < \phi^-(b), \text{ or} \\ \phi^+(a) = \phi^+(b) \text{ and } \phi^-(a) < \phi^-(b), \text{ or} \\ \phi^+(a) > \phi^+(b) \text{ and } \phi^-(a) = \phi^-(b); \end{cases} \\ aI^Ib & \text{iff} \quad \phi^+(a) = \phi^+(b) \text{ and } \phi^-(a) = \phi^-(b); \\ aR^Ib & \text{iff} \quad \begin{cases} \phi^+(a) > \phi^+(b) \text{ and } \phi^-(a) > \phi^-(b), \text{ or} \\ \phi^+(a) < \phi^+(b) \text{ and } \phi^-(a) < \phi^-(b); \end{cases} \end{cases} \tag{6.15}$$

where P^I, I^I, R^I respectively stand for preference, indifference and incomparability.

When aP^Ib, a higher power of a is associated to a lower weakness of a with regard to b. The information of both outranking flows is consistent and may therefore be considered as sure.

When aI^Ib, both positive and negative flows are equal.

When aR^Ib, a higher power of one alternative is associated to a lower weakness of the other. This often happens when a is good on a set of criteria on which b is weak and reversely b is good on some other criteria on which a is weak. In such a case the information provided by both flows is not consistent. It seems then reasonable to be careful and to consider both alternatives as incomparable. The PROMETHEE I ranking is prudent: it will not decide which action is best in such cases. It is up to the decision-maker to take his responsibility.

6.5.4 The PROMETHEE II Complete Ranking

PROMETHEE II consists of the (P^{II}, I^{II}) complete ranking. It is often the case that the decision-maker requests a complete ranking. The *net outranking flow* can then be considered.

$$\phi(a) = \phi^{+}(a) - \phi^{-}(a). \tag{6.16}$$

It is the balance between the positive and the negative outranking flows. The higher the net flow, the better the alternative, so that:

$$\begin{cases} aP^{II}b & \text{iff} \;\; \phi(a) > \phi(b), \\ aI^{II}b & \text{iff} \;\; \phi(a) = \phi(b). \end{cases} \tag{6.17}$$

When PROMETHEE II is considered, all the alternatives are comparable. No incomparabilities remain, but the resulting information can be more disputable because more information gets lost by considering the difference (6.16).

The following properties hold:

$$\begin{cases} -1 \leq \phi(a) \leq 1, \\ \sum_{x \in A} \phi(a) = 0. \end{cases} \tag{6.18}$$

When $\phi(a) > 0$, a is more outranking all the alternatives on all the criteria, when $\phi(a) < 0$ it is more outranked.

In real-world applications, we recommend to both the analysts and the decision-makers to consider both PROMETHEE I and PROMETHEE II. The complete ranking is easy to use, but the analysis of the incomparabilities often helps to finalize a proper decision.

As the net flow $\phi(\cdot)$ provides a complete ranking, it may be compared with a utility function. One advantage of $\phi(\cdot)$ is that it is built on clear and simple preference information (weights and preferences functions) and that it does rely on comparative statements rather than absolute statements.

6.5.5 The Profiles of the Alternatives

According to the definition of the positive and the negative outranking flows (6.13) and (6.14) and of the aggregated indices (6.10), we have:

$$\phi(a) = \phi^{+}(a) - \phi^{-}(a) = \frac{1}{n-1} \sum_{j=1}^{k} \sum_{x \in A} \left[P_j(a,x) - P_j(x,a) \right] w_j. \tag{6.19}$$

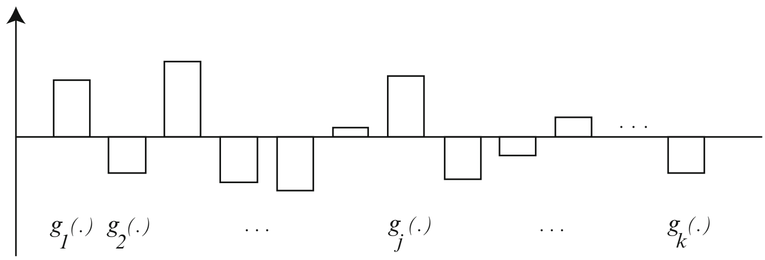


Fig. 6.4 Profile of an alternative

Consequently,

$$\phi(a) = \sum_{j=1}^{k} \phi_j(a) w_j \tag{6.20}$$

if

$$\phi_j(a) = \frac{1}{n-1} \sum_{x \in A} \left[P_j(a,x) - P_j(x,a) \right]. \tag{6.21}$$

$\phi_j(a)$ is the single criterion net flow obtained when only criterion $g_j(\cdot)$ is considered (100 % of the total weight is allocated to that criterion). It expresses how an alternative a is outranking ($\phi_j(a) > 0$) or outranked ($\phi_j(a) < 0$) by all the other alternatives on criterion $g_j(\cdot)$ only.

The profile of an alternative consists of the set of all the single criterion net flows: $\phi_j(a), j = 1, 2, \ldots, k$.

The profiles of the alternatives are particularly useful to appreciate their *"quality"* on the different criteria. It is extensively used by decision-makers to finalize their appreciation (Fig. 6.4).

According to (6.20), we observe that the global net flow of an alternative is the scalar product between the vector of the weights and the profile vector of this alternative. This property will be extensively used when building up the GAIA plane.

6.6 A Few Words About Rank Reversal

Pair-wise comparison methods, such as outranking methods, may suffer from the well-known rank reversal problem: the relative positions of two alternatives may be influenced by the presence of a third one. This phenomenon is not new and dates back from the beginning of social choice theory (see for instance the condition about irrelevant alternatives in the famous Arrow's theorem [3]).

A number of authors have already addressed this question in the context of multicriteria methods (see for instance [5] for the Analytic Hierarchy Process or [48] for ELECTRE methods). Let us stress that the debate is still very active and that a number of articles have been proposed to answer these issues. In the context of the PROMETHEE methods, W. De Keyser and P. Peeters [19] initially pointed out rank reversal occurrences in the context of the PROMETHEE I ranking. Following these observations, B. Mareschal et al. [35] and C. Verly et al. [45] have investigated conditions under which rank reversal could potentially occur in the PROMETHEE I and II rankings.

At first, it is important to stress that no unique definition of *rank reversal* exists. Some authors analyze if the positions of two alternatives can be affected by:

- the presence of a non-discriminating criterion;
- a copy of an alternative;
- a dominated alternative;
- any given alternative;
- ...

It is easy to prove that the PROMETHEE rankings will not be influenced by the presence or the elimination of a non discriminating criterion while it may be affected by copies of alternatives (see [45]). Furthermore, if a dominates b we will always have $\phi(a) \geq \phi(b)$ (whatever the other alternatives). No rank reversal could ever happen in such a situation.

If we investigate rank reversal occurrences induced by the deletion of a third alternative, we may come to the conclusion [35] that no rank reversal will occur in the PROMETHEE II ranking between a and b if

$$|\phi(a) - \phi(b)| > \frac{2}{n-1} \tag{6.22}$$

A direct corollary of this result is that rank reversal occurrences may only happen between alternatives which have close net flow scores. Additionally, C. Verly et al. [45] used computer simulations on artificial data sets to show that these rank reversal instances happened most of the time when the actual net flow differences were much lower than the $\frac{2}{n-1}$ threshold. This has led them to refine this bound. Finally, they extended the previous result in the context of the PROMETHEE I ranking and proved that no rank reversal will occur between a and b if the following conditions are satisfied:

$$|\phi^+(a) - \phi^+(b)| > \frac{1}{n-1} \tag{6.23}$$

$$|\phi^-(a) - \phi^-(b)| > \frac{1}{n-1} \tag{6.24}$$

6.7 The GAIA Visual Interactive Module

Let us first consider the matrix $M(n \times k)$ of the single criterion net flows of all the alternatives as defined in (6.21) (Table 6.4).

6.7.1 *The GAIA Plane*

The information included in matrix M is more extensive than the one in the evaluation Table 6.1, because the degrees of preference given by the generalized criteria are taken into account in M. Moreover the $g_j(a_i)$ are expressed on their own scale, while the $\phi_j(a_i)$ are dimensionless. In addition, let us observe, that M is not depending on the weights of the criteria. Consequently the set of the n alternatives can be represented as a cloud of n points in a k-dimensional space. According to (6.18) this cloud is centered at the origin. As the number of criteria is usually larger than two, it is impossible to obtain a clear view of the relative position of the points with regard to the criteria. We therefore project the information included in the k-dimensional space on a plane. Let us project not only the points representing the alternatives but also the unit vectors of the coordinate-axes representing the criteria.

The GAIA *plane* is the plane for which as much information as possible is preserved after projection. According to the *principal components analysis* technique it is defined by the two eigenvectors corresponding to the two largest eigenvalues of the covariance matrix $M'M$ of the single criterion net flows (Fig. 6.5).

Of course some information get lost after projection. The GAIA plane is a *meta model* (a model of a model). Let δ be the quantity of information preserved:

$$\delta = \frac{\lambda_1 + \lambda_2}{\sum_{j=1}^{k} \lambda_j} \tag{6.25}$$

Table 6.4 Single criterion net flows

	$\phi_1(\cdot)$	$\phi_2(\cdot)$	$\ldots$	$\phi_j(\cdot)$	$\ldots$	$\phi_k(\cdot)$
a_1	$\phi_1(a_1)$	$\phi_2(a_1)$	$\ldots$	$\phi_j(a_1)$	$\ldots$	$\phi_k(a_1)$
a_2	$\phi_1(a_2)$	$\phi_2(a_2)$	$\ldots$	$\phi_j(a_2)$	$\ldots$	$\phi_k(a_2)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$
a_i	$\phi_1(a_i)$	$\phi_2(a_i)$	$\ldots$	$\phi_j(a_i)$	$\ldots$	$\phi_k(a_i)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$
a_n	$\phi_1(a_n)$	$\phi_2(a_n)$	$\ldots$	$\phi_j(a_n)$	$\ldots$	$\phi_k(a_n)$

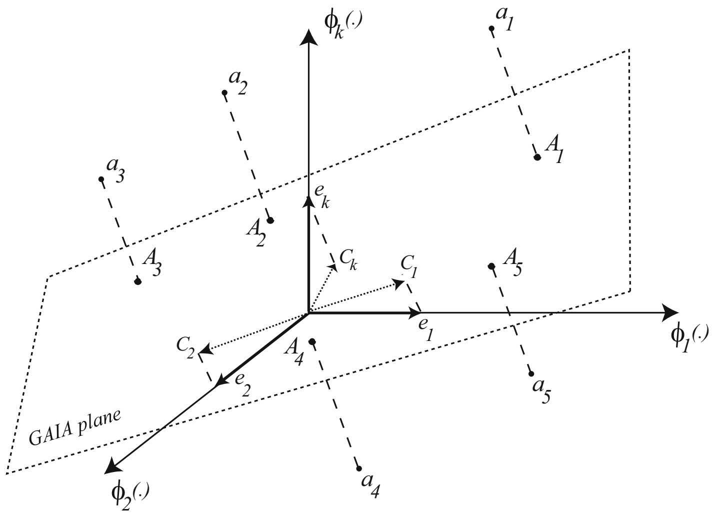


Fig. 6.5 Projection on the GAIA plane

where $\lambda_1, \lambda_2, \ldots, \lambda_j, \ldots, \lambda_k$ is the set of the k eigenvalues of $M'M$ ranked from the highest to the lowest one.

In most applications we have treated so far δ was larger than 60 % and in many cases larger than 80 %. This means that the information provided by the GAIA plane is rather reliable. This information is quite rich, it helps to understand the structure of a multicriteria problem. It is not often the case that δ is very small. When its value is too low (say $\delta < 0.5$) the GAIA plane becomes progressively useless.

6.7.2 Graphical Display of the Alternatives and of the Criteria

Let $(A_1, A_2, \ldots, A_i, \ldots, A_n)$ be the projections of the n points representing the alternatives and let $(C_1, C_2, \ldots, C_j, \ldots, C_k)$ be the projections of the k unit vectors of the coordinates axes of R^k representing the criteria. We then obtain a GAIA plane of the following type: Then the following properties hold (see [14, 33]) provided that δ is sufficiently high:

P 1: The longer a criterion axis in the GAIA plane, the more discriminating this criterion.

P 2: Criteria expressing similar preferences are represented by axes oriented in approximatively the same direction.

P 3: Criteria expressing conflicting preferences are oriented in opposite directions.

P 4: Criteria that are not related to each others in terms of preferences are represented by orthogonal axes.

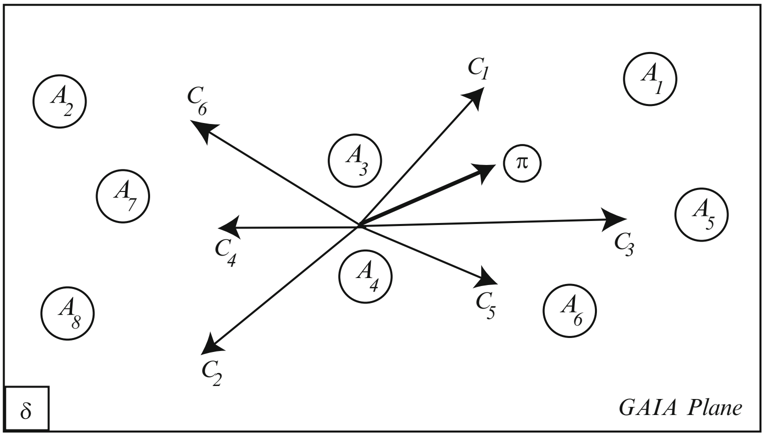


Fig. 6.6 Alternatives and criteria in the GAIA plane

P 5: Similar alternatives are represented by points located close to each other.
P 6: Alternatives being good on a particular criterion are represented by points located in the direction of the corresponding criterion axis.

On the example of Fig. 6.6, we observe:

- That the criteria $g_1(\cdot)$ and $g_3(\cdot)$ are expressing similar preferences and that the alternatives a_1 and a_5 are rather good on these criteria.
- That the criteria $g_6(\cdot)$ and $g_4(\cdot)$ are also expressing similar preferences and that the alternatives a_2, a_7, and a_8 are rather good on them.
- That the criteria $g_2(\cdot)$ and $g_5(\cdot)$ are rather independent
- That the criteria $g_1(\cdot)$ and $g_3(\cdot)$ are strongly conflicting with the criteria $g_4(\cdot)$ and $g_2(\cdot)$
- That the alternatives a_1, a_5 and a_6 are rather good on the criteria $g_1(\cdot)$, $g_3(\cdot)$ and $g_5(\cdot)$
- That the alternatives a_2, a_7 and a_8 are rather good on the criteria $g_6(\cdot)$, $g_4(\cdot)$ and $g_2(\cdot)$
- That the alternatives a_3 and a_4 are never good, never bad on all the criteria,
- ...

Although the GAIA plane includes only a percentage δ of the total information, it provides a powerful graphical visualisation tool for the analysis of a multicriteria problem. The discriminating power of the criteria, the conflicting aspects, as well as the "quality" of each alternative on the different criteria are becoming particularly clear.

6.7.3 The PROMETHEE Decision Stick. The PROMETHEE Decision Axis

Let us now introduce the impact of the weights in the GAIA plane. The vector of the weights is obviously also a vector of R^k. According to (6.20), the PROMETHEE net flow of an alternative a_i is the scalar product between the vector of its single criterion net flows and the vector of the weights:

$$\begin{aligned} a_i &: (\phi_1(a_i), \phi_2(a_i), \ldots, \phi_j(a_i), \ldots, \phi_k(a_i)), \\ w &: (w_1, w_2, \ldots, w_j, \ldots, w_k). \end{aligned} \tag{6.26}$$

This also means that the PROMETHEE net flow of a_i is the projection of the vector of its single criterion net flows on w. Consequently, the relative positions of the projections of all the alternatives on w provides the PROMETHEE II ranking. Clearly the vector w plays a crucial role. It can be represented in the GAIA plane by the projection of the unit vector of the weights. Let π be this projection, and let us call π the *PROMETHEE decision axis*.

On the example of Fig. 6.7, the PROMETHEE ranking is: $a_4 \succ a_3 \succ a_2 \succ a_1$. A realistic view of this ranking is given in the GAIA plane although some inconsistencies due to the projection can possibly occur.

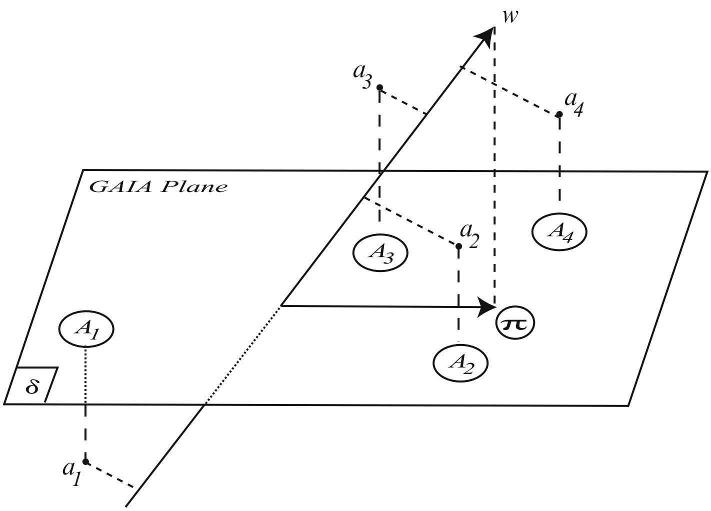


Fig. 6.7 PROMETHEE II ranking. PROMETHEE decision axis and stick

If all the weights are concentrated on one criterion, it is clear that the PROMETHEE decision axis will coincide with the axis of this criterion in the GAIA plane. Both axes are then the projection of a coordinate unit vector of $\mathbb{R}^k$. When the weights are distributed over all the criteria, the PROMETHEE decision axis appears as a weighted resultant of all the criterion axes $(C_1, C_2, \ldots, C_j, \ldots, C_k)$.

If π is long, the PROMETHEE decision axis has a strong decision power and the decision-maker is invited to select alternatives as far as possible in its direction.

If π is short, the PROMETHEE decision axis has no strong decision power. It means, according to the weights, that the criteria are strongly conflicting and that the selection of a good compromise is a hard problem.

When the weights are modified, the positions of the alternatives and of the criteria remain unchanged in the GAIA plane. The weight vector appears as a *decision stick* that the decision-maker can move according to his preferences in favour of particular criteria. When a sensitivity analysis is applied by modifying the weights, the PROMETHEE decision stick (w) and the PROMETHEE decision axis (π) are moving in such a way that the consequences for decision-making are easily observed in the GAIA plane (see Fig. 6.8).

Decision-making for multicriteria problems appears, thanks to this methodology, as a piloting problem. Piloting the decision stick over the GAIA plane. The PROMETHEE decision stick and the PROMETHEE decision axis provide a strong sensitivity analysis tool. Before finalising a decision we recommend to the decision-maker to simulate different weight distributions. In each case the situation can easily be appreciated in the GAIA plane, the recommended alternatives are located in the direction of the decision axis. As the alternatives and the criteria remain unchanged when the PROMETHEE decision stick is moving, the sensitivity analysis is particularly easy to manage. Piloting the decision stick is instantaneously operated by the PROMCALC, DECISION LAB and D-Sight software. The process is displayed graphically so that the results are easy to appreciate.

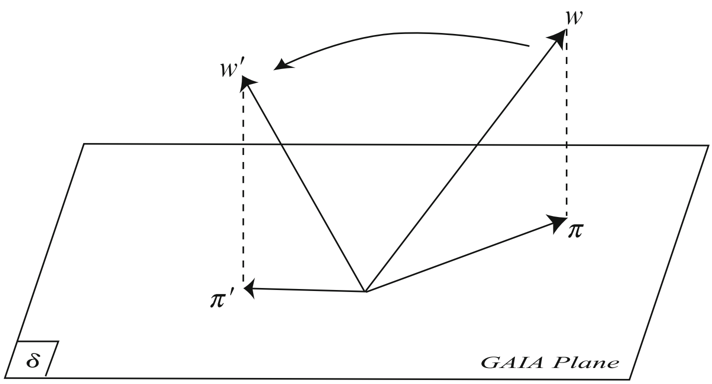


Fig. 6.8 Piloting the PROMETHEE decision stick

6.8 The PROMETHEE VI Sensitivity Tool (the "Human Brain")

The PROMETHEE VI module provides the decision-maker with additional information on his own personal view of his multicriteria problem. It allows to appreciate whether the problem is *hard or soft* according to his personal opinion.

It is obvious that the distribution of the weights plays an important role in all multicriteria problems. As soon as the weights are fixed, a final ranking is proposed by PROMETHEE II. In most of the cases the decision-maker is hesitating to allocate immediately precise values of the weights. His hesitation is due to several factors such as *indetermination, imprecision, uncertainty, lack of control,* ... on the real-world situation.

However the decision-maker has usually in mind some order of magnitude on the weights, so that, despite his hesitations, he is able to give some intervals including their correct values. Let these intervals be:

$$w_j^- \leq w_j \leq w_j^+, \; j = 1, \ldots, k. \tag{6.27}$$

Let us then consider the set of all the extreme points of the unit vectors associated to all allowable weights. This set is limiting an area on the unit hypersphere in $\mathbb{R}^k$ (Fig. 6.9). Let us project this area on the GAIA plane and let us call (*HB*) (*"Human Brain"*) the obtained projection. Obviously (*HB*) is the area including all the extreme points of the PROMETHEE decision axis (π) for all allowable weights. Two particular situations can occur (Fig. 6.10):

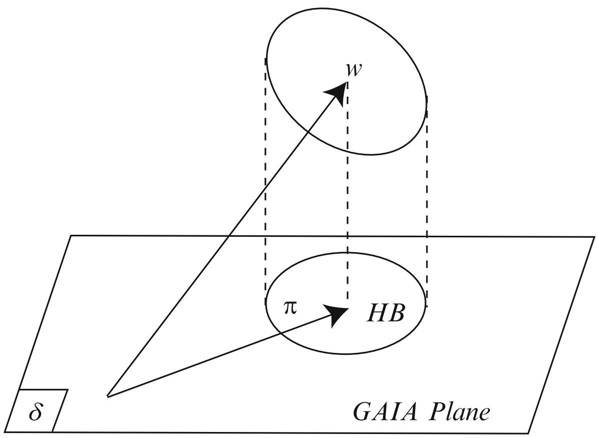


Fig. 6.9 "Human Brain"

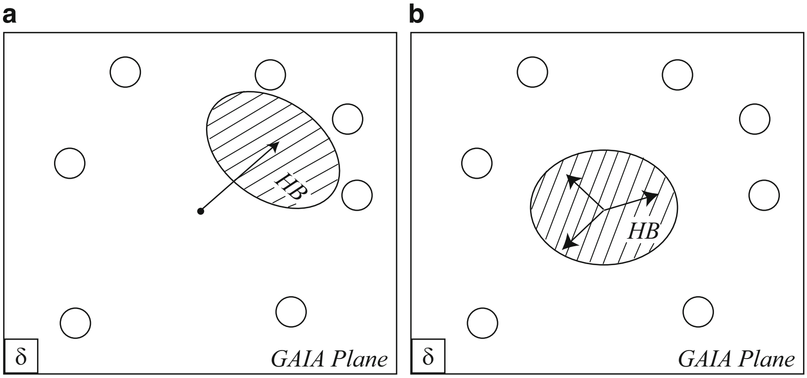


Fig. 6.10 Two types of decision problems. (**a**) Soft problem (S1). (**b**) Hard problem (S2)

S1: (*HB*) does not include the origin of the GAIA plane. In this case, when the weights are modified, the PROMETHEE decision axis (π) remains globally oriented in the same direction and all alternatives located in this direction are good. The multicriteria problem is rather easy to solve, it is a *soft* problem.

S2: Reversely if (*HB*) is including the origin, the PROMETHEE decision axis (π) can take any orientation. In this case compromise solutions can be possibly obtained in all directions. It is then actually difficult to make a final decision. According to his preferences and his hesitations, the decision-maker is facing a *hard* problem.

In most of the practical applications treated so far, the problems appeared to be rather soft and not too hard. This means that most multicriteria problems offer at the same time good compromises and bad solutions. PROMETHEE allows to select the good ones.

6.9 PROMETHEE V: MCDA Under Constraints

PROMETHEE I and II are appropriate to select one alternative. However in some applications a subset of alternatives must be identified, given a set of constraints. PROMETHEE V is extending the PROMETHEE methods to that particular case (see [11]).

Let $\{a_i, i = 1, 2, \ldots, n\}$ be the set of possible alternatives and let us associate the following boolean variables to them:

$$x_i = \begin{cases} 1 & \text{if } a_i \text{ is selected,} \\ 0 & \text{if not.} \end{cases} \tag{6.28}$$

The PROMETHEE V procedure consists of the two following steps:

Step 1: The multicriteria problem is first considered without constraints. The PROMETHEE II ranking is obtained for which the net flows $\{\phi(a_i), i = 1, 2, \ldots, n\}$ have been computed.

Step 2: The following $\{0, 1\}$ linear program is then considered in order to take into account the additional constraints (provided that they can be expressed linearly).

$$\max \left\{ \sum_{i=1}^{k} \phi(a_i) x_i \right\} \tag{6.29}$$

$$\sum_{i=1}^{n} \lambda_{p,i} x_i \sim \beta_p \quad p = 1, 2, \ldots, P \tag{6.30}$$

$$x_i \in \{0, 1\} \quad i = 1, 2, \ldots, n, \tag{6.31}$$

where $\sim$ holds for $=$, $\geq$ or $\leq$, and where the $\lambda_{p,i}$ are the coefficients of the constraints. The coefficients of the objective function (6.29) are the net outranking flows. The higher the net flow, the better the alternative. The purpose of the $\{0, 1\}$ linear program is to select alternatives collecting as much net flow as possible and taking the constraints into account.

The constraints (6.30) can include cardinality, budget, return, investment, marketing, … constraints. They can be related to all the alternatives or possibly to some clusters.

After having solved the $\{0, 1\}$ linear program, a subset of alternatives satisfying the constraints and providing as much net flow as possible is obtained. Classical 0–1 linear programming procedures may be used.

The PROMCALC software includes this PROMETHEE V procedure.

6.10 FlowSort

Recently, a number of researchers have proposed ways to extend the PROMETHEE methodology to sorting problems. Among them, we can cite PROMETHEE TRI [21] or PROMSORT [2]. In what follows, we describe a limited version of the FlowSort procedure developed by P. Nemery de Bellevaux in his Ph.D. thesis. From our point of view, this method constitutes the most natural extension of PROMETHEE to the sorting problematic.

The sorting problematic consists *in partitioning a set of alternatives into subsets with respect to pre-established norms* [47]. One way to interpret this definition is to assign a set of alternatives to predefined ordered groups (also called categories). For instance, one may think about the following applications:

- to assign a given patient to categories representing different disease grades according to a set of symptoms;
- to assign a company to categories representing different business failure risk levels according to financial criteria;
- ...

Let $Z_1, Z_2, \ldots, Z_V$ denote the V different categories. These are assumed to be ranked in order of preference: Z_1 is better than Z_2, Z_2 is better than Z_3, ... Consequently, Z_1 is considered to be best category while Z_V is the worst one. Let $\succ$ represent the preference order between the categories ($Z_1 \succ Z_2 \succ \ldots \succ Z_V$). We assume that each category Z_h is characterized by two limit profiles: the upper profile r_h and the lower profile r_{h+1} (let us note that the lower profile of Z_h corresponds to the upper profile of Z_{h+1}). Let $R = \{r_1, \ldots, r_{V+1}\}$ be the set of profiles. These are assumed to respect the following conditions:

Condition 1:

$$\forall a_i \in A : g_j(r_{V+1}) \leq g_j(a_i) \leq g_j(r_1) \ \forall j \in \{1, \ldots, q\} \tag{6.32}$$

Condition 2:

$$\forall r_h, r_l \in R | h < l : g_j(r_h) \geq g_j(r_l) \ \forall j \in \{1, \ldots, q\} \tag{6.33}$$

Condition 3:

$$\forall r_h, r_l \in R | h < l : \pi(r_h, r_l) > 0 \tag{6.34}$$

The first condition imposes that all the evaluations of the alternatives to be assigned are lying between r_{V+1} and r_1. As a natural consequence, no evaluation can be better than the one of the upper profile of the best category or worse than the lower profile of the worst category. Let us note that this condition is not restrictive since r_1 (respectively r_{V+1}) can always be defined as the ideal point of the problem (respectively the nadir point).

The two next conditions impose that some consistency should exist between the order of the categories and the preferences between the limit profiles:

- the evaluation of the upper limit profile of a better category should be at least as good as the evaluation of the upper profile of a worse category;
- the preference of the upper profile of a better category over the upper profile of a worse category should always be strictly positive.

Let us consider an alternative $a_i \in A$ to be sorted. The underlying idea of the FlowSort procedure is to compare a_i with respect to the elements of R by using the PROMETHEE I or PROMETHEE II ranking. Let us define $R_i = R \bigcup \{a_i\}$ (therefore $|R_i| = V + 2$). For all $x \in R_i$, the flow scores are computed as follows:

$$\phi_{R_i}^{+}(x) = \frac{1}{V+1} \sum_{y \in R_i} \pi(x, y) \tag{6.35}$$

$$\phi_{R_i}^{-}(x) = \frac{1}{V+1} \sum_{y \in R_i} \pi(y, x) \tag{6.36}$$

$$\phi_{R_i}(x) = \phi_{R_i}^{+}(x) - \phi_{R_i}^{-}(x) \tag{6.37}$$

The ranking based on the positive and negative flow scores can lead to two different situations:

$$Z_{\phi^+}(a_i) = Z_h \quad if \quad \phi_{R_i}^{+}(r_h) \geq \phi_{R_i}^{+}(a_i) > \phi_{R_i}^{+}(r_{h+1}) \tag{6.38}$$

$$Z_{\phi^-}(a_i) = Z_l \quad if \quad \phi_{R_i}^{-}(r_l) < \phi_{R_i}^{-}(a_i) \leq \phi_{R_i}^{-}(r_{l+1}) \tag{6.39}$$

where $Z_{\phi^+}(a_i)$ (respectively $Z_{\phi^-}(a_i)$) represents the assignment based on the positive (respectively negative) flow score only. Nevertheless, the assignment rule based on the PROMETHEE I ranking should integrate both of these aspects. As a consequence, let $b = \min\{h, l\}$ be the index of the category corresponding to the best assignment and let $w = \max\{h, l\}$ be the index of the category corresponding to the worst assignment. The first assignment rule will lead to conclude that a_i is assigned to the set of categories $[Z_b, \ldots, Z_w]$. Of course, if $w = b$ the assignment is unique.

Alternatively, the decision maker could force the assignment to a unique category by using a rule based on the net flow score:

$$Z_{\phi}(a_i) = Z_t \; if \; \phi_{R_i}(r_t) \geq \phi_{R_i}(a_i) > \phi_{R_i}(r_{t+1}) \tag{6.40}$$

As expected, the assignment procedures based on the PROMETHEE I and PROMETHEE II rankings are consistent. More formally [36]:

$$\forall a_i \in A : Z_b(a_i) \succeq Z_t(a_i) \succeq Z_w(a_i) \tag{6.41}$$

In other words, the assignment based on the net flow score will always lead to a category that is at least as good as ($\succeq$) the worst category and no better than the best category found by the first assignment rule.

These two assignment rules are the basics of FlowSort. Let us remind the reader that this section only constitutes a limited presentation of the method. We have to stress that a similar procedure exists when categories are represented by central profiles (instead of limit profiles) and that FlowSort is not limited to the PROMETHEE method [37] (even if the conditions imposed on the preference structure are close to it). Finally, it is worth noting that a number of theoretical properties have been analyzed to characterize the assignment rules. We refer the interested reader to [36] for a detailed analysis.

6.11 The PROMETHEE GDSS Procedure

The PROMETHEE Group Decision Support System has been developed to provide decision aid to a group of decision-makers (DM_1), (DM_2), ..., (DM_r), ... (DM_R) (see [29]). It has been designed to be used in a GDSS room including a PC, a printer and a video projector for the facilitator, and R working stations for the DM's. Each working station includes room for a DM (and possibly a collaborator), a PC and Tel/Fax so that the DM's can possibly consult their business base. All the PC's are connected to the facilitator through a local network.

There is no objection to use the procedure in the framework of teleconference or video conference systems. It this case the DM's are not gathering in a GDSS room, they directly talk together through the computer network.

One iteration of the PROMETHEE GDSS procedure consists in 11 steps grouped in three phases:

- Phase I: Generation of alternatives and criteria
- Phase II: Individual evaluation by each *DM*
- Phase III: Global evaluation by the group

Feedback is possible after each iteration for conflict resolution until a final consensus is reached.

6.11.1 Phase I: Generation of Alternatives and Criteria

Step 1: First contact Facilitator—DM's The facilitator meets the DM's together or individually in order to enrich his knowledge of the problem. Usually this step takes place in the business base of each DM prior to the GDSS room session.

Step 2: Problem description in the GDSS room The facilitator describes the computer infrastructure, the PROMETHEE methodology, and introduces the problem.

Step 3: Generation of alternatives It is a computer step. Each DM implements possible alternatives including their extended description. For instance strategies, investments, locations, production schemes, marketing actions, ...depending on the problem.

Step 4: Stable set of alternatives All the proposed alternatives are collected and displayed by the facilitator one by one on the video-screen, anonymously or not. An open discussion takes place, alternatives are canceled, new ones are proposed, combined ones are merged, until a stable set of n alternatives $(a_1, a_2, \ldots, a_i, \ldots, a_n)$ is reached. This brainstorming procedure is extremely useful, it often generates alternatives that were unforeseen at the beginning.

Step 5: Comments on the alternatives It is again a computer step. Each DM implements his comments on all the alternatives. All these comments are collected and displayed by the facilitator. Nothing gets lost. Complete minutes can be printed at any time.

Step 6: Stable set of evaluation criteria The same procedure as for the alternatives is applied to define a stable set of evaluation criteria $(g_1(\cdot), g_2(\cdot), \ldots, g_j(\cdot), \ldots g_k(\cdot))$. Computer and open discussion activities are alternating. At the end the frame of an evaluation table (Type Table 6.1) is obtained. This frame consists in a $(n \times k)$ matrix. This ends the first phase. Feedbacks are already possible to be sure a stable set of alternatives and criteria is reached.

6.11.2 Phase II: Individual Evaluation by Each DM

Let us suppose that each DM has a decision power given by a non-negative weight $(\omega_r, r = 1, 2, \ldots, R)$ so that:

$$\sum_{r=1}^{R} \omega_r = 1. \tag{6.42}$$

Step 7: Individual evaluation tables The evaluation table $(n \times k)$ has to be completed by each DM. Some evaluation values are introduced in advance by the facilitator if there is an objective agreement on them (prices, volumes, budgets, …). If not each DM is allowed to introduce his own values. All the DM's implement the same $(n \times k)$ matrix, if some of them are not interested in particular criteria, they can simply allocate a zero weight to these criteria.

Step 8: Additional PROMETHEE information Each DM develops his own PROMETHEE-GAIA analysis. Assistance is given by the facilitator to provide the PROMETHEE additional information on the weights and the generalized criteria.

Step 9: Individual PROMETHEE-GAIA analysis The PROMETHEE I and II rankings, the profiles of the alternatives and the GAIA plane as well as the net flow vector $\phi_r(\cdot)$ are instantaneously obtained, so that each DM gets his own clear view of the problem.

6.11.3 Phase III: Global Evaluation by the Group

Step 10: Display of the individual investigations The rankings and the GAIA plane of each DM are collected and displayed by the facilitator so that the group of all DM'S is informed of the potential conflicts.

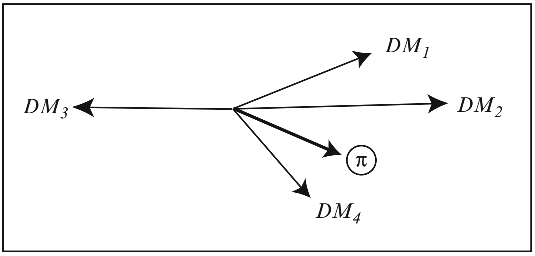


Fig. 6.11 Conflict between DM's

Step 11: Global evaluation The net flow vectors $\{\phi_r(\cdot), r = 1, \ldots, R\}$ of all the DM's are collected by the facilitator and put in a $(n \times R)$ matrix. It is a rather small matrix which is easy to analyzed. Each criterion of this matrix expresses the point of view of a particular DM.

Each of these criteria has a weight ω_r and an associated generalized criterion of Type 3 ($p = 2$) so that the preferences allocated to the deviations between the $\phi_i^r(\cdot)$ values will be proportional to these deviations.

A global PROMETHEE II ranking and the associated GAIA plane are then computed. As each criterion is representing a DM, the conflicts between them are clearly visualized in the GAIA plane. See for example Fig. 6.11 where DM_3 is strongly in conflict with DM_1, DM_2 and DM_4. The associated PROMETHEE decision axis (π) gives the direction in which to decide according to the weights allocated to the DM's. The alternatives (not represented on Fig. 6.11) to be considered are those in the direction of π.

If the conflicts are too sensitive the following feedbacks could be considered: Back to the weighting of the DM's. Back to the individual evaluations. Back to the set of criteria. Back to the set of alternatives. Back to the starting phase and to include an additional stakeholder ("DM") such as a social negotiator or a government mediator.

The whole procedure is summarized in the following scheme (Fig. 6.12):

6.12 The D-Sight Software

D-Sight [23] is the third generation of PROMETHEE based software; it has followed DECISION LAB 2000 and PROMCALC [12]. This application has been developed by Quantin Hayez at the CoDE-SMG laboratory. His work has been funded by the Walloon region under a First Spin-Off project supervised by Yves De Smet. Bertrand Mareschal initially acted as a scientific adviser. The software is available since February 2010 and despite the fact that it is quite new, many universities worldwide have already started to use it for educational and research purposes (http://www.d-sight.com/academic) [30]. Moreover, recent industrial projects testify its successful application in the fields of tenders evaluation, socio-economic assessment, infrastructure deployment . . . (http://www.d-sight.com/case-studies).

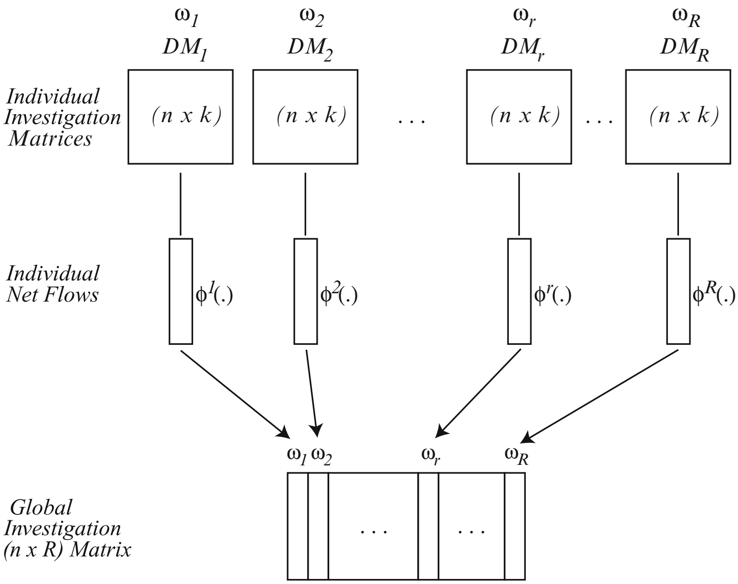


Fig. 6.12 Overview PROMETHEE GDSS procedure

D-Sight presents the same main functionalities as the preceding software (see Fig. 6.13). It is based on visual interactive tools that help the decision makers to better manage, understand and master their problems. The accustomed users of the PROMETHEE and GAIA methods will rediscover traditional tools such as an interactive GAIA plane, the PROMETHEE I and II rankings, the walking weights or weight stability intervals tools, in a new interface based on a flexible tabs system.

Additionally, D-Sight offers new features such as:

- the possibility to group criteria into a multiple layers hierarchy;
- an improved representation of the GAIA plane based on the explicit projections of the alternatives against the criteria or against the decision stick;
- a new representation of the PROMETHEE I ranking called the PROMETHEE Diamond (see Fig. 6.14);
- the PROMETHEE VI sensitivity tool (also called the “decision maker’s brain”) which was initially available in PROMCALC but not in Decision Lab 2000;
- the possibility to dynamically represent unicriterion net flow scores in a graph and, as a consequence, to better assess the impact of intra-criterion parameters;
- …

The software can easily be interfaced with other systems or databases and supports direct copy-paste with traditional applications. An automatic update procedure allows the users to always work with the latest release of the software. Finally, D-Sight offers a plugin system allowing the user to add features on the fly. These plugins are developed independently from the core system. They are available to

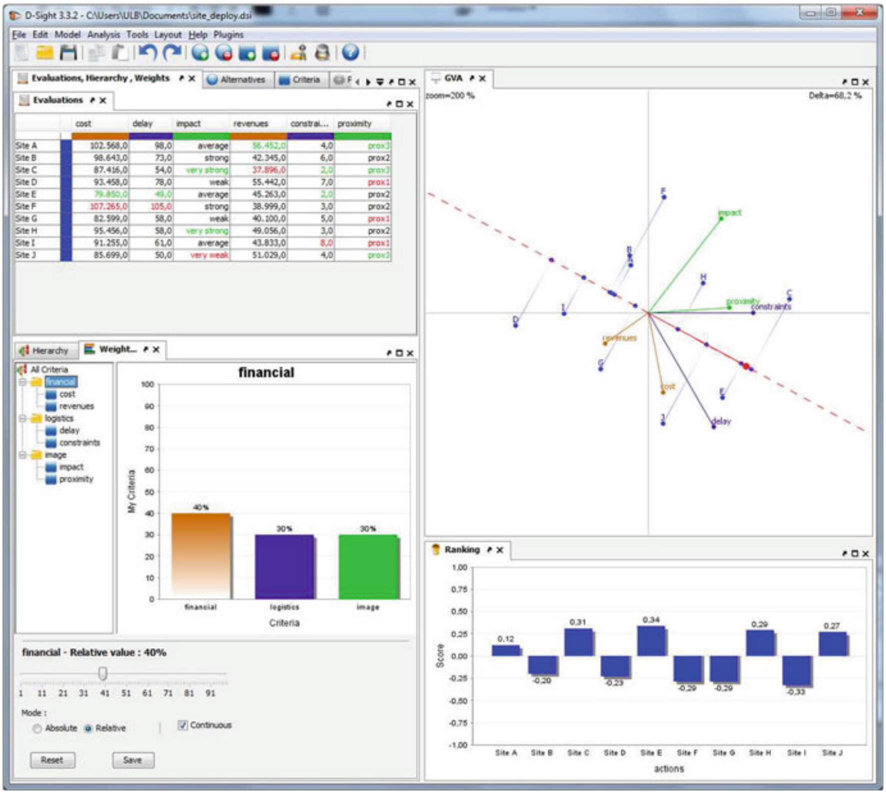


Fig. 6.13 Main functionalities of D-Sight

the user through an online plugin store accessible from D-Sight. With a single click, they are fully integrated in the software. Both D-Sight and the plugins are developed in Java. Some of the current available plugins are:

- a weights elicitation component based on an interactive tool;
- a module to geo-localize the alternatives in a complete interactive maps system directly connected to the mcda results (see Fig. 6.14);
- an optimization tool based on the PROMETHEE V procedure;
- a multi-actors plugin allowing decentralized decision making, while taking into account different stakeholders or scenarios;

Additional information about D-Sight can also be obtained on the website of the CoDE-SMG spin-off: http://www.d-sight.com.

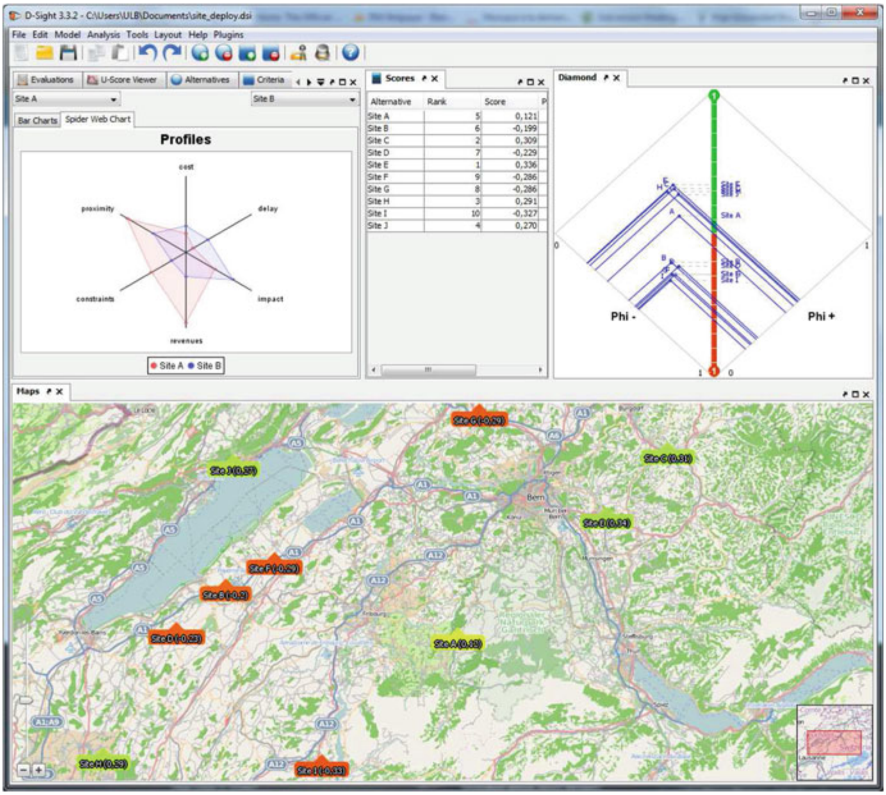


Fig. 6.14 D-Sight: geo-localization of the alternatives, PROMETHEE I diamond, comparisons of profiles

Acknowledgements The authors wish to address their most sincere acknowledgements to their colleague and friend Professor Johan Springael for his valuable assistance and comments.

References

1. Albadvi, A., Chaharsooghi, S.K., Esfahanipour, A.: Decision making in stock trading: an application of PROMETHEE. Eur. J. Oper. Res. **117**(2), 673–683 (2007)
2. Araz, C., Ozkarahan, I.: Supplier evaluation and management system for strategic sourcing based on a new multicriteria sorting procedure. Int. J. Prod. Econ. **106**(2), 585–606 (2007)
3. Arrow, K.J.: A difficulty in the concept of social welfare. J. Polit. Econ. **58**(4), 328–346 (1950)
4. Behzadian, M., Kazemzadeh, R.K., Albadvi, A., Aghdasi, M.: PROMETHEE: a comprehensive literature review on methodologies and applications Eur. J. Oper. Res. **100**(1), 198–215 (2010)
5. Belton, V., Gear, T.: On a short-comming of Saaty's method of analytic hierarchies. Omega **11**(3), 228–230 (1983)

6. Bouyssou, D., Perny, P.: Ranking methods for valued preference relations – a characterization of a method based on leaving and entering flows. Eur. J. Oper. Res. **61**(1–2), 186–194 (1992)
7. Brans, J.P.: L'ingénierie de la décision; Elaboration d'instruments d'aide à la décision. La méthode PROMETHEE. In: Nadeau, R., Landry, M. (eds.) L'aide à la décision: Nature, Instruments et Perspectives d'Avenir, pp. 183–213. Presses de l'Université Laval, Québec (1982)
8. Brans, J.P.: The space of freedom of the decision maker modelling the human brain. Eur. J. Oper. Res. **92**(3), 593–602 (1996)
9. Brans, J.P.: Ethics and decision. Eur. J. Oper. Res. **136**(2), 340–352 (2002)
10. Brans, J.P., Mareschal, B.: BANK ADVISER: Un système interactif multicritère pour l'évaluation financière des entreprises à l'aide des méthodes PROMETHEE. Actualité Economique **68**(4), 191–205 (1992)
11. Brans, J.P., Mareschal, B.: Promethee-V – MCDM problems with segmentation constraints. INFOR **30**(2), 85–96 (1992)
12. Brans, J.P., Mareschal, B.: The Promcalc and Gaia decision-support system for multicriteria decision aid. Decis. Support Syst. **12**(4–5), 297–310 (1994)
13. Brans, J.P., Mareschal, B.: The PROMETHEE VI procedure. How to differentiate hard from soft multicriteria problems. J. Decis. Syst. **4**, 213–223 (1995)
14. Brans, J.P., Mareschal, B.: PROMETHEE-GAIA. Une Méthodologie d'Aide à la Décision en Présence de Critères Multiples. Ellipses, Paris (2002)
15. Brans, J.P., Mareschal, B.: PROMETHEE methods. In: Figueira, J., Greco, S., Ehrgott, M. (eds.) Multiple Criteria Decision Analysis: State of the Art Surveys, pp. 163–196. Springer, Boston (2005)
16. Brans, J.P., Mareschal, B., Vincke, Ph.: PROMETHEE: a new family of outranking methods in multicriteria analysis. In: Brans, J.P. (ed.) Operational Research '84, pp. 477–490. North-Holland, Amsterdam (1984)
17. Brans, J.P., Macharis, C., Kunsch, P.L., Schwaninger, M.: Combining multicriteria decision aid and system dynamics for the control of socio-economic processes. an iterative real-time procedure. Eur. J. Oper. Res. **109**(2), 428–441 (1998)
18. Carmody, O., Kristóf, J., Frost, R.L., Makó, É., Kloprogge, J.T., Kokot, S.: A spectroscopic study of mechanochemically activated kaolinite with the aid of chemometrics. J. Colloid Interface Sci. **287**(1), 43–56 (2005)
19. De Keyser, W., Peeters, P.: A note on the use of PROMETHEE multicriteria methods. Eur. J. Oper. Res. **89**(3), 457–461 (1996)
20. Dhouib, D., Elloumi, S.: A new multi-criteria approach dealing with dependent and heterogeneous criteria for end-of-life product strategy. Appl. Math. Comput. **218**, 1668–1681 (2011)
21. Figueira, J.R., De Smet, Y., Mareschal, B., Brans, J.-P.: MCDA methods for sorting and clustering problems: PROMETHEE TRI and PROMETHEE CLUSTER. Technical Report IS-MG 2004/02, Université Libre de Bruxelles/SMG (2004)
22. Ghafghazi, S., Sowlati, T., Sokhansanj, S., Melin, S.: A multicriteria approach to evaluate district heating system options. Appl. Energy **87**(4), 1134–1140 (2010)
23. Hayez, Q., De Smet, Y., Bonney, J.: D-Sight: a new decision making software to address multi-criteria problems. Int. J. Decis. Support Syst. Technol. **4**(4), 1–23 (2012)
24. Hermans, C., Erickson, J., Noordewier, T., Sheldon, A., Kline, M.: Collaborative environmental planning in river management: an application of multicriteria decision analysis in the White River Watershed in Vermont. J. Environ. Manag. **84**(4), 534–546 (2007)
25. Hu, Y.C., Chen, C.J.: A PROMETHEE-based classification method using concordance and discordance relations and its application to bankruptcy prediction. Inf. Sci. **181**, 4959–4968 (2011)
26. Ilic, I., Bogdanovic, D., Zivkovic, D., Milosevic, N., Todorovic, B.: Optimization of heavy metals total emission, case study: Bor (Serbia). Atmos. Res. **101**(1–2), 450–459 (2011)
27. Ishizaka, A., Nemery de Bellevaux, P.: Selecting the best statistical distribution with PROMETHEE and GAIA. Comput. Ind. Eng. **61**, 958–969 (2011)

28. Li, W., Li, B.: An extension of the Promethee II method based on generalized fuzzy numbers. Expert Syst. Appl. **37**(7), 5314–5319 (2010)
29. Macharis, C., Brans, J.P., Mareschal, B.: The GDSS PROMETHEE procedure – a PROMETHEE-GAIA based procedure for group decision support. J. Decis. Syst. **7**, 283–307 (1998)
30. Macharis, C., Bernardini, A., De Smet, Y., Hayez, Q.: PROMETHEE in a multi actors setting: the use of the Multi actor Multi Criteria Analysis (MAMCA) methodology with D-SIGHT. In: Proceedings of the OR53 Conference, London (2010)
31. Madlener, R., Kowalski, K., Stagl, S.: New ways for the integrated appraisal of national energy scenarios: the case of renewable energy use in Austria. Energy Policy **35**(12), 6060–6074 (2007)
32. Marasovic, B., Babic, Z.: Two-step multi-criteria model for selecting optimal portfolio. Int. J. Prod. Econ. **134**, 58–66 (2011)
33. Mareschal, B., Brans, J.P.: Geometrical representations for MCDA. The GAIA module. Eur. J. Oper. Res. **34**, 69–77 (1988)
34. Mareschal, B., Brans, J.P.: BANK ADVISER. An industrial evaluation system. Eur. J. Oper. Res. **54**, 318–324 (1991)
35. Mareschal, B., De Smet, Y., Nemery de Bellevaux, P.: Rank reversal in the PROMETHEE II method: some new results. In: Proceedings of the IEEE 2008 International Conference on Industrial Engineering and Engineering Management, Singapore, pp. 959–963 (2008)
36. Nemery de Bellevaux, P.: On the use of multicriteria ranking methods in sorting problems Ph.D. thesis supervised by Prof. P. Vincke, ULB (2008)
37. Nemery de Bellevaux, P., Lamboray, C.: FlowSort: a flow-based sorting method with limiting or central profiles. TOP **16**, 90–113 (2008)
38. Ni, Y., Mei, M., Kokot, S.: Analysis of complex, processed substances with the use of NIR spectroscopy and chemometrics: classification and prediction of properties – the potato crisps example. Chemom. Intell. Lab. Syst. **105**(2), 147–156 (2011)
39. Nikolic, D., Jovanovic, I., Mihajlovic, I., Zivkovic, Z.: Multi-criteria ranking of copper concentrates according to their quality - an element of environmental management in the vicinity of copper - smelting complex in Bor, Serbia. J. Environ. Manag. **91**(2), 509–515 (2009)
40. Peng, Y., Kou, G., Wang, G., Shi, Y.: FAMCDM: a fusion approach of MCDM methods to rank multiclass classification algorithms. Omega **39**(6), 677–689 (2011)
41. Purcell, D.E., Leonard, G.J., O'Shea, M.G., Kokot, S.: A chemometrics investigation of sugarcane plant properties based on the molecular composition of epicuticular wax. Chemom. Intell. Lab. Syst. **76**(2), 135–147 (2005)
42. Roy, B.: Méthodologie d'Aide à la Décision. Economica, Paris (1985)
43. Roy, B., Bouyssou, D.: Aide Multicritère à la Décision. Méthodes et Cas. Economica, Paris (1993)
44. Vego, G., Kucar-Dragicevic, S., Koprivanac, N.: Application of multi-criteria decision-making on strategic municipal solid waste management in Dalmatia, Croatia. Waste Manag. **28**(11), 2192–2201 (2008)
45. Verly, C., De Smet, Y.: Some considerations about rank reversal occurrences in the PROMETHEE methods. Int. J. Multicrit. Decis. Making **3**(4), 325–345 (2013)
46. Vetschera, R., Teixeira de Almeida, A.: A PROMETHEE-based approach to portfolio selection problems. Comput. Oper. Res. **39**, 1010–1020 (2011)
47. Vincke, P.: Multicriteria Decision Aid. Wiley, New York (1989)
48. Wang, X., Triantaphyllou, E.: Ranking irregularities when evaluating alternatives by using some ELECTRE methods. Omega **36**, 45–63 (2008)
49. Yan, J., Dagang, T., Yue, P.: Ranking environmental projects model based on multicriteria decision-making and the weight sensitivity analysis. J. Syst. Eng. Electron. **18**(3), 534–539 (2007)
50. Zhang, G., Ni, Y., Churchill, J., Kokot, S.: Authentication of vegetable oils on the basis of their physico-chemical properties with the aid of chemometrics. Talanta **70**(2), 293–300 (2006)

Chapter 7
Other Outranking Approaches

Jean-M. Martel and Benedetto Matarazzo

Abstract In this chapter, we shortly describe some outranking methods other than ELECTRE and PROMETHEE. All these methods (QUALIFLEX, REGIME, ORESTE, ARGUS, EVAMIX, TACTIC and MELCHIOR) propose definitions and computations of particular binary relations, more or less linked to the basic idea of the original ELECTRE methods. Beside them, we will also describe other outranking methods (MAPPAC, PRAGMA, IDRA and PACMAN) that have been developed in the framework of the Pairwise Criterion Comparison Approach (PCCA) methodology, whose peculiar feature is to split the binary relations construction phase in two steps: in the first one, each pair of actions is compared with respect to two criteria a time; in the second step, all these partial preference indices are aggregated in order to obtain the final binary relations. Finally, one outranking method for stochastic data (the Martel and Zaras' method) is presented, based on the use of stochastic dominance relations between each pair of alternatives.

Keywords Multiple criteria decision analysis • Outranking methods • Pairwise criteria comparison approach

7.1 Introduction

The outranking methods constitute one of the most fruitful approach in MCDA. They main feature is to compare all feasible alternatives or actions by pair building up some binary relations, crisp or fuzzy, and then to exploit in an appropriate way these relations in order to obtain final recommendations. In this approach, the ELECTRE family and PROMETHEE methods (see Chaps. 5 and 6 in this book) are very well known; some interesting extensions of them have been recently

J.-M. Martel (✉)
Facultés des Sciences de l'Administration, University of Laval, Québec, QC, Canada
e-mail: jean-marc.martel@fsa.ulaval.ca

B. Matarazzo
Department of Economics and Business, University of Catania, Corso Italia, 55, Catania, Italy
e-mail: matarazz@unict.it

S. Greco et al. (eds.), *Multiple Criteria Decision Analysis*, International Series in Operations Research & Management Science 233,
DOI 10.1007/978-1-4939-3094-4_7

proposed, and they have been applied in a lot of real life problems. But beside them, there are also other outranking methods, interesting both from theoretical and operational points of view. All these methods propose definitions and computations of particular binary relations, more or less linked to the basic idea of the original ELECTRE methods, i.e. taking explicitly into account the reasons in favor and against an outranking relation (concordance-discordance analysis using appropriate veto thresholds). Some of these methods, moreover, present also a peculiar way to build up final recommendations, by exploiting the relations obtained in the previous step. In this chapter, we shortly describe some outranking methods other than ELECTRE and PROMETHEE. In Sect. 7.2, we present some outranking methods dealing with different kind of data (QUALIFLEX, REGIME, ORESTE, ARGUS, EVAMIX, TACTIC and MELCHIOR). Some of these methods are based on concordance-discordance analysis between the rankings of alternatives according to the considered criteria and the comprehensive ranking of them; others on direct comparison of each pair of alternatives, more or less strictly linked to the concordance-discordance analysis of ELECTRE type methods. In Sect. 7.4, some outranking methods (MAPPAC, PRAGMA, IDRA and PACMAN) are described. They have been developed in the framework of the Pairwise Criterion Comparison Approach (PCCA) methodology. Its peculiar feature is to split the binary relations construction phase in two steps: in the first one, each pair of actions is compared with respect to two criteria a time, among those considered in the problem, and partial preference indices are built up. In the second step, all these partial preference indices are aggregated in order to obtain the global indices and binary relations. An appropriated exploitation of these indices gives us the final recommendations. Finally, in Sect. 7.5 one outranking method for stochastic data (the Martel and Zaras' method) is presented. The main feature of this method is that the concordance-discordance analysis is based on the use of stochastic dominance relations on the set of feasible alternatives, comparing their cumulative distribution functions associated with each criterion. Some short conclusions are sketched in final Section.

7.2 Other Outranking Methods

The available information, about decision maker's (DM's) preferences, is not always of cardinal level; some times the evaluations of alternatives are ordinal scales, especially in social sciences. These evaluations may take the form of preorders. Several methods were been developed to aggregate this type of local (marginal) evaluation in order to obtain a comprehensive comparison of alternatives. For example, we can mention Borda, Condorcet, Copeland, Blin, Bowmam and Colantoni, Kemeny and Snell, etc. (see [36]). Some methods that we will recall in this Section drawn inspiration by some of them.

We present some outranking methods consistent with ordinal data, since they do not need to convert ordinal information to cardinal values, as it is the case, for example, in [20]. We will present some methods frequently mentioned in the literature on MCDA, where the general idea of outranking is globally implemented:

QUALIFLEX, REGIME, ORESTE, ARGUS, EVAMIX, TACTIC and MELCHIOR. These methods are not too complex and do not introduce the mathematical programming within their algorithm as it is the case, for example, in [7]. We present also EVAMIX even if it has been developed for ordinal and cardinal evaluations.

7.2.1 QUALIFLEX

The starting point of QUALIFLEX [33, 34] was a generalization of Jacquet-Lagrèze's permutation method [13] (see also [1] for a recent paper on this method).

It is a metric procedure and it is based on the evaluation of all possible rankings (permutations) of alternatives under consideration. Its mechanism of aggregation is based on Kemeny and Snell's rule.

This method is based on the comparison among the comprehensive ranking of the alternatives and the evaluations of alternatives according to each criterion from considered family F (impact matrix). These evaluations are ordinal and take the form of complete preorders. For each permutation, one computes a concordance/discordance index for each couple of alternatives, that reflects the concordance and the discordance of their ranks and their evaluation preorders from the impact matrix. This index is firstly computed at the level of single criterion, after at a comprehensive level with respect to all possible rankings. One tries to identify the permutation that maximizes the value of this index, i.e. the permutation whose ranking best reflects (the best compromise between) the preorders according to each criterion from F and the multi-criteria evaluation table.

The information concerning the coefficients of relative importance (weights) of criteria may be explicitly known (cardinal evaluations) or expressed as a ranking (for example a preorder). In this case, [33] has show that one can circumscribe the exploration to extreme points (the vertices) of polyhedron formed by the feasible weights.

Given the set of alternatives A, the concordance/discordance index for each couple of alternatives $(a, b), a, b \in A$, at the level of preorder according to the criterion $g_j \in F$ and the ranking corresponding to the kth permutation is:

$$I_{jk}(a,b) = \begin{cases} 1 & \text{if there is concordance} \\ 0 & \text{if there is ex aequo} \\ -1 & \text{if there is discordance.} \end{cases}$$

There is concordance (discordance) if a and b are ranked (not ranked) in the same order within the two preorders, and *ex aequo* if they have the same rank. The concordance/discordance index between the preorder according to the criterion g_j and the ranking corresponding to the kth permutation is:

$$I_{jk} = \sum_{a,b \in A} I_{jk}(a,b).$$

The comprehensive concordance/discordance index for the kth permutation is:

$$I_k = \sum_j \pi_j I_{jk}(a,b),$$

where π_j is the weight of criterion $g_j, j = 1, 2, \ldots, n$. The number of permutations k (Per_k) is $m!$, where $m = |A|$. The best compromise corresponds to the permutation that maximize I_k. If π_j are not explicitly known, but expressed by a ranking, then the best compromise is the permutation that:

$$\max_{P(\pi_j)} I_k,$$

where $P(\pi_j)$ is the set of feasible weights

Example 1. Given 3 alternatives $a_1, a_2, a_3 \in$ A; 3 criteria g_1, g_2, g_3 and the evaluation table (see Table 7.1, where a rank number 1 indicates the best outcome, while a rank 3 is assigned to the worst outcome with respect to each criterion), there are 3! possible permutations (comprehensive rankings):

$$\begin{aligned}
&Per_1: a_1 > a_2 > a_3\\
&Per_2: a_2 > a_1 > a_3\\
&Per_3: a_2 > a_3 > a_1\\
&Per_4: a_3 > a_2 > a_1\\
&Per_5: a_3 > a_1 > a_2\\
&Per_6: a_1 > a_3 > a_2.
\end{aligned}$$

One index is computed for each pair (g_j, Per_k), that, for our example, gives a total of 18 concordance/discordance indices. For example for the pair (g_1, Per_1), we have for the criterion g_1: $a_1 > a_2$, $a_2 \approx a_3$, $a_1 > a_3$, and for the Per_1: $a_1 > a_2$, $a_1 > a_3$, $a_2 > a_3$, that gives $+1$ for the couple (a_1, a_2), $+1$ for the couple (a_1, a_3) and 0 for the couple (a_2, a_3). Thus, the value of the index I_{11} is equal to 2.

The concordance/discordance indices are given in the Table 7.2.

Concerning the weights, for example:

1. If the three criteria have the same importance, i.e. $\pi_j = \frac{1}{3}, j = 1, 2, 3$, then we obtain that the maximum value of the index is $\frac{4}{3}$ for the permutations Per_1 and Per_6.

Table 7.1 Rank evaluation of alternatives (impact matrix)

		Criterion		
		g_1	g_2	g_3
Alternative	a_1	1	2	1
	a_2	2	1	3
	a_3	2	3	2

Table 7.2 The concordance/ discordance indices

		Criterion		
		g_1	g_2	g_3
Permutation	Per_1	2	1	1
	Per_2	0	3	−1
	Per_3	−2	1	−3
	Per_4	−2	−1	−3
	Per_5	0	−3	1
	Per_6	2	−1	3

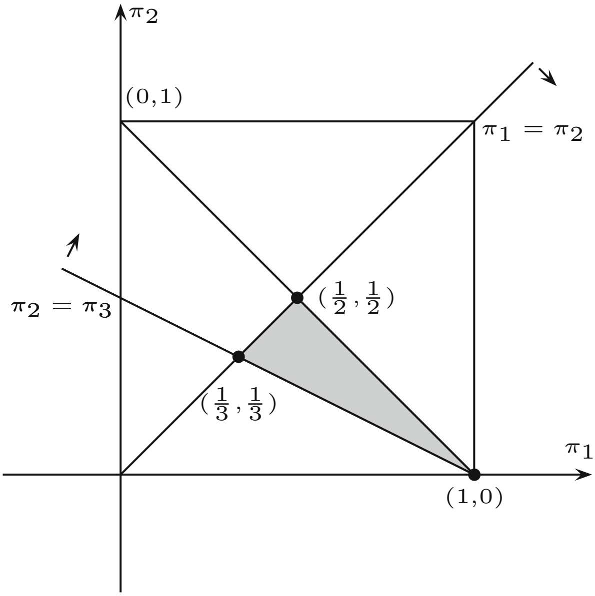


Fig. 7.1 Set of feasible weights

2. If we know that $\pi_1 \geq \pi_2$, $\pi_2 \geq \pi_3$ and $\pi_j \geq 0$ for all j, $\sum_j \pi_j = 1$, then $\pi_3 = 1 - \pi_1 - \pi_2$ (see Fig. 7.1).

Then, to obtain the permutation that maximizes the index I_k, we must check for the three vertices $(1, 0)$, $(\frac{1}{2}, \frac{1}{2})$ and $(\frac{1}{3}, \frac{1}{3})$ in the plane (π_1, π_2). The maximum value of the index is equal to 2 for the permutations Per_1 and Per_6, for the weights (1,0,0).

The result of this method is a ranking of alternatives under consideration. QUALIFLEX is based on comparison between the possible comprehensive rankings (permutations) of alternatives and each preorder corresponding to their criteria evaluations, but no outranking relation is constructed. An important limitation of this method concerns the fact that the number of permutations increases tremendously with the number of alternatives. This problem may be solved. Ancot [2] formulated this problem as a particular case of Quadratic Assignment Problem; this algorithm is implemented in the software MICROQUALIFLEX .

7.2.2 REGIME

The REGIME method [15, 16] can be viewed as an ordinal generalization of pairwise comparison methods such as concordance analysis. The starting point of this method is the concordance defined, roughly speaking, in the following way:

$$c(i,l) = \sum_{j \in \hat{C}_{il}} \pi_j,$$

where $\hat{C}_{il}$ is the concordance set, i.e. the set of criteria for which a_i is at least as good as a_l, a_i and $a_l \in A$, and π_j is the weight of criterion $g_j \in F$. The focus of this method is on the sign of $c(i,l) - c(l,i)$ for each pair of alternatives. If this sign is positive, alternative a_i is preferred to a_l; and the reverse if the sign is negative.

The first step of the REGIME method is the construction of the so-called regime matrix. The regime matrix is formed by pairwise comparison of alternatives in the multi-criteria evaluation table. Given a and $b \in A$, for every criterion we check whether a has a better rank than b, then on the corresponding place in the regime matrix the number $+1$ is noted, while if b is a better position than a, the number -1 is the result, and 0 in case of ex-aequo.

More explicitly, for each criterion $g_j, j = 1, 2, \cdots, n$, we can defined an indicator $c_{il,j}$ for each pair of alternatives (a_i, a_l).

$$c_{il,j} = \begin{cases} +1 & \text{if } r_{ij} < r_{lj} \\ 0 & \text{if } r_{ij} = r_{lj} \\ -1 & \text{if } r_{ij} > r_{lj}, \end{cases}$$

where r_{ij} (r_{lj}) is the rank of the alternative a_i (a_l) according to criterion g_j. When two alternatives are compared on all criteria, it is possible to form a vector

$$c_{il} = (c_{il,1}, \cdots, c_{il,j}, \cdots, c_{il,n})$$

that is called a regime and the regime matrix is formed of these regimes. These regimes will be used to determine rank order of alternatives.

The concordance index, in favor of the alternative a_i, is given by:

$$C_{il} = c(i,l) - c(l,i) = \sum_j \pi_j c_{il,j},$$

If the π_j are explicitly known, we can obtain a concordance matrix $\mathbf{C} = [C_{il}]$, with zero on the main diagonal (Table 7.3).

One half of this matrix can be ignored, since $C_{il} = -C_{li}$.

In general the available information concerning the weights is not explicit (not quantitative) and then it is not possible to compute the matrix $\mathbf{C}$. If the available

Table 7.3 Concordance matrix

$$
\begin{array}{c|ccccc|}
 & a_1 & \cdots\cdots & a_l & \cdots & a_m \\
\hline
a_1 & 0 & & \vdots & & \\
\vdots & & & \vdots & & \\
a_i & \cdots\cdots & \cdots\cdots & C_{il} & \cdots\cdots & \\
\vdots & & & \vdots & & \\
a_m & & & \vdots & & 0
\end{array}
$$

Table 7.4 Rank evaluation of alternatives (impact matrix)

		Criterion			
		g_1	g_2	g_3	g_4
Alternatives	a_1	3	1	1	2
	a_2	2	2	3	1
	a_3	1	3	2	3

Table 7.5 Regime matrix

		Criterion			
		g_1	g_2	g_3	g_4
Comparison	(a_1,a_2)	−1	+1	+1	−1
	(a_1,a_3)	−1	+1	+1	+1
	(a_2,a_1)	+1	−1	−1	+1
	(a_2,a_3)	−1	+1	−1	+1
	(a_3,a_1)	+1	−1	−1	−1
	(a_3,a_2)	+1	−1	+1	−1

information concerning the weights is ordinal, the sign of C_{il} may be determined with certainty only for some regimes [32]. For others regimes a such unambiguous result can not be obtained; such regime is called critical regime.

Example 2. We can illustrate this method on the basis of multi-criteria evaluation table with three alternatives and four criteria (Table 7.4, [16]).

For this example, the regime matrix is presented in Table 7.5.

If we make the hypothesis that $\pi_1 = \pi_2 = \pi_3 = \pi_4 = \frac{1}{4}$, we find $C_{12} = 0$, $C_{13} > 0$, $C_{21} = 0$, $C_{23} = 0$, $C_{31} < 0$ and $C_{32} = 0$. Thus a_1 is preferred to a_3, but we can not conclude between a_1 and a_2, a_2 and a_3. If we know for example that:

$$\pi_2 \geq \pi_4 \geq \pi_3 \geq \pi_1, \sum_j \pi_j = 1 \text{ and } \pi_j \geq 0,$$

then we find that $C_{12} = -\pi_1 + \pi_2 + \pi_3 - \pi_4 \geq 0$ in all cases, which means that, on the basis of a pairwise comparison, a_1 is preferred to a_2. In a similar way it can be shown that, given the same information on the weights, a_1 is preferred to a_3, and that a_2 is preferred to a_3. Thus we arrive at a transitive rank order of alternatives.

It is not possible to arrive at such definitive conclusions for all rankings of the weights. If we assume that:

$$\pi_1 \geq \pi_2 \geq \pi_3 \geq \pi_4, \sum_j \pi_j = 1 \text{ and } \pi_j \geq 0,$$

it is easy to see that from the first regime may result both positive and negative values of some C_{il}. For example if $\boldsymbol{\pi}$ = (0.40, 0.30, 0.25, 0.05), $C_{12} > 0$, whereas for $\boldsymbol{\pi}$ = (0.45, 0.30, 0.15, 0.10), $C_{12} < 0$. Therefore, the corresponding regime is called a critical regime. The main idea of regime analysis is to circumvent these difficulties by partitioning the set of feasible ordinal weights so that for each region a final conclusion can be drawn about the sign of C_{il}.

Let the ordinal information available about the weights be:

$$\pi_1 \geq \pi_2 \geq \pi_3 \geq \pi_4, \sum_j \pi_j = 1 \text{ and } \pi_j \geq 0.$$

The set of weights satisfying this information will be denoted as T. We have to check, for all regimes c_{il}, if c_{il} may assume both positive and negative values, given that π is an element of T. The total number of regimes to be examined is $2^n = 2^4 = 16$. For our example, the number of critical regimes is equal to the following four:

$$\begin{array}{cccc} -1 & +1 & +1 & -1 \\ +1 & -1 & -1 & +1 \\ -1 & +1 & +1 & +1 \\ +1 & -1 & -1 & -1 \end{array}$$

The number of critical regimes is even, since we known that if c_{il} is also a critical regime then $c_{li} = -c_{il}$ is also critical. The subsets of T can be characterized by means of the structure of the critical regimes. The four critical regimes of our example give two critical equations:

$$f_1(\boldsymbol{\pi}) = \pi_1 - \pi_2 - \pi_3 + \pi_4 = 0$$
$$f_2(\boldsymbol{\pi}) = \pi_1 - \pi_2 - \pi_3 - \pi_4 = 0.$$

The following subsets of T can be distinguished by means of these equations:

$$T_1 = T \cap \{\boldsymbol{\pi} : f_1(\boldsymbol{\pi}) > 0 \text{ and } f_2(\boldsymbol{\pi}) > 0\},$$
$$T_2 = T \cap \{\boldsymbol{\pi} : f_1(\boldsymbol{\pi}) > 0 \text{ and } f_2(\boldsymbol{\pi}) < 0\},$$
$$T_3 = T \cap \{\boldsymbol{\pi} : f_1(\boldsymbol{\pi}) < 0 \text{ and } f_2(\boldsymbol{\pi}) < 0\},$$
$$T_4 = T \cap \{\boldsymbol{\pi} : f_1(\boldsymbol{\pi}) < 0 \text{ and } f_2(\boldsymbol{\pi}) > 0\}.$$

An examination of $T_1, \ldots, T_4$ reveals that T_4 is empty, so that ultimately three relevant subsets remain. The subsets T_1, T_2 and T_3 are convex polyhedra, as it is the case for the set T. The extreme points of these polyhedra can be determined graphically in the case of four criteria. The extreme points for T are:

$$A\colon \quad (1,0,0,0)$$
$$B\colon \quad \left(\frac{1}{2},\frac{1}{2},0,0\right)$$
$$C\colon \quad \left(\frac{1}{3},\frac{1}{3},\frac{1}{3},0\right)$$
$$D\colon \quad \left(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\right)$$

In addition to these four points, the extreme points

$$E: \left(\frac{1}{2},\frac{1}{4},\frac{1}{4},0\right) \text{ and}$$
$$F: \left(\frac{1}{2},\frac{1}{6},\frac{1}{6},\frac{1}{6}\right)$$

are needed to characterize T_1, T_2 and T_3. The characterization of T_1, T_2 and T_3 by means of the extreme points are for T_1: A, B, E, F; for T_2: B, D, E, F and for T_3: B, C, D, E.

Once the partitioning of the weight set has been achieved, for each subset of T it is possible to indicate unambiguously the sign of C_{il} for each pair of alternatives. Let ν_{il} be defined as follows:

$$\nu_{il} = +1 \text{ if } C_{il} > 0,$$
$$\nu_{il} = -1 \text{ if } C_{il} < 0.$$

Then a pairwise comparison matrix $\mathbf{V}$ can be constructed consisting of elements equal to +1 or −1, and zeros on the main diagonal. A final ranking of alternatives can be achieved on the basis of $\mathbf{V}$.

For example, take an interior point of subset T_1 (e.g. the centroid computed as the mean of the extreme points). Determine the sign of C_{il} for all regimes occurring in the regime matrix (Table 7.5). Thus we find for the pairwise comparison matrix $\mathbf{V_1}$:

$$\mathbf{V_1} = \begin{bmatrix} 0 & -1 & -1 \\ +1 & 0 & -1 \\ +1 & +1 & 0 \end{bmatrix}$$

On the basis of $\mathbf{V_1}$ we may conclude that a_3 is preferred to a_2 which in turn is preferred to a_1. For the two other subsets of weights we find:

$$\mathbf{V_2} = \begin{bmatrix} 0 & -1 & +1 \\ +1 & 0 & -1 \\ -1 & +1 & 0 \end{bmatrix} \qquad \mathbf{V_3} = \begin{bmatrix} 0 & +1 & +1 \\ -1 & 0 & -1 \\ -1 & +1 & 0 \end{bmatrix}.$$

The second pairwise comparison matrix does not give a definitive ranking of alternatives, but on the basis of $\mathbf{V_3}$ we may conclude that a_1 is preferred to a_3 which is again preferred to a_2.

The relative size of subsets T_1, T_2 and T_3 are not equal. If we assume that the weights are uniformly distributed in T, the relative size of the subsets of T can be interpreted as the correspondent probability that alternative a_i is preferred to a_l. Probabilities are then aggregated to produce an overall ranking of alternatives. The relative sizes of the subsets can also be estimated using a random generator. This is recommended if there are seven criteria or more, since the number of subsets increases exponentially with the number of criteria [32].

The relevant subsets given an arbitrary number of criteria can be found in [16]. The REGIME method can be applied to mixed evaluations (ordinal and cardinal criteria) without losing the information contained in the quantitative evaluation. This requires a standardization of the quantitative evaluation. Israels and Keller [18] has been proposed a variant of REGIME method where the incomparability is accepted. The REGIME method is implemented in a system to support decision on a finite set of alternatives DEFINITE [19].

7.2.3 ORESTE

ORESTE (Organisation Rangement Et Synthése de donées relationelles) (see [37, 38]) has been developed to deal with the situation where the alternatives are ranked according to each criterion and the criteria themselves are ranked according to their importance. In fact, the ORESTE method can deal with the following multi-criteria problem. Let A be a finite set of alternatives a_i, $i = 1, 2, \cdots, m$. The consequences of the alternatives are analysed by a family F of n criteria. The relative importance of the criteria is given by a preference structure on the set of criteria F, which can be defined by a complete preorder S (the relation $S = I \cup P$ is strongly complete and transitive, the indifference I is symmetric and the preference P is asymmetric). For each criterion g_j, $j = 1, 2, \cdots, n$, we consider a preference structure on the set A, defined also by a complete preorder. The objective of the method is to find a global preference structure on A which reflects the evaluation of alternatives on each criterion and the preference structure among the criteria.

The ORESTE method operates in three distinct phases:

First phase. Projection of the position-matrix.
Second phase. Ranking the projections.
Third phase. Aggregation of the global ranks.

We start from n complete preorders of the alternatives from A related to the n criteria, (for each alternative is given a rank with respect to each criterion). Also for each criterion is given a rank related to its position in the complete preorder among the criteria. The mean rank discussed by Besson [5] is used. For example, if the following preorder is given for the criteria $g_1Pg_2Ig_3Pg_4$, then $r_1 = 1, r_2 = r_3 = 2.5$ and $r_4 = 4$, where r_j is the Besson-rank of criterion g_j; idem for the alternatives, $r_j(a)$ is the average (Besson) rank of alternative a with respect to the criterion g_j. Given $\{r_j(a), r_j\}$, ORESTE tries to build a preference structure $O = \{I, P, R\}$ on A such as:

- a_iPa_l if a_i is comprehensively preferred to a_l $(O_{il} = 1, O_{li} = 0)$,
- a_iIa_l if a_i is indifferent to a_l $(O_{il} = O_{li} = 1)$,
- a_iRa_l if a_i and a_l are comprehensively incomparable $(O_{il} = O_{li} = 0)$.

Projection Considering an arbitrary origin 0, a distance $d(0, a_j)$ is defined with the use of $\{r_j(a), r_j\}$ such that $d(0, a_j) < d(0, b_j)$ if aP_jb, where $a_j = g_j(a)$ is the evaluation of alternative a with respect to criterion g_j. When ties occur, an additional property is: if g_jIg_k and $r_j(a) = r_k(b)$, then $d(0, a_j) = d(0, b_k)$. For the author, the "city-block" distance is adequate:

$$d(0, a_j) = \alpha r_j(a) + (1 - \alpha) r_j,$$

where α stands for a suitable substitution rate $(0 < \alpha < 1)$. The projection may be performed in different ways [35, 38].

Example 3. Given 3 alternatives and 3 criteria (without ties), the complete preorders of alternatives are: aP_1bP_1c, bP_2cP_2a and cP_3aP_3b, and for the criteria: $g_1Pg_2Pg_3$. This example may be visualized by a position matrix (Table 7.6).

Being $r_1 = 1$, $r_2 = 2$, $r_3 = 3$, the city-block distance for this example is given in Table 7.7:

Table 7.6 Position-matrix

$$r_j(.) : \begin{matrix} \\ a \\ b \\ c \end{matrix} \begin{matrix} 1 & 2 & 3 \\ \end{matrix} \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix}$$

Table 7.7 City-block distance

$$d(0, a_j) : \begin{matrix} a \\ b \\ c \end{matrix} \overset{\begin{matrix} 1 & \quad 2 & \quad 3 \end{matrix}}{\begin{bmatrix} 1 & 1+\alpha & 1+2\alpha \\ 2+\alpha & 2-\alpha & 2 \\ 3-\alpha & 3 & 3-2\alpha \end{bmatrix}}$$

Ranking Since it is the relative position of projections that is important and not the exact value of $d(0, a_j)$, the projections will be ranked. To rank the projections, a mean rank $R(a_j)$ is assigned to a pair (a, g_j) such that $R(a_j) \leq R(b_k)$ if $d(0, a_j) \leq d(0, b_k)$. These ranks are called comprehensive ranks and are in the closed interval $[1, mn]$. For our example $R(a_1) < R(b_2)$ since $1 < 2 - \alpha$ $(0 < \alpha < 1)$.

Aggregation For each alternative one computes the summation of their comprehensive ranks over the set of criteria. For an alternative a this yields the final aggregation

$$R(a) = \sum_j R(a_j).$$

For our example, if $\frac{1}{3} < \alpha < \frac{1}{2}$ we obtain:

$$\begin{array}{ccccccccccccc} 1 & < & 1+\alpha & < & 2-\alpha & < & 1+2\alpha & < & 2 & < & 3-2\alpha & < & 2+\alpha \\ R(a_1) & < & R(b_1) & < & R(b_2) & < & R(c_1) & < & R(c_2) & < & R(c_3) & < & R(a_2) \\ 1 & & 2 & & 3 & & 4 & & 5 & & 6 & & 7 \\ & < & 3-\alpha & < & 3 & & & & & & & & \\ & < & R(a_3) & < & R(b_3), & & & & & & & & \\ & & 8 & & 9 & & & & & & & & \end{array}$$

and therefore:

$$R(a) = 16, R(b) = 14, R(c) = 15.$$

In the ORESTE method, the following index is also computed:

$$C(a, b) = \sum_{j:aP_jb} [R(b_j) - R(a_j)].$$

It is easily shown that $C(a, b) - C(b, a) = R(b) - R(a)$. Moreover, the maximum value of $R(b) - R(a)$ equals $n^2(m - 1)$.

For our example with $\frac{1}{3} < \alpha < \frac{1}{2}$, we obtain: $C(c, b) = 3$, $C(a, b) = 2$ and $C(a, c) = 3$. Thus, we may obtain the preference structure $O = \{I, P, R\}$ in such way that if $R(a) \leq R(b)$ then aIb or aPb or aRb, applying the following algorithm (see flow chart of Fig. 7.2) where β stands for an indifference level and γ for an incomparability level.

For our example with $\frac{1}{3} < \alpha < \frac{1}{2}$, we have $\frac{C(c,b)}{R(c)-R(b)} = 3$, $\frac{C(a,b)}{R(a)-R(b)} = 1$ and $\frac{C(a,c)}{R(a)-R(c)} = 3$. Thus, if $\beta = \frac{1}{18} = \frac{1}{n^2(m-1)}$ and $1 \leq \gamma \leq 3$, we obtain bPa, aRc and cRb.

These thresholds are interpreted in [35]. When $\gamma = \infty$, the outranking relation is a semi-order, which becomes a weak order if $\beta = 0$.

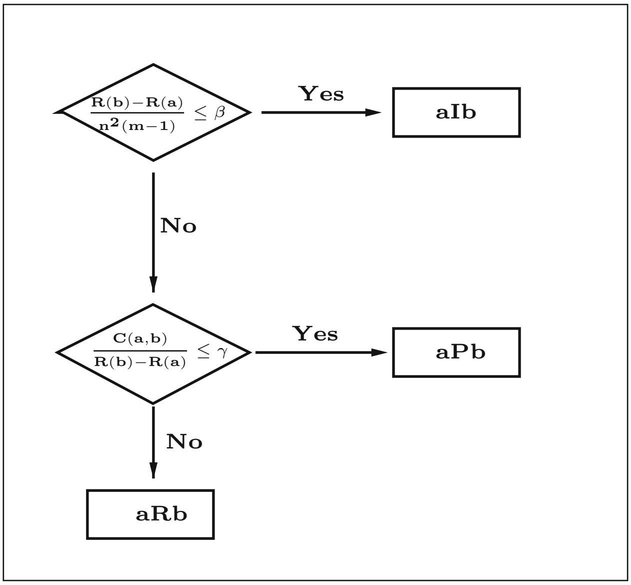


Fig. 7.2 ORESTE flow chart

The global preference relation P built by ORESTE is transitive [35]. The axiom known as the Pareto principle or citizen's sovereignty holds if $\beta < \frac{1}{n(m-1)}$, but the axiom of independence of irrelevant alternatives is generally violated [38].

7.2.4 ARGUS

The ARGUS method [8, 42] uses qualitative values for representing the intensity of preference on an ordinal scale. They express this intensity of preference between two alternatives $a, b \in A$ by selecting one of the following qualitative relations: indifference, small, moderate, strong or very strong preference. All evaluation on the criteria are treated as evaluations on an ordinal scale, but the evaluations of each alternative with respect to each criterion can be quantitative (interval or ratio scale) or qualitative (ordinal scale). We must indicate if the criterion must be MIN or MAX.

The way of obtaining the required information from the decision maker (DM) to model his/her preference structure, depends on the scale of measurement of each criterion. If the scale is *ordinal*, we may use the following possible values: very poor, poor, average, good, very good. To model the preference structure of the DM on this criterion, the DM must indicate his/her preference for each pair of values, constructing a preference matrix (Table 7.8).

Table 7.8 Preference matrix for a criterion with ordinal evaluation

$g_i(b)$		Very poor	Poor	Average	Good	Very good
$g_i(a)$	Very poor	Indiff.				
	Poor		Indiff.			
	Average			Indiff.		
	Good				Indiff.	
	Very good					Indiff.

Table 7.9 Preference matrix for a criterion (Max) with evaluation on a quantitative scale

$g_i(a) \geq g_i(b) > 0$	$d = g_i(a) - g_i(b)$	$\delta = \frac{g_i(a)-g_i(b)}{g_i(b)}$
Indifferent	$0 \leq d < d_1$	$0\,\% \leq \delta < \delta_1\,\%$
Small preference	$d_1 \leq d < d_2$	$\delta_1\,\% \leq \delta < \delta_2\,\%$
Moderate preference	$d_2 \leq d < d_3$	$\delta_2\,\% \leq \delta < \delta_3\,\%$
Strong preference	$d_3 \leq d < d_4$	$\delta_3\,\% \leq \delta < \delta_4\,\%$
Very strong preference	$d_4 \leq d$	$\delta_4\,\% \leq \delta$

In fact the DM must fill only the lower triangle of this matrix. The number of rows and columns of this matrix depends on the number of different values the ordinal criterion can have. The preference of the DM on an *interval scale* criterion will depend on $d = g_j(a) - g_j(b)$, while his/her preference on a *ratio scale* criterion will depend either on d only or on d, $g_j(a)$ and $g_j(b)$. For example, if his/her preference depends on d only, this means that only the absolute difference determines his/her preference. The preference structure of the DM for an interval scale criterion can be modeled by determining for which absolute difference d the DM is indifferent, for which d he/she has a moderate preference, for which d he/she has a strong and for which d he has a very strong preference. For a ratio scale criterion, he/she can also consider the relative difference δ (see Table 7.9).

The following ordinal scale may be used to reflect the importance of a criterion: not important, small, moderately, very and extremely important. The DM must indicate for each criterion, by selecting a value from this ordinal scale, how important the considered criterion is for him/her.

When the preference structure of the DM for each criterion is known as well as the importance of each criterion, the comparison of two alternatives a and b with respect to n criteria from F leads to a two-dimensional table (Table 7.10).

In a cell, f_{st} stands for the number of criteria of a certain importance for which a certain preference between the alternatives a and b occurs, $\sum_s \sum_t f_{st} = n$.

In order to get one overall appreciation of the comparison between the alternatives a and b, the DM must rank all cells of Table 7.10 where $g_j(a) > g_j(b)$. A ranking in eight classes is proposed to DM. Through this ranking one dimensional ordinal variable is created for each pair of alternatives. In fact there is a combined

Table 7.10 Preference importance table for g_j, a, b

	Criteria preference	Not imp.	Small imp.	Moderate imp.	Very imp.	Extremely imp.	w_j
$g_j(a) > g_j(b)$	Very strong	f_{11}	f_{12}	f_{13}	f_{14}	f_{15}	a
	Strong	f_{21}	f_{22}	f_{23}	f_{24}	f_{25}	b
	Moderate	f_{31}	f_{32}	f_{33}	f_{34}	f_{35}	$\vdots$
	Small	f_{41}	f_{42}	f_{43}	f_{44}	f_{45}	$\vdots$
$g_j(a) = g_j(b)$	No	f_{51}	f_{52}	f_{53}	f_{54}	f_{55}	$\vdots$
$g_j(a) < g_j(b)$	Small	f_{61}	f_{62}	f_{63}	f_{64}	f_{65}	$\vdots$
	Moderate	f_{71}	f_{72}	f_{73}	f_{74}	f_{75}	$\vdots$
	Strong	f_{81}	f_{82}	f_{83}	f_{84}	f_{85}	b
	Very strong	f_{91}	f_{92}	f_{93}	f_{94}	f_{95}	a

Table 7.11 Combined preferences with weights importance

	$g_j(a) > g_j(b)$	$g_j(a) < g_j(b)$
1	$u_1 = f_{15}$	$v_1 = f_{95}$
2	$u_2 = f_{14} + f_{25}$	$v_2 = f_{85} + f_{94}$
3	$u_3 = f_{13} + f_{24} + f_{45}$	$v_3 = f_{75} + f_{84} + f_{93}$
4	$u_4 = f_{12} + f_{23} + f_{34} + f_{45}$	$v_4 = f_{65} + f_{74} + f_{93} + f_{92}$
5	$u_5 = f_{11} + f_{22} + f_{33} + f_{44}$	$v_5 = f_{64} + f_{73} + f_{82} + f_{91}$
6	$u_6 = f_{21} + f_{32} + f_{43}$	$v_6 = f_{63} + f_{72} + f_{81}$
7	$u_7 = f_{31} + f_{42}$	$v_7 = f_{62} + f_{71}$
8	$u_8 = f_{41}$	$v_8 = f_{61}$

preference with respect to difference on evaluations and importance of weights where $g_j(a) > g_j(b)$ and where $g_j(a) < g_j(b)$ (see Table 7.11).

The decision maker can alter this ranking (by moving a cell from one class to another, by considering more or less classes) until it matches his/her personal conception. Based on those two variables, u_k and v_k, an outranking (S), indifference (I) or incomparability (R) relation between two alternatives is constructed:

$$\text{if } \sum_{k=1}^{h} u_k = \sum_{k=1}^{h} v_k \text{ for all } h = 1, \ldots, 8, \text{ then } aIb;$$

$$\text{if } \sum_{k=1}^{h} u_k \geq \sum_{k=1}^{h} v_k \text{ for all } h = 1, \ldots, 8, \text{ then } aSb;$$

$$\text{if } \sum_{k=1}^{h} u_k \leq \sum_{k=1}^{h} v_k \text{ for all } h = 1, \ldots, 8, \text{ then } bSa;$$

in all other cases aRb.

According to the basic idea of outranking, if alternative a is much better than alternative b on one (or more) criteria and b is much better than a on other criteria, there can be discordance between alternative b and alternative a, and b will not outrank a. The DM must explicitly indicate for each criterion when there is discordance between two evaluations on that particular criterion. For an ordinal criterion he/she can indicate in the upper triangle of the preference matrix (Table 7.8) when discordance occurs. For an interval or ratio criterion, the DM must indicate from which difference (absolute or relative), between the evaluations of two alternatives on that criterion, there is discordance.

Example 4. We have 4 alternatives, 4 criteria and the evaluation table (Table 7.12). In this example, the criteria g_1, g_2, g_3 are ordinal scales, and criterion g_4 is an interval scale to be minimized.

The following dominance relation can be observed from the data:

$a_4 D\, a_3$, so that after deleting a_3, the set of alternatives is $A = \{a_1, a_2, a_4\}$. It is necessary to make this pre-processing step.

The preference modeling of alternatives with respect to the criteria are given in Tables 7.13, 7.14, and 7.15.

Table 7.12 Evaluation of alternatives

	g_1	g_2	g_3	g_4
a_1	□	⊕	−	13
a_2	⊕	−	□	10
a_3	□	−	−	17
a_4	+	□	□	17

⊕ : verygood;
+ : good;
□ : acceptable;
− : moderate

Table 7.13 Criteria g_1 and g_3 (ordinal scales)

$g_2(b)$	⊖	−	□	+	⊕
$g_2(a)$ ⊖	Indifferent			Discordance	Discordance
−	Moderate	Indifferent			Discordance
□	Strong	Moderate	Indifferent		
+	Very strong	Strong	Moderate	Indifferent	
⊕	Very strong	Very strong	Strong	Small	Indifferent

Table 7.14 Criterion g_2 (ordinal scale)

$g_6(b)$	⊖	−	□	+	⊕
$g_6(a)$ ⊖	Indifferent				Discordance
−	Small	Indifferent			
□	Moderate	Small	Indifferent		
+	Strong	Moderate	Small	Indifferent	
⊕	Very strong	Strong	Moderate	Small	Indifferent

Table 7.15 Criterion g_4 (interval scale MIN)

Preference (a above b)	$d = g_j(a) > g_j(b)$
Indifferent	$0 \leq d < 1$
Small	$1 \leq d < 3.5$
Moderate	$3.5 \leq d < 6$
Strong	$6 \leq d < 9$
Very strong	$9 \leq d < \infty$
Discordance	$d < -\infty$

Table 7.16 Preference structure of weights

Weight	
Not important	
Small important	g_1, g_3
Moderately important	g_4
Very important	g_2
Extremely important	

Table 7.17 Pairwise comparison between a_1 and a_4

	$g_j(a_1) > g_j(a_4)$	$g_j(a_1) < g_j(a_4)$
1	0	0
2	0	0
3	0	0
4	1	0
5	1	0
6	0	2
7	0	0
8	0	0

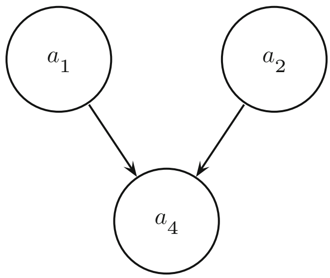


Fig. 7.3 Outranking graph

The preference structure of weights of the criteria is given in Table 7.16.

Suppose that the ranking in eight classes of the combined preference with weight of two alternatives presented in Table 7.11 is approved. Table 7.17 gives an example of a pairwise comparison between a_1 and a_4.

The pairwise comparison of all pair of alternatives from A permits to construct the following binary relations: $a_1 S a_4$, $a_1 R a_2$ and $a_2 S a_4$ (see Fig. 7.3).

The ARGUS method demands a relatively great effort from the DM to model his/her preferences.

7.2.5 EVAMIX

The EVAMIX (Evaluation Matrix) method [32, 45, 46] is a generalization of concordance analysis in the case of mixed information on the evaluation of alternatives on the judgment criteria. Thus a pairwise comparison is made for all pairs of alternatives to determine the so called concordance and discordance indices. The difference with standard concordance analysis is that separate indices are constructed for the qualitative and quantitative criteria. The comprehensive ranking of alternatives is the result of a combination of the concordance and discordance indices for the qualitative and quantitative criteria.

The set of criteria in the multi-criteria evaluation table is divided into a set of qualitative (ordinal) criteria O and a set of quantitative (cardinal) criteria C. It is assumed that the differences between alternatives can be expressed by means of two dominance measures: a dominance score $\alpha_{ii'}$ for the ordinal criteria, and a dominance score $a_{ii'}$ for the cardinal criteria. These scores represent the degree to which alternative a_i dominates alternative $a_{i'}$. They have the following structure:

$$\alpha_{ii'} = f(e_{ij}, e_{i'j}, \pi_j), \text{ for all } j \in O,$$
$$a_{ii'} = g(e_{ij}, e_{i'j}, \pi_j), \text{ for all } j \in C,$$

where e_{hj} represents the evaluation of alternative a_h on the criterion g_j and π_j the importance weight associated to this criterion ($\pi_j > 0$). These scores can be defined as follows:

$$\alpha_{ii'} = \left[\sum_{j \in O} \{\pi_j sgn(e_{ij} - e_{i'j})\}^c \right]^{\frac{1}{c}},$$

where

$$sgn(e_{ij} - e_{i'j}) = \begin{cases} +1 \text{ if } e_{ij} > e_{i'j} \\ 0 \text{ if } e_{ij} = e_{i'j} \\ -1 \text{ if } e_{ij} < e_{i'j}. \end{cases}$$

The symbol c denotes an arbitrary scaling parameter, for which any positive odd value may be chosen, $c = 1, 3, 5, \ldots$. In a similar manner, the quantitative dominance measure can be made explicit:

$$a_{ii'} = \left[\sum_{j \in C} \{\pi_j (e_{ij} - e_{i'j})\}^c \right]^{\frac{1}{c}}.$$

In order to be consistent, the same value for the scaling parameter c should be used as in formula for $\alpha_{ii'}$. It is assumed that the quantitative employed evaluation e_{ij} have been standardized ($0 \leq e_{ij} \leq 1$). Evidently, all standardized scores should have the same direction, i.e., a 'higher' score should (for instance) imply a 'larger' preference. It should be noted that the rankings e_{ij} ($j \in O$) of the qualitative criteria also have to represent 'the higher, the better'. Since $\alpha_{ii'}$ and $a_{ii'}$ will have different measurement units, a standardization into the same unit is necessary. The standardized dominance measures can be written as:

$$\delta_{ii'} = h(\alpha_{ii'}) \text{ and } d_{ii'} = h(a_{ii'}),$$

where h represents a standardization function.

Let us assume that weights π_j have quantitative properties. The overall dominance measure $D_{ii'}$ for each pair of alternatives $(a_i, a_{i'})$ is:

$$D_{ii'} = \pi_o \delta_{ii'} + \pi_c d_{ii'},$$

where $\pi_o = \sum_{j \in O} \pi_j$ and $\pi_c = \sum_{j \in C} \pi_j$. This overall dominance score reflects the degree to which alternative a_i dominates alternative $a_{i'}$ for the given set of criteria and the weights. The last step is to determine an appraisal score s_i for each alternative. In general the measure $D_{ii'}$ may be considered as function k of the constituent appraisal scores, or:

$$D_{ii'} = k(s_i, s_{i'}).$$

This expression represents a well-known pairwise comparison problem. Depending on the way function k is made explicit, the appraisal scores can be calculated. The most important assumptions behind the EVAMIX method concern the definition of the various functions. It is shown in [46], that at least three different techniques can be distinguished which are based on different definitions of $\delta_{ii'}$, $d_{ii'}$ and $D_{ii'}$. The most straightforward standardization is probably the additive interval technique. The overall dominance measure $D_{ii'}$ is defined as:

$$D_{ii'} = \frac{s_i}{s_i + s_{i'}},$$

which implies that $D_{ii'} + D_{i'i} = 1$. To arrive at such overall dominance measures with this additivity characteristic, the following standardization is used:

$$\delta_{ii'} = \frac{(\alpha_{ii'} - \alpha^-)}{(\alpha^+ - \alpha^-)} \text{ and } d_{ii'} = \frac{(a_{ii'} - a^-)}{(a^+ - a^-)}$$

where α^- (α^+) is the lowest (highest) qualitative dominance score of any pair of alternatives $(a_i, a_{i'})$ and a^- (a^+) is the lowest (highest) quantitative dominance score of any pair of alternatives $(a_i, a_{i'})$. The resulting appraisal score is:

$$s_i = \left[\sum_{i'} \frac{D_{i'i}}{D_{ii'}} \right]^{-1} .$$

This expression means that the appraisal scores add up to unity, i.e. $\sum_i s_i = 1$.

In the previous elaboration, quantitative weights π_j, $j = 1, 2, \ldots, n$, were assumed. In some circumstances, only qualitative priority expressions can be given. If only ordinal information is given, at least two different approaches may be followed: an expected value approach (see [32, Appendix 4.I]) or a random weight approach. The random weight approach implies that quantitative weights are created by a random selection out an area defined by the extreme weight sets. These random weights $\gamma_j, j = 1, \cdots, n$, have to fulfill the following conditions:

1. for each $\gamma_j, \gamma_{j'}, \omega_j \leq \omega_{j'} \Rightarrow \gamma_j \geq \gamma_{j'}$,
2. $\sum_j \gamma_j = 1$,

where ω_j denotes a ranking number expressing a qualitative weight with "lower" means "better". For each set of metric weights $\gamma_j, j = 1, \cdots, n$, generated during one run of the random number generator, a set of appraisal scores can be determined. By repeating this procedure many times, a frequency matrix can be constructed. Its element f_{ri} represents the number of times, alternative a_i was placed in the r-th position in the final ranking. A probability matrix with element p_{ri} can be constructed, where:

$$p_{ri} = \frac{f_{ri}}{\sum_i f_{ri}}.$$

So, p_{ri} represents the probability that a_i will receive an r-th position. We can make a comprehensive ranking of the alternatives in the following way:

$$\begin{aligned} a_i &= 1, \text{if } p_{1i} \text{ is maximal,} \\ a_{i'} &= 2, \text{if } p_{1i} + p_{2i'} \text{ is maximal and } i' \neq i, \\ a_{i''} &= 3, \text{if } p_{1i} + p_{2i'} + p_{3i''} \text{ is maximal and } i'' \neq i' \neq i, \end{aligned}$$

and so forth.

The EVAMIX method is based on important assumptions: (1) the definition of the various functions f, g, h and k; (2) the definition of the weights of the sets O and C and (3) the additive relationship of the overall dominance measure.

7.2.6 TACTIC

In the TACTIC method, proposed by Vansnick (see [6, 43]), the family of criteria F is composed of true-criteria or quasi-criteria (criteria with an indifference threshold $q_j > 0$) $g_j, j = 1, \ldots, n$, and the preference structures correspondent are (P, I) or (P, I, R), where R is the incomparability relation, if no veto-threshold $v_j(\cdot), j \in \mathcal{J} = \{1, 2, \cdots, n\}$ is considered or at least one v_j is introduced respectively.

To each criterion $g_j \in F$ an importance weight $\lambda_j > 0$ is associated, as in the ELECTRE methods (see Chap. 4 in this book). To model the preferences, the following subset of $\mathcal{J}$ is defined, $\forall a, b \in A, a \neq b$:

$$\mathcal{J}_T(a, b) = \{j \in | : g_j(a) > g_j(b) + q_j[g_j(b)]\},$$

where $q_j[g_j(b)]$ is the marginal indifference threshold as a function of the worst evaluation between $g_j(a)$ and $g_j(b)$, and therefore in this case we have aP_jb.

If in the set F only true criteria are considered, the statement aPb is true if and only if the following *concordance condition* is satisfied:

$$\sum_{j \in \mathcal{J}_T(a,b)} \lambda_j > \rho \sum_{j \in \mathcal{J}_T(b,a)} \lambda_j, \text{ i.e. } \frac{\sum_{j \in \mathcal{J}_T(a,b)} \lambda_j}{\sum_{j \in \mathcal{J}_T(b,a)} \lambda_j} > \rho \text{ if } \mathcal{J}_T(b, a) \neq \emptyset, \tag{7.1}$$

where the coefficient ρ is called required concordance level (usually, $1 \leq \rho \leq \frac{\sum_{j \in \mathcal{I}} \lambda_j}{\min_{j \in \mathcal{I}} \lambda_j} - 1$) and the two summations represent the absolute importance of the coalition of criteria in favor of a or b respectively.

If also some quasi-criterion is in the set F, in the preference structure (P, I, R) aPb is true if and only if both concordance condition (7.1) and the following *non-veto condition* are satisfied:

$$\forall j \in \mathcal{J}, g_j(a) + v_j[g_j(a)] \geq g_j(b), \tag{7.2}$$

where $v_j[g_j(a)]$ is the marginal veto threshold.

If the condition (7.2) is not satisfied by at least one criterion from F, we have aRb. On the other hand, we have aIb if and only if both pairs (a, b) and (b, a) do not satisfy condition (7.1) and no veto situation arises.

We remark that if $\rho = \rho^* = \frac{\sum_{j \in I} \lambda_j}{\min_{j \in I} \lambda_j} - 1$, the condition (7.1) is equivalent to the complete absence of criteria against the statement aPb, i.e. $\mathcal{J}_T(b, a) = \emptyset$ (and therefore in this case, (7.2) automatically holds). If $q_j = 0$ for each criterion g_j, the relation P is transitive for $\rho > \rho^*$. When ρ is decreasing from level ρ^*, we can have two types of intransitivity:

- aPb, bPc, aIc (or aRc),
- aPb, bPc, cPa.

If in Eq. (7.1) $\rho = 1$, we obtain the basic concordance-discordance procedure of Rochat type:

- for structures (P, I) (see [40]),

$$aPb \text{ iff } \sum_{j \in \mathcal{J}_T(a,b)} \lambda_j > \sum_{j \in \mathcal{J}_T(b,a)} \lambda_j;$$

$$aIb \text{ iff } \sum_{j \in \mathcal{J}_T(a,b)} \lambda_j = \sum_{j \in \mathcal{J}_T(b,a)} \lambda_j;$$

- for structures (P, I, R),

$$\begin{aligned} aPb \text{ iff } & \sum_{j \in \mathcal{J}_T(a,b)} \lambda_j > \sum_{j \in \mathcal{J}_T(b,a)} \lambda_j \text{ and} \\ & g_j(b) - g_j(a) \leq v_j[g_j(a)], \forall j \in \mathcal{J}; \\ aIb \text{ iff } & \sum_{j \in \mathcal{J}_T(a,b)} \lambda_j = \sum_{j \in \mathcal{J}_T(b,a)} \lambda_j \text{ and} \\ & g_j(b) - g_j(a) \leq v_j[g_j(a)], \forall j \in \mathcal{J}, \text{ and} \\ & g_j(a) - g_j(b) \leq v_j[g_j(a)], \forall j \in \mathcal{J}. \\ aRb \text{ iff } & \text{non}(aPb), \text{ non}(bPa) \text{ and non}(aIb). \end{aligned}$$

The main difference between the ELECTRE I and TACTIC preference modeling is that the latter method is based on the binary relation aPb, while the former aims to build up the outranking relation aSb, $a, b \in A$. Moreover, the validation of the preference relation is now based on a sufficiently large ratio between the importance of criteria in favor and against the statement aPb. Roy and Bouyssou [40] show that this second difference is actually just a formal one. They also remark that, as a consequence of the peculiar characterization of the statement aPb, in TACTIC method is difficult to split indifference and incomparability situations. No particular exploitation procedure is suggested in TACTIC method.

7.2.7 MELCHIOR

In the MELCHIOR method [21], the basic information is a family F of pseudo-criteria, i.e. criteria g_j with an indifference threshold q_j and a preference threshold p_j ($p_j > q_j \geq 0$) such that, $\forall j \in \mathcal{J}$ and $\forall a, b \in A$:

- a is strictly preferred to b (aP_jb) with respect to g_j iff $g_j(a) > g_j(b) + p_j[(g_j(b)]$,
- a is weakly preferred to b (aQ_jb) with respect to g_j iff $g_j(b) + p_j[(g_j(b)] \geq g_j(a) > g_j(b) + q[(g_j(b)]$,

- a and b are indifferent (aI_jb) iff there is no strict or weak preference between them.

No importance weights are attached to criteria, but a binary relation M is defined on F such that g_iMg_j means "criterion g_i is as least as important as criterion g_j". In order to state the comprehensive outranking relation aSb, the Author proposes to "match" in a particular way the criteria in favor and the criteria against the latter relation (concordance analysis) and to verify that no discordance situation exists, i.e. no criterion g_j from F exists such that $g_j(b) > g_j(a) + v_j$, where v_j is a suitable veto-threshold for criterion g_j (absence of discordance). In this method, a criterion $g_j \in F$ is said to be in favor of the outranking relation aSb if one of the following situations is verified:

- aP_jb (marginal strict preference of a over b) (1st condition)
- aP_jb or aQ_jb (marginal strict or weak preference of a over b) (2nd condition)
- $g_j(a) > g_j(b)$ (3rd condition).

A criterion $g_j \in F$ is said to be against the outranking relation aSb if one of the following situations is verified:

- bP_ja (marginal strict preference of b over a) (1st condition)
- bP_ja or bQ_ja (marginal strict or weak preference of b over a) (2nd condition)
- $g_j(b) > g_j(a)$ (3rd condition).

The *concordance analysis* with respect to the outranking relation aSb, $a, b \in A$, is made by checking if the family of criteria G in favor of this relation "hides" the family of criteria H that are against relation aSb. These subsets of criteria are compared just using the binary relation M on F. A subset G of criteria is said to "hide" a subset H of criteria ($G, H \subset F$, $F \cap G = \emptyset$) if, for each criterion g_i from H, there exists a criterion g_j from G such that

- g_jMg_i (1st condition) or
- g_jMg_i or not(g_iMg_j) (2nd condition),

where the same criterion g_j from G is allowed to hide at most one criterion from H.

By choosing two suitable combinations (see [21]) of the above conditions, the first stricter than the other, and verifying the concordance and the absence of discordance, a strong and a weak comprehensive outranking relation can be respectively built up. Then these relations are in turn exploited as in ELECTRE IV method (see Chap. 5 in this book). We remark that the latter in fact coincides with MELCHIOR if the same importance is assigned to each criterion.

We finally observe that in both TACTIC and MELCHIOR methods no possibility of interaction among criteria (see Chap. in this book) is taken into consideration, since the first one considers additive weights for the importance of each coalitions of criteria and the last one just "matches" one to one criteria in favor and against the comprehensive outranking relation aSb.

7.3 Pairwise Criterion Comparison Approach (PCCA)

In this approach, after the evaluations of potential alternatives with respect to different criteria, the phase of building up the outranking relations is split in two different steps, making comparisons at first level (partial aggregation) with respect to each subset of criteria $G_k \subset F$ ($|F| = m, G_k \neq \emptyset$, $|G| = k$, $k = 2, 3, \cdots, m-1$) and then aggregating at the second level these partial results (global aggregation).

With respect to weighting, this way of aggregating preferences allows to take into consideration the marginal *substitution rate* (trade-off) of each criterion from subset G_k at the first step and the *importance* of each coalition of criteria G_k at the second step, with the possibility to explicitly modeling the different meaning of these "weights" and the eventual *interaction* among criteria from each G_k (see Chap. in this book). Moreover, peculiar preference attitudes with respect to compensation, indifference and veto relations may be usefully introduced at each step of preference aggregation process; therefore, these particular options may be modelled at "local" and global level, when the partial and aggregated preferences indices respectively are built up.

For $k = 2$, (i.e. when two criteria a time are considered in the first phase of aggregation), we speak of Pairwise Criterion Comparison Approach (PCCA) , that is therefore a methodology in which first all the feasible actions are compared with respect to pairs of criteria from F, and then all the partial information so obtained are suitably aggregated.

Given $a, b \in A$, in the Multiple Attribute Utility Theory (see Chap. 8 in this book) the partial utility functions $u_i[g_i(a)]$, $i \in \mathcal{J}$, are aggregated in different ways to obtain the global utility $u(a)$ of each alternative and then the final recommendation.

In the outranking ELECTRE and PROMETHEE (see Chap. 5 in this book) families methods, from the evaluations of each action with respect to each criterion $g_i \in F$, some (crisp or fuzzy) marginal outranking or preference relations $\phi_i(a, b)$ are built up as elementary indices, or relations, with respect each criterion $i \in \mathcal{J}$ and each (ordered) pair of actions (a, b); then, using these marginal relations and other inter-criteria information, a comprehensive outranking relation or index $\phi(a, b)$ is obtained. In PCCA, in the first stage for each pair of actions (a, b) a fuzzy binary preference index $\delta_{ij}(a, b)$, $i, j \in \mathcal{J}$, is built up as elementary index taking into consideration two different criteria a time; then, by suitable aggregation of these partial indices, a global index $\delta(a, b)$ is obtained, expressing the comprehensive fuzzy preference of a over b.

As in all the other outranking methods, the exploitation of the indices expressing the comprehensive relation allows to obtain the recommendation for the decision problem at hand.

The main reasons that suggest this two levels aggregation procedure are the following:

- limited capacity of the human mind to compare a large number of elements at the same time, taking into consideration numerous and often conflicting evaluations simultaneously;

- limited ability of the DM for assessing a lot of parameters concerning subjective evaluations of general validity and considering all available information together.

Of course, this approach requires a larger number of computations and preference information, but:

- it actually helps in understanding and it supports the entire decision making process itself;
- it allows DM to use in an appropriate way all own preference information, requiring weaker coherence conditions, and to obtain further information about partial comparisons;
- it compares actions with respect to two criteria a time and then it is easier to set appropriate parameters reflecting the partial comparison at hand;
- it offers greater flexibility in the preference modeling, allowing explicitly the representation of specific preference framework and information DM wants to use each time in the considered comparison;
- it allows useful extensions of some well-known basic concepts, like weighting, compensation, dominance, indifference, incomparability, etc.
- it actually allows to model interaction between each couple of criteria (so called 2-level interaction), possibly the most important and really workable in an effective way.

Therefore, in our opinion the PCCA satisfies the following principles, relevant in any decision process, to build up realistic preference models and to obtain actual recommendations:

- *transparency*, making some light in any phase of the "black box" process (about the aggregation procedure in itself, the meaning of each parameter and index, their exploitation, etc.);
- *faithfulness*, respecting accurately the DM's preferences, without imposing too axiomatic constraints;
- *flexibility*, accepting and using any kind of information the DM wants and is able to give, neither more, nor less.

This means that DM will not be forced to "consistency" or "rationality". In other words, not too "external conditions" will be imposed to DM in expressing his/her preferences, but all actual information will be used. So, for example, not transitive trade-offs, $w_{i,k}$, (different from $w_{i,j} \cdot w_{j,k}$, where $w_{r,s}$ is the trade-off between criteria g_r and g_s), and or not complete importance weights (to some criterion no weight is associated) and also aggregated information (i.e., pooled importance weights, reflecting the interaction among criteria of each coalition) will be accepted as input.

Roughly speaking, the PCCA aggregation procedure can be applied to a lot of well-known compensatory or noncompensatory aggregation procedures resulting in binary preference indices. For each $j \in \mathcal{J}$, let $g_j \in F$ be an *interval scale* of measurement (i.e., unique up to a positive linear transformation) and $w_j, w_j \in \mathbb{R}^+$, be a suitable scale constant, called *trade-off* weight or *constant substitution rate*, reflecting (in a compensatory aggregation procedure) the increase on criterion value

g_j necessary to compensate a unitary decrease on other reference criterion from F in terms of global preference. In other words, w_j is used to transform the scale g_j for normalizing and weighting the criteria values in order to compare units on different criterion scales, for each $g_j \in F$. Often this normalization is made introducing two parameters g_j^* and $g_{*j}, j \in \mathcal{J}$, $(g_{*j} < g_j^*)$, usually fixed a priori by DM according to the specific decision problem at hand and related with the discrimination power of the criterion scales. These parameters represent, in the DM's view, respectively two suitable "levels" on criterion g_j to normalize its evaluations of feasible actions. For example, g_{*j} and g_j^* can be respectively the "neutral" and the "excellent" level or the minimum and maximum value that can be assumed on criterion g_j in case of DM's increasing preference with g_j; currently, $g_{*j} \leq \min\{g_j(x)\}$ and $g_j^* \geq \max\{g_j(x)\}$. Therefore we can write $w_j = \frac{t_j}{g_j^* - g_{*j}}$, where t_j represent the marginal weight ("importance") of criterion g_j after normalization of its scale.

Let consider the following subsets of $\mathcal{J}$:

$$\mathcal{J}_{a>b} = \{j \in \mathcal{J} : g_j(a) > g_j(b)\},$$
$$\mathcal{J}_{a=b} = \{j \in \mathcal{J} : g_j(a) = g_j(b)\},$$
$$\mathcal{J}_{a<b} = \{j \in \mathcal{J} : g_j(a) < g_j(b)\};$$

In this way, each doubleton $\{a, b\} \subseteq A$ determines a partition of $\mathcal{J}$, (possible an improper one, since some of the three subsets may be empty), whose elements are the subsets of criteria for which there is preference of a over b, indifference of a and b, preference of b over a, respectively.

Moreover, let be

$$\mathcal{J}_{a\geq b} = \{j \in \mathcal{J} \colon g_j(a) \geq g_j(b)\},$$

i.e. the subset of criteria for which there is a weak preference of a over b.

Let us remember, for example, the following elementary indices:

$$m(a,b) = |\mathcal{J}_{a>b}| \text{ (majority index)},$$
$$\lambda(a,b) = \sum_{(j\in\mathcal{J}_{a>b})} \lambda_j \text{ (Condorcet index)},$$

where $\lambda_j \in \mathbb{R}^+$ is the importance weight associated with criterion $g_j \in F$ and

$$w(a,b) = \sum_{(j\in\mathcal{J}_{a\geq b})} w_j \Delta_j(a,b) \text{ (weighted difference)},$$

where $\Delta_j(a,b) = g_j(a) - g_j(b)$ and all criteria are interval scales.

If we consider the subset of criteria $G = \{g_i, g_j\} \subseteq F$, indicating by f_{ij} any one of the above indices, computed with respect to G, it is possible to derive thence a new binary preference index $\delta_{ij}(a,b)$, defined as follows:

$$\delta_{ij}(a,b) = \begin{cases} \frac{f_{ij}(a,b)}{f_{ij}(a,b)+f_{ij}(b,a)} & \text{if } f_{ij}(a,b)+f_{ij}(b,a) > 0 \\ \frac{1}{2} & \text{if } f_{ij}(a,b)+f_{ij}(b,a) = 0 \end{cases} \tag{7.3}$$

The following properties hold, $\forall (a,b) \in A^2$:

$$\begin{aligned} 0 \le \delta_{ij} \le 1 &\Leftrightarrow \delta_{ij}(a,b) + \delta_{ij}(b,a) = 1, \\ \delta_{ij}(a,b) = 1 &\Leftrightarrow a \text{ partially dominates } b, \\ \delta_{ij}(a,b) = 0 &\Leftrightarrow b \text{ partially dominates } a, \end{aligned}$$

both being partial dominance relations defined with respect to the considered couple of criteria $\{g_i, g_j\} \subseteq F$.

Therefore, the general index $\delta_{ij}(a,b)$, obtained by the PCCA partial aggregation procedure, indicates the credibility of the dominance of a over b with respect to criteria g_i and g_j.

Let now λ_k, $\lambda_k \in \mathbb{R}^+$ be the normalized weight used in a noncompensatory aggregation procedure, called *importance weight*, associated with criterion $g_k \in F$, indicating the intrinsic importance of each criterion, independently by its evaluation scale. Then, we can aggregate the partial indices $\delta_{ij}(a,b)$ computed with respect to all the couples of different criteria g_i and g_j from F according to the PCCA logic, considering also the normalized *importance weight* λ_{ij} (i.e. $\sum_{i<j} \lambda_{ij} = 1$) *of the coalition* (couple) of criteria g_i and g_j, $i, j \in \mathcal{J}$.

We obtain the following aggregated index :

$$\delta(a,b) = \frac{1}{n-1} \sum_{ij(i<j)} \lambda_{ij} \delta_{ij}(a,b). \tag{7.4}$$

If there is no interaction between any couple of criteria, additive weights can be used in Eq. (7.4), i.e. $\lambda_{ij} = \lambda_i + \lambda_j$, otherwise $\lambda_{ij} > \lambda_i + \lambda_j$ in case of positive interaction (synergy) and $\lambda_{ij} < \lambda_i + \lambda_j$ in case of negative interaction (redundancy). The following properties hold, $\forall (a,b) \in A^2$:

$$\begin{aligned} 0 \le \delta(a,b) \le 1, &\quad \text{if and only if } \delta(a,b) + \delta(b,a) = 1, \\ \delta(a,b) = 1 &\quad \text{if and only if } a \text{ strictly dominates } b, \\ \delta(a,b) = 0 &\quad \text{if and only if } b \text{ strictly dominates } a \end{aligned}$$

(see Sect. 7.3.1).

Therefore, the particular meanings (credibility of dominance) of the partial and global indices $\delta_{ij}(a,b)$ and $\delta(a,b)$ respectively are results essentially linked to the *peculiar aggregation procedure* of PCCA and not to the specific bicriteria index considered each time.

In the framework of the PCCA methodology, different methods have been proposed: MAPPAC, PRAGMA, IDRA, PACMAN, each one with its own features to build up the correspondent outranking relations and indices.

7.3.1 MAPPAC

We recall that *a dominates b* (aDb), $a, b \in A$, with respect criteria from F if a is at least as good as b for the considered criteria and is strictly preferred to b for at least one criterion:

$$aDb \Leftrightarrow g_i(a) \geq g_i(b), \forall g_i \in F \text{ and } \exists j \in \mathcal{J} : g_j(a) > g_j(b)\,.$$

We say that *a weakly dominates b* (aD_wb) if a is at least as good as b for all the criteria from F:

$$aD_wb \Leftrightarrow g_i(a) \geq g_i(b), \forall\, g_i \in F.$$

We say that *a strictly dominates b* (aD_sb) iff $g_i(a) \geq g_i(a)$, $\forall i \in F$, where at most only one equality is valid. The binary relation D_w is a partial preorder (reflexive and transitive), while D (and D_s) is a partial order (irreflexive, asymmetric and transitive); the correspondent preference structures are partial order and strict partial order respectively. Of course, $D_s \subset D \subset D_w$, $aDb, bD_wc \Rightarrow aDc$ and $aD_wb, bDc \Rightarrow aDc, \forall a, b, c \in A$.

In PCCA, where a subset (couple) of criteria $G = \{g_i, g_j\} \subset F$, is considered at the first level of aggregation, we say that a partially dominates b ($aD_{ij}b$), if the relation of dominance is defined on G. We say that a is partially preferred or is partially indifferent to b ($aP_{ij}b_j$ and $aI_{ij}b$ respectively) if these relations hold with respect to the set of criteria $\{g_i, g_j\}$.

We observe that

$$aD_{ij}b \Rightarrow aP_{ij}b,$$

and

$$aD_{ij}b, \forall i, j \in \mathcal{J} \Leftrightarrow aD_sb \Rightarrow aDb \Rightarrow aPb,$$

if all criteria from F are true criteria.

In the MAPPAC method [30] the basic (or partial) indices $\pi_{ij}(a, b)$ can be interpreted as credibility indices of the partial dominance $aD_{ij}b$, indicating also the fuzzy degree of preference of a over b; the global index $\pi(a, b)$, corresponding to index $\delta(a, b)$ defined in (7.4), can be interpreted as the credibility index of strict dominance aD_sb, i.e. as the fuzzy degree of comprehensive preference of a over b.

If all criteria from F are interval scales, recalling that $\Delta_j(a, b) = g_j(a) - g_j(b)$, for each $j \in \mathcal{J}$ and $a, b \in A$, w_j, is the trade-off weight and λ_j the (normalized) importance weight of criterion $g_j, j \in \mathcal{J}$, the axiomatic system of MAPPAC partial indices can be summarized as follows (see Table 7.18) for each $a, b \in A$:

Table 7.18 Axiomatic system of MAPPAC basic indices

$\pi_{ij}(a,b)$	Binary relations	Signs of $\Delta_i(a,b)\cdot\Delta_j(a,b)$	Signs of $\Delta_i(a,b)+\Delta_j(a,b)$	Pair of signs of $\Delta_i(a,b),\ \Delta_j(a,b)$
$]0,1[$	$aP_{ij}b, bP_{ij}a, aI_{ij}b$	< 0	$\gtreqless$	$(+,-),(-,+)$
$\frac{1}{2}$	$aI_{ij}b$	$= 0$	$= 0$	$(0,0)$
1	$aD_{ij}b$	≥ 0	> 0	$(+,+),(+,0),(0,+)$
0	$bD_{ij}a$	≥ 0	< 0	$(-,-),(-,0),(0,-)$

- The basic indices $\pi_{ij}(a,b)$ are functions only of the signs of the differences in evaluations of a and b with respect to criteria g_i and g_j in case of concordant evaluations, i.e. iff $\Delta_i(a,b)\Delta_j(a,b) \geq 0$. In this case,

$$aD_{ij}b \Leftrightarrow \Delta_i(a,b) + \Delta_j(a,b) > 0$$

and

$$bD_{ij}a \Leftrightarrow \Delta_i(a,b) + \Delta_j(a,b) < 0$$

and then $\pi_{ij}(a,b) = 1$ and $\pi_{ij}(b,a) = 0$ in the first case, and $\pi_{ij}(a,b) = 0$ and $\pi_{ij}(b,a) = 1$ in the second case.
- The basic indices $\pi_{ij}(a,b)$ are functions of the values of the differences in evaluations of a and b with respect to criteria g_i and g_j and of trade-off weights w_i and w_j in case of discordant evaluations, i.e. iff $\Delta_i(a,b)\,\Delta_j(a,b) < 0$. In this case, the indices $\pi_{ij}(a,b)$ and $\pi_{ij}(b,a)$ will be of a compensatory type, lying in the interval $]0,1[$, and they will indicate the fuzzy degree of preference of a over b and of b over a respectively; if $w_i\Delta_i(a,b) + w_j\Delta_j(a,b) = 0$, $\pi_{ij}(a,b) = \pi_{ij}(b,a) = \frac{1}{2}$.
- The global index $\pi(a,b)$ is function of all the $\binom{m}{2}$ basic indices $\pi_{ij}(a,b)$ and of the importance weights λ_{ij} of all coalitions $\{g_i, g_j\}$ of criteria. If there is no interaction between criteria g_i and g_j, we have $\lambda_{ij} = \lambda_i + \lambda_j$. In case of strict dominance aD_sb or bD_sa, $\pi(a,b) = 1$ and $\pi(b,a) = 0$, or $\pi(a,b) = 0$ and $\pi(b,a) = 1$, respectively. Otherwise, $\pi(a,b)$ and $\pi(b,a)$ will lie in the interval $]0,1[$ and they will indicate the fuzzy degree of comprehensive preference of a over b and of b over a respectively.

Preference Indices We recall that $w_j\Delta_j(a,b) = w_j(g_j(a) - g_j(b)), j \in \mathcal{J}, a, b \in A$, is the normalized weighted difference of evaluations of actions a and b with respect to criterion g_j.

If we assume $f_{ij}(a,b) = \sum_{h\in\{i,j\}\cap\mathcal{J}_{a\geq b}} w_h\Delta_h(a,b)$ in Eq. (7.3) we obtain the partial index $\pi_{ij}(a,b)$ of MAPPAC, $a, b \in A$, $\{g_i, g_j\} \subset F$ ($|F| \geq 3$). This index can also be explicitly written as shown in Table 7.19.

Table 7.19 Basic preferences indices

$\pi_{ij}(a,b)$	$\pi_{ij}(b,a)$	
1	0	if $g_i(a) > g_i(b)$ and $g_j(a) > g_j(b)$
0	1	if $g_i(a) < g_i(b)$ and $g_j(a) < g_j(b)$
0.5	0.5	if $g_i(a) = g_i(b)$ and $g_j(a) = g_j(b)$
$\frac{w_i(g_i(a)-g_i(b))}{w_i(g_i(a)-g_i(b))+w_j(g_j(b)-g_j(a))}$	$\frac{w_j(g_j(b)-g_j(a))}{w_i(g_i(a)-g_i(b))+w_j(g_j(b)-g_j(a))}$	if $g_i(a) > g_i(b)$ and $g_j(a) \leq g_j(b)$ if $g_i(a) = g_i(b)$ and $g_j(a) < g_j(b)$
$\frac{w_j(g_j(a)-g_j(b))}{w_i(g_i(b)-g_i(a))+w_j(g_j(a)-g_j(b))}$	$\frac{w_i(g_i(b)-g_i(a))}{w_i(g_i(b)-g_i(a))+w_j(g_j(a)-g_j(b))}$	if $g_i(a) \leq g_i(b)$ and $g_j(a) > g_j(b)$ if $g_i(a) < g_i(b)$ and $g_j(a) = g_j(b)$

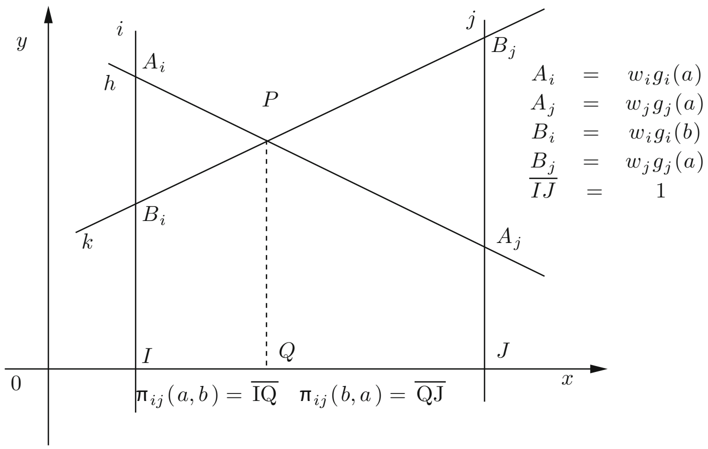


Fig. 7.4 Geometrical interpretation of basic preferences indices

It is invariant to the admissible transformation of any $g_j \in F$, i.e. all the positive affine transformations of the type $g'_j(\cdot) = \alpha g_j + \beta$, with $\alpha \in \mathbb{R}^+$ and $\beta \in \mathbb{R}$, being the criteria interval scales. It is the image of a valued binary relation, strictly complete, transitive and ipsodual (i.e. $\pi_{ij}(a,b) = 1 - \pi_{ij}(b,a)$) , that constitutes a complete preorder on A, and it indicates the fuzzy partial preference intensity of a over b.

The basic preference index $\pi_{ij}(a,b)$ may be immediately interpreted geometrically by considering the partial profiles of the actions a and b with respect to criteria g_i and g_j (see Fig. 7.4).

Let us consider the following subsets of F:

$$G^{+}(a,b) = \{g_h \in F : \Delta_h(a,b) > 0\} \quad \text{with} \quad |G^{+}(a,b)| = p,$$
$$G^{=}(a,b) = \{g_h \in F : \Delta_h(a,b) = 0\} \quad \text{with} \quad |G^{=}(a,b)| = o,$$
$$G^{-}(a,b) = \{g_h \in F : \Delta_h(a,b) < 0\} \quad \text{with} \quad |G^{-}(a,b)| = n,$$
$$D_1(a,b) = \{(g_i,g_j) \in F^2, g_i \neq g_j : aD_{ij}b\},$$
$$D_0(a,b) = \{(g_i,g_j) \in F^2, g_i \neq g_j : bD_{ij}a\}.$$

Of course, $G^{+}(a,b) \cup G^{=}(a,b) \cup G^{-}(a,b) = F$ and $|F| = m = p + o + n$, $(p,o,n \geq 0)$. Since (see [31])

$$\binom{m}{2} = \binom{p+o+n}{2} =$$
$$\binom{p+o}{2} + \binom{p+n}{2} + \binom{o+n}{2} - \binom{p}{2} - \binom{o}{2} - \binom{n}{2} =$$
$$\binom{p}{2} + \binom{o}{2} + \binom{n}{2} + po + pn + on,$$

we can split all the $\binom{m}{2}$ basic preference indices $\pi_{ij}(a,b)$ as follows:

$$\binom{m}{2} = |D_1(a,b)| + |D_o(a,b)| + \binom{o}{2} + pn.$$

Thus, $|D_1(a,b)| = \binom{m}{2}$ if and only if $p \geq m-1$ and $n = 0$ (i.e., $\Delta_h(a,b) \geq 0$ for each $h \in \mathcal{J}$, with at most only one equality), $|D_0(a,b)| = \binom{m}{2}$ if and only if $n \geq m-1$ and $p = 0$, $|D_1(a,b)| = |D_0(a,b)| = 0$ and $\pi_{ij}(a,b) = \frac{1}{2}$ for each $i,j \in \mathcal{J}$ if and only if $o = \frac{m}{2}$ (i.e., $\Delta_h(a,b) = 0$ for each $h \in \mathcal{J}$).

The *global preference index* $\pi(a,b)$ is the sum of all the $\binom{m}{2}$, $m > 2$, basic preference indices $\pi_{ij}(a,b)$, weighted each time by the normalized importance weights λ_{ij} of the considered couple of criteria g_i, g_j:

$$\pi(a,b) = \sum_{ij(i<j)} \pi_{ij}(a,b)\frac{\lambda_{ij}}{\Lambda},$$

where $\Lambda = \sum_{ij(i<j)} \lambda_{ij}$.

If there is no interaction between each couple of criteria, we have $\lambda_{ij} = \lambda_i + \lambda_j\ \forall i,j \in \mathcal{J}$, where λ_h is the normalized importance weight of criterion g_h, $h = 1,2\cdots m$, and therefore:

$$\pi(a,b) = \sum_{ij(i<j)} \pi_{ij}(a,b)\frac{\lambda_i + \lambda_j}{m-1},\quad i,j \in \mathcal{J}, \left(\sum_{ij(i<j)} \left(\lambda_i + \lambda_j\right) = m-1\right) \tag{7.5}$$

Therefore, in this case we can write $\pi(a,b)$ as:

$$\pi(a,b) = \pi_{PP}(a,b) + \pi_{P0}(a,b) + \pi_{NN}(a,b) + \pi_{NO}(a,b) + \pi_{OO}(a,b) + \pi_{PN}(a,b), \tag{7.6}$$

where:

$$\pi_{PP}(a,b) = \frac{p-1}{m-1} \sum_{i\in G^{+}(a,b)} \lambda_i;$$

$$\pi_{P0}(a,b) = \frac{1}{m-1}\left[p \sum_{i\in G^{=}(a,b)} \lambda_i + o \sum_{i\in G^{+}(a,b)} \lambda_i\right];$$

$$\pi_{NN}(a,b) = \pi_{NO}(a,b) = 0;$$

$$\pi_{OO}(a,b) = \frac{1}{2}\frac{o-1}{m-1} \sum_{i\in G^{=}(a,b)} \lambda_i;$$

$$\pi_{PN}(a,b) = \sum_{rs} \pi_{rs}(a,b)\frac{\lambda_{r+}\lambda_s}{m-1}, (g_r, g_s) \in G^{+}(a,b) \times G^{-}(a,b).$$

Let $S(a,b) = G^{+}(a,b) \cup G^{=}(a,b)$. We can write:

$$\pi_{D_1} = \pi_{PP}(a,b) + \pi_{PO}(a,b)$$

and, recalling Eq. (7.6),

$$\pi_S = \pi_{D_1}(a,b) + \pi_{OO}(a,b) = \frac{1}{m-1}\left[(p+o-1)\sum_{i\in S(a,b)} \lambda_i - \frac{1}{2}(o-1)\sum_{i\in G^{=}(a,b)} \lambda_i\right].$$

We observe that:

(a) if $G^{=}(a,b) = \emptyset$ or $|G^{=}(a,b)| = 1, \pi_{D_1}(a,b) = \pi_S(a,b)$;
(b) if $G^{-}(a,b) = \emptyset$, $\pi(a,b) = \pi_S(a,b)$;
(c) the index $\pi_S(a,b)$ is a linear combination of the *crisp* concordance index $c(a,b)$ of the ELECTRE methods (see Chap. 5 in this book) and the opposite of semi-sum of the importance weights of criteria from set $G^{=}(a,b)$; their coefficients

are respectively given by the ratios between the number of criteria belonging to the corresponding classes and the total number of criteria up to one unit (i.e., the number of *significant* criteria for a comparisons by means of pairs of criteria);

(d) if $|S(a,b)| \geq 2$, $\frac{1}{2} \leq \pi_S(a,b) \leq c(a,b) \leq 1$, and $\pi_S(a,b) = c(a,b) = 1$ if and only if aD_Sb (but $c(a,b) = 1$ does not imply $\pi_S(a,b) = 1$);

(e) $\pi_S(a,b) = 0$ if and only if $|S(a,b)| < 2$, and $\pi_S(a,b) = c(a,b) = 0$ if and only if $S(a,b) = \emptyset$ (but $\pi_S(a,b) = 0$ does not imply $c(a,b) = 0$);

(f) the compensatory component $\pi_{PN}(a,b)$ of $\pi(a,b)$ (see Eq. (7.6)) may be methodologically linked to the MAUT approach, in particular to the weighted sum with constant marginal substitution rates (trade-off weights);

(g) if the number o of the criteria g_h from F for which $g_h(a) = g_h(b)$ changes without modification in the sum of the relative importance weights of coalitions $G^+(a,b)$, $G^+(b,a)$ and $G^=(a,b)$, the value of the aggregate preference index $\pi(a,b)$ may vary, as a consequence of changing of its component $\pi_{PN}(a,b)$ value. More precisely:

  - $G^+(a,b) = \emptyset$ and $G^-(a,b) \neq \emptyset \Rightarrow \pi(a,b)$ increases with o, i.e. $\Delta_o\pi(a.b) > 0$,
  - $G^+(a,b) \neq \emptyset$ and $G^-(a,b) = \emptyset \Rightarrow \pi(a,b)$ decreases with o, i.e. $\Delta_o\pi(a.b) < 0$,
  - $\lim_{o \to +\infty} \Delta_o\pi(a,b) = 0$,
  - $\lim_{o \to +\infty} \pi(a,b) = \sum_{i \in G^+(a,b)} \lambda_i + \frac{1}{2} \sum_{i \in G^=(a,b)} \lambda_i$ (< 1 since $G^+(a,b) \subset F$),
  - if the relative importance weights of $G^+(a,b)$ and $G^-(a,b)$ are equal, the relation aIb is stable with respect to o;
  - $\forall\, o \geq 1, \frac{p}{p+n} \sum_{i \in S(a,b)} \lambda_i \geq \frac{1}{2} \Rightarrow aPb$ stable with respect to o,
  - if there is a perfect compensation between the normalized weighted differences in evaluations of opposite signs (i.e. neutral behavior of $\pi_{PN}(a,b)$), $\Delta_o\pi(a,b) > 0\ [< 0] \Leftrightarrow n \sum_{i \in S(a,b)} \lambda_i - p \sum_{i \in S(a,b)} \lambda_i > 0\ [< 0]$, i.e. if and only if $n > [<]\, p$,
  - the aggregate preference index $\pi(a,b)$ is an increasing function of p (i.e. $\Delta_p\pi(a,b) > 0$) if $\pi(a,b) < 1$,
  - $\lim_{p \to +\infty} \Delta_p\pi(a,b) = 0$,
  - $\lim_{p \to +\infty} \pi(a,b) = 1 - \frac{1}{2} \sum_{i \in G^-(a,b)} \lambda_i$.

Following the same principle of PCCA, it is possible to build up other partial and global preference indices, based on a logic of noncompensatory aggregation [29]. The common feature of all these indices is that they are based on bicriteria and global indices, measuring respectively the credibility of partial dominance and of strict dominance of a over b, $a, b \in A$. So, for example, if no 2-level interaction occurs among considered criteria, let us consider the following two aggregated indices:

$$\pi'(a,b) = \frac{1}{m-1}\left[(m-1)\sum_{i\in G^{+}(a,b)}\lambda_i - (p+\frac{o-1}{2})\sum_{i\in G^{=}(a,b)}\lambda_i\right],$$

$$\pi^*(a,b) = \frac{1}{m-1}\sum_{i,j;(i<j)}\left[\sum_{(i,j):\pi_{ij}(a,b)>0.5}(\lambda_i+\lambda_j) + 0.5\sum_{(i,j):\pi_{ij}(a,b)=0.5}(\lambda_i+\lambda_j)\right].$$

We can observe that index $\pi'(a,b)$ is totally noncompensatory and it is analogous to the concordance indices of ELECTRE I and II methods. On the other hand, index $\pi^*(a,b)$ is PCCA-totally noncompensatory (see [29]), depending on the *"coalition strength"* of the subsets (couples of criteria) of G^2 such that $aP_{ij}b$ or $aI_{ij}b$. Both these indices, like index $\pi(a,b)$, are also functions of p, n, o.

Taking into account the above properties and the peculiar features of the basic preference indices with respect to the dominance and compensation, MAPPAC and—more generally—PCCA may be considered as an *"intermediate"* MCDA methodology between the outranking (particularly ELECTRE) and MAUT methods.

Indifference Modelling Since the evaluations of actions a and b with respect to the couple of criteria g_i, g_j from F are compared each time to build up index $\pi_{ij}(a,b)$, and recalling that $\Delta_i(a,b)\ \Delta_j(a,b) > 0$ means by definition active or passive partial dominance of a over b (and then $\pi_{ij}(a,b) = 1$ or 0 respectively), it is useful to confine the dominance relation only if well founded situations will occur. Therefore, in order to take into account the inevitable inaccuracies and approximations in the actions evaluations, and in order to prevent small differences between these evaluations from creating partial dominance relations or preference intensities close to the maximum or minimum values, it is advisable to introduce suitable indifference areas on the plane $Og_i(a)g_j(a)$ in the neighborhood of point $I = (g_i(a) = g_i(b), g_j(a) = g_j(b))$.

These areas may be defined in various way, as functions of correspondent indifference thresholds, one for each criterion considered (see [27]). The marginal indifference threshold for criterion g_j, denoted by q_j, is not negative and unique for every couple of distinct actions $a, b \in A$ ($q_j(a,b) = q_j(b,a) \geq 0, \forall a,b \in A$) and it is a function of the evaluations of these actions according to the criterion considered:

$$q_j(a,b) = \alpha_j + \beta_j|\frac{g_j(a)+g_j(b)}{2}|, g_j \in F, \alpha_j, \beta_j \geq 0. \quad (7.7)$$

The first parameter α_j is expressed in the same scale of values as the criterion g_j, and q_j is a linear function of the arithmetical mean of the evaluations of the considered actions, being β_j the constant of proportionality. Then, if $\beta_j = 0$ or $\alpha_j = 0$, Eq. (7.7) supplies constant indifference thresholds, in absolute or relative value respectively. It is therefore possible to define an indifference area IA_{ij} for

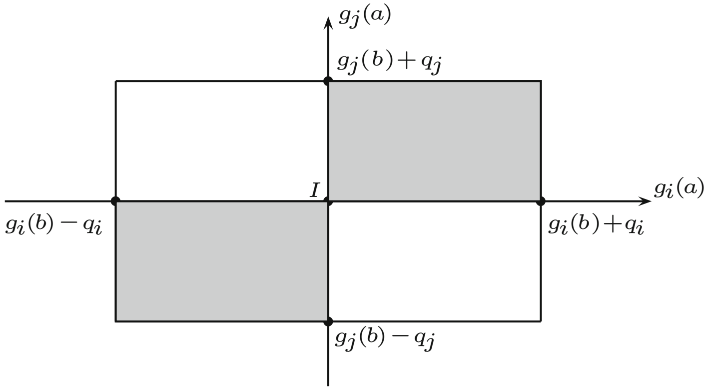


Fig. 7.5 Indifference areas: rectangular

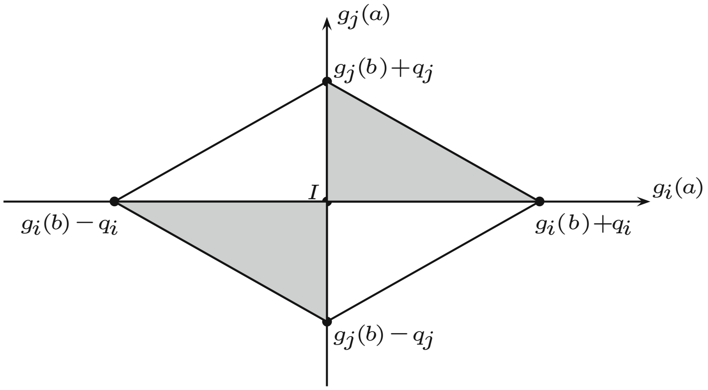


Fig. 7.6 Indifference areas: rhomboidal

each pair of actions $a, b \in A$ and criteria $g_i, g_j \in F$ as a function of the marginal indifference thresholds (7.7). This area may assume various shapes, for example:

- rectangular, if $\pi_{ij}(a,b) = \frac{1}{2}$ for $|g_i(a) - g_i(b)| \leq q_i(a,b)$ and $|g_j(a) - g_j(b)| \leq q_j(a,b)$ (see Fig. 7.5);
- rhomboidal, if $\pi_{ij}(a,b) = \frac{1}{2}$ for $\frac{|g_i(a)-g_i(b)|}{q_i(a,b)} + \frac{|g_j(a)-g_j(b)|}{q_j(a,b)} \leq 1$ (see Fig. 7.6);
- elliptical, if $\pi_{ij}(a,b) = \frac{1}{2}$ for $\frac{(g_i(a)-g_i(b))^2}{q_i^2(a,b)} + \frac{(g_j(a)-g_j(b))^2}{q_j^2(a,b)} \leq 1$ (see Fig. 7.7).

It also possible to introduce semi-rectangular, semi-rhomboidal and semi-elliptical indifference areas, corresponding to the shadowed areas in Figs. 7.5, 7.6, and 7.7 respectively, with the specific aim of eliminating the effect of partial dominance only, adding each time the further conditions:

$$\begin{cases} g_i(b) \leq g_i(a) \\ g_j(b) \leq g_j(a) \end{cases} \text{or} \begin{cases} g_i(a) \leq g_i(b) \\ g_j(a) \leq g_j(b). \end{cases}$$

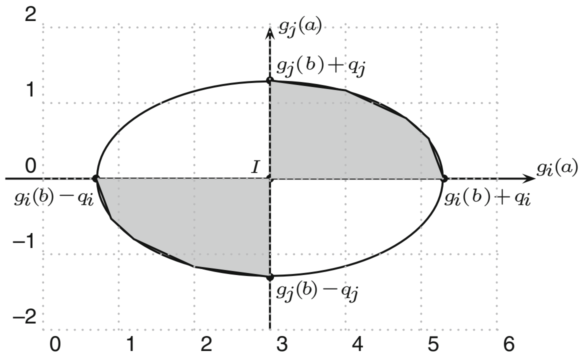


Fig. 7.7 Indifference areas: elliptical

Finally, it is also possible to consider mixed indifference areas, as a suitable combination of two or more of the cases considered above for each quadrant centered in point I. We can then modeling indifference in a flexible way, by setting different thresholds and/or shapes for each couple of criteria, according to the DM's preferential information.

Therefore, two separate indifference relations are obtained: strict indifference, denoted by $aI_{ij}b$, iff $\pi_{ij}(a,b) = \frac{1}{2}$ as a result of definition given in Table 7.19; large indifference, denoted by $aI^*_{ij}b$, iff a vector $\mathbf{q} \geq \mathbf{0}$ is introduced, $\mathbf{q} = [q_j(a,b)], j \in \mathcal{J}$, and some of the corresponding above indifference area conditions are satisfied, and thus $\pi_{ij}(a,b) = \frac{1}{2}$ is assumed.

Note that I_{ij} is an equivalence relation, whereas the relation $aI^*_{ij}b$ is not necessarily transitive.

Preference Structures Using the basic and global preference indices $\pi_{ij}(a,b)$ and $\pi(a,b)$ respectively, it is possible to immediately define the following correspondent binary relations of partial and comprehensive indifference and preference relations respectively, with the particular cases of dominance recalled above:

- Partial relations

$$\pi_{ij}(a,b) = 0.5 \Leftrightarrow aI_{ij}b,$$

$$0.5 < \pi_{ij}(a,b) \leq 1 \Leftrightarrow aP_{ij}b \quad (\pi_{ij}(a,b) = 1 \Leftrightarrow aD_{ij}b),$$

$$0 \leq \pi_{ij}(a,b) < 0.5 \Leftrightarrow bP_{ij}a \quad (\pi_{ij}(a,b) = 0 \Leftrightarrow bD_{ij}a).$$

- Comprehensive relations

$$\pi(a,b) = 0.5 \Leftrightarrow aIb,$$

$$0.5 < \pi(a,b) \leq 1 \Leftrightarrow aPb \quad (\pi(a,b) = 1 \Leftrightarrow aDb),$$

$$0 \leq \pi(a,b) < 0.5 \Leftrightarrow bPa \quad (\pi(a,b) = 0 \Leftrightarrow bDa).$$

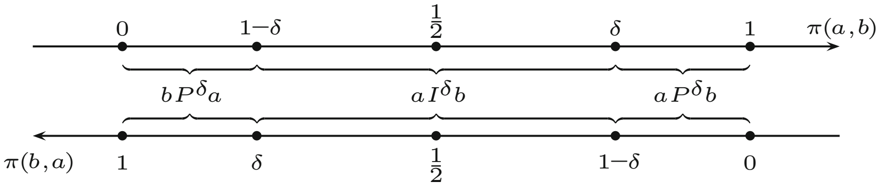


Fig. 7.8 Aggregated semiorder structure

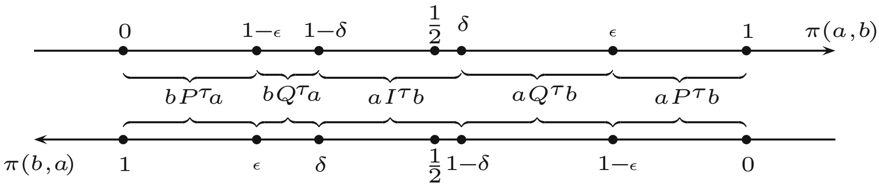


Fig. 7.9 Aggregated pseudo-order structure

Both these structures constitute a complete preorder on A. We observe that, if no indifference areas are introduced, will be $\pi_{ij}(a,b) + \pi_{ij}(b,a) = 1$ for each $i,j \in \mathcal{J}$ and $(a,b) \in A^2$ and therefore also $\pi(a,b) + \pi(b,a) = 1$.

Of course, by means of the same indices, we can also build up some other particular complete valued preference structures. For example, we may consider the structure of semiorder, obtained by introducing a real parameter $\delta \in [1/2, 1]$, which emphasizes the partial or global indifference relations (see Fig. 7.8).

In this case, the indifference relations are reflexive, symmetric and not transitive, while the preference relations are transitive, non reflexive and asymmetric. We note that if $\delta = \frac{1}{2}$ we obtain again a complete preorder with "punctual" indifference, i.e. only for $\pi(a,b) = \pi(b,a) = \frac{1}{2}$, while if $\delta = 1$, the binary preference relation is empty. Alternatively, by introducing two real parameters δ and ϵ, $\frac{1}{2} \leq \delta < \epsilon \leq 1$, it is possible to build a complete two-valued preference structure, assuming that there are two preference intensity levels, represented by the preference relations P^τ (strict preference) and Q^τ (weak preference) (see Fig. 7.9).

In this case the relations of indifference and of weak preference are not transitive and the preference model presents the properties of the well-known pseudo-order structure (see [44]).

Conflict Analysis Besides the concept of discordant criterion and veto threshold often used for building outranking relations, another interesting feature of PCCA approach is the possibility to consider a peculiar conflict analysis, taking into consideration the differences in evaluations of two actions with respect to each couple of criteria. The main aims of this analysis are the following:

- to explicitly define binary *incomparability relations* in presence of evaluations of two actions a and b in strong contrast on two criteria g_i and g_j, in the preference modeling phase (refusal to make a decision)
- to allow compensation only if differences in the conflicting evaluations are not too large; otherwise, to use non compensatory basic indices (functions only of importance weights), obtaining a *partially compensatory approach* (reduction of compensation) (see [29]).

These aims can be reached by defining a suitable *partial discordance index* $d_{ij}(a,b)$, $i,j \in \mathcal{J}$, $a,b \in A$, for each couple of criteria as a function of conflicting evaluations and entropy of information, and comparing this one with correspondent incomparability threshold r_{ij}, given by DM (see [27]). If we note by R_{ij} the partial incomparability relation with respect the couple of criteria g_i and g_j, we have:

$$d_{ij}(a,b) \geq r_{ij} \Leftrightarrow aR_{ij}b,\ (a,b) \in A^2, g_i, g_j \in F.$$

Then, considering all the possible couples of distinct criteria g_i, g_j from F, we have:

$$aRb \Leftrightarrow [aR_{ij}b \text{ for at least one couple } i,j \in \mathcal{J}].$$

This global incomparability relation R, symmetric but neither reflexive nor transitive, arise if at least one partial incomparability relation holds with respect to actions a and b.

The symmetric discordance index $d_{ij}(a,b)$, $i,j \in \mathcal{J}$, is defined as follows [26].

$$d_{ij} = |w_i\Delta_i(a,b) + w_j\Delta_j(b,a)|(1 - 2|\pi_{ij}(a,b) - 0.5|).$$

It lies in $[0, t_i + t_j]$ and reaches its maximum value only in case of maximum effective discordance of evaluations of a and b with respect to g_i and g_j (i.e. $g_i(a) = g_i^*$, $g_j(a) = g_{j*}$ and $g_i(b) = g_{i*}$, $g_j(b) = g_j^*$ or viceversa) and $t_i = t_j$ (equal normalized trade-off weights). Moreover, $d_{ij}(a,b) = 0$ if $\Delta_i(a,b) = \Delta_j(b,a) = 0$ or in case of partial dominance (evaluation concordance). Therefore, it is possible to set the incomparability thresholds r_{ij} according to the real preferential information of DM about the different level of compensation for each couple of criteria g_i and g_j:

$$r_{ij} \begin{cases} = 0 & \text{completely non compensatory approach} \\ \cong 0 & \text{low compensation is allowed} \\ \cong t_i + t_j & \text{high compensation is allowed} \\ > t_i + t_j & \text{totally compensatory approach.} \end{cases}$$

The concepts introduced above therefore permit also a modelling by means of the four binary relations I,P,Q,R, defined on A, which are exhaustive and mutually exclusive and constitute a fundamental relational preference system.

Exploitation Phase The results of the relational model in the form of fuzzy binary relations obtained can be presented in the form of suitable $\binom{m}{2}$ bicriteria $n \times n$ (i.e. $|A| \times |A|$) square matrices: $\mathbf{\Pi_{ij}} = [\pi_{ij}(a,b)]$, one for each couple of criteria g_i, g_j from F, containing the partial preference indices, and one aggregated matrix $\mathbf{\Pi} = [\pi(a,b)]$, with the comprehensive preference indices, $(a,b) \in A^2$.

The peculiar preference modeling flexibility of PCCA allows to respect accurately the DM's preference, without imposing too strong axiomatic constraints, and accepting and using any kind of information the DM is able to give. Therefore, DM is not forced to be *"consistent"*, *"rational"* or *"complete"*, but all information given by DM is accepted and used, neither more, nor less. Consequently, with respect to bi-criteria trade-offs w_{ij}, $i,j \in I$, it is possible to use as input not transitive (i.e. $w_{ij}w_{jk} = \frac{w_i}{w_j}\frac{w_j}{w_k} \neq \frac{w_i}{w_k} = w_{ik}$) or not complete (some w_{ij} not given by DM) trade-offs for some pairs of criteria (and therefore the component $\pi_{ij}(a,b)$ of index $\pi(a,b)$ correspondent to these criteria will be absent); and, with reference to importance weights λ_j, $j \in \mathcal{J}$, the DM may assign non additive weights λ_{ij} to some couple of criteria, modelling thus their interaction (i.e. weighting some index $\pi_{ij}(a,b)$ with a weight different from $\lambda_i + \lambda_j$). In all these cases, the aggregate index $\pi(a,b)$ will be computed taking into account the peculiar information actually used as input.

The indices of preference intensity contained in the aggregated matrix $\mathbf{\Pi}$ may, among other things, permit in the *exploitation phase* the building of specific partial or complete rankings of feasible actions as final prescription.

A first possible technique to build rankings can be based on the concept of qualification of a feasible action, introduced by Roy (see [39]). But, in order to take into consideration the most complete preference information given by the fuzzy relations, we can sum the global preference indices referred to each feasible action in comparison with others, obtaining its comprehensive preference index, aiming to build up the partition of A into S equivalence classes $C_1, C_2, \cdots, C_S, S \leq n$ (complete preorder), by means of a descending procedure (from the best action to the worst) or by an ascending procedure (from the worst to the best).

In either case, the peculiar feature of these techniques is that at every step they select the action(s) assigned to a certain position in the ranking considered and then repeat the procedure with respect to the subset of the remaining actions, eliminating at each iteration the action selected in the preceding one. Here is a brief description of one of the possible techniques.

Computation of the *comprehensive preference index*, $a \in A$:

$$\sigma_+^{(1)}(a) = \sum_{b \in A \setminus \{a\}} \pi(a,b).$$

This will be:

$$0 \leq \sigma_+^{(1)}(a) \leq n-1, \quad \forall a \in A.$$

In particular we obtain:

$$\sigma_{+}^{(1)}(a) = n - 1 \text{ or } \sigma_{+}^{(1)}(a) = 0,$$

if and only if a strictly dominates, or is strictly dominated by, respectively, all the remaining feasible actions. We then select the action(s) with the highest index $\sigma_{+}^{(1)}$. This action, or these actions, will occupy the first place in the decreasing ranking, forming class C_1. Then, given $A^{(1)} = A \setminus C_1$, we repeat the procedure with reference to the actions from this new subset, obtaining the indices:

$$\sigma_{+}^{(2)}(a) = \sum_{b \in A^{(1)} \setminus \{a\}} \pi(a, b),\ a \in A^{(1)}.$$

This iteration will make it possible to form class C_2, and so on (*descending procedure*).

The *increasing solution* may be obtained by calculating for each action a the comprehensive index

$$\sigma_{-}^{(1)}(a) = \sum_{b \in A \setminus \{a\}} \pi(b, a),$$

and placing in the last class C_s the action(s) which present the highest value for this index. We then proceed with the calculation of the indices $\sigma_{-}^{(2)}(a)$ related to the subset $A \setminus C_s$ and so on.

This way to build the rankings is suggested in order to reduce the risk that an action dominating or dominated by one or more feasible actions may assume a strongly discriminatory role over these. A dominated action has a distorting effect during the descending procedure, while a dominating action produces the same effect during the ascending procedure.

A useful geometrical interpretation on omometric axes of the complete preorders related to the actions considered each time in the k-th iteration may efficaciously express the different rankings with the corresponding comprehensive intensities of preference (see [27]). If the broken lines connecting the points representing the comprehensive preferences of each action at all different iterations prove to be more or less parallel, the relative comprehensive preferences tend to remain constant. On the other hand, if these broken lines intersect one another, the ranking will present inversion in terms of comprehensive preferences at the considered iterations.

Of course, in the building of all complete preorders it is possible to introduce suitable global indifferent thresholds, to prevent small differences in the comprehensive indices considered at every iteration from assuming a discriminating role (see [27]).

The building of preorders allows also to solve the choice problem. But it is also possible to directly use in a lot of different way the information about strict dominance (given by the comprehensive preference) indices to support DM in choice problem.

For example, let $P(a,b) = \max[\pi(a,b) - \pi(b,a), 0]$, that is $P(a,b) = T_L[\pi(a,b), 1-\pi(b,a)]$, where $T_L[.,.]$ means Lukasiewicz t-norm. Choice is usually based on the following scoring functions:

- non domination degree

$$\mu_{ND+}(a,\pi) = \min_{x\in A}[1 - P(x,a)] = \min_{x\in A} P^d(a,x),$$

  where $P^d(\cdot,\cdot)$ means "dual"of $P(\cdot,\cdot)$;
- non dominance degree

$$\mu_{ND-}(a,\pi) = \min_{x\in A}[1 - P(a,x)] = 1 - \max_{x\in A} P(a,x).$$

Let $A^{UND+} = \{a \in A : \mu_{ND+}(a,\pi) = 1\}$ (i.e. the subset of non-dominated actions from A) and $A^{UND-} = \{a \in A : \mu_{ND-}(\pi,a) = 1\}$ (i.e. the subset of non-dominating actions from A). Clearly, best action(s) will belong to set A^{UND+} and worst action(s) to set A^{UND-}. We observe that, if relation $\pi(a,b)$ is transitive, A^{UND+} and A^{UND-} are non empty.

7.3.2 PRAGMA

The Preference RAnking Global frequencies in Multicriteria Analysis (PRAGMA) [27–29] method is based on the peculiar PCCA aggregation logic (that is firstly on pairwise comparisons by means of couples of distinct criteria, and then on the aggregation of these partial results), and use the same data input and preferential information of MAPPAC, of which it constitutes a useful complement and presents the same flexibility in preference modeling. Moreover, it instrumentally uses the MAPPAC basic preferences indices to compute its specific information to support DM in his/her decision problem at hand. From the methodological point of view, PRAGMA is neither a classical outranking neither a MAUT method. In fact, the output of this approach are not binary outranking relations or scores. But, following the aggregation procedure of PCCA, in the first and in the second phase partial and global ranking frequencies are respectively built, one for each feasible action, and these frequencies are then exploited to give DM a useful recommendation (partial or complete preorders are the final output).

Partial and Global Frequencies Let the segment H_iH_j (see Fig. 7.10) be the *partial profile* of action $a_h \in A$, where the points H_i and H_j have as ordinates the weighted normalized evaluations of action a_h with respect to criteria g_i and g_j respectively, $g_i, g_j \in F$.

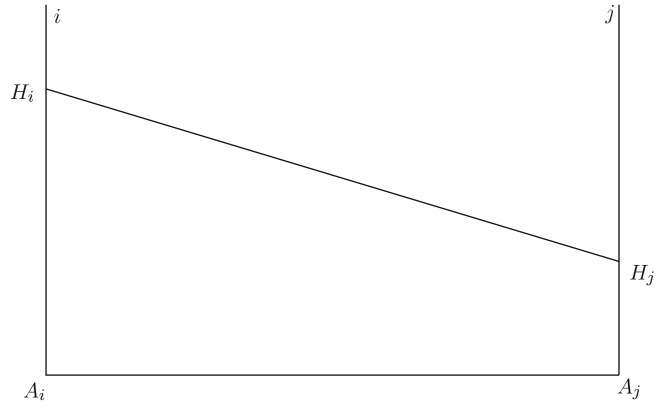


Fig. 7.10 Partial profile of action a_h

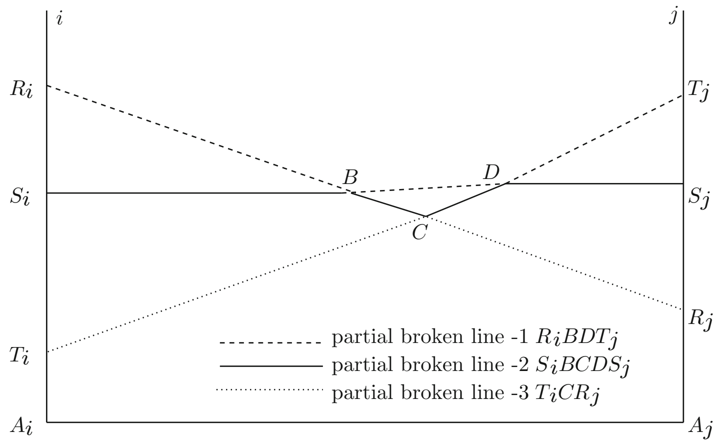


Fig. 7.11 Partial profiles and partial broken lines of a_r, a_s, a_t

Considering all couples of criteria, it is possible to obtain $\binom{m}{2}$ distinct partial profiles of a_h and we call *global profile* of a_h the set of these $\binom{m}{2}$ partial profiles.

We define as *partial broken line-k*, or partial broken line of level k of a_h, $k = 1, 2 \cdots, n$ the set of consecutive segments of its partial profiles, to which correspond, for each point, $k - 1$ partial profiles (distinct or coinciding) of greater ordinate. If, for example, it is $A = \{a_r, a_s, a_t\}$, we obtain the partial profiles and partial broken lines represented in Fig. 7.11.

We observe that the partial broken line-k, $k = 1, 2 \cdots, n$, coincides with the partial profiles of $a_h, a_h \in A$, if and only if a_h is partially dominated by d actions and dominates the remaining ones and/or if p couples of actions from A $(0 \leq p \leq k-1, d+p=k-1)$ exist such that, for each couple, their partial profiles come from opposite sides with respect to profile of a_h, and they intersect this profile at the same point.

Further, we define as *global broken line-k* or global broken lines of level k $(k = 1, 2, \cdots, n)$ the set of $\binom{m}{2}$ partial broken lines-k obtained by considering all the couples of distinct criteria $g_i, g_j \in G$. The global broken line-k coincides with the global profiles of a_h if and only if all the partial broken lines of level k, obtained by considering each of the $\binom{m}{2}$ couples of criteria, coincide with the corresponding partial profiles of a_h.

We define as the *partial frequency* of level k $(k = 1, 2 \cdots, n)$ of a_h, with reference to the criteria g_i and g_j, the value of the orthogonal projection on the straight line A_iA_j (given $\overline{A_iA_j} = 1$) of the intersection of the partial profile of a_h with the corresponding partial broken line of level k. If we indicate this frequency as $f_{ij}^k(a_h)$, it will be $0 \leq f_{ij}^k(a_h) \leq 1$, for all $a_h \in A$, $k = 1, 2 \cdots, n$. Thus, for example, from the graphics in Fig. 7.12.

$$\begin{aligned}
\overline{A_iA_j} &= 1; \quad \overline{A_iB} = 0.3; \quad \overline{BC} = 0.1;\\
\overline{CD} &= 0.2; \quad \overline{DA_j} = 0.4;\\
f_{ij}^{(1)}(a_r) &= 0.3; f_{ij}^{(2)}(a_r) = 0.1; f_{ij}^{(3)}(a_r) = 0.6;\\
f^{(1)})_{ij}(a_s) &= 0.3; f_{ij}^{(2)}(a_s) = 0.7; f_{ij}^{(3)}(a_s) = 0\\
f_{ij}^{(1)}(a_t) &= 0.4; f_{ij}^{(2)}(a_t) = 0.2; f_{ij}^{(3)}(a_t) = 0.4
\end{aligned}$$

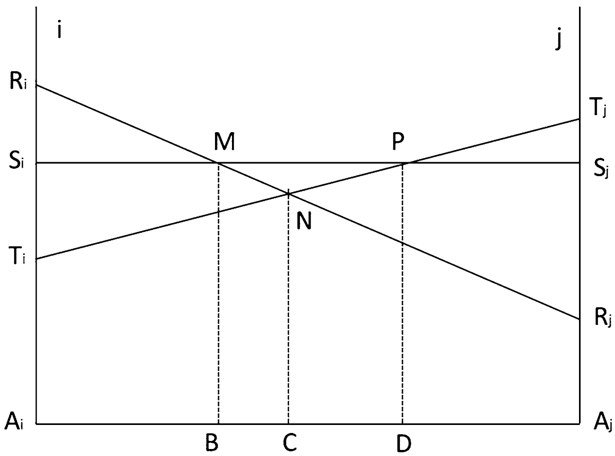


Fig. 7.12 Partial frequencies of a_r, a_s, a_t

The partial frequencies may be represented in matrix form, obtaining $\binom{m}{2}$ square $n \times n$ matrices $\mathbf{F_{ij}}$, which is the matrix of the partial ranking frequencies:

$$\mathbf{F_{ij}} = [f_{ij}^{(k)}(a_h)], a_h \in A;\ k = 1, 2 \cdots, n;\ i, j \in I. \tag{7.8}$$

The elements of the hth line of matrix (7.8) indicate in order the fractions of the interval unitary (A_i, A_j) for which the action a_h is in the kth position $(k = 1, 2 \cdots, n)$, while the elements of the kth column of the same matrix indicate those fractions for which the kth position (in the partial preference ranking considered) is assigned to the actions $a_1, a_2, \cdots, a_n$, respectively. Obviously:

$$\sum_{k=1}^{n} f_{ij}^{(k)}(a_h) = 1,\ \forall a_h \in A \text{ and } \sum_{h=1}^{n} f_{ij}^{(k)}(a_h) = 1\,, k = 1, 2, \cdots n.$$

If $f_{ij}^{(k)}(a_h) \in \{0, 1\}$, for all $a_h \in A$ and $k = 1, 2 \cdots, n$, the partial profiles of all the actions will be non intersecting and non-coinciding, and there will be no inversions with respect to the preference relation in the two complete preference preorders with respect to the criteria g_i and g_j, i.e. all actions from A partially dominate one another.

If v $(v = 2, 3 \cdots, n)$ partial profiles are coinciding, the corresponding partial broken lines-k must be built taking distinctly into account the coinciding profiles v times (see [28]).

Let us then define *global frequency of level k* , $(k = 1, 2 \cdots, n)$ of a_h as the weighted arithmetical mean of all the $\binom{m}{2}$ partial frequencies of level k of a_h, obtained by considering all the couples of distinct criteria g_i and g_j. Therefore, indicating this frequency by $f^{(k)}(a_h)$, we obtain, if no interaction between criteria is considered (see Sect. 7.4.1):

$$f^{(k)}(a_h) = \sum_{(i<j)i,j} f_{ij}^{(k)}(a_h) \frac{\lambda_i + \lambda_j}{m-1},\ a_h \in A,\ k = 1, 2 \cdots, n.$$

The linear combination of the matrices (7.8) with weights $\frac{\lambda_i+\lambda_j}{m-1}$ will therefore give the square $n \times n$ matrix $\mathbf{F} = [f^k(a_h)]$ $(h = 1, 2, \cdots, n;\ k = 1, 2, \cdots, n)$, called the global ranking frequency matrix. Its generic element $f^k(a_h)$ indicates the relative frequency with which $a_h \in A$ is present in the kth position $(k = 1, 2, \cdots, n)$ in the ranking obtained by considering all the criteria $g_j \in F$ and the global profiles of all the feasible actions. It will therefore be:

$$\sum_{k=1}^{n} f^{(k)}(a_h) = 1, \forall a_h \in A \text{ and } \sum_{h=1}^{n} f^{(k)}(a_h) = 1, k = 1, 2, \cdots, n.$$

It is possible to calculate the partial frequencies $f_{ij}^{(k)}(a_h)$ by means of an algorithm which uses the indices $\pi_{ij}(a_h, a_k)$ of the MAPPAC method (see [28]). It is therefore possible to consider marginal indifference thresholds and suitable indifference areas also when the PRAGMA method is implemented. In other words, the indices $\pi_{ij}(a_h, a_k)$ here instrumentally introduced, may be calculated in advance by using all the techniques and the preference modeling flexibility adopted with reference to the MAPPAC method (see Sect. 7.3.1).

Apart from these calculations, it is useful in any case to remember among others some particular features of the ranking frequencies obtained by the PRAGMA method:

1. The partial frequencies (and therefore also the global ones) of $a_h \in A$ are functions of the value of the normalized weighted differences between the evaluations of a_h and those of the remaining feasible actions with respect to the criteria considered. The values of these weighted differences may be overlooked only in the case of partial dominance (for partial frequencies) or strict dominance (for global frequencies), active or passive, of the action a_h.
2. If a_h partially dominates $n - k$ actions and it is partially dominated by the remaining $k - 1$ actions, $k = 1, 2 \cdots n$ the result is $f_{ij}^{(k)}(a_h) = 1$, whatever the values λ_i and λ_j.
3. If a_h strictly dominates $n - k$ actions and is strictly dominated by the remaining $k - 1$ actions, $k = 1, 2 \cdots, n$, the result is $f^{(k)}(a_h) = 1$, whatever the values of the weights $\lambda_j, j \in \mathcal{J}$.
4. If $f^{(k)}(a_h) = 1$, the action a_h occupies the kth position, $k = 1, 2 \cdots, n$, in every monocriterion ranking and a_h is preceded and followed by the same subset of actions in these rankings.

Therefore, the information obtained by means of analysis of the global frequencies $f^{(k)}(a_h)$ is more complete and more accurate than that obtained from an examination of all the distinct monocriterion rankings of the feasible actions, or from a mixture of these.

Exploitation and Recommendation In order to support DM in the decision problem at hand, it is often sufficient to analyze the elements of matrices $\mathbf{F_{ij}}$ and/or $\mathbf{F}$. For example, a straightforward reading of the global frequencies of matrix $\mathbf{F}$ could indicate which action(s) will finally be chosen. But the concise and accurate information regarding the frequencies of ranks each action may occupy can be extremely useful to build up final rankings.

If we want to obtain complete or partial rankings of the feasible actions in order to build up comprehensive evaluations and recommendations, it is possible, for example, to proceed in this way. Calculate for each action $a_h \in A$, the *accumulated frequencies of order* k, $k = 1, 2 \cdots, n$, summing the first k elements of the hth row of matrix $\mathbf{F}$, that is:

$$F^{(1)}(a_h) = f^{(1)}(a_h) \text{ and } F^{(k)}(a_h) = \sum_{i=1}^{k} f^{(i)}(a_h), k = 2, 3 \cdots, n.$$

Then establish the order q ($q = 1, 2, \cdots, n-1$) of the frequencies which are considered relevant to the building of the ranking, that is indicate to what order q and the respective importance α_k we intend to take into consideration the accumulated frequencies $F^k(a_h)$ for this purpose. The following comprehensive index is then built:

$$S^q(a_h) = \sum_{k=1}^{q} \alpha_k F^{(k)}(a_h),\ a_h \in A; 1 \geq \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_q > 0. \tag{7.9}$$

This gives the measure of the "strength" with which a_h occupies the first q positions in the aggregated ranking. This in practice will be $1 \leq q \leq \frac{n}{2}$, which regards the first positions in the ranking; the coefficients α_k indicate the relative importance (not increasing with k) of accumulated frequence of order k. In the first class C_1 of the decreasing ranking will be placed the action(s) to which the maximum value of $S^{(q)}(a_h)$ corresponds. In order to avoid ex aequo rankings, we can proceed by selecting whichever actions have obtained an equal value of $S^{(q)}$ on the basis of the values of the indices $S^{(q+1)}$ and, in the case of further equality, on those of the indices $S^{(q+2)}$ and so on. In this case ex aequo actions would be accepted only if their corresponding indices $S^{(i)}$ proved equal for $i = q, q+1, \cdots, n$. If, on the other hand, we desire to prevent small differences in the indices $S^{(q)}$ from having a discriminatory role in the building of the rankings, it is possible to consider global indifference thresholds (see [27]).

If we place $\alpha_k = 1$, for all k, in Eq. (7.9), we do not emphasize the greater importance of the global ranking frequencies of the first positions. On the other hand, if we accept $q = 1$, we take into account only the global frequencies of the first position for the purpose of building the rankings.

After building class C_1, with reference to the subset of the remaining actions $A^{(1)} = A - C_1$, we calculate again the partial, global and accumulated frequencies and the index (7.9), proceeding as above in order to build class C_2, and so on. We observe that at each iteration t the order q_t, on the basis of the which the index $S^{(q_t)}$ from (7.9) is to be calculated, must be restated so that it is a non increasing whole number and, taking into account the number $|A^{(t)}|$ of actions of the evaluation set, so that at each iteration t the ratio $\frac{q_t}{|A^{(t)}|}$ is as near as possible to the ratio $\frac{q}{n}$ of the first iteration (see [28]). In general, the rankings obtained are a function of the value of the order q originally selected (see [27]).

If at each useful iteration $F^{(k)}(a_r) \geq F^{(k)}(a_s)$ for all $k = 1, 2, \ldots, n$ and $F^{(k)}(a_r) > F^{(k)}(a_s)$ for some k, or if $\sum_{k=1}^{t}[F^{(k)}(a_r) - F^{(k)}(a_s)] \geq 0$ for all $t = 1, 2, \cdots, n$ and $F^{(k)}(a_r) \neq F^{(k)}(a_s)$ for some k, it is possible to speak of first degree or second degree *frequency dominance*, respectively, of a_r over a_s. In both cases, if $\alpha_1 > \alpha_2 > \cdots > \alpha_q$, a_r will precede a_s in any of the rankings obtained, whatever value may be chosen for q.

Besides the partition of the actions of A into equivalence classes (complete preorder) obtained with the descending procedure (or procedure *from above*) described, it is also possible to build another complete preorder in the same way using the ascending procedure (or procedure *from below*), that is selecting the action(s) to be placed in the last, next to last, $\cdots$ and finally in the first equivalence class.

In conclusion, it is possible to build a final ranking (partial preorder) of the feasible actions, as the intersections of the two decreasing and increasing rankings obtained by means of two separate procedures described. Using the PRAGMA method for the building of rankings, it is possible not only to establish any implicit incomparability deriving from the inversion of preferences in the preorders obtained by means of the two separate procedures, but also in this case it is possible to consider an explicit incomparability, obtained if the corresponding tests give a positive result, during the preference modeling phase. Since, as we have said, the PRAGMA method makes instrumental use of the basic preference indices, it is possible to use once again the same discordance indices already introduced in the MAPPAC method (see Sect. 7.3.1).

Besides these, moreover, it is also possible to consider other analogous discordance indices peculiar to the PRAGMA method, that is using the partial and global ranking frequencies. Thus, for example, with respect to a couple or all criteria simultaneously, a strengthening of the ranking frequencies of an action a_h, respectively partial or global, corresponding to the first and last positions in the ranking, can reveal strongly discordant evaluations of a_h by means of those criteria. Therefore this kind of situation, suitably analyzed, could lead the DM to reconsider the nature of a_h; since actually, in the building phase of the rankings, this situation may lead to a rapid choice of a_h both in the descending and in the ascending procedure, resulting in situations of conflictuality and implicit incomparability.

Software M&P (MAPPAC and PRAGMA) is a software to rank alternatives using the methods previously described. It presents a lot of options in order to be very flexible in the preference modeling, according to the PCCA philosophy. After loading or writing a file concerning the decisional problem at hand, in the Edit menu it is possible to set all the parameters required to compute the basic and global preference indices or ranking frequencies, i.e. trade-off and importance weights etc.. Some classical statistical analyses on the alternatives evaluations are also allowed (average values, standard deviations, correlations between criteria). The indifference areas can be performed in the Calculation menu. For each couple of criteria, suitable indifference thresholds and shapes can be defined. This option results in some non punctual indifference relations, that can also be seen on useful graphics, showing the indifference area and each pair of alternatives in the chosen plane Og_ig_j, g_i,$g_j \in F$. It is also possible to graphically represent the partial and global profiles and levels of the considered alternatives. Going to Solutions menu, after setting other optional parameters, we can firstly obtaining the (partial and global) preference matrices (MAPPAC) and frequencies matrices (PRAGMA); then, exploiting these data, the descending and ascending complete preorders and the final (partial) preorder (as their intersection) can be built up, respectively for MAPPAC and PRAGMA methods. On interesting geometrical interpretation on omometric axes of the complete preorders computation procedure, expresses with respect to each iteration the different rankings with the corresponding global preference intensities of the alternatives considered each time. This representation shows eventual inversion of preferences (as intersection of the corresponding straight lines) due to the presence of some strong dominance effect. Finally, it is possible to perform a suitable Conflict analysis among the alternatives, by setting the parameters needed to compute the bicriteria discordance indices and the incomparability relations, each time according to the corresponding compensation level established by the DM. The indifference and incomparability relations are also suitably presented in a geometrical way in the bicriteria planes Og_ig_j, for each $g_i, g_j \in F$, where the pairs of action are represented using different colours for different binary relation.

7.3.3 IDRA

A new MCDA methodology in the framework of PCCA was presented by Greco [12] in IDRA (Intercriteria Decision Rule Approach). Its main (and original) features are: to use mixed utility function (i.e. in the decision process both trade-off and importance intercriteria information are considered) and to allow bounded consistency, i.e. no hard constraint is imposed to the satisfaction of some axiomatic assumptions concerning intercriteria information obtained by DM. With respect to the last point, in a MCDA perspective two different kinds of coherence should be considered: the judgemental and the methodological. The first one concerns the intercriteria information supplied by DM and there is no room for technical judgement with respect to its internal coherence. The second one is related to the exploitation of intercriteria information in order to obtain the final recommendation and a coherence judgment based on some MCDA principles and axioms is allowed. Therefore, according to the judgemental coherence principle, within the IDRA method DM is allowed to give both trade-off and importance intercriteria information, without checking its not imposed coherence.

Let $g_j : A \to \mathbb{R}$, $\forall j \in \mathcal{J}$, an interval scale of measurement; a normalized value c_{hj} of $g_j(a_h)$, $a_h \in A$, can be obtained by introducing two suitable parameters $a(j)$, a minimum aspiration level, and $b(j)$, a maximum aspiration level, for each criterion $g_j \in F$, with $a(j) \leq \min g_j(x)$ and $b(j) \geq \max g_j(x)$, by defining

$$c_{hj} = \begin{cases} \frac{g_j(a_h)-(a_j)}{b(j)-a(j)} & \text{if } a(j) < b(j), \\ 0 & \text{if } a(j) = b(j). \end{cases}$$

In IDRA, as above emphasized, the compensatory approach and the noncompensatory approach are complementary, rather than alternative, aggregation procedures, following the line coming out from some well known experiments carried out by

Slovic [41] and others. The basic idea within IDRA is that matching (i.e. comparing two actions by making the action that is superior on one criterion to be so inferior in the other one that the previous advantage is canceled) is not a decision problem: it is rather a questioning procedure for obtaining the intercriteria information called, trade-off. On the contrary, choosing among equated (by matching) packing of actions is a typical decision problems, as ranking and sorting. Therefore, if this assumption is accepted, in each decision problem, like choice, there are two different types of intercriteria information: trade-off, which can be derived from a matching, and importance weights, linked to the intrinsic importance of each subset (also a singleton) of criteria from F.

As a consequence, there is only one utility function U^M, called mixed (see [12]), because both trade-off (α_j) and importance (λ_j) weights are considered, $j \in \mathcal{J}$; thus for each $a_h \in A$:

$$U^M(a_h) = \sum_{j=1}^{m} \lambda_j \alpha_j g_j(a_h).$$

The bounded consistency hypothesis:

- for trade-off weights, $w_{ik} w_{kj} = w_{ij}, i, j, k \in \mathcal{J}$, where, in general, w_{pq} is the tradeoff between the criteria g_p and g_q;
- for importance-weights, given $G_1, G_2 \subset F$,
  - if G_1 is more important than G_2, then $\sum_{g_j \in G_1} \lambda_j > \sum_{g_j \in G_2} \lambda_j$;
  - if G_2 is more important than G_1, then $\sum_{g_j \in G_1} \lambda_j < \sum_{g_j \in G_2} \lambda_j$;
  - if G_1 and G_2 are equally important, then $\sum_{g_j \in G_1} \lambda_j = \sum_{g_j \in G_2} \lambda_j$.

Very often these requirements are not satisfied by the answers given by the DM and the DM is said "incoherent". But, as remarked by Greco [12], most of these "inconsistencies" derive from the attempt to use information relative to partial comparisons (i.e. with respect to only some criteria from F) for global comparisons (i.e. where all the criteria from F are considered). In IDRA, the hypothesis of bounded consistency means that the information obtained from DM with respect to some criteria from F must be used only for comparisons with respect to the same criteria, according to the principle of judgemental coherence. Therefore, every above problem of intercriteria information consistency is "dissolved" in its origin. In IDRA the framework of PCCA is used to implement the bounded consistency hypothesis, considering therefore a couple of criteria at a time. We observe that, in particular, no requirement of completeness of the relations "more important than" and "equally important to" is assumed. As a consequence, for any couple of distinct criteria $g_i, g_j \in F$, one of the following intercriteria information can be obtained by the DM:

1. both the trade-off and the judgment about the relative importance of the criteria;
2. only the trade-off;
3. only the judgment about the relative importance of the criteria;
4. neither the trade-off nor the judgment about the relative importance of the criteria.

Using this information, a basic preference index $\pi^*_{ij}(a, b)$ can be suitably defined (see [12]). The index $\pi^*_{ij} : A \times A \rightarrow [0, 1]$ is the image of a valued binary relation, complete and ipsodual, and constitutes a complete valued preference structure (complete preorder) on set A. The index $\pi^*_{ij}(a, b)$ can be interpreted as the probability that a is preferred to b, with respect to a mixed utility function in which the trade-off and importance weights are randomly chosen in the set of intercriteria information furnished by the DM. In IDRA, each piece of intercriteria information concerning the trade-off or the relative importance of criteria can be considered a "decision rule" (tradeoff-rule or importance-rule respectively), since it constitutes a basis for an argumentation about the preference between the potential actions. The DM is asked to give a non negative credibility-weight to each decision rule, according to his/her judgment about the relevance of the corresponding pairwise criterion comparisons in order to establish a global preference [12]. Therefore, from the sum of the basic indices $\pi^*_{ij}(a, b)$, with respect all the considered couple of criteria, weighted by the correspondent credibility-weights for the tradeoff-rule or the importance-rule, the aggregated index $\pi(a, b)$ is obtained, for each $a, b \in A$. All these aggregated indices can be then exploited using the same procedure proposed for MAPPAC in order to obtain two complete preorders (decreasing and increasing solutions); the intersection of these two rankings gives the final ranking (partial preorder). The aggregated index of IDRA mainly differs from the analogous index of MAPPAC in this point: in MAPPAC all (i.e. with respect to each couple of criteria from F) basic indices are aggregated, while in IDRA only the elementary indices corresponding to couples of criteria about which the DM has given decision rules are aggregated (faithfulness principle). In IDRA there is a peculiar characteristic distinction between:

1. intercriteria information which is not supplied by the DM (i.e. the DM does not say anything about the relative importance between g_i and g_j);
2. intercriteria information by which the DM expresses his/her incapacity to say what is the trade-off or the relative importance between g_i and g_j (i.e. the DM says that he/she is not able to give this information).

In IDRA, in case **1.** the comparison with respect to criteria g_i and g_j plays no part; in case **2.** the same comparison contributes to the aggregated index by means of considering the corresponding basic index calculated taking into account all the possible importance-weights as equally probable, according to the "principle of insufficient reason" (so called Laplace criterion in the case of decision making under uncertainty).

7.3.4 PACMAN

A new DM-oriented approach to the concept of compensation in multicriteria analysis has been introduced by Giarlotta [10, 11] in PACMAN (Passive and Active Compensability Multicriteria ANalysis). The main feature of this approach is that the notion of compensability is analyzed by taking into consideration two criteria at a time, and distinguishing the compensating (or active) criterion from the compensated (or passive) one. Separating active and passive effects of compensation allows one to (1) point out a possible asymmetry of the notion of compensability, and (2) introduce a suitable valued binary relation of compensated preference.

The concept of compensation has been widely studied in many papers [40, 43, 44]. The literature on this topic is mainly concentrated on the analysis of decision methodologies, aggregation procedures and preference structures on the basis of this concept. Therefore definition and usage of compensation have essentiality been *method-oriented*, since this concept has been regarded as a theoretical device of classification.

On the contrary, the notion of compensation examined in PACMAN, namely *compensability*, is aimed at capturing the behavior of a decision maker towards the possibility to compensate among criteria. In fact, this approach, intercriteria compensability remains somehow "the possibility that an advantage on one criterion can offset a disadvantage on another one", but as it is determined by a DM and not by a method. Therefore, being more or less compensatory is not regarded here as the characteristic of a multicriteria methodology or of an aggregation procedure. Instead, it is an intrinsic feature of a DM. In this sense, a *DM-oriented* usage of the concept of compensation is introduced.

There are three steps in PACMAN:

- *compensability analysis*, the procedure aimed at modeling intercriteria relations by means of compensability;
- evaluation of the degree of active and passive preference of an alternative over another one by the construction (at several levels of aggregation) of *binary indices*;
- determination of a binary relation of strict preference, weak preference, indifference or incomparability for each couple of alternatives, on the basis of two valued relations of *compensated preference*.

At each step of the procedure PACMAN requires a strict interaction between the actors of the decision process. Therefore, also this approach allows application of the principles of faithfulness (to the information provided by DM), transparency (at each stage of the procedure) and flexibility (in preference modelization).

Compensability Analysis Let $g_j\colon A \to \mathbb{R}$ be an interval scale of measurement, representing the j-th criterion according to a non decreasing preference. For each $j \in \mathcal{J}$, let $\Delta_j\colon A \times A \to \mathbb{R}$ be the normalized difference function, defined by $\Delta_j(a,b) = (g_j(a) - g_j(b))/(\beta_j - \alpha_j)$, where α_j and β_j ($\alpha_j < \beta_j$) are respectively the minimum and the maximum value that can be assumed on $j \in \mathcal{J}$.

The aim of compensability analysis is to translate into numerical form the definition of bicriteria compensability for each pair of criteria. This is done by constructing, for each pair (i, j) of criteria, the compensatory function $CF_{i\rhd j}$ of i over j, which evaluates the compensating effect of a positive normalized difference Δ_i on the active criterion i over a negative normalized difference Δ_j on the passive criterion j.

Since a proper and complete estimation of the compensatory effect for every possible active and passive difference is too demanding in terms of amount and preciseness of the related information provided by the DM, a *fuzzy* function $CF_{i\rhd j}$ is built. This function associates to any pair of normalized differences $(\Delta_i, \Delta_j) \in]0,1] \times [-1,0[$ a number in $[0,1]$, which quantifies the degree of confidence that the positive difference Δ_i totally compensates the negative differences Δ_j. Extending the function in frontier by continuity, it is obtained a fuzzy compensatory function $CF_{i\rhd j}\colon [0,1] \times [-1,0] \to [0,1]$, which satisfies the following conditions:

Weak monotonicities

$$0 \le \Delta_{i_1} \le \Delta_{i_2} \le 1 \text{ and } -1 \le \Delta_j \le 0 \Rightarrow CF_{i\rhd j}(\Delta_{i_1}, \Delta_j) \le CF_{i\rhd j}(\Delta_{i_2}, \Delta_j),$$
$$0 \le \Delta_i \le 1 \text{ and } -1 \le \Delta_{j_1} \le \Delta_{j_2} \le 0 \Rightarrow CF_{i\rhd j}(\Delta_i, \Delta_{j_1},) \le CF_{i\rhd j}(\Delta_i, \Delta_{j_2}).$$

Continuity $CF_{i\rhd j}$ is continuous everywhere on $[0,1] \times [-1,0]$.

The rationale underlying this kind of fuzzy modeling is to minimize the amount of information required from the DM, without losing too much in content. The two conditions stated above are very helpful in this sense. In fact, in order to assess a compensatory function, the DM is asked to determine just the zones where the *degree of confidence* expressed by $CF_{i\rhd j}$ is *maximum* (usually equal to one) and *minimum* (usually equal to zero). Using monotonicity and continuity, it is possible to extend by linearization its definition to the whole domain $[0,1] \times [-1,0]$, without any further information. Some examples of compensatory functions are given in Fig. 7.13. By definition, $CF_{i\rhd i} \equiv 0$ for each $i \in \mathcal{J}$.

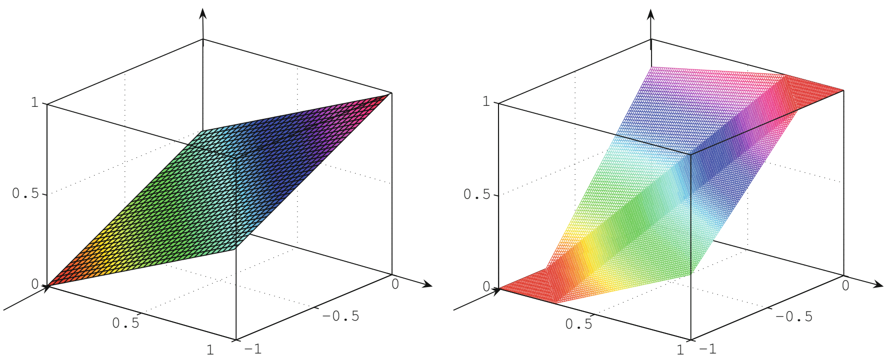


Fig. 7.13 Some examples of compensatory functions

The procedure for the construction of compensatory functions aims at simplifying the DM's task in providing meaningful information. On the other hand, this procedure requires the DM to provide a large amount of information. In fact, according to the PCCA philosophy, we estimate intercriteria compensability for each couple of criteria. Moreover, we still distinguish their compensatory reaction within the couple, according to whether they effect or endure compensation. This results in the necessity of assessing a compensatory function for each *ordered* pair of distinct criteria.

However, the large amount of information required by PACMAN allows one to model the relationships between each couple of criteria in a rather faithful and flexible way, according to the PCCA philosophy. Usually, an important criterion is relevant both actively (i.e., contributing to preference) and passively (i.e., opposing to preference). Therefore for each criterion passive resistance and active contribution are treated separately, being linked to the notions of, respectively, preference threshold and veto threshold in the outranking approach [40]. For a detailed description of the procedure used to construct compensatory functions, see [11] (Fig. 7.13).

Preference Modeling In PACMAN preferences are modelled on the basis of compensability analysis. This is accomplished in steps (2) and (3) of the procedure.

(2): Let $g_j \in G^+(a,b)$, i.e., $\Delta_j(a,b) > 0$. The positive difference $\Delta_j(a,b)$ has a double effect:

- active, because it gives some contribution to the (possible) overall preference of a over b (accept this global preference);
- passive, because it states a resistance to the (possible) overall preference of b over a (reject this global preference).

Active contribution and passive resistance of a over b are evaluated for each $g_j \in G^+(a,b)$, computing the *partial indices* $\Pi_j^+(a,b)$ and $\Pi_j^-(a,b)$, respectively. Successively, active and passive effects are separately aggregated, thus obtaining an evaluation of the total strength of the arguments in favour of a preference of a over b, and of those against a preference of b over a, respectively. Numerically, this is done by computing the two binary *global indices* $\Pi^+(a,b)$ and $\Pi^-(a,b)$. Clearly, the same evaluations are done for the pair (b,a), first computing the partial indices $\Pi_j^+(b,a)$ and $\Pi_j^-(b,a)$, and then the global indices $\Pi^+(b,a)$ and $\Pi^-(b,a)$.

The final output of this stage is a pair of *global net indices* $\Pi(a,b)$ and $\Pi(b,a)$ for each couple of alternatives $a,b \in A$. These indices express the degree of *compensated preference* of a over b and b over a, respectively. The index $\Pi(a,b)$ is obtained from the values of the indices $\Pi^+(a,b)$ and $\Pi^-(b,a)$; similarly, the index $\Pi(a,b)$ is obtained from the values of the indices $\Pi^+(b,a)$ and $\Pi^-(a,b)$. A formalization of the whole procedure can be found in [10].

(3) The last step of PACMAN is the construction of a fundamental system of preferences (P, Q, I, R). The relation between the alternatives a and b is determined from the values of the two global net indices $\Pi(a,b)$ and $\Pi(b,a)$ (see Fig. 7.14).

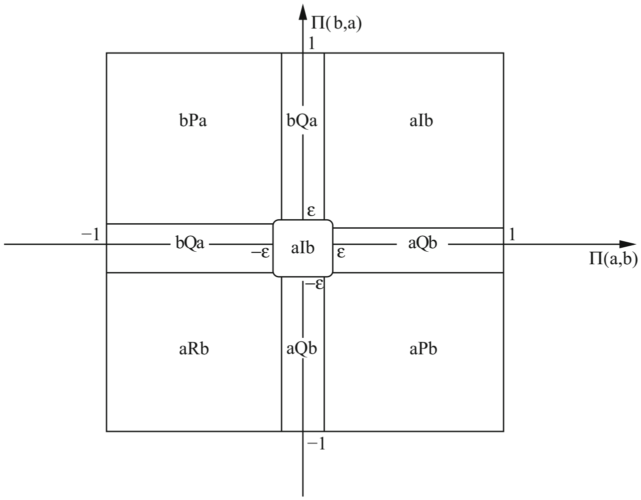


Fig. 7.14 Determination of a relation between the two alternatives $a,b \in A$ on the basis of the values of global indices

One of the main interesting features of PACMAN is that intercriteria compensability can be modelled with respect to the real scenarios, treating each pair of criteria in a peculiar way. Complexity and length of the related decision process is the price to pay for the attempt to satisfy the principles of faithfulness, transparency and flexibility.

Implementations of PACMAN Recently, a formal notion of implementation [3] has been provided for PACMAN. This general notion is designed with the aim of allowing the decision aider a more transparent interaction with the decision maker. The so-called regular implementations and their monotonicity properties are examined. Particular emphasis is given to those regular implementations of PACMAN which produce the lexicographic ordering as an output. This analysis sheds some light on the underlying philosophy of PACMAN.

A Linear Implementation of PACMAN In a more recent paper [4], a simplified implementation of PACMAN is proposed. This implementation consistently reduces the overall complexity of the methodology by employing the so-called linear compensatory functions.

Using Mathematica®, it is also developed a computer-aided graphical interface that eases the interaction among the actors of the decision process at each stage of PACMAN. Furthermore, this simplified version of PACMAN allows one to perform a sensitivity analysis in the form of a nonlinear optimization problem.

7.4 One Outranking Method for Stochastic Data

It frequently happens that we have to treat a decision context in which the performance of the alternatives according to each criterion/attribute is subject to various forms of imperfection of the available data. The form of imperfection that interests us here concerns the uncertainty, in the sense of probability (statistic or stochastic data). For example, frequently the decision maker calls upon several experts in order to obtain judgments which then forms the basic data. Since each alternative is not necessarily evaluated at the same level of anticipated performance by all experts, each combination of 'alternative-criterion' leads to a distribution of expert's evaluation. This type of distributional evaluation is considered as stochastic data.

Even if the multi-criteria analysis with stochastic data has so far been treated nearly exclusively in the theory of the multi-attributes utility framework, the outranking synthesis approach can be constituted an appropriate alternative. Some multi-criteria aggregation procedures belonging to this second approach have been developed specially to treat stochastic data. For example, we can mention the works by [9, 22–25, 48]. The majority of these methods construct outranking relations as in ELECTRE or PROMETHEE. In this chapter we have choose to present the Martel and Zaras' method that makes a link between the multi-attributes utility framework and the outranking approach.

7.4.1 Martel and Zaras' Method

We consider a multi-criteria problem which can be represented by the (A. A. E.) model (Alternatives, Attributes/Criteria, Evaluators). The elements of this model are as follows:

- $A = \{a_1, a_2, \ldots, a_m\}$ representing the set of all potential alternatives;
- $F = \{X_1, X_2, \ldots, X_n\}$ representing the set of attributes/criteria, any attribute X_j defined in the interval $[x_j^0, x_j^1]$, where x_j^0 is the worst value obtained with the attribute X_j and x_j^1 is the best value; $E = \{f_1, f_2, \cdots, f_n\}$ the set of evaluators, an evaluator $f_j(x_{ij})$ being a probability function associating to each alternative a_i a non-empty set of x_{ij} (a random variable) representing the evaluation of a_i relative to the attribute X_j.

In this method, it is assumed known the distributional evaluation of the alternatives according to each attribute and the weight of the attributes.

These attributes (criteria) are defined such that a larger value is preferred to a small value and that the probability functions are known. It is also assume that the attribute set F obeys the additive independence condition. Huang, Kira and Vertinsky (see [17]) showed in the case of the probability independence and the additive multi-attributes utility function, that the necessary condition for the multi-attributes stochastic dominance is to verify stochastic dominance on the level of each attribute. In practice, the essential characteristic of a multi-attributes problem is that the attributes are conflicting. Consequently, the Multi-attributes Stochastic Dominance relation results poor and useless to the DM. It seems to be reasonable to weaken this unanimity condition and accept a majority attribute condition.

Thus, Martel and Zaras' method [24] uses the stochastic dominance to compare the alternatives two by two, on each attribute. These comparisons are interpreted in terms of partial preferences. Next, the outranking approach is used for constructing outranking relations based on a concordance index and eventually on a discordance index. With this approach, a majority attribute condition (concordance test) replaces the unanimity condition of the classic dominance. Finally, these outranking relations are used in order to construct the prescription according to a specific problem statement.

Often, in order to conclude that alternative a_i is preferred or is at least as good as $a_{i'}$, with respect to the attribute X_j, it is unnecessary to make completely explicit all the decision-maker's partial preferences. In fact, it can be possible to conclude on the basis of stochastic dominance conditions of first, second and third order (i.e. FSD, SSD and TSD relations), for a class of increasing concave utility functions with decreasing absolute risk aversion (i.e. DARA utility functions class). If the decision-maker's (partial) preference for each attribute X_j can be related by the utility function $U_j \in$ DARA, then his preference for the $F_j(x_{ij})$ distribution associated with alternative a_i for each attribute X_j will be:

$$g_j(F_j(x_{ij})) = \int_{x_j^0}^{x_j^1} U_j(x_{ij})\, dF_j(x_{ij}).$$

Theorem 1 **[14].** *If $F_j(x_{ij})$ FSD $F_j(x_{i'j})$ or $F_j(x_{ij})$ SSD $F_j(x_{i'j})$ or $F_j(x_{ij})$ TSD $F_j(x_{i'j})$ and $F_j(x_{ij}) \geq F_j(x_{i'j})$, then $g_j(F_j(x_{ij})) \leq g_j(F_j(x_{i'j}))$ for all $U_j \in$ DARA, where $F_j(x_{ij})$ and $F_j(x_{i'j})$ represent cumulative distribution functions associated with a_i and $a_{i'}$ respectively.*

This theorem allows to conclude clearly that a_i is preferred to $a_{i'}$, with respect to the attribute X_j. We refer the reader to Zaras (see [47]) to review the concept of stochastic dominance.

In the MZ's model, two situations are identified; **clear situation**, where the conditions imposed by the theorem are verified (SD=FSD $\cup$ SDU $\cup$ TSD situations), and **unclear situation**, where none of the three stochastic dominance is verified. The value of the concordance index can be decomposed into two parts:

Explicable concordance, that corresponds to cases in which the expression of the decision-maker's preferences is trivial or clear.

$$C_E(a_i, a_{i'}) = \sum_{j=1}^{n} \pi_j \delta_j^E(a_i, a_{i'}),$$

where

$$\delta_j^E(a_i, a_{i'}) = \begin{cases} 1 \text{ if } F_j(x_{ij}) \; SD \; F_j(x_{i'j}) \\ 0 \text{ otherwise} \end{cases}$$

and π_j is the weight of attribute X_j, with $\pi_j \geq 0$ and $\sum_j^n \pi_j = 1$.

Non-explicable concordance that corresponds to the potential value of the cases in which the expression of the decision-maker's preferences is unclear.

$$C_{NE}(a_i, a_{i'}) = \sum_{j=1}^{n} \pi_j \delta_j^{NE}(a_i, a_{i'}),$$

where

$$\delta_j^{NE}(a_i, a_{i'}) = \begin{cases} 1 \text{ if no } F_j(x_{ij}) \; SD \; F_j(x_{i'j}) \text{ and} \\ \quad \text{no } F_j(x_{i'j}) \; SD \; F_j(x_{ij}) \\ 0 \text{ otherwise.} \end{cases}$$

This second part of the concordance is only a potential value, as it is not certain that for each of these attribute $F_j(x_{ij})$ will be preferred to $F_j(x_{i'j})$.

In these cases, it may be useful to state a condition which tries to make explicit the decision-maker's value functions $U_j(x_{ij})$. If the condition

$$0 \leq p - C_E(a_i, a_{i'}) \leq C_{NE}(a_i, a_{i'}),$$

where $p \in [0.5, 1]$ is the concordance threshold, is fulfilled, then the explication of the unclear cases leads to a value of the concordance index such that the concordance test is satisfied for the proposition that "a_i globally outranks $a_{i'}$". The objective is to reduce as far as possible, without increasing the risk of erroneous conclusions, the number of time where the $U_j(x_{ij})$ functions must be to make explicit. It is notably in the case of unclear situation that [24] uses the probabilistic dominance, as a complementary tool to the stochastic dominance, to build preference relationships.

A discordance index $D_j(a_i, a_i')$ for each attribute X_j may be eventually defined as the ratio between of the difference of the means of the distributions of a_i' and a_i and the range of the scale (if it is justified by the scale level of distributional evaluation):

$$D_j(a_i, a_{i'}) = \begin{cases} \frac{\mu(F_j(x_{i'j})) - \mu(F_j(x_{ij}))}{(x_i^1 - x_i^0)} & \text{if } F_j(x_{ij}) \; SD_j \; F_j(x_{i'j}) \\ 0 & \text{if } F_j(x_{ij}) \text{ not } SD_j \; F_j(x_{i'j}). \end{cases}$$

The difference between the average values of two distributions gives a good indication of the difference in performance of the two compared alternatives. If this difference is large enough in relation to the range of the scale, and SD is fulfilled on attribute X_j, then the chances are large that a_i is 'dominated' by $a_{i'}$. In this case, MZ assume a minimum level ν_j, called a veto threshold, of the discordance index $D_j(a_i, a_{i'})$, giving to a discordant attribute X_j the power of withdrawing all credibility that a_i globally outranks $a_{i'}$.

The discordance test is related to veto threshold ν_j for each attribute. The concordance and discordance relations for the potential alternatives from A are formulated in a classical manner:

$$\text{For all } (a_i, a_{i'}) \in A \times A, (a_i, a_{i'}) \in C_p \longleftrightarrow C(a_i, a_{i'}) \geq p$$

$$\text{For all } (a_i, a_{i'}) \in A \times A, (a_i, a_{i'}) \in D_\nu \longleftrightarrow \exists j / D_j(a_i, a_{i'}) \geq \nu_j.$$

The outranking relations result from the intersection between the concordance set and the complementary set of discordance set:

$$S(p, \nu_j) = C_p \cap \bar{D}_\nu = C_p \setminus D_\nu.$$

Therefore, like in ELECTRE I, we can conclude that a_{ij} globally outranks $a_{i'}$ $(a_i S a_{i'})$ if and only if $C(a_i, a_{i'}) \geq p$ and $D_j(a_i, a_{i'}) < \nu_j$ for all j. If we have no $a_i S a_{i'}$ and no $a_{i'} S a_i$, then a_i and $a_{i'}$ are incomparable, where S is a crisp outranking relation. On the basis on the level of overlapping of the compared distributions, Martel et al. [25] developed preference indices associated to the three types of stochastic dominance and constructed the valued outranking relations.

Depending on whether one is dealing with a choice or a ranking problematic, either the core of the graph of outranking relations is determined or the outranking relations are exploited as in ELECTRE II, for example.

Example 5. Given 6 alternatives a_1, a_2, a_3, a_4, a_5 and a_6, 4 attributes X_1, X_2, X_3 and X_4 and the stochastic dominance relation observed between each pair of alternatives $a_i \neq a_j$ according to each attribute (Table 7.20).

It is assumed that the weights of the attributes are respectively 0.09, 0.55, 0.27 and 0.09. The explicable concordance indices were calculated and they are presented in Table 7.21. The discordance indices are not considered in this example.

On the basis of the explicable concordance indices, we can build up the following outranking relations for a concordance threshold $p = 0.90$: $a_1 \mathrm{S} a_4$, $a_1 \mathrm{S} a_5$, $a_1 \mathrm{S} a_6$; $a_2 \mathrm{S} a_3$, $a_2 \mathrm{S} a_4$, $a_2 \mathrm{S} a_5$, $a_2 \mathrm{S} a_6$; $a_3 \mathrm{S} a_5$, $a_3 \mathrm{S} a_6$; $a_4 \mathrm{S} a_5$ and $a_6 \mathrm{S} a_5$. It is possible to construct the following partial pre-order graph (Fig. 7.15); within this graph, the transitivity is respected.

In Table 7.20 we observe that the relation between a_1 and a_3 according to attribute X_4 is **unclear** since no $F_4(x_{14})$ SD $F_4(x_{34})$ and no $F_4(x_{34})$ SD $F_4(x_{14})$. If the decision-maker can explicit U_4(x) and if a_1 is preferred to a_3 according to this attribute, then globally $a_1 \mathrm{S} a_3$ with a concordance thresholds $p = 0.90$ since $C(a_1, a_3) = 0.91$ (0.82 in Table 7.21 +0.09 (the weight of X_4)).

Table 7.20 Table of observed stochastic dominances

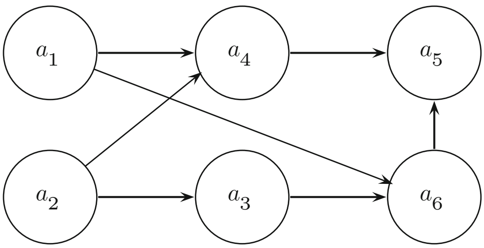
	X_1						X_2					
	a_1	a_2	a_3	a_4	a_5	a_6	a_1	a_2	a_3	a_4	a_5	a_6
a_1	×	–	–	–	TSD	–	×	FSD	FSD	FSD	FSD	FSD
a_2	FSD	×	–	–	FSD	–	–	×	FSD	FSD	FSD	FSD
a_3	FSD	FSD	×	SSD	FSD	FSD	–	–	×	FSD	FSD	FSD
a_4	FSD	FSD	–	×	SSD	FSD	–	–	–	×	FSD	–
a_5	–	–	–	–	×	–	–	–	–	–	×	–
a_6	FSD	FSD	–	–	FSD	×	–	–	–	–	FSD	×
	X_3						X_4					
	a_1	a_2	a_3	a_4	a_5	a_6	a_1	a_2	a_3	a_4	a_5	a_6
a_1	×	–	SSD	SSD	FSD	FSD	×	FSD	–[a]	FSD	FSD	SSD
a_2	FSD	×	SSD	SSD	FSD	FSD	–	×	FSD	FSD	FSD	FSD
a_3	–	–	×	–	FSD	FSD	–[a]	–	×	FSD	FSD	FSD
a_4	–	–	SSD	×	FSD	FSD	–	–	–	×	FSD	FSD
a_5	–	–	–	–	×	–	–	–	–	–	×	–
a_6	–	–	–	–	FSD	×	–	–	–	–	FSD	×

[a] No $a_1 SD a_3$ and no $a_3 SD a_1$ according to X_4

Table 7.21 Explicable concordances indices

	a_1	a_2	a_3	a_4	a_5	a_6
a_1	×	0.64	0.82*	0.91	1	0.91
a_2	0.36	×	0.91	0.91	1	0.91
a_3	0.09*	0.09	×	0.73	1	1
a_4	0.09	0.09	0.27	×	1	0.45
a_5	0	0	0	0	×	0
a_6	0.09	0.09	0	0.55	1	×

Fig. 7.15 Partial preorder



7.5 Conclusions

In this chapter some outranking methods different from ELECTRE and PROMETHEE family have been presented, able to manage different type of data (ordinal, cardinal and stochastic). Their description proved again the richness and flexibility of the outranking approach in preference modelling and in supporting DM in a lot of decisional problem at hand. Some properties of these approaches are common to all the outranking methods, others are peculiar features of some of them. In the following we recall the main characteristics of the considered methods.

(a) The input of these methods are alternative evaluations that can be given in the form of qualitative (ordinal scale), quantitative (with the particular case of interval scales) or stochastic (probability distribution) data with respect to all considered criteria. Sometimes also some technical parameters should be supplied by DM as infracriterion information (indifference, preference, veto thresholds).
(b) All these methods need as infracriterion information also the importance weights in numerical terms. In some of them, just a particular order of criteria is explicitly requested, and a random weight approach should be applied.
(c) The outranking methods within the PCCA approach need the elicitation of both importance and trade-off weights, but the information concerning weights does not need to respect completeness (i.e. all pairwise trade-off and/or importance weights given) and transitivity with respect to trade off weights.
(d) In their first step, all these methods (apart from PRAGMA) give as results some preference or outranking relations, crisp or fuzzy (preference relations and/or indices).
(e) The preference structure associated with these methods is usually P, I, R, obtained at global level (comprehensive evaluation). In the PCCA approach is also possible to obtain the same binary relations with respect to each couple or pair (g_i, g_j) of considered criteria (P_{ij}, I_{ij}, R_{ij}).
(f) Usually the final recommendation (complete or partial preorder) is obtained by the exploitation of the binary relations previously obtained. But in some ordinal method the complete final preorder is directly obtained as a result of the concordance-discordance analysis between different rankings.

Acknowledgements We are grateful to Prof. Silvia Angilella for her kind cooperation and great help in the editing of this chapter.

References

1. Alinezhad, A., Esfandiari, N.: Sensitivity analysis in the Qualiflex and Vikor methods. J. Optim. Ind. Eng. **10**, 29–34 (2012)
2. Ancot, J.: Micro-Qualiflex. Kluwer Academic Publishers, Dordrecht (1988)
3. Angilella, S., Giarlotta, A.: Implementations of PACMAN. Eur. J. Oper. Res. **194**(2), 474–495 (2009)

4. Angilella, S., Giarlotta, A., Lamantia, F.: A linear implementation of PACMAN. Eur. J. Oper. Res. **205**(2), 401–411 (2010)
5. Besson, M.: Rang moyen et agrégation de classements. Revue Française d'Automatique, d'Informatique et de Recherche Operationnelle **1**, 37–58 (1975)
6. Bouyssou, D., Pirlot, M.: An axiomatic approach to TACTIC. In: Cahiers du LAMSADE. Université Paris-Dauphine, Paris (2006)
7. Cook, W.D., Kress, M.: A multiple criteria decision model with ordinal preference data. Eur. J. Oper. Res. **54**, 191–198 (1991)
8. De Keyser, W.S.M., Peeters, P.H.M.: ARGUS - a new multiple criteria method based on the general idea of outranking. In: Paruccini, M. (ed.) Applying Multiple Criteria Aid for Decision to Environmental Management, pp. 263–278. Kluwer Academic Publishers, Dordrecht (1994)
9. Dendrou, B.A., Dendrou, S.A., Houtis, E.N.: Multiobjective decisions analysis for engineering systems. Comput. Oper. Res. **7**, 310–312 (1980)
10. Giarlotta, A.: Passive and active compensability multicriteria analysis (PACMAN). J. Multi-Criteria Decis. Anal. **7**(4), 204–216 (1998)
11. Giarlotta, A.: Multicriteria compensability analysis. Eur. J. Oper. Res. **133**(1), 190–209 (2001)
12. Greco, S.: A new PCCA method: IDRA. Eur. J. Oper. Res. **98**(3), 587–601 (1997)
13. Guigou, J.L.: Analyse des données et choix à critères multiples. Dunod, Paris (1973)
14. Hadar, J., Russell, W.R.: Rules for ordering uncertain prospects. Am. Econ. Rev. **59**(1), 25–34 (1969)
15. Hinloopen, E., Nijkamp, P.: Regime-methode voor ordinal multicriteria-analyses. Kwantitatieve Merhoden **7**(22), 61–78 (1986)
16. Hinloopen, E., Nijkamp, P., Rietveld, P.: Qualitative discrete multiple criteria choice models in regional planning. Reg. Sci. Urban Econ. **13**, 77–102 (1983)
17. Huang, C.C., Kira, D., Vertinsky, I.: Stochastic dominance rules for multiattribute utility functios. Rev. Econ. Stud. **41**, 611–616 (1969)
18. Israels, A.Z., Keller, W.J.: Multicriteria analyse voor kwalitatieve data. Kwantitative methoden **7**(22), 49–74 (1986)
19. Janssen, R.: Multiobjective Decision Support for Environmental Management. Kluwer Academic Publishers, Dordrecht (1992)
20. Köksalan, M., Karwan, M.H., Zionts, S.: An approach for solving discret alternative multiple criteria problems involving ordinal criteria. Nav. Res. Logist. **35**, 625–641 (1988)
21. Leclercq, J.P.: Propositions d'extensions de la notion de dominance en présen ce de relations d'ordre sur le pseudo-critéres: MELCHIOR. Revue Belge de Recherche Operationnelle, de Statistique et d' Informatique **24**(1), 32–46 (1984)
22. Mareschal, B.: Stochastic multicriteria decision making and uncertainty. Eur. J. Oper. Res. **26**, 58–64 (1986)
23. Martel, J.M., D'Avignon, G.R.: Projects ordering with multicriteria analysis. Eur. J. Oper. Res. **10**, 59–69 (1982)
24. Martel, J.M., Zaras, K.: Modeling preferences using stochastic and probabilistic dominances. In: International Conference on Methods and Applications of Multicriteria Decision Making, Mons, Belgium (1997)
25. Martel, J.M., Azondékon, S., Zaras, K.: Preference relations in multicriterion analysis under risk. JORBEL **31**(3–4), 55–83 (1992)
26. Matarazzo, B.: Multicriteria analysis of preferences by means of pairwise actions and criterion comparisons (MAPPAC). Appl. Math. Comput. **18**(2), 119–141 (1986)
27. Matarazzo, B.: A more effective implementation of the MAPPAC and PRAGMA methods. Found. Control Eng. **13**(4), 155–173 (1988)
28. Matarazzo, B.: Preference global frequencies in multicriterion analysis (PRAGMA). Eur. J. Oper. Res. **36**(1), 36–49 (1988)
29. Matarazzo, B.: PCCA and compensation. In: Korhonen, P., Lewandoski, A., Wallenius, J. (eds.) Multicriteria Decision Support. Lecture Notes in Economics and Mathematical Systems, vol. 356, pp. 99–108. Springer, Berlin (1989)

30. Matarazzo, B.: A pairwise criterion comparison approach: the MAPPAC and PRAGMA methods. In: Carlos A.B.C. (ed.) Readings in Multiple Criteria Decision Aid, pp. 253–273. Springer, Berlin (1990)
31. Matarazzo, B.: MAPPAC as a compromise between outranking methods and MAUT. Eur. J. Oper. Res. **54**, 48–65 (1991)
32. Nijkamp, P., Rietveld, P., Voogd, H.: Multicriteria Evaluation in Physical Planning. North Holland, Amsterdam (1990)
33. Paelinck, J.H.P.: Qualitative multiple criteria analysis, environmental protection and multiregional development. Pap. Reg. Sci. Assoc. **36**, 59–74 (1976)
34. Paelinck, J.H.P.: Qualitative multiple criteria analysis: an airport location. Environ. Plann. **9**, 883–895 (1977)
35. Pastijn, H., Leysen, J.: Constructing an outranking relation with ORESTE. Math. Comput. Model. **12**(10/11), 1255–1268 (1989)
36. Pomerol, J.C., Barba-Romero, S.: Choix multiple dans l'entreprise. Hermes, Paris (1993)
37. Roubens, M.: Analyse et agrégation des préférences: modélisation, ajustement et résumé de données relationnelles. Revue Belge de Recherche Operationnelle, de Statistique et d' Informatique. **20**(2), 36–67 (1980)
38. Roubens, M.: Preference relations on actions and criteria in multicriteria decision making. Eur. J. Oper. Res. **10**, 51–55 (1982)
39. Roubens, M., Vincke, P.: Preference Modelling. Springer, New York (1985)
40. Roy, B., Bouyssou, D.: Aide multicritère à la décision: Méthodes et cas. Economica, Paris (1993)
41. Slovic, P.: Choice between equally-valued alternatives. J. Exp. Econ. Hum. Percept. Perfom. **1**, 280–287 (1975)
42. Springael, J., De Keyser, W.: A new generation of multi-criteria based group support methods. In: Whistler, B.C. (ed.) Proceedings 17th International Conference on Multiple Criteria Decision Making, pp. 1–20, Canada (2004)
43. Vansnick, J.C.: On the problem of weighs in multiple criteria decision making (the noncompensatory approach). Eur. J. Oper. Res. **24**, 288–294 (1986)
44. Vincke, P.: Multicriteria Decision Aid. Wiley, New York (1992)
45. Voogd, H.: Multicriteria evaluation with mixed qualitative and quantitative data. Environ. Plan. B **9**, 221–236 (1982)
46. Voogd, H.: Multicriteria Evaluation for Urban and Regional Planning. Pion Ltd., London (1983)
47. Zaras, K.: Dominance stochastique pour deux classes de fonctions d'utilité: concaves et convexes. Recherche opérationnelle Oper. Res. **23**(1), 57–65 (1989)
48. Zaras, K., Martel, J.M.: Multiattribute analysis based on stochastic dominance. In: Munier, B., Machina, M.J. (eds.) Models and Experiments in Risk and Rationality, pp. 225–248. Kluwer Academic Publishers, Dordrecht (1994)

Part IV
Multiattribute Utility and Value Theories

Chapter 8
Multiattribute Utility Theory (MAUT)

James S. Dyer

Abstract In this chapter, we provide a review of multiattribute utility theory. We begin with a brief review of single-attribute preference theory, and explore preference representations that measure a decision maker's strength of preference and her preferences for risky alternatives. We emphasize the distinction between these two cases, and then explore the implications for multiattribute preference models. We describe the multiattribute decision problem, and discuss the conditions that allow a multiattribute preference function to be decomposed into additive and multiplicative forms under conditions of certainty and risk. The relationships among these distinct types of multiattribute preference functions are then explored, and issues related to their assessment and applications are surveyed.

Keywords Multiattribute utility theory • Additive value functions • Preference modeling

8.1 Introduction

In this chapter, we provide a review of multiattribute utility theory. As we shall discuss, multiattribute preference theory would be a more general term for this topic that covers several multiattribute models of choice. These models are based on alternate sets of axioms that have implications for their assessment and use. We begin with a brief review of single-attribute preference theory, and explore preference representations that measure a decision maker's preferences on an ordinal scale, her strength of preference and her preferences for risky alternatives. We emphasize the distinctions among these cases, and then explore their implications for multiattribute preference theory.

J.S. Dyer (✉)
Department of Information, Risk and Operations Management, The McCombs School of Business, The University of Texas at Austin, Austin, TX 78712, USA
e-mail: Jim.Dyer@mccombs.utexas.edu

S. Greco et al. (eds.), *Multiple Criteria Decision Analysis*, International Series in Operations Research & Management Science 233,
DOI 10.1007/978-1-4939-3094-4_8

In order to differentiate between theories for preference based on the notions of ordinal comparisons and strength of preference versus theories for risky choices, we will use the term value function to refer to the former and utility function to refer to the latter. This distinction was made by Keeney and Raiffa in 1976[1] and has been generally adopted in the literature. Further, we will use the term preference model or multiattribute preference model to include all of these cases.

We describe the multiattribute decision problem, and discuss the conditions that allow a multiattribute preference function to be decomposed into additive and multiplicative forms under conditions of certainty and risk. The relationships between multiattribute preference functions under conditions of certainty and risk are then explored, and issues related to their assessment and applications are surveyed.

We do not address the important issue of selecting the attributes for a multiattribute decision problem, which may influence the choice of the appropriate multiattribute preference model. This topic is explored in Keeney and Raiffa [32] and in more detail in Keeney [30]. More recent discussions include the contributions of Keeney and Gregory [31] and Butler et al. [6].

There are several important points related to the field of multi-criteria decision analysis that we wish to make. First, multiattribute preference theory provides an axiomatic foundation for choices involving multiple criteria. As a result, one can examine these axioms and determine whether or not they are reasonable guides to rational behavior. Most applications of the methods of multi-criteria decision analysis are developed for individuals who are making decisions on behalf of others, either as managers of publicly held corporations or as government officials making decisions in the best interests of the public. In such cases, one should expect these decision makers to use decision-making strategies that can be justified based on a reasonable set of axioms, rather than some ad hoc approach to decision making that will violate one or more of these axioms.

Often arguments are made that decision makers do not always make decisions that are consistent with the rational axioms of decision theory. While this may be true as a descriptive statement for individual decision making, it is much more difficult to identify situations involving significant implications for other parties where a cavalier disregard for normative theories of choice can be defended.

Second, multiattribute utility theory can be based on different sets of axioms that are appropriate for use in different contexts. Specifically, the axioms that are appropriate for risky choice do not have to be satisfied in order to use multiattribute models of preference for cases that do not explicitly involve risk. Much of the work on multiobjective mathematical programming, for example, does not require the consideration of risk, and many applications of the Analytical Hierarchy Procedure (AHP) are also developed in the context of certainty [40].

[1]The classic book *Decisions with Multiple Objectives* by R. L. Keeney and H. Raiffa was originally published by Wiley in 1976. The Cambridge University Press version was published in 1993.

The broad popularity of the award-winning book on multiattribute utility theory by Keeney and Raiffa [32] emphasized the use of multiattribute preference models based on the theories of von Neumann and Morgenstern [46], which rely on axioms involving risk. As a result, this approach has become synonymous in the view of many scholars with multiattribute preference theory. However, this theory is not the appropriate one for decisions involving multiple objectives when risk is not a consideration.[2] Instead, the multiattribute preference theories for certainty are based on ordinal comparisons of alternatives or on estimates of the strength of preference between pairs of alternatives.

Third, many existing approaches to multi-criterion decision analysis can be viewed as special cases or approximations to multiattribute preference models. We shall make this case for the popular methods of goal programming and the AHP as examples. By viewing these seeming disparate methods from this unifying framework, it is possible to gain new insights into these methodologies, recognize ways that these approaches might be sharpened or improved, and provide a basis for evaluating whether their application will result in solutions that are justified by a normative theory.

8.2 Preference Representations Under Certainty and Under Risk

Preference theory studies the fundamental aspects of individual choice behavior, such as how to identify and quantify an individual's preferences over a set of alternatives, and how to construct appropriate preference representation functions for decision making. An important feature of preference theory is that it is based on rigorous axioms which characterize an individual's choice behavior. These preference axioms are essential for establishing preference representation functions, and provide the rationale for the quantitative analysis of preference.

The basic categories of preference studies can be divided into characterizations of preferences under conditions of certainty or risk and over alternatives described by a single attribute or by multiple attributes. In the following, we will begin with the introduction of basic preference relations and then discuss preference representation under certainty and under risk for the single attribute case. We shall refer to a preference representation function under certainty as a *value function*, and to a preference representation function under risk as a *utility function*.

[2] "The important addition since 1976 concerns value functions that address strength of preference between pairs of consequences (see [4, 13])." A quote from the Preface to the Cambridge University Press Edition, R. L. Keeney and H. Raiffa, *Decisions with Multiple Objectives*, Cambridge University Press, 1993.

Preference theory is primarily concerned with properties of a binary preference relation $\succ$ on a choice set X, where X could be a set of commodity bundles, decision alternatives, or monetary gambles. For example, we might present an individual with a pair of alternatives, say x and y (e.g., two cars) where $x, y \in X$ (e.g., the set of all cars), and ask how they compare (e.g., do you prefer x or y?). If the individual says that x is preferred to y, then we write $x \succ y$, where $\succ$ means strict preference. If the individual states that he or she is indifferent between x and y, then we represent this preference as $x \sim y$. Alternatively, we can define $\sim$ as the absence of strict preference; that is, not $x \succ y$ and not $y \succ x$. If it is not the case that $y \succ x$, then we write $x \succsim y$, where $\succsim$ represents a weak preference (or preference-indifference) relation. We can also define $\succsim$ as the union of strict preference $\succ$ and indifference $\sim$; that is, both $x \succ y$ and $x \sim y$.

Preference studies begin with some basic assumptions (or axioms) of individual choice behavior. First, it seems reasonable to assume that an individual can state preference over a pair of alternatives without contradiction; that is, the individual does not strictly prefer x to y and y to x simultaneously. This leads to the following definition for *preference asymmetry*: preference is asymmetric if there is no pair x and y in X such that $x \succ y$ and $y \succ x$.

Asymmetry can be viewed as a criterion of preference consistency. Furthermore, if an individual makes the judgment that x is preferred to y, then he or she should be able to place any other alternative z somewhere on the ordinal scale determined by the following: either better than y, or worse than x, or both. Formally, we define *negative transitivity* by saying that preferences are negatively transitive if given $x \succ y$ in X and any third element z in X, it follows that either $x \succ z$ or $z \succ y$, or both.

If the preference relation $\succ$ is asymmetric and negatively transitive, then it is called a *weak order*. The weak order assumption implies some desirable properties of a preference ordering, and is a basic assumption in many preference studies. If the preference relation $\succ$ is a weak order, then the associated indifference and weak preference relationships are well behaved. The following statements summarize some of the properties of some of these relationships.

If strict preference $\succ$ is a weak order, then

1. strict preference $\succ$ is *transitive* (if $x \succ y$ and $y \succ z$, then $x \succ$ z);
2. indifference $\sim$ is transitive, *reflexive* ($x \sim x$ for all x); and *symmetric* ($x \sim y$ implies $y \sim x$);
3. exactly one of $x \succ y$, $y \succ x$, $x \sim y$ holds for each pair x and y; and
4. weak preference $\succsim$ is transitive and *complete* (for a pair x and y, either $x \succsim y$ or $y \succsim x$).

Thus, an individual whose strict preference can be represented by a weak order can rank all alternatives considered in a unique order. Further discussions of the properties of binary preference relations are presented in Fishburn [18, Chap. 2], Kreps [36, Chap. 2], and by Bouyssou and Pirlot in Chap. 4 of this volume.

8.2.1 Preference Functions for Certainty (Value Functions)

If strict preference $\succ$ on X is a weak order and X is finite or denumerable, then there exists a numeric representation of preference, a real-valued function $\overset{o}{v}$ on X such that $x \succ y$ if and only if $\overset{o}{v}(x) > \overset{o}{v}(y)$, for all x and y in X [18]. Since $\overset{o}{v}$ is a preference representation function under certainty, it is often called a *value function* [32]. This value function is said to be order-preserving since the real numbers $\overset{o}{v}(x), \overset{o}{v}(y), \ldots$ ordered by $>$ are consistent with the order of $x, y, \ldots$ under $\succ$. Thus, any monotonic transformations of $\overset{o}{v}$ will also be order-preserving for this binary preference relation. Since such a function only rank orders different outcomes, there is no added meaning of the values of $\overset{o}{v}$ beyond the order that they imply.

Notice that we use the symbol "*o*" to indicate that $\overset{o}{v}$ is an ordinal function. While the notion of an ordinal value function is very important for economic and decision theories, such a function is seldom assessed in practice. For example, if we know that preferences are monotonically increasing for some real-valued attribute x (e.g., more is better), then $\overset{o}{v}(x) = x$ is valid ordinal preference function. Therefore, we may choose an objective function of maximizing profits or minimizing costs, and be comfortable assuming implicitly that these objective functions are order-preserving preference functions for a decision maker. However, the notion of an ordinal value function does become important when we speak of multiattribute value functions, as we shall discuss.

In order to replicate the preferences of a decision maker with less ambiguity, we may wish to consider a "strength of preference" notion that involves comparisons of preference differences between pairs of alternatives. To do so, we need more restrictive preference assumptions, including that of a weak order over preferences between exchanges of pairs of alternatives [35, Chap. 4]. We use the term *measurable value function* for a value function that orders the differences in the strength of preference between pairs of alternatives or, more simply, the "preference differences" between the alternatives.

Once again, let X denote the set of all possible consequences in a decision situation, with $w, x, y, z, w', x', y' \in X$; define X^* as a nonempty subset of $X \times X$, and $\succsim^*$ as a binary relation on X^*. We shall interpret $wx \succsim^* yz$ to mean that the strength of preference for w over x is greater than or equal to the strength of preference for y over z. The notation $wx \sim^* yz$ means both $wx \succsim^* yz$ and $yz \succsim^* wx$, and $wx \succ^* yz$ means not $yz \succsim^* wx$.

There are several alternative axiom systems for measurable value functions, including the topological results of Debreu [11] and the algebraic development by Scott and Suppes [44]. Some of these systems allow both "positive" and "negative" preference differences and are called algebraic difference structures. For example, the "degree of preference" for x over w would be "negative" if w is preferred to x. Our development is based on an axiom system presented by Krantz et al. [[35], Definition 4.1] that does not allow negative differences; hence it is called a positive difference structure.

This set of axioms includes several technical assumptions that have no significant implications for behavior. However, a key axiom that does have an intuitive interpretation in terms of preferences is the following one: If $wx, xy, w'x', x'y' \in X^*$, $wx \succsim^* w'x'$, and $xy \succsim^* x'y'$, then $wy \succsim^* w'y'$. That is, if the difference in the strength of preference between w and x exceeds the difference between w' and x', and the difference in the strength of preference between x and y exceeds the difference between x' and y', then the difference in the strength of preference between w and y must exceed the difference between w' and y'. Some introspection should convince most readers that this would typically be true for preference comparisons of alternative pairs.

The axioms of Krantz et al. [35] imply that there exists a real-valued function v on X such that, for all $w, x, y, z \in X$, if w is preferred to x and y to z, then $wx \succsim^* yz$ if and only if

$$v(w) - v(x) \geq v(y) - v(z) \tag{8.1}$$

Further, v is unique up to a positive linear transformation, so it is a cardinal function (i.e., v provides an interval scale of measurement). That is, if v also satisfies (8.1), then there are real numbers $\mathrm{a} > 0$ and b such that $v'(x) = \mathrm{a}v(x) + \mathrm{b}$ for all $x \in X$ ([35], Theorem 4.1).

We define the binary preference relation $\succsim$ on X from the binary relation $\succsim^*$ on X^* in the natural way by requiring $wx \succsim^* yx$ if and only if $w \succsim y$ for all $w, x, y \in X$. Then from (8.1) it is clear that $w \succsim y$ if and only if $v(w) \geq v(y)$. Thus, v is a value function on X and, by virtue of (8.1), it is a measurable value function.

The ideas of strength of preference and of measurable value functions are important concepts that are often used implicitly in the implementation of preference theories in practice. Intuitively, it may be useful to think of a measurable value function as the unique preference function in the case of certainty that reveals the marginal value of additional units of the underlying commodity. For example, we would expect that the measurable value function over wealth for most individuals would be concave, since the first million dollars would be "worth" more to the individual than the second million dollars, and so on. This notion would be consistent with the traditional assumption in economics of diminishing marginal returns to scale.

Further, the measurable value function can be assessed using questions for subjects that do not require choices among lotteries, which may be artificial distractions in cases where subjects are trying to choose among alternatives that do not involve the consideration of risk. Examples of methods for assessing measurable value functions would include the direct rating of alternatives on a cardinal scale, or direct comparisons of preference differences. For a detailed discussion of these approaches, see Farquhar and Keller [16], von Winterfeldt and Edwards [47], and Kirkwood [34].

In addition, subjects can be asked to make ratio comparisons of preference differences. For example, they might be comparing automobiles relative to a "base case", say a Ford Taurus. Then, they could be asked to compare the improvement in acceleration offered by a BMW over a Taurus to the improvement offered by a Mercedes (relative to the same Taurus) in terms of a ratio. This ratio judgment could be captured and analyzed using the tools of the AHP, and this provides a link between measurable value functions and ratio judgments. This point has been made on numerous occasions, and is worth further exploration (e.g., see [12, 29, 41]).

8.2.2 Preference Functions for Risky Choice (Utility Functions)

We turn to preference representation for risky options, where the risky options are defined as lotteries or gambles with outcomes that depend on the occurrence from a set of mutually exclusive and exhaustive events. For example, a lottery could be defined as the flip of a fair coin, with an outcome of \$10 if heads occurs and an outcome of \$−2 if tails occurs.

Perhaps the most significant contribution to this area of concern was the formalization of expected utility theory by von Neumann and Morgenstern [46]. This development has been refined by a number of researchers and is most commonly presented in terms of three basic axioms [18].

Let P be a convex set of simple probability distributions or lotteries $\{p, q, r, \ldots\}$ on a nonempty set X of outcomes. We shall use p, q and r to refer to probability distributions and random variables interchangeably. For lotteries p, q, r in P and all λ, $0 < \lambda < 1$, the expected utility axioms are:

1. (Ordering) $\succ$ is a weak order;
2. (Independence) If $p \succ q$ then $(\lambda p + (1 - \lambda)\, r) \succ (\lambda q + (1 - \lambda)\, r)$ for all r in P;
3. (Continuity) If $p \succ q \succ r$ then there exist some $0 < \alpha < 1$ and $0 < \beta < 1$ such that $\alpha p + (1 - \alpha)\, r \succ q \succ \beta p + (1 - \beta)\, r$

The von Neumann–Morgenstern expected utility theory asserts that the above axioms hold if and only if there exists a real-valued function u such that for all p, q in P, $p \succsim q$ if and only if

$$\sum_{x \in X} p(x)u(x) \geq \sum_{x \in X} q(x)u(x) \tag{8.2}$$

Moreover, such a u is unique up to a positive linear transformation.

The expected utility model can also be used to characterize an individual's risk attitude [39], [32, Chap. 4]. If an individual's utility function over a closed interval is concave, linear, or convex, then the individual is risk averse, risk neutral, or risk seeking, respectively.

The von Neumann–Morgenstern theory of risky choice presumes that the probabilities of the outcomes of lotteries are provided to the decision maker. Savage [43] extended the theory of risky choice to allow for the simultaneous determination of subjective probabilities for outcomes and for a utility function u defined over those outcomes. Deduced probabilities in Savage's model are personal or subjective probabilities. The model itself is a subjective expected utility representation.

The assessment of von Neumann–Morgenstern utility functions will almost always involve the introduction of risk in the form of simple lotteries. For a discussion of these assessment approaches, see Keeney and Raiffa [32, Chap. 4], von Winterfeldt and Edwards [47], and Anderson and Clemen [3].

As a normative theory, the expected utility model has played a major role in the prescriptive analysis of decision problems. However, for descriptive purposes, the assumptions of this theory have been challenged by empirical studies [28]. Some of these empirical studies demonstrate that subjects may choose alternatives that imply a violation of the independence axiom. One implication of the independence axiom is that the expected utility model is "linear in probabilities." For a discussion, see Fishburn and Wakker [21]. A number of contributions have been made by relaxing the independence axiom and developing some nonlinear utility models to accommodate actual decision behavior [7, 20, 48].

8.2.3 Comment

Note that both the measurable value function $v(x)$ and the von Neumann and Morgenstern utility function $u(x)$ are cardinal measures, unique up to a positive linear transformation. However, the theory supporting the measurable function is based on axioms involving preferences differences, and it is assessed based on questions that rely on the idea of strength of preference. In contrast, the von Neumann and Morgenstern utility function is based on axioms involving lotteries, and it is assessed based on questions that typically involve lottery comparisons.

Suppose we find a subject and assess her measurable value function $v(x)$ and her utility function $u(x)$ over the same attribute (e.g., over monetary outcomes). Would these two functions be identical, except for measurement error? A quick reaction might be that they would be identical, since they are each unique representations of the subject's preferences, up to a positive linear transformation. However, that is not necessarily the case. Intuitively, a measurable value function $v(x)$ may be concave, indicating decreasing marginal value for the underlying attribute. However, a utility function $u(x)$ may be even more concave, since it will incorporate not only feelings regarding the marginal value of the attribute, but also it may incorporate psychological reactions to taking risks [14]. Empirical tests of this observation are provided by Krzysztofowicz [37] and Keller [33] and generally support this intuition. This is an important point, and one that we will emphasize again in the context of multiattribute preference functions (see [13, 15, 27, 42]).

8.3 Ordinal Multiattribute Preference Functions for the Case of Certainty

A decision maker uses the appropriate preference function, $v(x)$ or $v(x)$ in the case of certainty or $u(x)$ in the case of risk, to choose among available alternatives. The major emphasis of the work on multiattribute utility theory has been on questions involving conditions for the decomposition of a preference function into simple polynomials, on methods for the assessment of these decomposed functions, and on methods for obtaining sufficient information regarding the multiattribute preference functions so that the evaluation can proceed without its explicit identification with full precision.

Suppose that the alternatives defined for single attribute preference functions are now considered to be vectors. That is, suppose that $X = \prod_{i=1}^{n} X_i$ where X_i represents the performance of an alternative on attribute i. We will be interested in conditions allowing the determination that $(x_1, \ldots, x_n) \succsim (y_1, \ldots, y_n)$ if and only if $\overset{o}{v}(x_1, \cdots, x_n) \geq \overset{o}{v}(y_1, \cdots, y_n)$ for example. Essentially, all that is required is the assumption that the decision maker's preferences are a weak order on the vectors of attribute values.

In some cases, methods for multiattribute optimization do not need any additional information regarding a multiattribute preference function, other than perhaps invoking concavity to allow maximization. Geoffrion et al. [23] provide an example of an early approach to multiattribute optimization that does proceed with only local information regarding the implicit multiattribute preference function. Additional conditions are needed to decompose the multiattribute preference function into simple parts.

8.3.1 Preference Independence

The most common approach for evaluating multiattribute alternatives is to use an additive representation. For simplicity, we will assume that there exist a most preferred outcome x_i^* and a least preferred outcome x_i^0 on each attribute $i = 1$ to n. In the additive representation, a real value $\overset{o}{v}$ is assigned to each outcome $(x_1, \cdots, x_n)$ by

$$\overset{o}{v}(x_1, \cdots, x_n) = \sum_{i=1}^{n} \overset{o}{v}_i(x_i) \tag{8.3}$$

where the $\overset{o}{v}_i$ are single attribute value functions over X_i.[3] When it is convenient, we may choose the scaling $\overset{o}{v}_i\left(x_i^*\right) = 1, \overset{o}{v}_i\left(x_i^0\right) = 0$, and write $\overset{o}{v}(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} \overset{o}{\lambda}_i \overset{o}{v}_i(x_i)$ where $\sum_{i=1}^{n} \overset{o}{\lambda}_i = 1$.

If our interest is in simply rank-ordering the available alternatives then the key condition for the additive form in (8.3) is mutual preference independence. Suppose that we let $I \subset \{1, 2, \ldots, n\}$ be a subset of the attribute indices, and define X_I as the subset of the attributes designated by the subscripts in I. Also, we let $\overline{X}_I$ represent the complementary subset of the n attributes. Then,

1. 1. X_I is preference independent of $\overline{X}_I$ if $(w_I, \overline{w}_I) \succsim (x_I, \overline{w}_I)$ for any $w_I, x_I \in X_I$ and $\overline{w}_I \in \overline{X}_I$ implies $(w_I, \overline{x}_I) \succsim (x_I, \overline{x}_I)$ for all $\overline{x}_I \in \overline{X}_I$.
2. The attributes $X_1, \ldots, X_n$ are mutually preference independent if for every subset $I \subseteq \{1, \ldots, n\}$ the set X_I of these attributes is preference independent of $\overline{X}_I$

When coupled with a solvability condition and some technical assumptions, mutual preference independence implies the existence of an additive ordinal multiattribute value function for $n \geq 3$ attributes. Furthermore, this additive ordinal value function is unique up to a positive linear transformation.

Attributes X_i and X_j are preference independent if the tradeoffs (substitution rates) between X_i and X_j are independent of all other attributes. Mutual preference independence requires that preference independence holds for all pairs X_i and X_j. Essentially, mutual preference independence implies that the indifference curves for any pair of attributes are unaffected by the fixed levels of the remaining attributes. Debreu [11], Luce and Tukey [38], and Gorman [24] provide axiom systems and analysis for the additive form (8.3).

An example may help to illustrate the idea of preference independence. Suppose that a subject is attempting to evaluate automobiles based on the three criteria of cost, horsepower, and appearance. Assume that the subject decides that her preferences between two automobiles differing in cost and horsepower but with identical values for appearance are as follows: ($24,000, 150 hp, ugly) $\succ$ ($25,000, 170 hp, ugly). If the level of appearance does not affect the subject's indifference curve between cost and horsepower, then she will also prefer ($24,000, 150 hp, beautiful) to ($25,000, 170 hp, beautiful), and will maintain the same preference relation for any common value of appearance.

As a practical matter, it is only necessary for preference independence to hold for the n-1 pairs of criteria involving the first criterion and the other n-1 criteria taken one at a time. See Keeney and Raiffa [32, Chap. 3] for a discussion.

In Chap. 4 of this volume, Bouyssou and Pirlot provide an excellent discussion of the additive ordinal value function which they present as the use of conjoint measurement for multiple criteria decision making. In our development, we use

[3]Note that the $\overset{o}{v}_i$ are called partial value functions by Bouyssou and Pirlot in Chap. 4 of this volume.

the terminology ordinal additive value function instead in order to contrast this preference representation with other additive and non-additive preference models. We also use the term preference independence rather than simply independence to distinguish this key assumption from other forms of independence conditions that are appropriate for multiple criteria decision making in different contexts.

8.3.2 Assessment Methodologies

The additive ordinal value function would seem to be an attractive choice for practical applications of multiattribute decision making. However, the resulting additive function is, in general, difficult to assess. The problem arises because the single attribute functions $\overset{o}{v}_i$ cannot be assessed using the methods appropriate for the single-attribute measurable value functions. Instead, these functions can only be assessed through protocols that require tradeoffs between two attributes throughout the process, and these protocols are therefore burdensome for the decision makers. Further, the resulting additive function will only have an ordinal interpretation, rather than providing a measure of the strength of preference.

Keeney and Raiffa [32, Chap. 3] illustrate two assessment procedures for ordinal additive value functions. However, an example may be helpful to emphasize that the resulting additive value function may only provide an ordinal ranking of alternatives, since this important point is also a subtle one.

Suppose that an analyst is attempting to assess a preference function from a decision maker on three attributes X, Y, and Z that are related in the mind of the decision maker in a multiplicative form; that is, the decision maker's true preferences are represented by the product xyz where x, y, and z are attribute values. Of course, the analyst is not aware of this multiplicative form, and is attempting to develop an appropriate preference representation from the decision maker based on a verbal assessment procedure. Further, suppose that there is no risk involved, so the analyst would like to consider the use of an additive ordinal multiattribute value function.

An example of a situation that might involve this type of a preference function would be the ranking of oil exploration opportunities based on estimates of their oil reserves. Suppose that the decision maker thinks that these reserves can be estimated by multiplying the area (x) of the structure containing oil by its depth (y) to obtain the volume of the structure, and then multiplying this volume by its rate of recovery per volumetric unit (z). In practice, this is a simplification of the approach actually used in many cases to estimate oil reserves.

This multiplication of the relevant parameters could be done explicitly in this case, but this example should suggest that such a true preference structure could occur naturally. For simplicity, and to avoid complications associated with units of measurement, we will assume that $X = Y = Z = [1,10]$, which might occur if the analyst rescaled the actual units of measurement.

The analyst does not know the true underlying preference model of the decision maker, and so he might ask a series of questions to determine if mutual preference independence holds in this case. Consider alternative 1, with $x_1 = 2$, $y_1 = 3$, and $z_1 = 4$, versus alternative 2, with $x_2 = 4$, $y_2 = 2$, and $z_2 = 4$. The decision maker would be asked to compare (2,3,4) with (4,2,4), and would reply that she prefers alternative 2 (because $2 \times 3 \times 4 = 24$ and $4 \times 2 \times 4 = 32$, although these calculations are unknown to the analyst). It is easy to see that alternative 2 would remain preferred to alternative 1 for all common values of z_1 and z_2, so attributes X and Y are preference independent of Z. Likewise, a similar set of questions would reveal that X and Z are preference independent of Y, and Y and Z are preference independent of X, so these three attributes are mutually preference independent.

Therefore, the analyst concludes that the preferences of the decision maker can be represented by the ordinal additive multiattribute preference function

$$\overset{o}{v}(x, y, z) = \overset{o}{v}_x(x) + \overset{o}{v}_y(y) + \overset{o}{v}_z(z)$$

As we shall see, this is not a mistake even though the true preference function is multiplicative, and the assessment procedure will construct the correct ordinal additive function that will result in the same rank ordering of alternatives as the multiplicative preference function.

For this example only, we will abuse the notation and let subscripts of the attributes indicate the corresponding values of the single attribute functions. For examples, we will let x_0 indicate the value of attribute X such that $\overset{o}{v}_x(x_0) = 0$, and let y_1 indicate the value of attribute Y such that $\overset{o}{v}_y(y_1) = 1$, and so forth. Suppose the analyst begins the assessment procedure by letting $x_0 = y_0 = z_0 = 1$, which is allowable given the fact that the function is unique up to a linear transformation. That is, the analyst scales $\overset{o}{v}_x$ so that $\overset{o}{v}_x(x_0) = \overset{o}{v}_x(1) = 0$, and similarly scales $\overset{o}{v}_y(1) = \overset{o}{v}_z(1) = 0$. Therefore, we would have

$$\overset{o}{v}(1, 1, 1) = \overset{o}{v}_x(1) + \overset{o}{v}_y(1) + \overset{o}{v}_z(1) = 0 + 0 + 0 = 0.$$

The analyst then arbitrarily selects $x_1 = 2$; that is, he sets $\overset{o}{v}_x(x_1) = \overset{o}{v}_x(2) = 1$, which is also allowable by virtue of the scaling convention. Finally, the analyst involves the decision maker, and asks her to specify a value y_1 so that she is indifferent between the alternative (2,1,1) and the alternative (1,y_1,1). Based on her true multiplicative preference model unknown to the analyst, if she is indifferent between (2,1,1) and (1,y_1,1) it must be the case that $2 \times 1 \times 1 = 1 \times y_1 \times 1$, so she responds $y_1 = 2$. Based on this response, the analyst sets $\overset{o}{v}_y(y_1) = \overset{o}{v}_y(2) = 1$.

This means that $\overset{o}{v}(2, 1, 1) = \overset{o}{v}_x(2) + \overset{o}{v}_y(1) + \overset{o}{v}_z(1) = 1 + 0 + 0 = 1$, and that $\overset{o}{v}(1, 2, 1) = \overset{o}{v}_x(1) + \overset{o}{v}_y(2) + \overset{o}{v}_z(1) = 0 + 1 + 0 = 1$, which verifies to the analyst that the additive representation indicates that the decision maker is indifferent between the alternatives (2,1,1) and (1,2,1). In addition, the analyst knows that $\overset{o}{v}(2, 2, 1) = \overset{o}{v}_x(2) + \overset{o}{v}_y(2) + \overset{o}{v}_z(1) = 1 + 1 + 0 = 2$.

Now, the analyst asks the decision maker to specify a value for x_2 so that she is indifferent between the alternatives (2,2,1) and (x_2,1,1). This response will determine the value of x_2 such that $\overset{o}{v}_x(x_2) = 2$, because indifference between these two alternatives will require $\overset{o}{v}(x_2, 1, 1) = \overset{o}{v}_x(x_2) + \overset{o}{v}_y(1) + \overset{o}{v}_z(1) = 2 + 0 + 0 = 2$ also.

Using her implicit multiplicative preference function for the alternative (2,2,1), she obtains $2 \times 2 \times 1 = 4$, and since indifference requires $x_2 \times 1 \times 1 = 4$, she would identify $x_2 = 4$, so $\overset{o}{v}_x(4) = 2$. The reader should confirm that similar questions would determine $\overset{o}{v}_y(4) = \overset{o}{v}_z(4) = 2$, and that $\overset{o}{v}_x(8) = 3$, and so forth. Continuing in this fashion, and using similar questions to develop the assessments of $\overset{o}{v}_y(y)$ and $\overset{o}{v}_z(z)$, the analyst would develop graphs that would indicate $\overset{o}{v}_x(x) = \ln x/\ln 2$, $\overset{o}{v}_y(y) = \ln y/\ln 2$, and $\overset{o}{v}_z(z) = \ln z/\ln 2$, so that the ordinal additive multiattribute value function would be given by the sum of the logs of the variables. Notice that this ordinal value function is an order preserving transformation of the true underlying preference representation of the decision maker, which was never revealed explicitly to the analyst.

As this example illustrates, the assessment procedure will determine an additive ordinal value function that may be an order preserving transformation of a true preference relation that is not additive. The log function provides an example of such a transformation for a multiplicative preference relation, but other non-additive relationships may also be transformed to order preserving additive value functions. See Krantz et al. [35] for a discussion of other such transformations.

This example also illustrates the fact that the assessment methods required for accurately capturing an additive ordinal multiattribute value function may be tedious, and will require tradeoffs involving two or more attributes. This same point is made by Bouyssou and Pirlot in Chap. 4 of this volume. Thus, while this approach could be used in practice, it would be desirable to have simpler means of assessing the underlying preference functions. This can be accomplished if some additional preference conditions are satisfied, but the requirement of mutual preference independence will still be common to the preference models that are to follow.

8.4 Cardinal Multiattribute Preference Functions for the Case of Risk

When $X = \prod_{i=1}^{n} X_i$ in a von Neumann–Morgenstern utility model and the decision maker's preferences are consistent with some additional independence conditions, then $u(x_1, x_2, \ldots, x_n)$ can be decomposed into additive, multiplicative, and other well-structured forms that simplify assessment. In comparison with other sections, our coverage of this topic will be relatively brief since it is perhaps the most well known multiattribute preference model.

8.4.1 Utility Independence

An attribute X_i is said to be utility independent of its complementary attributes if preferences over lotteries with different levels of X_i do not depend on the fixed levels of the remaining attributes. Attributes $X_1, X_2, \ldots, X_n$ are mutually utility independent if all proper subsets of these attributes are utility independent of their complementary subsets. Further, it can be shown that if these same attributes are mutually preference independent, then they will also be mutually utility independent if any pair of the attributes is utility independent of its complementary attributes [32].

Returning to the automobile selection example, suppose that a decision maker is considering using the attributes of cost, horsepower, and appearance as before, but there is some uncertainty regarding some new environmental laws that may impact the cost and the horsepower of a particular automobile. Further, assume that the decision maker prefers more horsepower to less, lower costs and more attractive automobiles. The current performance levels of one of the alternatives may be ($25,000, 170 hp, ugly), but if the legislation is passed a new device will have to be fitted that will increase cost and decrease horsepower to ($25,700, 150 hp, ugly). An alternative automobile might have possible outcomes of ($28,000, 200 hp, ugly) and ($29,000, 175 hp, ugly) depending on this same legislation, which the decision maker estimates will pass with probability 0.5.

Therefore, the decision maker may consider choices between lotteries such as the one shown in Fig. 8.1. For example, the decision maker may prefer Auto 1 to Auto 2 because the risks associated with the cost and the horsepower for Auto 1 are more acceptable to her than the risks associated with the cost and horsepower of Auto 2. If the decision maker's choices for these lotteries do not depend on common values of the third attribute, then cost and horsepower are utility independent of appearance.

A multiattribute utility function $u(x_1, x_2, \ldots, x_n)$ can have the multiplicative form

$$1 + ku\left(x_1, x_2, \ldots, x_n\right) = \prod_{i=1}^{n}\left[1 + kk_i u_i\left(x_i\right)\right] \tag{8.4}$$

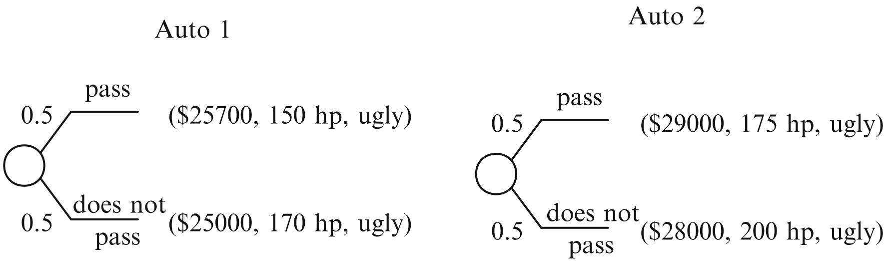


Fig. 8.1 Choice between two lotteries

if and only if the attributes $X_1, X_2, \ldots, X_n$ are mutually utility independent, where u_i is a single-attribute function over X_i scaled from 0 to 1, $0 \leq k_i \leq 1$ are positive scaling constants, and k is an additional scaling constant. If the scaling constant k is determined to be 0 through the appropriate assessment procedure, then (8.4) reduces to the additive form

$$\sum_{i=1}^{n} k_i u_i\left(x_i\right) \tag{8.5}$$

where $\sum_{i=1}^{n} k_i = 1$.

8.4.2 Additive Independence

A majority of the applied work in multiattribute utility theory deals with the case when the von Neumann–Morgenstern utility function is decomposed into the additive form (8.5). Fishburn [17] has derived necessary and sufficient conditions for a utility function to be additive. The key condition for additivity is the marginality condition which states that the preferences for any lotteries $p, q \in P$ should depend only on the marginal probabilities of the attribute values, and not on their joint probability distributions.

Returning to the automobile example once again, for additivity to hold, the decision maker must be indifferent between the two lotteries shown in Fig. 8.2, and for all other permutations of the attribute values that maintain the same marginal probabilities for each.

Notice that in either lottery, the marginal probability of receiving the most preferred outcome or the least preferred outcome on each attribute is identical (0.5). However, a decision maker may prefer the right-hand side lottery over the left-hand side lottery if the decision maker wishes to avoid a 0.5 chance of the poor outcome ($29,000, 150 hp, ugly) on all three attributes, or she may have the reverse

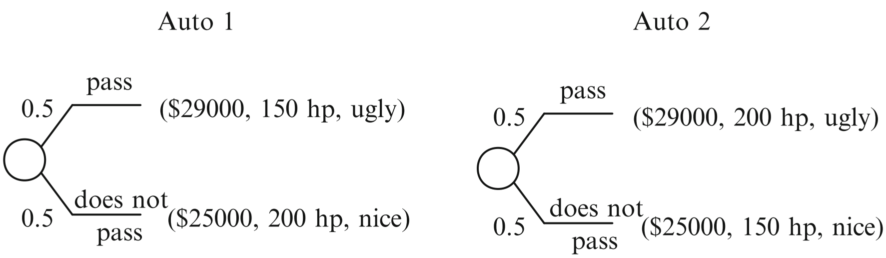


Fig. 8.2 Additive independence criterion for risk

preference if she is willing to accept some risk in order to have a chance at the best outcome on all three attributes. In either of the latter cases, utility independence may still be satisfied, and a multiplicative decomposition of the multiattribute utility function (8.4) may be appropriate.

Other independence conditions have been identified that lead to more complex non-additive decompositions of a multiattribute utility function. These general conditions are reviewed in Abbas and Bell [1].

8.4.3 Assessment Methodologies

The assessment of the multiplicative or additive form implied by the condition of mutual utility independence is simplified by the fact that each of the single-attribute utility functions may be assessed independently (more accurately, while all of the other attributes are held constant at arbitrarily selected values), using the well-known utility function assessment techniques suitable for single attribute utility functions. In addition, the constants k_{i} and k can be assessed using n relatively simple tradeoff questions. See Keeney and Raiffa [32] or Kirkwood [34] for additional details and examples.

8.5 Measurable Multiattribute Preference Functions for the Case of Certainty

We have delayed the discussion of measurable multiattribute preference functions until after the review of multiattribute utility theory because the latter may be more familiar to the reader. If so, this transposition of a more natural order of presentation may be helpful in providing the opportunity to discuss similarities between these models of preference, and therefore to enhance an intuitive understanding of the relationships among some important concepts.

Again let X denote the set of all possible consequences in a particular decision problem. In the multiattribute problem $X = \prod_{i=1}^{n} X_i$ where X_i is the set of possible consequences for the ith attribute. In this section, we use the letters $w, x, y,$ and z to indicate distinct elements of X. For example, $w \in X$ is represented by $(w_1, \ldots, w_n)$, where w_{i} is a level in the nonempty attribute set X_i for i = 1, …, n. Once again, we let $I \subset \{1, 2, \ldots, n\}$ be a subset of the attribute indices, define X_I as the subset of the attributes designated by the subscripts in I, and let $\overline{X}_I$ represent the complementary subset of the n attributes. We may write $w = (w_I, \overline{w}_I)$ or use the notation $(w_i, \overline{w}_i)$ and $(x_i, \overline{w}_i)$ to denote two elements of X that differ only in the level of the ith attribute. Finally, we also assume that the preference relation $\succsim$ on X is a weak order.

Next we introduce the notation necessary to define preferences based on strength of preference between vector-valued outcomes. We let $X^* = \{wx|\, w, x \in X\}$ be a nonempty subset of $X \times X$, and $\succsim^*$ denote a weak order on X^*. Once again, we may interpret wx $\succsim^*$ yz to mean that the preference difference between w and x is greater than the preference difference between y and z.

It seems reasonable to assume a relationship between $\succsim$ on X and $\succsim^*$ on X^* as follows. Suppose the attributes $X_1, \ldots, X_n$ are mutually preference independent. These two orders are difference consistent if, for all $w_i, x_i \in X_i$, $(w_i, \overline{w}_i) \succsim (x_i, \overline{w}_i)$ if and only if $(w_i, \overline{w}_i)\,(x_i{}^\circ, \overline{w}_i) \succsim^* (x_i, \overline{w}_i)\,(x_i{}^\circ, \overline{w}_i)$ for some $x_i^\circ \in X_i$ and some $\overline{w}_i \in \overline{X}_i$, and for any $i \in \{1, \ldots, n\}$, and if $w \sim$ x then $wy \sim^* xy$ or $yw \sim^* yx$ or both for any $y \in X$. Loosely speaking, this means that if one multiattributed alternative is preferred to another, then the preference difference between that alternative and some common reference alternative $\left(x_i^\circ, \overline{w}_i\right)$ will be larger than the preference difference between the alternative that is not preferred and this reference alternative.

8.5.1 Weak Difference Independence

In this section we identify a condition that we refer to as weak difference independence. This condition plays a role similar to the utility independence condition in multiattribute utility theory. We show how this condition can be exploited to obtain multiplicative and other nonadditive forms of the measurable multiattribute value function.

Specifically, the subset of attributes X_I is weak difference independent of $\overline{X}_I$ if, given any $w_I, x_I, y_I, z_I \in X_I$ and some $\overline{w}_I \in \overline{X}_I$ such that the subject's judgments regarding strength of preferences between pairs of multiattributed alternatives is as follows: $(w_I, \overline{w}_I)\,(x_I, \overline{w}_I) \succsim^* (y_I, \overline{w}_I)\,(z_I, \overline{w}_I)$, then the decision maker will also consider $(w_I, \overline{x}_I)\,(x_I, \overline{x}_I) \succsim^* (y_I, \overline{x}_I)\,(z_I, \overline{x}_I)$ for any $\overline{x}_I \in \overline{X}_I$. That is, the ordering of preference differences depends only on the values of the attributes X_I and not on the fixed values of $\overline{X}_I$.

The attributes are mutually weak difference independent if all proper subsets of these attributes are weak difference independent of their complementary subsets. Further, it can be shown that if these same attributes are mutually preference independent, then they will also be mutually weak difference independent if any pair of the attributes is weak difference independent of its complementary attributes [13].

Notice the similarity of the definition of weak difference independence to that of utility independence. In the latter case, preferences among lotteries depend only on the values of the attributes X_I and not on the fixed values of $\overline{X}_I$. In the case of certainty, the same notion applies to preference differences. Therefore, it should not be surprising that this condition leads to a decomposition of a measurable value function that is identical to the one implied by utility independence for utility functions.

This intuition may be formalized as follows. A measurable multiattribute value function $\overset{o}{v}(x_1, x_2, \ldots, x_n)$ on X can have the multiplicative form

$$1 + \lambda v(x) = \prod_{i=1}^{n} [1 + \lambda\lambda_i v_i(x_i)] \quad (8.6)$$

if and only if $X_1, \ldots, X_n$ are mutually weak difference independent, where $\overset{o}{v}_i$ is a single-attribute measurable value function over X_i scaled from 0 to 1, the λ_i are positive scaling constants, and λ is an additional scaling constant. If the scaling constant λ is determined to be 0 through the appropriate assessment procedure, then (8.6) reduces to the additive form

$$v(x) = \sum_{i=1}^{n} \lambda_i v_i(x_i) \quad (8.7)$$

where $\sum_{i=1}^{n} \lambda_i = 1$. Therefore, we obtain either an additive or a multiplicative measurable preference function that is based on notions of strength of preference.

8.5.2 Difference Independence

Finally, we are interested in the conditions that are required to ensure the existence of an additive multiattribute measurable value function. Recall that mutual preference independence guarantees the existence of an additive preference function for the case of certainty that will provide an ordinal ranking of alternatives, but it may not capture the underlying strength of preference of the decision maker. Further, the appropriate assessment technique will require tradeoffs that simultaneously consider two or more attributes as illustrated in Sect. 8.3.2.

Recall the example from Sect. 8.3.2 where the decision maker's true preferences were represented by the product of the attributes. If we were to ask the decision maker to express her preferences for the first attribute while holding the other attributes constant at some given values, she would respond in such a way that we would obtain a linear function for each attribute, rather than the correct logarithmic form. We would like to exclude this case, and be assured that the preference function that also measures strength of preference is additive.

Perhaps this point is worth some elaboration. Recall that the true preferences of the hypothetical decision maker introduced in Sect. 8.3.2 were consistent with the multiplicative representation xyz. Suppose we set y = z = 1, and ask the decision maker to consider the importance of changes in the attribute x while holding these other attribute values constant. Considering the alternatives (1,1,1), (3,1,1), and (5,1,1), she would indicate that the preference difference between (3,1,1) and (1,1,1) would be the same as the preference difference between (5,1,1) and (3,1,1). This is because her true preference relation gives $1 \times 1 \times 1 = 1$, $3 \times 1 \times 1 = 3$, $5 \times 1 \times 1 = 5$, and the preference difference between (3,1,1) and (1,1,1) is 3-1 = 2, which is also the preference difference between (5,1,1) and (3,1,1). If the analyst is not aware of the fact that this assessment approach cannot be used when only preference independence is satisfied, he might mistakenly conclude that $\overset{o}{v}(x, y, z) = x + y + z$ rather than the appropriate logarithmic transformation that we obtained earlier in Sect. 8.3.2.

The required condition for additivity that also provides a measurable preference function is called difference independence. The attribute X_i is difference independent of $\overline{X}_i$ if, for all $w_i, x_i \in X_i$ such that $(w_i, \overline{w}_i) \succsim (x_i, \overline{w}_i)$ for some $\overline{w}_i \in \overline{X}_i$, $(w_i, \overline{w}_i)\,(x_i, \overline{w}_i) \sim * (w_i, \overline{x}_i)\,(x_i, \overline{x}_i)$ for any $\overline{x}_i \in \overline{X}_i$. Intuitively, the preference difference between two multiattributed alternatives differing only on one attribute does not depend on the common values of the other attributes.

The attributes are mutually difference independent if all proper subsets of these attributes are difference independent of their complementary subsets. Again, it can be shown that if these same attributes are mutually preference independent, then they will also be mutually difference independent if X_1 is difference independent of $\overline{X}_1$ [13]. For the case of $n \geq 3$, mutual difference independence along with some additional structural and technical conditions[4] ensure that if wx, $yz \in X^*$, then $wx \succsim^* yz$ if and only if

$$\sum_{i=1}^{n} \lambda_i v_i(w_i) - \sum_{i=1}^{n} \lambda_i v_i(x_i) \geq \sum_{i=1}^{n} \lambda_i v_i(y_i) - \sum_{i=1}^{n} \lambda_i v_i(z_i) \tag{8.8}$$

and $x \succsim$ y if and only if

$$\sum_{i=1}^{n} \lambda_i v_i(x_i) \geq \sum_{i=1}^{n} \lambda_i v_i(y_i) \tag{8.9}$$

where v_i is a single-attribute measurable value function over X_i scaled from 0 to 1, and $\sum_{i=1}^{n} \lambda_i = 1$. Further, if v_i', i = 1, ..., n are n other functions with the same properties, then there exist constants $\alpha > 0, \beta_1, \ldots, \beta_n$ such that $v_i' = \alpha v_i + \beta_i$, $i = 1, \ldots, n$.

Result (8.9) is well known and follows immediately from the assumption that the attributes are mutually preference independent (Sect. 8.3.1). The significant result

[4]Specifically, we assume restricted solvability from below, an Archimedean property, at least three attributes are essential, and that the attributes are bounded from below. If n = 2, we assume that the two attributes are preferentially independent of one another and that the Thomsen condition is satisfied (see Krantz et al. [35] and the discussion by Bouyssou and Pirlot in Chap. 4 of this volume).

is (8.8), which means that $v(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} \lambda_i v_i(x_i)$ also provides difference measurement on X. Note that this latter result is obtained based on the observation that any arbitrarily selected attribute is difference independent of its complementary attributes.

8.5.3 Assessment Methodologies

Because the notion of a measurable multiattribute value function may not be familiar to many readers, we will briefly consider methods for the assessment of them. Further details and examples are provided by von Winterfeldt and Edwards [47], and by Kirkwood [34].

8.5.3.1 Verification of the Independence Conditions

The first issue to be considered is the verification of the independence conditions. Since methods for verifying mutual preference independence are discussed in Keeney and Raiffa [32], we focus on the independence conditions involving preference differences.

Difference consistency is so intuitively appealing that it could simply be assumed to hold in most practical applications. The following procedure could be used to verify difference independence. We determine $w_1, x_1 \in X_1$ such that $(w_1, \overline{w}_1) \succsim (x_1, \overline{w}_1)$ for some $\overline{w}_1 \in \overline{X}_1$. We then ask the decision maker to imagine that she is in situation 1: She already has $(x_1, \overline{w}_1)$ and she can exchange it for $(w_1, \overline{w}_1)$. Next, we arbitrarily choose $\overline{x}_1 \in \overline{X}_1$ and ask her to imagine situation 2: She already has $(x_1, \overline{x}_1)$, and she can exchange it for $(w_1, \overline{x}_1)$. Would she prefer to make the exchange in situation 1 or in situation 2, or is she indifferent between the two exchanges? If she is indifferent between the two exchanges for several different values of $w_1, x_1 \in X_1$ and $\overline{w}_1, \overline{x}_1 \in \overline{X}_1$ then we can conclude that X_1 is difference independent of $\overline{X}_1$.

For example, suppose we ask the decision maker to consider exchanging a car described by ($25,000, 150 hp, ugly) for a car described by ($25,000, 180 hp, ugly). Next, we ask her to consider exchanging ($35,000, 150 hp, nice) for ($35,000, 180 hp, nice). Would the opportunity to exchange a car with 150 hp for one with 180 hp be more important to the decision maker when the cost and appearance are $25,000 and ugly, or when they are $35,000 and nice? If the common values of these two attributes do not affect her judgments of the importance of these exchanges, then horsepower would be difference independent of cost and appearance.

Before using this procedure, we must ensure that the decision maker understands that we are asking her to focus on the exchange rather than on the final outcomes. For example, if she states that she prefers an exchange of $1,000,000 for $1,000,001 to an exchange of $5 for $500, then she undoubtedly is not focusing on the substitution of one outcome for another, but she is focusing instead on the final outcome. Thus, some training may be required before this approach to verification of difference independence is attempted.

To verify weak difference independence, partition X into X_I and $\overline{X}_I$, and choose $w_I, x_I, y_I, z_I \in X_I$ and $\overline{w}_I \in \overline{X}_I$ so that $(w_I, \overline{w}_I) \succ (x_I, \overline{w}_I)$, $(y_I, \overline{w}_I) \succ (z_I, \overline{w}_I)$ and the exchange of $(x_I, \overline{w}_I)$ for $(w_I, \overline{w}_I)$ is preferred to the exchange of $(z_I, \overline{w}_I)$ for $(y_I, \overline{w}_I)$. Then pick another value $\overline{x}_I$ of $\overline{X}_I$ and ask if the decision maker still prefers the exchange of $(x_I, \overline{x}_I)$ for $(w_I, \overline{x}_I)$ to the exchange of $(z_I, \overline{x}_I)$ for $(y_I, \overline{x}_I)$. This must be true if the subset X_I is weakly difference independent of $\overline{X}_I$. If the decision maker's response is affirmative, we repeat the question for other quadruples of consequences from X_I with the values of the criteria in $\overline{X}_I$ fixed at different levels. Continuing in this manner and asking the decision maker to verbally rationalize her responses, the analyst can either verify that X_I is weakly difference independent of $\overline{X}_I$ or discover that the condition does not hold.

Note that for the multiplicative measurable value function, it would only be necessary to verify weak difference independence for the special case of $I = \{i,j\}$, where i and j indicate the subscripts of an arbitrarily chosen pair of alternatives. This is true so long as the attributes are mutually preference independent.

For example, suppose we establish that the decision maker would prefer the exchange of the car ($25,000, 150 hp, ugly) for the car ($27,000, 200 hp, ugly) to the exchange of the car ($24,000, 130 hp, ugly) for the car ($25,000, 150 hp, ugly). If this preference for the first exchange over the second exchange does not depend on the common value of appearance, and if it also holds true for all other combinations of the values of cost and horsepower, then cost and horsepower are weak difference independent of appearance.

8.5.3.2 Assessment of the Measurable Value Functions

If difference independence or weak difference independence holds, each conditional measurable value function $\overset{o}{v}_i$ can be assessed while holding $\overline{x}_i$ constant at any arbitrary value (generally at $\overline{x}_i{}^\circ$). With the additive value function that does not provide difference measurement, this strategy cannot be used as illustrated above. As a result, any of the approaches for assessing a single attribute measurable value function referenced in Sect. 8.3.1 may be used, including the direct rating of attribute values on an arbitrary scale (e.g., from 0 to 100), or direct estimates of preference differences.

If the measurable value function is additive, the scaling constants may be assessed using the same trade-off approach suggested for estimating the scaling constants for the additive ordinal value function [32, Chap. 3]. For a discussion of other approaches to the assessment of the scaling constants for the additive and multiplicative cases, see also Dyer and Sarin [13].

Measurable multiattribute value functions may also be assessed using the ratio judgments and tools provided by the AHP methodology, and used as a basis for relating the AHP to formal preference theories that are widely accepted by economists and decision analysts. This point has been made by several authors, notably Kamenetzky [29] and Dyer [12].

Perhaps the best discussion of this important point is provided by Salo and Hamalainen [41]. As they observe, once a suitable range of performance [x_i°, x_i^*] has been defined for each attribute, the additive measurable value function representation may be scaled so that the values $v(x^\circ) = v\left(x_1^\circ, \ldots, x_n^\circ\right) = 0$ and $v(x^*) = v\left(x_1^*, \ldots, x_n^*\right) = 1$ are assigned to the worst and best conceivable consequences, respectively. By also normalizing the component value functions onto the [0,1] range, the additive representation can be written as

$$v(x) = \sum_{i=1}^{n} v_i(x_i) = \sum_{i=1}^{n} \left[v_i(x_i) - v_i\left(x_i^\circ\right)\right]$$

$$= \sum_{i=1}^{n} \left[v_i\left(x_i^*\right) - v_i\left(x_i^\circ\right)\right] \frac{v_i(x_i) - v_i\left(x_i^\circ\right)}{v_i\left(x_i^*\right) - v_i\left(x_i^\circ\right)} = \sum_{i=1}^{n} w_i s_i(x_i)$$

where $s_i(x_i) = v_i(x_i) - v_i\left(x_i^\circ\right) / v_i\left(x_i^*\right) - v_i\left(x_i^\circ\right) \in [0, 1]$ is the normalized score of x on the ith attribute and $w_i = v\left(x_i^*\right) - v\left(x_i^\circ\right)$ is the scaling constant or weight of the ith attribute.

A careful evaluation of this representation leads Salo and Hamalainen to the conclusion that pair wise comparisons in ratio estimation should be interpreted in terms of ratios of value differences between pairs of underlying alternatives. This, in turn, provides the link between traditional models of preference theory and the AHP, and reveals that the latter can be an alternative assessment technique for measurable multiattribute value functions (with some simple adjustments for normalization and scaling).

8.5.4 Goal Programming and Measurable Multiattribute Value Functions

Goal programming was originally proposed by Charnes et al. [10] as an ingenious approach to developing a scheme for executive compensation. As noted by Charnes and Cooper [9] in a review of the field, this approach to multiple objective optimization did not receive significant attention until the mid-1960s. However, during the past 40 years, we have witnessed a flood of professional articles and books dealing with applications of this methodology (e.g. see Ignizio [25], Trzaskalik and Michnik [45], Ignizio and Romero [26]).

This discussion is limited to the use of goal programming as a methodology for solving problems with multiple, compensatory objectives. That is, we do not address problems that do not allow tradeoffs among the objectives. These non-compensatory models involve the use of the non-Archimedian, or “preemptive priority”, weights.

An analysis of these models would be based on the theory of lexicographic orders, summarized by Fishburn [19]. The conditions that would justify the use of a non-compensatory model are very strict, and are unlikely to be met in a significant number of real-world applications.

8.5.4.1 Goal Programming as an Approximation to Multiattribute Preferences

Let us begin with a simple example. Suppose a manager has identified a problem that can be formulated as a traditional mathematical programming problem with one exception—there are two criterion functions, $f_1(x)$ and $f_2(x)$ where $x \in X$ is an n-vector of controllable and uncontrollable variables, and the non-empty feasible set X is defined by a set of constraints. For simplicity, and without loss of generality, we assume that our choice of X ensures $0 \le f_i(x) \le 1,\ i = 1, 2$.

To use goal programming, we ask the manager if she has any "goals" in mind for the criteria. She replies that she would be happy if $f_1(\cdot)$ were at least as large as b_1, but she does not feel strongly about increasing $f_1(\cdot)$ beyond b_1. However, she would like for $f_2(\cdot)$ to be somewhere between b_{2L} and b_{2U}. Finally, we ask her to assign "weights" of relative importance to the deviations of $f_1(\cdot)$ from b_1, and of $f_2(\cdot)$ from b_{2L} and b_{2U}, respectively. After some thought, she responds with the weights w_1, w_2 and w_3.

Now, we can immediately write down this problem as follows:

$$\begin{aligned} \min_{x \in X} \quad & w_1 y_1^- + w_2 y_2^- + w_3 y_3^+ \\ \textit{subject}\text{ to} \quad & f_1(x) - y_1^+ + y_1^- = b_1 \\ & f_2(x) - y_2^+ + y_2^- = b_{2L} \\ & f_2(x) - y_3^+ + y_3^- = b_{2U} \\ & y_i^+, y_i^- \ge 0,\ i = 1, 2, 3 \end{aligned} \tag{GP}$$

Notice that GP includes a "one-sided" formulation with respect to $f_1(\cdot)$, and a "goal interval" formulation with respect to $f_2(\cdot)$.

Let us pause a moment to reflect on this formulation. First, notice that $y_1^- = b_1 + y_1^+ - f_1(x)$. Suppose we introduce the relationship $f_1(x) - y_{12}^+ = 0$ as a new constraint for GP. Since b_1 is a constant, minimizing $w_1 y_1^-$ is obviously equivalent to minimizing $w_1 \left(y_1^+ - y_{12}^+\right)$.

Similarly, if we introduce the constraint $f_2(x) - y_{22}^+ = 0$, minimizing $w_2 y_2^-$ is equivalent to minimizing $w_2 \left(y_2^+ - y_{22}^+\right)$, and minimizing $w_3 y_3^+$ is equivalent to minimizing $w_3 \left(y_3^- + y_{22}^+ - b_{2U}\right)$. The constant b_{2U} is maintained in the last expression in order to facilitate a graphical portrayal of the objective function as we shall see. Combining the results and re-writing GP as a maximization problem, we have the equivalent problem statement:

$$\begin{array}{ll}\max\limits_{x\in X} & w_1\left(y_{12}^+ - y_1^+\right) + w_2\left(y_{22}^+ - y_2^+\right) + w_3\left(y_{22}^+ + y_3^- - b_{2U}\right) \\ \textit{subject to} & f_1(x) - y_1^+ + y_1^- = b_1 \\ & f_2(x) - y_2^+ + y_2^- = b_{2L} \\ & f_2(x) - y_3^+ + y_3^- = b_{2U} \\ & f_1(x) - y_{12}^+ = 0 \\ & f_2(x) - y_{22}^+ = 0 \\ & y_i^+, y_i^- \geq 0,\ i = 1,2,3 \\ & y_{12}^+, y_{22}^+ \geq 0 \end{array} \tag{VA}$$

where the objective function may be interpreted as the sum of two piecewise linear functions (e.g. see [8, pp. 351–355])

Figures 8.3 and 8.4 illustrate these two piecewise linear functions. Recall that piecewise linear transformations are commonly used to transform additive separable nonlinear programming problems into linear programming problems. The lines labeled $v_1(\cdot)$ and $v_2(\cdot)$ in Figs. 8.3 and 8.4 respectively suggest nonlinear preference functions that might be approximated by the bold piecewise linear functions.

Thus since VA is equivalent to GP, and VA may be viewed as a piecewise linear approximation to an additive separable nonlinear objective function, we are led to the conclusion that GP is an implicit approximation to the problem:

$$\max_{x\in X} \quad \overset{o}{v}_1(f_1(x)) + v_2(f_2(x)) \tag{V}$$

And how do we interpret V? Since the choice of goals and goal intervals in GP reflect the decision maker's preferences and no uncertainty is involved in the

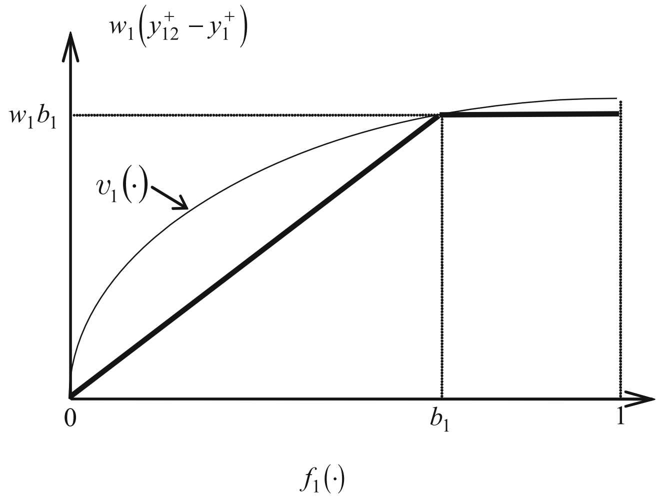


Fig. 8.3 Piecewise linear approximation of $v_1(\cdot)$

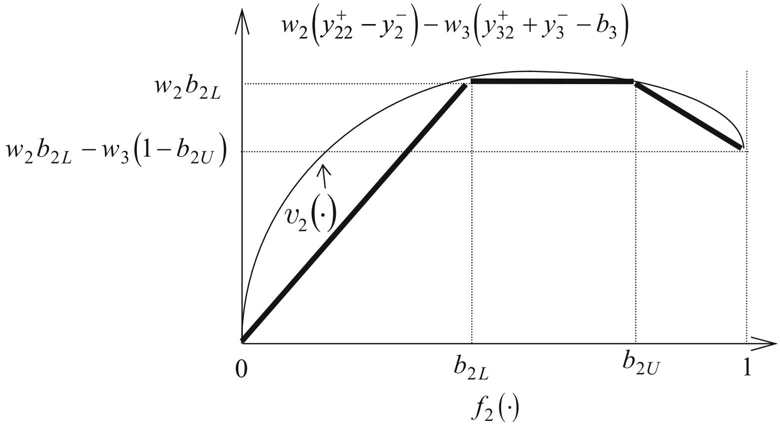


Fig. 8.4 Piecewise linear approximation of $v_2(\cdot)$

decision, $v_1(\cdot)$ and $v_2(\cdot)$ are measurable functions, and their sum, $v(f_1(\cdot), f_2(\cdot)) = v_1(f_1(\cdot)) + v_2(f_2(\cdot))$, is an additive separable measurable value function.

Goal programming is generally applied to problems where risk is not explicitly involved in the formulation. Therefore, the additive utility function theory developed for risky choice is not relevant for these applications. Likewise, the ordinal additive theories are not operational here because they require a simultaneous conjoint scaling of the separable terms. Goal programming applications generally allow the selection of each goal or goal interval independent of consideration of the values of the other criteria. This practice implies the existence of a measurable additive utility function under certainty.

This point has been made recently by Bordley and Kirkwood [5] in a general discussion of the relationship between goals and multiattribute preference models. Also see Abbas and Matheson [2]. This perspective provides some insights regarding the nature of goal programming, as well as some challenges. For example, how should the piecewise linear approximations to the nonlinear value functions be selected in order to minimize error? Geoffrion [22] provides some useful guidelines for choosing "goals" or "goal intervals" for each criterion so that the piecewise linear approximation to the implicit utility function provides the best fit.

One important implication of this point of view is that goal programming should not be considered an ad hoc, heuristic approach to solving multiple objective problems. Rather, the approach is based on a set of implicit, well-understood assumptions from multiattribute preference theory. Goal programming formulations should be either criticized or justified on the basis of these assumptions.

8.6 The Relationships among the Multiattribute Preference Functions

The necessary conditions for the additive and multiplicative measurable value functions and risky utility functions, notably mutual preference independence, are also necessary and sufficient for the ordinal additive value function that does not provide difference measurement. Therefore, it is natural to investigate the relationships among them. The following choice of scaling will be imposed. For $f = \overset{\circ}{v}, v$, or u, f is normalized by $f\,(x_1{}^*, \ldots, x_n{}^*) = 1$ and $f\,(x_1{}^\circ, \ldots, x_n{}^\circ) = 0$ and $f_i(x_i)$ is a conditional function on X_i scaled by $f_i\left(x_i^*\right) = 1$ and $f_i\left(x_i^\circ\right) = 0$. Finally, $\overset{o}{\lambda}, \lambda$ and k will be used as scaling constants for the ordinal and measurable value functions and the utility function, respectively.

8.6.1 The Additive Functions

The relationships among the alternative developments of the additive forms of real-valued functions on X follow immediately from their respective uniqueness properties [13]. This may be summarized as follows. Assume n $\geqq$ 3 and $X_1, \ldots, X_n$ are mutually preference independent. Then

1. if $X_1, \ldots, X_n$ are difference consistent and X_1 is difference independent of $\overline{X}_1$ then $\overset{\circ}{v} = v$;
2. if there exists a utility function u on X and if preferences over lotteries on $X_1, \ldots, X_n$ depend only on their marginal probability distributions and not on their joint probability distributions, then $\overset{\circ}{v} = u$.
3. if both 1 and 2 are satisfied, $\overset{\circ}{v} = v = u$

Note the implication of this result. In order for $\overset{\circ}{v} = v = u$ for a single decision maker, she must have preferences simultaneously consistent with mutual preference independence, difference independence, and additive independence for risky alternatives. Mutual preference independence will hold in all cases, but it may be the case that difference independence and/or additive independence for risky alternatives will not hold. Further, difference independence may hold for the preferences of a decision maker, implying that an additive measurable value function would provide a valid representation of her preferences, but additive independence for risky alternatives may not be satisfied, implying that an additive utility function would not be a valid representation of her preferences in decision scenarios involving risk.

8.6.2 *The Multiplicative Functions*

Throughout this section we assume that the following conditions are satisfied:

1. There are $n \geqq 3$ attributes, and $X_1, \ldots, X_n$ are mutually preference independent;
2. There exists a measurable value function v on X and X_1 is weak difference independent of $\overline{X}_1$, and
3. There exists a utility function u on X and X_1 is utility independent of $\overline{X}_1$.

Suppose we have assessed the additive value function $\mathring{v}$ and wish to obtain either v or u. Then the following relationships will hold ([13], Theorem 5). Either

1. 1. $\mathring{v}(x) = v(x)$ and $\mathring{v}_i(x_i) = v_i(x_i)$, i = 1, ..., n, or
2. 2. $\mathring{v}(x) \ln(1+\lambda) = \ln[1+\lambda v(x)]$ and $\mathring{\lambda}_i \mathring{v}_i(x_i) \ln(1+\lambda) = \ln[1+\lambda\lambda_i v_i(x_i)]$, i = 1, ..., n.

Either

1. 1. $\mathring{v}(x) = u(x)$ and $\mathring{v}_i(x_i) = u_i(x_i)$, i = 1, ..., n, or,
2. 2. $\mathring{v}(x) \ln(1+k) = \ln[1+ku(x)]$ and $\mathring{\lambda}_i \mathring{v}_i(x_i) \ln(1+k) = \ln[1+kk_i u_i(x_i)]$, $i = 1, \ldots, n$.

These relationships may be used to simplify the assessment of multiattribute preference functions. For example, suppose we define $x_i{}'$ as the equal difference point for attribute X_i if $(x_i{}', \bar{x}_i)\,(x_i{}^\circ, \bar{x}_i) \sim^* (x_i{}^*, \bar{x}_i)\,(x_i{}', \bar{x}_i)$ for any $\bar{x}_i \in \overline{X}_i$. Notice that $v_i(x_i{}') = {}^1\!/_2$ because of our choice of scaling. Given $\mathring{v}$, the assessment of $x_i{}'$ for any attribute X_i is enough to completely specify v, because if $v_i(x_i{}') = {}^1\!/_2$ for some $i \in \{1, \ldots, n\}$ then $v = \mathring{v}$. Otherwise, $1 + (1+\lambda)^{\mathring{\lambda}_i} = 2(1+\lambda)^{\mathring{\lambda}_i \mathring{v}_i(x_i')}$.

Finally, to derive u from $\mathring{v}$, find $x_i{}''$ for some attribute X_i such that the decision maker is indifferent between $x_i{}''$ and an equal chance lottery between $x_i{}^*$ and $x_i{}^\circ$ with the other criteria held fixed. A parallel result to the above relationship between ordinal and measurable value functions holds. Specifically, if $\mathring{v}_i(x_i{}'') = {}^1\!/_2$ for some $i \in \{1, \ldots, n\}$,, then $u = \mathring{v}$. Otherwise, $1 + (1+k)^{\mathring{\lambda}_i} = 2(1+k)^{\mathring{\lambda}_i \mathring{v}_i(x_i')}$.

These results can also be used to derive v after u has been assessed, or vice versa. For example, suppose u has been assessed using appropriate procedures. To obtain v, we find the equal difference point $x_i{}'$ for some criterion X_i. The second result above is used to obtain $\mathring{\lambda}_i$ and $\mathring{v}_i$ for each criterion, and we can obtain v. In a similar manner, u can be obtained from v after assessing $x_i{}''$ for some criterion X_i.

Since the AHP can be interpreted as ratios of preference differences, this relationship also allows the results from assessments based on the AHP to be suitably transformed into multiattribute utility functions appropriate for use in risky situations. This completes the circle required to synthesize ordinal multiattribute value functions, measurable multiattribute value functions, multiattribute utility functions, and multiattribute functions based on ratio judgments. As a result, the analyst is justified in choosing among a variety of assessment tools, and making the appropriate adjustments in order to calibrate the results into a coherent and theoretically sound representation of preferences.

8.7 Concluding Remarks

In this chapter, we have presented an informal discussion of "multiattribute utility theory". In fact, this discussion has emphasized that there is no single version of multiattribute utility theory that is relevant to multicriteria decision analysis. Instead, there are three distinct theories of multiattribute preference functions that may be used to represent a decision maker's preferences.

The ordinal additive multiattribute preference model requires the assumption of mutual preference independence, and is appropriate for use in the case of certainty. Most of the applications and methods of multicriteria decision analysis are presented in the context of certainty, and so this would seem to be an appealing theory to use for framing these approaches. However, as we have emphasized, the ordinal additive multiattribute preference model requires assessment techniques that are cumbersome in practice, and that force the decision maker to make explicit tradeoffs between two or more criteria in the assessment of the value functions defined on the individual criteria.

The measurable value functions also require the assumption of mutual preference independence, along with the stronger assumptions of weak difference independence or difference independence in order to obtain convenient decompositions of the model that are easy to assess. The assessment of these preference models is relatively easy, and they can be interpreted intuitively as providing a measure of strength of preference. In addition, the ratio judgments of the AHP can be interpreted as ratios of preference differences based on this theory, linking the AHP methodology to traditional models of preference accepted in the decision analysis and economics literatures.

Finally, multiattribute utility theory is an elegant and useful model of preference suitable for applications involving risky choice. The brilliant work of Keeney and Raiffa [32] has made this theory synonymous with multiple criterion decision making, and the ordinal and measurable theories are often overlooked or ignored as a result. In fact, these latter approaches may provide more attractive and appropriate theories for many applications of multicriteria decision analysis.

References

1. Abbas, A.E., Bell, D.: One-switch independence for multiattribute utility functions. Oper. Res. **59**(3), 764–771 (2011)
2. Abbas, A.E., Matheson, J.: Normative decision making with multiattribute performance targets. J. Multicrit. Decis. Anal. **16**, 67–78 (2010)
3. Anderson, R., Clemen, R.: Toward an improved methodology to construct and reconcile decision analytic preference judgments. Decis. Anal. **10**, 121–134 (2013)
4. Bell, D., Raiffa, H.: Risky choice revisited. In: Bell, D.E., Raiffa, H., Tversky, A. (eds.) Decision Making: Descriptive, Normative, and Prescriptive Interactions. Cambridge University Press, New York (1988)
5. Bordley, R., Kirkwood, C.: Multiattribute preference analysis with performance targets. Oper. Res. **52**(6), 823–835 (2004)
6. Butler, J.C., Dyer, J.S., Jia, J.: Using attributes to predict objectives in preference models. Decis. Anal. **3**(2), 100–116 (2006)
7. Camerer, C.: Individual decision making. In: Kagel, J.H., Roth, A.E. (eds.) Handbook of Experimental Economics. Princeton University Press, Princeton (1995)
8. Charnes, A., Cooper, W.W.: Management Models and Industrial Applications of Linear Programming. Wiley, New York (1961)
9. Charnes, A., Cooper, W.W.: Goal programming and multiple objective optimizations. Eur. J. Oper. Res. **1**(1), 37–54 (1977)
10. Charnes, A., Cooper, W.W., Ferguson, R.O.: Optimal estimation of executive compensation by linear programming. Manag. Sci. **1**(2), 138–151 (1955)
11. Debreu, G.: Topological methods in cardinal utility theory. In: Arrow, K.J., Karlin, S., Suppes, P. (eds.) Mathematical Methods in the Social Sciences. Stanford University Press, Stanford (1959)
12. Dyer, J.S.: Remarks on the analytic hierarchy process. Manag. Sci. **36**, 249–258 (1990)
13. Dyer, J.S., Sarin, R.A.: Measurable multiattribute value functions. Oper. Res. **22**, 810–822 (1979)
14. Dyer, J.S., Sarin, R.K.: Relative risk aversion. Manag. Sci. **28**, 875–886 (1982)
15. Ellsberg, D.: Classic and current notions of "Measurable Utility". Econometrica **64**, 528–556 (1954)
16. Farquhar, P.H., Keller, L.R.: Preference intensity measurement. Ann. Oper. Res. **19**, 205–217 (1989)
17. Fishburn, P.C.: Independence in utility with whole product sets. Oper. Res. **13**, 28–45 (1965)
18. Fishburn, P.C.: Utility Theory for Decision Making. Wiley, New York (1970)
19. Fishburn, P.C.: Lexicographic orders, utilities, and decision rules. Manag. Sci. **20**, 1442–1471 (1974)
20. Fishburn, P.C.: Nonlinear Preference and Utility Theory. Johns Hopkins University Press, Baltimore and London (1988)
21. Fishburn, P.C., Wakker, P.: The invention of the independence condition for preferences. Manag. Sci. **41**(7), 1130–1144 (1995)
22. Geoffrion, A.: Objective function approximations in mathematical programming. Math. Program. **13**(1), 23–37 (1977)
23. Geoffrion, A., Dyer, J.S., Feinberg, A.: An interactive approach for multi-criterion optimization with an application to the operation of an academic department. Manag. Sci. **19**, 357–368 (1972)
24. Gorman, W.M.: The structure of utility functions. Rev. Econ. Stud. **35**, 367–390 (1968)
25. Ignizio, J.P.: Introduction to Linear Goal Programming. Sage, Thousand Oaks (1986)
26. Ignizio, J., Romero, C.: Goal programming. In: Bidgoli, H. (ed.) Encyclopedia of Information Systems, vol. 2, pp. 489–500. Academic, San Diego (2003)
27. Jia, J., Dyer, J.S.: A standard measure of risk and risk-value models. Manag. Sci. **42**(12), 1691–1705 (1996)

28. Kahneman, D., Tversky, A.: Prospect theory: an analysis of decisions under risk. Econometrica **47**, 276–287 (1979)
29. Kamenetzky, R.D.: The relationship between the analytic hierarchy process and the additive value function. Decis. Sci. **13**, 702–713 (1982)
30. Keeney, R.L.: Value-Focused Thinking. Harvard University Press, Boston (1996)
31. Keeney, R.L., Gregory, R.S.: Selecting attributes to measure the achievement of objectives. Oper. Res. **53**(1), 1–11 (2005)
32. Keeney, R.L., Raiffa, H.: Decision with Multiple Objectives: Preference and Value Tradeoffs. Cambridge University Press, New York (1993)
33. Keller, L.R.: An empirical test of relative risk aversion. IEEE Trans. Man Syst. Cybern. **SMC-15**, 476–482 (1985)
34. Kirkwood, C.W.: Strategic Decision Making. Duxbury, Belmont (1996)
35. Krantz, D.H., Luce, R.D., Suppes, P., Tversky, A.: Foundations of Measurement, Volume I, Additive and Polynomial Representations. Academic, New York (1971)
36. Kreps, D.M.: A Course in Microeconomic Theory. Princeton University Press, Princeton (1990)
37. Krzysztofowicz, R.: Strength of preference and risk attitude in utility measurement. Organ. Behav. Hum. Perform. **31**, 88–113 (1983)
38. Luce, R.D., Tukey, J.W.: Simultaneous conjoint measurement: a new type of fundamental measurement. J. Math. Psychol. **1**, 1–27 (1964)
39. Pratt, J.W.: Risk aversion in the small and in the large. Econometrica **32**, 122–136 (1964)
40. Saaty, T.L.: The Analytic Hierarchy Process. McGraw-Hill, New York (1980)
41. Salo, A.A., Hamalainen, R.P.: On the measurement of preferences in the analytic hierarchy process. J. Multi-Criterion Decis. Anal. **6**, 309–319 (1997)
42. Sarin, R.K.: Strength of preference and risky choice. Oper. Res. **30**, 982–997 (1982)
43. Savage, L.J.: The Foundations of Statistics. Wiley, New York (1954)
44. Scott, D., Suppes, P.: Foundational aspects of theories of measurement. J. Symb. Log. **23**, 113–128 (1958)
45. Trzaskalik, T., Michnik, J. (eds.): Multiple Objective and Goal Programming: Recent Developments. Physica Verlag, Heidelberg (2002)
46. von Neumann, J., Morgenstern, O.: Theory of Games and Economic Behavior. Princeton University Press, Princeton (1947)
47. von Winterfeldt, D., Edwards, W.: Decision Analysis and Behavioral Research. Cambridge University Press, Cambridge (1986)
48. Wakker, P.: Prospect Theory: For Risk and Ambiguity. Cambridge University Press, Cambridge (2010)

Chapter 9
UTA Methods

Yannis Siskos, Evangelos Grigoroudis, and Nikolaos F. Matsatsinis

Abstract UTA methods refer to the philosophy of assessing a set of value or utility functions, assuming the axiomatic basis of MAUT and adopting the preference disaggregation principle. UTA methodology uses linear programming techniques in order to optimally infer additive value/utility functions, so that these functions are as consistent as possible with the global decision-maker's preferences (inference principle). The main objective of this chapter is to analytically present the UTA method and its variants and to summarize the progress made in this field. The historical background and the philosophy of the aggregation-disaggregation approach are firstly given. The detailed presentation of the basic UTA algorithm is presented, including discussion on the stability and sensitivity analyses. Several variants of the UTA method, which incorporate different forms of optimality criteria, are also discussed. The implementation of the UTA methods is illustrated by a general overview of UTA-based DSSs, as well as real-world decision-making applications. Finally, several potential future research developments are discussed.

Keywords UTA methods • Preference disaggregation • Ordinal regression • Additive utility • Multicriteria analysis

Y. Siskos (✉)
Department of Informatics, University of Piraeus, 80 Karaoli & Dimitriou Street, 18534 Piraeus, Greece
e-mail: ysiskos@unipi.gr

E. Grigoroudis • N.F. Matsatsinis
Decision Support Systems Laboratory, Technical University of Crete, University Campus, Kounoupidiana, 73100 Chania, Greece
e-mail: vangelis@ergasya.tuc.gr; nikos@ergasya.tuc.gr

S. Greco et al. (eds.), *Multiple Criteria Decision Analysis*, International Series in Operations Research & Management Science 233,
DOI 10.1007/978-1-4939-3094-4_9

9.1 Introduction

9.1.1 General Philosophy

In decision-making involving multiple criteria, the basic problem stated by analysts and Decision-Makers (DMs) concerns the way that the final decision should be made. In many cases, however, this problem is posed in the opposite way: assuming that the decision is given, how is it possible to find the rational basis for the decision being made? Or equivalently, how is it possible to assess the DM's preference model leading to exactly the same decision as the actual one or at least the most "similar" decision? The philosophy of preference disaggregation in multicriteria analysis is to assess/infer preference models from given preferential structures and to address decision-aiding activities through operational models within the aforementioned framework.

Under the term "multicriteria analysis" two basic approaches have been developed involving:

1. a set of methods or models enabling the aggregation of multiple evaluation criteria to choose one or more actions from a set A, and
2. an activity of decision-aid to a well-defined DM (individual, organization, etc.).

In both cases, the set A of potential actions (or objectives, alternatives, decisions) is analyzed in terms of multiple criteria in order to model all the possible impacts, consequences or attributes related to the set A.

Roy [108] outlines a general modeling methodology of decision-making problems, which includes four modeling steps starting with the definition of the set A and ending with the activity of decision-aid, as follows:

- *Level 1*: Object of the decision, including the definition of the set of potential actions A and the determination of a problem statement on A.
- *Level 2*: Modeling of a consistent family of criteria assuming that these criteria are non-decreasing value functions, exhaustive and non-redundant.
- *Level 3*: Development of a global preference model, to aggregate the marginal preferences on the criteria.
- *Level 4*: Decision-aid or decision support, based on the results of level 3 and the problem statement of level 1.

In level 1, Roy [108] distinguishes four reference problem statements, each of which does not necessarily preclude the others. These problem statements can be employed separately, or in a complementary way, in all phases of the decision-making process. The four problem statement are the following:

- *Problem statement* α: Choosing one action from A (choice).
- *Problem statement* β: Sorting the actions into predefined and preference ordered categories.
- *Problem statement* γ: Ranking the actions from the best one to the worst one (ranking).

– *Problem statement* δ: Describing the actions in terms of their performances on the criteria (description).

In level 2, the modeling process must conclude with a consistent family of criteria $\{g_1, g_2, \ldots, g_n\}$. Each criterion is a non-decreasing real valued function defined on A, as follows:

$$g_i : A \to [g_{i^*}, g_i^*] \subset \Re / a \to g(a) \in \Re \tag{9.1}$$

where $[g_{i^*}, g_i^*]$ is the criterion evaluation scale, g_{i^*} and g_i^* are the worst and the best level of the i-th criterion respectively, $g_i(a)$ is the evaluation or performance of action a on the i-th criterion and $\mathbf{g}(a)$ is the vector of performances of action a on the n criteria.

From the above definitions, the following preferential situations can be determined:

$$\begin{cases} g_i(a) > g_i(b) \Leftrightarrow a \succ b \;\; (a \text{ is preferred to } b) \\ g_i(a) = g_i(b) \Leftrightarrow a \sim b \;\; (a \text{ is indifferent to } b) \end{cases} \tag{9.2}$$

So, having a weak-order preference structure on a set of actions, the problem is to adjust additive value or utility functions based on multiple criteria, in such a way that the resulting structure would be as consistent as possible with the initial structure. This principle underlies the disaggregation-aggregation approach presented in the next section.

This chapter is devoted to UTA methods, which are regression based approaches that have been developed as an alternative to multiattribute utility theory (MAUT). UTA methods not only adopt the aggregation-disaggregation principles, but they may also be considered as the main initiatives and the most representative examples of preference disaggregation theory. Another, more recent example of the preference disaggregation theory is the dominance-based rough set approach (DRSA) leading to decision rule preference model via inductive learning (see Chap. 9.5 of this book).

9.1.2 The Disaggregation-Aggregation Paradigm

In the traditional aggregation paradigm, the criteria aggregation model is known a priori, while the global preference is unknown. On the contrary, the philosophy of disaggregation involves the inference of preference models from given global preferences (Fig. 9.1).

The disaggregation-aggregation approach [56, 116, 128, 130] aims at analyzing the behavior and the cognitive style of the DM. Special iterative interactive procedures are used, where the components of the problem and the DM's global judgment policy are analyzed and then they are aggregated into a value system (Fig. 9.2). The goal of this approach is to aid the DM to improve his/her knowledge

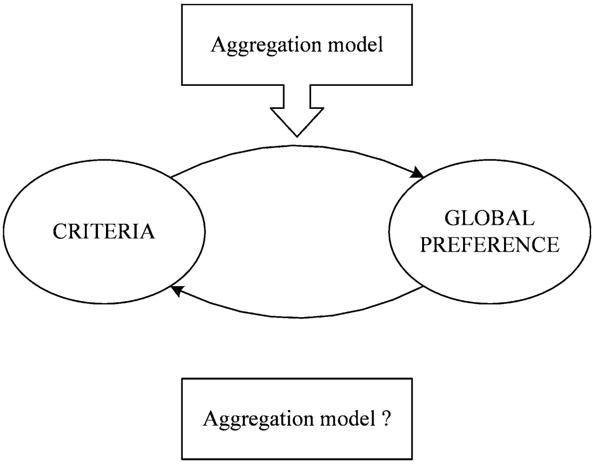


Fig. 9.1 The aggregation and disaggregation paradigms in MCDA [57]

about the decision situation and his/her way of preferring that entails a consistent decision to be achieved.

In order to use global preference given data, Jacquet-Lagrèze and Siskos [57] note that the clarification of the DM's global preference necessitates the use of a set of reference actions A_R. Usually, this set could be:

1. a set of past decision alternatives (A_R: past actions),
2. a subset of decision actions, especially when A is large ($A_R \subset A$),
3. a set of fictitious actions, consisting of performances on the criteria, which can be easily judged by the DM to perform global comparisons (A_R: fictitious actions).

In each of the above cases, the DM is asked to externalize and/or confirm his/her global preferences on the set A_R taking into account the performances of the reference actions on all criteria.

9.1.3 Historical Background

The history of the disaggregation principle in multidimensional/multicriteria analyses begins with the use of goal programming techniques, a special form of linear programming structure, in assessing/inferring preference/aggregation models or in developing linear or non-linear multidimensional regression analyses [118].

Charnes et al. [16] proposed a linear model of optimal estimation of executive compensation by analyzing or disaggregating pairwise comparisons and given measures (salaries); the model was estimated so that it could be as consistent as possible with the data from the goal programming point of view.

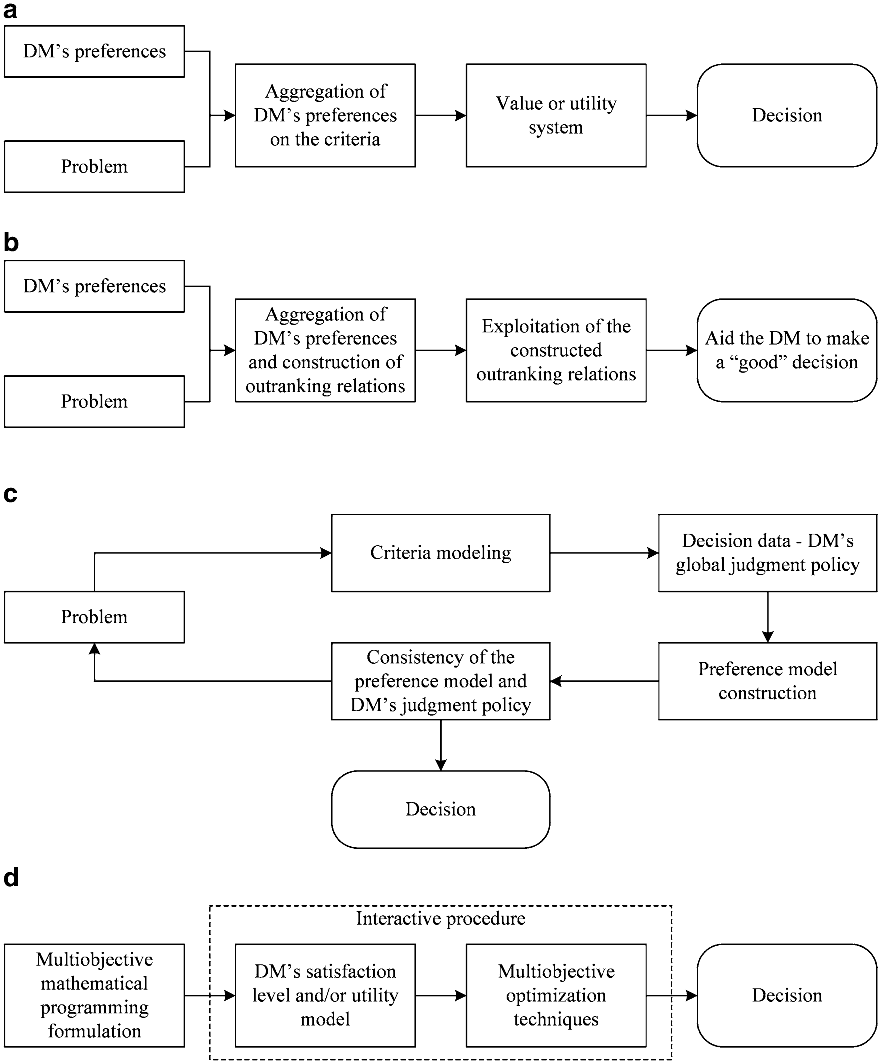


Fig. 9.2 The disaggregation-aggregation approach [127]. (**a**) The value system approach; (**b**) the outranking relation approach; (**c**) the disaggregation-aggregation approach; (**d**) the multiobjective optimization approach

Karst [65] minimized the sum of absolute deviations via goal programming in linear regression with one variable, while Wagner [147] generalizes the Karst's model in the multiple regression case. Later Kelley [68] proposed a similar model to minimize the Tchebycheff's criterion in linear regression.

Srinivasan and Shocker [143] outlined the ORDREG ordinal regression model to assess a linear value function by disaggregating pairwise judgments. Freed and

Glover [34] proposed goal programming models to infer the weights of linear value functions in the frame of discriminant analysis (problem statement β).

The research on handling ordinal criteria began with the studies of Young et al. [148], and Jacquet-Lagrèze and Siskos [55]. The latter research refers to the presentation of the UTA method in the "Cahiers du LAMSADE" series and indicates the actual initiation of the development of disaggregation methods. Both research teams faced the same problem: to infer additive value functions by disaggregating a ranking of reference alternatives. Young et al. [148] proposed alternating least squares techniques, without ensuring, however, that the additive value function is optimally consistent with the given ranking. In the case of the UTA method, optimality is ensured through linear programming techniques.

9.2 The UTA Method

9.2.1 Principles and Notation

The UTA (UTilité Additive) method proposed by Jacquet-Lagrèze and Siskos [56] aims at inferring one or more additive value functions from a given ranking on a reference set A_R. The method uses special linear programming techniques to assess these functions so that the ranking(s) obtained through these functions on A_R is (are) as consistent as possible with the given one.

The criteria aggregation model in UTA is assumed to be an additive value function of the following form [56]:

$$u(\mathbf{g}) = \sum_{i=1}^{n} p_i u_i(g_i) \tag{9.3}$$

subject to normalization constraints:

$$\begin{cases} \sum_{i=1}^{n} p_i = 1 \\ u_i(g_{i*}) = 0,\ u_i(g_i^*) = 1,\ \forall i = 1, 2, \ldots, n \end{cases} \tag{9.4}$$

where $u_i, i = 1, 2, \ldots, n$ are non decreasing real valued functions, named marginal value or utility functions, which are normalized between 0 and 1, and p_i is the weight of u_i (Fig. 9.3).

Both the marginal and the global value functions have the monotonicity property of the true criterion. For instance, in the case of the global value function the following properties hold:

$$\begin{cases} u[\mathbf{g}(a)] > u[\mathbf{g}(b)] \Leftrightarrow a \succ b \text{ (preference)} \\ u[\mathbf{g}(a)] = u[\mathbf{g}(b)] \Leftrightarrow a \sim b \text{ (indifference)} \end{cases} \tag{9.5}$$

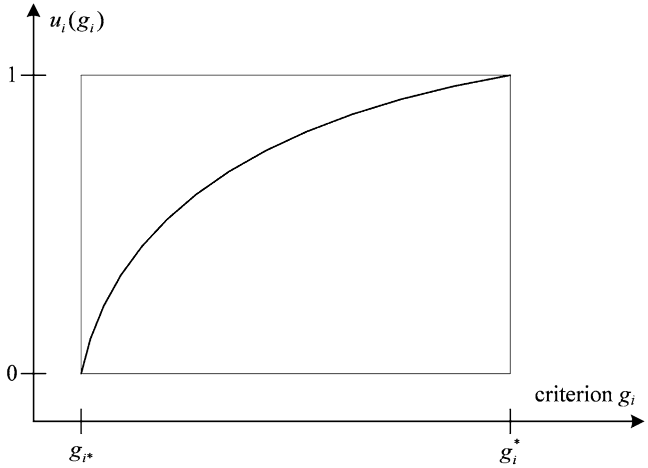


Fig. 9.3 The normalized marginal value function

The UTA method infers an unweighted form of the additive value function, equivalent to the form defined from relations (9.3) and (9.4), as follows:

$$u(\mathbf{g}) = \sum_{i=1}^{n} u_i(g_i) \tag{9.6}$$

subject to normalization constraints:

$$\begin{cases} \sum_{i=1}^{n} u_i(g_i^*) = 1 \\ u_i(g_{i*}) = 0, \qquad \forall i = 1, 2, \ldots, n \end{cases} \tag{9.7}$$

Of course, the existence of such a preference model assumes the preferential independence of the criteria for the DM [67], while other conditions for additivity can be found in [32, 33].

9.2.2 Development of the UTA Method

On the basis of the additive model (9.6)–(9.7) and taking into account the preference conditions (9.5), the value of each alternative $a \in A_R$ may be written as:

$$u'[\mathbf{g}(a)] = \sum_{i=1}^{n} u_i[g_i(a)] + \sigma(a) \ \forall a \in A_R \tag{9.8}$$

where $\sigma(a)$ is a potential error relative to $u'[\mathbf{g}(a)]$.

Moreover, in order to estimate the corresponding marginal value functions in a piecewise linear form, Jacquet-Lagrèze and Siskos [56] propose the use of linear interpolation. For each criterion, the interval $[g_{i*}, g_i^*]$ is cut into $(\alpha_i - 1)$ equal intervals, and thus the end points g_i^j are given by the formula:

$$g_i^j = g_{i*} + \frac{j-1}{\alpha_i - 1}(g_i^* - g_{i*}) \; \forall j = 1, 2, \ldots, \alpha_i \tag{9.9}$$

The marginal value of an action a is approximated by a linear interpolation, and thus, for $g_i(a) \in [g_i^j - g_i^{j+1}]$

$$u_i[g_i(a)] = u_i(g_i^j) + \frac{g_i(a) - g_i^j}{g_i^{j+1} - g_i^j}[u_i(g_i^{j+1}) - u_i(g_i^j)] \tag{9.10}$$

The set of reference actions $A_R = \{a_1, a_2, \ldots, a_m\}$ is also "rearranged" in such a way that a_1 is the head of the ranking (best action) and a_m its tail (worst action). Since the ranking has the form of a weak order R, for each pair of consecutive actions (a_k, a_{k+1}) it holds either $a_k \succ a_{k+1}$ (preference) or $a_k \sim a_{k+1}$ (indifference). Thus, if

$$\Delta(a_k, a_{k+1}) = u'[\mathbf{g}(a_k)] - u'[\mathbf{g}(a_{k+1})] \tag{9.11}$$

then one of the following holds:

$$\begin{cases} \Delta(a_k, a_{k+1}) \geq \delta \text{ iff } a_k \succ a_{k+1} \\ \Delta(a_k, a_{k+1}) = 0 \text{ iff } a_k \sim a_{k+1} \end{cases} \tag{9.12}$$

where δ is a small positive number so as to discriminate significantly two successive equivalence classes of R.

Taking into account the hypothesis on monotonicity of preferences, the marginal values $u_i(g_i)$ must satisfy the set of the following constraints:

$$u_i(g_i^{j+1}) - u_i(g_i^j) \geq s_i \; \forall j = 1, 2, \ldots, \alpha_i - 1, \; i = 1, 2, \ldots, n \tag{9.13}$$

with $s_i \geq 0$ being indifference thresholds defined on each criterion g_i. Jacquet-Lagrèze and Siskos [56] urge that it is not necessary to use these thresholds in the UTA model ($s_i = 0$), but they can be useful in order to avoid phenomena such as $u_i(g_i^{j+1}) = u_i(g_i^j)$ when $g_i^{j+1} \succ g_i^j$.

The marginal value functions are finally estimated by means of the following Linear Program (LP) with (9.6), (9.7), (9.12), (9.13) as constraints and with an objective function depending on the $\sigma(a)$ and indicating the amount of total deviation:

$$\begin{cases} [\min]F = \sum_{a \in A_R} \sigma(a) \\ \text{subject to} \\ \left.\begin{array}{l} \Delta(a_k, a_{k+1}) \geq \delta \text{ if } a_k \succ a_{k+1} \\ \Delta(a_k, a_{k+1}) = 0 \text{ if } a_k \sim a_{k+1} \end{array}\right\} \quad \forall k \\ u_i(g_i^{j+1}) - u_i(g_i^j) \geq 0 \qquad \forall i \text{ and } j \\ \sum_{i=1}^{n} u_i(g_i^*) = 1 \\ u_i(g_{i*}) = 0, u_i(g_i^j) \geq 0, \sigma(a) \geq 0 \ \forall a \in A_R, \forall i \text{ and } j \end{cases} \tag{9.14}$$

The stability analysis of the results provided by LP (9.14) is considered as a post-optimality analysis problem. As Jacquet-Lagrèze and Siskos [56] note, if the optimum $F^* = 0$, the polyhedron of admissible solutions for $u_i(g_i)$ is not empty and many value functions lead to a perfect representation of the weak order R. Even when the optimal value F^* is strictly positive, other solutions, less good for F, can improve other satisfactory criteria, like Kendall's τ.

As shown in Fig. 9.4, the post-optimal solutions space is defined by the polyhedron:

$$\begin{cases} F \leq F^* + k(F^*) \\ \text{all the constraints of LP (9.14)} \end{cases} \tag{9.15}$$

where $k(F^*)$ is a positive threshold which is a small proportion of F^*.

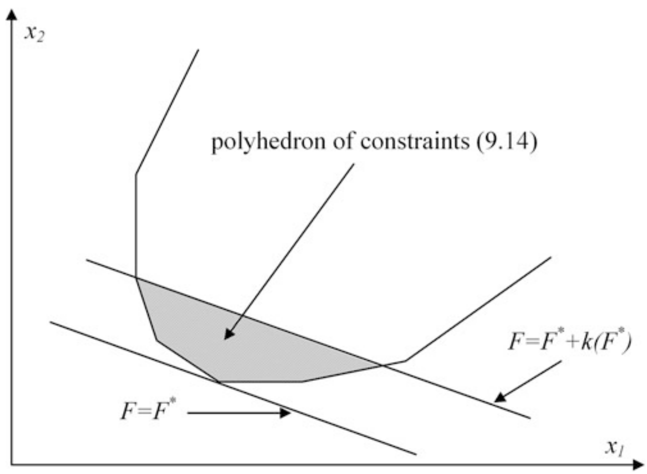


Fig. 9.4 Post-optimality analysis [56]

The algorithms which could be used to explore the polyhedron (9.15) are branch and bound methods, like reverse simplex method [146], or techniques dealing with the notion of the labyrinth in graph theory, such as Tarry's method [15], or the method of Manas and Nedoma [77]. Jacquet-Lagrèze and Siskos [56], in the original UTA method, propose the partial exploration of polyhedron (9.15) by solving the following LPs:

$$\begin{cases} [\min]u_i(g_i^*) \text{ and } [\max]u_i(g_i^*) \\ \text{in} & \forall i = 1, 2, \ldots, n \\ \text{polyhedron (9.15)} \end{cases} \tag{9.16}$$

The average of the previous LPs may be considered as the final solution of the problem. In case of instability, a large variation of the provided solutions appears, and this average solution is less representative. In any case, the solutions of the above LPs give the internal variation of the weight of all criteria g_i, and consequently give an idea of the importance of these criteria in the DM's preference system.

9.2.3 The UTASTAR Algorithm

The UTASTAR method [128] is an improved version of the original UTA model presented in the previous section. In the original version of UTA [56], for each packed action $a \in A_R$, a single error $\sigma(a)$ is introduced to be minimized. This error function is not sufficient to minimize completely the dispersion of points all around the monotone curve of Fig. 9.5. The problem is posed by points situated on the right of the curve, from which it would be suitable to subtract an amount of value/utility and not increase the values/utilities of the others.

In UTASTAR method, Siskos and Yannacopoulos [128] introduced a double positive error function, so that formula (9.8) becomes:

$$u'[g(a)] = \sum_{i=1}^{n} u_i[g_i(a)] - \sigma^+(a) + \sigma^-(a) \quad \forall\, a \in A_R \tag{9.17}$$

where σ^+ and σ^- are the underestimation and the overestimation error respectively.

Moreover, another important modification concerns the monotonicity constraints of the criteria, which are taken into account through the transformations of the variables:

$$w_{ij} = u_i(g_i^{j+1}) - u_i(g_i^j) \geq 0 \ \forall\, i = 1, 2, \ldots, n \text{ and } j = 1, 2, \ldots, \alpha_i - 1 \tag{9.18}$$

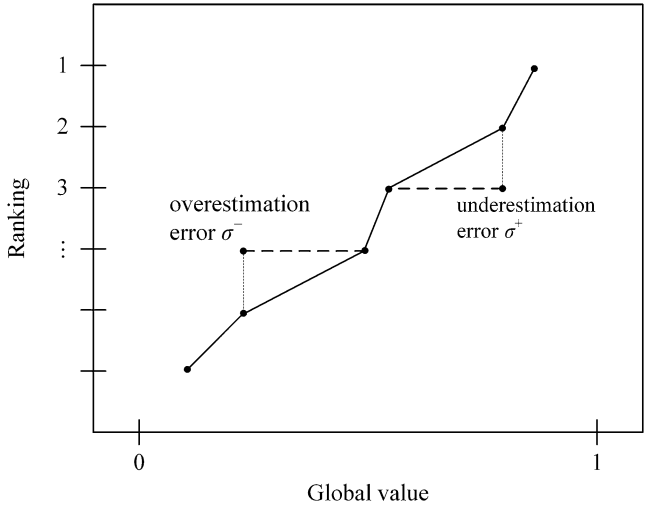


Fig. 9.5 Ordinal regression curve (ranking versus global value)

and thus, the monotonicity conditions (9.13) can be replaced by the non-negative constraints for the variables w_{ij} (for $s_i = 0$).

Consequently, the UTASTAR algorithm may be summarized in the following steps:

Step 1: Express the global value of reference actions $u[\mathbf{g}(a_k)], k = 1, 2, \ldots, m$, first in terms of marginal values $u_i(g_i)$, and then in terms of variables w_{ij} according to the formula (9.18), by means of the following expressions:

$$\begin{cases} u_i(g_i^1) = 0 & \forall i = 1, 2, \ldots, n \\ u_i(g_i^j) = \sum_{t=1}^{j-1} w_{it} & \forall i = 1, 2, \ldots, n \text{ and } j = 2, 3, \ldots, \alpha_i - 1 \end{cases} \tag{9.19}$$

Step 2: Introduce two error functions σ^+ and σ^- on A_R by writing for each pair of consecutive actions in the ranking the analytic expressions:

$$\begin{aligned} \Delta(a_k, a_{k+1}) = {} & u[\mathbf{g}(a_k)] - \sigma^+(a_k) + \sigma^-(a_k) \\ & -u[\mathbf{g}(a_{k+1})] + \sigma^+(a_{k+1}) - \sigma^-(a_{k+1}) \end{aligned} \tag{9.20}$$

Step 3: Solve the LP:

$$\begin{cases} [min]z = \sum_{k=1}^{m} [\sigma^{+}(a_k) + \sigma^{-}(a_k)] \\ \text{subject to} \\ \left.\begin{array}{l} \Delta(a_k, a_{k+1}) \geq \delta \text{ if } a_k \succ a_{k+1} \\ \Delta(a_k, a_{k+1}) = 0 \text{ if } a_k \sim a_{k+1} \end{array}\right\} \quad \forall k \\ \sum_{i=1}^{n} \sum_{j=1}^{\alpha_i - 1} w_{ij} = 1 \\ w_{ij} \geq 0, \sigma^{+}(a_k) \geq 0, \sigma^{-}(a_k) \geq 0 \ \forall i, j, \text{and } k \end{cases} \tag{9.21}$$

with δ being a small positive number.

Step 4: Test the existence of multiple or near optimal solutions of the LP (9.21) (stability analysis); in case of non uniqueness, find the mean additive value function of those (near) optimal solutions which maximize the objective functions:

$$u_i(g_i^*) = \sum_{j=1}^{\alpha_i - 1} w_{ij} \ \forall i = 1, 2, \ldots, n \tag{9.22}$$

on the polyhedron of the constraints of the LP (9.21) bounded by the new constraint:

$$\sum_{k=1}^{m} [\sigma^{+}(\alpha_k) + \sigma^{-}(\alpha_k)] \leq z^* + \varepsilon \tag{9.23}$$

where z^* is the optimal value of the LP in step 3 and ε is a very small positive number.

A comparison analysis between UTA and UTASTAR algorithms is presented in [128] through a variety of experimental data. UTASTAR method has provided better results concerning a number of comparison indicators, like:

1. The number of the necessary simplex iterations for arriving at the optimal solution.
2. The Kendall's τ between the initial weak order and the one produced by the estimated model.
3. The minimized criterion z (sum of errors) taken as the indicator of dispersion of the observations.

9.2.4 Robustness Analysis

UTA-based methods include robustness analysis to take account of the gap between the DM's "true" model and the model resulting from the disaggregation computational mechanism. Roy [109] considers robustness as an enabling tool for decision analysts to resist the phenomena of approximations and ignorance zones. It should be emphasized that robustness refers mainly to the decision model, in the light of the assertion "robust models produce a fortiori robust results".

However, robustness should also refer to the results and the decision support activities (e.g. conclusions, argumentation). In UTA methods robustness uses LP as the main inference mechanism. In this spirit, several UTA-type methods have been developed such as UTA-GMS [39], GRIP [31], and RUTA [64] to provide the DM with robust conclusions, Extreme Ranking Analysis [62] to determine the extreme ranking positions taken by the actions, and finally the robustness measurement control based on Monte Carlo sampling techniques (see [60, 61] for stochastic ordinal regression; see [41] for entropy measurement control).

Additional developments of robustness analysis in the context of UTA-type methods can be found in [17, 40, 63].

As presented in the previous section, in the UTA models, robustness refers to the post/near-optimality analysis. In the context of preference disaggregation approaches, Siskos and Grigoroudis [125] propose a general methodological framework for applying robustness analysis (Fig. 9.6).

The assessment of the robustness measures may depend on the post-optimality analysis results, and especially on the form and the extent of the polyhedron of the LP (9.14) or the LP (9.21). In particular, the observed variance in the post-optimality matrix indicates the degree of instability of the results. Following this approach, Siskos and Grigoroudis [125] proposed an Average Stability Index (*ASI*) based on the average of the normalized standard deviation of the estimated values $u_i(g_i^*)$ [42]. Instead of exploring only the extreme values $u_i(g_i^*)$, the post-optimality analysis may investigate every value $u_i(g_i^j)$ of each criterion. In this case, during the post-optimality stage, T LPs are formulated and solved, which maximize and minimize repeatedly $u_i(g_i^j)$, and the *ASI* for the i-th criterion is assessed as follows:

$$ASI(i) = 1 - \frac{1}{\alpha_i - 1} \sum_{j=1}^{\alpha_i - 1} \frac{\sqrt{T \sum_{k=1}^{T} (u_i^{jk})^2 - \left(\sum_{k=1}^{T} u_i^{jk} \right)^2}}{\frac{T}{\alpha_i - 1} \sqrt{\alpha_i - 2}} \qquad (9.24)$$

where $T = 2 \sum_i (\alpha_i - 1)$ and u_i^{jk} is the estimated value of $u_i(g_i^j)$ in the k-th post-optimality analysis LP ($j = 1, 2, \ldots, \alpha_i$).

The global robustness measure may be assessed as the average of the individual $ASI(i)$ values. Since *ASI* measures are normalized in the interval [0, 1], high levels of robustness are achieved when *ASI* is close to 1. However, if the analyst is not satisfied with the value of the *ASI* measures, several alternative rules of robustness

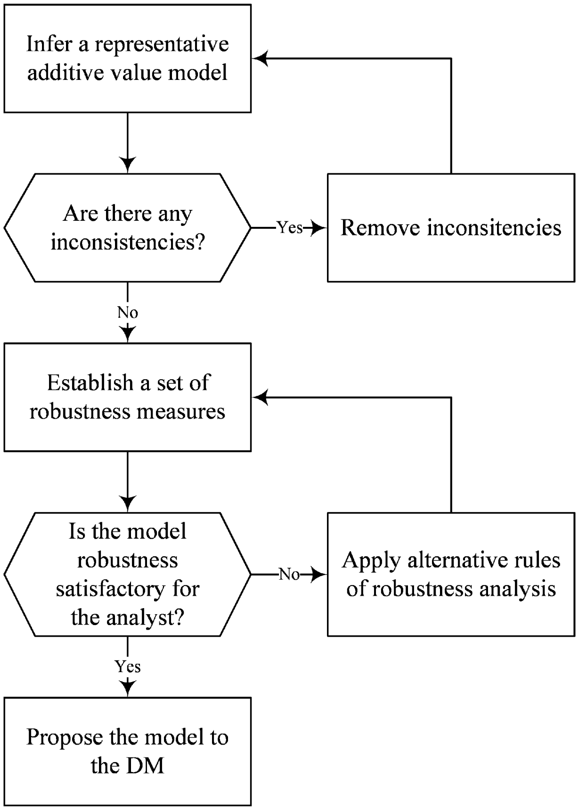


Fig. 9.6 Robustness analysis in preference disaggregation approaches [125]

analysis may be applied, including new global preference judgments, enumeration and management of the hyperpolyhedron vertices in post-optimality analysis, new preference relations on the set *A* during the extrapolation phase, etc. (see [125] for more details).

9.2.5 A Numerical Example

The implementation of the UTASTAR algorithm is illustrated by a practical example taken from [128]. The problem concerns a DM who wishes to analyze the choice of transportation means during the peak hours (home-work place). Suppose that the DM is interested only in the following three criteria:

1. price (in monetary units),
2. time of journey (in minutes), and
3. comfort (possibility to have a seat).

Table 9.1 Criteria values and ranking of the DM

Means of transportation	Price (μ)	Time (min)	Comfort	Ranking of the DM
RER	3	10	+	1
METRO (1)	4	20	++	2
METRO (2)	2	20	0	2
BUS	6	40	0	3
TAXI	30	30	+ + +	4

The evaluation in terms of the previous criteria is presented in Table 9.1, where it should be noted that the following qualitative scale has been used for the comfort criterion: 0 (no chance of seating), + (little chance of seating) ++ (great chance of finding a seating place), and +++ (seat assured). Also, the last column of Table 9.1 shows the DM's ranking with respect to the five alternative means of transportation.

The first step of UTASTAR, as presented in the previous section, consists of making explicit the utilities of the five alternatives. For this reason the following scales have been chosen:

$$[g_{1*}, g_1^*] = [30, 16, 2]$$
$$[g_{2*}, g_2^*] = [40, 30, 20, 10]$$
$$[g_{3*}, g_3^*] = [0, +, ++, + + +]$$

Using linear interpolation for the criterion according to formula (9.10), the value of each alternative may be written as:

$$u[\mathbf{g}(\text{RER})] = 0.07u_1(16) + 0.93u_1(2) + u_2(10) + u_3(+)$$
$$u[\mathbf{g}(\text{METRO1})] = 0.14u_1(16) + 0.86u_1(2) + u_2(20) + u_3(++)$$
$$u[\mathbf{g}(\text{METRO2})] = u_1(2) + u_2(20) + u_3(0) = u_1(2) + u_2(20)$$
$$u[\mathbf{g}(\text{BUS})] = 0.29u_1(16) + 0.71u_1(2) + u_2(40) + u_3(0)$$
$$= 0.29u_1(16) + 0.71u_1(2)$$
$$u[\mathbf{g}(\text{TAXI})] = u_1(30) + u_2(30) + u_3(+ + +) = u_2(30) + u_3(+ + +)$$

where the following normalization conditions for the marginal value functions have been used: $u_1(30) = u_2(40) = u_3(0) = 0$.

Also, according to formula (9.19), the global value of the alternatives may be expressed in terms of the variables w_{ij}:

$$u[\mathbf{g}(\text{RER})] = w_{11} + 0.93w_{12} + w_{21} + w_{22} + w_{23} + w_{31}$$
$$u[\mathbf{g}(\text{METRO1})] = w_{11} + 0.86w_{12} + w_{21} + w_{22} + w_{31} + w_{32}$$
$$u[\mathbf{g}(\text{METRO2})] = w_{11} + w_{12} + w_{21} + w_{22}$$

Table 9.2 Marginal value functions (initial solution)

Price	Time	Comfort
$u_1(30) = 0.000$	$u_2(40) = 0.000$	$u_3(0) = 0.000$
$u_1(16) = 0.500$	$u_2(30) = 0.050$	$u_3(+) = 0.000$
$u_1(2) = 0.500$	$u_2(20) = 0.050$	$u_3(++) = 0.000$
	$u_2(10) = 0.100$	$u_3(+++) = 0.400$

$$u[\mathbf{g}(\text{BUS})] = w_{11} + 0.71w_{12}$$
$$u[\mathbf{g}(\text{TAXI})] = w_{21} + w_{31} + w_{32} + w_{33}$$

According to the second step of the UTASTAR algorithm, the following expressions are written, for each pair of consecutive actions in the ranking:

$$\begin{aligned}
\Delta(\text{RER, METRO1}) &= 0.07w_{12} + w_{23} - w_{32} - \sigma^+_{RER} + \sigma^-_{RER} \\
&\quad + \sigma^+_{METRO1} - \sigma^-_{METRO1} \\
\Delta(\text{METRO1, METRO2}) &= -0.14w_{12} + w_{31} + w_{32} - \sigma^+_{METRO1} \\
&\quad + \sigma^-_{METRO1} + \sigma^+_{METRO2} - \sigma^-_{METRO2} \\
\Delta(\text{METRO2, BUS}) &= 0.29w_{12} + w_{21} + w_{22} \\
&\quad - \sigma^+_{METRO2} + \sigma^-_{METRO2} + \sigma^+_{BUS} - \sigma^-_{BUS} \\
\Delta(\text{BUS, TAXI}) &= w_{11} + 0.71w_{12} - w_{21} - w_{31} - w_{32} - w_{33} \\
&\quad - \sigma^+_{BUS} + \sigma^-_{BUS} + \sigma^+_{TAXI} - \sigma^-_{TAXI}
\end{aligned}$$

Based on the aforementioned expression, an LP according to (9.21) is formulated, with $\delta = 0.05$. An optimal solution is: $w_{11} = 0.5$, $w_{21} = 0.05$, $w_{23} = 0.05$, $w_{33} = 0.4$ with $[min]z = z^* = 0$. This solution corresponds to the marginal value functions presented in Table 9.2 and produces a ranking which is consistent with the DM's initial weak order.

It should be emphasized that this solution is not unique. Through post-optimality analysis (step 4), the UTASTAR algorithm searches for multiple optimal solutions, or more generally, for near optimal solutions corresponding to error values between z^* and $z^* + \epsilon$. For this reason, the error objective should be transformed to a constraint of the type (9.23).

In the presented numerical example, the initial LP has multiple optimal solutions, since $z^* = 0$. Thus, in the post-optimality analysis step, the algorithm searches for more characteristic solutions, which maximize the expressions (9.22), i.e. the weights of each criterion. Furthermore, in this particular case we have:

$$z^* = 0 \Longleftrightarrow \sigma^+(a_k) = \sigma^-(a_k) = 0 \; \forall \; k$$

so the error variables may be excluded from the LPs of the post-optimality analysis. Table 9.3 presents the formulation of the LP that has to be solved during this step.

Table 9.3 Linear programming formulation (post-optimality analysis)

w_{11}	w_{12}	w_{21}	w_{22}	w_{23}	w_{31}	w_{32}	w_{33}	RHS
0	0.07	0	0	1	0	−1	0	≥ 0.05
0	−0.14	0	0	0	1	1	0	= 0
0	0.29	1	1	0	0	0	0	≥ 0.05
1	0.71	−1	0	0	−1	−1	−1	≥ 0.05
1	1	1	1	1	1	1	1	= 1
1	1	0	0	0	0	0	0	$[max]u_1(g_1^*)$
0	0	1	1	1	0	0	0	$[max]u_2(g_2^*)$
0	0	0	0	0	1	1	1	$[max]u_3(g_3^*)$

Table 9.4 Post-optimality analysis and final solution

	w_{11}	w_{12}	w_{21}	w_{22}	w_{23}	w_{31}	w_{32}	w_{33}
$[max]u_1(g_1^*)$	0.7625	0.175	0	0	0.0375	0.025	0	0
$[max]u_2(g_2^*)$	0.05	0	0	0.05	0.9	0	0	0
$[max]u_3(g_3^*)$	0.3562	0.175	0	0	0.0375	0.025	0	0.4063
Average	0.3896	0.1167	0	0.0167	0.3250	0.0167	0	0.1354

Table 9.5 Marginal value functions (final solution)

Price	Time	Comfort
$u_1(30) = 0.000$	$u_2(40) = 0.000$	$u_3(0) = 0.000$
$u_1(16) = 0.390$	$u_2(30) = 0.000$	$u_3(+) = 0.017$
$u_1(2) = 0.506$	$u_2(20) = 0.017$	$u_3(++) = 0.017$
	$u_2(10) = 0.342$	$u_3(+++) = 0.152$

The solutions obtained during post-optimality analysis are presented in Table 9.4. The average of these three solutions is also calculated in the last row of Table 9.4. This centroid is taken as a unique utility function, provided that it is considered as a more representative solution of this particular problem.

This final solution corresponds to the marginal value functions presented in Table 9.5. Also, the utilities for each alternative are calculated as follows:

$$u[\mathbf{g}(\text{RER})] = 0.856$$
$$u[\mathbf{g}(\text{METRO1})] = 0.523$$
$$u[\mathbf{g}(\text{METRO2})] = 0.523$$
$$u[\mathbf{g}(\text{BUS})] = 0.473$$
$$u[\mathbf{g}(\text{TAXI})] = 0.152$$

where it is obvious that these values are consistent with the DM's weak order.

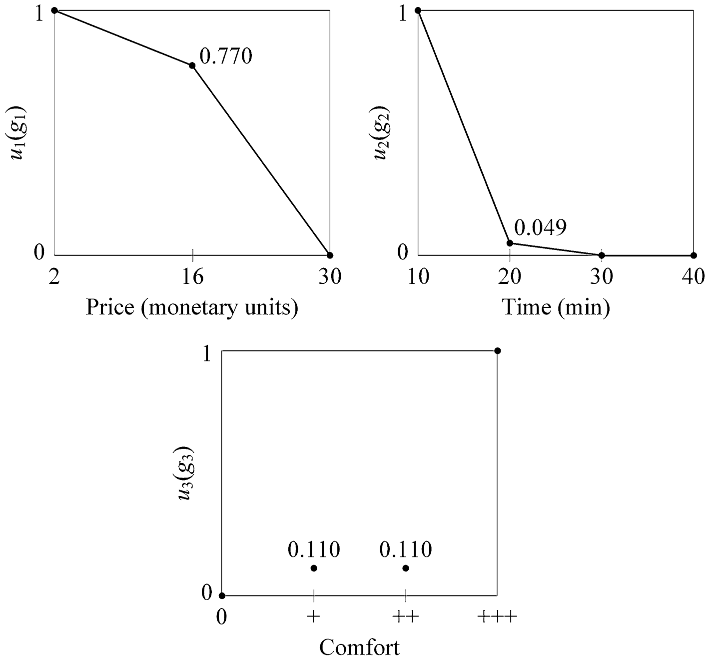


Fig. 9.7 Normalized marginal value functions

These marginal utilities may be normalized by dividing every value $u_i(g_i^j)$ by $u_i(g_i^*)$. In this case the additive utility can be written as:

$$u_{(\mathbf{g})} = 0.506u_1(g_1) + 0.342u_2(g_2) + 0.152u_3(g_3)$$

where the normalized marginal value functions are presented in Fig. 9.7.

9.3 Variants of the UTA Method

9.3.1 Alternative Optimality Criteria

Several variants of the UTA method have been developed, incorporating different forms of global preference or different forms of optimality criteria used in the linear programming formulation.

An extension of the UTA methods, where $u[\mathbf{g}(a)]$ is inferred from pairwise comparisons is proposed by Jacquet-Lagrèze and Siskos [56]. This subjective preference obtained by pairwise judgments is most often not transitive, and thus, the modified model may be written as in the following LP:

$$\begin{cases} [min]F = \sum\limits_{(a,b):a \succ b} \lambda_{ab} z_{ab} + \sum\limits_{(a,b):a \sim b} \lambda_{ab} z_{ba} \\ \text{subject to} \\ \sum\limits_{i=1}^{n} \{u_i[g_i(a)] - u_i[g_i(b)]\} + z_{ab} \geq 0 & \text{if } a \succ b \\ \sum\limits_{i=1}^{n} \{u_i[g_i(a)] - u_i[g_i(b)]\} + z_{ab} - z_{ba} = 0 & \text{if } a \sim b \ (\Rightarrow b \sim a) \\ u_i(g_i^{j+1}) - u_i(g_i^j) \geq s_i & \forall i, j \\ \sum\limits_{i=1}^{n} u_i(g_i^*) = 1 \\ u_i(g_{i*}) = 0, u_i(g_i^j) \geq 0, & \forall i, j \\ z_{ab} \geq 0 & \forall (a,b) \in R \end{cases} \tag{9.25}$$

λ_{ab} being a non negative weight reflecting a degree of confidence in the judgment between a and b.

An alternative optimality criterion would be to minimize the number of violated pairs of an order R provided by the DM in ranking R' given by the model, which is equivalent to maximize Kendall's τ between the two rankings. This extension is given by the mixed integer LP (9.26), where $\gamma_{ab} = 0$ if $u[\mathbf{g}(a)] - u[\mathbf{g}(b)] \geq \delta$ for a pair $(a,b) \in R$ and the judgment is respected, otherwise $\gamma_{ab} = 1$ and the judgment is violated. Thus, the objective function in this LP represents the number of violated pairs in the overall preference aggregated by $u(\mathbf{g})$.

$$\begin{cases} [min]F = \sum\limits_{(a,b) \in R} \gamma_{ab} \Leftrightarrow [max]\tau(R, R') \\ \text{subject to} \\ \sum\limits_{i=1}^{n} \{u_i[g_i(a)] - u_i[g_i(b)]\} + M \cdot \gamma_{ab} \geq \delta & \forall (a,b) \in R \\ u_i(g_i^{j+1}) - u_i(g_i^j) \geq s_i & \forall i, j \\ \sum\limits_{i=1}^{n} u_i(g_i^*) = 1 \\ u_i(g_{i*}) = 0, u_i(g_i^j) \geq 0 & \forall i, j \\ \gamma_{\alpha b} = 0 \text{ or } 1 & \forall (a,b) \in R \end{cases} \tag{9.26}$$

where M is a large number. Beuthe and Scannella [11] propose to handle separately the preference and indifference judgments, and modify the previous LP using the constraints:

$$\begin{cases} \sum_{i=1}^{n}\{u_i[g_i(a)] - u_i[g_i(b)]\} + M \cdot \gamma_{ab} \geq \delta & \text{if } a \succ b \\ \left.\begin{array}{l} \sum_{i=1}^{n}\{u_i[g_i(a)] - u_i[g_i(b)]\} + M \cdot \gamma_{ab} \geq 0 \\ \sum_{i=1}^{n}\{u_i[g_i(a)] - u_i[g_i(b)]\} + M \cdot \gamma_{ba} \geq 0 \end{array}\right\} & \text{if } a \sim b \end{cases} \tag{9.27}$$

The assumption of monotonicity of preferences, in the context of separable value functions, means that the marginal values are monotonic functions of the criteria. This assumption, although widely used, is sometimes not applicable to real-world situations. One way to deal with non-monotonic preferences is to divide the range of the criteria into intervals, so that the preferences are monotonic in each interval, and then treat each interval separately [67]. In the same spirit, Despotis and Zopounidis [22] present a variation of the UTASTAR method for the assessment of non-monotonic marginal value functions. In this model, the range if each criterion is divided into two intervals (see also Fig. 9.8):

$$\begin{cases} G_i^1 = \{g_{i*} = g_i^1, g_i^2, \ldots, g_i^{p_i} = d_i\} \\ G_i^2 = \{d_i = g_i^{p_i}, g_i^{p_i+1}, \ldots, g_i^{p_i+q_i} = g_i^*\} \end{cases} \tag{9.28}$$

where d_i is the most desirable value of g_i, and the parameters p_i and q_i are determined according to the dispersion of the input data; of course it holds that $p_i + q_i = \alpha_i$. In this approach, the main modification concerns the assessment of the decision variables w_{ij} of the LP (9.21). Hence, formula (9.19) becomes:

$$u_i(g_i^j) = \begin{cases} \sum_{t=1}^{j-1} w_{it} & \text{if } 1 < j \leq p_i \\ \sum_{t=1}^{p_i-1} w_{it} - \sum_{t=p_i}^{j-1} w_{it} & \text{if } p_i < j \leq \alpha_i \end{cases} \tag{9.29}$$

without considering the conditions $u_i(g_i^1) = 0$.

Another extension of the UTA methods refers to the intensity of the DM's preferences, similar to the context proposed in [143]. In this case, a series of constraints may be added during the LP formulation. For example, if the preference of alternative a over alternative b is stronger than the preference of b over c, then the following condition may be written:

$$[u'[\mathbf{g}(a)] - u'[\mathbf{g}(b)]] - [u'[\mathbf{g}(b)] - u'[\mathbf{g}(c)]] \geq \phi \tag{9.30}$$

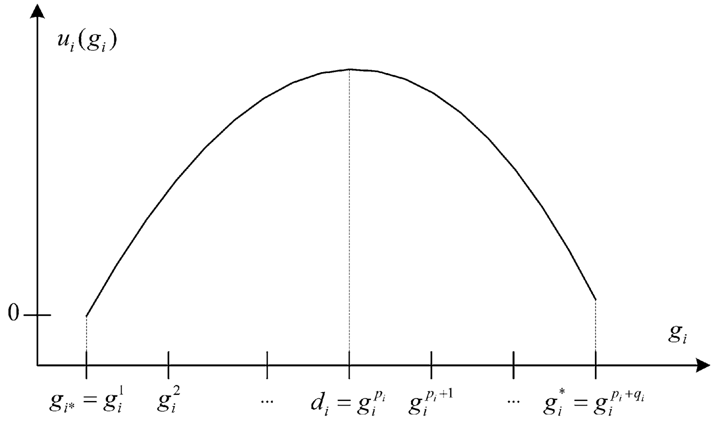


Fig. 9.8 A non-monotonic partial utility function [22]

where $\phi > 0$ is a measure of preference intensity and $u'(\mathbf{g})$ is given by formula (9.8). Thus, using formula (9.11), the following constraint should be added in LP (9.14):

$$\Delta(a,b) - \Delta(b,c) \geq \phi \tag{9.31}$$

In general, if the DM wishes to expand these preferences to the whole set of alternatives, a minimum number of $m-2$ constraints of type (9.34) is required.

Despotis and Zopounidis [22] consider the case where the DM ranks the alternatives using an explicit overall index I. Thus, formula (9.12) may be replaced by the following condition:

$$\Delta(a_k, a_{k+1}) = I_k - I_{k+1} \; \forall k = 1, 2, \ldots, m-1 \tag{9.32}$$

Besides the succession of the alternatives in the preference ranking, these constraints state that the difference of global value of any successive alternatives in the ranking should be consistent with the difference of their evaluation on the ratio scale.

In the same context, Oral and Kettani [103] propose the optimization of lexicographic criteria without discretisation of criteria scales G_i, where a ratio scale is used in order to express intensity of preferences.

Other variants of the UTA method concerning different forms of global preference are mainly focused on:

– additional properties of the assessed value functions, like concavity [22];
– construction of fuzzy outranking relations based on multiple value functions provided by UTA's post-optimality analysis [117].

The dimensions of the aforementioned UTA models affect the computational complexity of the formulated LPs. In most cases it is preferable to solve the dual

Table 9.6 LP size of UTA models

LP model	Constraints	Variables
LP (9.14)	$m + \sum_{i=1}^{n}(\alpha_i - 1)$	$m + \sum_{i=1}^{n}(\alpha_i - 1)$
LP (9.21)	m	$2m + \sum_{i=1}^{n}(\alpha_i - 1)$
LP (9.25)	$1 + [m(m-1)/2] + \sum_{i=1}^{n}(\alpha_i - 1)$	$\lvert P\rvert + \lvert I\rvert + \sum_{i=1}^{n}(\alpha_i - 1)$
LP (9.26)	$1 + [m(m-1)/2] + \sum_{i=1}^{n}(\alpha_i - 1)$	$[m(m-1)/2] + \sum_{i=1}^{n}(\alpha_i - 1)$

LP due to the structure of these LPs [56]. Table 9.6 summarizes the size of all LPs presented in the previous sections, where $|P|$ and $|I|$ denote the number of preference and indifference relations respectively, considering all possible pairwise comparisons in R. Also, it should be noted that LP (9.26) has $m(m-1)/2$ binary variables.

9.3.2 Meta-UTA Techniques

Other techniques, named meta-UTA, aimed at the improvement of the value function with respect to near optimality analysis or to its exploitation for decision support.

Despotis et al. [23] propose to minimize the dispersion of errors (Tchebycheff criterion) within the UTASTAR's step 4 (see Sect. 9.2.3). In case of a strictly positive error z^*, the aim is to investigate the existence of near optimal solutions of the LP (9.21) which give rankings R' such that $\tau(R', R) > \tau(R^*, R)$, with R^* being the ranking corresponding to the optimal value functions. The experience with the model [21] confirms that apart from the total error z^*, it is also the dispersion of the individual errors that is crucial for $\tau(R^*, R)$. Therefore, in the proposed post-optimality analysis, the difference between the maximum (σ_{max}) and the minimum error is minimized. As far as the individual errors are non-negative, this requirement can be satisfied by minimizing the maximum individual error (the L_∞ norm) according to the following LP:

$$\left\{\begin{array}{l} [min]\sigma_{max} \\ \text{subject to} \\ \text{all the constraints of LP (9.21)} \\ \sum_{k=1}^{m}[\sigma^+(a_k) + \sigma^-(a_k)] \le z^* + \varepsilon \\ \left.\begin{array}{l} \sigma_{max} - \sigma^+(a_k) \ge 0 \\ \sigma_{max} - \sigma^-(a_k) \ge 0 \end{array}\right\} \quad \forall k \\ \sigma_{max} \ge 0 \end{array}\right. \tag{9.33}$$

With the incorporation of the model (9.33) in UTASTAR, the value function assessment process becomes a lexicographic optimization process. That is, the final solution is obtained by minimizing successively the L_1 and the L_∞ norms.

Another approach concerning meta-UTA techniques refers to the UTAMP models. Beuthe and Scannella [9, 11] note that the values given to parameters s and δ in the UTA and UTASTAR methods, respectively, influence the results as well as the predictive quality of the models. Hence, in the framework of the research by Srinivasan and Shocker [143], they look for optimal values of s and/or δ in the case of positive errors ($z^* > 0$), as well as when UTA gives a sum of error equal to zero ($z^* = 0$).

In the post-optimality analysis step of UTASTAR (see Sect. 9.2.3), UTAMP1 model maximizes δ, which is the minimum difference between the global value of two consecutive reference actions. The name of the model denotes that, on the basis of UTA, maximizing δ leads to better identification for the relations of preference between actions.

Beuthe and Scannella [9] have also proposed to maximize the sum $(\delta + s)$ in order to stress not only the differences of utilities between actions, but also the differences between values at successive bounds. This more general approach was named UTAMP2. Note that s corresponds to the minimum of marginal value step w_{ij} in the UTASTAR algorithm. Although the simple addition of these parameters is legitimate since both of them are defined in the same value units, Beuthe and Scannella [11] note that a weighted sum formula may also be considered.

The UTAMP models, as well as the UTASTAR method, are based on the idea of centrality, although these approaches use a different interpretation of this notion. Bous et al. [13] propose an alternative method where the final solution is obtained by using an optimality criterion that directly implements the idea of centrality. They propose the ACUTA method, which is based on the computation of the analytic center of a polyhedron. In this approach, the product of the slack variables of constraints (9.12)–(9.13), or equivalently the sum of their logarithms is maximized. This non-linear objective function guarantees the uniqueness of the provided solution.

9.3.3 Stochastic UTA Method

Within the framework of multicriteria decision-aid under uncertainty, Siskos [118] developed a specific version of UTA (Stochastic UTA), in which the aggregation model to infer from a reference ranking is an additive utility function of the form:

$$u(\mathbf{d}^a) = \sum_{i=1}^{n} \sum_{j=1}^{\alpha_i} d_i^a(g_i^j) u_i(g_i^j) \tag{9.34}$$

subject to normalization constraints (9.7), where d_i^a is the distributional evaluation of action a on the i-th criterion, $d_i^a(g_i^j)$ is the probability that the performance of

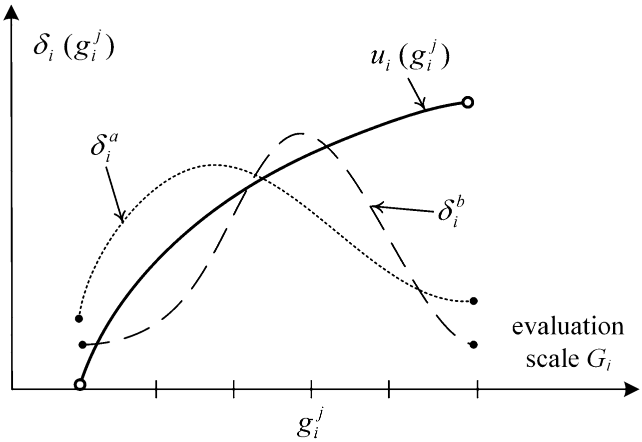


Fig. 9.9 Distributional evaluation and marginal value function

action a on the i-th criterion is g_i^j, $u_i(g_i^j)$ is the marginal value of the performance g_i^j, $\mathbf{d}^a$ is the vector of distributional evaluations of action a, and $u(\mathbf{d}^a)$and is the global utility of action a (see also Fig. 9.9).

This global utility is of the von Neumann-Morgenstern form [66], in the case of discrete g_i, where:

$$\sum_{j=1}^{\alpha_i} d_i^a(g_i^j) = 1 \tag{9.35}$$

Of course, the additive utility function (9.34) has the same properties as the value function:

$$\begin{cases} u(\mathbf{d}^a) > u(\mathbf{d}^b) \Leftrightarrow a \succ b \text{ (preference)} \\ u(\mathbf{d}^a) = u(\mathbf{d}^b) \Leftrightarrow a \sim b \text{ (indifference)} \end{cases} \tag{9.36}$$

Similarly to the cases of UTA and UTASTAR described in Sects. 9.2.2–9.2.3, the stochastic UTA method disaggregates a ranking of reference actions [122]. The algorithmic procedure could be expressed in the following way:

Step 1: Express the global expected utilities of reference actions $u(d^{\alpha_k})$, $k = 1, 2, \ldots, m$, in terms of variables:

$$w_{ij} = u_i(g_i^{j+1}) - u_i(g_i^j) \geq 0 \tag{9.37}$$

Step 2: Introduce two error functions σ^+ and σ^- by writing the following expressions for each pair of consecutive actions in the ranking:

$$\Delta(a_k, a_{k+1}) = u(\mathbf{d}^{a_k}) - \sigma^+(a_k) + \sigma^-(a_k) \\ - u(\mathbf{d}^{a_{k+1}}) + \sigma^+(a_{k+1}) - \sigma^-(a_{k+1}) \tag{9.38}$$

Step 3: Solve the LP (9.21) by using formulae (9.37) and (9.38).
Step 4: Test the existence of multiple or near optimal solutions.

Of course, the ideas employed in all variants of the UTA method are also applicable in the same way in the case of the stochastic UTA.

9.3.4 UTA-Type Sorting Methods

The extension of the UTA method in the case of a discriminant analysis model was firstly discussed by Jacquet-Lagrèze and Siskos [56]. The aim is to infer u from assignment examples in the context of problem statement β [108]. In the presence of two classes, if the model is without errors, the following inequalities must hold:

$$\begin{cases} a \in A_1 \Leftrightarrow u[\mathbf{g}(a)] \geq u_0 \\ a \in A_2 \Leftrightarrow u[\mathbf{g}(a)] < u_0 \end{cases} \tag{9.39}$$

with u_0 being the level of acceptance/rejection, which must be found in order to distinguish the set of accepted actions called A_1 and the set of rejected actions called A_2.

Introducing the error variables $\sigma(a)$, $a \in A_R$, the objective is to minimize the sum of deviations from the threshold u_0 for the ill classified actions (see Fig. 9.10). Hence, $u(\mathbf{g})$ can be estimated by means of the LP:

$$\begin{cases} [\min]F = \sum_{a \in A_R} \sigma(a) \\ \text{subject to} \\ \sum_{i=1}^{n} u_i[g_i(a)] - u_0 + \sigma(a) \geq 0 & \forall\, a \in A_1 \\ \sum_{i=1}^{n} u_i[g_i(a)] - u_0 - \sigma(a) \leq 0 & \forall\, a \in A_2 \\ u_i(g_i^{j+1}) - u_i(g_i^j) \geq s_i & \forall i \text{ and } j \\ \sum_{i=1}^{n} u_i(g_i^*) = 1 \\ u_i(g_{i*}) = 0, u_0 \geq 0, u_i(g_i^j) \geq 0, \sigma(a) \geq 0 \ \forall\, a \in A_R, \forall\, i \text{ and } j \end{cases} \tag{9.40}$$

In the general case, the DM's evaluation is expressed in terms of a classification of the reference alternatives into homogenous ordinal groups $A_1 \succ A_2 \succ \ldots \succ A_q$ (i.e. group A_1 includes the most preferred alternatives, whereas group A_q includes

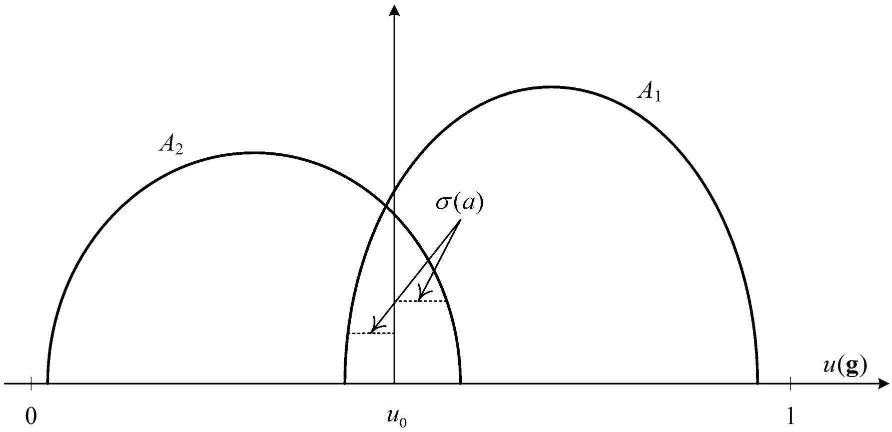


Fig. 9.10 Distribution of the actions A_1 and A_2 on $u(\mathbf{g})$ [56]

the least preferred ones). Within this context, the assessed additive value model will be consistent with the DM's global judgment, if the following conditions are satisfied:

$$\begin{cases} u[\mathbf{g}(a)] \geq u_1 & \forall\ a \in A_1 \\ u_l \leq u[\mathbf{g}(a)] < u_{l-1} & \forall\ a \in A_l \quad (l = 2, 3, \ldots, q-1) \\ u[\mathbf{g}(a)] < u_{q-1} & \forall\ a \in A_q \end{cases} \tag{9.41}$$

where $u_1 > u_2 > \ldots > u_{q-1}$ are thresholds defined in the global value scale [0, 1] to discriminate the groups, and u_l is the lower bound of group A_l.

This approach is named UTADIS method (UTilités Additives DIScriminantes) and is presented by Devaud et al. [24] (see also [28, 53, 152, 158]). Similarly to the UTASTAR method, two error variables are employed in the UTADIS method to measure the differences between the model's results and the predefined classification of the reference alternatives. The additive value model is developed to minimize these errors using a linear programming formulation of type (9.40). In this case, the two types of errors are defined as follows:

1. $\sigma_k^+ = \max\{0, u_l - u[\mathbf{g}(a_k)]\}\ \ \forall a_k \in A_l\ (l = 1, 2, \ldots, q-1)$ represents the error associated with the violation of the lower bound u_l of a group A_l by an alternative $a_k \in A_l$,
2. $\sigma_k^- = \max\{0, u[\mathbf{g}(a_k) - u_{l-1}]\}\ \ \forall a_k \in A_l (l = 2, 3, \ldots, q)$ represents the error associated with the violation of the upper bound u_{l-1} of a group A_l by an alternative $a_k \in A_l$.

Recently, several new variants of the original UTADIS method have been proposed (UTADIS I, II, III) to consider different optimality criteria during the development of the additive value classification model [28, 152, 158]. The UTADIS

I method considers both the minimization of the classification errors, as well as the maximization of the distances of the correctly classified alternatives from the value thresholds. The objective in the UTADIS II method is to minimize the number of misclassified alternatives, whereas UTADIS III combines the minimization of the misclassified alternatives with the maximization of the distances of the correctly classified alternatives from the value thresholds.

In the same context, Zopounidis and Doumpos [155] proposed the MHDIS method (Multi-group Hierarchical DIScrimination) extending the preference disaggregation analysis framework of the UTADIS method in complex sorting/classification problems involving multiple-groups. MHDIS addresses sorting problems through a hierarchical (sequential) procedure starting by discriminating group A_1 from all the other groups $\{A_2, A_3, \ldots, A_q\}$, and then proceeding to the discrimination between the alternatives belonging to the other groups. At each stage of this sequential/hierarchical process, two additive value functions are developed for the classification of the alternatives. Assuming that the classification of the alternatives should be made into q ordered classes, $A_1 \succ A_2 \succ \cdots \succ A_q$, $2(q-1)$ additive value functions are developed. These value functions have the following additive form:

$$\begin{cases} u_l(\mathbf{g}) = \sum_{i=1}^{n} u_{li}(g_i) \\ u_{\sim l}(\mathbf{g}) = \sum_{i=1}^{n} u_{\sim li}(g_i) \end{cases} \tag{9.42}$$

where u_l measures the value for the DM of a decision to assign an alternative into group A_l, whereas the $u_{\sim l}$ corresponds to the classification into the set of groups $A_{\sim l} = \{A_{l+1}, A_{l+2}, \ldots, A_q\}$ and both functions are normalized in the interval [0, 1].

The rules used to perform the classification of the alternatives have the following form:

$$\begin{cases} \text{if } u_1(a_k) > u_{\sim 1}(a_k) \text{ then } a_k \in A_1 \\ \quad \text{else if } u_2(a_k) > u_{\sim 2}(a_k) \text{ then } a_k \in A_2 \\ \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \qquad \text{else if } u_{q-1}(a_k) > u_{\sim(q-1)}(a_k) \text{ then } a_k \in A_{q-1} \\ \qquad\quad \text{else } a_k \in A_q \end{cases} \tag{9.43}$$

The development of all value functions in the MHDIS method is performed through the solution of three mathematical programming problems at each stage l of the discrimination process $l = 1, 2, \ldots, q-1$. Initially, an LP is solved to minimize the magnitude of the classification errors (in distance terms similarly to the UTADIS approach). Then, a mixed-integer LP is solved to minimize the total number of misclassifications among the misclassifications that occur after the solution of the initial LP, while retaining the correct classifications. Finally, a second LP is solved to maximize the clarity of the classification obtained from the solutions of the previous LPs.

9.3.5 Other Variants and Extensions

In all previous approaches, the value function was built in a one-step process by formulating an LP that requires only the DM's global preferences. In some cases, however, it would be more appropriate to build such a function from a two-step questioning process, by dissociating the construction of the marginal value functions and the assessment of their respective scaling constants.

In the first step, the various marginal value functions are built outside the UTA algorithm. These functions may be facilitated, for instance, by proposing specific parametrical marginal value functions to the DM and asking him/her to choose the one that matches his/her preferences on that specific criterion. Those functions should be normalized according to (9.4) conditions. Generally, the approaches applied in this construction step are:

(a) techniques based on MAUT theory [67, 70],
(b) the MACBETH method [3–5],
(c) the Quasi-UTA method [12], that uses "recursive exponential" marginal value functions, and
(d) the MIIDAS system (see Sect. 9.4) that combines artificial intelligence and visual procedures in order to extract the DM's preferences [135].

In the second step, after the assessment of these value functions, the DM is asked to give a global ranking of alternatives in a similar way as in the basic UTA method. From this information, the problem may be formulated via an LP, in order to assess only the weighting factors p_i of the criteria (scaling constants of criteria). Through this approach, initially named UTA II model [116], formula (9.11) becomes:

$$\begin{aligned}\Delta(a_k, a_{k+1}) = & \sum_{i=1}^{n} p_i\{u_i[g_i(a_k)] - u_i[g_i(a_{k+1})]\} \\ & -\sigma^+(a_k) + \sigma^-(a_k) + \sigma^+(a_{k+1}) - \sigma^-(a_{k+1})\end{aligned} \tag{9.44}$$

and the LP (9.14) is modified as follows:

$$\begin{cases} [min]F = \displaystyle\sum_{k=1}^{m}[\sigma^+(a_k) + \sigma^-(a_k)] \\ \text{subject to} \\ \left.\begin{array}{l}\Delta(a_k, a_{k+1}) \geq \delta \text{ if } a_k \succ a_{k+1} \\ \Delta(a_k, a_{k+1}) = 0 \text{ if } a_k \sim a_{k+1}\end{array}\right\} \quad \forall k \\ \displaystyle\sum_{i=1}^{n} p_i = 1 \\ p_i \geq 0, \sigma^+(a_k) \geq 0, \sigma^-(a_k) \geq 0 \quad \forall\, i, k. \end{cases} \tag{9.45}$$

The main principles of the UTA methods are also applicable in the specific field of multiobjective optimization, mainly in the field of linear programming with multiple objective functions. For instance, in the classical methods of Geoffrion et al. [35] and Zionts and Wallenius [150], the weights of the linear combinations of the objectives are inferred locally from trade-offs or pairwise judgments given by the DM at each iteration of the methods. Thus, these methods exploit in a direct way the DM's value functions and seek the best compromise solution through successive maximization of these assessed value functions.

Stewart [145] proposed a procedure of pruning the decision alternatives using the UTA method. In this approach a sequence of alternatives is presented to the DM, who places each new presented alternative in rank order relative to the earlier alternatives evaluated. This ranking of elements in a subset of the decision space is used to eliminate other alternatives from further consideration. In the same context, Jacquet-Lagrèze et al. [58] developed a disaggregation method, similar to UTA, to assess a whole value function of multiple objectives for linear programming systems. This methodology enables to find compromise solutions and is mainly based on the following steps:

1. Generation of a limited subset of feasible efficient solutions as representative as possible of the efficient set.
2. Assessment of an additive value function using PREFCALC system (see Sect. 9.4).
3. Optimization of the additive value function on the original set of feasible alternatives.

Finally, Siskos and Despotis [123], in the context of UTA-based approaches in multiobjective optimization problems, proposed the ADELAIS method. This approach refers to an interactive method that uses UTA iteratively, in order to optimize an additive value function within the feasible region defined on the basis of the satisfaction levels and determined in each iteration.

9.3.6 Other Disaggregation Methods

The main principles of the aggregation-disaggregation approach may be combined with outranking relation methods. The most important efforts concern the problem of determining the values of several parameters when using these methods. The set of these parameters is used to construct a preference model with which the DM accepts as a working hypothesis in the decision-aid study. In several real-world applications the assumption that the DM is able to give explicitly the values of each parameter is not realistic.

In this framework, the ELECCALC system has been developed [69], which estimates indirectly the parameters of the ELECTRE II method. The process is based on the DM's responses to questions of the system regarding his/her global preferences.

Furthermore, concerning problem statement β, several approaches consist in inferring the parameters of ELECTRE TRI through holistic information on DM's judgments. These approaches aim at substituting assignment examples for direct elicitation of the model parameters. Usually, the values of these parameters are inferred through a regression-type analysis on assignment examples.

Mousseau and Słowiński [99] propose an interactive aggregation-disaggregation approach that infers ELECTRE TRI parameters simultaneously starting from assignment examples. In this approach, the determination of the parameters' values (except the veto thresholds) that best restore the assignment examples is formulated through a non-linear optimization program.

Several efforts have tried to overcome the limitations of the aforementioned approach (computational difficulty, estimation of the veto threshold):

(a) Mousseau et al. [100, 101] consider the subproblem of the determination of the weights only, assuming that the thresholds and category limits have been fixed. This leads to formulate an LP (rather than non-linear in the global inference model). Through experimental analysis, they show that this approach is able to infer weights that restore in a stable way the assignment examples and it is also able to identify possible inconsistencies in these assignment examples.
(b) Doumpos and Zopounidis [29] use linear programming formulations in order to estimate all the parameters of the outranking relation classification model. However, in this approach, the parameters are estimated sequentially rather than through a global inference process. Thus, the proposed methodology does not specify the optimal parameters of the outranking relation (i.e. the ones that lead to a global minimum of the classification error). The results of this approach ("reasonable" specification of the parameters) serve rather as a basis for a thorough decision-aid process.

The problem of robustness and sensitivity analysis, through the extension of the previous research efforts is discussed in [26]. They consider the case where the DM can not provide exact values for the parameters of the ELECTRE TRI method, due to uncertain, imprecise or inaccurately determined information, as well as from lack of consensus among them. The proposed methodology combines the following approaches:

1. The first approach infers the value of parameters from assignment examples provided by the DM, as an elicitation aid.
2. The second approach considers a set of constraints on the parameter values reflecting the imprecise information that the DM is able to provide.

In the context of UTA-based ordinal regression analysis [119], the MUSA method has been developed in order to measure and analyze customer satisfaction [42, 134]. The method is used for the assessment of a set of marginal satisfaction functions in such a way that the global satisfaction criterion becomes as consistent as possible with customer's judgments. Thus, the main objective of the method is the aggregation of individual judgments into a collective value function.

The MUSA method assesses global and partial satisfaction functions Y^* and X_i^* respectively, given customers' ordinal judgments Y and X_i (for the i-th criterion). The ordinal regression analysis equation has the following form:

$$\hat{Y}^* = \sum_{i=1}^{n} b_i X_i^* - \sigma^+ + \sigma^- \tag{9.46}$$

where $\hat{Y}^*$ is the estimation of the global value function Y^*, n is the number of criteria, b_i is a positive weight of the i-th criterion, σ^+ and σ^- are the overestimation and the underestimation errors, respectively, and the value functions Y^* and X_i^* are normalized in the interval [0,100]. In the MUSA method the notation of ordinal regression analysis is adopted, where a criterion g_i is considered as a monotone variable X_i and a value function is denoted as X_i^*.

Similarly to the UTASTAR algorithm, the following transformation equations are used:

$$\begin{cases} z_m = y^{*m+1} - y^{*m} & \text{for } m = 1, 2, \ldots, \alpha - 1 \\ w_{ik} = b_i x_i^{*k+1} & \text{for } k = 1, 2, \ldots, \alpha_i - 1 \text{ and } i = 1, 2, \ldots, n \end{cases} \tag{9.47}$$

where y^{*m} is the value of the y^m satisfaction level, x_i^{*k} is the value of the x_i^* satisfaction level, and α and α_i are the number of global and partial satisfaction levels.

According to the previous definitions and assumptions, the MUSA estimation model can be written in an LP formulation, as follows:

$$\begin{cases} [min]F = \sum_{j=1}^{m} \sigma_j^+ + \sigma_j^- \\ \text{subject to} \\ \sum_{i=1}^{n} \sum_{k=1}^{x_i^j - 1} w_{ik} - \sum_{m=1}^{y^j - 1} z_m - \sigma_j^+ + \sigma_j^- \quad \text{for } j = 1, 2, \ldots, M \\ \sum_{m=1}^{\alpha - 1} z_m = 100 \\ \sum_{i=1}^{n} \sum_{k=1}^{\alpha_i - 1} w_{ik} = 100 \\ z_m, w_{ik}, \sigma_j^+, \sigma_j^- \forall m, i, j, k \end{cases} \tag{9.48}$$

where M is the size of the customer sample, and x_i^j and y^j are the j-th level on which variables X_i and Y are estimated (i.e. global and partial satisfaction judgments of the j-th customer). The MUSA method includes also a post-optimality analysis stage, similarly to step 4 of the UTASTAR algorithm.

An analytical development of the method and the provided results is given in [42], while the presentation of the MUSA DSS can be found in [43, 46].

The problem of building non-additive utility functions may also be considered in the context of aggregation-disaggregation approach. A characteristic case refers to positive interaction (synergy) or negative interaction among criteria (redundancy). Two or more criteria are synergic (redundant) when their joint weight is more (less) than the sum of the weights given to the criteria considered singularly.

In order to represent interaction among criteria, some specific formulations of the utility functions expressed in terms of fuzzy integrals have been proposed [38, 81, 102]. In this context, Angilella et al. [2] propose a methodology that allows the inclusion of additional information such as an interaction among criteria. The method aims at searching a utility function representing the DM's preferences, while the resulting functional form is a specific fuzzy integral (Choquet integral). As a result, the obtained weights may be interpreted as the "importance" of coalitions of criteria, exploiting the potential interaction between criteria. The method can also provide the marginal utility functions relative to each one of the considered criteria, evaluated on a common scale, as a consequence of the implemented methodology.

Hurson and Siskos [49] present a synergy of three complementary techniques to assess additive models on the whole criteria space. Their research includes a revised MACBETH technique, the standard MAUT trade-off analysis, and UTA-based methods for the assessment of both the marginal value functions, which are piecewise linear, and the weighting factors. The approach also uses a set of robustness measures and rules associated with MACBETH and UTA, in order to manage multiple LP solutions and extract robust conclusions from them. Several combinations of techniques are proposed which can facilitate the construction of the additive representation of DM's preferences. So, according to the properties of the DM's preferences and to the precise technical aspects of the decision-making problem, the analyst can choose the adequate combination of methods. Very recently Roy and Słowiński [110] presented a general framework to guide analysts and DMs in choosing the "right method".

The general scheme of the disaggregation philosophy is also employed in other approaches, including rough sets [27, 105, 140, 149], machine learning [106], and neural networks [76, 144]. All these approaches are used to infer some form of decision model (a set of decision rules or a network) from given decision results involving assignment examples, ordinal or measurable judgments.

9.4 Applications and UTA-Based DSS

The methods presented in the previous sections adopt the aggregation-disaggregation approach. This approach constitutes a basis for the interaction between the analyst and the DM, which includes:

– the consistency between the assessed preference model and the a priori preferences of the DM,
– the assessed values (values, weights, utilities, …), and
– the overall evaluation of potential actions (extrapolation output).

A general interaction scheme for this decision support process is given in Fig. 9.11.

Several decision support systems (DSSs), based on the UTA model and its variants, have been developed on the basis of disaggregation methods. These systems include:

(a) The PREFCALC system [52] is a DSS for interactive assessment of preferences using holistic judgments. The interactive process includes the classical aggregation phase where the DM is asked to estimate directly the parameters of the model (i.e. weights, trade-offs, etc.), as well as the disaggregation phase where the DM is asked to express his/her holistic judgments (i.e. global preference order on a subset of the alternatives) enabling an indirect estimation of the parameters of the model.
(b) MINORA (Multicriteria Interactive Ordinal Regression Analysis) is a multicriteria interactive DSS with a wide spectrum of supported decision making situations [130, 131]. The core of the system is based on the UTASTAR method and it uses special interaction techniques in order to guide the DM to reach a consistent preference system.
(c) MIIDAS (Multicriteria Interactive Intelligence Decision Aiding System) is an interactive DSS that implements the extended UTA II method [135]. In the first step of the decision-aid process, the system assess the DM's value functions, while in the next step, the system estimates the DM's preference model from his/her global preferences on a reference set of alternative actions. The system uses Artificial Intelligence and Visual techniques in order to improve the user interface and the interactive process with the DM.
(d) The UTA PLUS software [71] is an implementation of the UTA method, which allows the user to modify interactively the marginal value functions within limits set from a sensitivity analysis of the formulated ordinal regression problem. During all these modifications, a friendly graphical interface helps the DM to reach an accepted preference model.
(e) MUSTARD (Multicriteria Utility-based Stochastic Aid for Ranking Decisions) is an interactive DSS developed by Beuthe and Scannella [10], which incorporates several variants of the UTA method. The system provides several visual tools in order to structure the DM's preferences to a specific problem (see also [121]). The interactive process with the DM contains the following main steps: problem structuring, preference questionnaire, optimization solver-parameter computing, final results (full rankings and graphs).
(f) RUTA is a new UTA-based DSS proposed by Kadzinski et al. [64], which allows DMs to additionally exteriorize new types of preference information in terms of rank related statements (e.g. action *a* should be ranked in top 3, action *b* should be placed in bottom 5, etc.).

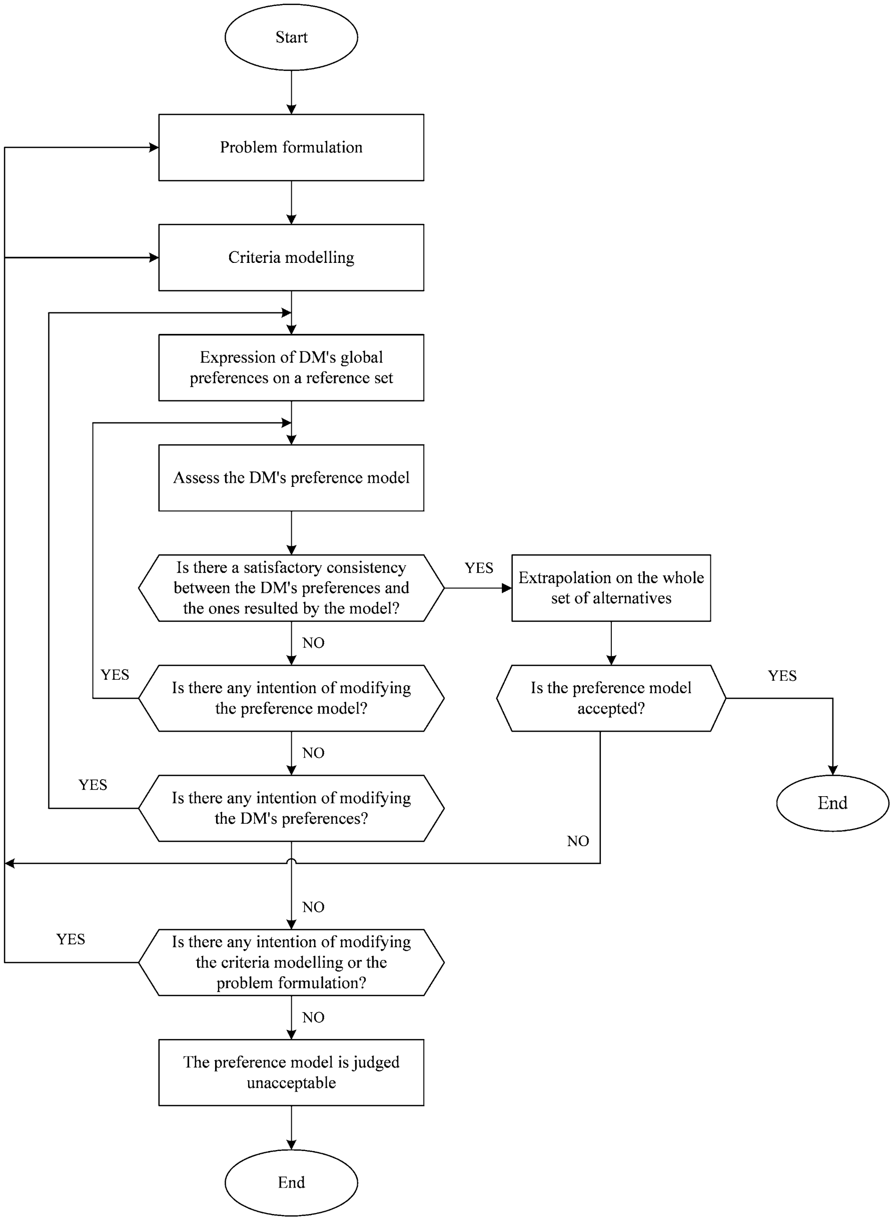


Fig. 9.11 Simplified decision support process based on disaggregation approach [57]

UTA methods have also been used in several works for conflict resolution in multi-actor decision situations [14, 54, 88]. In the same context, the MEDIATOR system was developed [59, 114, 115], which is a negotiation support system based on Evolutionary Systems Design (ESD) and database-centered implementation. ESD visualizes negotiations as a collective process of searching for designing a mutually acceptable solution. Participants are seen as playing a dynamical difference game in which a coalition of players is formed, if it can achieve a set of agreed upon goals. In MEDIATOR, negotiations are supported by consensus seeking through exchange of information and, where consensus is incomplete, by compromise. It assists in consensus seeking by aiding the players to build a group joint problem representation of the negotiations-in effect, joint mappings from control space to goal space (and through marginal utility functions) to utility space. Individual marginal utility functions are estimated by applying the UTA method. Players can arrive to a common coalition utility function through exchange of information and negotiation until players' marginal utility functions are identical. In addition to exchanging information and negotiating to expand targets, players can consider the use of axioms to contract the feasible region.

The UTA methods may be extended in the case of multiple DMs, taking into account different input information (criteria values) and preferences for a group of DMs. Two alternative approaches may be found in the literature [125]:

1. Application of the UTA/UTASTAR methods in order to optimally infer marginal value functions of individual DMs; the approach enables each DM to analyze his/her behavior according to the general framework of preference disaggregation.
2. Application of the UTA/UTASTAR methods in order to assess a set of collective additive value functions; these value functions are as consistent as possible with the preferences of the whole set of DMs, and thus, they are able to aggregate individual value systems.

In the context of the first approach, Matsatsinis et al. [96] propose a general methodology for collective decision-making combining different MCDA approaches and incorporating several criteria in order to measure the DMs' satisfaction over the aggregated rank-order of alternatives. Also, Matsatsinis and Delias [85] developed a general multicriteria protocol for multi-agent negotiations based on the UTA II method.

On the other hand, Siskos and Grigoroudis [125] propose the modification of the UTASTAR algorithm in order to infer a collective preference system for a group of DMs.

In the area of intelligent multicriteria DSSs, the MARKEX system has been proposed by Siskos and Matsatsinis [126] and Matsatsinis and Siskos [89, 91]. The system includes the UTASTAR algorithm and is used for the new product development process. It acts as a consultant for marketers, providing visual support to enhance understanding and to overcome lack of expertise. The data bases of the system are the results of consumer surveys, as well as financial information of the enterprises involved in the decision-making process. The system's model base encompasses statistical analysis, preference analysis, and brand choice models. Figure 9.12 presents a general methodological flowchart of the system. Also, MARKEX incorporates partial knowledge bases to support DMs in different stages of the product development process. The system incorporates three partial expert systems, functioning independently of each other. These expert systems use the following knowledge bases for the:

– selection of data analysis method,
– selection of brand choice model, and
– evaluation of the financial status of enterprises.

Furthermore, an intelligent web-based DSS, named DIMITRA, has been developed by Matsatsinis and Siskos [90]. The system is a consumer survey-based DSS, focusing on the decision-aid process for agricultural product development. Besides the implementation of the UTASTAR method in the preference analysis module, the DIMITRA system comprises several statistical analysis tools and consumer choice models. The system provides visual support to the DM (agricultural cooperatives, agribusiness firms, etc.) for several complex tasks, such as:

– evaluation of current and potential market shares,
– determination of the appropriate communication and penetration strategies, based on consumer attitudes and beliefs,
– adjustment of the production according to product's demand, and
– detection of the most promising markets.

In the same context, new research efforts have combined UTA-based DSSs with intelligent agents' technology. In general, the proposed methodologies engage the UTA models in a multi-agent architecture in order to assess the DM's preference system. These research efforts include mainly the following:

(a) An intelligent agent-based DSS, focusing on the determination of product penetration strategies has been developed [85, 93–95]. The system implements an original consumer-based methodology, in which intelligent agents operate in a functional and a structural level, simultaneously. Task, information and interface agents are included in the functional level in order to coordinate, collect necessary information and communicate with the DM. Likewise, the structural level includes elementary agents based on a generic reusable architecture and complex agents which aim to the development of a dynamical agent organization in a recursive way.
(b) A multi-agent architecture is proposed by Manouselis and Matsatsinis [80] for modeling electronic consumer's behavior. The implementation of the system refers to electronic marketplaces and incorporates a step-by-step methodology for intelligent systems analysis and design, used in the particular decision-aid process. The system develops consumer behavioral models for the purchasing and negotiation process adopting additional operational research tools and techniques. The presented application refers to the case of Internet radio.

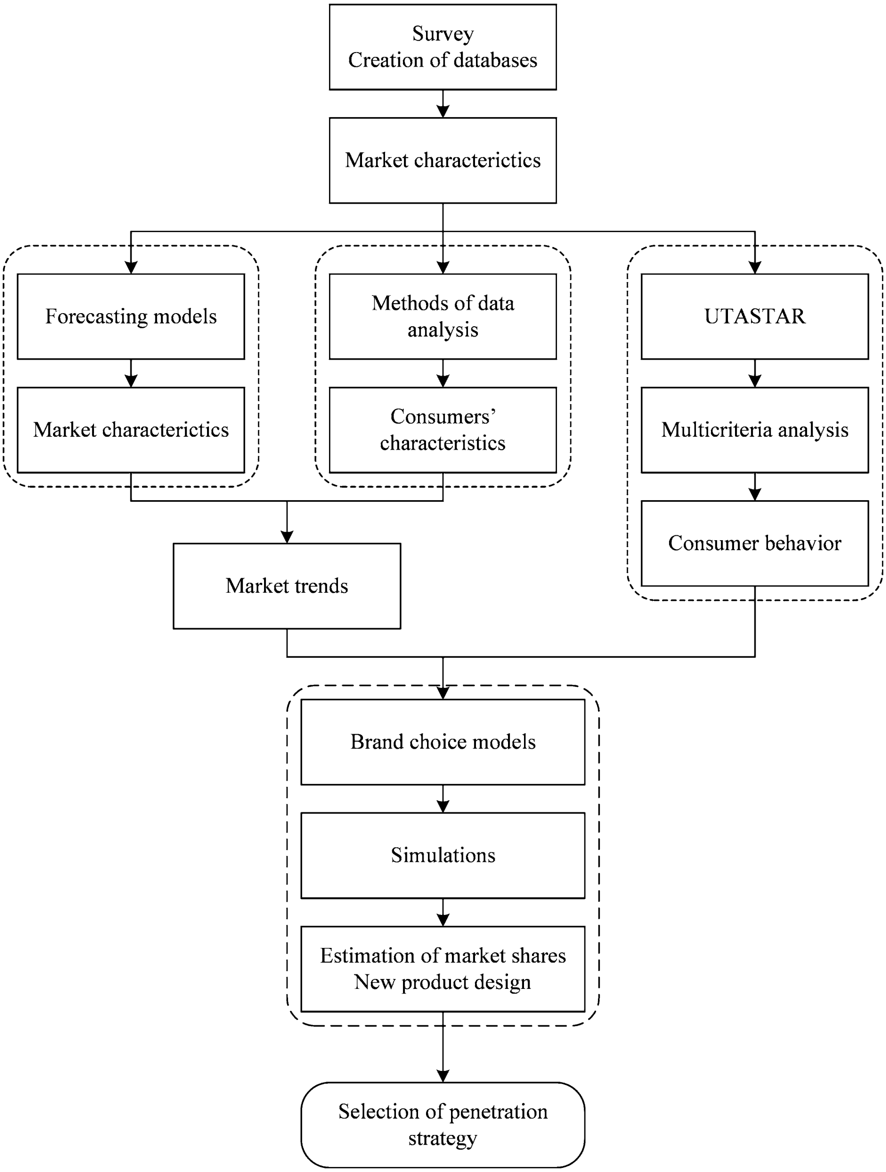


Fig. 9.12 Methodological flowchart of MARKEX [89]

(c) The AgentAllocator system [84] implements the UTA II method in the task allocation problem. These problems are very common to any multi-agent system in the context of Artificial Intelligence. The system is an intelligent agent DSS, which allows the DM to model his/her preferences in order to reach and employ the optimal allocation plan.

The need to combine data and knowledge in order to solve complex and ill-structured decision problems is a major concern in the modern marketing-management science. Matsatsinis [83] has proposed a DSS that implements the UTASTAR algorithm along with rule-induction data mining techniques. The main aim of the system is to derive and apply a set of rules that relate the global and the marginal value functions. A comparison between the original and the rule-based global values is used in the validity and stability analysis of the proposed methodology.

Furthermore, in the area of financial management, a variety of UTA-based DSSs has been developed, including mainly the following systems:

(a) The FINEVA system [159] is a multicriteria knowledge-based DSS developed for the assessment of corporate performance and viability. The system implements multivariate statistical techniques (e.g. principal components analysis), expert systems technology [92], and the UTASTAR method to provide integrated support in evaluating the corporate performance.
(b) The FINCLAS system [153] is a multicriteria DSS developed to study financial decision-making problems in which a classification (sorting) of the alternatives is required. The present form of the system is devoted to corporate credit risk assessment, and it can be used to develop classification models to assign a set of firms into predefined credit risk classes. The analysis performed by the system is based on the family of the UTADIS methods.
(c) The INVESTOR system [156] is developed to study problems related to portfolio selection and management. The system implements the UTADIS method, as well as goal programming techniques to support portfolio managers and investors in their daily practice.
(d) The PREFDIS system [157] is a multicriteria DSS developed to address classification problems. The system implements a series of preference disaggregation analysis techniques, namely the family of the UTADIS methods, in order to develop an additive utility function to be used for classification purposes.
(e) The INTELLIGENT INVESTOR system [111, 112] is an intelligent system which aims to support investment decision-making. The system integrates MCDA methods (UTASTAR algorithm) and artificial intelligence technologies (expert system), incorporating several portfolio management tools (Fundamental Analysis, Technical Analysis, and Market Psychology).

Also, as presented in Sect. 9.3.5, Siskos and Despotis [123] have developed the ADELAIS system, which is designed to decision-aid in multiobjective linear programming (MOLP) problems.

Table 9.7 Indicative applications of the UTA methods

Field	Scope	References
Financial management	Venture capital evaluation	[129]
	Portfolio selection and management	[50, 51, 160]
	Business failure prediction	[151, 154]
	Business financing	[131, 153, 159]
	Country risk assessment	[18, 104, 161]
Marketing	Marketing of new products	[141]
	Marketing of agricultural products	[6, 8, 89–91, 94, 97, 126, 137]
	Consumer behavior	[7, 74, 75, 80, 83, 87, 132, 133]
	Customer satisfaction	[44, 45, 47, 98, 113, 124, 136]
	Sales strategy problems	[107, 120]
Management (general)	Project evaluation	[12, 53]
	Environmental management	[25, 48, 122]
	Job evaluation	[37, 142]
	Healthcare & healthcare management	[30, 78, 79]
	E-government	[36, 138, 139]
	Recommender systems	[19, 73]
	Other	[1, 20, 72, 82, 86]

Over the past two decades UTA-based methods have been applied in several real-world decision-making problems from the fields of financial management, marketing, environmental management, as well as human resources management, as presented in Table 9.7. These applications have provided insight on the applicability of preference disaggregation analysis in addressing real-world decision problems and its efficiency.

Finally, the following real-world application, with emphasis on the synergy between UTA methods and other MCDA approaches, may be found in the literature:

(a) Hurson et al. [51] present a case study regarding the portfolio selection problem and the evaluation of stocks in the Athens stock exchange. The assessment of the additive value model is done by combining MACBETH on a single criterion level and MAUT for the determination of inter-criteria parameters.
(b) Siskos et al. [138, 139] propose a multicriteria methodology for e-government benchmarking in Europe. The proposed assessment procedure is supported by the MIIDAS DSS to visually determine the marginal value functions and elicit the set of admissible weights using the UTA II method. Finally, a set of complementary robustness analysis techniques is utilized to handle both the robustness of the evaluation model and the extreme ranking positions of the alternatives (i.e. countries).

(c) Demesouka et al. [20] present S-UTASTAR (spatial UTASTAR), a robust ordinal regression DSS for land-use suitability analyses. The S-UTASTAR is applied in a raster-based case study to identify appropriate municipal solid waste landfill sites. Moreover, the Stochastic Multiobjective Acceptability Analysis (SMAA) is applied, based on a probability distribution of the additive model parameters, to indicate the frequency that an alternative get the best ranks, aiding this way the decision making process.
(d) Doumpos et al. [30] present a UTADIS-based methodology for monitoring the postoperative behavior of patients that have received treatment for atrial fibrillation (AF). The model classifies the patients in seven categories according to their relapse risk, on the basis of seven criteria related to the AF type and pathology conditions, the treatment received by the patients, and their medical history. A two-stage robust multicriteria model development procedure is used to minimize the number and magnitude of the misclassifications.
(e) Lakiotaki and Matsatsinis [73] analyze movie user profiles as a result of a multi-criteria recommendation methodology, applied to real user data, in order to reveal any hidden aspect of user behavior that would eventually improve current system's performance.
(f) Delias et al. [19] propose a recommendation approach to match the customized needs of an organization against the existing technologies (innovative products or services). The system is able to create a profile based on the organization's needs and preferences. This profile is used to guide a recommendation process, according to which, available technologies are evaluated against the profile and proposed to the organization in a descending order.
(g) Krassadaki et al. [72] propose a methodological framework based on a multicriteria clustering approach that identifies different assessment behaviors, in order to adopt the most common student assessment policy.

9.5 Concluding Remarks and Future Research

The UTA methods presented in this chapter belong to the family of ordinal regression analysis models aiming to assess a value system as a model of the preferences of the DM. This assessment is implemented through an aggregation-disaggregation process. With this process the analyst is able to infer an analytical model of preferences, which is as consistent as possible with the DM' preferences. The acceptance of such a preference model is accomplished through a repetitive interaction between the model and the DM. This approach contributes towards an alternative reasoning for decision-aid (see Fig. 9.2).

Future research regarding UTA methods aims to explore further the potentials of the preference disaggregation philosophy within the context of multicriteria decision-aid. Jacquet-Lagrèze and Siskos [57] propose that potential research developments may be focused on:

(a) the inference of more sophisticated aggregation models by disaggregation, and
(b) the experimental evaluation of disaggregation procedures.

Finally, it would be interesting to explore the relationship of aggregation and disaggregation procedures in terms of similarities and/or dissimilarities regarding the evaluation results obtained by both approaches [57]. This will enable the identification of the reasons and the conditions under which aggregation and disaggregation procedures will lead to different or the same results.

References

1. Androulaki, S., Psarras, J., Angelopoulos, D.: Evaluating long term potential natural gas supply alternatives for Greece with multicriteria decision analysis. In: Paper Presented at the 77th Meeting of the EURO Working Group on MCDA, Rouen, 11–13 April 2013
2. Angilella, S., Greco, S., Lamantia, F., Matarazzo, B.: Assessing non-additive utility for multicriteria decision aid. Eur. J. Oper. Res. **158**(3), 734–744 (2004)
3. Bana e Costa, C.A., Vansnick, J.C.: MACBETH: an interactive path towards the construction of cardinal value functions. Int. Trans. Oper. Res. **1**(4), 489–500 (1994)
4. Bana e Costa, C.A., Vansnick, J.C.: Applications of the MACBETH approach in the framework of an additive aggregation model. J. Multicrit. Decis. Anal. **6**(2), 107–114 (1997)
5. Bana e Costa, C.A., Nunes Da Silva, F., Vansnick, J.C.: Conflict dissolution in the public sector: a case-study. Eur. J. Oper. Res. **130**(2), 388–401 (2001)
6. Baourakis, G., Matsatsinis, N.F., Siskos, Y.: Agricultural product design and development. In: Janssen, J., Skiadas, C.H. (eds.) Applied Stochastic Models and Data Analysis, pp. 1108–1128. World Scientific, Singapore (1993)
7. Baourakis, G., Matsatsinis, N.F., Siskos, Y.: Consumer behavioural analysis using multicriteria methods. In: Janssen, J., Skiadas, C.H., Zopounidis, C. (eds.) Advances in Stochastic Modelling and Data Analysis, pp. 328–338. Kluwer Academic, Dordrecht (1995)
8. Baourakis, G., Matsatsinis, N.F., Siskos, Y.: Agricultural product development using multidimensional and multicriteria analyses: the case of wine. Eur. J. Oper. Res. **94**(2), 321–334 (1996)
9. Beuthe, M., Scannella, G.: Applications comparées des méthodes d'analyse multicritère UTA. RAIRO Recherche Opérationelle **30**(3), 293–315 (1996)
10. Beuthe, M., Scannella, G.: MUSTARD user's guide. GTM, Facultés Universitaires Catholiques de Mons (FUCaM), Mons (1999)
11. Beuthe, M., Scannella, G.: Comparative analysis of UTA multicriteria methods. Eur. J. Oper. Res. **130**(2), 246–262 (2001)
12. Beuthe, M., Eeckhoudt, L., Scanella, G.: A practical multicriteria methodology for assessing risky public investments. Socio Econ. Plan. Sci. **34**(2), 121–139 (2000)
13. Bous, G., Fortemps, P., Glineur, F., Pirlot, M.: ACUTA: a novel method for eliciting additive value functions on the basis of holistic preference statements. Eur. J. Oper. Res. **206**(2), 435–444 (2010)
14. Bui, T.X.: Co-op: A Group Decision Support System for Cooperative Multiple Criteria Group Decision Making. Lecture Notes in Computer Science, vol. 290. Springer, Berlin (1987)
15. Charnes, A., Cooper, W.: Management Models and Industrial Applications of Linear Programming, vol. 1. Wiley, New York (1961)
16. Charnes, A., Cooper, W., Ferguson, R.O.: Optimal estimation of executive compensation by linear programming. Manag. Sci. **1**(2), 138–151 (1955)
17. Corrente, S., Greco, S., Kadziński, M., Słowiński, R.: Robust ordinal regression in preference learning and ranking. Mach. Learn. **93**(2–3), 381–422 (2013)

18. Cosset, J.C., Siskos, Y., Zopounidis, C.: Evaluating country risk: a decision support approach. Glob. Financ. J. **3**(1), 79–95 (1992)
19. Delias, P., Kyriakaki, G., Grigoroudis, E., Matsatsinis, N.: Innovation management software exploiting multiple criteria analysis: the case of innovation centre of Crete. Int. J. Decis. Support Syst. Technol. **4**(1), 30–42 (2013)
20. Demesouka, O.E., Anagnostopoulos, K.P., Siskos, Y.: Spatial robust ordinal regression for landfill site selection in northeast Greece. In: Paper Presented at the 10th Meeting of the Hellenic Working Group on MCDA, Thessaloniki, 21–23 November 2013
21. Despotis, D.K., Yannacopoulos, D.: Méthode d'estimation d'utilités additives concaves en programmation linéare multiobjectifs. RAIRO Recherche Opérationelle **24**, 331–349 (1990)
22. Despotis, D.K., Zopounidis, C.: Building additive utilities in the presence of non-monotonic preference. In: Pardalos, P.M., Siskos, Y., Zopounidis, C. (eds.) Advances in Multicriteria Analysis, pp. 101–114. Kluwer Academic, Dordrecht (1993)
23. Despotis, D.K., Yannacopoulos, D., Zopounidis, C.: A review of the UTA multicriteria method and some improvements. Found. Comput. Decis. Sci. **15**(2), 63–76 (1990)
24. Devaud, J.M., Groussaud, G., Jacquet-Lagrèze, E.: UTADIS: Une méthode de construction de fonctions d'utilité additives rendant compte de jugements globaux. In: European Working Group on Multicriteria Decision Aid, Bochum (1980)
25. Diakoulaki, D., Zopounidis, C., Mavrotas, G., Doumpos, M.: The use of a preference disaggregation method in energy analysis and policy making. Energy **24**(2), 157–166 (1999)
26. Dias, L., Mousseau, V., Figueira, J., Climaco, J.: An aggregation/disaggregation approach to obtain robust conclusions with ELECTRE TRI. Eur. J. Oper. Res. **138**(2), 332–348 (2002)
27. Dimitras, A.I., Słowiński, R., Susmaga, R., Zopounidis, C.: Business failure prediction using rough sets. Eur. J. Oper. Res. **114**(2), 263–280 (1999)
28. Doumpos, M., Zopounidis, C.: Multicriteria Decision Aid Classification Methods. Kluwer Academic, Dordrecht (2002)
29. Doumpos, M., Zopounidis, C.: On the development of an outranking relation for ordinal classification problems: an experimental investigation of a new methodology. Optim. Methods Softw. **17**(2), 293–317 (2002)
30. Doumpos, M., Xidonas, P., Xidonas, S., Siskos, Y.: Development of a robust multicriteria classification model for monitoring the postoperative behaviour of heart patients. Technical report, University of Piraeus (2014)
31. Figueira, J.R., Greco, S., Słowiński, R.: Building a set of additive value functions representing a reference preorder and intensities of preference: GRIP method. Eur. J. Oper. Res. **195**(2), 460–486 (2009)
32. Fishburn, P.: A note on recent developments in additive utility theories for multiple factors situations. Oper. Res. **14**, 1143–1148 (1966)
33. Fishburn, P.: Methods for estimating additive utilities. Manag. Sci. **13**, 435–453 (1967)
34. Freed, N., Glover, G.: Simple but powerful goal programming models for discriminant problems. Eur. J. Oper. Res. **7**, 44–60 (1981)
35. Geoffrion, A.M., Dyer, J.S., Feinberg, A.: An interactive approach for multi-criterion optimization, with an application to the operation of an academic department. Manag. Sci. **19**(4), 357–368 (1972)
36. Giannakopoulos, D., Manolitzas, P., Spyridakos, A.: Measuring e-government: a comparative study of the G2C online services progress using multi-criteria analysis. Int. J. Decis. Support Syst. Technol. **2**(4), 1–12 (2010)
37. González-Araya, M.C., Rangel, L.A.D., Lins, M.P.E., Gomes, L.F.A.M.: Building the additive utility functions for CAD-UFRJ evaluation staff criteria. Ann. Oper. Res. **116**(1–4), 271–288 (2002)
38. Grabisch, M.: The application of fuzzy integrals in multicriteria decision making. Eur. J. Oper. Res. **89**(3), 445–456 (1996)
39. Greco, S., Mousseau, V., Słowiński, R.: Ordinal regression revisited: multiple criteria ranking using a set of additive value functions. Eur. J. Oper. Res. **191**(2), 416–436 (2008)

40. Greco, S., Słowiński, R., Figueira, J., Mousseau, V.: Robust ordinal regression. In: Ehrgott, M., Greco, S., Figueira, J. (eds.) Trends in Multiple Criteria Decision Analysis, pp. 241–283. Springer, Berlin (2010)
41. Greco, S., Siskos, Y., Słowiński, R.: Controlling robustness in ordinal regression models. In: Paper Presented at the 75th Meeting of the EURO Working Group on MCDA, Tarragona, 12–14 April 2012
42. Grigoroudis, E., Siskos, Y.: Preference disaggregation for measuring and analysing customer satisfaction: the MUSA method. Eur. J. Oper. Res. **143**(1), 148–170 (2002)
43. Grigoroudis, E., Siskos, Y.: MUSA: A decision support system for evaluating and analysing customer satisfaction. In: Margaritis, K., Pitas, I. (eds.) Proceedings of the 9th Panhellenic Conference in Informatics, Thessaloniki, pp. 113–127 (2003)
44. Grigoroudis, E., Siskos, Y.: A survey of customer satisfaction barometers: results from the transportation-communications sector. Eur. J. Oper. Res. **152**(2), 334–353 (2003)
45. Grigoroudis, E., Malandrakis, J., Politis, J., Siskos, Y.: Customer satisfaction measurement: an application to the Greek shipping sector. In: Proceedings of the 5th Decision Sciences Institute's International Conference on Integrating Technology & Human Decisions: Global Bridges into the 21st Century, Athens, vol. 2, pp. 1363–1365 (1999)
46. Grigoroudis, E., Siskos, Y., Saurais, O.: TELOS: a customer satisfaction evaluation software. Comput. Oper. Res. **27**(7–8), 799–817 (2000)
47. Grigoroudis, E., Politis, Y., Siskos, Y.: Satisfaction benchmarking and customer classification: an application to the branches of a banking organization. Int. Trans. Oper. Res. **9**(5), 599–618 (2002)
48. Hatzinakos, I., Yannacopoulos, D., Faltsetas, C., Ziourkas, C.: Application of the MINORA decision support system to the evaluation of landslide favourability in Greece. Eur. J. Oper. Res. **50**(1), 60–75 (1991)
49. Hurson, Ch., Siskos, Y.: A synergy of multicriteria techniques to assess additive value models. Eur. J. Oper. Res. **238**, 540–551 (2014)
50. Hurson, Ch., Zopounidis, C.: Gestion de portefeuille et analyse multicritère. Economica, Paris (1997)
51. Hurson, Ch., Mastorakis, K., Siskos, Y.: Application of a synergy of MACBETH and MAUT multicriteria methods to portfolio selection in Athens stock exchange. Int. J. Multicrit. Decis. Mak. **2**(2), 113–127 (2012)
52. Jacquet-Lagrèze, E.: Interactive assessment of preferences using holistic judgments: the PREFCALC system. In: Bana e Costa, C.A. (ed.) Readings in Multiple Criteria Decision Aid, pp. 225–250. Springer, Berlin (1990)
53. Jacquet-Lagrèze, E.: An application of the UTA discriminant model for the evaluation of R&D projects. In: Pardalos, P.M., Siskos, Y., Zopounidis, C. (eds.) Advances in Multicriteria Analysis, pp. 203–211. Kluwer Academic, Dordrecht (1995)
54. Jacquet-Lagrèze, E., Shakun, M.F.: Decision support systems for semistructured buying decisions. Eur. J. Oper. Res. **16**, 48–56 (1984)
55. Jacquet-Lagrèze, E., Siskos, J.: Une méthode de construction de fonctions d' utilité additives explicatives d' une prèfèrence globale. In: Cahier du LAMSADE, vol. 16. Universitè de Paris-Dauphine, Paris (1978)
56. Jacquet-Lagrèze, E., Siskos, Y.: Assessing a set of additive utility functions for multicriteria decision making: the UTA method. Eur. J. Oper. Res. **10**(2), 151–164 (1982)
57. Jacquet-Lagrèze, E., Siskos, Y.: Preference disaggregation: 20 years of MCDA experience. Eur. J. Oper. Res. **130**(2), 233–245 (2001)
58. Jacquet-Lagrèze, E., Meziani, R., Słowiński, R.: MOLP with an interactive assessment of a piecewise linear utility function. Eur. J. Oper. Res. **31**(3), 350–357 (1987)
59. Jarke, M., Jelassi, M.T., Shakun, M.F.: MEDIATOR: toward a negotiation support system. Eur. J. Oper. Res. **31**(3), 314–334 (1987)
60. Kadziński, M., Tervonen, T.: Robust multi-criteria ranking with additive value models and holistic pair-wise preference statements. Eur. J. Oper. Res. **228**(1), 169–180 (2013)

61. Kadziński, M., Tervonen, T.: Stochastic ordinal regression for multiple criteria sorting problems. Decis. Support Syst. **55**(1), 55–66 (2013)
62. Kadziński, M., Greco, S., Słowiński, R.: Extreme ranking analysis in robust ordinal regression. Omega **40**(4), 488–501 (2012)
63. Kadziński, M., Greco, S., Słowiński, R.: Selection of a representative value function in robust multiple criteria ranking and choice. Eur. J. Oper. Res. **217**(3), 541–553 (2012)
64. Kadziński, M., Greco, S., Słowiński, R.: RUTA: a framework for assessing and selecting additive value functions on the basis of rank related requirements. Omega **41**(4), 735–751 (2013)
65. Karst, O.J.: Linear curve fitting using least deviations. J. Am. Stat. Assoc. **53**, 118–132 (1958)
66. Keeney, R.L.: Sitting Energy Facilities. Academic, New York (1980)
67. Keeney, R.L., Raiffa, H.: Decisions with Multiple Objectives: Preferences and Value Tradeoffs. Wiley, New York (1976)
68. Kelley, J.E.: An application of linear programming to curve fitting. J. Ind. Appl. Math. **6**(1), 15–22 (1958)
69. Kiss, L.N., Martel, J.M., Nadeau, R.: ELECCALC - an interactive software for modelling the decision maker's preferences. Decis. Support Syst. **12**(4–5), 311–326 (1994)
70. Klein, G., Moskowitz, H., Mahesh, S., Ravindran, A.: Assessment of multiattribute measurable value and utility functions via mathematical programming. Decis. Sci. **16**(3), 309–324 (1985)
71. Kostkowski, M., Słowiński, R.: UTA+ application (v. 1.20)-user's manual. In: Document du LAMSADE, vol. 95. Université de Paris-Dauphine, Paris (1996)
72. Krassadaki, E., Lakiotaki, K., Matsatsinis, N.: Students' behaviour in peer assessment: a multi-criteria clustering approach. Eur. J. Eng. Educ. **39**, 233–246 (2013)
73. Lakiotaki, K., Matsatsinis, N.: Analysing user behaviour in recommender systems. Int. J. Electron. Bus. **10**(1), 1–19 (2012)
74. Lakiotaki, K., Delias, P., Sakkalis, V., Matsatsinis, N.F.: User profiling based on multi-criteria analysis: the role of utility functions. Oper. Res. Int. J. **9**(1), 3–16 (2009)
75. Lakiotaki, K., Matsatsinis, N.F., Tsoukias, A.: Multi-criteria user profiling in recommender systems. IEEE Intell. Syst. **26**(2), 64–76 (2011)
76. Malakooti, B., Zhou, Y.Q.: Feedforward artificial neural networks for solving discrete multiple criteria decision making problems. Manag. Sci. **40**(11), 1542–1561 (1994)
77. Manas, M., Nedoma, J.: Finding all vertices of a convex polyhedron. Numer. Math. **12**, 226–229 (1968)
78. Manolitzas, P., Grigoroudis, E., Matsatsinis, N.: MEDUTA: integrating simulation modeling and multiple criteria analysis to improve emergency department performance. In: Paper Presented at the 2nd International Symposium and 24th National Conference on Operational Research, National Technical University of Athens, Athens, 26–28 September 2013
79. Manolitzas, P., Matsatsinis, N., Grigoroudis, E.: Reforming the hospitals in Greece: an integrated framework for improving the health care services in an emergency department. In: Paper Presented at the 6th Biennial Hellenic observatory Ph.D. Symposium on Contemporary Greece and Cyprus, London School of Economics (LSE), London, 6–7 June 2013
80. Manouselis, N., Matsatsinis, N.F.: Introducing a multi-agent, multi-criteria methodology for modeling electronic consumer's behavior: the case of internet radio. In: Klush, M., Zambonelli, F. (eds.) Cooperative Information Agents. Lecture Notes in Artificial Intelligence, vol. 2182, pp. 190–195. Springer, New York (2001)
81. Marichal, J.L., Roubens, M.: Determination of weights of interactive criteria from a reference set. Eur. J. Oper. Res. **124**(3), 641–650 (2000)
82. Mastorakis, K., Siskos, E.: Value focused pharmaceutical strategy determination with multicriteria decision analysis techniques. In: Paper presented at the 2nd International Symposium and 24th National Conference on Operational Research, National Technical University of Athens, Athens, 26–28 September 2013
83. Matsatsinis, N.F.: New agricultural product development using data mining techniques and multicriteria methods. In: Proceedings of the 1st Hellenic Association of Information and

Communication Technology in Agriculture, Food and Environment (HAICTA's Conference 2002), Athens, 6–7 June 2002
84. Matsatsinis, N.F., Delias, P.: AgentAllocator: An agent-based multi-criteria decision support system for task allocation. In: Marik, V., McFarlane, D., Valckenaers, P. (eds.) Holonic and Multiagent Systems for Manufacturing. Lecture Notes in Computer Science, pp. 225–235. Springer, Berlin (2003)
85. Matsatsinis, N.F., Delias, P.: A multi-criteria protocol for multi-agent negotiations. In: Ehrgott, M., Greco, S., Figueira, J. (eds.) Methods and Applications of Artificial Intelligence. Lectures Notes in Artificial Intelligence, vol. 3025, pp. 103–111. Springer, Berlin (2004)
86. Matsatsinis, N.F., Fortsas, V.C.: A multicriteria methodology for the assessment of distance education trainees. Oper. Res. Int. J. **5**(3), 419–434 (2005)
87. Matsatsinis, N.F., Samaras, A.P.: Brand choice model selection based on consumers' multicriteria preferences and experts' knowledge. Comput. Oper. Res. **27**(8), 689–707 (2000)
88. Matsatsinis, N.F., Samaras, A.P.: MCDA and preference disaggregation in group decision support systems. Eur. J. Oper. Res. **130**(2), 414–429 (2001)
89. Matsatsinis, N.F., Siskos, Y.: MARKEX: an intelligent decision support system for product development decision. Eur. J. Oper. Res. **113**(2), 363–354 (1999)
90. Matsatsinis, N.F., Siskos, Y.: DIMITRA: an intelligent decision support system for agricultural products development decisions. In: Proceedings of the 3rd European Conference of the European Federation for Information Technology in Agriculture, Food and the Environment (EFITA 2001), Montpellier, 18–20 June 2001
91. Matsatsinis, N.F., Siskos, Y.: Intelligent Support Systems for Marketing Decision. Kluwer Academic, Dordrecht (2003)
92. Matsatsinis, N.F., Doumpos, M., Zopounidis, C.: Knowledge acquisition and representation for expert systems in the field of financial analysis. Expert Syst. Appl. **12**(2), 247–262 (1997)
93. Matsatsinis, N.F., Moraitis, P., Psomatakis, V., Spanoudakis, N.: Intelligence software agents for products penetration strategy selection. In: Proceedings of the 9th European Workshop on Modelling Autonomous Agents in a Multi-Agent World (MAAMAW99), Valencia, 30 June–2 July 1999
94. Matsatsinis, N.F., Moraitis, P., Psomatakis, V., Spanoudakis, N.: Multi-agent architecture for agricultural products development. In: Proceedings of the 2nd European Conference of the European Federation for Information Technology in Agriculture, Food and the Environment (EFITA 2000), Bonn, 27–30 September 2000
95. Matsatsinis, N.F., Moraitis, P., Psomatakis, V., Spanoudakis, N.: An agent-based system for products penetration strategy selection. Appl. Artif. Intell. J. **17**(10), 901–925 (2001)
96. Matsatsinis, N.F., Grigoroudis, E., Samaras, A.P.: Aggregation and disaggregation of preferences for collective decision-making. Group Decis. Negot. **14**(3), 217–232 (2005)
97. Matsatsinis, N.F., Grigoroudis, E., Samaras, A.P.: Comparing distributors' judgements to buyers' preferences: a consumer value analysis in the Greek olive oil market. Int. J. Retail Distrib. Manag. **35**(5), 342–362 (2007)
98. Mihelis, G., Grigoroudis, E., Siskos, Y., Politis, Y., Malandrakis, Y.: Customer satisfaction measurement in the private bank sector. Eur. J. Oper. Res. **130**(2), 347–360 (2001)
99. Mousseau, V., Słowiński, R.: Inferring an ELECTRE-TRI model from assignment examples. J. Glob. Optim. **12**(2), 157–174 (1998)
100. Mousseau, V., Figueira, J., Naux, J.Ph.: Using assignment examples to infer weights for ELECTRE TRI method: some experimental results. Eur. J. Oper. Res. **130**(2), 263–275 (2000)
101. Mousseau, V., Słowiński, R., Zielniewicz, P.: A user-oriented implementation of the ELECTRE-TRI method integrating preference elicitation support. Comput. Oper. Res. **27**(7–8), 757–777 (2000)
102. Murofushi, T., Sugeno, M.: An interpretation of fuzzy measure and the Choquet integral as an integral with respect to a fuzzy measure. Fuzzy Sets Syst. **29**(2), 201–227 (1989)
103. Oral, M., Kettani, O.: Modelling the process of multiattribute choice. J. Oper. Res. Soc. **40**(3), 281–291 (1989)

104. Oral, M., Kettani, O., Cosset, J.C., Daouas, M.: An estimation model for country risk rating. Int. J. Forecast. **8**(4), 583–593 (1992)
105. Pawlak, Z.: Rough sets. Int. J. Inf. Comput. Sci. **11**(5), 341–356 (1982)
106. Quinlan, J.R.: Induction of decision trees. Mach. Learn. **1**(1), 81–106 (1986)
107. Richard, J.L.: Aide à la décision stratégique en P.M.E. In: Jacquet-Lagrèze, E., Siskos, Y. (eds.) Méthode de décision multicritère, pp. 119–142. Hommes et Techniques, Paris (1983)
108. Roy, B.: Méthodologie Multicritère d'Aide à la Décision. Economica, Paris (1985)
109. Roy, B.: Robustness in operational research and decision aiding: a multi-faceted issue. Eur. J. Oper. Res. **200**(3), 629–638 (2010)
110. Roy, B., Słowiński, R.: Questions guiding the choice of a multicriteria decision aiding method. EURO J. Decis. Process. **1**(1–2), 69–97 (2013)
111. Samaras, G.D., Matsatsinis, N.F.: INTELLIGENT INVESTOR: an intelligent decision support system for portfolio management. Oper. Res. Int. J. **4**(3), 357–372 (2005)
112. Samaras, G.D., Matsatsinis, N.F., Zopounidis, C.: A multicriteria DSS for stock evaluation using fundamental analysis. Eur. J. Oper. Res. **187**(3), 1380–1401 (2008)
113. Sandalidou, E., Baourakis, G., Grigoroudis, E., Siskos, Y.: Organic and conventional olive oil consumers: a comparative analysis using customer satisfaction evaluation approach. Cahiers Options Mediterraneennes **61**, 265–276 (2003)
114. Shakun, M.F.: Evolutionary Systems Design: Policymaking Under Complexity and Group Decision Support Systems. Holden-Day, San Francisco (1988)
115. Shakun, M.F.: Airline buyout: evolutionary systems design and problem restructuring in group decision and negotiation. Manag. Sci. **37**(10), 1291–1303 (1991)
116. Siskos, J.: Comment modéliser les préférences au moyen de fonctions d'utilité additives. RAIRO Recherche Opérationelle **14**, 53–82 (1980)
117. Siskos, J.: A way to deal with fuzzy preferences in multicriteria decision problems. Eur. J. Oper. Res. **10**(3), 314–324 (1982)
118. Siskos, J.: Analyse de systèmes de décision multicritère en univers aléatoire. Found. Control Eng. **10**(3–4), 193–212 (1983)
119. Siskos, J.: Analyses de regression et programmation lineaire. Revue de Statistique Appliquee **23**(2), 41–55 (1985)
120. Siskos, Y.: Evaluating a system of furniture retail outlets using an interactive ordinal regression method. Eur. J. Oper. Res. **23**, 179–193 (1986)
121. Siskos, Y.: MUSTARD: multicriteria utility-based stochastic aid for ranking decisions. J. Behav. Decis. Mak. **15**(5), 461–465 (2002)
122. Siskos, J., Assimakopoulos, N.: Multicriteria highway planning: a case study. Math. Comput. Model. **12**(10–11), 1401–1410 (1989)
123. Siskos, J., Despotis, D.K.: A DSS oriented method for multiobjective linear programming problems. Decis. Support. Syst. **5**(1), 47–55 (1989)
124. Siskos, Y., Grigoroudis, E.: Measuring customer satisfaction for various services using multicriteria analysis. In: Bouyssou, D., Jacquet-Lagreze, E., Perny, P., Słowiński, R., Vanderpooten, D., Vincke, P. (eds.) Aiding Decisions with Multiple Criteria: Essays in Honor of Bernard Roy, pp. 457–482. Kluwer Academic, Dordrecht (2002)
125. Siskos, Y., Grigoroudis, E.: New trends in aggregation-disaggregation approaches. In: Zopounidis, C., Pardalos, P.M. (eds.) Handbook of Multicriteria Analysis, pp. 189–214. Springer, Heidelberg (2010)
126. Siskos, J., Matsatsinis, N.F.: A DSS for market analysis and new product design. J. Decis. Syst. **2**(1), 35–60 (1993)
127. Siskos, Y., Spyridakos, A.: Intelligent multicriteria decision support: overview and perspectives. Eur. J. Oper. Res. **113**(2), 236–246 (1999)
128. Siskos, Y., Yannacopoulos, D.: UTASTAR: an ordinal regression method for building additive value functions. Investigação Operacional **5**(1), 39–53 (1985)
129. Siskos, J., Zopounidis, C.: The evaluation criteria of the venture capital investment activity: an interactive assessment. Eur. J. Oper. Res. **31**(3), 304–313 (1987)

130. Siskos, J., Spyridakos, A., Yannacopoulos, D.: MINORA: a multicriteria decision aiding system for discrete alternatives. J. Inf. Sci. Technol. **2**(2), 136–149 (1993)
131. Siskos, Y., Zopounidis, C., Pouliezos, A.: An integrated DSS for financing firms by an industrial development bank in Greece. Decis. Support. Syst. **112**(2), 151–168 (1994)
132. Siskos, Y., Grigoroudis, E., Matsatsinis, N.F., Baourakis, G.: Preference disaggregation analysis in agricultural product consumer behaviour. In: Pardalos, P.M., Siskos, Y., Zopounidis, C. (eds.) Advances in Multicriteria Analysis, pp. 185–202. Kluwer Academic, Dordrecht (1995)
133. Siskos, Y., Grigoroudis, E., Matsatsinis, N.F., Baourakis, G., Neguez, F.: Comparative behavioural analysis of european olive oil consumer. In: Janssen, J., Skiadas, C.H., Zopounidis, C. (eds.) Advances in Stochastic Modelling and Data Analysis, pp. 293–310. Kluwer Academic, Dordrecht (1995)
134. Siskos, Y., Grigoroudis, E., Zopounidis, C., Saurais, O.: Measuring customer satisfaction using a collective preference disaggregation model. J. Glob. Optim. **12**(2), 175–795 (1998)
135. Siskos, Y., Spiridakos, A., Yannacopoulos, D.: Using artificial intelligence and visual techniques into preference disaggregation analysis: the MIIDAS system. Eur. J. Oper. Res. **113**(2), 236–246 (1999)
136. Siskos, Y., Grigoroudis, E., Politis, Y., Malandrakis, Y.: Customer satisfaction evaluation: some real experiences. In: Colorni, A., Paruccini, M., Roy, B. (eds.) A-MCD-A: Multiple Criteria Decision Aiding, pp. 297–314. European Commission Joint Research Centre, Ispra (2001)
137. Siskos, Y., Matsatsinis, N.F., Baourakis, G.: Multicriteria analysis in agricultural marketing: the case of French olive oil market. Eur. J. Oper. Res. **130**(2), 315–331 (2001)
138. Siskos, E., Askounis, D., Psarras, J.: Robust e-government evaluation based on multiple criteria analysis techniques. In: Paper Presented at the 77th Meeting of the EURO Working Group on MCDA, Rouen, 11–13 April 2013
139. Siskos, E., Malafekas, M., Askounis, D., Psarras, J.: E-government benchmarking in European Union: a multicriteria extreme ranking approach. In: Douligeris, C., Polemi, N., Karantias, A., Lamersdorf, W. (eds.) Collaborative, Trusted and Privacy-Aware e/m-Services, pp. 338–348. Springer, New York (2013)
140. Słowiński, R.: Rough set approach to decision analysis. AI Expert Mag. **10**(3), 18–25 (1995)
141. Spiliopoulos, P.: Analyse et simulation du marché pour le lancement d'um nouveau produit – réalisation d'un siad. In: Thèse de 3e cycle. Universiré de Paris-Dauphine, Paris (1987)
142. Spyridakos, A., Siskos, Y., Yannakopoulos, D., Skouris, A.: Multicriteria job evaluation for large organisations. Eur. J. Oper. Res. **130**(2), 375–387 (2000)
143. Srinivasan, V., Shocker, A.D.: Linear programming techniques for multidimensional analysis of preferences. Psychometrika **38**(3), 337–396 (1973)
144. Stam, A., Sun, M., Haines, M.: Artificial neural network representations for hierarchical preference structures. Comput. Oper. Res. **23**(12), 1191–1201 (1996)
145. Stewart, T.J.: Pruning of decision alternatives in multiple criteria decision making, based on the UTA method for estimating utilities. Eur. J. Oper. Res. **28**(1), 79–88 (1987)
146. Van de Panne, C.: Methods for Linear and Quadratic Programming. North-Holland, Amsterdam (1975)
147. Wagner, H.M.: Linear programming techniques for regression analysis. J. Am. Stat. Assoc. **54**, 206–212 (1959)
148. Young, F.W., De Leeuw, J., Takane, Y.: Regression with qualitative and quantitative variables: an alternating least squares method with optimal scaling features. Phychometrica **41**(4), 505–529 (1976)
149. Zaras, K.: Rough approximation of a preference relation by a multi-attribute stochastic dominance for deterministic and stochastic evaluation problems. Eur. J. Oper. Res. **130**(2), 305–314 (2000)
150. Zionts, S., Wallenius, J.: An interactive programming method for solving the multiple criteria problem. Manag. Sci. **22**(6), 652–663 (1976)
151. Zopounidis, C.: A multicriteria decision-making methodology for the evaluation of the risk of failure and an application. Found. Control Eng. **12**(1), 45–67 (1987)

152. Zopounidis, C., Doumpos, M.: A multicriteria decision aid methodology for the assessment of country risk. Eur. Res. Manag. Bus. Econ. **3**(3), 13–33 (1997)
153. Zopounidis, C., Doumpos, M.: Developing a multicriteria decision support system for financial classification problems: the FINCLAS system. Optim. Methods Softw. **8**(3–4), 277–304 (1998)
154. Zopounidis, C., Doumpos, M.: Business failure prediction using UTADIS multicriteria analysis. J. Oper. Res. Soc. **50**(11), 1138–1148 (1999)
155. Zopounidis, C., Doumpos, M.: Building additive utilities for multi-group hierarchical discrimination: the MHDIS method. Optim. Methods Softw. **14**(3), 219–240 (2000)
156. Zopounidis, C., Doumpos, M.: INVESTOR: a decision support system based on multiple criteria for portfolio selection and composition. In: Colorni, A., Paruccini, M., Roy, B. (eds.) A-MCD-A: Multiple Criteria Decision Aiding, pp. 371–381. European Commission Joint Research Centre, Ispra (2000)
157. Zopounidis, C., Doumpos, M.: PREFDIS: a multicriteria decision support system for sorting decision problems. Comput. Oper. Res. **27**(7–8), 779–797 (2000)
158. Zopounidis, C., Doumpos, M.: A preference disaggregation decision support system for financial classification problems. Eur. J. Oper. Res. **130**(2), 402–413 (2001)
159. Zopounidis, C., Matsatsinis, N.F., Doumpos, M.: Developing a multicriteria knowledge-based decision support system for the assessment of corporate performance and viability: the FINEVA system. Fuzzy Econ. Rev. **1**(2), 35–53 (1996)
160. Zopounidis, C., Doumpos, M., Zanakis, S.H.: Stock evaluation using a preference disaggregation methodology. Decis. Sci. **30**(2), 313–336 (1999)
161. Zopounidis, C., Hurson, Ch., Doumpos, M.: Risque-Pays: Evaluation des Aspects economiques. Economica, Paris (2000)

Chapter 10
The Analytic Hierarchy and Analytic Network Processes for the Measurement of Intangible Criteria and for Decision-Making

Thomas L. Saaty

Abstract The Analytic Hierarchy Process (AHP) and its generalization to dependence and feedback, the Analytic Network Process (ANP), are theories of relative measurement of intangible criteria. With this approach to relative measurement, a scale of priorities is derived from pairwise comparison measurements only after the elements to be measured are known. The ability to do pairwise comparisons is our biological heritage and we need it to cope with a world where everything is relative and constantly changing. In traditional measurement one has a scale that one applies to measure any element that comes along that has the property the scale is for, and elements are measured one by one, not by comparing them with each other. In the AHP paired comparisons are made with judgments using numerical values taken from the AHP absolute fundamental scale of 1–9. A scale of relative values is derived from all these paired comparisons and it also belongs to an absolute scale that is invariant under the identity transformation like the system of real numbers. The AHP/ANP is useful for making multicriteria decisions involving benefits, opportunities, costs and risks. The ideas are developed in stages and illustrated with examples of real life decisions. The subject is transparent and despite some mathematics, it is easy to understand why it is done the way it is along the lines discussed here.

Keywords Analytic hierarchy process • Decision-making • Prioritization • Negative priorities • Rating • Benefits • Opportunities • Costs • Risks

T.L. Saaty (✉)
Katz Graduate School of Business, University of Pittsburgh, Pittsburgh, PA, USA
e-mail: saaty@katz.pitt.edu

S. Greco et al. (eds.), *Multiple Criteria Decision Analysis*, International Series in Operations Research & Management Science 233,
DOI 10.1007/978-1-4939-3094-4_10

10.1 Introduction

The purpose of decision-making is to help people make decisions according to their own understanding. They would then feel that they really made the decision themselves justified completely according to their individual or group values, beliefs, and convictions even as one tries to make them understand these better. Because decision-making is the most frequent activity of all people all the time, the techniques used today to help people make better decisions should probably remain closer to the biology and psychology of people than to the techniques conceived and circulated at a certain time and that are likely to become obsolete, as all knowledge does, even though decisions go on and on forever. This suggests that methods offered to help make better decisions should be closer to being descriptive and considerably transparent. They should also be able to capture standards and describe decisions made normatively. Natural science, like decision-making, is mostly descriptive and predictive to help us cope intelligently with a complex world.

Not long ago, people believed that the human mind is an unreliable instrument for performing measurement and that the only meaningful measurement is obtained on a physical scale like the meter and the kilogram invented by clever people who care about precision and objective truth. They did not think how the measurements came to have meaning for people and that this meaning depends on people's purpose each time they obtain a reading on that scale. In the winter, ice may be a source of discomfort but an ice drink in the summer can be a refreshing source of comfort. A number has no meaning except that assigned to it by someone. We may all agree on the numerical value of a reading on a physical scale, but not on what exactly that number means to each of us in practical terms. We tend to parrot abstractions that define a number but often forget that numbers are meant to serve some need that is inevitably subjective, which is ultimately more important for our survival. Thus it is our subjective values that are essential for interpreting the readings obtained through measurement. This interpretation depends on what one has in mind at the time and different people may interpret the same reading differently for the same situation depending on their goal. The reading may be called objective, but the interpretation is predominantly subjective. In this sense subjectivity is important, because without it objectivity has no intrinsic meaning. If the mind of an expert can produce measurement close to what we obtain through measuring instruments, then it has greater power than instruments to deal with a complexity for which we have no way to measure. What we have to do is examine the possibility and validity of this assumption as critically as we can. It turns out that when we have knowledge and experience, our brains are very good measuring instruments. That does not mean that we should discard what we use in science that enhances our understanding, but rather we should use it to support and strengthen what we do directly with our minds.

The subject of this chapter is the Analytic Hierarchy Process (AHP), the original theory of prioritization that derives relative scales of absolute numbers known as priorities from judgments expressed numerically on an absolute fundamental scale. It is also about a more general approach to decisions that is a generalization of hierarchies to networks with depndence and feedback, the Analytic Network Process (ANP). Both the AHP and ANP are descriptive approaches to decision-making. The AHP/ANP evolved out of my experience at the Arms Control and Disarmament Agency (ACDA) in the Department of State during the Kennedy and Johnson years. ACDA negotiated arms agreements with the Soviets in Geneva. I was invited to join ACDA, I think because of work I had done for the military using Operations Research mathematics. I published on it and wrote the first book on mathematical methods of operations research. At ACDA I supervised a team of foremost internationally known scientists, economists and game theorists (including four people who later won the Nobel Prize in economics: Aumann, Debreu, Harsanyi and Selten) who advised ACDA on arms tradeoffs, but we had some insurmountable difficulties in making lucid and usable recommendations to our highly intelligent and experienced negotiators who were guided by strong intuition deriving from long practice.

The basic problem is that we need to quantify intangibles of which there is nearly an infinite number and we can only do it by making comparison in relative terms. Even if everything were measurable, we would still need to compare the different types of measurements on the different scales and determine how important they are to us to make tradeoffs among them and reach a final answer. If we use tangibles and their measurements we would need to reduce them to a common relative frame of reference and then weight and combine them along with intangibles. Combining priorities of measurable quantities with those of non-measurable qualities needs ratio or even the stronger absolute scales, because we can then multiply and add the outcomes particularly when there is interdependence among all the elements involved in a decision.

The AHP is a theory of relative measurement on absolute scales of both tangible and intangible criteria based both on the judgment of knowledgeable and expert people and on existing measurements and statistics needed to make a decision. How to measure intangibles is the main concern of the mathematics of the AHP. The AHP has been mostly applied to multi-objective, multi-criteria and multiparty decisions because decision-making has this diversity. To make tradeoffs among the many intangible objectives and criteria, the judgments that are usually made in qualitative terms are expressed numerically. To do this, rather than simply assign a score out of a person's memory that is hard to justify, one must make reciprocal pairwise comparisons in a carefully designed scientific way. In the end, we must fit our entire world experience into our system of priorities if we are going to understand it. The AHP is based on four axioms: (1) reciprocal judgments, (2) homogeneous elements, (3) hierarchic or feedback dependent structure, and (4) rank order expectations. The synthesis of the AHP combines multidimensional scales of measurement into a single "unidimensional" scale of priorities. Decisions are determined by a single number for the best outcome or by a vector of priorities that gives a proportionate ordering of the different possible outcomes to which one can then allocate resources in an optimal way subject to both tangible and intangible constraints. We can also combine the judgments obtained from a group when several people are involved in a decision. It is known that with the reciprocal condition, the geometric mean is a necessary condition for combining individual judgments and that, contrary to the impossibility of combining individual judgments into a social welfare function when ordinals are used subject to certain conditions, with absolute judgments it is possible to construct with the AHP such a social welfare function that satisfies these conditions [9].

It is not idiosyncratic for one to believe that making a decision is more complex than just listing all the factors, good and bad, that one can think of and then plunge into numerical manipulations that surface a best outcome according to some plausible way of analysis. Nor is it less idiosyncratic to confine the analysis of decisions to risk and use risk aversion as a way to justify how to make a good choice. For every decision there are positive and negative factors to consider, usually interpreted psychologically in the form of benefits (gains) and opportunities (potential gains), and costs (losses) and risks (potential losses). How to evaluate a decision according to these merits (demerits) and how to combine them into a single overall answer is not easy to do and is something that leaders in business and government do qualitatively with the help of advisors to satisfy the broad goals that they serve. Multicriteria decision-making needs to provide meaningful quantitative assistance on this important, complex, and inevitable concern with its many intangibles.

10.2 Pairwise Comparisons; Inconsistency and the Principal Eigenvector

The psychologist Arthur Blumenthal writes in his book *The Process of Cognition*, Prentice-Hall Inc., Englewood Cliffs, New Jersey, 1977, that there are two types of judgment: "Comparative judgment which is the identification of some relation between two stimuli both present to the observer, and absolute judgment which involves the relation between a single stimulus and some information held in short term memory about some former comparison stimuli or about some previously experienced measurement scale using which the observer rates the single stimulus."

Comparative or relative judgment is made on pairs of elements to ensure accuracy. In paired comparisons, the smaller or lesser element is used as the unit, and the larger or greater element is estimated as a multiple of that unit with respect to the common property or criterion for which the comparisons are made. In this sense, measurement with many pairwise comparisons is made more scientifically than by assigning numbers more or less arbitrarily through guessing. What is really the scale to which such numbers belong so they can be operated on arithmetically in a legitimate way? For example, one cannot simply add numbers that belong to an ordinal or an interval scale. Because our brains are limited in size and the firings of their neurons are limited in intensity, it is clear that there is a limit to their ability to compare the very small with the very large. It is precisely for this reason that pairwise comparisons are made on elements or alternatives that are close or homogeneous and the more separated they are, the more need there is to put them in different groups and link these groups with a common element from one group to an adjacent group of slightly greater or slightly smaller elements. One can then compare the elements in each homogeneous group and then combine them through appropriate use of the measurement of the elements (pivots) that are common to consecutive groups.

We learn from making paired comparisons in the AHP that if A is 5 times larger in size than B and B is 3 times larger in size than C, then A is 15 times larger in size than C and thus we say that A dominates C 15 times. That is different from A having 5 dollars more than B and B having 3 dollars more than C implies that A has 8 dollars more than C. Defining intensity along the arcs of a graph and raising the resulting matrix of comparisons to powers measures the first kind of dominance precisely and never the second. It has definite meaning and as we shall see, because of the inconsistency inherent in making judgments, in the limit it is measured uniquely by the principal eigenvector. There is a useful connection between what we do with dominance priorities in the AHP and what is done with transition probabilities both of which use matrix algebra to find their answers. Transitions between states are multiplied and added. To compose the priorities of the alternatives of a decision with respect to different criteria, it is also necessary that the priorities of the alternatives with respect to each criterion be multiplied by the priority of that criterion and then added over all the criteria.

Paired comparisons deal with comparative judgment. However, in conformity with Blumenthal's observation above, the AHP also provides a way to rate alternatives one at a time to deal with absolute judgment. In absolute judgment the criteria are first prioritized through comparisons and then for each criterion one creates a scale of relative intensities possibly of widely ranging orders of magnitude. The priorities of these intensities are again appropriately derived through paired comparisons with respect to their criterion, and in the end the alternatives are rated one at a time by assigning each one an idealized intensity for each criterion, then weighting by the priorities of the criteria and adding to obtain their overall rating priority [5]. Thus rating applies only to alternatives taken one at a time and relies on standards (good or poor) in the memory of the decision maker to rate the alternatives. It is useful when the number of alternatives is large and we want to standardize our treatment of them. When alternatives are fundamentally new, different and not fully understood, paired comparisons are essential because there are no familiar and widely accepted standards on which they can be rated.

To derive priorities for criteria or attributes we either think of a need to be satisfied, or of a property of alternatives that we already have. In either case when there are several criteria we need to establish their priorities to select the best alternative that meets all the requirements.

Assume that one is given n stones, $A_1, \cdots, A_n$, with known weights $w_1, \cdots, w_n$, respectively, and suppose that a matrix of pairwise ratios is formed whose rows give the ratios of the weights of each stone with respect to all others. We have:

$$\begin{array}{c} \\ A_1 \\ \vdots \\ A_n \end{array} \begin{array}{c} A_1 \cdots A_n \\ \begin{bmatrix} w_1/w_1 & \cdots & w_1/w_n \\ \vdots & & \vdots \\ w_n/w_1 & \cdots & w_n/w_n \end{bmatrix} \end{array}.$$

To recover the vector $w = (w_1, \ldots, w_n)$ we introduce the system of equations:

$$Aw = \begin{bmatrix} w_1/w_1 & \cdots & w_1/w_n \\ \vdots & & \vdots \\ w_n/w_1 & \cdots & w_n/w_n \end{bmatrix} \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} = n \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} = nw,$$

where A has been multiplied on the right by the vector of weights w. The result of this multiplication is nw. To recover the scale from the matrix of ratios, one must solve the problem $Aw = nw$ or $(A - nI)w = 0$. This is a system of homogeneous linear equations. It has a nontrivial solution if and only if the determinant of $A - nI$ vanishes, that is, n is an eigenvalue of A. Now A has unit rank since every row is a constant multiple of the first row. As a result, all its eigenvalues except one are zero. The sum of the eigenvalues of a matrix is equal to its trace, the sum of its diagonal elements, and in this case the trace of A is equal to n. Thus n is an eigenvalue of A, and one has a nontrivial solution. The solution consists of positive entries and is unique to within a multiplicative constant.

To make w unique, we can normalize its entries by dividing by their sum. Thus, given the comparison matrix, we can recover the scale. In this case, the solution is any column of A normalized. Notice that in A the reciprocal property $a_{ji} = 1/a_{ij}$ holds; thus, also $a_{ii} = 1$. Another property of A is that it is consistent: its entries satisfy the condition $a_{jk} = a_{ik}/a_{ij}$. The entire matrix can be constructed from a set of n elements that form a chain across the rows and columns of A.

In the general case, the precise value of w_i/w_j cannot be given, but instead only an estimate of it as a judgment. For the moment, consider an estimate of these values by an expert whose judgments are small perturbations of the coefficients w_i/w_j. This implies small perturbations of the eigenvalues.

Let us for generality call $A_1, \ldots, A_n$ stimuli instead of stones. The quantified judgments on pairs of stimuli A_i, A_j, are represented by an n-by-n matrix $A' = (a_{ij})$, $ij = 1, 2, \ldots, n$. The entries a_{ij} are defined by the following entry rules.

Rule 1. If $a_{ij} = a$, then $a_{ji} = 1/a, a > 0$.
Rule 2. If A_i is judged to be of equal relative intensity to A_j then $a_{ij} = 1$, $a_{ji} = 1$; in particular, $a_{ii} = 1$ for all i.

Thus the matrix A' has the form:

$$A' = \begin{pmatrix} 1 & a_{12} & \cdots & a_{1n} \\ 1/a_{12} & 1 & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 1/a_{1n} & 1/a_{2n} & \cdots & 1 \end{pmatrix}.$$

Having recorded the quantified judgments on pairs of stimuli (A_i, A_j) as numerical entries a_{ij} in the matrix, the problem now is to assign to the n stimuli $A_1, A_2, \ldots, A_n$ a set of numerical weights that would "reflect the recorded judgments." In order to do that, the vaguely formulated problem must first be transformed into a precise mathematical one. This essential, and apparently harmless, step is the most crucial one in any problem that requires the representation of a real life situation in terms of an abstract mathematical structure. It is particularly crucial in the present problem where the representation involves a number of transitions that are not immediately discernible. It appears, therefore, desirable in the present problem to identify the major steps in the process of representation and to make each step as explicit as possible to enable the potential user to form his own judgment as to the *meaning and value* of the method in relation to *his* problem and *his* goal.

Why we must solve the principal eigenvalue problem in general has a simple justification based on the idea of dominance among the elements represented by the coefficients of the matrix. Dominance between two elements is obtained as the normalized sum of path intensities defined by the numerical judgments assigned to the arcs along a path. The overall dominance of an element is the sum of the entries in its row given by Ae, $e = (1, \ldots, 1)$ when A is consistent because then $A^k = n^{k-1}A$. When A is inconsistent, we must consider paths of dominance of all lengths between the two points. All the paths of a given length k are obtained by raising the matrix to the power k. According to Cesaro summability, the limit of the average or Cesaro sum $\lim_{N\to\infty} 1/N \sum_{k=0}^{N} A^k e$ that represents the average of all order dominance vectors up to N, is the same as the limit of the sequence of the powers of the matrix i.e. $(\lim_{k\to\infty} A^k)e$. Now we know from Perron theory that the sequence A^k converges to a matrix all whose columns are identical and are proportional to the principal right eigenvector of A. Thus $(\lim_{k\to\infty} A^k)e$ is also proportional to the principal right eigenvector of A.

Without the theory of Perron, the proof (not given here but known in eigenvalue theory) of how to go from $Aw = nw$ to $Aw = \lambda_{\max} w$, is related to small perturbation theory and the amount of inconsistency one allows. A modicum of inconsistency is necessary to change our mind about old relations when we learn new things.

Another way to prove the necessity of the principal eigenvector is based on the need for the invariance of priorities. No matter what method we use to derive the weights, by using them to weight and add the entries in each row to determine the dominance of the element represented in that row, we must get these priorities back as proportional to the expression $\sum_{j=1}^{n} a_{ij} w_j$ $i = 1, 2 \cdots n$, that is, we must solve $\sum_{j=1}^{n} a_{ij} w_j = c w_i$, $i = 1, 2 \cdots n$, because in the end they can be normalized.

Otherwise $\sum_{j=1}^{n} a_{ij}w_j$, $i = 1, \ldots, n$ would yield another set of different weights and they in turn can be used to form new expressions $\sum_{j=1}^{n} a_{ij}w_j$, $i = 1, 2 \ldots n$, and so on ad infinitum violating the need to have priorities that are invariant, unless in any case we solve the principal eigenvalue problem.

Our general problem takes the form:

$$A'w = \begin{bmatrix} 1 & a_{12} & \cdots & a_{1n} \\ 1/a_{12} & 1 & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 1/a_{1n} & 1/a_{2n} & \cdots & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = cw.$$

We now show that the perturbed eigenvalue from the consistent case is the principal eigenvalue of A'. Our argument involves both left and right eigenvectors of A'. Two vectors $x = (x_1, \cdots, x_n)$, $y = (y_1, \cdots, y_n)$ are orthogonal if their scalar product $x_1y_1 + \cdots + x_ny_n$ is equal to zero. It is known that any left eigenvector of a matrix corresponding to an eigenvalue is orthogonal to any right eigenvector corresponding to a different eigenvalue. This property is known as bi-orthogonality using which we can prove:

Theorem 1. *For a given positive matrix A, the only positive vector w and only positive constant c that satisfy $Aw = cw$, is a vector w that is a positive multiple of the principal eigenvector of A, and the only such c is the principal eigenvalue of A.*

Thus we see that both requirements of dominance and invariance lead us to the principal right eigenvector. The problem now is how good is the estimate of w. Notice that if w is obtained by solving this problem, the matrix whose entries are w_i/w_j is a consistent matrix. It is a consistent estimate of the matrix A'. The matrix itself need not be consistent. In fact, the entries of A' need not even be transitive; that is, A_1 may be preferred to A_2 and A_2 to A_3 but A_3 may be preferred to A_1. What we would like is a measure of the error due to inconsistency. It turns out that A' is consistent if and only if $\lambda_{\max} = n$ and that we always have $\lambda_{\max} \geq n$ when we solve the system of equations $Aw = \lambda_{\max}w$ for a non-negative reciprocal matrix A to obtain the priorities.

Thus the story is very different if the judgments are inconsistent, and as we said before, we need to allow inconsistent judgments for good reasons. In sports, team A beats team B, team B beats team C, but team C beats team A. How would we admit such an occurrence in our attempt to explain the real world if we do not allow inconsistency? So far we have legislated inconsistency, which is natural in making judgments, by assuming axiomatically that it should not exist particularly with regard to transitivity!

The priorities that we seek are concerned with the order to be captured from dominance judgments involving all order transitivity. Thus the problem of deriving unique priorities in decision-making by solving the principal eigenvalue problem of A' belongs to the field of mathematics known as *order topology*. In general priorities are not obtainable directly by the many methods of *metric topology* involving minimization of a metric such as the method of least squares (LSM) which determines a priority vector by minimizing the Frobenius norm of the difference between A and a positive rank one reciprocal matrix $[y_i/y_j]$:

$$\min_{y>0} \sum_{i,j=1}^{n} \left(a_{ij} - \frac{y_i}{y_j} \right)^2 \tag{10.1}$$

and the method of logarithmic least squares (LLSM) which determines a vector by minimizing the Frobenius norm of $[\log(a_{ij}x_j/x_i)]$:

$$\min_{x>0} \sum_{i,j=1}^{n} \left[\log a_{ij} - \log \left(\frac{x_i}{x_j} \right) \right]^2 . \tag{10.2}$$

Metric methods not only ignore transitivity, but also yield a variety of different answers thus violating the overall justification of the need for a single unique set of priorities. There is however a connection between order and optimization.

Solving the principal eigenvalue problem to obtain priorities is equivalent to the two problems of optimization that follow: Find w_i, $i = 1, \cdots, n$ which

1. maximize $\frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} w_j / w_i$, or, in the simpler linear optimization setting,
2. maximize $\sum_{j=1}^{n} w_j \sum_{i=1}^{n} a_{ij}$, obtained by multiplying the sum of each column j by its corresponding w_j and summing over j, subject to $\sum_{i=1}^{n} w_i = 1$.

10.3 Stimulus Response and the Fundamental Scale

What numbers should we use when we only have qualitative judgments to express our understanding in making pairwise comparisons of elements that are close or homogeneous? We note that to be able to perceive and sense objects in the environment our brains miniaturize them within our system of neurons so that we have a proportional relationship between what we perceive and what is out there. Without proportionality we cannot coordinate our thinking with our actions with the accuracy needed to control the environment. Proportionality with respect to a single stimulus requires that our response to a proportionately amplified or attenuated stimulus we receive from a source should be proportional to what our response would be to the original value of that stimulus. If $w(s)$ is our response to a stimulus of magnitude s, then the foregoing gives rise to the functional equation $w(as) = b\,w(s)$. This equation can also be obtained as the necessary condition for solving the Fredholm equation of the second kind:

$$\int_a^b K(s,t) w(t) dt = \lambda_{max} w(s)$$

obtained as the continuous generalization of the discrete formulation $Aw = \lambda_{max} w(s)$. The solution of this functional equation in the real domain is given by

$$w(s) = Ce^{\log b \frac{\log s}{\log a}} P\left(\frac{\log s}{\log a}\right),$$

where P is a periodic function of period 1 and $P(0) = 1$. One of the simplest such examples with $u = \log a / \log a$ is $P(u) = \cos(2\pi u)$ for which $P(0) = 1$ and from which the logarithmic law of response to stimuli can be obtained as a first order approximation as:

$$v(u) = C_1 e^{-\beta u} P(u) \approx C_2 \log s + C_3$$

$\log ab = -\beta, \beta > 0$. The expression on the right is the well-known Weber-Fechner law of logarithmic response $M = a \log s + b$, $a \neq 0$ to a stimulus of magnitude s. It belongs to an interval scale. The larger the stimulus, the larger a change in it is needed for that change to be detectable. The ratio of successive just noticeable differences (the well-known "jnd" in psychology) is equal to the ratio of their corresponding successive stimuli values. Proportionality is maintained. Thus, starting with a stimulus s_0 successive magnitudes of the new stimuli take the form:

$$\begin{aligned}
s_1 &= s_0 + \Delta s_0 = s_0 + \tfrac{\Delta s_0}{s_0} && = s_0(1 + r) \\
s_2 &= s_1 + \Delta s_1 = s_0(1 + r)^2 && \equiv s_0 \alpha^2 \\
&\vdots \\
s_n &= s_{n-1}\alpha \qquad = s_0 \alpha^n (n = 0, 1, 2, \ldots).
\end{aligned}$$

We consider the responses to these stimuli to be measured on a ratio scale ($b = 0$). A typical response has the form $M_i = a \log \alpha^i$, $i = 1, \cdots, n$, or one after another they have the form:

$$M_1 = a \log \alpha, M_2 = 2a \log \alpha, \cdots M_n = na \log \alpha.$$

We take the ratios $M_i/M_1, i = 1, \cdots, n$ of these responses in which the first is the smallest and serves as the unit of comparison, thus obtaining the integer values $1, 2, \cdots, n$ of the fundamental scale of the AHP.

A person may not be schooled in the use of numbers but still have feelings, judgment and understanding that enable him or her to make accurate comparisons (equal, moderate, strong, very strong and extreme and compromises between these intensities). Such judgments can be applied successfully to compare stimuli that are not too disparate but homogeneous in magnitude. By homogeneous we mean that they fall within specified bounds. The foregoing may be summarized to represent the fundamental scale for paired comparisons shown in Table 10.1.

Table 10.1 The fundamental scale of absolute numbers

Intensity of importance	Definition	Explanation
1	Equal importance	Two activities contribute equally to the objective
2	Weak	
3	Moderate importance	Experience and judgment slightly favor one activity over another
4	Moderate plus	
5	Strong importance	Experience and judgment strongly favor one activity over another
6	Strong plus	
7	Very strong or demonstrated importance	An activity is favored very strongly over another; its dominance demonstrated in practice
8	Very, very strong	
9	Extreme importance	The evidence favoring one activity over another is of the highest possible order of affirmation
Reciprocals of above	If activity *i* has one of the above nonzero numbers assigned to it when compared with activity *j*, then *j* has the reciprocal value when compared with *i*	A reasonable assumption
Rationals	Ratios arising from the scale	If consistency were to be forced by obtaining *n* numerical values to span the matrix

We know now that a judgment or comparison is the numerical representation of a relationship between two elements that share a common parent. We also know that the set of all such judgments can be represented in a square matrix in which the set of elements is compared with itself. Each judgment represents the dominance of an element in the column on the left over an element in the row on top. It reflects the answers to two questions: which of the two elements is more important with respect to a higher level criterion, and how strongly, using the 1–9 scale shown in Table 10.1 for the element on the left over the element at the top of the matrix. If the element on the left is less important than that on the top of the matrix, we enter the reciprocal value in the corresponding position in the matrix. It is important to note that the lesser element is always used as the unit and the greater one is estimated as a multiple of that unit. From all the paired comparisons we calculate the priorities and exhibit them on the right of the matrix. For a set of n elements in a matrix one needs $n(n-1)/2$ comparisons because there are n 1's on the diagonal for comparing elements with themselves and of the remaining judgments, half are reciprocals. Thus we have $(n^2-n)/2$ judgments. In some problems one may elicit only the minimum of $n-1$ judgments.

In a judgment matrix A, instead of assigning two numbers w_i and w_j (that generally we do not know), as one does with tangibles, and forming the ratio w_i/w_j we assign a single number drawn from the fundamental scale of absolute numbers shown in Table 10.1 to represent the ratio $(w_i/w_j)/1$. It is a nearest integer approximation to the ratio $(w_i/w_j)/1$. The ratio of two numbers from a ratio scale (invariant under multiplication by a positive constant) is an absolute number (invariant under the identity transformation). The derived scale will reveal what w_i and w_i are.

This is a central fact about the relative measurement approach. It needs a fundamental scale to express numerically the relative dominance relationship.

If one wishes to use actual measurements or use fractional values for judgments one of course can. In the end one needs to justify with care what one does.

Remark 1. The reciprocal property plays an important role in combining the judgments of several individuals to obtain a judgment for a group. Judgments must be combined so that the reciprocal of the synthesized judgments must be equal to the syntheses of the reciprocals of these judgments. It has been proved that the geometric mean is the unique way to do that. If the individuals are experts, they my not wish to combine their judgments but only their final outcome from a herarchy. In that case one takes the geometric mean of the final outcomes. If the individuals have different priorities of importance their judgments (final outcomes) are raised to the power of their priorities and then the geometric mean is formed [5].

10.3.1 Validation Example

Here is an example (one of many) which shows that the scale works well on homogeneous elements of a real life problem. A matrix of paired comparison judgments is used to estimate relative drink consumption in the United States as shown in Table 10.2. To make the comparisons, the types of drinks are listed on the left and at the top, and judgment is made as to how strongly the consumption of a drink on the left dominates that of a drink at the top. For example, when coffee on the left is compared with wine at the top, it is thought that it is consumed extremely more and a 9 is entered in the first row and second column position. A $1/9$ is automatically entered in the second row and first column position. If the consumption of a drink on the left does not dominate that of a drink at the top, the reciprocal value is entered. For example in comparing coffee and water in the first

Table 10.2 Which drink is consumed more in the U.S.? An example of estimation using judgments

Drink consumption in the U.S.	Coffee	Wine	Tea	Beer	Sodas	Milk	Water
Coffee	1	9	5	2	1	1	1/2
Wine	1/9	1	1/3	1/9	1/9	1/9	1/9
Tea	1/5	2	1	1/3	1/4	1/3	1/9
Beer	1/2	9	3	1	1/2	1	1/3
Sodas	1	9	4	2	1	2	1/2
Milk	1	9	3	1	1/2	1	1/3
Water	2	9	9	3	2	3	1
	The derived scale based on the judgments in the matrix is:						
	Coffee	Wine	Tea	Beer	Sodas	Milk	Water
	0.177	0.019	0.042	0.116	0.190	0.129	0.327
	with a consistency index of 0.022.						
	The actual consumption (from statistical sources) is:						
	0.180	0.010	0.040	0.120	0.180	0.140	0.330

row and eighth column position, water is consumed more than coffee slightly and a $1/2$ is entered. Correspondingly, a value of 2 is entered in the eighth row and first column position. At the bottom of Table 10.2, we see that the derived values and the actual values are close.

10.3.2 Clustering and Homogeneity; Using Pivots to Extend the Scale from 1–9 to 1–∞

Most real life decisions are not widely separated in ranges of criteria (one or two) because what is important to individuals or to groups to corporations and finally to governments needs to meet their most essential requirements. Note that the priorities in two adjacent categories would be sufficiently different, one being an order of magnitude smaller than the other, that in the synthesis, the priorities of the elements in the smaller set would ordinarily have little effect on the decision.

We note that our ability to make accurate comparisons of widely disparate objects on a common property is limited. We cannot compare with any reliability the very small with the very large. However, we can do it in stages by comparing objects of relatively close magnitudes and gradually increase their sizes until we reach the desired object of large size (see example later). In this process, we can think of comparing several close or homogeneous objects for which we obtain a scale of relative values, and then again pairwise compare the next set of larger objects that includes for example the largest object from the previous already compared collection, and then derive a scale for this second set. We then divide all the measurements in the second set by the value of the common object and multiply

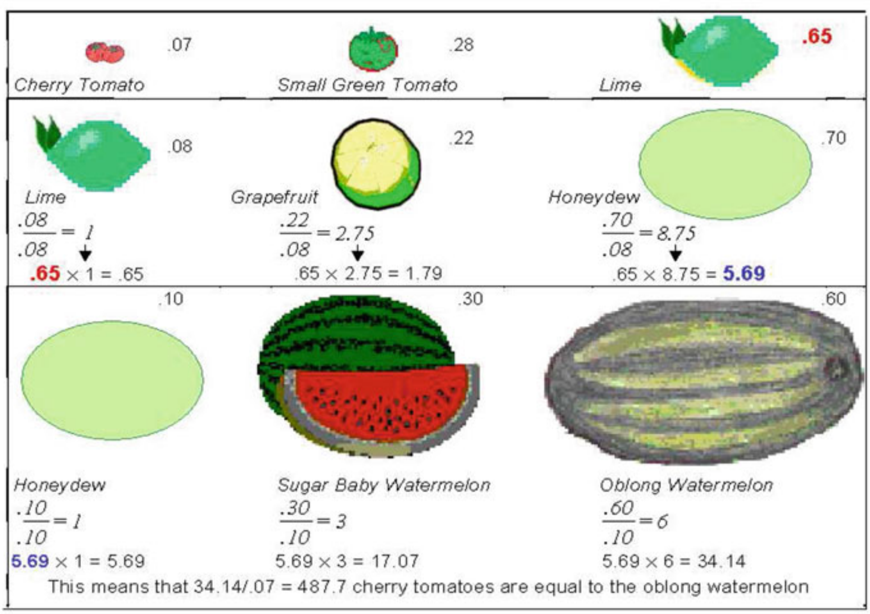


Fig. 10.1 Comparisons according to volume

all the resulting values by the weight of the common element in the first set, thus rendering the two sets to be measurable on the same scale and so on to a third collection of the objects using a common object from the second set.

In Fig. 10.1 a cherry tomato is eventually and indirectly compared with a large watermelon by first comparing it with a small tomato and a lime, the lime is then used again in a second cluster with a grapefruit and a honey dew where we then divide by the weight of the lime and then multiply by its weight in the first cluster, and then use the honey dew again in a third cluster and so on. In the end we have a comparison of the cherry tomato with the large watermelon and would accordingly extended the scale from 1–9 to 1–721.

10.4 Hospice Decision

Westmoreland County Hospital in Western Pennsylvania, like hospitals in many other counties around the United States, has been concerned with the costs of the facilities and manpower involved in taking care of terminally ill patients. Normally these patients do not need as much medical attention as do other patients. Those who best utilize the limited resources in a hospital are patients who require the medical attention of its specialists and advanced technology equipment, whose utilization depends on the demand of patients admitted into the hospital. The terminally ill need medical attention only episodically. Most of the time, such patients need psychological support. Such support is best given by the patient's family, whose members are able to supply the love and care the patients most need. For the mental health of the patient, home therapy is a benefit. From the medical standpoint, especially during a crisis, the hospital provides a greater benefit. Most patients need the help of medical professionals only during a crisis. Some will also need equipment and surgery. The planning association of the hospital wanted to develop alternatives and to choose the best one considering various criteria from the standpoint of the patient, the hospital, the community, and society at large.

In this problem, we need to consider the costs and benefits of the decision. Costs include economic costs and all sorts of intangibles, such as inconvenience and pain. Such disbenefits are not directly related to benefits as their mathematical inverses, because patients infinitely prefer the benefits of good health to these intangible disbenefits. To study the problem, one needs to deal with benefits and with costs separately.

I met with representatives of the planning association for several hours to decide on the best alternative. To make a decision by considering benefits and costs, one must first answer the question: In this problem, do the benefits justify the costs? If they do, then either the benefits are so much more important than the costs that the decision is based simply on benefits, or the two are so close in value that both the benefits and the costs should be considered. Then we use two hierarchies for the purpose and make the choice by forming the ratio from them of the benefits priority/costs priority for each alternative. One asks which is most beneficial in the benefits hierarchy (Fig. 10.2) and which is most costly in the costs hierarchy (Fig. 10.3).

If the benefits do not justify the costs, the costs alone determine the best alternative, which is the least costly. In this example, we decided that both benefits and costs had to be considered in separate hierarchies. In a risk problem, a third hierarchy is used to determine the most desired alternative with respect to all three: benefits, costs, and risks. In this problem, we assumed risk to be the same for all contingencies.

The planning association thought the concepts of benefits and costs were too general to enable it to make a decision. Thus, the planners and I further subdivided each (benefits and costs) into detailed subcriteria to enable the group to develop alternatives and to evaluate the finer distinctions the members perceived between the three alternatives. The alternatives were to care for terminally ill patients at the hospital, at home, or partly at the hospital and partly at home.

The two hierarchies are fairly clear and straightforward in their description. They descend from the more general criteria in the second level to secondary subcriteria in the third level and then to tertiary subcriteria in the fourth level on to the alternatives at the bottom or fifth level. At the general criteria level, each of the hierarchies, benefits or costs, involved three major interests. The decision should benefit the recipient, the institution, and society, and their relative importance is the prime determinant as to which outcome is more likely to be preferred. We located these three elements on the second level of the benefits hierarchy. As the decision would

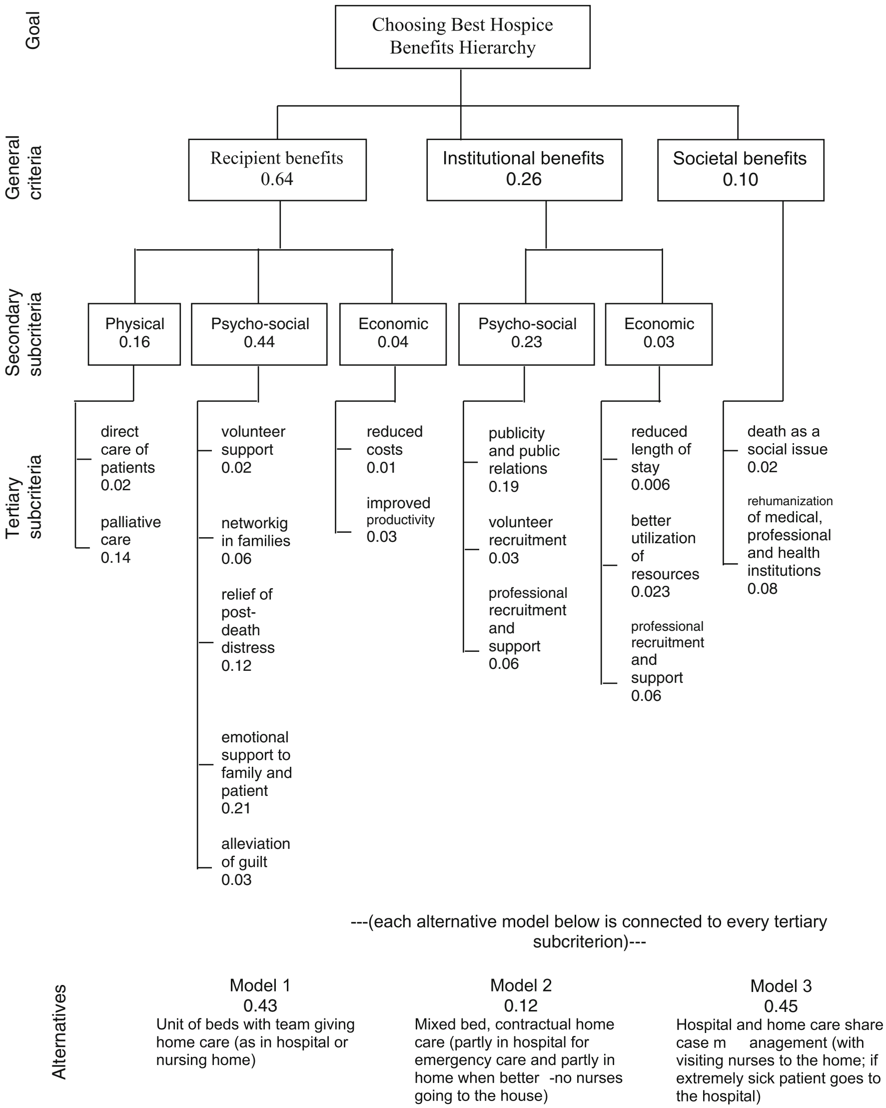


Fig. 10.2 To choose the best hospice plan, one constructs a hierarchy modeling the benefits to the patient, to the institution, and to society. This is the benefits hierarchy of two separate hierarchies

benefit each party differently and the importance of the benefits to each recipient affects the outcome, the group thought that it was important to specify the types of benefit for the recipient and the institution. Recipients want physical, psycho-social and economic benefits, while the institution wants only psycho-social and economic benefits. We located these benefits in the third level of the hierarchy. Each of these in turn needed further decomposition into specific items in terms

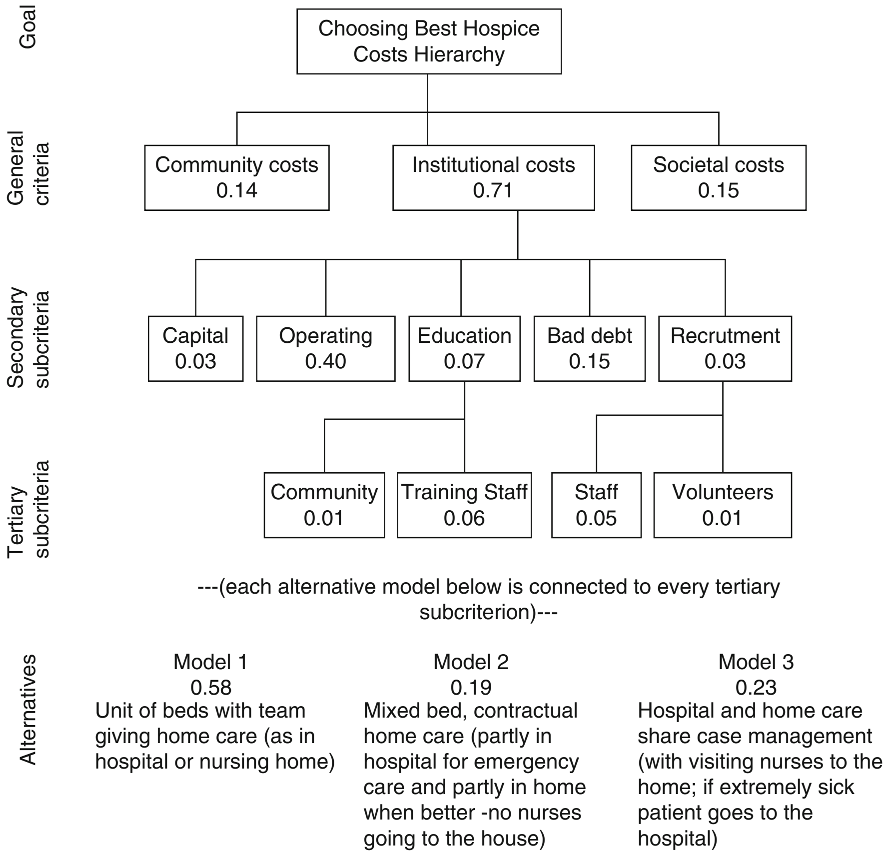


Fig. 10.3 To choose the best hospice plan, one constructs a hierarchy modeling the community, institutional, and societal costs. This is the costs hierarchy of two separate hierarchies

of which the alternatives could be evaluated. For example, while the recipient measures economic benefits in terms of reduced costs and improved productivity, the institution needed the more specific measurements of reduced length of stay, better utilization of resources, and increased financial support from the community. There was no reason to decompose the societal benefits into a third level subcriteria, hence societal benefits connects directly to the fourth level. The group considered three models for the alternatives, and they are at the bottom (or fifth level in this case) of the hierarchy: in Model 1, the hospital provided full care to the patients; in Model 2, the family cares for the patient at home, and the hospital provides only emergency treatment (no nurses go to the house); and in Model 3, the hospital and the home share patient care (with visiting nurses going to the home).

In the costs hierarchy there were also three major interests in the second level that would incur costs or pains: community, institution, and society. In this decision the costs incurred by the patient were not included as a separate factor. Patient and family could be thought of as part of the community. We thought decomposition was necessary only for institutional costs. We included five such costs in the third level: capital costs, operating costs, education costs, bad debt costs, and recruitment costs. Educational costs apply to educating the community and training the staff. Recruitment costs apply to staff and volunteers. Since both the costs hierarchy and the benefits hierarchy concern the same decision, they both have the same alternatives in their bottom levels, even though the costs hierarchy has fewer levels.

As usual with the AHP, in both the costs and the benefits models, we compared the criteria and subcriteria according to their relative importance with respect to the parent element in the adjacent upper level. For example, in the first matrix of comparisons of the three benefits criteria with respect to the goal of choosing the best hospice alternative, recipient benefits are moderately more important than institutional benefits and are assigned the absolute number 3 in the (1, 2) or first-row second-column position. Three signifies three times more. The reciprocal value is automatically entered in the (2, 1) position, where institutional benefits on the left are compared with recipient benefits at the top. Similarly a 5, corresponding to strong dominance or importance, is assigned to recipient benefits over social benefits in the (1, 3) position, and a 3, corresponding to moderate dominance, is assigned to institutional benefits over social benefits in the (2, 3) position with corresponding reciprocals in the transpose positions of the matrix.

Remark 2. In order to give the reader familiarity with the AHP without too much theory, we have delayed discussion of the measurement of the inconsistency and random inconsistency and of the ratio C.R. of the inconsistency of a given matrix and the corresponding random inconsistency to a later section. However, we have indicated the C.R. corresponding to each matrix immediately under that matrix (Table 10.3).

Judgments in a matrix may not be consistent. In eliciting judgments, one makes redundant comparisons to improve the validity of the answer, given that respondents may be uncertain or may make poor judgments in comparing some of the elements. Redundancy gives rise to multiple comparisons of an element with other elements and hence to numerical inconsistencies. For example, where we compare recipient benefits with institutional benefits and with societal benefits, we have the respective judgments 3 and 5. Now if $x = 3y$ and $x = 5z$ then $3y = 5z$ or $y = 5/3z$. If the judges were consistent, institutional benefits would be assigned the value 5/3

Table 10.3 The entries in this matrix respond to the question: which criterion is more important with respect to choosing the best hospice alternative and how strongly?

Choosing best hospice	Recipient benefits	Institutional benefits	Social benefits	Priorities
Recipient benefits	1	3	5	0.64
Institutional benefits	1/3	1	3	0.26
Social benefits	1/5	1/3	1	0.11
C.R. = 0.033				

instead of the 3 given in the matrix. Thus the judgments are inconsistent. In fact, we are not sure which judgments are the accurate ones and which are the cause of the inconsistency. Inconsistency is inherent in the judgment process. Inconsistency may be considered a tolerable error in measurement only when it is of a lower order of magnitude (10 %) than the actual measurement itself; otherwise the inconsistency would bias the result by a sizable error comparable to or exceeding the actual measurement itself.

When the judgments are inconsistent, the decision-maker may not know where the greatest inconsistency is. The AHP can show one by one in sequential order which judgments are the most inconsistent, and suggests the value that best improves consistency. However, this recommendation may not necessarily lead to a more accurate set of priorities that correspond to some underlying preference of the decision-maker. Greater consistency does not imply greater accuracy and one should go about improving consistency (if one can, given the available knowledge) by making slight changes compatible with one's understanding. If one cannot reach an acceptable level of consistency, one should gather more information or reexamine the framework of the hierarchy. For a 3-by-3 matrix this ratio should be about 5 %, for a 4-by-4 matrix about 8 %, and for larger matrices, about 10 %.

The process is repeated in all the matrices by asking the appropriate dominance or importance question. For example, for the matrix comparing the subcriteria of the parent criterion institutional benefits (Table 10.4), psycho-social benefits are regarded as very strongly more important than economic benefits, and 7 is entered in the (1, 2) position and 1/7 in the (2, 1) position.

In comparing the three models for patient care, we asked members of the planning association which model they preferred with respect to each of the covering or parent secondary criteria in level 3 or with respect to the tertiary criteria in level 4. For example, for the subcriterion direct care (located on the left-most branch in the benefits hierarchy), we obtained a matrix of paired comparisons (Table 10.5) in which Model 1 is preferred over Models 2 and 3 by 5 and 3 respectively and Model 3 is preferred by 3 over Model 2. The group first made all the comparisons using semantic terms for the fundamental scale and then translated them to the corresponding numbers.

For the costs hierarchy, I again illustrate with three matrices. First the group compared the three major cost criteria and provided judgments in response to the question: which criterion is a more important determinant of the cost of a hospice model (Table 10.6)?

Table 10.4 The entries in this matrix respond to the question: which subcriterion yields the greater benefit with respect to institutional benefits and how strongly?

Institutional benefits	Psycho-social	Economic	Priorities
Psycho-social	1	7	0.875
Economic	1/7	1	0.125
C.R. = 0.000			

Table 10.5 The entries in this matrix respond to the question: which model yields the greater benefit with respect to direct care and how strongly?

Direct care of patient	Model I	Model II	Model III	Priorities
Model I unit team	1	5	3	0.64
Model II mixed/home care	1/5	1	1/3	0.10
Model III case management	1/3	3	1	0.26
C.R. = 0.033				

Table 10.6 The entries in this matrix respond to the question: which criterion is a greater determinant of cost with respect to the care method and how strongly?

Choosing best hospice (costs)	Community	Institutional	Societal	Priorities
Community costs	1	1/5	1	0.14
Institutional costs	5	1	5	0.71
Societal costs	1	1/5	1	0.14
C.R. = 0.000				

Table 10.7 The entries in this matrix respond to the question: which criterion incurs greater institutional costs and how strongly?

Institutional costs	Capital	Operating	Education	Bad debt	Recruitment	Priorities
Capital	1	1/7	1/4	1/7	1	0.05
Operating	7	1	9	4	5	0.57
Education	4	1/9	1	1/2	1	0.01
Bad debt	7	1/4	2	1	3	0.21
Recruitment	1	1/5	1	1/3	1	0.07
C.R. = 0.08						

Table 10.8 The entries in this matrix respond to the question: which model incurs greater cost with respect to institutional costs for recruiting staff and how strongly?

Institutional costs for recruiting staff	Model I	Model II	Model III	Priorities
Model I unit team	1	5	3	0.64
Model II mixed/home care	1/5	1	1/3	0.10
Model III case management	1/3	3	1	0.26
C.R. = 0.000				

The group then compared the subcriteria under institutional costs and obtained the importance matrix shown in Table 10.7.

Finally, we compared the three models to find out which incurs the highest cost for each criterion or subcriterion. Table 10.8 shows the results of comparing them with respect to the costs of recruiting staff.

As shown in Table 10.9 we divided the benefits by the costs priorities for each alternative to obtain the best alternative, Model 3, that with the largest value for the ratio.

Table 10.9 Synthesis (P = Priorities, M = Model)

		Distributive mode			Ideal mode		
	P	M 1	M 2	M 3	M 1	M 2	M 3
Benefits							
Direct care of patient	0.02	0.64	0.10	0.26	1.00	0.16	0.41
Palliative care	0.14	0.64	0.10	0.26	1.00	0.16	0.41
Volunteer support	0.02	0.09	0.17	0.74	0.12	0.23	1.00
Networking in families	0.06	0.46	0.22	0.32	1.00	0.48	0.70
Relief of post death stress	0.12	0.30	0.08	0.62	0.48	0.13	1.00
Emotional support of family and patient	0.21	0.30	0.08	0.62	0.48	0.13	1.00
Alleviation of guilt	0.03	0.30	0.08	0.62	0.48	0.13	1.00
Reduced economic costs for patient	0.01	0.12	0.65	0.23	0.18	1.00	0.35
Improved productivity	0.03	0.12	0.27	0.61	0.20	0.44	1.00
Publicity and public relations	0.19	0.63	0.08	0.29	1.00	0.13	0.46
Volunteer recruitment	0.03	0.64	0.10	0.26	1.00	0.16	0.41
Professional recruitment and support	0.06	0.65	0.23	0.12	1.00	0.35	0.18
Reduced length of stay	0.01	0.26	0.10	0.64	0.41	0.41	1.00
Better utilization of resources	0.02	0.09	0.22	0.69	0.13	0.13	1.00
Increased monetary support	0.06	0.73	0.08	0.19	1.00	1.00	0.26
Death as a social issue	0.02	0.20	0.20	0.60	0.33	0.33	1.00
Rehumanization of institutions	0.08	0.24	0.14	0.62	0.39	0.23	1.00
Synthesis (taken from original AHP without approximations)		0.428	0.121	0.451	0.424	0.123	0.453
Costs							
Community costs	0.14	0.33	0.33	0.33	1.00	1.00	1.00
Institutional capital costs	0.03	0.76	0.09	0.15	1.00	0.12	0.20
Institutional operating costs	0.40	0.73	0.08	0.19	1.00	0.11	0.26
Institutional costs for educating the community	0.01	0.65	0.24	0.11	1.00	0.37	0.17
Institutional costs for training staff	0.06	0.56	0.32	0.12	1.00	0.57	0.21
Institutional bad debt	0.15	0.60	0.20	0.20	1.00	0.33	0.33
Institutional costs of recruiting staff	0.05	0.66	0.17	0.17	1.00	0.26	0.26
Institutional costs of recruiting volunteers	0.01	0.60	0.20	0.20	1.00	0.33	0.33
Societal costs	0.15	0.33	0.33	0.33	1.00	1.00	1.00
Synthesis (taken from original AHP without approximations)		0.583	0.192	0.224	0.523	0.229	0.249
Benefit/cost ratio		0.734	0.630	2.013	0.811	0.537	1.819

Table 10.9 shows two ways or modes of synthesizing the local priorities of the alternatives using the global priorities of their parent criteria: The distributive mode and the ideal mode. In the distributive mode, the weights of the alternatives sum to one. It is used when there is dependence among the alternatives and a unit priority is distributed among them. The ideal mode is used to obtain the single best alternative regardless of what other alternatives there are. In the ideal mode, the

local priorities of the alternatives under each criterion are divided by the largest value among them. This is done for each criterion; for each criterion one alternative becomes an ideal with value one. In both modes, the local priorities are weighted by the global priorities of the parent criteria and synthesized and the benefit-to-cost ratios formed. In Table 10.9 we rounded off the numbers to two decimal places. Unfortunately, that causes substantial difference from the actual results obtained in the AHP calculations. We request that the reader accept this as an illustration.

When the criteria priorities do not depend on the values of the alternatives with regard to those criteria, we need to derive their priorities by comparing them pairwise with each other with respect to higher-level criteria orgoal. It is a process of trading off one unit of one criterion against a unit of another, an ideal alternative from one against an ideal alternative from another. To determine the ideal, the alternatives are divided by the largest value among them for each criterion. In that case, the process of weighting and adding assigns each of the remaining alternatives a value that is proportionate to the value 1 given to the highest rated alternative. In this way the alternatives are weighted by the priorities of the criteria and summed to obtain the weights of the alternatives. This is the ideal mode of the AHP.

The distributive mode is essential for synthesizing the weights of alternatives with respect to tangible criteria with the same scale of measurement into a single criterion for that scale and then they are treated as intangibles and compared pairwise and combined with other intangibles with the ideal mode. The dominant mode of synthesis in the AHP where the criteria are independent from the alternatives is the ideal mode. The standard mode for synthesizing in the ANP where criteria depend on alternatives and also alternatives may depend on other alternatives is the distributive mode.

In this case, both modes lead to the same outcome for hospice, which is Model 3. As we shall see below, we need both modes to deal with the effect of adding (or deleting) alternatives on an already ranked set. The priorities of the alternatives in the benefits hierarchy belong to an absolute scale of relative numbers and the priorities of the alternatives in the costs hierarchy also belong to another absolute scale of relative numbers. These two relative scales cannot be arbitrarily combined. Later we provide another way to combine them. In this exercise they were assumed to be commensurate and were combined in the traditional way by forming benefit to cost ratios. To derive the answer we divide the benefits priority of each alternative by its costs priority. We then choose the alternative with the largest of these ratios.

Model 3 has the largest benefit to cost ratio in both the distributive and ideal modes, and the hospital selected it for treating terminal patients. This need not always be the case. In this case, there is dependence of the personnel resources allocated to the three models because some of these resources would be shifted based on the decision. Therefore the distributive mode is the appropriate method of synthesis. If the alternatives were sufficiently distinct with no dependence in their definition, the ideal mode would be the way to synthesize.

I also performed marginal analysis to determine where the hospital should allocate additional resources for the greatest marginal return. To perform marginal analysis, I first ordered the alternatives by increasing cost priorities and then formed the benefit-to-cost ratios corresponding to the smallest cost, followed by the ratios of the differences of successive benefits to differences in costs. If this difference in benefits is negative, the new alternative is dropped from consideration and the process continued. The alternative with the largest ratio is then chosen. For the costs and corresponding benefits from the synthesis rows in Table 10.9 one obtains:

- Benefits: 0.12, 0.45, 0.43;
- Costs: 0.20, 0.21, 0.59;
- Ratios: $0.12/0.20 = 0.60$, $(0.45 - 0.12)/(0.21 - 0.20) = 33$, $(0.43 - 0.45)/(0.59 - 0.21) = -0.051$.

The third alternative is not a contender for resources because its marginal return is negative. The second alternative is the best. In fact, in addition to adopting the third model, the hospital management chose the second model of hospice care for further development.

10.5 Rating Alternatives One at a Time in the AHP: Absolute Measurement

The AHP has a second way to derive priorities known as absolute measurement. It involves making paired comparisons but the criteria just above the alternatives, known as the covering criteria, are assigned intensities that vary in number and type. For example they can simply be: high, medium and low; or they can be: excellent, very good, good, average, poor and very poor; or for experience: more than 15 years, between 10 and 15, between 5 and 10 and less than 5 and so on. These intensities themselves are also compared pairwise to obtain their priorities as to importance, and they are then put in ideal form by dividing by the largest value. Finally each alternative is assigned an intensity, along with its accompanying priority, for each criterion. This process of assigning intensities is called rating the alternatives. The priority of each intensity is weighted by the priority of its criterion and summed over the weighted intensities for each alternative to obtain that alternative's final rating that also belongs to a ratio scale. It is often necessary to have categories of ratings for alternatives that are widely disparate so that one can rate the alternatives correctly. Ratings are useful when standards are established with which the alternatives must comply. They are also useful when the number of alternatives n is very large to perform pairwise comparisons on them for each criterion. In this case if the number of criteria is c, the number of rating operations in rating the alternatives is cn, whereas doing all the pairwise judgments involves $cn(n - 1)/2$ comparisons. Here is an example of absolute measurement.

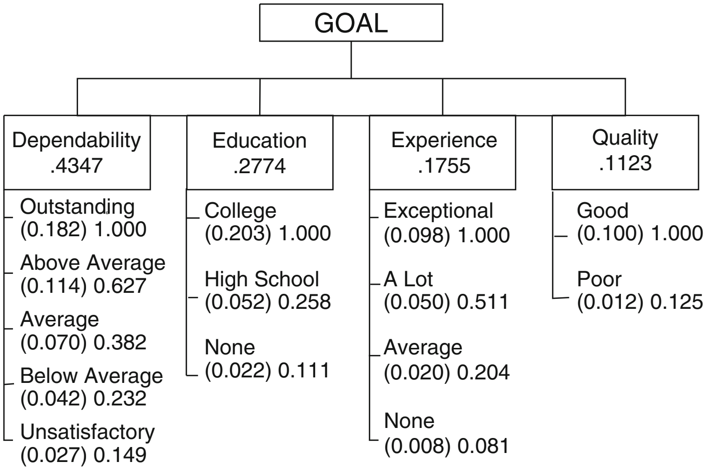


Fig. 10.4 Employee evaluation hierarchy

Table 10.10 Ranking intensities

	Outstanding	Above average	Average	Below average	Unsatisfactory	Priorities
Outstanding	1.0	2.0	3.0	4.0	5.0	0.419
Above average	1/2	1.0	2.0	3.0	4.0	0.263
Average	1/3	1/2	1.0	2.0	3.0	0.160
Below average	1/4	1/3	1/2	1.0	2.0	0.097
Unsatisfactory	1/5	1/4	1/3	1/2	1.0	0.062
Inconsistency ratio = 0.015						

10.5.1 Evaluating Employees for Salary Raises

Employees are evaluated for raises. The criteria are Dependability, Education, Experience, and Quality. Each criterion is subdivided into intensities, standards, or discrimination categories as shown in Fig. 10.4. Priorities are set for the criteria by comparing them in pairs. The intensities are then pairwise compared according to importance with respect to their parent criterion (example as in Table 10.10). Their priorities are often divided by the largest intensity for each criterion (second column of priorities in Fig. 10.4) particularly useful in preserving the ranks of the alternatives from the addition or deletion of other alternatives. Finally, each individual is rated in Table 10.11 by assigning the intensity rating that applies to him or her under each criterion and adding. The score is obtained by weighting the intensities by the priority of their criteria and then summing over the criteria to derive a total score for each individual. This approach can be used whenever it is possible to set priorities for intensities of the criteria, which is usually possible when sufficient experience with a given operation has been accumulated. The raises can be made in proportion to the normalized values on the right.

Table 10.11 Ranking alternatives

		Education	Experience	Quality		
	Dependability	0.2774	0.1775	0.1123	Total	Norm
1. Adams, V.	Outstanding	College	Excep.	Good	1.000	0.245
2. Becker, L.	Average	College	Average	Good	0.592	0.145
3. Hayat, F.	Average	College	A Lot	Good	0.645	0.158
4. Kesselman, S.	Above average	HighSchool	None	Poor	0.373	0.091
5. O'Shea, K.	Average	College	Average	Poor	0.493	0.121
6. Petres, T.	Average	College	None	Good	0.570	0.140
7. Tobias, K.	Average	None	A Lot	Poor	0.407	0.100

One needs to choose the intensities widely enough by putting them in different order-of-magnitude categories in which the elements can be compared with the fundamental scale, and then combine the categories with pivots as in the cherry with watermelon example. Any alternative can be appropriately rated and receives its correct final value no matter how large or how small. When rating widely contrasting alternatives and the rating of an alternative is exceedingly small with respect to a certain criterion, a zero value can be assigned to that alternative.

In ratings, adding new alternatives has no effect on the rank of existing alternatives. In paired comparisons the alternatives depend on each other and a new alternative can affect the relative ranks of existing alternatives. Using the ideal mode each time a new alternative is added prevents rank reversal with respect to irrelevant alternatives. However, if it is done only the first time and new alternatives are only compared with the first ideal so their values go above that ideal (more than one when necessary) there can be no rank reversal. It is clear that when alternatives are independent they can be rated one at a time and there would be no rank reversal. But even with independence, how many other alternatives of the same kind (sometimes also of a different kind) there are, can affect their rank. However, the number of alternatives cannot be used as a criterion for rating because it implies dependence of an alternative on how many others there are and a fortiori on their presence.

10.6 Paired Comparisons Imply Dependence

In most multicriteria decision problems the criteria are assumed independent of the alternatives and the alternatives independent of other alternatives. Paired comparisons imply dependence of a different kind. The common understanding is that when alternatives depend on each other it is according to their function like the electric industry depending on the coal industry for its output. In paired comparisons, the importance assigned to an alternative depends on what other alternatives it is compared with and how many there are. This is dependence not according to function but according to structure. This dependence happens even when the alternatives may be independent of each other according to function. Independence means that the rank of an alternative does not depend on what other alternatives there are and how many of them there may be. The situation with pairwise comparisons is that it automatically implies structural dependence. When a new alternative is added or an old one deleted the ranks of the other alternatives relative to each other may change. However one can preserve rank from adding new but irrelevant alternatives by creating an ideal alternative each time alternatives are added or deleted, or preserve it from any new alternative by simply idealizing the first time but never after and only comparing new alternatives with the first ideal and allowing the priority value of the new alternative to exceed one. Rating alternatives one at a time with appropriate and exhaustive orders of intensities for each criterion always preserves rank from structural effects, but is not always the best way to prioritize alternatives that may depend on the number and quality of other alternatives.

As we increase the number of copies of an alternative, it often loses (or conversely increases) its importance. For example, if gold, which is important, were to increase in quantity to fill the universe, it could lose its importance. No new criterion is added and no judgment is changed but only the quantity of gold. Relative measurement takes quantity into consideration. We often need to consider this kind of dependence known as structural dependence. When we add more alternatives, the ranks among old ones may change and what was preferred to another now because of the presence of new ones may no longer be preferred to the other. Another example is that of a company that sells cars A and B. Car B is better than car A but it costs more to make. It is more desirable all around for people to buy car B but they buy A because it is cheaper. The company advertises that it is going to make car C that is similar to B but much more expensive. People are now observed more and more to buy car B. The company never makes car C. This is a real life example from marketing. However, in some decision problems we may want to treat by fiat the alternatives of a decision as completely independent both in property and in number and quality and want to preserve the ranks of existing alternatives when new ones are added or old ones deleted. The AHP allows for both these possibilities. Actually, change in rank in the presence of relevant alternatives is a fact of our world. It is also a fact that when the number of irrelevant alternatives is very large, they can cause rank to change. Viruses are irrelevant in most decisions but they can eventually cause the death of all decision makers and make mockery of the decisions they thought were so important. In essence reality is much more interdependent than we have allowed for in our limited ways of thinking. Admittedly there are times when we wish to preserve rank no matter what the situation may be. We need to allow for both in our decision theories and not take the simple way out by always assuming independence.

10.7 When Is a Positive Reciprocal Matrix Consistent?

In light of the foregoing, for the validity of the vector of priorities to describe response, we need greater redundancy and therefore also a large number of comparisons. Because of the reciprocal relation, in all we need $n(n-1)/2$ comparisons. An expert may provide $(n-1)$ comparisons to fill one row or a spanning tree from which the matrix is consistent and the priorities are easily obtained. Let us relate the psychological idea of the consistency of judgments and its measurement to a central concept in matrix theory and also to the size of our channel capacity to process information. Let $A = [a_{ij}]$ be an n-by-n positive reciprocal matrix, so all $a_{ii} = 1$ and $a_{ij} = 1/a_{ji}$ for all $i, j = 1, \cdots, n$. Let $w = [w_i]$ be the principal right eigenvector of A, let $D = diag(w_1, \ldots, w_n)$ be the n-by-n diagonal matrix whose main diagonal entries are the entries of w, and set $E = D^{-1}AD = [a_{ij}w_j/w_i] = [\varepsilon_{ij}]$. Then E is similar to A and is a positive reciprocal matrix since $\varepsilon_{ji} = (a_{ji}w_i/w_j) = (a_{ij}w_j/w_i)^{-1} = 1/\varepsilon_{ij}$. Moreover, all the row sums of E are equal to the principal eigenvalue of A:

$$\sum_{j=1}^{n} \varepsilon_{ij} = \sum_{j} \frac{a_{ij}w_j}{w_i} = \frac{[Aw]_i}{w_i} = \frac{\lambda_{max} w_i}{w_i} = \lambda_{\max}.$$

The computation

$$\begin{aligned} n\lambda_{\max} &= \sum_{i=1}^{n}\Big(\sum_{j=1}^{n} \varepsilon_{ij}\Big) = \sum_{i=1}^{n} \varepsilon_{ii} + \sum_{i,j=1\ i\neq j}^{n} (\varepsilon_{ij} + \varepsilon_{ji}) \\ &= n + \sum_{i,j=1\ i\neq j}^{n} (\varepsilon_{ij} + \varepsilon_{ij}^{-1}) \\ &\geq n + (n^2 - n) = n^2 \end{aligned}$$

reveals that $\lambda_{\max} \geq n$. Moreover, since $(x+1)/x \geq 2$ for all $x > 0$, with equality if and only if $x = 1$, we see that $\lambda_{\max} = n$ if and only if all $\varepsilon_{ij} = 1$, which is equivalent to having all $a_{ij} = w_i/w_j$.

The foregoing arguments show that a positive reciprocal matrix A has $\lambda_{\max} \geq n$, with equality if and only if A is consistent. When A is consistent we have $A^k = n^{k-1}A$. As our measure of deviation of A from consistency, we choose the *consistency index*

$$\mu = \frac{\lambda_{max} - n}{n-1}.$$

We have seen that $\mu \geq 0$ and $\mu = 0$ if and only if A is consistent. We can say that as $\mu \to 0$, $a_{ij} \to w_i/w_j$, or $\varepsilon_{ij} = a_{ij}(w_j/w_i) \to 1$. These two desirable properties explain the term "n" in the numerator of μ; what about the term "$n-1$"

Table 10.12 Random index

Order	1	2	3	4	5	6	7	8	9	10
R.I.	0	0	0.52	0.89	1.11	1.25	1.35	1.40	1.45	1.49

in the denominator? Since trace $(A) = n$ is the sum of all the eigenvalues of A, if we denote the eigenvalues of A that are different from $\lambda_{\max}$ by $\lambda_2, \cdots, \lambda_{n-1}$, we see that $n = \lambda_{\max} + \sum_{i=2}^{n} \lambda_i$, so $n - \lambda_{\max} = \sum_{i=2}^{n} \lambda_i$ and $\mu = -(1/(n-1)) \sum_{i=2}^{n} \lambda_i$ is the average of the non-principal eigenvalues of A.

In order to get some feel for what the consistency index might be telling us about a positive n-by-n reciprocal matrix A, consider the following simulation: choose the entries of A above the main diagonal at random from the 17 values $\{1/9, 1/8, \cdots, 1, 2, \cdots, 8, 9\}$. Then fill in the entries of A below the diagonal by taking reciprocals. Put ones down the main diagonal and compute the consistency index. Do this many thousands of times and take the average, which we call the random index. Table 10.12 shows the values obtained from one set of such simulations and also their first order differences, for matrices of size $1, 2, \cdots, 10$.

A plot of the first two rows of Table 10.12 shows the asymptotic nature of random inconsistency. We also have shown that one should not compare more than about seven elements because increase in inconsistency is so small that it becomes difficult to perceive the ensuing small changes in the judgments needed to improve consistency [8]. In passing we note that there are several algorithms to change judgment to improve consistency, the best known among them is the gradient method of Patrick Harker [2, 3].

For a given positive reciprocal matrix $A = [a_{ij}]$ and a given pair of distinct indices $k > l$, define $A(t) = [a_{ij}(t)]$ by $a_{kl}(t) \equiv a_{kl} + t$, $a_{lk}(t) \equiv (a_{kl} + t)^{-1}$, and $a_{ij}(t) \equiv a_{ij}$ for all $i \neq k, j \neq l$, so $A(0) = A$. Let $\lambda_{\max}(t)$ denote the Perron eigenvalue of $A(t)$ for all t in a neighborhood of $t = 0$ that is small enough to ensure that all entries of the reciprocal matrix $A(t)$ are positive there. Finally, let $v = [v_i]$ be the unique positive eigenvector of the positive matrix A^T that is normalized so that $v^T w = 1$. Then a classical perturbation formula tells us that

$$\left.\frac{d\lambda_{\max}(t)}{da_{ji}}\right|_{t=0} = \frac{v^T A'(0) w}{v^T w} = v^T A'(0) w = v_k w_l - \frac{1}{a_{kl}^2} v_l w_k.$$

We conclude that

$$\frac{\partial \lambda_{\max}}{\partial a_{ij}} = v_i w_j - a_{ij}^2 v_j w_i, \quad i, j = 1, \cdots, n.$$

Because we are operating within the set of positive reciprocal matrices, $\partial\lambda_{\max}/\partial a_{ji} = -\partial\lambda_{\max}/\partial a_{ij}$ for all i and j. Thus, to identify an entry of A whose adjustment within the class of reciprocal matrices would result in the largest rate of change in $\lambda_{\max}$ we should examine the $n(n-1)/2$ values $\{v_i w_j - a_{ji}^2 v_j w_i\}$, $i > j$ and select (any) one of largest absolute value.

10.8 In the Analytic Hierarchy Process Additive Composition Is Necessary

Sometimes people have assigned criteria different weights when they are measured in the same unit. Others have used different ways of synthesis than multiplying and adding. An example should clarify what we must do. Synthesis in the AHP involves weighting the priorities of elements compared with respect to an element in the next higher level, called a parent element, by the priority of that element and adding over all such parents for each element in the lower level. Consider the example of two criteria C_1 and C_2 and three alternatives A_1, A_2 and A_3 measured in the same scale such as dollars. If the criteria are each assigned the value 1, then the weighting and adding process produces the correct dollar value as in Table 10.13.

However, it does not give the correct outcome if the weights of the criteria are normalized, with each criterion having a weight of 0.5. Once the criteria are given in relative terms, so must the alternatives also be given in relative terms. A criterion that measures values in pennies cannot be as important as another measured in thousands of dollars. In this case, the only meaningful importance of a criterion is the ratio of the total money for the alternatives under it to the total money for the alternatives under both criteria. By using these weights for the criteria, rather than 0.5 and 0.5, one obtains the correct final relative values for the alternatives.

What is the relative importance of each criterion? Normalization indicates relative importance. Relative values require that criteria be examined as to their relative importance with respect to each other. What is the relative importance of a criterion, or what numbers should the criteria be assigned that reflect their relative importance? Weighting each criterion by the proportion of the resource under it, as shown in Table 10.14, and multiplying and adding as in the additive synthesis of the AHP, we get the same correct answer. For criterion C_1 we have $(200+300+500)/[(200+300+500)+(150+50+100)] = 1000/1300$ and for criterion C_2 we have $(150+50+100)/[(200+300+500)+(150+50+100)] = 300/1300$. Here the criteria are automatically in normalized form, and their weights sum to one. We see that when the criteria are normalized, the alternatives must also be normalized to get the right answer. For example, if we look in Table 10.13 we have 350/1300 for the priority of alternative A_1. Now if we simply weight and add the values for alternative A_1 in Table 10.14 we get for its final value

Table 10.13 Calculating returns arithmetically

Alternatives	Criterion C_1 unnormalized weight = 1.0	Criterion C_2 unnormalized weight = 1.0	Weighted sum unnormalized	Normalized or relative values
A_1	200	150	350	350/1300 = 0.269
A_2	300	50	350	350/1300 = 0.269
A_3	500	100	600	600/1300 = 0.462
Column totals	1000	300	1300	1

Table 10.14 Normalized criteria weights and normalized alternative weights from measurements in the same scale (additive synthesis)

Alternatives	Criterion C_1 normalized weight $= 1000/1300 = 0.7692$	Criterion C_2 normalized weight $= 300/1300 = 0.2308$	Weighted sum
A_1	200/1000	150/300	350/1300 = 0.2692
A_2	300/1000	50/300	350/1300 = 0.2692
A_3	500/1000	100/300	600/1300 = 0.4615

$(200/1000)(1000/1300) + (150/300)(300/1300) = 350/1300$. It is clear that if the priorities of the alternatives are not normalized one does not get the desired outcome.

We have seen in this example that in order to obtain the correct final relative values for the alternatives when measurements on a measurement scale are given, it is essential that the priorities of the criteria be derived from the priorities of the alternatives. Thus when the criteria depend on the alternatives we need to normalize the values of the alternatives to obtain the final result. This procedure is known as the distributive mode of the AHP. It is also used in case of functional dependence of the alternatives on the alternatives and of the criteria on the alternatives. The AHP is a special case of the Analytic Network Process. The dominant mode of synthesis in the ANP with all its interdependencies is the distributive mode. The ANP automatically assigns the criteria the correct weights, if one only uses the normalized values of the alternatives under each criterion and also the normalized values for each alternative under all the criteria without any special attention to weighting the criteria.

10.9 Benefits, Opportunities, Costs and Risks

In many decision problems four kinds of concerns or merits are considered: benefits, opportunities, costs and risks, which we abbreviate as BOCR. The first two are advantageous and hence are positive and the second two are disadvantageous and are therefore negative [6, 7]. Later we show how to determine the relative importance of each of the BOCR.

There are two ways to combine BOCR priorities. The first is the traditional one (used by economists) in which one does not need the relative importance of the BOCR by simply forming their ratio BO/CR for each alternative obtained from a separate hierarchy for each of the four BOCR merits and selecting that alternative with the largest ratio. It is known as the ratio outcome. The second derives corresponding normalized weights b, o, c, and r obtained respectively by rating the best alternative (one at a time) for each of the BOCR with respect to strategic criteria illustrated with an example later. One then forms for the four values of each alternative the expression

$$bB + oO - cC - rR.$$

The first way is a tradeoff between a unit of BO against a unit of CR, a unit of the desirable against a unit of the undesirable. It may be advisable, for example, that if the costs are considered to be negligibly smaller than the benefits to use only the benefits for the best alternative of a decision and not form the ratio and vice versa. The second way simply subtracts the sum of the weighted undesirables from the sum of the weighted desirables to give the total gain or loss. It can give rise to negative priorities and when applied to measurements in dollars, for example, where the weights b, o, c, and r are the same, gives back the correct answer. We have seen examples in which numbers or differences of numbers are made so small that one faces the classical problem of dividing by zero or comparing things whose measurements are near zero.

Two other formulas have been considered and set aside. They are $bB + oO + c/C + r/R$, and $bB + oO + c(1-C) + r(1-R)$. The first with only $bB + c/C$ makes the benefits determine the outcome when the cost is very high, which is counter intuitive. The second is always positive and is equal to $bB + oO - cC - rR + (c+r)$, and adds a constant to the subtractive formula $bB + oO - cC - rR$.

Note that there is no advantage in using the weights b, o, c and r in the formula BO/CR because we would be multiplying the result for each alternative by the same constant bo/cr. Because all values lie between zero and one, we have from the series expansions of the exponential and logarithmic functions the approximation:

$$\begin{aligned}
\frac{bBoO}{cCrR} &= \exp(\log bB + \log oO - \log cC - \log rR) \\
&= 1 + (\log bB + \log oO - \log cC - \log rR) + \ldots \\
&\approx 1 + (bB - 1) + (oO - 1) - (cC - 1) - (rR - 1) \\
&= 1 + bB + oO - cC - rR.
\end{aligned}$$

Because one is added to the overall value of each alternative we can eliminate it. The approximate result is that the ratio formula is similar to the total formula with equal weights assumed for the B, O, C, R.

10.10 On the Admission of China to the World Trade Organization

This section was taken from an analysis done in 2000 carried out before the US Congress acted favorably on China joining the WTO and was hand-delivered to many of the members of the committee including its Chairperson. Since 1986, China had been attempting to join the multilateral trade system, the General Agreement on Tariffs and Trade (GATT) and, its successor, the World Trade Organization (WTO). According to the rules of the 135-member nations of WTO, a candidate member must reach a trade agreement with any existing member country that wishes to trade with it. By the time this analysis was done, China signed bilateral agreements with 30 countries—including the US (November 1999)—out of 37 members that had requested a trade deal with it [6].

As part of its negotiation deal with the US, China asked the US to remove its annual review of China's Normal Trade Relations (NTR) status, until 1998 called Most Favored Nation (MFN) status. In March 2000, President Clinton sent a bill to Congress requesting a Permanent Normal Trade Relations (PNTR) status for China. The analysis was done and copies sent to leaders and some members in both houses of Congress before the House of Representatives voted on the bill, May 24, 2000. The decision by the US Congress on China's trade-relations status will have an influence on US interests, in both direct and indirect ways. Direct impacts include changes in economic, security and political relations between the two countries as the trade deal is actualized. Indirect impacts will occur when China becomes a WTO member and adheres to WTO rules and principles. China has said that it would join the WTO only if the US gives it Permanent Normal Trade Relations status.

It is likely that Congress will consider four options. The least likely is that the US will deny China both PNTR and annual extension of NTR status. The other three options are:

1. **Passage of a clean PNTR bill**: Congress grants China Permanent Normal Trade Relations status with no conditions attached. This option would allow implementation of the November 1999 WTO trade deal between China and the Clinton administration. China would also carry out other WTO principles and trade conditions.
2. **Amendment of the current NTR status bill**: This option would give China the same trade position as other countries and disassociate trade from other issues. As a supplement, a separate bill may be enacted to address other matters, such as human rights, labor rights, and environmental issues.
3. **Annual extension of NTR status**: Congress extends China's Normal Trade Relations status for one more year, and, thus, maintains the status quo.

The conclusion of the study is that the best alternative is granting China PNTR status. China was granted that status by the US Congress, possibly influenced by this analysis.

Our analysis involves four steps. First, we prioritize the criteria in each of the benefits, costs, opportunities and risks hierarchies with respect to the goal. Figure 10.5 shows the resulting prioritization of these criteria. The alternatives and their priorities are shown under each criterion both in the distributive and in the ideal modes. The ideal priorities of the alternatives were used appropriately to synthesize their final values beneath each hierachy.

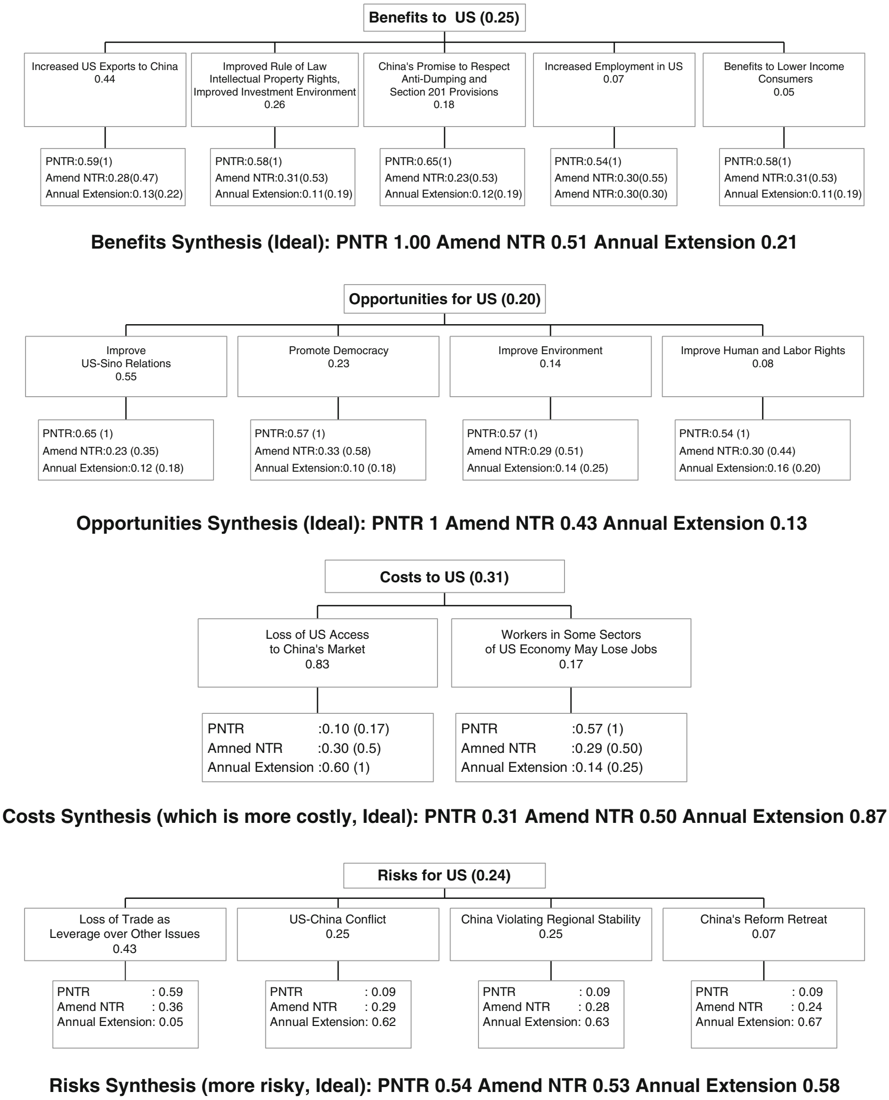


Fig. 10.5 Hierarchies for rating benefits, costs, opportunities, and risks

The priorities shown in Fig. 10.5 were derived from judgments that compared the elements involved in pairs. For readers to estimate the original pairwise judgments (not shown here) one forms the ratio of the corresponding two priorities shown, leave them as they are, or take the closest whole number, or its reciprocal if it is less than 1.0.

The idealized values are shown in parentheses after the original distributive priorities obtained from the eigenvector. The ideal values are obtained by dividing

Table 10.15 Priority ratings for the merits: benefits, costs, opportunities, and risks

		Benefits	Opportunities	Costs	Risks
Economic (0.56)	Growth (0.19)	High	Medium	Very low	Very low
	Equity (0.37)	Medium	Low	High	Low
Security (0.32)	Regional (0.03)	Low	Medium	Medium	High
	Non-proliferation (0.08)	Medium	High	Medium	High
	Threat to US (0.21)	High	High	Very high	Very high
Political (0.12)	Constituencies (0.1)	High	Medium	Very high	High
	American values (0.02)	Very low	Low	Low	Medium
Priorities		0.25	0.20	0.31	0.24

Intensities: very high (0.42), high (0.26), medium (0.16), low (0.1), very low (0.06)

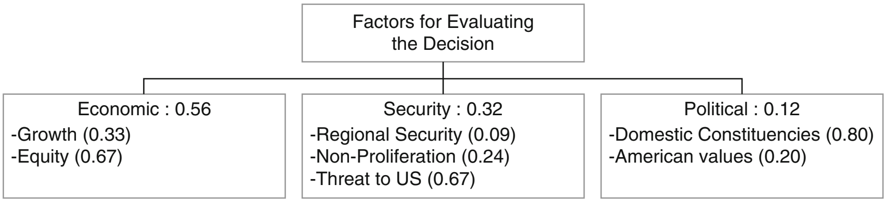


Fig. 10.6 Prioritizing the strategic criteria to be used in rating the BOCR

each of the distributive priorities by the largest one among them. For the Costs and Risks structures, the question is framed as to which is the *most* costly or risky alternative. That is, the most costly alternative ends up with the highest priority.

It is likely that, in a particular decision, the benefits, costs, opportunities and risks (BOCR) are not equally important, so we must also prioritize them. This is shown in Table 10.15. The priorities for the economic, security and political factors themselves were established as shown in Fig. 10.6 and used to rate the importance of the top ideal alternative for each of the benefits, costs, opportunities and risks from Table 10.15. Finally, we used the priorities of the latter to combine the synthesized priorities of the alternatives in the four hierarchies, using both formulas BO/CR and $bB + oO - cC - rR$ to obtain their final ranking, as shown in Table 10.11.

How to derive the priority shown next to the goal of each of the four hierarchies in Fig. 10.5 is outlined in Table 10.15. We rated each of the four merits: benefits, costs, opportunities and risks of the dominant PNTR alternative, as it happens to be in this case, in terms of intensities for each assessment criterion. The intensities, Very High, High, Medium, Low, and Very Low were themselves prioritized in the usual pairwise comparison matrix to determine their priorities. We then assigned the appropriate intensity for each merit on all assessment criteria. The outcome is as found in the bottom row of Table 10.15.

We are now able to obtain the overall priorities of the three major decision alternatives, given in the last two columns of Table 10.16. We see in bold that PNTR is the dominant alternative either way we synthesize as in the last two columns.

Table 10.16 Four methods of synthesizing BOCR using the ideal mode

Alternatives	Benefits (0.25)	Opportunities (0.20)	Costs (0.31)	Risks (0.24)	BO/CR	$bB + oO - cC - rR$
PNTR	1	1	0.31	0.54	**5.97**	**0.22**
Amend NTR	0.51	0.43	0.50	0.53	0.83	−0.07
Annual Exten.	0.21	0.13	0.87	0.58	0.05	−0.33

We have laid the basic foundation with hierarchies for what we need to deal with networks involving interdependencies. Let us now turn to that subject.

10.11 The Analytic Network Process

To simplify and deal with complexity, people who work in decision-making use mostly very simple hierarchic structures consisting of a goal, criteria, and alternatives. Yet, not only are decisions obtained from a simple hierarchy of three levels different from those obtained from a multilevel hierarchy, but also decisions obtained from a network can be significantly different from those obtained from a multilevel hierarchy. We cannot collapse complexity artificially into a simplistic structure of two levels, criteria and alternatives, and hope to capture the outcome of interactions in the form of highly condensed judgments that correctly reflect all that goes on in the world. For 30 years we have worked with people to decompose these judgments through more elaborate structures to organize our reasoning and calculations in sophisticated but simple ways to serve our understanding of the complexity around us. Experience indicates that it is not very difficult to do this although it takes more time and effort, but not too much more. We have consulted and lectured on this subject in many countries: extensively in the US, in Brazil, Chile, the Czech Republic, Turkey, Poland, Indonesia, Switzerland, and soon in England and in China. There seems to be worldwide interest in decisions with dependence and feedback. My book on this subject has been translated to two languages Russian and Chinese. *Indeed, we must use feedback networks to arrive at the kind of decisions needed to cope with the future.*

Many decision problems cannot be structured hierarchically because they involve the interaction and dependence of higher-level elements in a hierarchy on lower-level elements. Not only does the importance of the criteria determine the importance of the alternatives as in a hierarchy, but also the importance of the alternatives themselves determines the importance of the criteria. Two elephants chosen for work should have powerful trunks. One of them is slightly stronger but has only one ear. Strength alone would lead one to choose the strong but less attractive elephant unless the criteria of strength and attractiveness are evaluated in terms of the elephants, and strength receives a smaller value, and appearance a larger value because both elephants are strong. Feedback also enables us to factor the future into the present

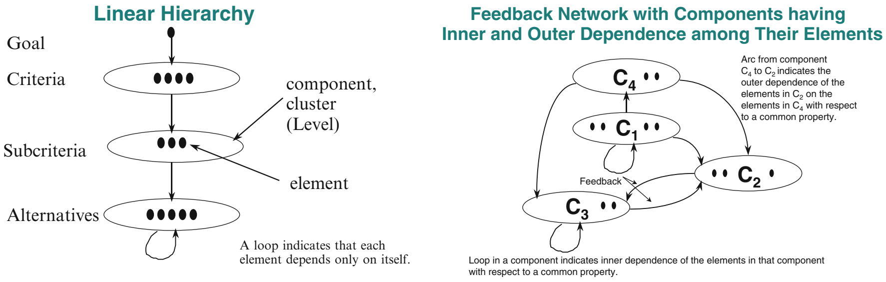


Fig. 10.7 How a hierarchy compares to a network

to determine what we have to do to attain a desired future. The Analytic Network Process is a generalization of the Analytic Hierarchy Process. The basic structures are networks. Priorities are established in the same way they are in the AHP using pairwise comparisons and judgments.

The feedback structure does not have the top-to-bottom form of a hierarchy but looks more like a network, with cycles connecting its components of elements, which we can no longer call levels, and with loops that connect a component to itself (see Fig. 10.7). It also has sources and sinks. A **source** node is an origin of paths of influence (importance) and never a destination of such paths. A **sink** node is a destination of paths of influence and never an origin of such paths. A full network can include source nodes; intermediate nodes that fall on paths from source nodes, lie on cycles, or fall on paths to sink nodes; and finally sink nodes. Some networks can contain only source and sink nodes. Still others can include only source and cycle nodes or cycle and sink nodes or only cycle nodes. A decision problem involving feedback arises often in practice. It can take on the form of any of the networks just described. The challenge is to determine the priorities of the elements in the network and in particular the alternatives of the decision and to justify the validity of the outcome. Because feedback involves cycles, and cycling is an infinite process, the operations needed to derive the priorities become more demanding than is familiar with hierarchies.

To obtain the overall dependence of elements such as the criteria, one proceeds as follows: Construct a zero-one matrix of criteria against criteria using the number one to signify dependence of one criterion on another, and zero otherwise. A criterion need not depend on itself as an industry, for example, may not use its own output. For each column of this matrix, construct a pairwise comparison matrix only for the dependent criteria, derive an eigenvector, and augment it with zeros for the excluded criteria. If a column is all zeros, then assign a zero vector to represent the priorities. The question in the comparison would be: For a given criterion, which of two criteria depends more on that criterion with respect to the goal or with respect to a higher-order controlling criterion?

In Fig. 10.7, a view is shown of a hierarchy and a network. A hierarchy is comprised of a goal, levels of elements and connections between the elements.

These connections go only to elements in lower levels. A network has clusters of elements, with the elements being connected to elements in another cluster (outer dependence) or the same cluster (inner dependence). A hierarchy is a special case of a network with connections going only in one direction. In a view of a hierarchy, such as that shown in Fig. 10.7, the levels in the hierarchy correspond to clusters in a network. One example of inner dependence in a component consisting of a father mother and baby is whom does the baby depend on more for its survival, its mother or itself. The baby depends more on its mother than on itself. Again suppose one makes advertising by newspaper and by television. It is clear that the two influence each other because the newspaper writers watch television and need to make their message unique in some way, and vice versa. If we think about it carefully everything can be seen to influence everything including itself according to many criteria. The world is far more interdependent than we know how to deal with using our existing ways of thinking and acting. We know it but how to deal with it. The ANP appears to be a plausible logical way to deal with dependence.

The priorities derived from pairwise comparison matrices are entered as parts of the columns of a supermatrix. The supermatrix represents the influence priority of an element on the left of the matrix on an element at the top of the matrix. A supermatrix along with an example of one of its general entry matrices is shown

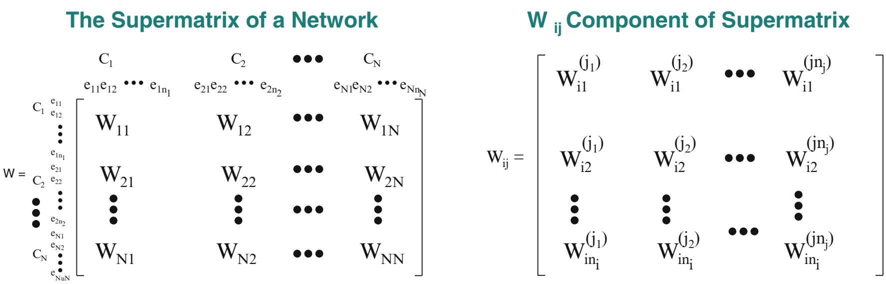


Fig. 10.8 The supermatrix of a network and detail of a component in it

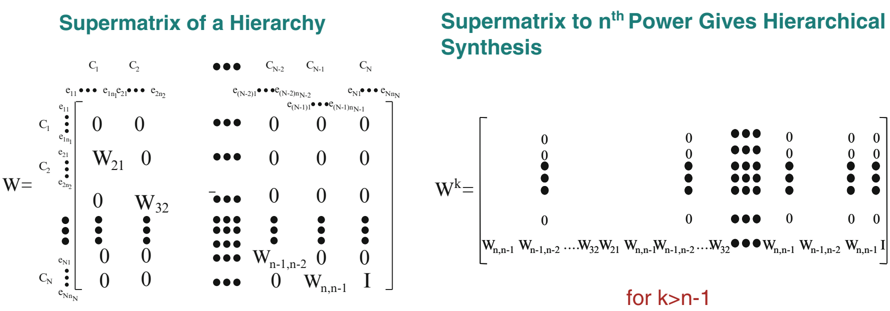


Fig. 10.9 The supermatrix of a hierarchy with the resulting limit matrix corresponding to hierarchical composition

in Fig. 10.8. The component C_i in the supermatrix includes all the priority vectors derived for nodes that are "parent" nodes in the C_i cluster. Figure 10.9 gives the supermatrix of a hierarchy along with the kth power that yields the principle of hierarchic composition in its $(k, 1)$ position.

Hierarchic composition yields multilinear forms that are of course nonlinear and have the form

$$\sum_{i_1,\cdots,i_p} x_1^{i_1} x_2^{i_2} \cdots x_p^{i_p}$$

where i_j indicates the jth level of the hierarchy and the x_j is the priority of an element in that level. The richer the structure of a hierarchy in breadth and depth, the more elaborate are the multilinear forms derived from it. There seems to be a good opportunity to investigate the relationship obtained by composition to covariant tensors and their algebraic properties. Powers of a variable allow for the possibility that the variable is repeated in the composition. Multilinear forms are related to polynomials and these by the Stone-Weierstrass theorem can be used to approximate arbitrarily close to continuous functions. Such functions may be assumed to underlie the representations of complex events in a decision. In this manner, mathematics and the apparent complicated use of numbers in decision-making can be related in a way that one can understand.

More concretely we have the covariant tensor

$$w_i^h = \sum_{i_2,\cdots,i_{h-1}=1}^{N_{h-1},\cdots,N_1} w_{i_1,i_2}^{h-1} \cdots w_{i_{h-2},i_{h-1}}^{2} w_{i_{h-1}}^{1} \qquad i_1 \equiv i$$

for the priority of the ith element in the hth level of the hierarchy. The composite vector W^h for the entire hth level is represented by the vector with covariant tensorial components. Similarly, the left eigenvector approach to a hierarchy gives rise to a vector with contravariant tensor components.

The classical problem of relating space (geometry) and time to subjective thought can perhaps be examined by showing that the functions of mathematical analysis (and hence also the laws of physics) are derivable as truncated series from the above tensors by composition in an appropriate hierarchy. The foregoing is reminiscent of the theorem in dimensional analysis that any physical variable is proportional to the product of powers of primary variables.

Multilinear forms are obviously nonlinear and are a powerful building stone to go from linearity to non-linearity through the use of complex structures (hierarchies and networks) and enable us to deal with the world according to our deepest ways of understanding and judgment.

In the ANP we look for steady state priorities from a limit supermatrix. To obtain the limit we must raise the matrix to powers. The reason for that is that to capture overall influence (dominance) one must consider all transitivities of different length. These are each represented by the corresponding power of the supermatrix. For each such matrix, the influence of an element on all others is obtained by taking the sum of its corresponding row. If we do that for all the elements, we obtain a vector of influence from that matrix. The sum of all such vectors gives the overall influence. Cesaro summability tells us that it is sufficient to obtain the outcome from the limiting power of the supermatrix.

The outcome of the ANP is nonlinear and rather complex. We know, from a theorem due to J.J. Sylvester that when the multiplicity of each eigenvalue of a matrix W is equal to one that an entire function $f(x)$ (power series expansion of $f(x)$ converges for all finite values of x) with x replaced by W, is given by

$$f(W) = \sum_{i=1}^{n} f(\lambda_i) Z(\lambda_i),$$

$$Z(\lambda_i) = \frac{\prod_{j \neq i} (\lambda_j I - A)}{\prod_{j \neq i} (\lambda_j - \lambda_i)},$$

$$\sum_{i=1}^{n} Z(\lambda_i) = 1, \quad Z(\lambda_i) Z(\lambda_j) = 0, \quad Z^2(\lambda_i) = Z(\lambda_i),$$

where I and 0 are the identity and null matrices respectively.

A similar expression is also available when some or all of the eigenvalues have multiplicities greater than one. We can easily see that if, as we need in our case, $f(W) = W^k$, then $f(\lambda_i) = \lambda_i^k$ and as $k \to \infty$ the only terms that give a finite nonzero value are those for which the modulus of λ_i is equal to one.

The fact that W is stochastic ensures this because

$$\max \sum_{j=1}^{n} a_{ij} \geq \sum_{j=1}^{n} a_{ij} \frac{w_j}{w_i} = \lambda_{\max} \text{ for } \max w_i$$

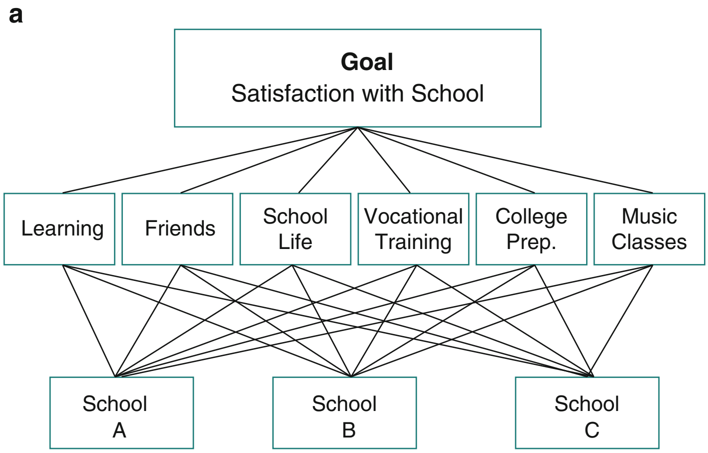
$$\min \sum_{j=1}^{n} a_{ij} \leq \sum_{j=1}^{n} a_{ij} \frac{w_j}{w_i} = \lambda_{\max} \text{ for } \min w_i$$

Thus for a row stochastic matrix we have

$$1 = \min \sum_{j=1}^{n} a_{ij} \leq \lambda_{\max} \leq \max \sum_{j=1}^{n} a_{ij} = 1,$$

and $\lambda_{\max} = 1$. See this author's 2001 book on the ANP [4], and also the manual for the ANP software [5]. Here are two examples that illustrate the validity of the supermatrix as a general framework for prioritization. The first as a generalization of hierarchies that gives back hierarchic answers, and the second as a method of computation and synthesis that carries the burden of computation with the user mostly providing judgments.

a



b

The School Hierarchy as Supermatrix

	Goal	Learning	Friends	School life	Vocational training	College preparation	Music classes	A	B	C
Goal	0	0	0	0	0	0	0	0	0	0
Learning	0.32	0	0	0	0	0	0	0	0	0
Friends	0.14	0	0	0	0	0	0	0	0	0
School life	0.03	0	0	0	0	0	0	0	0	0
Vocational training	0.13	0	0	0	0	0	0	0	0	0
College preparation	0.24	0	0	0	0	0	0	0	0	0
Music classes	0.14	0	0	0	0	0	0	0	0	0
Alternative A	0	0.16	0.33	0.45	0.77	0.25	0.69	1	0	0
Alternative B	0	0.59	0.33	0.09	0.06	0.50	0.09	0	1	0
Alternative C	0	0.25	0.34	0.46	0.17	0.25	0.22	0	0	1

Limiting Supermatrix & Hierarchic Composition

	Goal	Learning	Friends	School life	Vocational training	College preparation	Music classes	A	B	C
Goal	0	0	0	0	0	0	0	0	0	0
Learning	0	0	0	0	0	0	0	0	0	0
Friends	0	0	0	0	0	0	0	0	0	0
School life	0	0	0	0	0	0	0	0	0	0
Vocational training	0	0	0	0	0	0	0	0	0	0
College preparation	0	0	0	0	0	0	0	0	0	0
Music classes	0	0	0	0	0	0	0	0	0	0
Alternative A	0.3676	0.16	0.33	0.45	0.77	0.25	0.69	1	0	0
Alternative B	0.3781	0.59	0.33	0.09	0.06	0.50	0.09	0	1	0
Alternative C	0.2543	0.25	0.34	0.46	0.17	0.25	0.22	0	0	1

Fig. 10.10 (**a**) School choice hierarchy composition. (**b**) Supermatrix of school choice hierarchy gives same results as hierarchic composition

10.11.1 The Classic AHP School Example as an ANP Model

We show in Fig. 10.10a, b above, the hierarchy, and its corresponding supermatrix, and its limit supermatrix to obtain the priorities of three schools involved in a decision to choose one for the author's son. They are precisely what one obtains by hierarchic composition using the AHP. Figure 10.10a shows the priorities of the criteria with respect to the goal and those of the alternatives with respect to each criterion. There is an identity submatrix for the alternatives with respect to the alternatives in the lower right hand part of the matrix, because each alternative depends on itself. The level of alternatives in a hierarchy is a sink cluster of nodes that absorbs priorities but does not pass them on. This calls for using an identity submatrix for them in the supermatrix. The last three entries of column one of Fig. 10.10b give the overall priorities of the alternatives with respect to the goal.

10.11.2 Criteria Weights Automatically Derived from Supermatrix

Let us revisit the example we gave earlier in Table 10.13 of three alternatives and two criteria measured in the same unit. We use interdependence to determine what overall weight the criteria should have without computing the relative sum of the alternatives under each criterion to the total. Since we are dealing with tangibles we normalize each column to obtain the priorities for the alternatives under each criterion. We also normalize each row to obtain the priorities of the criteria with respect to each alternative. We enter these in a supermatrix as shown in Table 10.17; there is no need to weight the supermatrix because it is already column stochastic, so we can raise it to limiting powers right away and obtain the supermatrix in Table 10.18 in which, in this case it turns out that, all the columns are identical.

Table 10.17 The supermatrix

		Alternatives			Criteria	
		A_1	A_2	A_3	C_1	C_2
Alternatives	A_1	0.000	0.000	0.000	0.200	0.500
	A_2	0.000	0.000	0.000	0.300	0.167
	A_3	0.000	0.000	0.000	0.500	0.333
Criteria	C_1	0.571	0.857	0.833	0.000	0.000
	C_2	0.429	0.143	0.167	0.000	0.000

Table 10.18 The limit supermatrix

		Alternatives			Criteria	
		A_1	A_2	A_3	C_1	C_2
Alternatives	A_1	0.135	0.135	0.135	0.135	0.135
	A_2	0.135	0.135	0.135	0.135	0.135
	A_3	0.231	0.231	0.231	0.231	0.231
Criteria	C_1	0.385	0.385	0.385	0.385	0.385
	C_2	0.115	0.115	0.115	0.115	0.115

10.12 Two Examples of Estimating Market Share: The ANP with a Single Benefits Control Criterion

A market share estimation model is structured as a network of clusters and nodes. The object is to determine the relative market share of competitors in a particular business, or endeavor, by considering what affects market share in that business and introducing them as clusters, nodes and influence links in a network. No actual statistics are used in these examples, but only judgments by experts about relative influence. The decision alternatives are the competitors and the synthesized results are their relative dominance. The relative dominance results can then be compared against some outside measure such as dollars. If dollar income is the measure being used, the incomes of the competitors must be normalized to get it in terms of relative market share.

The clusters might include customers, service, economics, advertising, and quality of goods. The customers cluster might then include nodes for the age groups of the people that buy from the business: teenagers, 20–33 year olds, 34–55 year olds, 55–70 year olds, and over 70. The advertising cluster might include newspapers, TV, Radio, and Fliers. After all the nodes are created start by picking a node and linking it to the other nodes in the model that influence it. The "children" nodes will then be pairwise compared with respect to that node as a "parent" node. An arrow will automatically appear going from the cluster the parent node is in to the cluster with its children nodes. When a node is linked to nodes in its own cluster, the arrow becomes a loop on that cluster and we say there is inner dependence.

The linked nodes in a given cluster are pairwise compared for their influence on the node they are linked from (the parent node) to determine the priority of their influence on the parent node. Comparisons are made as to which is more important to the parent node in capturing "market share". These priorities are then entered in the supermatrix.

The clusters are also pairwise compared to establish their importance with respect to each cluster they are linked from, and the resulting matrix of numbers is used to weight the components of the original unweighted supermatrix to give the weighted supermatrix. This matrix is then raised to powers until it converges to give the limit supermatrix. The relative values for the companies are obtained from the columns of the limit supermatrix that in this case, with the help of Cesaro summability, are reduced in the software to be all the same. Normalizing these numbers yields the relative market share.

If comparison data in terms of sales in dollars, or number of members, or some other known measures are available, one can use their relative values to validate the outcome. The AHP/ANP has a compatibility metric to determine how close the ANP result is to the known measure. It involves taking the Hadamard product of the matrix of ratios of the ANP outcome and the transform of the matrix of ratios of the actual outcome summing all the coefficients and dividing by n^2. The requirement is that the value should be close to 1 and certainly not much more than 1.1.

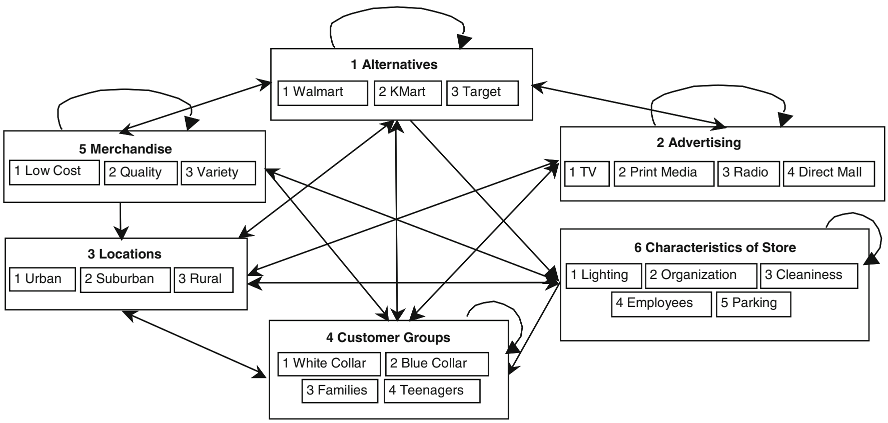


Fig. 10.11 The clusters and nodes of a model to estimate the relative market share of Walmart, Kmart and Target

We will give two examples of market share estimation showing details of the process in the first example and showing only the models and results in the second.

10.12.1 Example 1: Estimating the Relative Market Share of Walmart, Kmart and Target

The network for the ANP model shown in Fig. 10.11 describes quite well the influences that determine the market share of these companies. We will not use space in this chapter to describe the clusters and their nodes in greater detail.

10.12.1.1 The Unweighted Supermatrix

The unweighted supermatrix is constructed from the priorities derived from the different pairwise comparisons. The nodes, grouped by the clusters they belong to, are the labels of the rows and columns of the supermatrix. The column for a node *a* contains the priorities of the nodes that have been pairwise compared with respect to *a*. The supermatrix for the network in Fig. 10.11 is shown in Table 10.19. In Tables 10.19, 10.20, and 10.21 the following abbreviations have been used:

- Al—Alternatives, WM—Walmart, KM—KMart, Ta—Target;
- Ad—Advertising, TV, PM—Print Media, Ra—Radio, DM—Direct Mail;
- Lo—Location, Ur—Urban, Su—Suburban, Ru—Rural;
- CG—Customer Groups, WC—White Collar, BC—Blue Collar, Fa—Families, Te—Teenagers;

Table 10.19 The unweighted supermatrix

		1 Al			2 Ad				3 Lo		
		1 WM	2 KM	3 Ta	1 TV	2 PM	3 Ra	4 DM	1 Ur	2 Su	3 Ru
1 Al	1 WM	0.000	0.833	0.833	0.687	0.540	0.634	0.661	0.614	0.652	0.683
	2 KM	0.750	0.000	0.167	0.186	0.297	0.174	0.208	0.268	0.235	0.200
	3 Ta	0.250	0.167	0.000	0.127	0.163	0.192	0.131	0.117	0.113	0.117
2 Ad	1 TV	0.553	0.176	0.188	0.000	0.000	0.000	0.000	0.288	0.543	0.558
	2 PM	0.202	0.349	0.428	0.750	0.000	0.800	0.000	0.381	0.231	0.175
	3 Ra	0.062	0.056	0.055	0.000	0.000	0.000	0.000	0.059	0.053	0.048
	4 DM	0.183	0.420	0.330	0.250	0.000	0.200	0.000	0.273	0.173	0.219
3 Lo	1 Ur	0.114	0.084	0.086	0.443	0.126	0.080	0.099	0.000	0.000	0.000
	2 Su	0.405	0.444	0.628	0.387	0.416	0.609	0.537	0.000	0.000	0.000
	3 Ru	0.481	0.472	0.285	0.169	0.458	0.311	0.364	0.000	0.000	0.000
4 CG	1 WC	0.141	0.114	0.208	0.165	0.155	0.116	0.120	0.078	0.198	0.092
	2 BC	0.217	0.214	0.117	0.165	0.155	0.198	0.203	0.223	0.116	0.224
	3 Fa	0.579	0.623	0.620	0.621	0.646	0.641	0.635	0.656	0.641	0.645
	4 Te	0.063	0.049	0.055	0.048	0.043	0.045	0.041	0.043	0.045	0.038
5 Me	1 LC	0.362	0.333	0.168	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	2 Qu	0.261	0.140	0.484	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	3 Va	0.377	0.528	0.349	0.000	0.000	0.000	0.000	0.000	0.000	0.000
6 CS	1 Li	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	2 Or	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	3 Cl	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	4 Em	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	5 Pa	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

(continued)

- Me—Merchandise, LC—Low Cost, Qu—Quality, Va—Variety;
- CS—Characteristics of Store, Li—Lighting, Or—Organization, Cl—Cleanliness, Em—Employees, Pa—Parking.

10.12.1.2 The Cluster Matrix

The clusters themselves must be compared to establish their relative importance and use it to weight the supermatrix to make it column stochastic. A cluster impacts another cluster when it is linked from it, that is, when at least one node in the source cluster is linked to nodes in the target cluster. The clusters linked from the source cluster are pairwise compared for the importance of their impact on it with respect to market share, resulting in the column of priorities for that cluster in the cluster matrix. The process is repeated for each cluster in the network to obtain the matrix shown in Table 10.20. An interpretation of the priorities in the first column is that Merchandise (0.442) and Locations (0.276) have the most impact on Alternatives, the three competitors.

Table 10.19 (continued)

		4 CG				5 Me			6 CS				
		1 WC	2 BC	3 Fa	4 Te	1 LC	2 Qu	3 Va	1 Li	2 Or	3 Cl	4 Em	5 Pa
1 Al	1 WM	0.637	0.661	0.630	0.691	0.661	0.614	0.648	0.667	0.655	0.570	0.644	0.558
	2 KM	0.105	0.208	0.218	0.149	0.208	0.117	0.122	0.111	0.095	0.097	0.085	0.122
	3 Ta	0.258	0.131	0.151	0.160	0.131	0.268	0.230	0.222	0.250	0.333	0.271	0.320
2 Ad	1 TV	0.323	0.510	0.508	0.634	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	2 PM	0.214	0.221	0.270	0.170	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	3 Ra	0.059	0.063	0.049	0.096	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	4 DM	0.404	0.206	0.173	0.100	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
3 Lo	1 Ur	0.167	0.094	0.096	0.109	0.268	0.105	0.094	0.100	0.091	0.091	0.111	0.067
	2 Su	0.833	0.280	0.308	0.309	0.117	0.605	0.627	0.433	0.455	0.455	0.444	0.293
	3 Ru	0.000	0.627	0.596	0.582	0.614	0.291	0.280	0.466	0.455	0.455	0.444	0.641
4 CG	1 WC	0.000	0.000	0.279	0.085	0.051	0.222	0.165	0.383	0.187	0.242	0.165	0.000
	2 BC	0.000	0.000	0.649	0.177	0.112	0.159	0.165	0.383	0.187	0.208	0.165	0.000
	3 Fa	0.857	0.857	0.000	0.737	0.618	0.566	0.621	0.185	0.583	0.494	0.621	0.000
	4 Te	0.143	0.143	0.072	0.000	0.219	0.053	0.048	0.048	0.043	0.056	0.048	0.000
5 Me	1 LC	0.000	0.000	0.000	0.000	0.000	0.800	0.800	0.000	0.000	0.000	0.000	0.000
	2 Quality	0.000	0.000	0.000	0.000	0.750	0.000	0.200	0.000	0.000	0.000	0.000	0.000
	3 Variety	0.000	0.000	0.000	0.000	0.250	0.200	0.000	0.000	1.000	0.000	0.000	0.000
6 CS	1 Li	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.169	0.121	0.000	0.250
	2 Or	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.251	0.000	0.575	0.200	0.750
	3 Cl	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.673	0.469	0.000	0.800	0.000
	4 Em	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.308	0.304	0.000	0.000
	5 Pa	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.075	0.055	0.000	0.000	0.000

10.12.1.3 The Weighted Supermatrix

The weighted supermatrix shown in Table 10.21 is obtained by multiplying each entry in a block of the component at the top of the supermatrix by the priority of influence of the component on the left from the cluster matrix in Table 10.20. For example, the first entry, 0.137, in Table 10.20 is used to multiply each of the nine entries in the block (Alternatives, Alternatives) in the unweighted supermatrix shown in Table 10.19. This gives the entries for the (Alternatives, Alternatives) component in the weighted supermatrix of Table 10.21. Each column in the weighted supermatrix has a sum of 1, and thus the matrix is stochastic.

The limit supermatrix is not shown here to save space. It is obtained from the weighted supermatrix by raising it to powers until it converges so that all columns are identical. From the top part of the first column of the limit supermatrix we get the priorities we seek and normalize. We show what they are in Table 10.22.

Table 10.20 The cluster matrix

	1. Al	2. Ad	3. Li	4. CG	5. Me	6. CG
1. Alternatives (Al)	0.137	0.174	0.094	0.057	0.049	0.037
2. Advertising (Ad)	0.091	0.220	0.280	0.234	0.000	0.000
3. Locations (Li)	0.276	0.176	0.000	0.169	0.102	0.112
4. Customer groups (CG)	0.054	0.429	0.627	0.540	0.252	0.441
5. Merchandise (Me)	0.442	0.000	0.000	0.000	0.596	0.316
6. Characteristics of store (CS)	0.000	0.000	0.000	0.000	0.000	0.094

Table 10.21 The weighted supermatrix

		1 Al			2 Ad				3 Lo		
		1 WM	2 KM	3 Ta	1 TV	2 PM	3 Ra	4 DM	1 Ur	2 Su	3 Ru
1 Al	1 WM	0.000	0.114	0.114	0.120	0.121	0.110	0.148	0.058	0.061	0.064
	2 KM	0.103	0.000	0.023	0.033	0.066	0.030	0.047	0.025	0.022	0.019
	3 Ta	0.034	0.023	0.000	0.022	0.037	0.033	0.029	0.011	0.011	0.011
2 Ad	1 TV	0.050	0.016	0.017	0.000	0.000	0.000	0.000	0.080	0.152	0.156
	2 PM	0.018	0.032	0.039	0.165	0.000	0.176	0.000	0.106	0.064	0.049
	3 Ra	0.006	0.005	0.005	0.000	0.000	0.000	0.000	0.016	0.015	0.014
	4 DM	0.017	0.038	0.030	0.055	0.000	0.044	0.000	0.076	0.048	0.061
3 Lo	1 Ur	0.031	0.023	0.024	0.078	0.028	0.014	0.022	0.000	0.000	0.000
	2 Su	0.112	0.123	0.174	0.068	0.094	0.107	0.121	0.000	0.000	0.000
	3 Ru	0.133	0.130	0.079	0.030	0.103	0.055	0.082	0.000	0.000	0.000
4 CG	1 WC	0.008	0.006	0.011	0.071	0.086	0.050	0.066	0.049	0.124	0.058
	2 BC	0.012	0.011	0.006	0.071	0.086	0.085	0.112	0.140	0.073	0.141
	3 Fa	0.031	0.033	0.033	0.267	0.356	0.275	0.350	0.411	0.402	0.404
	4 Te	0.003	0.003	0.003	0.021	0.024	0.019	0.023	0.027	0.028	0.024
5 Me	1 LC	0.160	0.147	0.074	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	2 Qu	0.115	0.062	0.214	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	3 Va	0.166	0.233	0.154	0.000	0.000	0.000	0.000	0.000	0.000	0.000
6 CS	1 Li	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	2 Or	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	3 Cl	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	4 Em	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	5 Pa	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

(continued)

Table 10.21 (continued)

		4 CG				5 Me			6 CS				
		1 WC	2 BC	3 Fa	4 Te	1 LC	2 Qu	3 Va	1 Li	2 Or	3 Cl	4 Em	5 Pa
1 Al	1 WM	0.036	0.038	0.036	0.040	0.033	0.030	0.032	0.036	0.024	0.031	0.035	0.086
	2 KM	0.006	0.012	0.012	0.009	0.010	0.006	0.006	0.006	0.004	0.005	0.005	0.019
	3 Ta	0.015	0.007	0.009	0.009	0.006	0.013	0.011	0.012	0.009	0.018	0.015	0.049
2 Ad	1 TV	0.076	0.119	0.119	0.148	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	2 PM	0.050	0.052	0.063	0.040	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	3 Ra	0.014	0.015	0.012	0.023	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	4 DM	0.095	0.048	0.040	0.023	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
3 Lo	1 Ur	0.028	0.016	0.016	0.018	0.027	0.011	0.010	0.016	0.010	0.015	0.018	0.031
	2 Su	0.141	0.047	0.052	0.052	0.012	0.062	0.064	0.071	0.051	0.074	0.073	0.135
	3 Ru	0.000	0.106	0.101	0.098	0.063	0.030	0.029	0.076	0.051	0.074	0.073	0.295
4 CG	1 WC	0.000	0.000	0.151	0.046	0.013	0.056	0.042	0.247	0.082	0.156	0.107	0.000
	2 BC	0.000	0.000	0.350	0.096	0.028	0.040	0.042	0.247	0.082	0.134	0.107	0.000
	3 Fa	0.463	0.463	0.000	0.398	0.156	0.143	0.157	0.119	0.257	0.318	0.400	0.000
	4 Te	0.077	0.077	0.039	0.000	0.055	0.013	0.012	0.031	0.019	0.036	0.031	0.000
5 Me	1 LC	0.000	0.000	0.000	0.000	0.000	0.477	0.477	0.000	0.000	0.000	0.000	0.000
	2 Qu	0.000	0.000	0.000	0.000	0.447	0.000	0.119	0.000	0.000	0.000	0.000	0.000
	3 Va	0.000	0.000	0.000	0.000	0.149	0.119	0.000	0.000	0.316	0.000	0.000	0.000
6 CS	1 Li	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.016	0.017	0.000	0.097
	2 Or	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.035	0.000	0.079	0.027	0.290
	3 Cl	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.092	0.044	0.000	0.110	0.000
	4 Em	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.029	0.042	0.000	0.000
	5 Pa	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.010	0.005	0.000	0.000	0.000

Table 10.22 The synthesized results for the alternatives

Alternatives	Values from limit supermatrix	Actual values July 13, 1998	Normalized values from supermatrix	Actual market share as dollar sales normalized
Walmart	0.057	58 billion $	0.599	54.8
KMart	0.024	27.5 billion $	0.248	25.9
Target	0.015	20.3 billion $	0.254	19.2

10.12.1.4 Synthesized Results from the Limit Supermatrix

The relative market share of the alternatives Walmart, Kmart and Target from the limit supermatrix are: 0.057, 0.024 and 0.015. When normalized they are 0.599, 0.248 and 0.154.

The relative market share values obtained from the model were compared with the actual sales values by computing the compatibility index. The Compatibility Index, illustrated in the next example, is used to determine how close two sets of numbers from a ratio scale or an absolute scale are to each other. We form the matrix of ratios of each set and multiply element-wise one matrix by the transpose of the other (the Hadamard product), add all the entries of the resulting matrix and divide the outcome by n^2, where n is the order of the matrix which is the number of entries in each vector. The outcome should not exceed the value of 1.1. In this example the result is equal to 1.016 and falls below 1.1 and therefore is an acceptable outcome.

10.12.2 Example 2: US Athletic Footwear Market in 2000

My student Maria Lagasca has studied the US Athletic Footwear market. That market has seen tremendous growth over the years. Not only are these products used for specific athletic purposes but also they have been used as casual wear because of its ability to provide comfort and agility to consumers. Interest in the industry has grown to a large extent because of advances in research and development for durable yet comfortable materials. The industry is also considered as one of the heaviest advertisers based on a study made last year along with other industries such as apparel, beer/wine/liquor, computers and electronics. The study illustrated in Fig. 10.12 aims at estimating the market share using the ANP with the aid of SuperDecisions software. The estimates are then compared against the actual market share of various manufacturers in the year 2000. As the industry is fragmented (with many players holding fewer shares of the market), the other manufacturers have been lumped under the "Others" category as they are considered as homogeneous given the factors used in the analysis.

10.12.2.1 Clusters and Elements (Nodes)

1. Alternatives (brands competing against each other in the market)
   (a) Nike—Nike as an alternative brand for athletic footwear.
   (b) Reebok—Reebok as an alternative brand for athletic footwear.

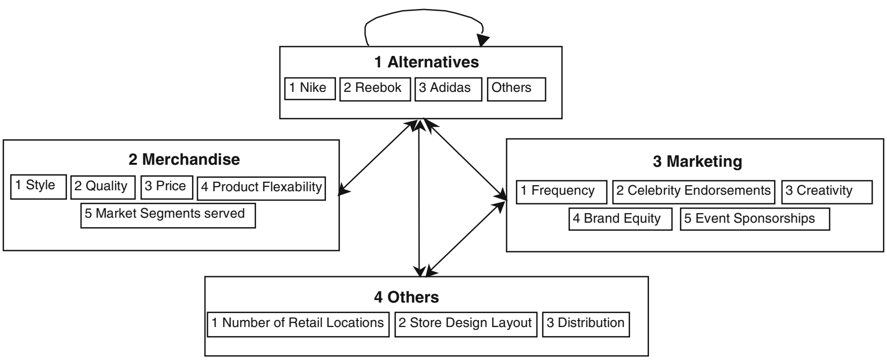


Fig. 10.12 The clusters and nodes of a model to estimate the relative market share of footware

(c) Adidas—Adidas as an alternative brand for athletic footwear.
(d) Others—Other alternative brands (And1, Skechers, New Balance, Timberland, etc.) for athletic footwear.

2. Merchandise (affects each brand and each brand affects the type of merchandising strategy)

   (a) Style—the ability of a manufacturer to immediately respond to customers tastes and needs or create demand by introducing new products to the market.
   (b) Quality—Quality includes the reliability / durability of products including the ability to withstand pressure and frequent use.
   (c) Price—defined as value for money.
   (d) Product Flexibility—Ability of the product to substitute for other footwear, i.e. Running shoes can be used for casual wear and other purposes.
   (e) Market Segments Served—Ability of the manufacturer to cover various target segments through their different product lines, i.e. men, women, children, basketball players, soccer players, etc.

3. Marketing (Marketing affects each of the brands and each brand affects the type of marketing strategy)

   (a) Frequency—Frequency of advertising regardless of media.
   (b) Celebrity Endorsements—Endorsement by a well-known popular sports celebrity.
   (c) Creativity—Creativity of marketing advertisements regardless of length.
   (d) Brand Equity—Ability to create brand awareness and recognition among various segments of the market.
   (e) Event Sponsorships—a marketing tool to advertise and create awareness for brand.

4. Others (Other factors affect the brand and each brand affects the type of strategy for these factors; also the Marketing strategy affects these factors)

   (a) Number of retail locations—The number and the coverage of retail locations across the United States.
   (b) Store design and layout—includes placement and effective layout of merchandise vis-à-vis competitors.
   (c) Distribution—shelf space and coverage of merchandise across the United States. Includes relationships with distributors and even with own distribution chain.

Comparisons were done based on information gathered for each individual manufacturer. Advertising was determined due to factors such as each manufacturer's relative selling, general, and administrative expenses from their annual reports. Advertisements (mostly in print) were also viewed and use of celebrity endorsements in the same period were also assessed relative to each brand to measure creativity as well as frequency. Brand equity was measured on more intuitive terms i.e. Nike's Swoosh logo is considered as one of the most recognized logos and brands, which gave them an advantage over the other brands.

Other factors, such as the number of retail locations, were assessed by counting the total number of such locations (from individual websites). Store layout and distribution information were gathered from the websites as well to assess the relative effectiveness of each factor. For instance, Reebok and Adidas, have fewer individual stores than Nike (Factory outlets and Niketown) and tend to be distributed in department stores or sporting goods stores facing more competition from other brands because of less exclusivity.

In terms of merchandise, prices are relatively the same for all brands although some like Adidas and Reebok may seem to be higher than other brands because of quality. Nike and other athletic footwear products tend to be more flexible in terms of how consumers use the products i.e. their basketball shoes are often substituted for casual wear and running shoes, which leads to a broader target segment. Also, Nike and the other brands seem to serve broader market segments specifically women and children. Their line extensions, e.g. Michael Jordan for men, have been extended to children.

As more and more people substitute athletic footwear for everyday use, Nike and the other brands seem to be stronger in catering to this need, thereby leading to more market share.

Table 10.23 gives the actual and the estimated market share for each brand. They are surprisingly close. This example was done as a take-home exercise. In this case the compatibility index obtained from the study is 1.001428, which is very good. We would be glad to provide the interested reader with at least a dozen such market share examples often worked out in class in about 1 hour without prior preparation or looking at numbers. They all have such close outcomes, because students, interested and familiar with the example, provided the judgments.

We now look at full blown decisions involving BOCR. First we give an outline of the steps recommended in applying the ANP.

10.13 Outline of the Steps of the ANP

1. Describe the decision problem in detail including its objectives, criteria and subcriteria, actors and their objectives and the possible outcomes of that decision. Give details of influences that determine how that decision may come out.
2. Determine the control criteria and subcriteria in the four control hierarchies one each for the benefits, opportunities, costs and risks of that decision and obtain their priorities from paired comparisons matrices. If a control criterion or subcriterion has a global priority of 3 % or less, you may carefully consider eliminating it from further consideration. The software automatically deals only with those criteria or subcriteria that have subnets under them. For benefits and opportunities, ask what gives the most benefits or presents the greatest opportunity to influence fulfillment of that control criterion. For costs and risks, ask what incurs the most cost or faces the greatest risk. Sometimes (very rarely),

Table 10.23 Footwear actual statistics and model results along with the compatibility index

Alternatives	A1	A2	A3	A4	
Actual market share	39.200	15.100	10.900	34.800	
Estimated market share from ANP model	40.670	15.040	11.330	32.970	
Pairwise comparison matrix from actual market share data					
A1	1	2.596026	3.59633	1.12643678	
A2	0.385204082	1	1.385321	0.43390805	
A3	0.278061224	0.721854	1	0.31321839	
A4	0.887755102	2.304636	3.192661	1	
Transpose of comparison matrix from estimated market share					
A1	1	0.369806	0.278584	0.81067126	
A2	2.70412234	1	0.753324	2.19215426	
A3	3.589585172	1.327449	1	2.90997352	
A4	1.233545648	0.456172	0.343646	1	
Result of Hadamard (element-wise) multiplication of previous two matrices					
	A1	A2	A3	A4	Row sums
A1	1	0.960026	1.001879	0.91316992	3.875075
A2	1.041638963	1	1.043596	0.95119337	4.036429
A3	0.998124448	0.958225	1	0.91145722	3.867807
A4	1.095086442	1.051311	1.097144	1	4.243542
				SUM =	16.02285
Number of alternatives: $n = 4$					
Compatibility index = (SUM/n^2) = 1.001428					

the comparisons are made simply in terms of benefits, opportunities, costs, and risks in the aggregate without using control criteria and subcriteria.

3. Determine the most general network of clusters (or components) and their elements that applies to all the control criteria. To better organize the development of the model as well as you can, number and arrange the clusters and their elements in a convenient way (perhaps in a column). Use the identical label to represent the same cluster and the same elements for all the control criteria.
4. For each control criterion or subcriterion, determine the clusters of the general feedback system with their elements and connect them according to their outer and inner dependence influences. An arrow is drawn from a cluster to any cluster whose elements influence it.
5. Determine the approach you want to follow in the analysis of each cluster or element, influencing (the preferred approach) other clusters and elements with respect to a criterion, or being influenced by other clusters and elements. The sense (being influenced or influencing) must apply to all the criteria for the four control hierarchies for the entire decision.

6. For each control criterion, construct the supermatrix by laying out the clusters in the order they are numbered and all the elements in each cluster both vertically on the left and horizontally at the top. Enter in the appropriate position the priorities derived from the paired comparisons as subcolumns of the corresponding column of the supermatrix.
7. Perform paired comparisons on the elements within the clusters themselves according to their influence on each element in another cluster they are connected to (outer dependence) or on elements in their own cluster (inner dependence). In making comparisons, you must always have a criterion in mind. Comparisons of elements according to which element influences a given element more and how strongly more than another element it is compared with are made with a control criterion or subcriterion of the control hierarchy in mind.
8. Perform paired comparisons on the clusters as they influence each cluster to which they are connected with respect to the given control criterion. The derived weights are used to weight the elements of the corresponding column blocks of the supermatrix. Assign a zero when there is no influence. Thus obtain the weighted column stochastic supermatrix.
9. Compute the limit priorities of the stochastic supermatrix according to whether it is irreducible (primitive or imprimitive [cyclic]) or it is reducible with one being a simple or a multiple root and whether the system is cyclic or not. Two kinds of outcomes are possible. In the first all the columns of the matrix are identical and each gives the relative priorities of the elements from which the priorities of the elements in each cluster are normalized to one. In the second the limit cycles in blocks and the different limits are summed and averaged and again normalized to one for each cluster. Although the priority vectors are entered in the supermatrix in normalized form, the limit priorities are put in idealized form because the control criteria do not depend on the alternatives.
10. Synthesize the limiting priorities by weighting each idealized limit vector by the weight of its control criterion and adding the resulting vectors for each of the four merits: Benefits (B), Opportunities (O), Costs (C) and Risks (R). There are now four vectors, one for each of the four merits. An answer involving ratio values of the merits is obtained by forming the ratio BO/CR for each alternative from the four vectors. The alternative with the largest ratio is chosen for some decisions. Companies and individuals with limited resources often prefer this type of synthesis.
11. Determine strategic criteria and their priorities to rate the top ranked (ideal) alternative for each of the four merits one at a time. The synthesized ideals for all the control criteria under each merit may result in an ideal whose priority is less than one for that merit. Only an alternative that is ideal for all the control criteria under a merit receives the value one after synthesis for that merit. Normalize the four ratings thus obtained and use them to calculate the overall synthesis of the four vectors. For each alternative, subtract the sum of the weighted costs and risks from the sum of the weighted benefits and opportunities.

12. Perform sensitivity analysis on the final outcome. Sensitivity analysis is concerned with "what if" kind of question to see if the final answer is stable to changes in the inputs whether judgments or priorities. Of special interest is to see if these changes change the order of the alternatives. How significant the change is can be measured with the Compatibility Index of the original outcome and each new outcome.

10.14 Complex Decisions with Dependence and Feedback

With the China example for hierarchies and with the market share examples it is now easier to deal with complex decisions involving networks. For each of the four BOCR merits we have criteria (and subcriteria where relevant) called control criteria that are prioritized under that merit through paired comparisons. For each of the control criteria we create a network of influences with respect to that control criterion as we did in the market share examples. We obtain the ideal outcome ranking for each control criterion and then synthesize these outcomes by weighting by the importance of the control criteria for each merit. We then rate the top alternative under each merit to obtain the weights b,o,c and r for the BOCR and use them to synthesize and obtain the final weights for the alternatives using the two formulas BO/CR and more importantly, $bB + oO - cC - rR$. Let us sketch out an example using as little space as possible.

10.14.1 The National Missile Defense Example

Not long ago, the United States government faced the crucial decision of whether or not to commit itself to the deployment of a National Missile Defense (NMD) system. Many experts in politics, the military, and academia had expressed different views regarding this decision. The most important rationale behind supporters of the NMD system was protecting the U.S. from potential threats said to come from countries such as North Korea, Iran and Iraq. According to the Central Intelligence Agency, North Korea's Taepo Dong long-range missile tests were successful, and it has been developing a second generation capable of reaching the U.S. Iran also tested its medium-range missile Shahab-3 in July 2000. Opponents expressed doubts about the technical feasibility, high costs (estimated at $60 billion), political damage, possible arms race, and the exacerbation of foreign relations. The idea for the deployment of a ballistic missile defense system has been around since the late 1960s but the current plan for NMD originated with President Reagan's Strategic Defense Initiative (SDI) in the 1980s. SDI investigated technologies for destroying incoming missiles. The controversies surrounding the project were intensified with

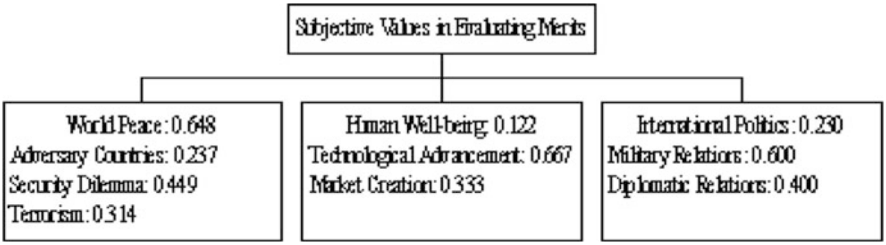


Fig. 10.13 Hierarchy for rating benefits, opportunities, costs and risks

the National Missile Defense Act of 1996, introduced by Senator Sam Nunn (D-GA) in June 25, 1996. The bill required Congress to make a decision on whether the U.S. should deploy the NMD system by 2000. The bill also targeted the end of 2003 as the time for the U.S. to be capable of deploying NMD.

The ANP was applied to analyze this decision. It was done in the usual three steps of the ANP process: (1) the BOCR merits and their control criteria and subcriteria prioritized with respect to each merit, (2) the network of influence for each control criterion from which priorities for the alternatives are derived as in the market share examples and then synthesized using the weights of the control criteria for each merit and finally, (3) the use of strategic criteria as in Fig. 10.13 to rate the merits one at a time as in Table 10.24 through their top alternative and use the resulting normalized ratings as priorities to weight and combine the priorities of each alternative with respect to the four merits to get the final answer.

On February 21, 2002 this author gave a half-day presentation on the subject to the National Defense University in Washington. In December 2002, President George W. Bush and his advisors decided to build the NMD. This study may have had no influence on the decision but still 2 years earlier (September 2000) it had arrived at the same decision produced by this analysis. The alternatives we considered for this analysis are: Deploy NMD, Global defense, R&D, Termination of the NMD program. Complete analysis of this example is given in the author's book on the ANP published in 2001. There were 23 criteria under the BOCR merits, including economic, terrorism, technological progress and everything else people were thinking about as important to develop or not to develop the NMD. After prioritization they were reduced to nine control criteria for all four merits. Each criterion was treated in a very similar way to the single market share examples (essentially economic benefits). Table 10.25 gives the final outcome. Here we see that the two formulas give the same outcome to deploy as the best alternative. The conclusion of this analysis is that pursuing the deployment of NMD is the best alternative. Sensitivity analysis indicates that the final ranks of the alternatives might change, but such change requires making extreme assumptions on the priorities of BOCR and of their corresponding control criteria.

Table 10.24 Priority ratings for the merits: benefits, opportunities, costs and risks

		Benefits	Opportunities	Costs	Risks
World peace	Adversary countries	Very high	Medium	High	Very low
	Security dilemma	Very low	Very low	Very high	Very low
	Terrorism	Medium	Very low	High	High
Human well-being	Technological advancement	High	High	Low	Very low
	Market creation	Medium	High	Very low	Very low
International politics	Military relations	High	High	Medium	Very low
	Diplomatic relations	Low	Low	Low	Very high
Priorities		$b = 0.264$	$o = 0.185$	$c = 0.361$	$r = 0.190$

Very high (0.419), High (0.263), Medium (0.160), Low (0.097), Very low (0.061)

Table 10.25 Overall syntheses of the alternatives

	BO/CR		$bB + oO - cC - rR$	
Alternatives	(from unweighted columns in Table 10.24)	Normalized	(from weighted columns in Table 10.24	Unitized[a]
Deploy	2.504	0.493	0.111	1.891
Global	1.921	0.379	0.059	1.000
R&D	0.560	0.110	−0.108	−1.830
Terminate	0.090	0.018	−0.278	−4.736

[a]Unitized means to divide each number in the column by the number with the smallest absolute value (it is recommended that one not unitize when such a number is close to zero)

10.15 Synthesis of Individual Judgments into a Representative Group Judgment

Kenneth Arrow's Impossibility Theorem, for which he received the Nobel Prize in 1972, stated that it was not possible to find a representative group judgment from the judgments of individuals using ordinal preferences. However, if one allows cardinal preferences, and uses the geometric mean to combine individual judgments as we do in the AHP, it is possible. Aczél and Saaty [1] proved, in a paper co-authored with Janos Aczél, that the unique way to combine reciprocal individual judgments into a corresponding reciprocal group judgment is by using their geometric mean. Arrow proved in his impossibility theorem, using ordinal preferences (either A is preferred to B or it is not), that there does not exist a social welfare function that satisfies all four conditions listed in Fig. 10.14, at once. We showed in [9], in a journal of which

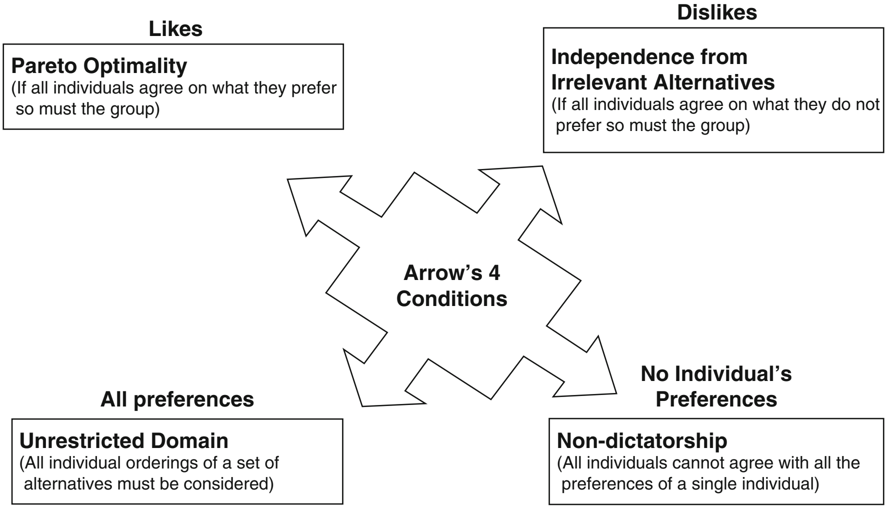


Fig. 10.14 Arrow's four conditions

Arrow is an editor, that *with cardinal intensities of preference and the geometric mean to combine the individual judgments into a representative group judgment, a social welfare function exists that satisfies these four conditions*. Thus we have a possibility theorem that won the Herbert Simon Award from the Chinese Academy of Sciences in 2011.

10.16 Conclusions

Numerous other examples along with the software Super Decisions for the ANP can be obtained from www.superdecisions.com. We hope that the reader now has a good idea as to how to use the AHP/ANP in making a complex decision. The AHP and ANP have found application in practice by many companies and governments. My book Decision Making for Leaders is now in nearly ten languages. Another policy study was done regarding whether the US should go to war with Iraq directly or through the UN, in September 2002. The analysis found that the US should go with the UN with priority more than double those of going alone or of going with a coalition. There is also the ongoing Middle East conflict. An ANP analysis showed that the best option is for Israel and the US to help the Palestinians both set up a state and in particular achieve a viable economy. My forthcoming book The Encyclicon has about 100 summarized examples of applications of the ANP. A list of more than a thousand references until the early 1990s on the AHP appears in [3].

Acknowledgements I am grateful to my colleague Professor Dr. Klaus Dellmann for his careful reading and suggestions to improve the paper and to my student Yeonmin Cho for her untiring efforts in coauthoring with me the case study about China and the WTO, and to my friend Ania Greda from Krakow Poland for her great help in making this manuscript more attractive for the reader. My thanks also go to my friend Kirti Peniwati for her valuable suggestions regarding the NMD example.

References

1. Aczél, J., Saaty, T.: Procedures for synthesizing ratio judgements. J. Math. Psychol. **27**, 93–102 (1983)
2. Harker, P.: Derivatives of the Perron root of a positive reciprocal matrix: with applications to the analytic hierarchy process. Appl. Math. Comput. **22**, 217–232 (1987)
3. Saaty, T.: Fundamentals of the Analytic Hierarchy Process. RWS Publications, Pittsburgh, PA (2000)
4. Saaty, T.: The Analytic Network Process. RWS Publications, Pittsburgh, PA (2001)
5. Saaty, R.: Decision Making in Complex Environments: The Analytic Network Process (ANP) for Dependence and Feedback; A Manual for the ANP Software SuperDecisions. Creative Decisions Foundation, Pittsburgh, PA (2002)
6. Saaty, T., Cho, Y.: The decision by the US Congress on China's trade status: a multicriteria analysis. Soc.-Econ. Plan. Sci. **35**(6), 243–252 (2001)
7. Saaty, T., Ozdemir, M.: Negative priorities in the analytic hierarchy process. Math. Comput. Model. **37**, 1063–1075 (2003)
8. Saaty, T., Ozdemir, M.: Why the magic number seven plus or minus two. Math. Comput. Model. **38**, 233–244 (2003)
9. Saaty, T., Vargas, L.: The possibility of group choice: pairwise comparisons and merging functions. Soc. Choice Welf. **38**, 481–496 (2012)

Chapter 11
On the Mathematical Foundations of MACBETH

Carlos A. Bana e Costa, Jean-Marie De Corte, and Jean-Claude Vansnick

Abstract MACBETH (Measuring Attractiveness by a Categorical Based Evaluation Technique) is a multicriteria decision analysis approach that requires only qualitative judgements about differences of value to help an individual or a group quantify the relative attractiveness of options. This chapter presents a new up-to-date survey of the mathematical foundations of MACBETH. Reference is also made to real-world applications and an extensive bibliography, spanning back to the early 1990s, is provided.

Keywords MACBETH • Questioning procedure • Qualitative judgements • Judgmental inconsistency • Cardinal value measurement • Interaction

11.1 Introduction

Let X (with $\#X = n \geq 2$) be a finite set of elements (alternatives, choice options, courses of action) that an individual or a group, J, wants to compare in terms of their relative attractiveness (desirability, value).

Ordinal value scales (defined on X) are quantitative representations of preferences that reflect, numerically, the order of attractiveness of the elements of X for J. The construction of an ordinal value scale is a straightforward process, provided that J is able to rank the elements of X by order of attractiveness—either directly or through pairwise comparisons of the elements to determine their relative attractiveness. Once the ranking is defined, one needs only to assign a real number $v(x)$ to each element x of X, in such a way that:

C.A. Bana e Costa (✉)
Centre for Management Studies of Instituto Superior Técnico (CEG-IST), Universidade de Lisboa, Av. Rovisco Pais, 1049-001 Lisbon, Portugal
e-mail: carlosbana@tecnico.ulisboa.pt

J.-M. De Corte • J.-C. Vansnick
Institut de Recherche en Développement Humain et des Organisations (HumanOrg), Université de Mons, Place du Parc, 20, 7000 Mons, Belgium
e-mail: Jean-Marie.DeCorte@umons.ac.be; Jean-Claude.Vansnick@umons.ac.be

S. Greco et al. (eds.), *Multiple Criteria Decision Analysis*, International Series in Operations Research & Management Science 233,
DOI 10.1007/978-1-4939-3094-4_11

1. $v(x) = v(y)$ if and only if J judges the elements x and y to be equally attractive;
2. $v(x) > v(y)$ if and only if J judges x to be more attractive than y.

The problem, however, is that, in a multiple criteria decision analysis, conclusions based on a additive value model may be quantitatively meaningless, because "to be quantitatively meaningful a statement should be unaffected by admissible transformations of all the quantities involved" [126, p. 91]. A necessary condition is that each value scale should be unique up to a positive affine transformation (an interval scale), as it is with a value difference scale. A value difference scale (defined on X) is a quantitative representation of preferences that is used to reflect, not only the order of attractiveness of the elements of X for J, but also the differences of their relative attractiveness, or in other words, the strength of J's preferences for one element over another. Unfortunately, the construction of an interval value scale is usually a difficult task.

Both numerical and non-numerical techniques have been proposed and used to build a value difference scale (hereafter, simply called a value scale)—see [113] for a survey. Examples of numerical techniques are direct rating and difference methods—see descriptions in [63, 195, 197]. They require J to be able to produce, either directly or indirectly, numerical representations of his or her strengths of preferences, which can be a difficult cognitive task—see [114]. Non-numerical techniques based on indifference judgements, such as the bisection method (also described by the same authors), force J to compare his or her strengths of preferences between two pairs of elements of X, therefore involving at least three different elements in each judgement. This requires J to perform an intensive cognitive task and is prone to be substantively meaningless—"substantive meaningfulness (...) requires that the qualitative relations (...) being modelled should be unambiguously understood by the decision maker" [126, p. 91].

The aforementioned difficulties inspired the development of MACBETH "**M**easuring **A**ttractiveness by a **C**ategorical **B**ased **E**valuation **T**echnique". The original research on the MACBETH approach was carried out in the early 1990s—see [6, 22, 27]—as a response to the following question:

> How can a value scale be built on X, both in a qualitatively and quantitatively meaningful way, without forcing J to produce direct numerical representations of preferences and involving only two elements of X for each judgement required from J?

Using MACBETH, J is asked to provide preferential information about two elements of X at a time, firstly by giving a judgement as to their relative attractiveness (ordinal judgement) and secondly, if the two elements are not deemed to be equally attractive, by expressing a qualitative judgement about the difference of attractiveness between the most attractive of the two elements and the other. Moreover, to ease the judgemental process, six semantic categories of difference of attractiveness, "very weak", "weak", "moderate", "strong", "very strong" or "extreme", or a succession of these (in case hesitation or disagreement arises) are offered to J as possible answers. This is somewhat in line with similar ideas previously proposed by Saaty [178] in a ratio measurement framework, or by Freeling [125] and Belton [62] in difference value measurement. By pairwise comparing the elements of X a matrix of qualitative judgements is filled in, with either only a few pairs of elements, or with all of them (in which case $n \cdot (n-1)/2$ comparisons would be made by J).

A brief review of the previous research on MACBETH is offered in Sect. 11.2, together with the evolution of its software's development. It shows that, on a technical level, MACBETH has evolved through the course of theoretical research and also through its extension to the multicriteria value measurement framework in numerous practical applications (see Sect. 11.10). Its essential characteristics, however, have never changed—see [57].

Section 11.3 through 11.9 of this chapter present an up-to-date survey of the mathematical foundations of MACBETH. Section 11.3 describes the two MACBETH modes of questioning mentioned above (both involving only two elements at a time) used to acquire preferential information from J, as well as the types of information that can be deduced from each of them. The subsequent sections are devoted to an up-to-date rigorous survey of the mathematical foundations of MACBETH. Section 11.4 addresses the numerical representation of those different types of information. These numerical representations are only possible if J's responses satisfy certain rational working hypotheses. Section 11.5 deals with the "consistency/inconsistency" of the preferential information gathered from J and Sect. 11.6 explores the practical problem of testing the consistency of preferential information. How should an inconsistency be dealt with? The answer to this question is the subject of Sect. 11.7. Sections 11.8 and 11.9 present what MACBETH proposes to J once the preference information provided by J is consistent. Finally, Sect. 11.10 lists several real-world applications of multicriteria value analysis in which the MACBETH approach was used.

This chapter will use the following notation:

- J is an evaluator, either a individual or group.
- X (with $\#X = n \geq 2$) is a finite set of elements (alternatives, choice options, courses of action) that J wants to compare in terms of their relative attractiveness (desirability, value).
- $\Delta att(x, y)$ is the "difference of attractiveness between x and y for J", where x and y are elements of X such that x is more attractive than y for J.
- $\Delta att(x, y) \succ \Delta att(z, w)$ means that $\Delta att(x, y)$ is greater than $\Delta att(z, w)$.
- ϕ is an empty set.
- $\mathbb{R}$ is the set of real numbers.
- $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geq 0\}$.
- $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$.
- $\mathbb{R}^*_+ = \mathbb{R}_+ \setminus \{0\}$.
- $\mathbb{Z}$ is the set of integer numbers.
- $\mathbb{N}$ is the set of non-negative integer numbers.
- $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$.
- $\mathbb{N}_{s,t} = \{s, s+1, \ldots, t\} = \{x \in \mathbb{N} \mid s \leq x \leq t\}$ where $s, t \in \mathbb{N}$, and $s < t$.
- The transpose of a matrix A will be denoted by ${}^tA$.

11.2 Previous Research and Software Evolution

In order to build an interval (value) scale based on the qualitative judgements of difference of attractiveness formulated by J, it is necessary that the six MACBETH categories “very weak”, “weak”, “moderate”, “strong”, “very strong” or “extreme” be represented by non-overlapping (disjoint) intervals of real numbers. The basic idea underlying the initial development of MACBETH was that the limits of these intervals should not be arbitrarily fixed a priori, but determined simultaneously with numerical value scores for the elements of X. Research was then conducted on how to test for the existence of such intervals and how to propose numerical values for the elements of X and for the limits of the intervals—see [6, Chap. IV]. This gave rise to the formulation of a chain of four linear programs—see [22–25]—that, implemented in GAMS, were used in the first real-world applications of MACBETH as a decision aiding tool to derive value scores and criteria weights in the framework of an additive aggregation model—see [27, 30, 76, 77]. Theoretical research conducted at the same time, and first presented in 1994 at the 11th International Conference on MCDM, demonstrated the equivalence of the approach by constant thresholds and the approach by measurement conditions—see [28].

The first MACBETH software was developed in 1994. In it, the objective function used in the GAMS implementation to determine a value scale was modified, on the basis of a simple principle—see [30, 31]—that makes it possible, for simple cases, to determine the scale “by hand” [57]. However, complete procedures to address and manage all cases of inconsistency were not available at that time. Therefore, the software offered its users the possibility of obtaining a compromise scale in the case of inconsistency. This initial software was used in several real world applications—see, for example, [14, 24, 29, 32, 33, 35, 103]. However, it had several important limitations:

1. The determination of suggestions was still heuristic and did not guarantee the minimal number of changes necessary to achieve consistency;
2. It was not possible for the evaluator to hesitate between several semantic categories when expressing judgements. It, therefore, did not enable one to facilitate the management of group judgemental disagreements;
3. It forced the evaluator to first provide all of the judgements before it could run any procedure. Consequently, judgemental inconsistency could only be detected for a full matrix of judgements. As a result, suggestions of changes to resolve inconsistency could only then be discussed, a restriction that did not lend itself to good interaction.

Subsequent theoretical research was therefore concentrated on resolving these problems. Results reported in [95, 159], allowing inconsistencies to be dealt with in a mathematically sound manner, were the turning point in the search for a more interactive formulation. Indeed, it was then possible to implement a procedure that automatically detects “inconsistency”, even for an incomplete matrix of judgements, in a new software called M-MACBETH—see www.m-macbeth.com and

[45]—which has been used to produce some of the figures in this paper. The objective of abandoning the suggestion of a compromise scale could also finally be achieved, since the origin of the inconsistency could now be found (detection of elementary incompatible systems) and explained to J. M-MACBETH finds the minimal number of necessary changes and, for any number of changes not greater than five, suggests all of the possible ways in which the inconsistency can be resolved. Furthermore, it is able to provide suggestions of multiple category changes, where a "k categories change" is considered to be equivalent to k "1 category changes".

Real-world applications in the specific context of bid evaluation (see references in Sect. 11.10) inspired research regarding the concepts of "robustness" [95] and sensitivity [10], the results of which were then included in the software, together with the possibility of addressing potential imprecision (uncertainty) associated with impacts of options, incorporating reference levels for one criterion at any time, and graphically representing comparisons of options on any two groups of criteria [57]. These issues are out of the scope of the present chapter and they are not also included in the version of the software, limited to scoring and weighting, embedded into the HIVIEW3 software in 2003—see [82] and www.catalyze.co.uk.

11.3 Types of Preferential Information

11.3.1 Type 1 Information

Type 1 information refers to preferential information obtained from J by means of Questioning Procedure 1.
Let x and y be two different elements of X.

Questioning Procedure 1. *A first question (Q1) is asked of J:*
Q1: Is one of the two elements more attractive than the other?
J's response (R1) can be: "Yes", or "No", or "I don't know".
If R1 = "Yes", a second question (Q2) is asked:
Q2: Which of the two elements is the most attractive?

The responses to Questioning Procedure 1 for several pairs of elements of X enable the construction of three binary relations on X:

$P = \{(x, y) \in X \times X : x$ is more attractive than $y\}$
$I = \{(x, y) \in X \times X : x$ is not more attractive than y and y is not more attractive than x, or $x = y\}$
$? = \{(x, y) \in X \times X : x$ and y are not comparable in terms of their attractiveness$\}$.

P is asymmetric, I is reflexive and symmetric, and ? is irreflexive and symmetric. Note that $? = X \times X \setminus (I \cup P \cup P^{-1})$, with $P^{-1} = \{(x, y) \in X \times X \mid yPx\}$.

Definition 1. Type 1 information about X is a structure $\{P, I, ?\}$ where P, I and $?$ are disjoint relations on X, P is asymmetric, I is reflexive and symmetric, and $? = X \times X \setminus (I \cup P \cup P^{-1})$.

11.3.2 Type 1+2 Information

Suppose that type 1 information $\{P, I, ?\}$ about X is available.

Questioning Procedure 2. *The following question (Q3) is asked, for all* $(x, y) \in P$*:*

Q3: How do you judge the difference of attractiveness between x and y?

J's response (R3) would be provided in the form "d_s" (where $d_1, d_2, \ldots, d_Q$ ($Q \in \mathbb{N} \setminus \{0, 1\}$) are semantic categories of difference of attractiveness defined so that, if $i < j$, the difference of attractiveness d_i is weaker than the difference of attractiveness d_j) or in the more general form (possibility of hesitation) "d_s to d_t", with $s \leq t$ (the response "I don't know" is assimilated to the response "d_1 to d_Q").

Remark 1. When $Q = 6$ and $d_1 =$ very weak, $d_2 =$ weak, $d_3 =$ moderate, $d_4 =$ strong, $d_5 =$ very strong, $d_6 =$ extreme, Questioning Procedure 1 is the mode of interaction used in the MACBETH approach and its M-MACBETH software.

R3 responses give rise to relations C_{st} ($s, t \in \mathbb{N}, 1 \leq s \leq t \leq Q$) where $C_{st} = \{(x, y) \in P \mid \Delta_{att}(x, y)$ is "d_s to d_t"$\}$. They enable the construction of an asymmetric relation on $P : \{((x, y), (z, w)) \in P \times P \mid \exists\, i, j, s, t \in \mathbb{N}$ with $1 \leq i \leq j < s \leq t \leq Q$, $(x, y) \in C_{st}, (z, w) \in C_{ij}\}$. Hereafter, C_{ss} will simply be referred to as C_s.

Definition 2. Type 1+2 information about X is a structure $\{P, I, ?, P^e\}$ where $\{P, I, ?\}$ is type 1 information about X and P^e is an asymmetric relation on P, the meaning of which is "$(x, y)P^e(z, w)$ when $\Delta_{att}(x, y) \succ \Delta_{att}(z, w)$".

11.4 Numerical Representation of the Preferential Information

11.4.1 Type 1 Scale

Suppose that type 1 information $\{P, I, ?\}$ about X is available.

Definition 3. A *type 1 scale* on X relative to $\{P, I\}$ is a function $\mu : X \to \mathbb{R}$ satisfying Condition 1.

Condition 1 $\forall\, x, y \in X,\ [xPy \Rightarrow \mu(x) > \mu(y)]$ *and* $[xIy \Rightarrow \mu(x) = \mu(y)]$.

Let $Sc_1(X, P, I) = \{\mu : X \to \mathbb{R} \mid \mu$ is a type 1 scale on X relative to $\{P, I\}\}$. When X, P and I are well determined, $Sc_1(X, P, I)$ will be noted Sc_1.

When $? = \phi$ and $Sc_1(X, P, I) \neq \phi$, each element of $Sc_1(X, P, I)$ is an ordinal scale on X.

11.4.2 Type 1+2 Scale

Suppose that type 1+2 information $\{P, I, ?, P^e\}$ about X is available.

Definition 4. A *type 1+2 scale* on X relative to $\{P, I, ?, P^e\}$ is a function $\mu : X \to \mathbb{R}$ satisfying Conditions 1 and 2.

Condition 2 $\forall\, x, y, z, w \in X,\ [(x, y)P^e(z, w) \Rightarrow \mu(x) - \mu(y) > \mu(z) - \mu(w)]$.

Let $Sc_{1+2}(X, P, I, P^e) = \{\mu : X \to \mathbb{R} \mid \mu$ is a type 1+2 scale on X relative to $\{P, I, P^e\}\}$. When X, P, I and P^e are well determined, $Sc_{1+2}(X, P, I, P^e)$ will be noted Sc_{1+2}.

11.5 Consistency: Inconsistency

Definition 5. Type 1 information $\{P, I, ?\}$ about X is

- *consistent* when $Sc_1(X, P, I) \neq \phi$
- *inconsistent* when $Sc_1(X, P, I) = \phi$.

Definition 6. Type 1+2 information $\{P, I, ?, P^e\}$ about X is

- *consistent* when $Sc_{1+2}(X, P, I, P^e) \neq \phi$
- *inconsistent* when $Sc_{1+2}(X, P, I, P^e) = \phi$.

When $Sc_{1+2}(X, P, I, P^e) = \phi$, one can have $Sc_1(X, P, I) = \phi$ or $Sc_1(X, P, I) \neq \phi$. In the first case, the message "no ranking" will appear in M-MACBETH; it occurs namely when J declares, in regards to elements x, y and z of X, that [xIy, yIz and xPz] or [xPy, yPz and zPx]. In the second case, the message "inconsistent judgement" will appear in M-MACBETH.

Although this is the only difference between the types of inconsistency introduced in M-MACBETH, it is interesting to mention, from a theoretical perspective, that one could further distinguish two sub-types of inconsistency (sub-type a and sub-type b) when $Sc_{1+2}(X, P, I, P^e) = \phi$ and $Sc_1(X, P, I) \neq \phi$.

Sub-type a inconsistency arises when there is a conflict between type 1 information and P^e that makes the simultaneous satisfaction of conditions 1 and 2 impossible. These kinds of conflicts are found essentially in four types of situations; namely when $x, y, z \in X$ exist such that

$$[\ xPy, yPz, xPz \text{ and } (y, z)P^e(x, z)\ ]$$

$$\text{or } [\ xPy, yPz, xPz \text{ and } (x, y)P^e(x, z)\ ]$$

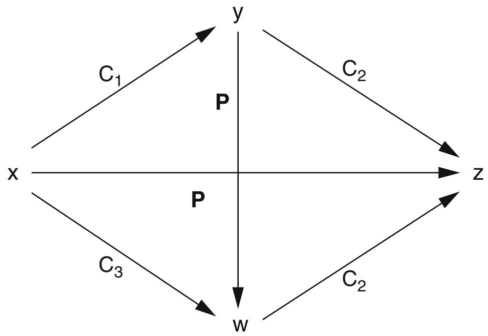


Fig. 11.1 Example of sub-type b inconsistency

$$\text{or } [\ xIy, yPz, xPz \text{ and } (x,z)P^e(y,z)\ ]$$
$$\text{or } [\ xIy, zPy, zPx \text{ and } (z,x)P^e(z,y)\ ].$$

Sub-type b inconsistency arises when there is no conflict between type 1 information and P^e but at least one conflict exists inside P^e that makes satisfying Condition 2 impossible. An example of this type of conflict is (see Fig. 11.1):

$$xPy, xPw, yPz, wPz, xPz, yPw$$
$$(x,y) \in C_1, (y,z) \in C_2$$
$$(x,w) \in C_3, (w,z) \in C_2.$$

In such a case, Condition 2 cannot be respected, because one should have

$$\begin{cases} \mu(x) - \mu(w) > \mu(y) - \mu(z) \ (1) \\ \mu(w) - \mu(z) > \mu(x) - \mu(y) \ (2) \end{cases}$$

which is impossible.

On the other hand, it is easily shown that the following two systems are compatible, that is, there is no conflict between type 1 information and P^e:

$$\begin{cases} \mu(x) - \mu(w) > \mu(y) - \mu(z) \\ \mu(x) - \mu(y) > 0 \\ \mu(x) - \mu(w) > 0 \\ \mu(x) - \mu(z) > 0 \\ \mu(y) - \mu(z) > 0 \\ \mu(w) - \mu(z) > 0 \\ \mu(y) - \mu(w) > 0 \end{cases} \quad \begin{cases} \mu(w) - \mu(z) > \mu(x) - \mu(y) \\ \mu(x) - \mu(y) > 0 \\ \mu(x) - \mu(w) > 0 \\ \mu(x) - \mu(z) > 0 \\ \mu(y) - \mu(z) > 0 \\ \mu(w) - \mu(z) > 0 \\ \mu(y) - \mu(w) > 0 \end{cases}$$

For a detailed study of inconsistency, see [95].

11.6 Consistency Test for Preferential Information

11.6.1 Testing Procedures

Suppose that $X = \{a_1, a_2, \ldots, a_n\}$.

During the interactive questioning process conducted with J, each time that a new judgement is obtained, the consistency of all the responses already provided is tested. This consistency test begins with a pre-test aimed at detecting the (potential) presence of cycles within the relation P and, if no such cycle exists, making a permutation of the elements of X in such a way that, in the matrix of judgements, all of the cells P or C_{ij} will be located above the main diagonal.

When there is no cycle in P, the consistency of type 1 information $\{P, I, ?\}$ is tested as follows:

- If $? \neq \phi$, a linear program named LP-test$_1$ is used;
- if $? = \phi$, rather than linear programming, a method named DIR-test$_1$ is used, which has the advantage of being easily associated with a very simple visualization of an eventual ranking within the matrix of judgements.

When $\{P, I, ?\}$ is consistent, the consistency of type 1+2 information $\{P, I, ?, P^e\}$ is tested with the help of a linear program named LPσ-test$_{1+2}$.

11.6.2 Pre-test of the Preferential Information

The pre-test of the preferential information is based on Property 1. (Evident because $\#X$ is finite.)

Property 1. Let $X^* \subset X$; if $\forall\, x \in X^*, \exists\, y \in X^*$ such that xPy, then $\exists\, x_1, x_2, \ldots, x_p \in X^*$ such that $x_1Px_2P \ldots Px_pPx_1$ (cycle).

The pre-test consists of seeking a permutation $\varphi : \mathbb{N}_{1,n} \rightarrow \mathbb{N}_{1,n}$ such that

$$\forall\, i, j \in \mathbb{N}_{1,n},\ [\, i > j \Rightarrow a_{\varphi(i)}(notP)a_{\varphi(j)}\,].$$

The permutation of the elements of X is made by the algorithm PRETEST, that detects cycles within P and sorts the elements(s) of X.
PRETEST:

1. $s \leftarrow n$;
2. among $a_1, a_2, \ldots, a_s$ find a_i which is not preferred over any other:

   if a_i exists, go to 3.;
   if not, return FALSE ($Sc_1 = \phi$, according to Property 1); finish.

3. permute a_i and a_s;
4. $s \leftarrow s-1$:

   if $s=1$, return TRUE; finish.
   If not, go to 2.

11.6.3 Consistency Test for Type 1 Information

Suppose that PRETEST detected no cycle within P and that the elements of X were renumbered as follows (to avoid the introduction of a permutation in the notation):

$$\forall\ i,j \in \mathbb{N}_{1,n},\ [\ i > j \Rightarrow a_i(notP)a_j\ ].$$

11.6.3.1 Consistency Test for Incomplete (? ≠ ϕ) Type 1 Information

Consider the linear program LP-test$_1$ with variables $x_1, x_2, \ldots, x_n$:

$$\begin{array}{ll} \min x_1 & \\ \text{subject to} & \\ x_i - x_j \geq d_{\text{min}} & \forall\ (a_i, a_j) \in P \\ x_i - x_j = 0 & \forall\ (a_i, a_j) \in I \text{ with } i \neq j \\ x_i \geq 0 & \forall\ i \in \mathbb{N}_{1,n} \end{array}$$

where d_{min} is a positive constant, and the variables $x_1, x_2, \ldots, x_n$ represent the numbers $\mu(a_1), \mu(a_2), \ldots, \mu(a_n)$ that should satisfy Condition 1 so that μ is a type 1 scale.

The objective function min x_1 of LP-test$_1$ is obviously arbitrary. It is trivial that $Sc_1 \neq \phi \Leftrightarrow$ LP-test$_1$ is feasible.

11.6.3.2 Consistency Test for Complete (? = ϕ) Type 1 Information

When $? = \phi$ and the elements of X have been renumbered (after the application of PRETEST), another simple test (DIR-test$_1$) allows one to verify if $P \cup I$ is a complete preorder on X. DIR-test$_1$ is based on Proposition 1 (Proved in [95]).

Proposition 1. *If* $[\ \forall\ i,j \in \mathbb{N}_{1,n}$ *with* $i < j,\ (a_i, a_j) \in P \cup I\ ]$ *then* $P \cup I$ *is a complete preorder on* X *if and only if* $\forall\ i,j \in \mathbb{N}_{1,n}$ *with* $i < j$: $\left[a_iPa_j \Rightarrow \begin{cases} \forall\ s \leq i, \forall\ t \geq j, a_sPa_t \\ \exists\ s : i \leq s \leq j-1 \text{ and } a_sPa_{s+1} \end{cases} \right].$

Proposition 1 means that when the "P cases" of the matrix of judgements forms a "staircase", a ranking exists such that each step of the "staircase" rests, at least partly, on the principal diagonal of the matrix.

11.6.4 Consistency Test for Type 1+2 Information

It would be possible to test the consistency of type 1+2 information with a linear program based on Conditions 1 and 2. However, the more efficient linear program LP-test$_{1+2}$, which includes "thresholds conditions" equivalent to Conditions 1 and 2, is used instead. LP-test$_{1+2}$ is based on Lemma 1 (Proved in [95]).

Lemma 1. *Let* $\mu : X \to \mathbb{R}$. μ *satisfies Conditions 1 and 2 if and only if there exist* Q *"thresholds"* $0 < \sigma_1 < \sigma_2 < \ldots < \sigma_Q$ *that satisfy Conditions 3, 4 and 5.*

Condition 3 $\forall\ (x, y) \in I,\ \mu(x) = \mu(y)$.

Condition 4 $\forall\ i, j \in \mathbb{N}_{1,Q}$ *with* $i \leq j,\ \forall\ (x, y) \in C_{ij},\ \sigma_i < \mu(x) - \mu(y)$.

Condition 5 $\forall\ i, j \in \mathbb{N}_{1,Q-1}$ *with* $i \leq j,\ \forall\ (x, y) \in C_{ij},\ \mu(x) - \mu(y) < \sigma_{j+1}$.

Program LP-test$_{1+2}$ has variables $x_1(= \mu(a_1)), \ldots, x_n(= \mu(a_n)), \sigma_1, \ldots, \sigma_Q$:

$$
\begin{array}{ll}
\min x_1 & \\
\text{subject to} & \\
x_p - x_r = 0 & \forall\ (a_p, a_r) \in I \text{ with } p < r \\
\sigma_j + d_{\min} \leq x_p - x_r & \forall\ i, j \in \mathbb{N}_{1,Q} \text{ with } i \leq j,\ \forall\ (a_p, a_r) \in C_{ij} \\
x_p - x_r \leq \sigma_{j+1} - d_{\min} & \forall\ i, j \in \mathbb{N}_{1,Q-1} \text{ with } i \leq j,\ \forall\ (a_p, a_r) \in C_{ij} \\
d_{\min} \leq \sigma_1 & \\
\sigma_{i-1} + d_{\min} \leq \sigma_i & \forall\ i \in \mathbb{N}_{2,Q} \\
x_i \geq 0 & \forall\ i \in \mathbb{N}_{1,n} \\
\sigma_i \geq 0 & \forall\ i \in \mathbb{N}_{1,Q}
\end{array}
$$

Taking into account Lemma 1, it is trivial that $Sc_{1+2} \neq \phi$ if and only if the linear program LP-test$_{1+2}$, which is based on Conditions 3–5, is feasible.

11.7 Dealing with Inconsistency

When a type 1+2 information $\{P, I, ?, P^e\}$ about X is inconsistent, it is convenient to be able to show J systems of constraints that render his or her judgements inconsistent and modifications of these judgements that would render LPσ-test$_{1+2}$ feasible.

11.7.1 Systems of Incompatible Constraints

Suppose that LP-test$_{1+2}$ is not feasible or, in other words, that the following system is incompatible (variables $x_1 (= \mu(a_1)), \ldots, x_n (= \mu(a_n))$, $\sigma_1, \ldots, \sigma_Q$ nonnegative):

$$\begin{cases} x_p - x_r = 0 & \forall\ (a_p, a_r) \in I \text{ with } \ p < r & \text{(t1)} \\ \sigma_i < x_p - x_r & \forall\ i, j \in \mathbb{N}_{1,Q} \text{ with } \ i \le j,\ \forall\ (a_p, a_r) \in C_{ij} & \text{(t2)} \\ x_p - x_r < \sigma_{j+1} & \forall\ i, j \in \mathbb{N}_{1,Q-1} \text{ with } \ i \le j,\ \forall\ (a_p, a_r) \in C_{ij} & \text{(t3)} \\ 0 < \sigma_1 & & \text{(t4)} \\ \sigma_{i-1} < \sigma_i & \forall\ i \in \mathbb{N}_{2,Q} & \text{(t5)} \end{cases}$$

Conventions:

- $\mathbb{R}^{m \times n}$ is the set of the real matrices with m lines and n columns.
- Matrix $M \in \mathbb{R}^{m \times n}$ is "non-zero" ($M \neq 0$) if at least one of its elements is not null.
- Matrix $M \in \mathbb{R}^{m \times n}$ is positive or null ($M \geq 0$) if all of its elements are positive or null.

The system of incompatible constraints can be written in the matrix format as follows:

$$\begin{cases} C \cdot Z > 0 \text{ (by grouping constraints (t2))} \\ D \cdot Z > 0 \text{ (by grouping constraints (t3))} \\ E \cdot Z > 0 \text{ (by grouping constraints (t4) and (t5))} \\ B \cdot Z = 0 \text{ (by grouping constraints (t1))} \end{cases}$$

where

$$Z = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ \sigma_1 \\ \sigma_2 \\ \vdots \\ \sigma_Q \end{pmatrix}$$

$$\begin{aligned} C &\in \mathbb{R}^{p_1 \times (n+Q)} \text{(where } p_1 \text{ is the number of constraints (t2))} \\ D &\in \mathbb{R}^{p_2 \times (n+Q)} \text{(where } p_2 \text{ is the number of constraints (t3))} \\ E &\in \mathbb{R}^{p_3 \times (n+Q)} \text{(where } p_3 \text{ is the number of constraints (t4) and (t5))} \\ B &\in \mathbb{R}^{r \times (n+Q)} \text{(where } r \text{ is the number of constraints (t1))} \end{aligned}$$

Note: if $r = 0$, one could consider that $B = 0 \in \mathbb{R}^{1\times(n+Q)}$ without losing generality.

Let A be the matrix $\begin{bmatrix} C \\ D \\ E \end{bmatrix} \in \mathbb{R}^{p\times(n+Q)}$ $(p = p_1 + p_2 + p_3)$. The system of incompatible constraints can be written more simply as

$$S \begin{cases} A \cdot Z > 0 \text{ (by grouping constraints (t2), (t3), (t4) and (t5))} \\ B \cdot Z = 0 \text{ (by grouping constraints (t1)).} \end{cases}$$

In order to detect incompatibilities between the constraints (t1), (t2), (t3), (t4) and (t5) and propose eventual corrections, we apply Proposition 2 (Proved in [95]), which is a corollary of Mangasarian's [150] version of the *Theorem of the Alternative*.

Proposition 2. *The system* $S \quad \{A \cdot Z > 0; B \cdot Z = 0\}$ *admits a solution* $Z \in \mathbb{R}^{(n+Q)\times 1}$ *or there exists* $Y \in \mathbb{R}^{p\times 1}, V, W \in \mathbb{R}^{r\times 1}$ *with* $Y \neq 0, Y \geq 0, V \geq 0, W \geq 0$ *such that* ${}^tA \cdot Y +{}^t B \cdot (V - W) = 0$ *and* $\forall i \in \mathbb{N}_{1,r},\ V_i \cdot W_i = 0$ *but never both.*

The interest of Proposition 2 is that vectors Y, V and W have positive or null components, thus making it compatible with linear programming (see Sects. 11.7.3 and 11.7.4)

11.7.2 Example 1

Suppose that $X = \{a_1, a_2, a_3, a_4\}$ and that J has formulated the following judgements:

- $P = \{(a_1, a_2), (a_1, a_3), (a_2, a_3), (a_3, a_4)\}$
- $(a_1, a_2) \in C_1,\ (a_1, a_3) \in C_4,\ (a_2, a_3) \in C_2,\ (a_3, a_4) \in C_2.$

Suppose that J also judges that a_2Pa_4 and that $(a_2, a_4) \in C_3$. LP-test$_1$ is feasible: the judgements are compatible with a ranking. LP-test$_{1+2}$ is not feasible: the software informs J that his or her judgements are "inconsistent".

Suppose now that J confirms his or her judgements. One must then have:

$$\begin{array}{llllll}
\sigma_1 < x_1 - x_2 & (1) & x_1 - x_2 < \sigma_2 & (2) & 0 < \sigma_1 & (11) \\
\sigma_2 < x_2 - x_3 & (3) & x_2 - x_3 < \sigma_3 & (4) & \sigma_1 < \sigma_2 & (12) \\
\sigma_2 < x_3 - x_4 & (5) & x_3 - x_4 < \sigma_3 & (6) & \sigma_2 < \sigma_3 & (13) \\
\sigma_3 < x_2 - x_4 & (7) & x_2 - x_4 < \sigma_4 & (8) & \sigma_3 < \sigma_4 & (14) \\
\sigma_4 < x_1 - x_3 & (9) & x_1 - x_3 < \sigma_5 & (10) & \sigma_4 < \sigma_5 & (15) \\
 & & & & \sigma_5 < \sigma_6 & (16)
\end{array}$$

or, in matrix format (which one can denote as $A \cdot Z > 0$):

$$\begin{pmatrix} 1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{pmatrix} > 0$$

Since it is known, according to Proposition 2, that the system has no solution, there necessarily exists $Y \in \mathbb{R}^{16\times 1} (Y \neq 0, Y \geq 0)$ such that ${}^tA \cdot Y = 0$. Thus, positive or null (but not all null) real numbers $y_1, y_2, \ldots, y_{16}$ exist such that $\sum_{i=1}^{16} y_i \cdot Col_i = 0$ (where Col_i is the column i of the matrix ${}^tA$).

In this simple example, one can see that it is enough to make $y_2 = y_5 = y_8 = y_9 = 1$ and $y_1 = y_3 = y_4 = y_6 = y_7 = y_{10} = y_{11} = y_{12} = y_{13} = y_{14} = y_{15} = y_{16} = 0$:

$$1 \cdot \underbrace{\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{Col_2} +1 \cdot \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{Col_5} +1 \cdot \underbrace{\begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{Col_8} +1 \cdot \underbrace{\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}}_{Col_9} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

These four vectors correspond to the four constraints (2), (5), (8) and (9) above:

$$\left.\begin{array}{l} \sigma_4 > x_2 - x_4 \ (8) \\ x_1 - x_3 > \sigma_4 \ (9) \end{array}\right\} \Rightarrow x_1 - x_3 > x_2 - x_4 \qquad (*)$$

Fig. 11.2 Example of incompatibility between (*) and (**)

Diff.	Couples		Couples	Diff.
strong	a1 - a3	>	a2 - a4	moderate
weak	a3 - a4	>	a1 - a2	very weak

$$\left.\begin{array}{l}\sigma_2 > x_1 - x_2 \ (2) \\ x_3 - x_4 > \sigma_2 \ (5)\end{array}\right\} \Rightarrow x_3 - x_4 > x_1 - x_2 \qquad (**)$$

(*) and (**) bring to the contradiction $x_1 - x_4 > x_1 - x_4$. The incompatibility between (*) and (**) is presented in M-MACBETH as shown in Fig. 11.2.

Note that the problem disappears if

$(a_1, a_3) \in C_3$ instead of C_4 ((*) disappears)
or $(a_2, a_4) \in C_4$ instead of C_3 ((*) disappears)
or $(a_3, a_4) \in C_1$ instead of C_2 ((**) disappears)
or $(a_1, a_2) \in C_2$ instead of C_1 ((**) disappears).

Note also that the inconsistency would not be eliminated for any modification of the judgement "$(a_2, a_3) \in C_2$".

If J confirms the judgement "$(a_2, a_4) \in C_3$", M-MACBETH calculates the different possibilities (four in example 1) that J can follow to make his or her judgements consistent with a "minimal" number of changes of category (one in Example 1). (We will specify in Sect. 11.7.4 the meaning of this notion).

In M-MACBETH, the "suggestions" of changes are presented (graphically) in the matrix of judgements. They are:

- to replace the judgement $(a_1, a_3) \in C_4$ with the judgement $(a_1, a_3) \in C_3$
- or to replace the judgement $(a_2, a_4) \in C_3$ with the judgement $(a_2, a_4) \in C_4$
- or to replace the judgement $(a_3, a_4) \in C_2$ with the judgement $(a_3, a_4) \in C_1$
- or to replace the judgement $(a_1, a_2) \in C_1$ with the judgement $(a_1, a_2) \in C_2$.

11.7.3 Identifying Constraints which Cause Inconsistency

Let us detail the various stages of our search for "suggestions". The first step consists of determining the constraints (t1), (t2) and (t3) which are "the origin of the incompatibilities" present in the system

$$S\left\{\begin{array}{l}A \cdot Z > 0 \\ B \cdot Z = 0\end{array}\right. \qquad \text{(see Sect. 11.7.1)}$$

We consider that a constraint is "at the origin of an incompatibility" when it is part of a system S' that

- is a “sub-system” of S,
- is incompatible,
- does not contain any incompatible “sub-system”.

Mathematically, this idea can be represented by Definition 7.

Definition 7. An incompatible elementary system (SEI) is a system

$$S' \begin{cases} A' \cdot Z > 0 \\ B' \cdot Z = 0 \end{cases}$$

such that

1. $A' \in \mathbb{R}^{p' \times (n+Q)}$ is a sub-matrix of A, and $B' \in \mathbb{R}^{r' \times (n+Q)}$ is a sub-matrix of B;
2. S' is incompatible;
3. If $\begin{cases} A'' \in \mathbb{R}^{p'' \times (n+Q)} \text{ is a sub-matrix of } A', \\ B'' \in \mathbb{R}^{r'' \times (n+Q)} \text{ is a sub-matrix of } B', \\ p'' + r'' < p' + r' \end{cases}$ then $\begin{cases} A'' \cdot Z > 0 \\ B'' \cdot Z = 0 \end{cases}$ is compatible.

However, our goal is not to determine all the SEI that could be extracted from the constraints using LPσ-test$_{1+2}$. We just want to find all of the judgements of the type $(a_s, a_t) \in C_{ij}$ that “generate” an incompatibility. In Sect. 11.7.4.3, we will explain how we use these judgements.

We know that an inconsistency occurs when the system

$$S \begin{cases} A \cdot Z > 0 \\ B \cdot Z = 0 \end{cases}$$

is incompatible; that is, $\exists Y \in \mathbb{R}^p$ and $V, W \in \mathbb{R}^r$ such that

$$\begin{cases} {}^tA \cdot Y +^t B \cdot (V - W) = 0 \\ Y \geq 0, \ V \geq 0, \ W \geq 0 \\ \forall \ i \in \mathbb{N}_{1,r}, \ V_i \cdot W_i = 0 \\ \exists \ i_0 \in \mathbb{N}_{1,p} \text{ such that } Y_{i_0} \neq 0 \end{cases}$$

In such a case, if $i_0 \leq p_1 + p_2$, where p_1 is the number of constraints (t2) and p_2 is the number of constraints (t3) (see Sect. 11.7.1), a constraint of the type $x_s - x_t < \sigma_j$ or $x_s - x_t > \sigma_j$ will correspond to S.

Consider, then, the system (with $i \leq p_1 + p_2$):

$$\text{Syst-}Y_i \begin{cases} {}^tA \cdot Y +^t B \cdot (V - W) = 0 \\ Y_i = 1 \end{cases}$$

If Syst-Y_i is compatible, for one of its solutions it corresponds to a system of incompatible constraints (t1), (t2), (t3), (t4) and (t5) where at least one constraint (that which corresponds to $Y_i = 1$) is of the type $x_s - x_t < \sigma_j$ or $x_s - x_t > \sigma_j$ and is part of a SEI. If Syst-Y_i is incompatible, the constraint that corresponds to Y_i is not part of any SEI.

To find all of the constraints (t2) and (t3) which are part of a SEI, it is sufficient to study the compatibility of all of the systems Syst-Y_i, for $i = 1, 2, \ldots, p_1 + p_2$.

We will proceed in a similar way, using the systems Syst-V_i and Syst-W_i, to find all of the constraints (t1) which are part of a SEI:

$$\text{Syst-}V_i \begin{cases} {}^tA \cdot Y + {}^t B \cdot (V - W) = 0 \\ W_i = 0 \\ V_i = 1 \end{cases}$$

and

$$\text{Syst-}W_i \begin{cases} {}^tA \cdot Y + {}^t B \cdot (V - W) = 0 \\ V_i = 0 \\ W_i = 1 \end{cases}$$

It is not necessary to examine all of the systems Syst-Y_i, Syst-V_i and Syst-W_i:

- If Syst-Y_i is compatible and has the solution Y, V, W, then
  - $\forall\, j > i$ such that $Y_j \neq 0$, Syst-Y_i is compatible;
  - $\forall\, j \in \mathbb{N}_{1,r}$ such that $V_j \neq 0$, Syst-V_i is compatible;
  - $\forall\, j \in \mathbb{N}_{1,r}$ such that $W_j \neq 0$, Syst-W_i is compatible.
- If Syst-V_i is compatible and has the solution Y, V, W, then
  - $\forall\, j > i$ such that $V_j \neq 0$, Syst-V_i is compatible;
  - $\forall\, j \in \mathbb{N}_{1,r}$ such that $W_j \neq 0$, Syst-W_i is compatible.
- If Syst-W_i is compatible and has the solution Y, V, W, then
  - $\forall\, j > i$ such that $W_j \neq 0$, Syst-W_i is compatible.

It is for this reason that a "witness-vector" $T \in \mathbb{N}^{p_1+p_2+2\cdot r}$ must be used, initially null, updated as follows:

- For any solution Y, V, W of a system Syst-Y_i, Syst-V_i or Syst-W_i do
  - $\forall\, j \in \mathbb{N}_{1,p_1+p_2},\ [\, Y_j \neq 0 \Rightarrow T_j = 1 \,]$
  - $\forall\, j \in \mathbb{N}_{1,r},\ [\, V_j \neq 0 \Rightarrow T_{p_1+p_2+j} = 1 \,]$
  - and $[\, W_j \neq 0 \Rightarrow T_{p_1+p_2+r+j} = 1 \,]$.

To find the interesting pairs, the compatibility of at most $p_1 + p_2 + 2r$ systems should be studied. The general algorithm to seek equations (t1) and inequalities (t2) and (t3) that are part of a SEI is the following:

- $T = (0, 0, \ldots, 0)$
- for $i = 1, 2, \ldots, p_1 + p_2$ do:
  - $T_i = 0$,
  - then if Syst-Y_i compatible and Y, V, W solution of Syst-Y_i

    then update T

- for $i = 1, 2, \ldots, r$ do:
  - if $T_{p_1+p_2+i} = 0$,
  - then if Syst-V_i compatible and Y, V, W solution of Syst-V_i

    then update T
- for $i = 1, 2, \ldots, r$ do:
  - if $T_{p_1+p_2+r+i} = 0$,
  - then if Syst-W_i compatible and Y, V, W solution of Syst-W_i

    then update T.

In this way one obtains the set of all of the equations and inequalities that make up the SEI.

11.7.4 Augmentation: Reduction in a Judgement with p Categories

11.7.4.1 Preliminaries

Notation:

- Judgement $(x, y) \in C_{ij}$ will be represented by element (x, y, i, j) of $X \times X \times \mathbb{N}_{1,Q} \times \mathbb{N}_{1,Q}$.
- Judgement $(x, y) \in I$ will be represented by element $(x, y, 0, 0)$ of $X \times X \times \mathbb{N} \times \mathbb{N}$.

Definition 8. A reduction in judgement (s, t, i, j) with p categories $(1 \leq p \leq Q + i)$ is the replacement of this judgement

- by the judgement $(s, t, i - p, i - p)$ if $i \geq p$
- by the judgement $(t, s, p - i, p - i)$ if $i < p$.

Definition 9. An augmentation of the judgement (s, t, i, j) with p categories $(1 \leq p \leq Q - j)$ is the replacement of this judgement by the judgement $(s, t, j + p, j + p)$.

Definition 10. A change of judgement (s, t, i, j) with p categories is an augmentation or a reduction of the judgement with p categories.

Comment: It is evident that one obtains the same final judgement as a result of "1 reduction of a judgement with p categories" or the "p successive reductions of a category of 1 judgement".

Convention: A "change in judgement (s, t, i, j) with p categories" will be represented by $(s, t, i, j, p) \in X \times X \times \mathbb{N}_{1,Q} \times \mathbb{N}_{1,Q} \times \mathbb{Z}$ (augmentation if $p > 0$, reduction if $p < 0$).

11.7.4.2 Exploitation of the Constraints of SEI

Let us recall from end of Sect. 11.7.3 (T is the "witness-vector") that

- if $T_i > 0$, it has a corresponding constraint (t2) or (t3) or (t1) that is part of an SEI;
- if $T_i = 0$, it has no corresponding constraint that is part of an SEI.

These variables, then, provide us with an indication as to the future "modification" to be made to the judgements associated with these constraints. Indeed, suppose that $T_i > 0$:

(a) if $1 \leq i \leq p_1$, a constraint $\sigma_u < x_s - x_t$ which is part of an SEI corresponds to variable T_i; if a change in its judgement $(s, t, \ldots, \ldots)$ can help to eliminate the SEI, it ensures that it will be a "reduction" (evident).
(b) if $p_1 + 1 \leq i \leq p_1 + p_2$, a constraint $x_s - x_t < \sigma_u$ which is part of an SEI corresponds to variable T_i; if a change in its judgement $(s, t, \ldots, \ldots)$ can help to eliminate the SEI, it ensures that it will be an "augmentation" (evident).
(c) if $p_1 + p_2 + 1 \leq i \leq p_1 + p_2 + r$, a constraint $x_s - x_t = 0$ which is part of an SEI corresponds to variable T_i; if a change in its judgement $(s, t, 0, 0)$ can help to eliminate the SEI, it ensures that it will be a "reduction".
(d) if $p_1 + p_2 + r + 1 \leq i \leq p_1 + p_2 + 2r$, a constraint $x_s - x_t = 0$ which is part of an SEI corresponds to variable T_i; if a change in its judgement $(s, t, 0, 0)$ can help to eliminate the SEI, it ensures that it will be an "augmentation" (proof similar to that of (c)).

Proof of (c):

Being $h = i - (p_1 + p_2)$, one knows (by the definition of T_i) that $\exists\ Y \in \mathbb{R}^p$, $\exists\ V, W \in \mathbb{R}^r$ with $Y \geq 0$, $V \geq 0$, $W \geq 0$, $Y \neq 0$, $V_h \neq 0$ and $W_h = 0$ such that ${}^t(A') \cdot Y +{}^t (B') \cdot (V - W) = 0$ or, if one notes $LineB_j$ the jth line of B',

$$ {}^t(A') \cdot Y + {}^tLineB_h \cdot V_h + \sum_{\substack{j=1 \\ j \neq h}}^{r} {}^tLineB_j \cdot V_j - \sum_{\substack{j=1 \\ j \neq h}}^{r} {}^tLineB_j \cdot W_j = 0 $$

(because $W_h = 0$).

The corresponding SEI $\begin{cases} A' \cdot Z > 0 \\ B' \cdot Z = 0 \end{cases}$ can be written $\begin{cases} A' \cdot Z > 0 \\ x_s - x_t = 0 \\ B'' \cdot Z = 0, \end{cases}$ where $B'' = \begin{bmatrix} LineB_1 \\ \vdots \\ LineB_{h-1} \\ LineB_{h+1} \\ \vdots \\ LineB_r \end{bmatrix}$ (the matrix B' without line $LineB_h$).

If one considers an "augmentation" of judgement $(s, t, 0, 0)$, the constraint $x_s - x_t = 0$ would be replaced by the constraint $x_s - x_t > 0$. The new system $\begin{cases} A' \cdot Z > 0 \\ x_s - x_t > 0 \\ B'' \cdot Z = 0 \end{cases}$ can be written $\begin{cases} A'' \cdot Z > 0 \\ B'' \cdot Z = 0, \end{cases}$ where $A'' = \begin{bmatrix} A' \\ LineB_h \end{bmatrix}$ (the matrix A' "augmented" with line $LineB_h$).

The system is still incompatible; indeed, if one poses

- $Y' = (Y_1, Y_2, \ldots, Y_p, V_h) \in \mathbb{N}^{p+1}$
- $V' = (V_1, \ldots, V_{h-1}, V_{h+1}, \ldots, V_r) \in \mathbb{N}^{r-1}$
- $W' = (W_1, \ldots, W_{h-1}, W_{h+1}, \ldots, W_r) \in \mathbb{N}^{r-1}$.

$$^t(A') \cdot Y + {}^tLineB_h \cdot V_h + \sum_{\substack{j=1 \\ j \neq h}}^{r} {}^tLineB_j \cdot V_j - \sum_{\substack{j=1 \\ j \neq h}}^{r} {}^tLineB_j \cdot W_j = 0$$

can be written: $^t(A'') \cdot Y' +^t (B'') \cdot (V' - W') = 0$, where $Y' \neq 0$ (since $Y \neq 0$), which proves the incompatibility of the system.

Each "suggestion" of a potential change $(T_i = 1)$ of a judgement $(s, t, \ldots, \ldots)$ can thus be stored in a vector S of $\mathbb{N}^4$ where

$$S_1 = s$$

$$S_2 = t$$

$$S_3 = \begin{cases} 1 \text{ if } \exists\, i \in \mathbb{N}_{1,p_1} \cup \mathbb{N}_{p_1+p_2+1,p_1+p_2+r} \text{ such that } T_i = 1 \\ \quad \text{(reduction)} \\ 0 \text{ otherwise} \end{cases}$$

$$S_4 = \begin{cases} 1 \text{ if } \exists\, i \in \mathbb{N}_{p_1+1,p_1+p_2} \cup \mathbb{N}_{p_1+p_2+r+1,p_1+p_2+2r} \text{ such that} \\ \quad T_i = 1 \quad \text{(augmentation)} \\ 0 \text{ otherwise} \end{cases}$$

We will denote by *PreSugg* the set of these "pre-suggestions". In the case of example 1 (see Sect. 11.7.3) one has

$$PreSugg = \{(a_1, a_3, 1, 0), (a_3, a_4, 1, 0), (a_1, a_2, 0, 1), (a_2, a_4, 0, 1)\}.$$

11.7.4.3 Search for Suggestions

Definition 11. *Changing judgements by m categories* is any set $Modif_m$ of the form $Modif_m = \{(s_1, t_1, i_1, j_1, p_1), (s_2, t_2, i_2, j_2, p_2), \ldots, (s_u, t_u, i_u, j_u, p_u) \mid \forall v \in \mathbb{N}_{1,u}, (s_v, t_v, i_v, j_v, p_v)$ is a change of judgement (s_v, t_v, i_v, j_v) with p_v categories$\}$ such that $\sum_{v=1}^{u} |p_v| = m$

Within Example 1, $\{(a_1, a_2, 1, 1, 2), (a_3, a_4, 2, 2, -1)\}$ is a "change of judgements with 3 categories", which consists of

- to replace the judgement $(a_1, a_2) \in C_1$ with the judgement $(a_1, a_2) \in C_3$ (augmentation of 2 categories)
- to replace the judgement $(a_3, a_4) \in C_2$ with the judgement $(a_3, a_4) \in C_1$ (reduction of 1 category)

Notation: the set of "judgement changes with m categories" which renders the judgements consistent will be denoted by $Sugg_m$.

Within Example 1,

- $\{(a_1, a_2, 1, 1, 2), (a_3, a_4, 2, 2, -1)\} \in Sugg_3$
- $\{(a_1, a_3, 4, 4, -1)\}$, $\{(a_3, a_4, 2, 2, -1)\}$, $\{(a_1, a_2, 1, 1, 1)\}$ and $\{(a_2, a_4, 3, 3, 1)\} \in Sugg_1$,

these are the 4 changes suggested in Sect. 11.7.3.

Once the PreSugg group is determined, the third step is to:

- determine the "minimum number of changes" (some possibly successive) necessary to render the judgements consistent;
- determine all of the combinations of such "minimal" changes.

More rigorously, this means

- find $m_0 = \min \{m \in \mathbb{N}^* | Sugg_m \neq \emptyset\}$
- clarify $Sugg_m$

In Example 1, we have already seen that $m_0 = 1$ (since $Sugg_1 \neq \emptyset$).

We will proceed as follows for all cases of inconsistency (see Fig. 11.3).

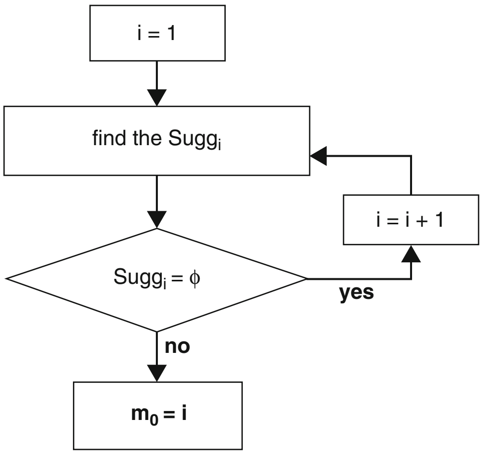


Fig. 11.3 Procedure for all cases of inconsistency

At each step i,

- the set of all "judgement changes of i categories", built on the basis of element PreSugg are considered;
- for each of the elements in this group:
  - carry out the modifications included in the selected item;
  - test the consistency of the new matrix of judgements; if it is consistent, store the element in $Sugg_i$;
  - restore the matrix to the initial judgements.

It is worth mentioning that we consider the possibility of changing a judgement by several categories.

This algorithm is always convergent since one can always give consistent judgements in a finite number of changes.

We emphasize that in practice, the cases of inconsistency that require more than 2 "changes of 1 category" are almost non-existent. The main reason being that any change in judgement that generates an inconsistency is immediately announced to J, who must then confirm or cancel his or her judgement.

This procedure allows one to avoid

- coarse errors of distraction (by cancelling the judgement);
- the "accumulation" of inconsistencies since, if J confirms his or her judgement, suggestions of changes that will eliminate the inconsistency are made.

11.7.5 Example 2

Suppose that $X = \{a_1, a_2, a_3, a_4\}$ and that J has formulated the following consistent judgements:

- $P = \{(a_1, a_2), (a_1, a_3), (a_2, a_3), (a_3, a_4)\}$
- $(a_1, a_2) \in C_1, (a_1, a_3) \in C_4, (a_2, a_3) \in C_2, (a_3, a_4) \in C_3$

Suppose that J adds that $a_2 P a_4$ and that $(a_2, a_4) \in C_3$: M-MACBETH informs J that his or her judgements are "inconsistent".

If J confirms the judgement $(a_2, a_4) \in C_3$, M-MACBETH will display the message: "Inconsistent judgements: MACBETH has found 6 ways to render the judgements matrix consistent with 2 category changes."

This time, it will be necessary to make at least 2 "changes of 1 category" to render the judgements consistent; there are 6 distinct combinations of such changes. Each of these 6 suggestions is presented graphically (see Fig. 11.4) within the table of judgements, accompanied by SEI which, moreover, shows why the suggestions made eliminate this incompatibility: Fig. 11.4 presents the first of six suggestions.

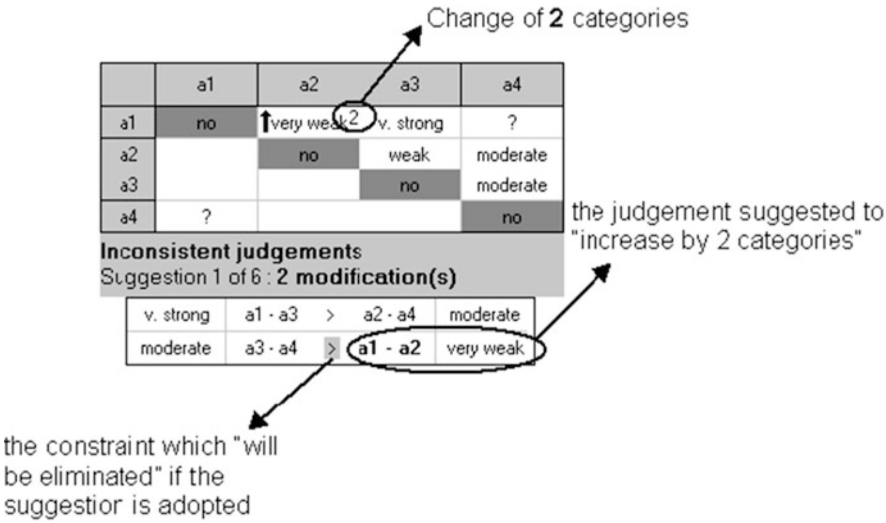


Fig. 11.4 Suggestion of change to resolve inconsistency

11.8 The MACBETH Scale

11.8.1 Definition of the MACBETH Scale

Suppose that $Sc_{1+2} \neq \phi$ and $a_1(P \cup I)a_2 \dots a_{n-1}(P \cup I)a_n$. The linear program LP-MACBETH with variables $x_1, \dots, x_n, \sigma_1, \dots, \sigma_Q$ is therefore feasible:

$$
\begin{array}{lll}
\min x_1 & & \\
\text{subject to} & & \\
x_p - x_r = 0 & \forall\ (a_p, a_r) \in I \text{ with } p < r & \text{(t1)} \\
\sigma_i + \frac{1}{2} \le x_p - x_r & \forall\ i, j \in N_{1,Q} \text{ with } i \le j, \forall (a_p, a_r) \in C_{ij} & \text{(t2')} \\
x_p - x_r \le \sigma_{j+1} - \frac{1}{2} & \forall\ i, j \in N_{1,Q-1} \text{ with } i \le j, \forall (a_p, a_r) \in C_{ij} & \text{(t3')} \\
\sigma_1 = \frac{1}{2} & & \text{(t4')} \\
\sigma_{i-1} + 1 \le \sigma_i & \forall\ i \in N_{2,Q} & \text{(t5')} \\
x_i \ge 0 & \forall\ i \in N_{1,n} & \\
\sigma_i \ge 0 & \forall\ i \in N_{1,Q} &
\end{array}
$$

Definition 12. Any function $EchMac : X \to \mathbb{R}$ such that $\forall\ i \in \mathbb{N}_{1,n}$, $EchMac(a_i) = x_i^*$—where $(x_1^*, \dots, x_n^*)$ is an optimal solution of LP-MACBETH—is called a *basic MACBETH scale*.

Definition 13. $\forall\ a \in \mathbb{R}_+^*, \forall\ b \in \mathbb{R}$ with $(a, b) \neq (1, 0)$, $a \cdot EchMac + b$ is a *transformed MACBETH scale*.

	a1	a2	a3	a4	a5	a6
a1	no	very weak	weak	moderate	moderate	strong
a2		no	very weak	moderate	moderate	moderate
a3			no	weak	moderate	moderate
a4				no	very weak	very weak
a5					no	very weak
a6						no

	Macbeth basic
a1	**8.00**
a2	6.50
a3	5.00
a4	2.00
a5	1.00
a6	**0.00**

Fig. 11.5 Matrix of judgements and basic MACBETH scale

11.8.2 Discussing the Uniqueness of the Basic MACBETH Scale

Nothing guarantees that a LP-MACBETH optimal solution is unique. For example, consider the matrix of judgements and the basic MACBETH scale shown is Fig. 11.5.

One can verify that, $\forall\, x \in [6, 7]$, $(8, x, 5, 2, 1, 0)$ is still an optimal solution of LP-MACBETH. Thus, a basic MACBETH scale is not necessarily unique. As long as the MACBETH scale is interpreted as a technical aid whose purpose is to provide the foundation for a discussion with J, this does not constitute a true problem. However, we have observed that in practice decision makers often adopt the MACBETH scale as the final scale. It is, therefore, convenient to guarantee the uniqueness of the MACBETH scale. This is obtained technically, as follows (where S_{mac} is the group of the constraints of LP-MACBETH):

Step (1) solution of LP-MACBETH

$\rightarrow$ optimal solution $x_1, x_2, \ldots, x_n$
$\rightarrow$ $\mu(a_1) = x_1, \mu(a_n) = x_n = 0$ (remark: $\mu(a_1)$ is unique)

Step (2) for $i = 2$ to $n - 1$

to solve $\max x_i$ under $\begin{cases} S_{mac} \\ x_1 = \mu(a_1), \ldots, x_{i-1} = \mu(a_{i-1}) \end{cases}$
$\rightarrow$ optimal solution $x_1, x_2, \ldots, x_n$
$\rightarrow$ $xmax = x_i$

to solve $\min x_i$ under $\begin{cases} S_{mac} \\ x_1 = \mu(a_1), \ldots, x_{i-1} = \mu(a_{i-1}) \end{cases}$
$\rightarrow$ optimal solution $x_1, x_2, \ldots, x_n$
$\rightarrow$ $xmin = x_i$

$$\mu(a_i) = \frac{xmin + xmax}{2}$$

Thus,

- to calculate $\mu(a_2)$, the variable x_1 is "fixed" to the value $\mu(a_1)$, the minimum and maximum values of x_2 are calculated and the average of the two results is taken as the value of $\mu(a_2)$;
- to calculate $\mu(a_3)$, the variable x_1 is "fixed" to the value of $\mu(a_1)$, the variable x_2 is "fixed" to the value of $\mu(a_2)$, the minimum and maximum values of x_3 are calculated and the average of the two values is taken as the value of $\mu(a_3)$;
- etc.

This method guarantees that $\mu(a_1), \mu(a_2), \ldots \mu(a_n)$ are unique for a given preferential information $\{P, I, ? = \phi, P^e\}$. It permits us to speak of "the" basic MACBETH scale, instead of "one" MACBETH scale.

11.8.3 Presentation of the MACBETH Scale

The MACBETH scale that corresponds to $\{P, I, ? = \phi, P^e\}$ consistent information is represented in two ways in M-MACBETH: a table and a "thermometer". In the example in Fig. 11.6, the transformed MACBETH scale represented in the thermometer was obtained by imposing the values of the elements d and c as 100 and 0 respectively.

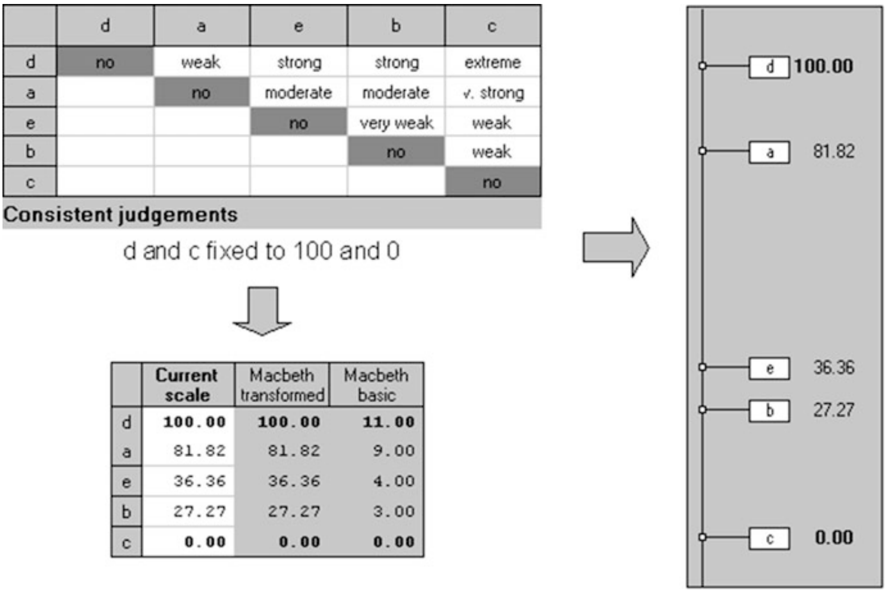


Fig. 11.6 Representations of the MACBETH scale

Even though the values attributed to c and d are fixed, in general an infinite number of scales that satisfy Conditions 1 and 2 exist. It is, thus, necessary to allow J to, should he or she want to, modify the values suggested. This is the subject of the Sect. 11.9.

11.8.4 Determining by Hand the Basic MACBETH Scale

In the case of small matrices of judgements, the basic MACBETH scale can be determined by hand.

Let us firstly present two alternative and equivalent formulations of the linear program LP-MACBETH (see Sect. 11.8.1): LP-MACBETH2008 [50] and LP-MACBETH2011 [55, 57].

1. LP-MACBETH2008 with variables $x_1, \cdots, x_n, \sigma_1, \cdots, \sigma_Q$

$$
\begin{aligned}
&\min \quad (x_1 - x_n) && \\
&\text{s. t. } x_p - x_r = 0 && \forall (a_p, a_r) \in \mathbb{I} \text{ with } p < r && (t_1)\\
&\sigma_i + \frac{1}{2} \leq x_p - x_r && \forall i, j \in \mathbb{N}_{1,Q} \text{ with } i \leq j, \ \forall (a_p, a_r) \in C_{ij} && (t_2')\\
&x_p - x_r \leq \sigma_{j+1} - \frac{1}{2} && \forall i, j \in \mathbb{N}_{1,Q-1} \text{ with } i \leq j, \ \forall (a_p, a_r) \in C_{ij} && (t_3')\\
&\sigma_1 = \frac{1}{2} && && (t_4')\\
&\sigma_{i-1} + 1 \leq \sigma_i && \forall i \in \mathbb{N}_{2,Q} && (t_5')\\
&x_n = 0 && && (t_6)\\
&x_i \geq 0 && \forall i \in \mathbb{N}_{1,n} &&\\
&\sigma_i \geq 0 && \forall i \in \mathbb{N}_{1,Q} &&
\end{aligned}
$$

2. LP-MACBETH2011 with variables $x_1, \cdots, x_n$

$$
\begin{aligned}
&\min \quad (x_1 - x_n) && \\
&\text{s. t. } x_p - x_r = 0 && \forall (a_p, a_r) \in \mathbb{I} \text{ with } p < r && (t_1)\\
&x_n = 0 && && (t_6)\\
&x_p - x_r \geq i && \forall i, j \in \mathbb{N}_{1,Q} \text{ with } i \leq j, \ \forall (a_p, a_r) \in C_{ij} && (t_7)
\end{aligned}
$$

$$x_p - x_r \geq x_k - x_m + i - j' \qquad \begin{array}{r} \forall i, j, i', j' \in \mathbb{N}_{1,Q} \text{ with} \\ i \leq j, i' \leq j' \text{ and} \\ i > j', \forall (a_p, a_r) \in C_{ij}, \\ \forall (a_k, a_m) \in C_{i',j'} \end{array} \qquad (t_8)$$

$$x_i \geq 0 \qquad \forall i \in \mathbb{N}_{1,n}$$

Indeed, it is easy to prove that :

$$\left[\exists \sigma_1, \cdots, \sigma_Q \text{ such that } (x_1, \cdots, x_n) \text{ is solution of } (t_1),\ (t'_2),\ (t'_3),\ (t'_4),\ (t'_5) \text{ and } (t'_6)\right]$$
$$\text{if and only if}$$
$$[(x_1, \cdots, x_n) \text{ is solution of } (t_1),\ (t_6),\ (t_7),\ \text{and } (t_s)]$$

If $[\forall i, j \in \mathbb{N}_{1,Q} \text{with} i < j, Cij = \emptyset]$ (i.e., there is no hesitation between two categories for any of the judgements elicited), the constraints (t_7) and (t_8) can be written simply as:

$$\begin{array}{r} \forall (a_p, a_r) \in C_i \text{and} \forall (a_k, a_m) \in C_{i'} \text{with} 0 \leq i < i' \leq Q : \\ x_p - x_r \geq x_k - x_m + i - i' \end{array} \qquad (t_9)$$

Consider the consistent matrix of judgements (with $n = 5$ and $Q = 6$) in Fig. 11.7. On the basis of constraint (t_9), the corresponding basic MACBETH scale can be determined by hand as follows:

As $x_5 = 0$ [constraint (t_6)], one only needs to determine the four “elementary differences”

$$x_1 - x_2, x_2 - x_3, x_3 - x_4 \text{and} x_4 - x_5.$$

- $\forall (a_i, a_{i+1}) \in C_1 \text{with} i \in \{1, 2, 3, 4\}, \text{take} x_i - x_{i+1} = 1$
  Here: $x_3 - x_4 = 1$

	a1	a2	a3	a4	a5
a1	no	weak	strong	strong	extreme
a2		no	moderate	moderate	v. strong
a3			no	very weak	weak
a4				no	weak
a5					no

Fig. 11.7 Consistent matrix of MACBETH qualitative judgements with no hesitation

- $\forall i, j \in \{1, 2, 3, 4, 5\}$ with $i < j$ and $j - i \geq 2$, calculate the difference $x_i - x_j$ whenever it is possible.

  Here, i.e. for the matrix in Fig. 11.7 no difference $x_i - x_j$ with $i < j$ and $j - i \geq 2$ can be calculated at this stage.
- If constraint (t_9) is not respected, modify the values of the elementary differences $x_i - x_{i+1}$.

  Here, the constraint is obviously respected at this stage.
- $\forall (a_i, a_{i+1}) \in C_2$ with $i \in \{1, 2, 3, 4\}$, take $x_i - x_{i+1} = \alpha_1 + 1$ where $\alpha_1 = \max\{x_i - x_j | i < j \text{and} (a_i, a_j) \in C_1\}$.

  Here, $\alpha_1 = 1$ and we take : $x_1 - x_2 = 2$ and $x_4 - x_5 = 2$.
- $\forall i, j \in \{1, 2, 3, 4, 5\}$ with $i < j$ and $j - i \geq 2$, calculate the difference $x_i - x_j$ whenever it is possible.

  Here, we have: $x_3 - x_5 = (x_3 - x_4) + (x_4 - x_5) = 3$.
- If constraint (t_9) is not respected, modify the values of the elementary differences $x_i - x_{i+1}$.

  Here, the constraint is respected at this stage.
- $\forall (a_i, a_{i+1}) \in C_3$ with $i \in \{1, 2, 3, 4\}$, take $x_i - x_{i+1} = \alpha_2 + 1$ where $\alpha_2 = \max\{x_i - x_j | i < j \text{and} (a_i, a_j) \in C_2\}$.

  Here, $\alpha_2 = 3$ and we take : $x_2 - x_3 = 4$.
- $\forall i, j \in \{1, 2, 3, 4, 5\}$ with $i < j$ and $j - i \geq 2$, calculate the difference $x_i - x_j$ whenever it is possible.

  Here, wehave :
$$\begin{aligned}
x_1 - x_3 &= (x_1 - x_2) + (x_2 - x_3) = 6 \\
x_1 - x_4 &= (x_1 - x_2) + (x_2 - x_3) + (x_3 - x_4) = 7 \\
x_1 - x_5 &= (x_1 - x_2) + (x_2 - x_3) + (x_3 - x_4) + (x_4 - x_5) = 9 \\
x_2 - x_4 &= (x_2 - x_3) + (x_3 - x_4) = 5 \\
x_2 - x_5 &= (x_2 - x_3) + (x_3 - x_4) + (x_4 - x_5) = 7 \\
x_3 - x_5 &= (x_3 - x_4) + (x_4 - x_5) = 3.
\end{aligned}$$
- If constraint (t_9) is not respected, modify the values of the elementary differences $x_i - x_{i+1}$.

  As can be observed in Fig. 11.8, there is a problem here: as $(a_2, a_5) \in C_5$ and $(a_1, a_4) \in C_4$, one must have $x_2 - x_5 \geq x_1 - x_4 + 1$ that is $(x_2 - x_3) + (x_3 - x_4) + (x_4 - x_5) \geq (x_1 - x_2) + (x_2 - x_3) + (x_3 - x_4) + 1$. So, in terms of the

	a1	a2	a3	a4	a5
a1	no 0	weak 2	strong 6	strong 7	extreme 9
a2		no 0	moderate 4	moderate 5	v. strong 7
a3			no 0	very weak 1	weak 3
a4				no 0	weak 2
a5					no 0

Fig. 11.8 First attempt to obtain the basic MACBETH scale

elementary differences $(x_1 - x_2)$, $(x_2 - x_3)$, $(x_3 - x_4)$ and $(x_4 - x_5)$, one must have $(x_4 - x_5) \geq (x_1 - x_2) + 1$. which implies, in our case : $(x_4 - x_5) \geq 3$ and one takes the smallest possible value, that is $x_4 - x_5 = 3$, and consequently $x_3 - x_5 = 4$.

- $\forall (a_i, a_{i+1}) \in C_3$ with $i \in \{1, 2, 3, 4\}$, take $x_i - x_{i+1} = \alpha_2^* + 1$ where $\alpha_2^* = \max\{x_i - x_j | i < j \text{and} (a_i, a_j) \in C_2\}$.
  Here, $\alpha_2^* = 4$ and one takes : $x_2 - x_3 = 5$.
- $\forall i, j \in \{1, 2, 3, 4, 5\}$ with $i < j$ and $j - i \geq 2$, calculate the difference $x_i - x_j$ whenever it is possible.

$$
\begin{aligned}
\text{Here, wehave :} x_1 - x_3 &= (x_1 - x_2) + (x_2 - x_3) = 7 \\
x_1 - x_4 &= (x_1 - x_2) + (x_2 - x_3) + (x_3 - x_4) = 8 \\
x_1 - x_5 &= (x_1 - x_2) + (x_2 - x_3) + (x_3 - x_4) + (x_4 \\
&\quad - x_5) = 11 \\
x_2 - x_4 &= (x_2 - x_3) + (x_3 - x_4) = 6 \\
x_2 - x_5 &= (x_2 - x_3) + (x_3 - x_4) + (x_4 - x_5) = 9 \\
x_3 - x_5 &= (x_3 - x_4) + (x_4 - x_5) = 4.
\end{aligned}
$$

As all the elementary differences are determined and the constraint (t_9) is respected (see Fig. 11.9), the basic MACBETH scale can be obtained:

$$
\begin{aligned}
x_5 &= 0 \\
x_4 &= x_5 + (x_4 - x_5) = 3 \\
x_3 &= x_4 + (x_3 - x_4) = 4 \\
x_2 &= x_3 + (x_2 - x_3) = 9 \\
x_1 &= x_2 + (x_1 - x_2) = 11.
\end{aligned}
$$

	a1	a2	a3	a4	a5
a1	no 0	weak 2	strong 7	strong 8	extreme 11
a2		no 0	moderate 5	moderate 6	v. strong 9
a3			no 0	very weak 1	weak 4
a4				no 0	weak 3
a5					no 0

Fig. 11.9 Second attempt to obtain the basic MACBETH scale

11.9 Discussion About a Scale

Suppose that, in the example in Fig. 11.6, J considers that the element a is badly positioned when compared to elements c and d and therefore J wants to redefine the value of a. It is then interesting to show J the limits within which the value of a can vary without violating the preferential information provided by J. Let us suppose in this section that we have a type 1+2 information about X which is consistent and that $? = \phi$.

Let μ_0 be a particular scale of Sc_{1+2}, L and H be two fixed elements of X with HPL (H more attractive than L) and a be an element of X (not indifferent to L and not indifferent to H) that J would like to have repositioned.

Let

- $Sc_{(\mu_0,H,L)} = \{\mu \in Sc_{1+2} \mid \mu(H) = \mu_0(H) \text{ and } \mu(L) = \mu_0(L)\}$ (scales for which values associated with H and L have been fixed)
- $Sc_{(\mu_0,\hat{a})} = \{\mu \in Sc_{1+2} \mid \forall\, y \in X$ with y not indifferent to a: $\mu(y) = \mu_0(y)\}$ (scales for which the values of all of the elements of X except a and its eventual equals have been fixed).

We call *free interval* associated to interval a :

$$\left]\inf_{\mu \in Sc_{(\mu_0,H,L)}} \mu(a), \sup_{\mu \in Sc_{(\mu_0,H,L)}} \mu(a)\right[$$

We call *dependent interval* associated to interval a :

$$\left]\inf_{\mu \in Sc_{(\mu_0,\hat{a})}} \mu(a), \sup_{\mu \in Sc_{(\mu_0,\hat{a})}} \mu(a)\right[$$

In the example in Fig. 11.6, if one selects a, two intervals are presented to J (see Fig. 11.10) which should be interpreted as follows:

$$\forall\, \mu \in Sc_{1+2}\,,\ [\,\mu(c) = 0, \mu(d) = 100\,] \Rightarrow 66.69 \leq \mu(a) \leq 99.98.$$

$$\forall\, \mu \in Sc_{1+2}\,,\ [\,\mu(c) = 0, \mu(d) = 100, \mu(e) = 36.36, \mu(b) = 27.27\,]$$

$$\Rightarrow 72.74 \leq \mu(a) \leq 90.9.$$

The closed intervals (in the example $[66.69, 99.98]$ and $[72.74, 90.9]$) that have been chosen to present to J are not the precise free and dependent intervals associated to a (which, by definition, are open); however, by taking a precision of 0.01 into account, they can be regarded as the "greatest" closed intervals included in the free and dependent intervals.

M-MACBETH permits the movement of element a with the mouse but, obviously, only inside of the dependent interval associated to a.

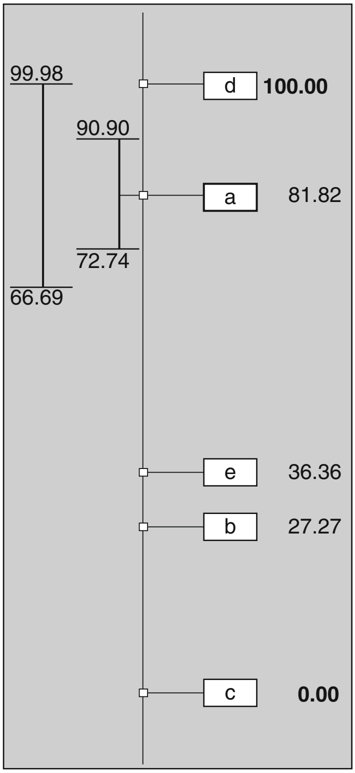


Fig. 11.10 "Greatest" closed intervals included in the free and dependent intervals

If J wants to give element a a value that is outside of the dependent interval (but still inside the free interval), the software points out that the values of the other elements must be modified. If J confirms the new value of a, a new MACBETH scale is calculated, taking into account the additional constraint that fix the new value of a.

The ("closed") free interval is calculated by integer linear programming. The ("closed") dependent interval could be also calculated in the same manner. However, M-MACBETH computes it by "direct" calculation formulas which make the determination of these intervals extremely fast—for details, see [95].

11.10 MACBETH and MCDA

The MACBETH approach and the M-MACBETH software have been used to build value functions and scoring and weighting scales, in the process of developing multicriteria decision aid models, in particular many simple additive value models.

In this framework, the MACBETH weighting procedure is presented in detail in [57]. A classification of applications of MACBETH reported in the literature is presented hereafter (further references to applications reported in Portuguese can be found in [58]).

Agriculture, Manufacturing & Services: Finance: [19, 20, 42, 49, 70, 72, 73, 78, 105, 116–119, 134, 185];
Information systems: [57, 124, 188];
Performance measurement: [64–67, 69, 71, 83, 88, 101, 107, 108, 127–129, 138, 142–145, 151, 189, 192];
Production & service planning: [2, 3, 29, 36, 79, 89, 90, 131, 139, 157, 171, 172, 174, 179];
Quality management: [12, 56, 80, 81, 112];
R&D project selection: [96];
Risk management: [84, 121, 161];
Strategy & resource allocation: [37, 38, 170];
Supply chain and logistics: [97–99, 133, 140, 141, 165, 183, 198];

Energy: Project prioritization and selection: [50, 100];
Technology choice: [60, 61, 75, 110, 111, 115, 120, 155, 156, 196];

Environment: Landscape management: [186, 190];
Climate change: [58, 85];
Risk management: [13, 51, 87, 135, 166, 184];
Sustainable development: [154, 176];
Water resource management: [4, 5, 39, 46];

Medical: [91–94, 148, 153, 158, 162–164, 175, 182];

Military: [15, 21, 136, 137, 146, 147, 191];

Public Sector: Conflict analysis and management: [8, 41, 68, 102, 177, 187];
Education: [1, 86];
Procurement: [7, 9, 11, 43, 57] [14, 15, 18, 24, 33, 40, 48, 50, 149, 173];
Project prioritization & resource allocation: [16, 26, 30, 32, 35, 47, 76, 77, 152, 167–169, 180, 181, 193];
Strategic planning & development: [34, 44, 52, 53, 59, 130, 160, 194];

Others: Human resource management: [17, 54, 103, 104, 106, 109, 122, 123, 132, 146, 147, 191];
Job selection: [10];
Sports: [74].

Acknowledgements The authors thank José Ferrão, Manuel Pedro Chagas, Mónica Oliveira, Teresa Rodrigues and two anonymous referees for their valuable comments on earlier versions of this chapter and Diana Lopes and Teresa Rodrigues for their invaluable collaboration in updating Sect. 11.10.

References

1. Abtahi, M.S.: The effectiveness in distance education for Iranian higher education. In: Procedia – Social and Behavioral Sciences, vol. 47, pp. 1315–1319, North Cyprus, 8–10 February 2012. Cyprus International Conference on Educational Research (CY-ICER) (2012)
2. Amora Silva, M.B.F., Bana e Costa, C.A.: Modelo multicritério de avaliação de capacidade empreendedora em empresas de base tecnológica. Engevista **10**(1), 4–14 (2008)
3. Andrade, A.C.Z.B., Bana e Costa, C.A.: Measuring security value: a multi-criteria decision analysis approach. Working paper cegist 20/2001, CEG-IST, Technical University of Lisbon (2001)
4. Antão da Silva, P., Bana e Costa, C.A., Correia, F.N.: Avaliação multicritério das incidências ambientais de medidas de controlo de cheias: Aplicação à bacia hidrográfica da Ribeira do Livramento. In: Trabalhos Técnicos do 1^o Congresso sobre Aproveitamento e Gestão de Recursos Hídricos em Países de Idioma Português, pp. 210–217, Rio de Janeiro (2000). Associação Brasileira de Engenharia Sanitária e Ambiental, CD-ROM edition
5. Ba, F., Bouchard, C., Abi-zeid, I.: Analyse multicritère pour la priorisation des interventions en matière d'approvisionnement en eau en milieu rural au Sénégal: cas de la région de Diourbel. Rev. Sci. Eau **24**(1), 9–22 (2011)
6. Bana e Costa, C.A.: Structuration, Construction et Exploitation d'un Modèle Multicritère d'Aide à la Décision. Ph.D. thesis, Technical University of Lisbon, Lisbon (1992)
7. Bana e Costa, C.A.: O modelo de apoio à avaliação de propostas nos concursos do Metro do Porto. FER XXI **XVIII**, 111, 113–115 (1999)
8. Bana e Costa, C.A.: The use of multicriteria decision analysis to support the search for less conflicting policy options in a multi-actor context: case-study. J. Multi-Criteria Decis. Anal. **10**(2), 111–125 (2001)
9. Bana e Costa, C.A.: Issues in facilitating bid evaluation in public call for tenders. In: Khosrowshahi, F. (ed.) Proceedings of the 3rd International Conference on Decision Making in Urban and Civil Engineering, CD-ROM edition, pp. 703–709. SOAS, London (2002)
10. Bana e Costa, C.A., Chagas, M.P.: A career choice problem: an example of how to use MACBETH to build a quantitative value model based on qualitative value judgments. Eur. J. Oper. Res. **153**(2), 323–331 (2004)
11. Bana e Costa, C.A., Corrêa, E.C.: Framing public calls for tenders through a multicriteria approach. In: Despotis, D.K., Zopounidis, C. (eds.) Proceedings of the 5th International Conference of the Decision Sciences Institute, vol. II, pp. 1353–1355, Athens University of Business and Economics, Athens (1999)
12. Bana e Costa, C.A., Corrêa, E.C.: Construction of a total quality index using a multicriteria approach: the case of Lisbon Gas Company. Research paper 10/2000, CEG-IST, Technical University of Lisbon (2000)
13. Bana e Costa, C.A., Corrêa, E.C.: O processo de construção do índice FUNCEME de vulnerabilidade à seca. Research paper 16/2000, CEG-IST, Technical University of Lisbon (2000)
14. Bana e Costa, C.A., Ferreira, J.A.: Concursos de obras públicas: Metodologia de avaliação de propostas. Aplicação a uma grande empreitada de infraestruturas de transportes. In: Actas do Congresso 94, Engenharia Portuguesa na Viragem do Século, pp. 47–w/9. Ordem dos Engenheiros, Lisbon (1994)
15. Bana e Costa, C.A., Marques, M.S.: Procurement decision support for the Portuguese MoD: The MACBETH approach and the acquisition of 8 × 8 AWV. In: Decision Support Methodologies for Acquisition of Military Equipment, NATO Research and Technology Organisation, Brussels (1-1, 1-14) (2009)
16. Bana e Costa, C.A., Oliveira, R.C.: Assigning priorities for maintenance, repair and refurbishment in managing a municipal housing stock. Eur. J. Oper. Res. **138**(2), 380–391 (2002)
17. Bana e Costa, C.A., Oliveira, M.D.: A multicriteria decision analysis model for faculty evaluation. OMEGA Int. J. Manag. Sci. **40**(4), 424–436 (2011)

18. Bana e Costa, C.A., Silva, F.N.: Concepção de uma 'boa' alternativa de ligação ferroviária ao porto de Lisboa: Uma aplicação da metodologia multicritério de apoio à decisão e à negociação. Investigação Operacional **14**(2), 115–131 (1994)
19. Bana e Costa, C.A., Soares, J.O.: Multicriteria approaches for portfolio selection: an overview. Rev. Financ. Mark. **IV**(1), 19–26 (2001)
20. Bana e Costa, C.A., Soares, J.O.: A multicriteria model for portfolio management. Eur. J. Financ. 10(3), 198–211 (2004)
21. Bana e Costa, C.A., Thomaz, J.F.: Locating centres of information and recruitment of volunteers for the Portuguese armed forces: a decision-analysis case-study. In: Proceedings of the 42nd Annual Conference of the International Military Testing Association, CD-ROM edition, pp. 173–180. International Military Testing Association, Edinburgh (2000)
22. Bana e Costa, C.A., Vansnick, J.C.: Sur la quantification des jugements de valeur: L'approche MACBETH. Cahiers du LAMSADE, 117, Université Paris-Dauphine, Paris (1993)
23. Bana e Costa, C.A., Vansnick, J.C.: MACBETH – an interactive path towards the construction of cardinal value functions. Int. Trans. Oper. Res. **1**(4), 489–500 (1994)
24. Bana e Costa, C.A., Vansnick, J.C.: Uma nova abordagem ao problema de construção de uma função de valor cardinal: MACBETH. Investigação Operacional **15**(1), 15–35 (1995)
25. Bana e Costa, C.A., Vansnick, J.C.: General overview of the MACBETH approach. In: Pardalos, P.M., Siskos, Y., Zopounidis, C. (eds.) Advances in Multicriteria Analysis, pp. 93–100. Kluwer, Dordrecht (1995)
26. Bana e Costa, C.A., Vansnick, J.C.: Applications of the MACBETH approach in the framework of an additive aggregation model. J. Multi-Criteria Decis. Anal. **6**(2), 107–114 (1997)
27. Bana e Costa, C.A., Vansnick, J.C.: The MACBETH approach: Basic ideas. In: Proceedings of the International Conference on Methods and Applications of Multicriteria Decision Making, pp. 86–88. FUCAM, Facultés Universitaires Catholiques de Mons (1997)
28. Bana e Costa, C.A., Vansnick, J.C.: A theoretical framework for measuring attractiveness by a categorical based evaluation technique (MACBETH). In: Clímaco, J. (ed.) Multicriteria Analysis, pp. 15–24. Springer, Berlin (1997)
29. Bana e Costa, C.A., Vansnick, A.C.: Construção multicritério da 'matriz de custos' de um modelo de afectação para gestão de stands de aeronaves. Investigação Operacional **17**(2), 179–199 (1997)
30. Bana e Costa, C.A., Vansnick, J.C.: The MACBETH approach: basic ideas, software and an application. In: Meskens, N., Roubens, M. (eds.) Advances in Decision Analysis, pp. 131–157. Kluwer, Dordrecht (1999)
31. Bana e Costa, C.A., Vansnick, J.C.: Cardinal value measurement with MACBETH. In: Zanakis, S.H., Doukidis, G., Zopounidis, C. (eds.) Decision Making: Recent Developments and Worldwide Applications, pp. 317–329. Kluwer, Dordrecht (2000)
32. Bana e Costa, C.A., Oliveira, R.C., Alvarez, M.L.: Concepção de um sistema multicritério de definição de prioridades de intervenção/conservação do património habitacional da Câmara Municipal de Lisboa. In: Comunicações ao 2^{o} Encontro sobre Conservação e Reabilitação de Edifícios, vol. II, pp. 925–934. LNEC, Laboratório Nacional de Engenharia Civil, Lisbon (1994)
33. Bana e Costa, C.A., Ferreira, J.A., Vansnick, J.C.: Avaliação multicritério de propostas: O caso de uma nova linha do Metropolitano de Lisboa. Revista de Transporte e Tecnologia **VII**(14), 31–65 (1995)
34. Bana e Costa, C.A., Costa-Lobo, M.L., Ramos, I.A.: Contributo da metodologia multicritério na elaboração do Plano Estratégico de Barcelos. Sociedade e Território **24**, 102–115 (1997)
35. Bana e Costa, C.A., Oliveira, R.C., Alvarez, M.L.: SMAI: Sistema multicritério de avaliação de intervenções num parque habitacional municipal. In: Bezelga, A.A., Leitão, A.B. (eds.) A Avaliação do Imobiliário, pp. 109–122. Associação Portuguesa dos Avaliadores de Engenharia, Lisbon (1997)

36. Bana e Costa, C.A., Ensslin, L., Costa, A.P.: Structuring the process of choosing rice varieties at the South of Brazil. In: Multicriteria Analysis for Land-Use Management, vol. 9: Environment and Management, pp. 33–45. Kluwer, Dordrecht (1998)
37. Bana e Costa, C.A., Ensslin, L., Zanella, I.J.: A real-world MCDA application in cellular telephony systems. In: Stewart, T.J., van den Honert, R.C. (eds.) Trends in Multicriteria Decision Making. Lecture Notes in Economics and Mathematical Systems, vol. 465, pp. 412–423. Springer, Berlin (1998)
38. Bana e Costa, C.A., Corrêa, E.C., Ensslin, L., Vansnick, J.C.: Decision support systems in action: integrated application in a multicriteria decision aid process. Eur. J. Oper. Res. **113**(2), 315–335 (1999)
39. Bana e Costa, C.A., Ensslin, L., Holz, E.: Avaliação ambiental em situação de conflito de interesses – Um estudo de caso. In: IX SILUBESA, Simpósio Luso-Brasileiro de Engenharia Sanitária e Ambiental, CD-ROM edition, pp. 1914–1923. Associação Brasileira de Engenharia Sanitária e Ambiental and Associação Portuguesa de Recursos Hídricos, Porto Seguro (BA) (2000)
40. Bana e Costa, C.A., Ferreira, J.A., Corrêa, E.C.: Metodologia multicritério de apoio à avaliação de propostas em concursos públicos. In: Antunes, C.H., Tavares, L.V. (eds.) Casos de Aplicação da Investigação Operacional, pp. 336–363. McGraw-Hill Portugal, Lisbon (2000)
41. Bana e Costa, C.A., Silva, F.N., Vansnick, J.C.: Conflict dissolution in the public sector: a case-study. Eur. J. Oper. Res. **130**(2), 388–401 (2001)
42. Bana e Costa, C.A., Barroso, L., Soares, J.O.: Qualitative modelling of credit scoring: a case study in banking. J. Eur. Res. Stud. **5**(1–2), 37–51 (2002)
43. Bana e Costa, C.A., Corrêa, E.C., De Corte, J.M., Vansnick, J.C.: Facilitating bid evaluation in public call for tenders: a socio-technical approach. OMEGA Int. J. Manag. Sci. **30**(3), 227–242 (2002)
44. Bana e Costa, C.A., da Costa-Lobo, M.L., Ramos, I.A., Vansnick, J.C.: Multicriteria approach for strategic town planning: the case of Barcelos. In: Bouyssou, D., Jacquet-Lagrèze, E., Perny, P., Sĺowiński, R., Vanderpooten, D., Vincke, Ph. (eds.) Aiding Decisions with Multiple Criteria. Essays in Honor of Bernard Roy, pp. 429–456. Kluwer, Boston (2002)
45. Bana e Costa, C.A., De Corte, J.M., Vansnick, J.C.: MACBETH. Working paper 03.56, London School of Economics (2003)
46. Bana e Costa, C.A., Antão da Silva, P., Correia, F.N.: Multicriteria evaluation of flood control measures: the case of Ribeira do Livramento. Water Resour. Manag. **18**(3), 263–283 (2004)
47. Bana e Costa, C.A., Fernandes, T.G., Correia, P.V.D.: Prioritisation of public investments in social infrastructures using multicriteria value analysis and decision conferencing: a case study. Int. Trans. Oper. Res. **13**(4), 279–297 (2006)
48. Bana e Costa, C.A., Lourenço, J.C., Bana e Costa, J.C.: Building MACBETH value-function models for bid evaluation at the Portuguese electric transmission company (REN). In: Antunes, C.H., Gomes, A. (eds.) Proceedings of the 19th Mini EURO Conference on Operational Research Models and Methods in the Energy Sector, CD-ROM edition. INESC Coimbra (2006)
49. Bana e Costa, C.A., Lourenço, J.C., Soares, J.O.: An interval weighting assignment model for credit analysis. J. Financ. Decis. Mak. **3**(2), 1–9 (2007)
50. Bana e Costa, C.A., Lourenço, J.C., Chagas, M.P., Bana e Costa, J.C.: Development of reusable bid evaluation models for the Portuguese electric transmission company. Decis. Anal. **5**(1), 22–42 (2008). doi:10.1287/deca.1080.0104
51. Bana e Costa, C.A., Oliveira, C.S., Vieira, V.: Prioritization of bridges and tunnels in earthquake risk mitigation using multicriteria decision analysis: application to Lisbon. Omega **36**(3), 442–450 (2008)
52. Bana e Costa, C.A., Lourenço, J.C., Bana e Costa, J.C.: Concepção de uma estratégia de desenvolvimento a médio prazo para Pernambuco. In: Actas do 15^o Congresso da APDR – Associação Portuguesa para o Desenvolvimneto Regional Cabo Verde (2009) APDR (1959–1681)

53. Bana e Costa, C.A., Craveiro, M.T., Rocha, A.S., Bana e Costa, J.C., Santos, P.S.: Avaliação multicritério na elaboração da matriz estratégica do programa local de habitação de Lisboa. In: Proceedings do 54^o IFHP World Congress, CD-ROM edition, Porto Alegre, 14–17 November 2010
54. Bana e Costa, C.A., Martins, P.A.F., Oliveira, M.D., Sernadas, A., Mota Soares, C.A.: Faculty evaluation using multicriteria value measurement. In: Rogozea, L. (ed.) Advances in Mathematical and Computational Methods – 12th WSEAS International Conference on Mathematical and Computational Methods in Science and Engineering (MACMESE '10), University of Algarve, Faro. WSEAS Press (2010)
55. Bana e Costa, C.A., De Corte, J.M., Vansnick, J.C.: MACBETH (measuring attractiveness by a categorical-based evaluation technique). In: Cochrane, J.J. (ed.) Wiley Encyclopedia in Operational Research and Management Science, pp. 2945–2950. Wiley, New York (2011)
56. Bana e Costa, C.A., Carnero, M.C., Oliveira, M.D.: A multi-criteria model for auditing a predictive maintenance programme. Eur. J. Oper. Res. **217**(2), 381–393 (2012)
57. Bana e Costa, C.A., De Corte, J.M., Vansnick, J.C.: MACBETH. Int. J. Inf. Technol. Decis. Mak. **11**(2), 359–387 (2012)
58. Bana e Costa, C.A., Angulo-Meza, L., Oliveira, M.D.: O método MACBETH e aplicação no Brasil. Engevista **15**(1), 3–27 (2013)
59. Bana e Costa, C.A., Lourenço, J.C., Oliveira, M.D., Bana e Costa, J.C.: A social–technical approach for group decision support in public strategic planning: the Pernambuco PPA case. Group Decis. Negot. **23**(1), 5–29 (2013)
60. Barin, A., Canha, L.N., Abaide, A.R., Orling, R.B., Martins, L.F.G.: Selection of hybrid renewable energy systems in landfills. In: 9th International Conference on the European Energy Market, pp. 1–5, Florence, 10–12 May 2012. IEEE, New York (2012)
61. Barin, A., Martins, L.F., Canha, L.N., Abaide, A., Oriling, R.B.: Selection of renewable energy technologies in landfills: the m-macbeth decision support. In: Mendez-Vilas, A. (ed.) Fuelling the Future: Advances in Science and Technologies for Energy Generation, Transmission and Storage, pp. 98–102. Universal-Publishers (2012)
62. Belton, V.: A Comparative Study of Methods for Multiple Criteria Decision Aiding. Ph.D. thesis, University of Cambridge, Cambridge (1986)
63. Belton, V., Stewart, T.J.: Multiple Criteria Decision Analysis: An Integrated Approach. Kluwer, Boston (2002)
64. Berrah, L., Clivillé, V.: Towards an aggregation performance measurement system model in a supply chain context. Comput. Ind. **58**(7), 709–719 (2007)
65. Berrah, L., Mauris, G., Vernadat, F.: Industrial performance measurement: an approach based on the aggregation of unipolar or bipolar expressions. Int. J. Prod. Res. **44**(18–19), 4145–4158 (2006)
66. Berrah, L., Mauris, G., Montmain, J.: Monitoring the improvement of an overall industrial performance based on a Choquet integral aggregation. Omega **36**(3), 340–351 (2008)
67. Berrah, L., Mauris, G., Montmain, J., Clivillé, V.: Efficacy and efficiency indexes for a multi-criteria industrial performance synthesized by Choquet integral aggregation. Int. J. Comput. Integr. Manuf. **21**(4), 415–425 (2008)
68. Bollinger, D., Pictet, J.: Potential use of e-democracy in MCDA processes. Analysis on the basis of a Swiss case. J. Multi-Criteria Decis. Anal. **12**(2–3), 65–76 (2003)
69. Bortuluzzi, S.C., Ensslin, S.R., Lyrio, M.V.L., Ensslin, L.: Proposta de um modelo multicritério de avaliação de desempenho econômico-financeiro para apoiar decisões de investimentos em empresas de capital aberto. CAP Account. Manag. **3**(3), 100–110 (2009)
70. Bortuluzzi, S.C., Ensslin, S.R., Ensslin, L.: Construção de um modelo de avaliacão de desempenho para a gestão financeira de uma empresa de informática. CAP Account. Manag. **4**(4), 12–22 (2010)
71. Bortuluzzi, S.C., Ensslin, S.R., Ensslin, L.: Avaliação de desempenho dos aspectos tangíveis e intangíveis da área de mercado: Estudo de caso em uma média empresa industrial. Revista Brasileira de Gestão de Negócios **12**(37), 425–446 (2010)

72. Bortuluzzi, S.C., Ensslin, S.R., Ensslin, L.: Avaliação de desempenho das variáveis financeiras e não financeiras que respondem pelo desempenho de uma indústria de móveis. Revista Gestão Industria **7**(2), 24–47 (2011)
73. Bortuluzzi, S.C., Ensslin, S.R., Lyrio, M.V.L., Ensslin, L.: Avaliação de desempenho econômico-financeiro: Uma proposta de integração de indicadores contábeis tradicionais por meio da metodologia multicritério de apoio à decisão construtivista (MCDA-C). Revista Alcance (Online) **18**(2), 200–218 (2011)
74. Brandão, L.C., Andrade, F.V.S., Soares de Mello, J.C.C.B.: 2012 UEFA Euro efficiency evaluation based on market expectations. In: 4th International Conference on Mathematics in Sport, pp. 32–37, Leuven, 5–7 June 2013
75. Burton, J., Hubacek, K.: Is small beautiful? A multicriteria assessment of small-scale energy technology applications in local governments. Energy Policy **35**(12), 6402–6412 (2007)
76. C3E. (ed.): Applying the Multi-criteria Method to the Evaluation of Structural Programmes. MEANS Handbook, vol. 4. European Commission, DG XVI/02, Brussels (1995)
77. C3E. (ed.): Evaluation Pilote Multicritère du Hainaut. Rapport d'Expérience. European Commission, DG XVI/02, Brussels (1995)
78. Cabello, J.M., Ruiz, F., Pérez-Gladish, B., Méndez-Rodríguez, P.: Synthetic indicators of mutual funds' environmental responsibility: an application of the reference point method. Eur. J. Oper. Res. (2014). doi:10.1016/j.ejor.2013.11.031
79. Cardoso, F.J.P., Gomes, L.F.A.M., Sanglard Júnior, M.: Administração das operações de telecomunicação: Uma análise de decisão. Revista Portuguesa e Brasileira de Gestão **2**(2), 91–103 (2003)
80. Carnero, M.C.: Evaluating a maintenance department in a service company. Int. J. Math. Models Methods Appl. Sci. **3**(3), 230–237 (2009)
81. Carnero, M.C.: Maintenance audit by means of an additive multicriteria model. In: Zafiris, V., Benavides, M., Gaoet, K. (eds.) Advances in Marketing, Management and Finances, pp. 147–152. WSEAS Press, Prague (2009)
82. Catalyze Ltd.: HIVIEW3 Starter Guide. Catalyze Ltd., London (2003)
83. Clivillé, V., Berrah, L., Mauris, G.: Quantitative expression and aggregation of performance measurements based on the MACBETH multi-criteria method. Int. J. Prod. Econ. **105**(1), 171–189 (2007)
84. Costa, H.R., Barros, M.D.O., Travassos, G.: Evaluating software project portfolio risks. J. Syst. Softw. **80**(1), 16–31 (2007)
85. Cox, R., Sanchez, J., Revie, C.W.: Multi-criteria decision analysis tools for prioritising emerging or re-emerging infectious diseases associated with climate change in Canada. PLoS ONE **8**(8), 1–16 (2013)
86. Cuadrado, M.R., Fernández, M.G.: Methodology to select the best business game in higher education. Am. J. Ind. Bus. Manag. **3**, 589–594 (2013)
87. Dall'Osso, F., Gonella, M., Gabbianelli, G., Withycombe, G., Dominey-Howes, D.: A revised (PTVA) model for assessing the vulnerability of buildings to tsunami damage. Nat. Hazards Earth Syst. Sci. **9**(5), 1557–1565 (2009)
88. de Azevedo, R., de Oliveira Lacerda, R., Ensslin, L., Jungles, A., Ensslin, S.: Performance measurement to aid decision making in the budgeting process for apartment-building construction: case study using MCDA-C. J. Constr. Eng. Manag. **139**(2), 225–235 (2013)
89. De Castro, A.K.A., Pinheiro, P.R., Souza, G.G.C.: A scheduling process applied to newspaper production. In: Proceedings of the 3rd International Conference on Service Systems and Service Management, Troyes, pp. 1245–1250. IEEE, New York (2006)
90. de Castro, A.K.A., Pinheiro, P.R., de Souza, G.G.C.: Application multicriteria decision analysis on tv digital. In: Sobh, T.M. (ed.) Advances in Computer and Information Sciences and Engineering, pp. 39–44. Springer, New York (2008) [ISBN 978-1-4020-8740-0]
91. de Castro, A.K.A., Pinheiro, P.R., de Souza, G.G.C.: A multicriteria model applied in the diagnosis of alzheimer's disease: a Bayesian network. In: 11th IEEE International Conference on Computational Science and Engineering, São Paulo, pp. 15–22 (2008)

92. de Castro, A., Pinheiro, P.R., Pinheiro, M.: An approach for the neuropsychological diagnosis of Alzheimer's disease: a hybrid model in decision making. In: Wen, P., Li, Y., Polkowski, L., et al. (eds.) 4th International Conference on Rough Sets and Knowledge Technology, Gold Coast, pp. 216–223. Springer, Berlin (2009)
93. de Castro, A., Pinheiro, P.R., Pinheiro, M.: Towards the neuropsychological diagnosis of Alzheimer's disease: a hybrid model in decision making. In: Lytras, M.D., DePablos, P.O., Damiani, E., et al. (eds.) 2nd World Summit on the Knowledge Society, Chania, pp. 522–531. Springer, Berlin (2009)
94. de Castro, A., Pinheiro, P.R., Pinheiro, M., Tamanini, I.: Towards the applied hybrid model in decision making: a neuropsychological diagnosis of Alzheimer's disease study case. Int. J. Comput. Intell. Syst. **4**(1), 89–99 (2011)
95. de Corte, J.M.: Un Logiciel d'Exploitation d'Informations Préférentielles pour l'Aide à la Décision. Bases Mathématiques et Algorithmiques. Ph.D. thesis, Presses Universitaires Montoises, University of Mons-Hainaut, Mons (2002)
96. de Lima, A.S., Damiani, J.H.S.: A proposed method for modeling research and development (R&D) project prioritization criteria. In: Kocaoglu, D.F., Anderson, T.R., Daim, T.U.; et al. (eds.) Conference of the Portland International Center for Management of Engineering and Technology. (PICMET'09), Portland, pp. 589–598. IEEE, New York (2009).
97. Della Bruna E. Jr., Ensslin, L., Ensslin, S.R.: Supply chain performance evaluation: a case study in a company of equipment for refrigeration. In: Proceedings of the 2011 IEEE International Technology Management Conference, San Jose (2011)
98. Dhouib, D.: Fuzzy Macbeth method to analyze alternatives in automobile tire wastes reverse logistics. In: International Conference on Advanced Logistics and Transport, pp. 321–326, Sousse, 29–31 May 2013
99. Dhouib, D.: An extension of MACBETH method for a fuzzy environment to analyze alternatives in reverse logistics for automobile tire wastes. OMEGA Int. J. Manag. Sci. **42**(1), 25–32 (2014)
100. Dias-Sardinha, I., Carolino, J., Matos, P.V., Mendes, I., Milheiras, S., Craveiro, D., Marques, M., Primdahl, J., Krutwagen, B.: Rehmine project - redevelopment of so domingos mine. Technical report, School of Economics & Management, Mértola, Portugal, June 2013
101. Dutra, A., Ensslin, S.R., Ensslin, L., Lima, M.V.A.: A incorporação da dimensão integrativa nos processos de avaliação do desempenho organizacional: Um estudo de caso. Revista Contemporânea de Contabilidade **6**(11), 109–136 (2009)
102. Eklund, P., Rusinowska, A., Swart, H.D.: A consensus model of political decision-making. Ann. Oper. Res. **158**(1), 5–20 (2008)
103. Ensslin, L., Ensslin, S.: Elaboração de um modelo construtivista para identificação de oportunidades de aperfeiçoamento de docentes do EPS-UFSC. Revista Produto & Produção **2**(3), 143–149 (1998)
104. Ensslin, L., Dutra, A., Ensslin, S.: MCDA: a constructivist approach to the management of human resources at a governmental agency. Int. Trans. Oper. Res. **7**(1), 79–100 (2000)
105. Ensslin, L., Montibeller, G.N., Lima, M.V.A.: Constructing and implementing a DSS to help evaluate perceived risk of accounts receivable. In: Haimes, Y.Y., Steuer, R.E. (eds.) Research and Practice in Multiple Criteria Decision Making. Lecture Notes in Economics and Mathematical Systems, vol. 487, pp. 245–259. Springer, Berlin (2000)
106. Ensslin, L., Montibeller, G.N., Noronha, S.: Apoio à Decisão: Metodologias para Estruturação de Problemas e Avaliação Multicritério de Alternativas. Editora Insular, Florianópolis (SC) (2001)
107. Ensslin, L., Giffhorn, E., Ensslin, S., Petri, S., Vianna, W.: Avaliação do desempenho de empresas terceirizadas com o uso da metodologia multicritério de apoio à decisão-construtivista. Revista Pesquisa Operacional **30**(1), 125–152 (2010)
108. Ensslin, L., Scheid, L.C.M., Ensslin, S.R., de Oliveira Lacerda, R.T.: Software process assessment and improvement using multicriteria decision aiding-constructivist. JISTEM: J. Inf. Syst. Technol. Manag. **9**(3), 473–496 (2012)

109. Ensslin, S.R., Ensslin, L., Back, F., de Oliveira Lacerda, R.T.: Improved decision aiding in human resource management: a case using constructivist multi-criteria decision aiding. Int. J. Product. Perform. Manag. **62**(7), 735–757 (2013)
110. Ertay, T., Kahraman, C.: Evaluation of renewable energy alternatives using MACBETH multicriteria method. In: 9th International FLINS Conference on Foundations and Applications of Computational Intelligence, Chengdu, pp. 937–943 (2010)
111. Ertay, T., Kahraman, C., Kaya, Äř: Evaluation of renewable energy alternatives using MACBETH and fuzzy AHP multicriteria methods: the case of Turkey. Technol. Econ. Dev. Econ. **19**(1), 38–62 (2013)
112. Fakhfakh, N., Pourraz, F., Verjus, H., Moreaux, P.: Client-oriented preferences model for QoS aggregation in service-based applications. In: Obaidat, M.S., Sevillano, J.L., Filipe, J. (eds.) E-Business and Telecommunications, vol. 314, pp. 141–155, Seville, 18–21 July 2011. International Joint Conference, ICETE. Springer, Berlin/Heidelberg (2011)
113. Farquhar, P.H., Keller, L.R.: Preference intensity measurement. Ann. Oper. Res. 19(1), 205–217 (1989)
114. Fasolo, B., Bana e Costa, C.A.: Tailoring value elicitation to decision makers' numeracy and fluency: expressing value judgments in numbers or words. OMEGA Int. J. Manag. Sci. **44**, 83–90 (2014)
115. Fernandes, M.B., Almeida, M.C., Henriques, A.G.: Assessing desalination as a sustainable alternative using a multiple criteria decision support model. Water Pract. Technol. **3**(3), 1–8 (2008)
116. Ferreira, F.A.F.: Bank branches' performance evaluation. In: IADIS International Conference e-Society, Dublin, pp. 392–394 (2006)
117. Ferreira, F.A.F., Santos, S.P., Rodrigues, P.M.M.: Adding value to bank branch performance evaluation using cognitive maps and MCDA: a case study. J. Oper. Res. Soc. **62**(7), 1320–1333 (2011)
118. Ferreira, F.A.F., Spahr, R.W., Santos, S.P., Rodrigues, P.M.M.: A multiple criteria framework to evaluate bank branch potential attractiveness. Int. J. Strateg. Prop. Manag. **16**(3), 254–276 (2012)
119. Ferreira, F., Santos, S.P., Marques, C., Ferreira, J.: Assessing credit risk of mortgage lending using MACBETH: a methodological framework. Manag. Decis. **52**(2), 1–36 (2014)
120. Ferreira, J., Pinheiro, M.D., de Brito, J.: Portuguese sustainable construction assessment tools benchmarked with BREEAM and LEED: an energy analysis. Energy Build. **69**, 451–463 (2014)
121. Figueiredo, M.S.M., Oliveira, M.D.: Prioritizing risks based on multicriteria decision aid methodology: Development of methods applied to ALSTOM power. In: IEEE International Conference on Industrial Engineering and Engineering Management, Hong Kong, pp. 1568–1572 (2009)
122. Filho, A.B., Marçal, A.S., Costa, G., Pinheiro, P.R.: Staff scheduling optimization in information technology projects. In: The Fourth IEEE Conference on Service Systems and Service Management, pp. 1–6, University of Electronic Science and Technology of China, Chengdu (2007)
123. Filho, A.B., Marçal, A.S., Costa, G., Pinheiro, P.R., Pinheiro, R.F.: A novel approach based on staff scheduling optimization in information technology projects. Int. J. Comput. Sci. Netw. Secur. **9**(9), 277–286 (2009)
124. Flix, B., Vasconcelos, A., Tribolet, J.: SimplexIS: evaluating the impact of e-Gov simplification measures in the information system architecture. World Acad. Sci. Eng. Technol. **63**, 636–641 (2012)
125. Freeling, A.N.S.: Belief and Decision Aiding. Ph.D. thesis, University of Cambridge (1983)
126. French, S.: Decision Theory: An Introduction to the Mathematics of Rationality. Ellis Horwood Limited, Chichester (1988)
127. Gallon, A.V., Ensslin, S.R., Ensslin, L.: Avaliação de desempenho organizacional em incubadoras de empresas por meio da metodologia multicritério de apoio à decisão construtivista (MCDA-C): a experiência do midi tecnológico. RAI: Revista de Administração e Inovação **8**(1), 37–63 (2011)

128. Giffhorn, E., Ensslin, L., Ensslin, S.R., Vianna, W.B.: Proposal of a multicriteria performance evaluation for outsourced project providers. In: Performance Measurement Association Conference 2009, University of Otago, New Zealand, 18 pp (2009)
129. Giffhorn, E., Ensslin, L., Ensslin, S.R., Petri, S.M., Vianna, W.B.: Avaliação do desempenho de empresas terceirizadas com o uso da metodologia multicritério em apoio à decisão - construtivista. Pesquisa Operacional **30**(1), 125–152 (2010)
130. Gomes, L.F.A.M., Rangel, L.A.D., Mello, J., Gomes, E., Fuks, S.: Apoio à administração universitária: Aplicação dos métodos Macbeth e Uta na universidade federal fluminense. Revista de Administração Mackenzie **3**(2), 155–174 (2002)
131. Grzebieluckas, C., Busson, M.A., Queiroz, S.G., Ensslin, L., Ensslin, S., Nickel, E., Balbim, A.J.: Instrumento para identificação das necessidades do consumidor no processo de desenvolvimento do design: Ilustrado ao caso de projeto de um automóvel. Gestão & produção **18**(2), 335–350 (2011)
132. Gurbuz, T.: Multiple criteria human performance evaluation using Choquet integral. Int. J. Computat. Intell. Syst. **3**(3), 290–300 (2010)
133. Gurbuz, T., Alptekin, S.E., Alptekin, G.I.: A hybrid MCDM methodology for ERP selection problem with interacting criteria. Decis. Support. Syst. **54**(1), 206–214 (2012)
134. Hurson, C., Mastorakis, K., Siskos, Y.: Application of a synergy of MACBETH and MAUT multicriteria methods to portfolio selection in Athens stock exchange. Int. J. Multicriteria Dec. Mak. **2**(2), 113–127 (2012)
135. Joerin, F., Cool, G., Rodriguez, M.J., Gignac, M., Bouchard, C.: Using multi-criteria decision analysis to assess the vulnerability of drinking water utilities.
136. Junior, H.V., Kienitz, K.H., Belderrain, M.C.N.: Metodologia de apoio a decisão para os processos de seleção de alvos e armamentos. Spectrum - Revista do Comando-Geral de Operações Aéreas **12**, 25–27 (2009) Environ. Monit. Assess. **166**(1–4), 313–330 (2010)
137. Junior, A.G.M., Gonçalves, T.J.M., Belderrain, M.C.N.: MACBETH aplicado ao cálculo da pena base do direito penal militar. INGEPRO - Inovação, Gestão e Produção **3**(1), 56–66 (2011)
138. Junior, A.G.M., Junior, M.M.C., Belderrain, M.C.N., Correia, A.R., Schwanz, S.H.: Multicriteria and multivariate analysis for port performance evaluation. Int. J. Prod. Econ. **140**(1), 450–456 (2012)
139. Karande, P., Chakraborty, S.: Evaluation and selection of flexible manufacturing systems using MACBETH method. Int. J. Ser. Oper. Manag. **16**(1), 123–144 (2013)
140. Karande, P., Chakraborty, S.: Using MACBETH method for supplier selection in manufacturing environment. Int. J. Ind. Eng. Comput. **4**, 259–272 (2013)
141. Khaled, A., Idrissi, M.A.J.: A semi-structured tailoring-driven approach for ERP selection. Int. J. Comput. Sci. Issues **9**(5), 71–80 (2012)
142. Lacerda, R.T.O., Ensslin, L., Ensslin, S.R.: Um estudo de caso sobre gerenciamento de portfólio de projectos e apoio à decisão multicritério. Revista Gestão Industrial **6**(1), 1–29 (2010)
143. Lacerda, R.T.O., Ensslin, L., Ensslin, S.R.: A performance measurement framework in portfolio management: a constructivist case. Manag. Decis. **49**(4), 648–668 (2011)
144. Lacerda, R.T.O., Ensslin, L., Ensslin, S.R.: A performance measurement view of IT project management. Int. J. Product. Perform. Manag. **60**(2), 132–151 (2011)
145. Lauras, M., Marques, G., Gourc, D.: Towards a multi-dimensional project performance measurement system. Decis. Support. Syst. **48**(2), 342–353 (2010)
146. Leitão, M.A.L.S., Thomaz, J.F., Bana e Costa, C.A.: Multimethodology in practice: an application in the Portuguese army performance appraisal model. In: Proceedings of the 44th Annual Conference of the International Military Testing Association, IMTA, pp. 511–527, Ottawa (2002)
147. Leitão, M.A.L.S., Thomaz, J.F., Freire, F.V., Bana e Costa, C.A.: Using decision conferencing and process consultation to facilitate the structuring of a performance appraisal model in the Portuguese army. Research paper 5/2002, CEG-IST, Technical University of Lisbon (2002)

148. Lopes, D.F.: Development of a multicriteria decision aiding model for monitoring and evaluating the performance of health care units. In: Oliveira, J.F., Vaz, C.B. (eds.) Atas do XVI Congresso da Associação Portuguesa de Investigação Operacional, CD-ROM edition, pp. 212–222, Bragança. Associação Portuguesa de Investigação Operacional (2013)
149. Lourenço, J.C.: Apoio à avaliação de propostas no concurso limitado por prévia qualificação para a realização dos projectos de especialidades (ao nível de projecto de execução) para a construção do parque de estacionamento subterrâneo na Praça D. Dinis em Coimbra. Working paper cegist n.14/2002, CEG-IST, Technical University of Lisbon (2002)
150. Mangasarian, O.L.: Nonlinear Programming. McGraw-Hill, New York (1969)
151. Marques, G., Gourc, D., Lauras, M.: Multi-criteria performance analysis for decision making in project management. Int. J. Proj. Manag. 29(8), 1057–1069 (2010)
152. Mateus, R., Ferreira, J.A., Carreira, J.: Multicriteria decision analysis (MCDA): Central Porto high-speed railway station. Eur. J. Oper. Res. **187**(1), 1–18 (2008)
153. Menezes, A.C., Pinheiro, P.R., Pinheiro, M.C.D., Cavalcante, T.P.: A hybrid model to support the diagnosis of disease: a case study for diabetes. In: 26th IEEE Canadian Conference of Electrical and Computer Engineering (CCECE), Regina (2013)
154. Merad, M., Dechy, N., Serir, L., Grabisch, M., Marcel, F.: Using a multi-criteria decision aid methodology to implement sustainable development principles within an organization. Eur. J. Oper. Res. **224**(3), 603–613 (2013)
155. Montignac, F.: An MCDA approach for evaluating hydrogen storage systems for future vehicles. In: 2nd Decision Deck Workshop. Université Paris Dauphine, 2 pp (2008)
156. Montignac, F., Noirot, I., Chaurdoune, S.: Multi-criteria evaluation of on-board hydrogen storage technologies using the MACBETH approach. Int. J. Hydrog. Energy **34**(10), 4561–4568 (2009)
157. Montmain, J., Sanchez, C., Vinches, M.: Multi criteria analyses for managing motorway company facilities: the decision support system SINERGIE. Adv. Eng. Inf. **23**(3), 265–287 (2009)
158. Moraes, L., Garcia, R., Ensslin, L., Conceição, M.J.: The multicriteria analysis for construction of benchmarkers to support the clinical engineering in the healthcare technology management. Eur. J. Oper. Res. **200**(3), 607–615 (2010)
159. Mousset, C., Vansnick, J.C.: About the representation of a precardinal information. Technical report, Université de Mons-Hainaut (2002)
160. Neves, A.R., Lourenço, J.C., Leal, V.: A multi-criteria approach to local energy planning - the case of Barreiro municipality. In: Procedia - Social and Behavioral Sciences, pp. 313–320, Vilamoura, Algarve, Portugal, 4–6 February 2012. 1st International Conference on Operations Research and Enterprise Systems (ICORES), SciTePress (2012)
161. Ntouskas, T., Polemi, N.: STORM-RM: a collaborative and multicriteria risk management methodology. Int. J. Multicriteria Dec. Mak. **2**(2), 159–177 (2012)
162. Nunes, L.C., Pinheiro, P.R., Pequeno, T.C.: An expert system applied to the diagnosis of psychological disorders. In: IEEE International Conference on Intelligent Computing and Intelligent Systems, Shanghai, pp. 363–367 (2009)
163. Nunes, L.C., Pinheiro, P.R., Pequeno, T.C.: Support tool in the diagnosis of major depressive disorder. In: 3rd World Summit on the Knowledge Society, WSKS (2) 2010, Corfu, pp. 136–145 (2010)
164. Nunes, L.C., Pinheiro, P.R., Pequeno, T.C.: Toward an application to psychological disorders diagnosis. Softw. Tools Algoritm. Biol. Syst. **696**, 573–580 (2011)
165. Oliveira, R.C., Lourenço, J.C.: A multicriteria model for assigning new orders to service suppliers. Eur. J. Oper. Res. **139**(2), 390–399 (2002)
166. Oliveira, C.S., Ferreira, M.A., Mota de Sá, F.: Seismic vulnerability and impact analysis: elements for mitigation policies. In: XI Congresso Nazionale "L'ingegneria Sismica in Italia, Genova, 29 pp (2004)

167. Oliveira, M.D., Rodrigues, T.C., Bana e Costa, C.A., Brito de Sá, A.: Prioritizing health care interventions: A multicriteria resource allocation model to inform the choice of community care programmes. In: 15^{o} Congresso da Associação Portuguesa de Investigação Operacional, Coimbra, pp. 177–186. INESC, Coimbra (2011)
168. Oliveira, M.D., Rodrigues, T.C., Bana e Costa, C.A., Brito de Sá, A.: Prioritizing health care interventions: A multicriteria resource allocation model to inform the choice of community care programmes. In: 37th International Conference on Operational Research Applied to Health Services, Cardiff, pp. 255–265 (2011)
169. Oliveira, M.D., Rodrigues, T.C., Bana e Costa, C.A., Brito de Sá, A.: Prioritizing health care interventions: A multicriteria resource allocation model to inform the choice of community care programmes. In: Tanfani, E., Testi, A. (eds.) Advanced Decision Making Methods Applied to Health Care, pp. 139–152. Springer, New York (2012)
170. Omann, I.: Product service systems and their impacts on sustainable development: a multi-criteria evaluation for Austrian companies. Frontiers **16**, 1–34 (2003)
171. Palacios, O.Y.S., Rincón, P.C.N., Corriou, J.P., Pardo, M.C., Fonteix, C.: Multicriteria optimization of production conditions for a new phthalate-free PVC plasticizer. J. Ind. Eng. Chem. (2013). doi:10.1016/j.jiec.2013.09.021
172. Pedro, M.I., Cabral, F.L.: A new innovative model using RFID: a system design and its implementation. Int. J. Latest Trends Financ. Econ. Sci. **3**(2), 508–219 (2013)
173. Porto, J.L.: Analyse multicritère dans le cadre des appels d'offres pour la constrution de travaux publics et prives: Le cas du métro de Porto au Portugal. Newsletter of the European Working Group 'Multiple Criteria Decision Aiding' **Serie II**(15), 1–2 (1999)
174. Rodrigues, A., Pinheiro, P.R., Rodrigues, M.M., Albuquerque, A.B., Gonçalves, F.M.: Towards the selection of testable use cases and a real experience. In: WSKS (2), pp. 513–521 (2009)
175. Rodrigues, T., Oliveira, M.D., Brito de Sá, A.: Multicriteria model to allocate human resources in community care programmes. In: 1^{st} Portuguese Meeting in Bioengineering (ENBENG), Lisbon, Portugal, pp. 1–6, Lisbon, 1–4 March 2011. Bioengineering (ENBENG). doi:10.1109/ENBENG.2011.6026087
176. Romeiro, A.R., Bernasconi, P., Puga, B.P., Andrade, D.C., Sobrinho, R.P.: Assessment of existing and proposed policy instruments for biodiversity conservation in São Paulo - Brazil: a coarse grain analysis. Brazil Mata Atlantice Coarse Grain Case Study Report 3, POLICYMIX (2012)
177. Roubens, M., Rusinowska, A., de Swart, H.: Using MACBETH to determine utilities of governments to parties in coalition formation. Eur. J. Oper. Res. **172**(2), 588–603 (2006)
178. Saaty, T.L.: The Analytic Hierarchy Process. McGraw-Hill, New York (1980)
179. Sanchez, C., Montmain, J., Vinches, M., Mahieu, B.: Planning of maintenance operations for a motorway operator based upon multicriteria evaluations over a finite scale and sensitivity analyses. In: Conference on Informatics in Control, Automation and Robotics, pp. 23–35 (2007)
180. Sanchez-Lopez, R., Bana e Costa, C.A.: El enfoque macbeth para la incorporación de temas transversales en la evaluación de proyectos de desarrollo. In: Leyva López, J.C., Avilés Ochoa, E., Zepeda Rodriguez, J.J. (eds.) Herramientas Operativas para el Análisis Multicriterio del Desarrollo Económico Local, pp. 335–356. Plaza y Valdés, México (2009)
181. Sanchez-Lopez, R., Bana e Costa, C.A., De Baets, B.: The MACBETH approach for multicriteria evaluation of development projects on cross-cutting issues. Ann. Oper. Res. **199**(1), 393–408 (2012)
182. Santos, F.A., Margotti, A.E., Garcia, R.: Multi-Criteria Decision Aid (MCDA) as a tool to support health technology incorporation process. In: Long, M. (ed.) World Congress on Medical Physics and Biomedical Engineering, vol. 39, pp. 692–695, Beijing, 26–31 May 2012. IFMBE Proceedings, Springer, Berlin/Heidelberg (2012)

183. Shah, L.A., Vernadat, F., Etienne, A., Siadat, A.: Value-Risk Graph: a decision-making tool for supply chain and industrial system engineering. In: 6th IFAC Conference on Management and Control of Production and Logistics, vol. 6, pp. 414–419. Center for Information Technology Renato Archer, Fortaleza (2013)
184. Silva, J.S.V., Bana e Costa, C.A.: Especificação de uma ferramenta de apoio à decisão para gestão pública em regiões semi-áridas. Research paper 23/2001, CEG-IST, Technical University of Lisbon (2001)
185. Soares, J.O., Bana e Costa, C.A.: Value judgements in portfolio management. Working paper cegist 28/2001, CEG-IST, Technical University of Lisbon (2001)
186. Soguel, N., Martin, M.J., Tangerini, A.: The impact of housing market segmentation between tourists and residents on the hedonic price for landscape quality. Swiss J. Econ. Stat. **144**(IV), 655–678 (2008)
187. Soguel, N.C., Chatagny, F.: Analysing Municipalities Amalgamation and Collaboration through the Principle of Fiscal Equivalence and the Principal-Agent Model. Contributions á L'Action Publique. J.L. Chappelet, PPUR & Haupt Verlag, 18 pp (2006)
188. Sousa, J.V.M., de v Silva, J.S., Fernandez, M.P.: MAC-OLSR: improving OLSR protocol in mesh networks using multiple metrics. In: 6th Euro American Conference on Telematics and Information Systems, pp. 1–8, Valencia, 23–25 May 2012
189. Stolt, R., Ensslin, L.: Avaliação de pilotos very ligth jets utilizando-se a MCDA-C. Conexão Sipaer **1**(1), 85–103 (2009)
190. Tangerini, A., Pictet, J., Soguel, N.: Using a MultiPLE criteria decision analysis approach for landscape quality assessment. Working paper de l'IDHEAP 17/2006, Chavannes-près-Renens, Chaire de Finances publiques (2006)
191. Thomaz, J.F., Bana e Costa, C.A.: Decision conferencing within a multimethodological framework: Developing a performance appraisal model for the Portuguese army officers. In: 46th Annual Conference of the International Military Testing Association (IMTA) & NATO's Research Task Group on Recruiting and Retention of Military Personnel, IMTA, Brussels, 13 pp (2004)
192. Vernadat, F., Shah, L., Etienne, A., Siadat, A.: VR-PMS: a new approach for performance measurement and management of industrial systems. Int. J. Prod. Res. **51**(23–24), 7420–7438 (2013)
193. Vieira, V., Oliveira, C.S., Bana e Costa, C.A.: A methodology to evaluate the strategic importance of bridges and tunnels considering seismic vulnerability: Application to Lisbon. In: EuroConference on Global Change and Catastrophe Risk Management: Earthquake Risks in Europe, Laxenburg, IIASA, 10 pp (2000)
194. Vindigni, G., Caniglia, E., D'Amico, M., di Vita, G.: Supporting small producers in mexico's economic integration process. A case study on rural michoacan based on an MCDA approach. In: 119th EAAE Seminar 'Sustainability in the Food Sector: Rethinking the Relationship between the Agro-Food System and the Natural, Social, Economic and Institutional Environments', Capri (2010)
195. von Winterfeldt, D., Edwards, W.: Decision Analysis and Behavioral Research. Cambridge University Press, Cambridge (1986)
196. Wancura, H., Mubbala, R., Steinberger-Wilckens, R., Novak, A.: A technology monitoring and assessment tool for technology road mapping. In: 4th European PEFC and H2 Forum, Lucerne, pp. 1–5 (2013)
197. Watson, S.R., Buede, D.M.: Decision Synthesis: The Principles and Practice of Decision Analysis. Cambridge University Press, Cambridge (1987)
198. Zamcopé, C.F., Ensslin, L., Ensslin, S.R., Dutra, A.: Modelo para avaliar o desempenho de operadores logísticos - Um estudo de caso na indústria têxtil. Gestão & Produção **17**(4), 693–705 (2010)

Part V
Non-classical MCDA Approaches

Chapter 12
Dealing with Uncertainties in MCDA

Theodor J. Stewart and Ian Durbach

Abstract This chapter presents various approaches to incorporating formal modelling of risks and uncertainties into multi-criteria decision analysis, in a theoretically valid but also operationally meaningful manner. We consider both internal uncertainties (in the formulation and modelling of the decision problem), and external uncertainties arising from exogenous factors, but with greater attention paid to the latter. After a broad discussion on the meaning of uncertainty, we first review approaches to sensitivity analysis, which is particularly, although not exclusively, relevant to internal uncertainties. We discuss the role, but also some limitations, of representing uncertainties in formal probabilistic structures, linked also to concepts of expected (multi-attribute) utility theory. Such probability/utility approaches may be used in explicitly identifying a most preferred solution, or simply to eliminate certain courses of action when stochastically dominated (in various senses) by others. In some contexts it may be useful to view minimization of various risk measures as additional criteria in more standard MCDA models, and we comment on advantages and disadvantages of such approaches. Finally we discuss the integration of MCDA with scenario planning, in order to deal with deeper uncertainties (not easily if at all representable by probability models), particularly in a strategic planning context. The emphasis throughout is on the practice of MCDA rather than on esoteric theoretical results.

Keywords Multicriteria decision analysis • Risk • Uncertainty • Sensitivity analysis • Utility theory • Scenario planning

---

T.J. Stewart (✉)
Department of Statistical Sciences, University of Cape Town, Rondebosch 7701, South Africa
e-mail: theodor.stewart@uct.ac.za

I. Durbach
Department of Statistical Sciences, University of Cape Town, Rondebosch 7701, South Africa

Research Center, African Institute for Mathematical Sciences, 6–8 Melrose Road, Muizenberg 7945, South Africa
e-mail: ian.durbacj@uct.ac.za

S. Greco et al. (eds.), *Multiple Criteria Decision Analysis*, International Series in Operations Research & Management Science 233,
DOI 10.1007/978-1-4939-3094-4_12

12.1 What is Uncertainty?

The term uncertainty can have many different meanings. The Chambers Dictionary (1998 edition) defines "uncertain" as not definitely known or decided; subject to doubt or question. Klir and Folger [61] quote six different definitions for "uncertainty" from Webster's Dictionary. In the context of practical applications in multicriteria decision analysis, however, the definition given by Zimmermann [109] would appear to be particularly appropriate. With minor editing, this is as follows:

> Uncertainty implies that in a certain situation a person does not possess the information which quantitatively and qualitatively is appropriate to describe, prescribe or predict deterministically and numerically a system, its behaviour or other characteristics.

At a most fundamental level, uncertainty relates to a state of the human mind, i.e. lack of complete knowledge about something. Many writers also use the term "risk", although the definition of the term varies widely. Some earlier work tended to apply the term "risk" to situations in which probabilities on outcomes are (to a large extent) known objectively (cf. [39, p. 389], [76] for some reference to this view). More recently, the concept of risk has come to refer primarily to the desirability or otherwise of uncertain outcomes, in addition to simple lack of knowledge. Thus, for example, [34] refers to risk as "a chance of something bad happening", and in fact separates uncertainty (alternatives with several possible outcome values) from the fundamental concept of risk as a bad outcome. Sarin and Weber [87] state that "judgements about riskiness depend on both the probability and the *magnitude of adverse effects*" (our emphasis), while [54] also discuss the psychological aspects of establishing a preference order on risks.

For the most part in this chapter, we shall make use of the value-neutral term "uncertainty", referring to "risk" only when direct preference orderings of the uncertainty *per se* are relevant (for example, in Sect. 12.5). It is interesting to note in passing that while the thrust of the present discussion is to give consideration to the effects of uncertainty on MCDA, there has also been work on applying multicriteria concepts to the measurement of risk for other purposes, as for example in credit risk assessment ([28], who make use of a rough sets approach).

A number of authors (e.g. [35, 109]) have attempted to categorize types or sources of uncertainty in the context of decision making. French [35], for example, identifies no less than ten different sources of uncertainty which may arise in model building for decision aid, which he classifies into three groups referring broadly to uncertainties in the modelling (or problem structuring) process, in the use of models for exploring trends and options, and in interpreting results. The common theme underlying such categorizations, as well as those of other authors, such as Friend [37] and Levary and Wan [68], is the need at very least distinguish between *internal uncertainty*, relating to the process of problem structuring and analysis, and *external uncertainty*, regarding the nature of the environment and thereby the consequences of a particular course of action which may be outside of the control of the decision maker. Let us briefly examine each of these broad categories of uncertainty.

12.1.1 Internal Uncertainty

This refers to both the structure of the model adopted and the judgmental inputs required by those models, and can take on many forms, some of which are resolvable and others which are not. Resolvable uncertainties relate to imprecision or ambiguity of meaning—for example, what exactly may be meant by a criterion such as "quality of life"? Less easily resolvable problems may arise when different stakeholders generate different sets of criteria which are not easily reconciled; or perceive alternatives in such different ways that they differ fundamentally on how they contribute to the same criterion.

Imprecisions in human judgments, whether these relate to specifications of preferences or values (for example importance weights in many models), or to assessments of consequences of actions, have under certain circumstances been modelled by fuzzy set (see, for example, Chaps. 4 and 5 of Klir and Folger [61]) and related approaches (such as the use of rough sets as described by Greco et al. [41–43]. From the point of view of practical decision aid, such models of imprecision add complexity to an already complex process, and the result may often be a loss of transparency to the decision maker, contrary to the ethos of MCDA. For this reason, the view espoused here is that internal uncertainties should ideally be resolved as far as is possible by better structuring of the problem (cf. [11, Chap. 3]) and/or by appropriate sensitivity and robustness analysis where not resolvable, which will further be discussed in Sect. 12.2. The *evidential reasoning (ER)* approach described by Xu [102], to which we shall refer again at the end of Sect. 12.3, does provide a more formal model for integrating imprecise preference information that cannot fully be resolved.

12.1.2 External Uncertainty

This refers to lack of knowledge about the consequences of a particular choice. Friend [37] and French [35] both recognize a further distinction between uncertainty about the environment and uncertainty about related decision areas, as described below.

- *Uncertainty about the environment* represents concern about issues outside the control of the decision maker. Such uncertainty may be a consequence of a lack of understanding or knowledge (in this sense it is similar to uncertainty about related decision areas) or it may derive from the randomness inherent in processes (for example the chance of equipment failure, or the level of the stock market). For example, the success of an investment in new production facilities may rest on the size of the potential market, which may depend in part on the price at which the good will be sold, which itself depends on factors such as the cost of raw materials and labour costs. A decision about whether or not to invest in the new facilities must take all of these factors into account. This kind of uncertainty may be best handled by responses of a technical nature such as market research, or forecasting.

- *Uncertainty about related decision areas* reflects concern about how the decision under consideration relates to other, interconnected decisions. For example, suppose a company which supplies components to computer manufacturers is looking to invest in a management information system. They would like their system to be able to communicate directly with that of their principal customers; however, at least one of these customers may be planning to install a new system in the near future. This customer's decision could preclude certain of the options open to the supplier and would certainly have an impact on the attractiveness of options. The appropriate response to uncertainty of this kind may be to expand the decision area to incorporate interconnected decisions, or possibly to collaborate or negotiate with other decision makers.

Under many circumstances, both internal and external uncertainties can be treated in much the same manner, for example by appropriate sensitivity analyses *post hoc*. In other words, the approach might be to make use of a crisp deterministic MCDA methodology, and to subject the results and conclusions to extensive sensitivity studies. Indeed, we would assert that such sensitivity studies should routinely be part of any MCDA application, and some approaches are discussed in Sect. 12.2.

Where uncertainties are of sufficient magnitude and importance to be modelled explicitly as part of the MCDA methodology, however, the modelling approaches for internal and external uncertainties may often become qualitatively different in nature. It seems, therefore, that the treatment of the two types of uncertainty should preferably be discussed in separate papers or chapters. In order to provide focus for the present paper, our attention will be focussed primarily, apart from Sect. 12.2, on consideration of the *external uncertainties* as defined above. Without in any way minimizing the importance of dealing with internal uncertainties, our choice of the problem of external uncertainties as the theme for this chapter is in part due to the present authors' practical experience, which suggests that it is the external uncertainties which are often of sufficient magnitude and importance to require more explicit modelling. The present chapter complements in many ways the survey paper by Durbach and Stewart [32] which does include more on internal uncertainties and the behavioural models of uncertainty and risk perception. (It should perhaps be acknowledged that there is also some inevitable overlap between Durbach and Stewart [32] and the current chapter, but the thrusts are still distinct.)

Admittedly, the boundary between external uncertainty and imprecision is, well, fuzzy! To this extent, some of the material in this chapter is appropriate to internal uncertainties as well, while some methods formulated to deal with human imprecision might equally well be useful in dealing with external uncertainties. We leave it to the reader to decide where this may be true. We do not attempt here a comprehensive review of literature related primarily to internal uncertainties, but the interested reader may wish to consult some of the following references:

- Fuzzy set approaches: [22, 23, 61, 104]; some discussion may also be found in [32];
- Rough set approaches: [41–44].

Our approach is pragmatic in intention, motivated by practical needs of real-world decision analysis. In particular, the fundamental philosophical point of departure is a belief in the over-riding need for *transparency* in any MCDA: it is vitally and critically important that any approaches to MCDA are fully understandable to all participants in the process. Elegant mathematical models which are inaccessible to such participants are of very little practical value.

Within the context of the opening discussion, let us now define a notational framework within which to consider MCDA under uncertainty (primarily "external uncertainty" as defined earlier). Let X be the set of actions or decision alternatives. When there is no uncertainty about the outcomes, there exists a one-to-one correspondence between elements of X and consequences in terms of the criteria, and X may written as the product space $\prod_{i=1}^{n} X_i$, where X_i is the set of evaluations with respect to criterion i. In other words, any $x \in X$ may be viewed as an n-dimensional vector with elements $x_i \in X_i$, where x_i represents the evaluation of x with respect to the criterion i.

Under uncertainty, however, the one-to-one correspondence between actions and evaluations or consequences breaks down. It may be possible to postulate or to conceptualize an ultimate set of consequences $Z_1(x), \ldots, Z_n(x)$ corresponding to each of the criteria, but at decision time there will still exist many possible values for each $Z_i(x)$. For ease of notation, we shall use $\mathbf{Z}(x)$ to indicate the vector of $Z_i(x)$ values.

In some cases, it may be possible and useful to structure $Z_i(x)$ (or $\mathbf{Z}(x)$) in the form $Z_i(x, \xi)$ (or $\mathbf{Z}(x, \xi)$), where $\xi \in \Xi$ fully characterizes the external conditions, sometimes termed the "states of nature", and Ξ represents the set of all possible states of nature. The assumption is then that once ξ (the state of nature) is established or revealed, then the consequences in terms of each criterion will also be known. We observe, however, that even Ξ might not be fully known or understood at decision time, and that Ξ could possibly depend upon the action x (although, for ease of notation, we shall not show this explicitly).

The question to be addressed in this chapter is that of constructing some form of (possibly partial) preference ordering on X, when the consequences are incompletely known or understood in the sense described in the previous paragraph.

As indicated earlier, one approach may be initially to ignore the uncertainty, and to conduct the analysis on the basis of a nominal set of consequences $z_1, z_2, \ldots, z_n$ chosen to be representative of the possible $Z_i(x)$, followed by extensive sensitivity analysis which takes into account the range of uncertainty in each $Z_i(x)$. Under many circumstances this may be adequate. Care needs to be exercised in undertaking sensitivity analyses, however, as simple "one-at-a-time" variations in unknown parameter values may fail to identify effects of higher order interactions. Some of the complications inherent in undertaking properly validated sensitivity analyses, and suggestions as to how these may be addressed, are discussed by Rios Insua

[83], Parnell et al. [81], and Saltelli [86]. Section 12.2 describes some practical approaches for managing such sensitivity studies.

In the remainder of this chapter, the focus will be on situations in which the ranges of uncertainty are too substantial to be handled purely by sensitivity analysis. In Sect. 12.3 we discuss the use of probability models and related methods to represent the uncertainties formally, emphasizing particularly the comprehensively axiomatized approach of multiattribute utility theory. The potential for relaxing the needs to specify complete utility functions are addressed in Sect. 12.4, which leads naturally to the use of pairwise comparison models for MCDA. In many practical situations, decision maker preferences for various types of risk (magnitude and impact of the uncertainties) may be modelled by defining explicit risk-avoidance criteria, and these are discussed in Sect. 12.5. Finally, links between MCDA and scenario planning for dealing with uncertainties are presented in Sect. 12.6, before concluding with some general implications for practice.

12.2 Sensitivity Analysis and Related Methods

For the purposes of this section, we postulate the existence of an "evaluation function" $\Psi(\mathbf{Z}(x,\xi),\phi)$, which indicates a degree of satisfaction associated with the outcome of the decision. In this formulation:

- The function $\Psi(\mathbf{Z}(x,\xi),\phi)$ could be a utility, a distance from a desired outcome, etc.;
- The factors ξ and ϕ represent respectively the external influences (incompletely known, and outside of the decision makers' control) on consequences of the decision, and the internal uncertainties as to how these consequences should be evaluated in terms of decision maker goals (e.g. importance weights, tradeoffs).

The aim of sensitivity analysis is typically to identify potentially optimal solutions amongst uncertainty ranges in ξ (external) and ϕ (internal). Sensitivity analysis is aimed at providing insights into:

1. whether the outcome of the decision model changes as ξ and/or ϕ take on different values within the stated bounds. For simplicity of presentation here, we shall assume a choice problematique i.e. the selection of a single preferred alternative;
2. the values of ξ and ϕ for which each alternative may be deemed to be the best.

Sensitivity analysis is most appropriately applied when the uncertainties are essentially subjective in nature, i.e. either internal uncertainties (ϕ) or situations in which the state is already determined (not subject to future random fluctuations) but still unknown. For ease of presentation we shall denote the combination of subjective uncertainties in state (typically state probabilities) and internal uncertainties by $\psi = (\xi,\phi)$, and assume that there are no other external random influences. In this case, we shall express the evaluation function simply as $\Psi(\mathbf{Z}(x),\psi)$.

If the decision maker has provided a precise specification of elements of ψ, sensitivity analysis involves varying ψ away from these specified values and examining the impact on results. This can be done in an *ad hoc* fashion, although a preferable approach is to use one of the many well-known methods for systematically exploring the space of possible preference parameters (see the review in [52]). Many of the so-called "interactive" or "progressive articulation of preferences" methods (e.g. [42] and especially [92]) may also be useful as tools for sensitivity analysis.

If no precise specification of ψ can be given, alternative forms of sensitivity analysis are provided by inverse-preference and preference disaggregation models. (Interval-based decision models [75] may also be used, but fall outside the scope of the aims of present section.) Inverse preference models typically work by providing information about the volume and types of values for ψ (if any) that would lead to the selection of each alternative. Effectively, instead of asking 'which alternative is best given a particular ψ?', one asks for example 'what ranges of or possible values for ψ would result in a particular alternative being considered the best?'. Partial or total ignorance about possible values for ψ is incorporated through appropriate probability distributions defined over these inputs.

One such inverse-preference method is stochastic multi-criteria acceptability analysis (SMAA). The original SMAA method [66] analysed the combinations of attribute weights (internal uncertainties) that result in each of a set of prospective alternatives being selected when using an additive utility function. Subsequently a number of variants have been developed. These differ in terms of the preference model used and the type of information that is imprecisely known, but are all based upon Monte Carlo simulation from distributions which indicate the extent of the uncertainty in ψ. For example, SMAA variants are available for value functions [64, 66], outranking [50], reference point methods [29, 67], and prospect theory [65] methods. Several probabilistic AHP models [7, 69] also use Monte Carlo simulation to randomly generate pairwise evaluations from the distributions specified by decision makers, in similar fashion to SMAA.

For illustration, the process described here relates to uncertain *importance weight* information, but can readily be extended to other subjective uncertainties. SMAA in this context is based on simulating a large number of random weight vectors from a probability distribution defined over the weight space and observing the proportion and distinguishing features of weight vectors which result in each alternative obtaining a particular rank r (usually the "best" rank, $r = 1$). Other uncertain evaluations, e.g. partial value assessments in value function methods, are also conventionally treated in SMAA using probability distributions, with each simulation run drawing values at random from these distributions. Adapting SMAA models to use other uncertainty formats, however, is generally straightforward [30]. In any case, in order to illustrate the process for uncertain weights, let the set of (randomly generated) weight vectors that result in alternative a_i obtaining rank r be denoted by W_i^r. SMAA is based on an analysis of these sets of weights using a number of descriptive measures, the most important of which are:

Acceptability indices The rank-r acceptability index b_i^r measures the proportion of all simulation runs i.e. weight vectors, that make alternative a_i obtain rank r. A cumulative form of the acceptability index called the k-best ranks acceptability index is defined as $\mathcal{B}_i^k = \sum_{r=1}^{k} b_i^r$ and measures the proportion of all weight vectors for which alternative a_i appears anywhere in the best k ranks.

Central weight vectors The central weight vector $\mathbf{w}_i^c$ is defined as the center of gravity of the favourable weight space W_i^1. It gives a concise description of the "typical" preferences supporting the selection of a particular alternative a_i, and in practice is computed from the empirical (element-wise) averages of all weight vectors supporting the selection of a_i as the best alternative.

Ranges on favourable weights These simply indicate the minima and maxima of the observed favourable weights supporting alternative a_i.

Preference disaggregation models also aim to provide information on conditions under which one or more alternatives may be preferred to others, particularly with regard to internal uncertainty. These models typically use a set of global preference statements to infer the parameters of a preference model before applying that model to a larger set of alternatives to arrive at a choice or ranking or classification. In the original UTA method (see [53]), the breakpoints of piecewise linear marginal value functions are estimated by a linear program whose main elements are the constraints $U(a) > U(b) \iff a \succ b$ and $U(a) = U(b) \iff a \sim b$, along with some technical constraints (e.g. imposing montonicity and a zero-point). Of course, more than one set of value functions may be compatible with the specified global preference statements. The robust ordinal regression approach [46] addresses this issue by providing "necessary" preference relations indicating support from *all* compatible value functions, and "possible" preference relations indicating support from *at least one* compatible value function. A linear programming model is constructed to determine whether a given alternative a is possibly or necessarily preferred to an alternative b in the light of available preference information.

In addition, preference statements can be in the form of ranking preference differences as well as alternatives, and value functions are not constrained to be piecewise linear. A number of extensions of the basic robust ordinal regression approach have been made to accommodate sorting problems [45], nonadditive functions [4], and outranking methods [47].

This section on sensitivity analysis has focussed on subjective and particularly internal uncertainties. We shall now, for the remainder of the chapter focus on external uncertainties.

12.3 Probabilistic Models and Expected Utility

The most thoroughly axiomatized mathematical treatment of uncertainty is that of probability theory, and possibly extensions such as Dempster-Shafer theory [88]. The application of probability concepts requires the specification of a (multivariate) probability distribution on $\mathbf{Z}(x)$ for each action x, so that in effect the decision requires a comparison of probability distributions (sometimes called "lotteries" in this context). Let $\mathbf{P}^x(\mathbf{z})$ denote the probability distribution function on $\mathbf{Z}(x)$, i.e.:

$$\mathbf{P}^x(\mathbf{z}) = \Pr[Z_1(x) \leq z_1, Z_2(x) \leq z_2, \ldots, Z_n(x) \leq z_n].$$

Define $P_i^x(z_i)$ as the corresponding marginal probability distribution function for $Z_i(x)$.

Where uncertainties are structured in terms of "states of nature", the probability distributions may be defined on the ξ (rather than on the $\mathbf{Z}(x)$ directly). In some situations, the probability distribution on ξ may be independent of the action which would make the application of probability models much more tractable, but this will not necessarily always be the case.

A possibility at this stage is to construct a deterministic MCDA model based only on expectations, and to subject the results to some form of (possibly interactive) sensitivity analysis, such as described in the previous section, guided by the known distributional properties. Examples of this are in the PROTRADE method described by Goicoechea et al. [39, Chap. 7], dealing with an interactive method for multiobjective mathematical programming problems, and in the stochastic extensions to outranking proposed by Mareschal [72].

Although simulations reported in [31] suggest that simple expectation models can often return similar results to models taking the full ranges of outcomes into account, this conclusion clearly cannot be generalized to all situations. Multiattribute utility theory (MAUT) extends the concept of expectation to include explicit modelling of risk preferences, i.e. of the magnitudes of dispersion that may occur. MAUT is discussed by Dyer in Chap. 8 of this volume, and also more comprehensively in the now classic texts of Keeney and Raiffa [57] and von Winterfeldt and Edwards [97]. In essence, MAUT seeks to construct a "*utility function*" $U(\mathbf{Z})$, such that for any two actions x and y in X, $x \succsim y$ if and only if $\mathrm{E}[U(\mathbf{Z}(x))] \geq \mathrm{E}[U(\mathbf{Z}(y))]$, where expectations are taken with respect to the probability distributions on $\mathbf{Z}(x)$ and on $\mathbf{Z}(y)$ respectively.

Practically, the construction of the global utility function $U(\mathbf{Z})$ starts with the construction of partial or marginal utility functions individually for each attribute, say $u_i(Z_i)$, satisfying the expected utility hypothesis for variations in Z_i only. The axioms underlying the existence of such marginal utility functions and the methods for their construction are well-known from univariate decision analysis (see, for example, Chap. 8, or [40, Chap. 6]). It is well-established that these axioms are not descriptively valid, in the sense that decision makers do systematically violate them (see, for example, the various paradoxes described by Kahneman and Tversky [56], or in the text of Bazerman [9]). Attempts have been made to extend the utility models to account for observed behaviour (see, for example, [78] for a review of such extensions in the multicriteria context). Nevertheless, as we have argued elsewhere (e.g.,[11, Sect. 4.3.1]), descriptive failures do not lessen the value of the simpler axiomatically based theory of MAUT as a coherent discipline within which to construct preferences in a simple, transparent and yet defensible manner.

The real challenge relates to the aggregation of the $u_i(Z_i)$ into a $U(\mathbf{Z})$ still satisfying the expected utility hypothesis for the multivariate outcomes. The two simplest forms of aggregation are the *additive* and *multiplicative*, which we shall now briefly review (although a full description can be found in Chap. 8).

Additive aggregation. In this case, we define:

$$U(\mathbf{Z}) = \sum_{i=1}^{n} k_i u_i(Z_i). \tag{12.1}$$

This model is only justifiable if the criteria are *additively independent*, i.e. if preferences between the multivariate lotteries depend only on the marginal probability distributions. That this is not an entirely trivial assumption may be seen by considering two-dimensional lotteries ($n = 2$) in which there are only two possible outcomes on each criterion, denoted by z_i^0 and z_i^1 for $i = 1, 2$. Suppose that $z_i^1 \succ z_i^0$. Then without loss of generality, the partial utility functions can be standardized such that $u_1(z_1^0) = u_2(z_2^0) = 0$ and $u_1(z_1^1) = u_2(z_2^1) = 1$. Consider then a choice between two lotteries defined as follows:

- The lottery giving equal chances on $(z_1^0 \ ; \ z_2^0)$ and $(z_1^1 \ ; \ z_2^1)$; and
- The lottery giving equal chances on $(z_1^0 \ ; \ z_2^1)$ and $(z_1^1 \ ; \ z_2^0)$.

We note that both lotteries give the same marginal distributions on each Z_i, i.e. equal chances on each of z_i^0 and on z_i^1 for each i. It is easily verified that with additive aggregation defined by (12.1), both of these lotteries yield an expected utility of $(k_1 + k_2)/2$. The additive model thus suggests that the decision maker should always be indifferent between these two lotteries. There seems, however, to be no compelling axiomatic reason for forcing indifference between the above two options. Where there is some measure of compensation between the criteria (in the sense that good performance on one can compensate for poorer outcomes on the other), the second option may be preferred as it ensures that one always gets some benefit (a form of multivariate risk aversion). On the other hand, if there is need to ensure equity between the criteria (if they represent benefits to conflicting social groups, for example), then the first lottery (in which loss or gain is always shared equally) may be preferred.

Multiplicative aggregation. Now we define $U(\mathbf{Z})$ such that:

$$1 + kU(\mathbf{Z}) = \prod_{i=1}^{n} [1 + kk_i u_i(Z_i)] \tag{12.2}$$

where the multivariate risk aversion k parameter satisfies:

$$1 + k = \prod_{i=1}^{n} [1 + kk_i] \tag{12.3}$$

Use of the multiplicative model requires that the condition of *mutual utility independence* be satisfied. A subset of criteria, say $C \subset \{1, 2, \ldots, n\}$ is said to be utility independent of its complement $\bar{C} = \{1, 2, \ldots, n\} \setminus C$, if preferences for lotteries involving only Z_i for $i \in C$ *for fixed values* of Z_i for $i \in \bar{C}$ are independent of these fixed values. The criteria are said to be mutually utility independent if every subset of the criteria is utility independent of its complement.

In principle, however, there are no compelling reasons why criteria *should necessarily be* mutually utility independent, and in fact it can be difficult in practice to verify that the condition holds. Good problem structuring for MCDA would seek to ensure preferential independence of some form between criteria (for example, such that trade-offs between pairs of criteria are independent of outcomes on other criteria), but mutual utility independence is a stronger assumption and more elusive concept than simple preferential independence.

Models based on weaker preference assumptions have been developed, such as the multilinear model given by:

$$U(\mathbf{Z}) = \sum_{i=1}^{n} k_i u_i(Z_i) + \sum_{i=1}^{n} \sum_{i<j\leq n} k_{ij} u_i(Z_i) u_j(Z_j) + \ldots + k_{12\ldots n} u_1(Z_1) u_2(Z_2) \ldots u_n(Z_n) \quad (12.4)$$

The large number of parameters which have to fitted to decision maker preferences is prohibitive in most real world applications. Even the multiplicative model is far from trivial to apply in practice. Its construction involves the following steps:

- *Assessment of the partial utilities* $u_i(Z_i)$ by standard single attribute lottery procedures.
- *Parameter estimation:* The multiplicative model includes $n + 1$ parameters which have in principle to be estimated, although in the light of (12.3), only n independent parameters need estimation. Estimates thus require at least n preference statements concerning hypothetical choices to be made by the decision maker. Some of these can be based on deterministic trade-off assessments, but at least one of the hypothetical choices must involve consideration of preferences between multivariate lotteries.

In exploring the literature, it is difficult to find many reported applications even of the multiplicative model, let alone the multilinear model. Some of the practical complications of properly implementing these models are illustrated by Rosqvist [85] and Yilmaz [105].

Such difficulties of implementation raise the question as to how sensitive the results of analysis may be to the use of the additive model (12.1) instead of the more theoretically justifiable aggregation models given by (12.2) or (12.4). We have seen earlier that situations can be constructed in which the additive model may generate misleading results. But how serious is this in practice? Construction of the additive model requires much less demanding inputs from the decision maker, and it may be that the resultant robustness or stability of the model will compensate for biases introduced by use of the simpler model. In [89] a number of simulation studies are reported in which the effects are studied of using the additive aggregation model when "true preferences" follow a multiplicative aggregation model. Details may be found in the cited reference, but in essence it appeared that the errors introduced by using the additive model were generally small for realistic ranges of problem settings. The errors were in any case substantially smaller than those introduced by incorrect modelling of the partial utility functions (such as by over-linearization of the partial functions which appears to be a frequent but erroneous simplification). Related work [90] has also demonstrated that more fundamental violations of preferential independence may also introduce substantial errors.

Concerns about the validity of the fundamental axiomatic foundations of utility theory, even for single criterion problems, have led other writers to formulate alternative models to circumvent these. From the standpoint of prescriptive decision aid, a particular concern is that several utility techniques for eliciting the marginal value functions $u_i(z_i)$ (e.g. certainty-equivalence and probability-equivalence methods) assume that the axioms of EUT hold during the elicitation process [17], even though these axioms are known not to be descriptively valid. Utility function assessments based on elicited responses from decision makers who do not follow EUT may thus be systematically biased. Importantly, this concern for the validity of estimated marginal utility functions relates to observed or descriptive behaviour, and is thus independent of any debate around the desirability of the axioms in a normative decision aiding sense. Wakker and Deneffe [98] propose an alternative assessment method—the gamble trade-off method—that does not depend on the actual probability values, and is thus robust to the kinds of probability transformations that decision makers often use. These procedures are extended in [1, 16] to allow for the assessment of both non-expected utility and probability weighting functions, and in [3] to allow the full assessment of the prospect theory utility function i.e. one that is defined over the whole domain of losses and gains. A number of authors [15, 78, 106] have reviewed generalizations to utility theory and developed procedures for the decomposition of multi-attribute non-expected utility functions, while others (e.g. [13, 103]) relax the demands of probability theory by invoking concepts from Dempster-Shafer theory of evidence.

Unfortunately, these generalizations tend often to make the models even more complex and thus less transparent to decision makers, further aggravating difficulties of implementation. Our overall conclusion is thus that in the practical application of expected utility theory to decision making under uncertainty, the use of the additive aggregation model is likely to be adequate in a many settings. The imprecisions and uncertainties involved in constructing the partial utilities, which need in any case to be addressed by careful sensitivity analysis, are likely to outweigh any distinctions between the additive and multiplicative models. In fact, given that marginal utility functions based on preferences between hypothetical lotteries may generally not differ markedly from deterministic value functions based on relative strengths of preference (e.g. [97, Chap. 10]), we conjecture that even the first step of the model construction could be based on the latter (e.g. by use of the SMART methodology, [97, Sect. 8.2]). Some recent evidence in support of this view has been provided by Abdellaoui et al. [2]. Nevertheless, situations may arise when simplified utility models are simply inadequate, and some of the other models discussed below may need to be considered.

12.4 Pairwise Comparisons

As indicated in the previous section, the requirements of fitting a complete utility function can be extremely demanding both for the decision maker (in providing the necessary judgemental inputs) and for the analysts (in identifying complete multivariate distributions). We have seen how the assumption of a simple additive model may substantially reduce these demands without serious penalty in many practical situations. Nevertheless, other attempts at avoiding the construction of the full utility model have been made.

Even for single criterion models, the construction and validation of the complete utility model may be seen as too burdensome. Quite early work recognized, however, that it may often not be necessary to construct the full utility function in order to confirm whether one alternative is preferred to another. The conclusions may be derived from the concepts of *stochastic dominance* introduced by Hadar and Russell [48], and extended (to include third order stochastic dominance) by Whitmore [101].

For purposes of defining stochastic dominance, suppose for the moment that there is only one criterion which we shall denote by $Z(x)$ (i.e. unsubscripted). Then let $P^x(z)$ be the (univariate) probability distribution function of $Z(x)$, i.e.: $P^x(z) = \Pr[Z(x) \leq z]$. With some abuse of notation, we shall use P^x (without argument) to denote the probability distribution described by the function $P^x(z)$. Suppose also that values for $Z(x)$ are bounded between z^L and z^U.

Three degrees of stochastic dominance may then be defined as follows.

First degree stochastic dominance (*FSD*): P^x stochastically dominates P^y in the *first degree* if and only $P^x(z) \leq P^y(z)$ for all $z \in [z^L, z^U]$ [48].

Second degree stochastic dominance (*SSD*): P^x stochastically dominates P^y in the *second degree* if and only:

$$\int_{z^L}^{\zeta} P^x(z)dz \leq \int_{z^L}^{\zeta} P^y(z)dz$$

for all $\zeta \in [z^L, z^U]$ [48].

Third degree stochastic dominance (*TSD*): P^x stochastically dominates P^y in the *third degree* if and only $\mathrm{E}[Z(x)] \geq \mathrm{E}[Z(y)]$ and:

$$\int_{z^L}^{\eta} \int_{z^L}^{\zeta} P^x(z)dzd\zeta \leq \int_{z^L}^{\eta} \int_{z^L}^{z} P^y(z)dzd\zeta$$

for all $\eta \in [z^L, z^U]$ [101].

In this single-criterion case, the standard axioms of expected utility theory imply the existence of a utility function $u(z)$ such that $x \succ y$ if and only if:

$$\int_{z^L}^{z^U} u(z)dP^x(z > \int_{z^L}^{z^U} u(z)dP^y(z).$$

Without having explicitly to identify the utility function, however, considerations of stochastic dominance allow us to conclude the following [8]:

1. If P^x stochastically dominates P^y in the first degree (P^x *FSD* P^y), then $x \succ y$ provided that $u(z)$ is an increasing function of z (which can be generally be assumed to be true in practical problems).
2. If P^x *SSD* P^y, then $x \succ y$ provided that $u(z)$ is a concave increasing function of z (i.e. the decision maker is risk averse).
3. If P^x *TSD* P^y, then $x \succ y$ provided that $u(z)$ is a concave increasing function of z with positive third derivative (corresponding to a risk averse decision maker exhibiting decreasing absolute risk aversion).

The potential importance of the above results lies in the claim which has been made that in practice some form of stochastic dominance may hold between many pairs of probability distributions. In other words, we may often be able to make pairwise comparisons between alternatives according to a particular criterion on the basis of stochastic dominance considerations, without needing to establish the partial value function for comparison of lotteries. In fact, we may often argue that FSD provides a strict pairwise preference, while SSD and TSD provide weaker forms of pairwise preference. Only in the absence of any stochastic dominance would we be unable to determine a preference without obtaining much stronger preference information from the decision maker.

Many of the more recent developments in this area have focussed on the problem of continuous optimization under stochastic dominance constraints (see, for example, [27]), often in the context of (single-criterion) portfolio optimization [84]. However, for discrete decision problems the existence of pairwise preferences at the level of a single criterion under uncertainty suggests that some form of outranking approach may be appropriate to aggregation across multiple criteria under uncertainty. A number of approaches [24, 26, 33, 71, 74] compare distributions by constructing a matrix $\mathbf{P}^j$ whose entries P^j_{ik} denote the probability that alternative a_i is superior to alternative a_k on criterion c_j i.e. $\Pr[Z_{ij} \geq Z_{kj}]$. The models differ with respect to the subsequent exploitation of the probabilities. Dendrou et al. [26] and Liu et al. [71] both aggregate the P^j_{ik} using a weighted sum over attributes to arrive at a global index for each pairwise comparison P_{ik}. Fan et al. [33] compute joint probabilities associated with each of 2^J possible permutations of binary indicators denoting (attribute-specific) outranking between a pair of alternatives. Each of these is taken as evidence in favour of the 'superiority', 'inferiority', or 'indifference' of a_i relative to a_k, based on a comparison with a user-defined threshold. A further algorithm is required to exploit the results. Martel [74] incorporate more sophisticated outranking concepts such as indifference and preference thresholds, but subsequent aggregation and exploitation proceeds in a similar fashion to ELECTRE III. D'Avignon and Vincke [24] compute stochastic "preference indices" measuring the degree of preference for one lottery over another in terms of one criterion, to be aggregated according to an outranking philosophy. Their preference indices may not be easily interpretable by many decision makers however, and perhaps with this problem in mind, [73] (but see also [5]) suggested an alternative outranking approach in which preferences according to individual criteria were established as far as possible by stochastic dominance considerations.

Martel and Zaras found it useful to introduce two forms of concordance index, which they term "explicable" and "non-explicable". For the "explicable" concordance, x is judged at least as good as y according to criterion i if P_i^x stochastically dominates P_i^y at first, second or third degrees. This can be quite a strong assumption, as the preference assumption under TSD requires decreasing absolute risk aversion. The "non-explicable" concordance arises if neither of P_i^x or P_i^y stochastically dominates the other. The authors concede that in this case it is not certain that x is at least as good as y, but they do combine the two indices under certain conditions. The discordance when comparing x to y is only non-zero in their model if P_i^y FSD P_i^x. The extensions of Azondékon and Martel [5], Nowak [80], and Zhang et al. [108] are largely concerned with constructing more fine-grained indices of stochastic dominance. Dominance-based methods have also been extended to make use of other data types, notably fuzzy numbers, and possibilistic and evidentiary evaluations [12, 20, 107]. These initially transform uncertain quantities so that they assume some of the properties of probability distributions before applying standard dominance concepts. Notably, this allows for the possibility of using several different data types in the same decision problem.

Although the implementation of many of the dominance-based approaches remain untested, they may have potential as an approach to dealing with uncertainty in MCDA using quite minimal preference information from the decision maker. This might at least be valuable for a first-pass screening of alternatives. Two problems may, however, limit wide applicability, especially in the MCDA context:

- Strong independence assumptions are implicitly made: The approach is based entirely on the marginal distributions of the elements of $\mathbf{Z}(x)$. This would only be valid if these elements (i.e. the criteria) were stochastically independent, or if the decision maker's preferences were additively independent in the sense of Keeney and Raiffa [57]. Either assumption would need to be carefully justified.
- Strong risk aversion assumptions are made: As indicated above, the method as proposed bases concordance measures on risk aversion and on decreasing absolute risk aversion. Especially the latter assumption may not always be easy to verify. The method can be weakened by basing concordance either only on FSD or on FSD and SSD, but this may not generate such useful results.

There is clear scope for further research aimed at addressing the above problems.

12.5 Risk Measures as Surrogate Criteria

In this and the next sections, we move to more pragmatic approaches to dealing with uncertainty in the multicriteria context.

One obvious modelling approach is to view avoidance of risks as decision criteria in their own right. For example, the standard Markowitz portfolio theory (cf. [54]) represents a risky single-criterion objective (monetary reward) in terms of what are effectively two non-stochastic measures, namely expectation and standard deviation of returns. In this sense a single criterion decision problem under uncertainty is structured as a deterministic bi-criterion decision problem. The extension to risk components for each of number of fundamental criteria is obvious (see, for example, [76, p. 104], in the context of AHP).

There has, in fact, been a considerable literature on the topic of measuring risk for purposes of decision analysis, much of it motivated by the descriptive failures of expected utility theory. Papers by Sarin and Weber [87], Jia and Dyer [54], and Krokhmal et al. [63] contain many useful references. This literature is virtually entirely devoted to the single criterion case (typically financial returns), but it is worth recalling some of the key results with a view to extending the approaches to the multicriteria case.

The common theme has been that of developing axiomatic foundations for representation of psychological perceptions of risk (including consideration of importance and impact in addition to simple uncertainty), often based on some form of utility model. For example, Bell [10] considers situations in which, if a decision maker switches from preferring one (typically more risky) lottery to another as his/her wealth increases, then he/she never switches back to preference for the first as wealth further increases. This he terms the "one-switch" rule for risk preferences, and demonstrates that if the decision maker is decreasingly risk averse, obeys the one switch rule, and approaches risk neutrality as total wealth tends to infinity, then the utility as a function of wealth w must take on the form $w - be^{-cw}$ for some positive parameters b and c. Taking expectations results in an additive aggregation of two criteria, namely:

- The expectation of wealth (to be maximized); and
- The expectation of be^{-cw} (to be minimized), which can be viewed as a measure of risk.

Sarin and Weber [87] and Jia and Dyer [54] provide arguments for general moments of the distribution of returns (including but not restricted to variance) and/or expectations of terms such as be^{-cw}, as measures of risk. While these may be useful as descriptive measures of risk behaviour, from the point of view of practical decision aid the use of variances to measure risk has been criticised for its symmetric treatment of gains are losses as well as its "ineffective" treatment of low-probability events [63]. It also seems doubtful whether a decision maker would be able to interpret anything but variance (or standard deviation) for purposes of providing necessary preference information (to establish tradeoffs, relative weights, goals, etc.).

More recent attention has focussed on a number of "downside" risk measures which consider only the impact of negative events. These include the semi-variance $\mathrm{E}[(X - \mathrm{E}[X])^2 | X < \mathrm{E}[X]]$, which measures the risk associated with obtaining a below-average performance and has been extended to an expected regret measure [25] using an arbitrary threshold t rather than mean performance i.e. $\mathrm{E}[(X - t)^2 | X < t]$. Two further measures of risk can be obtained by either defining an *a priori* desired probability level and assessing the associated quantile of performance (often referred to as 'variance-at-risk' in financial applications), or by defining an *a priori* target and assessing the probability of this target not being met. The use of quantiles (and, by extension, probabilities) for single-attribute risk modelling has been criticised, however, for (a) not accounting for extreme losses beyond the specified cut-off, (b) non-convexity, implying that the risk of a portfolio of alternatives may exceed the sum of the risks of its constituents, and (c) discontinuity with respect to the specified probability level [63]. The implications of these criticisms for MCDA have yet to be established, but it seems clear that the use of any more complex risk measures designed in response to these criticisms—in particular, 'conditional variance-at-risk' measuring expected losses conditional on losses exceeding a specified quantile—runs the risk of placing unrealistic demands on the decision maker's ability to assess inputs and interpret outputs. Limited empirical and simulation work which we have undertaken in the context of fisheries management [91] suggested that perceptions of risk of fishery collapse might be modelled better by probabilities of achieving one or more goals (in that case, periods of time before a collapse of the fishery). One advantage of such measures is that they might be much more easily interpreted by decision makers for purposes of expressing preferences or value judgements.

Given the apparent modelling success in representing preferences for single criterion problems under uncertainty by a simple additive aggregation of expected return and one or more risk measures (such as variance), there seems to be merit in exploring the extension of these results to the general multicriteria problem under uncertainty. In other words, each criterion (not necessarily financial) for which there exists substantial uncertainties might be restructured in terms of two separate criteria, viz. expected return and a measure of risk. Many of the above results produce an axiomatic justification for an additive aggregation of expected return and risk, so that these sub-criteria would be preferentially independent under the same axiomatic assumptions.

In spite of how obvious such multicriteria extensions might be, there seems to be little reference in the literature to explicit multicriteria modelling in which each criterion subject to uncertainty is decomposed into subcriteria representing expected return and risk. It is our experience, however, that various risk-avoidance criteria arise almost naturally during the structuring phase of decision modelling, so that in practice risk avoidance criteria may in fact be more common than is apparent from the literature.

Kirkwood [59] has shown that evaluating alternatives by $\sum_{i=1}^{n}[w_i u_i(\mathrm{e}[Z_i]) - w_i^R \sigma_i^2]$ with "risk weights" defined by $w_i^R = (1/2) w_i u_i''(\mathrm{E}[Z_i])$ can lead to close approximations of expected utility under the important conditions that the Z_i be normally distributed and the underlying utility functions “do not deviate too much from linear”. Other results [31], however, suggest that under strongly non-linear preferences this model can perform poorly.

Some of the few explicit references to multicriteria modelling in terms of a risk-return decomposition appear in the context of goal programming. For example, [6] expresses a stochastic multicriteria problem in terms of goals on combinations of risks and returns which are then solved by goal programming, but he does not separate out the risk and return components which may have led to a simpler model structure. Korhonen [62] develops a multicriteria model for financial management, in which a number of different financial performance measures are used as criteria, some of which have a risk interpretation. Details of the solution procedure are not given, but the formulation clearly lends itself to a goal programming structure.

A somewhat earlier paper by Keown and Taylor [58] describes an integer goal programming model for capital budgeting, which can be viewed (together with the STRANGE method of Teghem et al. [94]) as an extension of chance-constrained stochastic programming [see, for example, 14]. Keown and Taylor define goals in terms of desired probability levels, which may generically be expressed in the form:

$$\Pr\left[g(Z) \leq \beta\right] \geq \alpha$$

where $g(Z)$ is some performance function based on the unknown attribute values, β the desired level of performance, and α a desired probability of achieving such performance. By using normal approximations, however, Keown and Taylor reduce the probability goal to one expressed in terms of a combination of mean and standard deviation which is subsequently treated in a standard goal programming manner. This suggests opportunity for research into investigation of generalized goal programming models which deal directly with deviations from both the desired performance levels (b, above) and the desired probability levels (α, above).

Some work on fuzzy multiobjective programming (e.g. [22, 23]) can be viewed in a similar manner, in the sense that a degree of anticipated level of goal achievement, measured in a fuzzy membership sense, may be interpreted as a risk measure.

Despite the attractiveness of using a single fixed target for each criteria, [21] show that this implies that an equivalent utility function formulation cannot be guaranteed. In order for such an equivalence to exist, the target must be probabilistic—an alternative formulation of the expected utility model is to assume a decision maker who has only two different utility levels depending on whether an uncertain target is met or not. The ‘target-oriented’ decision maker assesses probabilities $p(x)$ that the target is achieved given an attribute performance of x, rather than a utility function $u(x)$. Bordley and Kirkwood [19] argue that in some circumstances this may be a “more intuitively appealing task”, and extend the single-attribute results in [21] to show that for each multi-linear (or multiplicative or additive) utility function, there is an equivalent multi-linear (or multiplicative or additive) target-oriented formulation. In fact both the variance-based and probability-based goal programming models can be shown to be special cases of the target-oriented preference model [19].

More generally, the structuring of MCDA problems under uncertainty in terms of expected value and risk sub-criteria for each main criterion does have the advantage of being relatively simple and transparent to users. Such an approach appears to be easily integrated into any of the main MCDA methodologies, namely value measurement, outranking and goal programming/reference point methods. As indicated earlier, however, a decidedly open research question relates to the manner in which risk is most appropriately measured *for this purpose*.

A further practical issue is the extent to which the necessary independence properties can be verified. In other words, to what extent can "risk" on one criterion be measured and assessed without taking into consideration ranges of uncertainties on the other criteria. Once again, this offers much scope for further research.

12.6 Scenario Planning and MCDA

Scenario planning ([96], but see also [36] for the decision support context) was developed as a technique for facilitating the process of identifying uncertain and uncontrollable factors which may impact on the consequences of decisions in the strategic management context. Scenario analysis has been widely accepted as an important component of strategic planning, and it is thus somewhat surprising how little appears to have been written concerning links between MCDA and scenario planning. A discussion of the link between scenario planning and decision making is provided by Harries [49], but does not place this in an MCDA framework. Some multiobjective mathematical programming models, for example [70], do include some scenario concepts in an MCDM framework, but these scenarios tend to focus on technical and easily quantified components such as demands, rather than the richer "strategic conversation" espoused by van der Heiden. A broader review of the interrelationships between scenario planning and MCDA is given by Stewart et al. [93].

One of the problems which arise in discussing scenarios is the lack of clear and agreed definitions of what is meant by a "scenario". Stewart et al. [93] identified at least four distinctly different concepts which were summarized as follows:

Shell Scenario Planning Approach: This approach is well-documented by Van der Heijden [96]. The emphasis is on constructing a coherent story of the future context against which the consequences of policies or strategies will be worked out. The intention of having alternative scenarios is primarily seen to be that of providing the basis for a "strategic conversation" concerning pros and cons of strategic decision options. The scenario relates to external events against which policies are compared and evaluated. It has been stressed in this approach that policy options *do not* form part of the scenario.

Scenarios for exploring uncertainty: Scenarios may be used to explore how different uncertainties may play out, i.e. to explore a range of possible outcomes: see, e.g., [99]. In some senses this use of scenarios is similar to that within the Shell scenario planning context described above. The key difference is that there are no identified strategies needing to be evaluated against them. One simply explores possible futures, maybe to stimulate thinking about whether a change in strategy is necessary or whether there are opportunities that might be capitalized upon. Government Foresight studies are a good example of such a use: precursors to subsequent development and deliberation of specific strategies.

Scenarios for advocacy or political argument: This approach is allied to the previous two, but policy decisions or directions which are either being advocated or opposed are now explicitly integrated into the scenario, in order to emphasize plausible consequences of the policy directions. The purpose in producing the scenario is to create a story which highlights either the benefits or dangers of following one or other policy. Hughes [51] refers to utopian or dystopian perspectives being embedded in such uses of scenarios. The scenarios developed for South African political futures at a workshop involving a number of significant players during 1991/1992 are often held up as an example of this use of scenarios (and suggested as a major driver in the relatively peaceful transition which followed).[1] Even the names chosen to describe the scenarios ("ostrich", "lame duck", "Icarus" and "flight of the flamingos") were chosen to evoke strong emotive responses. However significant these scenarios were in influencing the direction of negotiations in South Africa, they did not involve any analytical comparison of policy options ... the "flight of the flamingos" was embraced as self-evidently the only desirable future.

Representative sample of future states: This is a more technical approach. Future states are conceptualized in terms of a multivariate probability distribution on the state space. It is, however, recognized that the complete distribution may never be fully identified, and may in any case be too mathematically complicated to permit clear analysis of management options. For this reason, analysis will be based on a small number of representative outcomes in the sample space, but designed for good coverage as in experimental design, rather than selected randomly or because they seem "interesting".

The primary goal of scenario planning, at least in the first three perspectives above, is in the first instance to provide a structured "conversation" to sensitize decision makers to external and uncontrollable uncertainties, and to develop a shared understanding of such uncertainties. The approach is, however, naturally extended to the more analytical process of designing, evaluating and selecting courses of action on the basis of robustness to these uncertainties, which suggests close parallels with MCDA (as discussed, for example, by Goodwin and Wright [40]). We shall explore these parallels shortly.

Scenarios are meant to represent fairly extreme futures than can still be viewed as plausible. As to what constitutes sufficiently "extreme" would depend on the

[1] For a detailed description, see Global Business Network, paper accessed on 4 Jan 2011 from http://www.generonconsulting.com/publications/papers/pdfs/MontFleur.pdf.

facilitator, as in a very real sense, there will always be a possible future more extreme (and thus with greater potential impact on the consequences of decisions) than any which is incorporated into formal scenarios.

Van der Heijden suggests five principles which should guide scenario construction:

- At least two scenarios are required to reflect uncertainty, but more than four has proved (in his experience) to be impractical;
- Each scenario must be plausible, meaning that it can be seen to evolve in a logical manner from the past and present;
- Each scenario must be internally consistent;
- Scenarios must be relevant to the client's concerns and they must provide a useful, comprehensive and challenging framework against which the client can develop and test strategies and action plans;
- The scenarios must produce a novel perspective on the issues of concern to the client.

Once scenarios are constructed, they may be used to explore and to evaluate alternative strategies for the organization. Most proponents of scenario planning seem to avoid formal evaluation and analysis procedures, preferring to leave the selection of strategy to informed judgement. For example, [96] (pp. 232–235) rejects "traditional rationalistic decision analysis" as an approach which seeks to find a "right answer". This, however, represents are rather limited and technocratic perception of decision analysis, contrary to the constructive and learning view espoused by most in the MCDA field. The constructivist perspective is discussed at a number of places by Belton and Stewart [11] (see particularly Chaps. 3, 4 and 11), where it is argued that the underlying axioms are not meant to suggest a "right answer", but to provide a coherent discipline within which to construct preferences and strategies. Within such a view, the aims of scenario planning and MCDA share many commonalities, suggesting the potential for substantial synergies in seeking to integrate MCDA and scenario planning. On the one hand, MCDA can enrich the evaluation process in scenario planning, while the scenario planning approach can contribute to deeper understanding of the effects of external uncertainties in MCDA.

Various authors have hinted at the concept of scenarios in MCDA. These include, for example, [55, 60], although this is largely in the context of a two state stochastic programming model; [100], also in a stochastic programming context; [76], Sect. 3, who refer to "states of nature"; [95] in the context of multiple objective linear programming; [70, 77] in the context of power systems planning. These authors do not in general refer directly to the philosophical basis of scenario planning, however, and in some cases at least, the models are structured to suggest that the scenarios or states of nature constitute a complete sample space (see later).

Pomerol [82] is one of the few to discuss scenario planning in the context of decision theory or decision analysis, but without substantive link to MCDA. He does however warn (page 199) of the danger that what might appear to be a robust choice of action (perhaps through unstructured and unsupported use of scenarios) may in fact be an illusion resulting from the fact that some events have simply been ignored. Such a danger suggests another perspective on the potential for two-way synergistic advantage between scenario planning and formal decision analysis: not only may scenario planning provide a means of dealing with uncertainties in MCDA, but decision analysis might contribute to avoiding of illusions of robustness or control in decision making. In the latter context, MCDA might contribute to the choice of scenarios as well as to the formal analysis of alternative courses of action.

Perhaps the closest formal link between MCDA and scenario planning is given in Chap. 14 of Goodwin and Wright [40], which we sought to extend in [93]. In the remainder of this section, we outline these later extensions. Suppose that a set of p scenarios indexed as $r = 1, 2, \ldots, p$ have been identified for purposes of evaluating alternatives. Let us then define $z_{ir}(x)$ (expressed by a lower case letter to emphasize that this is no longer viewed as a random variable) as the consequence of action x in terms of criterion i, under the conditions defined by scenario r. As before, $\mathbf{z}_r(x)$ will represent the corresponding vector of consequences. We assume for each criterion i and scenario r that preferences are monotonically increasing with values of $z_{ir}(x)$, but we do not by any means imply that preferences are linear in the $\mathbf{z}_r(x)$. All that can be inferred is that an alternative x, say, is preferred to alternative y (say) according to criterion i under the assumptions of scenario r if and only if $z_{ir}(x) > z_{ir}(y)$. If the scenarios are sufficiently rich to characterize the effects of uncertainties, then each alternative x will to the same degree be sufficiently characterized by the 2-dimensional $(n \times p)$ array of performance measures $z_{ir}(x)$.

For the remainder of this section, we shall assume that the action space is finite, i.e. $X = \{x^1, x^2, \ldots, x^q\}$, say. For this case, Goodwin and Wright [40] propose a three stage process based on a value function model:

1. Create an additive (multiattribute) value function model for the n criteria, say $\sum_{i=1}^{n} w_i v_i(z_i)$, where the partial value functions $v_i(z_i)$ are defined over the range of $z_{ir}(x)$ values occurring across all scenarios.
2. For each alternative x and scenario r, calculate $V_r(x) = \sum_{i=1}^{i} w_i v_i(z_{ir}(x))$.
3. Display the $p \times q$ table of $V_r(x)$ values to the decision maker for a final selection, although Goodwin and Wright do not discuss modes of decision support for this final choice (implying that perceive it to be a relatively straightforward cognitive task, which we find difficult to accept in general).

A critical assumption in the above approach is that of a scenario-independent value function, i.e. that value trade-offs between criteria are the same under all scenarios, which again we find far from self-evidently true in general. See for example [18] for a discussion on the dangers of assuming overly strong independence between scenarios.

Montibeller et al. [79] discuss practical problems which do arise in comparison of outcomes for all alternative-scenario combinations on a single basis. They proposed application of multiattribute value theory *within* each scenario, but accepting, for example, that weights associated with different criteria may, and quite typically do vary between scenarios. Their approach provides an evaluation of alternatives separately for each scenario, but they do not seek formal aggregation across

Table 12.1 Description of consequences for the simple example

	Scen. S_1		Scen. S_2	
Alternative	Crit. C_1	Crit. C_2	Crit. C_1	Crit. C_2
x_1	0	0	1	1
x_2	1	0	0	1

scenarios. Rather, they seek to identify alternatives which are robust across scenarios in some sense.

The following example, which is a slight extension of that discussed in Sect. 12.3, illustrates the difficulties in selecting between alternatives on the basis of the table of $V_r(x)$ values. In particular, it demonstrates that "robustness" across scenarios is not necessarily either well-defined or desirable when defined mainly in terms of variability in the $V_r(x)$ values.

Example: We have two alternatives (x^1 and x^2), two criteria (C_1 and C_2), two scenarios (S_1 and S_2) and two possible outcomes (expressed as 0 or 1) on each criterion. Consequences for each action and scenario in terms of each criterion are given in Table 12.1.

The important distinction between the two alternatives is that x^1 results in equal performance on both criteria under either scenario, while x^2 results in diametrically opposing performances on the two criteria under either scenario. As discussed in Sect. 12.3, there is no fundamental reason why one alternative should be preferred to the other. Concerns for equity between criteria would favour x^1, while an acceptance that good performance on one criterion might compensate for poorer outcomes on the other criterion might favour choice of x^2. A complete MAUT analysis would resolve the conflicts, but it is not clear that simpler aggregation methodologies would capture the relevant preferences. In the context of this example, any methodology should in its structure allow keep the door open to accept either x^1 or x^2 depending on the specific decision preferences which unfold.

Without loss of generality, the marginal value functions for each of the two criteria can be defined such that $v_i(0) = 0$ and $v_i(1) = 1$ for both criteria. For the Goodwin-Wright approach, the $V_r(x)$ table becomes:

	Scenarios	
Alternative	S_1	S_2
x^1	0	1
x^2	w_1	w_2

This representation tends to obscure equity issues, and conventional robustness considerations seem likely to bias evaluation towards a form of risk aversion which would favour x^2.

There is a clear recognition that preference aggregation needs to be carried out across both the criteria and scenarios [40, 70, 77]. The view espoused by Stewart et al. [93] is that in a scenario-based MCDA structure, alternatives do fundamentally need to be evaluated and compared in terms of all $p \times q$ performance measures identified earlier. In other words, at some point attention needs to be given to how well an alternative performs in terms of each criterion under the conditions of each scenario. In [93], we make this recognition explicit by reference to each criterion-scenario combination as a *metacriterion*. Each metacriterion represents a dimension on which preferences can and need to be formed and stated. In the above simple example, there are thus 4 metacriteria, corresponding to the last four columns of Table 12.1. Assuming that there is no alternative that is simultaneously best in terms of all $p \times q$ metacriteria, any decision made will reflect a balance between better performance on some metacriteria and lesser performance on others, i.e. there is an inevitable tradeoff between performances on each metacriterion, even if this may sometimes be difficult to express explicitly.

The scenario-based MCDA is thus equivalent to a standard multicriteria problem with $p \times q$ criteria (which we have termed metacriteria. In principle, any technique of MCDA could be applied to this metacriterion structure, but we illustrate the approach in terms of a value function methodology. Provided that the metacriteria are preferentially independent, standard results [e.g., 57, Chap. 5] imply that the alternatives may be ordered on the basis of an additive value function which can here be expressed in the form:

$$V(x) = \sum_{i=1}^{n} \sum_{r=1}^{p} w_{ir} v_{ir}(z_{ir}(x)) \tag{12.5}$$

where according to our structure, separate partial value functions need to be established for each criterion-scenario combination. This approach is illustrated below for our previous simple example.

Example (Continued). We can without loss of generality scale each marginal value function such that $v_{ir}(0) = 0$ and $v_{ir}(1) = 1$. Thus $V(x^1) = w_{12} + w_{22}$ and $V(x^2) = w_{11} + w_{22}$, so that x^1 is preferred to x^2 if and only if $w_{12} > w_{11}$, and *vice versa* (with indifference if $w_{12} = w_{11}$).

Consider how the assessment of w_{12} and w_{11} might now proceed. We have that performance on criterion 2 is independent of action within each scenario, so that the performance on criterion 2 becomes a defining feature of the scenarios. The question to the decision maker is thus whether good performance on criterion 1 is more important in scenario 1 (characterized by poor outcomes on criterion 2 irrespective of action taken) or in scenario 2 (characterized by good outcomes on criterion 2). When inter-criterion compensation is beneficial, the first is more important; under concerns for equity, the second is more important. The necessity for such value judgements regarding compensation and equity concerns are clearly surfaced directly by the proposed methodology.

More generally, consider how metacriterion weights may be established. Swing-weighting is an established procedure for weight elicitation, but we need to recognize that the number of metacriteria will typically be too large to perform all swing-weighting comparisons simultaneously. Some form of hierarchical assessment may be needed, and two potential approaches may be recognized:

Approach 1.
- For each scenario r, compare the importance swings for each of the n criteria within this scenario, giving estimates of the ratios w_{ir}/w_{kr} for all pairs of criteria i, k;
- Then for one or two of the more important criteria, compare the relative importance of the swings for these criteria across each of the p scenarios.

Approach 2.
- For each criterion i, compare the importance swings of criterion i within each of the p scenarios, giving estimates of the ratios w_{ir}/w_{is} for all pairs of scenarios r, s;
- Then for one or two selected scenarios, compare the relative importance of the swings for each of the n criteria.

Neither approach differentiates in essence between the evaluation of importance of metacriteria *within* scenarios (comparisons of the initial criteria in a standard MCDA approach), or *between* scenarios (comparisons of scenarios). The distinction between the approaches is a matter of the timing of the comparisons during the analytical process. At this stage, we have not formed any clear conclusions as to which approach is preferable, which should form the topic of future empirical research. In the above simple example, however, either approach would recognize that $w_{2r} = 0$ for both scenarios (a zero swing having zero importance), leaving just the comparison of w_{11} and w_{12} to be undertaken, as indicated in the example (with the implied focus on importance of equity versus compensation).

12.7 Implications for Practice

It should be evident from the preceding discussion that there still remains considerable scope for research into the treatment of substantive external uncertainties within an MCDA framework. It is hoped that such research will lead to ever-improved methodologies. Nevertheless, for the practitioner, certain guidelines can be given at the present time. These may be summarized as follows.

1. There is always a role for systematic sensitivity analysis for moderate levels of uncertainty, especially internal uncertainties, but care needs to be taken to avoid simple "one-at-a-time" variations in assumptions, as such an approach may miss interacting effects.
2. For those working within a value or utility function framework, the expectation of a simple additive value function can generate quite useful insights for the decision maker, *provided that* due attention is given to the shape (changing marginal values) of the function (cf. Stewart [89]). On the other hand, complete multiplicative or multilinear multiattribute utility functions may be difficult to implement correctly.

3. With any MCDA approach, there may be value and some theoretical justification in decomposing those criteria for which there is substantial uncertainty regarding outcomes, into two subcriteria of expected value and a risk measure respectively. An open question remains as to whether variance or standard deviation (which are conventionally used in this context) are the most appropriate risk measures for all problem types.
4. The integration of MCDA and scenario planning appears to be a potentially powerful tool, and may be particularly transparent to many decision makers. The approach is relevant to any methodology of MCDA. There do, nevertheless, remain some open questions, especially as regards the number of scenarios to be used and the means by which they are constructed or selected.

References

1. Abdellaoui, M.: Parameter-free elicitation of utility and probability weighting functions. Manag. Sci. **46**(11), 1497–1512 (2000)
2. Abdellaoui, M., Barrios, C., Wakker, P.: Reconciling introspective utility with revealed preference: experimental arguments based on prospect theory. J. Econ. **138**(1), 356–378 (2007)
3. Abdellaoui, M., Bleichrodt, H., Paraschiv, C.: Loss aversion under prospect theory: a parameter-free measurement. Manag. Sci. **53**(10), 1659–1674 (2007)
4. Angilella, S., Greco, S., Matarazzo, B.: Non-additive robust ordinal regression: a multiple criteria decision model based on the choquet integral. Eur. J. Oper. Res. **201**(1), 277–288 (2010)
5. Azondékon, S.H., Martel, J.M.: "Value" of additional information in multicriterion analysis under uncertainty. Eur. J. Oper. Res. **117**, 45–62 (1999)
6. Ballestero, E.: Stochastic goal programming: a mean-variance approach. Eur. J. Oper. Res. **131**, 476–481 (2001)
7. Banuelas, R., Antony, J.: Application of stochastic analytic hierarchy process within a domestic appliance manufacturer. Eur. J. Oper. Res. Soc. **58**(1), 29 (2007)
8. Bawa, V.S.: Optimal rules for ordering uncertain prospects. J. Financ. Econ. **2**, 95–121 (1975)
9. Bazerman, M.H.: Judgment in Managerial Decision Making, 5th edn. Wiley, New York (2002)
10. Bell, D.E.: One-switch utility functions and a measure of risk. Manag. Sci. **34**, 1416–1424 (1988)
11. Belton, V., Stewart, T.J.: Multiple Criteria Decision Analysis: An Integrated Approach. Kluwer Academic Publishers, Boston (2002)
12. Ben, A.S., Jabeur, K., Martel, J.: Multiple criteria aggregation procedure for mixed evaluations. Eur. J. Oper. Res. **181**(3), 1506–1515 (2007)
13. Beynon, M., Curry, B., Morgan, P.: The Dempster-Shafer theory of evidence: an alternative approach to multicriteria decision modelling. OMEGA: Int. J. Manag. Sci. **28**, 37–50 (2000)
14. Birge, J.R., Louveaux, F.: Introduction to Stochastic Programming, 2nd edn. Springer, New York (2011)
15. Bleichrodt, H., Miyamoto, J.: A characterization of quality-adjusted life-years under cumulative prospect theory. Math. Oper. Res. **28**(1), 181–193 (2003)
16. Bleichrodt, H., Pinto, J.: A parameter-free elicitation of the probability weighting function in medical decision analysis. Manag. Sci. **46**(11), 1485–1496 (2000)
17. Bleichrodt, H., Pinto, J., Wakker, P.: Making descriptive use of prospect theory to improve the prescriptive use of expected utility. Manag. Sci. **47**(11), 1498–1514 (2001)

18. Bordley, R.F., Hazen, G.: Nonlinear utility models arising from unmodelled small world intercorrelations. Manag. Sci. **38**, 1010–1017 (1992)
19. Bordley, R., Kirkwood, C.: Multiattribute preference analysis with performance targets. Oper. Res. **52**(6), 823 (2004)
20. Boujelben, M., Smet, Y., Frikha, A., Chabchoub, H.: Building a binary outranking relation in uncertain, imprecise and multi-experts contexts: the application of evidence theory. Int. J. Approx. Reason. **50**(8), 1259–1278 (2009)
21. Castagnoli, E., Calzi, M.: Expected utility without utility. Theory Decis. **41**(3), 281–301 (1996)
22. Chang, N.B., Wang, S.: A fuzzy goal programming approach for the optimal planning of metropolitan solid waste management systems. Eur. J. Oper. Res. **99**, 303–321 (1997)
23. Chang, N.B., Wen, C., Chen, Y.: A fuzzy multi-objective programming approach for optimal management of the reservoir watershed. Eur. J. Oper. Res. **99**, 289–302 (1997)
24. D'Avignon, G.R., Vincke, P.: An outranking method under uncertainty. Eur. J. Oper. Res. **36**, 311–321 (1988)
25. Dembo, R., Rosen, D.: The practice of portfolio replication. a practical overview of forward and inverse problems. Ann. Oper. Res. **85**, 267–284 (1999)
26. Dendrou, B., Dendrou, S., Houstis, E.: Multiobjective decision analysis for engineering systems. Comput. Oper. Res. **7**, 301–312 (1980)
27. Dentcheva, D., Ruszczynski, A.: Optimization with stochastic dominance constraints. SIAM J. Optim. **14**(2), 548–566 (2003)
28. Dimitras, A., Slowinski, R., Susmaga, R., Zopounidis, C.: Business failure prediction using rough sets. Eur. J. Oper. Res. **114**, 263–280 (1999)
29. Durbach, I.: A simulation-based test of stochastic multicriteria acceptability analysis using achievement functions. Eur. J. Oper. Res. **170**, 923–934 (2006)
30. Durbach, I., Davis, S.: Decision support for selecting a shortlist of electricity-saving options: a modified SMAA approach. ORiON **28**(2), 99–116 (2012)
31. Durbach, I., Stewart, T.: Using expected values to simplify decision making under uncertainty. Omega **37**(2), 312–330 (2009)
32. Durbach, I.N., Stewart, T.J.: Modelling uncertainty in multi-criteria decision analysis. Eur. J. Oper. Res. **223**, 1–14 (2012)
33. Fan, Z., Liu, Y., Feng, B.: A method for stochastic multiple criteria decision making based on pairwise comparisons of alternatives with random evaluations. Eur. J. Oper. Res. **207**(2), 906–915 (2010)
34. Fishburn, P.C.: Foundations of risk measurement. I. Risk as probable loss. Manag. Sci. **30**, 396–406 (1984)
35. French, S.: Uncertainty and imprecision: modelling and analysis. J. Oper. Res. Soc. **46**, 70–79 (1995)
36. French, S., Maule, J., Papamichail, N.: Decision Behaviour, Analysis and Support. Cambridge University Press, Cambridge (2009)
37. Friend, J.: The strategic choice approach. In: Rosenhead, J., Mingers, J. (eds.) Rational Analysis for a Problematic World Revisited, 2nd edn. pp. 115–149. Wiley, Chichester (2001)
38. Gal, T., Stewart, T.J., Hanne, T. (eds.): Multicriteria Decision Making: Advances in MCDM Models, Algorithms, Theory, and Applications. Kluwer Academic Publishers, Boston (1999)
39. Goicoechea, A., Hansen, D.R., Duckstein, L.: Multiobjective Decision Analysis with Engineering and Business Applications. Wiley, New York (1982)
40. Goodwin, P., Wright, G.: Decision Analysis for Management Judgement, 4th edn. Wiley, Chichester (2009)
41. Greco, S., Matarazzo, B., Slowinski, R.: Rough approximation of a preference relation by dominance relations. Eur. J. Oper. Res. **117**, 63–83 (1999)
42. Greco, S., Matarazzo, B., Slowinski, R.: The use of rough sets and fuzzy sets in MCDM. In: Gal, T., Stewart, T.J., Hanne, T. (eds.) Multicriteria Decision Making: Advances in MCDM Models, Algorithms, Theory, and Applications, Chap. 14. Kluwer Academic Publishers, Boston (1999)

43. Greco, S., Matarazzo, B., Slowinski, R.: Rough sets theory for multicriteria decision analysis. Eur. J. Oper. Res. **129**, 1–47 (2001)
44. Greco, S., Matarazzo, B., Slowinski, R.: Rough sets methodology for sorting problems in presence of multiple attributes and criteria. Eur. J. Oper. Res. **138**, 247–259 (2002)
45. Greco, S., Mousseau, V., Słowiński, R.: Multiple criteria sorting with a set of additive value functions. Eur. J. Oper. Res. **207**(3), 1455–1470 (2010)
46. Greco, S., Słowiński, R., Figueira, J., Mousseau, V.: Robust ordinal regression. In: Greco, S., Ehrgott, M., Figuera, J. (eds.) Trends in Multiple Criteria Decision Analysis, Chap 9. Springer, New York (2010)
47. Greco, S., Kadziński, M., Mousseau, V., Słowiński, R.: ELECTRE-GKMS: robust ordinal regression for outranking methods. Eur. J. Oper. Res. **214**(1), 118–135 (2011)
48. Hadar, J., Russell, W.R.: Rules for ordering uncertain prospects. Am. Econ. Rev. **59**, 25–34 (1969)
49. Harries, C.: Correspondence to what? Coherence to what? What is good scenario-based decision making. Technol. Forecast. Soc. Chang. **70**, 797–817 (2003)
50. Hokkanen, J., Lahdelma, R., Miettinen, K., Salminen, P.: Determining the implementation order of a general plan by using a multicriteria method. J. Multi-Criteria Decis. Anal. **7**(5), 273–284 (1998)
51. Hughes, N.: A historical overview of strategic scenario planning. Technical Report. UKERC and EON.UK/EPSRC Project on Transition Pathways to a Low Carbon Economy (2009)
52. Insua, D., French, S.: A framework for sensitivity analysis in discrete multi-objective decision-making. Eur. J. Oper. Res. **54**(2), 176–190 (1991)
53. Jacquet-Lagrèze, E., Siskos, Y.: Preference disaggregation: 20 years of MCDA experience. Eur. J. Oper. Res. **130**, 233–245 (2001)
54. Jia, J., Dyer, J.S.: A standard measure of risk and risk-value models. Manag. Sci. **42**, 1691–1705 (1996)
55. Jiménez, A., Mateos, A., Ríos-Insua, S.: Missing consequences in multiattribute utility theory. Omega Int. J. Manag. Sci. **37**, 395–410 (2009)
56. Kahneman, D., Tversky, A.: Prospect theory: an analysis of decision under risk. Econometrica **47**, 263–291 (1979)
57. Keeney, R.L., Raiffa, H.: Decisions with Multiple Objectives. Wiley, New York (1976)
58. Keown, A.J., Taylor III, B.W.: A chance-constrained integer goal programming model for capital budgeting in the production area. J. Oper. Res. Soc. **31**, 579–589 (1980)
59. Kirkwood, C.: Estimating the impact of uncertainty on deterministic multiattribute evaluation. Manag. Sci. **38**(6), 819–826 (1992)
60. Klein, G., Moskowitz, H., Ravindran, A.: Interactive multiobjective optimization under uncertainty. Manag. Sci. **36**, 58–75 (1990)
61. Klir, G.J., Folger, T.A.: Fuzzy Sets, Uncertainty and Information. Prentice Hall, Englewood Cliffs, NJ (1988)
62. Korhonen, A.: Strategic financial management in a multinational financial conglomerate: a multiple goal stochastic programming approach. Eur. J. Oper. Res. **128**, 418–434 (2001)
63. Krokhmal, P., Zabarankin, M., Uryasev, S.: Modeling and optimization of risk. Surv. Oper. Res. Manag. Sci. **16**(2), 49–66 (2011)
64. Lahdelma, R., Salminen, P.: SMAA-2: stochastic multi-criteria acceptability analysis for group decision making. Oper. Res. **49**(3), 444–454 (2001)
65. Lahdelma, R., Salminen, P.: Prospect theory and stochastic multicriteria acceptability analysis (SMAA). Omega **37**(5), 961–971 (2009)
66. Lahdelma, R., Hokkanen, J., Salminen, P.: SMAA – Stochastic multiobjective acceptability analysis. Eur. J. Oper. Res. **106**, 137–143 (1998)
67. Lahdelma, R., Miettinen, K., Salminen, P.: Reference point approach for multiple decision makers. Eur. J. Oper. Res. **164**(3), 785–791 (2005)
68. Levary, R.R., Wan, K.: A simulation approach for handling uncertainty in the analytic hierarchy process. Eur. J. Oper. Res. **106**, 116–122 (1998)

69. Levary, R.R., Wan, K.: An analytic hierarchy process based simulation model for entry mode decision regarding foreign direct investment. Omega **27**(6), 661–677 (1999)
70. Linares, P.: Multiple criteria decision making and risk analysis as risk management tools for power systems planning. IEEE Trans. Power Syst. **17**, 895–900 (2002)
71. Liu, Y., Fan, Z., Zhang, Y.: A method for stochastic multiple criteria decision making based on dominance degrees. Inform. Sci. **181**(19), 4139–4153 (2011)
72. Mareschal, B.: Stochastic multicriteria decision making and uncertainty. Eur. J. Oper. Res. **26**, 58–64 (1986)
73. Martel, J.M., Zaras, K.: Stochastic dominance in multicriterion analysis under risk. Theory Decis. **39**, 31–49 (1995)
74. Martel, J., d'Avignon, G., Couillard, G.: A fuzzy outranking relation in multicriteria decision making. Eur. J. Oper. Res. **25**, 258–271 (1986)
75. Mateos, A., Ríos-Insua, S., Jiménez, A.: Dominance, potential optimality and alternative ranking in imprecise multi-attribute decision making. J. Oper. Res. Soc. **58**(3), 326–336 (2006)
76. Millet, I., Wedley, W.C.: Modelling risk and uncertainty with the analytic hierarchy process. J. Multi-Criteria Decis. Anal. **11**, 97–107 (2002)
77. Miranda, V., Proença, L.M.: Why risk analysis outperforms probabilistic choice as the effective decision support paradigm for power system planning. IEEE Trans. Power Syst. **13**, 643–648 (1998)
78. Miyamoto, J.M., Wakker, P.: Multiattribute utility theory without expected utility foundations. Oper. Res. **44**, 313–326 (1996)
79. Montibeller, G., Gummer, H., Tumidei, D.: Combining scenario planning and multi-criteria decision analysis in practice. J. Multi-Criteria Decis. Anal. **14**, 5–20 (2006)
80. Nowak, M.: Preference and veto thresholds in multicriteria analysis based on stochastic dominance. Eur. J. Oper. Res. **158**(2), 339–350 (2004)
81. Parnell, G.S., Jackson, J.A., Burk, R.C., Lehmkuhld, L.J., Engelbrecht, Jr. J.A.: R&D concept decision analysis: using alternate futures for sensitivity analysis. J. Multi-Criteria Decis. Anal. **8**, 119–127 (1999)
82. Pomerol, J.C.: Scenario development and practical decision making under uncertainty. Decis. Support Syst. **31**, 197–204 (2001)
83. Rios Insua, D.: Sensitivity Analysis in Multi-Objective Decision Making. Lecture Notes in Economics and Mathematical Systems, vol. 347. Springer, Berlin (1990)
84. Roman, D., Darby-Dowman, K., Mitra, G.: Portfolio construction based on stochastic dominance and target return distributions. Math. Program. **108**(2), 541–569 (2006)
85. Rosqvist, T.: Simulation and multi-attribute utility modelling of life cycle profit. J. Multi-Criteria Decis. Anal. **10**, 205–218 (2001)
86. Saltelli, A., Tarantola, A.S., Chan, K.: A role for sensitivity analysis in presenting the results from MCDA studies to decision makers. J. Multi-Criteria Decis. Anal. **8**, 139–145 (1999)
87. Sarin, R.K., Weber, M.: Risk-value models. Eur. J. Oper. Res. **70**, 135–149 (1993)
88. Shafer, G.:A Mathematical Theory of Evidence. Princeton University Press, Princeton (1976)
89. Stewart, T.J.: Simplified approaches for multi-criteria decision making under uncertainty. J. Multi-Criteria Decis. Anal. **4**, 246–258 (1995)
90. Stewart, T.J.: Robustness of additive value function methods in MCDM. J. Multi-Criteria Decis. Anal. **5**, 301–309 (1996)
91. Stewart, T.J.: Measurements of risk in fisheries management. ORiON **14**, 1–15 (1998)
92. Stewart, T.J.: Concepts of interactive programming. In: Gal, T., Stewart, T.J., Hanne, T. (eds.) Multicriteria Decision Making: Advances in MCDM Models, Algorithms, Theory, and Applications, Chap. 10. Kluwer Academic Publishers, Boston (1999)
93. Stewart, T.J., French, S., Rios, J.: Integrating multicriteria decision analysis and scenario planning – review and extension. Omega Int. J. Manag. Sci. **41**, 679–688 (2013). doi:10.1016/j.omega.2012.09.003
94. Teghem, Jr. J., Dufrane, D., Thauvoye, M.: STRANGE: an interactive method for multi-objective linear programming under uncertainty. Eur. J. Oper. Res. **26**, 65–82 (1986)

95. Urli, B., Nadeau, R.: PROMISE/scenarios: an interactive method for multiobjective stochastic linear programming under partial uncertainty. Eur. J. Oper. Res. **155**, 361–372 (2004)
96. Van der Heijden, K.: Scenarios: The Art of Strategic Conversation. Wiley, Chichester (1996)
97. von Winterfeldt, D., Edwards, W.: Decision Analysis and Behavioral Research. Cambridge University Press, Cambridge (1986)
98. Wakker, P., Deneffe, D.: Eliciting von Neumann-Morgenstern utilities when probabilities are distorted or unknown. Manag. Sci. **42**(8), 1131–1150 (1996)
99. Walker, W.E., Harremoes, P., Rotmans, J., van der Sluijs, J.P., van Asselt, M.B.A., Janssen, P., Krayer von Krausswalker, M.P.: Defining uncertainty: a conceptual basis for uncertainty management in model based decision support. Integr. Asess. **4**(1), 5–17 (2003)
100. Watkins, Jr. D.W., McKinney, D.C., Lasdon, L.S., Nielsen, S.S., Martin, Q.W.: A scenario-based stochastic programming model for water supplies from the highland lakes. Int. Trans. Oper. Res. **7**, 211–230 (2000)
101. Whitmore, G.A.: Third order stochastic dominance. Am. Econ. Rev. **60**, 457–459 (1970)
102. Xu, D.: An introduction and survey of the evidential reasoning approach for multiple criteria decision analysis. Ann. Oper. Res. **195**(1), 163–187 (2012)
103. Yang, J.B.: Rule and utility based evidential reasoning approach for multiattribute decision analysis under uncertainties. Eur. J. Oper. Res. **131**, 31–61 (2001)
104. Yeh, C., Deng, H., Pan, H.: Multi-criteria analysis for dredger dispatching under uncertainty. J. Oper. Res. Soc. **50**, 35–43 (1999)
105. Yilmaz, M.R.: An information-expectation framework for decisions under uncertainty. J. Multi-Criteria Decis. Anal. **1**, 65–80 (1992)
106. Zank, H.: Cumulative prospect theory for parametric and multiattribute utilities. Math. Oper. Res. **26**(1), 67–81 (2001)
107. Zaras, K.: Rough approximation of a preference relation by a multi-attribute dominance for deterministic, stochastic and fuzzy decision problems. Eur. J. Oper. Res. **159**(1), 196–206 (2004)
108. Zhang, Y., Fan, Z., Liu,Y.: A method based on stochastic dominance degrees for stochastic multiple criteria decision making. Comput. Ind. Eng. **58**(4), 544–552 (2010)
109. Zimmermann, H.: An application-oriented view of modeling uncertainty. Eur. J. Oper. Res. **122**, 190–198 (2000)

Chapter 13
Decision Rule Approach

Salvatore Greco, Benedetto Matarazzo, and Roman Słowiński

Abstract In this chapter we present the methodology of Multiple-Criteria Decision Aiding (MCDA) based on preference modelling in terms of "*if…, then…*" decision rules. The basic assumption of the decision rule approach is that the decision maker (DM) accepts to give preference information in terms of examples of decisions and looks for simple rules justifying her decisions. An important advantage of this approach is the possibility of handling inconsistencies in the preference information, resulting from hesitations of the DM. The proposed methodology is based on the elementary, natural and rational principle of dominance. It says that if action x is at least as good as action y on each criterion from a considered family, then x is also comprehensively at least as good as y. The set of decision rules constituting the preference model is induced from the preference information using a knowledge discovery technique properly adapted in order to handle the dominance principle. The mathematical basis of the decision rule approach to MCDA is the Dominance-based Rough Set Approach (DRSA) developed by the authors. We present some basic applications of this approach, starting from multiple-criteria classification problems, and then going through decision under uncertainty, hierarchical decision making, classification problems with partially missing information, problems with imprecise information modelled by fuzzy sets, until multiple-criteria choice and

S. Greco (✉)
Department of Economics and Business, University of Catania, Corso Italia 55, 95129 Catania, Italy

Portsmouth Business School, Centre of Operations Research and Logistics (CORL), University of Portsmouth, Richmond Building, Portland Street, Portsmouth PO1 3DE, UK
e-mail: salgreco@unict.it

B. Matarazzo
Department of Economics and Business, University of Catania, Corso Italia 55, 95129 Catania, Italy
e-mail: matarazz@unict.it

R. Słowiński
Institute of Computing Science, Poznan University of Technology, 60-965 Poznan, Poland

Systems Research Institute, Polish Academy of Sciences, 01-447 Warsaw, Poland
e-mail: roman.slowinski@cs.put.poznan.pl

S. Greco et al. (eds.), *Multiple Criteria Decision Analysis*, International Series in Operations Research & Management Science 233,
DOI 10.1007/978-1-4939-3094-4_13

ranking problems, and some classical problems of operations research. All these applications are illustrated by didactic examples whose aim is to show in an easy way how DRSA can be used in various contexts of MCDA.

Keywords Rough set • Dominance-based Rough Set Approach • Rough approximations • Decision rules

13.1 Introduction

Multiple-criteria decision support aims at giving the decision maker (DM) a recommendation [82] in terms of the best action(s) (choice), or in terms of the assignment of actions to pre-defined and preference-ordered classes (classification, called also sorting), or in terms of the ranking of actions from the best to the worst (ranking). None of these recommendations can be elaborated before the DM provides some preference information. Based on this preference information, a preference model of the DM is constructed with the aim of getting a recommendation satisfying the DM's preferences.

Two major preference models have been used until now in multiple criteria decision analysis: Multi-Attribute Utility Theory (MAUT; see [17, 69]) and the outranking approach [18, 81]. These models require specific preference information, more or less directly related to the model parameters. For example, the DM is often asked for pairwise comparisons of actions from which one can assess the substitution rates for a MAUT model or the importance weights for an outranking model (see [19, 73]). This kind of preference information seems to be close to the natural reasoning of the DM. She is typically more confident exercising her decisions than explaining them. The transformation of this information into MAUT or outranking models seems, however, less evident. According to Slovic [84], people make decisions by searching for *rules* which provide good justification of their choices. So, after getting the preference information in terms of exemplary decisions, it would be natural to use this information for building the preference model in terms of "*if*…, *then* …" decision rules. Examples of such rules are the following:

- "*if* maximum speed of car x is at least 175 km/h *and* its price is at most $12,000, *then* car x is comprehensively at least medium",
- "*if* car x is at least weakly preferred to car y with respect to acceleration *and* the price of car x is no more than slightly worse than that of car y, *then* car x is at least as good as car y".

The rules induced from exemplary decisions represent a preference attitude of the DM and enable her understanding of the reasons of her preference. The acceptance of these rules by the DM justifies, in turn, their use for decision support. This is concordant with the principle of posterior rationality by March [72] and with aggregation-disaggregation logic by Jacquet-Lagrèze [67].

The set of decision rules accepted by the DM can be applied to a set of potential actions in order to obtain either an assignment of actions to classes (sorting) or specific preference relations in the set of actions (choice and ranking). From exploitation of these results, a suitable recommendation can be obtained to support the DM in the decision problem at hand.

So, the preference model in the form of decision rules induced from examples fulfils both representation and recommendation tasks (see [82]).

The induction of rules from examples is a typical approach of artificial intelligence. This explains our interest in rough set theory [76, 77, 80, 85] which proved to be a useful tool for analysis of vague description of decision situations [78, 86]. The aim of rough set analysis is the explanation of the dependence between the values of some decision attributes, playing the role of "dependent variables", by means of the values of other condition attributes, playing the role of "independent variables". For example, in a diagnostic context, data about the presence of some diseases are given by decision attributes, while data about symptoms are given by condition attributes. An important advantage of the rough set approach is that it can deal with partly inconsistent examples, i.e. objects indiscernible by condition attributes but discernible by decision attributes (for example, cases where the presence of different diseases is associated with the presence of the same symptoms). Moreover, it provides useful information about the role of particular attributes and their subsets, and prepares the ground for representation of knowledge hidden in the data by means of "*if. . . , then . . .*" decision rules, relating values of some condition attributes with values of decision attributes (for example "if symptom A and B are present, then there is disease X").

For a long time, however, the use of the rough set approach and, in general, of data mining techniques, has been restricted to classification problems where the preference order of evaluations is not considered. Typical examples of such problems come from medical diagnostics. In this context, symptom A is not better or worse than symptom B, or disease X is not preferable to disease Y. It is thus sufficient to consider A as different from B, and X as different from Y. There are, however, situations, where discernibility is not sufficient to handle all relevant information. Consider, for example, two firms, α and β, evaluated for assessment of bankruptcy risk by a set of criteria including the "debt ratio" (total debt/total assets). If firm α has a low value of the debt ratio while firm β has a high value of the debt ratio, then, within data mining and classical rough set theory, α is different (discernible) from β with respect to the considered attribute (debt ratio). However, from the viewpoint of preference analysis and, say, bankruptcy risk evaluation, the debt ratio of α is not simply different from the debt ratio of β but, clearly, the former is better than the latter.

The basic principle of the classical rough set approach and data mining techniques is the following "*indiscernibility principle*": if x is indiscernible with y, i.e. x has the same characteristics as y, then x should belong to the same class as y; if not, there is an inconsistency between x and y. According to this principle, if a patient has symptom A and disease X while another patient has symptom B and disease Y, there is not any inconsistency and one can draw a simple conclusion that symptom A is associated with disease X, while symptom B is associated with disease Y.

In multiple-criteria decision analysis, "*indiscernibility principle*" is not sufficient to convey all relevant semantics of the available information. Consider again the above two firms: α having a low value of debt ratio and β having a high value of the debt ratio. Suppose that evaluations of these firms on other attributes (profitability indices, quality of managers, market competitive situation, etc.) are equal. Suppose, moreover, that firm α has been assigned by a DM to a class of higher risk than firm β. According to the indiscernibility principle, one can say that α and β are discernible, and it follows that low debt ratio is associated with high risk while high debt ratio is associated with low risk. This is contradictory, of course. The reason is that, within multiple-criteria decision analysis, the "indiscernibility principle" has to be substituted by the following "dominance principle": if x dominates y, i.e. x is at least as good as y with respect to all considered criteria, then x should belong to a class not worse than the class of y; if not, there is an inconsistency between x and y. Applying the dominance principle to α and β, one can state an inconsistency between the values of their debt ratio and the assessed risk of their bankruptcy, which leads to a paradoxical conclusion that the lower the debt ratio the higher the risk of bankruptcy.

For this reason, Greco et al. ([28, 32, 34, 35, 38, 49, 90]; for an elementary introduction see [37]; for some detailed survey see [91–94]) have proposed an extension of rough set theory based on the dominance principle, which permits to deal with MCDA problems. This innovation is mainly based on substitution of the indiscernibility relation by a dominance relation in the rough approximation of decision classes (sorting) or preference relations (choice and ranking). An important consequence of this fact is a possibility of inferring from exemplary decisions the preference model in terms of decision rules being logical statements of the type "*if. . . , then. . .*". The separation of certain and doubtful knowledge about the DM's preferences is done by distinction of different kinds of decision rules, depending whether they are induced from examples consistent with the dominance principle or from examples inconsistent with the dominance principle. The latter rules are very important, because they represent situations of hesitation in the DM's expression of preferences.

This is to say that, using a properly modified technique of artificial intelligence, one can construct a preference model in form of decision rules, by induction from exemplary decisions. Such a preference model has some interesting properties: it is expressed in a natural language without using any complex analytical formulation, its interpretation is straightforward and does not depend on technical parameters, often it uses only a subset of the considered attributes in each rule, and, finally, it can represent situations of hesitation, typical for a real expression of preferences. The decision rule preference model is therefore a new approach to MCDA, which becomes a strong alternative for MAUT and outranking approaches.

Indeed building any MCDA preference models requires information about aggregation of multi-attribute characteristics of actions. This information has to be provided by the DM, possibly assisted by an analyst. This information is often processed in a way which is not clear for the DM, such that (s)he cannot see what are the exact relations between the provided information and the final recommendation. Consequently, very often the decision model is perceived by the DM as a *black box* whose result has to be accepted because the analyst's authority guarantees that the result is right. In this context, the aspiration of the DM to find good reasons to make decision is frustrated and rises the need for a more transparent methodology in which the relation between the original information and the final recommendation is clearly shown. Such a transparent methodology searched for has been called *glass box* [57] and we shall show that this is the main advantage of the preference model in form of decision rules induced on the basis of the rough set theory based on the dominance principle.

In this chapter, we present the Dominance-based Rough Set Approach (DRSA) to multiple-criteria decision problems, starting from multiple-criteria classification problems, and then going through decision under uncertainty, hierarchical decision making, classification problems with partially missing information, problems with imprecise information modelled by fuzzy sets, until multiple-criteria choice and ranking problems, and some classical problems of operations research.

13.2 Dominance-Based Rough Set Approach (DRSA)

13.2.1 Data Table

For the sake of a didactic exposition, we introduce the main ideas of DRSA (for a detailed exposition see [51]) through a very simple example. Let us suppose that students of a technical college are evaluated taking into account their marks in Mathematics, Physics and Literature. Let us suppose that one is interested in finding some general rules for a comprehensive evaluation of the students. These general rules can be inferred from some previous examples of decisions, i.e. comprehensive evaluations of students made in the past. Let us consider the examples presented in Table 13.1.

Table 13.1 Data table presenting examples of comprehensive evaluations of students

Student	Mathematics	Physics	Literature	Comprehensive evaluation
S1	*Good*	*Medium*	*Bad*	*Bad*
S2	*Medium*	*Medium*	*Bad*	*Medium*
S3	*Medium*	*Medium*	*Medium*	*Medium*
S4	*Good*	*Good*	*Medium*	*Good*
S5	*Good*	*Medium*	*Good*	*Good*
S6	*Good*	*Good*	*Good*	*Good*
S7	*Bad*	*Bad*	*Bad*	*Bad*
S8	*Bad*	*Bad*	*Medium*	*Bad*

Each application of the DRSA is based on a data table having the form of Table 13.1. In general, a data table can be described as follows. Each row of the table corresponds to an *object*. In multiple-criteria decision analysis objects are usually called *actions*. In the considered example, the objects (actions) are students. Each column of the table corresponds to an *attribute*, i.e. to a different type of information. In our example, the attributes are: Mathematics, Physics, Literature, Comprehensive evaluation. Each cell of this table indicates an *evaluation* (quantitative or qualitative) of the object placed in the corresponding row by means of the attribute in the corresponding column. In the above example, the evaluation is a mark of the considered student in a given course (Mathematics, Physics, Literature) or in the Comprehensive evaluation.

Therefore, formally, a *data table* is the 4-tuple $\boldsymbol{S} =< U, Q, V, f >$, where U is a finite set of *objects* (universe), $Q = \{q_1, q_2, \ldots, q_m\}$ is a finite set of *attributes*, V_q is the domain (value set) of attribute q, $V = \bigcup_{q \in Q} V_q$ and $f : U \times Q \to V$ is a total function such that $f(x, q) \in V_q$ for each $q \in Q, x \in U$, called *information function.*

In our example, $U = \{$S1,S2,S3,S4,S5,S6,S7,S8$\}$, $Q = \{$Mathematics, Physics, Literature, Comprehensive evaluation$\}$, $V_{Mathematics} = V_{Physics} = V_{Literature} = V_{Comprehensive\_evaluation} = V = \{Bad, Medium, Good\}$, the information function $f : U \times Q \to V$ can be rebuilt from Table 13.1 such that, for example, f(S1,Math.) $= Good$, f(S1,Phys.) $= Medium$, f(S1,Lit.) $= Bad$ and so on.

Let us remark that the domain of each attribute is monotonically ordered according to preference. In multiple-criteria decision analysis, such an attribute is called *criterion*. As one can see in Table 13.1, *Good* is better than *Medium* and *Medium* is better than *Bad*. In general, this fact can be formally expressed as follows. Let $\succeq_q$ be a *weak preference* relation on U with reference to criterion $q \in Q$, such that $x \succeq_q y$ means "x is at least as good as y with respect to criterion q". Suppose that $\succeq_q$ is a complete preorder, i.e. a strongly complete (which means that for each $x, y \in U$, at least one of $x \succeq_q y$ and $y \succeq_q x$ is verified, and thus x and y are always comparable with respect to criterion q) and transitive binary relation. In the following we shall denote by $\succ_q$ the asymmetric part of $\succeq_q$ and by $\sim_q$, its symmetric part. The meaning of $x \succ_q y$ is "x is preferred to y with respect to criterion q" and the meaning of $x \sim_q y$ is "x is indifferent to y with respect to criterion q". For example, in Table 13.1, we see that, with respect to Mathematics, S1, being *Good*, is preferred to S2, being *Medium*, which is denoted by S1$\succ_{Mathematics}$S2. Analogously, with respect to Physics, S1, being *Medium*, is indifferent to S2, being also *Medium*, which is denoted by S1$\sim_{Physics}$S2.

The attributes from Q are divided in two sets, C and D with $C \neq \emptyset$, $D \neq \emptyset$, $C \cap D = \emptyset$ and $C \cup D = Q$. The attributes from C are called condition attributes, while the attributes from D are called decision attributes. This distinction is made with the aim of explaining the evaluations on D using the evaluations on C. Very often $D = \{d\}$ and then its evaluations are considered in terms of a classification of objects from U. The case in which there are more than one decision attribute, i.e. $D = \{d_1, \ldots, d_p\}$, is also very interesting because it is related to decisions with multiple decision makers, $\mathrm{DM}_1, \ldots, \mathrm{DM}_p$, expressing different classifications corresponding to particular decision attributes, i.e. d_1 represents classification of

$\mathrm{DM}_1, \ldots, d_p$ represents classification of DM_p. For the sake of simplicity, in the following, we shall consider the case of a single decision attribute, i.e. $D = \{d\}$. More formally, let $\boldsymbol{Cl} = \{Cl_t, t \in \{1, \ldots, n\}\}$, be a classification of U, such that each $x \in U$ belongs to one and only one class $Cl_t \in \boldsymbol{Cl}$. In the above example, the set of condition attributes is $C = \{$Mathematics, Physics, Literature$\}$ and the decision attribute is $d =$ Comprehensive evaluation. In this case, the aim is to explain evaluation on d, using evaluations on C. Consequently, the classification is referred to the comprehensive evaluation, $\boldsymbol{Cl} = \{Cl_1, Cl_2, Cl_3\}$, $Cl_1 = \{$*Bad* students$\} = \{$S1,S7,S8$\}$, $Cl_2 = \{$*Medium* students$\} = \{$S2,S3$\}$ and $Cl_3 = \{$*Good* students$\} = \{$S4,S5,S6$\}$.

We assume that, for all $r, s \in \{1, \ldots, n\}$, such that $r > s$, each element of Cl_r is preferred to each element of Cl_s. More formally, if $\succeq$ is a comprehensive weak preference relation on U, i.e. $x \succeq y$ means: "x is at least as good as y" for any $x, y \in U$, then it is supposed that

$$[x \in Cl_r, y \in Cl_s, r > s] \Rightarrow x \succ y$$

where $x \succ y$ means $x \succeq y$ and *not* $y \succeq x$.

In our example, each element of Cl_2 is preferred to each element of Cl_1 (each *Medium* student is better than each *Bad* student) and each element of Cl_3 is preferred to each element of Cl_2 (each *Good* student is better than each *Medium* student).

13.2.2 Dominance Principle

A natural question with respect to Table 13.1 arises: what classification patterns can be induced from the data table? They represent *knowledge* which may be useful for explanation of a policy of comprehensive evaluation and for prediction of future decisions. In this sense, it is a preference model of a DM who made the comprehensive evaluations and provided the exemplary decisions. Knowledge discovery from Table 13.1 will respect the following *dominance principle*: given $x, y \in U$, if x is at least as good as y with respect to all criteria from a subset $P \subseteq C$, then x should have a comprehensive evaluation at least as good as y. If this is not the case, the reasons for that may be as follows:

(1) some aspects relevant to the comprehensive evaluation are ignored, i.e. some significant criteria are missing in subset P, or
(2) given the evaluations of students on criteria from P, the DM hesitates with the comprehensive evaluation.

The following two examples drawn from Table 13.1 explain reasons (1) and (2).

Example 1, relative to reason (1). Let us consider students S1 and S3 with respect to their evaluations on Mathematics and Physics. Remark that student S1 is not worse than S3, in both Mathematics and Physics, however, S1 is comprehensively evaluated as *Bad*, while S3 is comprehensively evaluated as *Medium*. This contradicts the dominance principle with respect to Mathematics and Physics. This inconsistency with the dominance principle is solved by taking into account the Literature, which gives advantage to S3 (whose evaluation is *Medium*) over S1 (whose evaluation is *Bad*). Thus, considering Mathematics, Physics and Literature, S1 does not dominate S3, i.e. it is no more true that S1 has an evaluation at least as good as S3 on all considered criteria (Mathematics, Physics and Literature). In consequence, after including the Literature in the set of criteria, the dominance principle is no more contradicted. In other words, the evaluation on Literature is necessary to avoid contradiction with the dominance principle while giving comprehensive evaluation to S1 and S3.

Example 2, relative to reason (2). Let us consider students S1 and S2 with respect to evaluations on Mathematics, Physics and Literature. Student S1 is not worse than S2 with respect to all the considered criteria, however, S1 is comprehensively evaluated as *Bad* while S2 is comprehensively evaluated as *Medium*. This contradicts the dominance principle. This inconsistency with the dominance principle cannot be solved by taking into account one criterion more, because all the available information has been used. In consequence, given all the available information, the comprehensive evaluation of S1 and S2 contradicts the dominance principle. This contradiction may be interpreted as a hesitation of the DM.

DRSA permits to detect all the inconsistencies with the dominance principle following from hesitations, but this is not the sole interesting feature. The main advantage of DRSA is its capacity of discovering certain and doubtful knowledge from the data table, that is a preference model which has also its certain and doubtful part; the certain part is inferred from decision examples consistent with the dominance principle, while the doubtful part is inferred from decision examples inconsistent with the dominance principle. The preference model is useful for both explanation of past decisions and recommendation for new decisions.

13.2.3 Decision Rules

To have a first idea of the multiple-criteria decision analysis performed with DRSA, let us take into account the following example relative to Table 13.1. Consider student S3 with comprehensive evaluation *Medium* and the set of students classified as *at least Medium*, that is *Medium* or *Good*. Taking into account all three criteria—Mathematics, Physics and Literature—the comprehensive evaluation of S3 is not inconsistent with the dominance principle. Indeed, in Table 13.1 there is no other student dominated by S3 and having a better comprehensive evaluation. Remark, however, that less than three criteria are sufficient to ensure the consistency. In fact, the evaluations on Mathematics and Literature are sufficient for the comprehensive evaluation of S3 consistent with the dominance principle. Further reduction of criteria (to Mathematics or Literature only) makes the comprehensive evaluation of S3 inconsistent. For example, considering only Mathematics, we can see that S1 dominates S3, but it has a worse comprehensive evaluation. Therefore, {Mathematics, Literature} is a minimal set of criteria ensuring the consistent evaluation of S3. In other words, one can induce from Table 13.1 a minimal conclusion that each student having not worse evaluations than S3, has also a not worse comprehensive evaluation; this conclusion creates the following decision rule:

ρ : "*if* Mathematics≥*Medium and* Literature≥*Medium*,
then the comprehensive evaluation is at least *Medium*
(that is *Medium* or *Good*)".

It is interesting to remark that this decision rule, possibly useful as an element of a preference model, is a result of search of a boundary line between consistency and inconsistency with the dominance principle. In general, we can say that the preference model, and all the decision analysis using DRSA, can be seen as a search of this boundary line between consistency and inconsistency.

13.2.4 Rough Approximations

Let us continue the presentation of the most important concepts relative to DRSA.

As it was told before, the considered objects are evaluated by criteria from set C from one side, and by the comprehensive decision d from the other side. Using the dominance relation with respect to d, we can define unions of classes relative to a particular dominated or dominating class—these unions of classes are called *upward* and *downward unions of classes*, defined, respectively, as:

$$Cl_t^{\geq} = \bigcup_{s \geq t} Cl_s, \quad Cl_t^{\leq} = \bigcup_{s \leq t} Cl_s.$$

Observe $Cl_1^{\geq} = Cl_n^{\leq} = U$, $Cl_n^{\geq} = Cl_n$ and $Cl_1^{\leq} = Cl_1$.

In the above example, $Cl_1^{\geq}$=$Cl_1 \cup Cl_2 \cup Cl_3$ ={students comprehensively *Bad*, *Medium* or *Good*}={all students in Table 13.1 }, $Cl_2^{\geq}$=$Cl_2 \cup Cl_3$ ={students comprehensively *Medium* or *Good*}={S2,S3,S4,S5,S6} and $Cl_3^{\geq} = Cl_3$ ={students comprehensively *Good*}={S4,S5,S6}.

On the other hand, using the dominance relation with respect to criteria from set C, we can define sets of objects dominating or dominated by a particular object. It is said that object x *P-dominates* object y with respect to $P \subseteq C$ (denotation xD_Py) if $x \succeq_q y$ for all $q \in P$. For example, in Table 13.1, S1 dominates S3 with respect to P ={Mathematics, Physics} because S1$\succeq_{Mathematics}$S3 and S1$\succeq_{Physics}$S3. Since the intersection of complete preorders is a partial preorder and $\succeq_q$ is a complete preorder for each $q \in P$, and $D_P = \bigcap_{q \in P} \succeq_q$, then the dominance relation D_P is a partial preorder. Given $P \subseteq C$ and $x \in U$, let

$$\begin{aligned} D_P^+(x) &= \{y \in U : yD_Px\}, \\ D_P^-(x) &= \{y \in U : xD_Py\} \end{aligned}$$

represent, so-called, *P−dominating set* and *P−dominated set* with respect to x, respectively. For example in Table 13.1, for P ={Mathematics, Physics},

$$D^{+}_{\{Mathematics,Physics\}}(\text{S1}) = \{\text{S1,S4,S5,S6}\},$$

$$D^{-}_{\{Mathematics,Physics\}}(\text{S1}) = \{\text{S1,S2,S3,S5,S7,S8}\}.$$

Given a set of criteria $P \subseteq C$, an object $x \in U$ creates an *inconsistency with the dominance principle* with respect to the upward union of classes $Cl_t^{\geq}$, $t = 2, \ldots, n$, if one of the following conditions holds:

(1) x belongs to class Cl_t or better but it is P-dominated by an object y belonging to a class worse than Cl_t, i.e. $x \in Cl_t^{\geq}$ but $D_P^+(x) \cap Cl_{t-1}^{\leq} \neq \emptyset$ (for example, considering P = {Mathematics, Physics} and the upward union of classes composed of students comprehensively evaluated as "at least *Medium*", student x = S3 creates an inconsistency with the dominance principle: in fact x = S3 belongs to the class of *Medium* students, but there is another student y = S1, P-dominating x and belonging to the class of *Bad* students, that is to a worse class),

(2) x belongs to a worse class than Cl_t but it P-dominates an object y belonging to class Cl_t or better, i.e. $x \notin Cl_t^{\geq}$ but $D_P^-(x) \cap Cl_t^{\geq} \neq \emptyset$ (consider the example from point (1), but taking x = S1 and y = S3).

If for a given set of criteria $P \subseteq C$, the assignment of $x \in U$ to $Cl_t^{\geq}$, $t = 2, \ldots, n$, creates an inconsistency with the dominance principle, we say that x belongs to $Cl_t^{\geq}$ *with some ambiguity*. Thus, x belongs to $Cl_t^{\geq}$ *without any ambiguity* with respect to $P \subseteq C$, if $x \in Cl_t^{\geq}$ and there is no inconsistency with the dominance principle. This means that all objects P-dominating x belong to $Cl_t^{\geq}$, i.e. $D_P^+(x) \subseteq Cl_t^{\geq}$.

Furthermore, *x possibly belongs to $Cl_t^{\geq}$* with respect to $P \subseteq C$ if one of the following conditions holds:

– according to decision attribute d, x belongs to $Cl_t^{\geq}$ (in the example from point (1) above, this is the case of x = S3 which, taking into account criteria from P = {Mathematics, Physics}, could belong to $Cl_2^{\geq}$, i.e. to the set of student comprehensively evaluated as "at least *Medium*"),

– according to decision attribute d, x does not belong to $Cl_t^{\geq}$ but it is inconsistent in the sense of the dominance principle with an object y belonging to $Cl_t^{\geq}$ (in the example from point (1) above, this is the case of x = S1 which, taking into account criteria from P = {Mathematics, Physics}, could belong to $Cl_2^{\geq}$, even if S1 is comprehensively evaluated as *Bad*).

In terms of ambiguity, x possibly belongs to $Cl_t^{\geq}$ with respect to $P \subseteq C$, if x belongs to $Cl_t^{\geq}$ with or without any ambiguity. Due to reflexivity of the dominance relation D_P, conditions (1) and (2) can be summarized as follows: *x possibly belongs* to class Cl_t or better, with respect to $P \subseteq C$, if among the objects P-dominated by x there is an object y belonging to class Cl_t or better, i.e. $D_P^-(x) \cap Cl_t^{\geq} \neq \emptyset$.

In DRSA, the sets to be approximated are upward and downward unions of classes and the items (granules of knowledge) used for this approximation are P-dominating and P-dominated sets.

The P-lower and the P-upper approximation of upward union $Cl_t^{\geq}$, $t \in \{1, \ldots, n\}$, with respect to $P \subseteq C$ (denotation $\underline{P}(Cl_t^{\geq})$ and $\overline{P}(Cl_t^{\geq})$, respectively), are defined as:

$$\underline{P}(Cl_t^{\geq}) = \{x \in U : D_P^+(x) \subseteq Cl_t^{\geq}\},$$

$$\overline{P}(Cl_t^{\geq}) = \bigcup_{x \in Cl_t^{\geq}} D_P^+(x) = \{x \in U : D_P^-(x) \cap Cl_t^{\geq} \neq \emptyset\}.$$

Let us comment the above definitions. The P-lower approximation of an upward union $Cl_t^{\geq}$, $\underline{P}(Cl_t^{\geq})$, is composed of all objects x from the universe such that all objects y having at least the same evaluations on all the considered criteria from P also belong to class Cl_t or better. Thus, one can say that if an object y has at least as good evaluations on criteria from P as object x belonging to $\underline{P}(Cl_t^{\geq})$, then, certainly, y belongs to class Cl_t or better. Therefore, taking into account all decision examples from the considered data table (exemplary decisions), one can conclude that the evaluations on criteria from $P \subseteq C$ of an object x belonging to $\underline{P}(Cl_t^{\geq})$ create a *partial profile* (partial, because $P \subseteq C$), such that for an object y it is sufficient to dominate this partial profile in order to belong to class Cl_t or better. This is the case of above decision rule (ρ), using the partial profile built on criteria from $P = \{\text{Mathematics, Literature}\}$; this profile corresponds to $x =$ S3, which belongs to $\underline{P}(Cl_2^{\geq})$. The assignment of a decision rule is true for all objects from the considered data table, but it can also be used by induction for objects that are not in U. Indeed, it is rather natural to admit such a working hypothesis that, if for a new object z, its evaluations on criteria from P are not worse than the evaluations of x, then z should be assigned to class Cl_t or better. This is because we can consider the data table as a record of experience of the DM. Thus, if according to the experience of the DM—i.e. according to the data table at hand—all objects having evaluations on criteria from P not worse than x are assigned to class Cl_t or better, then the simplest classification strategy following from the previous experience is to assign any other object z, having evaluations on criteria from P not worse than x, to class Cl_t or better.

Let us come back to Table 13.1. Given $P = \{\text{Mathematics, Physics}\}$, $\underline{P}(Cl_3^{\geq}) = \underline{P}(Cl_2^{\geq}) = \{\text{S4,S6}\}$ and $\underline{P}(Cl_1^{\geq}) = \{\text{all the students}\}$. Precisely, given the information provided by evaluations on Mathematics and Physics, S4 and S6 belong to $\underline{P}(Cl_3^{\geq})$, the P-lower approximation of the union of (at least) *Good* students, because there is no other student having at least the same evaluations on Mathematics and Physics and belonging to a class comprehensively worse. This is not the case of student S5, even if she belongs to the class of students comprehensively evaluated as *Good*. Indeed, S5 does not belong to P-lower approximation of (at least) *Good* students because there is another student, S1, having not worse evaluations (in this case, exactly the same) on Mathematics and Physics and, nevertheless, not belonging to the class of (at least) *Good* students. The fact that S4 and S6 belong to $\underline{P}(Cl_3^{\geq})$

permits to conclude that, according to the information given by Table 13.1, an evaluation (at least) *Good* in Mathematics and (at least) *Good* in Physics is enough to assign a student to the union of (at least) *Good* students.

The P-upper approximation of an upward union $Cl_t^{\geq}$, $\overline{P}(Cl_t^{\geq})$, is composed of all objects x from the universe which, in comparison with an object y belonging to union $Cl_t^{\geq}$, have at least the same evaluations on all the considered criteria from P. In other words, the P-upper approximation of an upward union $Cl_t^{\geq}$, $\overline{P}(Cl_t^{\geq})$, is composed of all objects x from the universe, whose evaluations on criteria from P are not worse than evaluations of at least one other object y belonging to class Cl_t or better. Thus, one can say that, if an object z has not worse evaluations on criteria from P than an object x belonging to $\overline{P}(Cl_t^{\geq})$, then z *possibly* belongs to class Cl_t or better. Therefore, taking into account all decision examples from the considered data table, one can conclude that the evaluations of an object x belonging to $\overline{P}(Cl_t^{\geq})$, on criteria from P, create a *partial profile*, such that an object z dominating this profile possibly belongs to class Cl_t or better. This conclusion is true for all objects from the considered data table, but it can also be used by induction for objects that are not in U. Indeed, it is again natural to admit such a working hypothesis that, if for a new object z, its evaluations on criteria from P are not worse than the evaluations of x, then z could be assigned to class Cl_t or better.

Coming back to our example, one can see that, given P = {Mathematics, Literature}, $\overline{P}(Cl_3^{\geq})$ = {S4,S5,S6}, $\overline{P}(Cl_2^{\geq})$ = {S1,S2,S3,S4,S5,S6} and $\overline{P}(Cl_1^{\geq})$ = {all the students}. Precisely, given the information provided by the comprehensive evaluation, S1 is not (at least) *Medium* student, i.e. S1 does not belong to union $Cl_2^{\geq}$. However, S1 belongs to $\overline{P}(Cl_2^{\geq})$, the upper approximation of the union of at least *Medium* students, because there is another student, S2, having at most the same evaluations on Mathematics and Literature (i.e. S2 is dominated by S1) and belonging to the set of students comprehensively evaluated as at least *Medium*.

Analogously, the P-lower and the P-upper approximation of downward union $Cl_t^{\leq}$, $t \in \{1,\ldots,n\}$, with respect to $P \subseteq C$ (denotation $\underline{P}(Cl_t^{\leq})$ and $\overline{P}(Cl_t^{\leq})$, respectively), are defined as:

$$\underline{P}(Cl_t^{\leq}) = \{x \in U : D_P^-(x) \subseteq Cl_t^{\leq}\},$$

$$\overline{P}(Cl_t^{\leq}) = \bigcup_{x \in Cl_t^{\leq}} D_P^-(x) = \{x \in U : D_P^+(x) \cap Cl_t^{\leq} \neq \emptyset\}.$$

The P-lower and P-upper approximation of $Cl_t^{\leq}$ have analogous interpretation of the P-lower and P-upper approximation of $Cl_t^{\geq}$.

13.2.5 Properties of Rough Approximations

The P-lower and P-upper approximations defined as above satisfy the following properties for all $t \in \{1,\ldots,n\}$ and for any $P \subseteq C$:

$$\underline{P}(Cl_t^{\geq}) \subseteq Cl_t^{\geq} \subseteq \overline{P}(Cl_t^{\geq}), \quad \underline{P}(Cl_t^{\leq}) \subseteq Cl_t^{\leq} \subseteq \overline{P}(Cl_t^{\leq}).$$

This property means that all the objects belonging to $Cl_t^{\geq}$ without any ambiguity belong also to $Cl_t^{\geq}$, and all the objects from $Cl_t^{\geq}$ are among the objects that belong to $Cl_t^{\geq}$ with some possible ambiguity. With respect to Table 13.1, one can see that, given $P = C = \{\text{Mathematics, Physics, Literature}\}$ and the union of at least *Medium* students, $Cl_2^{\geq}$, we have:

$$\underline{P}(Cl_2^{\geq}) = \{\text{S3,S4,S5,S6}\},$$

$$Cl_2^{\geq} = \{\text{S2,S3,S4,S5,S6}\},$$

$$\overline{P}(Cl_2^{\geq}) = \{\text{S1,S2,S3,S4,S5,S6}\}.$$

Let us observe that all objects belonging to $\underline{P}(Cl_2^{\geq})$ are from $Cl_2^{\geq}$. However, S2 from $Cl_2^{\geq}$ does not belong to $\underline{P}(Cl_2^{\geq})$ because comprehensive evaluation of S2 is inconsistent with comprehensive evaluation of S1 (classified as *Bad*) in the sense of the dominance principle. This inconsistency creates an ambiguity because S1 dominates S2 on criteria from P and, nevertheless, S1 has been classified worse than S2. Let us also observe that all objects belonging to $Cl_2^{\geq}$, belong also to $\overline{P}(Cl_2^{\geq})$. However, S1 from $\overline{P}(Cl_2^{\geq})$ does not belong to $Cl_2^{\geq}$ because, basing on the available information, it only possibly belongs to $Cl_2^{\geq}$ due to the above ambiguity with S2.

The above examples point out that the differences between $Cl_2^{\geq}$ and $\underline{P}(Cl_2^{\geq})$ from one side, and between $\overline{P}(Cl_2^{\geq})$ and $Cl_2^{\geq}$ from the other side, are related to the inconsistency (or ambiguity) of information. This observation can be generalized: the set difference between upper and lower approximation is composed of objects, whose assignment to the considered upward union $Cl_t^{\geq}$ or downward union $Cl_t^{\leq}$ is ambiguous, that is inconsistent, with the dominance principle. This justifies the following definition. The P-boundaries (P-doubtful regions) of $Cl_t^{\geq}$ and $Cl_t^{\leq}$ are defined as:

$$Bn_P(Cl_t^{\geq}) = \overline{P}(Cl_t^{\geq}) - \underline{P}(Cl_t^{\geq}), \quad Bn_P(Cl_t^{\leq}) = \overline{P}(Cl_t^{\leq}) - \underline{P}(Cl_t^{\leq}).$$

In the above example, we have: $Bn_P(Cl_2^{\geq}) = \overline{P}(Cl_2^{\geq}) - \underline{P}(Cl_2^{\geq}) = \{\text{S1,S2}\}$; indeed, the ambiguity of S1 and S2 with respect to $Cl_2^{\geq}$ was already explained.

The P-lower and P-upper approximations satisfy the following specific complementarity properties:

$$\underline{P}(Cl_t^{\geq}) = U - \overline{P}(Cl_{t-1}^{\leq}), \ t = 2, \ldots, n,$$

$$\underline{P}(Cl_t^{\leq}) = U - \overline{P}(Cl_{t+1}^{\geq}), \ t = 1, \ldots, n-1,$$

$$\overline{P}(Cl_t^{\geq}) = U - \underline{P}(Cl_{t-1}^{\leq}), \ t = 2, \ldots, n,$$

$$\overline{P}(Cl_t^{\leq}) = U - \underline{P}(Cl_{t+1}^{\geq}), \ t = 1, \ldots, n-1.$$

The first expression above has the following interpretation: if object x belongs without any ambiguity to class Cl_t or better, it is impossible that it could belong, even with some ambiguity, to class Cl_{t-1} or worse, i.e. $\underline{P}(Cl_t^{\geq}) = U - \overline{P}(Cl_{t-1}^{\leq})$. Let us consider the set of at least *Medium* students, i.e. $Cl_2^{\geq}$, and the set of (at most) *Bad* students, i.e. $Cl_1^{\leq}$. The above complementarity property means that a student is at least *Medium* without any ambiguity if and only if, basing on available information, it is impossible that she could be comprehensively evaluated as *Bad*, even with some ambiguity. According to this definition, if we consider $P = C =$ {Mathematics, Physics, Literature}, we have: $\underline{P}(Cl_2^{\geq}) =$ {S3,S4,S5,S6} and $\overline{P}(Cl_1^{\leq})$ = {S1,S2,S7,S8}, thus, $\underline{P}(Cl_2^{\geq}) = U - \overline{P}(Cl_1^{\leq})$.

Due to the complementarity property, $Bn_P(Cl_t^{\geq}) = Bn_P(Cl_{t-1}^{\leq})$, for $t = 2, \ldots, n$, which means that if x belongs with ambiguity to class Cl_t or better, it also belongs with ambiguity to class Cl_{t-1} or worse. In our example, $Bn_P(Cl_2^{\geq}) = Bn_P(Cl_1^{\leq}) =$ {S1,S2}. In simple words, this can be expressed as follows: the students, whose assignment to at least *Medium* class is ambiguous, are the same as students, whose assignment to at most *Bad* class is also ambiguous.

A very important property related to the value of information is the following monotonicity of rough approximations with respect to the considered set of attributes: given $R \subseteq P \subseteq C$,

$$\underline{R}(Cl_t^{\geq}) \subseteq \underline{P}(Cl_t^{\geq}), \quad \overline{R}(Cl_t^{\geq}) \supseteq \overline{P}(Cl_t^{\geq})$$

$$\underline{R}(Cl_t^{\leq}) \subseteq \underline{P}(Cl_t^{\leq}), \quad \overline{R}(Cl_t^{\leq}) \supseteq \overline{P}(Cl_t^{\leq})$$

$$Bn_R(Cl_t^{\geq}) \supseteq Bn_P(Cl_t^{\geq}), \quad Bn_R(Cl_t^{\leq}) \supseteq Bn_P(Cl_t^{\leq}).$$

This property has the following interpretation. When the considered information is augmented, then the ambiguity decreases or, at least, does not increase. This means that, if object x is ambiguous with respect to a set of criteria R, then with respect to another set of criteria $P \supseteq R$ the same object x may become non-ambiguous, because the new information conveyed by criteria from $P - R$ may remove this ambiguity. Let us consider the assignment of student S5, in our example, to the union $Cl_3^{\geq}$ of (at least) *Good* students. If $P =$ {Physics}, then S5 is ambiguous and does not belong to $\underline{P}(Cl_3^{\geq})$. Indeed, S5 creates an ambiguity with students S1, S2 and S3 that are evaluated at least as good as S5 with respect to Physics and, nevertheless, their comprehensive evaluation is worse than S5. If we augment the available information by evaluation on Mathematics, considering therefore $P =$ {Physics, Mathematics}, then S5 is still ambiguous and does not belong to $\underline{P}(Cl_3^{\geq})$. In this case, however, the set of students ambiguous with S5 is reduced to S1 only. Finally, if $P =$ {Physics, Mathematics, Literature}, then S5 is no more ambiguous in comparison with other students and thus S5 belongs to $\underline{P}(Cl_3^{\geq})$.

13.2.6 Quality of Approximation, Reducts and Core

The ratio

$$\gamma_P(\boldsymbol{Cl}) = \frac{|U - (\bigcup_{t\in\{2,\ldots,n\}} Bn_P(Cl_t^{\geq}))|}{|U|} =$$

$$= \frac{|U - (\bigcup_{t\in\{1,\ldots,n-1\}} Bn_P(Cl_t^{\leq}))|}{|U|}$$

defines the *quality of approximation of the classification* $\boldsymbol{Cl}$ by means of criteria from set $P \subseteq C$, or, briefly, *quality of classification*, where $|\cdot|$ means cardinality of a set. This ratio expresses the proportion of all P-correctly classified objects, i.e. all the non-ambiguous objects, to all the objects in the data table. Let us calculate this ratio for Table 13.1; taking $P = \{$Mathematics, Physics$\}$, we have

$$\underline{P}(Cl_2^{\geq}) = \{S4,S6\}, \underline{P}(Cl_3^{\geq}) = \{S4,S6\}, \overline{P}(Cl_2^{\geq})=\{S1,S2,S3,S4,S5,S6\},$$

$$\overline{P}(Cl_3^{\geq})=\{S1,S4,S5,S6\}, \underline{P}(Cl_1^{\leq}) = \{S7,S8\}, \underline{P}(Cl_2^{\leq})=\{S2,S3,S7,S8\},$$

$$\overline{P}(Cl_1^{\leq})=\{S1,S2,S3,S5,S7,S8\}, \overline{P}(Cl_2^{\leq}=\{S1,S2,S3,S5,S7,S8\}.$$

This means that

$$Bn_P(Cl_2^{\geq})=\{S1,S2,S3,S5\}, Bn_P(Cl_3^{\geq})=\{S1,S5\},$$

$$Bn_P(Cl_1^{\leq})=\{S1,S2,S3,S5\}, Bn_P(Cl_2^{\leq})=\{S1,S5\}.$$

Thus, the quality of classification with respect to $\boldsymbol{Cl}$ and criteria from set P is

$$\gamma_P(\boldsymbol{Cl}) = \frac{|U - (Bn_P(Cl_2^{\geq}) \cup Bn_P(Cl_3^{\geq}))|}{|U|} =$$

$$= \frac{|U - (Bn_P(Cl_1^{\leq}) \cup Bn_P(Cl_2^{\leq}))|}{|U|} = \frac{|\{S4, S6, S7, S8\}|}{|U|} = \frac{4}{8}.$$

Due to the above monotonicity property, for all $R, P \subseteq C$ the following implication is true

$$R \subseteq P \Rightarrow \gamma_R(\boldsymbol{Cl}) \leq \gamma_P(\boldsymbol{Cl}).$$

This property is illustrated for Table 13.1 by the results of calculation presented in Table 13.2.

Every minimal subset of criteria $P \subseteq C$ such that $\gamma_P(\boldsymbol{Cl}) = \gamma_C(\boldsymbol{Cl})$ is called a *reduct* of C with respect to $\boldsymbol{Cl}$ and is denoted by $RED_{\boldsymbol{Cl}}$(P). $\gamma_P(\boldsymbol{Cl}) = \gamma_C(\boldsymbol{Cl})$

Table 13.2 Quality of classification and Shapley value for classification ***Cl*** and set of criteria *P*

Set of criteria *P*	Ambiguous objects	Non-ambiguous objects	Quality of classification	Shapley value
{Mathematics}	S1,S2,S3,S4,S5,S6	S7,S8	0.25	0.167
{Physics}	S1,S2,S3,S5	S4,S6,S7,S8	0.5	0.292
{Literature}	S1,S2,S3,S4,S7,S8	S5,S6	0.25	0.292
{Mathematics, Physics}	S1,S2,S3,S5	S4,S6,S7,S8	0.5	−0.375
{Mathematics, Literature}	S1,S2	S3,S4,S5,S6,S7,S8	0.75	0.125
{Physics, Literature}	S1,S2	S3,S4,S5,S6,S7,S8	0.75	−0.125
{Mathematics, Physics, Literature}	S1,S2	S3,S4,S5,S6,S7,S8	0.75	−0.125

means that, if P is a reduct, then no object which is non-ambiguous with respect to C, is ambiguous with respect to P. In other words, reducing the information from the set of all criteria C to the subset P, no new ambiguity arises. The condition $\gamma_P(\boldsymbol{Cl}) = \gamma_C(\boldsymbol{Cl})$ is not sufficient for declaring P a reduct. The other important condition in the definition of reduct is the minimality. Supposing that P is a reduct, minimality means that, for any $q \in P$, $\gamma_{P-\{q\}}(\boldsymbol{Cl}) < \gamma_C(\boldsymbol{Cl})$. Therefore, the reducts are all the subsets $P \subseteq C$ which keep the same number of ambiguous objects as C, and such that removing any criterion from P one creates new ambiguous objects.

Looking at the results presented in Table 13.2, one can conclude that in our example there are two reducts: $RED^1_{\boldsymbol{Cl}}$={Mathematics, Literature} and $RED^2_{\boldsymbol{Cl}}$ = {Physics, Literature}.

A data table may have more than one reduct. The intersection of all the reducts is known as the *core*, denoted by $\mathrm{CORE}_{\boldsymbol{Cl}}$. In our example, the core is

$$\mathrm{CORE}_{\boldsymbol{Cl}} = RED^1_{\boldsymbol{Cl}} \cap RED^2_{\boldsymbol{Cl}} =$$

$$\{\text{Mathematics, Literature}\} \cap \{\text{Physics, Literature}\} = \{\text{Literature}\}.$$

The criteria from the core are indispensable for keeping the quality of classification at the level attained for set C. Other criteria from different reducts are *exchangeable*, in the sense that they can substitute each other and their joint presence is not necessary to keep the quality of classification at the level attained for set C. The criteria which do not appear in any reduct are *superfluous* and they have no influence on the quality of approximation of the classification. In our example, the criterion of Literature is indispensable because, for all $P \subseteq C$ such that Literature does not belong to P, we have $\gamma_P(\boldsymbol{Cl}) < \gamma_C(\boldsymbol{Cl}) = 0.75$. This means that removing a core criterion from C creates new ambiguous objects. For the monotonicity of rough approximations with respect to the considered set of attributes, these new ambiguous objects will be present in all the subsets of criteria P which do not include the core criterion. One can see in Table 13.2 that S3 and S5 are ambiguous objects for all the subsets of criteria not including Literature. This is not the case for other criteria belonging to the reducts. In our example, Mathematics and Physics are exchangeable, so it is sufficient that one of them stays with Literature in order to keep the number of ambiguous objects unchanged. This means that the information supplied by the core criteria cannot be substituted by the information supplied by other criteria.

13.2.7 Importance and Interaction Among Criteria

In [26, 42], the information about the quality of classification with respect to all subsets of the considered set of criteria was analysed in view of finding the relative importance and the interaction among criteria. The main idea is based on observation that the quality of classification with respect to all subsets of criteria is a fuzzy measure with the property of Choquet capacity [10]. Such a measure can be used to calculate some specific indices introduced in cooperative game theory (for example the Shapley value [83]) and in the fuzzy measure theory ([24, 74]; see also [25]). Using the quality of classification from Table 13.2, the Shapley value indicating the importance of particular criteria is equal to 0.167 for Mathematics and to 0.292 for Physics and Literature. Therefore, Physics and Literature are quite more important than Mathematics. The Shapley interaction index for pairs of criteria is equal, respectively, to −0.375 for Mathematics and Physics, 0.125 for Mathematics and Literature, and −0.125 for Physics and Literature. It follows that there is a redundancy of information between Mathematics and Physics, and between Physics and Literature, while there is a synergy of information between Mathematics and Literature.

Such a type of analysis can be conducted also on the decision rules in order to determine the importance of each condition and the interaction among different conditions in the considered rules [47]. Let us consider again the rule

ρ : “*if* Mathematics≥*Medium and* Literature≥*Medium*,
then the comprehensive evaluation is at least *Medium*”.

Consider now the two following rules having only one of the two conditions with the same conclusion:

ρ’: “*if* Mathematics≥*Medium*,
then the comprehensive evaluation is at least *Medium*”,

ρ”: “*if* Literature≥*Medium*,
then the comprehensive evaluation is at least *Medium*”.

In Table 13.1, rule ρ is always verified, while rule ρ’ is verified in 5 on 6 cases (the one counterexample is student S1) and rule ρ” is verified in 4 on 5 cases (the one counterexample is student S8). Thus, the credibility of rule ρ is 1, while the credibility of rules ρ' and ρ" is $\frac{5}{6}$ and $\frac{4}{5}$, respectively. This means that in rule ρ the importance of condition "Mathematics$\geq$*Medium*" is $\frac{5}{6}$ and the importance of condition "Literature$\geq$*Medium*" is $\frac{4}{5}$. Moreover, there is a negative interaction between the two conditions which can be measured as the difference between the credibility of rule ρ on one side and the sum of the credibility of rules ρ' and ρ" on the other side, that is $1 - \frac{5}{6} - \frac{4}{5} \approx -0.63$. This means that the general tendency is such that students at least *Medium*, who are at least *Medium* in Mathematics, are also at least *Medium* in Literature. Observe that the sign of interaction among criteria can be different at the global level of the whole decision table and at the local level of a specific decision rule. In the case considered, Mathematics and Literature present a positive synergy at the global level, measured by the Shapley value equal to 0.125 (see Table 13.2), while at the local level of the decision rule ρ, there is a negative interaction between conditions concerning the same criteria (-0.63). Let us remark that the core of this analysis is the same as the analysis of the importance and interaction among criteria based on the quality of classification, that is the Shapley value and the Shapley interaction indices.

13.3 Variable Consistency Dominance-Based Rough Set Approach (VC-DRSA)

The definitions of rough approximations introduced in Sect. 13.2 are based on a strict application of the dominance principle. However, when defining non-ambiguous objects, it is reasonable to accept a limited proportion of negative examples, particularly for large data tables. Such extended version of DRSA is called Variable-Consistency DRSA model (VC-DRSA) [46]. It is presented below.

For any $P \subseteq C$, we say that $x \in U$ belongs to $Cl_t^{\geq}$ *without any ambiguity at consistency level* $l \in (0, 1]$, if $x \in Cl_t^{\geq}$ and at least $l * 100\,\%$ of all objects $y \in U$ dominating x with respect to P also belong to $Cl_t^{\geq}$, i.e.

$$\frac{\left|D_P^+(x) \cap Cl_t^{\geq}\right|}{\left|D_P^+(x)\right|} \geq l.$$

For example, student S3 in Table 13.1 does not belong to P-lower approximation of the union of at least *Medium* students, where $P = \{\text{Mathematics, Physics}\}$, because there is student S1 who is at least as good as S3 both in Mathematics and Physics but comprehensively evaluated as Bad. Anyway, if we fix $l \leq \frac{5}{6}$, then S3 belongs to P-lower approximation of the union of at least *Medium* students, because there are no more counterexamples than $(1 - l) * 100\,\%$ of all students being not worse than S3 on Mathematics and Physics.

The level l is called *consistency level* because it controls the degree of consistency between objects qualified as belonging to $Cl_t^{\geq}$ without any ambiguity. In other words, if $l < 1$, then at most $(1–l)*100\,\%$ of all objects $y \in U$ dominating x with respect to P do not belong to $Cl_t^{\geq}$ and thus contradict the inclusion of x in $Cl_t^{\geq}$.

Analogously, for any $P \subseteq C$ we say that $x \in U$ belongs to $Cl_t^{\leq}$ *without any ambiguity at consistency level* $l \in (0, 1]$, if $x \in Cl_t^{\leq}$ and at least $l*100\,\%$ of all the objects $y \in U$ dominated by x with respect to P also belong to $Cl_t^{\leq}$, i.e.

$$\frac{\left|D_P^-(x) \cap Cl_t^{\leq}\right|}{\left|D_P^-(x)\right|} \geq l.$$

For example, student S1 in Table 13.1 does not belong to P-lower approximation of the union of at most *Bad* students, where $P = \{\text{Physics, Literature}\}$, because there is student S2 who is at most as good as S1 both on Physics and Literature, but comprehensively evaluated as Medium. Anyway, if we fix $l \leq \frac{2}{3}$, then S3 belongs to P-lower approximation of the union of at most *Bad* students, because there are no more counterexamples than $(1 - l)*100\,\%$ of all students being not better than S1 on Physics and Literature.

The concept of non-ambiguous objects at some consistency level l leads naturally to the definition of P-lower approximations of the unions of classes $Cl_t^{\geq}$ and $Cl_t^{\leq}$, respectively:

$$\begin{aligned}\underline{P}^l(Cl_t^{\geq}) &= \{x \in Cl_t^{\geq} : \frac{\left|D_P^+(x) \cap Cl_t^{\geq}\right|}{\left|D_P^+(x)\right|} \geq l\} \\ \underline{P}^l(Cl_t^{\leq}) &= \{x \in Cl_t^{\leq} : \frac{\left|D_P^-(x) \cap Cl_t^{\leq}\right|}{\left|D_P^-(x)\right|} \geq l\}.\end{aligned}$$

Given $P \subseteq C$ and consistency level$l$, we can define the *P-upper approximations* of $Cl_t^{\geq}$ and $Cl_t^{\leq}$, denoted by $\overline{P}^l(Cl_t^{\geq})$ and $\overline{P}^l(Cl_t^{\leq})$, respectively, by complementation of $\underline{P}^l(Cl_{t-1}^{\leq})$ and $\underline{P}^l(Cl_{t+1}^{\geq})$ with respect to U:

$$\overline{P}^l(Cl_t^{\geq}) = U - \underline{P}^l(Cl_{t-1}^{\leq}), \quad \overline{P}^l(Cl_t^{\leq}) = U - \underline{P}^l(Cl_{t+1}^{\geq}).$$

$\overline{P}^l(Cl_t^{\geq})$ can be interpreted as a set of all the objects belonging to $Cl_t^{\geq}$, *possibly ambiguous* at consistency level l. Analogously, $\overline{P}^l(Cl_t^{\leq})$ can be interpreted as a set of all the objects belonging to $Cl_t^{\leq}$, *possibly ambiguous* at consistency level l. The *P-boundaries* (P-doubtful regions) of $Cl_t^{\geq}$ and $Cl_t^{\leq}$ at consistency level l are defined as:

$$Bn_P^l(Cl_t^{\geq}) = \overline{P}^l(Cl_t^{\geq}) - \underline{P}^l(Cl_t^{\geq}),$$

$$Bn_P^l(Cl_t^{\leq}) = \overline{P}^l(Cl_t^{\leq}) - \underline{P}^l(Cl_t^{\leq}), t = 1, \ldots, n.$$

The *variable consistency* model of the dominance-based rough set approach provides some degree of flexibility in assigning objects to lower and upper approximations of the unions of decision classes. The following property can be easily proved: for $0 < l' < l \leq 1$ and $t = 2, \ldots, n$,

$$\underline{P}^{l}(Cl_t^{\geq}) \subseteq \underline{P}^{l'}(Cl_t^{\geq}) \text{ and } \overline{P}^{l'}(Cl_t^{\geq}) \subseteq \overline{P}^{l}(Cl_t^{\geq}).$$

The *variable consistency* model is inspired by the *variable precision* model proposed by Ziarko [102, 103] within the classical, indiscernibility-based rough set approach. An extensive study of variable consistency rough set approaches has been made in [101], and all probabilistic rough set approaches have been summarized in [100].

13.4 Induction of Decision Rules from Rough Approximations of Upward and Downward Unions of Decision Classes

13.4.1 A Syntax of Decision Rules Involving Dominance with Respect to Partial Profiles

The end result of DRSA is a representation of the information contained in the considered data table in terms of simple "*if. . . , then. . .* " decision rules. Considering Table 13.1, one can induce, for example, the following decision rules (within parentheses there are symbols of students supporting the corresponding rule):

Rule (1): "*if* the evaluations in Physics *and* Literature are at least *Medium, then* the student is comprehensively at least *Medium*" (S3,S4,S5,S6)

Rule (2): "*if* the evaluation in Physics is at most *Medium and* the evaluation in Literature is at most *Bad, then* the student is comprehensively at most *Medium*" (S1,S2,S7)

or

Rule (3): "*if* the evaluation in Physics is at least *Medium and* the evaluation in Literature is at most *Bad, then* the student is comprehensively *Bad* or *Medium* (due to ambiguity of information)" (S1,S2).

In fact, the decision rules are not induced directly from the data table but from lower and upper approximations of upward and downward unions of decision classes. For a given upward or downward union of classes, $Cl_t^{\geq}$ or $Cl_s^{\leq}$, the decision rules induced under a hypothesis that objects belonging to $\underline{P}(Cl_t^{\geq})$ or $\underline{P}(Cl_s^{\leq})$ are *positive* (i.e. must be covered by the induced rules) and all the others *negative* (i.e. must not be covered by the induced rules), suggest an assignment to "class Cl_t or better", or to "class Cl_s or worse", respectively. For example, Rule (1) is based on the observation that student S3 belongs to $\underline{P}(Cl_2^{\geq})$, while Rule (2) is based on the observation that S1 belongs to $\underline{P}(Cl_2^{\leq})$, where $P = \{\text{Physics, Literature}\}$.

On the other hand, the decision rules induced under a hypothesis that, for $s < t$, objects belonging to the intersection $\overline{P}(Cl_s^{\leq}) \cap \overline{P}(Cl_t^{\geq})$ are *positive* and all the others *negative*, are suggesting an assignment to some classes between Cl_s and Cl_t. For example, Rule (3) is based on the observation that students S1 and S2 belong to $\overline{P}(Cl_1^{\leq}) \cap \overline{P}(Cl_2^{\geq})$.

Generally speaking, in case of preference-ordered data it is meaningful to consider the following five types of decision rules:

(1) *certain $D_{\geq}$-decision rules*, providing lower profile descriptions for objects belonging to union $Cl_t^{\geq}$ without ambiguity: "*if* $x_{q1} \succeq_{q1} r_{q1}$ *and* $x_{q2} \succeq_{q2} r_{q2}$ *and* $\ldots x_{qp} \succeq_{qp} r_{qp}$, *then* $x \in Cl_t^{\geq}$", where for each $w_q, z_q \in V_q$, "$w_q \succeq_q z_q$ "means "w_q is <u>at least</u> as good as z_q"; this is the case of Rule (1) which can be re- written as

$$\textit{if } x_{Physics} \succeq_{Physics} Medium \textit{ and}$$

$$x_{Literature} \succeq_{Literature} Medium, \textit{ then } x \text{ belongs to } Cl_2^{\geq};$$

(2) *possible $D_{\geq}$-decision rules*, providing lower profile descriptions for objects belonging to union $Cl_t^{\geq}$ with or without any ambiguity: "*if* $x_{q1} \succeq_{q1} r_{q1}$ *and* $x_{q2} \succeq_{q2} r_{q2}$ *and* $\ldots x_{qp} \succeq_{qp} r_{qp}$, *then* x possibly belongs to $Cl_t^{\geq}$"; this is the case of the following

Rule (4): "*if* the evaluation in Physics is at least *Medium, then* the student could be comprehensively at least *Medium*" (S1,S2,S3,S4,S5,S6).

Let us remark that the conclusion of Rule (4), "the student could be comprehensively at least *Medium*" should be read as "it is not completely certain that the student is *Bad*, so it is possible that she is at least *Medium*". Rule (4) can also be re-written as

$$\textit{if } x_{Physics} \succeq_{Physics} Medium, \textit{ then } x \text{ possibly belongs to } Cl_2^{\geq};$$

(3) *certain $D_{\leq}$-decision rules*, providing upper profile descriptions for objects belonging to union $Cl_t^{\leq}$ without ambiguity: "*if* $x_{q1} \preceq_{q1} r_{q1}$ *and* $x_{q2} \preceq_{q2} r_{q2}$ *and* $\ldots$ $x_{qp} \preceq_{qp} r_{qp}$, *then* $x \in Cl_t^{\leq}$, where for each $w_q, z_q \in V_q$, "$w_q \preceq_q z_q$" means "w_q is <u>at most</u> as good as z_q"; this is the case of Rule (2) which can be re- written as

$$\textit{if } x_{Physics} \preceq_{Physics} Medium \textit{ and } x_{Literature} \preceq_{Literature} Bad,$$

$$\textit{then } x \text{ belongs to } Cl_2^{\leq};$$

(4) *possible $D_{\leq}$-decision rules*, providing upper profile descriptions for objects belonging to union $Cl_t^{\leq}$ with or without any ambiguity: "*if* $x_{q1} \preceq_{q1} r_{q1}$ *and* $x_{q2} \preceq_{q2} r_{q2}$ *and* $\ldots$ $x_{qp} \preceq_{qp} r_{qp}$, *then* x possibly belongs to $Cl_t^{\leq}$", this is the case of the following

Rule (5): "*if* the evaluation in Literature is at most *Bad, then* the student could be comprehensively at most *Medium*" (S1,S2,S7).

Let us remark that the conclusion of Rule (5) "the student could be comprehensively at most *Medium*" should be read as "it is not completely certain that the student is *Good*, so it is possible that she is at most *Medium*". Rule (5) can also be re-written as

$$\textit{if } x_{Literature} \preceq_{Literature} Bad,$$

$$\textit{then } x \text{ possibly belongs to } Cl_2^{\leq};$$

(5) *approximate $D_{\geq\leq}$-decision rules*, providing simultaneously lower and upper profile descriptions for objects belonging to classes $Cl_s \cup Cl_{s+1} \cup \ldots \cup Cl_t$, without possibility of discerning to which class: "*if* $x_{q1} \succeq_{q1} r_{q1}$ *and* ... $x_{qk} \succeq_{qk} r_{qk}$ *and* $x_{qk+1} \preceq_{qk+1} r_{qk+1}$ *and* ... $x_{qp} \preceq_{qp} r_{qp}$, *then* $x \in Cl_s \cup Cl_{s+1} \cup \ldots \cup Cl_t$"; this is the case of Rule (3) which can be re-written as

$$\textit{if } x_{Physics} \succeq_{Physics} Medium \textit{ and} x_{Literature} \preceq_{Literature} Bad,$$

$$\textit{then } x \text{ belongs to } Cl_1 \text{ or } Cl_2.$$

In the left hand side of a $D_{\geq\leq}$-decision rule we can have "$x_q \succeq_q r_q$" and "$x_q \preceq_q r'_q$", where $r_q \geq r'_q$, for the same $q \in C$. Moreover, if $r_q = r'_q$, the two conditions boil down to "$x_q \sim_q \text{r}_q$", where for each $w_q, z_q \in V_q$, "$w_q \sim_q z_q$" means "w_q is indifferent to z_q".

The rules of type (1) and (3) represent *certain knowledge* induced from the data table, while the rules of type (2), (4) represent *possible knowledge*, and rules of type (5) represent *doubtful knowledge*.

The rules of type (1) and (3) are *exact*, if they do not cover negative examples, and they are *probabilistic* otherwise. In the latter case, each rule is characterized by a *confidence ratio*, representing the probability that an object matching the left hand side (LHS) of the rule matches also its right hand side (RHS). Probabilistic rules are concordant with the VC-DRSA model presented above. To give an example, consider the following probabilistic $D_{\geq}$*-decision rules* and $D_{\leq}$*-decision rules* obtained from Table 13.1 (within parentheses, the symbols of students supporting the corresponding rule but not concordant with its RHS are underlined):

Rule (6): "*if* the evaluation in Mathematics is at least *Good*, *then* the student is comprehensively at least *Good* in 75 % of cases (*confidence*)", (<u>S1</u>,S4,S5,S6),

Rule (7): "*if* the evaluation in Physics is at most *Medium*, *then* the student is at most *Medium* in 83.3 % of cases (*confidence*)", (S1,S2,S3,<u>S5</u>,S7,S8).

Let us remark that the probabilistic decision rules are very useful when large data tables are considered. In large data tables, an ambiguity typically exists and prevents finding some very strong patterns because the certain decision rules are contradicted by the ambiguous examples. Probabilistic decision rules, permitting a limited number of counterexamples, may represent these strong patterns.

Since a decision rule is a kind of implication, by a *minimal* rule we understand such an implication that there is no other implication with the antecedent (the LHS of the rule) of at least the same weakness (in other words, a rule using a subset of its elementary conditions and/or weaker elementary conditions) and the consequent (the RHS of the rule) of at least the same strength (in other words, a $D_{\geq}$ or a $D_{\leq}$-*decision rule* assigning objects to the same union or sub-union of classes, or a $D_{\geq\leq}$-*decision rule* assigning objects to the same or larger set of classes). Consider, for example, the following decision rules which both are true for objects from Table 13.1:

Rule (A): "*if* the evaluation in Mathematics, Physics and Literature is at least *Good, then* the student is comprehensively at least *Medium*" (S6)

Rule (B): "*if* the evaluation in Mathematics is at least *Good* and the evaluation in Literature is at least *Medium, then* the student is comprehensively at least *Good*" (S4,S5,S6).

Comparison of decision rules (A) and (B) shows that rule (A) is not minimal indeed:

(1) rule (B) has weaker conditions than rule (A) because rule (A) has conditions on all three criteria, while rule (B) has conditions on Mathematics and Literature only; moreover, rule (A) has a stronger requirement than rule (B) with respect to Literature (at least *Good* instead of at least *Medium*), while the requirement with respect to Mathematics is not weaker (both rules require an evaluation at least *Good*);
(2) rule (B) has a stronger conclusion than rule (A) because "comprehensively at least *Good*" is more precise than "comprehensively at least *Medium*".

A set of decision rules is *complete* if it is able to cover all objects from the data table in such a way that consistent objects are re-classified to their original classes and inconsistent objects are classified to clusters of classes referring to this inconsistency. An example of a complete set of decision rules induced from Table 13.1 is given below (between parentheses there are symbols of students supporting the considered rule):

Rule (α): "*if* the evaluation in Mathematics *and* Physics is at most *Bad*, *then* the student is comprehensively at most *Bad*", (S7,S8);

Rule (β): "*if* the evaluation in Physics *and* Literature is at most *Medium*, *then* the student is comprehensively at most *Medium*", (S1,S2, S3,S7,S8);

Rule (γ): "*if* the evaluation in Mathematics *and* Physics is at most *Medium*, *then* the student is comprehensively at most *Medium*", (S2,S3,S7,S8);

Rule (δ): "*if* the evaluation in Physics *and* Literature is at least *Medium*, *then* the student is comprehensively at least *Medium*", (S3,S4, S5,S6);

Rule (ε): "*if* the evaluation in Physics is at least *Good and* the evaluation in Literature is at least *Medium*, *then* the student is comprehensively at least *Good*", (S4,S6);

Rule (ζ): "*if* the evaluation in Physics is at least *Medium and* the evaluation in Literature is at least *Good*, *then* the student is comprehensively at least *Good*", (S5,S6);

Rule (η): "*if* the evaluation in Physics is at least *Medium* and the evaluation in Literature is at most *Bad*, *then* the student is comprehensively *Bad* or *Medium* (due to ambiguity of information)", (S1,S2).

13.4.2 Different Strategies of Decision Rule Induction

We call *minimal* each set of decision rules that is complete and non-redundant, i.e. exclusion of any rule from this set makes it non-complete. Remark that our set of decision rules, (α)–(η), presented at the end of Sect. 4.1 is not minimal. Indeed, one can remove rule (γ) and the remaining set of rules is still complete and minimal; elimination of any other rule does not permit a proper reclassification of at least one student from Table 13.1.

One of three induction strategies can be adopted to obtain a set of decision rules [95]:

- *minimal* description, i.e. generation of a minimal set of rules,
- *exhaustive* description, i.e. generation of all rules for a given data table,
- *characteristic* description, i.e. generation of a set of "strong" rules covering relatively many objects each, however, all together not necessarily all objects from U.

Let us also remark that, contrary to traditional rule induction in machine learning, within DRSA the domains of the considered criteria need not to be discretized, because the syntax of dominance-based rules makes them much less specific than the traditional rules with elementary conditions of the type "attribute=value". This is particularly true for the minimal description strategy of induction because, in the case of exhaustive description, the number of all rules may also augment exponentially with the number of different evaluations on particular criteria. Specific algorithms for induction of decision rules consistent with the dominance principle have been proposed in [3, 4, 45, 48, 98].

13.4.3 Application of Decision Rules

A set of decision rule can be seen as a preference model and used to support future decisions. Let us suppose that two new students, S9 and S10, not considered in above Table 13.1, are to be evaluated comprehensively. Evaluations of these students in Mathematics, Physics and Literature are given in Table 13.3.

Using decision rules proposed within DRSA, different types of preference models can be considered. In general, one can consider the following models:

Table 13.3 Evaluations of new students

Student	Mathematics	Physics	Literature
S9	*Medium*	*Good*	*Good*
S10	*Bad*	*Bad*	*Good*

(1) preference model composed of $D_{\geq}$-*decision rules* only,
(2) preference model composed of $D_{\leq}$-*decision rules* only,
(3) preference model composed of $D_{\geq}$-*decision rules*, $D_{\leq}-$*decision rules* and $D_{\geq\leq}-$*decision rules*.

In case of model (1), when applying $D_{\geq}-$*decision rules* to object x, it is possible that x either matches LHS of at least one decision rule or does not match LHS of any decision rule. Let us consider a preference model composed of three $D_{\geq}-$*decision rules*: rule (δ), rule (ε) and rule (ζ) presented at the end of Sect. 4.1. One can see that S9 matches the LHS of all three rules while S10 does not match the LHS of any of the three rules.

According to rule (δ), S9 is comprehensively at least *Medium*. According to rules (ε) and rule (ζ), S9 is comprehensively at least *Good*. Thus, it is reasonable to assign S9 to the class of *Good* students. In general, in the case of at least one matching of $D_{\geq}-$*decision rules*, it is reasonable to conclude that x belongs to class Cl_t, being the lowest class of the upward union $Cl_t^{\geq}$, where $Cl_t^{\geq}$ is the upward union resulting from intersection of all RHS of rules matching x. Precisely, if x matches LHS of rules ρ_1, $\rho_2,\ldots,\rho_u$, whose RHS are $x \in Cl_{t1}^{\geq}$, $x \in Cl_{t2}^{\geq},\ldots$, $x \in Cl_{tu}^{\geq}$, respectively, then x is assigned to class Cl_t, where $Cl_t^{\geq} = \bigcap_{i=1}^{u} Cl_{ti}^{\geq}$ or, equivalently, $t = max\{t1, t2, \ldots, tu\}$.

Since S10 does not match any $D_{\geq}-$*decision rules*, decision rule among rule (δ), rule (ε) and rule (ζ), it is reasonable to conclude that S10 is neither at least *Medium* nor at least *Good*. Therefore, S10 is classified as comprehensively *Bad*. The idea behind is that the induced $D_{\geq}-$*decision rules* are considered as arguments for assignment of new objects to classes Cl_t, where $t > 1$. Therefore, if there is no $D_{\geq}-$*decision rules* matching a new object x, there is no argument to assign x to Cl_t with $t > 1$; it remains to conclude that x belongs to Cl_1, i.e. to the worst class. In general, in the case of no matching of $D_{\geq}-$*decision rules*, it is concluded that x belongs to Cl_1, i.e. to the worst class, since no rule with RHS suggesting a better classification of x is matching this object.

Now, let us consider model (2) composed of $D_{\leq}-$*decision rules* only. Let us assume that it is composed of three rules: rule (α), rule (β) and rule (γ) from Sect. 4.1. One can see that S9 does not match the LHS of any rule while S10 matches the LHS of rule (α) and rule (γ). Since S9 does not match any decision rule from the considered preference model, it is reasonable to conclude that S9 is neither at most *Medium* nor at most *Bad*. Therefore, S9 is classified as comprehensively *Good*. In general, in the case of no matching of $D_{\leq}-$*decision rules*, it is concluded that x belongs to the best class Cl_n because no rule with RHS suggesting a worse classification of x is matching this object.

Table 13.4 Evaluations of new students

Student	Mathematics	Physics	Literature
S11	*Bad*	*Medium*	*Good*
S12	*Bad*	*Medium*	*Bad*

According to rule (α), S10 is comprehensively at most *Bad*, while, according to rule (γ), S10 is comprehensively at most *Medium*. Thus, it is reasonable to assign S10 to the class of *Bad* students. In general, in the case of at least one matching of $D_{\leq}$*−decision rules*, it is reasonable to conclude that x belongs to class Cl_z, being the highest class of the downward union $Cl_z^{\leq}$ resulting from intersection of all RHS of rules matching x. Precisely, if x matches the LHS of rules $\rho_1, \rho_2,\ldots,\rho_v$, whose RHS are $x \in Cl_{t1}^{\leq}, x \in Cl_{t2}^{\leq},\ldots, x \in Cl_{tv}^{\leq}$, respectively, then x is assigned to class Cl_z, where $Cl_z^{\leq} = \bigcap_{i=1}^{v} Cl_{ti}^{\leq}$ or, equivalently, $z = min\{t1, t2, \ldots, tv\}$.

In model (3), $D_{\geq}$*−decision rules*, $D_{\leq}$*−decision rules* and $D_{\geq\leq}$*−decision rules* are used. For example, let us consider a preference model composed of five rules: rule (α), rule (γ), rule (δ), rule (ζ) and rule (η) from Sect. 4.1, and suppose that two new students, S11 and S12, are to be evaluated comprehensively. Evaluations of these students in Mathematics, Physics and Literature are given in Table 13.4.

S11 matches the LHS of $D_{\leq}-$*decision* rule (γ) and of two $D_{\geq}-$*decision rules*, (δ) and (ζ). Thus, on the basis of rule (γ), S11 is at most *Medium*, while according to rule (δ), S11 is at least *Medium*, and according to rule (ζ), S11 is at least *Good*. In other words, rule (γ) suggests that S11 is comprehensively at most *Medium*, while rules (δ) and (ζ) suggest that S11 is comprehensively at least *Good*. This means that there is an ambiguity in the comprehensive evaluation of S11 by rule (γ) from one side, and rules (δ) and (ζ) from the other side. In this situation the classes *Medium* and *Good* fix the range of the ambiguous classification and it is reasonable to conclude that student S11 is comprehensively *Medium* or *Good*. In general, for this kind of preference model, the final assignment of an object x matching both, $D_{\geq}$*−decision rules* $\rho_1^{\geq}, \rho_2^{\geq}, \ldots, \rho_u^{\geq}$, whose RHS are $x \in Cl_{t1}^{\geq}, x \in Cl_{t2}^{\geq},\ldots, x \in Cl_{tu}^{\geq}$, and $D_{\leq}$*−decision rules* $\rho_1^{\leq}, \rho_2^{\leq},\ldots,\rho_v^{\leq}$, whose RHS are $x \in Cl_{z1}^{\leq}, x \in Cl_{z2}^{\leq},\ldots, x \in Cl_{zv}^{\leq}$, is made to the union of all classes between Cl_t and Cl_z, i.e. to $Cl_t \cup Cl_{t+1} \cup \ldots Cl_z$ *if* $t \leq z$, or to $Cl_z \cup Cl_{z+1} \cup \ldots Cl_t$ if $t \geq z$, such that $t = max\{t1, t2, \ldots, tu\}$ and $z = min\{z1, z2, \ldots, zv\}$. Remark that if only $D_{\geq}$-decision rules or only $D_{\leq}$-decision rules are matching x, then the above union boils down to a single class, Cl_t or Cl_z, respectively.

S12 matches the LHS of $D_{\leq}$-decision rule (γ) and of $D_{\geq\leq}$-decision rule (η). Thus, on the basis of rule (γ), S12 is at most *Medium*, while according to rule (η), S12 is *Bad* or *Medium*, without possibility of discerning to which one of the two classes it must be assigned. In this situation, it is reasonable to conclude that student S12 is comprehensively *Bad* or *Medium*. In general, for this kind of preference model, the final assignment of an object x is made as follows. Let us suppose that x matches, on one hand, $D_{\geq}$*−decision rules* $\rho_1^{\geq}, \rho_2^{\geq},\ldots,\rho_u^{\geq}$, whose RHS are $x \in Cl_{t1}^{\geq}$, $x \in Cl_{t2}^{\geq},\ldots, x \in Cl_{tu}^{\geq}$, and $D_{\leq}$*−decision rules* $\rho_1^{\leq}, \rho_2^{\leq},\ldots,\rho_v^{\leq}$, whose RHS are

$x \in Cl_{z1}^{\leq}$, $x \in Cl_{z2}^{\leq}$,..., $x \in Cl_{zv}^{\leq}$, and, on the other hand, $D_{\geq\leq}$*−decision rules* $\rho_1^{\geq\leq}$, $\rho_2^{\geq\leq}$,...,$\rho_w^{\geq\leq}$, whose RHS are $x \in Cl_{a1}^{\geq} \cap Cl_{b1}^{\leq}$, $x \in Cl_{a2}^{\geq} \cap Cl_{b2}^{\leq}$, ..., $x \in Cl_{aw}^{\geq} \cap Cl_{bw}^{\geq}$, $ai \leq bi$ for all $i = 1, 2, \ldots, w$. Then, let $t = max\{t1, t2, \ldots, tu\}$, $z = min\{z1, z2, \ldots, zv\}$, $k = min\{a1, a2, \ldots, aw\}$ and $h = max\{b1, b2, \ldots, bw\}$. Now, define A and B as follows:

$$A = \begin{cases} Cl_t \cup Cl_{t+1} \cup \ldots Cl_z \ \textit{if}\ t \leq z \\ Cl_z \cup Cl_{z+1} \cup \ldots Cl_t \ \textit{if}\ t > z \end{cases}, \ B = Cl_k \cup Cl_{k+1} \cup \ldots Cl_h.$$

Finally, x is assigned to $A \cup B$.

A new classification procedure for dominance-based probabilistic decision rules coming from VC-DRSA model has been proposed in [1].

13.4.4 Decision Trees: An Alternative to Decision Rules

The dominance-based rough approximations can also serve to induce *decision trees* representing knowledge discovered from preference-ordered data. Several forms of decision trees, useful for representation of classification patterns, have been proposed by Giove et al. [23]. One of these trees, representing knowledge discovered from Table 13.1, is presented in Fig. 13.1.

The decision tree presented in Fig. 13.1 can be interpreted as follows. The root (node 1) of the tree is a test node. The test formulates the following question with respect to all the students: "is the evaluation in Mathematics at least *Medium*?". The root has two child nodes (nodes 2 and 3). The right child node (node 2) concerns

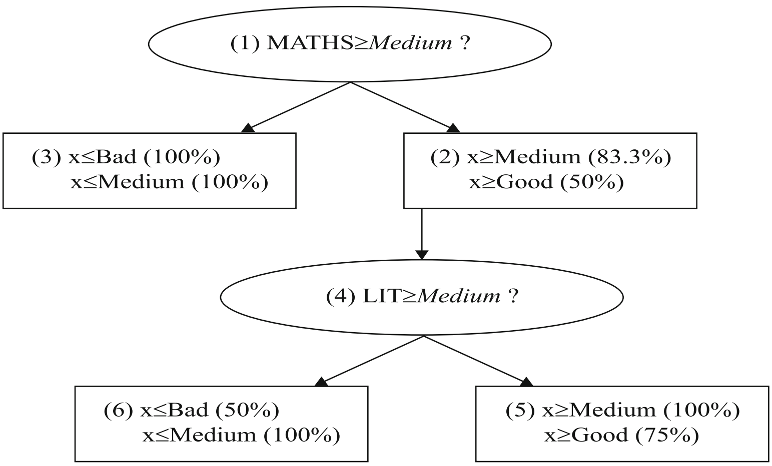


Fig. 13.1 Decision tree representing knowledge included from Table 13.1

the students who passed the test, i.e. all the students being at least *Medium* in Mathematics (S1,S2,S3,S4,S5,S6), while the left child node (node 3) concerns the students who did not pass the test, i.e. all the students being worse than *Medium* in Mathematics (S7,S8). According to node 2, 83.3 % of students being at least *Medium* in Mathematics are comprehensively at least *Medium* (S2,S3,S4,S5,S6); moreover, 50 % of students being at least *Medium* in Mathematics are comprehensively at least *Good* (S4,S5,S6). According to node 3, 100 % of students being worse than *Medium* in Mathematics are comprehensively at most *Bad* (S7,S8). Node 3 says also that the students having an evaluation worse than *Medium* in Mathematics are comprehensively at most *Medium* in 100 %. Let us observe that the information that students are at most *Medium* in 100 % of cases is redundant with respect to the information that the same students are at most *Bad* in 100 % of cases, because a student at most *Bad* is of course also at most *Medium*. Node 4 is the second test node. The test formulates the following question with respect to the students who passed the first test: "is the evaluation in Literature at least *Medium*?". The right node 5 concerns all the students being at least *Medium* in Mathematics and Literature (S3,S4,S5,S6); 100 % of these students are comprehensively at least *Medium* (S3,S4,S5,S6) and 75 % of them are at least *Good* (S4,S5,S6). The left node 6, in turn, concerns all the students being at least *Medium* in Mathematics and worse than *Medium* in Literature (S1,S2); 50 % of these students are comprehensively at most *Bad* (S1) and 100 % of them are at most *Medium* (S1,S2).

Let us show how decision tree classifies new objects. Consider the above decision tree and students S9 and S10 from Table 13.3. As evaluation of S9 in Mathematics is *Medium*, according to the test node from the root of the tree, we can conclude that S9 is comprehensively at least *Medium* with credibility=83.3 % and at least *Good* with credibility=50 %. Furthermore, as evaluation of S9 in Literature is *Good*, according to the second test node, the student is for sure comprehensively at least *Medium* (credibility=100 %) and at least *Good* with credibility=75 %. With respect to student S10, the decision tree says that she is for sure comprehensively *Bad* (credibility=100 %). Let us remark the great transparency of the classification decision provided by the decision tree: indeed it explains in detail how the comprehensive evaluation is reached and what is the impact of particular elementary conditions on the confidence of classification.

13.5 Extensions of DRSA

13.5.1 DRSA with Joint Consideration of Dominance, Indiscernibility and Similarity Relations

Very often in data tables describing realistic decision problems, there are data referring to a preference order (criteria) and data not referring to any specific preference order (attributes).

Table 13.5 Information table of the illustrative example

Warehouse	Attributes			
	A_1	A_2	A_3	A_4
C1	High	700	A	Profit
C2	High	420	A	Loss
C3	Medium	500	B	Profit
C4	Medium	555	B	Loss
C5	Low	400	A	Loss
C6	Low	100	B	Loss

The following example illustrates the point. In Table 13.5, six companies are described by means of four attributes:

- A_1, capacity of management,
- A_2, number of employees,
- A_3, localization,
- A_4, company profit or loss.

The objective is to induce decision rules explaining profit or loss on the basis of attributes A_1, A_2 and A_3. Let us observe that

- attribute A_1 is a criterion, because the evaluation with respect to the capacity of management is preferentially ordered (high is better than medium, and medium is better than low);
- attribute A_2 is a quantitative attribute, because the values of the number of employees are not preferentially ordered (neither the high number of employees is in general better than the small number, nor the inverse); for quantitative attributes it is reasonable to use a similarity relation, which, in general, is a binary relation, only reflexive and neither transitive nor symmetric; for example, with respect to the data from Table 13.5, similarity between companies can be defined as follows: company a is similar to company b with respect to the attribute "number of employees" if

$$\frac{|number\ of\ employees\ of\ a\ - number\ of\ employees\ of\ b|}{number\ of\ employees\ of\ b} \leq 10\,\%;$$

  Let us remark that C3 is similar to C4 because $\frac{|500-555|}{555} \leq 10\,\%$, while C4 is not similar to C3 because $\frac{|555-500|}{500} > 10\,\%$. This shows how similarity relation may not satisfy symmetry. Let us suppose now that there is another company, C7, having 530 employees. Then, C4 is similar to C7 because $\frac{|555-530|}{530} \leq 10\,\%$ and C7 is similar to C3 because $\frac{|530-500|}{500} \leq 10\,\%$. However, we have already verified that C4 is not similar to C3. This shows how similarity may not satisfy transitivity;

- attribute A_3 is a qualitative attribute, because there is no preference order between different types of localization: two companies are indiscernible with respect to localization if they have the same localization;
- decision classes defined by attribute A_4 are preferentially ordered (obviously, profit is better than loss).

Let us remark that indiscernibility is the typical binary relation considered within the classical rough set approach (CRSA) while an extension of rough sets to the similarity relation has been proposed by Słowiński and Vanderpooten [87, 88] (for a fuzzy extension of this approach see [27, 41]). Greco et al. [29, 33] proposed an extension of DRSA to deal with data table like Table 13.5, where preference, indiscernibility and similarity are to be considered jointly. Applying this approach to Table 13.5, several decision rules can be induced; the following set of decision rules covers all the examples (within parentheses there are symbols of companies supporting the corresponding decision rule):

Rule (1): "*if* capacity of management is medium, *then* the company makes profit or loss", (C3,C4),
Rule (2): "*if* capacity of management is (at least) high *and* the number of employees is similar to 700, *then* the company makes profit", (C1),
Rule (3): "*if* capacity of management is (at most) low, *then* the company makes loss", (C5,C6),
Rule (4): "if the number of employees is similar to 420, then the company makes loss", (C2,C5).

13.5.2 DRSA and Interval Orders

In the previous sections we considered precise evaluations of objects on particular criteria and precise assignment of each object to one class. In practice, however, due to imprecise measurement, random variation of some parameters, unstable perception or incomplete definition of decision classes and preference scales of criteria, the evaluations and/or assignment may not be univocal. This was not the case in our example considered above; however, it is realistic to ask how DRSA should change in order to handle Table 13.1 augmented by students S13, S14 and S15 presented in Table 13.6.

Table 13.6 Students with interval evaluations

Student	Mathematics	Physics	Literature	Comprehensive evaluation
S13	*Medium-Good*	*Medium*	*Bad-Medium*	*Bad*
S14	*Medium*	*Good*	*Medium*	*Medium-Good*
S15	*Medium-Good*	*Medium-Good*	*Medium*	*Medium-Good*

This adaptation of DRSA has been considered in [15, 16]. Its basic idea consists in approximation of an interval order of the comprehensive evaluation by means of interval orders on particular criteria; the key concept of this approximation is a specially defined dominance relation.

13.5.3 Fuzzy DRSA: Rough Approximations by Means of Fuzzy Dominance Relations

The concept of dominance can be refined by introducing gradedness through the use of fuzzy sets in the sense of semantics expressing preferences for pairs of objects (for a detailed presentation of fuzzy preferences see [20]; see also [75]). The gradedness introduced by the use of fuzzy sets refines the classic crisp preference structures. The idea is the following. Let us consider the problem of classifying some enterprises according to their profitability. Let us suppose that the DM decides that the enterprises should be classified according to their ROI (Return On Investment). More precisely, she considers that enterprise x with ROI not smaller than 12 % should be assigned to the class of profitable enterprises $Cl_2^{\geq}$ and, otherwise, it should be assigned to the class of non profitable enterprises $Cl_1^{\leq}$. Now, consider enterprise a with ROI equal to 12 % and enterprise b with ROI equal to 11.9 %. The difference between the ROI of a and b is very small, however, it is enough to make a radically different assignment of these two enterprises. This example shows that it would be more reasonable to consider a smooth transition from $Cl_1^{\leq}$ to $Cl_2^{\geq}$. Such a transition can be controlled by a graded credibility $Cl_2^{\geq}(x)$, telling to what degree enterprise x belongs to $Cl_2^{\geq}$, defined as follows:

$$Cl_2^{\geq}(x) = \begin{cases} 0 & \textit{if } ROI(x) < 10\,\% \\ (ROI(x) - 10\,\%)/2 & \textit{if } 10\,\% \leq ROI(x) < 12\,\% \\ 1 & \textit{if } ROI(x) \geq 12\,\% \end{cases}$$

The correlative credibility that enterprise x belongs to $Cl_1^{\leq}$ can be defined as: $Cl_1^{\leq}(x) = 1 – Cl_2^{\geq}(x)$.

According to the above definition, we get $Cl_2^{\geq}(a) = 1$ and $Cl_2^{\geq}(b) = 0.95$, which means that a is for sure a profitable enterprise while b is profitable with a credibility of 95 %. Thus, the small difference between ROI(a) and ROI(b) does not lead to radically different classification of the two enterprises with respect to profitability.

The above reasoning about a smooth transition from truth to falsity of an inclusion relation can be applied to the dominance relation considered in the rough approximations. In Sect. 13.3, the dominance relation xD_Py has been declared true if evaluations of object x on all criteria from set P are not worse than those of object y. Continuing our example of classification with respect to profitability, let us consider among criteria the percentage growth of the sales, denoted by GS. Let us also define the weak preference relation $\succeq_{GS}$ with respect GS as

$$(1)\; x \succeq_{GS} y \Leftrightarrow GS(x) \geq GS(y).$$

(1) can be read as "enterprise x is at least as good as y with respect to GS if and only if GS of x is greater than or equal to GS of y". Considering two enterprises, a and b, one can remark that if the difference between $GS(a)$ and $GS(b)$ is small, for example $GS(a) = 12\,\%$ and $GS(b) = 12.1\,\%$, then definition (1) is too restrictive; in this situation it is hard to say that enterprise b is definitely better than enterprise a. It is thus realistic to assume an indifference threshold $q > 0$ on criterion GS, so that the following definition would replace (1)

$$(2)\; x \succeq_{GS} y \Leftrightarrow GS(x) \geq GS(y) - q.$$

(2) can be read as "enterprise x is at least as good as y with respect to GS if and only if GS of x is greater than or equal to GS of y decreased by an indifference threshold q". For example, if $q = 1\,\%$, then enterprise a is indifferent to enterprise b. Definition (2) is not yet completely satisfactory. Let us consider enterprises c and d such that $GS(c) = 13\,\%$ and $GS(d) = 13.1\,\%$. Using (2) with $q = 1\,\%$ we have to conclude that $a \succeq_{GS} c$ while *non* $a \succeq_{GS} d$. This is counterintuitive because the difference between $GS(c)$ and $GS(d)$ is very small and one would expect a similar result of comparison of c and d with a. Therefore, the following reformulation of the definition of weak preference $\succeq_{GS}$ seems reasonable:

(3) "x is at least as good as y with a credibility $\succeq_{GS}(x, y)$"

where

$$\succeq_{GS}(x, y) = \begin{cases} 0 & \text{if } GS(x) < GS(y) - p \\ \frac{p-(GS(y)-GS(x))}{p-q} & \text{if } -p \leq GS(x) - GS(y) < -q \\ 1 & \text{if } GS(x) \geq GS(y) - q \end{cases}$$

and p is a preference threshold such that $p > q$.

(3) has the following interpretation:

- it is completely true ($\succeq_{GS}(x, y) = 1$) that enterprise x is at least as good as y with respect to GS under the same condition as (2);
- it is completely false ($\succeq_{GS}(x, y) = 0$) that enterprise x is at least as good as y with respect to GS when $GS(x)$ is smaller than $GS(y)$ by at least p;
- between the two extremes ($0 < \succeq_{GS}(x, y) < 1$), the credibility that enterprise x is at least as good as y with respect to GS increases linearly with the opposite of the difference $GS(y)$–$GS(x)$.

Applying (3) with $q = 1\,\%$ and $p = 2\,\%$ to enterprises a, c, and d described above, one gets $\succeq_{GS}(a, c) = 1$ and $\succeq_{GS}(a, d) = 0.9$. Thus, considering comparison of c and d with a on GS, the small difference between $GS(c)$ and $GS(d)$ does not give as radically different results as before.

In [35, 39, 40, 52], DRSA was extended by using in two different ways *fuzzy dominance relation*. These extensions of the rough approximation into the fuzzy context maintain the same desirable properties of the crisp rough approximation of preference-ordered decision classes. These generalizations follow the traditional line of using fuzzy logical connectives in definitions of lower and upper approximation. In fact, there is no rule for the choice of the "right" connective, so this choice is always arbitrary to a certain extent. For this reason, in [54], a new fuzzy rough approximation was proposed. It avoids the use of fuzzy connectives, such as T-norm, T-conorm and fuzzy implication, which extend "and", "or" and "if ..., then ..." operators within fuzzy logic (see, for example, [20]), but at the price of introducing a certain degree of subjectivity related to the choice of one or another of their functional form. The proposed approach solves this problem because it is based on the ordinal properties of fuzzy membership functions only.

13.5.4 DRSA with Missing Values: Multiple-Criteria Classification Problem with Missing Values

In practical applications, the data table is often incomplete because some data are missing. For example, let us consider the profiles of students presented in Table 13.7, where "*" means that the considered evaluation is missing (for example students S16 and S17 have not yet passed the examination in Literature and Physics, respectively).

An extension of DRSA enabling the analysis of incomplete data tables has been proposed in [31, 36]. In this extension it is assumed that the dominance relation between two objects is a directional statement, where a subject object is compared to a referent object having no missing values on considered criteria. With respect to Table 13.7, one can say that, taking into account all the criteria,

(1) subject S17 dominates referent S18: in fact, referent S18 has no missing value;
(2) it is unknown if subject S18 dominates referent S17: in fact, referent S17 has no evaluation in Physics.

From (1) we can derive the following decision rule: "*if* a student is at least *Medium* in Mathematics, Physics and Literature, *then* the student is comprehensively at least *Medium*". This rule can be simplified into one of the following rules: "if a student is at least *Medium* in Physics, *then* the student is comprehensively at least *Medium*" or "if a student is at least *Medium* in Literature, *then* the student is comprehensively at least *Medium*".

Table 13.7 Example of missing values in the evaluation of students

Student	Mathematics	Physics	Literature	Comprehensive evaluation
S16	*Medium*	*Bad*	$*$	*Bad*
S17	*Medium*	$*$	*Good*	*Good*
S18	*Medium*	*Medium*	*Medium*	*Medium*

Table 13.8 Substitution of missing values in the evaluation of students

Student	Mathematics	Physics	Literature	Comprehensive evaluation
S16A	*Medium*	*Bad*	*Bad*	*Bad*
S16B	*Medium*	*Bad*	*Medium*	*Bad*
S16C	*Medium*	*Bad*	*Good*	*Bad*
S17A	*Medium*	*Bad*	*Good*	*Good*
S17B	*Medium*	*Medium*	*Good*	*Good*
S17C	*Medium*	*Good*	*Good*	*Good*
S18	*Medium*	*Medium*	*Medium*	*Medium*

The advantage of this approach is that the rules induced from the rough approximations defined according to the extended dominance relation are *robust*, i.e. each rule is supported by at least one object with no missing value on the criteria represented in the condition part of the rule. To better understand this feature, let us compare the above approach with another approach suggested to deal with missing values [70, 71]. In the latter it is proposed to substitute an object having a missing value by a set of objects obtained by putting all possible evaluations in the place of the missing value. Thus, from Table 13.7 one would obtain the following Table 13.8.

From Table 13.8, one can induce the rule: "*if* a student is at least *Medium* in Physics *and* at least *Good* in Literature, *then* the student is comprehensively at least *Good*". However, this rule is not robust because in the original Table 13.7, no student has such a profile.

DRSA extended to deal with missing values maintains all good characteristics of the dominance-based rough set approach and boils down to the latter when there are no missing values. This approach can also be used to deal with decision table in which dominance, similarity and indiscernibility must be considered jointly with respect to criteria and attributes. Another extension of DRSA for dealing with imprecise or missing evaluations of objects, and imprecise assignments of objects to classes, has been presented in [16].

13.5.5 DRSA for Decision Under Uncertainty

In [44] we opened a new avenue for applications of the rough set concept to analysis of preference-ordered data. We considered the classical problem of decision under uncertainty extending DRSA by using *stochastic dominance*. In a risky context, an act A stochastically dominates an act B if, for all possible levels *k* of gain or loss, the probability of obtaining an outcome at least as good as *k* with A is not smaller than with B. In this context we have an ambiguity if an act A stochastically dominates an act B, but, nevertheless, B has a comprehensive evaluation better than A. On this basis, it is possible to restate all the concepts of DRSA and adapt this approach to preference analysis under risk and uncertainty. We considered the case of traditional additive probability distribution over the set of future states of the world; however, the model is rich enough to handle non-additive probability distributions and even qualitative ordinal distributions. The rough set approach gives a representation of DM's preferences under uncertainty in terms of "*if…, then…*" decision rules induced from rough approximations of sets of exemplary decisions (preference-ordered classification of acts described in terms of outcomes in uncertain states of the world). This extension is interesting with respect to MCDA from two different viewpoints:

(1) each decision under uncertainty can be viewed as a multicriteria decision, where the criteria are the outcomes in different states of the world;
(2) DRSA adapted to decision under uncertainty can be applied to deal with multicriteria decision under uncertainty, i.e. decision problem where in each future state of the world the outcomes are expressed in terms of a set of criteria (see [17, 96]).

13.5.6 DRSA for Hierarchical Structure of Attributes and Criteria

In many real life situations, the process of decision-making is decomposable into sub-problems; this decomposition may either follow from a natural hierarchical structure of the evaluation or from a need of simplification of a complex decision problem. These situations are referred to *hierarchical decision problems*. The structure of a hierarchical decision problem has the form of a *tree* whose *nodes* are attributes and criteria describing objects. An example structure of a hierarchical classification problem is shown in Fig. 13.2. The cars are sorted into three classes: acceptable, hardly acceptable and non-acceptable, on the basis of three criteria (Price, Max speed, Fuel consumption) and two regular attributes (Colour and Country of production); one of criteria—Fuel consumption—is further composed of four sub-criteria.

In [15], hierarchical decision problems are considered where the decision is made in a finite number of steps due to hierarchical structure of regular attributes and criteria. The proposed methodology is based on decision rule preference model induced from examples of hierarchical decisions made by the DM on a set of reference objects. To deal with inconsistencies appearing in decision examples, DRSA has been adapted to hierarchical classification problems. In these problems, the main difficulty consists in *propagation* of inconsistencies along the tree, i.e. taking into account at each node of the tree the inconsistent information coming from lower level nodes. In the proposed methodology, the inconsistencies are propagated from the bottom to the top of the tree in the form of *subsets* of possible attribute values instead of single values. In the case of hierarchical criteria, these subsets are *intervals* of possible criterion values. Subsets of possible values may also appear in leafs of the tree, i.e. in evaluations of objects by the lowest-level attributes

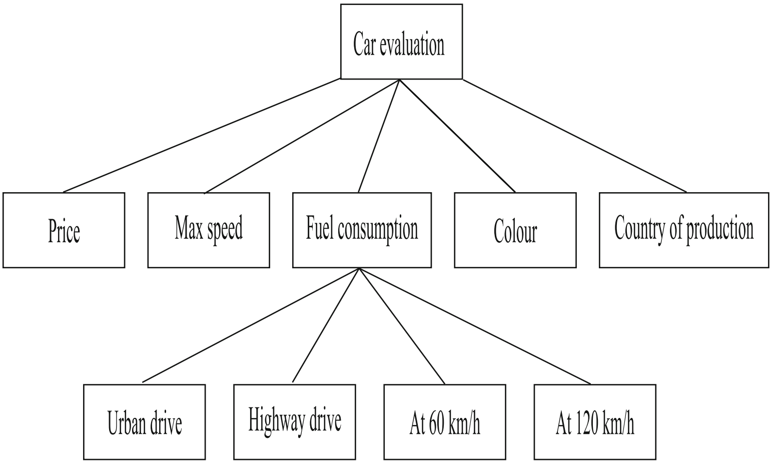


Fig. 13.2 The hierarchy of attributes and criteria for a car classification problem

and criteria. To deal with multiple values of attributes in description of objects the classical rough set approach has been adapted adequately. Interval evaluations of objects on particular criteria can be handled by DRSA extended to interval orders (see point 5.2).

13.6 DRSA for Multiple-Criteria Choice and Ranking

DRSA can also be applied to multiple-criteria choice and ranking problems. However, there is a basic difference between classification problems from one side and choice and ranking from the other side. To give a didactic example, consider a set of companies A for evaluation of a risk of failure, taking into account the debt ratio criterion. To assess the risk of failure of company x, we will not compare the debt ratio of x with the debt ratio of all the other companies from A. The comparison will be made with respect to a fixed risk threshold on the debt ratio criterion. Indeed, the debt ratio of x can be the highest of all companies from A and, nevertheless, x can be classified as a low risk company if its debt ratio is below the fixed risk threshold. Consider, in turn, the situation, in which we must choose the lowest risk company from A or we want to rank the companies from A from the less risky to the most risky one. In this situation, the comparison of the debt ratio of x with a fixed risk threshold is not useful and, instead, a pairwise comparison of the debt ratio of x with the debt ratio of all other companies in A is relevant for the choice or ranking. Thus, in general, while classification is based on absolute evaluation of objects (e.g. comparison of the debt ratio with the fixed risk threshold), choice and ranking refer to relative evaluation, by means of pairwise comparisons of objects (e.g. comparisons of the debt ratio of pairs of companies).

The necessity of pairwise comparisons of objects in multiple-criteria choice and ranking problems requires some further extensions of DRSA. Simply speaking, in this context we are interested in the approximation of a binary relation, corresponding to a comprehensive preference relation, using other binary relations, corresponding to marginal preference relations on particular criteria, for pairs of objects. In the above example, we would approximate the binary relation "from the viewpoint of the risk of failure, company x is comprehensively preferred to company y" using binary relations on the debt ratio criterion, like "the debt ratio of x is *much better* than that of y" or "the debt ratio of x is *weakly better* than that of y", and so on.

Technically, the modification of DRSA necessary to approach the problems of choice and ranking are twofold:

(1) *pairwise comparison table* (PCT) is considered instead of the simple data table [32]: PCT is a decision table whose rows represent pairs of objects for which multiple-criteria evaluations and a comprehensive preference relation are known;
(2) *dominance principle* is considered for *pairwise comparisons* instead of simple objects: if object x is preferred to y at least as strongly as w is preferred to z on all the considered criteria, then x must be comprehensively preferred to y at least as strongly as w is comprehensively preferred to z.

The application of DRSA to the choice or ranking problems proceeds as follows. First, the DM gives some examples of pairwise comparisons with respect to some reference objects, for example a complete ranking from the best to the worst of a limited number of objects—well known to the DM. From this set of examples, a preference model in terms of "*if…, then…*" decision rules is induced. These rules are applied to a larger set of objects. A proper exploitation of the results so obtained gives a final recommendation for the decision problem at hand. Below, we present more formally and in greater detail this methodology.

13.6.1 Pairwise Comparison Table (PCT) as a Preference Information and a Learning Sample

Let A be the set of objects for the decision problem at hand. Let us also consider a set of reference objects $B \subseteq A$ on which the DM is expressing her preferences by pairwise comparisons. Let us represent the comprehensive preference by a function $P : A \times A \to \boldsymbol{R}$. In general, for each $x, y \in A$,

- if $P(x, y) > 0$, then $P(x, y)$ can be interpreted as a degree to which x is evaluated better than y,
- if $P(x, y) < 0$, then $P(x, y)$ can be interpreted as a degree to which x is evaluated worse than y,
- if $P(x, y) = 0$, then x is evaluated equivalent to y.

The semantic value of preference P can be different. We remember two possible interpretations:

(1) $P(x, y)$ represents a *degree of outranking* of x over y, i.e. $P(x, y)$ is the credibility of the proposition "x is at least as good as y";
(2) $P(x, y)$ represents a *degree of net preference* of x over y, i.e. $P(x, y)$ is the strength with which x is preferred to y.

In case (1), $P(x, y)$ measures the strength of arguments in favor of x and against y, while $P(y, x)$ measures the arguments in favor of y and against x. Thus, there is no relation between values of $P(x, y)$ and $P(y, x)$. In case (2), $P(x, y)$ synthesizes arguments in favor of x and against y together with arguments in favor of y and against x. $P(y, x)$ has a symmetric interpretation and the relation $P(x, y) = -P(y, x)$ is expected.

Let us suppose that objects from set A are evaluated by a consistent family of n criteria $g_i : A \to \boldsymbol{R}$, $i = 1, 2, \dots, n$, such that, for each object $x \in A$, $g_i(x)$ represents the evaluation of x with respect to criterion g_i. Using the terms of the rough set approach, the family of criteria constitutes the set C of condition attributes. With respect to each criterion $g_i \in C$ one can consider a particular preference function $P_i : \boldsymbol{R} \times \boldsymbol{R} \to \boldsymbol{R}$, such that for all $x, y \in A$, $P_i[g_i(x), g_i(y)]$ for criterion g_i has an interpretation analogous to comprehensive preference relation $P(x, y)$, i.e.

- if $P_i[g_i(x), g_i(y)] > 0$, then $P_i[g_i(x), g_i(y)]$ is a degree to which x is better than y on criterion g_i,
- if $P_i[g_i(x), g_i(y)] < 0$, then $P_i[g_i(x), g_i(y)]$ is a degree to which x is worse than y on criterion g_i,
- if $P_i[g_i(x), g_i(y)] = 0$, then x is equivalent to y on criterion g_i.

Let us suppose that the DM expresses her preferences with respect to pairs (x, y) from $E \subseteq B \times B$, $|E| = e$. These preferences are represented in an $e \times (m + 1)$ Pairwise Comparison Table $\boldsymbol{S}_{PCT}$. The m rows correspond to the pairs from E. For each $(x, y) \in E$ in the corresponding row, the first n columns include information about preferences $P_i[g_i(x), g_i(y)]$ on particular criteria from set C, while the last, $(m + 1) - th$ column represents the comprehensive preference $P(x, y)$.

13.6.2 Multigraded Dominance

Given subset $P \subseteq C$ ($P \neq \emptyset$) of criteria and pairs of objects $(x, y), (w, z) \in A \times A$, the pair (x, y) is said to P-*dominate* the pair (w, z) (denotation $(x, y)D_P(w, z)$), if $P_i[g_i(x), g_i(y)] > P_i[g_i(w), g_i(z)]$ for all $g_i \in P$, i.e. if x is preferred to y at least as strongly as w is preferred to z with respect to each criterion $g_i \in P$. Let us remark that the dominance relation D_P is a partial preorder on $A \times A$; as, in general, it involves different grades of preference on particular criteria, it is called *multigraded* dominance relation.

Given $P \subseteq C$ and $(x,y) \in E$, we define:

- a set of pairs of objects *P−dominating* (x,y), called *P−dominating set*, $D_P^+(x,y) = \{(w,z) \in E : (w,z)D_P(x,y)\}$,
- a set of pairs of objects *P−dominated* by (x,y), called *P−dominated set*, $D_P^-(x,y) = \{(w,z) \in E : (x,y)D_P(w,z)\}$.

The *P−dominating sets* and the *P−dominated sets* defined on E for considered pairs of reference objects from E are "granules of knowledge" that can be used to express *P−lower* and *P−upper* approximations of set $B_k^{\geq} = \{(x,y) \in E : P(x,y) \geq k\}$, corresponding to comprehensive preference of degree *at least k*, and set $B_k^{\leq} = \{(x,y) \in E : P(x,y) \leq k\}$, corresponding to comprehensive preference of degree *at most k*, respectively:

$$\underline{P}(B_k^{\geq}) = \{(x,y) \in E : D_P^+(x,y) \subseteq B_k^{\geq}\},$$

$$\overline{P}(B_k^{\geq}) = \bigcup_{(x,y)\in B_k^{\geq}} D_P^+(x,y) = \{(x,y) \in E : D_P^-(x,y) \cap B_k^{\geq} \neq \emptyset\},$$

$$\underline{P}(B_k^{\leq}) = \{(x,y) \in E : D_P^-(x,y) \subseteq B_k^{\leq}\},$$

$$\overline{P}(B_k^{\leq}) = \bigcup_{(x,y)\in B_k^{\leq}} D_P^-(x,y) = \{(x,y) \in E : D_P^+(x,y) \cap B_k^{\leq} \neq \emptyset\}.$$

The set difference between *P−lower* and *P−upper* approximations of sets $B_k^{\geq}$ and $B_k^{\leq}$ contains all the ambiguous pairs (x,y):

$$Bn_P(B_k^{\geq}) = \overline{P}(B_k^{\geq}) - \underline{P}(B_k^{\geq}),\; Bn_P(B_k^{\leq}) = \overline{P}(B_k^{\leq}) - \underline{P}(B_k^{\leq}),$$

The above rough approximations of $B_k^{\geq}$ and $B_k^{\leq}$ satisfy properties analogous to the rough approximations of upward and downward unions of classes $Cl_t^{\geq}$ and $Cl_t^{\leq}$; precisely, these are:

- inclusion: $\underline{P}(B_k^{\geq}) \subseteq B_k^{\geq} \subseteq \overline{P}(B_k^{\geq})$, $\quad \underline{P}(B_k^{\leq}) \subseteq B_k^{\leq} \subseteq \overline{P}(B_k^{\leq})$;
- complementarity:
  $\underline{P}(B_k^{\geq}) = E - \overline{P}(B_k^{\leq} - 1)$, $\quad \overline{P}(B_k^{\geq}) = E - \underline{P}(B_k^{\leq} - 1)$,
  $\underline{P}(B_k^{\leq}) = E - \overline{P}(B_k^{\geq} + 1)$, $\quad \overline{P}(B_k^{\leq}) = E - \underline{P}(B_k^{\geq} + 1)$,
  where $B_k^{>} = E - B_k^{\leq}$ and $B_k^{<} = E - B_k^{\geq}$ and the rough approximation of $B_k^{>}$ and $B_k^{<}$ are analogous to those of $B_k^{\geq}$ and $B_k^{\leq}$, for example, $\underline{P}(B_k^{>}) = \{(x,y) \in E : D_P^+(x,y) \subseteq B_k^{>}\}$;
- monotonicity: for each $R, P \subseteq C$, such that $R \subseteq P$,
  $\underline{R}(B_k^{\geq}) \subseteq \underline{P}(B_k^{\geq})$, $\quad \overline{R}(B_k^{\geq}) \supseteq \overline{P}(B_k^{\geq})$,
  $\underline{R}(B_k^{\leq}) \subseteq \underline{P}(B_k^{\leq})$, $\quad \overline{R}(B_k^{\leq}) \supseteq \overline{P}(B_k^{\leq})$.

The concepts of the quality of approximation, reducts and core can be extended also to the approximation of the comprehensive preference relation by multigraded dominance relations. In particular, the coefficient

$$\gamma_P = \frac{\left| E - \bigcup_k Bn_P\left(B_k^{\geq}\right) \right|}{|E|} = \frac{\left| E - \bigcup_k Bn_P\left(B_k^{\leq}\right) \right|}{|E|}.$$

defines the *quality of approximation of comprehensive preference* $P(x,y)$ by criteria from $P \subseteq C$. It expresses the ratio of all pairs $(x,y) \in E$ whose degree of preference of x over y is correctly assessed using set P of criteria, to all the pairs of objects contained in E. Each minimal subset $P \subseteq C$, such that $\gamma_P = \gamma_C$, is called *reduct* of C (denoted by $RED_{S_{PCT}}$). Let us remark that $\boldsymbol{S}_{PCT}$ can have more than one reduct. The intersection of all reducts is called the *core* (denoted by $CORE_{S_{PCT}}$).

It is also possible to use the Variable Consistency Model on $\boldsymbol{S}_{PCT}$ ([89]; see also [22]) relaxing the definitions of $P-$*lower* approximations of graded comprehensive preference relations represented by sets $B_k^{\geq}$ and $B_k^{\leq}$, such that some pairs in $P-$*dominated* or $P-$*dominating* sets belong to the opposite relation but at least $l*100\,\%$ of pairs belong to the correct one. Then, the definition of $P-$*lower* approximations of $B_k^{\geq}$ and $B_k^{\leq}$ with respect to set $P \subseteq C$ of criteria boils down to:

$$\underline{P}^l\left(B_k^{\geq}\right) = \left\{ (x,y) \in E : \frac{\left|D_P^{+}(x,y)) \cap B_k^{\geq}\right|}{\left|D_P^{+}(x,y)\right|} \geq l \right\},$$

$$\underline{P}^l\left(B_k^{\leq}\right) = \left\{ (x,y) \in E : \frac{\left|D_P^{-}(x,y)) \cap B_k^{\leq}\right|}{\left|D_P^{-}(x,y)\right|} \geq l \right\}.$$

13.6.3 Induction of Decision Rules from Rough Approximations of Graded Preference Relations

Using the rough approximations of sets $B_k^{\geq}$ and $B_k^{\leq}$, that is rough approximations of comprehensive preference relation $P(x,y)$ of degree at least or at most k, respectively, it is possible to induce a generalized description of the preference information contained in a given $\boldsymbol{S}_{PCT}$ in terms of decision rules with a special syntax. We are considering decision rules of the following types:

(1) $D_{\geq}-$*decision rules*:

$$\textit{if } P_{i1}[g_{i1}(x), g_{i1}(y)] \geq k_{i1} \textit{ and} \ldots\ P_{ir}[g_{ir}(x), g_{ir}(y)] \geq k_{ir},$$
$$\textit{then } P(x,y) \geq k$$

where $\{g_{i1}, \ldots, g_{ir}\} \subseteq C$; for example: "*if* car x is much better than y with respect to maximum speed *and* at least weakly better with respect to acceleration, *then* x is comprehensively better than y"; these rules are supported by pairs of objects from the $P-$*lower* approximation of sets $B_k^{\geq}$ only;

(2) $D_{\leq}$*−decision rules*:

$$\textit{if } P_{i1}[g_{i1}(x), g_{i1}(y)] \leq k_{i1} \textit{ and} \ldots\ P_{ir}[g_{ir}(x), g_{ir}(y)] \leq k_{ir},$$
$$\textit{then } P(x,y) \leq k$$

where $\{g_{i1}, \ldots, g_{ir}\} \subseteq C$; for example: "*if* car x is much worse than y with respect to price *and* weakly worse with respect to comfort, *then* x is comprehensively worse than y"; these rules are supported by pairs of objects from the *P−lower* approximation of sets $B_k^{\leq}$ only;

(3) $D_{\geq\leq}$*−decision rules*:

$$\textit{if } P_{i1}[g_{i1}(x), g_{i1}(y)] \geq k_{i1} \textit{ and} \ldots\ P_{ir}[g_{ir}(x), g_{ir}(y)] \geq k_{ir} \textit{ and}$$
$$P_{j1}[g_{j1}(x), g_{j1}(y)] \leq k_{j1} \textit{ and} \ldots P_{js}[g_{js}(x), g_{js}(y)] \leq k_{js},$$
$$\textit{then } h \leq P(x,y) \leq k$$

where $\{g_{i1}, \ldots, g_{ir}\}, \{g_{j1}, \ldots, g_{js}\} \subseteq C$; for example: "*if* car x is much worse than y with respect to price *and* much better with respect to comfort, *then* x is indifferent or better than y, and there is not enough information to distinguish between the two situations"; these rules are supported by pairs of objects from the intersection of the *P−upper* approximation of sets $B_k^{\geq}$ and $B_h^{\leq}$ $(h < k)$ only.

13.6.4 Use of Decision Rules for Decision Support

The decision rules induced from rough approximations of sets $B_k^{\geq}$ and $B_k^{\leq}$ for a given $\boldsymbol{S}_{PCT}$, describe the comprehensive preference relations $P(x,y)$ either exactly ($D_{\geq}$ and $D_{\leq}$*−decision rules*) or approximately ($D_{\geq\leq}$*−decision rules*). A set of these rules covering all pairs of $\boldsymbol{S}_{PCT}$ represent a preference model of the DM who gave the pairwise comparison of reference objects. Application of these decision rules on a new subset $M \subseteq A$ of objects induces a specific preference structure on M.

For simplicity, in the following we consider the case where $P(x,y)$ is interpreted as outranking and assumes two values only: $P(x,y) = 1$, which means that x is at least as good as y, and $P(x,y) = -1$, which means that x is *not* at least as good as y. In the following $P(x,y) = 1$ will be denoted by xSy and $P(x,y) = -1$ will be denoted by xS^cy.

In fact, any pair of objects $(u,v) \in M \times M$ can match the decision rules in one of four ways:

- at least one $D_{\geq}$*−decision rule* and neither $D_{\leq}$ nor $D_{\geq\leq}$*−decision rules*,
- at least one $D_{\leq}$*−decision rule* and neither $D_{\geq}$ nor $D_{\geq\leq}$*−decision rules*,
- at least one $D_{\geq}$*−decision rule* and at least one $D_{\leq}$*−decision rule*, or at least $D_{\geq\leq}$*−decision rules*,
- no *decision rule*.

These four ways correspond to the following four situations of outranking, respectively:

- uSv and *not* $uS^c v$, that is *true outranking* (denoted by $uS^T v$),
- $uS^c v$ and *not* uSv, that is *false outranking* (denoted by $uS^F v$),
- uSv and $uS^c v$, that is *contradictory outranking* (denoted by $uS^K v$),
- *not* uSv and *not* $uS^c v$, that is *unknown outranking* (denoted by $uS^U v$).

The four above situations, which together constitute the so-called *four-valued outranking* [30], have been introduced to underline the presence and absence of *positive* and *negative* reasons for the outranking. Moreover, they make it possible to distinguish contradictory situations from unknown ones.

A final *recommendation* (choice or ranking) can be obtained upon a suitable exploitation of this structure, i.e. of the presence and the absence of outranking S and S^c on M. A possible exploitation procedure consists in calculating a specific score, called Net Flow Score, for each object $x \in M$:

$$S_{nf}(x) = S^{++}(x) - S^{+-}(x) + S^{-+}(x) - S^{--}(x),$$

where

$S^{++}(x) =$
card($\{y \in M$: there is at least one decision rule which affirms $xSy\}$),
$S^{+-}(x) =$
card($\{y \in M$: there is at least one decision rule which affirms $ySx\}$),
$S^{-+}(x) =$
card($\{y \in M$: there is at least one decision rule which affirms $yS^c x\}$),
$S^{--}(x) =$
card($\{y \in M$: there is at least one decision rule which affirms $xS^c y\}$).

The recommendation in ranking problems consists of the total preorder determined by $S_{nf}(x)$ on M; in choice problems, it consists of the object(s) $x^* \in M$, such that $S_{nf}(x^*) = \max_{x \in M}\{S_{nf}(x)\}$.

The above procedure has been characterized with reference to a number of desirable properties in [30, 99, 100].

13.6.5 Illustrative Example

Let us suppose that a company managing a chain of warehouses wants to buy some new warehouses. To choose the best proposals or to rank them all, the managers of the company decide to analyze first the characteristics of eight warehouses already owned by the company (reference objects). This analysis should give some indications for the choice and ranking of the new proposals. Eight warehouses belonging to the company have been evaluated by three following criteria: capacity of the sales staff (g_1), perceived quality of goods (g_2) and high traffic location (g_3).

Table 13.9 Decision table with reference objects

Warehouse	g_1	g_2	g_3	d (ROE %)
1	*good*	*medium*	*good*	10.35
2	*good*	*sufficient*	*good*	4.58
3	*medium*	*medium*	*good*	5.15
4	*sufficient*	*medium*	*medium*	−5
5	*sufficient*	*medium*	*medium*	2.42
6	*sufficient*	*sufficient*	*good*	2.98
7	*good*	*medium*	*good*	15
8	*good*	*sufficient*	*good*	−1.55

The domains (scales) of these attributes are presently composed of three preference-ordered echelons: $V_1 = V_2 = V_3 = \{sufficient, medium, good\}$. The decision attribute (d) indicates the profitability of warehouses, expressed by the Return On Equity (ROE) ratio (in %). Table 13.9 presents a decision table with the considered reference objects.

With respect to the set of criteria $C = \{g_1, g_2, g_3\}$, the following numerical representation is used for criterion $g_i, i = 1, 2, 3$: $g_i(x) = 1$ if x is *sufficient*, $g_i(x) = 2$ if x is *medium*, $g_i(x) = 3$ if x is *good*.

The degree of preferences with respect to pairs of actions are defined as $P_i[(x), g_i(y)] = g_i(x) - g_i(y), i = 1, 2, 3$, and they are coded as follows:

$P_i[g_i(x), g_i(y)] = -2 \Leftrightarrow P_i[g_i(y), g_i(x)] = 2$,
which means that "x is worse than y" and
"y is better than x",
$P_i[g_i(x), g_i(y)] = -1 \Leftrightarrow P_i[g_i(y), g_i(x)] = 1$,
which means that "x is weakly worse than y" or
"y is weakly better than x",
$P_i[g_i(x), g_i(y)] = P_i[g_i(y), g_i(x)] = 0$,
which means that "x is equivalent to y".

Using the decision attribute, the comprehensive outranking relation was build as follows: warehouse x is at least as good as warehouse y with respect to profitability ($P(x, y) = 1 \Leftrightarrow xSy$) if

$$\text{ROE}(x) \geq \text{ROE}(y) - 2\,\%.$$

Otherwise, i.e. if $\text{ROE}(x) < \text{ROE}(y) - 2\,\%$, warehouse x is *not* at least as good as warehouse y with respect to profitability ($P(x, y) = -1 \Leftrightarrow xS^c y$).

The pairwise comparisons of reference objects are gathered $\boldsymbol{S}_{PCT}$. In Table 13.10, there is a small fragment of $\boldsymbol{S}_{PCT}$.

The rough set analysis of the $\boldsymbol{S}_{PCT}$ leads to the conclusion that the set of decision examples on reference objects is inconsistent. The quality of approximation of S and S^c by all criteria from set C is equal to 0.44. Moreover, $RED_{S_{PCT}} = CORE_{S_{PCT}} = \{g_1, g_2, g_3\}$; this means that no criterion is superfluous.

Table 13.10 A fragment of S_{PCT}

Pair(x,y) of warehouses	$P_1[g_1(x),g_1(y)]$	$P_2[g_2(x),g_2(y)]$	$P_3[g_3(x),g_3(y)]$	S or S^c
(1,2)	0	1	0	S
(4,7)	−2	0	−1	S^c
(6,3)	−1	−1	0	S^c
...	...	...	...	...

The C-lower approximations and the C-upper approximations of S and S^c, obtained by means of multigraded dominance relations, are as follows:

$$\underline{C}(S) = \{(1,2),(1,4),(1,5),(1,6),(1,8),(3,2),(3,4),(3,5),(3,6),(3,8),(7,2),(7,4),(7,5),(7,6),(7,8)\},$$

$$\underline{C}(S^c) = \{(2,1),(2,7),(4,1),(4,3),(4,7),(5,1),(5,3),(5,7),(6,1),(6,3),(6,7),(8,1),(8,7)\}.$$

All the remaining 36 pairs of reference objects belong to the *C−boundaries* of S an S^c, i.e. $Bn_C(S) = Bn_C(S^c)$.

The following minimal $D_{\geq}$*−decision rules* and $D_{\leq}$*−decision rules* can be induced from lower approximations of S and S^c, respectively (within parentheses there are the pairs of objects supporting the corresponding rules):

if $P_1[g_1(x),g_1(y)] \geq 1$ *and* $P_2[g_2(x),g_2(y)] \geq 1$, *then* xSy;
$((1,6),(3,6),(7,6))$,
if $P_2[g_2(x),g_2(y)] \geq 1$ *and* $P_3[g_3(x),g_3(y)] \geq 0$, *then* xSy;
$((1,2),(1,6),(1,8),(3,2),(3,6),(3,8),(7,2),(7,6),(7,8))$,
if $P_2[g_2(x),g_2(y)] \geq 0$ *and* $P_3[g_3(x),g_3(y)] \geq 1$, *then* xSy;
$((1,4),(1,5),(3,4),(3,5),(7,4),(7,5))$,
if $P_1[g_1(x),g_1(y)] \leq -1$ *and* $P_2[g_2(x),g_2(y)] \leq -1$, *then* xS^cy;
$((6,1),(6,3),(6,7))$,
if $P_2[g_2(x),g_2(y)] \leq 0$ *and* $P_3[g_3(x),g_3(y)] \leq -1$, *then* xS^cy;
$((4,1),(4,3),(4,7),(5,1),(5,3),(5,7))$,
if $P_1[g_1(x),g_1(y)] \leq 0$ *and* $P_2[g_2(x),g_2(y)] \leq -1$, *and*
$P_3[g_3(x),g_3(y)] \leq 0$ *then* xS^cy;
$((2,1),(2,7),(6,1),(6,3),(6,7),(8,1),(8,7))$.

Moreover, it was possible to induce five minimal $D_{\geq\leq}$*−decision rules* from the boundary of approximation of S and S^c:

if $P_2[g_2(x),g_2(y)] \leq 0$ *and* $P_2[g_2(x),g_2(y)] \geq 0$
(i.e. $P_2[g_2(x),g_2(y)] = 0$) *and* $P_3[g_3(x),g_3(y)] \leq 0$
and $P_3[g_3(x),g_3(y)] \geq 0$ (i.e. $P_3[g_3(x),g_3(y)] = 0$), *then* xSy *or* xS^cy;
$((1,1),(1,3),(1,7),(2,2),(2,6),(2,8),(3,1),(3,3),(3,7),(4,4),$
$(4,5),(5,4),(5,5),(6,2),(6,6),(6,8),(7,1),(7,3),(7,7),$

Table 13.11 Ranking of warehouses for sale by decision rules and the Net Flow Score procedure

Warehouse	g_1	g_2	g_3	Net Flow Score	Ranking
1'	*good*	*sufficient*	*medium*	1	5
2'	*sufficient*	*good*	*good*	11	1
3'	*sufficient*	*medium*	*sufficient*	−8	8
4'	*sufficient*	*good*	*sufficient*	0	6
5'	*sufficient*	*sufficient*	*medium*	−4	7
6'	*sufficient*	*good*	*good*	11	1
7'	*medium*	*sufficient*	*sufficient*	−11	9
8'	*medium*	*medium*	*medium*	7	3
9'	*medium*	*good*	*sufficient*	4	4
10'	*medium*	*sufficient*	*sufficient*	−11	9

$(8,2),(8,6),(8,8)),$
if $P_2[g_2(x),g_2(y)] \leq 1$ *and* $P_3[g_3(x),g_3(y)] \geq 1,$ *then* xSy *or* xS^cy;
$((2,4),(2,5),(6,4),(6,5),(8,4),(8,5)),$
if $P_2[g_2(x),g_2(y)] \geq 1$ *and* $P_3[g_3(x),g_3(y)] \leq -1,$ *then* xSy *or* xS^cy;
$((4,2),(4,6),(4,8),(5,2),(5,6),(5,8)),$
if $P_1[g_1(x),g_1(y)] \geq 1$ *and* $P_2[g_2(x),g_2(y)] \leq 0,$
and $P_3[g_3(x),g_3(y)] \leq 0$ *then* xSy *or* xS^cy;
$((1,3),(2,3),(2,6),(7,3),(8,3),(8,6)),$
if $P_1[g_1(x),g_1(y)] \geq 1$ *and* $P_2[g_2(x),g_2(y)] \leq -1,$ *then* xSy *or* xS^cy;
$((2,3),(2,4),(2,5),(8,3),(8,4),(8,5)).$

Using all above decision rules and the Net Flow Score exploitation procedure on ten other warehouses proposed for sale, the managers obtained the result presented in Table 13.11. The dominance-based rough set approach gives a clear recommendation:

- for the **choice problem** it suggests to **select warehouse 2' and 6'**, having maximum score (11),
- for the **ranking problem** it suggests the **ranking** presented in the last column of Table 13.11, as follows:

$$(2',6') \rightarrow (8') \rightarrow (9') \rightarrow (1') \rightarrow (4') \rightarrow (5') \rightarrow (3') \rightarrow (7',10')$$

13.6.6 Fuzzy Preferences

Let us consider the case where the preferences $P_i[g_i(x),g_i(y)]$ with respect to each criterion $g_i \in C$, as well as the comprehensive preference $P(x,y)$, can assume values from a finite set. For example, given $x,y \in A$, the preferences $P_i[g_i(x),g_i(y)]$

and $P(x, y)$ can assume the following qualitatively ordinal values: x is much better than y, x is better than y, x is equivalent to y, x is worse than y, x is much worse than y. Let us suppose, moreover, that each possible value of $P_i[g_i(x), g_i(y)]$ and $P(x, y)$ is fuzzy in the sense that it is true at some level of credibility between 0 and 100 %, e.g. "x is better than y on criterion g_i with credibility 75 %", or "x is comprehensively worse than y with credibility 80 %". Greco et al. [35] proved that the fuzzy comprehensive preference $P(x, y)$ can be approximated by means of fuzzy preferences $P_i[g_i(x), g_i(y)]$ after translating the dominance-based rough approximations of S_{PCT} defined for the crisp case, by means of fuzzy operators.

13.6.7 Preferences Without Degree of Preferences

The values of $P_i[g_i(x), g_i(y)]$ considered in the dominance-based rough approximation of $\boldsymbol{S}_{PCT}$ represent a degree (strength) of preference. It is possible, however, that in some cases, the concept of degree of preference with respect to some criteria is meaningless for a DM. In these cases, there does not exist a function $P_i[g_i(x), g_i(y)]$ expressing how much x is better than y with respect to criterion g_i and, on the contrary, we can directly deal with values $g_i(x)$ and $g_i(y)$ only. For example let us consider the car decision problem and four cars x, y, w, z with the maximum speed of 210 km/h, 180 km/h, 150 km/h and 140 km/h, respectively. Even if the concept of degree of preference is meaningless, it is possible to say that with respect to the maximum speed, x is preferred to z at least as much as y is preferred to w. On the basis of this observation, Greco et al. [35] proved that comprehensive preference $P(x, y)$ can be approximated by means of criteria with only ordinal scales, for which the concept of degree of preference is meaningless. An example of decision rules obtained in this situation is the following:

"*if* car x has a maximum speed of at least 180km/h while car y has a maximum speed of at most 140km/h *and* the comfort of car x is at least good while the comfort of car y is at most medium, *then* car x is at least as good as car y".

13.7 DRSA and Operations Research Problems

DRSA is also a useful instrument in the toolbox of Operations Research (OR). DRSA has been applied to the following OR problems:

(1) interactive multiobjective optimization (IMO-DRSA) [57];
(2) interactive evolutionary multiobjective optimization under risk and uncertainty [60];
(3) decision under uncertainty and time preference [59].

13.7.1 DRSA to Interactive Multiobjective Optimization (IMO-DRSA)

DRSA to interactive multiobjective optimization (IMO-DRSA) [57] permits to deal with many optimization problems considered within OR (ranging from inventory management to scheduling, passing through portfolio management) in a way which is very much oriented towards interaction with the users. In fact, in IMO-DRSA a sample of representative solutions to a multiobjective optimization problem is presented to the DM who is asked to indicate a subset of relatively "good" solutions in the sample. Applying DRSA to the sample of representative solutions classified into "good" and "others" by the DM, a set of decision rules is induced in the form: "if objective $f_{i_1}(x) \geq \alpha_{i_1}$ and $\ldots f_{i_p}(x) \geq \alpha_{i_p}$, then x is a good solution". The DM selects the rule that in his/her opinion is the most representative of his/her preferences and the constraints coming from that rule are joined to the previous set of constraints imposed on the Pareto optimal set, in order to focus on a part interesting from the point of view of DM's preferences in the next iteration. For example, if the DM selects the rule "if objective $f_{i_1}(x) \geq \alpha_{i_1}$ and $\ldots f_{i_p}(x) \geq \alpha_{i_p}$, then x is a good solution", then the constraints $f_{i_1}(x) \geq \alpha_{i_1}$ and $\ldots f_{i_p}(x) \geq \alpha_{i_p}$ are joined to the set of constraints of the multiobjective optimization problem, such that the new set of constraints implies a Pareto optimal set being the subset of the original Pareto optimal set. This subset satisfies the requirements of the selected rule, so that it is composed of solutions that are at this stage considered as relatively good by the DM. The procedure continues iteratively until the DM is satisfied with one solution from the current sample - this is the most preferred solution.

13.7.2 DRSA to Interactive Evolutionary Multiobjective Optimization

Very often real life optimization problems are so complex that exact methods fail to find an optimal solution. In these cases some heuristics have to be applied. Within multiobjective optimization, Evolutionary Multiobjective Optimization (EMO) appeared to be particularly efficient; see, e.g., [11, 12]. The underlying reasoning behind the EMO search of an approximation of the Pareto optimal frontier is that, in the absence of any preference information, all Pareto optimal solutions have to be considered equivalent. On the other hand, if the DM (alternatively called user) is involved in the multiobjective optimization process, then the preference information provided by the DM can be used to focus the search on the most preferred part of the Pareto optimal frontier. This idea stands behind Interactive Multiobjective Optimization (IMO) methods proposed long time before EMO has emerged. Recently, it became clear that merging the IMO and EMO methodologies should be beneficial for the multiobjective optimization process [8]. Several approaches have been presented in this context; see, e.g., [7, 9, 11, 13, 14, 21, 66, 68, 79]. The methodology of interactive EMO based on DRSA [60, 62], involves application of decision rules, which are induced from easily elicited preference information by DRSA, according to two general schemes, called DRSA-EMO and DRSA-EMO-PCT. This aim is to focus the search of the Pareto optimal frontier on the most preferred region. More specifically, DRSA is used for structuring preference information obtained through interaction with the user, and then a set of decision rules representing user's preferences is induced from this information. These rules are used to rank solutions in the current population of EMO, which has an impact on the selection and crossover. Within interactive EMO, one can also apply DRSA for decision under uncertainty. This permits to take into account robustness concerns in the multiobjective optimization. In fact, two methods of robust optimization methods combining DRSA and interactive EMO have been proposed: DARWIN [60] (Dominance-based rough set Approach to handling Robust Winning solutions in INteractive multiobjective optimization) and DARWIN-PCT [62] (DARWIN using Pairwise Comparison Tables). DARWIN and DARWIN-PCT can be considered as two specific instances of DRSA-EMO and DRSA-EMO-PCT, respectively.

13.7.3 DRSA to Decision Under Uncertainty and Time Preference

DRSA can also be applied to preference modeling for decision under uncertainty with consequences distributed over time, using the idea of time-stochastic dominance, i.e. putting together the concept of time dominance and stochastic dominance [59]. Preference information provided by the DM is a set of decision examples specifying the quality of some chosen acts, i.e. assigning these acts to preference-ordered classes. The resulting preference model expressed in terms of "*if. . . , then. . .*" decision rules is much more intelligible than any utility function. Moreover, it permits to handle inconsistent preference information. Let us observe that the approach handles an additive probability distribution as well as a non-additive probability, and even a qualitative ordinal probability. Furthermore, in case the elements of sets of possible probability values and of time epochs were very numerous (in real life applications they are very often infinite), it would be enough to consider a subset of the most significant probability values (e.g., $0, 0.1, 0.2, \ldots, 0.9, 1$) and a subset of the most significant epochs (e.g., each month). Applying DRSA to decision under uncertainty and time preference we get decision rules of the type:

"if the cumulated outcome at t_1 is at least 50 with a probability of .5, and the cumulated outcome at t_2 is at least 300 with a probability of .7, then act a is (at least) good",

or

"if the cumulated outcome at t_1 is at most 100 with a probability of .25, and the cumulated outcome at t_2 is at most 150 with a probability of .8, then act a is (at most) medium".

This method can be extended on the case of pairwise comparisons [61] obtaining rules whose syntax is:

"if the difference between the cumulated outcome of act a and act b is not smaller than 50 at t_1 with a probability of .6, and not smaller than 300 at t_2 with a probability of .4, then act a is at least weakly preferred to act b".

The above methodology can be very useful for dealing with many OR problems where uncertainty of outcomes and their distribution over the time play a fundamental role, such as portfolio selection, scheduling with time-resource interactions and inventory management. Indeed, putting together the decision rules produced by this methodology with IMO-DRSA and DRSA applied to EMO, provides an important tool for dealing with even more OR problems. An example of a recent application of this methodology to typical OR problems, which are inventory control and portfolio selection, can be found in [64, 65], respectively.

13.8 Conclusions

The content of this chapter shows that the "decision rule approach to MCDA" has been possible due to the multi-layered development of the Dominance-based Rough Set Approach.

Some remarks relative to comparison of DRSA with other MCDA methodologies will be useful to fully appreciate the decision rule approach:

(1) for multiple criteria sorting problems, a set of DRSA decision rules is equivalent to a general utility function, simply increasing with respect to each criterion, with a set of thresholds corresponding to frontiers between preference-ordered decision classes [43, 49]; more generally, for multicriteria decision problems, using a utility function is equivalent to adopt a set of DRSA decision rules; these rules have a specific syntax when the utility function assumes specific formulations (for example an associative operator) [55];
(2) for multiple-criteria choice and ranking problems, a set of DRSA decision rules is equivalent to a general conjoint measurement model (see [6]), non-additive and non-transitive, proposed by Bouyssou and Pirlot ([5]; see also [50]);
(3) decision rules obtained by DRSA are more general than Sugeno integral ([97]; see also [25]), considered to be the most general max-min ordinal aggregator; in fact, Sugeno integral is equivalent to a set of single-graded decision rules where evaluations with respect to conditions and conclusion of a rule are of the same degree, for example,

ρ_1 : "*if* Mathematics$\geq$*Medium and* Literature$\geq$*Medium*,
then the comprehensive evaluation is at least *Medium*",

is a single-grade decision rule, while a rule following from DRSA,

ρ_2 : "*if* Physics$\geq$*Good and* Literature$\geq$*Medium*,
then the comprehensive evaluation is at least *Good*"

is not a single-graded decision rule; this means that if in the set of decision rules there is at least one rule which is not single graded, then the DM's preferences cannot be represented by the Sugeno integral; for example, if DM's preferences are represented by a set of decision rules containing rule ρ_2, then these preferences cannot be represented by the Sugeno integral [43, 55];

(4) preferences modelled by outranking methods from ELECTRE family can be represented by a set of specific DRSA decision rules based on PCT [50, 53].

The main features of DRSA can be summarized as follows:

- preference information necessary to deal with a multiple-criteria decision problem is asked to the DM in terms of exemplary decisions;
- rough set analysis of preference information supplies some useful elements of knowledge about the decision situation; these are: the relevance of attributes and/or criteria, information about their interaction (from quality of approximation and its analysis using fuzzy measures theory), minimal subsets of attributes or criteria (reducts) conveying the relevant knowledge contained in the exemplary decisions, the set of the non-reducible attributes or criteria (core);
- preference model induced from the preference information is expressed in a natural and comprehensible language of "*if*..., *then*..." decision rules, fulfilling the postulate of transparency and interpretability of preference models in decision support; each decision rule can be clearly identified with those parts of the preference information (decision examples) which support the rule; the rules inform the DM about the relationships between conditions and decisions; in this way, the rules permit *traceability* of the decision support process and give *understandable justifications* for the decision to be made, so that the resulting preference model constituted for the DM a glass box;
- heterogeneous information (qualitative and quantitative, preference-ordered or not, crisp and fuzzy evaluations, and ordinal and cardinal scales of preferences, with a hierarchical structure and with missing values) can be processed within DRSA, while classical MCDA methods consider only quantitative ordered evaluations with rare exceptions,
- decision rule preference model resulting from the rough set approach is more general than all existing models of conjoint measurement due to its capacity of handling inconsistent preferences;
- apart from their clear meaning, the decision rules are characterized by some interestingness measures, among which Bayesian confirmation measures seem to be the most appropriate, as shown in the studies [56, 63]. These interestingness measures can be also used to generalize the concept of rough approximations [58];

- proposed methodology is based on elementary concepts and mathematical tools (sets and set operations, binary relations), without recourse to any algebraic or analytical structures; the main idea is very natural and the key concept of dominance relation is even objective.

There is no doubt that the use of the decision rule model and the capacity of handling inconsistent preference information with DRSA opened a fascinating research field to MCDA and moved it towards artificial intelligence, knowledge discovery, data analytics and preference learning.

Acknowledgements The research of the third author has been supported by the Polish National Science Centre, grant no. DEC-2011/01/B/ST6/07318.

References

1. Blaszczynski, J., Greco, S., Słowiński, R.: Multi-criteria classification—a new scheme for application of dominance-based decision rules. Eur. J. Oper. Res. **181**, 1030–1044 (2004)
2. Blaszczynski, J., Greco, S., Słowiński, R., Szelag, M.: Monotonic variable consistency rough set approaches. Int. J. Approx. Reason. **50**, 979–999 (2009)
3. Blaszczynski, J., Słowiński, R., Szelag, M.: Sequential covering rule induction algorithm for variable consistency rough set approaches. Inf. Sci. **181**, 987–1002 (2011)
4. Blaszczynski, J., Greco, S., Matarazzo, B., Słowiński, R., Szelag, M.: jMAF—dominance-based rough set data analysis framework. In: Suraj, Z., Skowron, A. (eds.) Rough Sets and Intelligent Systems—Professor Zdzisław Pawlak in Memoriam, pp. 185–209. Springer, Berlin (2013)
5. Bouyssou, D., Pirlot, M.: Nontransitive decomposable conjoint measurement. J. Math. Psychol. **46**, 677–703 (2002)
6. Bouyssou, D., Pirlot, M.: Conjoint measurement tools for MCDM: a brief introduction In: Figueira, J., Greco, S., Ehrgott, M. (eds.) Multiple Criteria Decision Analysis. State of the Art Surveys, pp. 97–151. Springer, New York (2015)
7. Branke, J., Kausler, T., Schmeck, H.: Guidance in evolutionary multi-objective optimization. Adv. Eng. Softw. **32**, 499–507 (2001)
8. Branke, J., Deb, K., Miettinen, K., Słowiński, R. (eds.): Multiobjective Optimization: Interactive and Evolutionary Approaches. Springer, Berlin (2008)
9. Branke, J., Greco, S., Słowiński, R., Zielniewicz, P.: Interactive evolutionary multiobjective optimization using robust ordinal regression. In: Ehrgott, M., Fonseca, C.M., Gandibleux, X., Hao, J.K., Sevaux, M. (eds.) Evolutionary Multi-Criterion Optimization (EMO'09), pp. 554–568. Springer, Berlin (2009)
10. Choquet, G.: Theory of capacities. Ann. I. Fourier **5**, 131–295 (1953)
11. Coello, C.A.C., Van Veldhuizen, D.A., Lamont, G.B.: Evolutionary Algorithms for Solving Multi-Objective Problems. Kluwer, New York (2002)
12. Deb, K.: Multi-Objective Optimization Using Evolutionary Algorithms. Wiley, Chichester (2001)
13. Deb, K., Chauduri, S.: I-mode: an interactive multi-objective optimization and decision-making using evolutionary methods. Appl. Soft. Comput. **10**, 496–511 (2010)
14. Deb, K., Sundar, J., Rao, N., Chauduri, S.: Reference point based multi-objective optimization using evolutionary algorithms. Int. J. Comput. Intell. Res. **2**, 273–286 (2006)
15. Dembczynski, K., Greco, S., Słowiński, R.: Methodology of rough-set-based classification and sorting with hierarchical structure of attributes and criteria. Control. Cybern. **31**, 891–920 (2002)

16. Dembczynski, K., Greco, S., Słowiński, R.: Methodology of rough-set-based classification and sorting with hierarchical structure of attributes and criteria. Eur. J. Oper. Res. **198**, 626–636 (2009)
17. Dyer, J.: MAUT-multiattribute utility theory. In: Figueira, J., Greco, S., Ehrgott, M. (eds.) Multiple Criteria Decision Analysis. State of the Art Surveys, Chap. 8. Springer, New York (2015). doi:10.1007/978-1-4939-3094-4
18. Figueira, J., Mousseau, V., Roy, B.: ELECTRE methods. In: Figueira, J., Greco, S., Ehrgott, M. (eds.) Multiple Criteria Decision Analysis. State of the Art Surveys, Chap. 5. Springer, New York (2015). doi:10.1007/978-1-4939-3094-4
19. Fishburn, P.: Methods for estimating additive utilities. Manag. Sci. **13**, 435–453 (1967)
20. Fodor, J., Roubens, M.: Fuzzy Preference Modelling and Multicriteria Decision Support. Kluwer, Dordrecht (1994)
21. Fonseca, C.M., Fleming, P.J.: Genetic algorithms for multiobjective optimization: formulation, discussion, and generalization. In: Proceedings of the Fifth International Conference on Genetic Algorithms, pp. 416–423 (1993)
22. Fortemps, P., Greco, S., Słowiński, R.: Multicriteria decision support using rules that represent rough-graded preference relations. Eur. J. Oper. Res. **188**, 206–223 (2008)
23. Giove, S., Greco, S., Matarazzo, B., Słowiński, R.: Variable consistency monotonic decision trees. In: Alpigini, J., Peters, J., Skowron, A., Zhong, Z. (eds.) Rough Sets and Current Trends in Computing, pp. 247–254. Springer, Berlin (2002)
24. Grabisch, M.: k-order additive discrete fuzzy measures and their representation. Fuzzy Sets Syst. **92**, 167–189 (1997)
25. Grabisch, M., Labreuche, C.: Fuzzy measures and integrals in MCDA. In: Figueira, J., Greco, S., Ehrgott, M. (eds.) Multiple Criteria Decision Analysis. State of the Art Surveys, Chap. 14. Springer, New York (2015). doi:10.1007/978-1-4939-3094-4
26. Greco, S., Matarazzo, B., Słowiński, R.: Fuzzy measures technique for rough set analysis. In: Proceeding of the 6th European Congress on Intelligent Techniques & Soft Computing (1998)
27. Greco, S., Matarazzo, B., Słowiński, R.: Fuzzy similarity relation as a basis for rough approximations. In: Polkowski, L., Skowron, A. (eds.) Rough Sets and Current Trends in Computing, pp. 283–289. Springer, Berlin (1998)
28. Greco, S., Matarazzo, B., Słowiński, R.: A new rough set approach to evaluation of bankruptcy risk. In: Zopounidis, C. (ed.) Rough Fuzzy and Fuzzy Rough Sets, pp. 121–136. Kluwer, Dordrecht (1998)
29. Greco, S., Matarazzo, B., Słowiński, R.: A new rough set approach to multicriteria and multiattribute classification. In: Polkowski, L., Skowron, A. (eds.) Rough Sets and Current Trends in Computing, pp. 60–67. Springer, Berlin (1998)
30. Greco, S., Matarazzo, B., Słowiński, R., Tsoukias, A.: Exploitation of a rough approximation of the outranking relation in multicriteria choice and ranking. In: Stewart, T.J., van den Honert, R. (eds.) Trends in Multicriteria Decision Making, pp. 45–60. Springer, Berlin (1998)
31. Greco, S., Matarazzo, B., Słowiński, R.: Handling missing values in rough set analysis of multi-attribute and multi-criteria decision problems. In: Zhong, N., Skowron, A., Ohsuga, S. (eds.) New Directions in Rough Sets, Data Mining and Granular-Soft Computing, pp. 146–157. Springer, Berlin (1999)
32. Greco, S., Matarazzo, B., Słowiński, R.: Rough approximation of a preference relation by dominance relations. Eur. J. Oper. Res. **117**, 63–83 (1999)
33. Greco, S., Matarazzo, B., Słowiński, R.: Rough sets methodology for sorting problems in presence of multiple attributes and criteria. Eur. J. Oper. Res. **138**, 247–259 (1999)
34. Greco, S., Matarazzo, B., Słowiński, R.: Rough sets theory for multicriteria decision analysis. Eur. J. Oper. Res. **129**, 1–47 (1999)
35. Greco, S., Matarazzo, B., Słowiński, R.: The use of rough sets and fuzzy sets in MCDM. In: Gal, T., Stewart, T., Hanne, T. (eds.) Advances in Multiple Criteria Decision Making, pp. 14.1–14.59. Kluwer, Dordrecht (1999)

36. Greco, S., Matarazzo, B., Słowiński, R.: Dealing with missing data in rough set analysis of multi-attribute and multi-criteria decision problems. In: Zanakis, S., Doukidis, G., Zopounidis, C. (eds.) Decision Making: Recent Developments and Worldwide Applications, pp. 295–316. Kluwer, Dordrecht (2000)
37. Greco, S., Matarazzo, B., Słowiński, R.: Decision rules. In: Encyclopedia of Management, 4th edn., pp. 178–183. Gale Group, Farmington Hills, MI (2000)
38. Greco, S., Matarazzo, B., Słowiński, R.: Extension of the rough set approach to multicriteria decision support. INFOR **38**, 161–196 (2000)
39. Greco, S., Matarazzo, B., Słowiński, R.: Fuzzy dominance-based rough set approach. In: Masulli, F., Parenti, R., Pasi, G. (eds.) Advances in Fuzzy Systems and Intelligent Technologies, pp. 56–66. Shaker Publishing, Maastricht (2000)
40. Greco, S., Matarazzo, B., Słowiński, R.: Fuzzy extension of the rough set approach to multicriteria and multiattribute sorting. In: Fodor, J., Baets, B.D., Perny, P. (eds.) Preferences and Decisions under Incomplete Knowledge, pp. 131–151. Physica-Verlag, Heidelberg (2000)
41. Greco, S., Matarazzo, B., Słowiński, R.: Rough set processing of vague information using fuzzy similarity relations. In: Calude, C., Paun, G. (eds.) Finite Versus Infinite—Contributions to an Eternal Dilemma, pp. 149–173. Springer, London (2000)
42. Greco, S., Matarazzo, B., Słowiński, R.: Assessment of a value of information using rough sets and fuzzy measures. In: Chojcan, J., Leski, J. (eds.) Fuzzy Sets and their Applications, pp. 185–193. Silesian University of Technology Press, Gliwice (2001)
43. Greco, S., Matarazzo, B., Słowiński, R.: Conjoint measurement and rough set approach for multicriteria sorting problems in presence of ordinal criteria. In: Colorni, A., Paruccini, M., Roy, B. (eds.) A-MCD-A: Aide Multi Critère à la Décision—Multiple Criteria Decision Aiding, Ispra, pp. 117–144. European Commission Report, EUR 19808 EN (2001)
44. Greco, S., Matarazzo, B., Słowiński, R.: Rough set approach to decisions under risk. In: Ziarko, W., Yao, Y. (eds.) Rough Sets and Current Trends in Computing, pp. 160–169. Springer, Berlin (2001)
45. Greco, S., Matarazzo, Słowiński, R., Stefanowski, J.: An algorithm for induction of decision rules consistent with dominance principle. In: Ziarko, W., Yao, Y. (eds.) Rough Sets and Current Trends in Computing, pp. 304–313. Springer, Berlin (2001)
46. Greco, S., Matarazzo, Słowiński, R., Stefanowski, J.: Variable consistency model of dominance-based rough set approach. In: Ziarko, W., Yao, Y. (eds.) Rough Sets and Current Trends in Computing, pp. 170–181. Springer, Berlin (2001)
47. Greco, S., Matarazzo, Słowiński, R., Stefanowski, J.: Importance and interaction of conditions in decision rules. In: Alpigini, J., Peters, J.F., Skowron, A., Zhong, N. (eds.) Rough Sets and Current Trends in Computing, pp. 255–262. Springer, Berlin (2002)
48. Greco, S., Matarazzo, Słowiński, R., Stefanowski, J.: Mining association rules in preference-ordered data. In: Hacid, M.S., Ras, Z., Zighed, D., Kodratoff, Y. (eds.) Foundations of Intelligent Systems, pp. 442–450. Springer, Berlin (2002)
49. Greco, S., Matarazzo, B., Słowiński, R.: Multicriteria classification, Chap. 16.1.9. In: Kloesgen, W., Zytkow, J. (eds.) Handbook of Data Mining and Knowledge Discovery, pp. 318–328. Oxford University Press, New York (2002)
50. Greco, S., Matarazzo, B., Słowiński, R.: Preference representation by means of conjoint measurement and decision rule model. In: Bouyssou, D., Jacquet-Lagrèze, E., Perny, P., Słowiński, R., Van-derpooten, D., Vincke, P. (eds.) Aiding Decisions with Multiple Criteria—Essays in Honor of Bernard Roy, pp. 263–313. Kluwer, Boston (2002)
51. Greco, S., Matarazzo, B., Słowiński, R.: Rough approximation by dominance relations. Int. J. Intell. Syst. **17**, 153–171 (2002)
52. Greco, S., Inuiguchi, M., Słowiński, R.: Dominance-based rough set approach using possibility and necessity measures. In: Alpigini, J., Peters, J., Skowron, A., Zhong, N. (eds.) Rough Sets and Current Trends in Computing, pp. 85–92. Springer, Berlin (2002)
53. Greco, S., Predki, B., Słowiński, R.: Searching for an equivalence between decision rules and concordance-discordance preference model in multicriteria choice problems. Control Cybern. **31**, 921–935 (2002)

54. Greco, S., Inuiguchi, M., Słowiński, R.: A new proposal for fuzzy rough approximations and gradual decision rule representation. In: Dubois, D., Grzymala-Busse, J., Inuiguchi, M., Polkowski, L. (eds.) Rough Fuzzy and Fuzzy Rough Sets. Springer, Berlin (2004)
55. Greco, S., Matarazzo, B., Słowiński, R.: Axiomatic characterization of a general utility function and its particular cases in terms of conjoint measurement and rough-set decision rules. Eur. J. Oper. Res. **158**, 271–292 (2004)
56. Greco, S., Pawlak, Z., Słowiński, R.: Can Bayesian confirmation measures be useful for rough set decision rules? Eng. Appl. Artif. Intell. **17**, 345–361 (2004)
57. Greco, S., Matarazzo, B., Słowiński, R.: Dominance-based rough set approach to interactive multiobjective optimization. In: Branke, J., Deb, K., Miettinen, K., Słowiński, R. (eds.) Multiobjective Optimization: Interactive and Evolutionary Approaches, pp. 121–155. Springer, Berlin (2008)
58. Greco, S., Matarazzo, B., Słowiński, R.: Parameterized rough set model using rough membership and bayesian confirmation measures. Int. J. Approx. Reason. **49**, 285–300 (2008)
59. Greco, S., Matarazzo, B., Słowiński, R.: Dominance-based rough set approach to decision under uncertainty and time preference. Ann. Oper. Res. **176**, 41–75 (2010)
60. Greco, S., Matarazzo, B., Słowiński, R.: Dominance-based rough set approach to interactive evolutionary multiobjective optimization. In: Greco, S., Pereira, R.A.M., Squillante, M., Kacprzyk, J., Yager, R.R. (eds.) Preferences and Decisions: Models and Applications, pp. 225–260. Springer, Berlin (2010)
61. Greco, S., Matarazzo, B., Słowiński, R.: Dominance-based rough set approach to preference learning from pairwise comparisons in case of decision under uncertainty. In: Hüllermeier, E., Kruse, R., Hoffmann, F. (eds.) Computational Intelligence for Knowledge-Based Systems Design. 13th International Conference on Information Processing and Management of Uncertainty. Institute for the Physics and Mathematics of the Universe 2010. Lecture Notes in Computer Science, vol. 6178, pp. 584–594. Springer, Berlin (2010)
62. Greco, S., Matarazzo, B., Słowiński, R.: Interactive evolutionary multiobjective optimization using dominance-based rough set approach. In: Proceedings of WCCI 2010, IEEE World Congress on Computational Intelligence, July 18–23, Barcelona, Spain, pp. 3026–3033 (2010)
63. Greco, S., Słowiński, R., Szczech, I.: Properties of rule interestingness measures and alternative approaches to normalization of measures. Inf. Sci. **216**, 1–16 (2012)
64. Greco, S., Matarazzo, B., Słowiński, R.: Beyond Markowitz with multiple criteria decision aiding. J. Bus. Econ. **83**, 29–60 (2013)
65. Greco, S., Matarazzo, B., Słowiński, R., Vaccarella, G.: Inventory control using interactive multiobjective optimization guided by dominance-based rough set approach (submitted, 2015)
66. Greenwood, G.W., Hu, X., D'Ambrosio, J.G.: Fitness functions for multiple objective optimization problems: combining preferences with pareto rankings. In: Belew, R., Vose, M. (eds.) Foundations of Genetic Algorithms, pp. 437–455. Morgan Kaufmann, San Francisco, CA (1997)
67. Jacquet-Lagrèze, E.: Systèmes de décision et acteurs multiples—contribution à une théorie de l'action pour les sciences des organisations. Thèse de doctorat d'état, Université de Paris-Dauphine, Paris (1981)
68. Jaszkiewicz, A.: Interactive multiobjective optimization with the Pareto memetic algorithm. Found. Comput. Decis. Sci. **32**, 15–32 (2007)
69. Keeney, R.L., Raiffa, H.: Decision with Multiple Objectives—Preferences and Value Trade-offs. Wiley, New York (1976)
70. Kryszkiewicz, M.: Rough set approach to incomplete information systems. Inf. Sci. **112**, 39–49 (1998)
71. Kryszkiewicz, M.: Rules in incomplete information systems. Inf. Sci. **113**, 271–292 (1999)
72. March, J.G.: Bounded rationality, ambiguity, and the engineering of choice. In: Bell, D.E., Raiffa, H., Tversky, A. (eds.) Decision Making, Descriptive, Normative and Prescriptive Interactions, pp. 33–58. Cambridge University Press, New York (1988)

73. Mousseau, V.: Problèmes liés à l'évaluation de l'importance en aide multicritère à la décision: Réflexions théoriques et expérimentations. Thèse de doctorat d'état, Université de Paris-Dauphine, Paris (1993)
74. Murofushi, T., Soneda, S.: Techniques for reading fuzzy measures (iii): interaction index. In: Proceedings of 9th Fuzzy Systems Symposium, Sapporo, May 1993, pp. 693–696 (2002)
75. Ozturk, M., Tsoukias, A., Vincke, P.: Preference modelling. In: Figueira, J., Greco, S., Ehrgott, M. (eds.) Multiple Criteria Decision Analysis. State of the Art Surveys, Chap. 3. Springer, New York (2015). doi:10.1007/978-1-4939-3094-4
76. Pawlak, Z.: Rough Sets. Int. J. Inf. Comput. Sci. **11**, 341–356 (1982)
77. Pawlak, Z.: Rough Sets. Theoretical Aspects of Reasoning about Data. Kluwer, Dordrecht (1991)
78. Pawlak, Z., Słowiński, R.: Rough set approach to multi-attribute decision analysis. Eur. J. Oper. Res. **72**, 443–459 (1994)
79. Phelps, S., Köksalan, M.: An interactive evolutionary metaheuristic for multiobjective combinatorial optimization. Manag. Sci. **49**, 1726–1738 (2003)
80. Polkowski, L.: Rough Sets: Mathematical Foundations. Physica-Verlag, Heidelberg (2002)
81. Roy, B.: The outranking approach and the foundation of ELECTRE methods. Theor. Decis. **31**, 49–73 (1991)
82. Roy, B.: Decision science or decision aid science? Eur. J. Oper. Res. **66**, 184–203 (1993)
83. Shapley, L.S.: A value for n-person games. In: Kuhn, H., Tucker, A. (eds.) Contributions to the Theory of Games II, pp. 307–317. Princeton University Press, Princeton (1953)
84. Slovic, P.: Choice between equally-valued alternatives. J. Exp. Psychol. Hum. Percept. Perform. **1**, 280–287 (1975)
85. Słowiński, R.: Intelligent Decision Support. Handbook of Applications and Advances of the Rough Sets Theory. Kluwer, Dordrecht (1992)
86. Słowiński, R.: Rough set learning of preferential attitude in multi-criteria decision making. In: Komorowski, J., Ras, Z. (eds.) Methodologies for Intelligent Systems, pp. 642–651. Springer, Berlin (1993)
87. Słowiński, R., Vanderpooten, D.: Similarity relation as a basis for rough approximations. In: Wang, P. (ed.) Advances in Machine Intelligence & Soft-Computing, vol. IV, pp. 17–33. Duke University Press, Durham, NC (1997)
88. Słowiński, R., Vanderpooten, D.: A generalized definition of rough approximations based on similarity. IEEE Trans. Data Knowl. Eng. **12**, 331–336 (2000)
89. Słowiński, R., Greco, S., Matarazzo, B.: Mining decision-rule preference model from rough approximation of preference relation. In: Proceedings of the 26th IEEE Annual International Conference on Computer Software & Applications (COMPSAC 2002), pp. 1129–1134. Oxford (2002)
90. Słowiński, R., Greco, S., Matarazzo, B.: Rough set analysis of preference-ordered data. In: Alpigini, J., Peters, J., Skowron, A., Zhong, N. (eds.) Rough Sets and Current Trends in Computing, pp. 44–59. Springer, Berlin (2002)
91. Słowiński, R., Greco, S., Matarazzo, B.: Rough set based decision support. In: Burke, E., Kendall, G. (eds.) Search Methodologies: Introductory Tutorials in Optimization and Decision Support Techniques, 2nd ed. pp. 557–609. Springer, New York (2014)
92. Słowiński, R., Greco, S., Matarazzo, B.: Rough sets in decision making. In: Meyers, R.A. (ed.) Encyclopedia of Complexity and Systems Science, pp. 7753–7786. Springer, New York (2009)
93. Słowiński, R., Greco, S., Matarazzo, B.: Rough set and rule-based multicriteria decision aiding. Pesquisa Operaciona **32**, 213–269 (2012)
94. Słowiński, R., Greco, S., Matarazzo, B.: Rough Set Methodology for Decision Aiding. Chapter 22 [in]: Kacprzyk, J. and Pedrycz, W. (eds.), Handbook of Computational Intelligence, pp. 349–370, Springer, Berlin (2015)
95. Stefanowski, J.: On rough set based approaches to induction of decision rules. In: Polkowski, L., Skowron, A. (eds.) Rough Sets in Data Mining and Knowledge Discovery, vol. 1, pp. 500–529. Physica-Verlag, Heidelberg (1998)

96. Stewart, T.: Dealing with uncertainties in MCDA. In: Figueira, J., Greco, S., Ehrgott, M. (eds.) Multiple Criteria Decision Analysis. State of the Art Surveys, Chap. 12. Springer, New York (2015). doi:10.1007/978-1-4939-3094-4
97. Sugeno, M.: Theory of fuzzy integrals and its applications. Doctoral thesis, Tokyo Institute of Technology, Paris, France (1974)
98. Susmaga, R., Słowiński, R., Greco, S., Matarazzo, B.: Generation of reducts and rules in multi-attribute and multi-criteria classification. Control Cybern. **29**, 969–988 (2000)
99. Szelag, M., Greco, S., Słowiński, R.: Rule-based approach to multicriteria ranking, Chapter 6. In: Doumpos, M., Grigoroudis, E. (eds.) Multicriteria Decision Aid and Artificial Intelligence: Links, Theory and Applications, pp. 127–160. Wiley-Blackwell, London (2013)
100. Szelag, M., Greco, S., Słowiński, R.: Variable consistency dominance-based rough set approach to preference learning in multicriteria ranking. Information Sciences **277**, 525–552 (2014)
101. Yao, Y., Greco, S., Słowiński, R.: Probabilistic Rough Sets. Chapter 24. In: Kacprzyk, J. and Pedrycz, W. (eds.), Handbook of Computational Intelligence, pp. 387–411. Springer, Berlin (2015)
102. Ziarko, W.: Variable precision rough sets model. J. Comput. Syst. Sci. **1993**, 39–59 (1993)
103. Ziarko, W.: Rough sets as a methodology for data mining. In: Polkowski, L., Skowron, A. (eds.) Rough Sets in Knowledge Discovery. Vol. 1, pp. 554–576. Physica-Verlag, Heidelberg (1998)

Chapter 14
Fuzzy Measures and Integrals in MCDA

Michel Grabisch and Christophe Labreuche

Abstract This chapter aims at a unified presentation of various methods of MCDA based on fuzzy measures (capacity) and fuzzy integrals, essentially the Choquet and Sugeno integral. A first section sets the position of the problem of multicriteria decision making, and describes the various possible scales of measurement (cardinal unipolar and bipolar, and ordinal). Then a whole section is devoted to each case in detail: after introducing necessary concepts, the methodology is described, and the problem of the practical identification of fuzzy measures is given. The important concept of interaction between criteria, central in this chapter, is explained in detail. It is shown how it leads to k-additive fuzzy measures. The case of bipolar scales leads to the general model based on bi-capacities, encompassing usual models based on capacities. A general definition of interaction for bipolar scales is introduced. The case of ordinal scales leads to the use of Sugeno integral, and its symmetrized version when one considers symmetric ordinal scales. A practical methodology for the identification of fuzzy measures in this context is given.

Keywords Choquet integral • Fuzzy measure • Interaction • Bi-capacities

14.1 Introduction

MultiCriteria Decision Aid (MCDA) aims at modeling the preferences of a Decision Maker (DM) over alternatives described by several points of view, which are denoted by $X_1, \ldots, X_n$. An alternative is characterized by a value w.r.t. each point of view and is thus identified with a point in the Cartesian product X of the points of view: $X = X_1 \times \cdots \times X_n$. We denote by $N := \{1, \ldots, n\}$ the index set of points

M. Grabisch (✉)
Paris School of Economics, Université Paris I - Panthéon-Sorbonne, 106-112, Bd de l'Hôpital, 75013 Paris, France
e-mail: michel.grabisch@univ-paris1.fr

C. Labreuche
Thales Research & Technology, 1 avenue Augustin Fresnel, 91767 Palaiseau Cedex, France
e-mail: christophe.labreuche@thalesgroup.com

S. Greco et al. (eds.), *Multiple Criteria Decision Analysis*, International Series in Operations Research & Management Science 233,
DOI 10.1007/978-1-4939-3094-4_14

of view. The preference relation of the DM over alternatives is denoted by $\succeq$. For $x, y \in X$, "$x \succeq y$" means that the DM prefers alternative x to y. The symmetric and asymmetric parts of $\succeq$ are denoted by $\succ$ and $\sim$ respectively.

The main concern in practice is to come up with the knowledge of $\succeq$ on $X \times X$ from a relatively small amount of questions asked to the DM on $\succeq$. The information provided by the DM can be composed of examples of comparisons between alternatives, which gives $\succeq$ on a subset of $X \times X$, as well as more qualitative judgments, whose modelling is more complex, and depends on the kind of representation of $\succeq$ we choose. In general, we look for a *numerical representation* [68] $u : X \to \mathbb{R}$ such that:

$$\forall x, y \in X \;\;,\;\; x \succeq y \;\Leftrightarrow\; u(x) \geq u(y). \tag{14.1}$$

It is classical to write u in the following way [63]:

$$u(x) = F\,(u_1(x_1), \ldots, u_n(x_n)) \quad \forall x \in X, \tag{14.2}$$

where the u_i's : $X_i \to \mathbb{R}$ are called the *utility functions* (also called *value functions*) and $F : \mathbb{R}^n \to \mathbb{R}$ is an *aggregation function*. A result by Krantz et al. gives the axioms that characterize the representation of $\succeq$ by (14.2) [68]. As it will be detailed in Sect. 14.2.1, the *weak separability* axiom is the key axiom that justifies the construction of utility functions, that is partial preference relations over the points of view, from the overall preference relation $\succeq$. A criterion is defined as a preference relation $\succeq_i$ over one point of view X_i. Thus a criterion is the association of one point of view X_i with its related utility function u_i.

In practice, we restrict ourself to a family $\mathcal{F}$ of aggregation functions (parameterized by some coefficients). The justification of the use of a special family is based on an axiomatic approach. The axioms that characterize the family should be in accordance with the problem in consideration and the behaviour of the decision maker. The DM has then to provide the needed information to set the parameters of the model. The more restrictive the family is, the less representative it is, but the less information the DM shall give.

The most classical functions used to aggregate the criteria are the weighted sums $F(u_1, \ldots, u_n) = \sum_{i=1}^{n} \alpha_i\, u_i$. As an aggregation operator, they are characterized by an independence axiom [63, 122]. This property implies some limitations in the way the weighted sum can model typical decision behaviours. To make this more precise, let us consider the example of two criteria having the same importance, an example which we will consider in more details in Sect. 14.3.5. We are interested in the following four alternatives: x is bad in both criteria, y is bad in the first criterion but good at the second one, z is good in the first criterion but bad in the second one, and t is good in both. Clearly $x \prec t$ and the DM is equally satisfied by y and z since the two criteria have the same importance. However, the comparison of y, z with x and t leads to several cases. First, the DM may say that $x \sim y \sim z \prec t$, where $\sim$ means indifference. This depicts a DM who is *intolerant*, since both criteria have to be satisfied in order to get a satisfactory alternative. In the opposite way, the DM may think that $x \prec y \sim z \sim t$, which depicts a *tolerant* DM, since only one criterion has to be satisfactory in order to get a satisfactory alternative. Finally, we may have all intermediate cases, where $x \prec y \sim z \prec t$. An important fact is that, due to additivity, the weighted sum is unable to distinguish among all these cases, in particular, all decision behaviours related to tolerance or intolerance are missed. These phenomena are called *interaction* between criteria. They encompass also other phenomena such as *veto*. We will show in this chapter that the notions of capacity and fuzzy integrals enable to model previous phenomena.

The construction of the utility functions and the determination of the parameters of the aggregation function are often carried out in two separate steps. The utility functions are generally set up first, that is without the knowledge of the precise aggregation function F within $\mathcal{F}$. However, the utility functions have no intrinsic meaning to the DM and shall be determined from questions regarding only the overall preference relation $\succeq$. It is not assumed that the DM can isolate attributes and give information directly on u_i. This point is generally not considered in the literature. The main reason is probably that due to the use of a weighted sum as an aggregation function, the independence assumption (preferential or cardinal independence) makes it possible in some sense to separate each attribute and thus construct the utility functions directly. This becomes far more complicated when this assumption is removed. Besides, these approaches are not relevant from a theoretical standpoint. To our knowledge, the only approach that addresses this problem with the use of a weighted sum is the so-called MACBETH approach designed by Bana e Costa and Vansnick [2–4]. A generalization of this approach to more complex aggregation operators has been proposed by Grabisch et al. [54] and Mayag et al. [93]. These approaches are considered in this chapter.

The determination of the utility function is not concerned only with measurement considerations. The main difficulty is to ensure commensurability between criteria. Commensurateness means that one shall be able to compare any element of one point of view with any element of any other point of view. This is inter-criteria comparability:

> For $x_i \in X_i$ and $x_j \in X_j$, we have $u_i(x_i) \geq u_j(x_j)$ iff x_i is considered at least as good as x_j by the DM.

Commensurateness implies the existence of a preference relation over $\bigcup_{i=1}^{n} X_i$. This assumption, considered by Modave and Grabisch [97], is very strong. Taking a simple example involving two criteria (for instance consumption and maximal speed), this amounts to know whether the DM prefers a consumption of 5 L/100 km to a maximum speed of 200 km/h. This does not generally make sense to the DM, so that he or she is not generally able to make this comparison directly.

In Sects. 14.3 and 14.4 we push the previous method one step further by considering on top of intra-criteria information some natural inter-criteria information to determine the aggregation functions as well. We will show that the requirements induced by measurement considerations naturally imply the use of fuzzy integrals as aggregation operators. In Sect. 14.5, we deal with the case of ordinal information. It will be seen that this induces difficulties, so that the previous construction no more applies.

14.2 Measurement Theoretic Foundations

As explained in the introduction, we focus on a model called *decomposable* given by Eq. (14.2), involving an aggregation function $F : \mathbb{R}^n \longrightarrow \mathbb{R}$, and utility functions $u_i : X_i \longrightarrow \mathbb{R}$, $i = 1, \dots, n$.

In this section we will give some considerations coming from measurement theory as well as more practical considerations coming from the MACBETH approach around this kind of model. This will help us in giving a firm theoretical basis to our construction.

14.2.1 Basic Notions of Measurement, Scales

This section is based on [68, 107], to which the reader is referred for more details.

The fundamental aim of measurement theory is to build homomorphisms f between a relational structure $\mathcal{A}$ coming from observation, and a relational structure $\mathcal{B}$ based on real numbers (or more generally, some totally ordered set). Doing so, we get a numerical *representation* of our observation. A *scale* (of measurement) is the triplet $(\mathcal{A}, \mathcal{B}, f)$. If no ambiguity occurs, f alone denotes the scale.

A simple example is when $\mathcal{A} = (A, \succeq)$, where $\succeq$ is a binary relation expressing e.g. the preference of the DM on some set A, and $\mathcal{B}$ is simply $(\mathbb{R}, \geq)$. As usual, $\sim$ and $\succ$ denote respectively the symmetric and asymmetric parts of $\succeq$, and $A/\sim$ is the set of equivalence classes of $\sim$ (when defined). This measurement problem is called *ordinal measurement*. The homomorphism satisfies the following condition

(Ord[A**])** $a \succeq b$ iff $f(a) \geq f(b), \quad \forall a, b \in A.$

Obviously, f is not unique since any strictly increasing transform $\phi \circ f$ of f is also a homomorphism. Generally speaking, the set of functions $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ such that $\phi \circ f$ remains a homomorphism is called the *set of admissible transformations*.

Types of scale are defined by their set of admissible transformations. The most common ones are:

- *ordinal scales*, where the set of admissible transformations are all strictly increasing functions. Examples: scale of hardness, of earthquakes intensity.
- *interval scales*, where all $\phi(t) = \alpha t + \beta$, $\alpha > 0$ are admissible (positive affine transformations). Example: temperature in Celsius.
- *ratio scales*, where the admissible transformations are of the form $\phi(t) = \alpha t$, $\alpha > 0$. Examples: temperature in Kelvin, mass.

Thus, our condition **(Ord[**A**])** defines an ordinal scale. The conditions under which such a f exists are well known. A necessary condition is that $\succeq$ is a weak order (reflexive, complete, transitive). A second condition (and then both are necessary and sufficient) is that $A/\sim$ contains a countable order-dense subset (this is known as the Birkhoff-Milgram theorem, we do not enter further into details).

An ordinal scale is rather poor, and does not really permit to handle numbers, since usual arithmetic operations are not invariant under admissible transformations. It would be better to build an interval scale in the above sense. This is related to the *difference measurement* problem: in this case, $\mathcal{A} = (A, \succeq^*)$, where $\succeq^*$ is a quaternary relation. The meaning of $ab \succeq^* st$ is the following: the difference of intensity (e.g. of preference) between a and b is larger than the difference of intensity between s and t. Then, the homomorphism f should satisfy:

$$ab \succeq^* st \Leftrightarrow f(a) - f(b) \geq f(s) - f(t). \tag{14.3}$$

It is shown that under several conditions on $\mathcal{A}$, such a function f exists, and that it defines an interval scale. Thus the ratio $\frac{f(a)-f(b)}{f(s)-f(t)}$ is meaningful (invariant under any admissible transformation).

Based on this remark, we express the interval scale condition under a form which is suitable for our purpose.

(Inter[A]). $\forall a, b, s, t \in A$ such that $a \succ b$ and $s \succ t$, we have

$$\frac{f(a) - f(b)}{f(s) - f(t)} =: k(a, b, s, t)\ , \quad k(a, b, s, t) \in \mathbb{R}_+$$

if and only if the difference of satisfaction degree that the DM feels between a and b is $k(a, b, s, t)$ times as large as the difference of satisfaction between s and t.

The conditions of existence of f amounts to verify the following condition.

(C-Inter[A]). $\forall a, b, s, t, u, v \in A$ such that $a \succ b$, $s \succ t$ and $u \succ v$,

$$k(a, b, s, t) \times k(s, t, u, v) = k(a, b, u, v).$$

We end this section by addressing the case where A is a product space, as for $X = X_1 \times \cdots \times X_n$. Conditions for an ordinal representation by $u : X \longrightarrow \mathbb{R}$ are given by the Birkhoff-Milgram theorem. However, we are interested in a decomposable form of u (see (14.2)). If F is one-to-one in each place, then necessarily $\sim$ satisfies *substitutability*:

$$(x_i, z_{-i}) \sim (y_i, z_{-i}) \Leftrightarrow (x_i, z'_{-i}) \sim (y_i, z'_{-i}), \quad \forall x, y, z, z' \in X. \tag{14.4}$$

Notation $z = (x_A, y_{-A})$ means that z is defined by $z_i = x_i$ if $i \in A$, else $z_i = y_i$ (hence, $-A$ stands for $N \setminus A$). This property implies the existence of equivalence relations $\sim_i$ on each X_i. If F is strictly increasing, then $\sim$ has to be replaced by $\succeq$ in (14.4) (this is called *weak separability*), and relations $\succeq_i$ are obtained on each X_i.

Reciprocally, substitutability (or weak separability) and the conditions of the Birkhoff-Milgram theorem lead to an *ordinal* representation: hence, u is unique up to a strictly increasing function.

This result remains of theoretical interest, since not verifiable in practice, and moreover, it does not lead to an interval scale. The MACBETH methodology will serve as a basis for such a construction, whose essence is briefly addressed below. Before that, some words on unipolar and bipolar scales are in order.

14.2.2 Bipolar and Unipolar Scales

Let us view scales under a different point of view. Let $(A, \succeq)$ be a relational system, and f a scale, which is supposed to be numerical, without loss of generality. It may exists in A a particular element or level e, called *neutral level*, such that if $a \succ e$, then a is considered as "good", while if $e \succ a$, then a is considered as "bad" for the DM. We may choose for convenience f such that $f(e) = 0$.

Such a neutral level exists whenever relation $\succeq$ corresponds to two opposite notions of common language. For example, this is the case when $\succeq$ means "more attractive than", "better than", etc., whose pairs of opposite notions are respectively "attractiveness/repulsiveness", and "good/bad". By contrast, relations as "more prioritary than", "more allowed than", "belongs more to category C than" do not clearly exhibit a neutral level.

A scale is said to be *bipolar* if A contains such a neutral level. A *unipolar scale* has no neutral level, but has a least level, i.e. an element or level a_0 in A such that $a \succeq a_0$ for all $a \in A$. We may for convenience choose f so that $f(a_0) = 0$.

A scale has a greatest element if there exists an element or level $a_1 \in A$ such that $a_1 \succeq a$, for all $a \in A$. We say that a unipolar scale is *bounded* if it has a greatest level. A bipolar scale is bounded if it has a least and a greatest level (since there is an inherent symmetry in bipolar scales, the existence of a greatest level implies the existence of a least level).

Taking our previous examples, the relations "more attractive than", "better than", "more prioritary than" may not be bounded, while "more allowed than" and "belongs more to category C than" are clearly bounded, the greatest levels being respectively "fully authorized" and "fully belongs to C".

Typically, f maps on $\mathbb{R}$ (resp. $\mathbb{R}_+$) when the scale is unbounded bipolar (resp. unipolar). In the case of bounded scales, f maps respectively to a closed interval centered on 0, and an interval such as $[0, \beta]$.

It is convenient to denote by $\mathbb{1}$ the neutral level of a bipolar scale, or the least level of a unipolar scale. We denote by $\mathbb{O}$ the greatest level when it exists, and by $-\mathbb{O}$ the least level of a bipolar scale.

When the scale is unbounded, it may be convenient to introduce another particular level, called the *satisfactory level*. We may also use $\mathbb{O}$ to denote the satisfactory level. This level is considered as *good and completely satisfactory* if the DM could obtain it, even if more attractive elements could exist in A (due to unboundness). The existence of such a level has been the main argument of H. Simon in his theory of *satisficing bounded rationality* [114], and a fundamental assumption in the MACBETH methodology, as described in next section. For convenience, we may fix $f(\mathbb{O}) = 1$. If in addition the scale is bipolar, the same considerations lead to a level also denoted $-\mathbb{O}$ (unsatisfactory level).

Finally, let us remark that there is no direct relation between unipolar/bipolar scales and the types of scales given in Sect. 14.2.1 (interval, ratio, etc.). For example, the temperature scales are clearly unipolar with a least level (at least in the physical sense), but may be of the ratio type (in Kelvin) or of the interval type (in Celsius, Fahrenheit). However, the neutral level of a bipolar scale clearly plays the role of the zero in a ratio scale, since it cannot be shifted.

14.2.3 Construction of the Measurement Scales and Absolute References Levels

The MACBETH methodology [2–4], described in Chap. 10, permits to build interval scales from a questionnaire. We limit ourselves here to necessary notions.

We consider A a finite set on which the decision maker is able to express some preference (the finiteness assumption is necessary for the method. If A is infinite, then a finite subset $\tilde{A}$ of representative objects should be chosen). The decision maker is asked for any pair $(a, b) \in A^2$:

1. Is a more attractive than b?
2. If yes, is the difference of attractivity between a and b *very weak*, *weak*, *moderate*, *strong*, *very strong*, or *extreme*?

The first question concerns ordinal measurement: we are looking for a function $f : A \longrightarrow \mathbb{R}$ satisfying condition **(Ord[A])**. The second question is related to difference measurement. The six ordered categories *very weak,...,extreme* define a quaternary relation on A, as defined in Sect. 14.2.1. MACBETH is able to test in a simple way if f as in (14.3) exists, and if yes, produces such a function, unique up to a positive affine transformation. In summary, we get an interval scale satisfying conditions **(Inter[A])** and **(C-Inter[A])**.

As explained in Sect. 14.2.2, we may have a unipolar or a bipolar scale, in which case a $\mathbb{1}$ level exists. It is convenient to choose f such that $f(\mathbb{1}) = 0$. If several sets $A_1, \dots, A_n$ are involved, then commensurability between the scales $f_1, \dots, f_n$ may be required, as it will be seen later.

We say that scales f_i, f_j are *commensurate* if $f_i(a_i) = f_j(a_j)$ means that the DM has the same intensity of attractiveness (or satisfaction, etc.) for a_i and a_j. A set of scales is commensurate if any pair is commensurate. Under the assumption that all f_i's are interval scales, it is sufficient to find two levels on each A_i, $i = 1, \dots, n$ for which the DM feels an equal satisfaction for all i (they are in a sense *absolute levels*), and to impose equality of the scales for those levels.

Obviously, the levels $\mathbb{1}_i$ of each A_i have an identical absolute meaning, provided the A_i's are either all bipolar or all unipolar, but not mixed. We fix $f_i(\mathbb{1}_i) = 0$, $i = 1, \dots, n$.

The second absolute levels could be the levels $\mathbb{O}_i$ (satisfactory levels in case of unbounded scales, and greatest elements otherwise). As suggested in Sect. 14.2.2, we may fix $f_i(\mathbb{O}_i) = 1$, $i = 1, \ldots, n$.

The same considerations apply to the absolute levels $-\mathbb{O}_i$.

To conclude this section, let us stress the fact that the underlying assumptions on which MACBETH (and hence, the method presented here) is based is that the DM is able to deliver information concerning difference measurement, and that the DM is able to exhibit on A two elements or levels with an absolute meaning, denoted $\mathbb{1}$ and $\mathbb{O}$, the precise meaning of them being dependent on the type of scale. We adopt throughout the paper the convention that

$$f(\mathbb{1}) = 0, \quad f(\mathbb{O}) = 1. \tag{14.5}$$

14.3 Unipolar Scales

We address in this section the construction of our model in the case of unipolar scales. As explained in Sect. 14.2, we have on each X_i two absolute levels $\mathbb{1}_i$ and $\mathbb{O}_i$ given by the DM.

14.3.1 Notion of Interaction: A Motivating Example

To introduce more precisely the idea of interaction and show some flaws of the weighted sum, let us give an example. The director of a university decides on students who are applying for graduate studies in management where some prerequisites from school are required. Students are indeed evaluated according to mathematics (M), statistics (S) and language skills (L). All the marks with respect to the scores are given on the same scale from 0 to 20. These three criteria serve as a basis for a preselection of the candidates. The best candidates have then an interview with a jury of members of the university to assess their motivation in studying in management. The applicants have generally speaking a strong scientific background so that mathematics and statistics have a big importance to the director. However, he does not wish to favor too much students that have a scientific profile with some flaws in languages. Besides, mathematics and statistics are in some sense *redundant*, since, usually, students good at mathematics are also good at statistics. As a consequence, for students good in mathematics, the director prefers a student good at languages to one good at statistics. Consider the following student A

	Mathematics (M)	Statistics (S)	Languages (L)
Student A	16	13	7

Student A is highly penalized by his performance in languages. Hence, the director would prefer a student (with the same mark in mathematics) that is a little bit better in languages even if the student would be a little bit worse in statistics. This means that the director prefers the following student to A

	Mathematics (M)	Statistics (S)	Languages (L)
Student B	16	11	9

We have thus

$$A \prec B \tag{14.6}$$

Consider now a student that has a weakness in mathematics. In this case, since the applicants are supposed to have strong scientific skills, a student good in statistics is now preferred to one good in languages. Consider the following two students

	Mathematics (M)	Statistics (S)	Languages (L)
Student C	6	13	7
Student D	6	11	9

Following above arguments, C is preferred to D even though C has poor language skills.

$$C \succ D \tag{14.7}$$

Satisfying (14.6) and (14.7) at the same time leads to the following requirement

$$F(16, 13, 7) > F(16, 11, 9) \quad \text{and} \quad F(6, 13, 7) < F(6, 11, 9).$$

No weighted sum can model such preferences since (14.6) implies that languages is more important than statistics whereas (14.7) tells exactly the contrary. There is an inversion of preferences between (14.6) and (14.7) in the sense that the relative importance of languages compared to statistics depends on the satisfaction level in mathematics. This behaviour is a typical example of *interaction* between criteria.

14.3.2 Capacities and Choquet Integral

The natural generalization of giving weights on criteria is to assign weights on *coalitions* (i.e. groups, subsets) of criteria. This can be achieved by introducing particular functions on $\mathcal{P}(N)$, called fuzzy measures or capacities. We recall that $N := \{1, \ldots, n\}$ is the index set of criteria.

A *fuzzy measure* [118] or *capacity* [11] is a set function $\mu : 2^N \to \mathbb{R}$ satisfying

(FM$_a$) $A \subseteq B \Rightarrow \mu(A) \leq \mu(B)$,
(FM$_b$) $\mu(\emptyset) = 0$,
(FM$_c$) $\mu(N) = 1$.

We denote by $\mathcal{M}$ the set of capacities. Property **(FM$_a$)** is called *monotonicity* of the capacity. In MCDA, $\mu(A)$ is interpreted as the overall assessment of the binary alternative $(1_A, 0_{-A})$. A set function satisfying only **(FM$_b$)** is called a *game* or a *non-monotonic fuzzy measure*.

The *conjugate* μ^* of a capacity μ is defined by $\mu^*(S) = \mu(N) - \mu(N \setminus S)$. The capacity is said to be *additive* if $\mu(A \cup B) = \mu(A) + \mu(B)$, whenever $A \cap B = \emptyset$, while it is said to be *symmetric* if $\mu(A)$ depends only on $|A|$.

The set of set functions on N is homomorphic to the set of *pseudo-Boolean functions* $f : \{0,1\}^n \to \mathbb{R}$. More precisely, if we define for any set $A \subseteq N$ the vector $\delta_A \in \{0,1\}^n$ by $\delta_A(i) = 1$ if $i \in A$ and 0 otherwise, then for any set function μ we can define its associated pseudo-Boolean function f by $f(\delta_A) := \mu(A)$ for all $A \subseteq N$, and reciprocally.

Several possible extensions of f on $\mathbb{R}^n_+$ can be defined. The first one called *multilinear extension* [61, 63] takes the form

$$\mathcal{M}_\mu(a) = \sum_{A \subseteq N} m(A) \cdot \prod_{i \in A} a_i\,, \quad \forall a = (a_1, \ldots, a_n) \in \mathbb{R}^n_+ \tag{14.8}$$

where $m(A)$ corresponds to the Möbius transform (see e.g. [108]) of μ, associated to f, which is defined by

$$m(A) = \sum_{B \subseteq A} (-1)^{|A \setminus B|} \mu(B).$$

Reciprocally, μ can be recovered from the Möbius transform by

$$\mu(A) = \sum_{B \subseteq A} m(B).$$

The second extension called *Lovász extension* is defined by

$$\mathcal{C}_\mu(a) = \sum_{A \subseteq N} m(A) \cdot \bigwedge_{i \in A} a_i\,, \quad \forall a \in \mathbb{R}^n_+ \tag{14.9}$$

where m is the Möbius transform of μ, and $\wedge$ is the min operator. Later, we will also use the notation $\vee$ to denote the max operator. This expression corresponds to the Choquet integral [11]. An equivalent expression in terms of the capacity μ is

$$\mathcal{C}_\mu(a) = a_{\tau(1)}\, \mu(N) + \sum_{i=2}^{n} \left(a_{\tau(i)} - a_{\tau(i-1)}\right) \mu\left(\{\tau(i), \cdots, \tau(n)\}\right), \tag{14.10}$$

where τ is a permutation on N such that $a_{\tau(1)} \leq a_{\tau(2)} \leq \cdots \leq a_{\tau(n)}$. Note that the Choquet integral is also well-defined w.r.t. set functions which are games.

When the capacity is additive, the Choquet integral reduces to a weighted sum.

We say that $a, b \in \mathbb{R}^n_+$ are *comonotone* if $a_i < a_j \Rightarrow b_i \leq b_j$ for any $i, j \in N$. In other words, a, b are comonotone if they belong to $\Gamma_\tau := \{a \in \mathbb{R}^n_+ \mid a_{\tau(1)} \leq a_{\tau(2)} \leq \cdots \leq a_{\tau(n)}\}$ for the same permutation τ. Thus, it is clear from (14.10) that for comonotone a, b we have $\mathcal{C}_\mu(a + b) = \mathcal{C}_\mu(a) + \mathcal{C}_\mu(b)$. This property, called *comonotonic additivity*, is characteristic of the Choquet integral, as shown by Schmeidler [111].

For other properties and characterizations of the Choquet integral, we refer the reader to survey papers [16, 85, 103], and also to Sect. 14.3.4.

Taking F as the Choquet integral, let us see whether it exists some capacity μ such that $\mathcal{C}_\mu$ is able to model relation (14.6) and (14.7). The modeling of (14.6) implies that $2\mu(\mathrm{M}, \mathrm{S}) > \mu(\mathrm{M}) + 1$, while (14.7) gives $2\mu(\mathrm{S}) > \mu(\mathrm{S}, \mathrm{L})$. There is no contradiction between previous two inequalities, hence the Choquet integral can model the preferences of the DM.

14.3.3 Construction of Utility Functions

14.3.3.1 Difficulty of the Construction of Utility Functions

In order to construct the utility functions, the DM is only supposed to provide information regarding options in X (in particular, part of the binary relation $\succeq$ and the quaternary relation $\succeq^*$ on X). However, in practice, a utility function u_i is often constructed by the DM, by isolating attribute X_i and asking questions directly regarding the preference of the DM on the set X_i (independently on the values on the other attributes). This practice is justified only when

$$(x_i, t_{-i})(y_i, t_{-i}) \succeq^\star (z_i, t_{-i})(w_i, t_{-i})$$
$$\Longleftrightarrow \quad (x_i, s_{-i})(y_i, s_{-i}) \succeq^\star (z_i, s_{-i})(w_i, s_{-i})$$

for all $x_i, y_i, z_i, w_i \in X_i$ and all $t_{-i}, s_{-i} \in X_{-i}$. This condition is called *weak difference independence* [24]. Note that it contains the weak separability (see Sect. 14.1) as particular case when $z_i = t_i$.

Proposition 1 (Corollary 2 in [24] Applied to Theorem 6.3 in [63]). *Assume that $n \geq 2$. $\succeq^\star$ satisfies weak difference independence for all $i \in N$ and is representable by a function u if and only if there exists a capacity μ and utility functions $u_1, \ldots, u_n$ such that u is the multi-linear value model $u(x) = \mathcal{M}_\mu(u_1(x_1), \ldots, u_n(x_n))$ for all $x \in X$.*

From this result, the Choquet integral does not fulfill the weak difference independent property. Hence the construction of the utility functions through the Choquet integral is not an easy task.

14.3.3.2 General Method for Building Utility Functions

Let us describe now a general method to construct the utility functions u_i without the prior knowledge of F [54, 77]. The utility functions shall be determined through questions regarding elements of X. Following the MACBETH approach [2–4], the subset $X\rfloor_i$ (for $i \in N$) of X will serve as a basis for the determination of u_i:

$$X\rfloor_i = \{(x_i, \mathbb{1}_{-i}) \ , \ x_i \in X_i\} .$$

We apply the MACBETH methodology to each set $X\rfloor_i$, which amounts to satisfy conditions **(Ord[$X\rfloor_i$])**, **(Inter[$X\rfloor_i$])**, **(C-Inter[$X\rfloor_i$])**. This gives the numerical representation $u_{X\rfloor_i}$ of $X\rfloor_i$. It is uniquely determined if (14.5) is applied. Since $\mathbb{1}_i$ is a least level of X_i, the utility function u_i is non-negative.

For $(x_i, \mathbb{1}_{-i}) \in X\rfloor_i$, one has by (14.2) and (14.5), since $u_{X\rfloor_i}(x_i, \mathbb{1}_{-i})$ corresponds to the overall utility of the act $(x_i, \mathbb{1}_{-i})$:

$$u_{X\rfloor_i}(x_i, \mathbb{1}_{-i}) = F(u_i(x_i), u_{-i}(\mathbb{1}_{-i})) = F(u_i(x_i), 0_{-i}) .$$

Assume that the family $\mathcal{F}$ of aggregation functions satisfies

$$\exists \alpha_i \in \mathbb{R}_+^* \ , \quad F(a_i, 0_{-i}) = \alpha_i \, a_i \ \text{ for all } a_i \in \mathbb{R}_+. \tag{14.11}$$

Since $u_{X\rfloor_i}(\mathbb{O}_i, \mathbb{1}_{-i}) = F(1_i, 0_{-i}) = \alpha_i$, we get for any $x_i \in X_i$:

$$u_i(x_i) = \frac{F(u_i(x_i), 0_{-i})}{F(1_i, 0_{-i})} = \frac{u_{X\rfloor_i}(x_i, \mathbb{1}_{-i})}{u_{X\rfloor_i}(\mathbb{O}_i, \mathbb{1}_{-i})}. \tag{14.12}$$

This shows that if all aggregation functions belonging to $\mathcal{F}$ satisfy (14.11) then u_i can be determined by (14.12) from cardinal information related to $X\rfloor_i$.

Note that we do not need to assume weak separability, thanks to (14.11).

Considering the case of the Choquet integral, it is easy to see that whenever $\mu(\{i\}) > 0$ for any $i \in N$, condition (14.11) is fulfilled so that the utility functions can be constructed with $\mathcal{F}$ being equal to the Choquet integral w.r.t. capacities satisfying previous condition.

14.3.3.3 Construction of Utility Functions Without any Commensurability Assumption

The approach described in the previous section assumes that the DM provides on each attribute the two elements $\mathbb{1}_i$ and $\mathbb{O}_i$. When this is not possible, an alternative method is proposed in [73] without any assumption about commensurability among criteria.

The main idea of Labreuche [73] is now summarized. Commensurateness between the criteria is required for the Choquet integral since this aggregation function is based on a ranking of the values of the criteria. Yet, the Choquet integral is a piecewise weighted sum function, in which the weights of the criteria are the same whenever the criteria are ranked in the same order. Considering two criteria i and k, the weight of criterion i depends on the relative values of criteria i and k. This means that, if the value of criterion k varies and the other criteria are fixed, then one may observe that the weight of criterion i suddenly changes when the value of criterion k is equal to that of criterion i (see also Sect. 14.3.4.3 and Fig. 14.1). From this remark, it is possible to construct, from an element of attribute i, an element of attribute k that is commensurate to the previous element. This construction does not work if the weight of criterion i does not depend on criterion k. If this holds for any value on the other criteria, one can show that this implies that the criteria i and k are independent. Applying this construction to any pair i, k of criteria, one obtains a partition of the set of criteria. In each set, the criteria interact one with another, and it is thus possible to construct vectors of values on the attributes that are commensurate. There is complete independence between the criteria of any two sets in this partition. Hence one cannot ensure commensurability between two sets in the partition. But this is not a problem since the Choquet integral is additive between groups of criteria that are independent.

Within each partition, one can construct two vectors of commensurate elements denoted by $\mathbb{1}$ and $\mathbb{O}$ respectively. Then the utility functions are obtained by applying the approach of Sect. 14.3.3.2 on $\mathbb{1}$ and $\mathbb{O}$.

14.3.4 Justification of the Use of the Choquet Integral

We now review the major characterizations of the Choquet integral.

14.3.4.1 Justification Through Information on the Binary Alternatives

We show that if we consider natural information that allow the modeling of interaction between criteria on top of information regarding $X\rfloor_i$, the Choquet integral comes up as a natural aggregation function. The justification of the use of the Choquet integral does not come from a pure axiomatic approach but rather from some reasonable information asked to the DM.

The information regarding the aggregation of the criteria can be limited to alternatives whose scores on criteria are either $\mathbb{1}_i$ or $\mathbb{O}_i$. This leads to defining the following set:

$$X\rceil_{\{0,1\}} := \{(\mathbb{O}_A, \mathbb{1}_{-A}) \ , A \subseteq N\},$$

called the set of *binary alternatives*. The application of the MACBETH methodology leads to the interval scale $u_{X\rceil_{\{0,1\}}}$, which requires the satisfaction of conditions **(Ord[**$X\rceil_{\{0,1\}}$**])**, **(Inter[**$X\rceil_{\{0,1\}}$**])**, and **(C-Inter[**$X\rceil_{\{0,1\}}$**])**. Applying (14.5) to this scale, it becomes uniquely determined:

$$u_{X\rceil_{\{0,1\}}}(\mathbb{1}_N) = 0, \quad u_{X\rceil_{\{0,1\}}}(\mathbb{O}_N) = 1. \tag{14.13}$$

The second condition in (14.13) says that an alternative which is completely satisfactory on each criteria should be completely satisfactory, and similarly for the first condition.

From $u_{X\rceil_{\{0,1\}}}$, it is natural to define a capacity μ by, for all $A \subseteq N$, $\mu(A) := u_{X\rceil_{\{0,1\}}}(\mathbb{O}_A, \mathbb{1}_{-A})$. Consequently, we write u as follows:

$$u(x) = F_\mu (u_1(x_1), \ldots, u_n(x_n)), \tag{14.14}$$

where F_μ is the aggregation function that depends on μ in a way that is not known for the moment.

The u_i's correspond to interval scales, whose admissible transformations are the positive affine transformations (see Sect. 14.2.1). Hence, one could change all u_i's in $\alpha u_i + \beta$, for any $\alpha > 0$ and $\beta \in \mathbb{R}$, without any change in the model. On the other hand, $\mu(A)$ corresponds in fact to the difference of the satisfaction degrees between the alternatives $(\mathbb{O}_A, \mathbb{1}_{-A})$ and $\mathbb{1}_N$. Applying this to $A = \emptyset$, the value $\mu(\emptyset)$ shall always be equal to zero, whatever the interval scale attached to $X\rceil_{\{0,1\}}$ may be. Hence, μ corresponds to a ratio scale, and can be replaced by $\gamma\mu$, with $\gamma \in \mathbb{R}_+$, since these are the admissible transformations for ratio scales. Hence one shall have [77]:

(Meas-Inter) The preference relation $\succeq$ and the ratio $\frac{u(x)-u(y)}{u(z)-u(t)}$ for $x, y, z, t \in X\rfloor_i$ (for all $i \in N$) and for $x, y, z, t \in X\rceil_{\{0,1\}}$ shall not be changed if all the u_i's are changed into $\alpha u_i + \beta$ with $\alpha > 0$ and $\beta \in \mathbb{R}$, and μ is changed into $\gamma\mu$ with $\gamma \in \mathbb{R}_+$.

From this property and idempotency of F_μ (i.e. $F_\mu(\beta, \ldots, \beta) = \beta$ for all $\beta \in \mathbb{R}$) [27], one can show the following two properties [77]

Properly Weighted (PW): If μ satisfies conditions **(FM**$_b$**)** and **(FM**$_c$**)**, then $F_\mu(1_A, 0_{-A}) = \mu(A)$, $\forall A \subseteq N$.

Stability for the admissible Positive Linear transformations (weak SPL): If μ satisfies conditions **(FM**$_b$**)** and **(FM**$_c$**)**, then for all $A \subseteq N$, $\alpha > 0$, and $\beta \in \mathbb{R}$,

$$F_\mu((\alpha + \beta)_A, \beta_{-A}) = \alpha F_\mu(1_A, 0_{-A}) + \beta$$

Since F_μ aggregates satisfaction scales, it is natural to assume that $x \mapsto F_\mu(x)$ is increasing.

Increasingness (In): If μ satisfies conditions **(FM$_a$)** and **(FM$_b$)**, then $\forall x, x' \in \mathbb{R}^n$,

$$x_i \leq x'_i \ \forall i \in N \ \Rightarrow F_\mu(x) \leq F_\mu(x')$$

Measurement considerations yield linearity of the mapping $\mu \mapsto F_\mu(x)$ [77].

Linearity w.r.t. the Measure (LM): If μ satisfies condition **(FM$_b$)**, then for all $x \in \mathbb{R}^n$ and $\gamma, \delta \in \mathbb{R}$,

$$F_{\gamma\mu+\delta\mu'}(x) = \gamma F_\mu(x) + \delta F_{\mu'}(x).$$

The following result can be shown.

Theorem 1 (Theorem 1 in [77]). *F_μ satisfies **(LM)**, **(In)**, **(PW)** and **(weak SPL)** if and only if $F_\mu \equiv \mathcal{C}_\mu$ in $\mathbb{R}^n$.*

We have seen that the measurement conditions we have on u_i and $u_{X\rceil_{\{0,1\}}}$ lead naturally to axioms **(LM)**, **(In)**, **(PW)** and **(weak SPL)**. There is only one aggregation function that satisfies these axioms, namely the Choquet integral w.r.t. μ.

Let us remark that Theorem 1 is a weak version of an axiomatic characterization obtained by Marichal [85].

14.3.4.2 Axiomatization of the Choquet Integral as an Aggregation Function

We already presented in the previous section an extension of the axiomatic characterization by Marichal [85]. The first axiomatization of the Choquet integral is due to Schmeidler [111]. An aggregation function $F : \mathbb{R}^n \to \mathbb{R}$ is a Choquet integral iff $F(1, \dots, 1) = 1$, F is monotone, and F satisfies to *comonotone additivity* (see Sect. 14.3.2). Two other properties can also be used. F is said to be *horizontally min-additive* (originally called *horizontally additive*) [5] if for all $t \in \mathbb{R}$ and all $c \in \mathbb{R}$

$$F(t) = F(t \wedge c) + F(t - (t \wedge c)).$$

F is said to be *horizontally max-additive* [13] if for all $t \in \mathbb{R}$ and all $c \in \mathbb{R}$

$$F(t) = F(t \vee c) + F(t - (t \vee c)).$$

These two properties are particular cases of the comonotone additivity as the two options $t \wedge c$ and $t - (t \wedge c)$ (resp. $t \vee c$ and $t - (t \vee c)$), which sum up to x, are comonotone. The three previous properties are shown to be equivalent [13]. Hence, in the axiomatic characterization by Schmeidler, comonotone additivity can be replaced by horizontally min-additivity or horizontally max-additivity.

14.3.4.3 Axiomatizations of the Choquet Integral with Utility Functions

There are very few axiomatization of the Choquet integral with the utility functions. The first one is done in the context of *Decision Under Uncertainty*, i.e. when $u_1 = \cdots = u_n =: \varphi$ and $X_1 = \cdots = X_n =: Y$, by Couceiro and Marichal [14]. Without loss of generality assume that $[0,1] \subseteq Y$. According to [14], there exists μ, φ and $A \subseteq N$ such that $u(x) = C_v(\varphi(x_1), \ldots, \varphi(x_n))$ and $u(1_A, 0_{-A}) \neq u(0, \ldots, 0)$ iff u is *comonotonically modular* (i.e. $u(x) + u(x') = u(x \wedge x') + u(x \vee x')$ for all $x, x' \in X$ that are comonotone) and u is *weakly homogeneous* (i.e. there exists a non-decreasing function $\varphi : Y \to \mathbb{R}$ with $\varphi(0) = 0$ such that $u(r_A, 0_{-A}) - u(0, \ldots, 0) = \varphi(r)(u(1_A, 0_{-A}) - u(0, \ldots, 0))$ for all $r \in Y$ and all $A \subseteq N$).

An axiomatization of the overall utility function u described by both the utility functions $u_1, \ldots, u_n$ and the Choquet integral (see (14.2)) is proposed in [75] without any assumption on the commensurability between the criteria. Up to date, this is the only axiomatization of the Choquet integral in the context of MCDA. We assume that u is known on X. In this chapter, we only describe the main axiom. The example presented in Sect. 14.3.1 to motivate the interest of the Choquet integral in MCDA, is based on the idea that the relative weight between two criteria is conditional on a third one being good or bad (see also Sect. 14.4.1). The importance of criterion i at an alternative $x \in X$ is proportional to the partial derivative $\frac{\partial u}{\partial x_i}(x)$. Translating comonotone additivity in terms of importance, the importance of a criterion depends on the relative ordering of the criteria, but not on the precise values on the criteria. Consider the importance of criterion i. Let us fix x_i. If we vary the value of an attribute $k \neq i$, the weight of criterion i will take only two values (see Fig. 14.1): (1) when attribute k is ranked below attribute i, and (2) when the opposite holds.

Commensurateness Through Interaction (CTI): *Let $i, k \in N$ with $i \neq k$. We have the alternative:*

(i) Either for all $x_i \in X_i$, there exists $x_k^\star \in X_k$ and $\mathcal{X}_{-i,k} \subseteq X_{-i,k}$ non empty, such that for all $x_{-i,k} \in \mathcal{X}_{-i,k}$, the function $x_k \mapsto \frac{\partial U(x)}{\partial x_i}$ is not constant. More precisely, for all $x_{-i,k} \in \mathcal{X}_{-i,k}$, there exists $c_0, c_1 \in \mathbb{R}$ with $c_0 \neq c_1$ such that

$$\forall x_k < x_k^* \quad \frac{\partial U}{\partial x_i}(x_i, x_k, x_{-i,k}) = c_0 \tag{14.15}$$

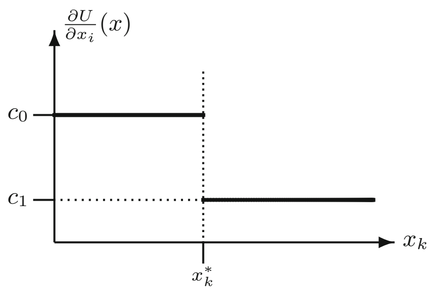


Fig. 14.1 The two values that $x_k \mapsto \frac{\partial U}{\partial x_i}(x)$ can take

$$\forall x_k > x_k^* \quad \frac{\partial U}{\partial x_i}(x_i, x_k, x_{-i,k}) = c_1 \tag{14.16}$$

Moreover, for all $x_{-i,k} \in X_{-i,k} \setminus \mathcal{X}_{-i,k}$*, the function* $x_k \mapsto \frac{\partial U(x)}{\partial x_i}$ *is constant.*

(ii) *Or for all* $x_i \in X_i$ *and all* $x_{-i,k} \in X_{-i,k}$*, the function* $x_k \mapsto \frac{\partial U(x)}{\partial x_i}$ *is constant.*

In this axiom, the element x_k^* satisfies $u_k(x_k^*) = u_i(x_i)$ and is thus commensurate to x_i. This holds only in condition (i). It is shown that this holds only when there is some interaction between criteria i and k [75].

14.3.5 Shapley Value and Interaction Index

By construction, the capacity μ expresses the score of binary alternatives. Since there are 2^n such alternatives, it may be difficult to analyse or explain the behaviour of the decision maker through the values taken by μ.

A first question of interest is: "What is the importance of a given criterion for the decision?". We may say that a criterion i is important if whenever added to some coalition A of criteria, the score of $(\mathbb{O}_{A\cup i}, \mathbb{1}_{-(A\cup i)})$ is significantly larger than the score of $(\mathbb{O}_A, \mathbb{1}_{-A})$. Hence, an importance index should compute an average value Δ_i of the quantity $\mu(A \cup i) - \mu(A)$ for all $A \subseteq N \setminus i$. A second requirement is that the sum of importance indices for all criteria should be a constant, say 1. Lastly, the importance index should not depend on the numbering of the criteria. Strangely enough, these three requirements plus a linearity assumption, which imposes that the average Δ_i is a weighted arithmetic mean, suffices to determine uniquely the importance index, known as the *Shapley importance index* [112]

$$\phi^\mu(i) := \sum_{K \subseteq N \setminus i} \frac{(n-k-1)!k!}{n!} \big[\mu(K \cup i) - \mu(K)\big] \tag{14.17}$$

with $k := |K|$. We omit the superscript if no ambiguity occurs. The *Shapley value* is the vector $(\phi(1), \ldots, \phi(n))$. As said above, we have $\sum_{i=1}^n \phi(i) = \mu(N) = 1$. Another fundamental property is that $\phi(i) = \mu(\{i\})$ if μ is additive.

We have shown by an example in Sect. 14.3.1 that interaction may occur among criteria, and that the Choquet integral was able to deal with situations where interaction occurs. We define this notion more precisely. Let us consider for simplicity 2 criteria and the following alternatives (see Fig. 14.2):

- $x = (\mathbb{1}_1, \mathbb{1}_2)$
- $y = (\mathbb{O}_1, \mathbb{1}_2)$
- $z = (\mathbb{1}_1, \mathbb{O}_2)$
- $t = (\mathbb{O}_1, \mathbb{O}_2)$

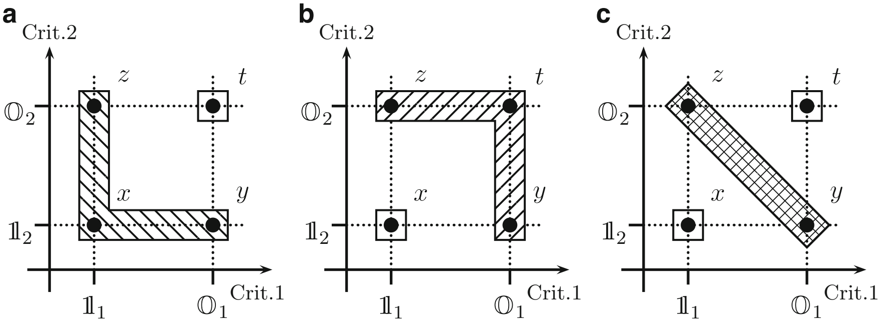


Fig. 14.2 Different cases of interaction: complementary criteria (**a**), substitutive criteria (**b**), independent criteria (**c**)

Clearly, t is more attractive than x, but preferences over other pairs may depend on the decision maker. Due to monotonicity ($\mathbf{FM}_a$), we can range from the two extremal following situations (recall that $\mu(\{1,2\}) = 1$ and $\mu(\emptyset) = 0$):

extremal situation 1 (lower bound): we put $\mu(\{1\}) = \mu(\{2\}) = 0$, which is equivalent to the preferences $x \sim y \sim z$ (Fig. 14.2a) (strictly speaking, $\mu(\{i\})$ cannot attain the value 0: see Sect. 14.3.3). This means that for the DM, both criteria have to be satisfactory in order to get a satisfactory alternative, the satisfaction of only one criterion being useless. We say that the criteria are *complementary*.

extremal situation 2 (upper bound): we put $\mu(\{1\}) = \mu(\{2\}) = 1$, which is equivalent to the preferences $y \sim z \sim t$ (Fig. 14.2b). This means that for the DM, the satisfaction of one of the two criteria is sufficient to have a satisfactory alternative, satisfying both being useless. We say that the criteria are *substitutive*.

Clearly, in these two situations, the criteria are not independent, in the sense that the satisfaction of one of them acts on the usefulness of the other in order to get a satisfactory object (necessary in the first case, useless in the second). We say that there is some *interaction* between the criteria.

A situation without interaction is such that the satisfaction of each criterion brings its own contribution to the overall satisfaction, hence:

$$\mu(\{1,2\}) = \mu(\{1\}) + \mu(\{2\}) \tag{14.18}$$

(additivity) (see Fig. 14.2c). In the first situation, $\mu(\{1,2\}) > \mu(\{1\}) + \mu(\{2\})$, while the reverse inequality holds in the second situation. This suggests that the interaction I_{12} between criteria 1 and 2 should be defined as:

$$I_{12}^{\mu} := \mu(\{1,2\}) - \mu(\{1\}) - \mu(\{2\}) + \mu(\emptyset). \tag{14.19}$$

This is simply the difference between binary alternatives on the diagonal (where there is strict dominance) and on the anti-diagonal (where no dominance relation exists). The interaction is positive when criteria are complementary, while it is negative when they are substitutive. This is consistent with intuition considering that when criteria are complementary, they have no value by themselves, but put together they become important for the DM.

In the case of more than two criteria, the definition of interaction follows the same idea as with the Shapley index, in the sense that all coalitions of N have to be taken into account. The following definition has been first proposed by Murofushi and Soneda [102], for a pair of criteria i, j:

$$I_{ij}^{\mu} := \sum_{K \subseteq N \setminus \{i,j\}} \frac{(n-k-2)!k!}{(n-1)!} \big[\mu(K \cup \{i,j\}) - \mu(K \cup \{i\}) - \mu(K \cup \{j\}) + \mu(K)\big], \quad (14.20)$$

The definition of this index has been extended to any coalition $\emptyset \neq A \subseteq N$ of criteria by Grabisch [33]:

$$I^{\mu}(A) := \sum_{K \subseteq N \setminus A} \frac{(n-k-|A|)!k!}{(n-|A|+1)!} \sum_{L \subseteq A} (-1)^{|A|-|L|} \mu(K \cup L), \forall A \subseteq N, A \neq \emptyset. \quad (14.21)$$

We have $I_{ij} = I(\{i,j\})$. When $A = \{i\}$, $I(\{i\})$ coincides with the Shapley index $\phi(i)$. It is easy to see that when the fuzzy measure is additive, we have $I(A) = 0$ for all A such that $|A| > 1$. Also $I_{ij} > 0$ (resp. $< 0, = 0$) for complementary (resp. substitutive, non-interactive) criteria.

The definition can be extended to $A = \emptyset$, just putting $\sum_{L \subseteq A} (-1)^{|A|-|L|} \mu(K \cup L) = \mu(K)$. Hence I defines a set function $I : \mathcal{P}(N) \longrightarrow \mathbb{R}$. Properties of this set function has been studied and related to the Möbius transform [17, 50]. In particular, it is possible to recover μ if I is given for each $A \subseteq N$, which means that the interaction index can be viewed as a particular *transform* of a fuzzy measure, which is invertible, as the Möbius transform. Also, I has been characterized axiomatically by Grabisch and Roubens [48], in a way similar to the Shapley index.

Another important property is that the interaction index can be obtained recursively from the Shapley importance index, by considering sub-problems with less criteria [48]. For I_{ij}^{μ}, the relation writes:

$$I_{ij}^{\mu} = \phi^{\mu^{[ij]}}([ij]) - \phi^{\mu_{N \setminus i}}(j) - \phi^{\mu_{N \setminus j}}(i), \quad (14.22)$$

where $[ij]$ stands for an artificial criterion (i and j taken together), $\mu^{[ij]} : \mathcal{P}((N \setminus \{i,j\}) \cup \{[ij]\}) \longrightarrow [0,1]$, with $\mu^{[ij]}(A) := \mu(A \cup \{i,j\})$ if $A \ni [ij]$, and $\mu(A)$ else, and $\mu_{N\setminus i}$ is the restriction of μ to $N \setminus i$.

14.3.6 k-Additive Measures

14.3.6.1 Definition of the *k*-Additive Measures

Although we have shown that our construction is able to model in a clear way interaction, this has to be paid by an exponential complexity, since the number of binary alternatives is 2^n. There exists a way to cope with complexity by defining sub-families of fuzzy measures, which require less than 2^n coefficients to be defined. The first such family which has been defined is the one of *decomposable measures* [20, 125], which includes the well-known class of λ-measures proposed by Sugeno [118]. These fuzzy measures are defined by a kind of density function, and thus need only $n-1$ coefficients. However, they have a very limited ability to represent interaction since e.g. I_{ij} has the same sign for all i,j.

A second family is given by the concept of k-additive measure, which is detailed in this section.

Definition 1 ([33]). Let $k \in \{1,\dots,n-1\}$. A fuzzy measure μ is said to be k-additive if $I(A) = 0$ whenever $|A| > k$, and there exists some $A \subseteq N$ with $|A| = k$ such that $I(A) \neq 0$.

The set of k-additive capacities is denoted by $\mathcal{M}^k$. From the properties of interaction cited in Sect. 14.3.5, a 1-additive measure is simply an additive measure, hence the name. Also, since μ is completely determined by the values of I on $\mathcal{P}(N)$, a k-additive measure is determined by $1 + n + \binom{n}{2} + \dots + \binom{n}{k}$ parameters, among which 2 are not free.

The 2-additive measure, which needs only $\frac{n(n+1)}{2} - 1$ parameters, permits to model interaction between pair of criteria, which is in general sufficient in practice (it is in fact fairly difficult to have a clear understanding of interaction among more than two criteria).

The Choquet integral can be expressed using I instead of μ in a very instructive way when the measure is 2-additive [32]:

$$\mathcal{C}_\mu(a_1,\dots,a_n) = \sum_{I_{ij}>0}(a_i \wedge a_j)I_{ij} + \sum_{I_{ij}<0}(a_i \vee a_j)|I_{ij}| + \sum_{i=1}^{n} a_i\Big(\phi_i - \frac{1}{2}\sum_{j\neq i}|I_{ij}|\Big), \quad \forall a \in [0,1]^n, \quad (14.23)$$

for all $(a_1, \ldots, a_n) \in \mathbb{R}^n_+$, with the property that $\phi_i - \frac{1}{2}\sum_{j \neq i} |I_{ij}| \geq 0$ for all i. It can be seen that the Choquet integral for 2-additive measures is the sum of a conjunctive, a disjunctive and an additive part, corresponding respectively to positive interaction indices, negative interaction indices, and the Shapley value. Equation (14.23) shows clearly the disjunctive and conjunctive effects of negative and positive interaction between criteria, which has been explained in Sect. 14.3.5. It is important to notice that, due to the normalization $\sum_{i=1}^n \phi_i = 1$, (14.23) is a convex combination of disjunctions, conjunctions, and a linear part. Hence, as illustrated in [35] in a graphical way, the Choquet integral is the convex closure of all conjunctions and all disjunctions of pair of criteria, and of all dictators (single criteria).

Before ending this section, we mention a third family of fuzzy measures introduced by Miranda and Grabisch, the *p-symmetric fuzzy measures* [96]. The idea is to generalize symmetric fuzzy measures (see Sect. 14.3.2), by considering a partition $\{A_1, \ldots, A_p\}$ of N into subsets of indifference: taking elements in $A_1, \ldots, A_p$, the value of μ does not depend on the particular elements which are chosen in each A_i, but only on their number. Hence a symmetric measure corresponds to a 1-symmetric measure (i.e. the partition is N itself). The number of parameters needed to define a p-symmetric measure is $\prod_{i=1}^p (|A_i| + 1) - 2$.

14.3.6.2 Axiomatic Characterization of the Choquet Integral w.r.t. 2-Additive Measures

For a 2-additive capacity, one can construct the capacity from a subset of $X\rceil_{\{0,1\}}$ (see Sect. 14.3.4.1), restricted to the binary alternatives that take the satisfactory level $\mathbb{O}$ on at most two criteria:

$$X\rceil^{\text{Two}}_{\{0,1\}} := \{a_\emptyset\} \cup \{a_i\,,\ i \in N\} \cup \{a_{i,j}\,,\ \{i,j\} \subseteq N\},$$

where $a_\emptyset := (\mathbb{1}_N)$, $a_i = (\mathbb{O}_i, \mathbb{1}_{-i})$ and $a_{i,j} := (\mathbb{O}_{\{i,j\}}, \mathbb{1}_{-\{i,j\}})$. The DM is asked to provide strict preference $\rhd$ and indifference $\sim$ over $X\rceil^{\text{Two}}_{\{0,1\}}$. Moreover, we define a natural non-strict ordering $\succeq_M$ between elements of $X\rceil^{\text{Two}}_{\{0,1\}}$ due to the monotonicity conditions. For $(x, y) \in \{(a_i, a_0), i \in N\} \cup \{(a_{ij}, a_i), i, j \in N, i \neq j\}$, $x \succeq_M y$ if neither $x \rhd y$ nor $x \sim y$. We write $x \gg y$ if there is a strict path of the union of $\rhd$, $\sim$ and $\succeq_M$ (i.e. the strict path shall use $\rhd$ at least once) between x and y. Moreover, we denote by $\equiv$ the non-strict cycles of the union of $\rhd$, $\sim$ and $\succeq_M$. Relation $\sim$ (resp. $\gg$) is indifference (resp. strict preference) that can be deduced from $\rhd$, $\sim$ and $\succeq_M$.

Theorem 2 ([93]). *The preferential information $\{\rhd, \sim\}$ is representable by a 2-additive Choquet integral on $X\rceil^{\text{Two}}_{\{0,1\}}$ if and only if the following conditions are satisfied:*

1 *the union of $\rhd$, $\sim$ and $\succeq_M$ contains no strict cycle;*
2 Monotonicity of Preferential Information (MOPI). *For all $\{i, j, k\} \subseteq N$, we have the following property*

$$\left.\begin{array}{l} a_{i,j} \equiv a_{p_{i,j}} \\ a_{i,k} \equiv a_{p_{i,k}} \\ p_{i,j} \neq p_{i,k} \end{array}\right\} \Rightarrow [not(a_l \gg a_\emptyset), l \in \{i,j,k\} \setminus \{p_{i,j}, p_{i,k}\}]$$

where $p_{i,j} = i$ *or* j*, and* $p_{i,k} = i$ *or* k*.*

Roughly speaking, the MOPI property says that when the two indifferences in the left hand side hold, the criterion not appearing alone in the left hand side (i.e. criterion $\{i,j,k\} \setminus \{p_{i,j}, p_{i,k}\}$) does not count, and thus shall not be strictly preferred to zero.

14.3.7 Final Recommendation and Identification of Capacities

We assume here that the utility functions u_i are known (See Sect. 14.3.4). So, we focus in this section on the determination of the capacity.

The DM is usually not interested in all options in X. We denote by $Y \subseteq X$ the set of alternatives of interest for the DM. The goal of decision aid is to propose a recommendation regarding the options in Y.

14.3.7.1 Preferential Information

In order to give a recommendation to the DM, some preferential information is asked to the DM. Many types of preferential information (PI) are considered in the literature [1, 55, 65, 71]. In practice, it turns out that the most meaningful PI for the DM is composed of comparisons of options. For the sake of simplicity, we assume thus that the PI is composed of a partial order $\trianglerighteq$ over $\mathbb{R}^N$. For $a, b \in \mathbb{R}^N$, relation $a \trianglerighteq b$ means that the DM finds a at least as good as b. We denote by $\mathcal{M}^k(\trianglerighteq)$ the set of (k-additive) fuzzy measures in $\mathcal{M}^k$ that satisfy the PI $\trianglerighteq$:

$$\mathcal{M}^k(\trianglerighteq) = \Big\{\mu \in \mathcal{M}^k \text{ , } \forall a, b \in \mathbb{R}^N \quad a \trianglerighteq b \;\Rightarrow\; C_\mu(a) \geq C_\mu(b)\Big\}.$$

The set $\mathcal{M}^k(\trianglerighteq)$ is a polytope. It may be the case that $\mathcal{M}^k(\trianglerighteq) = \emptyset$, which means that the PI cannot be represented by a Choquet integral. We say in this case that the PI is inconsistent. An algorithm based on mixed-integer linear programming can be used to identify the elements of the PI that can be modified or removed in order to restore consistency, together with an associated explanation [83].

14.3.7.2 Identification of a Capacity, and Associated Recommendation

Most of the elicitation methods based on the Choquet integral consist in selecting one particular capacity in $\mathcal{M}^k(\trianglerighteq)$ (fulfilling the PI) that maximizes some functional, and then make the recommendations to the DM regarding the comparison of the options in Y or the assessment of the options in Y.

The capacity is chosen as a (the) solution to an optimization problem

$$\max_{\mu \in \mathcal{M}^k(\trianglerighteq)} G_{\trianglerighteq}(\mu)$$

where $G_{\trianglerighteq}$ is a function depending on $\trianglerighteq$ to be maximized [55]. Functional $G_{\trianglerighteq}$ can be the entropy [65], the opposite of the variance [65] or a linearized entropy [71].

As remarked by Marichal and Roubens in [88], when the DM states that $a \trianglerighteq b$, he or she generally means that a is significantly preferred to b. If the overall utilities of the two alternatives a and b are almost the same, it will probably not represent the DM's intention. Hence, among all elements of $\mathcal{M}^k(\trianglerighteq)$, one should prefer the ones with the highest margin. This leaded Marichal and Roubens to introducing a positive coefficient ϵ in the right-hand side of constraints, and to maximize ϵ:

Maximize ϵ
under the constraints $\mu \in \mathcal{M}^k$, $\epsilon \geq 0$, and $a \trianglerighteq b \;\Rightarrow\; C_\mu(a) \geq C_\mu(b) + \varepsilon$.

This is a linear programming problem. It is a simplified version of a linear method proposed by Marichal and Roubens [88].

When one is interested in a 2-additive capacity, there exists also an extension of the MACBETH approach [90–92]. Following and extending Sect. 14.3.6.2, the idea is to ask the DM to compare the elements of $X\rceil^{\text{Two}}_{\{0,1\}}$, and also to provide information regarding intensities of preferences between pairs of elements of $X\rceil^{\text{Two}}_{\{0,1\}}$. Compared to the traditional MACBETH approach, the difficulty is to include constraints related to the specificity of the 2-additive Choquet integral. An original interactive approach is to propose recommendations when the PI is inconsistent [92].

Other learning methods have been tried, principally using quadratic programming (see e.g. Grabisch [31, 49]), heuristics (see e.g. Ishii and Sugeno [62], Mori and Murofushi [99] and Grabisch [30]), genetic algorithms (see in particular Wang [123], Kwon and Sugeno [69], Combarro and Miranda [12], Grabisch [37], Verkeyn et al. [121]), self-organizing feature maps (see Soria-Frisch [117]), and particle swarm optimization (see Wang [124]). An overview on the subject can also be found in reference [119].

14.3.7.3 Robust Preference Relations

The major difficulty the facilitator faces is that there usually does not exist a single capacity that fulfills his PI, i.e. $\mathcal{M}^k(\trianglerighteq)$ is in practice far from being reduced to a point. The approaches presented in the previous section are not quite satisfactory for the DM since he does not usually understand what maximizing the functional really means. The use of a maximization problem introduces some additional information that does not come from the DM.

The facilitator shall rather stick strictly to what the DM says and add no further information. One then looks for a robust way to recommend some comparisons among the options from the preferential information [110]. The concept of *necessity preference relation* has been recently introduced for robust quantitative multi-criteria decision models [59, 60]. It has been applied to the Choquet integral in [1]. An option $x \in X$ is *necessarily* preferred to another option $y \in X$ (denoted by $x \succsim_{\mathrm{N}} y$) according to the necessity preference relation, if the first option is preferred to the second one according to all models that fulfill the PI provided by the DM. More precisely, $x \succsim_{\mathrm{N}} y$ iff [1]:

$$\forall \mu \in \mathcal{M}^k(\trianglerighteq) \quad C_\mu(u_1(x_1), \ldots, u_n(x_n)) \geq C_\mu(u_1(y_1), \ldots, u_n(y_n)). \qquad (14.24)$$

This necessity preference relation is usually incomplete, unless the model is completely specified from the preferential information of the DM [72]. Note that (14.24) can be easily extended to the case where the utility functions are not precisely set [84].

14.3.7.4 Explanation of the Recommendation

Providing convincing explanations to accompany recommendations is a key issue in decision-aiding. Indeed, *explaining* the recommended choice(s) to the decision-maker is crucial to improve the acceptance of the recommendation [10, 64, 94], but also sometimes to allow the decision-maker to justify in turn the decision against other stakeholders. In the context of decision models such as the Choquet integral, the problem is made very difficult since the quantitative models are not designed to be easily explainable.

There are a few works which aim at generating an explanation of the outcome of a quantitative multi-criteria decision model [10, 64, 70, 74, 98]. Note that two of these references consider a Choquet integral and the other ones focus on the weighted sum. The three references [10, 64, 98] use the same idea. It consists in selecting the k (where k is a parameter) criteria that have the largest contribution to the overall utility. This approach suffers some limitations as there is no formal justification of the arguments that are selected, and the textual explanation does not mention the importance of criteria.

The idea of references [70, 74] is that, due to the complexity of explaining a preference model based on utility theory, several explanation reasonings (argumentation schema) are necessary to cover all cases—ranging from situations where the prescription is trivial to situations where the prescription is much tighter. A subset of criteria can be selected for the explanation if these criteria are decisive in some sense depending on the explanation reasoning.

14.4 Bipolar Scales

We address now the construction of the model in the case of bipolar scales. As explained in Sect. 14.2, we have on each X_i one neutral level $\mathbb{1}_i$ and another absolute level $\mathbb{O}_i$ given by the DM.

14.4.1 A Motivating Example

Let us go a little deeper in the example described in Sect. 14.3.1. We have seen in Sect. 14.3.1 that for students good in mathematics, the director prefers someone good at languages to one good at statistics. In other words, when the mark with respect to mathematics is good, the director thinks that languages is more important than statistics. This leads to the following rule

(R1): For a student good at mathematics (M), L is more important than S.

The comparison between students A and B in Sect. 14.3.1 are governed by this rule. Let us consider now another set of students. Consider the following students E and F

	Mathematics (M)	Statistics (S)	Languages (L)
Student E	14	16	7
Student F	14	15	8

According to rule **(R1)**, the director prefers student F to E

$$E \prec F \tag{14.25}$$

As justified in Sect. 14.3.1, when the score w.r.t. mathematics is bad, a student good in statistics is now preferred to one good in languages. More precisely, we have the following statement

(R2): For a student bad in mathematics M, S is more important than L.

Consider the following two students

	Mathematics (M)	Statistics (S)	Languages (L)
Student G	9	16	7
Student H	9	15	8

Following rule **(R2)**, G is preferred to H even though G is very bad in languages.

$$G \succ H \tag{14.26}$$

Relations (14.25) and (14.26) look similar to (14.6) and (14.7). However, we will see that they exhibit a weakness of the Choquet integral. Let us indeed try to model (14.25) and (14.26) with the help of the Choquet integral. We have $\mathcal{C}_\mu(E) = 7 + 7\mu(\{\mathrm{M}, \mathrm{S}\}) + 2\mu(\{\mathrm{S}\})$ and $\mathcal{C}_\mu(F) = 8 + 6\mu(\{\mathrm{M}, \mathrm{S}\}) + \mu(\{\mathrm{S}\})$. This shows that (14.25) is equivalent to

$$\mu(\{\mathrm{M}, \mathrm{S}\}) + \mu(\{\mathrm{S}\}) < 1.$$

Similarly, relation (14.26) is equivalent to $\mu(\{\mathrm{M}, \mathrm{S}\}) + \mu(\{\mathrm{S}\}) > 1$, which contradicts previous inequality. Hence, the Choquet integral cannot model (14.25) and (14.26).

It is no surprise that the Choquet integral cannot model both **(R1)** and **(R2)**. This is due to the fact that the Choquet integral satisfies comonotonic additivity (see Sect. 14.3.2). In our example, the marks of the four students E, F, G and H are ranked in the same way: languages is the worst score, mathematics is the second best score, and statistics is the best score. Those four students are comonotonic. The Choquet integral is able to model rules of the following type:

(R1'): If M is the best satisfied criteria, L is more important than S.

(R2'): If M is the worst satisfied criteria, S is more important than L.

On the other hand, rules **(R1)** and **(R2)** make a reference to absolute values (good/bad in mathematics). The Choquet integral does not allow to model this type of property. The Choquet integral fails to represent the expertise that makes an explicit reference to an absolute value. This happens quite often in applications.

Let us study the meaning of the reference point used in rules **(R1)** and **(R2)**. In our example, the satisfaction level is either rather good (good in mathematics) or rather bad (bad in mathematics). This makes an implicit reference to a neutral level that is neither good nor bad. This suggests to construct criteria on ratio scales. In such scales, the zero element is the neutral element. It has an absolute meaning and cannot be shifted. Values above this level are attractive (good) whereas values below the zero level are repulsive (bad).

14.4.2 The Symmetric Choquet Integral and Cumulative Prospect Theory

14.4.2.1 Definitions

Let $f : N \longrightarrow \mathbb{R}$ be a real-valued function, and let us denote by $f^+(i) := f(i) \vee 0$, $\forall i \in N$, and $f^- := (-f)^+$ the positive and negative parts of f.

The *symmetric Choquet integral* [15] (also called the *Šipoš integral* [115]) of f w.r.t. μ is defined by:

$$\check{\mathcal{C}}_\mu(f) := \mathcal{C}_\mu(f^+) - \mathcal{C}_\mu(f^-).$$

This differs from the usual definition of Choquet integral for real-valued functions, sometimes called *asymmetric Choquet integral* [15], which is

$$\mathcal{C}_\mu(f) := \mathcal{C}_\mu(f^+) - \mathcal{C}_{\mu^*}(f^-)$$

where μ^* is the conjugate of μ (see Sect. 14.3.2). The Cumulative Prospect Theory model [120] generalizes these definitions, by considering different capacities for the positive and negative parts of the integrand.

$$\mathrm{CPT}_{\mu_1,\mu_2}(f) := \mathcal{C}_{\mu_1}(f^+) - \mathcal{C}_{\mu_2}(f^-).$$

14.4.2.2 Application to the Example

Let us go back to the example of Sect. 14.4.1. In this example, value 10 for the marks seems to be the appropriate neutral value. Hence, in order to transform the regular marks given in the interval [0, 20] to a ratio scale, it is enough to subtract 10 to each mark yielding the mark 10 to the zero level. This gives:

	Mathematics (M)	Statistics (S)	Languages (L)
Student E'	4	6	−3
Student F'	4	5	−2
Student G'	−1	6	−3
Student H'	−1	5	−2

Modeling our example with the Šipoš integral, a straightforward calculation shows that (14.25) is equivalent to $\mu(\{S\}) < \mu(\{L\})$ whereas relation (14.26) is equivalent to $\mu(\{S\}) > \mu(\{L\})$, which contradicts previous inequality. Hence, the Šipoš integral is not able to model both (14.25) and (14.26).

Trying now the representation of our example with the CPT model, it is easy to see that (14.25) is equivalent to $\mu_1(\{S\}) < \mu_2(\{L\})$, and relation (14.26) is equivalent to $\mu_1(\{S\}) > \mu_2(\{L\})$. Therefore, the CPT model too fails to model both (14.25) and (14.26).

14.4.3 Bi-capacities and the Corresponding Integral

The Choquet, Šipoš and CPT models are limited by the fact that they are constructed on the notion of capacity. The idea is thus to generalize the notion of capacity. Such generalizations have first been introduced in the context of game theory. The concept of ternary voting games has recently been defined by D. Felsenthal and M. Machover as a generalization of binary voting games [26]. Binary voting games model the result of a vote when some voters are in favor of the bill and the other voters are against [113]. The main limitation of such games is that they cannot represent decision rules in which *abstention* is an alternative option to the usual *yes* and *no* opinions. This leaded D. Felsenthal and M. Machover to introduce *ternary voting games* [26]. These voting games can be represented by a function v with two arguments, one for the *yes* voters and the other one for the *no* voters. This concept of ternary voting game has been generalized by Bilbao et al. in [6], yielding the definition of *bi-cooperative game*. Let

$$\mathcal{Q}(N) = \{(A, B) \in \mathcal{P}(N) \times \mathcal{P}(N) \mid A \cap B = \emptyset\}.$$

A bi-cooperative game is a function $\nu : \mathcal{Q}(N) \to \mathbb{R}$ satisfying $\nu(\emptyset, \emptyset) = 0$. In the context of game theory, the first argument A in $\nu(A, B)$ is called the *defender* part (*positive contributors*), and the second argument B in $\nu(A, B)$ is called the *defeater* part (*negative contributors*).

This generalization has recently been rediscovered independently by the authors in the context of MCDA [44, 80]. A bi-capacity is a function $\nu : \mathcal{Q}(N) \to \mathbb{R}$ satisfying

(BFM$_a$) $A \subseteq A' \Rightarrow \nu(A, B) \leq \nu(A', B)$,
(BFM$_b$) $B \subseteq B' \Rightarrow \nu(A, B) \geq \nu(A, B')$,
(BFM$_c$) $\nu(\emptyset, \emptyset) = 0$,
(BFM$_d$) $\nu(N, \emptyset) = 1$, $\nu(\emptyset, N) = -1$

Conditions **(BFM$_a$)** and **(BFM$_b$)** together define *monotonic* bi-capacities. Bi-capacities are special cases of bi-cooperative games. In MCDA, $\nu(A, B)$ is interpreted as the overall assessment of the ternary alternative $\left(1_A, -1_B, 0_{-(A\cup B)}\right)$. Thanks to that interpretation, the first argument A in $\nu(A, B)$ is called the *positive* part, and the second argument B in $\nu(A, B)$ is called the *negative* part.

The *conjugate* or *dual* ν^* of a bi-capacity ν can be defined by $\nu^*(S, T) = -\nu(T, S)$ for all $(S, T) \in \mathcal{Q}(N)$ [76, 78]. In the context of Game Theory, it means that the defenders and the defeaters are switched, and the abstentionists are untouched. This definition of dual bi-capacity coincides with that proposed in [26] for ternary voting games.

A bi-capacity ν is of the *CPT type* if it can be written $\nu(A, B) = \mu_1(A) - \mu_2(B)$, for all $(A, B) \in \mathcal{Q}(N)$, where μ_1, μ_2 are capacities. If $\mu_1 = \mu_2$, we say that the bi-capacity is *symmetric*. If μ_1 and μ_2 are additive, then ν is said to be *additive*.

A similar concept has also been introduced by S. Greco et al. leading to the concept of *bipolar capacity* [57]. A bipolar capacity is a function $\zeta : \mathcal{Q}(N) \to [0,1] \times [0,1]$ with $\zeta(A,B) =: \big(\zeta^+(A,B), \zeta^-(A,B)\big)$ such that

- If $A \supset A'$ and $B \subseteq B'$ then $\zeta^+(A,B) \geq \zeta^+(A',B')$ and $\zeta^-(A,B) \leq \zeta^-(A',B')$.
- $\zeta^-(A,\emptyset) = 0$, $\zeta^+(\emptyset,A) = 0$ for any $A \subseteq N$.
- $\zeta(N,\emptyset) = (1,0)$ and $\zeta(\emptyset,N) = (0,1)$.

$\zeta^+(A,B)$ can be interpreted as the importance of coalition A of criteria in the presence of B for the positive part. $\zeta^-(A,B)$ can be interpreted as the importance of coalition B of criteria in the presence of A for the negative part.

The Choquet integral w.r.t. a bi-capacity ν proposed in [44, 47] is now given. For any $a \in \mathbb{R}^n$,

$$\mathcal{BC}_\nu(a) := \mathcal{C}_{\mu_{N^+}}(|a|)$$

where $\mu_{N^+}(C) := \nu\left(C \cap N^+, C \cap N^-\right)$, $N^+ = \{i \in N \mid a_i \geq 0\}$, $N^- := N \setminus N^+$, and $|a|$ stands for $(|a_1|, \dots, |a_n|)$. Note that μ_{N^+} is a non-monotonic capacity.

The Choquet integral w.r.t. a bipolar capacity can also be defined [57]. For $a \in \mathbb{R}^n$, let τ be a permutation on N such that

$$\left|a_{\tau(1)}\right| \leq \cdots \leq \left|a_{\tau(n)}\right|. \tag{14.27}$$

Let

$$\begin{aligned} A_i^+ &:= \left\{\tau(j)\;,\; j \in \{i,\dots,n\} \;\text{ such that } a_{\tau(j)} \geq 0\right\} \\ A_i^- &:= \left\{\tau(j)\;,\; j \in \{i,\dots,n\} \;\text{ such that } a_{\tau(j)} < 0\right\} \end{aligned}$$

and

$$\begin{aligned} C^+(a;\zeta) &= \textstyle\sum_{i \in N} \left(a^+_{\tau(i)} - a^+_{\tau(i-1)}\right) \zeta^+\left(A_i^+, A_i^-\right) \\ C^-(a;\zeta) &= \textstyle\sum_{i \in N} \left(a^-_{\tau(i)} - a^-_{\tau(i-1)}\right) \zeta^-\left(A_i^+, A_i^-\right) \end{aligned}$$

where $a_{\tau(0)} := 0$ and for $a \in \mathbb{R}$ we set $a^+ = \max(a,0)$ and $a^- = (-a)^+$. Finally the Choquet integral w.r.t. ζ is defined by

$$C(a;\zeta) := C^+(a;\zeta) - C^-(a;\zeta).$$

For $a \in \mathbb{R}^n$ for which several permutations τ satisfy (14.27), it is easy to see that the previous expression depends on the choice of the permutation. This is not the case of the usual Choquet integral or the Choquet integral w.r.t. a bi-capacity. Enforcing that the results are the same for all permutations satisfying (14.27), we obtain the following constraints on the bipolar capacity:

$$\forall (A,B) \in \mathcal{Q}(N)\;,\;\; \zeta^+(A,B) - \zeta^-(\emptyset,B) = \zeta^+(A,\emptyset) - \zeta^-(A,B).$$

It can be shown then that the bipolar capacity ζ reduces exactly to a bi-capacity ν defined by Grabisch and Labreuche [47]

$$\nu(A,B) := \zeta^+(A,B) - \zeta^-(\emptyset,B).$$

One has indeed $\zeta^+(A,B) = \nu(A,B) - \nu(\emptyset,B)$ and $\zeta^-(A,B) = \nu(A,\emptyset) - \nu(A,B)$. Moreover, it can be shown that the Choquet integral w.r.t. ζ is equal to $\mathcal{BC}_\nu$. As a consequence, the concept of bipolar capacity reduces to bi-capacities when the Choquet integral is used. For this reason, we will consider only bi-capacities from now on. Note however that the concept of bipolar capacities has some interests in itself for other domains than MCDA.

The concept of bi-capacities is now applied to the example of Sect. 14.4.2.2.

Let us try to model (14.25) and (14.26) with the extension of the Choquet integral to bi-capacities. We have $\mathcal{BC}_\nu(4,6,-3) = \mathcal{C}_{\mu_{N^+}}(4,6,3) = 3\mu(\{\mathrm{M},\mathrm{S},\mathrm{L}\}) + \mu(\{\mathrm{M},\mathrm{S}\}) + 2\mu(\{\mathrm{S}\}) = 3\nu(\{\mathrm{M},\mathrm{S}\},\{\mathrm{L}\}) + \nu(\{\mathrm{M},\mathrm{S}\},\emptyset) + 2\nu(\{\mathrm{S}\},\emptyset)$ and $\mathcal{BC}_\nu(4,5,-2) = 2\nu(\{\mathrm{M},\mathrm{S}\},\{\mathrm{L}\}) + 2\nu(\{\mathrm{M},\mathrm{S}\},\emptyset) + \nu(\{\mathrm{S}\},\emptyset)$. Hence (14.25) is equivalent to

$$\nu(\{\mathrm{M},\mathrm{S}\},\emptyset) - \nu(\{\mathrm{M},\mathrm{S}\},\{\mathrm{L}\}) > \nu(\{\mathrm{S}\},\emptyset)$$

Similarly, relation (14.26) is equivalent to

$$\nu(\{\mathrm{S}\},\{\mathrm{L}\}) > 0.$$

There is no contradiction between these two inequalities. Therefore, $\mathcal{BC}_\nu$ is able to model the example. This aggregation operator models the expertise that makes an explicit reference to an absolute value.

14.4.4 Representation of the Motivating Example

We would like to stress that bi-capacities cannot account for *all* decision behaviours involving bipolar scales. To illustrate this, let us change the scores of E' and F' as follows.

	Mathematics (M)	Statistics (S)	Languages (L)
Student E''	2	6	−4
Student F''	2	5	−3

It is easy to check that maintaining $E'' \prec F''$ is equivalent to

$$\nu(\{\mathrm{S}\},\{\mathrm{L}\}) < 0,$$

a contradiction with $G' \succ H'$. The fact is that with E'', F'', the score on mathematics is now too weak with respect to the score on languages. Hence E'' should be preferred to F'' since the latter one is better in statistics. Hence the Choquet integral w.r.t. a bi-capacity fails to represent **(R1)** and **(R2)** in general [81].

One may seek for a more general model than bi-capacity able to represent rules **(R1)** and **(R2)**. Such a model, that can be described by an aggregation function $F : \mathbb{R}^n \to \mathbb{R}$, must be continuous and non-decreasing. It should also be piecewise linear as a natural generalization of the Choquet integral. As a matter of fact, there does not exist any aggregation function satisfying both the previous three conditions and rules **(R1)** and **(R2)** [81]. To formalize Axioms **(R1)** and **(R2)**, criteria Mathematics, Languages and Statistics are denoted by i, j^+ and j^- respectively. More precisely, the following result holds.

Proposition 2 ([81]). *Assume that* $F : \mathbb{R}^n \to \mathbb{R}$ *is continuous, non-decreasing and piecewise linear. Let* $\Phi^+ \subseteq \{f \in \mathbb{R}^n\ ,\ f_i \geq 0\}$, $\Phi^- \subseteq \{f \in \mathbb{R}^n\ ,\ f_i \leq 0\}$ *such that* F *is linear in* Φ^+ *and in* Φ^-. *If there exists a nonempty open set* $B \subseteq \mathbb{R}^2$ *and* $f_{-\{i,j^+,j^-\}} \in \mathbb{R}^{n-3}$ *such that*

$$\Phi^+ \cap \Phi^- \supseteq \{(0_i, g_{j^+}, g_{j^-}, f_{-\{i,j^+,j^-\}})\ ,\ \forall (g_{j^+}, g_{j^-}) \in B\}$$

then rules **(R1)** *and* **(R2)** *cannot be represented by* F *in the two domains* Φ^+ *and in* Φ^- *(i.e., criterion* j^+ *is more important than criterion* j^- *in* Φ^+*, and criterion* j^+ *is less important than criterion* j^- *in* Φ^-*).*

This proposition proves that if, for two neighbor domains Φ^+ and Φ^- such that the value of criterion i can be arbitrarily small independently of criteria j^+ and j^- in both Φ^+ and Φ^-, then rules **(R1)** and **(R2)** cannot be satisfied in both Φ^+ and Φ^-. This explains why bi-capacities cannot represent both $E'' \prec F''$ and $G' \succ H'$. In short, rules **(R1)** and **(R2)** cannot be satisfied if criterion i is the one closest to the neutral level among criteria i, j^+, j^-.

One cannot gain a lot by extending bi-capacities to more complex models. Actually, bi-capacities enable to represent the following rules.

(R1"): If the value w.r.t. criterion i is attractive (> 0), and i is not the closest to the neutral level among criteria i, j^+, j^-, then criterion j^+ is more important than criterion j^-.

(R2"): If the value w.r.t. criterion i is repulsive (< 0), and i is not the closest to the neutral level among criteria i, j^+, j^-, then criterion j^+ is less important than criterion j^-.

One can interpret this restriction in the following way. When criterion i has the closest value to the neutral level among criteria i, j^+ and j^-, the distinction between i being attractive or repulsive is not so meaningful to the DM and shall be removed from rules **(R1)** and **(R2)**.

14.4.5 General Method for Building Utility Functions

Let us now describe a general method to construct the utility functions u_i without the prior knowledge of F. It is possible to extend the method described in Sect. 14.3.3 in a straightforward way. Due to the existence of a neutral level, utility functions can now take positive and negative values. Hence assumption (14.11) is replaced by the following one:

$$\exists \alpha_i \in \mathbb{R}_+^* \ , \quad F\,(a_i, 0_{-i}) = \alpha_i\, a_i \ \text{ for all } a_i \in \mathbb{R}. \tag{14.28}$$

Then the utility function can be derived from (14.12). It has been shown in [54] that the Šipoš integral satisfies (14.28). However, this condition is too restrictive since the usual Choquet does not fulfill it [54]. As a consequence, we are looking for a more general method.

Since the neutral level has a central position, the idea is to process separately elements which are "above" the neutral level (attractive part), and "below" it (repulsive part). Doing so, we may avoid difficulties due to some asymmetry between attractive and repulsive parts [44, 80]. The positive part of the utility function of X_i will be based on the two absolute levels $\mathbb{1}_i$ and $\mathbb{O}_i$, while the negative part is based on the absolute levels $\mathbb{1}_i$ and $-\mathbb{O}_i$, as defined in Sect. 14.2.3.

Generalizing (14.5), we set

$$u_i(\mathbb{1}_i) = 0\ , \ u_i(\mathbb{O}_i) = 1 \ \text{ and } \ u_i(-\mathbb{O}_i) = -1. \tag{14.29}$$

The two values 1 and -1 are opposite to express the symmetry between $\mathbb{O}_i$ and $-\mathbb{O}_i$.

The construction of the positive and negative parts of the utility function u_i is performed through the MACBETH methodology from the following two sets $X\rfloor_i^+$ and $X\rfloor_i^-$:

$$X\rfloor_i^{\pm} = \left\{(x_i, \mathbb{1}_{-i})\ ,\ x_i \in X_i^{\pm}\right\},$$

where $X_i^+ = \{x_i \in X_i\ ,\ (x_i, \mathbb{1}_{-i}) \succeq \mathbb{1}_N\}$ and $X_i^- = \{x_i \in X_i\ ,\ (x_i, \mathbb{1}_{-i}) \preceq \mathbb{1}_N\}$. Interval scales $u_{X\rfloor_i^+}, u_{X\rfloor_i^-}$ are obtained for $i = 1, \ldots, n$, provided conditions **(Ord[$X\rfloor_i^+$])**, **(Inter[$X\rfloor_i^+$])**, **(C-Inter[$X\rfloor_i^+$])**, **(Ord[$X\rfloor_i^-$])**, **(Inter[$X\rfloor_i^-$])**, and **(C-Inter[$X\rfloor_i^-$])** are satisfied for $i = 1, \ldots, n$. Now the scales are uniquely determined if one applies (14.5) to all positive scales, and the symmetric condition

$$u_{X\rfloor_i^-}(\mathbb{1}_N) = 0 \ \text{ and } \ u_{X\rfloor_i^-}(-\mathbb{O}_i, \mathbb{1}_{-i}) = -1. \tag{14.30}$$

to all negative scales. Like for interval scales, one has for $x_i \in X_i^{\pm}$

$$u_{X\rfloor_i^{\pm}}\,(x_i, \mathbb{1}_{-i}) = F\,(u_i(x_i), 0_{-i})\,.$$

The assumption on the family $\mathcal{F}$ becomes

$$\exists \alpha_i^{\pm} \in \mathbb{R}_+^* \ , \quad F(a_i, 0_{-i}) = \alpha_i^{\pm}\, a_i \ \text{ for all } a_i \in \mathbb{R}_{\pm}. \tag{14.31}$$

Hence by (14.29), one has for any $x_i \in X_i^{\pm}$:

$$u_i(x_i) = \frac{F(u_i(x_i), 0_{-i})}{F(\pm 1_i, 0_{-i})} = \frac{u_{X\rfloor_i^{\pm}}(x_i, \mathbb{1}_{-i})}{u_{X\rfloor_i^{\pm}}(\pm \mathbb{O}_i, \mathbb{1}_{-i})}. \tag{14.32}$$

Hence, under assumption (14.31), the positive and negative parts of the utility functions can be constructed in two separate steps by (14.32) from cardinal information related to $X\rfloor_i^{\pm}$.

It can be shown that the Choquet integral, Šipoš integral, the CPT model and the generalized Choquet integral fulfills (14.31).

14.4.6 Justification of the Use of the Generalized Choquet Integral

14.4.6.1 Required Information

For any $i \in N$, the utility function u_i is built from $u_{X\rfloor_i^+}$ and $u_{X\rfloor_i^-}$ like in Sect. 14.4.5.

Inter-criteria information is a generalization of the set $X\rceil_{\{0,1\}}$. The three reference levels $-\mathbb{O}_i$, $\mathbb{1}_i$ and $\mathbb{O}_i$ are now used to build the set of *ternary alternatives*:

$$X\rceil_{\{-1,0,1\}} := \left\{ \left(\mathbb{O}_A, -\mathbb{O}_B, \mathbb{1}_{-(A\cup B)} \right) \ , \ (A, B) \in \mathcal{Q}(N) \right\}.$$

Let $u_{X\rceil_{\{-1,0,1\}}}$ be a numerical representation of $X\rceil_{\{-1,0,1\}}$. In the previous set, three special points can be exhibited: $\mathbb{O}_N$, $\mathbb{1}_N$ and $-\mathbb{O}_N$. Thanks to commensurability between the $\mathbb{O}_i$ levels, between the $\mathbb{1}_i$ levels and between the $-\mathbb{O}_i$ levels, it is natural to set

$$u_{X\rceil_{\{-1,0,1\}}}(-\mathbb{O}_N) = -1 \ , \ u_{X\rceil_{\{-1,0,1\}}}(\mathbb{1}_N) = 0 \ \text{ and } \ u_{X\rceil_{\{-1,0,1\}}}(\mathbb{O}_N) = 1. \tag{14.33}$$

Relation $u_{X\rceil_{\{-1,0,1\}}}(\mathbb{O}_N) = 1$ means that the alternative which is satisfactory on all attributes is also satisfactory. Relation $u_{X\rceil_{\{-1,0,1\}}}(\mathbb{1}_N) = 0$ means that the alternative which is neutral on all attributes is also neutral. Finally, relation $u_{X\rceil_{\{-1,0,1\}}}(-\mathbb{O}_N) = -1$ means that the alternative which is unsatisfactory on all attributes is also unsatisfactory. Since there are only two degrees of freedom in a scale of difference, one of these three points must be removed for the practical construction of the scale. We decide to remove the act $-\mathbb{O}_i$. Let $X\rceil_{\{-1,0,1\}}^* := X\rceil_{\{-1,0,1\}} \setminus \{-\mathbb{O}_N\}$.

The numerical representation $u_{X\rceil_{\{-1,0,1\}}^*}$ on $X\rceil_{\{-1,0,1\}}^*$ is ensured by **(Ord[**$X\rceil_{\{-1,0,1\}}^*$**])**, **(Inter[**$X\rceil_{\{-1,0,1\}}^*$**])**, **(C-Inter[**$X\rceil_{\{-1,0,1\}}^*$**])** and the last two conditions in (14.33). $u_{X\rceil^*_{\{-1,0,1\}}}$ is uniquely determined by previous requirements. In summary

$$u_{X\rceil_{\{-1,0,1\}}}\left(\mathbb{O}_A, -\mathbb{O}_B, \mathbb{1}_{-(A\cup B)}\right)$$
$$= \begin{cases} u_{X\rceil^*_{\{-1,0,1\}}}\left(\mathbb{O}_A, -\mathbb{O}_B, \mathbb{1}_{-(A\cup B)}\right) & \text{if } (A,B) \neq (\emptyset, N) \\ -1 & \text{otherwise} \end{cases}$$

14.4.6.2 Measurement Conditions

$u_{X\rceil_{\{-1,0,1\}}}$ can be described by a bi-capacity ν defined by: $\nu(A,B) := u_{X\rceil_{\{0,1\}}}(\mathbb{O}_A, -\mathbb{O}_B, \mathbb{1}_{-(A\cup B)})$. Consequently, it is natural to write u as follows:

$$u(x) = F_\nu\left(u_1(x_1), \ldots, u_n(x_n)\right), \tag{14.34}$$

where F_ν is the aggregation function.

We introduce the following axioms.

(Bi-LM): For any bi-capacities ν, ν' on $\mathcal{Q}(N)$ satisfying **(BFM$_c$)**, for all $x \in \mathbb{R}^n$ and $\gamma, \delta \in \mathbb{R}$,

$$F_{\gamma\nu+\delta\nu'}(x) = \gamma F_\nu(x) + \delta F_{\nu'}(x)$$

(Bi-In): For any bi-capacity ν on $\mathcal{Q}(N)$ satisfying **(BFM$_a$)**, **(BFM$_b$)** and **(BFM$_c$)**, $\forall x$, $x' \in \mathbb{R}^n$,

$$x_i \leq x'_i\,,\ \forall i \in N \ \Rightarrow F_\nu(x) \leq F_\nu(x')$$

(Bi-PW): For any bi-capacity ν satisfying **(BFM$_a$)**, **(BFM$_b$)**, **(BFM$_c$)**, $F_\nu\left(1_A, -1_{A'}, 0_{-A\cup A'}\right) = \nu(A, A')$, $\forall (A, A') \in \mathcal{Q}(N)$.

(Bi-weak SPL$^+$): For any bi-capacity v on $\mathcal{Q}(N)$ satisfying **(BFM$_a$)**, **(BFM$_b$)**, **(BFM$_c$)**, for all $A, C \subseteq N$, $\alpha > 0$, and $\beta \geq 0$,

$$F_\nu\left((\alpha+\beta)_A, \beta_{-A}\right) = \alpha F_v\left(1_A, 0_{-A}\right) + \beta v(N, \emptyset).$$

These axioms are basically deduced from the measurement conditions on $u_{X\rfloor^\pm_i}$ and ν. This is done exactly as in Sect. 14.3.4.1 [44, 80].

For $A \subseteq N$, consider the following application $\Pi_A : \mathbb{R}^n \to \mathbb{R}^n$ defined by $(\Pi_A(x))_i = x_i$ if $i \in A$ and $-x_i$ otherwise. By **(Bi-PW)**, $\nu(B, B')$ corresponds to the point $(1_B, -1_{B'}, 0_{(B\cup B')})$. Define $\Pi_A \circ \nu(B, B')$ as the term of the bi-capacity associated to the point $\Pi_A(1_B, -1_{B'}, 0_{-B\cup B'}) = (1_{(B\cap A)\cup(B'\setminus A)}, -1_{(B\setminus A)\cup(B'\cap A)}, 0_{-B\cup B'})$. Hence we set

$$\Pi_A \circ \nu(B, B') := \nu\left((B\cap A)\cup(B'\setminus A), (B\setminus A)\cup(B'\cap A)\right).$$

By symmetry arguments, it is reasonable to have $F_{\Pi_A \circ \nu}(\Pi_A(x))$ being equal to $F_\nu(x)$.

(Bi-Sym): For any $\nu : \mathcal{Q}(N) \to \mathbb{R}$ satisfying **(BFM$_c$)**, we have for all $A \subseteq N$

$$F_\nu(x) = F_{\Pi_A \circ \nu}(\Pi_A(x)) .$$

We have the following result.

Theorem 3 (Theorem 1 in [80]). *$\{F_\nu\}_\nu$ satisfies* **(Bi-LM)***,* **(Bi-In)***,* **(Bi-PW)***,* **(Bi-weak SPL$^+$)** *and* **(Bi-Sym)** *if and only if for any $\nu : \mathcal{Q}(N) \to \mathbb{R}$ satisfying* **(BFM$_a$)***,* **(BFM$_b$)***,* **(BFM$_c$)** *and* **(BFM$_d$)***, and for any $a \in \mathbb{R}^n$,*

$$F_\nu(a) = \mathcal{BC}_\nu(a).$$

The measurement conditions we have on u_i and $u_{X\rceil_{\{-1,0,1\}}}$ lead to axioms **(Bi-LM)**, **(Bi-In)**, **(Bi-PW)**, **(Bi-weak SPL$^+$)** and **(Bi-Sym)**. The Choquet integral w.r.t a bi-capacity ν is the only aggregation operator satisfying the previous set of axioms. So the generalized Choquet integral comes up very naturally when one works with information related to a bi-capacity.

14.4.7 Shapley Value and Interaction Index

As for capacities, due to the complexity of the bi-capacity model, involving 3^n coefficients, it is important in practice to be able to analyze a bi-capacity in terms of decision behaviours, namely importance of criteria and interaction among them.

We address first the importance index $\phi_i(v)$ for a bi-capacity v. Unlike capacities where we have previously seen that we come up quite easily to a unique definition, many definitions seem suitable for bi-capacities (see [7, 26, 46, 66, 79]). Note that the last two proposals are identical. Cooperative Game Theory is a good approach to select the most appropriate definition. In this setting, the bi-capacity v is interpreted as a bi-cooperative game. More precisely, in the context of cost sharing problems, $v(A,B)$ is the stand alone price of serving agents in $A \cup B$ when A have decided to contribute positively to the game and B have decided to contribute negatively to the game [82]. Unlike usual games, where at the end all players join the grand coalition, it is not assumed here that all players have decided to be positive contributors. We denote by S the set of players that have decided to be positive contributors, and by T the set of players that have decided to be negative contributors. The remaining players $N \setminus (S \cup T)$ have chosen not to participate to the game. As a result, the share of the total cost among the players depends on the bi-coalition (S,T). We denote by $\varphi_i^{S,T}(v)$ the payoff allotted to player i. This share is uniquely obtained by extending the requirements characterizing the Shapley value, and by adding a monotonicity requirement [82]

$$\varphi_i^{S,T}(v) = \sum_{K\subseteq(S\cup T)\setminus\{i\}} \frac{k!(s+t-k-1)!}{(s+t)!}$$
$$\times\, [v(S\cap(K\cup\{i\}), T\cap(K\cup\{i\})) - v(S\cap K, T\cap K)].$$

From this expression, the payoff for positive contributors (i.e., players in S) is non-negative, the payoff for negative contributors (i.e., players in T) is non-positive, and the payoff for the remaining players is zero. The idea is thus to define the importance of criterion i relatively to bi-coalition (S,T) as $\phi_i^{S,T}(v) = \left|\varphi_i^{S,T}(v)\right|$ in order to obtain non-negative values. One can then define the mean importance of criterion i as the average value of $\phi_i^{S,T}(v)$ over all bi-coalitions (S,T) such that $S\cup T = N$ [79]:

$$\phi_i(v) = \frac{1}{2^{n-1}} \sum_{S\subseteq N\,,\, i\in S} \phi_i^{S,N\setminus S}(v)$$
$$= \sum_{(A,B)\in\mathcal{Q}(N\setminus\{i\})} \frac{(a+b)!(n-a-b-1)!}{2^{a+b}\, n!} \times (v(A\cup\{i\},B) - v(A,B\cup\{i\}))$$

This value turns out to be exactly the average weight of criterion i in the bipolar Choquet integral [66, 79].

The interaction index $I_{ij}(v)$ can be obtained from the importance indices by using the recursive axiom of [48]:

$$I_{ij}(v) = \sum_{(A,B)\in\mathcal{Q}(N\setminus\{i\})} \frac{(a+b)!(n-a-b-2)!}{2^{a+b}\,(n-1)!} \times \left(\delta^{A,B}_{\{i,j\},\emptyset}(v) - \delta^{A,B}_{\emptyset,\{i,j\}}(v)\right)$$

where $\delta^{A,B}_{\{i,j\},\emptyset}(v) = v(A\cup\{i,j\},B) - v(A\cup\{i\},B) - v(A\cup\{j\},B) + v(A,B)$ and $\delta^{A,B}_{\emptyset,\{i,j\}}(v) = v(A,B\cup\{i,j\}) - v(A,B\cup\{i\}) - v(A,B\cup\{j\}) + v(A,B)$. The interaction index $I_{ij}(v)$ can be interpreted in terms of the variation of the mean weight of criterion i in the bipolar Choquet integral when criterion j varies [66].

14.4.8 Particular Models

As for capacities, one is interested in particular sub-models of bi-capacities to reduce the number of parameters. We only give references in this section.

The Möbius transform of bi-capacities has been defined in [46]. An alternative definition—called bipolar Möbius transform—has been proposed by Fujimoto and Murofushi [28] and Fujimoto et al. [29]. A linear transform relates these two proposals [28]. The concept of a k-additive bi-capacity is derived from the Möbius transform [46]. A similar definition can also be obtained from bipolar Möbius transform [29].

Let us now turn to another interesting particular model. As noted in Sect. 14.4.1, in most MCDA problems with sign-dependent decision strategies, the bipolar nature is not generally compulsory on all criteria. Let us denote by $P \subseteq N$ the set of criteria for which the DM's behavior is clearly of bipolar nature. In the example given in Sect. 14.4.1, P is reduced to criterion Mathematics. The approach proposed in [67] is to allow more degrees of freedom on the criteria P compared to the remaining criteria $N \setminus P$ that do not need bipolarity. This is done by enforcing some symmetry conditions on the criteria $N \setminus P$, which state that the interaction between positive and negative values vanishes for these criteria. Several sub-models can be constructed: partially CPT bi-capacities (ranging from usual bi-capacities to the CPT model), partially symmetric bi-capacities (ranging from usual bi-capacities to the Sipos integral), and partially asymmetric bi-capacities (ranging from usual bi-capacities to capacities) [67].

Finally, the concept of p-symmetry, as well as decomposable capacities, has also been generalized to bi-capacities [43, 95].

14.4.9 Identification of Bi-capacities

For $a \in \mathbb{R}^n$ fixed, the mapping $\nu \mapsto \mathcal{BC}_\nu(a)$ is linear. Therefore, the methods described in Sect. 14.3.7 for the determination of a capacity can be extended with no change to the case of bi-capacities. In particular, this enables the determination of ν with a quadratic method from a set of alternatives with the associated scores, and with a linear method from a set of comparisons between alternatives. The constraints on the bi-capacity are composed of conditions **(BFM$_a$)**, **(BFM$_b$)**, **(BFM$_c$)** and **(BFM$_d$)**.

However, we are faced here to another difficulty. A bi-capacity contains 3^n unknowns which makes its determination quite delicate. As an example, with 5 criteria, a capacity has $2^5 = 32$ coefficients whereas a bi-capacity holds $3^5 = 243$ coefficients. Ten well-chosen learning examples are generally enough to determine a capacity with 5 criteria. It would require maybe 80 learning examples to determine a bi-capacity with 5 criteria. This is obviously beyond what a human being could stand.

The way out to this problem is to reduce the complexity of the model. The first idea is to restrict to sub-classes of bi-capacities, such as the k-additive bi-capacities described above. For instance, there are $2\,n^2 - 3 = 47$ unknowns for a 2-additive bi-capacity with 5 criteria. Other approaches are also possible.

14.5 Ordinal Scales

14.5.1 Introduction

So far, we have supposed that the quantities we deal with (score, utilities, ...) are defined on some numerical scale, either an interval or a ratio scale, let us say a *cardinal scale*. In practical applications, most of the time it is not possible to have directly cardinal information, but merely ordinal information. The MACBETH methodology we presented in Sect. 14.2.3 is a well-founded means to produce cardinal information from ordinal information. In some situations, this method may not apply, the decision maker being not able to give the required amount of information or being not consistent. In such a case, there is nothing left but to use the ordinal information as such, coping with the poor structure behind ordinal scales. We try in this section to define a framework, point out the difficulties, give the known axiomatizations and indicate the main tools to handle ordinal scales in the viewpoint of capacities.

In the sequel, ordinal scales are denoted by L or similar, and are supposed to be finite totally ordered sets, with top and bottom denoted $\mathbb{1}$ and $\mathbb{O}$.

Since ordinal scales forbid the use of usual arithmetic operations (see Sect. 14.2.1), minimum ($\wedge$) and maximum ($\vee$) become the main operations. Hence, decision models are more or less limited to combinations of these operations. We call *Boolean polynomials* expressions $P(a_1, \ldots, a_n)$ involving n variables and coefficients valued in L, linked by $\wedge$ or $\vee$ in an arbitrary combination of parentheses, e.g. $((\alpha \wedge a_1) \vee (a_2 \wedge (\beta \vee a_3))) \wedge a_4$. An important result by Marichal [87] says that the *Sugeno integral* w.r.t. a capacity coincides with the class of Boolean polynomials such that $P(\mathbb{O}, \mathbb{O}, \ldots, \mathbb{O}) = \mathbb{O}$, $P(\mathbb{1}, \mathbb{1}, \ldots, \mathbb{1}) = \mathbb{1}$, and P is non-decreasing w.r.t. each variable. Since these conditions are natural in decision making, this shows that the Sugeno integral plays a central role when scales are ordinal, and the whole section is devoted to it.

Before entering into details, we wish to underline the fact that however, this is not the only way to deal with ordinal information. Roubens has proposed a methodology based on the Choquet integral (which has far better properties than the Sugeno integral, as we will show), where scores of an alternative on criteria are related to the number of times this alternative is better or worse than the others on the same criteria [109].

Let us begin by pinpointing fundamental difficulties linked to the ordinal context.

- **finiteness of scales:** sticking to a decomposable model of the type (14.2), the function F is now defined from L^n to L. Clearly it is impossible that F be strictly increasing due to the finiteness of L. A solution may be to map F on L', with $|L'| > |L|$. A systematic study of this point has been given in [40, 42], giving rise to algorithms building the scale L' and the aggregation function F, given a preference profile. On the other hand, most measurement theoretic results are based on a solvability condition and Archimedean axioms, which cannot hold on a finite set.
- **ordinal nature:** the Sugeno integral, even defined as a function from $\mathbb{R}^n$ to $\mathbb{R}$, can never be strictly increasing, and large domains of indifference exist. Hence, the decomposable model cannot satisfy weak separability (see Sect. 14.2.1). Specifically, Marichal [87] has shown that the Sugeno integral satisfies weak separability if and only if there is a dictator criterion. However, any Sugeno integral

induces a preference relation $\succeq$ which satisfies *directional weak separability*, defined by:

$$(x_i, z_{-i}) \succ (y_i, z_{-i}) \Rightarrow (x_i, z'_{-i}) \succeq (y_i, z'_{-i}), \quad \forall x, y, z, z' \in X.$$

This weaker condition ensures that no preference reversal occurs.

- **construction of utility functions** u_i**:** since on ordinal scales arithmetical operations are not permitted, the method described in Sects. 14.3.3 and 14.4.5 cannot be applied directly. The ordinal counterpart of the multiplication being the minimum operator ($\wedge$), Eq. (14.11) becomes:

$$F(a_i, \mathbb{O}_{-i}) = \alpha_i \wedge a_i.$$

The term α_i acts as a saturation level, hiding all utilities a_i larger than α_i. Hence relation (14.11) cannot be satisfied and the previous method cannot be applied to build the utility functions. As a consequence, to avoid this problem most of works done in this area suppose that the attributes are defined on a common scale L, although this is not in general a realistic assumption.

However, Grabisch et al. [52] have proposed a practical method to build a model based on the symmetric Sugeno integral (see Sect. 14.5.3) from a preference profile, which is able to build the utility functions and the capacity. Also, Greco et al. [58] have shown from a theoretical standpoint that it is possible to build utility functions in a Sugeno integral model (see Sect. 14.5.4).

14.5.2 The Sugeno Integral

We consider a capacity μ on N taking its value in L, with $\mu(\emptyset) = \mathbb{O}$ and $\mu(N) = \mathbb{1}$. Let $a := (a_1, \ldots, a_n)$ be a vector of scores in L^n. The *Sugeno integral* of a w.r.t. μ is defined by Sugeno [118]:

$$\mathcal{S}_\mu(a) := \bigvee_{i=1}^{n} [a_{\tau(i)} \wedge \mu(A_{\tau(i)})], \tag{14.35}$$

where τ is a permutation on N so that $a_{\tau(1)} \leq a_{\tau(2)} \leq \cdots \leq a_{\tau(n)}$, and $A_{\tau(i)} := \{\tau(i), \ldots, \tau(n)\}$. One can notice the similarity with the Choquet integral. Taking $L = [0, 1]$, Choquet and Sugeno integrals coincide when either the capacity or the integrand is 0-1 valued, specifically:

$$\mathcal{S}_\mu(1_A, 0_{-A}) = \mu(A) = \mathcal{C}_\mu(1_A, 0_{-A}), \quad \forall A \subseteq N$$
$$\mathcal{S}_\mu(a) = \mathcal{C}_\mu(a) \quad \forall a \in [0, 1]^n \text{ iff } \mu(A) \in \{0, 1\} \quad \forall A \subseteq N.$$

We refer the reader to survey papers [22, 103] and to [86, 87] for properties of the Sugeno integral, especially in a decision making perspective. We mention that in the context of decision under uncertainty, an axiomatic construction similar to the one of Savage has been done by Dubois et al. [21, 23].

As said in the introduction, making decision with the Sugeno integral has some drawbacks, which are clearly put into light with the following results [86, 101]. Let $\succeq$ be a weak order (complete, reflexive, transitive) on $[0,1]^n$, and for $a, b \in [0,1]^n$, denote $a \geq b$ if $a_i \geq b_i$ for all $i \in N$, and $a > b$ if $a \geq b$ and $a_i > b_i$ for some $i \in N$, and $a \gg b$ if $a_i > b_i$ for all $i \in N$. We say that $\succeq$ satisfies *monotonicity* if $a \geq b$ implies $a \succeq b$, the *strong Pareto condition* if $a > b$ implies $a \succ b$, and the *weak Pareto condition* if $a \gg b$ implies $a \succ b$. Then the following holds.

Proposition 3. *Let μ be a capacity on N, and $\succeq_\mu$ the weak order induced by the Sugeno integral $\mathcal{S}_\mu$.*

(i) *$\succeq_\mu$ always satisfies monotonicity.*
(ii) *$\succeq_\mu$ satisfies the weak Pareto condition iff μ is 0-1 valued.*
(iii) *$\succeq_\mu$ never satisfies the strong Pareto condition.*

Note that the Choquet integral always satisfies the weak Pareto condition, and the strong one iff μ is strictly monotone.

Since arithmetic operations cannot be used with ordinal scales, our definitions of importance and interaction indices cannot work, and alternatives must be sought. Grabisch [34] has proposed definitions which more or less keep mathematical properties of the original Shapley value and interaction index. However, these indices, especially the interaction index, do not seem to convey the meaning they are supposed to have.

14.5.3 Symmetric Ordinal Scales and the Symmetric Sugeno Integral

This section introduces *bipolar ordinal scales*, i.e. ordinal scales with a central neutral level, and a symmetry around it, and is based on [36, 38, 39]. The aim is to have a structure similar to cardinal bipolar scales, so as to build a counterpart of the CPT model, using a Sugeno integral for the "positive" part (above the neutral level), and another one for the "negative" part (below the neutral level):

$$\mathrm{OCPT}_{\mu_1,\mu_2}(a) := \mathcal{S}_{\mu_1}(a^+) \ominus \mathcal{S}_{\mu_2}(a^-)$$

("O" stands for "ordinal") where $a^+ := a \vee 0$, $a^- := (-a)^+$, and $\ominus$ is a suitable difference operator. We will show that this task is not easy.

Let us call L^+ some ordinal scale, and define $L := L^+ \cup L^-$, where L^- is a reversed copy of L^+, i.e. for any $a, b \in L^+$, we have $a \leq b$ iff $-b \leq -a$, where

$-a, -b$ are the copies of a, b in L^-. We want to endow L with operations $\varovee, \varowedge$ satisfying (among possible other conditions):

(**C1**) $\varovee, \varowedge$ coincide with $\vee, \wedge$ respectively on L^+
(**C2**) $-a$ is the symmetric of a, i.e. $a \varovee (-a) = \mathbb{O}$.

Hence we may extend to L what exists on L^+ (e.g. the Sugeno integral), and a difference operation could be defined. The problem is that conditions (**C1**) and (**C2**) imply that $\varovee$ would be non-associative in general. Take $\mathbb{O} < a < b$ and consider the expression $(-b) \varovee b \varovee a$. Depending on the place of parentheses, the result differs since $((-b) \varovee b) \varovee a = \mathbb{O} \varovee a = a$, but $(-b) \varovee (b \varovee a) = (-b) \varovee b = \mathbb{O}$.

It can be shown that the best solution (i.e. associative on the largest domain) for $\varovee$ is given by:

$$a \varovee b := \begin{cases} -(|a| \vee |b|) & \text{if } b \neq -a \text{ and } |a| \vee |b| = -a \text{ or } = -b \\ \mathbb{O} & \text{if } b = -a \\ |a| \vee |b| & \text{else.} \end{cases} \quad (14.36)$$

Except for the case $b = -a$, $a \varovee b$ equals the absolutely larger one of the two elements a and b.

The extension of $\wedge$, viewed as the counterpart of multiplication, is simply done on the principle that the rule of sign should hold: $-(a \varowedge b) = (-a) \varowedge b$, $\forall a, b \in L$. It leads to an associative operator, defined by:

$$a \varowedge b := \begin{cases} -(|a| \wedge |b|) & \text{if sign}\, a \neq \text{sign}\, b \\ |a| \wedge |b| & \text{else.} \end{cases} \quad (14.37)$$

Based on these definitions, the OCPT model writes:

$$\text{OCPT}_{\mu_1,\mu_2}(a) := \mathcal{S}_{\mu_1}(a^+) \varovee (-\mathcal{S}_{\mu_2}(a^-)).$$

When $\mu_1 = \mu_2 =: \mu$, we get the *symmetric Sugeno integral*, denoted $\check{\mathcal{S}}_\mu$.

Going a step further, it is possible to define the Sugeno integral w.r.t. bi-capacities, following the same way as with the Choquet integral. One can show that, defining $\mathcal{BS}_\nu(a) := \mathcal{S}_{\mu_{N^+}}(|a|)$, with same notations as in Sect. 14.4.3 and replacing in the definition of Sugeno integral $\vee, \wedge$ by $\varovee, \varowedge$, the expression is [45] (see also Greco et al. [57] for a similar definition):

$$\mathcal{S}_\nu(a) = \langle \mathop{\varovee}_{i=1}^{n} \Big[|a_{\tau(i)}| \varowedge \nu(A_{\tau(i)} \cap N^+, A_{\tau(i)} \cap N^-) \Big] \rangle, \quad (14.38)$$

where τ is a permutation on N so that $|a_{\tau(1)}| \leq \cdots \leq |a_{\tau(n)}|$, $N^+ := \{i \in N \mid a_i \geq \mathbb{O}\}$, $N^- := N \setminus N^+$, and the expression $\langle \mathop{\varovee}_{i=1}^{n} b_i \rangle$ is a shorthand for $(\mathop{\varovee}_{i=1}^{n} b_i^+) \varovee (-\mathop{\varovee}_{i=1}^{n} b_i^-)$. It can be shown that if ν is of the CPT type, one recovers the OCPT model.

Lastly, we mention Denneberg and Grabisch, who have proposed a general formulation of the Sugeno integral on arbitrary bipolar spaces [18].

14.5.4 A Model of Decision Based on the Sugeno Integral

We present in this section an axiomatization of a decision model based on the Sugeno integral, obtained independently by Bouyssou and Marchant [8, 9], and by Słowiński et al. [116] (see also Greco et al. [58]), the latter work giving also a connection with decision based rules, while the former gives a connection with noncompensatory sorting models.

We partition X into nonempty r categories $C^1, \dots, C^r$, each category containing alternatives which are considered as indifferent for the decision maker, ranked in such a way that the desirability of a category increases with its label, i.e., C^1 contains the less preferred objects, while C^r contains the most preferred ones. We introduce the notation $C^{\geq k} = \bigcup_{i=k}^{r} C^k$.

We say that the partition $\{C^1, \dots, C^r\}$ is representable by a Sugeno integral if there exist utility functions $u_i : X_i \to \mathbb{R}_+$, $\forall i \in N$, real numbers $0 < \rho_1 < \cdots < \rho_r$, and a capacity μ (not necessarily normalized) such that, for all $x \in X$,

$$x \in C^{\geq k} \Leftrightarrow \mathcal{S}_\mu(u_1(x_1), \dots, u_n(x_n)) > \rho_k.$$

Theorem 4 ([9, 116]). *A partition $\{C^1, \dots, C^r\}$ is representable by a Sugeno integral if and only if it satisfies for all $i \in N$*

$$\left.\begin{matrix} (x_i, a_{-i}) \in C^{\geq k} \\ \textit{and} \\ (y_i, b_{-i}) \in C^{\geq \ell} \end{matrix}\right\} \Rightarrow \left\{\begin{matrix} (x_i, b_{-i}) \in C^{\geq \ell} \\ \textit{or} \\ (z_i, a_{-i}) \in C^{\geq k} \end{matrix}\right.$$

for all $k, \ell \in \{2, \dots, r\}$ with $\ell \leq k$, all $x_i, y_i, z_i \in X_i$, and all $a_{-i}, b_{-i} \in X_{-i}$.

Bouyssou and Marchant proved that the above Sugeno integral model is equivalent to a noncompensatory model (i.e., a partition of X is representable by a Sugeno integral if and only if it is representable by a noncompensatory model). We say that a partition $\{C^1, \dots, C^r\}$ is representable by a noncompensatory model if there exist subsets $A_i^r \subseteq \cdots \subseteq A_i^2 \subseteq X_i$ for all $i \in N$, upwards collections of subsets (i.e., collections $\mathcal{F}$ satisfying the property: if $A \in \mathcal{F}$ and $A \subseteq B$, then $B \in \mathcal{F}$) $\mathcal{F}^r \subseteq \cdots \subseteq \mathcal{F}^2 \subseteq 2^N$, such that

$$x \in C^{\geq k} \Leftrightarrow \{i \in N \mid x_i \in A_i^k\} \in \mathcal{F}^k.$$

The interpretation of the model goes as follows: A_i^k contains all elements of X_i which are judged "satisfactory at the level k". In order for an alternative to belong to a category at least k, it is necessary that its evaluations are judged satisfactory at level k on a subset of attributes which is judged sufficiently important for level k (as indicated by $\mathcal{F}^k$).

Lastly, we mention that Greco, Słowiński et al. [56, 116] proved that the Sugeno integral model can be equivalently represented by a set of particular decision rules of the following form:

$$\text{If } u_{i_1}(x_{i_1}) \geq \ell \text{ and } \cdots \text{ and } u_{i_q}(x_{i_q}) \geq \ell, \text{ then } x \in C^{\geq \ell}$$

with $\{i_1, \ldots, i_q\} \subseteq N$, $\ell \in \{2, \ldots, r\}$, the u_i's are utility functions as before, and the rules must satisfy the two following properties:

- For each $x \in C^\ell$, there exists a rule implying that $x \in C^{\geq \ell}$ and no rule implying $x \in C^{\geq k}$ with $k > \ell$;
- If a rule with level ℓ is satisfied, the same rules with levels $k < \ell$ must also be satisfied.

This result shows that a decision model by a set of rules can be more general than a Sugeno integral model.

14.5.5 The Lexicographic Sugeno Integral

In the whole section, we consider that utility functions are the identity function, i.e., we compare directly scores of alternatives, denoted by the vectors $f, g, \ldots$ on L^n.

The weaknesses of the Sugeno integral, as pointed out in Proposition 3, can be overcome if one uses a lexicographic approach. Indeed, it is well known that the lexicographic approach can refine preorders produced by some aggregation function, like the minimum. We elaborate on this point. For two vectors f, g, the *lexicographic order* is defined as follows:

$$f \leq^l g \Leftrightarrow [f = g \text{ or } f_j < g_j, j = \min\{i \mid f_i \neq g_i\}].$$

Next, the *leximin* and *leximax* [100] are defined as follows:

$$f \preceq_{\text{lmin}} g \Leftrightarrow (f_{(1)}, \ldots, f_{(n)}) \leq^l (g_{(1)}, \ldots, g_{(n)})$$
$$f \preceq_{\text{lmax}} g \Leftrightarrow (f_{(n)}, \ldots, f_{(1)}) \leq^l (g_{(n)}, \ldots, g_{(1)}),$$

with $f_{(1)} \leq f_{(2)} \leq \cdots \leq f_{(n)}$, and similarly for g. The leximin and leximax satisfy the strong Pareto condition, and hence the weak Pareto condition and monotonicity. Moreover, f and g are indifferent for the leximin and leximax if and only if the reordered vectors are identical. Hence, f is indifferent to any of its permuted versions. More importantly, the leximim (resp., leximax) refines the minimum (resp., maximum), in the sense that it has less pairs of indifferent alternatives and agrees with the strict order given by the minimum (resp., maximum).

We present now a refinement of the order induced by the Sugeno integral, due to Dubois and Fargier [19] and Fargier and Sabbadin [25]. For some positive integer m, let us consider elements of $(L^m)^n$, where L is a finite scale, i.e., vectors whose components are themselves vectors of L^m (in other words, these are $n \times m$ matrices). We define on them the *maxmin order relation* as follows. For any $u, v \in (L^m)^n$

$$u \preceq_{\text{maxmin}} v \Leftrightarrow \max_{i=1}^{n} \min_{j=1}^{m} u_{ij} \leq \max_{i=1}^{n} \min_{j=1}^{m} v_{ij}.$$

It is a complete preorder, and we can define similarly $\preceq_{\text{minmax}}$. Let us consider now a complete preorder $\trianglerighteq$ on vectors of L^m. It is then possible to apply the definitions of leximin and leximax to matrices of $(L^m)^n$, since rows can be rearranged in increasing or decreasing order. We denote these complete preorders on $(L^m)^n$ by $\preceq_{\text{lmin}(\trianglerighteq)}$ and $\preceq_{\text{lmax}(\trianglerighteq)}$. Then $\preceq_{\text{lmax}(\preceq_{\text{lmin}})}$ is a refinement of $\preceq_{\text{maxmin}}$, and $\preceq_{\text{lmin}(\preceq_{\text{lmax}})}$ is a refinement of $\preceq_{\text{minmax}}$.

Two matrices u, v of $(L^m)^n$ are indifferent by $\preceq_{\text{lmax}(\preceq_{\text{lmin}})}$ or $\preceq_{\text{lmin}(\preceq_{\text{lmax}})}$ if and only if the rows of u are those of v, up to a permutation of the rows, and up to a permutation of the elements in each row. For example, with $n = 4$ and $m = 3$:

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 5 & 2 \\ 2 & 3 & 4 \\ 2 & 5 & 4 \end{bmatrix} \qquad \begin{bmatrix} 4 & 3 & 2 \\ 2 & 4 & 5 \\ 2 & 3 & 1 \\ 2 & 5 & 1 \end{bmatrix}$$

these two matrices are indifferent for $\preceq_{\text{lmax}(\preceq_{\text{lmin}})}$ and $\preceq_{\text{lmin}(\preceq_{\text{lmax}})}$.

There is a particular case of interest. Let us take $m = 2$, and consider a capacity μ. Then the Sugeno integral can be expressed as a maxmin order relation as follows: defining the matrix $M_{f,\mu} := ((\lambda, \mu(\{i \mid f_i \geq \lambda\})_{\lambda \in L})$, it is easy to check that

$$\mathcal{S}_\mu(f) \leq \mathcal{S}_\mu(g) \Leftrightarrow M_{f,\mu} \preceq_{\text{maxmin}} M_{g,\mu}.$$

Hence, by previous results, $\preceq_{\text{lmax}(\preceq_{\text{lmin}})}$ is a lexicographic refinement of the Sugeno integral.

For a given capacity μ, two acts are indifferent if and only if the matrices are the same up to permutations as explained above. This amounts to say that the decumulative functions $\mu(f \geq \lambda), \mu(g \geq \lambda)$ are identical.

It is known from Moulin [100] that the leximin and leximax can be coded by a sum, provided that acts are defined on some finite scale $L := \{l_0, l_1, \ldots, l_m\}$. For example, the leximax is recovered by performing the transformation $\phi(l_k) := (n+1)^k$ for any $l_k \in L$. Then $f \preceq_{\text{lmax}} g$ if and only if $\sum_{i=1}^{n} \phi(f_i) \leq \sum_{i=1}^{n} \phi(g_i)$. This is because $n\phi(l_k) < \phi(l_{k+1})$, for all k. Using this procedure, it is shown in [19] that the $\preceq_{\text{lmax}(\preceq_{\text{lmin}})}$ ordering amounts to compare acts by a Choquet integral w.r.t. μ (up to a transformation of the scale). This proves that $\preceq_{\text{lmax}(\preceq_{\text{lmin}})}$ always satisfies monotonicity and the weak Pareto condition, and satisfies the strong Pareto condition if and only if μ is strictly monotone.

Another approach has been proposed by Murofushi [101], see also a discussion and a comparison of various approaches in [41].

14.5.6 Identification of Capacities

In situations where utility functions are known, the problem of the identification of capacities when the model is a Sugeno integral (or OCPT, bipolar Sugeno integral) in an ordinal context, or even when $L = [0, 1]$ or $[-1, 1]$, appears to be rather different from the case of the Choquet integral. The main reason is that we are not able to write the identification problem as a minimization problem *stricto sensu* (see Sect. 14.3.7), since the notion of difference between values, hence of error, is not defined in a way which is suitable on an ordinal scale, to say nothing about "squared errors" and "average values".

Even if we take L as a real interval, which permits to define a squared error criterion as for the Choquet integral, the minimization problem obtained is not easy to solve, since it involves non-linear, non-differentiable operations $\vee, \wedge, \varovee, \varowedge$. In such cases, only meta-heuristic methods can be used, as genetic algorithms, simulated annealing, etc. There exist some works in this direction, although most of the time used for the Choquet integral, which is questionable [37, 123].

What can be done without error criterion to minimize? The second option, also used for the Choquet integral (see Sect. 14.3.7), is to find capacities which enable the representation of the preference of the DM over a set of alternatives of interest by the Sugeno integral (or OCPT,...). A detailed study of this problem has been done by Rico et al. [106] for the Sugeno integral. We mention also the work of Greco et al. based on decision rules, which can be found in Chap. 12 of this book (see also [56]).

14.6 Concluding Remarks

This chapter has tried to give a unified presentation of MCDA methods based on fuzzy integrals. It has shown that the concepts of capacity and bi-capacity naturally arise as overall utility of binary and ternary alternatives, and that the Choquet integral appears to be the unique solution for aggregating criteria, under a set of natural axioms.

This methodology has been applied in various fields of MCDA from a long time, particularly in subjective evaluation, and seems to receive more and more attention. Following the pioneering works of Sugeno [118], many researchers in the eighties in Japan have applied in practical problems the Sugeno integral, for example to opinion poll [104], and later the Choquet integral (see a summary of main works in [49]). More recent applications can be found in [37, 51], see also [53, 89, 105].

References

1. Angilella, S., Greco, S., Matarazzo, B.: Non-additive robust ordinal regression: a multiple criteria decision model based on the Choquet integral. Eur. J. Oper. Res. **201**, 277–288 (2010)
2. Bana e Costa, C.A., Vansnick, J.-C.: A theoretical framework for Measuring Attractiveness by a Categorical Based Evaluation TecHnique (MACBETH). In: Proceedings of XIth International Conference on MultiCriteria Decision Making, Coimbra, August 1994, pp. 15–24
3. Bana e Costa, C.A., Vansnick, J.-C.: Applications of the MACBETH approach in the framework of an additive aggregation model. J. Multicrit. Decis. Anal. **6**, 107–114 (1997)
4. Bana e Costa, C.A., Vansnick, J.-C.: The MACBETH approach: basic ideas, software and an application. In: Meskens, N., Roubens, M. (eds.) Advances in Decision Analysis, pp. 131–157. Kluwer, Dordrecht (1999)
5. Benvenuti, P., Mesiar, R., Vivona, D.: Monotone set functions-based integrals. In: Pap, E. (ed.) Handbook of Measure Theory, pp. 1329–1379. Elsevier Science, Amsterdam (2002)
6. Bilbao, J.M., Fernandez, J.R., Jiménez Losada, A., Lebrón, E.: Bicooperative games. In: Bilbao, J.M. (ed.) Cooperative Games on Combinatorial Structures. Kluwer, Boston (2000)
7. Bilbao, J.M., Fernández, J.R., Jiménez, N., López, J.J.: The Shapley value for bi-cooperative games. Ann. Oper. Res. **158**, 99–115 (2008)
8. Bouyssou, D., Marchant, T.: An axiomatic approach to noncompensatory sorting methods in MCDM I: the case of two categories. Eur. J. Oper. Res. **178**(1), 217–245 (2007)
9. Bouyssou, D., Marchant, Th.: An axiomatic approach to noncompensatory scoring methods in MCDM, II: more than two categories. Eur. J. Oper. Res. **178**, 246–276 (2007)
10. Carenini, G., Moore, J.D.: Generating and evaluating evaluative arguments. Artif. Intell. **170**, 925–952 (2006)
11. Choquet, G.: Theory of capacities. Annales de l'Institut Fourier **5**, 131–295 (1953)
12. Combarro, E.F., Miranda, P.: Identification of fuzzy measures from sample data with genetic algorithms. Comput. Oper. Res. **33**, 3046–3066 (2006)
13. Couceiro, M., Marichal, J.L.: Axiomatizations of Lovász extensions of pseudo-Boolean functions. Fuzzy Sets Syst. **181**, 28–38 (2011)
14. Couceiro, M., Marichal, J.L.: Axiomatizations of quasi-Lovász extensions of pseudo-Boolean functions. Aequationes Math. **82**, 213–231 (2011)
15. Denneberg, D.: Non-additive Measure and Integral. Kluwer, Dordrecht (1994)
16. Denneberg, D.: Non-additive measure and integral, basic concepts and their role for applications. In: Grabisch, M., Murofushi, T., Sugeno, M. (eds.) Fuzzy Measures and Integrals—Theory and Applications, pp. 42–69. Physica Verlag, Heidelberg (2000)
17. Denneberg, D., Grabisch, M.: Interaction transform of set functions over a finite set. Inf. Sci. **121**, 149–170 (1999)
18. Denneberg, D., Grabisch, M.: Measure and integral with purely ordinal scales. J. Math. Psychol. **48**, 15–27 (2004)
19. Dubois, D., Fargier, H.: Bridging the gap between discrete Sugeno and Choquet integrals. Presented at RUD'2005 (workshop on Risk, Utility and Decision), Heidelberg, 22–24 June 2005
20. Dubois, D., Prade, H.: A class of fuzzy measures based on triangular norms. Int. J. Gen. Syst. **8**, 43–61 (1982)
21. Dubois, D., Prade, H., Sabbadin, R.: Qualitative decision theory with Sugeno integrals. In: Grabisch, M., Murofushi, T., Sugeno, M. (eds.) Fuzzy Measures and Integrals—Theory and Applications, pp. 314–332. Physica Verlag, Heidelberg (2000)
22. Dubois, D., Marichal, J.-L., Prade, H., Roubens, M., Sabbadin, R.: The use of the discrete Sugeno integral in decision making: a survey. Int. J. Uncertainty Fuzziness Knowledge Based Syst. **9**(5), 539–561 (2001)
23. Dubois, D., Prade, H., Sabbadin, R.: Decision-theoretic foundations of qualitative possibility theory. Eur. J. Oper. Res. **128**, 459–478 (2001)

24. Dyer, J.S., Sarin, R.K.: Measurable multiattribute value functions. Oper. Res. **27**, 810–822 (1979)
25. Fargier, H., Sabbadin, R.: Qualitative decision under uncertainty: back to expected utility. Artif. Intell. **164**, 245–280 (2005)
26. Felsenthal, D., Machover, M.: Ternary voting games. Int. J. Game Theory **26**, 335–351 (1997)
27. Fodor, J., Roubens, M.: Fuzzy Preference Modelling and Multi-Criteria Decision Aid. Kluwer, Boston (1994)
28. Fujimoto, K., Murofushi, T.: Some characterizations of k-monotonicity through the bipolar Möbius transform in bi-capacities. J. Adv. Comput. Intell. Intell. Inf. **9**(5), 484–495 (2005)
29. Fujimoto, K., Murofushi, T., Sugeno, M.: k-additivity and $\mathcal{C}$-decomposability of bi-capacities and its integral. Fuzzy Sets Syst. **158**, 1698–1712 (2007)
30. Grabisch, M.: A new algorithm for identifying fuzzy measures and its application to pattern recognition. In: International Joint Conference of the 4th IEEE International Conference on Fuzzy Systems and the 2nd International Fuzzy Engineering Symposium, Yokohama, March 1995, pp. 145–150
31. Grabisch, M.: The application of fuzzy integrals in multicriteria decision making. Eur. J. Oper. Res. **89**, 445–456 (1996)
32. Grabisch, M.: Alternative representations of discrete fuzzy measures for decision making. Int. J. Uncertainty Fuzziness Knowledge Based Syst. **5**, 587–607 (1997)
33. Grabisch, M.: k-Order additive discrete fuzzy measures and their representation. Fuzzy Sets Syst. **92**, 167–189 (1997)
34. Grabisch, M.: On the representation of k-decomposable measures. In: 7th IFSA World Congress, Prague, June 1997, pp. 478–483
35. Grabisch, M.: A graphical interpretation of the Choquet integral. IEEE Trans. Fuzzy Syst. **8**, 627–631 (2000)
36. Grabisch, M.: The Möbius function on symmetric ordered structures and its application to capacities and integrals. In: 9th International Conference on Information Proc. and Management of Uncertainty in Knowledge-Based systems (IPMU'2002), Annecy, July 2002, pp. 755–762
37. Grabisch, M.: Modelling data by the Choquet integral. In: Torra, V. (ed.) Information Fusion in Data Mining, pp. 135–148. Physica Verlag, Heidelberg (2003)
38. Grabisch, M.: The symmetric Sugeno integral. Fuzzy Sets Syst. **139**, 473–490 (2003)
39. Grabisch, M.: The Möbius function on symmetric ordered structures and its application to capacities on finite sets. Discret. Math. **287**(1–3), 17–34 (2004)
40. Grabisch, M.: Representation of preferences over a finite scale by a mean operator. Math. Soc. Sci. **52**, 131–151 (2006)
41. Grabisch, M.: Some lexicographic approaches to the Sugeno integral. In: Proceedings of the International Conference on Information Processing and Management of Uncertainty (IPMU'06), Paris, July 2006, pp. 572–579
42. Grabisch, M.: How to score alternatives when criteria are scored on an ordinal scale. J. Multi-Criteria Decis. Anal. **15**, 31–44 (2008)
43. Grabisch, M., Labreuche, Ch.: Bi-capacities. In: Joint International Conference on Soft Computing and Intelligent Systems and 3d International Symposium on Advanced Intelligent Systems, Tsukuba, October 2002
44. Grabisch, M., Labreuche, Ch.: Bi-capacities for decision making on bipolar scales. In: EUROFUSE Workshop on Informations Systems, Varenna, September 2002, pp. 185–190
45. Grabisch, M., Labreuche, Ch.: Bi-capacities. Technical Report 2003/002, LIP6, UPMC. http://www.lip6.fr/reports/lip6.2003.002.html (2003)
46. Grabisch, M., Labreuche, Ch.: Bi-capacities—I: definitions, Möbius transform and interaction. Fuzzy Sets Syst. **151**, 211–236 (2005)
47. Grabisch, M., Labreuche, Ch.: Bi-capacities—II: the Choquet integral. Fuzzy Sets Syst. **151**, 237–259 (2005)
48. Grabisch, M., Roubens, M.: An axiomatic approach to the concept of interaction among players in cooperative games. Int. J. Game Theory **28**, 547–565 (1999)

49. Grabisch, M., Nguyen, H.T., Walker, E.A.: Fundamentals of Uncertainty Calculi, with Applications to Fuzzy Inference. Kluwer, Dordrecht (1995)
50. Grabisch, M., Marichal, J.-L., Roubens, M.: Equivalent representations of set functions. Math. Oper. Res. **25**(2), 157–178 (2000)
51. Grabisch, M., Murofushi, T., Sugeno, M.: Fuzzy Measures and Integrals. Theory and Applications (edited volume). Studies in Fuzziness. Physica Verlag, Heidelberg (2000)
52. Grabisch, M., Dia, S., Labreuche, Ch.: A multicriteria decision making framework in ordinal context based on Sugeno integral. In: Joint 9th IFSA World Congress and 20th NAFIPS International Conference, Vancouver, July 2001
53. Grabisch, M., Duchêne, J., Lino, F., Perny, P.: Subjective evaluation of discomfort in sitting position. Fuzzy Optim. Decis. Mak. **1**(3), 287–312 (2002)
54. Grabisch, M., Labreuche, Ch., Vansnick, J.-C.: On the extension of pseudo-Boolean functions for the aggregation of interacting bipolar criteria. Eur. J. Oper. Res. **148**, 28–47 (2003)
55. Grabisch, M., Kojadinovic, I., Meyer, P.: A review of capacity identification methods for Choquet integral based multi-attribute utility theory—applications of the Kappalab R package. Eur. J. Oper. Res. **186**, 766–785 (2008)
56. Greco, S., Matarazzo, B., Słowinski, R.: Conjoint measurement and rough set approach for multicriteria sorting problems in presence of ordinal criteria. In: Colorni, A., Paruccini, M. (eds.) A-MCD-A Aide Multicritère à la Décision—Multiple Criteria Decision Aiding, pp. 117–144. JRC (2001)
57. Greco, S., Matarazzo, B., Słowinski, R.: Bipolar Sugeno and Choquet integrals. In: EURO-FUSE Workshop on Informations Systems, Varenna, September 2002, pp. 191–196
58. Greco, S., Matarazzo, B., Słowinski, R.: The axiomatic basis of Sugeno integral and associative aggregation operator. In: 4th International Workshop on Preferences and Decisions, pp. 33–38, Trento, September 2003
59. Greco, S., Matarazzo, B., Słowinski, R.: Ordinal regression revisited: multiple criteria ranking with a set of additive value functions. Eur. J. Oper. Res. **191**, 416–436 (2008)
60. Greco, S., Słowinski, R., Figueira, J., Mousseau, V.: Robust ordinal regression. In: Ehrgott, M., Figueira, J., Greco, S. (eds.) New Trends in Multiple Criteria Decision Analysis. Springer (2016)
61. Hammer, P.L., Rudeanu, S.: Boolean Methods in Operations Research and Related Areas. Springer, New York (1968)
62. Ishii, K., Sugeno, M.: A model of human evaluation process using fuzzy measure. Int. J. Man-Mach. Stud. **22**, 19–38 (1985)
63. Keeney, R.L., Raiffa, H.: Decision with Multiple Objectives. Wiley, New York (1976)
64. Klein, D.A.: Decision Analytic Intelligent Systems: Automated Explanation and Knowledge Acquisition. Lawrence Erlbaum Associates, Mahwah (1994)
65. Kojadinovic, I.: Minimum variance capacity identification. Q. J. Oper. Res. (4OR) **12**, 23–36 (2006)
66. Kojadinovic, I.: A weight-based approach to the measurement of the interaction among criteria in the framework of aggregation by the bipolar Choquet integral. Eur. J. Oper. Res. **179**, 498–517 (2007)
67. Kojadinovic, I., Labreuche, Ch.: Partially bipolar Choquet integrals. IEEE Trans. Fuzzy Syst. **17**, 839–850 (2009)
68. Krantz, D.H., Luce, R.D., Suppes, P., Tversky, A.: Foundations of Measurement. Additive and Polynomial Representations, vol. 1. Academic, New York (1971)
69. Kwon, S.H., Sugeno, M.: A hierarchical subjective evaluation model using non-monotonic fuzzy measures and the Choquet integral. In: Grabisch, M., Murofushi, T., Sugeno, M. (eds.) Fuzzy Measures and Integrals—Theory and Applications, pp. 375–391. Physica Verlag, Heidelberg (2000)
70. Labreuche, Ch.: Argumentation of the results of a multi-criteria evaluation model in individual and group decision aiding. In: International Conference of the Euro Society for Fuzzy Logic and Technology (EUSFLAT), pp. 482–487, Barcelona, 7–9 September 2005

71. Labreuche, Ch.: Identification of a fuzzy measure with an L^1 entropy. In: International Conference on Information Processing and Management of Uncertainty in Knowledge-Based Systems (IPMU), Malaga, 22–27 June 2008
72. Labreuche, Ch.: On the robustness for the Choquet integral. In: International Conference on Information Processing and Management of Uncertainty in Knowledge-Based Systems (IPMU), pp. 484–493, Dortmund, 28 June to 02 July 2010
73. Labreuche, Ch.: Construction of a Choquet integral and the value functions without any commensurateness assumption in multi-criteria decision making. In: International Conference of the Euro Society for Fuzzy Logic and Technology (EUSFLAT), Aix Les Bains, 18–22 July 2011
74. Labreuche, Ch.: A general framework for explaining the results of a multi-attribute preference model. Artif. Intell. **175**, 1410–1448 (2011)
75. Labreuche, Ch.: An axiomatization of the Choquet integral and its utility functions without any commensurateness assumption. In: International Conference on Information Processing and Management of Uncertainty in Knowledge-Based Systems (IPMU), Catania, 9–13 July 2012
76. Labreuche, Ch., Grabisch, M.: Bi-cooperative games and their importance and interaction indices. In: 14th Mini-EURO Conference on Human Centered Processes (HCP'2003), pp. 287–291, Luxembourg, May 2003
77. Labreuche, Ch., Grabisch, M.: The Choquet integral for the aggregation of interval scales in multicriteria decision making. Fuzzy Sets Syst. **137**, 11–26 (2003)
78. Labreuche, Ch., Grabisch, M.: The Shapley value and the interaction indices for bi-cooperative games. Technical Report 2003/001, LIP6, UPMC, http://www.lip6.fr/reports/lip6.2003.001.html (2003)
79. Labreuche, Ch., Grabisch, M.: Definition of importance indices for bicapacities in mcda. In: International Conference on Information Processing and Management of Uncertainty in Knowledge-Based Systems (IPMU), Paris, 2–7 July 2006, pp. 292–299
80. Labreuche, Ch., Grabisch, M.: Generalized Choquet-like aggregation functions for handling bipolar scales. Eur. J. Oper. Res. **172**, 931–955 (2006)
81. Labreuche, Ch., Grabisch, M.: The representation of conditional relative importance between criteria. Ann. Oper. Res. **154**, 93–122 (2007)
82. Labreuche, Ch., Grabisch, M.: A value for bi-cooperative games. Int. J. Game Theory **37**, 409–438 (2008)
83. Labreuche, Ch., Le Huédé, F.: Inconsistencies in the determination of a capacity. In: European Conference on Artificial Intelligence (ECAI) - Workshop on Advances in Preference Handling, Riva del Garda, 28 August to 1 September 2006, pp. 73–80
84. Labreuche, Ch., Miranda, P., Lehuédé, F.: Computation of the robust preference relation combining a Choquet integral and utility functions. In: European Conference on Artificial Intelligence (ECAI) - Workshop on Advances in Preference Handling, Lisbon, 28 August to 1 September 2010, pp. 49–54
85. Marichal, J.-L.: An axiomatic approach of the discrete Choquet integral as a tool to aggregate interacting criteria. IEEE Trans. Fuzzy Syst. **8**(6), 800–807 (2000)
86. Marichal, J.-L.: On Sugeno integral as an aggregation function. Fuzzy Sets Syst. **114**, 347–365 (2000)
87. Marichal, J.-L.: An axiomatic approach of the discrete Sugeno integral as a tool to aggregate interacting criteria in a qualitative framework. IEEE Trans. Fuzzy Syst. **9**(1), 164–172 (2001)
88. Marichal, J.-L., Roubens, M.: Dependence between criteria and multiple criteria decision aid. In: 2nd International Workshop on Preferences and Decisions, Trento, 1998, pp. 69–75
89. Mauris, G., Montmain, J., Akharraz, A.: Knowledge management as a support for collective decision-making and argumentation processes. In: 9th International Conference on Information Processing and Management of Uncertainty in Knowledge-Based systems (IPMU'2002), Annecy, July 2002, pp. 91–98
90. Mayag, B.: Elaboration d'une démarche constructive prenant en compte les interactions entre critères en aide multicritère à la décision. Ph.D. thesis, University of Paris I (2010)

91. Mayag, B., Grabisch, M., Labreuche, Ch.: Un algorithme de détermination de la capacité pour l'intégrale de Choquet 2-additive. In: LFA'2008, Actes des Rencontres Francophones sur la Logique Floue et ses Applications, Lens, 16–17 October 2008
92. Mayag, B., Grabisch, M., Labreuche, Ch.: An interactive algorithm to deal with inconsistencies in the representation of cardinal information. In: International Conference on Information Processing and Management of Uncertainty in Knowledge-Based Systems (IPMU), Dortmund, 28 June to 02 July 2010, pp. 148–157
93. Mayag, B., Grabisch, M., Labreuche, Ch.: A representation of preferences by the Choquet integral with respect to a 2-additive capacity. Theor. Decis. **71**, 297–324 (2011)
94. Mcsherry, D.: Explanation in recommender systems. AIR **24**, 179–197 (2005)
95. Miranda, P., Grabisch, M.: *p*-Symmetric bi-capacities. In: International Summer School on Aggregation Operators and their Applications, Alcala, July 2003, pp. 123–129
96. Miranda, P., Grabisch, M., Gil, P.: *p*-Symmetric fuzzy measures. Int. J. Uncertainty Fuzziness Knowledge Based Syst. **10**(Suppl.), 105–123 (2002)
97. Modave, F., Grabisch, M.: Preference representation by a Choquet integral: the commensurability hypothesis. In: 7th International Conference on Information Processing and Management of Uncertainty in Knowledge-Based Systems (IPMU'98), Paris, July 1998, pp. 164–171
98. Montmain, J., Mauris, G., Akharraz, A.: Elucidation and decisional risk in a multi criteria decision based on a Choquet integral aggregation: a cybernetic framework. Int. J. Multi-Criteria Decis. Anal. **13**, 239–258 (2005)
99. Mori, T., Murofushi, T.: An analysis of evaluation model using fuzzy measure and the Choquet integral. In: 5th Fuzzy System Symposium, Kobe, 1989, pp. 207–212 (in Japanese)
100. Moulin, H.: Axioms of Cooperative Decision Making. Cambridge University Press, Cambridge (1988)
101. Murofushi, T.: Lexicographic use of Sugeno integrals and monotonicity conditions. IEEE Trans. Fuzzy Syst. **9**(6), 783–794 (2001)
102. Murofushi, T., Soneda, S.: Techniques for reading fuzzy measures (III), interaction index. In: 9th Fuzzy System Symposium, Sapporo, May 1993, pp. 693–696
103. Murofushi, T., Sugeno, M.: Fuzzy measures and fuzzy integrals. In: Grabisch, M., Murofushi, T., Sugeno, M. (eds.) Fuzzy Measures and Integrals – Theory and Applications, pp. 3–41. Physica Verlag, Heidelberg (2000)
104. Onisawa, T., Sugeno, M., Nishiwaki, Y., Kawai, H., Harima, Y.: Fuzzy measure analysis of public attitude towards the use of nuclear energy. Fuzzy Sets Syst. **20**, 259–289 (1986)
105. Pignon, J.P., Labreuche, Ch.: A methodological approach for operational and technical experimentation based evaluation of systems of systems architectures. In: International Conference on Software & Systems Engineering and their Applications (ICSSEA), Paris, 4–6 December 2007
106. Rico, A., Grabisch, M., Labreuche, Ch., Chateauneuf, A.: Preference modelling on totally ordered sets by the Sugeno integral. Discret. Appl. Math. **147**, 113–124 (2005)
107. Roberts, F.S.: Measurement Theory. Addison-Wesley, Reading (1979)
108. Rota, G.C.: On the foundations of combinatorial theory I. Theory of Möbius functions. Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete **2**, 340–368 (1964)
109. Roubens, M.: Ordinal multiattribute sorting and ordering in the presence of interacting points of view. In: Bouyssou, D., Jacquet-Lagrèze, E., Perny, P., Slowinsky, R., Vanderpooten, D., Vincke, Ph. (eds.) Aiding Decisions with Multiple Criteria: Essays in Honor of Bernard Roy, pp. 229–246. Kluwer, Dordrecht (2001)
110. Roy, B.: Multicriteria Methodology for Decision Aiding. Kluwer, Dordrecht (1996)
111. Schmeidler, D.: Integral representation without additivity. Proc. Am. Math. Soc. **97**(2), 255–261 (1986)
112. Shapley, L.S.: A value for *n*-person games. In: Kuhn, H.W., Tucker, A.W. (eds.) Contributions to the Theory of Games. Number 28 in Annals of Mathematics Studies, vol. II, pp. 307–317. Princeton University Press, Princeton (1953)
113. Shapley, L.S.: Simple games: an outline of the descriptive theory. Behav. Sci. **7**, 59–66 (1962)

114. Simon, H.: Rational choice and the structure of the environment. Psychol. Rev. **63**(2), 129–138 (1956)
115. Šipoš, J.: Integral with respect to a pre-measure. Math. Slovaca **29**, 141–155 (1979)
116. Slowiński, R., Greco, S., Matarazzo, B.: Axiomatization of utility, outranking and decision-rule preference models for multi-criteria classification. Control Cybern. **31**(4), 1005–1035 (2002)
117. Soria-Frisch, A.: Unsupervised construction of fuzzy measures through self-organizing feature maps and its application in color image segmentation. Int. J. Approx. Reason. **41**, 23–42 (2006)
118. Sugeno, M.: Theory of Fuzzy Integrals and Its Applications. Ph.D. thesis, Tokyo Institute of Technology (1974)
119. Torra, V.: Learning aggregation operators for preference modeling. In: Fürnkranz, J., Hüllermeier, E. (eds.) Preference Learning, pp. 317–333. Springer, New York (2010)
120. Tversky, A., Kahneman, D.: Advances in prospect theory: cumulative representation of uncertainty. J. Risk Uncertain. **5**, 297–323 (1992)
121. Verkeyn, A., Botteldooren, D., De Baets, B.: Genetic learning of fuzzy integrals accumulating human-reported environmental stress. Appl. Soft Comput. **11**, 305–314 (2011)
122. Wakker, P.: Additive Representations of Preferences. Kluwer, Dordrecht (1989)
123. Wang, Z., Leung, K.S., Wang, J.: A genetic algorithm for determining nonadditive set functions in information fusion. Fuzzy Sets Syst. **102**, 462–469 (1999)
124. Wang, X.-Z., He, Y.-L., Dong, L.-C., Zhao, H.-Y.: Particle swarm optimization for determining fuzzy measures from data. Inf. Sci. **181**, 4230–4252 (2011)
125. Weber, S.: $\perp$-Decomposable measures and integrals for Archimedean t-conorms $\perp$. J. Math. Anal. Appl. **101**, 114–138 (1984)

Chapter 15
Verbal Decision Analysis

Helen Moshkovich, Alexander Mechitov, and David Olson

Abstract Verbal Decision Analysis is a new methodological approach for the construction of decision methods with multiple criteria. The approach is based on cognitive psychology, applied mathematics, and computer science. Problems of eliciting exact quantitative estimations from the decision makers may be overcome by using preferential information from the decision makers in the ordinal form (e.g., "more preferable", "less preferable",...). This type of judgments is known to be much more stable and consistent. Ways of how to obtain and use ordinal judgments for alternatives' evaluation on multiple criteria are discussed. The family of decision methods based on the approach is described.

Keywords Decision analysis • Multiple criteria • Ordinal judgments • Preference elicitation • ZAPROS • ORCLASS

15.1 Introduction

Decisions involving multiple criteria should be based upon human decision maker preference. Unstructured problems present humans with challenges when trade-offs exist among available alternatives. A key feature of this problem domain is human cognitive limitations. Scientific validity is challenged by multiple criteria methods that ask too much in the way of model input. For instance, the accuracy of trade-offs in criteria using vastly different scales called for in lottery trade-offs can be quite challenging and the reliability of weights calculated on the basis of such input can be dubious. At its best, such inputs (and similar inputs used by other multiple choice methods) may be unreliable and unstable.

H. Moshkovich • A. Mechitov (✉)
College of Business, The University of Montevallo, Montevallo, AL 35115, USA
e-mail: MoshHM@montevallo.edu; Mechitov@montevallo.edu

D. Olson
Department of Management, University of Nebraska, Lincoln, Lincoln, NE 68588-0491, USA
e-mail: dolson3@unl.edu

S. Greco et al. (eds.), *Multiple Criteria Decision Analysis*, International Series in Operations Research & Management Science 233,
DOI 10.1007/978-1-4939-3094-4_15

Humans psychologically are more capable of expressing ordinal relationships than cardinal data. Verbal Decision Analysis methods were developed with the primary motivation of basing analysis on the most reliable human input available. Larichev [24] identified admissible in terms of reliability and accuracy as: ordering alternatives with respect to value, qualitative comparison of two estimates on two criteria scales, and qualitative comparison of probabilities of two alternatives. Other inputs typically used in multiple criteria analysis were rated as complex or only admissible for small dimensions. Verbal Decision Analysis (VDA) is based on such reliable and valid input data expressing human preference.

15.1.1 Features of Unstructured Decision Problems

According to Simon [55], decision problems may be divided into three main groups: (1) well-structured problems, (2) ill-structured problems, and (3) unstructured problems.

Well-structured problems are problems where the essential dependencies between parameters are known and may be expressed in a formal way. Problems of this class are being rather successfully solved by operations management methods.

Ill-structured or mixed problems have both qualitative and quantitative elements, but unknown and undefined problem elements tend to dominate these tasks. Problems in this class are rather diversified and methods from different areas may be used to work with them including "cost-benefit" analysis, as well as multiple criteria decision making and multiple criteria decision aids.

Unstructured problems are the problems with mostly qualitative parameters with no objective model for their aggregation. We can see examples of such tasks in policy making and strategic planning in different fields, as well as in personal decisions. These problems are in the area of multicriteria decision aids but require some special considerations in the methods used.

Larichev and Moshkovich [30, 31] proposed the following list of general features for the unstructured problems:

- the problems in this class are unique in the sense that each problem is new to the decision maker and has characteristics not previously experienced;
- parameters (criteria) in these problems are mostly qualitative in nature, most often formulated in a natural language;
- in many cases evaluations of alternatives against these parameters may be obtained only from experts (or the decision maker him/her self);
- an overall evaluation of alternatives' quality may be obtained only through subjective preferences of the decision maker.

Human judgment is the basic source of information in unstructured problems. Being interested in the result, the decision maker would like to control the whole process, including selection of experts and formation of the decision rule(s). Verbal Decision Analysis (VDA) was proposed as a framework for the unstructured problems [31].

15.2 Main Principles of Verbal Decision Analysis

The role of decision making methods applied to unstructured problems should be to help the decision maker to structure the problem (form a set of alternatives and elaborate a set of relevant criteria) and work out a consistent policy for evaluating/comparing multiple criteria alternatives.

As human judgment is the central source of information in unstructured problems, the proposed methods should consider the constraints of the human information processing system as well as the psychological validity of input data in decision analysis. This requires that the methods should: (1) use language for problem description that is natural to the decision maker; (2) implement psychologically valid measurement of criteria and psychologically valid preference elicitation procedures; (3) incorporate means for consistency check of the decision maker's information; (4) be "transparent" to the decision maker and provide explanations of the result.

Verbal Decision Analysis is oriented on construction of a set of methods for different types of decision tasks within the stated framework.

15.2.1 Natural Language of a Problem Description

Verbal Decision Analysis tries to structure a decision problem by using the natural language commonly used by a decision maker and other parties participating in the decision process [27]. The goal of problem structuring is to define alternatives and the primary criteria to be used for evaluation.

In unstructured practical decision tasks most decisions involve qualitative criteria with no natural numerical equivalents [24, 31].

People are known to be poor at estimating and comparing objects that are close in value. It is reasonable for qualitative as well as for originally quantitatively measured criteria to have scales with several distinct levels, possibly differentiated in words and examples [16, 18, 62]. For example, experts were found to have much closer estimates of applicants over separate criteria using scales with a small number of verbal estimates than when using a 1–10 quality scale [44].

Verbal descriptions over criteria scale levels instead of numerical values, not only allow the decision maker to be more confident in his(her) own evaluations, but also should lead to information from experts that is more stable. Therefore, Verbal Decision Analysis uses scales with verbal descriptions of criteria levels for unstructured problems.

15.2.2 Psychological Basis for Decision Rules Elaboration

The measurements discussed in the previous section may be referred to as primary measurements. These primary measurements structure the problem to allow construction of a decision rule for overall evaluation and/or comparison of alternatives. Construction of the decision rule for unstructured problems includes elicitation of the decision maker's preferences as there are almost no objective dependencies between decision criteria.

The complexity involved in eliciting preference information from human subjects has been widely recognized. The process of eliciting necessary information for such decisions is one of the major challenges facing the field [17, 23, 24, 56].

The limitations in human ability to evaluate and to compare multiple criteria options can lead to inconsistencies in human judgments [51, 59] or to application of simplified rules that do not consider essential aspects of the options under consideration [28, 43, 48].

It is important to understand what types of input information are reliable. Larichev [24] attempted to collect and classify all elementary operations in information processing used in normative decision-making. Twenty-three operations were defined and analyzed from the perspective of their complexity for human subjects. The study concluded that quantitative evaluation and comparison of different objects was much more difficult for subjects than conducting the same operations through qualitative ordinal expression of preference.

The following operations were found admissible on the basis of the known research results [31]:

- rank ordering of criteria importance;
- qualitative comparison of attribute values for one criterion or two criteria;
- qualitative evaluation of probabilities.

Experiments in [13] demonstrated that people can somewhat consistently compare attribute values against three criteria if they are helped with the presentation of those. The main idea was that first, subjects compared attribute values against all possible combinations against two criteria. This information about their preferences was used to color code more and less preferable parts of each combination. The results are preliminary and require additional prove to be considered admissible.

Some other operations are expected to be admissible although not enough research has been obtained to date to be sure of admissibility.

Qualitative judgments are preferable for the majority of operations. Therefore, Verbal Decision Analysis uses ordinal (cardinal) judgments as compared to interval data.

15.2.3 Theoretical Basis for Decision Rules Elaboration

Ordinal comparisons are always the first practical step in preference elicitation procedures in multiple criteria analysis. Rather often, scaling procedures follow this step (resulting in quantitative values for all elements of the model). There are ways to analyze the decision on the basis of ordinal judgments, sometimes leading to the preferred decision without resort to numbers [2, 20, 21, 31]. Possible types of available ordinal preference information can be grouped as follows:

- rank ordering of separate levels upon criterion scales (ordinal scales);
- rank ordering of criteria upon their importance;
- pairwise comparison of real alternatives;
- ordinal tradeoffs: pairwise comparison of hypothetical alternatives differing in estimates of only two criteria.

Ordinal Scales are used in the rule of dominance (Pareto Principle). This rule states that one alternative is more preferable than another if it has criterion levels that are not less preferable on all attributes and is more preferable on at least one. This rule does not utilize criterion importance and is not necessarily connected with an additive form of a value function but it requires preferential independence of each separate criterion from all other criteria.

Rank Ordering of Criteria upon Importance does not provide any decision rule by itself. In combination with ordinal scales and lexicographical criterion ranking, the rule for selection of the best alternative may be as follows: first select alternatives with the best possible level upon the most important criterion. From the resulting subset select alternatives with the best possible level upon the next important criterion and so on. This rule is based on the assumption that in the criterion ranking one attribute is more important than all the other attributes, which follow it in the ranking. This preemptive rule does not necessarily imply the additive value function, but has the obvious drawback of its non-compensatory nature, and is theoretically unpopular.

Pairwise Comparison of real alternatives may be directly used in some methods (see, e.g. [22]). In general this information by itself will lead to the solution (if you compare all pairs of alternatives then you can construct a complete rank order of alternatives). But the whole area of multiple criteria decision analysis has evolved from the notion that this task is too difficult for the decision maker. This approach is mostly used in multiple criteria mathematical programming (in which there is not a finite number of alternatives for consideration). Still this information is considered to be highly unstable [24, 59].

Ordinal Tradeoffs [30] exploit the idea of tradeoffs widely used in decision analysis for deriving criterion weights, but is carried out in a verbal (ordinal) form for each pair of criteria and for all possible criterion levels. To find the tradeoff we have to ask the decision maker to consider two criteria and choose which he(she) prefers to sacrifice to some lower level of attainment. When levels are changed from the best to the worst attribute level, this corresponds to the questions in the “swing” procedure for criterion weights [8, 62], but does not require quantitative estimation of the preference.

The use of such tradeoffs is valid if there is preferential independence of pairs of criteria from all other criteria. Two of these preference elicitation methods provide the safest basis for preference identification: ordinal criterion scales and ordinal tradeoffs.

15.2.4 Consistency Check of Decision Maker’s Information

Valid implementation of both ordinal criterion scales and ordinal tradeoffs requires preferential independence of one or two criteria (for all practical purposes if there is pairwise criterion independence, there exists an additive value function and it is reasonable to conclude that any group of criteria is independent from the rest—see [63]). In addition, in many practical cases the decision rule would require transitivity of preferences. It is necessary to check for these conditions for the method to be valid.

The use of preferential independence conditions stems from the desire to construct an efficient decision rule from relatively weak information about the decision maker’s preferences. On the other hand complete checking for this condition will require an exhaustive number of comparisons. Therefore it is reasonable to carry out a partial check of the independence condition over pairs of alternatives [30, 31]. First all necessary tradeoff comparisons are carried out with all criterion levels except those being considered held at their most preferable level. Then, the same tradeoffs are carried out with all other criteria held at their least preferable level. If preferences are the same in both cases, those two criteria are considered to be preferentially independent from all other criteria.

This check is considered to be profound as the change in criterion levels is the most drastic (from the best to the worst) and stability of preferences under those conditions is good evidence of independence.

In case of dependency Verbal Decision Analysis recommends trying to reformulate the problem: group some criteria if they seem to be dependent, or decompose some criteria if their dependence seems to have a root in some essential characteristic combining several others that should be considered separately (see [31] for more details).

To be able to check for consistency of the information elicited (for ordinal information in the form of transitivity of preferences), Verbal Decision Analysis applies “closed procedures” where subsequent questions can be used to check information over all previous questions. For instance, if we ask the decision maker to compare A and B, then B and C, it’s a good idea to ask the decision maker to compare A and C as well. If A is preferred to B, B is preferred to C, and A is preferred to C, then everything is consistent. If C is preferred to A, the preferences are intransitive. Within our approach, transitivity of preferences is assumed, so the decision maker is asked to reconsider comparisons from which intransitivity arises.

15.2.5 Explanation of the Analysis Result

The last but not the least requirement for Verbal Decision Analysis is to demonstrate the results of the analysis to the decision maker in a way that connects the problem structure and the elicited information with the resulting recommended alternative or alternatives.

It should be possible for the decision maker to see how information provided by him(her) lead to the result obtained. This is a necessary condition for the decision maker to rely on the result and to have the necessary information for re-analysis in case the result does not seem plausible. Methods based on Verbal Decision Analysis principles provide the ability to give explanations due to their logical and valid elicitation and their use of qualitative information.

In the next two sections methods based on these principles are presented for two important decision problems: rank ordering of multiple criteria alternatives and ordinal classification/sorting [66] of multiple criteria alternatives.

15.3 Decision Methods for Multiple Criteria Alternatives' Ranking

15.3.1 Problem Formulation

The problems of ranking alternatives evaluated against a set of criteria are wide spread in real life. There are many decision aiding methods oriented on the solution of these problems [19, 31, 50, 52].

Within the Verbal Decision Analysis framework, we consider an unstructured problem where there is a number of alternatives with mostly qualitative characteristics evaluated by human experts. The task is to elaborate a subjective decision rule able to establish at least a partial order on the set of alternatives.

Alternatives are evaluated against a set of criteria with *verbal* formulations of quality grades along their scales.

Formal presentation of the problem under consideration is as follows:

Given:

1. There is a set of n criteria for evaluation of alternatives.
2. X_i is a finite set of possible verbal values on the scale of criterion $i = 1, \ldots, n$, where $|X_i| = n_i$.
3. $X = \prod_{i=1}^{n} X_i$ is a set of all possible vectors in the space of n criteria.
4. $A = \{a_1, \ldots, a_i, \ldots, a_m\} \subseteq X$ is a subset of vectors from X describing real alternatives.

Required: to rank order alternatives from the set A on the basis of the decision-maker's preferences.

We will use the following notations for relationships between alternatives:

- $\succeq_i$ is the weak preference relationship with respect to criterion i: for $a, b \in A, a \succeq_i b$ means a is at least as good as b with respect to criterion i;
- $\succ_i$ is the strict preference relationship with respect to criterion i: $a, b \in A, a \succ_i b$ iff $a \succeq_i b$ and not $b \succeq_i a$;
- $\sim_i$ is the indifference relationship with respect to criterion i: $a, b \in A$, $a \sim_i b$ iff $a \succeq_i b$ and $b \succeq_i a$;
- $\succeq$ is the weak preference relationship: for $a, b \in A$, $a \succeq b$ means a is at least as good as b;
- $\succ$ is the strict preference relationship: $a, b \in A$, $a \succ b$ iff $a \succeq b$ and not $b \succeq a$;
- $\sim$ is the indifference relationship with respect to criterion i: $a, b \in A$, $a \sim b$ iff $a \succeq b$ and $b \succeq a$.

A good example of such a problem is selection of applicants for an interview for a faculty position [44]. A variant of a set of criteria with simple ordinal scales for evaluation of an applicant for a position in Management Information Systems is presented in Table 15.1.

There are two major types of the stated problems. The classical VDA approach assumes that the set of alternatives is big or we may have different sets of alternatives to rank order over time. In this case the idea is to construct a decision rule in the

Table 15.1 Criteria for applicant evaluation

Criteria	Scale
A. Ability to teach our students	A1. Above average
	A2. Average
	A3. Below average
B. Ability to teach SA&D and DBMS	B1. Above average
	B2. Average
	B3. Below average
C. Evaluation of completed research and scholarship	C1. Above average
	C2. Average
	C3. Below average
D. Potential in publications	D1. Above average
	D2. Average
	D3. Below average
E. Potential leadership in research	E1. Above average
	E2. Average
	E3. Below average
F. Match of research interests	F1. Above average
	F2. Average
	F3. Below average

criteria space X and then use it on any set of real alternatives A. Methods ZAPROS-LM [31] and ZAPROS III [25] are oriented on this type of problem.

The second type of problem deals with a relatively small number of alternatives in a unique task. In this case the process is oriented on limited information needed to resolve the current situation. Methods STEP-ZAPROS [45] and UniCombos [1] are devoted to these problems. In the following sections we will discuss the main ideas of each method.

15.3.2 The Joint Ordinal Scale: Method ZAPROS-LM

Methods ZAPROS and ZAPROS-LM [30, 31] are based on the implementation of ordinal verbal scales and ordinal tradeoffs on the scales of criterion pairs near two reference situations. The goal is the construction of the Joint Ordinal Scale for all criteria. The name ZAPROS is the abbreviation of Russian words: Closed Procedures near Reference Situations.

The first step in any decision analysis is to form the set of alternatives, form the set of criteria, and to evaluate alternatives against criteria. As we have decided to use only ordinal judgments for comparison of alternatives, the first step in this direction is to elaborate ordinal scales.

Formally, ordering criterion values along one criterion scale requires the decision maker to select the preferred alternative out of two hypothetical vectors from X differing in values with respect to one criterion (with all other values being at the same level).

This information allows formation of a strict preference relation $\succ_i$ for each criterion $i = 1, \ldots, n$.

Ordinal scales allow pairwise comparison of real alternatives according to the rule of *dominance*.

Definition 1. Alternative a is not less preferable than alternative b, if for each criterion i alternative a is not less preferable than alternative b ($a \succeq_i b$ for $i = 1, \ldots, n$).

The next level of preference elicitation is based on comparison in an *ordinal form* of combinations of values with respect to two criteria.

To carry out such a task we need to ask a decision maker questions of the kind: "what do you prefer: to have this (better) level with respect to criterion i and that (inferior) level with respect to criterion j, or this (better) level for criterion j and that (inferior) level for criterion i if all other criteria are at the same level?"

Possible responses in this case are: more preferable, less preferable or equally preferable [30].

The decision-maker may be asked to make these "ordinal tradeoffs" for each pair of criteria and for each pair of possible values in their scales.

The same information may be obtained with far fewer questions by comparing two hypothetical vectors from X differing in values with respect to two criteria (with

Table 15.2 Comparison of hypothetical alternatives

Criteria	Alternative 1		Alternative 2	
A. Ability to teach our students	Above average	A1	Above average	A1
B. Ability to teach SA&D and DBMS	Above average	B1	Above average	B1
C. Evaluation of completed research and scholarship	Above average	C1	Above average	C1
D. Potential in publications	**Average**	**D2**	**Above average**	**D1**
E. Potential leadership in research	Above average	E1	Above average	E1
F. Match of research interests	**Above average**	**F1**	**Below average**	**F3**
Possible answers				
1. Alt.1 is more preferable than Alt.2				
2. Alt.1 and Alt.2 are equally preferable				
3. Alt.1 is less preferable than Alt.2				

all other values being at the same level). Still the number of the comparisons for all possible combinations of criterion values may be quite large.

ZAPROS [30, 31] uses only part of this information for the construction of the Joint Ordinal Scale (JOS). The decision-maker is asked to compare pairs of hypothetical vectors from $Y \subset X$, each vector with the *best possible values* for all criteria but one. The number of these vectors is not large $|Y| = \sum_{i=1}^{n}(n_i - 1) + 1$.

The goal is to construct a complete rank ordering of all vectors from Y on the basis of the decision maker's preferences. An example of a possible preference elicitation question is presented in Table 15.2.

Definition 2. Joint Ordinal Scale (JOS) is a complete rank order of vectors from Y, where Y is a subset of vectors from X with all the best values but one. Complete rank order means that for each $x, y \subseteq Y$ $x \succ y$ or $y \succ x$ or $x \sim y$.

If the comparisons do not violate transitivity of preferences, we are able to construct a complete rank order of the vectors from Y on the basis of this information, forming the Joint Ordinal Scale. An example of the JOS for the applicants' problem is presented in Table 15.3 with the JOS rank for the most preferred vector marked as 1.

Construction of the Joint Ordinal Scale provides a simple rule for comparison of multi-attribute alternatives. The correctness of rule 3 in case of pairwise preferential independence of criteria was proven in [30]. The crucial difference between the rule of dominance and this rule is that we are able now to compare criterion values with respect to *different* criteria.

Definition 3. Alternative a is not less preferable than alternative b, if for each criterion value of a there may be found a not more preferable unique criterion value of alternative b.

There is an easy way to implement this rule, introduced and proven correct in [45]. Let us substitute a criterion value in each alternative by the corresponding rank in the Joint Ordinal Scale ($JOS(a)$). Then rearrange them in the ascending order (from the most preferred to the least preferred one), so that

Table 15.3 An example of a joint ordinal scale

Equal criterion values	Rank in JOS	Corresponding vector(s)
A1,B1,C1,D1,E1,F1	1	A1B1C1D1E1F1
C2, E2	2	A1B1C2D1E1F1
		A1B1C1D1E2F1
A2, D2, F2	3	A2B1C1D1E1F1
		A1B1C1D2E1F1
		A1B1C1D1E1F2
B2	4	A1B2C1D1E1F1
B3, E3, F3	5	A1B3C1D1E1F1
		A1B1C1D1E3F1
		A1B1C1D1E1F3
A3, C3, D3	6	A3B1C1D1E1F1
		A1B1C3D1E1F1
		A1B1C1D3E1F1

$$JOS_1(a) \leq JOS_2(a) \leq \ldots \leq JOS_n(a).$$

Then the following rule for comparison of two alternatives may be presented.

Definition 4. Alternative a is not less preferable than alternative b if for each $i = 1, \ldots, n$ $JOS_i(a) \leq JOS_i(b)$.

Let use our Joint Ordinal Scale presented in Table 15.3 to compare the following two applicants, incomparable on the basis of the dominance rule:

$$a = (A1, B2, C1, D1, E1, F2)$$

and

$$b = (A1, B1, C1, D2, E2, F1).$$

Let substitute each criterion value in alternatives a and b with corresponding rank from the JOS. We'll get for a vector (1,4,1,1,1,3) and for b vector (1,1,1,3,2,1). When rearranged in an ascending order, the following two vectors can be easily compared:

$$JOS(a) = (1, 1, 1, 1, 3, 4)$$

and

$$JOS(b) = (1, 1, 1, 1, 2, 3).$$

It is clear now that alternative b is preferred to alternative a.

ZAPROS suggests using Joint Ordinal Scale for pairwise comparison of alternatives from A, thus constructing a partial order on this set.

The construction and implementation of Joint Ordinal Scale, as stated above, is based on two assumptions: transitivity of the decision maker's preferences and preferential independence of pairs of criteria (the last condition leads to an additive value function in the decision maker's preferences [30, 31]). This is the basis for the correctness of rule 4.

For the decision method to be valid within the paradigm of Verbal Decision Analysis it should provide means for verification of underlying assumptions. ZAPROS provides these means as follows.

15.3.2.1 Verification of the Structure of the Decision Maker's Preferences

When comparing vectors from Y (for JOS construction) the decision maker can give contradictory responses. In the problem under consideration these responses may be determined as violations of transitivity in the constructed preference relation.

Possible responses of the decision maker in comparison of hypothetical vectors y_i and y_j from Y (see Table 15.2) reflect the binary relation of strict preference ($\succ$) or indifference ($\sim$) between these two alternatives. The following conditions should be met as a result of the decision maker's responses:

if $y_i \succ y_j$ and $y_j \succ$ or $\sim y_k$ then $y_i \succ y_k$

if $y_i \sim y_j$ and $y_j \sim y_k$ then $y_i \sim y_k$

if $y_i \sim y_j$ and $y_j \succ y_k$ then $y_i \succ y_k$.

These conditions are checked in the process of preference elicitation, the intransitive pairs are presented to the decision maker for reconsideration.

The procedure for transitivity verification is described in details in [30, 31], is implemented in a corresponding computerized system and was used in a number of different tasks [31, 35, 44].

The next assumption necessary to check is the pairwise preferential independence of criteria.

Definition 5. Criteria i and j are preferentially independent from the other criteria, if preference between vectors with equal values with respect to all criteria but i and j, does not depend on the values of equal components.

As it is impossible to carry out preference elicitation for all possible combinations of equal values, it was proposed to check preferential independence for pairs of criteria near two very different "reference situations". One variant is based on all the best values for equal components (used in the construction of JOS). The second with the worst possible values for equal components.

If the decision maker's preferences among criterion values are the same when elicited using these two different points, then it is assumed the criteria are preferentially independent.

Although this check is not comprehensive, the preferential stability when using essentially different criterion values as the "reference" point suggests it would hold with the intermediate levels as well [31].

15.3.3 Joint Scale for Quality Variation: ZAPROS III

The construction of the Joint Ordinal Scale (see Sect. 15.3.2), only a relatively small number of comparisons are carried out, limited to vectors with all the best criterion values but one.

In general, the decision-maker may be asked to compare any two hypothetical vectors from X differing in values with respect to two criteria (with all other values being at the same level).

Larichev [25] proposed just that in a method called ZAPROS III . The method requires comparing all criterion values for all pairs of criteria and using this information for comparison of real alternatives.

ZAPROS III introduces a notion of *Quality Variation (QV)* which is the result of changing one value on the scale of one criterion (e.g., from *Average ability to teach our students* to *Below Average* level).

The decision maker is to compare all possible QVs for each pair of criteria with the assumption that all other criterion values are at the same level (reference situation). The number of QVs for each scale is $n_i(n_i-1)/2$, where n_i is the number of values on the criterion scale.

Once all comparisons for two criteria are carried out all QVs for them are rank ordered forming the Joint Scale for Quality Variation (JSQV). For example, let assume that the JSQV for the first two criteria in applicants' evaluation example are as follows (we will use A1A2 to show changing value from A1 to A2):

$$A1A2 \succ B1B2 \succ A1A3 \succ B1B3 \succ A2A3 \succ B2B3.$$

It is proposed to carry out these comparisons at two reference situations (as in ZAPROS): with all the best and all the worst values with respect to other criteria. If the comparisons provide the same JSQV, these criteria are considered to be preferentially independent.

Let look at a simple example for three criteria: A, B, and C. Suppose JSQV for criteria A & B, B & C, and A & C are as follows:

$$A1A2 \succ B1B2 \succ A1A3 \succ B1B3 \succ A2A3 \succ B2B3.$$

$$C1C2 \succ B1B2 \succ B1B3 \succ C1C3 \succ B2B3 \succ C2C3.$$

$$C1C2 \succ A1A2 \succ A1A3 \succ C1C3 \succ A2A3 \succ C2C3.$$

Table 15.4 Ranks for JSQV

Pair	C1C2	A1A2	B1B2	A1A3	B1B3	C1C3	A2A3	B2B3	C2C3
Rank	1	2	3	4	5	6	7	8	9

If we combine all this information together the JSQV is:

$$C1C2 \succ A1A2 \succ B1B2 \succ A1A3 \succ B1B3 \succ C1C3 \succ A2A3 \succ B2B3 \succ C2C3.$$

If in the process violations of transitivity of preferences are discovered, they are presented to the decision maker, and resolved.

Each QV for each criterion gets a rank (e.g., C1C2 has rank 1, A1A2 has rank 2, etc.). The resulting JSQV ranks are presented in Table 15.4.

These ranks may be used to compare alternatives. In ZAPROS III [25] it is proposed to present each real alternative as a combination of JSQV ranks. This is not always possible. For example, in alternative *a*=(A3,B1,C2) it is not clear if A3 should be presented as A1A3 or A2A3. In ZAPROS we have only information on A1A2 and A1A3. We do not have information on A2A3 and so there is no question about the rank to use. With JSQV we need to differentiate these two cases.

To overcome this, ranks describing two alternatives at the same time should be used. In this case the following rule for comparison is correct:

Definition 6. Alternative *a* is not less preferable than alternative *b* if for each $i = 1, \ldots, n$ $JSQV_i(a) \leq JSQV_i(b)$.

Let demonstrate this rule for alternatives:

$$a = (A3, B1, C2)$$

and

$$b = (A2, B2, C1).$$

For criterion A the change is from A2 to A3, so we change A3 in alternative *a* to rank 7 (see Table 15.4 for A2A3) and A2 in alternative *b* to rank 0; for criterion B change is from B1 to B2, so we change B2 to rank 3 and B1 to rank 0; for criterion C change is from C1 to C2, so we change C2 to rank 1 and C1 for rank 0. As a result alternative *a* is presented as vector (7,0,1) or JSQV(*a*) = (0,1,7), and alternative *b* is presented as a vector (0,3,0) or JSQV(*b*) = (0,0,3). Vector (0,0,3) dominates vector (0,1,7), so alternative *b* is preferred to alternative *a*.

Some pairs of real alternatives may still be incomparable. In ZAPROS III it is proposed to sequentially select non-dominated nuclei (analogous to ZAPROS [30]). Alternatives from the first nucleus are assigned rank 1. An alternative has a rank r if it is dominated by an alternative ranked r−1 and itself dominates alternative ranked r+1. As a result some alternatives can have a "fuzzy" rank (e.g., 5–7).

While constructing JSQV the number of required comparisons by the decision maker may be quite large. It is reasonable to use this approach for relatively small problems (small number of criteria and small number of possible criterion values) with a relatively large number of real alternatives.

In [45] the authors proposed to use additional comparisons only after applying Joint Ordinal Scale for comparison of real alternatives. The goal is to elicit only information necessary to compare alternatives left incomparable after that. The process is iterative (as needed), that is why it was named STEP-ZAPROS and is described in the next section.

15.3.4 Goal Oriented Process for Quality Variations: STEP-ZAPROS

This approach views the general application of ordinal preferences for comparison of real alternatives as a three-step procedure:

1. use rule of dominance to compare real alternatives on the basis of ordinal scales. If required decision accuracy is obtained, stop here
2. construct Joint Ordinal Scale and use it to compare real alternatives. If required decision accuracy is obtained, stop here
3. use additional ordinal tradeoffs to compare real alternatives as necessary. Use restructuring procedures if the necessary accuracy is not achieved.

Additional comparisons are carried out only when necessary and only the necessary comparisons are carried out. Thus, the procedure is oriented on efficient acquisition of necessary information when small number of real alternatives needs to be compared.

When comparing real alternatives using Joint Ordinal Scale, alternatives are presented through JOS ranks: JOS(a) and JOS(b) (see Sect. 15.3.2). If alternatives a and b have been left incomparable it means we have at least two ranks such that $JOS_i(a) < JOS_i(b)$ while $JOS_j(a) > JOS_j(b)$. These ranks represent some criterion values in JOS.

The idea is to form two vectors from X different in values with respect to only two criteria (with all the best values with respect to all other criteria). Different criterion values represent the "contradicting" ranks in JOS(a) and JOS(b).

Let our JOS(a) $= (1, 1, 1, 2, 3, 3)$ and JOS(b) $= (1, 1, 1, 1, 1, 5)$. They are incomparable according to JOS as rank 5 is less preferable than rank 2 or 3. If, for example, rank 5 is more preferable than ranks 3 and 3 *together*, then alternative b would be preferable to alternative a.

Rank 3 is presented in the JOS (see Table 15.3) by corresponding criterion values A2, D2, and F2. Rank 5 corresponds to criterion values B3, E3, and F3.

Table 15.5 Effectiveness of STEP-ZAPROS

Parameters								
Number of criteria	5	5	5	5	7	7	7	7
Number of criterion values	3	3	5	5	3	3	5	5
Number of alternatives	30	50	30	50	30	50	30	50
% of compared alternatives	76	76	73	74	63	64	56	59
Additional comparisons	14	17	63	96	21	30	86	141

It allows formation of the following vectors, representing combination of ranks (3,3) and (1,5) and differing in only two criterion values: (A1,B1,C1,D2,E1,F2) and (A1,B1,C1,D1, E1,F3). Comparison of these two vectors will compare D1D2 with F2F3 (see Sect. 15.3.3).

If the second vector is preferred to the first one then alternative *b* is preferred to alternative *a*. If not, the alternatives may be left incomparable.

As the comparison of such specially formed vectors reflects comparison of pairs of ranks in the Joint Ordinal Scale, it is referred to as Paired Joint Ordinal Scale (PJOS) and allows the following rule for comparison of real alternatives:

Definition 7. Alternative *a* is not less preferable than alternative *b* if for each pair of criterion values (a_i, a_j) of alternative *a* there exists a pair of values (b_k, b_l) of alternative *b* such that PJOS$(a_i, a_j) \leq$ PJOS(b_k, b_l).

The proof of the correctness of the rule in case of an additive value function is given in [45].

Preferential independence of criteria is checked while constructing the Joint Ordinal Scale (see Sect. 15.3.2). Transitivity of preferences at the third step is checked only partially in the process of comparisons (as we have previous information on preferences among some of pairs of JOS ranks). It is technically possible to carry out auxiliary comparisons (as in ZAPROS) to ensure transitive closure. It can be applied as necessary at the discretion of the consultant.

To demonstrate the potential of these three steps, simulation results were presented in [45]. Partial information for different problem sizes is presented in Table 15.5.

Data show that (1) the number of real alternatives does not influence the efficiency of the procedure very much; (2) the number of criteria to some extent influences overall comparability of alternatives; (3) the number of criterion values has a *crucial* influence on the number of additional comparisons carried out in the third step.

Overall the data show that method ZAPROS is most efficient for tasks where number of criteria is relatively small and number of alternatives for comparison is relatively large.

15.3.5 Working in the Space of Real Alternatives: UniCombos

UniCombos [1] is a computerized system which is based on the ideas of the described VDA methods but it has three major differences:

- the approach assumes that we need to rank order only small number of real alternatives;
- a decision maker can consistently compare alternatives' values differing against more than two criteria;
- the ability of the decision maker to compare complex combinations of alternatives' values is helped by means to visualize those values.

The first statement leads to an opportunity to elicit only limited preferential information from the decision maker pertaining only to the comparison of real alternatives while the other two allow much higher level of comparability among the real alternatives.

As Step-ZAPROS, UniCombos assumes three steps in the process. The first step, as in all VDA methods is connected with comparison of alternatives' values on one criterion, resulting in ordinal scales and the dominance rule for pairwise comparison. If the rank ordering is not achieved, the second step is to compare all alternatives' values against two criteria (the so called "dyads"). It is the same as constructing JSQV in ZAPROS III but the comparisons are limited to combinations, present among the alternatives and necessary to compare real alternatives.

As an interactive system, the UniCombos checks the comparability of the alternatives after each additional piece of preferential information is obtained. It will stop as soon as all alternatives are compared and/or the best alternative is found.

Once all possible pairs of alternatives' values are compared but there are still incomparable alternatives in the set, the system will present the decision maker with the so-called, "tryads" of alternatives' values—values different against three criteria. The application of the step is based on the relatively positive feedback about stability of subjects' preferences from am experiment described in [13]. The main conclusion was that the decision maker will be consistent if presented with the "tryads" in a way which incorporates previous information on preferability of "dyads". Let us illustrate the process using the following simple example.

Let assume we have two alternatives for comparison $a = (\mathrm{A3}, \mathrm{B1}, \mathrm{C1})$ and $b = (\mathrm{A2}, \mathrm{B2}, \mathrm{C2})$. We already made the comparison of "dyads" which resulted in the comparisons presented as JSQV ranks in Table 15.4. Using the ranks to re-write these alternatives we obtain $\mathrm{JSQV}(a) = (0, 0, 7)$ while $\mathrm{JSQV}(b) = (0, 1, 3)$. There is no dominance in the vectors, so alternatives are left incomparable using "dyads".

To overcome this incomparability the decision maker is presented with a "tryad" of values for comparison. In this case it will be presentation of the alternatives a and b (as we have only three criteria in the problem). The peculiarity of UniCombos is how it presents these values. The main idea is that we divide alternatives' values into two groups: one combines a pair of values of alternative a which is known to be more preferable than the pair of values in alternative b and present this part in

Table 15.6 Presentation of a "tryad" to the decision maker

Alternative *a*	Alternative *b*
Ability to teach is below average	Ability to teach is average
Ability to teach DBMS is above average	Ability to teach DBMS is average
Completed research is above average	**Completed research is average**

a lighter color. The value left is less preferable and is presented in a darker color. As a result each alternative is presented as a combination two parts—lighter and darker—to make it easier for the decision maker to carry out the comparison.

Table 15.6 shows an example where boldfaced test presents less preferable (darker) part of the "tryad".

There are several ways how these two alternatives may be presented: e.g., A2B2 $\succ$ A3B1 but C1 $\succ$ C2 (presented) or A2C2 $\succ$ A3C1 but B1 $\succ$ B2, and so on. The system uses this quality to present the decision maker with the same two alternatives in different ways to check the consistency of the comparison.

UniCombos is able to continue the process technically with any number of alternatives' values (4 or 5) until the desired comparability of alternatives is achieved. The system was used in [61] for the selection of a construction contract. There were three alternatives and seven criteria involved. The demo of the system is accessible at http://iva.isa.ru.

15.4 Decision Methods for Multiple Criteria Alternatives' Classification

Along with multiple criteria choice/ranking problems, people may face multiple criteria classification problems [66]. Rather a large number of classification tasks in business applications may be viewed as tasks with classes which reflect the levels of the same property. Evaluating creditworthiness of clients is rather often measured on an ordinal level as, e.g., "excellent", "good", "acceptable", or "poor" [4]. Articles submitted to the journals in the majority of cases are divided into four groups: "accepted", "accepted with minor revisions", "may be accepted after revision and additional review", "rejected" [31]. Applicants for a job are divided into accepted and rejected, but sometimes there may be also a pool of applicants left for further analysis as they may be accepted in some circumstances [3, 57]. Buildings of the "old town" are divided into the ones of high, average or low historical value [60].

Multiple criteria problems with ordinal criterion scales and ordinal decision classes were named problems of *ordinal classification* (ORCLASS)[29].

15.4.1 Problem Formulation

Formal presentation of the problem under consideration is close to the one in Sect. 15.3.1 as we use criteria scales with finite set of verbal values and analyze the criterion space. Thus items 1–4 are the same while item 5 and what is required in the problem differ.

Given:

1. There is a set of n criteria for evaluation of alternatives.
2. X_i is a finite set of possible verbal values on the scale of criterion $i = 1, \ldots, n$, where $|X_i| = n_i$.
3. $X = \prod_{i=1}^{n} X_i$ is a set of all possible vectors in the space of n criteria.
4. $A = \{a_1, \ldots, a_i, \ldots, a_m\} \subseteq X$ is a subset of vectors from X describing real alternatives
5. $C = \{C_1, \ldots, C_i, \ldots, C_k\}$ is a set of decision classes.

Required: distribute alternatives from A among decision classes C on the basis of the decision-maker's preferences.

For example, the applicants' problem presented in Table 15.1 may be viewed as a classification problem if we need to divide all applicants into three classes: (1) accepted for an interview, (2) left for further consideration, (3) rejected.

We will use the same notation for preferences as in Sect. 15.3.1. In addition, notation C(a) means class for alternative a, e.g., C(a)=C_2 means alternative a belongs to the second class.

15.4.2 An Ordinal Classification Approach: ORCLASS

As in ZAPROS the VDA framework assumes ordinal criterion scales establishing a *dominance* relationship among vectors from X (see Definition 1). In ordinal classification there is an *ordinal relationship among decision classes* as well. This means that alternatives from class C_1 are preferred to alternatives in class C_2 and so on. The least preferable alternatives are presented in class C_k. As a result alternatives with "better" qualities (criterion values) should be placed in a "better" class.

These ordinal qualities allow formation of an effective decision maker's preference elicitation approach [26, 29, 31, 33, 38–41].

The decision maker is presented with vectors from X and asked directly to define an appropriate decision class. The cognitive validity of this form of preference elicitation was thoroughly investigated and found admissible [28, 32].

It is possible to present the decision maker with all possible vectors from X to construct a universal classification rule in the criterion space. However, it is impractical even for relatively small problem sizes. The ordinal nature of criterion scales and decision classes allows formulation of a strict preference relation: if vector x is placed in a better class than vector y, then vector x is more preferable than vector y.

Definition 8. For any vectors $x, y \in X$ where $\mathrm{C}(x) = C_i$ and $\mathrm{C}(y) = C_j$ if $i < j$ then $x \succ y$.

As a result we can formulate a condition for a non-contradictory classification of vectors x and y: if vector x dominates vector y and is placed into ith class, then vector y should be placed into a class not more preferable than the ith class.

Definition 9. For any vectors $x, y \in X$ if y is dominated by x ($x \succ y$) and $\mathrm{C}(x) = C_i$, then $\mathrm{C}(y) = C_j$ where $j \geq i$.

Using this quality we can introduce a notion of *expansion by dominance* [29].

Definition 10. If vector $x \in X$ is assigned class C_i by a decision maker, then for all $y \in X$ such that $x \succ y$ possible classes are C_j where $j \geq i$. For all $y \in X$ such that $y \succ x$ possible classes are C_j where $j \leq i$.

Each classification of a vector from X by a decision maker limits possible classes for all dominating it and dominated by it vectors from X. Let mark as $G(x) = \{l, l+1, l+2, .., m\}$ a subset of classes admissible at this moment for vector x. In the beginning $G(x)$ for each $x \in X$ contains all possible classes from 1 to k. When the number of admissible classes for the x becomes equal to one, a unique class is assigned to x.

Using expansion by dominance we can obtain classification for some vectors from X not presented to the decision maker (there are some results [29, 31, 33, 38] showing that between 50 and 75 % of vectors may be classified indirectly using this rule).

In addition, there is a simple way to discover possible errors in the decision maker's classifications: if an assigned class is outside the admissible range, there is a contradiction in the ordinal classification. Contradictory classifications may be presented to the decision maker for reconsideration.

For more details on the procedure see [29, 31].

The efficiency of the *indirect* classification of vectors from set X depends on the vectors presented to the decision maker as well as on the class assigned [29, 31]. Ideally, we would like to present the decision maker with as few questions as possible and still be able to construct a complete classification of vectors from set X. Different heuristic approaches were proposed to deal with this problem, based on the desire to find the most "informative" vectors to be presented to the decision maker for classification.

15.4.3 Effectiveness of Preference Elicitation

The first approach was proposed in ORCLASS [31] and is based on the maximum "informativeness" of unclassified vectors from X . Each class is presented by its "center": average of criterion values of vectors already in the class. For each unclassified vector x for each its admissible class i from $G(x)$ "similarity" measure $p_i(x)$ is calculated. The "similarity" measure evaluates how *probable* the class is for this vector. To calculate p_i, the normalized distance between the vector x and the center of the class is calculated. Also, for each admissible class the number of indirectly classified vectors $g_i(x)$ if x is assigned class C_i is evaluated.

Informativeness $F(x)$ for vector x is calculated as sum of products of probability by the number of indirectly classified vectors for all admissible classes:

$$F(x) = \sum p_i(x)g_i(x).$$

The vector with the largest "informativeness" value is selected for classification by the decision maker. After the decision maker classifies this vector, the expansion by dominance is carried out and informativeness of all vectors is recalculated. The approach favors vectors from X which define approximately equal number of other vectors by indirect classification in case of any of the admissible classes.

Simulations show high effectiveness of the procedure with only 5–15 % of all vectors from X necessary to be classified by the decision maker [31]. The drawback of the approach is its high computational complexity.

Another approach was proposed in [33]. It is based on a maxmin principle. For each unclassified vector the minimum number of indirectly classified vectors in case of admissible classes is defined and the vector with the maximum number is selected for classification by the decision maker. The computational complexity of the approach is a bit lower than in the previous case.

Another algorithm called CYCLE was presented in [38]. The idea is to construct "chains" of vectors between vectors x and y which belong to different classes. The "chain" is constructed sequentially by changing one criterion value in vector x by one level until we obtain criterion values of vector y. Then the most "informative" vector is searched only in the chain, thus essentially lowering the computational complexity of the algorithm. The process is dynamic and searches for the "longest" chain between two vectors.

The effectiveness of the approach was compared to the algorithms of monotone function decoding and appeared much more effective for smaller problems and simpler borders while being somewhat less effective in more complex cases.

15.4.4 Class Boundaries

Ordinal classification allows not only a convenient method of preference elicitation, but also an efficient way to present the final classification of set X.

Let assume we have a classification of set X into classes C. We will view C_i as a subset of vectors from X, assigned to the ith class.

Two special groups of vectors may be differentiated among them: *lower border* of the class LB_i and the *upper border* UB_i. Upper border includes all *non-dominated* vectors in the class, while lower border includes all *non-dominating* vectors in this class.

These two borders accurately represent the ith class: we can classify any other vector as belonging to class C_i if its criterion values are between values of vectors from LB_i and UB_i.

Let us look at vector C(x)=C_i which is not in the upper or lower border of the class. It means there is a vector $y \in UB_i$ for which $y \succ x$, thus $C(y) \leq C(x)$. Analogously there is object $z \in LB_i$ for which $x \succ z$. Thus $C(x) \geq C(z)$. But $C(y) = C(z) = C_i$. This leads to $C(x) = C_i$.

Borders summarize classification rules. If we know classification of vectors in the class borders only, it would be enough to classify any vector from set X [29, 31, 33]. That is why, heuristic methods are oriented on finding potential "border vectors" for presentation to the decision maker.

15.4.5 Real Alternatives Classification: SAC and CLARA

In cases when it is necessary to classify a relatively small number of alternatives only once (not to construct a classification rule in the criteria space) a modified approach may be used to decrease the number of vectors the decision maker has to classify. Method SAC (Subset of Alternatives Classification) [37] and CLARA (Classification of Real Alternatives) [60] are designed for this type of a problem.

In the SAC method the principle of evaluating "informativeness" is the same as in ORCLASS (see Sect. 15.4.3) but only the indirectly classified *real alternatives* are taken into account (not all alternatives from the set X). This makes the process less complex. Another difference is that in SAC you can evaluate "relative informativeness" of a vector in the form of $F(x)/[(1 + v/F(x)]$ where v is the variance in the number of indirectly classified alternatives. The variance is dependent on the number of classes and the number of criteria. The recommended values are between 2.2 and 3.5. If v is equal to zero, the informativeness is calculated the same as in ORCLASS.

Method CLARA is also oriented on classification of real alternatives but selection of alternatives to be presented to the decision maker for classification is based on the CYCLE approach (see Sect. 15.4.3). The algorithm is based on dichotomy of alternatives' chains, beginning with the longest one. For example, let us consider we have only two classes. We know that real alternative A1B1C2D1 belongs to class C_1 and real alternative A3B3C3D1 belongs to class C_2. We construct a "chain" of dominating real alternatives between these two alternatives, e.g.,

$$A1B1C2D1 \succ A2B1C2D1 \succ A2B2C2D1 \succ A3B2C2D1 \succ A3B3C3D1$$

If there are no other real alternatives to form any other chains in this case, the alternative to be presented to the decision maker is A2B2C2D1 because it is in the middle of the chain (the best candidate for the "border between two classes). There may be several chains we can form: if we have real alternative A1B2C2D1, another chain of alternatives is

$$A1B1C2D1 \succ A1B2C2D1 \succ A2B2C2D1 \succ A3B2C2D1 \succ A3B3C3D1$$

In this case the chains are of the same length and produce the same alternative for classification.

15.4.6 Hierarchical Ordinal Classification

Verbal Decision Analysis is a powerful approach which produces transparent methods and processes attractive to the decision makers. On another hand the majority of applications of these methods concentrate on small size problems. For examples, two cases—application of ORCLASS to marketing decisions in a small firm [14] as well as application of ZAPROS III to the rank ordering of ways to diagnose Alzheimer's Disease [58] both used just three criteria in alternatives' evaluation. Applications dealing with high number of criteria are usually presented as a hierarchy of criteria.

Method CLARA [53] and computerized system VERBA [47] directly address the possibility and sometimes necessity to use a hierarchy of criteria in real applications. Ordinal classification is a logical way of constructing such hierarchies. As we have ordinal verbal scales for criteria, they may be viewed as decision classes for criteria of a lower level.

Let return to our applicants' evaluation example in Table 15.1. We can present the problem using only three criteria instead of six: "ability to teach", "ability for research", "match of research interests" with scales "above average", "average", and "below average". We can use ordinal classification approach to combine all possible value combinations for two criteria "ability to teach students" and "ability to teach SA&D and DBMS" into the three "decision classes" of "ability to teach": "above average", "average", and "below average". Criterion "ability for research" is a combination of three low level criteria: "evaluation of complete research", "potential in publications", and "potential leadership in research". This way we can solve two small ORCLASS problems and then solve a small ZAPROS problem, making the process less exhausting and more attractive to the decision maker.

Method CLARA used two-level hierarchy of criteria to construct a classification rule in the criterion space for evaluating investment risk in construction projects [53]. They had six final decision classes, with six criteria at the highest level with 3–4 criterion values. Each of these criteria was a combination of 3–4 criteria at the lower level. The problem of this size can be solved only by a hierarchy.

System VERBA is an attempt to combine all these VDA approaches in one place. It allows decomposition of any problems into a hierarchy of subproblems and a flexible implementation of ordinal classification and/or rank ordering of alternatives (ZAPROS style) for any subproblem or its parts. All preferential information is stored, checked for transitivity and may be used at any time for any type of a problem. Partial illustration of the system may be found in [47].

15.5 Place of Verbal Decision Analysis in MCDA

The decision maker is the central figure in decision making based on multiple criteria. Elicitation of the decision makers' preferences should take into account peculiarities of human behavior in the decision processes. This is the central goal of Verbal Decision Analysis.

Like outranking methods (e.g., ELECTRE, PROMETHEE) VDA provides outranking relationships among multiple criteria alternatives. At the same time, VDA is designed to elicit a sound preference relationship that can be applied to future cases while outranking methods are intended to compare a given set of alternatives. VDA is more oriented on tasks with rather large number of alternatives while number of criteria is usually relatively small. Outranking methods deal mostly with reverse cases.

VDA bases its outranking on axiomatic relationships, to include direct assessment, dominance, transitivity, and preferential independence. Outranking methods use weights as well as other parameters, which serve an operational purpose but also introduce heuristics and possible intransitivity of preferences. VDA is based on the same principles as multi-attribute utility theory (MAUT), but is oriented on using the verbal form of preference elicitation and on evaluation of alternative decisions without resort to numbers. That is why we consider that VDA is oriented on the same tasks as MAUT and will be compared in a more detail to this approach to multiple criteria decision making.

15.5.1 Multi Attribute Utility Theory and Verbal Decision Analysis Methods

The central part of MAUT concentrates on deriving numeric scores for criterion values and relative criterion weights which are combined in an overall evaluation of an alternative's value.

There are a number of methods and procedures for eliciting criterion weights and scores. Some of these methods are based on sound theory, while others use simplified heuristic approaches.

Experiments show that different techniques may lead to different weights [5, 54], but in modeling situations varying criterion weights often does not change the result thus leading to the conclusion that equal weights work sufficiently well [7, 9]. However, the situation may not be the same for real decision tasks when differences between alternatives are small. Slight differences in weights can lead to reversals in the ranking of alternatives [34, 35, 64].

Two approaches (MAUT and VDA) were applied to the same decision making problems [12, 27, 36]. Positive and negative features of each approach were analyzed, the circumstances under which one or the other would be favored were examined.

Three groups of criteria for comparison were considered: methodological, institutional and personal [12, 27].

Methodological criteria characterize an approach from the following perspectives:

- measurements of alternatives with respect to criteria;
- consideration of alternatives;
- complexity reduction;
- quality of output;
- cognitive burden.

Measurements. VDA uses verbal scales, while MAUT is oriented on obtaining numerical values.

People use verbal communication much more readily than quantitative communication. Words are perceived as more flexible and less precise, and therefore seem better suited to describe vague opinions. Erev and Cohen stated that " forcing people to give numerical expressions for vague situations where they can only distinguish between a few levels of probability may result in misleading assessments" [10].

But there are positive factors in utilization of quantitative information: people attach a degree of precision, authority and confidence to numerical statements that they do not ordinarily associate with verbal statements, and it is possible to use quantitative methods of information processing.

The experiments made over many years by Prof. T. Wallsten and his colleagues demonstrated no essential differences in the accuracy of evaluations [6, 10], but there was essential difference in the number of preference reversals. The frequency of reversals was significantly decreased when using the verbal mode [15].

The two methods differ considerably in whether they *force consideration of alternatives*. If the best alternative is not found by using "verbal" comparisons, VDA seeks to form another alternative that has not previously been considered (generating new knowledge) by acknowledging the fact that there is no best alternative among presented. VDA assumes that if it is not possible to find better alternative on an ordinal level, there is either no satisfactory alternative or alternatives are too close in quality to differentiate between them.

The numerical approach does not force thorough consideration of alternatives, as it is capable to evaluate even very small differences among alternatives. It is always possible to find the best alternative in this case. The question is if the result is reliable enough.

Complexity. VDA diminishes complexity of judgments required from the decision maker as it concentrates only on *essential* differences. The MAUT method requires very exact (numerical) comparisons of differences among criteria and/or alternatives in majority of cases.

Quality of output. MAUT provides overall utility value for each alternative. This makes it possible to not only identify the best alternative but also to define the difference in utility between alternatives. This means that the output of MAUT methods is rich enough to give the decision-maker the basis for detailed evaluation and comparison of any set of alternatives.

VDA attempts to construct a binary relation between alternatives which may lead to incomparable alternatives, but assures that comparisons are based on sound information elicitation.

Cognitive burden. A goal of all decision methods is reducing the confusing effect of ambiguity in preferences. Methods deal with this phenomenon in very different ways. VDA alters ambiguity and corresponding compensations into levels (rather than exact numbers).

MAUT attempts to estimate the exact amount of uncertainty. The payoff is that the analysis can derive a single estimate of uncertainty to go with the single estimate of utility.

Institutional criteria include: the ease of using the approach within organizations, and consequences of cultural differences.

Both MAUT and VDA can be considered improvements over confounding cost-benefit analysis based upon data with little hope of shared acceptance. Achieving greater clarity does, to some extent, provide improved communication within organizations. However, the information upon which MAUT develops utility is of suspect reliability.

The VDA approach uses more direct communication and active groups are used to assign the verbal quality grades on criteria scales. The VDA approach does not require the decision-maker or expert to have previous knowledge in decision methods. On the other hand, MAUT findings can be presented graphically and provide sensitivity analysis because of its numerical basis.

Some cultural differences may influence the applicability of different approaches. Americans tend to use numerical evaluations more often than in some other countries (e.g., Russia). American analysts are usually required "to put a price tag on goods not traded in any market place" [11]. That is not always the case in Europe.

Personal criteria include: the educational level required of decision-makers to use methods; and how the professional habits of analysts influence the selection of an approach.

The practical experience and intellectual ability of the decision-maker are presuppositions for the utilization of any analytical technique. MAUT requires more detailed trade-off balancing, calling for deeper ability to compare pairs of criteria performances. VDA is designed to focus on more general concepts.

Training in decision analysis helps decision-makers to understand and accept the MAUT approach. VDA methods do not require any special knowledge in decision analysis on the part of the decision-maker. The VDA approach is especially useful when a decision is made under new circumstances or in conditions of high ambiguity.

Comparison: The MAUT approach has a strong mathematical basis. MAUT provides a strong justification of the type of utility function used for aggregation of single-attribute utilities over criteria. Different kinds of independence conditions can be assumed [19]. In the case of criteria dependence, a nonlinear form quite different from the simple additive linear model is available. The involvement of the decision-maker is needed to elaborate a utility function. But after this is accomplished, it is possible to compare many alternatives. Should a new alternative appear, no additional decision-maker efforts are needed. Possible inaccuracy in the measurements could be compensated for by sensitivity analysis.

Conversely, the questions posed to decision-makers have no psychological justification. Some questions could be very difficult for humans to completely understand. Decision-makers require special training or orientation in order for MAUT methods to be used. Possible human errors in evaluating model parameters are not considered. Sensitivity analysis is recommended to evaluate stability of the result.

Verbal Decision Analysis has both psychological and mathematical basis. In all stages of the method natural language is used to describe concepts and information gathered relating to preference. Preferential criteria independence is checked. If criteria are dependent, we may try to transform the verbal description of a problem to obtain independence [31]. For example, sometimes criteria (or their scales) may be too detailed (not necessary information) or too general (not possible to differentiate). In these cases introducing two or three more detailed criteria instead of one too general for evaluation or collapsing a couple of criteria into one on a more general level may lead to preferential independence. In addition VDA has special procedures for the identification of contradictions in the information provided by the decision-maker.

Conversely, there are some cases when incomparability (due to lack of reliable information) does not guarantee identification of one best alternative. There may be more than one alternative ranked at the best level. The decision rule might not be decisive enough in cases when a decision must be reached quickly. There is no guarantee that experts could find a better alternative after formulation of directions for improvements of existing alternatives.

15.5.2 Practical Value of the Verbal Decision Analysis Approach

VDA has positive features of:

- Using psychologically valid preference input;
- Providing checks for input consistency;
- Implementing mathematically sound rules.

VDA was used in a number of applications for different types of decision problems. ZAPROS (and its variations) was used in R&D planning [30, 31], applicants' selection [44], job selection [34, 35], and pipeline selection [12, 27, 36].

R&D planning problem was connected with a state agency financing different research projects. Number of applications for funding was around several thousand each year, approximately 70 % of them were awarded required (or reduced) funding. The decisions were to be made rather quickly after the deadline for applications (couple of months). To be able to cope with this level of complexity, it was decided to construct a decision rule in the criterion space and apply it to alternatives' descriptions against the criteria which were obtained through experts. ZAPROS was used to construct Joint Ordinal Scale in the criterion space which was used to form ordered groups of alternatives (for sequential distribution of funds). The number of criteria ranged from 5 to 7 for different subgroups of projects.

The task of applicants selection was implemented in one of the American universities where there could be more than 100 applicants for a faculty position [44]. Six criteria with three level (verbal) scales were used to construct the Joint Ordinal Scale to be used to select s subset of better applicants for further analysis and an interview. The department chair was the decision maker in this case.

Pipeline selection was a somewhat different type of problem where there were relatively small number of very complicated alternatives: possible routes for a new gas pipeline. Modified variant of ZAPROS was used to elicit preferences from the decision maker in this case and use it to analyze the quality of presented alternatives. All alternatives were found out to be not good enough for implementation. The analysis was directed towards "redefining" the problem through a more detailed and/or less detailed criteria and formation of a new "adjusted" alternative acceptable for the authorities.

Tamanini et al. [58] applied ZAPROS III to rank order tools in of Alzheimer's disease diagnosis. In this work, preferences were obtained through questionnaires from experts and postmortem patient diagnosis. The study enabled identification of tests that would more quickly detect patients with Alzheimer's disease. Ustinovichius et al. [61] used UniCombos [1] to compare construction contracts using seven criteria for three real alternatives. ZAPROS-LM was used in rank ordering real retailer commercialization decisions in Brazil based on discussions and analysis with key managers [49]. Mendes et al. [42] demonstrated use of VDA in the design of mobile television application, applying ZAPROS to the characteristics of prototypes based on user experience and intentions.

The ordinal classification approach was used for R&D planning and journals' evaluation, as well as for job selection [31, 39]. In addition, this approach was found to be very useful in the area of knowledge base construction for expert systems and is often used in medical diagnostics [26, 33, 40, 41, 46]. Yevseyeva et al. [65] applied a SAC like method for neuropsychology patient diagnosis. CLARA was used in several applications concerned with decision making in the area of construction [53, 60]. Gomes et al. [14] applied ORCLASS to marketing decisions for a small business in Brazil engaged in the distribution of dental products.

15.6 Conclusion

MCDA is an applied science. The primary goal of research in MCDA is to develop tools to help people to make more reasonable decisions. In many cases the development of such tools requires combination of knowledge derived from such areas as applied mathematics, cognitive psychology, and organizational behavior. Verbal Decision Analysis is an example of such a combination. It is based on valid mathematical principles, takes into account peculiarities of human information processing system, and places the decision process within the organizational environment of the decision making.

This chapter has reviewed the basic underpinnings of Verbal Decision Analysis. They were demonstrated with early VDA methods, such as ZAPROS and ORCLASS, and their later modifications, such as ZAPROS III, UniCombos, CLARA, VERBA, and others. There is active research in further development of VDA methods, both in Russia, the home of VDA, and in the Americas. A number of published applications to real decision problems were discussed here, demonstrating the maturity of VDA.

References

1. Ashikmin, I., Furems, E.: UniCombos: intelligent decision support system for multi-criteria comparison and choice. J. Multi-Criteria Decis. Anal. **13**(2–3), 147–157 (2005)
2. Barron, F.H., Barrett, B.E.: Decision quality using ranked attribute weights. Manag. Sci. **42**, 1515–1523 (1996)
3. Ben-David, A.: Automated generation of symbolic multiattribute ordinal knowledge-based DSSs: methodology and applications. Decis. Sci. **23**, 1357–1372 (1992)
4. Ben David, A., Sterling, L.A., Pao, Y.H.: Learning and classification of monotonic ordinal concepts. Comput. Intell. **5**(1), 45–49 (1989)
5. Borcherding, K., Eppel, T., von Winterfeldt, D.: Comparison of weighting judgments in multiattribute utility measurement. Manag. Sci. **27**, 1603–1619 (1991)
6. Budescu, D., Wallsten, T.: Processing linguistic probabilities: general principles and empirical evidence. Psychol. Learn. Motiv. **32**, 275–318 (1995)
7. Dawes, R.M., Corrigan, B.: Linear models in decision making. Psychol. Bull. **81**, 95–106 (1974)

8. Edwards, W., Barron, F.H.: SMARTS and SMARTER: improved simple methods for multiattribute utility measurement. Organ. Behav. Hum. Decis. Process. **60**, 306–325 (1994)
9. Einhorn, H., Hogarth, R.: Unit weighting schemes for decision making. Organ. Behav. Hum. Decis. Process. **13**, 171–192 (1975)
10. Erev, I., Cohen, B.: Verbal versus numerical probabilities: efficiency, biases, and the preference paradox. Organ. Behav. Hum. Decis. Process. **45**, 1–18 (1990)
11. Fischoff, B.: The real world: what good is it? Organ. Behav. Hum. Decis. Process. **65**, 232–248 (1996)
12. Flanders, N.E., Brown, R.V., Andre'eva, Y., Larichev, O.: Justifying public decisions in Arctic oil and gas development: American and Russian approaches. Arctic **51**(3), 262–279 (1998)
13. Furems, E.M., Larichev, O.I., Roizenson, G.V., Lotov, A.V., Miettinen, K.: Human behavior in a multi-criteria choice problem with individual tasks of different difficulties. Int. J. Inf. Technol. Decis. Mak. **2**(1), 29–40 (2003)
14. Gomez, L., Moshkovich, H., Torres, H.: Marketing decisions in small business: how verbal decision analysis can help. Int. J. Manag. Decis. Mak. **11**(1), 19–36 (2010)
15. Gonzalez-Vallejo, C., Wallsten, T.: The effects of communication mode on preference reversal and decision quality. J. Exp. Psychol. Learn. Mem. Cogn. **18**, 855–864 (1992)
16. Goodwin, P., Wright, G.: Decision Analysis for Management Judgments. Wiley, New York (1991)
17. Kahneman, D., Slovic, P., Tversky, A. (eds.) Judgment Under Uncertainty: Heuristics and Biases. Cambridge University Press, Cambridge (1982)
18. Keeney, R.: Value-Focused Thinking. Harvard University Press, Cambridge (1992)
19. Keeney, R.L., Raiffa, H.: Decisions with Multiple Objectives: Preferences and Value Tradeoffs. Wiley, New York (1976)
20. Kirkwood, C.W., Corner, J.L.: The effectiveness of partial information about attribute weights for ranking alternatives in multiattribute decision making. Organ. Behav. Hum. Decis. Process. **54**, 456–476 (1993)
21. Kirkwood, C.W., Sarin, R.K.: Ranking with partial information: a method and an application. Oper. Res. **33**, 38–48 (1985)
22. Korhonen, P.: A visual reference direction approach to solving discrete multiple criteria problems. Eur. J. Oper. Res. **34**(2), 152–159 (1988)
23. Larichev, O.I. : Psychological validation of decision methods. J. Appl. Syst. Anal. **11**, 37–46 (1984)
24. Larichev, O.I.: Cognitive validity in design of decision-aiding techniques. J. Multi-Criteria Decis. Anal. **1**(3), 127–138 (1992)
25. Larichev, O.I.: Ranking multicriteria alternatives: the method ZAPROS III. Eur. J. Oper. Res. **131**, 550–558 (2001)
26. Larichev, O.I.: Close imitation of expert knowledge: the problem and methods. Int. J. Inf. Technol. Decis. Mak. **1**(1), 27–42 (2002)
27. Larichev, O., Brown, R.: Numerical and verbal decision analysis: comparison on practical cases. J. Multi-Criteria Decis. Anal. **9**, 263–273 (2000)
28. Larichev, O.I., Moshkovich, H.M.: Limits to decision-making ability in direct multiattribute alternative evaluation. Organ. Behav. Hum. Decis. Process. **42**, 217–233 (1988)
29. Larichev, O.I., Moshkovich, H.M.: An approach to ordinal classification problems. Int. Trans. Oper. Res. **1**(3), 375–385 (1994)
30. Larichev, O.I., Moshkovich, H.M.: ZAPROS-LM - a method and system for ordering multiattribute alternatives. Eur. J. Oper. Res. **82**, 503–521 (1995)
31. Larichev, O.I., Moshkovich, H.M.: Verbal Decision Analysis for Unstructured Problems. Kluwer Academic Press, Boston (1997)
32. Larichev, O.I., Moshkovich, H.M., Rebrik, S.B.: Systematic research into human behavior in multiattribute object classification problems. Acta Psychol. **68**, 171–182 (1988)
33. Larichev, O.I., Moshkovich, H.M., Furems, E.M., Mechitov, A.I., Morgoev, V.K.: Knowledge Acquisition for the Construction of Full and Contradiction Free Knowledge Bases. IecProGamma, Groningen (1991)

34. Larichev, O.I., Moshkovich, H.M., Mechitov, A.I., Olson, D.L.: Experiments comparing qualitative approaches to rank ordering of multiattribute alternatives. J. Multi-Criteria Decis. Anal. **2**(1), 5–26 (1993)
35. Larichev, O.I., Olson, D.L., Moshkovich, H.M., Mechitov, A.I.: Numeric vs. cardinal measurements in multiattribute decision making: (How exact is enough?) Organ. Behav. Hum. Decis. Process. **64**, 9–21 (1995)
36. Larichev, O.I., Brown, R., Andre'eva, E., Flanders, N.: Categorical decision analysis for environmental management: a Siberian gas distributing case. In: Caverni, J.-P., Bar-Hillel, M., Barron, F.H., Jungermann, H. (eds.) Contribution to Decision Making, pp. 255–286. North-Holland, Amsterdam (1995)
37. Larichev, O.I., Kortnev, A., Kochin, D.: Decision support system for classification on of a finite set of multicriteria alternative. Decis. Support Syst. **33**(1), 13–21 (2002)
38. Larichev, O., Asanov, A., Naryzhny, Y.: Effectiveness evaluation of expert classification methods. Eur. J. Oper. Res. **138**, 260–273 (2002)
39. Mechitov, A.I., Moshkovich, H.M., Olson, D.L.: Problems of decision rules elicitation in a classification task. Decis. Support Syst. **12**, 115–126 (1994)
40. Mechitov, A.I., Moshkovich, H.M., Olson, D.L., Killingsworth, B.: Knowledge acquisition tool for case-based reasoning systems. Expert Syst. Appl. **9**(2), 201–212 (1995)
41. Mechitov, A.I., Moshkovich, H.M., Bradley, J.H., Schellenberger, R.E.: An ordinal model for case-based reasoning in a classification task. Int. J. Expert Syst. Res. Appl. **9**(2), 225–242 (1996)
42. Mendes, M.S., Carvalho, A.L., Furtado, E., Pinheiro, P.R.: Towards analyzing alternatives of interaction design based on verbal decision analysis of user experience. Int. J. Interactive Mob. Technol. **4**(2), 17–231 (2010)
43. Montgomery, H.: A study of intransitive preferences using a think aloud procedure. In: Jungerman, H., de Zeeuw, G. (eds.) Decision Making and Change in Human Affairs. Riedel, Dordrecht (1977)
44. Moshkovich, H.M., Schellenberger, R., Olson, D.L.: Data influences the result more than preferences: some lessons from implementation of multiattribute techniques in a real decision task. Decis. Support Syst. **22**, 73–84 (1998)
45. Moshkovich, H.M., Mechitov, A., Olson, D.L.: Ordinal judgments for comparison of multiattribute alternatives. Eur. J. Oper. Res. **137**, 625–641 (2002)
46. Moshkovich, H.M., Mechitov, A.I., Olson, D.: Rule induction in data mining: effect of ordinal scales. Expert Syst. Appl. J. **22**, 303–311 (2002)
47. Oleynikov, D., Butenko, L., Moshkovich, H., Mechitov, A.: ARACE – a new method for verbal decision analysis. Int. J. Inf. Technol. Decis. Mak. **14**(1), 115–140 (2015)
48. Payne, J.W.: Task complexity and contingent processing in decision making: an information search and protocol analysis. Organ. Behav. Hum. Process. **16**, 366–387 (1976)
49. Rodrigues, G., Cohen, E.: Optimizing marketing segmentation through an information system using verbal decision analysis in the Chamber of Commerce in Rio de Janeiro (CDLRIO). In: Academy of World Business, Marketing & Management Development Conference Proceedings, vol. 3.1, pp. 432–436 (2008)
50. Roy, B.: Multicriteria Methodology for Decision Aiding. Kluwer Academic Publisher, Dordchert (1996)
51. Russo, J.E., Rosen, L.D.: An eye fixation analysis of multiattribute choice. Mem. Cogn. **3**, 267–276 (1975)
52. Saaty, T.L.: The Analytic Hierarchy Process. McGraw-Hill, New York (1980)
53. Shevchenko, G., Ustinovichius, L., Andruskevichius, A.: Multi-attribute analysis of investments risk alternatives in construction. Technol. Econ. Dev. Econ. **14**(3), 428–443 (2008)
54. Shoemaker, P.J.H., Waid, C.C.: An experimental comparison of different approaches to determining weights in additive utility models. Manag. Sci. **28**(2), 182–196 (1982)
55. Simon, H. The New Science of Manaqement Decision. Harper and Row, New York (1960)
56. Slovic, P., Fishhoff, B., Lichtenstein, S.: Behavioral decision theory. Ann. Psychol. Rev. **28**, 1–39 (1977)

57. Slowinski, R., Rough set approach to decision analysis. AI Expert **9**(2), 19–25 (1995)
58. Taminini, I., De Castro, A.K., Pinheiro, P.R., Pinheiro, M.C.D.: Applied neuroimaging to the diagnosis of Alzheimer's disease: a multicriteria model. Commun. Comput. Inf. Sci. **49**, 532–541 (2009)
59. Tversky, A.: Intransitivity of preferences. Psychol. Rev. **76**, 31–48 (1969)
60. Ustinovichius, L., Kutut, V., Kochin, D., Shevchenko, G.: Verbal analysis of renovation investment strategy of old town. In: Zavadskas, E.K., Kaklauskas, A., Skibniewski, M.J. (eds.) The 25th International Symposium on Automation and Robotics in Construction. ISARC-2008. Selected papers, pp. 397–411 (2008)
61. Ustinovichius, L., Barvidas, A., Vishnevskaja, A., Ashikmin, I.: Multicriteria verbal analysis for the decision of construction problems. Technol. Econ. Dev. Econ. **15**(2), 326–340 (2009)
62. von Winterfeldt, D., Edwards, W.: Decision Analysis and Behavioral Research. Cambridge University Press, Cambridge (1986)
63. von Winterfeldt, D., Fischer, G.W.: Multiattribute utility theory: models and assessment procedures. In: Wendt, D., Vlek, C. (eds.) Utility, Probability, and Human Decision Making. Reidel, Dordrecht (1975)
64. Wainer, H.: Estimating coefficients in linear models: it don't make no nevermind. Psychol. Bull. **85**, 213–217 (1976)
65. Yevseyeva, I., Miettinen, K., Rasanen, P.: Verbal ordinal classification with multicriteria decision aiding. Eur. J. Oper. Res. **185**(3), 964–983 (2008)
66. Zopounidis, C., Doumpos, M.: Multicriteria classification and sorting methods: a literature review. Eur. J. Oper. Res. **138**, 229–246 (2002)

Chapter 16
A Review of Fuzzy Sets in Decision Sciences: Achievements, Limitations and Perspectives

Didier Dubois and Patrice Perny

Abstract We try to provide a tentative assessment of the role of fuzzy sets in decision analysis. We discuss membership functions, aggregation operations, linguistic variables, fuzzy intervals and valued preference relations. The importance of the notion of bipolarity and the potential of qualitative evaluation methods are also pointed out. We take a critical standpoint on the state of the art, in order to highlight the actual achievements and try to better assess what is often considered debatable by decision scientists observing the fuzzy decision analysis literature.

Keywords Decision • Qualitative value scales • Aggregation • Linguistic variables • Preference relations • Fuzzy intervals • Ranking methods

16.1 Introduction

The idea of using fuzzy sets in decision sciences is not surprising since decision analysis is a field where human-originated information is pervasive. The seminal paper in this area was written by Bellman and Zadeh [9] in 1970, highlighting the role of fuzzy set connectives in criteria aggregation. That pioneering paper makes three main points:

1. Membership functions can be viewed as a variant of utility functions or rescaled objective functions, and optimized as such.
2. Combining membership functions, especially using the minimum, can be one approach to criteria aggregation.
3. Multiple-stage decision-making problems based on the minimum aggregation connective can then be stated and solved by means of dynamic programming.

D. Dubois (✉)
IRIT, CNRS & Université de Toulouse, Toulouse, France
e-mail: dubois@irit.fr

P. Perny
LIP6, Université P. et M. Curie, Paris, France
e-mail: patrice.perny@lip6.fr

S. Greco et al. (eds.), *Multiple Criteria Decision Analysis*, International Series in Operations Research & Management Science 233,
DOI 10.1007/978-1-4939-3094-4_16

This view was taken over by Tanaka et al. [115] and Zimmermann [132] who developed popular multicriteria linear optimisation techniques in the seventies. The idea is that constraints are soft and can be viewed as criteria. Then any linear programming problem becomes a max-min fuzzy linear programming problem.

Other ingredients of fuzzy set theory like fuzzy ordering relations, linguistic variables and fuzzy intervals have played a major role in the diffusion of fuzzy set ideas in decision sciences. We can especially point out the following:

1. Gradual or valued preference relations (stemming from Zadeh's fuzzy orderings [127]) further studied by Orlovsky [94], Fodor and Roubens [63], Van De Walle et al. [119], Díaz et al. [34, 36], and others [12]. This notion was applied to outranking methods for multicriteria decision-making.
2. Many other aggregation operations are used so as to refine the multicriteria aggregation technique of Bellman and Zadeh: t-norms and conorms, symmetric sums, uninorms, leximin, Sugeno and Choquet integrals etc. This trend is testified by three recent books (Beliakov et al. [8], Torra and Narukawa [116], Grabisch et al. [74]).
3. Fuzzy interval computations so as to cope with uncertainty in numerical aggregation schemes. Especially, extensions of the weighted average with uncertain weights [98, 125].
4. Fuzzy interval comparison techniques enable the best option in a set of alternatives with fuzzy interval ratings to be selected [124].
5. Linguistic variables [128] are supposed to model human originated information, so as to get decision methods closer to the user cognition [78].

What has been the contribution of fuzzy sets to decision sciences? Following the terminology of the original Bellman-Zadeh paper, fuzzy decision analysis (FDA) is supposed to take place in a "fuzzy environment", in contrast with probabilistic decision analysis, taking place "under uncertainty". But, what is a fuzzy environment? It seems that many authors take it an environment where the major source of information is linguistic, so that linguistic variables are used, which does not correspond to Bellman and Zadeh's proposal. One should nevertheless not oppose "fuzzy environment" to "uncertain environment": the former in fact often means "using fuzzy sets", while the latter refers to an actual decision situation: there is epistemic uncertainty due to missing information, not always related to linguistic imprecision.

Actually, for many decision theory specialists, it is not clear that fuzzy sets have ever led to a new decision paradigm. Indeed, some have argued that either such techniques already existed under a different terminology, or that fuzzy decision methods are but fuzzifications of standard decision techniques. More precisely,

- Fuzzy optimization following Bellman and Zadeh [9], Tanaka et al. [115] and Zimmermann [132] comes down to max-min bottleneck optimization. But bottleneck optimisation and maximin decisions already existed independently of fuzzy sets.

- In many cases, fuzzy sets have just been added to existing techniques (fuzzy AHP methods, fuzzy weighted averages, fuzzy extensions of ELECTRE-style Multiple Criteria Decision Making (MCDM) methods) with no clear benefits (especially when fuzzy information is changed into precise numbers at the preprocessing level, via defuzzification, which can be observed sometimes).
- Fuzzy preference modelling is an extension of standard preference modelling and must be compared to probabilistic or measurement-based preference modeling.

In fact, contrary to what is often claimed in FDA papers, it is not always the case that adding fuzzy sets to an existing method improves it in a significant way. It does needs to be articulated by convincing arguments, based on sufficient knowledge of state-of-the-art existing techniques.

To make a real contribution one must show that the new technique

- either addresses in a correct way an issue not previously handled by existing methods: e.g., criterion dependence using Choquet integral.
- or proposes a new setting for expressing decision problems more in line with the information provided by users: for instance using qualitative information instead of numerical.
- or yet possesses a convincing rationale (e.g., why such an aggregation method? why model uncertainty by a fuzzy set?) and a sound formal setting amenable to some axiomatization.
- in any case, is amenable to a validation procedure.

Unfortunately, it is not always clear that any such contribution appears in many proposals and published papers on FDA. The validation step, especially, seems to be neglected. It is not enough to propose a choice recipe illustrated on a single example.

This position paper takes a skeptical viewpoint on the fuzzy decision literature, so as to help laying bare what is its actual contribution. Given the large literature available there is no point to providing a complete survey. However we shall try to study various ways fuzzy sets were instilled in decision methods, and provide a tentative assessment of the cogency of such proposals. More specifically, we shall first deal with membership functions viewed as evaluating utility, and discuss the underlying scaling problem in Sect. 16.2. We also consider the issue of linguistic variables. In Sect. 16.3 we consider valued preference relations and their possible meanings, and, in Sect. 16.4, how they have been used in outranking methods for multicriteria decision-making. Then, in Sect. 16.5, it is the contribution of aggregation operations in multifactorial evaluation that is analysed, focusing on the refinement of qualitative scales and the issue of bipolar preference. Section 16.6 deals with the role of fuzzy intervals in sensitivity analysis for decision evaluation, focusing on fuzzy weighted averages, and fuzzy extensions of Saaty's Analytic Hierarchical Process method. Finally, Sect. 16.7 outlines a classification of fuzzy interval ranking methods, driven by the possible interpretations of the membership functions, in connection with interval orderings and stochastic dominance.

16.2 Membership Functions in Decision-Making

First we discuss the role played by membership functions in decision techniques. Then we consider the use of membership grades and linguistic terms for rating the worth of decisions and evaluating pairwise preference.

16.2.1 Membership Functions and Truth Sets in Decision Analysis

A membership function is, like sets in general, an abstract notion, a mathematical tool. It only introduces grades in the abstract Boolean notion of set-membership. So, using the terminology of membership functions in a decision problem does not necessarily enrich its significance. In order to figure out the contribution of fuzzy sets, one must always declare what a given membership function accounts for in a given problem or context. Indeed, there is not a unique semantic interpretation of membership functions. Several ones have been laid bare [49] and can be found in the literature:

- A measure of similarity to prototypes of a linguistic concept (then membership degrees are related to distance); this is used when linguistic terms are modeled by membership functions, and naturally obtained as an output of fuzzy clustering methods (Ruspini [110]).
- A possibility distribution [130] representing our incomplete knowledge of a parameter, state of nature, etc., that we cannot control. Possibility distributions can be numerical or qualitative [50]. In the numerical case, such a membership function can encode a family of probability functions (see [39] for a survey).
- A numerical encoding of a preference relation over feasible options, similar to a utility or an objective function. This is really the idea at the core of the Bellman-Zadeh paradigm of decision-making in a fuzzy environment. In decision problems, membership functions introduce grades in the traditionally Boolean notion of feasibility. In the latter case, a membership function models a fuzzy constraint [54, 129]. A degree of feasibility differs from the degree of attainment of a non-imperative goal.

In the scope of decision under uncertainty, membership functions offer an alternative to both probability distributions and utility functions, especially when only qualitative value scales are used. But these two interpretations of membership functions should not be confused nor should we use one for the other in problems involving both fuzzy constraints and uncertainty [54].

Then the originality of the fuzzy approach may lie:

- either in its capacity to translate linguistic terms into quantitative ones in a flexible way;
- or to explicitly account for the lack of information, avoiding the questionable use of unique, often uniform probability distributions [39];

- or in its set-theoretic view of numerical functions. Viewing a utility function as a fuzzy set, a wider range of aggregation operations becomes available, some of which generalize the standard weighted average, some of which generalize logical connectives.

However, not only must a membership function be interpreted in the practical context under concern, the scale in which membership degrees lie must also be well-understood and its expressive power made clear.

16.2.2 Truth-Sets as Value Scales: The Meaning of End-Points

The often totally ordered set of truth-values, we shall denote by $(L, \geq)$, is also an abstract construct. Interpretive assumptions must be laid bare if it is used as a value scale for a decision problem. The first issue concerns the meaning of the end-points of the scale; and whether a mid-point in the scale exists and has any meaning. Let us denote by 0 the least element in L and by 1 the greatest element. Let us define a mid-point of L as an element $e \in L$ such that

1. $\exists \lambda^-, \lambda^+ \in L, \lambda^- < e < \lambda^+$;
2. there is an order-reversing bijection $n : L \to L$ such that $n(1) = 0; n(e) = e$ (n is a strong negation function such that if $\lambda < e$, then $n(\lambda) > e$).

Three kinds of scales can be considered depending on the existence and the meaning of these landmark points [52]:

- *Negative unipolar scales*: then, 0 has a totally negative flavour while 1 has a neutral flavour. For instance, a possibility distribution, a measure of loss.
- *Positive unipolar scales*: 0 has a neutral flavour while 1 has a fully positive flavour. For instance degrees of necessity, a measure of gain.
- *Bipolar scales*: when 1 has a totally positive flavour while 0 has a totally negative flavour. Then, contrary to the two other cases, the scale contains a mid-point e that has a neutral flavour and that plays the role of a boundary between positive and negative values.

For instance, the unit interval viewed as a probability scale is bipolar since 0 means impossible, 1 means certain and $1/2$ indicates a balance between the probable and the improbable (however the role of $1/2$ becomes less obvious when there are more than two alternatives involved). The membership scale of a fuzzy set is in principle bipolar, insofar as $1/2$ represents the cross-over point between membership and non-membership. However if a membership function is used as a possibility distribution as suggested by Zadeh [130], the scale becomes negative unipolar since then while 0 means impossible, 1 only means possible, which is neutral. The dual scale of necessity degrees in possibility theory [50] is on the contrary positive unipolar, since the top value of L expresses full certainty while the bottom represents full uncertainty, hence neutral. In these latter cases, the midpoint, even if it exists, plays no role in the representation.

Finally, if membership grades express preference, the degree of satisfaction of a goal is often bipolar (in order to express satisfaction, indifference and dissatisfaction) [73]. However, another approach is to use two separate unipolar scales, one to express the degree of feasibility of a solution (it is a negative unipolar scale ranging from not feasible to feasible), one to express the attractiveness of solutions (a positive unipolar scale ranging from indifference to full satisfaction). More generally, loss functions map to negative unipolar scales, while gain functions map to positive unipolar scales. Gains and losses can be separately handled as in cumulative prospect theory [118].

This information about landmark points in the scale captures ideas of good or bad in the absolute. A simple preference relation cannot express this kind of knowledge: ranking solutions to a decision problem from the best to the worst without making the meaning of the value scale explicit, nothing prevents the best solution found from being judged rather bad, or on the contrary the worst solution from being somewhat good.

The choice of landmark points also has strong impact on the proper choice of aggregation operations (t-norms, co-norms, uninorms) [51]. Especially landmark points in the scale are either neutral or absorbing elements of such aggregation operations.

16.2.3 Truth-Sets as Value Scales: Quantitative or Qualitative?

The second issue pertains to the expressive power of grades in a scale L and has to do with its algebraic richness. One can first decide if an infinite scale makes sense or not. It clearly makes sense when representing preference about a continuous measurable attribute. Then, the reader is referred to the important literature on measurement theory (see [82] and Chap. 16 in [17]) whose aim is to represent preference relations by means of numerical value functions. According to this literature, there are three well-known kinds of continuous value scales

- *Ordinal scales*: The numerical values are defined up to a monotone increasing transformation. Only the ordering on L matters. It makes no sense to add degrees in such scales.
- *Interval scales*: The numerical values are defined up to a positive affine transformation ($\lambda \in L \mapsto a\lambda + b, a > 0$). Interestingly, in decision theory the most popular kind of value scales are interval scales. On such scales, only the difference of ratings makes sense. But they cannot express the idea of good and bad (they are neither bipolar nor even unipolar) since the value 0 plays no specific role, and these scales can be unbounded.
- *Ratio scales*: The numerical values are defined up to a positive linear transformation $a\lambda, a > 0$. On such scales, only the quotient of two ratings makes sense. This kind of scale is often unipolar positive as the bottom value 0 lies at the bottom of the scale.

Another option is to go for a finite, or *classificatory* scale: $L = \{0 < \lambda_1 < \lambda_2 < \cdots < \lambda_m = 1\}$ where elements of the scale are not numerical but are labels of classes forming a totally ordered set. In the following we shall also speak of a *qualitative* scale. It underlies the assumption that the values in the scale are significantly distinct from one another (hence they cannot be too numerous): in particular, the value λ_i is significantly better that λ_{i-1}. Each value may have a precise meaning in the context of the application. For instance, it may correspond to a linguistic term, that people understand in the specific context. Or, each value can be encoded as an integer, and refers to a precise reference case (like in the classification of earthquakes), with a full-fledged description of its main features. This kind of scale is often neglected both in usual measurement theory and in fuzzy set theory. Yet a qualitative scale is more expressive than a simple ordering relation because of the presence of absolute landmarks that can have a positive, negative or neutral flavour.

Clearly the nature of the scale also affects the kind of aggregation function that can be used to merge degrees. An aggregation operation $*$ on an ordinal scale must satisfy an ordinal invariance property such as the following:

$$a * b > c * d \iff \varphi(a) * \varphi(b) > \varphi(c) * \varphi(d)$$

for all monotonic increasing transformations φ of an ordinal scale L. See [74, Chap. 8], on this problem. Basically only operations based on maximum and minimum remain meaningful on such scales. Averaging operations then make no sense.

More often than not in decision problems, people are asked to express their preferences by ticking a value on a continuous line segment. Then such values are handled as if they were genuine real numbers, computing averages or variances. This kind of technique is nearly as debatable as asking someone to explicitly provide a real number expressing preference. All we can assume is that the corresponding scale is an ordinal scale. In particular, there is a problem of commensurateness between scales used by several individuals: the same numerical value provided by two individuals may fail to bear the same meaning. On the other hand qualitative scales can better handle this problem: landmark values correspond to classes of situations that can be identically understood by several individuals and may be compared across several criteria. A small qualitative scale is cognitively easier to grasp than a continuous value scale and has thus more chance to be consensual.

In summary there is a whole literature on numerical utility theory that should be exploited if fuzzy set decision researchers wish to justify the use of numerical membership grades in decision techniques. From this point of view, calling a utility function *a membership function* is not a contribution. Yet, fuzzy set theory offers a framework to think of aggregation connectives in a broader perspective than the usual weighted averaging schemes. But there is no reason to move away from the measurement tradition of standard decision analysis. Fuzzy set tools should essentially enrich it. A few researchers tried to address the issue of membership function measurement (Türksen and Bilgic [117], Marchant [37, 86, 87]), but this problem should be further investigated.

16.2.4 From Numerical to Fuzzy Value Scales

Being aware that precise numerical techniques in decision evaluation problems are questionable, because they assume more information than what can actually be supplied by individuals, many works have been published that claim to circumvent this difficulty by means of fuzzy set-related tools. The rationale often goes as follows: If a precise value in the real line provided by an expert is often ill-known, it can be more faithfully represented by an interval or a fuzzy interval. Moreover, the elements in a qualitative scale may encode linguistic value judgments, which can be modeled via linguistic variables [128].

16.2.4.1 Evaluations by Pairs of Values

When an individual ticks a value in a value scale or expresses a subjective opinion by means of a number x, it sounds natural to admit that this value has limited precision. The unit interval is far too refined to faithfully interpret subjective value judgments. It is tempting to use an interval $[a, b]$ in order to describe this imprecision. However, it is not clear that this approach takes into account the ordinal nature of the numerical encoding of the value judgment. It is natural to think that the width of an interval reflects the amount of imprecision of this interval. However in an ordinal scale, width of intervals make no sense: if $[a, b] = [c, d]$, in general $[\varphi(a), \varphi(b)] \neq [\varphi(c), \varphi(d)]$ for a monotonic scale transformation φ. So the use of interval-valued ratings presupposes an assumption on the nature of the value scale, which must be more expressive than an ordinal scale, for instance an interval scale. It must be equipped with some kind of metric. Justifying it may again rely on suitable (e.g., preference difference) measurement techniques. Moving from an interval to a fuzzy interval with a view to cope with the uncertainty of the interval boundaries, one is not better off, since on an ordinal scale, the shape of the membership function is meaningless: there is no such thing as a triangular fuzzy number on an ordinal scale.

Some authors use pairs of values $(\mu, \nu) \in [0, 1]^2$ with $\mu + \nu \leq 1$ following Atanassov's convention [4]. Not only the latter encoding looks problematic in the light of the above considerations,[1] but this representation technique is moreover ambiguous: it is not clear whether this pair of values corresponds to *more information* or *less information* than a single value [53]:

1. Using an uncertainty semantics, it expresses less information than point values because it encodes an ill-known value $\lambda \in [\mu, 1 - \nu]$. Then the uncertainty

[1]Indeed, the addition $\mu + \nu$ is questionable on an ordinal scale. One may replace $\mu + \nu \leq 1$ by $\mu \leq n(\nu)$, for a strong negation on L, but then $\mu \leq n(\nu)$ implies $\varphi(\mu) \leq \varphi(n(\nu))$ while we need $\varphi(\mu) \leq n(\varphi(\nu))$.

interval representation is more explicit. Moreover, the aggregation operations proposed by Atanassov are fully compatible with the interval-extension of pointwise aggregation operations [33].

2. Or it expresses more information than point values: then μ is the strength in favour of a decision, ν in disfavour of this decision. This is a unipolar bivariate convention that fits argumentation semantics, and departs from the Savagean utility theory tradition. However it is not clear that researchers adopting Atanassov convention refer to pioneering works, like Cumulative Prospect Theory [118] (CPT), when adopting this kind of bipolar view. This setting is also more information-demanding than using single evaluations, so that it does not address at all the concerns raised by the debatable richness assumption of numerical ratings.

The proper choice of a semantics of Atanassov style value pairs affects the way information will be processed [53]:

1. The standard injection $L \to L^2$ is not the same: $\lambda \mapsto (\lambda, 1-\lambda)$ in the interval case, $\lambda \mapsto (\lambda, 0)$ for a positive unipolar scale in the bipolar case (then the non-trivial pairs add negative information to single positive evaluations).
2. Under the uncertainty semantics, you need to apply interval analysis methods to see the impact of uncertainty on the global evaluation (insofar as the numerical scale is meaningful).
3. Under the argumentation semantics, you may first separately aggregate positive and negative information by appropriate (possibly distinct) methods and then aggregate the results as done in CPT.

16.2.4.2 Linguistic vs. Numerical Scales

Quite a number of papers on FDA have been published in the last 15 years or so, with the aim of exploiting linguistic information provided by decision-makers. Namely a criterion is viewed as mapping decisions on a finite linguistic term set forming a qualitative scale. A number of authors then consider a criterion as a linguistic variable after Zadeh [128], namely they represent fuzzy linguistic terms in the value scale as fuzzy intervals, that form a fuzzy partition of the unit interval. Contrary to the free use of any value on a numerical scale surrounded by imprecision in the form of a fuzzy interval, the linguistic approach only uses a finite set of prescribed fuzzy intervals, and the decision-maker uses them as ratings. More often than not the unit interval, taken as a value scale, is shared into overlapping intervals of equal length that form the supports of the fuzzy intervals. It corresponds here to what Zadeh calls a *granulation* of the unit interval [131]. One advantage of this framework is that criteria aggregations can be modelled by means of fuzzy if-then rules that can be processed using any fuzzy inference method (like in fuzzy control). However this approach is debatable for a number of reasons:

- It is not clear that the granular scale thus used is qualitative any more since each linguistic term is viewed as a fuzzy interval on some numerical scale. If the scale is in fact ordinal, sharing this scale into fuzzy intervals with the same shape clearly makes no sense. Such linguistic scales are thus either not qualitative or meaningless.
- Arithmetic aggregation operations that do not make sense on the underlying numerical scale will not make more sense when applied to fuzzy intervals.
- Combining fuzzy intervals from a partition (crisp or fuzzy) generally does not yield elements of the partition. One has to resort to some form of linguistic approximation in order to construct a closed operation on the linguistic scale. But if this operation is abstracted from a numerical one, properties of the latter (for instance associativity) will be often lost (see [47]).
- If moreover one applies fuzzy control interpolation methods to build an aggregation function (using the standard Mamdani fuzzification-inference - defuzzification scheme), what is constructed is a numerical function which highly depends, for instance, on the choice of a defuzzification method.

In fact, linguistic variables proposed by Zadeh are meaningful if the underlying numerical scale corresponds to an objective measurable attribute, like height, temperature, etc. A linguistic variable on an abstract numerical scale is all the more meaningless because, prior to the membership function measurement problems that are already present on measurable attributes, the question of how to make sense of ratings on this abstract scale is to be solved first. So this trend leads to debatable techniques that are neither more meaningful nor more robust to a change of numerical encoding (of the linguistic values) than purely numerical techniques (see the last chapter of [16] for a detailed critique of this line of works).

Besides, due to the above difficulties, there have been some attempts at directly using linguistic labels, like the 2-tuple linguistic representation [77]. The 2-tuple method handles pairs (i, σ) where i denotes the rank of label $\lambda_i \in L$ in a finite qualitative scale and $\sigma \in [-0.5, 0.5)$. The purpose is to easily go from a numerical value x lying between 0 and n to a symbolic one in L by means of the integer closest to x, interpreting $\lambda_i \in L$ as $x = i + \sigma \in [0, n]$. Then any numerical aggregation function $*$ can be applied to the qualitative scale L: $\lambda_i * \lambda_j = \lambda_k$ where $i * j = k + \sigma$. In this view, σ is a numerical value expressing the precision of the translation from the original result to the linguistic scale.

This kind of so-called linguistic approaches are as quantitative as any standard number-crunching method. It just uses a standard rounding technique as a linguistic approximation tool, for the sake of practical convenience. It no longer accounts for the imprecision of linguistic evaluations. Moreover the idea that a qualitative scale should be mapped to a sequence of adjacent integers is debatable. First one must justify the choice of equally distributed integers. Then, one must study how the ranking of decisions obtained by aggregation of partial ratings on the integer scale depends on the choice of the monotonic mapping $L \to \mathbb{N}$ encoding the linguistic values.

16.3 The Two Meanings of Fuzzy Preference Relations

An important stream of works was triggered by the book by Fodor and Roubens [63], that extend preference modelling to the gradual situation [12, 27, 34, 36, 119]. In classical preference modelling, an outranking relation provided by a decision-maker is decomposed into strict preference, indifference and incomparability components in order to be used. Fuzzy preference relations are valued extensions of relations expressing preference, i.e., of variants of ordering or preordering relations [17, Chap. 2]. There is no point discussing the current state of this literature in detail here (see Fodor and De Baets [62] for a recent survey). However most existing works in this vein develop mathematical aspects of fuzzy relations, not so much its connection to actual preference data. This is probably due to the fact that the meaning of membership grades to fuzzy relations is not so often discussed.

16.3.1 Unipolar vs. Bipolar Fuzzy Relations

The notion of fuzzy ordering originated in Zadeh's early paper [127] has been improved and extensively studied in recent years (Bodenhofer et al. [12]). A fuzzy relation on a set S is just a mapping $R : S \times S \rightarrow [0, 1]$ (or to any totally ordered scale). One assumption that pervades the fuzzy relational setting is not often emphasized: a fuzzy relation makes sense only if it is meaningful to compare $R(x, y)$ to $R(z, w)$ for 4-tuples of acts (x, y, z, w), that is, in the scope of preference modelling, to decide whether x is preferred (or not) to y in the same way or not as z is preferred to w. A fuzzy preference relation should thus be viewed as the result of a measurement procedure reflecting the expected or observed properties of crisp quaternary relations $Q(x, y, z, w)$ that should be specified by the decision-maker (see Fodor [61] for preliminary investigations).

In this case one may argue that $R(x, y)$ reflects the intensity of the preference of x over y. Nevertheless, the mathematical properties of R will again be dictated by the meaning of the extreme values of the preference scale, namely when $R(x, y) = 0$ or 1. If the unit interval is viewed as a bipolar scale, then $R(x, y) = 1$ means full strict preference of x over y, and is equivalent to $R(y, x) = 0$, which expresses full negative preference. It suggests indifference be modelled by $R(x, y) = R(x, y) = 1/2$, and more generally the property

$$R(x, y) + R(y, x) = 1$$

is naturally assumed. This property generalizes completeness, and $R(x, y) > 1/2$ expresses a degree of strict preference. Antisymmetry then reads $R(x, y) = 1/2 \implies x = y$. These are reciprocal (or tournament) relations that do not fit with the usual encoding of reflexive crisp relations, and exclude incomparability. Indeed, in the usual convention of the crisp case, reflexivity reads $R(x, x) = 1$, while incomparability reads $R(x, y) = R(y, x) = 0$.

In order to stick to the latter convention, the unit interval must then be viewed as a negative unipolar scale, with neutral upper end, and the preference status between x and y cannot be judged without checking the pair $(R(x,y), R(y,x))$. In this case, $R(x,y)$ evaluates weak preference, and completeness means:

$$\max(R(x,y), R(y,x)) = 1,$$

while indifference is modelled by $R(x,y) = R(y,x) = 1$. On the contrary, $R(x,y) = R(y,x) = 0$ captures incomparability. In other words, this convention allows the direct extension of usual preference relations to valued ones on a unipolar scale where 1 has a neutral flavour. Conventions of usual outranking relations are retrieved when restricting to Boolean values. The expression of antisymmetry must be handled with care in connection to the underlying similarity relation on the preference scale, as shown by Bodenhofer [11].

16.3.2 Fuzzy Strict Preference Relations

The definition of fuzzy preference relations is often driven by the need of some desirable mathematical properties of preference relations (e.g. completeness, asymmetry, transitivity) but sometimes, the initial meaning of valuations is forgotten in the construction. Hence, the way fuzzy binary relations are handled in decision procedures is not always meaningful with respect to the nature of the initial preference information. For example, the definition of a strict preference relation P from a large preference relation R has received much attention in the literature (see e.g. [63, 97, 101]) is often based on the following equation:

$$P(x,y) = T(R(x,y), 1 - R(y,x)) \tag{16.1}$$

where T is a t-norm. For instance, it is well known that if R is min-transitive, i.e. for all x, y, z we have $R(x,y) \geq \min\{R(x,z), R(z,y)\}$, and T is the Łukasiewicz t-norm defined by $T(x,y) = \max\{x + y - 1, 0\}$ then P is also min-transitive. This argument is used by several authors to justify the definition of valued strict preference by Eq. (16.1) with the Łukasiewicz t-norm for T (see e.g. [97]). This leads to the following definition of strict preference: $P(x,y) = \max\{R(x,y) - R(y,x), 0\}$ which is necessarily asymmetric, i.e. either $P(x,y) = 0$ or $P(y,x) = 0$. Obviously, this definition is not adequate when R takes its values in an ordinal scale. For example, consider x, y, z, w such that $R(x,y) = 1, R(z,w) = 0.8, R(y,x) = 0.7$ and $R(w,z) = 0.4$, we have $P(x,y) = 0.3 < P(z,w) = 0.4$. However, if the R valuation is expressed on an ordinal scale, the only relevant information provided by its numerical representation is that $R(x,y) > R(z,w) > R(y,x) > R(w,z)$. Hence another admissible numerical representation would be $R(x,y) = 1, R(z,w) = 0.64, R(y,x) = 0.49$ and $R(w,z) = 0.16$, which gives a rank-reversal in valuations since now $P(x,y) = 0.51 > P(z,w) = 0.48$. Therefore, such a definition of P is not meaningful when membership in R is valued on an ordinal scale.

Interestingly enough, although it is commonly believed that using minimum for conjunction is natural to handle ordinal valuations, the choice of min or nilpotent minimum for T in Eq. 16.1 does not solve the problem either. The nilpotent minimum [81, 102] $\underline{\min}(x, y)$ returns $\min(x, y)$ if $x > 1 - y$ and 0 otherwise. It has properties similar to Łukasiewicz t-norm in the sense that its induced material implication of the form $1 - \underline{\min}(x, 1 - y)$ is also its residuation $x \to y = 1$ if $x \leq y$ and $\max(1 - x, y)$ otherwise. Yet, for example, if $R(x, y) = 1, R(z, w) = 0.8, R(y, x) = 0.5$ and $R(w, z) = 0.4$ we have $P(x, y) = 0.5 < P(z, w) = 0.6$ whereas with $R(x, y) = 1, R(z, w) = 0.64, R(y, x) = 0.25$ and $R(w, z) = 0.16$ we get now $P(x, y) = 0.75 > P(z, w) = 0.64$. Other definitions such as

$$P(x, y) = \begin{cases} R(x, y) & \text{if } R(x, y) > R(y, x) \\ 0 & \text{otherwise} \end{cases}$$

proposed in [95] would better fit to ordinal membership values. Quite surprisingly, this construction is less frequent than (16.1), probably because it induces discontinuities in the definition of strict preferences.

16.3.3 Fuzzy Preference Relations Expressing Uncertainty

The above type of fuzzy relations presupposes that objects to be compared are known precisely enough to allow for a precise quantification of preference intensity. However there is another possible explanation of why preference relations should be valued, and this is when the objects to be compared are ill-known even if the preference between them remains crisp. Then $R(x, y)$ reflects the *likelihood of a crisp weak preference* $x \succeq y$. Under this interpretation, some valued relations directly refer to probability. Probability of preference is naturally encoded by valued tournament relations [26], letting

$$R(x, y) = Prob(x \succ y) + \frac{1}{2} Prob(x \sim y),$$

where $x \sim y \iff x \succeq y$ and $x \succeq y$, which implies $R(x, y) + R(y, x) = 1$. Uncertainty about preference can be defined by a probability distribution P over possible preference relations, i.e., $T_i \subset S \times S$ with $x \succeq_i y \iff (x, y) \in T_i$ and $P(T_i) = p_i, i = 1, \ldots, N$. Then

$$R(x, y) = \sum_{i: x \succ_i y} p_i + \sum_{i: x \sim_i y} \frac{1}{2} p_i.$$

More details can be found in [28]. This comes close to the setting of voting theory, historically the first suggested framework for interpreting (what people thought could be understood as) fuzzy relations [10].

This approach also applies when the merit of alternatives x and y can be quantified on a numerical scale and represented by a probability distribution on this scale. Then, $R(x, y) = P(u(x) > u(y))$, where $u : S \to \mathbb{R}$ is a utility function. Calling such valued tournament relations fuzzy can be misleading in the probabilistic setting, unless one considers that fuzzy just means gradual. However, what is gradual here is the likelihood of preference, the latter remaining a crisp notion, as opposed to the case when shades of preference are taken into account. Only when modelling preference intensity does a valued relation fully deserve to be called fuzzy.

Other uncertainty theories can be used as well to quantify uncertain preference, including possibility theory, i.e., $R(x, y) = \Pi(x \succeq y)$ is the degree of possibility of preference. It is such that $\max(R(x, y), R(y, x)) = 1$ since $\max(\Pi(x \succeq y), \Pi(y \succeq x)) = 1$ in possibility theory. In this case again, the underlying scale for $R(x, y)$ is negative unipolar, which does correspond to similar conventions as the gradual extension of outranking relations outlined above. However, in the possibilistic uncertainty setting, $1 - R(x, y) = N(y \succ x)$ corresponds to the degree of certainty of a strict preference. This kind of valued relations is closely akin to interval orderings [106] and the comparison of fuzzy intervals is discussed later on in this paper.

It is clear that when defining transitivity and extracting a ranking of elements from a fuzzy preference relation, the above considerations play a crucial role. According to whether valued relations refer to preference intensity or likelihood of strict preference, whether the value scale is bipolar or not, transitivity will take very different forms (e.g., compare cycle transitivity [26, 28] and usual fuzzy relation transitivity). Also the procedures for choosing best elements in the sense of a valued preference relation [80] should depend on the nature of the value scale and on the way preference degrees are interpreted.

16.3.4 Transitivity and Arrow's Theorem

Arrow's theorem [3] shows the impossibility of aggregating n preference relations $R_1, \ldots, R_n$ into a complete preference weak-order R while respecting some desirable conditions. Any relation R_i can be seen as an individual ranking of candidates in a multi-agent decision problem. Alternatively relation R_i can be seen as a ranking of alternatives in a multicriteria problem. Hence Arrow's theorem has a significant impact both in the context of voting and in multicriteria analysis. The result admits several versions, but the standard conditions at the origin of the impossibility result are *universality* (every n-tuple of preference orders is admissible), *unanimity* (x must be preferred to y when the n preference relations unanimously support this assertion), *independence* (preference between two alternatives only depend on their relative position in the n preference relations), *transitivity and completeness* (the result of the aggregation must be a complete weak-order) and *non-dictatorship* (for all i there exists an input profile $(R_1, \ldots, R_n)$ such that $R_i \neq R$). After the publication of this result, various attempts to escape Arrow's framework in order to solve ordinal aggregation problems have been proposed in the literature on Social Choice. The reader is referred to [113] for a survey on the main impossibility results obtained in Social Choice Theory. There are several reasons that can explain these negative results. One of the main arguments given is that the preference information contained in the n preference orders R_i, $i = 1, \ldots, n$ is too poor to properly solve conflicts in the aggregation process. If we use richer structures, we can hope to escape Arrow's framework. Another argument is that the overall preference must be a weak order, which is a strong constraint. Resorting to fuzzy relations either in the description of individual preferences or in the expression of the overall preferences resulting from the aggregation process can be seen as possible ways of introducing more flexibility and escaping Arrow's framework. We briefly discuss below these two options.

An example of positive aggregation result obtained with fuzzy relations is due to Ovchinnikov [96]. The idea is to relax the transitivity and completeness axiom of overall preferences by admitting *fuzzy* preference orders as possible results of the aggregation. More precisely, the overall preference relation must be a fuzzy binary relation satisfying a T-transitivity condition, defined for any t-norm T as follows:

$$R(x, y) \geq T(R(x, z), R(z, y)) \tag{16.2}$$

In his paper, Ovchinnikov suggests resorting to the Łukasiewicz t-norm and define the transitivity of the fuzzy social preference R obtained by aggregation of individual preferences R_i. This amounts to defining transitivity as follows: $R(x, y) \geq R(x, z) + R(z, y) - 1$. Under this relaxed transitivity constraint, he shows that Arrow's impossibility result does not hold anymore: a "transitive" social ordering can be obtained from any profile $(R_1, \ldots, R_n)$ of crisp relations while preserving all usual desirable properties in the aggregation (universality, independence, unanimity, non-dictatorship). Suppose indeed we want to aggregate the preferences of n voters. For all pair (x, y) of candidates, we define $R_i(x, y) = 1$ if agent i prefers x to y and $R_i(x, y) = 0$ otherwise. Hence we can define the collective fuzzy preference relation by $R(x, y) = 1/n \sum_{i=1}^{n} R_i(x, y)$. Since individual preferences are transitive we have $R_i(x, y) \geq R_i(x, z) + R_i(z, y) - 1$ for all x, y, z and all $i = 1, \ldots, n$ and therefore $R(x, y) \geq R(x, z) + R(z, y) - 1$ by addition. However, this apparently positive result is misleading: it turns out that the transitivity condition used is so weak that is not incompatible with Condorcet cycles, as shown in the following example.

Example 1. Consider a decision problem with 3 candidates $\{a, b, c\}$ and assume that $n = 3p$ for some positive integer p and consider a preference profile with the following relation:

p voters have preferences aR_ibR_ic,
p voters have preferences bR_icR_ia,
p voters have preferences cR_iaR_ib.

We obtain: $R(a, b) = 2/3, R(b, c) = 2/3$ and $R(c, a) = 2/3$ but also $R(b, a) = 1/3, R(c, b) = 1/3$ and $R(a, c) = 1/3$ a typical cyclic preference with respect to majority. We remark indeed that the α-cut of R with $\alpha = 1/2$ includes a directed cycle; this is usually considered as a typical problematic situation in which majority is not decisive. It does not permit us to designate a winner nor to rank the candidates. Yet, such a relation is "transitive" since $R(x,y) \geq R(x,z) + R(z,y) - 1$ holds for any triple of alternatives.

This example shows that this notion of transitivity used for fuzzy relations is so weak that it subsumes Condorcet effects (usual observed as cyclic preferences). This weakening of transitivity does not provide any real practical solution to the problem raised by Arrow's theorem.

Besides, several authors have considered a stronger version of T-transitivity called min-transitivity ($T = \min$) for fuzzy social preferences [6, 7, 59]. This condition is more significant because any α-cut of a min-transitive relation is a transitive relation. Hence, being able to construct min-transitive social preferences should be a way of getting partial orders for social preference. Unfortunately the results obtained under the min-transitivity constraint are largely negative: either there exists a dictator or an oligarchy that concentrate all the decisive power. This shows that considering social fuzzy preferences does not help that much in overcoming problems related to ordinal aggregation and Arrow's theorem.

Another attempt to incorporate fuzzy preferences in these problems is to let individuals express richer preferences with valued relations, so as to take preference intensities into account. For example, let us assume that one-dimensional preferences are defined from individual utility functions u_i, for any dimension i by $R_i(x,y) = u_i(x) - u_i(y)$. By construction we have $R_i(x,y) + R_i(y,z) = R_i(x,z)$ for all i. Hence if $R(x,y)$ is defined as a weighted average of preference indices $R_i(x,y)$ then we get $R(x,y) + R(y,z) = R(x,z)$ and therefore $\min\{R(x,y), R(y,z)\} \leq R(x,z)$ for all x, y, z. Hence we get a min transitive relation by weighted aggregation. Unfortunately this cannot be seen as an original solution to the aggregation problem because, in this case, the transitivity of social preferences directly derives from the existence of an additive utility function defined by an average $u(x) = \frac{\sum_{i=1}^{n} u_i(x)}{n}$. We have indeed, by construction, $R(x,y) = u(x) - u(y)$ and the fuzzy relation R is just another presentation of a classical additive utility model.

16.4 Fuzzy Outranking Relations in Multicriteria Decision Problems

We briefly review here some basic techniques used to construct fuzzy binary outranking relations from ratings according to various criteria. We will then discuss their use to derive recommendations. For the sake of illustration, we consider a multicriteria decision problem characterized by a finite set X of alternatives and $f_1, \ldots, f_n$, n objective functions (modelling criteria) to be maximized. Any solution $x \in X$ is characterized by a vector $(x_1, \ldots, x_n)$ where $x_i = f_i(x)$ is a value measuring the attractiveness of x with respect to criterion i.

16.4.1 Fuzzy Concordance Relations

The standard definition of a rating function consists in stating that x is strictly preferred to y with respect to criterion i (denoted by xP_iy) as soon $x_i > y_i$. When $x_i = y_i$, alternatives x and y are seen as indifferent (denoted by xI_iy) with respect to criterion i. However, in most cases, small differences of performance are not sufficient to justify a strict preference and it is commonly assumed that differences of evaluations that remain below a given threshold q_i are not characteristic of a strict preference. Hence a more general model is often used in which xP_iy as soon as $x_i - y_i > q_i$ and xI_iy whenever $|x_i - y_i| \leq q_i$. Threshold q_i is named the indifference threshold, it is positive but not necessarily constant and may vary along the criterion axis, making structure (I_i, P_i) an interval-order [68, 103]. It is used to partition the set of pairs of alternatives for a given i into two sets of pairs, pairs concordant with strict preference and pairs concordant with indifference. However the precise definition of such a threshold is difficult, especially on continuous criterion scales, because it imposes to define a crisp separation on a continuum of situations characterized by more or less important preference differences. The introduction of fuzzy sets here allows a more cautious construction, enabling a gradual transition from indifference to strict preference. More precisely three fuzzy preferences relations are constructed for every criterion $i \in \{1, \ldots, n\}$:

$$\begin{array}{lll} \text{strict preference: } & P_i(x,y) & = t_i(x_i, y_i) \\ \text{weak preference: } & S_i(x,y) & = 1 - t_i(y_i, x_i) \\ \text{indifference: } & I_i(x,y) & = 1 - \max\{t_i(x,y), t_i(y,x)\} \end{array} \quad (16.3)$$

where t_i is a function from $\mathbb{R}^2$ to $[0, 1]$, non-decreasing of the first argument and non-increasing of the second argument, such that $t_i(x, x) = 0$. Within the interval $[q_i^-, q_i^+]$ of admissible values for q_i, any difference of type $x_i - y_i$ corresponds to some hesitation between indifference and strict preference. Outside this interval it is expected that $t_i(x_i, y_i) = 0$ when $x_i - y_i \leq q_i^-$ and $t_i(x_i, y_i) = 1$ when $x_i - y_i \geq q_i^+$. Such a construction appears for example in the Promethee method for P_i [18] and in Electre methods [109] for S_i and in various variants [63, 100, 102] under the following form:

$$t_i(x,y) = \begin{cases} 1 & \text{if } x_i - y_i > q_i^+ \\ \frac{x_i - y_i - q_i^-}{q_i^+ - q_i^-} & \text{if } q_i^- < x_i - y_i \leq q_i^+ \\ 0 & \text{if } x_i - y_i \leq q_i^- \end{cases} \quad (16.4)$$

Then an overall fuzzy relation C_P (resp. C_I, C_S) can be obtained by aggregation of one-dimensional preference relations P_i (resp. I_i, S_i). More precisely $C_P(x,y) = \psi(P_1(x,y), \ldots, P_n(x,y))$ where ψ is an aggregation function defined from $[0, 1]^n$ to $[0, 1]$. A standard choice for ψ is the weighted sum but other aggregation functions could be considered as well, including median and other order-statistics, quasi-arithmetic means, ordered weighted averages, Choquet or Sugeno integrals [74].

Relation C_P gives, for every pair of alternatives (x, y), the degree to which the criteria support the preference judgement xPy. Other constructions in the same spirit have also been proposed, with some variations, for instance in Electre III [107] and Mappac methods [89]. The main advantage of all these constructions is that fuzzy preference indices resulting from the aggregation process continuously depend on rating values for individual criteria. Hence slight variations of these values cannot entail drastic changes in preference judgements.

One of the key questions with this approach concerns the choice of function ψ to aggregate fuzzy relations P_i. This choice is rarely justified in methodological papers. In particular, in Electre-like methods, P_i is presented as a credibility index with an ordinal semantics. An inequality of type $P_i(x, y) > P_i(z, y)$ only relies on the fact that $x_i > z_i$ and another inequality of type $P_i(x, y) > P_i(z, w)$ only relies on the fact that $x_i - y_i > z_i - w_i$. Hence the valuation scale used for fuzzy relations P_i should be interpreted ordinally, P_i being fuzzy because it represents a nested family of semi-orders, in which every α-cut of P_i corresponds to a given value of q_i within $[q_i^-, q_i^+]$. Hence any strictly increasing automorphism ϕ of the unit interval should preserve the information contained in fuzzy relations P_i. However, under such assumptions, one should not choose a weighted sum for ψ because we would no longer be able to derive meaningful conclusions from fuzzy relation C_P resulting from the aggregation of relations P_i, as shown in the following example:

Example 2. Consider a decision problem with two alternatives $\{a, b\}$ and three criteria leading to the following one-dimensional preference relations:

$$P_1 = \begin{pmatrix} 0 & 0 \\ 0.4 & 0 \end{pmatrix} \quad P_2 = \begin{pmatrix} 0 & 0 \\ 0.6 & 0 \end{pmatrix} \quad P_3 = \begin{pmatrix} 0 & 0.8 \\ 0 & 0 \end{pmatrix}$$

$$C_P = \frac{P_1 + P_2 + P_3}{3} = \begin{pmatrix} 0 & 0.27 \\ 0.33 & 0 \end{pmatrix}$$

If C_P is defined by a weighted sum as above, then we get $C_P(a, b) = 0.27 < C_P(b, a) = 0.33$ and therefore it seems natural to conclude that b is preferred to a because the preference of b over a is better supported in the family of criteria considered. However, if we apply a transformation of the valuation scale $\phi(x) = x^2$, which seems to preserve fuzzy relations seen as nested family of fuzzy orders, we get the following relations $P_i'(x, y) = \phi(P_i(x, y))$ and their average $C_{P'}$:

$$P_1' = \begin{pmatrix} 0 & 0 \\ 0.16 & 0 \end{pmatrix} \quad P_2' = \begin{pmatrix} 0 & 0 \\ 0.36 & 0 \end{pmatrix} \quad P_3' = \begin{pmatrix} 0 & 0.64 \\ 0 & 0 \end{pmatrix}$$

$$C_{P'} = \frac{P_1' + P_2' + P_3'}{3} = \begin{pmatrix} 0 & 0.21 \\ 0.17 & 0 \end{pmatrix}$$

Hence we obtain $C_{P'}(a, b) = 0.21 > 0.17 = C_{P'}(b, a)$ and now it seems natural to conclude that a is preferred to b because a gets more support.

This example of preference reversal shows that the weighted sum is not so natural to aggregate fuzzy relations defined from criteria since the final decision may depend of the particular valuation scale we have used for grading preferences. One may wonder if a median or any other order statistics compatible with an ordinal scale (even Sugeno integrals) would not be better in place of the (weighted) average at this stage. Moreover, this example shows that functions t_i used in the construction of relations P_i and I_i [see Eq. (16.3)] cannot be chosen independently of each other because they actually are used as a tool to map to a common scale the differences of performances observed on various criteria.

16.4.2 Fuzzy Discordance Relations and the Veto Principle

Preference aggregation methods are largely inspired by voting theory. As the notion of concordance relation recalled above refers to the notion of weighted majority, the notion of veto is also present in several multicriteria aggregation procedures to limit the possibility of compensating pros and cons in the comparison of two alternatives having conflicting profiles. For example, in Electre methods [109], the idea of providing every criterion with a right of veto is implemented through the use of veto thresholds. To any criterion i is assigned a veto threshold v_i which is defined as the largest difference of type $y_i - x_i$ that is compatible with the overall preference xPy. Formally, we have:

$$\exists i \in \{1, \ldots, n\} : y_i - x_i > v_i \Rightarrow \text{not}(xPy)$$

When veto thresholds are used, we can penalise alternatives presenting good but irregular profiles in favour of average but well-balanced profiles. We can also favour good alternatives presenting at least one top-level quality. Moreover, we can clearly distinguish between the situation where x and y have more or less the same scores on all criteria (indifference) and the situation where each has enough advantage on one criterion to veto the other (incomparability due to the presence of conflicting criteria).

A veto threshold v_i is not necessarily constant, it can vary along the criterion axis, just as the indifference threshold. It can also be set to infinity when the criterion is not sufficiently critical to have a veto. As for indifference thresholds, assigning a proper value to a veto threshold is quite difficult because it has a drastic impact on overall preferences. It is easier to obtain an interval $[v_i^-, v_i^+]$ of values in which the difference $y_i - x_i$ becomes gradually more discordant with preference xPy as it grows from v_i^- to v_i^+. In this respect, fuzzy sets and fuzzy relations are quite useful because they allow the definition of gradual transitions from non-veto to veto situations, from absence of conflict to presence of conflict. For example, in the Electre III method [107], gradual transitions are modelled by discordance indices defined, for any criterion $i \in \{1, \ldots, n\}$, as follows:

$$d_i(x,y) = \begin{cases} 1 & \text{if } x_i - y_i > v_i^+ \\ \frac{x_i - y_i - v_i^-}{v_i^+ - v_i^-} & \text{if } v_i^- < x_i - y_i \leq v_i^+ \\ 0 & \text{if } x_i - y_i \leq v_i^- \end{cases} \tag{16.5}$$

Note that this is just an example since any strictly increasing ϕ-transformed of d_i such that $\phi(0) = 0$ and $\phi(1) = 1$ would model the same nested family of relations. To be consistent with the definition of preference P_i given in (16.3) it is necessary to have $v_i^- \geq q_i^+$ so that criterion i cannot be discordant with preference xPy if it is not fully concordant with yPx (i.e. $d_i(x,y) > 0$ implies $t_i(y,x) = 1$).

Then, to measure the strength of discordance in the set of criteria, an overall discordance index with respect to preference P is defined by:

$$D_P(x,y) = 1 - \prod_{i=1}^{n} (1 - d_i(x,y))^{w_i} \tag{16.6}$$

where w_i are positive weights that can be used to take into account the importance of criteria when rejecting a preference. The main desirable property of this construction is that $D_P(x,y) = 1$ as soon as one criterion (at least) vetoes the preference xPy. This could be achieved by a t-conorm as well, that could be used in place of the dual geometric mean in (16.6). On the other hand, $D_P(x,y)$ must be non-decreasing with respect to discordance indices $d_i(x,y)$ defined by (16.5). This discordance index can be used to define an overall fuzzy preference as follows:

$$P(x,y) = T(C_P(x,y), 1 - D_P(x,y)) \tag{16.7}$$

where T is a t-norm allowing to translate numerically the “concordance and non-discordance principle” stating that x is preferred to y if and only if the set of concordant criteria is strong enough and there is no discordant criterion. Standard choices for T are continuous t-norms, in particular the product t-norm (used in Electre III) and the minimum.

Here also, the important property resulting from Eqs. (16.4)–(16.7) is that $P(x,y)$ is a continuous function of rating values x_i and y_i for $i = 1,\ldots,n$ non-decreasing with the x_i's and non-increasing with the y_i's. The advantage of fuzziness here is to avoid prior truncation of information by enabling graded preferences instead of reducing preference analysis to all or nothing judgements. On the other hand, the definition of proper valuation scales for fuzzy concordance and discordance relations is not completely clear. Just as for concordance relations (see Example 2) the definition of an overall discordance relation D_P from fuzzy relations d_i in Eq. (16.6) assumes that values $d_i(x,y), i = 1,\ldots,n$ are expressed on a common absolute scale measuring strength of discordance. Moreover, the aggregation of values $C_P(x,y)$ and $D_P(x,y)$ to define $P(x,y)$ also requires some commensurability assumption between concordance and discordance indices, an assumption that could be questioned. One cannot indeed define properly C_P values independently of D_P

values due to their interaction in the definition of P. This point is generally not discussed precisely in papers introducing preference models based on concordance and non-discordance concepts and needs further investigation.

16.4.3 Choosing, Ranking and Sorting with Fuzzy Preference Relations

Fuzzy preference relations provide a graded information on preference or similarities between alternatives of a decision problems. They are quite useful to discriminate alternatives and derive recommendations in choice or ranking problems, but also in preference-based supervised classification problems. We mention now some examples showing how fuzzy preference relations can be used advantageously to support decision making in multicriteria decision aid. More details can be found in [63, 64, 80].

Choice Problems Choice problems consist in determining, within a set of alternatives, a subset, as small as possible, of best elements. The advantage of fuzzy preferences is to improve discrimination possibilities in defining such a set. For example a standard way of performing a selection from a fuzzy pairwise preference or weak-preference matrix R is to define the non-domination score of any alternative x in X, according to Orlovski as follows [94]:

$$\mathrm{ND}(x, X, R) = 1 - \max_{y \in X} \max\{R(y, x) - R(x, y), 0\} \tag{16.8}$$

ND can be seen as the membership function of the fuzzy set of non-dominated elements in X and defines a nested family of sets corresponding to increasing levels of requirement. Let us illustrate its use in multicriteria choice problems.

Example 3. Let us consider a choice problem on $X = \{x, y, z\}$ with three criteria $\{1, 2, 3\}$ of equal importance, and the performance table given below. If we define relation P_i using function t_i as defined in (16.3)–(16.4) we get the following C_P relation:

Criterion i	x	y	z	q_i^-	q_i^+
1	15	10	5	1	5
2	6	14	10	1	5
3	10	7	13	1	5

$$C_P = \frac{P_1 + P_2 + P_3}{3} = \begin{pmatrix} 0 & 0.49 & 0.33 \\ 0.33 & 0 & 0.58 \\ 0.42 & 0.33 & 0 \end{pmatrix}$$

If we use a standard crisp preference relation like $x \succ y$ iff $C_P(x, y) > C_P(y, x)$ (relative majority), we get a cyclic preference relation which is not very helpful to make a selection. On the contrary, if we compute a fuzzy set from ND scores, we get: $\mathrm{ND}(x, X, C_P) = 0.91$, $\mathrm{ND}(y, X, C_P) = 0.84$, $\mathrm{ND}(z, X, C_P) = 0.75$. Looking at the

α-cuts of the fuzzy set of non-dominated elements we can say that the result of the choice problem is either $\{x\}$, $\{x, y\}$, $\{x, y, z\}$ depending on the number of alternatives needed or on level of confidence required to accept a candidate.

This example shows that fuzzy relations provide better discrimination possibilities. However, the price to pay is that valued preferences need to be expressed on an absolute numerical scale to allow the computation of ND as above, which requires more information than just seeing a fuzzy relation as the set of its α-cuts. Note that score ND is only given here for illustration and many other scores defined from a preference matrix R have been proposed in the literature. In any case, the score $s(x, X, R)$ of x in X with respect to relation R must be defined as a non-decreasing function of indices $R(x, y), y \in X$ and a non-increasing function of indices $R(y, x), y \in X$, for more details see e.g. [64].

Ranking Problems Ranking problems consist in defining a partial or total weak-order on alternatives representing their relative values. It is used in recommender systems to provide the user with the k most relevant items. It is also a possible answer to choice problems in which the number of alternatives to be selected is fixed to a given k. Ranking is also natural when items or tasks must be ordered in a waiting line before being processed.

The non-domination score ND defined in (16.8) is not really appropriate to derive a ranking. Although it can be used to separate top elements from dominated elements, it does not really help in discriminating lower elements. For instance, for a matrix encoding the crisp order $a \succ b \succ c \succ d$ we have $\text{ND}(a, \{a, b, c, d\}) = 1$ but $\text{ND}(y, \{a, b, c, d\}) = 0$ for all y in $\{b, c, d\}$. For this reason, ranking on the basis of ND is usually performed by an iterated choice sequence. The first iteration consists in setting $Y_1 = X$ and defining the first equivalence class of the ranking by $X_1 = \text{ND}(Y_1) = \arg\max_{x \in X} \text{ND}(x, Y_1)$; then the iteration is performed by setting $Y_{i+1} = Y_i \setminus X_i$ and $X_{i+1} = \text{ND}(Y_{i+1})$ until no alternative remains unranked. This ranking procedure seems natural. However it hides some unexpected properties that make it questionable as illustrated by the following example:

Example 4. Let us consider a choice problem on $X = \{a, b, c, d\}$ with six criteria $\{1, \ldots, 6\}$. The performance table, indifference thresholds and criteria weights are given below. If we define relation P_i using function t_i as defined in (16.3)–(16.4) we get the following C_P relation:

Criterion i	a	b	c	d	q_i^-	q_i^+	w_i
1	15	0	10	5	0	1	0.20
2	0	15	5	10	0	1	0.20
3	15	10	5	0	0	1	0.28
4	0	10	5	15	0	1	0.16
5	5	0	15	10	0	1	0.12
6	5	0	10	15	0	1	0.04

$$C_P = \sum_{i=1}^{6} w_i P_i = \begin{pmatrix} 0.00 & 0.64 & 0.48 & 0.48 \\ 0.36 & 0.00 & 0.64 & 0.48 \\ 0.52 & 0.36 & 0.00 & 0.60 \\ 0.52 & 0.52 & 0.40 & 0.00 \end{pmatrix}$$

Hence a is the most preferred element with non-domination score $\mathrm{ND}(a, \{a, b, c, d\}, C_P) = 0.96$. Then, after removing a, we get b as second best element with $\mathrm{ND}(b, \{b, c, d\}, C_P) = 0.96$, then c and finally d yielding the following ordering: $a \succ b \succ c \succ d$. Now, if we increase the grade of c on criterion 1 from 10 to 20, then matrix C_P is only impacted on row 3 column 1 where we get 0.44 instead of 0.04. If we apply again the ranking by iterated choices on the modified matrix, we get first d with $\mathrm{ND}(d, \{a, b, c, d\}, C_P) = 0.8$ then, after removing d we get b, c indifferent and finally a which gives $d \succ b \sim c \succ a$. We can remark that initially we had $c \succ d$ and now we have $d \succ c$. The position of c in the final ranking decreased while one of its grades increases. This non-monotonicity of the ranking procedure is a bad property that illustrates the difficulties that may occur in designing ranking procedures from choice functions. This problem is quite general and could be observed with other scoring functions. Fortunately, some scoring functions do not need to be recomputed within iterated choice sequences and are easier to incorporate in ranking procedure. This is the case, for instance, of the net flow ranking method that consists in directly using score $s(x, X, R) = \sum_{y \in X} R(x, y) - \sum_{y \in X} R(y, x)$ as an overall performance index. This is the option chosen in the Promethee method [18].

Another possibility to build a ranking or at least a partial order is to construct from R a fuzzy relation Z which is min-transitive, i.e. $Z(x, z) \geq \min\{Z(x, y), Z(y, z)\}$. The advantage of such a fuzzy relation is that its α-cuts form a nested family of partial orders corresponding to more discriminating relations as the cutting level increases. Such min-transitive relations can be obtained by constructing covering relations or transitive closures from the initial relation R. Here are some examples using relation P defined from R by $P(x, y) = \max\{R(x, y) - R(y, x), 0\}$:

- *transitive closure:* $Z(x, y) = \max\{P^*(x, y) - P^*(y, x), 0\}$ where P^* is the transitive closure[2] of P
- *forward covering:* $Z(x, y) = \min_{z \in Y} \min\{1 - P(y, z) + P(x, z), 1\}$
- *backward covering:* $Z(x, y) = \min_{z \in Y} \min\{1 - P(z, x) + P(z, y), 1\}$

for more details see e.g. [64].

Sorting Problems Preference-based sorting problems consist in assigning the alternatives to predefined categories on the basis of their intrinsic qualities. Sorting procedures are usually based on the comparison of the alternatives with respect to reference points. These reference points represent either the (lower or upper) frontier of a preference class, or typical examples of the category under consideration. So, preference-based sorting procedures can be seen as supervised classification methods making use of preference or indifference relations to assign the alternatives to categories. Several examples of such procedures have been proposed in multicriteria decision analysis, see e.g. [75, 76, 88, 100, 108, 112].

Fuzzy preferences are quite useful also in this area because they allow the definition of fuzzy categories that partially overlap each other, thus leaving room

[2]For more details on the computation of the max-min closure of a fuzzy relation see [38].

for hesitation or ambiguity in the assignment. This offers a quite natural descriptive possibility. Between two typical alternatives from two distinct categories there is indeed a continuum of less typical intermediary situations, and deciding whether an alternative should belong to one category or the other is often difficult. Let us give some very simple examples of procedures using fuzzy preference and indifference relations to define membership functions characterizing categories:

- **The case of ordered categories:** assume that a sequence of reference points $r^1, \ldots, r^q$ ($r^i \in \mathbb{R}^n$) has been defined in the space of criteria such that $r_i^{k+1} - r_i^k > q_i$ for all criteria i and categories k (q_i is the indifference threshold) so as to define frontiers between ordered categories. For example, we can define category C_k as the set of alternatives that are preferred to r^k but not preferred to r^{k+1} one can define membership to categories by $\mu_{C_k}(x) = \min\{P(x, r^k), 1 - P(x, r^{k+1})\}$ which is a direct translation through a standard multivalued logic of the definition of the category.
- **The case of non-ordered categories:** assume that q sets of reference points $R^1, \ldots, R^q$ have been defined in the space of criteria, R^k representing typical elements of category C_k. Then if C_k represents the category of elements that are indifferent to some element of R^k one can define membership to categories by $\mu_{C_k}(x) = \max_{y \in R^k}\{I(x, y)\}$ where I is a fuzzy indifference relation constructed from rating values. For example I may be a concordance relation C_I as previously defined. The max operation can be replaced by a t-co-norm so as to enable some reinforcement when an alternative x is indifferent to several reference points in the same set R^k.

Membership functions $\mu_{C_k}(x)$ can be cut at a given threshold to obtain a crisp (but possibly ambiguous) assignment of alternatives to categories. Alternatively, any solution x can be assigned to the most likely category, i.e. the category that maximizes $\mu_{C_k}(x)$ over all possible k. Further details on these methods are available in [100].

16.5 Fuzzy Connectives for Decision Evaluation in the Qualitative Setting

Fuzzy sets connectives have triggered a considerable development of aggregation operators for decision evaluation [8, 74, 116]. It was the pioneering Bellman-Zadeh's paper that popularized a non-compensatory operation (the minimum), in place of averaging, for aggregation processes in multi-objective problems. Yet, this mode of aggregation had been already extensively used since the 1940s in non-cooperative game theory, and more recently in bottleneck optimisation. However, Bellman-Zadeh's proposal has sometimes been misunderstood, as to its actual role. In fact this non-compensatory approach also pioneered later developments in the literature on soft constraint satisfaction methods [54]. In particular, the non-compensatory max-min approach and its refinements stands in opposition to the traditional optimisation literature where constraints are crisp, as well as to the systematic use of averages and their extensions for aggregating criteria. The framework of aggregation operations takes a very general view that subsumes the two traditions. Rather than providing another survey of aggregation operations that are already documented in the above-cited recent books, we discuss the issue of qualitative approaches that can be developed in the light of current developments so as to overcome the above critique of linguistic scales in Sect. 2.4.2.

16.5.1 Aggregation Operations: Qualitative or Quantitative

The nature of the value scale employed for rating the worth of decisions dictates whether an aggregation operation is legitimate or not. Should we use a qualitative or a quantitative approach? There are pros and cons. We are faced with a modeling dilemma.

Using quantitative scales, we dispose of a very rich framework:

- We can account for very refined aggregation attitudes, especially trade-off, compensation and dependence between criteria
- A very fine-grained ranking of alternatives can be obtained.
- The aggregation technique can be learned from data.
- However, numerical preference data are not typically what decision-makers provide.

Qualitative approaches (ordinal or qualitative scales) may look more anthropomorphic. Indeed, contrary to what classical decision theory suggests, people can make decisions in the face of several criteria, sometimes without numerical utility nor criteria importance assessments (see the works by Gigerenzer [70], for instance). However it is well-known that people make little sense of refined absolute value scales (not more than seven levels). More precisely, in a qualitative setting:

- We are closer to the information humans can actually supply.
- We can nevertheless model preference dependence using graphical models (see the recent literature on CP-nets [15]).
- Making small qualitative scales commensurate with each other is easier.
- But the choice of aggregation operations is very limited (it ranges from impossibility theorems in the ordinal case, to only min and max and their combinations in the qualitative case).
- Finite value scales induce a strong lack of discrimination: the set of potential decisions will be clustered into as many groups of indifferent alternatives as the number of levels in the absolute scale.
- It is not clear how to handle qualitative bipolar information (pros and cons).

In fact there are discrete t-norms other than minimum on finite scales. The main alternative is Łukasiewicz discrete t-norm, that is a truncated sum. This choice underlies assumptions on the meaning of a qualitative scale $L = \{\lambda_0, \lambda_1, \dots, \lambda_n\}$,

1. Like with the 2-tuple method, L is mapped to the integers: $\lambda_i = i$. In particular, λ_i is understood as being i times stronger than λ_1.
2. There is a saturation effect that creates counterintuitive ties when aggregating objective functions in this setting.

So this approach is not really qualitative, and not very attractive altogether at the practical level. In fact it is important to better lay bare the meaning of a qualitative value scale, and point out the assumptions motivating the restriction of aggregation operations to min and max. Using a qualitative scale, two effects can be observed:

1. **Negligibility effect:** Steps in the evaluation scale are far away from each other. It implies a strong focus on the most likely states of nature, on the most important criteria. This is what implies a lack of compensation between attributes. For instance, aggregating five ratings by the minimum, $\min(5, 5, 5, 5, 1) < \min(2, 2, 2, 2, 2)$: many 5's cannot compensate for a 1 and beat as many 2's.
2. **Drowning effect:** There is no comparison of the number of equally satisfied attributes. The rating vector $(5, 5, 5, 5, 1)$ is worth the same as $(1, 1, 1, 1, 1)$ if compared by means of the min operation. It means that we refrain from counting.

It is clear that focusing on important criteria is something expected from human behavior [70]. However the equivalence between $(5, 5, 5, 5, 1)$ and $(1, 1, 1, 1, 1)$ is much more debatable and conflicts with the intuition, be it because the latter Pareto-dominates the former. The main idea to improve the efficiency of qualitative aggregation operations is to preserve the negligibility effect, while allowing for counting. Note that if we build a preference relation on the set of alternatives rated on an absolute scale L on the basis of pairwise comparisons made by the decision-maker, one may get chains of strictly preferred alternatives with length $m >| L |$. So, humans discriminate better on pairwise comparisons than using absolute value scales.

16.5.2 Refinements of Qualitative Aggregation Operations

Let V be a set of alternatives, and assume a unique finite value scale L for rating n criteria, L being small enough to ensure commensurability. At one extreme, one may consider the smallest possible value scale $L = \{0, 1\}$. So each alternative is modelled by a Boolean vector $\vec{u} = (u_1, u_2, \dots, u_n) \in \{0, 1\}^n$. Let $\succeq$ denote the overall preference relation over $\{0, 1\}^n$, supposed to be a weak order. Suppose without loss of generality that criteria are ranked in the order of their relative importance (criterion i as at least as important as criterion $i + 1$). Three principles for a qualitative aggregation operations should be respected for the aggregation to be rational in a pairwise comparison

1. **Focus effect**: If one alternative satisfies the most important criterion where the ratings of the two alternatives differ then it should be preferred. Formally it reads as follows: given two vectors of ratings $\vec{u}$ and $\vec{v}$, if $u_i = v_i, i = 1, \dots, k-1$, and $u_k = 1, v_k = 0$, where criterion k is strictly more important than criterion $k+1$, then $\vec{u} \succ \vec{v}$
2. **Compatibility with strict Pareto-dominance** (CSPD): If an alternative satisfies only a subset of criteria satisfied by another then the latter should be preferred.
3. **Restricted compensation**: If an alternative satisfies a number of equally important criteria greater than the number of criteria of the same importance satisfied by another alternative, on the most important criteria where some ratings differ, the former alternative should be preferred.

Strict Pareto-Dominance is defined for any value scale as $\vec{u} >_P \vec{v}$ if and only if $\forall i = 1, \dots n, u_i \geq v_i$ and $\exists j, u_j > v_j$. Then the CSPD principle reads:

$$\vec{u} >_P \vec{v} \text{ implies } \vec{u} \succ \vec{v}.$$

Clearly, the basic aggregation operations min and max violate strict Pareto-Dominance. Indeed we may have $\min_{i=1,\dots,n} u_i = \min_{i=1,\dots,n} v_i$ while $\vec{u} >_P \vec{v}$. In fact, in the finite case, there is no strictly increasing function $f : L^n \to L$. So any aggregation function on a finite scale will violate strict Pareto-Dominance. But just applying the latter to L^n, the obtained partial order on V contains chains $\vec{v_1} >_P \vec{v_2} >_P \cdots >_P \vec{v_m}$ much longer that the numbers of elements in the value scale. Given that we take the negligibility effect for granted, the approach to mend these basic operations is thus not to change them, but to refine them. Two known methods recover Pareto-dominance by refining the min-ordering (see [43] for a bibliography):

- Discrimin: $\vec{u} >_{dmin} \vec{v}$ if and only if $\min_{i:u_i \neq v_i} u_i > \min_{i:u_i \neq v_i} v_i$
- Leximin: Rank $\vec{u}$ and $\vec{v}$ in increasing order: let $\vec{u^\sigma} = (u_{\sigma(1)} \leq u_{\sigma(2)} \leq \cdots \leq u_{\sigma(n)})$ and $\vec{v^\tau} = (v_{\tau(1)} \leq v_{\tau(2)} \leq \cdots \leq v_{\tau(n)}) \in L^n$, then $\vec{u} >_{lmin} \vec{v}$ if and only if $\exists k, \forall i < k, u_{\sigma(i)} = v_{\tau(i)}$ and $u_{\sigma(k)} > v_{\tau(k)}$

The Discrimin method deletes vector positions that bear equal values in $\vec{u}$ and $\vec{v}$ prior to comparing the remaining components. The leximin method is similar but it cancel pairs of equal entries, one from each vector, regardless of their positions. Similar refinements of the maximum operation, say Discrimax and Leximax can be defined.

Clearly, $\vec{u} >_P \vec{v}$ implies $\vec{u} >_{dmin} \vec{v}$ which implies $\vec{u} >_{lmin} \vec{v}$. So by constructing a preference relation that refines a qualitative aggregation operation, we recover a good behavior of the aggregation process without needing a more refined absolute scale.

The minimum and the maximum aggregation operations can be extended so as to account for criteria importance. Consider a weight distribution $\vec{\pi}$ that evaluates the priority of criteria, with $\max \pi_i = 1$. Consider the order-reversing map $\nu(\lambda_i) = \lambda_{m-i}$ on a scale with $m + 1$ steps. The following extensions of the minimum and the maximum are now well-known:

- **Prioritized Maximum**: $P\max(\vec{u}) = \max_{i=1,\ldots n} \min(\pi_i, u_i)$ Here $P\max(\vec{u})$ is high as soon as there is an important criterion with high satisfaction rating.
- **Prioritized Minimum**: $P\min(\vec{u}) = \min_{i=1,\ldots,n} \max(\nu(\pi_i), u_i)$ Here $P\min(\vec{u})$ is high as soon as all important criteria get high satisfaction ratings.
- **Sugeno Integral**: $S_{\gamma,u}(f) = \max_{\lambda_i \in L} \min(\lambda_i, \gamma(U_{\lambda_i}))$ where $U_{\lambda_i} = \{i, u_i \geq \lambda_i\}$ and $\gamma : 2^S \mapsto L$ ranks groups of criteria.

In the last aggregation scheme, $\gamma(A)$ is the priority degree of the group of criteria $A \subseteq \{1, \ldots, n\}$. It is a capacity, i.e., if $A \subseteq B$ then $\gamma(A) \leq \gamma(B)$. When γ is a possibility (resp. necessity) measure, i.e., $\gamma(A) = \max_{i \in A} \gamma(\{i\})$ (resp. $\min_{i \notin A} \nu(\gamma(\{i\}))$ then the Prioritized Maximum $P\max$ (resp. Minimum $P\min$) operation is retrieved. These operations have been used in decision under uncertainty (as a substitute to expected utility [57]) and for criteria aggregation with finite value scales [105].

The leximin and leximax operations can be extended in order to refine $P\max$ and $P\min$. The idea is as follows (Fargier and Sabbadin [60]). Given a totally ordered set $(\Omega, \trianglerighteq)$ the leximin and leximax relations $>_{lmin}$ and $>_{lmax}$ compare vectors in Ω^n, based on comparing values using the relation $\trianglerighteq$. Call these techniques Leximax($\trianglerighteq$), Leximin($\trianglerighteq$). For the leximin and leximax comparison of utility vectors, we use $\Omega = L$ and $\trianglerighteq = \geq$.

Iterating this scheme allows for a comparison of matrices with entries in L. Namely let $H = [h_{i,j}]$ be such a matrix, $\Omega = L^n$. The Leximax($\succeq_{lmin}$) relation (where $\trianglerighteq = \succeq_{lmin}$) can be used for comparing the rows $H_{j\cdot}$ of the matrix:

$$F \succeq_{lmax(lmin)} G \Leftrightarrow \begin{cases} \forall j, F_{(j)\cdot} \sim_{lmin} G_{(j)\cdot} \text{ or} \\ \exists i \text{ s.t. } \forall j > i, F_{(j)\cdot} \sim_{lmin} G_{(j)\cdot} \text{ and } F_{(i)\cdot} \succ_{lmin} G_{(i)\cdot} \end{cases}$$

where $H_{(i)\cdot} = i^{th}$ row of H, reordered increasingly. It takes the minimum on elements inside the rows, and the leximax across rows. To compute this ordering, we must shuffle the entries of each matrix so as to rank values on each line in increasing order, and rows top-down in decreasing lexicographic order. Then compare the two matrices lexicographically, first the top rows, then, if equal, the second top rows, etc.

Example 5. : Consider the comparison of the two matrices.

$F =$

7	**3**	4	8	5
6	**3**	7	4	9
5	6	**3**	7	7

$G =$

8	**3**	**3**	5	9
3	7	**3**	8	4
7	**3**	8	5	5

It is clear that $\max_i \ \min_j \ f_{i,j} = \max_i \ \min_j \ g_{i,j}$. Reordering increasingly inside lines:

$$F' = \begin{array}{|c|c|c|c|c|}\hline 3&4&5&7&8\\\hline 3&4&6&7&9\\\hline 3&5&6&7&7\\\hline\end{array} \qquad G' = \begin{array}{|c|c|c|c|c|}\hline 3&3&5&8&9\\\hline 3&3&4&7&8\\\hline 3&5&5&7&8\\\hline\end{array}$$

Then rows are rearranged reordered top down in the sense of leximax. It is clear that (see bold face entries):

$$\begin{array}{|c|c|c|c|c|}\hline 3&5&\mathbf{6}&7&7\\\hline 3&4&6&7&9\\\hline 3&4&5&7&8\\\hline\end{array} \quad \succ_{lmax(lmin)} \quad \begin{array}{|c|c|c|c|c|}\hline 3&5&\mathbf{5}&7&8\\\hline 3&3&5&8&9\\\hline 3&3&4&7&8\\\hline\end{array}$$

The Leximax($\succeq_{lmin}$) relation is a (very discriminative) complete and transitive relation. Two matrices are equally preferred if and only if they have the same coefficients up to an increasing reordering inside rows and a decreasing reordering of columns. It refines the maximin comparison of matrices based on computing $\max_i \ \min_j \ h_{i,j}$. Likewise we can define Leximin($\succeq_{lmin}$), Leximax($\succeq_{lmax}$); Leximin($\succeq_{lmax}$).

These notions are instrumental to refine the prioritized maximum and minimum. In the prioritized case, alternatives $\vec{u}$ are encoded in the form of $n \times 2$ matrices $F^u = [f_{ij}]$ with $f_{i1} = \pi_i$ and $f_{i2} = u_i, i = 1, \ldots, n$: It is then clear that $P\max(\vec{u}) = \max_{i=1,n} \min_{j=1,2} f_{ij}$. Hence $P\max$ is refined by the *Leximax*(*Leximin*($\geq$)) procedure:

$$P\max(\vec{u}) > P\max(\vec{v}) \implies F^u \succ_{lmax(\succeq lmin)} F^v.$$

The prioritized minimum can be similarly refined applying Leximin($\succeq_{lmax}$) to matrices $F^u = [f_{ij}]$ with $f_{i1} = n(\pi_i)$ and $f_{i2} = u_i$. It is easy to verify that the Leximin($\succeq_{lmax}$) and Leximax($\succeq_{lmin}$) obey the three principles of Focus Effect, Strict Pareto-Dominance, and Restricted Compensation.

The same kind of trick can be applied to refine the ordering induced by Sugeno integrals. But the choice of the matrix encoding alternatives depends on the form chosen for expressing this aggregation operation. For instance, using the form proposed above, you can choose $f_{i1} = \lambda_i, f_{i2} = \gamma(U_{\lambda_i})$. This choice is not the best one, as shown in [41]. Moreover, part of the lack of discrimination is due to the capacity γ itself that estimates the importance of groups of criteria. In order to refine the capacity γ one idea is to generalize the leximax refinement of possibility measures [55]. To this end, it is useful to consider the so-called "Qualitative" Moebius transform [72] of γ:

$$\gamma_{\#}(A) = \gamma(A) \text{ if } \gamma(A) > \max_{B \subsetneq A} \gamma(B)$$
$$= 0 \ \text{ otherwise.}$$

It is such that $\gamma(A) = \max_{E\subseteq A}\gamma_\#(E)$, that is $\gamma_\#$ contain the minimal amount of information to recover γ. It is clear that if γ is a possibility measure, then $\gamma_\#(E) > 0$ only if E is a singleton, i.e., $\gamma_\#$ is a possibility distribution. The extension of the leximax refinement of possibility measures to capacities γ is then just obtained by comparing the sets $\{\gamma_\#(E), E \subseteq A\}$ and $\{\gamma_\#(E), E \subseteq B\}$ for two groups of criteria A and B, using leximax. More details on these issues can be found in [41, 42].

An interesting question is to define counterparts to discrimin and leximin procedures for any (monotonic) aggregation operation $f : L^2 \to L$ on qualitative scales. The refinement question also makes sense for continuous scales (see [83]).

16.5.3 *Numerical Encoding of Qualitative Aggregation Functions*

Additive encodings of the leximax and leximin procedures have existed for a long time in the optimisation literature, when the number of alternatives to be compared or the evaluation scale is finite (such an encoding is not possible in the continuous case). A mapping $\phi : L \to [0,1]$, where $L = \{0 < \lambda_1 < \lambda_2 < \cdots < \lambda_m = 1\}$, is said to be n-super-increasing if and only if $\phi(\lambda_i) > n\phi(\lambda_{i-1}), \forall i = 2, m$. We also assume $\phi(0) = 0$ and $\phi(\lambda_m) = 1$. Mapping ϕ is also called *big-stepped*. It is clear that for any such mapping,

$$\max_{i=1,\dots n} u_i > \max_{i=1,\dots n} v_i \text{ implies } \sum_{i=1,\dots n} \phi(u_i) > \sum_{i=1,\dots n} \phi(v_i)$$

e.g., $\phi(\lambda_i) = k^{i-m}$ for $k > n$ achieves this goal. The worst case is when

$$\max(0, 0, \dots, 0, \lambda_i) > \max(\lambda_{i-1}, \dots, \lambda_{i-1}).$$

This is a numerical representation of the leximax ordering:

Property: $\vec{u} \succ_{lmax} \vec{v}$ if and only if $\sum_{i=1,\dots n} \phi(u_i) > \sum_{i=1,\dots n} \phi(v_i)$.
Now consider the big-stepped mapping $\psi(\lambda_i) = \frac{1-k^{-i}}{1-k^{-m}}, k > n$. Again it holds that:

$$\min_{i=1,\dots n} u_i > \min_{i=1,\dots n} v_i \text{ implies } \sum_{i=1,\dots n} \psi(u_i) > \sum_{i=1,\dots n} \psi(v_i)$$

And it offers a numerical representation of the leximin ordering.
Property: $\vec{u} \succ_{lmin} \vec{v}$ if and only if $\sum_{i=1,\dots n} \psi(u_i) > \sum_{i=1,\dots n} \psi(v_i)$.
These representation results have been extended to the above refinements of the prioritized maximum and minimum [60] by means of weighted averages involving super-increasing sequences of numerical values. For instance, there exists a weighted average, say $AV_+(\vec{u})$, representing $\succeq_{lmax(\succeq_{lmin})}$ and thus

refining Pmax. Namely consider a super-increasing transformation χ of the scale L such that:

$$\max_i \min(\pi_i, u_i) > \max_i \min(\pi_i, v_i) \implies \sum_{i=1,\dots n} \chi(\pi_i) \cdot \chi(u_i) > \sum_{i=1,\dots n} \chi(\pi_i) \cdot \chi(v_i).$$

The worst case is when:

$$\max(\min(\lambda_j, \lambda_j), 0, \dots, 0) > \max(\min(\lambda_j, \lambda_{j-1}), \min(1_L, \lambda_{j-1}), \dots, \min(1_L, \lambda_{j-1})).$$

Hence the sufficient condition:

$$\forall j \in \{0, \dots, m-1\}, \chi(\lambda_j)^2 > (n+1)\chi(\lambda_{j-1}) \cdot \chi(1_L)$$

The following result then holds:

$$\vec{u} \succ_{lmax(lmin)} \vec{v} \text{ if and only if } \sum_{i=1,\dots n} \chi(\pi_i)\chi(u_i) > \sum_{i=1,\dots n} \chi(\pi_i)\chi(v_i).$$

The values $\chi(\pi_i), i = 1, \dots, n$ can be normalized in such a way as to satisfy $\sum_{i=1}^{n} \chi(\pi_i) = 1$ so that we do use a weighted average to represent $\succ_{lmax(lmin)}$.

The same kind of refinement by mapping to a numerical scale can be considered for Sugeno integrals. The idea is to use a Choquet integral. However, in order to get a minimally redundant expression of Sugeno integral, it can be put in the following form:

$$S_\gamma(\vec{u}) = \max_{A \subseteq N} \min(\gamma_\#(A), \min_{i \in A} u_i),$$

where $\gamma_\#(A)$ is the above defined qualitative Moebius transform. We can use a super-increasing transformation of $\gamma_\#$ into a mass function $m_\# : 2^S \mapsto [0,1] : m_\#(E) = \chi(\gamma_\#(E))$ in the sense of Shafer [114], such that $\sum_{E \subseteq C} m_\#(E) = 1$. The above leximax refinement of the ranking induced by γ can then be represented by means of the belief function $Bel(A) = \sum_{E \subseteq A} m_\#(E)$. When γ is a possibility measure, the refining belief function is a probability measure. The Sugeno integral can then be refined by a Choquet integral of the form (see [41, 42] for details):

$$Ch_\#^{lsug}(\vec{u}) = \sum_{A \subseteq S} m_\#(A) \cdot \min_{s \in A} \chi(u_i).$$

The lessons drawn from this line of study is that the discrimination power of qualitative aggregation methods (which in some sense are the off-springs of the Bellman-Zadeh decision framework) can be drastically increased by lexicographic refinement techniques that respect the qualitative nature of the preference information as well as the focus effect on most important issues observed in human decision behavior. Moreover these refinement techniques bring us back to standard numerical aggregation methods, that, through the use of super-increasing transformations, are robust (because qualitative in essence) contrary to many number crunching-preference aggregation methods.

16.5.4 Bipolarity in Qualitative Evaluation Processes

Cumulative Prospect Theory, due to Tversky and Kahneman [118] was motivated by the empirical finding that people, when making decisions, do not perceive the worth of gains and losses in the same way. This approach assesses the importance of positive affects and negative affects of decisions *separately*, by means of two monotonic set functions $g^+(A^+)$, $g^-(A^-)$, which respectively evaluate the importance of the set of criteria A^+ where the alternatives a score favorably, and the importance of the set of criteria A^- where they score unfavorably. For instance, one can separately compute the expected utility of the losses and of the gains, using different utility functions. Then they suggest to compute the so-called net predisposition $N(a) = g^+(A^+) - g^-(A^-)$ of each decision a in order to rank-order them in terms of preference. This view is at odds with classical decision theory where there is no distinction between gains and losses. Couched in terms of fuzzy sets, the CPT approach, which relies on the idea of bipolar information, is akin to a form of independence between membership and non-membership grades in the spirit of Atanassov [4]. However, decision works inspired by the misleadingly called intuitionistic fuzzy sets never make the connection with CPT.

This line of thought, was recently extended to the case where positive and negative affects are not independent (see [73]):

- Using bi-capacities on a bipolar scale in the form of functions $N(a) = g(A^+, A^-)$ monotonic in the first place, antimonotonic with the second one.
- So-called bipolar capacities $N(a) = (g^+(A^+, A^-), g^-(A^+, A^-))$ living on bivariate unipolar scales, keeping the positive and the negative evaluations separate.

An interesting question is then how to evaluate decisions from *qualitative* bipolar information, namely how to extend the min and max rules if there are both positive and negative arguments? This question was recently discussed by the first author with colleagues [13, 58]. In the proposed simplified setting, a set of Boolean criteria (C) is used, each of which has a polarity $p = +$ (positive) or $-$ (negative). Such criteria are called *affects*. For instance, when buying a house, the presence of a garden is a positive affect; the location of the house in a noisy or dangerous environment is a negative affect. Each affect is supposed to possess an importance level in a qualitative scale L. The focus effect is assumed in the sense that the order of magnitude of the importance of a group A of affects with a prescribed polarity is the one of the most important affect, in the group ($\gamma(A) = \max_{x \in A} \pi_i$ is a possibility measure).

Preference between two alternatives a and b is then achieved by comparing the pairs $(\Pi(A^-), \Pi(A^+))$ and $(\Pi(B^-), \Pi(B^+))$, evaluating separately positive and negative affects of a (respectively $A^+ = \{i, p(i) = +, u_i = 1\}$ and $A^- = \{i, p(i) = -, u_i = 1\}$), based on the relative importance of these affects. The most natural rule that comes to mind is called the Bipolar Possibility Relation. The principle at work (that plays the role of computing the net predisposition in the qualitative setting) is:
Comparability of positive and negative affects: When comparing Boolean vectors $\vec{u}$ and $\vec{v}$, a negative affect against $\vec{u}$ (resp. against $\vec{v}$) is a positive argument pro $\vec{v}$ (resp. pro $\vec{u}$).

Along with the way groups of affects are weighted, the decision-maker is supposed to focus on the most important affect regardless of its polarity. The following decision rule follows [58]:

$$a \succeq^{Biposs} b \iff \max(\Pi(A^+), \Pi(B^-)) \geq \max(\Pi(B^+), \Pi(A^-))$$

The relation $\succeq^{Biposs}$ is complete, but only its strict part is transitive. This relation collapses to the Bellman-Zadeh minimum aggregation rule if all affects are negative and to the maximum rule if all affects are positive (which also has something to do with Atanassov connectives). This is similar to the CPT approach where: $a > b \iff g^+(A^+) + g^-(B^-) > g^+(B^+) + g^-(A^-)$, the possibilistic rule being obtained by changing $+$ into max. This decision rule is sound and cognitively plausible but it is too rough as it creates too many indifference situations.

Refinements of this decision rule can be based on an idea originally due to B. Franklin: canceling arguments of equal importance for $\vec{a}$ or against $\vec{b}$, by arguments for b or against a until we find a difference on each side. This leads to a complete and transitive refinement of $\succeq^{Biposs}$. Let $A^+_\lambda = \{i \in A^+, \pi_i = \lambda\}$ be the arguments for a with strength λ. (resp. A^-_λ the arguments against a with strength λ.). The following decision rule checks the positive and negative affects for each element of a pair (a, b) of alternatives top-down in terms of importance:

$$a \succeq^{Lexi} b \iff \exists \lambda \in L \text{ such that } \begin{cases} \forall \beta > \lambda, & |A^+_\beta| - |A^-_\beta| = |B^+_\beta| - |B^-_\beta| \\ \text{and} & |A^+_\lambda| - |A^-_\lambda| > |B^+_\lambda| - |B^-_\lambda|. \end{cases}$$

It focuses on the maximal level of importance when there are more arguments for one than for the other (using the comparability postulate). This rule generalizes Gigerenzer's "take the best" heuristic [70] and can be encoded in the CPT framework using super-increasing transformations. It has been empirically tested, and proves to be the one people often use when making decisions according to several criteria when subject to the focus effect [13].

16.6 Uncertainty Handling in Decision Evaluation Using Fuzzy Intervals

Fuzzy intervals have been widely used in FDA so as to account for the fact that subjective evaluations are imprecise. In many cases, it comes down to applying the extension principle to existing evaluation tools: weighted averages or expected utilities using fuzzy interval weights [125], fuzzy extensions of Saaty's Analytical Hierarchical Process [85], and other numerical or relational MCDM techniques (like TOPSIS, PROMETHEE, ELECTRE, etc. [67]). Many of these techniques, in their original formulations are ad hoc, or even debatable (see [16] for a critique of many of them). So their fuzzy-valued extensions are often liable of the same defects.

The problem with fuzzy set methods extending existing ones is that more often than not the proposed handling of fuzzy intervals is itself ad hoc or so approximate that the benefit of the fuzzification is lost. Moreover the thrust of fuzzy interval analysis is to provide information about the uncertainty pervading the results of the decision process. Some authors make an unjustified use of defuzzification that erases all information of this type. For instance the decision-maker is asked for some figures in the form of fuzzy intervals so as to account for the difficulty to provide precise ratings, and then these ratings are defuzzified right away. Or the fuzzy intervals are propagated through the decision process but the final results are defuzzified in order to make the final decision ranking step easier. In such procedures, it is not clear why fuzzy intervals were used. The uncertainty pervading the ratings should play a role in the final decision-making process, namely to warn the decision maker when information is not sufficient for justifying a clear ranking of alternatives.

In this section we discuss two examples where fuzzy intervals have been extensively used, but where some intrinsic technical or computational difficulties need to be properly addressed in decision evaluation techniques: fuzzy weighted averages and fuzzy AHP methods.

16.6.1 Fuzzy Weighted Averages

An obvious way of introducing fuzzy sets in classical aggregation techniques is to assume that local evaluations are ill-known and represented by fuzzy intervals. The question is then how to aggregate fuzzy evaluations and how to provide a ranking of alternatives. Fuzzy weighted averages are a good example of such a problem, that dates back to Baas and Kwakernaak [5]. There have been numerous papers on fuzzy weighted averages since then (see [56] for a bibliography before 2000 and [125] for a recent survey). The key technique involved here is fuzzy interval analysis [56]. Two important points need to be stressed, that are not always acknowledged:

- Computing arithmetic expressions with fuzzy intervals cannot always be done by means of plain fuzzy arithmetics (that is, combining partial results obtained by means of the four operations).
- Before considering a fuzzy interval approach, one must understand how to solve the problem with plain intervals.

The first point is actually a corollary of the second one. In particular the name "fuzzy number" seems to have misled many authors who seem not to realize that what is often called a fuzzy number is a generalized (gradual) interval. For instance some authors have tried to equip fuzzy addition with a group structure, which is already lost for intervals. Many expect the defuzzification of fuzzy intervals to yield a precise number, while stripping a fuzzy set from its fuzziness naturally yields a set. These points are discussed at length in [66] where a genuine fuzzy extension of a number (a gradual number) is suggested, such that a fuzzy interval is a standard interval of such gradual numbers. In this paper we systematically use the name *fuzzy interval* to remove all ambiguity.

As a consequence, fuzzy interval analysis inherits all difficulties encountered when computing with intervals [91]. In particular if a given ill-known quantity appears several times in some arithmetic expression, one must be very careful to observe that substituting this quantity with the same interval in several places does not preserve the constraint stating that behind this interval lies the same quantity. For instance, if $x \in [a, b]$, then $[a, b] - [a, b] \neq 0$, while $x - x = 0$ regardless of the value of x. More generally, interval analysis (hence fuzzy interval analysis) requires an optimization problem to be solved. Computing fuzzy weighted averages has more to do with constraint propagation than with the arithmetics of fuzzy intervals.

The difficulty is already present with imprecise (interval) weights. The problem of computing interval-valued averages can be posed in two ways:

1. Maximise and minimise $\frac{\sum_{i=1}^{n} x_i \cdot w_i}{\sum_{i=1}^{n} w_i}$ under the constraints: $w_i \in [a_i, b_i] \subset [0, +\infty), x_i \in [c_i, d_i], i = 1, \ldots, n$;
2. Maximise and minimise $\sum_{i=1}^{n} x_i \cdot p_i$ under the constraints: $p_i \in [u_i, v_i] \subseteq [0, 1], x_i \in [c_i, d_i], i = 1, \ldots, n, \sum_{i=1}^{n} p_i = 1$.

The two problems yield different results if the same intervals $[u_i, v_i] = [a_i, b_i]$ are used. In both cases, the maximal (resp. minimal) solution is attained for $x_i = d_i$ (resp. $x_i = c_i$). In the second problem, one issue is: what does normalization mean when only intervals $([u_1, v_1], [u_2, v_2] \ldots, [u_n, v_n])$ are available? It is clear that the sum of these intervals is an interval, hence never equal to 1. One way out is to view a vector of interval weights as a set of standard normalized vectors $\vec{p} = (p_1, p_2 \ldots, p_n)$.

Specific conditions must be satisfied if all bounds are to be reachable by such weight vectors, that is $\forall i = 1 \ldots n, \exists (p_1, p_2 \ldots, p_n) \in \prod_{i=1}^{n} [u_i, v_i]$ such that $p_i = u_i$ and another such vector such that $p_i = v_i$. The question is completely solved by de Campos et al. [30] in the case of imprecise probability weights. Necessary and sufficient conditions are (the second one implies the first one)

1. $\sum_{i=1}^{n} u_i \leq 1 \leq \sum_{i=1}^{n} v_i$ (non-emptiness)
2. $u_i + \sum_{j \neq i} v_j \geq 1$; $v_i + \sum_{j \neq i} u_j \leq 1$ (attainability).

Attainability ensures a form of arc-consistency (optimally short intervals $[u_i, v_i]$ for the probabilities p_i) in the terminology of constraint satisfaction. It also comes down to the coherence condition of Walley [121] for probability bounds. Updating the bounds of the intervals $[u_i, v_i]$ so that this condition be satisfied can be viewed as a form of normalization.

A fast method has been proposed a long time ago, to solve the second problem, with explicit expressions whose computation is linear in the number of arguments [44]. Now the first problem can be connected to the second one by comparing the intervals

$$I_w = \{\frac{\sum_{i=1}^{n} x_i \cdot w_i}{\sum_{i=1}^{n} w_i}, w_i \in [a_i, b_i]\} = \{\sum_{i=1}^{n} x_i \cdot p_i, p_i = \frac{w_i}{\sum_{i=1}^{n} w_i}, w_i \in [a_i, b_i]\}$$

and

$$I_p = \{\sum_{i=1}^{n} x_i \cdot p_i, p_i \in [\frac{a_i}{a_i + \sum_{j \neq i} b_j}, \frac{b_i}{b_i + \sum_{j \neq i} a_j}], \sum_{i=1}^{n} p_i = 1\}.$$

That the bounds of these intervals are attainable for normalized vectors $(p_1, p_2 \ldots, p_n)$ is established in [123]. However as pointed out in [99] (and contrary to what is suggested in the previous version of this paper [40]) the latter interval strictly contains the former one, except if $n = 2$. From a geometrical point of view, the problem is to study the relative position of the sets of probability vectors

$$\mathbf{W} = \{(\frac{w_1}{\sum_{i=1}^{n} w_i}, \ldots, \frac{w_n}{\sum_{i=1}^{n} w_i}), w_i \in [a_i, b_i], i = 1, \ldots, n\}$$

and

$$\mathbf{W}^N = \{(p_1, \ldots, p_n) : p_i \in [\frac{a_i}{a_i + \sum_{j \neq i} b_j}, \frac{b_i}{b_i + \sum_{j \neq i} a_j}], i = 1, \ldots, n, \sum_{i=1}^{n} p_i = 1\}.$$

For $n = 2$ these sets are equal to the segment $\{(p, 1 - p) : p \in [\frac{a_1}{a_1 + b_2}, \frac{b_1}{b_1 + a_2}]\}$. Consider the case $n > 2$ and for simplicity we assume $a_i > 0, \forall i$. It can be observed that the polyhedra $\mathbf{W}$ and $\mathbf{W}^N$ both lie in the hyperplane $\mathbf{H} = \{\vec{p} = (p_1, \ldots, p_n) : \sum_{i=1}^{n} p_i = 1\}$ of normalized vectors, but no facet of one is parallel to a facet of the other. Let $\mathbf{P} = \{\vec{p} \in \mathbf{H} : p_i \geq 0, i = 1, \ldots, n\}$ be the positive part of $\mathbf{H}$, that is, the set of probability assignments in $\mathbf{H}$. More precisely, $\mathbf{W}^N$ is the intersection between $\mathbf{P}$ and the Cartesian product $\times_{i=1,\ldots,n}[\frac{a_i}{a_i+\sum_{j \neq i} b_j}, \frac{b_i}{b_i+\sum_{j \neq i} a_j}]$ [98].

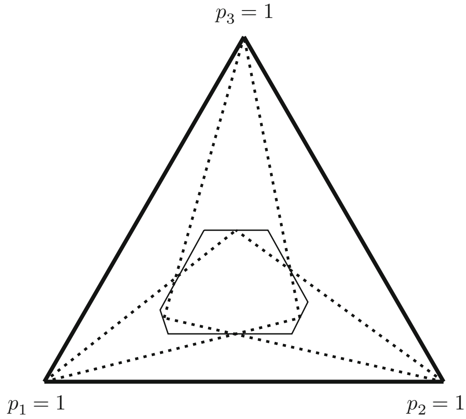


Fig. 16.1 Interval-weighted average vs. interval convex sum

Each pair of parallel facets of $\mathbf{W}^N$ is of the form $p_i = p$, where $p = \frac{a_i}{a_i + \sum_{j \neq i} b_j}$ and $p = \frac{b_i}{b_i + \sum_{j \neq i} a_j}$; they are also parallel to the facet of $\mathbf{P}$ such that $p_i = 0$, and this for $i = 1, \ldots, n$. In contrast, $\mathbf{W}$ is obtained by homothetic projection of the Cartesian product $\times_{i=1,\ldots,n}[a_i, b_i]$ on $\mathbf{P}$. The vertices of this polyhedron lie inside the facets of $\mathbf{W}^N$ as shown in [99], so that both $\mathbf{W}^N$ and $\mathbf{W}$ have the same marginal projections of the form $[\frac{a_i}{a_i + \sum_{j \neq i} b_j}, \frac{b_i}{b_i + \sum_{j \neq i} a_j}]$. Moreover, each edge of $\mathbf{W}$ is the homothetic projection of one edge of the hyper-rectangle $\times_{i=1,\ldots,n}[a_i, b_i]$ and lies in the same hyperplane as one axis of the referential, hence on a line containing the vertex of $\mathbf{P}$ on that axis. The set $\mathbf{W}$ is defined via constraints of the form $\frac{p_i}{p_j} = \frac{w_i}{w_j}$, while $\mathbf{W}^N$ is generated from bounds on individual p_i. This type of representation is common in the area of imprecise probabilities [121], where credal sets of the form $\mathbf{W}^N$ are obtained by constraints on the probabilities of elementary events [30] while those of the form $\mathbf{W}$ are obtained by constraining conditional probabilities.

For instance consider the case where $n = 3$ pictured by Fig. 16.1 [99], where $\mathbf{P}$ is an equilateral triangle in bold line with vertices $(1, 0, 0), (0, 1, 0), (0, 0, 1)$. The set $\mathbf{W}$ of renormalized weights is a hexagon whose sides are obtained by lines (dotted on the figure) drawn from the vertices of the triangle (two per vertex). It is clear that $\mathbf{W}^N$ is the smallest hexagon, with opposite edges parallel to the edges of the triangle $\mathbf{P}$, containing $\mathbf{W}$ (in continuous thin lines in the figure). In the fuzzy case, we get fuzzy subsets of $\mathbf{P}$ consisting of nested hexagons.

It should be clear from [99] and the above discussion that the vertices of $\mathbf{W}$ are generally not the same as the vertices of $\mathbf{W}^N$. As a consequence, the intervals

$$\mathbf{X} = \{\frac{\sum_{i=1}^{n} x_i \cdot w_i}{\sum_{i=1}^{n} w_i}, w_i \in [a_i, b_i], i = 1, \ldots, n\}$$

and

$$\mathbf{X}^N = \{\sum_{i=1}^{n} x_i \cdot p_i, p_i \in [\frac{a_i}{a_i + \sum_{j \neq i} b_j}, \frac{b_i}{b_i + \sum_{j \neq i} a_j}], i = 1, \ldots, n, \sum_{i=1}^{n} p_i = 1\}$$

have little chance to be equal because their bounds are attained on vertices of $\mathbf{W}$ and $\mathbf{W}^N$ respectively. And as $\mathbf{W} \subset \mathbf{W}^N$ it follows that, in general, $\mathbf{X} \subset \mathbf{X}^N$.

A consequence of this finding of Pavlačka [99] is that in multicriteria decision-making based on fuzzy weighted averages, constraining ill-known normalized vectors and normalizing ill-known weights will give different results. Note that in the precise case, one may indifferently compute a convex sum of ratings according to the various criteria either using normalized weight vectors or using a weighted sum with suitable non-normalized weight vectors. As the results without normalization will be proportional to the results using a normalized vector, this choice of method is immaterial for determining the ranking of available options. However in the interval-valued case (let alone in the fuzzy case) there is no guarantee that the rankings of options will be the same with the two approaches, respectively using intervals $\mathbf{X}$ and $\mathbf{X}^N$ for comparing the options, as they rely on imprecision polyhedra having different vertices. All we know is that $\mathbf{X}$ will be strictly included in $\mathbf{X}^N$. Even worse, there is generally no way of having the polyhedron of normalized vectors induced by unnormalized interval weights equal to a polyhedron induced by suitable ranges of imprecision bearing on components of a normalized weight vector, due to the general incompatibility of their respective configurations.

This dilemma is very problematic when applying fuzzy weighted averages to multicriteria decision analysis. Namely, how to model the imprecision on the weights bearing on each criterion? Usually a weighted average is defined as a convex sum of ratings. Should we elicit imprecision on normalized weight vectors and compute a fuzzy convex sum? or elicit imprecision on un-normalized weights and compute the image of the fuzzy weights via a rational fraction as done by most authors since Baas and Kwakernaak [5] proposed it?

The natural way out of this dilemma may come from the following considerations. Note that the two intervals $\mathbf{X}_1$ and $\mathbf{X}_2$ (likewise $\mathbf{X}_1^N$ and $\mathbf{X}_2^N$) evaluating two options 1 and 2 do not constrain independent quantities: Each assignment of weights w_i in $[a_i, b_i]$ determines single values, one in $\mathbf{X}_1$ and one in $\mathbf{X}_2$, that are linked to each other. So the final ranking of options based on (fuzzy) interval global ratings cannot be achieved by applying any (fuzzy) interval ranking method to one of the sets of ill-known ratings $\mathbf{X}_k$ or $\mathbf{X}_k^N$. The interval ratings obtained for each option are not independent. A natural ranking criterion is then the following: If one option is better than another one for each assignment of weights in their ranges, then the former option is better than the latter. Note that this ranking criterion yields results that the renormalization procedure will leave unchanged: with this criterion, comparing $\mathbf{X}_1$ and $\mathbf{X}_2$, or $\mathbf{X}_1^N$ and $\mathbf{X}_2^N$, would give the same results . This fact pleads in favor of extending, to intervals or fuzzy intervals, the whole decision procedure at once, including the aggregation and the ranking steps, so as to take the interactivity between the fuzzy global evaluation intervals into account.

In the case of fuzzy weight vectors, the above reasoning must be applied to α-cuts of the fuzzy intervals involved. It should be clear that a fuzzy weight vector $\vec{\tilde{p}} = (\tilde{p}_1, \tilde{p}_2 \ldots, \tilde{p}_n)$ should be actually viewed as a fuzzy set of normalized weight vectors $\vec{p}$, where attainability conditions are met for all interval weight vectors $((\tilde{p}_1)_\alpha, (\tilde{p}_2)_\alpha \ldots, (\tilde{p}_n)_\alpha)$ formed by cuts. The degree of membership of $\vec{p}$ in $\vec{\tilde{p}}$ is equal to $\min_{i=1}^{n} \mu_{\tilde{p}_i}(p_i)$.

The above problems have been also considered more recently by scholars dealing with type 2 fuzzy sets for calculating (fuzzy-valued) centroids, for instance [79, 84], but these authors do not seem to rely on existing results in fuzzy interval analysis and sometimes reinvent their own terminology. Extensions of such calculations to Choquet integrals with fuzzy-valued importance weights for groups of criteria are more difficult to carry out, as the issue of ranking intervals $[c_i, d_i]$, let alone fuzzy intervals, must be addressed within the calculation [90].

16.6.2 Fuzzy Extensions of the Analytical Hierarchy Process

A number of multicriteria decision-making methods have been extended so as to deal with fuzzy data. Here we confine ourself to the case of Saaty's Analytical Hierarchy Process [111]. Numerous fuzzy versions of Saaty's methods have been proposed (since Van Laaroven and Pedrycz, [120], see the bibliography in [56]). Many such proposals seem to pose and solve questionable fuzzy equations, as we shall argue.

The principle of the AHP method relies on the following ideal situation

- The pairwise relative preference of n items (alternatives, criteria) is modelled by a $n \times n$ *consistent* preference matrix A, where each coefficient a_{ij} is supposed to reflect how many more times item i is preferred to item j.
- A consistent preference matrix is one that is reciprocal in the sense that $\forall i, j, a_{ij} = 1/a_{ji}$ and product-transitive ($\forall i, j, k, a_{ij} = a_{ik} \cdot a_{kj}$).
- Then its largest eigenvalue is $\lambda = n$ and there exists a corresponding eigenvector $\vec{p} = (p_1, p_2 \ldots, p_n)$ with $\forall i, j, a_{ij} = \frac{p_i}{p_j}$, yielding relative importance weights.

Even if widely used, this method is controversial and has been criticised by MCDM scholars as being ill-founded at the measurement level, and having paradoxical lacks of intuitive invariance properties (the scale used is absolute, with no degree of freedom, see [16], for instance). Moreover in practice, pairwise comparison data do not provide consistent matrices. Typically, the decision-maker provides, for each pair (i,j), a value $v_{ij} \in \{2, \ldots 9\}$ if i is preferred to j v_{ij} times, $v_{ij} = 1$ if there is indifference. The matrix A with coefficients $a_{ij} = v_{ij}, a_{ji} = 1/a_{ij}$ if $v_{ij} \geq 1$ is then built. Generally, product transitivity is not empirically achieved. A preference matrix A is considered all the more consistent as the largest eigenvalue of A is close to n.

Asking for precise values v_{ij} is debatable, because these coefficients are arguably imprecisely known.[3] So many researchers have considered fuzzy valued pairwise comparison data. The fuzzification of Saaty's AHP has consisted to extend the computation scheme of Saaty with fuzzy intervals. However this task turned out to be difficult for several reasons.

- Replacing a consistent preference matrix by a fuzzy-valued preference matrix loses the properties of the former. The reciprocal condition $\tilde{a}_{ij} = 1/\tilde{a}_{ji}$ no longer implies $\tilde{a}_{ij} \cdot \tilde{a}_{ji} = 1$, nor can the product transitivity property hold for all entries of the fuzzy matrix in the form $\tilde{a}_{ij} = \tilde{a}_{ik} \cdot \tilde{a}_{kj}$ when $\tilde{a}_{ij}$ are fuzzy intervals.
- Fuzzy eigenvalues or vectors of fuzzy-valued matrices are hard to define in a rigorous way: writing the usual equation $A\vec{p} = \lambda\vec{p}$ replacing vector and matrix entries by fuzzy intervals leads to overconstrained equations.
- It is tempting to solve the problems with each interval-matrix defined from α-cuts $(\tilde{a}_{ij})_\alpha = [\underline{a_{ij}}_\alpha, \overline{a_{ij}}_\alpha]$ of the fuzzy coefficients, as done by Csutora and Buckley [25] for instance. These authors suggest to solve the eigenvalue problem for the two extreme preference matrices with respective coefficients $\underline{a_{ij}}_\alpha$ and $\overline{a_{ij}}_\alpha$, with the view to find an interval-valued eigenvalue. However these boundary matrices are not even reciprocal since if $\tilde{a}_{ij} = 1/\tilde{a}_{ji}$, $\underline{a_{ij}}_\alpha = 1/\overline{a_{ji}}_\alpha$, not $1/\underline{a_{ji}}_\alpha$. So the meaning of normalized weights computed from these boundary matrices is totally unclear.

The bottom-line is that the natural extension of a simple crisp equation $ax = b$ (let alone an eigenvalue problem) is not necessarily a fuzzy equation of the form $\tilde{a}\tilde{x} = \tilde{b}$ where fuzzy intervals are substituted to real numbers and equality of membership functions on each side is required:

- The first equation $ax = b$ refers to a constraint to be satisfied by a model referring to a certain reality.
- But fuzzy intervals $\tilde{a}, \tilde{x}, \tilde{b}$ only represent *knowledge* about actual values a, x, b
- Even if $ax = b$ is taken for granted, it is not clear why the knowledge about ax should be equated to the knowledge about b (pieces of knowledge $\tilde{a}$, and $\tilde{b}$ may derive from independent sources). The objective constraint $ax = b$ only enforces a consistency condition $\tilde{a}\tilde{x} \cap \tilde{b} \neq \emptyset$.
- If indeed a fuzzy set $\tilde{x}$ is found such that $\tilde{a}\tilde{x} = \tilde{b}$, it does not follow that the actual quantities a, x, b verify $ax = b$. Moreover, equation $\tilde{a}\tilde{x} = \tilde{b}$ may fail to have solutions.

Recently, Ramik and Korviny [104] proposed to compute the degree of consistency of a fuzzy preference matrix $\tilde{A}$ whose entries are triangular fuzzy intervals $\tilde{a}_{ij}$ and $\tilde{a}_{ji}$ with respective supports and modes:

- for $\tilde{a}_{ij}$: $[\underline{a_{ij}}, \overline{a_{ij}}]$ and a^M_{ij};
- for $\tilde{a}_{ji}$: $[1/\overline{a_{ij}}, 1/\underline{a_{ij}}]$ and $1/a^M_{ij}$

[3]In fact, one may even consider that the very question of determining how many more times a criterion is important than another one is meaningless.

Their consistency degree is defined as the minimal distance between $\tilde{A}$ and a fuzzy consistent matrix $\tilde{X}$ understood as a so-called ratio matrix with coefficients of the form $\tilde{x}_{ij} = \frac{\tilde{x}_i}{\tilde{x}_j}$, where $\tilde{x}_i$ is a triangular fuzzy interval with mode x_i^M, and the normalisation condition $\sum_{i=1}^n x_i^M = 1$ is assumed. The distance $d(\tilde{A}, \tilde{X})$ used is a scalar distance between the vectors formed by the three parameters of the triangular fuzzy intervals, chosen such that the solution of the problem can be analytically obtained in agreement with the geometric mean method of computation of weights, already used in Van Laaroven and Pedrycz [120]. Fuzzy weights $\tilde{x}_i$ are thus obtained.

Let alone the fact that the formulation of the problem is partially ad hoc due to triangular approximation of inverses of triangular fuzzy intervals, and due to the choice of the normalisation condition (see the previous subsection), this approach also suffers from the above epistemological flaw consisting in considering a fuzzy interval as a simple substitute to a precise number, whereby a direct extension of the standard method consists just in replacing numbers by fuzzy intervals and running a similar computation as in the precise case. In particular the distance $d(\tilde{A}, \tilde{X})$ arguably evaluates an *informational* proximity between epistemic states (states of knowledge) about preference matrices, and says little about the scalar distance between the underlying precise ones.

Instead of viewing fuzzy interval preference matrices as fuzzy *substitutes* to precise ones, one may on the contrary acknowledge fuzzy pairwise preference data as *imprecise knowledge about regular preference information.* The fuzzy interval preference matrix is then seen as constraining an ill-known precise *consistent* comparison matrix. Inconsistencies in comparison data are thus explicitly explained by the imprecise nature of human-originated information. Such a constraint-based view of fuzzy AHP has been explained by Ohnishi and colleagues [93].

Namely consider a fuzzy matrix $\tilde{A}$ with entries $\tilde{a}_{ij} = 1/\tilde{a}_{ji}$ and $\tilde{a}_{ii} = 1$ and denote by μ_{ij} the membership function of $\tilde{a}_{ij}$. The relevance of a consistent preference matrix A to the user preference data described by $\tilde{A}$ can be evaluated without approximation as

$$\mu_{\tilde{A}}(A) = \min_{i,j:i<j} \mu_{ij}(a_{ij}).$$

A given normal weight vector $\vec{p} = (p_1, p_2 \ldots, p_n)$ satisfies the fuzzy preference matrix $\tilde{A}$ to degree $\mu(\vec{p}) = \min_{i<j} \mu_{ij}(\frac{p_i}{p_j})$. The degree of consistency of the preference data is

$$Cons(\tilde{A}) = \sup \mu(\vec{p}) = \sup_{\vec{p}:a_{ij}=\frac{p_i}{p_j}, \forall i,j:i<j} \mu_{\tilde{A}}(A).$$

The best induced weight vectors are the Pareto maximal elements among $\{\vec{p}, \mu(\vec{p}) = Cons(\tilde{A})\}$. The reader is referred to [93] for details. It is interesting to contrast this methodology, where the problem can be posed without approximation, and the otherwise elegant one by Ramik and Korviny [104]. In their method, the fuzzy matrix $\tilde{X}$ can be viewed as an approximation of the imprecise information matrix $\tilde{A}$ such that $Cons(\tilde{X}) = 1$ in the sense of Ohnishi et al. (the core matrix X^M is consistent in the sense of Saaty by construction). But the constraint-based approach seems to be more respectful of the original imprecise preference data, that can be directly exploited.

The constrained-based approach does not solve the difficulties linked to the critiques addressed to Saaty's method, but its interpretation is much clearer than its direct fuzzification, in the sense that it does not require a new theory of fuzzy eigenvalues, nor does it pose fuzzy interval equations with a debatable meaning. It makes sense if it is taken for granted that human preference can be ideally modelled by consistent preferences matrices in the sense of Saaty. The constraint-based approach outlined above only tries to cope with the problem of inconsistency of human judgments by acknowledging their lack of precision.

16.7 Comparing Fuzzy Intervals: A Constructive Setting

There is an enormous literature on fuzzy interval ranking methods, but very few attempts at proposing a rational approach to the definition of ranking criteria. This section tries to suggest one possible approach towards a systematic classification of ranking indices and fuzzy relations induced by the comparison of fuzzy intervals. The issue of ranking objects rated by fuzzy intervals should be discussed in the perspective of decision under uncertainty. The connection between the ranking of fuzzy intervals and fuzzy preference relations should be emphasized. There are many ranking methods surveyed elsewhere [56, 124]. Here we suggest a unifying principle: such ranking methods should be directly based on probabilistic notions of dominance (see Chap. 8 in [17]) or their possibilistic counterparts on the one hand, and interval orders [68, 103] on the other hand.

While many ranking methods have been proposed (and still are), most of the time they are derived on an ad hoc basis: an often clever index is proposed, sometimes for triangular fuzzy intervals only, and its merits are tested on a few examples. Systematic comparison studies are not so numerous (except for Bortolan and Degani [14], Lee [22], for instance). Moreover these comparison are based on intuitive feelings of what a good ranking method should do, tested on a few examples and counterexamples. There is a lack of first principles for devising well-founded techniques. However Wang and Kerre [124] proposed an interesting set of axioms that any preference relation $\succeq$ between fuzzy intervals $\tilde{a}, \tilde{b}$. should satisfy. For instance:

- *Reflexivity*: $\tilde{a} \succeq \tilde{a}$.
- *Certainty of dominance*: If $\tilde{a} \cap \tilde{b} = \emptyset$ then $\tilde{a} \succ \tilde{b}$ or $\tilde{b} \succ \tilde{a}$.
- *Consistency with fuzzy interval addition*: $\tilde{a} \succeq \tilde{b}$ implies $\tilde{a} + \tilde{c} \succeq \tilde{b} + \tilde{c}$

Dubois et al. [56] also classified existing methods distinguishing between:

1. *scalar indices* (based on defuzzification understood by the replacement of a fuzzy interval by a representative number);
2. *goal-based indices*: computing the degree of attainment of a fuzzy goal by each fuzzy interval, which bears some similarity to the expected utility approach, understanding a fuzzy goal as a utility function;
3. *relational indices*: based on computing to what extent a fuzzy interval dominates another. In this case, some methods are based on metric considerations (those based on computing possibility and necessity of dominance), and others are based on computing areas limited by the membership functions of the fuzzy intervals to be compared.

Another view of the comparison of fuzzy intervals can exploit links between fuzzy intervals and other settings: possibility, probability theories and interval orders. This kind of idea can be found in some previous papers in the literature, but it has never been systematically explored. Yet, it might provide a systematic way to produce comparison indices and classifying them. In this section, we outline such a research program.

16.7.1 Four Views of Fuzzy Intervals

A fuzzy interval $\tilde{a}$, like any fuzzy set is defined by a membership function $\mu_{\tilde{a}}$. This fuzzy set is normalized ($\exists x \in \mathbb{R}, \mu_{\tilde{a}}(x) = 1$) and its cuts are bounded closed intervals. Let us denote by $[\underline{a}, \overline{a}]$ its core and $[a_*, a^*]$ its support.

Like any fuzzy set it needs to be cast inside an interpretive setting in order to be usefully exploited. To our knowledge there are four existing views of fuzzy intervals understood as a representation of uncertainty

1. *Metric possibility distributions*: in this case the membership function of $\tilde{a}$ is viewed as a possibility distribution π_x, following the suggestion of Zadeh [130]. A fuzzy interval represent gradual incomplete information about some ill-known precise quantity x: $\pi_x(r)$ is all the greater as $r \in \mathbb{R}$ is close to a totally possible value. Moreover, we consider the interval $[0, 1]$ as a similarity scale and the membership function as a rescaling of the distance between elements on the real line.
2. *One point-coverage functions of nested random intervals*: In this case, a fuzzy interval $\tilde{a}$ is induced by the Lebesgue measure ℓ on $[0, 1]$, and the cut multi-mapping with range in the set of closed intervals of the real line $\mathcal{I}(\mathbb{R})$:

$$[0, 1] \to \mathbb{R} : \alpha \mapsto \tilde{a}_\alpha \in \mathcal{I}(\mathbb{R}).$$

Then $\pi_x(u) = \ell(\{\alpha, x \in \tilde{a}_\alpha\})$. This view comes from the fact that a numerical necessity measure is a special case of belief functions and a possibility distribution is the one-point coverage of a random set [71]. In this case, the membership function of $\tilde{a}$ is viewed as the contour function of a consonant continuous belief function. This line has been followed by Dubois and Prade [46] and Chanas and colleagues [19]. The latter also envisaged a more general framework in the form of random intervals limited by two random variables $\dot{x} \leq \ddot{x}$ with disjoint support, such that $\pi_x(u) = Prob(\dot{x} \leq u \leq \ddot{x})$. Then the nestedness property is lost and the possibility distribution thus obtained is no longer equivalent to the knowledge of the two random variables: they lead to a belief function on the real line in the sense of Smets [122]

3. *Families of probability functions*: As formally a possibility measure is a special case of belief function, and a belief function is a special case of (coherent) lower probability in the sense of Walley [121], a fuzzy interval $\tilde{a}$ also encodes a special family of probability measures

$$\mathcal{P}_{\tilde{a}} = \{P : P(\tilde{a}_\alpha) \geq 1 - \alpha, \alpha \in [0, 1]\}.$$

Namely it can be shown that for fuzzy intervals $\tilde{a}$, $\Pi(A) = \sup\{P(A), P \in \mathcal{P}_{\tilde{a}}\}$ for all measurable subsets A of the real line. This approach proposed by Dubois and Prade [48] was studied in the general case by De Cooman and Aeyels [31]. It is clear that it allows probabilistic inequalities to be interpreted in terms of fuzzy intervals (with cuts symmetric around the mean for Chebychev's inequality [39]).

4. *Intervals bounded by gradual numbers*: This is a more recent view advocated by Fortin et al. [66]. The idea is to view a fuzzy interval as a regular interval of functions. To this end, the interval component of the fuzzy interval must be disentangled from its fuzzy (or gradual) component. A gradual number $\breve{r}$ is a mapping from the positive unit interval to the reals: $\alpha \in (0, 1] \mapsto r_\alpha \in \mathbb{R}$. For instance, the mid-point of a fuzzy interval $\tilde{a}$ with cuts $[\overline{a}_\alpha, \underline{a}_\alpha]$ is a gradual number $\breve{a}_\alpha = \frac{\underline{a}_\alpha + \overline{a}_\alpha}{2}$. It is clear that a fuzzy interval can be viewed as an interval of gradual numbers lower bounded by $\underline{a}_\alpha$ and upper-bounded by $\overline{a}_\alpha$. Gradual numbers inside the fuzzy interval $\tilde{a}$ can be generated by selecting a number inside each cut of $\tilde{a}$. In fact, they are the selection functions of the cut-mapping. Although the idea of using a pair of functions to represent a fuzzy interval is not new (e.g., the so-called *L-R fuzzy numbers*, that enable closed forms of arithmetic operations on fuzzy intervals to be derived in terms of inverses of shape functions L and R [56]), the key novelty here is to treat a fuzzy interval as a regular one.

What is clear from the above classification is that ranking fuzzy intervals should be related to techniques for ranking intervals or for ranking random quantities. There are well-known methods for comparing intervals, namely

1. *Interval orders*: $[a, b] >_{IO} [c, d]$ if and only if $a > d$ (Fishburn [68], Pirlot and Vincke [103]). Note that, interpreting intervals as possibility distributions $[a, b]$ and $[c, d]$, respectively restricting ill-known quantities x and y, the statement $[a, b] >_{IO} [c, d]$ can be interpreted as by means of the necessity degree as $N(x > y) = 1$, given that $(x, y) \in [a, b] \times [c, d]$. Likewise, $\Pi(x > y) = 1$ if and only if $b > c$.

2. *Interval lattice extension of the usual ordering*: If we extend the maximum and minimum operations on the real line to intervals, it yields

$$\max([a,b],[c,d]) = \{z = \max(x,y) : x \in [a,b], y \in [c,d]\}$$
$$= [\max(a,c), \max(b,d)]$$

and likewise $\min([a,b],[c,d]) = [\min(a,c), \min(b,d)]$. The set of closed intervals equipped with min and max forms a lattice, and the canonical ordering in this lattice is

$$[a,b] \geq_{lat} [c,d] \iff \max([a,b],[c,d]) = [a,b]$$
$$\iff \min([a,b],[c,d]) = [c,d] \iff a \geq c \text{ and } b \geq d.$$

3. *Subjective approach*: this is Hurwicz criterion that uses a coefficient of optimism $\lambda \in [0,1]$ for describing the attitude of the decision-maker. Intervals can be compared via a selection of precise substitutes to intervals:

$$[a,b] \geq_\lambda [c,d] \iff \lambda a + (1-\lambda)b \geq \lambda c + (1-\lambda)d.$$

It is clear that $[a,b] >_{IO} [c,d]$ implies $[a,b] \geq_{lat} [c,d]$, which is equivalent to $\forall\lambda \in [0,1], [a,b] \geq_\lambda [c,d]$.

There are also well-known methods for comparing random variables

1. *1st Order Stochastic Dominance*: $x \geq_{SD} y$ if and only if $\forall\theta, P(x \geq \theta) \geq P(y \geq \theta)$ ([17] Chap. 8)
2. *Probabilistic preference relations*: $R(x,y) = P(x \geq y)$ (e.g., $x > y$ if and only if $R(x,y) > \alpha > 0.5$) [26].
3. *Scalar substitutes*: Comparing x and y by their expectations, more generally the expectation of their utilities $u(x)$ and $u(y)$.

For independent random variables, $P(x \geq y) = 1$ implies $x \geq_{SD} y$, which is equivalent to $\int u(t)dP_x(t) \geq \int u(t)dP_y(t)$ for any monotonic increasing utility functions u. For monotonically related random variables with joint distribution function $\min(F_x(r), F_y(r'))$ it is clear that $P(x \geq y) = 1$ expresses 1st order stochastic dominance exactly.

16.7.2 Constructing Fuzzy Interval Ranking Methods

According to the chosen interpretation of fuzzy intervals, the above methods for comparing intervals and probabilities can be extended, possibly conjointly and thus define well-founded ranking techniques for fuzzy intervals. So doing, a number of existing ranking methods can be retrieved, and make sense in a particular setting.

16.7.2.1 Metric Approach

If fuzzy intervals are viewed as mere possibility distributions $\pi_x = \mu_{\tilde{a}}$ and $\pi_y = \mu_{\tilde{b}}$, it is natural to exploit counterparts to probabilistic ranking techniques, turning probability measures into possibility and necessity measures. One gets methods that are well-known:

1. *Interval lattice extension of stochastic dominance*: Clearly there are two cumulative distributions one can derive from a continuous fuzzy interval:

   - the upper distribution $F^*(\theta) = \Pi(x \leq \theta) = \mu_{\tilde{a}}(\theta)$ if $\theta \leq \underline{a}$ and 1 otherwise;
   - the lower distribution $F_*(\theta) = N(x \leq \theta) = 1 - \mu_{\tilde{a}}(\theta)$ if $\theta \geq \overline{a}$ and 0 otherwise;

   Then, we can

   - either combine stochastic dominance and interval ordering:

   $$\tilde{a} \geq_{IO} \tilde{b} \iff \forall\theta, N(x \geq \theta) \geq \Pi(y \geq \theta),$$

   which is a very demanding criterion that basically requires that $\overline{b} \leq \underline{a}$ and $1 - \mu_{\tilde{a}}(\theta) \geq \mu_{\tilde{b}}(\theta), \forall\theta \in [\overline{b}, \underline{a}]$
   - or combine stochastic dominance and the lattice interval ordering:

   $$\tilde{a} \geq_{lat} \tilde{b} \iff \forall\theta, \Pi(x \geq \theta) \geq \Pi(y \geq \theta) \text{ and } N(x \geq \theta) \geq N(y \geq \theta).$$

   It comes down to comparing cuts of $\tilde{a}$ and $\tilde{b}$ using the lattice interval ordering or yet the well-known comparison method via the extended minimum or maximum $\tilde{a} \geq_{\tilde{c}} \tilde{b}$ if and only if $\widetilde{\max}(\tilde{a}, \tilde{b}) = \tilde{a}$ (or $\widetilde{\min}(\tilde{a}, \tilde{b}) = \tilde{b}$)

2. *Counterparts of expected utility*: compute the possibility and the necessity of reaching a fuzzy goal G using possibility and necessity of fuzzy events. In this case, the membership function μ_G represents a preference profile that stands for a utility function, and special cases of Sugeno integrals can be computed:

   - The degree of possibility of reaching the goal:

   $$\Pi_{\tilde{a}}(G) = \sup_{\theta} \min(\mu_{\tilde{a}}(\theta), \mu_G(\theta)).$$

   - The degree of necessity of reaching the goal:

   $$N_{\tilde{a}}(G) = \inf_{\theta} \max(1 - \mu_{\tilde{a}}(\theta), \mu_G(\theta)).$$

   These criteria are possibilistic counterparts of expected utility functions (optimistic and pessimistic, respectively). They have been axiomatized as such by Dubois et al. [57]. When μ_G is an increasing function, one can compare fuzzy intervals $\mu_{\tilde{a}}$ and $\mu_{\tilde{b}}$ by comparing pairs $(N_{\tilde{a}}(G), \Pi_{\tilde{a}}(G))$ and $(N_{\tilde{b}}(G), \Pi_{\tilde{b}}(G))$ using interval ordering techniques. This approach systematizes the one of Chen [23].

3. *Possibilistic valued relations*: compute valued preference relations obtained as the degrees of possibility or necessity that x, restricted by $\tilde{a}$, is greater than y restricted by $\tilde{b}$. For instance the index of certainty of dominance $R(x, y) = N(x \geq y) = 1 - \sup_{v>u} \min(\pi_x(u), \pi_y(v))$. This line of thought goes back the seventies [5] and was systematized by Dubois and Prade [45]. It extends interval orderings [106] since $N(x \geq y) = 1 - \inf\{\alpha : \tilde{a}_\alpha >_{IO} \tilde{b}_\alpha\}$.

16.7.2.2 Random Interval Approach

One may wish to probabilize interval ranking methods, interpreting a fuzzy interval as a nested random interval. For instance, one may use the valued relation approach to comparing random numbers, extended to intervals:

1. The random interval order yields a valued relation of the form: $R_{IO}(\tilde{a}, \tilde{b}) = Prob(\tilde{a}_\alpha \geq_{IO} \tilde{b}_\beta)$; this kind of approach has been especially proposed by Chanas and colleagues [20, 21]. The randomized form of the canonical lattice interval extension of the usual order of reals > reads: $R_C(\tilde{a}, \tilde{b}) = Prob(\tilde{a}_\alpha \geq_{lat} \tilde{b}_\beta)$; both expressions presuppose some assumption be made regarding the dependence structure between the parameters α and β viewed as random variables on the unit interval.
2. The probabilistic version of the subjective approach leads to the following valued relation that depends on the coefficient of optimism: $R_\lambda(\tilde{a}, \tilde{b}) = Prob(\lambda \underline{a}_\alpha + (1 - \lambda)\overline{a}_\alpha \geq \lambda \underline{a}_\alpha + (1 - \lambda)\overline{b}_\alpha)$

One may also generalize stochastic dominance to random intervals [1]. To this end, we must notice that $Prob(\underline{a}_\alpha \leq \theta) = \Pi_{\tilde{a}}(x \leq \theta)$ and $Prob(\overline{a}_\alpha \leq \theta) = N_{\tilde{a}}(x \leq \theta)$. Hence we get the same approach as in the ordinal case when we compare any among $Prob(\underline{a}_\alpha \leq \theta)$ or $Prob(\overline{a}_\alpha \leq \theta)$ to any of $Prob(\underline{b}_\alpha \leq \theta)$ or $Prob(\overline{b}_\alpha \leq \theta)$. One gets a special case of stochastic dominance between belief functions studied by Denoeux [32].

Finally one may also compute the average interval using the Aumann integral: $E(\tilde{a}) = \int_0^1 \tilde{a}_\alpha$ [92], and compare $E(\tilde{a})$ and $E(\tilde{b})$ using interval comparison methods. For instance, the Hurwicz method then coincides with the subjective approach of de Campos and Gonzales [29] and subsumes Yager's [126] and Fortemps and Roubens [65] techniques.

16.7.2.3 Imprecise Probability Approach

Viewing fuzzy intervals as families of probability measures yields techniques close to the random set approach [24]:

- The extension of 1st Order Stochastic Dominance to fuzzy intervals remains the same since $\Pi_{\tilde{a}}(x \leq \theta)$ (resp. $N_{\tilde{a}}(x \leq \theta)$) is also the upper (resp. lower) probability of the event "$x \leq \theta$" in the sense of the probability family $\mathcal{P}_\alpha$.
- Comparing upper and lower expected values of x and y, namely $E^*(\tilde{a}) = \int_0^1 \overline{a}_\alpha d\alpha$ and $E_*(\tilde{a}) = \int_0^1 \underline{a}_\alpha d\alpha$ comes down to comparing mean intervals since $E(\tilde{a}) = [E_*(\tilde{a}), E^*(\tilde{a})]$ [46].
- One may also construct interval- valued preference relations obtained as upper and lower probabilities of dominance: $\begin{cases} R^*(x,y) = P^*(x \geq y), \\ R_*(x,y) = P_*(x \geq y). \end{cases}$ and exploit them.

  A different interval-valued quantity that is relevant in this context is $[E_*(\tilde{a}-\tilde{b}), E^*(\tilde{a}-\tilde{b})]$ to be compared to 0. In imprecise probability theory, comparing lower expectations of x and y is not equivalent to comparing the lower expectation to $x-y$ to 0, generally more details can be found in [35].

16.7.2.4 Gradual Number Approach

Viewing fuzzy intervals as intervals of gradual numbers, we first need a method for comparing gradual numbers: again three natural techniques come to mind [2]. They extend the comparison of random variables to some extent, because the inverse of a cumulative distribution function is a special case of gradual number:

1. *Levelwise comparison*: $\breve{r} \geq \breve{s}$ if and only if $\forall\alpha, r_\alpha \geq s_\alpha$. It is clear that this definition reduces to 1st order stochastic dominance when the gradual number is the inverse of a distribution function.
2. *Area comparison method*:

$$\breve{r} >^S \breve{s} \iff \int_0^1 \max(0, r^\alpha - s^\alpha) d\alpha > \int_0^1 \max(0, s^\alpha - r^\alpha) d\alpha.$$

3. *Comparing defuzzified values*: the natural way of defuzzifying a gradual number is to compute the number $m(\breve{r}) = \int_0^1 r^\alpha d\alpha$. This expression reduces to standard expectation using inverses of distribution functions. And clearly $\breve{r} >^S \breve{s} \iff m(\breve{r}) > m(\breve{s})$.

The connection between gradual numbers and the dominance index $P(x > y)$ is worth exploring. In fact, under suitable dependence assumptions, $P(x > y)$ is related to the Lebesgue measure $\ell(\{\alpha, r_\alpha \geq s_\alpha\})$.

On this ground one can compare fuzzy intervals $\tilde{a}$ and $\tilde{b}$, viewed as genuine intervals of functions $[\underline{\breve{a}}, \overline{\breve{a}}]$ and $[\underline{\breve{b}}, \overline{\breve{b}}]$ limited by gradual numbers, where $\underline{\breve{a}}_\alpha = \underline{a}_\alpha$ and $\overline{\breve{a}}_\alpha = \overline{a}_\alpha$, so that $\tilde{a} = \{\breve{r} : \underline{\breve{a}} \leq \breve{r} \leq \overline{\breve{a}}.\}$ One retrieves some ranking methods already found by the above previous approaches:

- Lattice interval extension of $>$: $\tilde{a} \geq_{lat} \tilde{b}$ if and only if $\underline{\tilde{a}} \geq \underline{\tilde{b}}$ and $\overline{\tilde{a}} \geq \overline{\tilde{b}}$.

- Stochastic dominance with subjective approach: $\tilde{a} \geq_{\lambda} \tilde{b}$ if and only if $\lambda\underline{\breve{a}}+(1-\lambda)\overline{\breve{a}} \geq \lambda\underline{\breve{b}} + (1-\lambda)\overline{\breve{b}}$.
- Subjective approach by comparing expectations: $\tilde{a} \geq_{\lambda} \tilde{b}$ if and only if $\int_0^1 (\lambda\underline{a}_{\alpha} + (1-\lambda)\overline{a}_{\alpha})d\alpha \geq \int_0^1 (\lambda\underline{b}_{\alpha} + (1-\lambda)\overline{b}_{\alpha})d\alpha$.

Note that $\tilde{a} \geq_{IO} \tilde{b}$ reads $\underline{\breve{a}} \geq \overline{\breve{b}}$, which is equivalent to the comparison of the interval supports of the corresponding fuzzy intervals.

This typology only aims at emphasizing the impact of attaching an interpretation to fuzzy intervals on the search for ranking methods. It can serve as a tool for constructing well-founded ranking methods for fuzzy intervals and to study their properties.

16.8 Conclusion

The use of fuzzy sets in decision analysis remains somewhat debatable so long as proposals for using fuzzy intervals, fuzzy preference relations, linguistic value scales are not better positioned in the stream of current research in measurement theory [82] and decision sciences [17, 67]. There does not seem to exist a niche for an isolated theory of fuzzy decision-making. However the use of fuzzy sets may focus the attention of scholars of traditional decision theory on some issues that were not otherwise considered, like encompassing averaging approaches to criteria aggregation and soft constraint satisfaction, a non-probabilistic view of gradual preference in relational approaches, a refined handling of incomplete information, and a well-founded basis for qualitative approaches to evaluation.

Several messages result from the above analysis of the literature:

- Fuzzy set theory offers a bridge between numerical approaches and qualitative approaches to decision analysis, but:

  1. The use of linguistic variables encoded by fuzzy intervals does not always make a numerical method more qualitative or meaningful.
  2. Replacing numerical values by fuzzy intervals rather corresponds to a kind of sensitivity analysis, not to a move toward the qualitative.
  3. The right question is: how to faithfully encode qualitative techniques on numerical scales, rather than using linguistic terms to extend already existing ad hoc numerical techniques.

- There is a strong need to develop original fuzzy set-based approaches to multicriteria decision analysis that are not a rehashing of existing techniques with ad hoc fuzzy interval computations.
- Fuzzy set theory and its mathematical environment (aggregation operations, graded preference modeling, and fuzzy interval analysis) provide a general framework to pose decision problems in a more open-minded way, towards a unification of existing techniques.

Open questions remain, such as:

- Refine any qualitative aggregation function using discri-schemes or lexi-schemes
- Computational methods for finding discrimin-leximin solutions to fuzzy optimization problems.
- Devise a behavioral axiomatization of new aggregation operations in the scope of MCDM, decision under uncertainty and fuzzy voting methods
- Develop a general axiomatic framework for ranking fuzzy intervals based on first principles.
- Study the impact of semantics of fuzzy preference relations (probabilistic, possibilistic, distance-based,.) on how they should be exploited for ranking purposes.
- Provide a unified framework for fuzzy choice functions [69].

Acknowledgements This paper is an extended version of a position paper written by the first author [40]. The latter is grateful to O. Pavlačka for pointing out a mistake in that paper, and to E. Huellermeier for suggestions on ranking fuzzy intervals stemming from fuzzy weighted averages.

References

1. Aiche, F., Dubois, D.: An extension of stochastic dominance to fuzzy random variables. In: Hüllermeier, E., Kruse, R., Hoffmann, F. (eds.) Computational Intelligence for Knowledge-Based Systems Design (IPMU 2010). Lecture Notes in Computer Science, vol. 6178, pp.159–168. Springer, Berlin (2010)
2. Aiche, F., Dubois, D.: Possibility and gradual number approaches to ranking methods for random fuzzy intervals. In: Greco, S., et al. (eds.) Advances in Computational Intelligence (IPMU 2012), vol. 3. Communications in Computer and Information Science, vol. 299, pp. 9–18. Springer, Berlin (2012)
3. Arrow, K.J.: Social Choice and Individual Values. Wiley, New York (1951)
4. Atanassov, K.: Intuitionistic fuzzy sets. Fuzzy Sets Syst. **20**, 87–96 (1986)
5. Baas, S.M., Kwakernaak, H.: Rating and ranking of multiple aspect alternatives using fuzzy sets. Automatica **13**, 47–58 (1977)
6. Barrett, C.R., Pattanaik, P.K., Salles, M.: On the structure of fuzzy social welfare functions. Fuzzy Sets Syst. **19**, 1–10 (1986)
7. Barrett, C.R., Pattanaik, P.K., Salles, M.: Aggregation of fuzzy preferences. In: Multiperson Decision Making Models Using Fuzzy Sets and Possibility Theory, pp. 155–162. Springer, Berlin (1990)
8. Beliakov, G., Pradera, A., Calvo, T.: Aggregation Functions: A Guide for Practitioners Studies in Fuzziness and Soft Computing, vol. 221. Springer, Berlin (2008)
9. Bellman, R.E., Zadeh, L.A.: Decision making in a fuzzy environment. Manag. Sci. **17**, B141–B164 (1970)
10. Blin, J.M., Whinston, A.B.: Fuzzy sets and social choice. J. Cybern. **3**(4), 17–22 (1973)
11. Bodenhofer, U.: Representations and constructions of similarity-based fuzzy orderings. Fuzzy Sets Syst. **137**, 113–136 (2003)
12. Bodenhofer, U., De Baets, B., Fodor, J.C.: A compendium of fuzzy weak orders: representations and constructions. Fuzzy Sets Syst. **158**(8), 811–829 (2007)
13. Bonnefon, J.-F., Dubois, D., Fargier, H., Leblois, S.: Qualitative heuristics for balancing the pros and the cons. Theory Decis. **65**, 71–95 (2008)

14. Bortolan, G., Degani, R.: A review of some methods for ranking fuzzy subsets. Fuzzy Sets Syst. **15**, 1–19 (1985)
15. Boutilier, C., Brafman, R., Domshlak, C., Hoos, H.H., Poole, D.: CP-nets: a tool for representing and reasoning with conditional ceteris paribus preference statements. J. Artif. Intell. Res. **21**, 135–191 (2004)
16. Bouyssou, D., Marchant, T., Pirlot, M., Perny, P., Tsoukias, A., Vincke, P.: Evaluation Models: A Critical Perspective. Kluwer Academic Publishers, Boston (2000)
17. Bouyssou, D., Dubois, D., Pirlot, M., Prade, H.: Decision-Making Process. ISTE, London & Wiley, Hoboken, NJ (2009)
18. Brans, J.-P., Mareschal, B., Vincke, P.: PROMETHEE: a new family of outranking methods in multicriteria analysis. Manag. Sci. **31**(6), 647–656 (1985)
19. Chanas, S.S., Nowakowski, M.: Single value simulation of fuzzy variable. Fuzzy Sets Syst. **25**, 43–57 (1988)
20. Chanas, S., Zielinski, P.: Ranking fuzzy real intervals in the setting of random sets: further results. Inf. Sci. **117**, 191–200 (1999)
21. Chanas, S., Delgado, M., Verdegay, J.L., Vila, M.A.: Ranking fuzzy real intervals in the setting of random sets. Inf. Sci. **69**, 201–217 (1993)
22. Chang, P.T., Lee, E.S.: Fuzzy arithmetics and comparison of fuzzy numbers, In: Delgado, M., Kacprzyk, J., Verdegay, J. L., Vila, M.A. (eds.) Fuzzy Optimization: Recent Advances, pp. 69–81. Physica-Verlag, Heidelberg (1994)
23. Chen, S.H.: Ranking fuzzy numbers with maximising sets and minimising sets. Fuzzy Sets Syst. **17**, 113–129 (1985)
24. Couso, I., Dubois, D.: An imprecise probability approach to joint extensions of stochastic and interval orderings. In: Greco, S., et al. (eds.) Advances in Computational Intelligence (IPMU 2012), vol. 3. Communications in Computer and Information Science, vol. 299, pp. 388–399. Springer, Berlin (2012)
25. Csutora, R., Buckley, J.: Fuzzy hierarchical analysis: the Lambda-Max method. Fuzzy Sets Syst. **120**, 181–195 (2001)
26. De Baets, B., De Meyer, H.: Transitivity frameworks for reciprocal relations: cycle-transitivity versus FG-transitivity. Fuzzy Sets Syst. **152**, 249–270 (2005)
27. De Baets, B., Fodor, J.: Twenty years of fuzzy preference structures (1978–1997). Belgian J. Oper. Res. Stat. Comput. Sci. **37**, 61–82 (1997)
28. De Baets, B., De Meyer, H., De Schuymer, B., Jenei, S.: Cyclic evaluation of transitivity of reciprocal relations. Soc. Choice Welf. **26**, 217–238 (2006)
29. de Campos, L., Gonzalez, A.: A subjective approach for ranking fuzzy numbers. Fuzzy Sets Syst. **29**, 145–153 (1989)
30. de Campos, L.M., Huete, J.F., Moral, S.: Probability intervals: a tool for uncertain reasoning. Int. J. Uncertainty Fuzziness Knowledge Based Syst. **2**, 167–196 (1994)
31. De Cooman, G., Aeyels, D.: Supremum-preserving upper probabilities. Inf. Sci. **118**, 173–21 (1999)
32. Denoeux, T.: Extending stochastic order to belief functions on the real line. Inf. Sci. **179**, 1362–1376 (2009)
33. Deschrijver, G., Kerre, E.: On the relationship between some extensions of fuzzy set theory. Fuzzy Sets Syst. **133**, 227–235 (2004)
34. Díaz, S., De Baets, B., Montes, S.: Additive decomposition of fuzzy pre-orders. Fuzzy Sets Syst. **158**(8), 830–842 (2007)
35. Destercke, S., Couso, I.: Ranking of fuzzy intervals seen through the imprecise probabilistic lens. Fuzzy Sets Syst. **278**, 20–39 (2015)
36. Díaz, S., De Baets, B., Montes, S.: On the compositional characterization of complete fuzzy pre-orders. Fuzzy Sets Syst. **159**(17), 2221–2239 (2008)
37. Desimpelaere, C., Marchant, T.: An empirical test of some measurement-theoretic axioms for fuzzy sets. Fuzzy Sets Syst. **158**, 1348–1359 (2009)
38. Duan, R., Pettie, S.: Fast algorithms for (max,min)-matrix multiplication and bottleneck shortest paths. In: Proceedings of the Nineteenth Annual ACM-SIAM Symposium on Discrete Algorithms, pp. 384–391. ACM Press, New York (2009)

39. Dubois, D.: Possibility theory and statistical reasoning. Comput. Stat. Data Anal. **51**(1), 47–69 (2006)
40. Dubois, D.: The role of fuzzy sets in decision sciences: old techniques and new directions. Fuzzy Sets Syst. **184**, 3–28 (2011)
41. Dubois, D., Fargier, H.: Making Sugeno integrals more discriminant. Int. J. Approx. Reason. **50**, 880–898 (2009)
42. Dubois, D., Fargier, H.: Capacity refinements and their application to qualitative decision evaluation. In: Claudio, S., Gaetano, C. (eds.) Symbolic and Quantitative Approaches to Reasoning with Uncertainty (ECSQARU 2009), Verona (Italy). Lecture Notes in Artificial Intelligence, vol. 5590, pp. 311–322. Springer, Berlin (2009)
43. Dubois, D., Fortemps, P.: Computing improved optimal solutions to max-min flexible constraint satisfaction problems. Eur. J. Oper. Res. **118**, 95–126 (1999)
44. Dubois, D., Prade H.: Additions of interactive fuzzy numbers. IEEE Trans. Autom. Control **26**, 926–936 (1981)
45. Dubois, D., Prade, H.: Ranking fuzzy numbers in the setting of possibility theory. Inf. Sci. **30**, 183–224 (1983)
46. Dubois, D., Prade, H.: The mean value of a fuzzy number. Fuzzy Sets Syst. **24**, 279–300 (1987)
47. Dubois, D., Prade, H.: Fuzzy arithmetics in qualitative reasoning, In: Blaquière, A. (ed.) Modeling and Control of Systems in Engineering, Quantum Mechanisms, Economics and Biosciences. Lecture Notes in Control and Information Sciences, pp. 457–467. Springer, Berlin (1989)
48. Dubois, D., Prade, H.: When upper probabilities are possibility measures. Fuzzy Sets Syst. **49**, 65–74 (1992)
49. Dubois, D., Prade, H.: The three semantics of fuzzy sets. Fuzzy Sets Syst. **90**, 141–150 (1997)
50. Dubois, D., Prade, H.: Possibility theory: qualitative and quantitative aspects. In: Smets, P. (ed.) Quantified Representation of Uncertainty and Imprecision. Handbook on Defeasible Reasoning and Uncertainty Management Systems, vol. 1, pp. 169–226. Kluwer Academic Publishers, Dordrecht (1998)
51. Dubois, D., Prade, H.: On the use of aggregation operations in information fusion processes. Fuzzy Sets Syst. **142**, 143–161 (2004)
52. Dubois, D., Prade, H.: An introduction to bipolar representations of information and preference. Int. J. Intell. Syst. **23**(8), 866–877 (2008)
53. Dubois, D., Prade, H.: Gradualness, uncertainty and bipolarity: making Sense of fuzzy sets. Fuzzy Sets Syst. **192**, 3–24 (2012)
54. Dubois, D., Fargier, H., Prade, H.: Possibility theory in constraint satisfaction problems: handling priority, preference and uncertainty. Appl. Intell. **6**, 287–309 (1996)
55. Dubois, D., Fargier, H., Prade, H.: Possibilistic likelihood relations. In: Proceedings of IPMU'98, pp. 1196–1203. EDK, Paris (1998)
56. Dubois, D., Kerre, E., Mesiar, R., Prade, H.: Fuzzy interval analysis. In: Dubois, D., Prade, H. (eds.) Fundamentals of Fuzzy Sets. The Handbooks of Fuzzy Sets Series, pp. 483–581. Kluwer Academic Publishers, Boston, MA (2000)
57. Dubois, D., Prade, H., Sabbadin, R.: Decision theoretic foundations of qualitative possibility theory. Eur. J. Oper. Res. **128**, 459–478 (2001)
58. Dubois, D., Fargier, H., Bonnefon, J.-F.: On the qualitative comparison of decisions having positive and negative features. J. Artif. Intell. Res. **32**, 385–417 (2008)
59. Dutta, B.: Fuzzy preferences and social choice. Fuzzy Sets Syst. **13**, 215–219 (1987)
60. Fargier, H., Sabbadin, R.: Qualitative decision under uncertainty: back to expected utility. Artif. Intell. **164**, 245–280 (2005)
61. Fodor, J.: Some remarks on difference measurement. In: Proceedings of 7th International Symposium on Applied Intelligence and Informatics, Budapest, pp. 11–14 (2009)

62. Fodor, J.C., De Baets, B.: Fuzzy preference modelling. In: Bustince, H., Herrera, F., Montero, J. (eds.) Fundamentals and Recent Advances. Fuzzy Sets and Their Extensions: Representation, Aggregation and Models. Studies in Fuzziness and Soft Computing, vol. 220, pp. 207–217. Springer, Berlin (2008)
63. Fodor, J., Roubens, M.: Fuzzy Preference Modelling and Multicriteria Decision Support. Kluwer Academic Publishers, Dordrecht (1994)
64. Fodor, J., Orlovski, S., Perny, P., Roubens, M.: The use of fuzzy preference models in multiple criteria choice, ranking and sorting. In: Slowinski, R. (ed.) Fuzzy Sets in Decision Analysis, Operations Research and Statistics, pp. 69–101. Kluwer Academic Publishers, Dordrecht (1998)
65. Fortemps, P., Roubens, P.M.: Ranking and defuzzification methods based on area compensation. Fuzzy Sets Syst. **82**, 319–330 (1996)
66. Fortin, J., Dubois, D., Fargier, H.: Gradual numbers and their application to fuzzy interval analysis. IEEE Trans. Fuzzy Syst. **16**, 388–402 (2008)
67. Figueira, J., Greco, S., Ehrgott, M.M. (eds.): Multiple Criteria Decision Analysis: State of the Art. International Series in Operations Research & Management Science, vol. 78. Springer, Boston (2005)
68. Fishburn, P.: Interval Orderings. Wiley, New York (1987)
69. Georgescu, I.: Fuzzy Choice Functions: A Revealed Preference Approach. Springer, Berlin (2007)
70. Gigerenzer, G., Todd, P.M., the ABC group: Simple Heuristics that Make Us Smart. Oxford University Press, New York (1999)
71. Goodman, I.R.: Fuzzy sets as equivalence classes of random sets. In: Yager R.R. (ed.) Fuzzy Sets and Possibility Theory, pp. 327–342. Pergamon Press, Oxford (1982)
72. Grabisch, M.: The Moebius transform on symmetric ordered structures and its application to capacities on finite sets. Discrete Math. **287**, 17–34 (2004)
73. Grabisch, M., Greco, S., Pirlot, M.: Bipolar and bivariate models in multicriteria decision analysis: descriptive and constructive approaches. Int. J. Intell. Syst. **23**(9), 930–969 (2008)
74. Grabisch, M., Marichal, J.-L., Mesiar, R., Pap, E.: Aggregation Functions. Cambridge University Press, Cambridge (2009)
75. Greco, S., Matarazzo, B., Slowinski, R.: Rough sets theory for multicriteria decision analysis. Eur. J. Oper. Res. **129**(1),1–47 (2001)
76. Greco, S., Benedetto, M., Slowinski, R.: Fuzzy set extensions of the dominance-based rough set approach. In: Humberto, B.S., Francisco H., Javier, M. (eds.) Fuzzy Sets and Their Extensions: Representation, Aggregation and Models. Studies in Fuzziness and Soft Computing, vol. 220, pp. 239–261. Springer, Berlin (2008)
77. Herrera, F., Martinez, L.: A model based on linguistic 2-tuples for dealing with multigranular hierarchical linguistic contexts in multi-expert decision-making. IEEE Trans. Syst. Man Cybern. Part B **31**(2), 227–234 (2001)
78. Herrera, F., Herrera-Viedma, E., Martinez, L.: A fusion approach for managing multi-granularity linguistic term sets in decision making. Fuzzy Sets Syst. **114**, 43–58 (2000)
79. Karnik, N.N., Mendel, J.M.: Centroid of a type-2 fuzzy set. Inf. Sci. **132** (1–4), 195–220 (2001)
80. Kitainik, L.: Fuzzy Decision Procedures With Binary Relations. Kluwer Academic Publishers, Dordrecht (1993)
81. Klement, E.P., Mesiar, R., Pap, E.: Triangular Norms. Kluwer Academic Publishers, Dordrecht (2000)
82. Kranz, D.H,. Luce, R.D., Suppes, P., Tverski, A.: Foundations of Measurement, Volume 1 Additive and Polynomial Representations. Academic, New York (1971)
83. Kyselovà, D., Dubois, D., Komorníkovà, M., Mesiar, R.: Refining aggregation operator-based orderings in multifactorial evaluation - Part I: continuous scales. IEEE Trans. Fuzzy Syst. Spec. Issue Aggregation Oper. **15**, 1100–1106 (2007)
84. Liu, X., Mendel, J.M., Wu, D.: Analytical solution methods for the fuzzy weighted average. Inf. Sci. **187**, 151–170 (2012)

85. Lootsma, F.: Fuzzy Logic for Planning and Decision-Making. Kluwer Academic Publishers, Dordrecht (1997)
86. Marchant, T.: The measurement of membership by comparisons. Fuzzy Sets Syst. **148**, 157–177 (2004)
87. Marchant, T.: The measurement of membership by subjective ration estimation. Fuzzy Sets Syst. **148**, 179–199 (2004)
88. Massaglia, R., Ostanello, A.: N-tomic: a support system for multicriteria segmentation problems. In: Korhonen, P., Lewandowski, A., Wallenius, J. (eds.) Multiple Criteria Decision Support. Lecture Notes in Economics and Mathematical Systems, vol. 356, pp. 167–174. Springer, Berlin (1991)
89. Matarazzo, B.: Multicriterion analysis by means of pairwise actions and criterion comparisons (MAPPAC). Appl. Math. Comput. **18**(2), 119–141 (1986)
90. Meyer, P., Roubens, M.: On the use of the Choquet integral with fuzzy numbers in multiple criteria decision support. Fuzzy Sets Syst. **157**, 927–938 (2006)
91. Moore, R.: Methods and Applications of Interval Analysis. SIAM Studies on Applied Mathematics. Society for Industrial and Applied Mathematics, Philadelphia (1979)
92. Ogura, Y., Li, S.M, Ralescu, D.A.: Set defuzzification and Choquet integral. Int. J. Uncertainty Fuzziness Knowledge Based Syst. **9**, 1–12 (2001)
93. Ohnishi, S., Dubois, D., Prade, H., Yamanoi, T.: A Fuzzy Constraint-Based Approach to the Analytic Hierarchy Process. In: Bouchon-Meunier, B., et al. (eds.) Uncertainty and Intelligent Information Systems, pp. 217–228. World Scientific, Singapore (2008)
94. Orlovsky, S.: Decision-making with a fuzzy preference relation. Fuzzy Sets Syst. **1**, 155–168 (1978)
95. Ovchninnikov, S.V.: Structure of fuzzy binary relations. Fuzzy Sets Syst. **1**, 155–167 (1981)
96. Ovchninnikov, S.V.: Social choice and Łukasiewicz logic. Fuzzy Sets Syst. **43**, 275–289 (1991)
97. Ovchninnikov, S.V., Roubens, M.: On strict preference relations. Fuzzy Sets Syst. **43**, 319–326 (1991)
98. Pavlačka, O.: Modeling uncertain variables of the weighted average operation by fuzzy vectors. Inf. Sci. **181**, 4969–4992 (2011)
99. Pavlačka, O.: Note on the lack of equality between fuzzy weighted average and fuzzy convex sum. Fuzzy Sets Syst. **213**, 102–105 (2013)
100. Perny, P.: Multicriteria filtering methods based on concordance and non-discordance principles. Ann. Oper. Res. **80**, 137–165 (1998)
101. Perny, P., Roubens, M.: Preference modelling. In: Slowinski, R. (ed.) Fuzzy Sets in Decision Analysis, Operations Research and Statistics, Chap. 1, pp. 3–30. Kluwer Academic Publishers, Dordrecht (1998)
102. Perny, P., Roy, B.: The use of fuzzy outranking relations in preference modelling. Fuzzy Sets Syst. **49**, 33–53 (1992)
103. Pirlot, M., Vincke, P.: Semiorders: Properties, Representations, Applications. Kluwer Academic Publishers, Dordrecht (1994)
104. Ramík, J., Korviny, P.: Inconsistency of pair-wise comparison matrix with fuzzy elements based on geometric mean. Fuzzy Sets Syst. **161**, 1604–1613 (2010)
105. Rico, A., Grabisch, M., Labreuche, C., Chateauneuf, A.: Preference modeling on totally ordered sets by the Sugeno integral. Discrete Appl. Math. **147**(1), 113–124 (2005)
106. Roubens, M., Vincke, P.: Fuzzy possibility graphs and their application to ranking fuzzy numbers. In: Kacprzyk, J., Roubens, M. (eds.) Non-conventional Preference Relations in Decision Making, pp. 119–128. Springer, Berlin (1988)
107. Roy, B.: ELECTRE III: un algorithme de classement fondé sur une représentation floue des préférences en présence de critères multiples. Cahiers du CERO **20**(1), 3–24 (1978)
108. Roy, B.: A Multicriteria Analysis for Trichotomic Segmentation Problems, vol. 9, pp. 245–257. Gower Press, Aldershot (1981)
109. Roy, B.: Multicriteria Methodology for Decision Analysis. Kluwer Academic Publishers, Dordrecht (1996)

110. Ruspini, E.H.: On the semantics of fuzzy logic. Int. J. Approx. Reason. **5**(1), 45–88 (2008)
111. Saaty, T.L.: The Analytic Hierarchy Process: Planning, Priority Setting, Resource Allocation. McGraw-Hill, New York (1980)
112. Schader, M., Bertrand, P., Diday, E., Lechevallier, Y., Burtschy, B. (eds.): Rough Classification with Valued Closeness Relation. Springer, Berlin (1994)
113. Sen, A.K.: Social choice theory. In: Intrilagor, M.D., Arrow, K.J. (eds.) Handbook of Mathematical Economics. North-Holland, Amsterdam (1986)
114. Shafer, G.: A Mathematical Theory of Evidence. Princeton University Press, Princeton (1976)
115. Tanaka, H., Okuda, T., Asai, K.: On fuzzy-mathematical programming. J. Cybern. **3**(4), 37–46 (1974)
116. Torra, V., Narukawa, Y.: Modeling Decisions: Information Fusion and Aggregation Operators. Springer, Berlin (2007). ISBN 978-3-540-68789-4
117. Türksen, I.B., Bilgic, T.: Measurement of membership functions: theoretical and empirical work. In: Dubois, D., Prade, H. (eds.) Fundamentals of Fuzzy Sets. The Handbooks of Fuzzy Sets, pp. 195–230. Kluwer Academic Publishers, Boston (2000)
118. Tversky, A., Kahneman, D.: Advances in prospect theory: cumulative representation of uncertainty. J. Risk Uncertain. **5**, 297–323 (1992)
119. Van De Walle, B., De Baets, B., Kerre, E.: Charaterizable fuzzy preference structures. Ann. Oper. Res. **80**, 105–136 (1998)
120. Van Laaroven, P.J.M., Pedrycz, W.: A fuzzy extension of Saaty's priority theory. Fuzzy Sets Syst. **11**, 229–241 (1983)
121. Walley, P.: Statistical Reasoning with Imprecise Probabilities. Chapman and Hall, London (1991)
122. Walley, P.: Belief functions on real numbers. Int. J. Approx. Reason. **40**, 181–223 (2005)
123. Wang, Y.-M., Elhag, T.M.S.: On the normalization of interval and fuzzy weights. Fuzzy Sets Syst. **157**, 2456–2471 (2006)
124. Wang, X., Kerre, E.: Reasonable properties for the ordering of fuzzy quantities (2 parts). Fuzzy Sets Syst. **118**, 375–406 (2001)
125. Wang, Y.-M., Luo, Y.: Generalised fuzzy weighted mean and its applications. Int. J. Gen. Syst. **38**, 533–546 (2009)
126. Yager, R.R.: A procedure for ordering fuzzy subsets of the unit interval. Inf. Sci. **24**, 143–161 (1981)
127. Zadeh, L.A.: Similarity relations and fuzzy orderings. Inf. Sci. **3**, 177–200 (1971)
128. Zadeh L.A.: The concept of a linguistic variable and its application to approximate reasoning. Inf. Sci. Part 1: **8**, 199–249, Part 2: **8**, 301–357, Part 3: **9**, 43–80 (1975)
129. Zadeh, L.A.: Calculus of fuzzy restrictions. In: Zadeh, L.A., Fu, K.S., Tanaka, K., Shimura, M. (eds.) Fuzzy Sets and Their Applications to Cognitive and Decision Processes, pp. 1–39. Academic, New York (1975)
130. Zadeh, L.A.: Fuzzy sets as a basis for a theory of possibility. Fuzzy Sets Syst. **1**, 3–28 (1978)
131. Zadeh, L.A.: Toward a theory of fuzzy information granulation and its centrality in human reasoning and fuzzy logic. Fuzzy Sets Syst. **90**, 111–127 (1997)
132. Zimmermann, H.-J.: Fuzzy programming and linear programming with several objective functions. Fuzzy Sets Syst. **1**, 45–55 (1978)
133. D. Dubois, H. Prade (2001) Advances in the egalitarist approach to decision-making in a fuzzy environment. In *Dynamical Aspect in Fuzzy Decision Making* (Y. Yoshida, Ed.) Physica-Verlag, Heidelberg Germany, V. 73, Studies in Fuzziness and Soft Computing, 213-240.

Part VI
Multiobjective Optimization

Chapter 17
Vector and Set Optimization

Gabriele Eichfelder and Johannes Jahn

Abstract This chapter is devoted to recent developments in vector and set optimization. Based on the concept of a pre-order optimal elements are defined. In vector optimization properties of optimal elements and existence results are gained. Further, an introduction to vector optimization with a variable ordering structure is given. In set optimization basic concepts are summed up.

Keywords Vector optimization • Set optimization • Existence results • Variable ordering structure

17.1 Introduction

In vector optimization one investigates optimal elements of a set in a, in generally pre-ordered, space. The problem of determining these optimal elements, if they exist at all, is called a vector optimization problem. Problems of this type can be found not only in mathematics but also in engineering and economics. There, these problems are also called multiobjective (or multi criteria or Pareto) optimization problems or one speaks of multi criteria decision making. Vector optimization problems arise, for example, in functional analysis (the Hahn-Banach theorem, the Bishop-Phelps lemma, Ekeland's variational principle), statistics (Bayes solutions, theory of tests, minimal covariance matrices), approximation theory (location theory, simultaneous approximation, solution of boundary value problems) and cooperative game theory (cooperative n player differential games and, as a special case, optimal control problems). In the last decades vector optimization has been extended to

G. Eichfelder (✉)
Institute for Mathematics, Technische Universität Ilmenau, PF 10 05 65,
98684 Ilmenau, Germany
e-mail: gabriele.eichfelder@tu-ilmenau.de

J. Jahn
Department of Mathematics, University of Erlangen-Nuremberg, Cauerstr. 11,
91058 Erlangen, Germany
e-mail: johannes.jahn@fau.de

S. Greco et al. (eds.), *Multiple Criteria Decision Analysis*, International Series
in Operations Research & Management Science 233,
DOI 10.1007/978-1-4939-3094-4_17

problems with set-valued maps. This field, called set optimization, has important applications to variational inequalities and optimization problems with multivalued data. Recently, also vector optimization problems with the space equipped with a variable ordering structure instead of a pre-order have gained interest, as such problems arise for instance in medical image registration or portfolio optimization.

In the applied sciences F.Y. Edgeworth [19] (1881) and V. Pareto [39] (1906) were probably the first who introduced an optimality concept for vector optimization problems. Both have given the standard optimality notion in multiobjective optimization. Therefore, optimal points are called Edgeworth-Pareto optimal points in the modern special literature.

We give a brief historical sketch of the early works of Edgeworth and Pareto.

Edgeworth introduces notions in his book [19] on page 20: "Let P, the utility of X, one party, $= F(x\,y)$, and Π, the utility of Y, the other party, $= \Phi(x\,y)$". Then he writes on page 21: "It is required to find a point $(x\,y)$ such that, *in whatever direction* we take an infinitely small step, P and Π do not increase together, but that, while one increases, the other decreases". Hence, Edgeworth presents the definition of a minimal solution, compare Definition 8, for the special case of $Y = \mathbb{R}^2$ partially ordered by the natural ordering, i.e. for two objectives $f_1 : S \to \mathbb{R}$ and $f_2 : S \to \mathbb{R}$ and with $K = \mathbb{R}^2_+$.

In the English translation of Pareto's book [39] one finds on page 261: "We will say that the members of a collectivity enjoy *maximum ophelimity* in a certain position when it is impossible to find a way of moving from that position very slightly in such a manner that the ophelimity enjoyed by each of the individuals of that collectivity increases or decreases. That is to say, any small displacement in departing from that position necessarily has the effect of increasing the ophelimity which certain individuals enjoy, and decreasing that which others enjoy, of being agreeable to some and disagreeable to others". The concept of "ophelimity" used by Pareto, is explained on page 111: "In our Cours we proposed to designate economic *utility* by the word *ophelimity*, which some other authors have since adopted", and it is written on page 112: "For an individual, the ophelimity of a certain quantity of a thing, added to another known quantity (it can be equal to zero) which he already possesses, is the pleasure which this quantity affords him". In our modern terms "ophelimity" can be identified with an objective function and so, the definition of a minimal solution given in Definition 8 actually describes what Pareto explained.

These citations show that the works of Edgeworth and Pareto concerning vector optimization are very close together and, therefore, it makes sense to speak of *Edgeworth-Pareto optimality* as proposed by Stadler [44]. It is historically not correct that optimal points are called Pareto optimal points as it is done in various papers.

In mathematics this branch of optimization has started with a paper of H.W. Kuhn and A.W. Tucker [34]. Since about the end of the 1960s research is intensively made in vector optimization.

In the following sections we first recall the concepts of a pre-order and a partial order of a set which naturally induce the notion of minimal and maximal elements for vector and set optimization problems. In Sect. 17.3, we present different optimality notions such as minimal, weakly minimal, strongly minimal and properly minimal elements in a pre-ordered linear space and discuss the relations among these notions. We further give conditions guaranteeing the existence of minimal, weakly minimal and properly minimal elements in linear spaces and we present an engineering application in magnetic resonance systems. This section concludes with an introduction to vector optimization with a variable ordering structure. Finally, in Sect. 17.4, we discuss set optimization problems and different approaches for defining optimal elements in set optimization.

17.2 Pre- and Partial Orders

Minimizing a scalar valued function $f: X \to \mathbb{R}$ on some set X, two objective function values are compared by saying $f(x)$ is better than $f(y)$ if $f(x) \leq f(y)$. In vector optimization problems, i.e. in optimization problems with a vector-valued objective function, and even more general in set optimization problems, i.e. in optimization problems with a set-valued objective function, we need relations for comparing several vectors or even sets. For that, let us recall some concepts from the theory of ordered sets [42].

Definition 1. Let $\mathcal{Q}$ be an arbitrary nonempty set with a binary relation $\leq$. Let $A, B, D \in \mathcal{Q}$ be arbitrarily chosen. The binary relation $\leq$ is said to be

(i) *reflexive* if $A \leq A$.
(ii) *transitive* if $A \leq B$ and $B \leq D$ imply $A \leq D$.
(iii) *symmetric* if $A \leq B$ implies $B \leq A$.
(iv) *antisymmetric* if $A \leq B$ and $B \leq A$ imply $A = B$.

Definition 2. The binary relation $\leq$ is said to be

(i) a *pre-order* if it is reflexive and transitive.
(ii) a *partial order* if it is reflexive, transitive and antisymmetric or in other words, if it is a pre-order that is antisymmetric.
(iii) an *equivalence relation* if it is reflexive, transitive and symmetric.

When the relation $\leq$ is a pre-order/a partial order, we say that $\mathcal{Q}$ is a pre-ordered/partially ordered set.

It is important to note that in a pre-ordered set two arbitrary elements cannot be compared, in general, in terms of the pre-order.

Throughout this section let Y be an arbitrary real linear space. For $\mathcal{Q} = Y$ we say that the pre-order is compatible with the linear structure of the space if it is compatible with addition, i.e. for $x, y, w, z \in Y$ and $x \leq y,\ w \leq z$ we obtain $x + w \leq y + z$, and compatible with multiplication with a nonnegative real number, i.e. for $x, y \in Y$, $\alpha \in \mathbb{R}_+$ and $x \leq y$ we obtain $\alpha x \leq \alpha y$. For introducing this definition in a more general setting we need the power set of Y,

$$\mathcal{P}(Y) := \{A \subset Y \mid A \text{ is nonempty}\} .$$

Notice that the power set $\mathscr{P}(Y)$ of Y is a conlinear space introduced by Hamel [26]. In a conlinear space addition and multiplication with a nonnegative real number are defined but in contrast to the properties of a linear space the second distributive law is not required.

Definition 3. Suppose that $\mathscr{Q}$ is a subset of the power set $\mathscr{P}(Y)$. We say that the binary relation $\leq$ is

(i) *compatible with the addition* if $A \leq B$ and $D \leq E$ imply $A + D \leq B + E$ for all $A, B, D, E \in \mathscr{Q}$.
(ii) *compatible with the multiplication* with a nonnegative real number if $A \leq B$ implies $\lambda A \leq \lambda B$ for all scalars $\lambda \geq 0$ and all $A, B \in \mathscr{Q}$.
(iii) *compatible with the conlinear structure* of $\mathscr{P}(Y)$ if it is compatible with both the addition and the multiplication with a nonnegative real number.

By setting $\mathscr{Q} := \{\{y\} \mid y \in Y\} \subset \mathscr{P}(Y)$, Definition 3 includes as a special case the compatibility of a pre-order with the linear structure of the space Y as discussed above.

The connection between a pre-order (a partial order) in a linear space and a (pointed) convex cone is given in the following theorem.

Theorem 1. *Let Y be a real linear space.*

(a) If $\leq$ is a pre-order which is compatible with the addition and the multiplication with a nonnegative real number, then the set

$$K := \{y \in Y \mid 0_Y \leq y\}$$

is a convex cone. If, in addition, $\leq$ is antisymmetric, i.e. $\leq$ is a partial order, then K is pointed, i.e. $K \cap (-K) = \{0_Y\}$.
(b) If K is a convex cone, then the binary relation

$$\leq_K := \{(x, y) \in Y \times Y \mid y - x \in K\}$$

is a pre-order on Y which is compatible with the addition and the multiplication with a nonnegative real number. If, in addition, K is pointed, then $\leq_K$ is a partial order.

If the convex cone K introduces some pre-order we speak of an *ordering cone*.
Let us consider some examples illustrating the above concepts.

Example 1.

(a) Let Y be the linear space of all $n \times n$ real symmetric matrices. Then the pointed convex cone $\mathscr{S}_+^n$ of all positive semidefinite matrices introduces a partial order on Y.
(b) Let $K \subset Y$ be an ordering cone. For $\mathscr{Q} = \mathscr{P}(Y)$ we define a binary relation by the following: Let $A, B \in \mathscr{P}(Y)$ be arbitrarily chosen sets. Then

$$A \preccurlyeq_s B \;:\Longleftrightarrow\; \big(\forall\, a \in A\ \exists\, b \in B : a \leq_K b\big) \text{ and } \big(\forall\, b \in B\ \exists\, a \in A : a \leq_K b\big).$$

This relation is called *set less* or *KNY order relation* $\preccurlyeq_s$ and has been independently introduced by Young [50] and Nishnianidze [38]. It has been presented by Kuroiwa [35] in a slightly modified form. This relation is a pre-order and compatible with the conlinear structure of the space.

Based on a pre-order we can define minimal and maximal elements of some set $\mathcal{Q}$.

Definition 4. Let $\mathcal{Q}$ be a pre-ordered set. Let $\mathcal{A}$ be a nonempty subset of $\mathcal{Q}$, $T \in \mathcal{Q}$ and $\bar{A} \in \mathcal{A}$. We say that

(i) $\bar{A}$ is a *minimal element* of $\mathcal{A}$ if $A \leq \bar{A}$ for some $A \in \mathcal{A}$ implies $\bar{A} \leq A$.
(ii) $\bar{A}$ is a *maximal element* of $\mathcal{A}$ if $\bar{A} \leq A$ for some $A \in \mathcal{A}$ implies $A \leq \bar{A}$.
(iii) T is a *lower bound* of $\mathcal{A}$ if $T \leq A$ for all $A \in \mathcal{A}$.
(iv) T is an *upper bound* of $\mathcal{A}$ if $A \leq T$ for all $A \in \mathcal{A}$.

When the binary relation $\leq$ is a partial order, $\bar{A}$ is a minimal element of $\mathcal{A}$ if $A \not\leq \bar{A}$ for all $A \in \mathcal{A}$, $A \neq \bar{A}$, and $\bar{A}$ is a maximal element of $\mathcal{A}$ if $\bar{A} \not\leq A$ for all $A \in \mathcal{A}$, $A \neq \bar{A}$. If $K \subset Y$ denotes a pointed convex cone that introduces a partial order in Y we thus have that some element $\bar{y} \in A$ is a minimal element of $A \subset Y$ if

$$(\{\bar{y}\} - K) \cap A = \{\bar{y}\} . \tag{17.1}$$

Minimal elements are also known as *Edgeworth-Pareto-minimal* or *efficient* elements and will be discussed more detailed in the following section together with variations of this definition. Similar, some element $\bar{y} \in A$ is a maximal element of $A \subset Y$ if

$$(\{\bar{y}\} + K) \cap A = \{\bar{y}\} . \tag{17.2}$$

Moreover, $\bar{y} \in Y$ is a lower bound of A if $A \subset \{\bar{y}\} + K$ and an upper bound if $A \subset \{\bar{y}\} - K$.

Let $\min A$ and $\max A$ denote the sets of minimal elements and maximal elements of A w.r.t. the partial order $\leq_K$, i.e.

$$\min A = \{\bar{a} \in A \mid A \cap (\{\bar{a}\} - K) = \{\bar{a}\}\} ,$$
$$\max A = \{\bar{a} \in A \mid A \cap (\{\bar{a}\} + K) = \{\bar{a}\}\} .$$

We end this section with the definition of a chain and the famous Zorn's lemma, which is the most important result which provides a sufficient condition for the existence of a minimal element of a set, see Sect. 17.3.2.

Definition 5. Let $\mathcal{Q}$ be a pre-ordered set.

(i) $A, B \in \mathcal{Q}$ are said to be *comparable* if either $A \leq B$ or $B \leq A$ holds.
(ii) A nonempty subset $\mathcal{A}$ of $\mathcal{Q}$ is called a *chain* if any pair A, $B \in \mathcal{A}$ is comparable.

Lemma 1 (Zorn's Lemma). *Every pre-ordered set, in which every chain has an upper (lower) bound, contains at least one maximal (minimal) element.*

17.3 Vector Optimization

In this section we discuss more detailed optimality notions in vector optimization. We also give conditions guaranteeing the existence of (weakly, properly) minimal elements in a linear space. Further, we present an application in medical engineering, the field design of a magnetic resonance system. The section concludes with an introduction to vector optimization with a variable ordering structure: instead of a pre-ordered space, there, a relation is defined on the space which is in general not transitive, not antisymmetric and also not compatible with the linear structure of the space.

17.3.1 Optimality Concepts

In the following, let Y denote a real linear space that is pre-ordered by some convex cone $K \subset Y$ and let A denote some nonempty subset of Y. In general, one is mainly interested in minimal and maximal elements of the set A, but in certain situations it also makes sense to study variants of these concepts. For instance weakly minimal elements are often of interest in theoretical examinations whereas properly minimal elements are sometimes more of interest for applications.

Part (i) and (ii) in the definition below coincide with Definition 4(i) and (ii) in the case of $\mathcal{Q}$ a linear space and a pre-order given by the convex cone K.

Definition 6.

(i) An element $\bar{y} \in A$ is called a *minimal element* of the set A, if

$$(\{\bar{y}\} - K) \cap A \subset \{\bar{y}\} + K . \tag{17.3}$$

(ii) An element $\bar{y} \in A$ is called a *maximal element* of the set A, if

$$(\{\bar{y}\} + K) \cap A \subset \{\bar{y}\} - K . \tag{17.4}$$

(iii) An element $\bar{y} \in A$ is called a *strongly minimal element* of the set A, if

$$A \subset \{\bar{y}\} + K . \tag{17.5}$$

(iv) Let K have a nonempty algebraic interior, i.e. $\mathrm{cor}K \neq \emptyset$. An element $\bar{y} \in A$ is called a *weakly minimal element* of the set A, if

$$(\{\bar{y}\} - \mathrm{cor}K) \cap A = \emptyset . \tag{17.6}$$

If the ordering cone K is pointed, then the inclusions (17.3) and (17.4) can be replaced by (17.1) and (17.2), respectively. Of course, corresponding concepts as *strongly maximal* and *weakly maximal* can be defined analogously. Since every maximal element of A is also minimal w.r.t the pre-order induced by the convex cone $-K$, without loss of generality it is sufficient to study the minimality notion. In terms of lattice theory a strongly minimal element of a set A is also called *zero* element of A. It is a lower bound of the considered set, compare Definition 4(iii). As this notion is very restrictive it is often not applicable in practice. Notice that the notions "minimal" and "weakly minimal" are closely related. Take an arbitrary weakly minimal element $\bar{y} \in A$ of the set A, that is $(\{\bar{y}\} - \mathrm{cor}(K)) \cap A = \emptyset$. The set $\hat{K} := \mathrm{cor}(K) \cup \{0_Y\}$ is a convex cone and it induces another pre-order in Y. Consequently, $\bar{y}$ is also a minimal element of the set A with respect to the pre-order induced by $\hat{K}$. Figures 17.1 and 17.2a illustrate the different optimality notions.

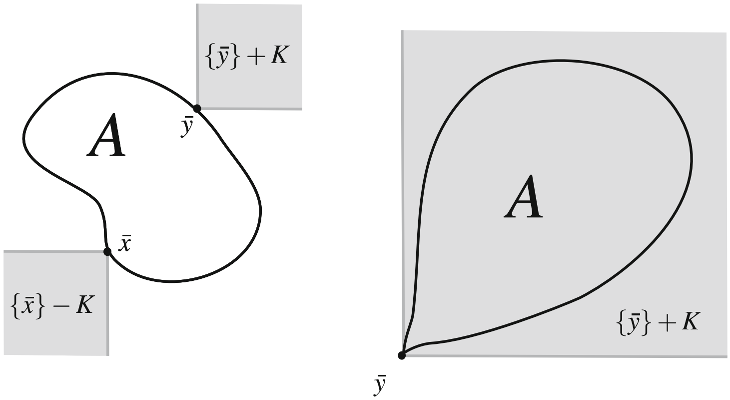


Fig. 17.1 (**a**) Minimal element $\bar{x}$ and maximal element $\bar{y}$ of a set A. (**b**) Strongly minimal element $\bar{y}$ of a set A

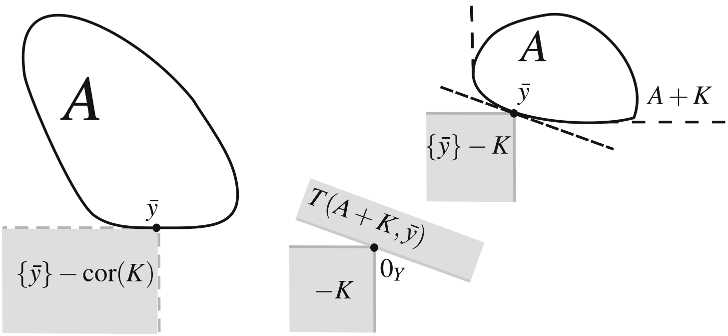


Fig. 17.2 (**a**) Weakly minimal element $\bar{y}$ of a set A. (**b**) Properly minimal element $\bar{y}$ of a set A

Example 2.

(a) Let Y be the real linear space of functionals defined on a real linear space X and pre-ordered by a pointwise order. Moreover, let A denote the subset of Y which consists of all sublinear functionals on X. Then the algebraic dual space X' is the set of all minimal elements of A. This is proved in [31, Lemma 3.7] and is a key for the proof of the basic version of the Hahn-Banach theorem.

(b) Let X and Y be pre-ordered linear spaces with the ordering cones K_X and K_Y, and let $T : X \to Y$ be a given linear map. We assume that there is a $q \in Y$ so that the set $A := \{x \in K_X \mid T(x) + q \in K_Y\}$ is nonempty. Then an *abstract complementary problem* leads to the problem of finding a minimal element of the set A. For further details we refer to [10, 17]. Obviously, if $q \in K_Y$, then 0_X is a strongly minimal element of the set A.

The next lemma gives relations between the different optimality concepts.

Lemma 2.

(a) Every strongly minimal element of the set A is also a minimal element of A.

(b) Let K have a nonempty algebraic interior and $K \neq Y$. Then every minimal element of the set A is also a weakly minimal element of the set A.

Proof.

(a) It holds $A \subset \{\bar{y}\} + K$ for any strongly minimal element $\bar{y}$ of A. Thus

$$(\{\bar{y}\} - K) \cap A \subset A \subset \{\bar{y}\} + K\,.$$

(b) The assumption $K \neq Y$ implies $(-\mathrm{cor}(K)) \cap K = \emptyset$. Therefore, for an arbitrary minimal element $\bar{y}$ of A it follows

$$\begin{aligned}\emptyset &= (\{\bar{y}\} - \mathrm{cor}(K)) \cap (\{\bar{y}\} + K)\\ &= (\{\bar{y}\} - \mathrm{cor}(K)) \cap (\{\bar{y}\} - K) \cap A\\ &= (\{\bar{y}\} - \mathrm{cor}(K)) \cap A\end{aligned}$$

which means that $\bar{y}$ is also a weakly minimal element of A.

In general, the converse statement of Lemma 2 is not true. This fact is illustrated by

Example 3. Let $Y = \mathbb{R}^2$ and let a partial order be induced by the cone $K = \mathbb{R}^2_+$. Consider the set $A = [0,1] \times [0,1]$. The unique minimal element is $0_{\mathbb{R}^2}$ while all elements of the set $\{(y_1, y_2) \in A \mid y_1 = 0 \vee y_2 = 0\}$ are weakly minimal elements. Note that $0_{\mathbb{R}^2}$ is also a strongly minimal element.

Minimal elements are a subset of the (algebraic) boundary ∂A of the set A.

Lemma 3. *Let K be nontrivial and pointed. Then every minimal element of the set A is an element of the algebraic boundary ∂A of A.*

Proof. Assume $\bar{y}$ is a minimal element of A but $\bar{y} \in \text{cor}(A)$. Then for any $k \in K \setminus \{0_Y\}$ there exists some $\lambda > 0$ with $\bar{y} - \lambda\, k \in A$. Then $\bar{y} - \lambda\, k \in A \cap (\{\bar{y}\} - K)$ in contradiction to $\bar{y}$ a minimal element.

The following lemma, compare [45], indicates that the minimal elements of a set A and the minimal elements of the set $A + K$ where K denotes the ordering cone are closely related. This result is of interest for further theoretical examinations, for instance regarding duality results. Especially if the set $A + K$ is convex while the set A is not convex, the consideration of $A + K$ instead of A is advantageous, e.g. when necessary linear scalarization results as given in [31, Theorems 5.11, 5.13] should be applied.

Lemma 4.

(a) If the ordering cone K is pointed, then every minimal element of the set $A + K$ is also a minimal element of the set A.
(b) Every minimal element of the set A is also a minimal element of the set $A + K$.

Proof.

(a) Let $\bar{y} \in A + K$ be an arbitrary minimal element of the set $A + K$. If we assume that $\bar{y} \notin A$, then there is an element $y \neq \bar{y}$ with $y \in A$ and $\bar{y} \in \{y\} + K$. Consequently, we get $y \in (\{\bar{y}\} - K) \cap (A + K)$ which contradicts the assumption that $\bar{y}$ is a minimal element of the set $A + K$. Hence, we obtain $\bar{y} \in A \subset A + K$ and, therefore, $\bar{y}$ is also a minimal element of the set A.
(b) Take an arbitrary minimal element $\bar{y} \in A$ of the set A, and choose any $y \in (\{\bar{y}\} - K) \cap (A + K)$. Then there are elements $a \in A$ and $k \in K$ so that $y = a + k$. Consequently, we obtain $a = y - k \in \{\bar{y}\} - K$, and since $\bar{y}$ is a minimal element of the set A, we conclude $a \in \{\bar{y}\} + K$. But then we get also $y \in \{\bar{y}\} + K$. This completes the proof.

If the cone K has a nonempty algebraic interior, the statement of Lemma 4 is also true if we replace minimal by weakly minimal [31, Lemma 4.13].

Another refinement of the minimality notion is helpful from a theoretical point of view. These optima are called properly minimal. Until now there are various types of concepts of proper minimality. The notion of proper minimality (or proper efficiency) was first introduced by Kuhn–Tucker [34] and modified by Geoffrion [25], and later it was formulated in a more general framework (Benson–Morin [3], Borwein [7], Vogel [45], Wendell–Lee [46], Wierzbicki [48], Hartley [27], Benson [2], Borwein [8], Nieuwenhuis [37], Henig [28], and Zhuang [53]). We present here a definition introduced by Borwein [7] and Vogel [45]. For a collection of other definitions of proper minimality see for instance [31, p. 113f].

Recall that the *contingent cone* (or *Bouligand tangent cone*) $T(A, \bar{y})$ to a subset A of a real normed space $(Y, \|\cdot\|)$ in $\bar{y} \in \text{cl}(A)$ is the set of all tangents h which are defined as follows: An element $h \in Y$ is called a tangent to A in $\bar{y}$, if there are a sequence $(y_n)_{n\in\mathbb{N}}$ of elements $y_n \in A$ and a sequence $(\lambda_n)_{n\in\mathbb{N}}$ of positive real numbers λ_n so that

$$\bar{y} = \lim_{n\to\infty} y_n \text{ and } h = \lim_{n\to\infty} \lambda_n(y_n - \bar{y}) .$$

Here, cl(A) denotes the closure of A.

Definition 7. Let $(Y, \|\cdot\|)$ be a real normed space. An element $\bar{y} \in A$ is called a *properly minimal element* of the set A, if $\bar{y}$ is a minimal element of the set A and the zero element 0_Y is a minimal element of the contingent cone $T(A + K, \bar{y})$ (see Fig. 17.2b).

It is evident that a properly minimal element of a set A is also a minimal element of A.

Example 4. Let Y be the Euclidean space $\mathbb{R}^2$ and let a partial order be induced by the cone $K = \mathbb{R}^2_+$. Consider $A = \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1^2 + y_2^2 \leq 1\}$. Then all elements of the set $\{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 \in [-1, 0],\ y_2 = -\sqrt{1 - y_1^2}\}$ are minimal elements of A. The set of all properly elements of A reads as $\{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 \in (-1, 0),\ y_2 = -\sqrt{1 - y_1^2}\}$.

The optimality concepts for subsets of a real linear space naturally induce concepts of optimal solutions for *vector optimization problems*. Let X and Y be real linear spaces, and let K, as before, be a convex cone in Y. Furthermore, let S be a nonempty subset of X, and let $f : S \to Y$ be a given map. Then the *vector optimization problem*

$$\min_{x\in S} f(x) \tag{VOP}$$

is to be interpreted in the following way: Determine a (weakly, strongly, properly) minimal solution $\bar{x} \in S$ which is defined as the inverse image of a (weakly, strongly, properly) minimal element $f(\bar{x})$ of the image set $f(S) = \{f(x) \in Y \mid x \in S\}$.

Definition 8. An element $\bar{x} \in S$ is called a *(weakly, strongly, properly) minimal solution* of problem (VOP) w.r.t. the pre-order induced by K, if $f(\bar{x})$ is a (weakly, strongly, properly) minimal element of the image set $f(S)$ w.r.t. the pre-order induced by K.

For $Y = \mathbb{R}^m$ partially ordered by the natural ordering, i.e. $K = \mathbb{R}^m_+$, we call (VOP) also a *multiobjective optimization* problem, as the m objectives $f_i : S \to \mathbb{R}$, $i = 1, \ldots, m$, are minimized simultaneously. A minimal solution, then also called Edgeworth-Pareto optimal, compare page 696, is thus a point $\bar{x} \in S$ such that there exists no other $x \in S$ with

$$f_i(x) \leq f_i(\bar{x}) \text{ for all } i = 1, \ldots, m,$$

and

$$f_j(x) < f_j(\bar{x}) \text{ for at least one } j \in \{1, \ldots, m\} .$$

Example 5. Let $X = \mathbb{R}^2$ and Y be the Euclidean space $\mathbb{R}^2$ and let a partial order be induced by the cone $K = \mathbb{R}^2_+$. Consider the constraint set

$$S := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 - x_2 \le 0,\ \ x_1 + 2x_2 - 3 \le 0\}$$

and the vector function $f : S \to \mathbb{R}^2$ with

$$f(x_1, x_2) = \begin{pmatrix} -x_1 \\ x_1 + x_2^2 \end{pmatrix} \quad \text{for all} \ \ (x_1, x_2) \in S\,.$$

The point $\left(\frac{3}{2}, \frac{57}{16}\right)$ is the only maximal element of $T := f(S)$, and the set of all minimal elements of T reads

$$\left\{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 \in \left[-1, \frac{1}{2}\sqrt[3]{2}\right] \ \ \text{and} \ \ y_2 = -y_1 + y_1^4\right\}.$$

The set of all minimal solutions of the vector optimization problem $\min_{x \in S} f(x)$ is given as

$$\left\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \in \left[-\frac{1}{2}\sqrt[3]{2}, 1\right] \ \ \text{and} \ \ x_2 = x_1^2\right\}$$

(see Fig. 17.3).

17.3.2 Existence Results

In this subsection we give assumptions which guarantee that at least one optimal element of a subset of a pre-ordered linear space exists. These investigations will be

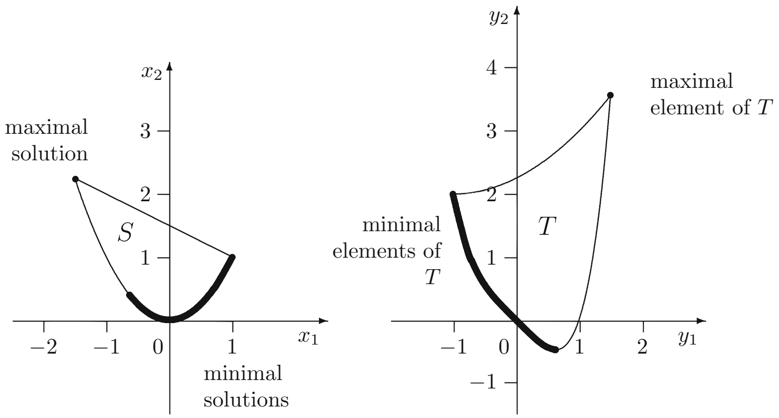


Fig. 17.3 Minimal and maximal elements of $T = f(S)$

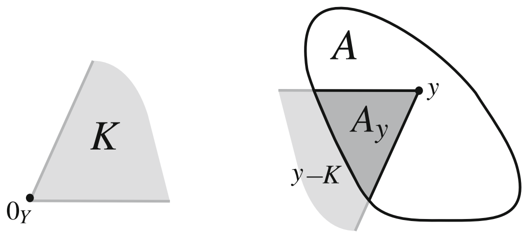


Fig. 17.4 Section A_y of a set A

done for the minimality, the properly minimality and the weakly minimality notion. Strongly minimal elements are not considered because this optimality notion is too restrictive.

In order to get existence results under weak assumptions on a set we introduce the following

Definition 9. Let A be a nonempty subset of a pre-ordered linear space Y where the pre-order is introduced by a convex cone $K \subset Y$. If for some $y \in Y$ the set $A_y = (\{y\} - K) \cap A$ is nonempty, A_y is called a *section* of the set A (see Fig. 17.4).

The assertion of the following lemma is evident.

Lemma 5. *Let A be a nonempty subset of a pre-ordered linear space Y with an ordering cone K.*

(a) Every minimal element of a section of the set A is also a minimal element of the set A.
(b) If $cor(K) \neq \emptyset$*, then every weakly minimal element of a section of the set A is also a weakly minimal element of the set A.*

It is important to remark that for the notion of proper minimality a similar statement is not true in general. We begin now with a discussion of existence results for the notion of minimal elements. The following existence result is a consequence of Zorn's lemma (Lemma 1). Recall that an ordering cone in a real topological linear space is called *Daniell* if every decreasing net (i.e. $i \leq j \Rightarrow y_j \leq_K y_i$) which has a lower bound converges to its infimum. And a real topological linear space Y with an ordering cone K is called *boundedly order complete*, if every bounded decreasing net has an infimum.

Theorem 2. *Let Y be a topological linear space which is pre-ordered by a closed ordering cone K. Then we have:*

(a) If the set A has a closed section which has a lower bound and the ordering cone K is Daniell, then there is at least one minimal element of the set A.
(b) If the set A has a closed and bounded section and the ordering cone K is Daniell and boundedly order complete, then there is at least one minimal element of the set A.
(c) If the set A has a compact section, then there is at least one minimal element of the set A.

Proof. Let A_y (for some $y \in Y$) be an appropriate section of the set A. If we show that every chain in the section A_y has a lower bound, then by Zorn's lemma (Lemma 1) A_y has at least one minimal element which is, by Lemma 5(a), also a minimal element of the set A.

Let $\{a_i\}_{i \in I}$ be any chain in the section A_y. Let $\mathscr{F}$ denote the set of all finite subsets of I which are pre-ordered with respect to the inclusion relation. Then for every $F \in \mathscr{F}$ the minimum

$$y_F := \min \{a_i \mid i \in F\}$$

exists and belongs to A_y. Consequently, $(y_F)_{F \in \mathscr{F}}$ is a decreasing net in A_y. Next, we consider several cases.

(a) A_y is assumed to have a lower bound so that $(y_F)_{F \in \mathscr{F}}$ has an infimum. Since A_y is closed and K is Daniell, $(y_F)_{F \in \mathscr{F}}$ converges to its infimum which belongs to A_y. This implies that any chain in A_y has a lower bound.

(b) Since A_y is bounded and K is boundedly order complete, the net $(y_F)_{F \in \mathscr{F}}$ has an infimum. The ordering cone K is Daniell and, therefore, $(y_F)_{F \in \mathscr{F}}$ converges to its infimum. And since A_y is closed, this infimum belongs to A_y. Hence, any chain in A_y has a lower bound.

(c) Now, A_y is assumed to be compact. The family of compact subsets A_{a_i} $(i \in I)$ has the finite intersection property, i.e., every finite subfamily has a nonempty intersection. Since A_y is compact, the family of subsets A_{a_i} $(i \in I)$ has a nonempty intersection (see Dunford–Schwartz [18, p. 17]), that is, there is an element

$$\hat{y} \in \bigcap_{i \in I} A_{a_i} = \bigcap_{i \in I} (\{a_i\} - K) \cap A_y \, .$$

Hence, $\hat{y}$ is a lower bound of the subset $\{a_i\}_{i \in I}$ and belongs to A_y. Consequently, any chain in A_y has a lower bound.

Notice that the preceding theorem remains valid, if "section" is replaced by the set itself. Theorem 2 as well as the following example is due to Borwein [10], but Theorem 2(c) was first proved by Vogel [45] and Theorem 2(a) can essentially be found, without proof, in a survey article of Penot [40].

Example 6. We consider again the problem formulated in Example 2(b). Let X and Y be pre-ordered topological linear spaces with the closed ordering cones K_X and K_Y where K_X is also assumed to be Daniell. Moreover, let $T : X \to Y$ be a continuous linear map and let $q \in Y$ be given so that the set $A := \{x \in K_X \mid T(x) + q \in K_Y\}$ is nonempty. Clearly the set A is closed and has a lower bound (namely 0_X). Then by Theorem 2(a) the set A has at least one minimal element.

For the next existence result we need the so-called *James theorem* [33].

Theorem 3 (James Theorem). *Let A be a nonempty bounded and weakly closed subset of a real quasi-complete locally convex space Y. If every continuous linear functional $l \in Y^*$ (with Y^* the topological dual space of Y) attains its supremum on A, then A is weakly compact.*

Using this theorem together with Theorem 2(c) we obtain the following result due to Borwein [10].

Theorem 4. *Let A be a nonempty subset of a real locally convex space Y.*

(a) If A is weakly compact, then for every closed convex cone K in Y the set A has at least one minimal element with respect to the pre-order induced by K.
(b) In addition, let Y be quasi-complete (for instance, let Y be a Banach space). If A is bounded and weakly closed and for every closed convex cone K in Y the set A has at least one minimal element with respect to the pre-order induced by K, then A is weakly compact.

Proof.

(a) Every closed convex cone K is also weakly closed [31, Lemma 3.24]. Since A is weakly compact, there is a compact section of A. Then, by Theorem 2(c), A has at least one minimal element with respect to the pre-order induced by K.
(b) It is evident that the functional 0_{Y^*} attains its supremum on the set A. Therefore, take an arbitrary continuous linear functional $l \in Y^* \backslash \{0_{Y^*}\}$ (if it exists) and define the set $K := \{y \in Y \mid l(y) \leq 0\}$ which is a closed convex cone. Let $\bar{y} \in A$ be a minimal element of the set A with respect to the pre-order induced by K, i.e.

$$(\{\bar{y}\} - K) \cap A \subset \{\bar{y}\} + K \,. \tag{17.7}$$

Since

$$\{\bar{y}\} - K = \{y \in Y \mid l(y) \geq l(\bar{y})\}$$

and

$$\{\bar{y}\} + K = \{y \in Y \mid l(y) \leq l(\bar{y})\} \,,$$

the inclusion (17.7) is equivalent to the implication

$$y \in A, \; l(y) \geq l(\bar{y}) \implies l(y) = l(\bar{y}) \,.$$

This implication can also be written as

$$l(\bar{y}) \geq l(y) \text{ for all } y \in A \,.$$

This means that the functional l attains its supremum on A at $\bar{y}$. Then by the James theorem (Theorem 3) the set A is weakly compact.

The preceding theorem shows that the weak compactness assumption on a set plays an important role for the existence of minimal elements.

Next, we study existence theorems which follow from scalarization results, compare [30]. Recall that a nonempty subset A of a real normed space $(Y, \|\cdot\|)$ is called *proximinal*, if every $y \in Y$ has at least one best approximation from A, that is, for every $y \in Y$ there is an $\bar{y} \in A$ with

$$\|y - \bar{y}\| \leq \|y - a\| \text{ for all } a \in A .$$

Any nonempty weakly closed subset of a real reflexive Banach space is proximinal [31, Corollary 3.35]. A functional $f : A \to \mathbb{R}$ with A a nonempty subset of a linear space pre-ordered by K is called *strongly monotonically increasing* on A, if for every $\bar{y} \in A$

$$y \in (\{\bar{y}\} - K) \cap A,\ y \neq \bar{y} \implies f(y) < f(\bar{y}) .$$

If $\text{cor}(K) \neq \emptyset$, then f is called *strictly monotonically increasing*, if for every $\bar{y} \in A$

$$y \in (\{\bar{y}\} - \text{cor}(K)) \cap A \implies f(y) < f(\bar{y}) .$$

Theorem 5. *Assume that either assumption (a) or assumption (b) below holds:*

(a) Let A be a nonempty subset of a partially ordered normed space $(Y, \|\cdot\|_Y)$ with a pointed ordering cone K, and let Y be the topological dual space of a real normed space $(Z, \|\cdot\|_Z)$. Moreover, for some $y \in Y$ let a weak-closed section A_y be given.*

(b) Let A be a nonempty subset of a partially ordered reflexive Banach space $(Y, \|\cdot\|_Y)$ with a pointed ordering cone K. Furthermore, for some $y \in Y$ let a weakly closed section A_y be given.

If, in addition, the section A_y has a lower bound $\hat{y} \in Y$, i.e. $A_y \subset \{\hat{y}\} + K$, and the norm $\|\cdot\|_Y$ is strongly monotonically increasing on K, then the set A has at least one minimal element.

Proof. Let the assumptions of (a) be satisfied. Take any $z \in Z^* \setminus A_y = Y \setminus A_y$ and any $a \in A_y$. Since every closed ball in $Z^* = Y$ is weak*-compact, the set

$$A_y \cap \{w \in Y \mid \|w\|_Y \leq \|a\|_Y\}$$

is weak*-compact as well. Notice that the functional mapping from Y to $\mathbb{R}$ given by $w \mapsto \|z - w\|_Y$ is weakly* lower semicontinuous. Thus the section A_y is proximinal. On the other hand, if the assumption (b) is satisfied, then the section A_y is proximinal as well. Consequently, there is an $\bar{y} \in A_y$ with

$$\|\bar{y} - \hat{y}\|_Y \le \|a - \hat{y}\|_Y \text{ for all } a \in A_y \,. \tag{17.8}$$

The norm $\|\cdot\|_Y$ is strongly monotonically increasing on K and because of $A_y - \{\hat{y}\} \subset K$ the functional $\|\cdot - \hat{y}\|_Y$ is strongly monotonically increasing on A_y, compare [31, Theorem 5.15(b)].

Next we show that $\bar{y}$ is a minimal element of A_y. Assume this is not the case. Then there is an element $a \in (\{\bar{y}\} - K) \cap A_y$ with $a \neq \bar{y}$. This implies $\|a - \hat{y}\|_Y < \|\bar{y} - \hat{y}\|_Y$ in contradiction to (17.8).

Finally, an application of Lemma 5(a) completes the proof.

Example 7. Let A be a nonempty subset of a pre-ordered Hilbert space $(Y, \langle .,. \rangle)$ with an ordering cone K_Y. Then the norm on Y is strongly monotonically increasing on K_Y if and only if $K_Y \subset K_Y^*$ with $K_Y^* = \{y^* \in Y^* \mid y^*(y) \ge 0 \text{ for all } y \in K_Y\}$ the dual cone of K_Y [41, 47]. Thus, if the ordering cone K_Y has the property $K_Y \subset K_Y^*$ and A has a weakly closed section bounded from below, then A has at least one minimal element.

For the minimality notion a scalarization result concerning positive linear functionals leads to an existence theorem which is contained in Theorem 4(a). But for the proper minimality notion such a scalarization result is helpful. We recall the important *Krein-Rutman theorem*. For a proof see [9, p. 425] or [31, Theorem 3.38].

Theorem 6 (Krein-Rutman Theorem). *In a real separable normed space $(Y, \|\cdot\|)$ with a closed pointed convex cone $K \subset Y$ the quasi-interior*

$$K_{Y^*}^{\#} := \{y^* \in Y^* \mid y^*(y) > 0 \ \textit{for all}\ y \in K \setminus \{0_Y\}\}$$

of the topological dual cone is nonempty.

Theorem 7. *Let A be a weakly compact subset of a partially ordered separable normed space $(Y, \|\cdot\|)$ with a closed pointed ordering cone K. Then there exists at least one properly minimal element $\bar{y} \in A$.*

Proof. According to the Krein-Rutman theorem, Theorem 6, the quasi-interior of the topological dual cone is nonempty. Then every continuous linear functional which belongs to that quasi-interior attains its infimum on the weakly compact set A. So there exist some $\bar{y} \in A$ and some $l \in K_{Y^*}^{\#}$ with

$$l(\bar{y}) \le l(y) \ \text{ for all } \ y \in A \,. \tag{17.9}$$

As $l \in K_{Y^*}^{\#}$, l is strongly monotonically increasing on A. First we assume $\bar{y}$ is not a minimal element of A. Then there is an element $a \in (\{\bar{y}\} - K) \cap A$ with $a \neq \bar{y}$. This implies $l(a) < l(\bar{y})$ which is a contradiction to (17.9). Thus $\bar{y}$ is a minimal element of A and it remains to show that 0_Y is a minimal element of the contingent cone $T(A + K, \bar{y})$.

Take any tangent $h \in T(A + K, \bar{y})$. Then there are a sequence $(y_n)_{n\in\mathbb{N}}$ of elements in $A + K$ and a sequence $(\lambda_n)_{n\in\mathbb{N}}$ of positive real numbers with $\bar{y} = \lim_{n\to\infty} y_n$

and $h = \lim_{n\to\infty} \lambda_n(y_n - \bar{y})$. The linear functional l is continuous and, therefore, we get $l(\bar{y}) = \lim_{n\to\infty} l(y_n)$. Since the functional l is also strongly monotonically increasing on Y, the inequality (17.9) implies

$$l(\bar{y}) \le l(y) \text{ for all } y \in A + K .$$

Then it follows

$$l(h) = \lim_{n\to\infty} l(\lambda_n(y_n - \bar{y})) = \lim_{n\to\infty} \lambda_n(l(y_n) - l(\bar{y})) \ge 0 .$$

Hence we obtain

$$l(0_Y) = 0 \le l(h) \text{ for all } h \in T(A + K, \bar{y}) .$$

With the same arguments as before we conclude that 0_Y is a minimal element of $T(A + K, \bar{y})$. This completes the proof.

A further existence theorem for properly minimal elements is given by

Theorem 8. *Assume that either assumption (a) or assumption (b) below holds:*

(a) Let A be a nonempty subset of a partially ordered normed space $(Y, \|\cdot\|_Y)$ with a pointed ordering cone K which has a nonempty algebraic interior, and let Y be the topological dual space of a real normed space $(Z, \|\cdot\|_Z)$. Moreover, let the set A be weak-closed.*

(b) Let A be a nonempty subset of a partially ordered reflexive Banach space $(Y, \|\cdot\|_Y)$ with a pointed ordering cone K which has a nonempty algebraic interior. Furthermore, let the set A be weakly closed.

If, in addition, there is an $\hat{y} \in Y$ with $A \subset \{\hat{y}\} + cor(K)$ and the norm $\|\cdot\|_Y$ is strongly monotonically increasing on K, then the set A has at least one properly minimal element.

Proof. The proof is similar to that of Theorem 5. Since the norm $\|\cdot\|_Y$ is strongly monotonically increasing on K and $A - \{\hat{y}\} \subset \mathrm{cor}(K)$ we get with the same arguments that there is some $\bar{y} \in A$ with

$$\|\bar{y} - \hat{y}\|_Y \le \|y - \hat{y}\|_Y \text{ for all } y \in A \tag{17.10}$$

and that $\bar{y}$ is a minimal element of A. It remains to show that 0_Y is a minimal element of $T(A + K, \bar{y})$.

Since the norm $\|\cdot\|_Y$ is assumed to be strongly monotonically increasing on K, we obtain from (17.10)

$$\|\bar{y} - \hat{y}\|_Y \le \|y - \hat{y}\|_Y \le \|y + k - \hat{y}\|_Y \text{ for all } y \in A \text{ and all } k \in K .$$

This results in

$$\|\bar{y} - \hat{y}\|_Y \leq \|y - \hat{y}\|_Y \text{ for all } y \in A + K . \tag{17.11}$$

It is evident that the functional $\| \cdot - \hat{y}\|_Y$ is both convex and continuous in the topology generated by the norm $\| \cdot \|_Y$. Then, see for instance [31, Theorem 3.48], the inequality (17.11) implies

$$\|\bar{y} - \hat{y}\|_Y \leq \|\bar{y} - \hat{y} + h\|_Y \text{ for all } h \in T(A + K, \bar{y}) . \tag{17.12}$$

With $T := T(A + K, \bar{y}) \cap (\{\hat{y} - \bar{y}\} + K)$ the inequality (17.12) is also true for all $h \in T$, i.e.

$$\|0_Y - (\hat{y} - \bar{y})\|_Y \leq \|h - (\hat{y} - \bar{y})\|_Y \text{ for all } h \in T$$

and $\| \cdot - (\hat{y} - \bar{y})\|_Y$ is because of $T - \{\hat{y} - \bar{y}\} \subset K$ strongly monotonically increasing on T. With the same arguments as in Theorem 5 0_Y is a minimal element of T.

Next we assume that 0_Y is not a minimal element of the contingent cone $T(A + K, \bar{y})$. Then there is an element $y \in (-K) \cap T(A + K, \bar{y})$ with $y \neq 0_Y$. Since $A \subset \{\hat{y}\} + \text{cor}(K)$ and $\bar{y} \in A$, there is a $\lambda > 0$ with $\bar{y} + \lambda y \in \{\hat{y}\} + K$ or $\lambda y \in \{\hat{y} - \bar{y}\} + K$. Consequently, we get

$$\lambda y \in (-K) \cap T(A + K, \bar{y}) \cap (\{\hat{y} - \bar{y}\} + K)$$

and therefore, we have $\lambda y \in (-K) \cap T$ which contradicts the fact that 0_Y is a minimal element of the set T. Hence, 0_Y is a minimal element of the contingent cone $T(A + K, \bar{y})$ and the assertion is obvious.

Example 8. Let A be a nonempty subset of a partially ordered Hilbert space $(Y, \langle ., . \rangle)$ with an ordering cone K_Y which has a nonempty algebraic interior and for which $K_Y \subset K_Y^*$ (compare Example 7). If A is weakly closed and there is an $\hat{y} \in Y$ with $A \subset \{\hat{y}\} + \text{cor}(K_Y)$, then the set A has at least one properly minimal element.

Finally, we turn our attention to the weak minimality notion. Using Lemma 2(b) we can easily extend the existence theorems for minimal elements to weakly minimal elements, if we assume additionally that the ordering cone $K \subset Y$ does not equal Y and that it has a nonempty algebraic interior. This is one possibility in order to get existence results for the weak minimality notion. In the following theorems we use directly appropriate scalarization results for this optimality notion.

Theorem 9. *Let A be a nonempty subset of a pre-ordered locally convex space Y with a closed ordering cone $K_Y \neq Y$ which has a nonempty algebraic interior. If A has a weakly compact section, then the set A has at least one weakly minimal element.*

Proof. Applying a separation theorem we get that, since the ordering cone K_Y is closed and does not equal Y, there is at least one continuous linear functional $l \in K_Y^* \setminus \{0_{Y^*}\}$ with $K_Y^* = \{y^* \in Y^* \mid y^*(y) \geq 0 \text{ for all } y \in K_Y\}$. This functional attains its infimum on a weakly compact section of A, i.e. there is some $\bar{y} \in A$ and some $y \in Y$ with

$$l(\bar{y}) \leq l(a) \text{ for all } a \in A_y \,. \tag{17.13}$$

Assume $\bar{y}$ is not a weakly minimal element of A_y. Then there is some $a \in (\{\bar{y}\} - \operatorname{cor}(K)) \cap A_y$, and as l is strictly monotonically increasing on A_y due to $l \in K_Y^* \setminus \{0_Y\}$ we get $l(a) < l(\bar{y})$ in contradiction to (17.13). Thus $\bar{y}$ is a weakly minimal element of A_y and because of Lemma 5(b) also of A.

Notice that Theorem 9 could also be proved using Theorem 4(a) and Lemma 2(b).

Theorem 10. *Assume that either assumption (a) or assumption (b) below holds:*

(a) Let A be a nonempty subset of a pre-ordered normed space $(Y, \|\cdot\|_Y)$ with an ordering cone K which has a nonempty algebraic interior, and let Y be the topological dual space of a real normed space $(Z, \|\cdot\|_Z)$. Moreover, for some $y \in Y$ let a weak-closed section A_y be given.*

(b) Let A be a nonempty subset of a pre-ordered reflexive Banach space $(Y, \|\cdot\|_Y)$ with an ordering cone K which has a nonempty algebraic interior. Furthermore, for some $y \in Y$ let a weakly closed section A_y be given.

If, in addition, the section A_y has a lower bound $\hat{y} \in Y$, i.e. $A_y \subset \{\hat{y}\} + K$, and the norm $\|\cdot\|_Y$ is strictly monotonically increasing on K, then the set A has at least one weakly minimal element.

Proof. The proof is similar to that of Theorem 5.

Example 9. Let A be a nonempty subset of $L_\infty(\Omega)$, the linear space of all (equivalence classes of) essentially bounded functions $f\colon \Omega \to \mathbb{R}$ ($\emptyset \neq \Omega \subset \mathbb{R}^n$) with the norm $\|\cdot\|_{L_\infty(\Omega)}$ given by

$$\|f\|_{L_\infty(\Omega)} := \operatorname{ess\,sup}_{x\in\Omega}\{|f(x)|\} \text{ for all } f \in L_\infty(\Omega) \,.$$

The ordering cone $K_{L_\infty(\Omega)}$ is defined as

$$K_{L_\infty(\Omega)} := \{f \in L_\infty(\Omega) \mid f(x) \geq 0 \text{ almost everywhere on } \Omega\} \,.$$

It has a nonempty topological interior and it is weak* Daniell. We show that if the set A has a weak*-closed section bounded from below, then A has at least one weakly minimal element:

If we consider the linear space $L_\infty(\Omega)$ as the topological dual space of $L_1(\Omega)$, then the assertion follows from Theorem 10, if we show that the norm $\|\cdot\|_{L_\infty(\Omega)}$ is strictly monotonically increasing on the ordering cone $K_{L_\infty(\Omega)}$. It is evident that

$$\begin{aligned}\operatorname{int}(K_{L_\infty(\Omega)}) = \{f \in L_\infty(\Omega) \mid &\text{there is an } \alpha > 0 \text{ with} \\ &f(x) \geq \alpha \text{ almost everywhere on } \Omega\} \neq \emptyset \,.\end{aligned}$$

As $K_{L_\infty(\Omega)}$ is convex with a nonempty topological interior, $\text{int}(K_{L_\infty(\Omega)})$ equals the algebraic interior of $K_{L_\infty(\Omega)}$. Take any functions $f, g \in K_{L_\infty(\Omega)}$ with $f \in \{g\} - \text{int}(K_{L_\infty(\Omega)})$. Then we have $g - f \in \text{int}(K_{L_\infty(\Omega)})$ which implies that there is an $\alpha > 0$ with

$$g(x) - f(x) \geq \alpha \text{ almost everywhere on } \Omega$$

and

$$g(x) \geq \alpha + f(x) \text{ almost everywhere on } \Omega \ .$$

Consequently, we get

$$\text{ess sup}_{x\in\Omega} \{g(x)\} \geq \alpha + \text{ ess sup}_{x\in\Omega} \{f(x)\}$$

and

$$\|g\|_{L_\infty(\Omega)} > \|f\|_{L_\infty(\Omega)} \ .$$

Hence, the norm $\|\cdot\|_{L_\infty(\Omega)}$ is strictly monotonically increasing on $K_{L_\infty(\Omega)}$.

We conclude this section with the *Bishop-Phelps lemma* [6], which is a special type of an existence result for maximal elements. First we recall that in a real normed space $(Y, \|\cdot\|_Y)$ for an arbitrary continuous linear functional $l \in Y^*$ and an arbitrary $\gamma \in (0, 1)$ the cone

$$C(l, \gamma) := \{y \in Y \mid \gamma \|y\|_Y \leq l(y)\}$$

is called *Bishop-Phelps cone*. Notice that this cone is convex and pointed and, therefore, it can be used as an ordering cone in the space Y.

Lemma 6 (Bishop-Phelps Lemma). *Let A be a nonempty closed subset of a real Banach space* $(Y, \|\cdot\|_Y)$*, and let a continuous linear functional* $l \in Y^*$ *be given with* $\|l\|_{Y^*} = 1$ *and* $\sup_{y\in A} l(y) < \infty$*. Then for every* $y \in A$ *and every* $\gamma \in (0, 1)$ *there is a maximal element* $\bar{y} \in \{y\} + C(l, \gamma)$ *of the set A with respect to the Bishop-Phelps ordering cone* $C(l, \gamma)$*.*

For the proof we refer to [6] as well as to [29, p. 164].

17.3.3 Application: Field Design of a Magnetic Resonance System

In this subsection we discuss a vector optimization problem of the type (VOP) which is of importance in magnetic resonance systems in medical engineering. Magnetic resonance (MR) systems are significant devices in medical engineering which may produce images of soft tissue of the human body with high resolution and good contrast. Among others, it is a useful device for cancer diagnosis. The images are physically generated by the use of three types of magnetic fields: the main field, the gradient field and the radio frequency (RF) field, compare [43].

MR uses the spin of the atomic nuclei in a human body and it is the hydrogen proton whose magnetic characteristics are used to generate images. One does not consider only one spin but a collection of spins in a voxel being a small volume element. Without an external magnetic field the spins in this voxel are randomly oriented and because of their superposition their effects vanish (see Fig. 17.5a). By using the main field which is generated by super-conducting magnets, the spin magnets align in parallel or anti-parallel to the field (see Fig. 17.5b). There is a small majority of up spins in contrast to down spins and this difference leads to a very weak magnetization of the voxel. The spin magnet behaves like a magnetic top used by children; this is called the spin precession (see Fig. 17.6).

With an additional RF pulse the magnetization flips. This stimulation with an RF pulse leads to magnetic resonances in the body. In order to get the slices that give us the images, we use a so-called gradient field with the effect that outside the defined slice the nuclear spins are not affected by the RF pulse. The obtained voxel information in a slice can then be used for the construction of MR images via a two-dimensional Fourier transform. A possible MR image of a human head is given in Fig. 17.7.

There are various optimization problems in the context of the improvement of the quality of MR images. We restrict ourselves to the description of the following bicriterial optimization problem, i.e. we consider a vector optimization problem as

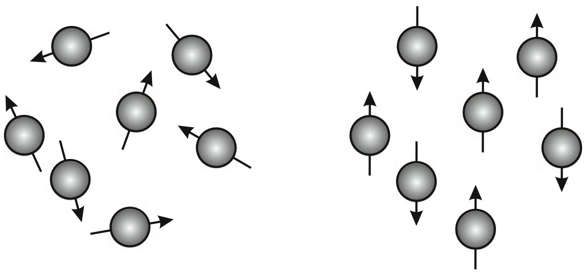


Fig. 17.5 (**a**) Arbitrary spins. (**b**) Parallel and anti-parallel aligned spins

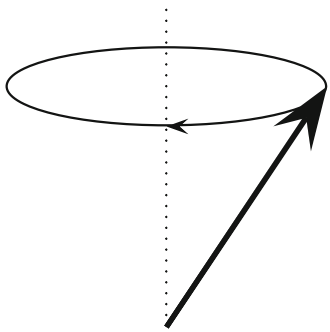


Fig. 17.6 Spin precession

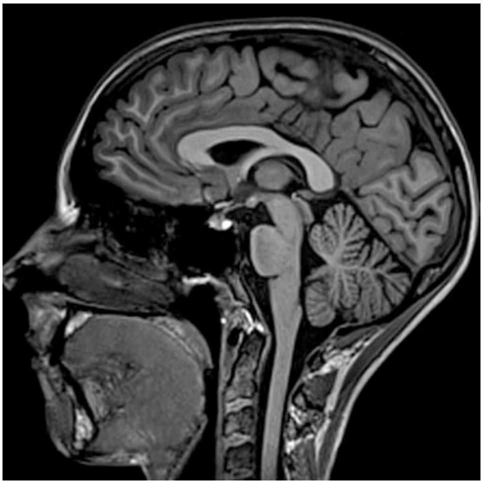


Fig. 17.7 A so-called sagittal T1 MP-RAGE image taken up by the 3 tesla system MAGNETOM Skyra produced by Siemens AG. With kind permission of Siemens AG Healthcare sector

presented in (VOP) with $Y = \mathbb{R}^2$ the Euclidean space. This problem was already considered by Bijick (Schneider), Diehl and Renz [4, 5]. We assume that in the image space a partial order is introduced by the cone $K = \mathbb{R}^2_+$. For good MR images it is important to improve the homogeneity of the RF field for specific slices. Here we assume that the MR system uses $n \in \mathbb{N}$ antennas. The complex design variables $x_1, \ldots, x_n \in \mathbb{C}$ are the so-called scattering variables. Thus we choose $X = \mathbb{C}^n$. For a slice with $p \in \mathbb{N}$ voxels let $H^x_{k\ell},\ H^y_{k\ell} \in \mathbb{C}$ (for $k \in \{1, \ldots, p\}$ and $\ell \in \{1, \ldots, n\}$) denote the cartesian components of the RF field of the k-th antenna in the ℓ-th voxel, if we work with a current of amplitude 1 ampere and phase 0. Then the objective function f_1, which is a standard deviation, reads as follows

$$f_1(x) := \frac{\sqrt{\frac{1}{p-1}\sum_{k=1}^{p}\left(H^-_k(x)\overline{H^-_k(x)} - \sum_{k=1}^{p} w_k H^-_k(x)\overline{H^-_k(x)}\right)^2}}{\sum_{k=1}^{p} w_k H^-_k(x)\overline{H^-_k(x)}}$$

for all $x \in \mathbb{C}^n$ with

$$H^-_k(x) := \frac{1}{2}\sum_{\ell=1}^{n} \overline{x}_\ell \overline{(H^x_{kl} - i\,H^y_{kl})} \ \text{ for all } \ x \in \mathbb{C}^n \ \text{ and } \ k \in \{1, \ldots, p\}$$

(here i denotes the imaginary unit and the overline means the conjugate complex number). Moreover, we would like to reduce the specific absorption rate (SAR) which is the RF energy absorbed per time unit and kilogram. Global energy absorption in the entire body is an important value for establishing safety thresholds. If $m > 0$ denotes the mass of the patient and $M \in \mathbb{R}^{(n,n)}$ denotes the so-called scattering matrix, then the second objective function f_2 is given by

$$f_2(x) := \frac{1}{2m} x^\top (I - M^\top M)x \text{ for all } x \in \mathbb{C}^n$$

where I denotes the (n, n) identity matrix. f_2 describes the global SAR.

The constraints of this bicriterial problem describing the set S in (VOP) are given by upper bounds for the warming of the tissue within every voxel. The HUGO body model which is a typical human body model based on anatomical data of the Visible Human Project®, has more than 380,000 voxels which means that this bicriterial optimization problem has more than 380,000 constraints. A discussion of these constraints cannot be done in detail in this text. Using the so-called modified Polak method [31, Algorithm 12.1] one obtains an approximation of the image set of the set of minimal solutions of this large-scale bicriterial problem. The numerical results qualitatively illustrated in Fig. 17.8 are obtained by Bijick (Schneider, University of Erlangen-Nürnberg, Erlangen, 2010, private communication). These results are better than the realized parameters in an ordinary MR system which uses a symmetric excitation pulse.

Notice in Fig. 17.8 that the global SAR measured in $\frac{\text{w}}{\text{kg}}$ is considered per time unit which may be very short because one considers only short RF pulses.

17.3.4 Vector Optimization with a Variable Ordering Structure

In vector optimization one assumes in general, as we have seen in the subsections above, that a pre-order is given by some nontrivial convex cone K in the considered space Y. But already in 1974 in one of the first publications [51] related to the

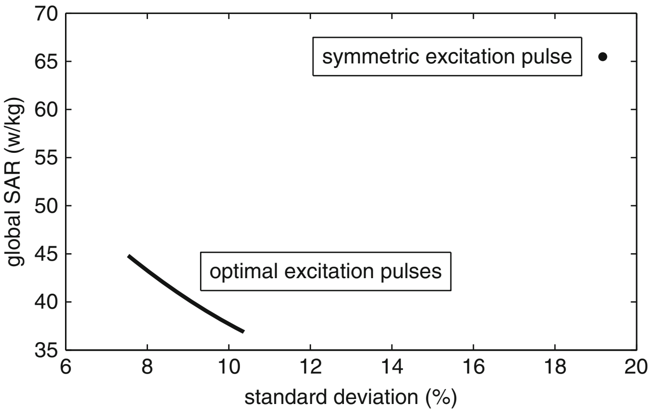


Fig. 17.8 Qualitative illustration of the image points of minimal solutions and the image point of the standard excitation pulse

definition of optimal elements in vector optimization also the idea of variable ordering structures was given: to each element of the space a cone of dominated (or preferred) directions is defined and thus the ordering structure is given by a set-valued map. In [51] a candidate element was defined to be *nondominated* if it is not dominated by any other reference element w.r.t. the corresponding cone of this other element. Later, also another notion of optimal elements in the case of a variable ordering structure was introduced [11–13]: a candidate element is called a *minimal* (or *nondominated-like*) element if it is not dominated by any other reference element w.r.t. the cone of the candidate element.

Recently there is an increasing interest in such variable ordering structures motivated by several applications for instance in medical image registration [21, 23] or in portfolio optimization [1, 20]. For a study of such vector optimization problems with a variable ordering structure it is important to differentiate between the two mentioned optimality concepts as well as to examine the relation between the concepts. In view of applications it is also important to formulate characterizations of optimal elements by scalarizations for allowing numerical calculations.

In the following we assume Y to be a real topological linear space and A to be a nonempty subset of Y. Let $\mathscr{D}\colon Y \rightrightarrows Y$ be a set-valued map with $\mathscr{D}(y)$ a pointed convex cone for all $y \in Y$ and let $\mathscr{D}(A) := \bigcup_{y\in A} \mathscr{D}(y)$ denote the image of A under $\mathscr{D}$.

Based on the cone-valued map $\mathscr{D}$ one can define two different relations: for $y, \bar{y} \in Y$ we define

$$y \leq_1 \bar{y} \ \text{ if } \ \bar{y} \in \{y\} + \mathscr{D}(y) \tag{17.14}$$

and

$$y \leq_2 \bar{y} \ \text{ if } \ \bar{y} \in \{y\} + \mathscr{D}(\bar{y}) \ . \tag{17.15}$$

We speak here of a variable ordering (structure), given by the ordering map $\mathscr{D}$, despite the binary relations given above are in general not transitive nor compatible with the linear structure of the space, to express that the pre-order given by a cone in most vector optimization problems in the literature is replaced by a relation defined by $\mathscr{D}$.

Relation (17.14) implies the concept of nondominated elements defined in [51, 52]. We also state the definitions of weakly and strongly nondominated elements which can easily be derived from the original definition of nondominated elements.

Definition 10.

(a) An element $\bar{y} \in A$ is a *nondominated element* of A w.r.t. the ordering map $\mathscr{D}$ if there is no $y \in A \setminus \{\bar{y}\}$ such that $\bar{y} \in \{y\} + \mathscr{D}(y)$, i.e., $y \not\leq_1 \bar{y}$ for all $y \in A \setminus \{\bar{y}\}$.

(b) An element $\bar{y} \in A$ is a *strongly nondominated* element of A w.r.t. the ordering map $\mathscr{D}$ if $\bar{y} \in \{y\} - \mathscr{D}(y)$ for all $y \in A$.

(c) Let $\mathscr{D}(y)$ have a nonempty interior, i.e. $\text{int}(\mathscr{D}(y)) \neq \emptyset$, for all $y \in A$. An element $\bar{y} \in A$ is a *weakly nondominated* element of A w.r.t. the ordering map $\mathscr{D}$ if there is no $y \in A$ such that $\bar{y} \in \{y\} + \text{int}(\mathscr{D}(y))$.

Example 10. Let $Y = \mathbb{R}^2$, the cone-valued map $\mathscr{D}\colon \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ be defined by

$$\mathscr{D}(y_1, y_2) := \begin{cases} \text{cone conv}\{(y_1, y_2), (1, 0)\} & \text{if } (y_1, y_2) \in \mathbb{R}^2_+,\ y_2 \neq 0, \\ \mathbb{R}^2_+ & \text{otherwise,} \end{cases}$$

and

$$A := \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 \geq 0,\ y_2 \geq 0,\ y_2 \geq 1 - y_1\}.$$

Here cone and conv denote the conic hull and the convex hull, respectively. Then $\mathscr{D}(y_1, y_2) \subset \mathbb{R}^2_+$ for all $(y_1, y_2) \in \mathbb{R}^2$ and one can check that $\{(y_1, y_2) \in A \mid y_1 + y_2 = 1\}$ is the set of all nondominated elements of A w.r.t. $\mathscr{D}$ and that all elements of the set $\{(y_1, y_2) \in A \mid y_1 + y_2 = 1 \ \vee y_1 = 0 \ \vee \ y_2 = 0\}$ are weakly nondominated elements of A w.r.t. $\mathscr{D}$.

In Definition 10 the cone $\mathscr{D}(y) = \{d \in Y \mid y + d \text{ is dominated by } y\} \cup \{0_Y\}$ can be seen as the set of dominated directions for each element $y \in Y$. Note that when $\mathscr{D}(y) \equiv K$, where K is a pointed convex cone, and the space Y is partially ordered by K, the concepts of nondominated, strongly nondominated and weakly nondominated elements w.r.t. the ordering map $\mathscr{D}$ reduce to the classical concepts of minimal, strongly minimal and weakly minimal elements w.r.t. the cone K, compare Definition 6. Strongly nondominated is a stronger concept than nondominatedness, as it is not only demanded that $\bar{y} \in \{y\} + (Y \setminus \{\mathscr{D}(y)\})$ for all $y \in A \setminus \{\bar{y}\}$, but even $\bar{y} \in \{y\} - \mathscr{D}(y)$ for all $y \in A \setminus \{\bar{y}\}$ for $\bar{y}$ being strongly nondominated w.r.t. $\mathscr{D}$. This can be interpreted as the requirement of being far away from being dominated.

The second relation, relation (17.15), leads to the concept of minimal, also called nondominated-like, elements [11–13].

Definition 11.

(a) An element $\bar{y} \in A$ is a *minimal element* of A w.r.t. the ordering map $\mathscr{D}$ if there is no $y \in A \setminus \{\bar{y}\}$ such that $\bar{y} \in \{y\} + \mathscr{D}(\bar{y})$, i.e., $y \not\leq_2 \bar{y}$ for all $y \in A \setminus \{\bar{y}\}$.
(b) An element $\bar{y} \in A$ is a *strongly minimal* element of A w.r.t. the ordering map $\mathscr{D}$ if $A \subset \{\bar{y}\} + \mathscr{D}(\bar{y})$.
(c) Let $\text{int}(\mathscr{D}(y)) \neq \emptyset$ for all $y \in A$. An element $\bar{y} \in A$ is a *weakly minimal* element of A w.r.t. the ordering map $\mathscr{D}$ if there is no $y \in A$ such that $\bar{y} \in \{y\} + \text{int}(\mathscr{D}(\bar{y}))$.

For an illustration of both optimality notions see Fig. 17.9.

The concepts of strongly minimal and strongly nondominated elements w.r.t. an ordering map $\mathscr{D}$ are illustrated in the following example.

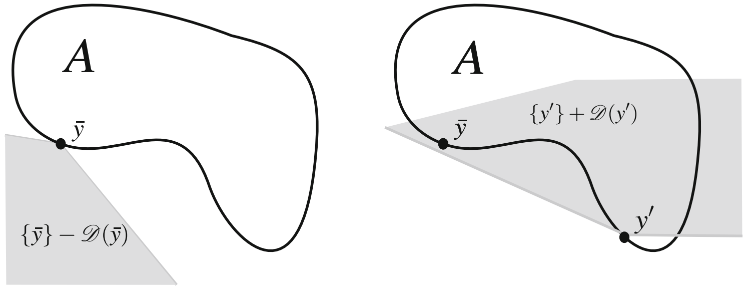


Fig. 17.9 The element $\bar{y} \in A$ is a minimal element of A w.r.t. the ordering map $\mathscr{D}$ whereas $\bar{y}$ is not a nondominated element of A w.r.t. the ordering map $\mathscr{D}$ because of $\bar{y} \in \{y'\} + \mathscr{D}(y') \setminus \{0_Y\}$, cf. [21, 23]

Example 11. Let $Y = \mathbb{R}^2$, the cone-valued map $\mathscr{D}\colon \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ be defined by

$$\mathscr{D}(y_1, y_2) := \begin{cases} \mathbb{R}^2_+ & \text{if } y_2 = 0, \\ \text{cone conv}\{(|y_1|, |y_2|), (1, 0)\} & \text{otherwise,} \end{cases}$$

and

$$A := \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 \leq y_2 \leq 2y_1\} .$$

One can check that $(0, 0) \in A$ is a strongly minimal and also a strongly nondominated element of A w.r.t. $\mathscr{D}$.

Regarding the notion of minimal elements, the cone $\mathscr{D}(y)$ for some $y \in Y$ can be viewed as the set of preferred directions: $\mathscr{D}(y) := \{d \in Y \mid y - d \text{ is preferred to } y\} \cup \{0_Y\}$. Observe that $\bar{y}$ is a minimal element of some set $A \subset Y$ w.r.t. $\mathscr{D}$ if and only if it is a minimal element of the set A with Y partially ordered by $K := \mathscr{D}(\bar{y})$.

Replacing $\mathscr{D}$ by $\tilde{\mathscr{D}}$ with $\tilde{\mathscr{D}}(y) := -\mathscr{D}(y)$ for all $y \in Y$ in the Definitions 10 and 11, we obtain corresponding concepts of (weakly, strongly) *max-nondominated* and *maximal* elements of a set A w.r.t. the ordering map $\mathscr{D}$.

The following example illustrates that the concepts of nondominated and of minimal elements w.r.t. an ordering map $\mathscr{D}$ are not directly related.

Example 12. Let $Y = \mathbb{R}^2$, the cone-valued map $\mathscr{D}_1\colon \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ be defined by

$$\mathscr{D}_1(y_1, y_2) := \begin{cases} \text{cone conv}\{(-1, 1), (0, 1)\} & \text{if } y_2 \geq 0, \\ \mathbb{R}^2_+ & \text{otherwise,} \end{cases}$$

and

$$A := \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1^2 + y_2^2 \leq 1\} .$$

Then $(-1, 0)$ is a nondominated but not a minimal element of A w.r.t. $\mathscr{D}_1$.

Considering instead the cone-valued map $\mathscr{D}_2 \colon \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ defined by

$$\mathscr{D}_2(y_1, y_2) := \begin{cases} \text{cone conv}\{(1,-1),(1,0)\} & \text{if } y_2 \geq 0, \\ \mathbb{R}^2_+ & \text{otherwise,} \end{cases}$$

then $(0, -1)$ is a minimal but not a nondominated element of A w.r.t. $\mathscr{D}_2$.

Considering instead the cone-valued map $\mathscr{D}_3 \colon \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ defined by

$$\mathscr{D}_3(y_1, y_2) := \begin{cases} \mathbb{R}^2_+ & \text{if } y \in \mathbb{R}^2 \setminus \{(0,-1),(-1,0)\}, \\ \{(z_1, z_2) \in \mathbb{R}^2 \mid z_1 \leq 0,\ z_2 \geq 0\} & \text{if } y = (0,-1), \\ \{(z_1, z_2) \in \mathbb{R}^2 \mid z_1 \geq 0,\ z_2 \leq 0\} & \text{if } y = (-1,0), \end{cases}$$

then all elements of the set $\{(y_1, y_2) \in \mathbb{R}^2 \mid y_1^2 + y_2^2 = 1,\ y_1 \leq 0,\ y_2 \leq 0\}$ are minimal elements of A w.r.t. $\mathscr{D}$ but there is no nondominated element of the set A w.r.t. $\mathscr{D}$.

The two optimality concepts are only related under strong assumptions on $\mathscr{D}$:

Lemma 7.

(a) If $\mathscr{D}(y) \subset \mathscr{D}(\bar{y})$ for all $y \in A$ for some minimal element $\bar{y}$ of A w.r.t. $\mathscr{D}$, then $\bar{y}$ is also a nondominated element of A w.r.t. $\mathscr{D}$.

(b) If $\mathscr{D}(\bar{y}) \subset \mathscr{D}(y)$ for all $y \in A$ for some nondominated element $\bar{y}$ of A w.r.t. $\mathscr{D}$, then $\bar{y}$ is also a minimal element of A w.r.t. $\mathscr{D}$.

Besides considering optimal elements of a set, all concepts apply also for a vector optimization problem with the image space equipped with a variable ordering structure, analogously to Definition 8.

Many properties of minimal elements in a partially ordered space are still valid for optimal elements w.r.t. a variable ordering, whereas others, see for instance Lemma 10, hold in general only under additional assumptions. For both optimality concepts, for minimal and for nondominated elements w.r.t. an ordering map $\mathscr{D}$, and for the related concepts of strongly and weakly optimal elements, we can easily derive the following properties.

Lemma 8.

(a) Any strongly nondominated element of A w.r.t. $\mathscr{D}$ is also a nondominated element of A w.r.t. $\mathscr{D}$. Any strongly minimal element of A w.r.t. the ordering map $\mathscr{D}$ is also a minimal element of A w.r.t. $\mathscr{D}$.

(b) If $\mathscr{D}(A)$ is pointed, then there is at most one strongly nondominated element of A w.r.t. $\mathscr{D}$.

(c) Let $int(\mathscr{D}(y)) \neq \emptyset$ for all $y \in A$. Any nondominated element of A w.r.t. $\mathscr{D}$ is also a weakly nondominated element of A w.r.t. $\mathscr{D}$. Any minimal element of A w.r.t. $\mathscr{D}$ is also a weakly minimal element of A w.r.t. $\mathscr{D}$.

(d) *If $\bar{y}$ is a strongly nondominated element of A w.r.t. $\mathcal{D}$, then the set of minimal elements of A w.r.t. $\mathcal{D}$ is empty or equals $\{\bar{y}\}$. If $\mathcal{D}(A)$ is pointed, then $\bar{y}$ is the unique minimal element of A w.r.t. $\mathcal{D}$.*

(e) *If $\bar{y} \in A$ is a strongly minimal element of A w.r.t. $\mathcal{D}$ and if $\mathcal{D}(\bar{y}) \subset \mathcal{D}(y)$ for all $y \in A$, then $\bar{y}$ is also a strongly nondominated element of A w.r.t. $\mathcal{D}$.*

Proof.

(a) Let $\bar{y}$ be a strongly nondominated element of A w.r.t. $\mathcal{D}$. Then $\bar{y} \in \{y\} + \mathcal{D}(y)$ for some $y \in A$ together with $\bar{y} \in \{y\} - \mathcal{D}(y)$ implies that $\bar{y} - y \in \mathcal{D}(y) \cap (-\mathcal{D}(y))$ and due to the pointedness of $\mathcal{D}(y)$ we obtain $\bar{y} = y$. The same for a strongly minimal element of A w.r.t. $\mathcal{D}$.

(b) Let $\bar{y}$ be a strongly nondominated element of A w.r.t. $\mathcal{D}$. If $\mathcal{D}(A)$ is pointed, then $\bar{y} - y \in -\mathcal{D}(y) \subset -\mathcal{D}(A)$ implies $\bar{y} - y \notin \mathcal{D}(A)$ for all $y \in A \setminus \{\bar{y}\}$, i.e. $y \notin \{\bar{y}\} - \mathcal{D}(\bar{y})$ for all $y \in A \setminus \{\bar{y}\}$ and thus no other element of A can be strongly nondominated w.r.t. $\mathcal{D}$.

(c) Follows directly from the definitions.

(d) As $\bar{y}$ is a strongly nondominated element of A w.r.t. $\mathcal{D}$ it holds for all $y \in A \setminus \{\bar{y}\}$

$$\bar{y} \in \{y\} - \mathcal{D}(y) \tag{17.16}$$

and hence, y cannot be a minimal element of A w.r.t. $\mathcal{D}$. Next, assume there exists $y \in A$ such that $y \in \{\bar{y}\} - \mathcal{D}(\bar{y})$. Together with (17.16) we conclude

$$y - \bar{y} \in \mathcal{D}(y) \cap (-\mathcal{D}(\bar{y})) \subset \mathcal{D}(A) \cap (-\mathcal{D}(A)) .$$

If $\mathcal{D}(A)$ is pointed then $y = \bar{y}$ and thus $\bar{y}$ is a minimal element of A w.r.t. $\mathcal{D}$.

(e) As $\bar{y}$ is a strongly minimal element of A w.r.t. $\mathcal{D}$ it holds under the assumptions here that $\bar{y} \in \{y\} - \mathcal{D}(\bar{y}) \subset \{y\} - \mathcal{D}(y)$ for all $y \in A$ and hence $\bar{y}$ is a strongly nondominated element of A w.r.t. $\mathcal{D}$.

A well-known result is that the minimal elements of a set in a partially ordered space are a subset of the boundary of that set, see Lemma 3. The result remains true for variable ordering structures.

Lemma 9.

(a) *(i) Let $int(\mathcal{D}(y)) \neq \emptyset$ for all $y \in Y$. If $\bar{y} \in A$ is a weakly minimal element of the set A w.r.t. the ordering map $\mathcal{D}$, then $\bar{y} \in \partial A$.*

(ii) If $\bar{y} \in A$ is a minimal element of the set A w.r.t. the ordering map $\mathcal{D}$ and $\mathcal{D}(\bar{y}) \neq \{0_Y\}$, then $\bar{y} \in \partial A$.

(b) *(i) If $\bigcap_{y \in A} int(\mathcal{D}(y)) \neq \emptyset$ and $\bar{y} \in A$ is a weakly nondominated element of the set A w.r.t. the ordering map $\mathcal{D}$, then $\bar{y} \in \partial A$.*

(ii) If $\bigcap_{y \in A} \mathcal{D}(y) \neq \{0_Y\}$ and $\bar{y} \in A$ is a nondominated element of the set A w.r.t. the ordering map $\mathcal{D}$, then $\bar{y} \in \partial A$.

Proof.

(a) (i) If $\bar{y} \in \operatorname{int}(A)$ then for any $d \in \operatorname{int}(\mathscr{D}(\bar{y}))$ there exists some $\lambda > 0$ with $\bar{y} - \lambda d \in A$. Then

$$\bar{y} - \lambda d \in A \cap (\{\bar{y}\} - \operatorname{int}(\mathscr{D}(\bar{y})))$$

in contradiction to $\bar{y}$ a weakly minimal element of A w.r.t. $\mathscr{D}$.

(ii) Similar to (i) but choose $d \in \mathscr{D}(\bar{y}) \setminus \{0_Y\}$.

(b) Similar to (a)(i): if $\bar{y} \in \operatorname{int}(A)$ then choosing $d \in \bigcap_{y \in A} \operatorname{int}(\mathscr{D}(y))$ there exists $\lambda > 0$ such that

$$\bar{y} - \lambda\, d \in A \cap (\{\bar{y}\} - \operatorname{int}(\mathscr{D}(\bar{y} - \lambda\, d)))$$

in contradiction to $\bar{y}$ a weakly nondominated element of A w.r.t. $\mathscr{D}$.

(ii) Similar to (b)(i), but choose $d \in \left(\bigcap_{y \in A} \mathscr{D}(y)\right) \setminus \{0_Y\}$.

The following example demonstrates that we need for instance in (b)(i) in Lemma 9 an assumption like

$$\bigcap_{y \in A} \operatorname{int}(\mathscr{D}(y)) \neq \emptyset\,. \tag{17.17}$$

Example 13. For the set $A = [1,3] \times [1,3] \subset \mathbb{R}^2$ and the ordering map $\mathscr{D} \colon \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$,

$$\mathscr{D}(y) := \begin{cases} \mathbb{R}^2_+ & \text{for all } y \in \mathbb{R}^2 \text{ with } y_1 \geq 2, \\ \{(z_1, z_2) \in \mathbb{R}^2 \mid z_1 \leq 0,\ z_2 \geq 0\} & \text{else,} \end{cases}$$

the point $\bar{y} = (2,2)$ is a weakly nondominated element of A w.r.t. $\mathscr{D}$ but $\bar{y} \notin \partial A$.

In vector optimization in a pre-ordered space ordered by some convex cone K one often considers the set $A + K$ instead of the set A, cf. Lemma 4. The advantage may be that $A + K$ is convex while the set A is not. Further more, the set $A + K$ is interesting for considering dual problems. That also applies to vector optimization problems with a variable ordering structure [21] and thus we also study the relation of the optimal elements of some set A and of the set

$$M := \bigcup_{y \in A} \{y\} + \mathscr{D}(y) \tag{17.18}$$

w.r.t. the ordering map $\mathscr{D}$.

Lemma 10. *Let M be defined as in (17.18).*

(a) (i) If $\bar{y} \in A$ is a minimal element of the set M w.r.t. $\mathscr{D}$, then it is also a minimal element of the set A w.r.t. $\mathscr{D}$.

(ii) If $\bar{y} \in A$ is a minimal element of the set A w.r.t. $\mathscr{D}$ and if $\mathscr{D}(y) \subset \mathscr{D}(\bar{y})$ for all $y \in A$, then $\bar{y}$ is also a minimal element of the set M w.r.t. $\mathscr{D}$.

(b) (i) If $\bar{y} \in M$ is a nondominated element of the set M w.r.t. $\mathscr{D}$, then $\bar{y} \in A$ and $\bar{y}$ is also a nondominated element of the set A w.r.t. $\mathscr{D}$.

(ii) If $\bar{y} \in A$ is a nondominated element of the set A w.r.t. $\mathscr{D}$, and if

$$\mathscr{D}(y + d) \subset \mathscr{D}(y) \text{ for all } y \in A \text{ and for all } d \in \mathscr{D}(y)\ , \tag{17.19}$$

then $\bar{y}$ is a nondominated element of M w.r.t. $\mathscr{D}$.

Proof.

(a) The first implication (i) follows from $A \subset M$. Next we assume for (ii) that $\bar{y}$ is a minimal element of A but not of M w.r.t. $\mathscr{D}$, i.e. there exists some $y \in A$ and $d_y \in \mathscr{D}(y) \setminus \{0_Y\}$ with $y + d_y \in \{\bar{y}\} - (\mathscr{D}(\bar{y}) \setminus \{0_Y\})$. As $\mathscr{D}(\bar{y})$ is a pointed convex cone and $\mathscr{D}(y) \subset \mathscr{D}(\bar{y})$ this implies

$$\begin{aligned} y &\in \{\bar{y}\} - (\mathscr{D}(y) \setminus \{0_Y\}) - (\mathscr{D}(\bar{y}) \setminus \{0_Y\}) \\ &\subset \{\bar{y}\} - (\mathscr{D}(\bar{y}) \setminus \{0_Y\}) - (\mathscr{D}(\bar{y}) \setminus \{0_Y\}) \\ &\subset \{\bar{y}\} - (\mathscr{D}(\bar{y}) \setminus \{0_Y\})\ , \end{aligned}$$

in contradiction to $\bar{y}$ a minimal element of A w.r.t. $\mathscr{D}$.

(b) (i) If $\bar{y} \in M \setminus A$ then $\bar{y} \in \{y\} + (\mathscr{D}(y) \setminus \{0_Y\})$ for some $y \in A \subset M$ in contradiction to $\bar{y}$ a nondominated element of M w.r.t. $\mathscr{D}$. Thus $\bar{y} \in A$. Due to $A \subset M$, $\bar{y}$ is then also a nondominated element of A w.r.t. $\mathscr{D}$. Next we assume for (ii) that $\bar{y}$ is a nondominated element of A w.r.t. $\mathscr{D}$ but not of M, i.e. there exists some $y \in A$ and $d_y \in \mathscr{D}(y) \setminus \{0_Y\}$ with $\bar{y} \in \{y + d_y\} + (\mathscr{D}(y + d_y) \setminus \{0_Y\})$. As $\mathscr{D}(y)$ is a pointed convex cone and $\mathscr{D}(y + d_y) \subset \mathscr{D}(y)$ this implies

$$\begin{aligned} \bar{y} &\in \{y\} + (\mathscr{D}(y) \setminus \{0_Y\}) + (\mathscr{D}(y + d_y) \setminus \{0_Y\}) \\ &\subset \{y\} + (\mathscr{D}(y) \setminus \{0_Y\}) + (\mathscr{D}(y) \setminus \{0_Y\}) \\ &\subset \{y\} + (\mathscr{D}(y) \setminus \{0_Y\})\ , \end{aligned}$$

in contradiction to $\bar{y}$ a nondominated element of A w.r.t. $\mathscr{D}$.

The condition (17.19) can be replaced by $\mathscr{D}(y + d) + \mathscr{D}(y) \subset \mathscr{D}(y)$ for all $y \in Y$ and all $d \in \mathscr{D}(y)$ and corresponds to the property of transitivity of the binary relation, cf. [14], as this implies: If y^1 is dominated by y^2 [in the sense of (17.14)], i.e. $y^1 \in \{y^2\} + \mathscr{D}(y^2)$, and if y^2 is dominated by y^3, i.e. $y^2 \in \{y^3\} + \mathscr{D}(y^3)$, then $y^1 \in \{y^2\} + \mathscr{D}(y^2) \subset \{y^3\} + \mathscr{D}(y^3)$, i.e. y^1 is dominated by y^3. A variable ordering structure satisfying the condition (17.19) and defining a transitive relation $\leq_1$ is given in the following example.

Example 14. Define the cone-valued map $\mathscr{D}\colon \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ by

$$\mathscr{D}(y_1,y_2) := \begin{cases} \{(r\cos\varphi, r\sin\varphi) \mid r \geq 0,\ \varphi \in [0,\pi/8]\} & \text{if } y_1 \geq \pi/2, \\ \{(r\cos\varphi, r\sin\varphi) \mid r \geq 0,\ \varphi \in [0,\frac{\pi}{2}+\frac{\pi}{8}-y_1]\} & \text{if } y_1 \in (\pi/8,\pi/2), \\ \mathbb{R}^2_+ & \text{if } y_1 \leq \pi/8. \end{cases}$$

Then $\mathscr{D}$ depends only on y_1 and for $y_1 \geq \bar{y}_1$ for some $y, \bar{y} \in \mathbb{R}^2$ we conclude $\mathscr{D}(y) \subset \mathscr{D}(\bar{y})$. As for any $y \in \mathbb{R}^2$ and any $d \in \mathscr{D}(y)$ we have $d_1 \geq 0$ and thus $y_1 + d_1 \geq y_1$ we conclude that (17.19) is satisfied and $\leq_1$ is transitive.

In general only the cones $\mathscr{D}(y)$ for $y \in A$ are of interest for modeling a decision making problem. Thus we have the freedom of setting $\mathscr{D}(y) := \{0_Y\}$ for all $y \in Y \setminus A$. This allows us to make the assumption (17.19) dispensable for the result in Lemma 10(b):

Lemma 11. *Let $\mathscr{D}\colon Y \rightrightarrows Y$ be given with $\mathscr{D}(y) = \{0_Y\}$ for all $y \in Y \setminus A$ and let M be defined as in (17.18). Then an element $\bar{y} \in Y$ is a nondominated element of the set A w.r.t. $\mathscr{D}$ if and only if it is a nondominated element of the set M w.r.t. $\mathscr{D}$.*

Proof. First assume $\bar{y}$ is a nondominated element of the set A w.r.t. $\mathscr{D}$. If it is not also nondominated of M w.r.t. $\mathscr{D}$, then there exist some $y \in A$ and some $d \in \mathscr{D}(y)$ such that

$$\bar{y} \in \{y + d\} + \mathscr{D}(y + d) \setminus \{0_Y\} \ \text{ with } \ y + d \notin A\ . \tag{17.20}$$

Thus $y + d \in M \setminus A$ and $\mathscr{D}(y + d) = \{0_Y\}$ in contradiction to (17.20). The other implication follows from Lemma 10(b)(i).

Next, we give some scalarization results for (weakly) nondominated and minimal elements w.r.t. a variable ordering structure. Scalarization means the replacement of the vector optimization problem by a scalar-valued optimization problem. A basic scalarization technique in vector optimization is based on continuous linear functionals l from the topological dual space Y^*, compare for instance the proof of Theorem 7. Then one examines the scalar-valued optimization problems

$$\min_{y \in A}\ l(y)\ .$$

In finite dimensions, $Y = \mathbb{R}^m$, this scalarization is also well known as weighted sum approach, and the components $l_i \in \mathbb{R}$, $i = 1, \ldots, m$, are then denoted as weights. We get the following sufficient conditions for optimal elements w.r.t. a variable ordering [21, 24]:

Theorem 11. *Let $\bar{y} \in A$.*

(a) (i) If for some $l \in (\mathscr{D}(\bar{y}))^*$

$$l(\bar{y}) < l(y) \text{ for all } y \in A \setminus \{\bar{y}\}\ ,$$

then $\bar{y}$ is a minimal element of A w.r.t. the ordering map $\mathscr{D}$.

(ii) If for some $l \in (\mathscr{D}(\bar{y}))^{\#}$

$$l(\bar{y}) \leq l(y) \text{ for all } y \in A\,,$$

then $\bar{y}$ *is a minimal element of A w.r.t. the ordering map* $\mathscr{D}$*.*

(iii) Let $int(\mathscr{D}(y)) \neq \emptyset$ *for all* $y \in A$*. If for some* $l \in (\mathscr{D}(\bar{y}))^{*} \setminus \{0_{Y^*}\}$

$$l(\bar{y}) \leq l(y) \text{ for all } y \in A\,,$$

then $\bar{y}$ *is a weakly minimal element of A w.r.t. the ordering map* $\mathscr{D}$*.*

(b) (i) If for some $l \in (\mathscr{D}(A))^{*}$

$$l(\bar{y}) < l(y) \text{ for all } y \in A \setminus \{\bar{y}\}\,,$$

then $\bar{y}$ *is a nondominated element of A w.r.t. the ordering map* $\mathscr{D}$*.*

(ii) If for some $l \in (\mathscr{D}(A))^{\#}$

$$l(\bar{y}) \leq l(y) \text{ for all } y \in A\,,$$

then $\bar{y}$ *is a nondominated element of A w.r.t. the ordering map* $\mathscr{D}$*.*

(iii) Let $int(\mathscr{D}(y)) \neq \emptyset$ *for all* $y \in A$ *and let* $\mathscr{D}(A)$ *be convex. If for some* $l \in (\mathscr{D}(A))^{*} \setminus \{0_{Y^*}\}$

$$l(\bar{y}) \leq l(y) \text{ for all } y \in A\,,$$

then $\bar{y}$ *is a weakly nondominated element of A w.r.t. the ordering map* $\mathscr{D}$*.*

Proof.

(a) (i) If $\bar{y}$ is not a minimal element of A w.r.t. $\mathscr{D}$, then $\bar{y} - y \in \mathscr{D}(\bar{y}) \setminus \{0_Y\}$ for some $y \in A$ and as $l \in (\mathscr{D}(\bar{y}))^{*}$ this implies $l(\bar{y}) \geq l(y)$ in contradiction to the assumption.

(ii) If $\bar{y} - y \in \mathscr{D}(\bar{y}) \setminus \{0_Y\}$ for any $y \in A$ then we get by $l \in (\mathscr{D}(\bar{y}))^{\#}$ that $l(\bar{y}) > l(y)$, in contradiction to the assumption.

(iii) If $\bar{y} - y \in \text{int}(\mathscr{D}(\bar{y}))$ for any $y \in A$ then $l \in (\mathscr{D}(\bar{y}))^{*} \setminus \{0_{Y^*}\}$ implies, compare [31, Lemma 3.21], $l(\bar{y}) > l(y)$, in contradiction to the assumption.

(b) (i) If $\bar{y}$ is not a nondominated element of A w.r.t. $\mathscr{D}$, then $\bar{y} - y \in \mathscr{D}(y) \setminus \{0_Y\}$ for some $y \in A$. As $l \in (\mathscr{D}(A))^{*}$ also $l \in (\mathscr{D}(y))^{*}$ and thus $l(\bar{y}) \geq l(y)$ in contradiction to the assumption.

(ii) If $\bar{y} - y \in \mathscr{D}(y) \setminus \{0_Y\}$ for any $y \in A$ then $l \in (\mathscr{D}(A))^{\#}$ and thus $l \in (\mathscr{D}(y))^{\#}$ implies $l(\bar{y}) > l(y)$, in contradiction to the assumption.

(iii) If $\bar{y} - y \in \text{int}(\mathscr{D}(\bar{y}))$ for any $y \in A$ then $l \in (\mathscr{D}(A))^{*} \setminus \{0_{Y^*}\}$ and thus $l \in (\mathscr{D}(y))^{*} \setminus \{0_{Y^*}\}$ implies $l(\bar{y}) > l(y)$ using again [31, Lemma 3.21], in contradiction to the assumption.

Because of $(\mathscr{D}(A))^* \subset (\mathscr{D}(\bar{y}))^*$ and $(\mathscr{D}(A))^\# \subset (\mathscr{D}(\bar{y}))^\#$ for any $\bar{y} \in A$ it suffices in (a) to consider functionals l in $(\mathscr{D}(A))^*$ and in $(\mathscr{D}(A))^\#$, respectively. A necessary condition for the quasi-interior of the dual cone of a convex cone to be nonempty is the pointedness of the cone [31, Lemma 1.27]. This shows the limitation of the above results if the variable ordering structure varies too much, i.e., if $\mathscr{D}(A)$ is no longer a pointed cone. Then the quasi-interior of the dual cone $(\mathscr{D}(A))^\#$ is empty and the above characterizations can no longer be applied. For that reason also nonlinear scalarization have to be considered, compare [20, 22, 23].

Under the additional assumption that A is a convex set also necessary conditions for weakly optimal elements and hence also for optimal elements w.r.t. a variable ordering can be formulated with the help of linear functionals.

Theorem 12. *Let A be convex and let $\mathrm{int}(\mathscr{D}(y)) \neq \emptyset$ for all $y \in A$.*

(a) For any weakly minimal element $\bar{y} \in A$ of A w.r.t. the ordering map $\mathscr{D}$ there exists some $l \in (\mathscr{D}(\bar{y}))^* \setminus \{0_{Y^*}\}$ with

$$l(\bar{y}) \leq l(y) \text{ for all } y \in A\ .$$

(b) Set

$$\hat{D} := \bigcap_{y \in A} \mathscr{D}(y)$$

and let $\mathrm{int}(\hat{D})$ be nonempty. For any weakly nondominated element $\bar{y} \in A$ of A w.r.t. the ordering map $\mathscr{D}$ there exists some $l \in \hat{D}^* \setminus \{0_{Y^*}\}$ with

$$l(\bar{y}) \leq l(y) \quad \text{for all} \quad y \in A\ .$$

Proof.

(a) Since $\bar{y}$ is a weakly minimal element of A w.r.t. the ordering map $\mathscr{D}$ the intersection of the sets $\{\bar{y}\} - \mathrm{int}(\mathscr{D}(\bar{y}))$ and A is empty. Applying a separation theorem there exists a continuous linear functional $l \in Y^* \setminus \{0_{Y^*}\}$ and a real number α with

$$l(\bar{y} - d) \leq \alpha \leq l(y) \text{ for all } d \in \mathscr{D}(\bar{y}) \text{ and for all } y \in A\ .$$

As $\mathscr{D}(\bar{y})$ is a cone we conclude $l(d) \geq 0$ for all $d \in \mathscr{D}(\bar{y})$ and thus $l \in (\mathscr{D}(\bar{y}))^* \setminus \{0_{Y^*}\}$, and due to $0_Y \in D(\bar{y})$ we obtain $l(\bar{y}) \leq l(y)$ for all $y \in A$.

(b) Since $\bar{y} \in A$ is a weakly nondominated element of A w.r.t. the ordering map $\mathscr{D}$ it holds $\bar{y} \notin \{y\} + \mathrm{int}(\mathscr{D}(y))$ for all $y \in A$ and thus $\bar{y} \notin \{y\} + \mathrm{int}(\hat{D})$ for all $y \in A$. Then $(\{\bar{y}\} - \mathrm{int}(\hat{D})) \cap A = \emptyset$ and again with a separation theorem this results in $l(\bar{y}) \leq l(y)$ for all $y \in A$ for some $l \in \hat{D}^* \setminus \{0_{Y^*}\}$.

The necessary condition for weakly nondominated elements w.r.t. the ordering map $\mathscr{D}$ is very weak if the cones $\mathscr{D}(y)$ for $y \in A$ vary too much, because then the cone $\hat{D}$ is very small (or even trivial) and the dual cone is very large.

Example 15. Let $Y \in \mathbb{R}^2$ and let $\mathscr{D}$ and A be defined as in Example 13. The unique nondominated element w.r.t. $\mathscr{D}$ is $(2, 1)$ and all the elements of the set

$$\{(2,t) \in \mathbb{R}^2 \mid t \in [1,3]\} \cup \{(t,1) \in \mathbb{R}^2 \mid t \in [1,3]\}$$

are weakly nondominated w.r.t. $\mathscr{D}$. Further, $\mathscr{D}(A) = \{(z_1, z_2) \in \mathbb{R}^2 \mid z_2 \geq 0\}$ and thus $(\mathscr{D}(A))^* = \{(z_1, z_2) \in \mathbb{R}^2 \mid z_1 = 0,\ z_2 \geq 0\}$, i.e. $(\mathscr{D}(A))^{\#} = \emptyset$. Let $l \in (\mathscr{D}(A))^* \setminus \{0_{Y*}\}$ be arbitrarily chosen, i.e. $l_1 = 0$, $l_2 > 0$, and consider the scalar-valued optimization problem $\min_{y \in A} l^\top y$. Then all elements of the set $\{(t, 1) \in \mathbb{R}^2 \mid t \in [1, 3]\}$ are minimal solutions and hence are weakly nondominated elements of A w.r.t. $\mathscr{D}$ according to Theorem 11(b)(iii). All the other weakly nondominated elements w.r.t. $\mathscr{D}$ cannot be found by the sufficient condition. Because of $\text{int}(\hat{D}) = \emptyset$, the necessary condition of Theorem 12(b) cannot be applied.

17.4 Set Optimization

Now we introduce *set optimization problems* as special vector optimization problems. Various optimality concepts are discussed for these problems.

In this section let S be a nonempty set, let Y be a real linear space, let $K \subset Y$ be a convex cone which defines a partial ordering $\leq := \leq_K$ and let $F\colon S \rightrightarrows Y$ be a set-valued map. Then we consider the set optimization problem

$$\min_{x \in S} F(x)\,. \tag{SOP}$$

Up to now many authors have used a vector approach for the formulation of optimality notions for this problem. First, we discuss this approach and then we present a more suitable set approach.

17.4.1 Vector Approach

In this subsection we give a short overview on some concepts of optimal solutions of problem (SOP) based on a vector approach. For simplicity we assume in this subsection that the convex cone K is pointed. Then, similar to minimal solutions of a vector optimization problem, see Definition 8, we can say that a pair $(\bar{x}, \bar{y})$ with $\bar{x} \in S$ and $\bar{y} \in F(\bar{x})$ is a *minimizer* of (SOP) if

$$F(S) \cap (\{\bar{y}\} - (K \setminus \{0_Y\})) = \emptyset$$

for $F(S) := \bigcup_{x \in S} F(x)$, which means that $\bar{y} \in \min F(S)$. In general only one element does not imply that the whole set $F(\bar{x})$ is in a certain sense minimal with respect to all sets $F(x)$ with $x \in S$.

Another optimality notion has been recently introduced in [15, Definition 1.3]. An element $\bar{x} \in S$ is called a *feeble (multifunction) minimal point* of problem (SOP) if

$$\exists\, \bar{y} \in F(\bar{x}) : F(S \backslash \{\bar{x}\}) \cap (\{\bar{y}\} - (K \backslash \{0_Y\})) = \emptyset \,.$$

The equality means that $\bar{y}$ is not dominated by any arbitrary point in the set $F(S \backslash \{\bar{x}\})$. It is not required that the element $\bar{y}$ is a minimal element of the set $F(\bar{x})$. Obviously, this optimality notion is even weaker than the concept of a minimizer because the set $F(\bar{x})$ is not considered in the definition. The following simple example illustrates possible difficulties with this notion.

Example 16. For $S := \{1, 2, 3\}$ consider $F(1) = \{1\}$, $F(2) = \{2\}$, $F(3) = [1, 3]$ and $K := \mathbb{R}_+$. It is evident that $\bar{x} = 3$ (with $\bar{y} = 1$) is a feeble minimal point of problem (SOP), although $F(1)$ would be the "better" set because $F(2), F(3) \subset F(1) + K$.

A variation of this feeble notion is given in [15, Definition 1.3] in the following way: An element $\bar{x} \in S$ is called a *(multifunction) minimum point* of problem (SOP) if

$$F(S \backslash \{\bar{x}\}) \cap (\{y\} - (K \backslash \{0_Y\})) = \emptyset \text{ for all } y \in F(\bar{x}) \,.$$

This condition is equivalent to the equality

$$F(S \backslash \{\bar{x}\}) \cap (F(\bar{x}) - (K \backslash \{0_Y\})) = \emptyset \,,$$

which means that the set $F(\bar{x})$ is not dominated by any set $F(x)$ with $x \in S,\ x \neq \bar{x}$. The next example shows that this optimality notion is too strong in set optimization.

Example 17. For $S := \{1, 2\}$ consider for arbitrary real numbers a, b, c, d with $-\infty < a < b < \infty$ and $-\infty < c < d < \infty$ the intervals $F(1) = [a, b]$ and $F(2) = [c, d]$, and set $K := \mathbb{R}_+$. In this case $\bar{x} = 1$ is a minimum point of problem (*SOP*) if and only if

$$F(2) \cap (F(1) - (K \backslash \{0\})) = \emptyset \,,$$

which means that $[c, d] \cap (-\infty, b) = \emptyset$ or $b \leq c$. So, this is a very strong requirement.

Variants of the discussed notions have also been mentioned in [49, p. 10], where the nondomination concept is again used for different optimality notions (but notice that the utilized solution concept in [49] uses the set less relation of interval analysis, which is covered by the unified set approach).

17.4.2 Set Approach

Although the concept of a minimizer is of mathematical interest, it cannot often be used in practice. In order to avoid this drawback it is necessary to work with practically relevant order relations for sets. In Example 1(b) the set less order relation $\preccurlyeq_s$ has been already defined for the comparison of sets. In interval analysis there are even more order relations in use, like the *certainly less* $\preccurlyeq_c$ or the *possibly less* $\preccurlyeq_p$ relations (see [16]), i.e. for arbitrary nonempty sets $A, B \subset Y$ one defines

$$A \preccurlyeq_c B \;\; :\Longleftrightarrow \;\; (A = B) \text{ or } (A \neq B, \forall\, a \in A\; \forall\, b \in B : a \leq b)$$

and

$$A \preccurlyeq_p B \;\; :\Longleftrightarrow \;\; (\exists\, a \in A\; \exists\, b \in B : a \leq b)\ .$$

From a practical point of view the order relation $\preccurlyeq_s$ seems to be more appropriate in applications. In the case of order intervals the order relations $\preccurlyeq_s$ and $\preccurlyeq_c$ are described by a pre-order of the minimal and maximal elements of these intervals. But for general nonempty sets A and B, which possess minimal elements and maximal elements, this property may not be fulfilled. Figure 17.10 illustrates two sets $A, B \in \mathscr{P}(Y)$ with $A \preccurlyeq_s B$ and the properties $\max A \subset \max B - K$ but $\max B \not\subset \max A + K$. This means that there may be elements $b \in \max B$ and $a \in \max A$ which are not comparable with respect to the pre-order $\leq$.

In order to avoid this drawback we discuss new concepts involving the minimal and maximal elements of a set. This leads to various definitions of "minmax less" order relations. In the following let

$$\mathscr{M} := \{A \in \mathscr{P}(Y) \mid \min A \text{ and } \max A \text{ are nonempty}\}.$$

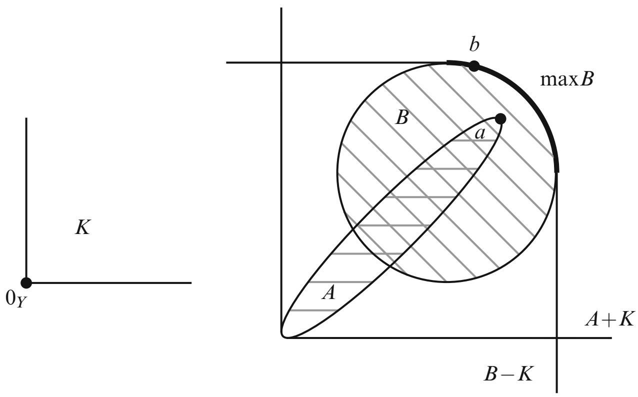


Fig. 17.10 Illustration of two sets A and B with $A \preccurlyeq_s B$, and $a \in \max A$ and $b \in \max B$ with $a \not\leq b$ and $b \not\leq a$

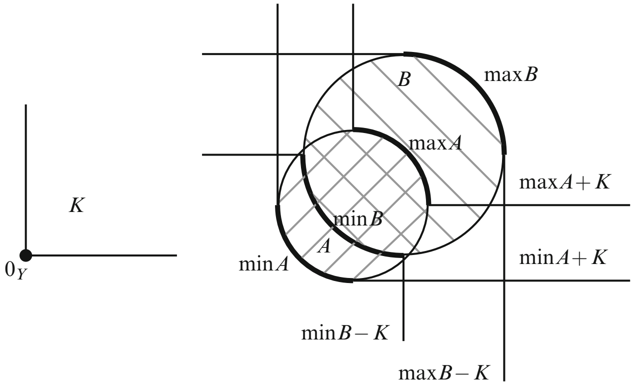


Fig. 17.11 Illustration of two sets $A, B \in \mathscr{M}$ with $A \preccurlyeq_m B$

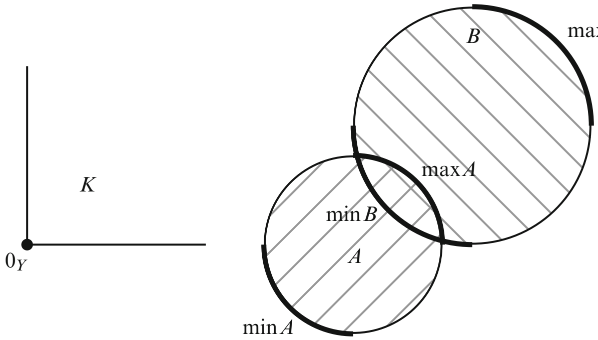


Fig. 17.12 Illustration of two sets $A, B \in \mathscr{M}$ with $A \preccurlyeq_{mc} B$

Definition 12. Let $A, B \in \mathscr{M}$ be arbitrarily chosen sets. Then the *minmax less order relation* $\preccurlyeq_m$ is defined by

$$A \preccurlyeq_m B \quad :\Longleftrightarrow \quad \min A \preccurlyeq_s \min B \text{ and } \max A \preccurlyeq_s \max B$$

(the subscript m stands for minmax).

Figure 17.11 illustrates the minmax less order relation.

Our second new order relation is defined as follows.

Definition 13. Let $A, B \in \mathscr{M}$ be arbitrarily chosen sets. Then the *minmax certainly less order relation* $\preccurlyeq_{mc}$ is defined by

$$A \preccurlyeq_{mc} B \;:\Longleftrightarrow\; (A = B) \text{ or } (A \neq B, \min A \preccurlyeq_c \min B \text{ and } \max A \preccurlyeq_c \max B).$$

(the subscript mc stands for minmax certainly).

Figure 17.12 illustrates this order relation.

Our third new order relation is defined as follows.

Definition 14. Let $A, B \in \mathscr{M}$ be arbitrarily chosen sets. Then the *minmax certainly nondominated order relation* $\preccurlyeq_{mn}$ is defined by

$$A \preccurlyeq_{mn} B \ :\Longleftrightarrow \ (A = B) \text{ or } (A \neq B, \max A \preccurlyeq_s \min B).$$

(the subscript *mn* stands for minmax nondominated).

Remark 1. When $Y = \mathbb{R}$, the order relation $\preccurlyeq_{mn}$ reduces to the order relation for unequal intervals given in [36, p. 9].

To illustrate the order relation $\preccurlyeq_{mn}$ let us consider two teams A and B of football players. If the best ones of B play not better than the worse ones of A, then one could say that A is better than B w.r.t. the order relation $\preccurlyeq_{mn}$.

With the following proposition we compare the afore-mentioned order relations.

Proposition 1 (Comparing Known and New Order Relations). *Let* $A, B \in \mathscr{M}$ *with* $A \neq B$ *be arbitrarily given. Suppose that* A *and* B *have the quasi domination property, i.e.* $\min A + K = A + K$ *and* $\max A - K = A - K$. *Then*

(i) $A \preccurlyeq_c B \Rightarrow A \preccurlyeq_{mc} B \Rightarrow A \preccurlyeq_m B \Rightarrow A \preccurlyeq_s B$.
(ii) $A \preccurlyeq_c B \Rightarrow A \preccurlyeq_{mn} B \Rightarrow A \preccurlyeq_m B$.
(iii) $A \preccurlyeq_{mn} B$ *does not always imply* $A \preccurlyeq_{mc} B$ *and* $A \preccurlyeq_{mc} B$ *does not always imply* $A \preccurlyeq_{mn} B$.

Proof.

(i) "$A \preccurlyeq_c B \ \Rightarrow \ A \preccurlyeq_{mc} B$": By the definition, $A \preccurlyeq_c B$ and $A \neq B$ mean $B - A \subset K$. As $\min A$, $\max A$ are subsets of A and $\min B$, $\max B$ are subsets of B, it is immediate that $\min B - \min A \subset K$ and $\max B - \max A \subset K$ being equivalent to the inequality $A \preccurlyeq_{mc} B$.

"$A \preccurlyeq_{mc} B \ \Rightarrow \ A \preccurlyeq_m B$": By the definition, $A \preccurlyeq_{mc} B$ and $A \neq B$ imply $\min B - \min A \subset K$ and $\max B - \max A \subset K$. Then for any $a_{\min} \in \min A$, $b_{\min} \in \min B$, $a_{\max} \in \max A$ and $b_{\max} \in \max B$ we have $a_{\min} \leq b_{\min}$ and $a_{\max} \leq b_{\max}$. Therefore, $\min B \subset \min A + K$, $\min A \subset \min B - K$ and $\max B \subset \max A + K$, $\max A \subset \max B - K$, being equivalent to the inequality $A \preccurlyeq_m B$.

"$A \preccurlyeq_m B \ \Rightarrow \ A \preccurlyeq_s B$": It is simple to see that the inequality $A \preccurlyeq_m B$ implies that

$$\min A \subset \min B - K \text{ and } \min B \subset \min A + K = A + K$$

and

$$\max A \subset \max B - K = B - K \text{ and } \max B \subset \max A + K.$$

Therefore, we have

$$\min B \subset A + K \text{ and } \max A \subset B - K,$$

which leads to

$$B + K = \min B + K \subset A + K \text{ and } A - K = \max A - K \subset B - K.$$

Consequently, we have

$$B \subset A + K \text{ and } A \subset B - K$$

being equivalent to the inequality $A \preccurlyeq_s B$.

(ii) "$A \preccurlyeq_c B \Rightarrow A \preccurlyeq_{mn} B$": By the definition, $A \preccurlyeq_c B$ and $A \neq B$ imply $B - A \subset K$. Then for any $a_{\max} \in \max A$ and $b_{\min} \in \min B$ we have $a_{\max} \leq b_{\min}$. Therefore, $\max A \subset \min B - K$ and $\min B \subset \max A + K$, i.e. $\max A \preccurlyeq_s \min B$ being equivalent to the inequality $A \preccurlyeq_{mn} B$.

"$A \preccurlyeq_{mn} B \Rightarrow A \preccurlyeq_m B$": By the definitions of the order relations $\preccurlyeq_{mn}$ and $\preccurlyeq_s$, $A \preccurlyeq_{mn} B$ and $A \neq B$ imply $\max A \subset \min B - K$ and $\min B \subset \max A + K$. As the sets A and B satisfy the quasi domination property, we have

$$\min A \subset A \subset A - K = \max A - K \subset (\min B - K) - K = \min B - K$$

and

$$\min B \subset \max A + K \subset A + K = \min A + K$$

which imply $\min A \preccurlyeq_s \min B$. Analogously,

$$\max A \subset \min B - K \subset B - K = \max B - K$$

and

$$\max B \subset B \subset B + K = \min B + K \subset (\max A + K) + K = \max A + K$$

which imply $\max A \preccurlyeq_s \max B$. These two relations imply that $A \preccurlyeq_m B$.

(iii) See the following Example 18.

We illustrate the relations in Proposition 1 by some examples. We also show that the implications in this proposition are strict, i.e. the converse implications do not hold.

Example 18. Let $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+$ and consider the sets

$$\begin{aligned}
A_1 &= \{(x,y) \mid x^2 + y^2 \leq 1\} \\
A_2 &= \{(x,y) \mid (x-1)^2 + (y-1)^2 \leq 1\} \\
A_3 &= \{(x,y) \mid (x-1)^2 + y^2 \leq 1\} \\
A_4 &= \{(x,y) \mid (x-1)^2 + (y-1)^2 \leq 1,\ x^2 + y^2 \geq 1\} \\
A_5 &= \text{conv}\{(-2,0), (-3,-1), (0,-2)\} \\
A_6 &= \text{conv}\{(4,2), (0,2), (4,-2\}
\end{aligned}$$

(see Fig. 17.13).
One can check that

$$A_1 \preccurlyeq_{mc} A_2 \text{ but } A_1 \not\preccurlyeq_{mn} A_2,\ A_1 \not\preccurlyeq_c A_2,$$
$$A_1 \preccurlyeq_m A_3 \text{ but } A_1 \not\preccurlyeq_{mn} A_3 \text{ and } A_1 \not\preccurlyeq_{mc} A_3,$$
$$A_1 \preccurlyeq_{mn} A_4 \text{ but } A_1 \not\preccurlyeq_c A_4$$

and

$$A_5 \preccurlyeq_{mn} A_6 \text{ but } A_5 \not\preccurlyeq_{mc} A_6$$

($A_5 \not\preccurlyeq_{mc} A_6$ because $(-3,-1) \in \min A_5$, $(4,-2) \in \min A_6$ but $(4,-2) \not\geq (-3,-1)$ which means that $\min A_5 \not\preccurlyeq_c \min A_6$).

Figure 17.10 illustrates that the minmax less relation $\preccurlyeq_m$ is stronger than the set less relation $\preccurlyeq_s$.

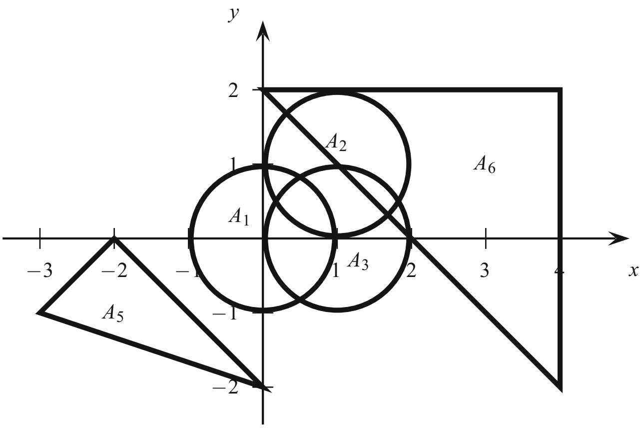


Fig. 17.13 Illustration of the sets A_1, A_2, A_3, A_5 and A_6 in Example 18

Now we assume that the power set $\mathscr{P}(Y)$ or its subset $\mathscr{M}$ is equipped with order relations considered in this chapter, except the order relation $\preccurlyeq_p$. Observing that these order relations are pre-orders, we discuss a unifying approach for problem (SOP) w.r.t. these various order relations.

Let us begin with the concepts of optimal solutions w.r.t. the pre-order $\preccurlyeq$.

Definition 15. Suppose that F is a set-valued map from S to $(\mathscr{Q}, \preccurlyeq)$ and let $\bar{x} \in S$.

(i) $\bar{x}$ is an optimal solution of (SOP) w.r.t. the pre-order $\preccurlyeq$ iff

$$F(x) \preccurlyeq F(\bar{x}) \text{ for some } x \in S \Rightarrow \; F(\bar{x}) \preccurlyeq F(x).$$

(ii) $\bar{x}$ is a strongly optimal solution of (SOP) w.r.t. the pre-order $\preccurlyeq$ iff

$$F(\bar{x}) \preccurlyeq F(x) \text{ for all } x \in S.$$

Now, let us consider the case, if F takes values on $\mathscr{P}(Y)$. We will use the following notations. Namely, let $(\mathscr{Q}, \preccurlyeq) = (\bar{\mathscr{N}}, \preccurlyeq_*)$ with the pre-order $\preccurlyeq_*$, where the subscript $*$ is one of $s,\ c,\ m,\ mc,\ mn$, and sets under consideration belong to

$$\bar{\mathscr{N}} = \begin{cases} \mathscr{P}(Y) & \text{if } * \text{ is } s,\ c \\ \mathscr{M} & \text{if } * \text{ is } m,\ mc,\ mn. \end{cases}$$

Remark 2. If $\preccurlyeq$ is a partial order, i.e. it is anti-symmetric, then $\bar{x}$ is an optimal solution of (SOP) if and only if

$$F(x) \preccurlyeq F(\bar{x}) \text{ for some } x \in S \Rightarrow \; F(\bar{x}) = F(x).$$

Finally we establish the existence of optimal solutions to the set optimization problem. A proof of this result is given in [32, Theorem 5.1].

Theorem 13. *Suppose that S is compact, F take values on $\mathscr{Q}$ and is semicontinuous w.r.t. the pre-order $\preccurlyeq$ on S. Then the problem (SOP) has an optimal solution w.r.t. the pre-order $\preccurlyeq$.*

References

1. Baatar, D., Wiecek, M.M.: Advancing equitability in multiobjective programming. Comput. Math. Appl. **52**, 225–234 (2006)
2. Benson, H.P.: An improved definition of proper efficiency for vector maximization with respect to cones. J. Math. Anal. Appl. **71**, 232–241 (1979)
3. Benson, H.P., Morin, T.L.: The vector maximization problem: proper efficiency and stability. SIAM J. Appl. Math. **32**, 64–72 (1977)
4. Bijick (Schneider), E.: Optimierung der Homogenität von HF-Feldern bei der Magnetresonanzbildgebung. Diplomarbeit, University of Erlangen-Nürnberg, Erlangen (2005)

5. Bijick (Schneider), E., Diehl, D., Renz, W.: Bikriterielle Optimierung des Hochfrequenzfeldes bei der Magnetresonanzbildgebung. In: Küfer, K.-H., Rommelfanger, H., Tammer, C., Winkler, K. (eds.) Multicriteria Decision Making and Fuzzy Systems: Theory, Methods and Applications, pp. 85–98. Shaker, Aachen (2006)
6. Bishop, E., Phelps, R.R.: The support functionals of a convex set. Proc. Sympos. Pure Math. **7**, 27–35 (1963)
7. Borwein, J.M.: Proper efficient points for maximizations with resprect to cones. SIAM J. Control Optim. **15**, 57–63 (1977)
8. Borwein, J.M.: The geometry of Pareto efficiency over cones. Math. Oper. Stat. Ser. Optim. **11**, 235–248 (1980)
9. Borwein, J.M.: Continuity and differentiability properties of convex operators. Proc. Lond. Math. Soc. (3) **44**, 420–444 (1982)
10. Borwein, J.M.: On the existence of Pareto efficient points. Math. Oper. Res. **8**, 64–73 (1983)
11. Chen, G.Y.: Existence of solutions for a vector variational inequality: an extension of the Hartmann-Stampacchia Theorem. J. Optim. Theory Appl. **74**, 445–456 (1992)
12. Chen, G.Y., Yang, X.Q.: Characterizations of variable domination structures via nonlinear scalarization. J. Optim. Theory Appl. **112**, 97–110 (2002)
13. Chen, G.Y., Huang, X., Yang, X.: Vector Optimization, Set-Valued and Variational Analysis. Springer, Berlin (2005)
14. Chew, K.L.: Domination structures in abstract spaces. In: Southeast Asian Bulletin of Mathematics. Proceedings of 1st Franco-Southeast Asian Mathematical Conference, pp. 190–204 (1979)
15. Chinaie, M., Zafarani, J.: Image space analysis and scalarization of multivalued optimization. J. Optim. Theory Appl. **142**, 451–467 (2009)
16. Chiriaev, A., Walster, G.W.: Interval arithmetic specification. Technical Report (1998)
17. Cryer, C.W., Dempster, M.A.H.: Equivalence of linear complementarity problems and linear programs in vector lattice Hilbert spaces. SIAM J. Control Optim. **18**, 76–90 (1980)
18. Dunford, N., Schwartz, J.T.: Linear Operators, Part I. Interscience Publishers, New York (1957)
19. Edgeworth, F.Y.: Mathematical Psychics. Kegan Paul, London (1881)
20. Eichfelder, G.: Variable ordering structures in vector optimization, chapter 4. In: Ansari, Q.H., Yao, J.-C. (eds.) Recent Developments in Vector Optimization, 32 p. Springer, Berlin (2011)
21. Eichfelder, G.: Optimal elements in vector optimization with a variable ordering structure. J. Optim. Theory Appl. **151**(2), 217–240 (2011)
22. Eichfelder, G., Ha, T.X.D.: Optimality conditions for vector optimization problems with variable ordering structures. Optimization **62**(5), 597–627 (2013)
23. Eichfelder, G.: Variable Ordering Structures in Vector Optimization. Springer, Heidelberg (2014)
24. Engau, A.: Variable preference modeling with ideal-symmetric convex cones. J. Glob. Optim. **42**, 295–311 (2008)
25. Geoffrion, A.M.: Proper efficiency and the theory of vector maximization. J. Math. Anal. Appl. **22**, 618–630 (1968)
26. Hamel, A.: Variational principles on metric and uniform spaces. Habilitation Thesis, University of Halle-Wittenberg, Germany (2005)
27. Hartley, R.: On cone-efficiency, cone-convexity and cone-compactness. SIAM J. Appl. Math. **34**, 211–222 (1978)
28. Henig, M.I.: Proper efficiency with respect to cones. J. Optim. Theory Appl. **36**, 387–407 (1982)
29. Holmes, R.B.: Geometric Functional Analysis and Its Applications. Springer, New York (1975)
30. Jahn, J.: Existence theorems in vector optimization. J. Optim. Theory Appl. **50**, 397–406 (1986)
31. Jahn, J.: Vector Optimization: Theory, Applications, and Extensions, 2nd edn. Springer, Heidelberg (2011)
32. Jahn, J., Ha, T.X.D.: New order relations in set optimization. J. Optim. Theory Appl. **148**, 209–236 (2011)
33. James, R.: Weak compactness and reflexivity. Israel J. Math. **2**, 101–119 (1964)

34. Kuhn, H.W., Tucker, A.W.: Nonlinear programming. In: Neyman, J. (ed.) Proceedings of the 2nd Berkeley Symposium on Mathematical Statistics and Probability, pp. 481–492. University of California Press, Berkeley (1951)
35. Kuroiwa, D.: Natural Criteria of Set-Valued Optimization. Manuscript. Shimane University, Shimane (1998)
36. Moore, R.E., Kearfott, R.B., Cloud, M.J.: Introduction to Interval Analysis. SIAM, Philadelphia (2009)
37. Nieuwenhuis, J.W.: Properly efficient and efficient solutions for vector maximization problems in Euclidean space. J. Math. Anal. Appl. **84**, 311–317 (1981)
38. Nishnianidze, Z.G.: Fixed points of monotonic multiple-valued operators. Bull. Georgian Acad. Sci. **114**, 489–491 (1984) (in Russian)
39. Pareto, V.: Manuale di Economia Politica. Societa Editrice Libraria, Milano, (1906). English Translation: Pareto, V.: Manual of Political Economy, translated by Schwier, A.S.. Augustus M., Kelley Publishers, New York (1971)
40. Penot, J.-P.: L'optimisation à la Pareto: Deux ou trois choses que je sais d'elle. Publications Mathématiques de Pau, Paris (1978)
41. Rolewicz, S.: On a norm scalarization in infinite dimensional Banach spaces. Control Cybern. **4**, 85–89 (1975)
42. Schröder, B.S.W.: Ordered Sets: An Introduction. Birkhäuser, Boston (2001)
43. Siemens AG: Magnets, Spins, and Resonances: An Introduction to the basics of Magnetic Resonance. Manuscript, Siemens Medical Solutions (Healthcare), Magnetic Resonance, Erlangen (2003)
44. Stadler, W.: Fundamentals of multicriteria optimization. In: Stadler, W. (ed.) Multicriteria Optimization in Engineering and in the Sciences, pp. 1–25. Plenum Press, New York (1988)
45. Vogel, W.: Vektoroptimierung in Produkträumen. Anton Hain, Meisenheim am Glam (1977)
46. Wendell, R.E., Lee, D.N.: Efficiency in multiple objective optimization problems. Math. Program. **12**, 406–414 (1977)
47. Wierzbicki, A.P.: Penalty methods in solving optimization problems with vector performance criteria. Technical Report of the Institute of Automatic Control, TU Warsaw, Warsaw (1974)
48. Wierzbicki, A.P.: Basic properties of scalarizing functionals for multiobjective optimization. Math. Oper. Stat. Ser. Optim. **8**, 55–60 (1977)
49. Wu, H.-C.: Duality theory in interval-valued linear programming problems. J. Optim. Theory Appl. **150**, 298–316 (2011)
50. Young, R.C.: The algebra of many-valued quantities. Math. Ann. **104**, 260–290 (1931)
51. Yu, P.L.: Cone convexity, cone extreme points, and nondominated solutions in decision problems with multiobjectives. J. Optim. Theory Appl. **14**, 319–377 (1974)
52. Yu, P.L.: Multiple-Criteria Decision Making: Concepts, Techniques and Extensions. Plenum Press, New York (1985)
53. Zhuang, D.: Regularity and maximality properties of set-valued structures in optimization. Dissertation, Dalhousie University, Halifax (1989)

Chapter 18
Continuous Multiobjective Programming

Margaret M. Wiecek, Matthias Ehrgott, and Alexander Engau

Abstract We present our view of the state of the art in continuous multiobjective programming. After an introduction we formulate the multiobjective program (MOP) and define the most important solution concepts in Sect. 18.2. In Sect. 18.3 we summarize properties of efficient and nondominated sets. Optimality conditions are reviewed in Sect. 18.4. The main part of the chapter consists of Sects. 18.5 and 18.6 that deal with solution techniques for MOPs and approximation of efficient and nondominated sets. In Sect. 18.7 we discuss specially-structured problems including linear, nonlinear, parametric, and bilevel MOPs. In Sect. 18.8 we present our perspective on future research directions.

Keywords Multiobjective programming • Efficient solution • Pareto point • Nondominated solution • Scalarization • Approximation • Representation • Parametric programming • Bilevel programming

18.1 Introduction

Multiobjective programming is a part of mathematical programming dealing with decision problems characterized by multiple and conflicting objective functions that are to be optimized over a feasible set of decisions. Such problems, referred to as multiobjective programs (MOPs), are commonly encountered in many areas

M.M. Wiecek
Department of Mathematical Sciences, Clemson University, Clemson, SC 29634-0975, USA
e-mail: wmalgor@clemson.edu

M. Ehrgott (✉)
Department of Management Science, Lancaster University, Bailrigg, Lancaster LA1 4YX, UK
e-mail: m.ehrgott@lancaster.ac.uk

A. Engau
Rowe School of Business, Dalhousie University, Halifax, Nova Scotia B3H 4R2, Canada
Department of Mathematical and Statistical Sciences, University of Colorado Denver, Denver, CO 80217-3364, USA
e-mail: alexander.engau@ucdenver.edu

S. Greco et al. (eds.), *Multiple Criteria Decision Analysis*, International Series in Operations Research & Management Science 233,
DOI 10.1007/978-1-4939-3094-4_18

of human activity including engineering, management, and others. Throughout the chapter we understand multiobjective programming as pertaining to situations where feasible alternatives are available implicitly, through constraints in the form of mathematical functions. An optimization problem (typically a mathematical program) has to be solved to explicitly find those alternatives of interest to a decision maker. Decision problems with multiple criteria and explicitly available alternatives are treated within multicriteria decision analysis (MCDA). This view constitutes the difference between multiobjective programming and MCDA which complement each other within multicriteria decision-making (MCDM).

In the last 60 years, a great deal of theoretical, methodological and applied studies have been undertaken in the area of multiobjective programming. This chapter presents a review of the theory and methodology of finite-dimensional MOPs over continuous Euclidean domains; the more general settings of vector or set-valued optimization and the more specific focus on combinatorial problems are treated separately in Chapters 17 and 19 in this book. The content of our review is based on the understanding that the primary (although not necessarily the ultimate) goal of multiobjective programming is to seek solutions of MOPs. Consequently, exact methods suitable for finding these solutions are considered the most fundamental tools for dealing with MOPs and therefore given special attention. Heuristic methods or metaheuristics are not included in this chapter but covered in Chapter 23 in this book. The selection of a preferred solution of the MOP performed by the decision maker can be considered the ultimate goal of MCDM. However, the modeling of decision maker preferences is outside the scope of this chapter and belongs to the domain of MCDA.

Similar to the first edition of this volume, we begin this chapter by providing a theoretical foundation of multiobjective programming in Sects. 18.2–18.4. In Sect. 18.2, we define MOPs and relevant solution concepts based on binary relations and cones, that are now extended to the notions of more general domination structures and variable cones. Sects. 18.3 and 18.4 contain an updated summary of properties of the solution sets and conditions for efficiency, respectively. The subsequent sections focus on methodological aspects of multiobjective programming and have seen major revisions to accurately reflect some new developments in the field. In Sect. 18.5, numerous methods for generating individual elements or subsets of the solution sets of MOPs are collected including scalarization approaches and nonscalarizing methods. The latter include methods based on different optimality concepts in the Euclidean vector space, algorithms borrowed from nonlinear programming, and set-oriented methods that have been designed specifically for MOPs. Sects. 18.6 on approximation techniques has been extended as well and now also addresses representation and measures of its quality. Sects. 18.7 on specially structured problems includes updated overviews of linear and nonlinear MOPs followed by new results for parametric and bilevel MOPs. The chapter is concluded in Sect. 18.8 with our view of current and future research directions.

We point out that contributions are not always presented chronologically but rather with respect to the order implied by the content of this chapter and with respect to their level of generality. Due to their scope, some articles could be referred to in two or more sections of this chapter. Based on our judgment, we typically refer to them once in the section that we find the most relevant.

18.2 Problem Formulation and Solution Concepts

Let $\mathbb{R}^n$ and $\mathbb{R}^p$ be Euclidean vector spaces referred to as the decision space and the objective space. Let $X \subset \mathbb{R}^n$ be a feasible set and let f be a vector-valued objective function $f : \mathbb{R}^n \to \mathbb{R}^p$ composed of p real-valued objective functions, $f = (f_1, \ldots, f_p)$, where $f_k : \mathbb{R}^n \to \mathbb{R}$ for $k = 1, \ldots, p$. A multiobjective program (MOP) is given by

$$\begin{aligned} &\min \quad (f_1(x), \ldots, f_p(x)) \\ &\text{subject to } x \in X. \end{aligned} \tag{18.1}$$

Throughout this chapter we refer to problem (18.1) as the MOP. When $p = 2$ the problem is referred to as the biobjective program (BOP). We usually assume that the set X is given implicitly in the form of constraints, i.e., $X := \{x \in S \subseteq \mathbb{R}^n : g_j(x) \leq 0, j = 1, \ldots, l;\ h_j(x) = 0, j = 1, \ldots, m\}$ for some given set S. A feasible solution $x \in X$ is evaluated by p objective functions producing the outcome $f(x)$. We define the set of all attainable outcomes or criterion vectors for all feasible solutions in the objective space, $Y := f(X) \subset \mathbb{R}^p$. We use $\operatorname{bd} Y$, $\operatorname{int} Y$, $\operatorname{ri} Y$, and $\operatorname{cl} Y$ to denote the boundary, interior, relative interior, and closure of Y. Furthermore $\operatorname{cone} Y$ and $\operatorname{conv} Y$ denote the conical and convex hulls of Y.

Occasionally, we will deal with a special case of the MOP with the feasible set defined by $X' := \{x \in S' \subseteq \mathbb{R}^n : g_j(x) \leq 0, j = 1, \ldots, l\}$. This MOP only with inequality constraints will be referred to as the MOP$'$.

18.2.1 Partial Orders and Pareto Optimality

The symbol "min" in the MOP is generally understood as finding optimal or preferred outcomes in Y and their pre-images in X. Let $y^1 \succ y^2$ denote that an outcome y^1 is preferred to an outcome y^2 and let $y^1 \succeq y^2$ denote preference of y^1 over y^2 or indifference between y^1 and y^2. Given a binary relation $\mathcal{R}$, we say that an outcome y^1 is preferred (or indifferent) to an outcome y^2 with respect to this relation if and only if y^1 is in relation with y^2, i.e., $y^1 \succeq_{\mathcal{R}} y^2$ if and only if $y^1 \mathcal{R} y^2$.

The following notation is used. Let $\mathbb{R}^p$ be a Euclidean vector space and $y, y' \in \mathbb{R}^p$.

- $y < y'$ denotes $y_k < y'_k$ for all $k = 1, \ldots, p$.
- $y \leqq y'$ denotes $y_k \leq y'_k$ for all $k = 1, \ldots, p$.
- $y \leq y'$ denotes $y \leqq y'$ but $y \neq y'$.
- Let $\mathbb{R}^p_{\geqq} := \{y \in \mathbb{R}^p : y \geqq 0\}$. The sets $\mathbb{R}^p_{\geq}, \mathbb{R}^p_{>}$ are defined accordingly.

Due to the lack of a canonical order of vectors in $\mathbb{R}^p$, the notion of optimality allows for some flexibility. Three ordering relations commonly chosen are the Pareto relation denoted as *Par*, the lexicographic relation denoted as *lex*, and the max-ordering relation denoted as *MO*.

Definition 1. Let $\mathbb{R}^p$ be a Euclidean vector space and $y^1, y^2 \in \mathbb{R}^p$.

- $y^1 \succeq_{Par} y^2$ if and only if $y^1 \leqq y^2$.
- $y^1 \succeq_{lex} y^2$ if and only if $y^1 = y^2$ or there is some $k, 1 \leq k \leq p$, such that $y^1_i = y^2_i, i = 1, \ldots, k-1$ and $y^1_k < y^2_k$.
- $y^1 \succeq_{MO} y^2$ if and only if $\max_{k=1,\ldots,p} y^1_k \leq \max_{k=1,\ldots,p} y^2_k$.

In Definition 1, only $\succeq_{lex}$ is a total order whereas $\succeq_{Pareto}$ and $\succeq_{MO}$ are partial orders in $\mathbb{R}^p$. The most commonly used ordering relation in multiobjective programming is the Pareto relation for which the corresponding solutions in the decision space are called *Pareto optimal* or *efficient*. The definition of three variations of efficient solutions is given below.

Definition 2. Consider the MOP. A point $x \in X$ is called

- a weakly efficient solution if there is no $x' \in X$ such that $f(x') < f(x)$;
- an efficient solution if there is no $x' \in X$ such that $f(x') \leq f(x)$;
- a strictly efficient solution if there is no $x' \in X, x' \neq x$, such that $f(x') \leqq f(x)$.

We shall denote the weakly efficient solutions, efficient solutions, and strictly efficient solutions by X_{wE}, X_E, X_{sE}, respectively, and we shall call their images weak Pareto points and Pareto points, respectively. The latter are denoted by Y_{wN}, Y_N. Note that strictly efficient solutions correspond to unique efficient solutions, and therefore they do not have a counterpart in the objective space. For many of the solution approaches presented in Sect. 18.5, statements of the form "If $\hat{x} \in X$ is a unique optimal solution of the approach then $\hat{x} \in X$ is an efficient solution" are presented. Uniqueness of a solution actually implies that $\hat{x}$ is strictly efficient for the MOP.

18.2.2 Cones and Nondominated Outcomes

More general preference and indifference relations between outcomes result from partial orders that are described by a binary relation $\mathcal{R}$ defined on Y. To derive a definition for a class of preferences between outcomes, we consider cones. A set $\mathcal{C} \subset \mathbb{R}^p$ is a cone, if $\alpha y \in \mathcal{C}$ whenever $y \in \mathcal{C}$ and $0 \leq \alpha \in \mathbb{R}$. Given a cone $\mathcal{C}$, we say that an outcome y^1 dominates (is preferred to) an outcome y^2 with respect to this cone, $y^1 \succ_{\mathcal{C}} y^2$, if and only if $y^2 - y^1 \in \mathcal{C} \setminus \{0\}$, or equivalently, if there exists a direction $d \in \mathcal{C}, d \neq 0 : y^2 = y^1 + d$. Then $y^1 \succeq_{\mathcal{C}} y^2$ if $y^1 \succ_{\mathcal{C}} y^2$ or $y^1 = y^2$. We observe that the set $\mathcal{C} = \mathbb{R}^p_{\geqq}$ is the cone that is associated with the Pareto relation in Definition 1, and in this context it is therefore also referred to as the Pareto cone.

Given a cone $\mathcal{C}$, we can also define a relation $\mathcal{R}_\mathcal{C}$ on $\mathbb{R}^p$ by $y^1 \mathcal{R}_\mathcal{C} y^2$ if and only if $y^2 - y^1 \in \mathcal{C}$. This relation is compatible with addition and scalar multiplication (i.e., $y^1 \mathcal{R} y^2$ implies $(y^1 + z)\mathcal{R}(y^2 + z)$ for all $z \in \mathbb{R}^p$ and $y^1 \mathcal{R} y^2$ implies $\alpha y^1 \mathcal{R} \alpha y^2$ for all $0 < \alpha \in \mathbb{R}$). Conversely, given a relation $\mathcal{R}$ on $\mathbb{R}^p$ we can define a set $\mathcal{C}_\mathcal{R}$ as $\mathcal{C}_\mathcal{R} := \{d \in \mathbb{R}^p : d = y^2 - y^1 \text{ and } y^1 \mathcal{R} y^2\}$, see Ehrgott [114]. If $\mathcal{R}$ is compatible with scalar multiplication and $0 \in \mathcal{C}_\mathcal{R}$ then $\mathcal{C}_\mathcal{R}$ is a cone; (if $0 \notin \mathcal{C}_\mathcal{R}$ then $\mathcal{C}_\mathcal{R} \cup \{0\}$ is a cone). If $\mathcal{R}$ is compatible with addition then $d \in \mathcal{C}_\mathcal{R}$ and $d = y^2 - y^1$ imply that $y^1 \mathcal{R} y^2$. Theorem 1 summarizes these relationships between binary relations and cones. Note that a reflexive, antisymmetric, and transitive binary relation is a partial order on $\mathbb{R}^p$.

Theorem 1. *1. Let $\mathcal{R}$ be a binary relation on $\mathbb{R}^p$ which is compatible with addition. Then $0 \in \mathcal{C}_\mathcal{R}$ if and only if $\mathcal{R}$ is reflexive; $\mathcal{C}_\mathcal{R}$ is pointed (i.e., $\mathcal{C} \cap (-\mathcal{C}) = \{0\}$) if and only if $\mathcal{R}$ is antisymmetric; $\mathcal{C}_\mathcal{R}$ is convex if and only if $\mathcal{R}$ is transitive.*
2. Let $\mathcal{C}$ be a cone. Then $\mathcal{R}_\mathcal{C}$ is reflexive if and only if $0 \in \mathcal{C}$; $\mathcal{R}_\mathcal{C}$ is antisymmetric if and only if $\mathcal{C}$ is pointed; $\mathcal{R}_\mathcal{C}$ is transitive if and only if $\mathcal{C}$ is convex.

Thus some binary relations and cones are equivalent concepts, and we can define a notion of nondominated solutions for MOPs (Yu [407]).

Definition 3. Let $\mathcal{C} \subset \mathbb{R}^p$ be a cone and $Y \subset \mathbb{R}^p$. Then $y \in Y$ is called a nondominated outcome of the MOP if

- there does not exist $y^1 \in Y$ and $d \in \mathcal{C}, d \neq 0 : y = y^1 + d$, or equivalently,
- $(y - \mathcal{C}) \cap Y = \{y\}$.

We shall denote the set of all nondominated outcomes of the MOP by $N(X, f, \mathcal{C})$ or $N(Y, \mathcal{C})$. One typically assumes that the cone $\mathcal{C}$ is proper (i.e., $\{0\} \neq \mathcal{C} \neq \mathbb{R}^p$) and pointed. The pre-images of the nondominated outcomes are called efficient solutions and are denoted by $E(X, f, \mathcal{C})$. We recall that for the Pareto cone $\mathbb{R}^p_\geqq$, we also simply write X_E for $E(X, f, \mathbb{R}^p_\geqq)$, and Y_N for $N(Y, \mathbb{R}^p_\geqq)$. We also define weakly nondominated solutions in the objective space, the pre-images of which in the decision space are called weakly efficient.

Definition 4. Let $\mathcal{C} \subset \mathbb{R}^p$ be a cone and $Y \subset \mathbb{R}^p$. Then $y \in Y$ is called a weakly nondominated outcome of the MOP if $(y - \operatorname{int} \mathcal{C}) \cap Y = \emptyset$.

We state some basic properties of nondominated sets, see Sawaragi et al. [321].

Theorem 2. *Let Y, Y_1, Y_2 be subsets of $\mathbb{R}^p$, $\mathcal{C}$, $\mathcal{C}_1$ and $\mathcal{C}_2$ be cones in $\mathbb{R}^p$.*

- *If $0 \in \mathcal{C}$ then $N(Y, \mathcal{C}) \supset N(Y + \mathcal{C}, \mathcal{C})$. If, additionally, $\mathcal{C}$ is pointed and convex then the inclusion becomes an equality.*
- $N(Y, \mathcal{C}) \subset \operatorname{bd}(Y)$.
- $N(Y_1 + Y_2, \mathcal{C}) \subset N(Y_1, \mathcal{C}) + N(Y_2, \mathcal{C})$.
- $N(\alpha Y, \mathcal{C}) = \alpha N(Y, \mathcal{C})$ *for* $0 \leq \alpha \in \mathbb{R}$.
- *If $\mathcal{C}_1 \subset \mathcal{C}_2$ then $N(Y, \mathcal{C}_2) \subset N(Y, \mathcal{C}_1)$.*
- $N(Y, \mathcal{C}_1 \cup \mathcal{C}_2) = N(Y, \mathcal{C}_1) \cap N(Y, \mathcal{C}_2)$.

The following results relate efficient and nondominated points with respect to a convex polyhedral cone $\mathcal{C}$ to efficient and Pareto points.

Theorem 3. *Let $\mathcal{C}$ be a convex polyhedral cone represented by $\{d \in \mathbb{R}^p : Ld \geqq 0\}$ where L is a $q \times p$ matrix. Then*

1. *[379]* $E(X, f, \mathcal{C}) \subseteq E(X, Lf, \mathbb{R}^q_{\geq})$.
2. *[204]* $L[N(Y, \mathcal{C})] \subseteq N(L[Y], \mathbb{R}^q_{\geqq})$.

The effect of general linear transformations on the efficient and nondominated set is investigated by Cambini et al. [69]. The nondominated set is examined with respect to a class of nonpolyhedral cones by Engau and Wiecek [134]. Properties of efficient sets for collections of MOPs are studied by Gardenghi et al. [163, 164]. The collections include a variety of configurations incorporating separable or composite objective functions, local or global variables, and linking variables modeling interacting MOPs.

18.2.3 Domination Sets and Variable Cones

Nondominated solutions are also defined with respect to more general domination sets and structures. Given an outcome $y \in Y$, Yu [406–409] introduces domination sets $D(y) = \{d \in \mathbb{R}^p : y \succ y + d\}$ where each element $d \in D(y)$ is called a dominated direction or domination factor. Nondominated outcomes then depend on the collection $\mathcal{D} = \{D(y) : y \in Y\}$ of these sets that for different outcomes y can also be different and form what is called a (variable) domination structure.

Definition 5. Let $\mathcal{D} = \{D(y) \subset \mathbb{R}^p\}$ be a domination structure for $Y \subset \mathbb{R}^p$. Then $y \in Y$ is called a nondominated outcome of the MOP with respect to $\mathcal{D}$ if

- there does not exist $y^1 \in Y$ and $d^1 \in D(y^1), d^1 \neq 0 : y = y + d$, or equivalently,
- $Y \cap \bigcup_{y^1 \in Y} \left(y - D(y^1)\right) \subseteq \{y\}$.

If each $D(y)$ is defined using an underlying binary relation that is reflexive, antisymmetric, and transitive then each $D(y)$ is a pointed convex cone and $\mathcal{D}$ is also called a variable ordering relation by Huang et al. [203] or a variable cone by Engau [131]. Moreover, for relations that are compatible with addition, a direction d is dominated if and only if the reversed direction $-d$ is preferred. In this case, Definition 5 can equivalently be stated in terms of a preference structure $\mathcal{P}$ using sets $P(y) = \{d \in \mathbb{R}^p : y + d \succ y\}$. This is not true for general domination structures, however, and Definitions 5 and 6 are different in general.

Definition 6. Let $\mathcal{P} = \{P(y) \subset \mathbb{R}^p\}$ be a preference structure for $Y \subset \mathbb{R}^p$. Then $y \in Y$ is called a nondominated outcome of the MOP with respect to $\mathcal{P}$ if

- there does not exist $y^1 \in Y$ and $d \in \mathcal{D}(y), d \neq 0 : y^1 = y + d$, or equivalently,
- $Y \cap (y + D(y)) \subseteq \{y\}$.

Yu and Leitmann [410] and Bergstresser et al. [45, 46] consider domination and preference structures using convex sets rather than convex cones. Lin [250] provides a comparison of the defined optimality concepts and Chew [77] proposes a reformulation for general vector spaces. Takeda and Nishida [355] introduce fuzzy domination structures for MOP while Hazen and Morin [192, 193] Hazen [191] study optimality conditions for MOP with a nonconical order. Many of these earlier results are collected in the monograph by Yu [409]. The extension of dominance sets from Euclidean to more general vector spaces is also considered by Weidner [377, 378] who later studies scalarization approaches to MOPs with preferences modeled by parameter-depending sets [380, 381]. Chen and Yang [74] also relate a variable domination structure to a nonlinear scalarization for MOP, Chen et al. [75] examine variable dominations structures for set-valued optimization problems, and Ceng and Huang [71] establish a series of existence theorems for generalized vector variational inequalities with a variable ordering relation. Wu [396] further examines the relevance of convex cones for a solution concept in fuzzy MOP.

Baatar and Wiecek [17] develop the structure of domination for the equitability preference (cf. Sect. 18.5.2.4) and show that the underlying domination structure is variable and consists of certain polyhedral cones that depend upon the location of an outcome y in certain sectors of the outcome space. Mut and Wiecek [283] generalize this structure with other convex polyhedral cones. Other examples of preference models based on variable cones are reviewed by Wiecek [387] and further studied by Engau [131] who also addresses several practical limitations of constant cones for the disambiguitation of desirable tradeoffs. Most recently, Eichfelder [128, 129] and Eichfelder and Ha [130] derive several further results including Fermat and Lagrange multiplier rules that are motivated from two specific applications of variable ordering structures in medical image registration and intensity modulated radiation therapy.

18.2.4 Local, Proper, and Approximate Solutions

All the classes of solutions defined above are global solutions. However, we also define local solutions of the MOP. A point $x \in X$ is called a locally efficient solution of the MOP if there exists a neighborhood $N(x)$ such that there is no $x' \in N(x) \cap X$ such that $f(x') \leq f(x)$. Similarly, all other classes of local solutions in the decision space and the objective space can be defined. In this chapter, all solutions of optimization problems are global unless stated otherwise.

Additionally, the following authors define properly efficient solutions: Kuhn and Tucker [242], Klinger [228], Geoffrion [167], Borwein [55], Benson [28], Wierzbicki [390] and Henig [196]. Borwein and Zhuang define super efficient solutions [57, 58].

Definition 7. A point $\hat{x} \in X$ is called a properly efficient solution of the MOP′ in the sense of Kuhn-Tucker if $\hat{x} \in X_E$ and if there does not exist a $d \in \mathbb{R}^n$ such that $\nabla f_k(\hat{x})^T d \leq 0$ for all $k = 1, \ldots, p$ with a strict inequality for some k and $\nabla g_j(\hat{x})^T d \leq 0$ for all $j \in I(\hat{x}) = \{j : g_j(\hat{x}) = 0\}$.

Definition 8. A point $\hat{x} \in X$ is called a properly efficient solution of the MOP in the sense of Geoffrion if $\hat{x} \in X_E$ and if there exists $M > 0$ such that for each $k = 1, \dots, p$ and each $x \in X$ satisfying $f_k(x) < f_k(\hat{x})$ there exists an $l \neq k$ with $f_l(x) > f_l(\hat{x})$ and $(f_k(\hat{x}) - f_k(x))/(f_l(x) - f_l(\hat{x})) \leq M$.

The sets of all properly efficient solutions and properly nondominated outcomes (in the sense of Geoffrion) are denoted by X_{pE} and Y_{pN}, respectively. Approximate efficient solutions are initially defined by Kutateladze [243] and later by Loridan [255] in the following way.

Definition 9. Let $\epsilon \in \mathbb{R}^p_{\geq}$. A point $\hat{x} \in X$ is called an ϵ-efficient solution of the MOP if there is no $x' \in X$ such that $f(x') \leq f(\hat{x}) - \epsilon$.

Letting $E(X, f, \mathcal{C}, \epsilon)$ and $N(Y, \mathcal{C}, \epsilon)$ denote the ϵ-efficient solution in decision and outcome space, respectively, Engau and Wiecek [133] generalize Theorem 3 for approximate solutions.

Theorem 4. *Let $\mathcal{C}$ be a convex polyhedral cone represented by $\{d \in \mathbb{R}^p : Ld \geqq 0\}$ where L is a $q \times p$ matrix. Let $\epsilon \in \mathbb{R}^p_{\geqq}$, $b = L\varepsilon \in \mathbb{R}^q$ and $\mathbb{R}^q_{\geqq b} := \{z \in \mathbb{R}^q : z \geqq b\}$. Then*

1. $E(X, f, \mathcal{C}, \epsilon) \subseteq E(X, Lf, \mathbb{R}^q_{\geqq b})$;
2. $L[N(Y, \mathcal{C}, \epsilon)] \subseteq N(L[Y], \mathbb{R}^q_{\geqq b})$.

Other types of approximate efficient solutions are defined by White [384]. Later, Gutiérrez et al. [184] show that many of these definitions can be seen as particular instances of their more general definition [182], in terms of certain coradiant subsets of the underlying ordering cone.

Similarly, weakly ϵ-efficient solutions and strictly ϵ-efficient solutions and their images can be defined. Let $y^I_k := \min\{f_k(x) : x \in X\}$ be the (global) minimum of $f_k(x), k = 1, \dots, p$. The point $y^I \in \mathbb{R}^p, y^I = (y^I_1, \dots, y^I_p)$ is called the ideal point for the MOP. A point y^U where $y^U_k := \min\{f_k(x) : x \in X\} - \epsilon_k, k = 1, \dots, p$, where the components of $\epsilon = (\epsilon_1, \dots, \epsilon_p) \in \mathbb{R}^p$ are small positive numbers, is called a utopia point for the MOP. Furthermore, the point y^N with $y^N_k := \max\{f_k(x) : x \in X_E\}$ is called the nadir point for the MOP. For each $x \in X_E$ it holds: $y^U < y^I \leq f(x) \leq y^N$. We shall assume that $y^I < y^N$ for the MOP.

18.3 Properties of the Solution Sets

In this section we discuss properties of the nondominated and efficient sets including existence, stability, convexity, and connectedness of the solution sets $N(Y, \mathcal{C})$ and $E(X, f, \mathcal{C})$ of the MOP. Here we assume that $\mathcal{C}$ is a pointed, closed, convex cone. We first consider existence of nondominated points and efficient solutions.

Theorem 5 ([56]). *Let $Y \neq \emptyset$ and suppose there exists a $y^0 \in Y$ such that $Y^0 = (y^0 - \mathcal{C}) \cap Y$ is compact. Then $N(Y, \mathcal{C}) \neq \emptyset$.*

An earlier result by Corley requires Y to be $\mathcal{C}$-semicompact, i.e., every open cover $\{Y \setminus (y^\tau - \operatorname{cl} \mathcal{C}) : y^\tau \in Y,\ \tau \in \mathcal{T}\}$ of Y, where $\mathcal{T}$ is an index set, has a finite subcover.

Theorem 6 ([86]). *If $Y \neq \emptyset$ and Y is $\mathcal{C}$-semicompact then $N(Y, \mathcal{C}) \neq \emptyset$.*

Corollary 1 ([189]). *If $Y \neq \emptyset$ and Y is $\mathcal{C}$-compact (i.e., $(y - \mathcal{C}) \cap Y$ is compact for all $y \in Y$) then $N(Y, \mathcal{C}) \neq \emptyset$.*

Sawaragi et al. [321] also give necessary and sufficient conditions for $N(Y, \mathcal{C})$ to be nonempty for the case of a nonempty, closed, convex set Y. Essentially, the existence of nondominated points can be guaranteed under some compactness assumption. Consistently, the existence of efficient solutions can be guaranteed under appropriate continuity assumptions on the objective functions and compactness assumptions on X. The function $f : \mathbb{R}^n \to \mathbb{R}^p$ is said to be $\mathcal{C}$-semicontinuous if the pre-image of $y - \operatorname{cl} \mathcal{C}$ is a closed subset of $\mathbb{R}^n$ for all y in $\mathbb{R}^p$.

Theorem 7. *Let $\emptyset \neq X \subset \mathbb{R}^n$ be a compact set and assume $f : \mathbb{R}^n \to \mathbb{R}^p$ is C-semicontinuous. Then $E(X, f, \mathcal{C}) \neq \emptyset$.*

A review of existence results for nondominated and efficient sets is provided by Sonntag and Zălinescu [346].

Stability of MOPs is studied, among others, in the following context. Let $y \in Y$ be a feasible solution. If y is dominated then there exists a $y' \in Y, y' \neq y$, such that $y' \succ_{\mathcal{C}} y$. The question arises whether y' is nondominated. In this case, $N(Y, \mathcal{C})$ is called externally stable. Note that the external stability condition can also be written as $Y \subset N(Y, \mathcal{C}) + \mathcal{C}$.

Theorem 8 ([321]). *Let $\mathcal{C}$ be a pointed, closed, convex cone and let $Y \neq \emptyset$ be a $\mathcal{C}$-compact set. Then $N(Y, \mathcal{C})$ is externally stable.*

In addition, Sawaragi et al. [321] prove that the necessary and sufficient conditions for existence of nondominated points for nonempty, closed, convex sets Y are also necessary and sufficient for external stability of $N(Y, \mathcal{C})$ in that case.

We now state some relationships between the various nondominated and efficient sets. From Definitions 2 and 8 it is clear that $X_{pE} \subseteq X_E \subseteq X_{wE}$ and $X_{sE} \subseteq X_E$, and therefore $Y_{pN} \subseteq Y_N \subseteq Y_{wN}$. Again, for convex sets a stronger result holds. Hartley [189] proves that if Y is $\mathcal{C}$-closed ($Y + \mathcal{C}$ is closed) and $\mathcal{C}$-convex ($Y + \mathcal{C}$ is convex) then $Y_{pN} \subseteq Y_N \subseteq \operatorname{cl} Y_{pN}$ and that equality holds if Y is polyhedral. The $\mathcal{C}$-convexity condition on Y is satisfied if, e.g., the set X is convex and the objective functions f_k are convex. Therefore it makes sense to define the convex MOP.

If all the objective functions $f_k, k = 1, \ldots, p$ of the MOP are convex and the feasible set X is convex then the problem is called convex MOP. The outcome set Y of the convex MOP is $\mathbb{R}^p_{\geqq}$-convex, i.e., $Y + \mathbb{R}^p_{\geqq}$ is a convex set.

The last property of the efficient and the nondominated sets that we discuss is connectedness.

Theorem 9 ([376]). *Assume that $f_1, \dots, f_p$ are continuous and that X satisfies one of the following conditions.*

1. *$X \subset \mathbb{R}^n$ is a compact, convex set.*
2. *X is a closed, convex set and for all $y \in Y, X(y) := \{x \in X : f(x) \leqq y\}$ is compact.*

Then the following statements hold:

1. *If $f_k : \mathbb{R}^n \to \mathbb{R}$ are quasiconvex on X for $k = 1, \dots, p$ then X_{wE} is connected.*
2. *If $f_k : \mathbb{R}^n \to \mathbb{R}$ are strongly quasiconvex on X for $k = 1, \dots, p$ then X_E is connected.*

Warburton [376] also gives examples showing that X_E may be disconnected if X is compact and convex but f_k only quasiconvex, and that X_E (respectively X_{wE}) may be disconnected if $X(y)$ is not compact for some y.

Because convex functions are continuous and the image of a connected set under a continuous function is connected, it follows immediately that the sets Y_{wN} and Y_N are connected under the assumptions stated in Theorem 9, if the objective functions $f_k, k = 1, \dots, p$ are continuous. However, connectedness of Y_N can also be proved under more general assumptions.

Theorem 10 ([284]). *Let $\mathcal{C}$ be a closed, convex, nonempty cone that does not contain a nontrivial subspace of $\mathbb{R}^p$ and let Y be a closed, convex, and $\mathcal{C}$-compact set. Then $N(Y, \mathcal{C})$ is connected.*

In the remaining sections we will mainly consider nondominance and efficiency with respect to the Pareto cone, i.e., $\mathcal{C} = \mathbb{R}^p_{\geqq}$. Thus, throughout the rest of the chapter, efficiency is meant for the MOP (18.1) in the sense of Definition 2.

18.4 Conditions for Efficiency

Conditions for efficiency are powerful theoretical tools for determining whether a feasible point is efficient. Denote the set of indices of active inequality constraints at $\hat{x} \in X$ by $I(\hat{x}) = \{j \in \{1, \dots, l\} : g_j(\hat{x}) = 0\}$ (compare Definition 7).

18.4.1 First Order Conditions

Assume that the objective functions $f_k, k = 1, \dots, p$, and the constraint functions $g_j, j = 1, \dots, l$; $h_j, j = 1, \dots, m$ of the MOP are continuously differentiable.

Theorem 11. *Fritz-John necessary conditions for efficiency [90]. If $\hat{x}$ is efficient then there exist vectors $w \in \mathbb{R}^p_{\geqq}$, $u \in \mathbb{R}^l_{\geqq}$, and $v \in \mathbb{R}^m$, $(w, u, v) \neq 0$ such that*

$$\sum_{k=1}^{p} w_k \nabla f_k(\hat{x}) + \sum_{j=1}^{l} u_j \nabla g_j(\hat{x}) + \sum_{j=1}^{m} v_j \nabla h_j(\hat{x}) = 0$$

$$u_j g_j(\hat{x}) = 0 \textit{ for all } j = 1, \ldots, l.$$

Kuhn-Tucker type conditions for efficiency have also been studied for MOP′.

Definition 10. The MOP′ is said to satisfy the Kuhn-Tucker constraint qualification at $\hat{x} \in X'$ if for any $d \in \mathbb{R}^n$ such that $\nabla g_j(\hat{x})^T d \leq 0$ for all $j \in I(\hat{x})$, there exist a continuously differentiable function $a : [0,1] \to \mathbb{R}^n$ and a real scalar $\alpha > 0$ such that $a(0) = \hat{x}, g(a(\beta)) \leqq 0$ for all $\beta \in [0,1]$ and $a'(0) = \alpha d$.

Theorem 12. *Kuhn-Tucker necessary conditions for efficiency [276]. Let the Kuhn-Tucker constraint qualification hold at $\hat{x} \in X'$. If $\hat{x}$ is efficient for the MOP′ then there exist vectors $w \in \mathbb{R}^p_{\geq}$ and $u \in \mathbb{R}^l_{\geqq}$ such that*

$$\sum_{k=1}^{p} w_k \nabla f_k(\hat{x}) + \sum_{j=1}^{l} u_j \nabla g_j(\hat{x}) = 0$$

$$u_j g_j(\hat{x}) = 0 \textit{ for all } j = 1, \ldots, l.$$

The feasible points of MOP′ satisfying the Kuhn-Tucker necessary conditions are called the critical points of this problem.

Theorem 13. *Kuhn-Tucker necessary conditions for proper efficiency [242, 321]. If $\hat{x} \in X'$ is properly efficient (in the sense of Kuhn-Tucker) for the MOP′ then there exist vectors $w \in \mathbb{R}^p_{>}$ and $u \in \mathbb{R}^l_{\geqq}$ such that*

$$\sum_{k=1}^{p} w_k \nabla f_k(\hat{x}) + \sum_{j=1}^{l} u_j \nabla g_j(\hat{x}) = 0$$

$$u_j g_j(\hat{x}) = 0 \textit{ for all } j = 1, \ldots, l.$$

If the Kuhn-Tucker constraint qualification is satisfied at a point $\hat{x} \in X'$ then the condition in Theorem 13 is also necessary for $\hat{x}$ to be properly efficient in the sense of Geoffrion for the MOP′, as shown by Sawaragi et al. [321].

Theorem 14. *Kuhn-Tucker sufficient conditions for proper efficiency [242, 276, 321]. Let the MOP′ be convex and let $\hat{x} \in X'$. If there exist vectors $w \in \mathbb{R}^p_{>}$ and $u \in \mathbb{R}^l_{\geqq}$ such that*

$$\sum_{k=1}^{p} w_k \nabla f_k(\hat{x}) + \sum_{j=1}^{l} u_j \nabla g_j(\hat{x}) = 0$$

$$u_j g_j(\hat{x}) = 0 \textit{ for all } j = 1, \ldots, l$$

then $\hat{x}$ *is properly efficient in the sense of Kuhn-Tucker for the MOP*$'$.

18.4.2 Second Order Conditions

Various types of second-order conditions for efficiency have been developed. For this type of conditions it is usually assumed that the objective functions $f_k, k = 1, \ldots, p$ and the constraint functions $g_j, j = 1, \ldots, l; h_j, j = 1, \ldots, m$ of the MOP are twice continuously differentiable.

Several necessary and sufficient second-order conditions for the MOP are developed by Wang [375]. Cambini et al. [67] establish second order conditions for MOPs with general convex cones while Cambini [66] develops second order conditions for MOPs with the Pareto cone. Aghezzaf [4] and Aghezzaf and Hachimi [5] develop second-order necessary conditions for the MOP$'$. Additional works include Bolintinéanu and El Maghri [54], Bigi and Castellani [49], Jiménez and Novo [215], and Gutiérrez et al. [183].

18.5 Generation of the Solution Sets

There are two general classes of approaches to generating solution sets of MOPs: scalarization methods and nonscalarizing methods. These approaches convert the MOP into a single objective program (SOP), a sequence of SOPs, or another MOP. Under some assumptions solution sets of these new programs yield solutions of the original problem. Scalarization methods explicitly employ a scalarizing function to accomplish the conversion while nonscalarizing methods use other means. Solving the SOP typically yields one solution of the MOP so that a repetitive solution scheme is needed to obtain a subset of solutions of the MOP. Among the nonscalarizing approaches there are methods using other optimality concepts in $\mathbb{R}^p$ than Pareto, descent and homotopy methods transferred from nonlinear programming, and a new class of set-oriented methods.

18.5.1 Scalarization Methods

The traditional approach to solving MOPs is by scalarization which involves formulating an MOP-related SOP by means of a real-valued scalarizing function typically being a function of the objective functions of the MOP, auxiliary scalar or vector variables, and/or scalar or vector parameters. Sometimes the feasible set of the MOP is additionally restricted by new constraint functions related to the objective functions of the MOP and/or the new variables introduced.

In this section we review the most well-known scalarization techniques and list related results on the generation of various classes of solutions of the MOP.

18.5.1.1 The Weighted-Sum Approach

In the weighted-sum approach a weighted sum of the objective functions is minimized:

$$\begin{aligned} \min \quad & \sum_{k=1}^{p} \lambda_k f_k(x) \\ & \text{subject to } x \in X, \end{aligned} \tag{18.2}$$

where $\lambda \in \mathbb{R}^p_{\geq}$.

Theorem 15 ([167]).

1. *Let $\lambda \in \mathbb{R}^p_{\geq}$. If $\hat{x} \in X$ is an optimal solution of problem (18.2) then $\hat{x} \in X_{wE}$. If $\hat{x} \in X$ is a unique optimal solution of problem (18.2) then $\hat{x} \in X_E$.*
2. *Let $\lambda \in \mathbb{R}^p_{>}$. If $\hat{x} \in X$ is an optimal solution of problem (18.2) then $\hat{x} \in X_{pE}$.*
3. *Let the MOP be convex. A point $\hat{x} \in X$ is an optimal solution of problem (18.2) for some $\lambda \in \mathbb{R}^p_{>}$ if and only if $\hat{x} \in X_{pE}$.*

18.5.1.2 The Weighted *t*-th Power Approach

In the weighted t-th power approach a weighted sum of the objective functions taken to the power of t is minimized:

$$\begin{aligned} \min \quad & \sum_{k=1}^{p} \lambda_k (f_k(x))^t \\ & \text{subject to } x \in X, \end{aligned} \tag{18.3}$$

where $\lambda \in \mathbb{R}^p_{>}$ and $t > 0$.

Theorem 16 ([385]). *Let $\lambda \in \mathbb{R}^p_{>}$.*

1. *For all $t > 0$, if $\hat{x} \in X$ is an optimal solution of problem (18.3) then $\hat{x} \in X_E$.*
2. *If a point $\hat{x} \in X$ is efficient then there exists a $\hat{t} > 0$ such that for every $t \geq \hat{t}$, the point $\hat{x}$ an optimal solution of problem (18.3).*

Under certain conditions, applying the t-th power to the objective functions of nonconvex MOPs may convexify the set $Y_N + \mathbb{R}^p_{\geqq}$ so that the weighting approach can be successfully applied to generate efficient solutions of these MOPs (Li [247]).

18.5.1.3 The Weighted Quadratic Approach

In the weighted quadratic approach a quadratic function of the objective functions is minimized:

$$\begin{aligned} \min \quad & f(x)^T Qf(x) + q^T f(x) \\ & \text{subject to } x \in X, \end{aligned} \tag{18.4}$$

where Q is a $p \times p$ matrix and q is a vector in $\mathbb{R}^p$.

Theorem 17 ([367]). *Under conditions of quadratic Lagrangian duality, if $\hat{x} \in X$ is efficient then there exist a symmetric $p \times p$ matrix Q and a vector $q \in \mathbb{R}^p$ such that $\hat{x}$ is an optimal solution of problem (18.4).*

More recently, results of quadratic scalarizations have been investigated by Fliege [145] who also shows which parameter sets can be used to recover all solutions of an MOP where the ordering in the image space is induced by an arbitrary convex cone.

18.5.1.4 The Guddat et al. Approach

Let x^0 be an arbitrary feasible point for the MOP. Consider the following problem:

$$\begin{aligned} \min \quad & \sum_{k=1}^{p} \lambda_k f_k(x) \\ \text{subject to} \quad & f_k(x) \leq f_k(x^0), \ k = 1, \dots, p \\ & x \in X \end{aligned} \tag{18.5}$$

where $\lambda \in \mathbb{R}^p_{\geq}$.

Theorem 18 ([175]). *Let $\lambda \in \mathbb{R}^p_{>}$. A point $x^0 \in X$ is an optimal solution of problem (18.5) if and only if $x^0 \in X_E$.*

In [175], this result is also generalized for scalarizations in the form of problem (18.5) with an objective function being strictly increasing on $\mathbb{R}^p$ (cf. Definition 12).

18.5.1.5 The ε-Constraint Approach

In the ε-constraint method one objective function is retained as a scalar-valued objective while all the other objective functions generate new constraints. The k-th ε-constraint problem is formulated as:

$$\begin{aligned} \min \quad & f_k(x) \\ \text{subject to} \quad & f_i(x) \leq \varepsilon_i,\ i = 1, \ldots, p;\ i \neq k \\ & x \in X. \end{aligned} \tag{18.6}$$

Let $\varepsilon_{-k} \in \mathbb{R}^{p-1}, \varepsilon_{-k} = (\varepsilon_1, \ldots, \varepsilon_{k-1}, \varepsilon_{k+1}, \ldots, \varepsilon_p)$. Let the set $\Psi = \{\varepsilon \in \mathbb{R}^p$: Problem (18.6) is feasible for $\varepsilon_{-k} = (\varepsilon_1, \ldots, \varepsilon_{k-1}, \varepsilon_{k+1}, \ldots, \varepsilon_p)$ for all $k = 1, \ldots, p\}$.

Theorem 19 ([72]).

1. *If, for some $k, k \in \{1, \ldots, p\}$, there exists $\varepsilon_{-k} \in \mathbb{R}^{p-1}$ such that $\hat{x}$ is an optimal solution of problem (18.6) then $\hat{x} \in X_{wE}$.*
2. *If, for some $k, k \in \{1, \ldots, p\}$, there exists $\varepsilon_{-k} \in \mathbb{R}^{p-1}$ such that $\hat{x}$ is a unique optimal solution of problem (18.6) then $\hat{x} \in X_E$.*
3. *A point $\hat{x} \in X$ is efficient if and only if there exists $\varepsilon \in \Psi$ such that $\hat{x}$ is an optimal solution of problem (18.6) for every $k = 1, \ldots, p$ and with $f_i(\hat{x}) = \varepsilon_i, i = 1, \ldots, p, i \neq k$.*

The method of proper equality constraints is a modification of the ε-constraint method in which the constraints with the right-hand side parameters ε_i are equalities (Lin [249]).

18.5.1.6 The Improved ε-Constraint Approach

The ε-constraint approach has numerical disadvantages when applied to problems with a specific structure, in particular discrete multiobjective problems, see Ehrgott and Ryan [119]. The improved constraint approach tries to overcome those difficulties using the following two scalarizations:

$$\begin{aligned} \min \quad & f_k(x) - \sum_{i \neq k} \lambda_i l_i \\ \text{subject to} \quad & f_i(x) + l_i \leq \varepsilon_i,\ i = 1, \ldots, p;\ i \neq k \\ & l_i \geq 0,\ i = 1, \ldots, p;\ i \neq k \\ & x \in X, \end{aligned} \tag{18.7}$$

and

$$\begin{aligned} \min \quad & f_k(x) + \sum_{i \neq k} \lambda_i l_i \\ \text{subject to} \quad & f_i(x) - l_i \leq \varepsilon_i, \; i = 1, \ldots, p; \; i \neq k \\ & l_i \geq 0, \; i = 1, \ldots, p; \; i \neq k \\ & x \in X, \end{aligned} \tag{18.8}$$

where weights $\lambda_i \geq 0, i \neq k$. Here $l_i, i \neq k$ are slack variables for the inequality constraints in problem (18.6). Problem (18.7) with all weights $\lambda_i = 0$ corresponds to the original ε-constraint problem (18.6). In problem (18.8), the ε-constraints of problem (18.6) are allowed to be violated and the violation is penalized in the objective function.

Theorem 20 ([118]).

1. *Let $\lambda \in \mathbb{R}^p_>$. If $(\hat{x}, \hat{l})$ is an optimal solution of problem (18.7) then $\hat{x} \in X_E$.*
2. *Let $\lambda \in \mathbb{R}^p_>$. If $(\hat{x}, \hat{l})$ is an optimal solution of problem (18.7) with $\hat{l} > 0$ then $\hat{x} \in X_{pE}$.*
3. *If $\hat{x} \in X_{pE}$ then, for every $k \in \{1, \ldots, p\}$, there exist $\varepsilon, \hat{l}$, and $\lambda \in \mathbb{R}^p_>$ such that $(\hat{x}, \hat{l})$ is an optimal solution of problem (18.7).*
4. *Let $\lambda \in \mathbb{R}^p_>$. If $(\hat{x}, \hat{l})$ is an optimal solution of problem (18.8) with $\hat{l} > 0$ then $\hat{x} \in X_{pE}$.*
5. *If $\hat{x} \in X_{pE}$ then, for every $k \in \{1, \ldots, p\}$, there exist $\varepsilon, \hat{l}$, and λ^k with $\lambda^k_i > 0$ for all $i \neq k$ such that $(\hat{x}, \hat{l})$ is an optimal solution of problem (18.8) for all $\lambda \in \mathbb{R}^{p-1}, \lambda \geqq \lambda^k$.*

18.5.1.7 The Penalty Function Approach

Meng et al. [269] use penalty terms for the objective functions of the MOP and for the constraints of the MOP′. The scalarized MOP assumes the form:

$$\begin{aligned} \min \quad & \sum_{k=1}^{p} \lambda_k \max\{f_k(x) - M, 0\}^2 \\ & \text{subject to } x \in X, \end{aligned} \tag{18.9}$$

where $\lambda \in \mathbb{R}^p_>$, M is a scalar such that $M < f_k(x^0), k = 1, \ldots, p$, and $x^0 \in X$. The scalarized MOP′ becomes:

$$\begin{aligned} \min \quad & \sum_{k=1}^{p} \lambda_k \max\{f_k(x) - M, 0\}^2 + M^2 \sum_{k=1}^{p} \max\{g_j(x), 0\} \\ & \text{subject to } x \in S', \end{aligned} \tag{18.10}$$

where $\lambda \in \mathbb{R}^p_>$, $M < 0$ and such that $M < f_k(\hat{x}), k = 1, \ldots, p$, and $\hat{x} \in X$.

Theorem 21 ([269]).

1. *If x^0 is an optimal solution of problem (18.9) then $x^0 \in X_E$ for the MOP.*
2. *Let $\hat{x}$ be an optimal solution of problem (18.10). If $\hat{x}$ is a feasible solution of the MOP′ then $x^0 \in X_E$ for the MOP′.*

Exact penalty terms and the weighted-sum approach are used by Bernau [47].

18.5.1.8 The Benson Approach

Benson [27] introduces an auxiliary vector variable $l \in \mathbb{R}^p$ and uses a known feasible point x^0 in the following scalarization:

$$\begin{aligned} \max \quad & \sum_{k=1}^{p} l_k \\ \text{subject to}\; & f_k(x) + l_k = f_k(x^0),\ \ k = 1, \ldots, p \\ & l \geqq 0 \\ & x \in X. \end{aligned} \tag{18.11}$$

Not only can this approach find an efficient solution but it can also check whether the available point x^0 is efficient.

Theorem 22 ([27]).

1. *The point $x^0 \in X$ is efficient if and only if the optimal objective value of problem (18.11) is equal to zero.*
2. *If $(\hat{x}, \hat{l})$ is an optimal solution of problem (18.11) with a positive optimal objective value then $\hat{x} \in X_E$.*
3. *Let the MOP be convex. If no finite optimal objective value of problem (18.11) exists then $X_{pE} = \emptyset$.*

Earlier, Charnes and Cooper [73] have proposed problem (18.11) and proved part 1 of Theorem 22.

18.5.1.9 Reference Point Approaches

The family of reference point approaches includes a variety of methods in which a feasible or infeasible reference point in the objective space is used. A reference point in the objective space, $r \in \mathbb{R}^p$, is typically a vector of satisfactory or desirable criterion values referred to as aspiration levels, $r_k, k = 1, \ldots, p$. However, it may also be a vector representing a currently available outcome or a worst outcome.

Distance-Function-Based Approaches These methods employ a distance function, typically based on a norm, to measure the distance between a utopia (or ideal) point and the points in the Pareto set. Let $d : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}_{\geq}$ denote a distance function. The generic problem is formulated as:

$$\begin{aligned} \min \quad & d(f(x), r) \\ & \text{subject to } x \in X, \end{aligned} \tag{18.12}$$

where $r \in \mathbb{R}^p$ is a reference point.

Under suitable assumptions satisfied by the distance function, problem (18.12) yields efficient solutions which in this case are often called compromise solutions. Let d be a distance function derived from a norm, i.e., $d(y^1, y^2) = ||y^1 - y^2||$ for some norm $|| \cdot || : \mathbb{R}^p \to \mathbb{R}_{\geq}$.

Definition 11. 1. A norm $\| \cdot \| : \mathbb{R}^p \to \mathbb{R}_{\geq}$ is called monotonically increasing if $\|y^1\| \leq \|y^2\|$ holds for all $y^1, y^2 \in \mathbb{R}^p$ with $|y^1_k| \leq |y^2_k|,\ k = 1, \ldots, p$ and $\|y^1\| < \|y^2\|$ holds if $|y^1_k| < |y^2_k|,\ k = 1, \ldots, p$.
2. A norm $\| \cdot \|$ is called strictly monotonically increasing, if $\|y^1\| < \|y^2\|$ holds for all $y^1, y^2 \in \mathbb{R}^p$ with $|y^1_k| \leq |y^2_k|,\ k = 1, \ldots, p$ and $|y^1_k| \neq |y^2_k|$ for some k.

Theorem 23 ([114]).

1. *Let $\| \cdot \| : \mathbb{R}^p \to \mathbb{R}_{\geq}$ be monotonically increasing and assume $r = y^I$.*
   - *If $\hat{x}$ is an optimal solution of problem (18.12) then $\hat{x} \in X_{wE}$.*
   - *If $\hat{x}$ is a unique optimal solution of problem (18.12) then $\hat{x} \in X_E$.*
2. *Let $\| \cdot \|$ be strictly monotonically increasing and assume $r = y^I$. If $\hat{x}$ is an optimal solution of problem (18.12) then $\hat{x} \in X_E$.*

The norms studied in the literature include the family of weighted l_p-norms for $1 \leq p \leq \infty$ (Yu [405], Zeleny [415], Bowman [59]), a family of norms proposed by Gearhart [165], composite norms (Bárdossy et al. [22], Jeyakumar and Yano [214]) and oblique norms (Schandl et al. [327]).

Among the l_p-norms, the weighted l_∞-norm (also known as the Chebyshev or Tchebycheff norm) has been extensively studied. Since it produces all weakly efficient solutions of convex and nonconvex MOPs, it has been modified to ensure that efficient rather than weakly efficient solutions are found. The modified norms include the augmented l_∞-norm (Steuer and Choo [351], Steuer [348]) and the modified l_∞-norm (Kaliszewski [219]).

Scalarizations based on more general distance functions such as gauges have also been considered and proved to generate weakly efficient or properly efficient solutions for convex and nonconvex MOPs (Klamroth et al. [227]). Since these approaches implicitly use not only the utopia point but also gauge-related directions, they are discussed later. Together with some other gauge-based approaches in Section 18.5.1.11.

The Achievement Function Approach A certain class of real-valued functions $s_r : \mathbb{R}^p \to \mathbb{R}$, referred to as achievement functions, is used to scalarize the MOP. The scalarized problem is given by

$$\begin{aligned} \min \quad & s_r(f(x)) \\ \text{subject to } & x \in X. \end{aligned} \tag{18.13}$$

Similar to distance functions discussed above, certain properties of achievement functions guarantee that problem (18.13) yields (weakly) efficient solutions.

Definition 12. An achievement function $s_r : \mathbb{R}^p \to \mathbb{R}$ is said to be

1. increasing if for $y^1, y^2 \in \mathbb{R}^p$, $y^1 \leqq y^2$ then $s_r(y^1) \leq s_r(y^2)$,
2. strictly increasing if for $y^1, y^2 \in \mathbb{R}^p$, $y^1 < y^2$ then $s_r(y^1) < s_r(y^2)$,
3. strongly increasing if for $y^1, y^2 \in \mathbb{R}^p$, $y^1 \leq y^2$ then $s_r(y^1) < s_r(y^2)$.

Theorem 24 ([391, 392]).

1. *Let an achievement function s_r be increasing. If $\hat{x} \in X$ is a unique optimal solution of problem (18.13) then $\hat{x} \in X_E$.*
2. *Let an achievement function s_r be strictly increasing. If $\hat{x} \in X$ is an optimal solution of problem (18.13) then $\hat{x} \in X_{wE}$.*
3. *Let an achievement function s_r be strongly increasing. If $\hat{x} \in X$ is an optimal solution of problem (18.13) then $\hat{x} \in X_E$.*

Among many achievement functions satisfying the above properties we mention the strictly increasing function

$$s_r(y) = \max_{k=1,\ldots,p} \{\lambda_k(y_k - r_k)\}$$

and the strongly increasing functions

$$s_r(y) = \max_{k=1,\ldots,p} \{\lambda_k(y_k - r_k)\} + \rho_1 \sum_{k=1}^{p} \lambda_k(y_k - r_k)$$

$$s_r(y) = -\|y - r\|^2 + \rho_2 \|(y - r)_+\|^2,$$

where $r \in \mathbb{R}^p, \lambda \in \mathbb{R}^p_>$ is a vector of positive weights, $\rho_1 > 0$ is sufficiently small, $\rho_2 > 1$ is a penalty parameter, and $(y-r)_+$ is a vector with components $\max\{0, y_k - r_k\}$ (Wierzbicki [391, 392]).

The Weighted Geometric Mean Approach Consider the weighted geometric mean of the differences between the nadir point y^N and the objective functions with the weights in the exponents

$$\begin{aligned} \max \quad & \Pi_{k=1}^{p}(y_k^N - f_k(x))^{\lambda_k} \\ \text{subject to}\, & f_k(x) \leq y_k^N,\ k = 1,\dots,p \\ & x \in X, \end{aligned} \tag{18.14}$$

where $\lambda \in \mathbb{R}^p_>$. Let the MOP be convex. According to Lootsma et al. [254], an optimal solution of problem (18.14) is efficient.

Goal Programming In (GP) one is interested in achieving a desirable goal or target established for the objective functions of the MOP. The vector of these goals produces a reference point in the objective space and therefore goal programming can be viewed as a variation of the reference point approaches. Let $r \in \mathbb{R}^p$ be a goal. The general formulation of GP is

$$\begin{aligned} \min \quad & a(\delta^-, \delta^+) \\ \text{subject to}\, & f_k(x) + \delta_k^- - \delta_k^+ = r_k,\ k = 1,\dots,p \\ & x \in X, \end{aligned} \tag{18.15}$$

where $\delta^-, \delta^+ \in \mathbb{R}^p$ are variables representing negative and positive deviations from the goal r, and $a(\delta^-, \delta^+)$ is an achievement function.

A real-valued achievement function, $a : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}$, is typically defined as the weighted sum of deviations (weighted or non-preemptive GP) or the maximum deviation from among the weighted deviations (l_∞-GP). Solving problem (18.15) with this function results in minimizing all deviations simultaneously. A vector-valued achievement function, $a : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}^L$, is associated with L priority levels to which the objective functions are assigned, and results in a new GP-related MOP typically solved with the lexicographic approach based on the priority ranking (see Sect. 18.5.2.1). In this case the deviations are minimized sequentially and the technique is known as lexicographic GP. Jones and Tamiz [217] and Chapter 21 of this book provide recent bibliographies of GP while Romero [313] presents a general achievement function including the weighted, maximum and lexicographic functions as special cases.

Whether an optimal solution of problem (18.15) is efficient for the MOP depends on the achievement function a and the goal r. In fact, when solving problem (18.15) non-Pareto criterion vectors $f(\hat{x})$ are quite common. Tamiz and Jones [356] develop tests for efficiency and methods to restore efficiency in case problem (18.15) produces a solution $\hat{x}$ that is not efficient.

Dependent on the type of goals (i.e., whether over- and/or underachievement of r_k is penalized, or only values outside a certain interval are penalized) one can define a subset of the objective space $G := \{y \in \mathbb{R}^p : y_k \geq r_k, k \in K_1;\ y_k \leq r_k, k \in K_2;\ y_k = r_k, k \in K_3;\ y_k \in [r_k^l, r_k^u], k \in K_4\}$, where $K_1 \cup K_2 \cup K_3 \cup K_4 = \{1,\dots,p\}$, and interpret goal programming as finding a feasible solution x that is in or close to G (Steuer [348]). In this way, G can be understood as a reference set for the MOP.

The Reference Set Approach The concept of a reference point has been generalized by some authors. Michalowski and Szapiro [275] use two reference points to search the Pareto set of multiobjective linear programs. Skulimowski [342] studies the notion of a reference set. Simple examples of reference sets include the sets $r-C$ or $(r^1 - C) \cap (r^2 + C)$ for $r^2 \succ_C r^1$, where r, r^1, r^2 are reference points and C is a closed, convex, and pointed cone. Under suitable conditions, a solution to the MOP obtained by means of minimizing the distance from a reference set is nondominated. Cases in which a reference set can be reduced to a reference point are also examined in [342].

18.5.1.10 Direction-Based Approaches

This group of scalarizing approaches employs a reference point $r \in \mathbb{R}^p$, a direction in the objective space along which a search is performed, and a real variable α measuring the progress along the direction.

The Roy Approach Perhaps the first approach of that kind has been proposed by Roy [315, p. 242] which (slightly reformulated) can be written as

$$\begin{aligned} \max \quad & \alpha \\ \text{subject to}\; & f_k(x) + \alpha e \leq r_k,\; k = 1, \ldots, p \\ & x \in X, \end{aligned} \tag{18.16}$$

where $e \in \mathbb{R}^p$ is a vector of ones and determines the fixed direction of search. Depending on the choice of the reference point r the approach finds a (weakly) efficient solution.

The Goal-Attainment Approach Given a (feasible or infeasible) goal vector r and a direction $d \leq 0$ along which the search is performed the goal-attainment approach is formulated as

$$\begin{aligned} \max \quad & \alpha \\ \text{subject to}\; & f_k(x) - \alpha d_k \leq r_k, k = 1, \ldots, p \\ & x \in X, \end{aligned} \tag{18.17}$$

and produces a weakly efficient point (Gembicki and Haimes [166]).

The Pascoletti and Serafini Approach This is a more general approach with an unrestricted search direction $d \in \mathbb{R}^p$ and an auxiliary vector variable $l \in \mathbb{R}^p$:

$$\begin{aligned} \max \quad & \alpha \\ \text{subject to}\; & f_k(x) - \alpha d_k + l_k = r_k,\; k = 1, \ldots, p \\ & l \geqq 0 \\ & x \in X. \end{aligned} \tag{18.18}$$

Theorem 25 (**[295]**).

1. *If* $(\hat{\alpha}, \hat{x}, \hat{l})$ *is a finite optimal solution of problem (18.18) then* $\hat{x} \in X_{wE}$.
2. *If* $(\hat{\alpha}, \hat{x}, \hat{l})$ *is a unique finite optimal solution of problem (18.18) then* $\hat{x} \in X_E$.

The Reference Direction Approach Independently of [166], Korhonen and Wallenius [230] propose an approach analogous to the goal-attainment method and refer to it as a generalized goal programming model. With the inclusion of nonnegative slack variables they arrive at problem (18.18) with a feasible reference point $r \in Y$ and a search direction $d \leq 0$. Additionally, to move on the Pareto set they parametrize the reference point r or the search direction d, and obtain:

$$\begin{aligned} &\max \quad \alpha \\ &\text{subject to } f_k(x) - \alpha(d_k + \alpha_1 \Delta d_k) + l_k = (r_k + \alpha_2 \Delta r_k),\ k = 1, \ldots, p \\ &l \geqq 0 \\ &x \in X, \end{aligned} \tag{18.19}$$

where $\Delta d = [\Delta d_1, \ldots, \Delta d_p]$ and $\Delta r = [\Delta r_1, \ldots, \Delta r_p]$ are auxiliary vectors used for the parametrization with the parameters $\alpha_1 \geq 0$ and $\alpha_2 \geq 0$. The vector Δr is called the reference direction and problem (18.19) is called the reference direction approach (see also [222, 231]).

The Modified Pascoletti and Serafini Approach Since a solution to problem (18.18) may not be finite, the following modification has been developed:

$$\begin{aligned} &\text{lex max} \quad (\alpha, \|l\|_p) \\ &\text{subject to } f_k(x) - \alpha d_k + l_k = r_k, k = 1, \ldots, p \\ &l \geqq 0 \\ &x \in X. \end{aligned} \tag{18.20}$$

Theorem 26 (**[328]**). *Let* $r \in Y + \mathbb{R}^p_{\geqq}$, $d \in \mathbb{R}^p \setminus \mathbb{R}^p_{\geqq}$ *and* $1 \leq p \leq \infty$. *Then problem (18.20) has a finite optimal solution* $(\hat{\alpha}, \hat{x}, \hat{l})$ *where* $\hat{x} \in X_E$.

The Normal Boundary Intersection Approach The approach by Das and Dennis [92] is motivated by the interest in obtaining an evenly distributed set of Pareto solutions. Let y^I be the ideal point and let Φ be the $p \times p$ matrix the k-th column of which is given by $f(x^k) - y^I$, where x^k is a global minimizer of the objective function $f_k, k = 1, \ldots, p$. The set of points in $\mathbb{R}^p$ that are convex combinations of $f(x^k) - y^I$, i.e., $\{\Phi\lambda : \lambda \in \mathbb{R}^p_{\geq}, \sum_{k=1}^p \lambda_k = 1\}$, is referred to as the convex hull of the individual global minima of the objective functions (CHIM). Let a search direction be given by the unit normal, denoted by ξ, to the CHIM pointing toward the origin. Consider the following SOP:

$$\begin{aligned} \max \quad & \alpha \\ \text{subject to}\ & f(x) - \alpha\xi - \Phi\lambda = y^I \\ & x \in X. \end{aligned} \tag{18.21}$$

When the approach is iteratively applied to convex MOPs with evenly distributed coefficients λ_k of the convex combination, the authors claim and demonstrate by examples that an evenly distributed set of Pareto solutions is produced. However, the efficiency of the optimal solutions of problem (18.21) is not guaranteed. A scalarization using the CHIM and a direction normal to it is developed by Ismail-Yahaya and Messac [208]. Using the concept of goal attainment of Gembicki [166], Shukla [340] modifies problem (18.21) to guarantee that its optimal solution is weakly efficient for the MOP.

18.5.1.11 Gauge-Based Approaches

Assume without loss of generality that $0 \in Y + \mathbb{R}^p_{\geqq}$. Let B be a polytope in $\mathbb{R}^p$ containing the origin in its interior and let $y \in \mathbb{R}^p$. The polyhedral gauge $\gamma : \mathbb{R}^p \to \mathbb{R}$ of y is defined as $\gamma(y) = \min\{\alpha \geq 0 : y \in \alpha B\}$. The vectors defined by the extreme points of the unit ball B of γ are called fundamental vectors and are denoted by ν.

Gauges are used to measure the distance in the objective space either in the interior or the exterior of the outcome set. The former leads to the inner scalarization while the latter is related to the outer scalarization.

The Inner Gauge-Based Approach Consider the gauge problem

$$\begin{aligned} \max \quad & \gamma(y) \\ \text{subject to}\ & y = f(x) \\ & y \in Y \cap (-\mathbb{R}^p_{\geqq}). \end{aligned} \tag{18.22}$$

Theorem 27 ([227]). *Let the set Y be strictly* $\operatorname{int}\mathbb{R}^p_{\geqq}$*-convex, i.e., $Y + \operatorname{int}\mathbb{R}^p_{\geqq}$ is strictly convex. If $\hat{x}$ is an optimal solution of problem (18.22) then $\hat{x} \in X_{pE}$.*

The Outer Gauge-Based Approach Let B be the unit ball of a polyhedral gauge γ such that the fundamental vectors $\nu^1, \ldots, \nu^t$ of $B \cap (-\mathbb{R}^p_{\geqq})$ satisfy $Y \cap (-\mathbb{R}^p_{\geqq}) \subseteq \{y \leqq 0 : y \geqq \sum_{i=1}^t \lambda_i \nu^i, \sum_{i=1}^t \lambda_i = 1, \lambda_i \geq 0\}$ and consider the problem

$$\begin{aligned} \max \quad & \alpha \\ \text{subject to} \quad & \alpha v^i - f(x) \geqq 0, \ i = 1, \ldots, t \\ & \alpha \geq 0 \\ & x \in X. \end{aligned} \tag{18.23}$$

Theorem 28 ([227]). *Let the set Y be strictly* $\operatorname{int} \mathbb{R}^p_{\geqq}$*-convex. If* $(\hat{\alpha}, \hat{x})$ *is an optimal solution of problem (18.23) then* $\hat{x} \in X_{pE}$.

A combination of each of the gauge-based approaches together with the weighted l_∞ norm method results in two other approaches generating weakly efficient solutions of nonconvex MOPs (Klamroth et al. [227]).

18.5.1.12 Composite and Other Approaches

In order to achieve certain properties of scalarizations, some authors develop composite approaches involving a combination of methods. The hybrid method is composed of the weighted sum and the ϵ-constraint approaches (Wendell and Li [383]). More generally, an increasing scalarizing function can be combined with constraints on some or all objective functions (Soland [344]). Dual approaches include the ε-constraint approach coupled with Lagrangian duality (Chankong and Haimes [72]) or generalized Lagrangian duality (TenHuisen and Wiecek [358, 359]).

A very general scalarization method using continuous functionals has been proposed by Gerth and Weidner [168] and Ester and Tröltzsch [136]. (Weakly, properly) nondominated points of (nonconvex) MOPs are characterized through the existence of functionals with certain properties.

18.5.2 Approaches Based on Non-Pareto Optimality

In contrast to scalarizing approaches discussed in Sect. 18.5.1, methods using other optimality concepts first replace the Pareto relation with another suitable relation. Even that they may then scalarize the MOP that is now governed by the new relation, they can be seen as nonscalarizing methods because they do not explicitly employ a scalarizing function. In effect, there are usually strong links to efficiency. In the following sections we summarize these approaches.

18.5.2.1 The Lexicographic Approach

The lexicographic approach makes use of the lexicographic relation and assumes a ranking of the objective functions according to their importance. Let π be a permutation of $\{1, \ldots, p\}$ and assume that $f_{\pi(k)}$ is more important than $f_{\pi(k+1)}$, $k = 1, \ldots, p-1$. Let $f^\pi : \mathbb{R}^n \to \mathbb{R}^p$ be $(f_{\pi(1)}, \ldots, f_{\pi(p)})$. The lexicographic problem is formulated as

$$\begin{aligned} \text{lex min} \quad & f^\pi(x) \\ \text{subject to } & x \in X. \end{aligned} \tag{18.24}$$

A feasible solution $\hat{x} \in X$ is said to be optimal for problem (18.24) if there is no $x \in X$ such that $f(x) \succ_{lex} f(\hat{x})$. Optimal solutions of problem (18.24) are called lexicographically optimal. This problem is solved as follows. Let $X_0^\pi = X$ and define recursively $X_k^\pi := \{\hat{x} \in X_{k-1}^\pi : f_{\pi(k)}(\hat{x}) = \min f_{\pi(k)}(x), x \in X_{k-1}^\pi\}$ for $k = 1, \ldots, p$.

Theorem 29. *Let π be a permutation of $\{1, \ldots, p\}$.*

1. *If $X_k^\pi = \{\hat{x}\}$ is a singleton then $\hat{x}$ is an optimal solution of problem (18.24) and $\hat{x} \in X_E$.*
2. *All elements of X_p^π are optimal solutions of problem (18.24) and $X_p^\pi \subset X_E$.*

Note that the inclusion $\bigcup_\pi X_p^\pi \subset X_E$ is usually strict.

18.5.2.2 The Max-Ordering Approach

The max-ordering approach makes use of the max-ordering relation and does only consider the objective function f_k which has the highest (worst) value. The max-ordering problem is formulated as

$$\begin{aligned} \min \max_{k=1,\ldots,p} \quad & f_k(x) \\ \text{subject to } & x \in X. \end{aligned} \tag{18.25}$$

A feasible solution $\hat{x} \in X$ is said to be optimal for problem (18.25) if there is no $x \in X$ such that $f(x) \succ_{MO} f(\hat{x})$. Optimal solutions of problem (18.25) are called max-ordering optimal.

An optimal solution of problem (18.25) is weakly efficient. Furthermore, if $X_E \neq \emptyset$ and problem (18.25) has an optimal solution then there exists an optimal solution of problem (18.25) which is efficient, and consequently a unique optimal solution of problem (18.25) is efficient.

It is possible to include a weight vector $\lambda \in \mathbb{R}^p_{\geq}$ so that the weighted max-ordering problem becomes

$$\begin{aligned} \min \max_{k=1,\ldots,p} \quad & \lambda_k f_k(x) \\ \text{subject to } & x \in X. \end{aligned} \tag{18.26}$$

Theorem 30 ([239]). *Let $Y \subset \mathbb{R}^p_>$.*

1. *Let $\lambda \in \mathbb{R}^p_\geq$. If $\hat{x} \in X$ is an optimal solution of problem (18.26) then $\hat{x} \in X_{wE}$.*
2. *If $\hat{x} \in X_{wE}$ there exists a $\lambda \in \mathbb{R}^p_>$ such that $\hat{x}$ is an optimal solution of problem (18.26)*

A result similar to Theorem 30 can be proved for efficient solutions when an additional SOP to minimize $\sum_{k=1}^{p} f_k(x)$ subject to $\lambda_k f_k(x) \leq y^*$, where y^* is the optimal value of problem (18.26), is solved.

The max-ordering approach plays an important role in robust optimization, where each objective function f_k is interpreted as the objective for making a decision in a scenario k, see Kouvelis and Yu [239].

18.5.2.3 The Lexicographic Max-Ordering Approach

The idea of the max-ordering approach can be extended to consider the second worst, third worst, etc., objective. That means, for $x \in X$ one reorders the components of $f(x)$ in nonincreasing order. For $y \in \mathbb{R}^p$ let $\Theta(y) = (\theta_1(y), \ldots, \theta_p(y))$ be a permutation of y such that $\theta_1(y) \geq \ldots \geq \theta_p(y)$. The lexicographic max-ordering approach then seeks to lexicographically minimize $\Theta(f(x))$ over the feasible set X. This corresponds to seeking preferred outcomes according to the lexicographic max-ordering relation $y^1 \succeq_{lex-MO} y^2$ if and only if $\theta(y^1) \succeq_{lex} \theta(y^2)$.

It is easy to see that an optimal solution of the lexicographic max-ordering problem is also an optimal solution of the max-ordering problem and efficient for the MOP, which strengthens the corresponding result for the max-ordering approach. When a weight vector $\lambda \in \mathbb{R}^p_>$ is introduced (again assuming that $Y \subset \mathbb{R}^p_>$), the problem becomes

$$\begin{aligned} \text{lex min} \quad & \big(\lambda_1\theta_1(f(x)), \ldots, \lambda_p\theta_p(f(x))\big) \\ & \text{subject to } x \in X. \end{aligned} \tag{18.27}$$

Theorem 31 ([113]). *A point $\hat{x} \in X$ is efficient if and only if there exists a $\lambda \in \mathbb{R}^p_>$ such that $\hat{x}$ is an optimal solution of (18.27).*

For problems with a special structure further results can be obtained. For convex problems, Behringer [25] shows that problem (18.27) can be solved through solving a sequence of max-ordering problems. A solution of problem (18.27) is also called a nucleolar solution by Marchi and Oviedo [262]. Ehrgott and Skriver [120] give an algorithm for the bicriteria discrete case and Sankaran [320] provides one for the convex case.

18.5.2.4 The Equitability Approach

The concept of equitability was first known as the generalized Lorenz dominance [264] and its formalization in the context of multiobjective programming is accomplished by Kostreva and Ogryczak [233] and Kostreva et al. [238]. This concept is applicable if the objective functions are measured on a common scale (or normalized to a common scale).

While the Pareto preference assumes a binary relation between outcome vectors that is reflexive, transitive and strictly monotone ($y - \epsilon e_i$ is preferred to y for $\epsilon > 0$ and $k = 1, \ldots, p$, where $e_i \in \mathbb{R}^p$ is the i-th unit vector), equitability makes two further assumptions. It is assumed that there is indifference between y and y' if there is a permutation π such that $y' = (y_{\pi(1)}, \ldots, y_{\pi(p)})$. The second assumption is the Pigou-Dalton principle of transfers: Let $y \in \mathbb{R}^p$ and $y_{k'} > y_{k''}$ then $y - \epsilon e_{k'} + \epsilon e_{k''}$ is preferred to y for $0 < \epsilon < y_{k'} - y_{k''}$. A preference relation $\succeq_{equi}$ satisfying these assumptions is derived from a variable domination structure as mentioned in Sect. 18.2.3.

In order to obtain equitably efficient solutions of the MOP one proceeds as follows. For $y \in \mathbb{R}^p$ let $\Theta(y) = (\theta_1(y), \ldots, \theta_p(y))$ be as in Sect. 18.5.2.3. Next, using this vector of ordered outcomes define the cumulative ordered outcome vector $\bar{\Theta}(y) = (\bar{\theta}_1(y), \ldots, \bar{\theta}_p(y))$, where

$$\bar{\theta}_k(y) = \sum_{i=1}^{k} \theta_i(y).$$

The equitability preference can then be defined by $y^1 \succeq_{equi} y^2$ if and only if $\bar{\Theta}(y^1) \succeq_{\mathbb{R}^p_{\geqq}} \bar{\Theta}(y^2)$. An equitable MOP can be written as

$$\begin{aligned} \min \quad & \left(\bar{\theta}_1(f(x)), \ldots, \bar{\theta}_p(f(x))\right) \\ & \text{subject to } x \in X. \end{aligned} \tag{18.28}$$

The relationship between equitably efficient solutions and efficient solutions is provided by the following theorem.

Theorem 32. *An efficient solution of problem (18.28) is an equitably efficient solution of the MOP.*

To generate equitably efficient solutions, scalarization using strictly convex functions can be applied. Kostreva and Ogryczak [233] show that any optimal solution of $\min_{x \in X} \sum_{k=1}^{p} s(f_k(x))$, where $s : \mathbb{R} \to \mathbb{R}$ is strictly convex and increasing, is equitably efficient. Ogryczak and his co-authors apply the equitability approach to portfolio optimization [291], location [290, 294], telecommunication [293], and second order stochastic dominance [292].

18.5.3 Descent Methods

Descent methods are well established tools of nonlinear programming for computing optimal solutions to SOPs. During the last decade researchers have undertaken efforts to develop descent methods for solving MOPs. The methods compute points satisfying the Kuhn-Tucker conditions for efficiency without making use of scalarizing approaches. Their performance, being supported with convergence proofs, mimics the performance of their counterparts in nonlinear programming.

Fliege and Svaiter [147] seem to be the first to propose a steepest descent algorithm for unconstrained MOPs and a feasible descent method, as an extension of Zoutendijk's method of feasible directions, for the constrained case. The steepest descent direction is derived from the gradient of the objective functions of the MOP. Graña Drummond and Svaiter [106] extend this method for the unconstrained case with convex cones. Both of these methods are later used to prove convergence in the steepest descent algorithm by Bento et al. [43] that uses Armijo's rule to generate a well-defined sequence of iterates that, under suitable assumptions, converges to a critical point of the MOP. Vieira et al. [373] combine the method of Fliege and Svaiter [147] with a new golden section line search algorithm that converges to a point at which the necessary first-order KT conditions for efficient solutions of the MOP are satisfied. Other steepest descent methods include the Multiple-Gradient Descent Algorithm (MGDA) by Désidéri [104], and the more recent method by Chuong and Yao [80].

A projected gradient methods is developed by Graña Drummond and Iusem [105] for convex and nonconvex MOPs with convex cones generalizing the Pareto cone. Fukuda and Graña Drummond [153] later extend the results in this earlier paper by showing that every sequence of iterates converges to a weakly efficient point for MOP, under suitable assumptions. Meantime, Fliege et al. [149] present Newton's method for unconstrained MOPs. An auxiliary quadratic MOP, that carries the gradient and Hessian information of the objective functions of the MOP, is solved with the max-ordering approach. Its solution provides the Newton direction along which the search is performed. Recently, Fukuda and Graña Drummond [154] propose an inexact projected gradient method for constrained MOPs as an extension of the algorithms in [105] and [153].

18.5.4 Set-Oriented Methods

In mathematical programming, the concept of an optimization algorithm has relied on the computation of a series of iterates (or points) converging to an optimal solution of the mathematical program. Many years ago this concept was carried over to multiobjective programming giving rise to numerous scalarization approaches. The context of multiobjective programming and the desire to capture the solution set have motivated researchers to come up with algorithms working iteratively on sets rather than points.

Set-oriented methods have been designed specifically for MOPs. In contrast to all previously presented approaches, they find a solution set of the MOP without using scalarizing functions or other optimality concepts. They are developed with the philosophy of solving the MOP by capturing part of or the whole solution set rather than approximating it or sequentially computing its individual points. They resemble approximation methods reviewed in Sect. 18.6 because they provide a set representing or approximating the true solution set. However, since they originate from global optimization they have been placed in its own category of methods.

18.5.4.1 The Balance and Level Set Approaches

Galperin [160] introduces the balance set approach. The approach is based on sets of points with a bounded deviation from global minima of the individual objective functions. For $k = 1, \dots, p$ define the sets

$$X_k(\eta) := \{x \in X : f_k(x) - y^I_k \le \eta_k\}. \tag{18.29}$$

Then $\eta \in \mathbb{R}^p$ is called a balance point if the intersection $\cap_{k=1}^p X_k(\eta) \neq \emptyset$ but for any $\eta', \eta' \le \eta$, the intersection $\cap_{k=1}^p X_k(\eta') = \emptyset$. The set of all balance points for the MOP is called the balance set denoted by Υ. Galperin and Wiecek [161] demonstrate the applicability of this approach on example problems. Ehrgott et al. [121] show that the balance set is equal to the Pareto set translated by the ideal point: $\Upsilon = Y_N - y^I$.

It is possible to require all η_k to be equal to one another in (18.29). The smallest $\eta \in \mathbb{R}$ such that the intersection $\cap_{k=1}^p X_k(\eta) \neq \emptyset$ is then called the balance number which, according to Ehrgott and Galperin [116], can be found by solving the problem

$$\begin{aligned} \min \max_{k=1,\dots,p} \quad & (f_k(x) - y^I_k) \\ & \text{subject to } x \in X. \end{aligned} \tag{18.30}$$

The definition of balance number can be extended to include weights, i.e., $\eta_k = \eta\lambda_k$ in (18.29). For positive weights $\lambda \in \mathbb{R}^P_>$ the smallest η with the nonempty intersection of $X_k(\eta)$ (called the apportioned balance number $\eta(\lambda)$) can again be computed via a min-max problem. Ehrgott [115] compares the sets Υ and $\{\eta(\lambda)\lambda : \sum_{k=1}^p \lambda_k = 1, \lambda \in \mathbb{R}^p_>\}$ and gives conditions for them being equal.

The balance space approach is closely related to the level set approach. The level set of objective function f_k with respect to $\bar{x} \in X$ is

$$L^k_{\le}(\bar{x}) := \{x \in X : f_k(x) \le f_k(\bar{x})\},$$

the strict level set is

$$L^k_{<}(\bar{x}) := \{x \in X : f_k(x) < f_k(\bar{x})\},$$

and the level curve is

$$L^k_{=}(x) := \{x \in X : f_k(x) = f_k(\bar{x})\}.$$

Therefore the sets $X_k(\eta)$ used in the balance space approach are level sets of f_k with respect to levels $y^I_k + \eta_k$. The main result on level sets is the following theorem.

Theorem 33 ([121]).

1. *A point $\hat{x} \in X$ is weakly efficient if and only if $\cap_{k=1}^{p} L^k_{<}(\hat{x}) = \emptyset$.*
2. *A point $\hat{x} \in X$ is efficient if and only if $\cap_{k=1}^{p} L^k_{\leq}(\hat{x}) = \cap_{k=1}^{p} L^k_{=}(\hat{x})$.*
3. *A point $\hat{x} \in X$ is strictly efficient if and only if $\cap_{k=1}^{p} L^k_{\leq}(\hat{x}) = \{\hat{x}\}$.*

This geometric characterization of efficiency can be exploited when dealing with problems that have a geometric structure, e.g., location problems as in Ehrgott et al. [122].

18.5.4.2 The ϵ-Efficiency Approach

Let $\{\epsilon^\tau\}$ be a sequence in $\mathbb{R}^p$ with $\lim_{\tau \to \infty} \epsilon^\tau = 0$. Lemaire [244] introduces a notion of a sequence of auxiliary MOPs converging to the original MOP and studies properties of the sequence of ϵ^τ-efficient sets of these problems with respect to the efficient set X_E of the original problem. In particular, he shows that every weakly efficient point of the MOP can be obtained as a limit of a sequence of ϵ^τ-efficient points of the auxiliary problems.

18.5.4.3 Continuation Methods

Continuation (homotopy or path-following) methods had originally been developed for computing approximate solutions of a system of parameterized nonlinear equations. They have been applied for solving a variety of nonlinear SOPs and later MOPs.

Guddat et al. [175] introduce these methods to multiobjective programming. They reduce the weighted-sum scalarization of the MOP (being a multiparametric problem due to p weights) to a sequence of one-parametric SOPs and develop theory and algorithms of a continuation approach for computing Karush-Kuhn-Tucker (KKT) points of the SOPs that are efficient for the MOP. Similar work with different scalarization approaches is continued by Guddat et al. [176]. Rakowska et al. [308, 309] solve the weighted-sum scalarization of a biobjective structural optimization problem by tracing its efficient set using a software package HOMPACK containing several homotopy curve tracking algorithms.

An in-depth study of continuation methods in multiobjective programming is provided by Hillermeier [198]. Taking up the viewpoint of differential geometry, he develops a generalized, multidimensional continuation (homotopy) method for MOPs that is based on the characterization of Pareto optimal points as KKT points of the SOPs. The solutions to the KKT system with decisions, objectives, and weights as variables are viewed as the elements of the null space of the associated nonlinear mapping, that under certain assumptions can be characterized as a submanifold of dimension $p - 1$. Its local parametrization by means of a graph representation via the tangent-normal space splitting can be exploited for the development of numerical procedures that allow the local exploration of this manifold and its solutions for the MOP.

A path-following algorithm for computing critical points of box-constrained nonconvex MOPs is designed by Miglierina et al. [278]. The authors prove that the limit points of the solutions of a dynamical system associated with the MOP are the critical points of the MOP.

18.5.4.4 Covering Methods

Covering methods belong to another class of set-oriented methods developed during the last decade. They include techniques based on stochastic principles, subdivision algorithms of the initial feasible set of the MOP, or continuation methods also called recover techniques.

Schäfler et al. [326] develop a stochastic covering algorithm for unconstrained MOPs. They use the gradients of the objective functions to formulate a quadratic SOP that implies a function whose zeros fulfill the Kuhn-Tucker conditions for efficiency and therefore identify efficient points. The function yields a stochastic differential equation whose numerical solution leads to a numerical cover of all or a large number of efficient points.

Covering methods with subdivision techniques globally solve MOPs that do not possess any specific structure or properties. These techniques start with a compact subset of the feasible set of the MOP and generate outer approximations of the efficient set that create a tight box-cover of this set. In effect, the efficient set is covered with n-dimensional small boxes. In practice, however, these methods are restricted to moderate dimensions of the feasible space. Dellnitz et al. [102] make use of the differential equation of Schäfler et al. [326] but keep their approach deterministic. The discretized differential equation yields a discrete dynamical system that possesses the efficient set as an attractor. An application of subdivision techniques results in three covering algorithms for unconstrained MOPs. Jahn [211] combines a subdivision technique of Dellnitz et al. [102] with a search strategy to come up with a multiobjective search algorithm with subdivision technique (MOSAST) for constrained MOPs.

The covering method of Schütze et al. [331] uses a local generalized continuation algorithm. Starting with a collection of n-dimensional boxes where every box contains at least one point satisfying the Kuhn-Tucker conditions for efficiency, the algorithm successively extends the collection by new boxes that also contain points satisfying these conditions. An overview of covering methods is recently provided by Schütze et al. [333].

18.6 Approximation of the Pareto Set

It is of interest to design methods for obtaining a complete description of the Pareto and efficient sets since solving MOPs is understood as finding these sets. An exact description might be available analytically as a closed-form formula, numerically as a set of points, or in a mixed form as a parameterized set of points.

Unfortunately, for a majority of MOPs it is not easy to obtain an exact description of the Pareto set that typically includes an infinite number of points. Even when it is theoretically possible to find these points exactly, this may be computationally challenging and expensive and is therefore usually abandoned. For some other problems finding elements of the Pareto set is even impossible due to the numerical complexity of the resulting optimization problems.

Since the exact solution set is very often not attainable, an approximated description of this set becomes an appealing alternative. Approximating approaches have been developed for the following purposes: to represent the Pareto set when this set is numerically available (linear or convex MOPs); to approximate the Pareto set when some but not all Pareto points are numerically available (nonlinear MOPs); and to approximate the Pareto set when Pareto points are not numerically available (computationally expensive MOPs).

For any MOP, the approximation requires less effort and often may be accurate enough to play the role of the solution set. Additionally, if the approximation represents this set in a simplified, structured, and understandable way, it may effectively support the decision-maker.

Even though we refer to all following methods as approximation, we distinguish between discrete representations (collections of points) of the Pareto set and approximated Pareto sets that use some sort of approximating structure in addition to the original points. Discrete representations present a finite number of solutions that are available explicitly, whereas approximations do not limit the number of solutions that are only implicitly available through an approximating structure. Additionally, all the points in a discrete representation are optimal for the MOP; this is not necessarily true for the points inferred from an approximated Pareto set.

We refer to Ruzika and Wiecek [317] for a review of exact representation and approximation methods, and a discussion of their quality and measures for evaluating it. In this chapter, we specifically focus on quality measures for representation approaches.

18.6.1 Quality Measures for Representations

Within the past 15 years, many authors have suggested quality measures for providing a "good" representation of the Pareto set. The meaning of "good", here, is ambiguous because no definite consensus has been reached in the mathematical and operations research community on what qualities a good representation of the Pareto set should possess. In this section, we present a brief review of the measures proposed in the literature. We use the symbols $\overline{Y}_N$ and $\overline{X}_E$ to denote the actually computed Pareto set Y_N and efficient set X_E, respectively. For a detailed survey on this subject we refer to Faulkenberg and Wiecek [142].

The measures can be sorted into three main groups as suggested by Sayin [323]: measures of *cardinality*, *coverage*, and *spacing*. Cardinality refers to the number of points in a representation. In general, one desires enough points to fully represent the outcome set Y. Measures of coverage seek to ensure that all regions of the outcome set are represented, that is, no portion of the outcome set should be neglected. Measures of spacing quantify the distance between points in the representation. Typically, a representation is expected to have uniform, or equidistant, spacing, so that all portions of the outcome set are represented to an equal degree. In some papers, measures are defined for arbitrary sets. In these cases, we present the measures in the context of the Pareto set.

18.6.1.1 Measures of Cardinality

Measures of cardinality count the number of efficient (or Pareto) points in the representation. Clearly, two straightforward candidates for measures of cardinality are the size of the generated Pareto set, $|\overline{Y}_N|$, and the size of the generated efficient set, $|\overline{X}_E|$. Van Veldhuizen [371] proposes the former measure as "overall nondominated vector generation", while Sayin [323] proposes the latter. In this group we also have the measure proposed by Wu and Azarm [397] called the "number of distinct choices". This measure is similar to the previous two but takes additional preferences into account: Pareto outcomes within a certain distance of each other are counted as a single point.

Since the cardinality of a discrete representation of the Pareto set is easily controlled, this category of measures is less critical than the following two. In general, the cardinality of a representation should be minimized while still maintaining good coverage and spacing.

18.6.1.2 Measures of Coverage

Measures of coverage face the challenge of trying to assess an unknown set because, in general, the true Pareto set is unknown *a priori*. However, one has to ensure that no region of Y_N is neglected. Because of this, the coverage of a discrete representation is typically maximized, and unless otherwise noted, the following measures should be maximized as well.

Czyżak and Jaszkiewicz [89] introduce the measure "D1", that is defined as a weighted average of the distances between a point in Y_N and the closest point in the representation, and the measure "D2", that is defined as the largest weighted distance between a point in Y_N and the closest point in the representation. Since these distances should be as small as possible, both of these measures should be minimized.

Zitzler and Thiele [424] propose the "S-measure" to determine the size of the region dominated by $\overline{Y}_N$. Each point $y \in \overline{Y}_N$ dominates a (hyper)cube with one corner at y and another at $y^{\max}$ where $y^{\max} = (f_1^{\max}, \ldots, f_p^{\max})$ and $f_i^{\max} = \max_{x \in X} f_i(x)$. Thus, the region dominated by $\overline{Y}_N$ is found by taking the union of these cubes for all $y \in \overline{Y}_N$. The value of the S-measure is the volume of this union. Zitzler [423] also defines a measure called "M_3" to determine the overall range of the representation which is calculated as an average of the ranges of the objective functions.

Sayin [323] suggests the measure "coverage" which determines the maximum distance, ϵ, between a point in Y_N and its closest neighbor in the representation. Because it is expected that every point in Y_N be represented in the discrete representation, this measure is minimized rather than maximized.

Wu and Azarm [397] introduce three measures, "overall Pareto spread," "i^{th} Pareto spread," and "hyperarea difference." The first two measures calculate the range of the entire representation and of each individual criterion, respectively. The third measure is a slight variation on the S-measure of Zitzler and Thiele [424] and calculates the difference (in terms of volume) between the portions of the objective space which are dominated by the true Pareto set and a given representation of the Pareto set.

Meng et al. [268] propose a measure of coverage called "extension". Let $\{y^1, \ldots, y^p\}$ be a set of reference outcomes where $y^i = (L_1, \ldots, L_{i-1}, U_i, L_{i+1}, \ldots, L_p)$ and $U_i = max_{x \in X_E} f_i(x)$ and $L_i = min_{x \in X_E} f_i(x)$. The extension measures an average distance between the points in $\overline{Y}_N$ and the reference outcomes. A small value of the extension is preferred to a larger value because the latter could indicate that the representation is mainly in the center of the true Pareto set with the outskirts being neglected.

Zitzler et al. [425] suggest a measure of the "outer diameter" of a discrete representation which measures the maximum weighted range over all the objective functions.

18.6.1.3 Measures of Spacing

A discrete representation of the Pareto set with equally-spaced Pareto points is desired so that each region of the true Pareto set is represented to an equal degree.

However, having equidistant Pareto points does not guarantee a good coverage as well, so measures of spacing should always be used in conjunction with a coverage measure.

Schott [330] proposes a measure for bicriteria problems called "spacing" which takes the standard deviation of the distances between nearest-neighbor points and for which small values are desired. Zitzler [423] proposes the "M_2" measure that calculates the average cardinality of the set of points in $\overline{Y}_N$ whose distance from each other is greater than a fixed value. This measure gives a sense of the number of redundancies that are contained in $\overline{Y}_N$.

Sayin [323] proposes the measure "uniformity" which is defined as the minimum distance between any two distinct points in $\overline{Y}_N$. A measure called "cluster", which measures the average size of a redundant cluster of points (with respect to a parameter) in $\overline{Y}_N$, is suggested by Wu and Azarm [397]. Messac and Mattson [272] present a measure of spacing called "evenness". For each point y in $\overline{Y}_N$, two (hyper)spheres are constructed: the smallest and the largest spheres that can be formed between y and any other point in the set such that no other points are within the spheres. The evenness is calculated as the ratio between the standard deviation and the mean of the set of minimum and maximum diameters for each point in $\overline{Y}_N$.

A measure called "uniformity" inspired by wavelet analysis is developed by Meng et al. [268] for comparing two different representations of Y_N. Collette and Siarry [85] define two different spacing measures for bicriteria problems: "spacing", which is a modification of Schott's measure [330], and the "hole relative size" measure that gives the ratio of the largest gap between two adjacent points to an average gap in $\overline{Y}_N$.

18.6.1.4 Hybrid Measures

Several authors propose measures which overlap the above three categories. Deb et al. [99] suggest the "Δ- measure" for bicriteria problems which takes into account both the spacing between generated Pareto points and the coverage of the true Pareto set by the generated representation. This measure calculates the distance between each point and its nearest neighbors (a spacing-type measure) as well as the distance between the individual objective minima and their respective single nearest neighbor (a coverage-type measure).

Leung and Wang [245] suggest the "U-measure" which measures both coverage and spacing, similar to Deb's measure [99]. This measure calculates the average deviation from the ideal point so that a small U-measure indicates a representation that is close to equidistant and covers the entire Pareto set.

Ideas from the field of information theory are used by Farhang-Mehr and Azarm [141] who introduce a measure called "entropy" that assesses all three of the quality categories: cardinality, coverage, and spacing. A high entropy value is desired because a set with high entropy maximizes coverage and minimizes redundancies for a given cardinality.

Bozkurt et al. [60] introduce a so-called “integrated preference functional” measure to evaluate the quality of a finite approximate solution set for the MOP, that is based on the weighted Tchebycheff norm as the underlying value function. Although computationally restricted to BOPs, this measure can be extended conceptually to the case of more than two objectives.

18.6.2 Representation and Approximation Approaches

Representation and approximation approaches typically employ an iterative method to produce points or objects approximating the Pareto set. A majority of approaches employ a scalarization technique as an integrated component of the resulting algorithm. The scalarization is used to generate Pareto points that either become the representation or are used to construct other approximating objects such as polyhedral sets and functions, curves, and rectangles.

Discrete representation approaches produce point-wise approximations of the Pareto set. Some researchers have also integrated quality measures into algorithms for generating discrete representations that meet a prespecified quality criterion.

18.6.2.1 Representation for BOPs

Approaches for biobjective programs (BOPs) are proposed by Jahn and Merkel [212] who use the ε-constraint scalarization and a tunneling technique, by Helbig [195] who discretizes the convex hull if individual minima of the objective functions and uses the max-ordering method, and by Zhang and Gao [417] who use the weighted-Chebyshev scalarization to find a new Pareto point in the direction perpendicular to the weight vector used for the generation of the current point and with a predefined step length to take care of spacing. Faulkenberg and Wiecek [143] develop two methods for generating discrete representations with equidistant points for BOPs with solution sets determined by convex, polyhedral cones. The Constraint Controlled-Spacing method is based on the ε-constraint method with an additional constraint to control the spacing of generated points. The Bilevel Controlled-Spacing method has a bilevel structure with the lower-level generating the nondominated points and the upper-level controlling the spacing, and is extended to MOPs. A homotopy-based approach is proposed by Pereyra [300] for unconstrained BIPs. The approach makes use of the Newton’s method to solve the Kuhn-Tucker necessary conditions for efficiency. A nonlinear constraint is added to yield equally spaced Pareto points. The approach is extended to the constrained case by Pereyra et al. [301].

18.6.2.2 Representation for MOPs

As early as in 1980 filtering techniques were proposed to produce discrete representative subsets of the outcome sets of MOPs. Steuer and Harris [352] suggest using a forward and reverse interactive filtering scheme to produce a representative subset of the Pareto extreme points for linear MOPs. For the same class of problems, Morse [281] proposes a filtering method involving cluster analysis for reducing redundancy in the Pareto sets, with redundancy measuring the decision maker's indifference between two Pareto points. Mattson et al. [267] develop a Smart Pareto filter to produce representations with good cardinality and complete coverage which emphasize areas with high tradeoffs more than areas with low or insignificant tradeoffs.

Representation by means of a finite set of elements is studied by Nefedov [285] with special attention given to convergence of the set approximating the Pareto set. Armann [15] develops a method for choosing parameters in the hybrid weighted-sum and ε-constraint scalarization for general MOPs. Given the desired number of points in the representation, an integer program is solved to determine the values of ε to be used in the hybrid scalarization so that in the resulting representation, the distance between neighboring points is maximized. Helbig [194] presents an approach to approximate the nondominated set of general MOPs with convex cones.

A global shooting procedure to find a representation of the Pareto set for problems with compact outcome sets is proposed by Benson and Sayin [41] while an approach producing representative subsets of the Pareto set for linear MOPs is introduced by Sayin [323]. A method based on an interior point algorithm for convex quadratic MOPs is given by Fliege and Heseler [146]. Churkina [81] proposes a target-level method using an infinite set of reference points and the l_∞-norm.

Sayin [325] develops a method for linear MOPs to produce a representation with a given target coverage value or the maximum coverage possible given a target cardinality when the set of efficient faces is known a priori.

Buchanan and Gardiner [62] perform a comparative study of two versions of the weighted Chebyshev method [351], one using the ideal point as a reference point and the other using the nadir point. When choosing weights from the uniform distribution, discrete representations produced using the nadir outcome as the reference point have better coverage than those produced with the ideal point.

Messac and Mattson [271] show that the earlier developed method of Physical Programming [270] can generate well spaced Pareto points. Messac et al. [273] and Messac and Mattson [272] develop the Normal Constraint method for producing representations with good coverage. Shao and Ehrgott [334] combine the global shooting procedure of Benson and Sayin [41] and the Normal Boundary Intersection method [92] to produce a revised Normal Boundary Intersection method for linear MOPs. The advantages of Physical Programming [270], the Normal Boundary Intersection [92] and Normal Constraint [272, 273] methods are combined by Utyuzhnikov et al. [370] to produce well distributed Pareto points for nonconvex MOPs. A filtering procedure excludes local Pareto points from the obtained representation.

A variation of the ε-constraint method for general MOPs is used by Fu and Diwekar [152]. Changing the parameter ε in a pseudo-random manner, they produce representations that have more complete coverage than those produced by the traditional method of using uniformly spaced ε values. Kim and de Weck [225] propose the Adaptive Weighted-Sum method that is able to generate Pareto points in nonconvex regions, thus improving the coverage of the weighted-sum method as well as making it applicable to nonconvex MOPs.

Masin and Bukchin [266] present the Diversity Maximization Approach to produce representations with good coverage for general MOPs. At each iteration, the most diverse outcome, that is defined as the one that maximizes the minimum coordinate-wise distance between the new point and all the points already in the representation, is added to the representation. Although this method is applicable to general MOPs, it is recommended predominantly for mixed-integer and combinatorial problems.

Stochastic search algorithms based on (additive) ε-dominance are developed by Schütze et al. [332]. They construct ε-representations of the Pareto set and derive bounds on the representation quality and cardinality.

For MOPs with bounded and connected Pareto sets, Müller-Gritschneder et al. [282] introduce the concept of the ith criterion tradeoff limit as the set of those Pareto points for which a deterioration of the ith objective does not lead to an improvement in the other criteria. The boundary of the Pareto set is then defined as the union of the criterion limit. The authors use the goal attainment scalarization [166] and construct a Pareto set representation in p successive steps. Starting with p individual global minima in the first step, in the successive steps, a representation of the Pareto set for each combination of the objective functions $i = 1, i = 2, \ldots, p$, is computed and the union of the successive representations is taken.

Eichfelder [126] seems to be the first author to attempt to control the spacing of generated nondominated points. Her method is based on the Pascoletti and Serafini scalarization [295], making it applicable to general MOPs and notions of optimality defined by general cones. She derives sensitivity information in a neighborhood at a nondominated point and uses this information to determine input parameters for the scalarization so that the produced nondominated point is a prespecified distance from the previous point.

The coverage and spacing defined by Sayin [323] motivate Karasakal and Köksalan [223] to develop a method for producing discrete representations of the Pareto set for general MOPs, and Leyffer [246] who presents a bilevel optimization method to maximize the coverage of a discrete representation of the Pareto set of convex MOPs. The latter is also applied to nonconvex problems.

Kamenev [221] constructs a representation for the augmented set of outcomes, $Y + \mathbb{R}^p_{\geqq}$, referred to as the Edgeworth-Pareto hull (EPH), for general MOPs. In every iteration of the algorithm, a new sample of points in the feasible set X are generated and mapped into the objective space. An outcome having the largest distance to the closest point in the current representation is added to the representation. Convergence of the method is established.

Daskilewicz and German [93] first sample the Pareto set to obtain its point-wise representation and then map the points into a p-dimensional barycentric coordinate system. Each coordinate of a point is calculated based on the nondomination level of that point with respect to $p - 1$ objectives.

18.6.2.3 Polyhedral Approximation

Approaches producing polyhedral approximations first generate Pareto points using a scalarization method and then construct approximating polyhedral sets or functions.

For BOPs, the generated Pareto points are connected with line segments. For convex BOPs, Cohon [83] and Poliščuk [306] develop similar inner approximations while Cohon et al. [84], Fruhwirth et al. [151], Yang and Goh [400], and Siem et al. [341] propose sandwich approximations composed of inner and outer approximations.

Approaches for linear MOPs are proposed by Voinalovich [374], whose method yields a system of linear inequalities as an outer approximation, by Solanki et al. [345], who extends the sandwich approach of Cohon et al. [84], and by Benson [37], whose algorithm produces the Pareto set exactly.

Craft et al. [88] introduce a sandwich algorithm for convex MOPs that is similar to the sandwich algorithms of Solanki et al. [345] and Klamroth et al. [227] and is accompanied by a closed-form algebraic solution to the problem of finding distances between the inner and outer approximation. The algorithm is applied to intensity-modulated radiation therapy inverse planning problems.

Motivated by a beam intensity optimization problem in radiotherapy treatment planning, Shao and Ehrgott [335] come up with an approximation version of Benson's algorithm [37], while Ehrgott et al. [124] extend this algorithm to convex MOPs. Both algorithms provide sandwich approximations. In [335] and [125], the same authors develop two additional dual variants of Benson's algorithm [37] for the dual linear MOP.

Rennen et al. [310] enhance the sandwich algorithms of Solanki et al. [345], Klamroth et al. [227], and Craft et al. [88] to approximate the Pareto set of convex MOPs. They also extend the results of Siem et al. [341] to MOPs and show that by using transformation functions the sandwich algorithms can also be applied to nonconvex MOPs. More recently, Bokrantz and Forsgren [52] present another sandwich algorithm for convex MOPs which retains the quality of the algorithm by Rennen et al. [310] but achieves a more attractive ratio between the computational effort and the number of objectives.

Chernykh [76] approximates the EPH of the convex set Y in the form of a system of linear inequalities. A cone-based approach for general MOPs is proposed by Kaliszewski [220]. Kostreva et al. [237] develop a method using simplices, which is applicable to MOPs with discontinuous objective functions or a disconnected feasible set. A brief outline of an approach based on the normal-boundary intersection technique is offered by Das [91].

Schandl et al. [328] and Klamroth et al. [227] base their approaches on polyhedral distance functions that are constructed successively during the execution of the algorithm and utilize both to evaluate the quality of the approximation and to generate additional Pareto points. Norms and gauges are used for convex MOPs while nonconvex functions are used for nonconvex MOPs. The approximation itself is used to define a problem-dependent distance function and is independent of objective function scaling. In Klamroth et al. [227], inner and outer approximation approaches are proposed for convex and nonconvex MOPs and in all cases the approximation improves where it is currently the worst, a unique property among the approximation approaches.

Similar to Klamroth et al. [227], the methods by Gourion and Luc [173, 174] also achieve an outer approximation for nonconvex MOPs by so-called free disposal nonconvex polyhedra. The main distinction of their new method resides in the possible variation of direction in which the approximation is performed, and in its convergence for both efficient and weakly efficient sets.

Adaptive and nonadaptive methods have been developed for the polyhedral approximation of the EPH. In the former, at each iteration the approximation is refined and new approximation directions are computed. In the latter, a collection of approximation directions are assumed a priori. These types of methods are fully developed for convex MOPs by Lotov et al. [258] who include mathematical details and offer the visualization of the multidimensional Pareto set as the Pareto sets of two-dimensional slices (decision maps) of the EPH. Lotov et al. [257] apply these ideas also to nonconvex MOPs. Adaptive methods are proposed by Berezkin et al. [44] who use the union of a finite number of cones attached at Pareto points and contained in the (nonconvex) EPH for its approximation. Their methods are supported with a statistical estimation of the approximation quality. Additional properties of the polyhedral approximation with respect to the number of facets of approximating polyhedra are provided by Efremov and Kamenev [112]. Continuing in this direction, Lotov and Maiskaya [256] develop nonadaptive methods using coverings on the direction sphere.

Hartikainen et al. [186–188] propose the concept of an inherently nondominated set, i.e., a set all of whose points are Pareto. They construct a polyhedral approximation of the Pareto set as a subcomplex of a Delaunay triangulation of an initial finite set of Pareto points. The approximation is inherently nondominated and available as a solution set of a mixed-integer MOP that plays the role of a surrogate problem for the original computationally expensive nonconvex MOP. Based on the global analysis theory of Smale [343], Lovison [259] introduces a complementary approach also using simplicial complexes to approximate the efficient set (rather than the Pareto set) of nonconvex MOPs. A quadratic convergence result is included. In a follow-up paper [260], Lovison develops a globally convergent approximation algorithm for BOPs.

18.6.2.4 Nonlinear Approximation

This group of methods includes approaches producing quadratic, cubic, and other approximations.

For BOPs, Wiecek et al. [389] use piecewise quadratic approximating functions while Payne et al. [298] and Polak [305] use cubic functions to interpolate the Pareto set. Other structures used for approximation purposes include rectangles (Payne and Polak [297] and Payne [296]) and the hyper-ellipse (Li et al. [248]). Each of these nonlinear functions may provide a closed-form approximating formula. Fernández and Tóth [144] make use of the ε-constraint ε-method and construct an outer approximation of the efficient set of BOPs.

For linear MOPs, Gorissen and den Hertog [172] propose an inner approximation in the form of an arbitrary degree polynomial. They treat the ϵ-constraint scalarization of the original MOP as an uncertain SOP and apply adjustable robust optimization of Ben-Tal et al. [26]. Approximation of the weakly efficient set of convex MOPs by means of a convergent sequence of scalarizing functions is proposed by Luc et al. [261]. For linear MOPs the approach yields the entire set of weakly efficient solutions. For MOPs, Martin et al. [265] first apply a random search to generate a discrete approximation of the Pareto set and then use a regression method to fit a linear or quadratic surface through the earlier generated points. The method is customized for nonconvex problems with disconnected Pareto sets. Haanpää [185] approximates the Pareto set of nonconvex MOPs with a vector valued-function constructed by means of the ideal point and a scalar-valued surrogate function satisfying certain conditions. Radial basis functions, kriging functions, or regression functions can be used as the surrogate.

18.7 Specially Structured Problems

Many solution techniques for solving general MOPs can be modified or further improved to exploit known problem characteristics or special structures. This section gives an overview of such methods for linear, nonlinear, parametric, and bilevel MOPs.

18.7.1 Multiobjective Linear Programming

A multiobjective linear program (MOLP) is the following problem:

$$\begin{aligned} \min \quad & Cx \\ \text{subject to } & Ax = b \\ & x \geqq 0, \end{aligned} \tag{18.31}$$

where C is a $p \times n$ objective function matrix, A is an $l \times n$ constraint matrix, and $b \in \mathbb{R}^l$. It is usually assumed that the rows of A are linearly independent. For ease of exposition we shall assume that X is bounded, therefore compact. In consequence Y is also a compact polyhedron. Throughout this section we do not discuss issues arising from degeneracy which is important for simplex type algorithms to solve MOLPs. Degeneracy is addressed in some of the papers referred to in Sect. 18.7.1.1.

In this section, we review methods for finding efficient solutions of MOLPs of type (18.31). Because problem (18.31) is a special case of a convex MOP, all efficient solutions can be found by the weighted-sum scalarization. For $\lambda \in \mathbb{R}^p_{\geq}$, let LP($\lambda$) denote the linear program $\min\{\lambda^T Cx : Ax = b, x \geqq 0\}$.

Theorem 34 ([207]). *$\hat{x} \in X_E$ if and only if $\hat{x}$ is an optimal solution of LP(λ) for some $\lambda \in \mathbb{R}^p_{>}$.*

In view of Geoffrion's Theorem 15, Theorem 34 implies that $X_E = X_{pE}$ for MOLPs. The polyhedral structure of X and Y allows a more thorough investigation of the efficient and Pareto sets. Fruhwirth and Mekelburg [150] present a detailed analysis of the structure of Y_N for the case of $p = 3$ criteria. Let F be a face of X. If $F \subset X_E$, it is called an efficient face. F is called a maximal efficient face if it is an efficient face and for all faces F' such that $F \subset F'$, F' is not an efficient face.

Theorem 35. *1. $X_E = X$ if and only if there exists $x^0 \in \operatorname{ri} X$ such that $x^0 \in X_E$. Otherwise $X_E \subset \operatorname{bd} X$ and $X_E = \cup_{j \in J} F_j$, where F_j are maximal efficient faces and J is a finite index set.*

2. Let F be a face of X. F is an efficient face if and only if there exists $\hat{x} \in \operatorname{ri} F$ such that $\hat{x} \in X_E$.

3. Let F be a face of X and $F = \operatorname{conv}\{x^1, \ldots, x^q\}$. Then $F \subset X_E$ if and only if $\{x^1, \ldots, x^q\} \subset X_E$.

4. For each maximal efficient face F_i there is a subset $\Lambda_i \subset \{\lambda \in \mathbb{R}^p_{>} : \sum_{k=1}^p \lambda_k = 1\}$ such that all $x \in F_i$ are optimal for LP(λ) for all $\lambda \in \Lambda_i$.

The decomposition of the weight space indicated in the last point of Theorem 35 can be further elaborated. Let $\Lambda := \{\lambda \in \mathbb{R}^P_{>} : \sum_{i=1}^p \lambda_i = 1\}$ denote the set of weights. Theorem 34 suggests a decomposition of Λ into subsets, such that for each λ in a subset LP(λ) has the same optimal solutions. Such a partition can be attempted with respect to efficient bases (see Sect. 18.7.1.1) of the MOLP or with respect to extreme points of X_E or Y_N. Some of the simplex based algorithms mentioned below use such decompositions. The main results for weight set decomposition with respect to extreme points of Y_N are summarized in the following theorem. We assume that Y is of dimension p.

Theorem 36 ([42]). *Let $\{y^1, \ldots, y^q\}$ be the Pareto extreme points of Y and $\Lambda(y) = \{\lambda \in \mathbb{R}^p : \lambda^T y \leq \lambda^T y' \text{ for all } y' \in Y\}$. Then the following statements hold.*

1. $\Lambda \subset \cup_{i=1}^q [\Lambda(y^i) \cap \Lambda]$.
2. $\operatorname{int} \Lambda(y^i) \neq \emptyset$, $\Lambda \cap \operatorname{int} \Lambda(y^i) = \operatorname{int}(\Lambda \cap \Lambda(y^i)) \neq \emptyset$.
3. *If $\lambda \in \Lambda \cap \operatorname{int} \Lambda(y^i)$ then y^i is a unique optimal solution of the problem* $\min\{\lambda^T y : y \in Y\}$.

4. $[\Lambda \cap \operatorname{int} \Lambda(y^i)] \cap [\Lambda \cap \Lambda(y^j)] = \emptyset$ *and therefore* $\Lambda \cap \Lambda(y^i) \neq \Lambda \cap \Lambda(y^j)$ *when* $i \neq j$.
5. *If* F *is a proper face of* Y *and* $\bar{y}, y^* \in \operatorname{ri} F$ *then* $\Lambda(\bar{y}) = \Lambda(y^*)$.

Due to Theorem 34, any MOLP can in principle be solved using parametric linear programming. However, simplex, interior point, and objective-space methods have also been developed to deal with MOLPs directly.

18.7.1.1 Multicriteria Simplex Methods

Some notation is first needed to explain multicriteria simplex algorithms.

- An extreme point (zero dimensional face) of X that is efficient is called efficient extreme point.
- A basis B of (18.31) (an index set of l linearly independent columns of A) is called efficient basis if there exists a $\lambda \in \mathbb{R}^p_>$ such that B is an optimal basis of LP(λ).
- Let B be a basis and $N := \{1, \ldots, n\} \setminus B$. Let C_B and C_N be the columns of C indexed by B and N, respectively. A_B, A_N, x_B and x_N are defined accordingly. Then $\bar{C} = C - C_B A_B^{-1} A$ is called the reduced cost matrix with respect to B. $\bar{C}_B$ and $\bar{C}_N$ are defined analogously to C_B and C_N.
- Let B be an efficient basis. A variable x_j is called efficient nonbasic variable if there exists a $\lambda \in \mathbb{R}^p_>$ such that $\lambda^T \bar{C}_N \geqq 0$ and $\lambda^T \bar{c}^j = 0$, where $\bar{c}^j$ is the jth column of $\bar{C}$, $j \in N$.
- Let B be an efficient basis and x_j be an efficient nonbasic variable. A feasible pivot from B with x_j as entering variable is called an efficient pivot.

If B is a basis then (x_B, x_N) with $x_N = 0$ and $x_B = A_B^{-1} b$ is a basic solution, and it is a basic feasible solution if additionally $x_B \geqq 0$. A basic feasible solution is an extreme point of X.

Theorem 37. *1. If* $X_E \neq \emptyset$ *then there exists an efficient extreme point.*
2. *Let* B *be an efficient basis. Then* $(x_B, 0) \in X_E$.
3. *Let* $x \in X$ *be an efficient extreme point. Then there exists an efficient basis* B *such that* x *is a basic feasible solution for* B.
4. *Let* B *be an efficient basis and* x_j *be an efficient nonbasic variable. Then any efficient pivot leads to an efficient basis.*

Multicriteria Simplex algorithms proceed in three phases. First, an auxiliary single objective LP $\min\{e^T z : Ax + z = b; x, z \geqq 0\}$ is solved to check feasibility (assuming, without loss of generality, $b \geqq 0$). $X \neq \emptyset$ if and only if the optimal value of this LP is 0. If $X \neq \emptyset$, in Phase 2 an initial efficient extreme point is found or the algorithm stops with the conclusion that $X_E = \emptyset$. Finally in Phase 3, efficient pivots are performed to explore all efficient extreme points or efficient bases.

Theorem 38. *Tests for efficient nonbasic variables. Let B be an efficient basis and $\bar{C}$ be the reduced cost matrix with respect to B. Let $j \in N$. Let e be the vector of all ones and let I be the identity matrix of appropriate dimension.*

1. *[138] Consider the LP $\min\{e^T z : -\bar{C}_N y + \bar{c}^j \delta + Iz = 0; y, \delta, z \geqq 0\}$. x_j is an efficient nonbasic variable if and only if the optimal objective value of this LP is zero.*
2. *[206] Consider a subset $J \subseteq N$ and the LP $\min\{e^T z : -\bar{C}_N y + \bar{C}_J \delta + Iz = e; y, \delta, v \geqq 0\}$. Each variable $x_j, j \in J$, is efficient if and only if this LP has an optimal solution.*
3. *[109] Variable x_j is efficient if and only if there is a solution $(\hat{z}, \hat{y})$ of the linear system $\bar{C}_N^T(z + e) - y = 0; z, y \geqq 0$ with $\hat{y}_j = 0$.*
4. *[422] Variable x_j is efficient if and only if the LP $\{\min 0^T z : \sum_{i \neq j} \bar{C}_{ki} z_i \geq \bar{C}_{kj}, k = 1, \ldots, p; z \geqq 0\}$ is infeasible.*

Theorem 39. *Finding an efficient (extreme) point.*

1. *[108] Let $x^0 \in X$ and consider the LP $\{\max e^T z : Cx - Iz = Cx^0; Ax = b; x, z \geqq 0\}$. If $(\hat{x}, \hat{z})$ is an optimal solution of this LP then $\hat{x} \in X_E$. If the LP is unbounded $X_E = \emptyset$.*
2. *[29] Let $x^0 \in X$. If the LP $\{\min -z^T Cx^0 + u^T b : z^T C - u^T A + w^T = -e^T C; w, z \geqq 0\}$ has no optimal solution then $X_E = \emptyset$. Otherwise let $(\hat{z}, \hat{u}, \hat{w})$ be an optimal solution. Then an optimal extreme point of the LP $\{\max(\hat{z} + e)^T Cx : x \in X\}$ is an efficient extreme point of the MOLP.*

Algorithms based on the simplex method are proposed by Armand and Malivert [13, 14], Evans and Steuer [138], Ecker and Kouada [109, 111], Isermann [206], Gal [158], Philip [302, 303], Schönfeld [329], Strijbosch [353], Yu and Zeleny [411, 412], and Zeleny [416]. Some numerical experiments for an implementation of Steuer's [348] multicriteria simplex method (called ADBASE [350]) on randomly generated problems are available in [349, 388]. Ehrgott et al. [123] use the weighted-sum scalarization of MOLPs (Theorem 34) and single objective duality to generalize the single objective primal-dual simplex algorithm for the multiobjective case. An improved method specifically for biobjective network flow problems is also given by Eusébio et al. [137].

In order to determine the whole efficient set, it is necessary to find subsets of efficient extreme points, the convex hulls of which determine maximal efficient faces. This process can be considered an additional phase of the multicriteria simplex method. Approaches that follow this strategy can be considered bottom up, as they build efficient faces starting from faces of dimension 0 (extreme points).

Sayin [322] proposes a top-down approach instead. Consider an MOLP, where X is given in the form $X = \{x \in \mathbb{R}^n : \bar{A}x \leqq \bar{b}\}$, i.e., the nonnegativity constraints are included in $\bar{A}$. Let $M = \{1, \ldots, n + l\}$ denote the set of indices of constraints and $\mathcal{M} := \{J : J \subseteq M\}$. Then each $J \in \mathcal{M}$ represents a face $F(J)$ of X, $|J_1| \leq |J_2|$ implies $\dim F(J_1) \geq \dim F(J_2)$, and $J_1 \subseteq J_2$ implies $F(J_2) \subseteq F(J_1)$. She solves the

LP (18.32) to check whether or not a face is efficient. For $J \in \mathcal{M}$, let $\bar{A}^J$ and $\bar{b}^J$ denote submatrices (subvectors) of $\bar{A}$ and $\bar{b}$ containing only rows with indices in J.

$$\begin{aligned} \max \quad & e^T Cx - e^T Cy \\ \text{subject to } & \bar{A}x \leqq \bar{b} \\ & \bar{A}y \leqq \bar{b} \\ & \bar{A}^J y = \bar{b}^J \\ & -Cx + Cy \leqq 0. \end{aligned} \tag{18.32}$$

Theorem 40 ([322]).

1. *$F(J)$ is a proper face of X if and only if the LP (18.32) is feasible.*
2. *$F(J)$ is an efficient proper face of X if and only if the optimal objective value of the LP (18.32) is 0.*

In conjunction with the first statement (1) of Theorem 35 it is now shown that there are subsets $\mathcal{E}^s$ of $\mathcal{M}^s := \{J \in \mathcal{M} : |J| = s\}$ such that $X_E = \cup_{s=0}^{n+l} \cup_{J \in \mathcal{E}^s} F(J)$. Sayin's algorithm checks whether $J = \emptyset$, i.e., whether $X = X_E$, and then proceeds to larger sets J, i.e., faces of smaller and smaller dimension. In this process, supersets of sets already checked can be eliminated (as they correspond to subsets of faces already classified as efficient or nonefficient).

18.7.1.2 Interior Point Methods

Interior point methods are not easy to adapt for MOLPs since they construct a sequence of points that converges to a single point on the boundary of X. Thus most interior point methods proposed in the literature are of an interactive nature, see, e.g., Arbel [12], Aghezzaf and Ouaderhman [6], and the references therein.

Abhyankar et al. [1] propose a method of centers of polytopes to find a nondominated point of Y. Assume that $Y = \{y : Gy \geqq H\}$, where $H \in \mathbb{R}^p$, is bounded and full dimensional and given in the form $Y = \{G_s^T x \geq H_s, s = 1, \ldots, q_1\}$, where G_s denotes the s-th row of G. Let $\mathcal{C}$ be a polyhedral cone given in the form $\mathcal{C} = \{d \in \mathbb{R}^p : Ld \geqq 0\} = \{d : L_s^T d \geq 0, s = 1, \ldots, q_2\}$, where L_s denotes the s-th row of L. Starting from $y^0 \in \operatorname{int} Y$, a sequence of points $\{y^\tau\} \subseteq \operatorname{int} Y$ is constructed so that $y^\tau \to \hat{y} \in N(Y, \mathcal{C})$. In this sequence, $y^{\tau+1}$ is the center of a polytope $Y^\tau := (y^\tau - \mathcal{C}) \cap Y$. Therefore $y^{\tau+1}$ can be determined as a unique maximizer of the potential function

$$f_{Y^\tau}(y) := \sum_{s=1}^{q_1} \ln(G_s^T y - H_s) + \sum_{s=1}^{q_2} (L_s^T y^\tau - L_s^T y)$$

by solving

$$\sum_{s=1}^{q_1} \frac{G_s^T}{G_s^T y - H_s} - \sum_{s=1}^{q_2} \frac{L_s^T}{L_s^T y^s - L_s^T y} = 0.$$

Theorem 41 ([1]). *Every subsequence of* $\{y^\tau\}$ *converges to some point* $\hat{y} \in N(Y, \mathcal{C})$.

Abhyankar et al. [1] also construct an approximation of a portion of the nondominated faces of Y by constructing a sequence of algebraic surfaces (ellipsoids) $\{\tilde{Y}^\tau\}$ that approaches a part of $N(Y, \mathcal{C})$. In this way, they are able to parameterize the nondominated set.

To obtain a description of X_E they consider the polar cone of $\mathcal{C}$, $\mathcal{C}^* := \{d \in \mathbb{R}^n d = \sum_{i=1}^q \lambda_i v^i, \lambda_i \geq 0\}$ and the cone of increasing directions $\mathcal{C}^> = \{d \in \mathbb{R}^n : (CV)d \geq 0\}$, where C is the objective function matrix, and $CV = (Cv^1, \ldots, Cv^q)$ and $v^i, \ldots, v^q$ are the generators of $\mathcal{C}^*$. Then the methodology above can be applied to X with cone $\mathcal{C}^>$.

The question whether the entire efficient set of an MOLP can be found using interior point methods has only recently been answered in the affirmative. Blanco et al. [51] accomplish this with a semidefinite programming approach. They derive a polynomial system of inequalities, which encodes the efficient set, and obtain the following result relying on the theory of moment matrices.

Theorem 42. *The entire set of efficient extreme points of an MOLP is encoded in the set of optimal solutions of a semidefinite program.*

This theorem implies that all efficient extreme points can be computed using interior point methods. Since the size of the semidefinite program is not polynomial in the input size of the MOLP, this theorem does not imply that the semidefinite programming approach to MOLP is computationally efficient.

18.7.1.3 Objective Space Methods

Objective space methods for solving MOLPs are based on the assumption that the dimension of the objective space is typically smaller than the dimension of the decision space and therefore the number of Pareto extreme points of the set $Y := \{y \in \mathbb{R}^p : y = Cx \text{ for some } x \in X\} = C[X]$ to examine should be smaller than the number of efficient extreme points of the set X.

Dauer and Liu [97] propose a simplex-like method performed at those extreme points of X that correspond to the extreme points of Y. Let $\bar{C} = [\bar{C}_B, \bar{C}_N]$ be the reduced cost matrix associated with an extreme point $x \in X$. Define the cone spanned by the columns of $\bar{C}_N$ as cone $\bar{C}_N := \{d \in \mathbb{R}^p : d = \sum_{j \in N} \lambda_j \bar{c}^j, \lambda_j \geq 0\}$. A frame of cone $\bar{C}_N$ is a minimal collection of vectors selected from among the columns of $\bar{C}_N$ that determine this cone.

Theorem 43 ([97]). *Let $y = Cx$ be an extreme point of Y and let $\bar{C}$ be the reduced cost matrix associated with the extreme point $x \in X$. Let E^j be the edge of X determined by a column $\bar{c}^j$ in $\bar{C}_N$. The image of E^j under C is contained in an edge of Y if and only if $\bar{c}^j$ is in a frame of the* cone $\bar{C}_N$.

Dauer and Saleh [98] develop an algebraic representation of $Y = \{y \in \mathbb{R}^p : Gy \geqq H\}$ and propose an algorithm to construct this set. Additionally, they develop an algebraic representation of a polyhedral set $\tilde{Y}$ that has the same Pareto structure as that of Y and that has no extreme points that are not Pareto. In this way, any method designed for finding all vertices of convex polyhedral sets becomes suitable to find the set of Pareto extreme points of Y. However, they also propose an algorithm using a single objective linear program in $\mathbb{R}^{p+1}$, the set of optimal basic solutions of which corresponds to the set of extreme points of $\tilde{Y}$ (and to the set of Pareto extreme points of Y). Dauer [94] presents an improved version of the algorithm of Dauer and Liu [97] to generate the set of all Pareto extreme points and edges of Y. He gives special attention to degenerate extreme points and the collapsing effect that reduces the number of extreme points of X that are necessary to analyze in order to fully determine the structure of Y. Almost a parallel effort to represent the set Y in terms of a set of inequalities is undertaken by White [386]. Yan et al. [399] also give an algorithm for finding all of the solutions of an MOLP by suitable representation of the structure of the solution set.

Dauer and Gallagher [96] develop an algorithm called MEF for determining high-dimensional maximal Pareto faces of Y. Algorithm MEF requires as input a nonredundant system of linear inequalities representing Y (or $\tilde{Y}$).

Theorem 44 ([96]).

1. *If F_X is a maximal efficient face of X then $C[F_X] := \{y \in \mathbb{R}^p : y = Cx$ for some $y \in F_X\}$ is a maximal Pareto face of Y.*
2. *If F_Y is a maximal Pareto face of Y then $C^{-1}[F_Y] \cap X := \{x \in \mathbb{R}^n : Cx \in F_Y\}$ is a maximal efficient face of X.*

Theorem 45 ([96]). *Let $Y = \{y \in \mathbb{R}^p : Gy \geqq H\}$ and G_k be the k-th row of the matrix G. Let F be a nonempty face of Y and let $I(F)$ denote the set of indices of the active constraints defining F (i.e., $I(F) = \{i : G_i y = H_i$ for all $y \in F\}$). Then*

1. *$F \subseteq Y_N$ if and only if $\sum_{j \in I(F)} \lambda_j G_j > 0$ for some collection of scalars $\lambda_j \geq 0, j \in I(F)$.*
2. *F is a maximal Pareto face of dimension $p - |I(F)|$ if and only if*
   - (a) *there exist scalars $\lambda_j \geq 0, j \in I(F)$, such that $\sum_{j \in I(F)} \lambda_j G_j > 0$ and*
   - (b) *$\sum_{j \in I(F)} \mu_j G_j \not> 0$ for every $J \subsetneqq I(F)$ and every collection of scalars $\mu_j \geq 0, j \in J$.*

The work of Benson follows on the work of Dauer et al. He describes the outcome set-based algorithm in [37]. Its purpose is to generate all efficient extreme points of Y_N. Let $\bar{Y} := \{y \in \mathbb{R}^p : Cx \leqq y \leqq \hat{y}\}$, where $\hat{y}$ is such that $y_k < \hat{y}_k, k = 1, \ldots, p$ for all $y \in Y = C[X]$. Then $\bar{Y}_N = Y_N$. The method starts by finding a simplex S^0

containing $\bar{Y}$ and its vertices. S^0 is given by $S^0 := \{y \in \mathbb{R}^p : y \leq \hat{y}, \beta \leq e^T y\}$, where $\beta = \min\{e^T y : y \in \bar{Y}\}$. Given S^τ, an extreme point y^τ of S^τ is chosen that is not contained in $\bar{Y}$. Then w^τ is chosen as a unique point on the boundary of $\bar{Y}$ on the line segment connecting an interior point y^0 of Y with y^τ. Now, $S^{\tau+1}$ can be obtained by

$$S^{\tau+1} = S^\tau \cap \{y \in \mathbb{R}^p : (u^\tau)^T y \leq b^T v^\tau\}$$

where $u^\tau, v^\tau \in \mathbb{R}^p$ are such that $F^\tau = \{y \in \bar{Y} : (u^\tau)^T y = b^T v^\tau\}$ is a face of Y containing w^τ. The computation of the extreme points of $S^{\tau+1}$ completes the iteration.

Theorem 46 ([37]). *The outcome set-based algorithm is finite and at termination $S^K = \bar{Y}$, where K is the number of iterations.*

All Pareto extreme points of Y are found by eliminating from the extreme points of $\bar{Y}$ those for which $y_k = \hat{y}_k$ for some k. Benson [35] shows that also all weak Pareto extreme points of Y are found. In [36], Benson combines the algorithm with a simplicial partitioning technique, which makes the computation of the extreme points of $S^{\tau+1}$ more efficient. Ehrgott et al. [125] employ results of Heyde and Löhne [197] from a geometric duality theory for MOLPs and develop a dual variant of Benson's algorithm [37].

A new approach to determining maximal efficient faces in MOLPs is also described by Pourkarimi et al. [307], who decide between efficiency and weak efficiency of faces using relative interior points that are computed from a sequence of generated points that are affinely independent.

MOLPs make up a class of problems for which the computation of efficient (extreme) points is very well developed. Despite the availability of many effective algorithms for MOLPs, only few of them can compute all efficient extreme points without supplementary information on the weight vector λ, which conveniently eliminates the task of choosing that vector. The multicriteria simplex methods provide information on the weight vectors as a byproduct, while the outer approximation method of Benson [37], its improved version by Ehrgott et al. [125], the primal-dual simplex algorithm by Ehrgott et al. [123], and the semidefinite programming approach by [51] treat the weights as variables and integrate their computation with the computation of efficient points.

18.7.2 Nonlinear MOPs

In this section we review results on some classes of MOPs with nonlinear objective functions including piecewise linear, quadratic and polynomial, and fractional functions.

18.7.2.1 MOPs with Piecewise Linear Objectives

MOPs with piecewise linear objective functions are not much studied. Achary [2] develops a simplex-type method to enumerate all efficient solutions of a biobjective transportation problem. Nickel and Wiecek [286] propose an approach in which the task of finding efficient solutions of the original problem is replaced by tasks of finding efficient solutions of simpler subproblems starting with subsets of the highest dimension. More recently, Yang and Yen [401] investigate the structure of the Pareto solution set of a piecewise linear MOP in a normed space and describe it as union of finitely many semiclosed polyhedra. Similar structural results are derived by Zhen and Yang [420, 421] for a polyhedral ordering cone, including a bounded weak sharp minimum property for the problems that satisfy a cone-convexity assumption, and by Fang et al. [140] for both continuous and more general discontinuous cases. The sensitivity of piecewise linear parametric MOPs is also studied by Fang and Yang [139]. Extending the approach developed by Thuan and Luc [366] for linear MOPs to the convex piecewise linear case, these authors prove that the efficient and weakly efficient sets satisfy similar structural properties and are locally represented by finite unions of polyhedra whose vertices and recession directions are smooth functions of the parameter, under some suitable smoothness assumptions.

18.7.2.2 Quadratic MOPs

MOPs with quadratic functions have been of interest to many authors. Unconstrained quadratic MOPs with strictly convex objective functions are analyzed by Beato-Moreno et al. [23]. They obtain an explicit characterization of the efficient set for the biobjective case and show that the p-objective case can be reduced to the $(p-1)$-objective case. The set of weakly efficient solutions of quadratic MOPs with convex objective functions is examined by Beato-Moreno et al. [24]. A method to produce an analytic description of the efficient set of linearly constrained convex quadratic MOPs is proposed by Goh and Yang [171]. The stability of quadratic MOPs is addressed by Badra and Maaty [18], and by Hirschberger [199] who shows that in the convex quadratic case, there exist an efficient "compromise" arc that connects any two efficient points and lies completely in the set of efficient points. This author also describes a method to construct such an arc computationally.

Quadratic MOPs with fuzzy parameters and decision variables in objective and constraints are studied by Saad [318], and by Ammar [9, 10] who shows that the efficient solutions can be found using fuzzy scalarization problems as demonstrated for a fuzzy portfolio problem with convex quadratic objectives [8]. Moosavian et al. [280] apply sequential quadratic programming to nonlinear MOPs and apply their new methodology to water management and optimal annual scheduling of power generation in hydropower plants, using the Analytic Hierarchy Process as underlying decision support system. Another application is described by Rhode and

Weber [311] who formulate an economic vector optimization problem that contains one quadratic and several linear objective functions and discuss ways to solve such problems.

Some researchers also relate quadratic MOPs to linear complementarity problems (LCPs). Kostreva and Wiecek [234] demonstrate and Isac et al. [205] exploit equivalence between the LCP and a nonconvex quadratic MOP with linear constraints. Later, Kostreva and Yang [235] use the same equivalence to convert LCP into an equivalent MOP and a minimax optimization problem, for which they use global optimality conditions to achieve certain (un)solvability conditions for the original LCP. Generalized linear complementarity problems are related to MOPs by Ebiefung [107] and Bhatia and Gupta [48]. Korhonen and Yu [231] also use a linear complementarity approach to MOPs with one quadratic and several linear objective functions.

18.7.2.3 Polynomial MOPs

MOPs with polynomial objective and constraint functions are studied by Kostreva et al. [236] who use the Benson approach to develop a method for finding efficient solutions of those problems. The resulting SOP is solved with a continuation method. Stanimirović and Stanimirović [347] describe an implementation of several scalarization methods for solving polynomial MOPs in Mathematica®. The chosen methods include weighting, ε-constraints, lexicographic orders, and goal programming, among others, and techniques for the verification of Pareto optimality including the method by Benson.

18.7.2.4 Fractional MOPs

Multiobjective fractional problems (MOFPs), objective functions of which are fractional functions, have been extensively studied and this review covers only a small part of available articles. A survey on biobjective problems in this class is given by Cambini et al. [68]. If numerators and denominators of the objective functions of MOFPs are affine functions, the problems are referred to as multiobjective linear fractional programs (MOLFPs). Kornbluth and Steuer [232] and Benson [30] develop a simplex-based procedure to find weakly efficient vertices of MOLFPs. Gupta [180] relates efficient points of these problems to efficient points of an MOLP and to efficient points of a number of biobjective linear programs. Connectedness of the weakly efficient set of MOLFPs is examined by Choo and Atkins [79]. Scalarizations have also been applied by Metev and Gueorguieva [274] to generate weakly efficient solutions of MOLFPs. An algorithm to find all efficient solutions of MOLFPs with zero-one variables is proposed by Gupta [179] while MOLFPs with integer variables are examined by Gupta and Malhotra [181] and, more recently, by Sharma [336]. Conditions for efficiency for MOLFPs with convex constraints are developed by Gulati and Islam [177, 178]. An interactive method for computing nondominated and finding preferred solutions using the ε-constrained scalarization is presented in Costa [87]. If numerators and denominators of the objective functions are nonlinear functions, the problems are referred to as multiobjective nonlinear fractional programs. For these problems, conditions for the existence of efficient solutions are developed by Kaul and Lyall [224], while Fritz-John and Kuhn-Tucker type conditions for efficiency are proposed by Gulati and Ahmad [3]. In addition, in the last few years a plethora of specialized optimality conditions for these problems with objectives that satisfy one of many generalized convexity notions have been published by, among a good gross of other authors, Long [253], Zalmai and Zhang [413], Gao and Rong [162], Niculescu [287], and Chinchuluun et al. [78]. Unfortunately, hardly any of these papers present solution techniques and relevant examples or applications so that we do not summarize these contributions here but refer the reader to the extensive list of references collected by Engau [132].

Among numerous new articles that have appeared since then, one of the more recent trends is the increased focus on MOFP duality that includes the papers by Mishra et al. [279] and Zalmai and Zhang [414] for nonsmooth semi-infinite MOFP, by Suneja and Kohli [354], and by Ying [404] on higher-order symmetric duality. Another trend is the adoption of approximate solution concepts: new papers that address notions of and conditions for (weak) ϵ-efficiency for MOFP are Kim et al. [226] and Verma [372]. New sufficiency results are provided by Sharma and Ahmed [337]. Finally, Liu et al. [252] introduce generalized definitions of the Lagrangean function and its saddle points for a class of multiobjective fractional optimal control problems.

18.7.3 Parametric Multiobjective Programming

There are two general ways of introducing parameters into MOPs. Parameters can be introduced into the original MOP to parametrize the feasible set or the objective functions, or the MOP is scalarized which yields a parametrized SOP. These two directions significantly differ from each other. In the former, one obtains a parametric family of solution sets and in the latter the solution set of the original problem is parametrized.

18.7.3.1 Parametric MOPs

We consider a family of parametric MOPs formulated as

$$\begin{aligned} \min \quad & (f_1(x,\chi),\ldots,f_p(x,\chi)) \\ & \text{subject to } x \in X(\chi) \subset \mathbb{R}^n, \end{aligned} \tag{18.33}$$

where $\chi \in \mathbb{R}^s$ is a parameter vector and the set X is a set-valued mapping from $\mathbb{R}^s$ to $\mathbb{R}^n$.

The first studies on parametrization of MOPs go back to the works of Bitran [50] and Benson [31] on linear MOPs with parametrized objective functions. A series of theoretical results follows. A variety of findings are provided by Sawaragi et al. [321] in which stability results for problems with parametric feasible sets, objective functions, and domination structures are obtained. Studies on continuity and closedness of multifunctions are performed by Penot and Sterna-Karwat [299]. A review of stability and sensitivity results is given by Tanino and Kuk [357]. Sensitivity of efficient solutions and Pareto outcomes is also examined by Balbas et al. [19] for MOPs with parametric objective and constraint functions.

The theoretical studies have been accompanied by few recent applied papers. Unconstrained MOPs with a scalar parameter in the objective functions are studied by Witting [393], Dellnitz and Witting [101], and Witting et al. [395]. When varying the parameter, the authors obtain a family of solution sets and develop methods to compute paths within this family. The methods make use of the Kuhn-Tucker conditions for efficiency which are parametric and therefore solved with a continuation method or calculus of variations.

Parametric MOPs are applied by Witting et al. [394] to mechatronic systems where a scalar parameter plays the role of time. Ross et al. [314] work on a low-earth orbiting satellite case study that provides several contexts for parametric MOPs: changes in the objective vector, in priorities among the objectives, in a scalarization function, and in the number of decision makers.

18.7.3.2 Parametrization of the Scalarized MOP

The first reference on the parametrization of the scalarized MOP seems to be the book by Guddat et al. [175]. The authors propose interactive methods based on the scalarization of the MOP with the weighted-sum and ε-constraint methods simultaneously and also based on the parametrization of the scalarization parameters (cf. Sect. 18.5.4.3).

Jin and Sendhoff [216] construct a collection of test MOPs in which the objectives are scalarized and the scalarizing weights change.

Buryak and Insarov [63] develop a model of a system whose state is determined by direct and indirect parameters, and a parametric optimization method for solving it. Direct parameters change freely within a feasible set while indirect parameters depend upon the direct parameters (due to a functional relationship or statistical dependencies of a known type). A vector of objective functions characterizes the effectiveness of the system goal to choose values of direct parameters so that the obtained objective values are acceptable by the decision maker.

Enkhbat et al. [135] study weighted-sum scalarizations of linear and convex MOPs with linear parametric weights, and apply algorithms available in the literature to this class of problems.

Romanko et al. [312] explore relationships between MOPs and their weighted-sum and ε-constraint scalarization which they treat as single objective parametric programs. For three-objective problems with linear, quadratic, and second-order conic objectives, they develop parametric (single objective) optimization algorithms to compute the efficient set of the MOP. They also provide closed-form descriptions of the Pareto sets.

18.7.4 Bilevel Multiobjective Programming

Bilevel MOPs (BLMOPs) constitute a special class of constrained MOPs whose feasible set consists of solutions to another MOP. Bilevel problems exhibit a two-level structure. In the upper-level problem, optimal values of upper-level optimization variables, which become parameters for the lower-level problem, are determined so as to minimize the upper-level vector objective. In the lower-level problem, its vector objective is minimized with respect to the lower-level optimization variables under the given parameters. In other words, the set of feasible points of the upper-level problem is given by the solution set of the lower-level parametric optimization problem, while the solutions of the lower-level problem influence the upper-level objective values.

With a scalar-valued objective function on each level, the problem is referred to as a bilevel single-objective program (BLSOP). Such problems have been researched for a few decades and many solution methods have been proposed for their various formulations; see the monographs by Bard [21] and Dempe [103], and also the multiple-author books [277, 338] for in-depth tutorials and reviews on this subject. Despite this vast research effort, BLSOPs remain challenging even today if they fall in the class of global optimization problems.

In BLSOP, the upper-level decision maker (leader) cannot restrict the lower-level decision maker (follower) as long as the latter makes a decision that is feasible for the former. In effect, one can consider two approaches to the lower-level decision maker's behavior with respect to the upper-level decision maker [103, 338]. The first is to assume that the follower cooperates with the leader and makes a decision that also optimizes the latter's objective function. The second approach assumes a conservative strategy under which there is no cooperation between the decision makers. These approaches are often referred to as the optimistic (cooperative) and pessimistic (noncooperative) option, respectively. In a big majority of the reviewed papers, the authors assume the optimistic option while in some papers this assumption is not explicitly made but still used.

18.7.4.1 Relationships between Bilevel Single Objective and Multiobjective Programming

Exploiting MOPs for solving BLSOPs seems to be the earliest research activity connecting multiobjective and bilevel programming. In the nineteen eighties researchers tried to establish a link between a linear BLSOP and a related linear biobjective program. A relationship of this kind is first investigated by Bard [20], and then continued by Ünlü [369], Clark and Westerberg [82], Candler [70], Wen and Hsu [382], and Haurie et al. [190]. Marcotte and Savard [263] establish that the bilevel optimality and the efficiency with respect to the upper and lower objective functions are different concepts [263].

Fülop [155] is the first to observe that more than two objective functions are needed in a multiobjective formulation equivalent to a BLSOP. He establishes the equivalence between linear BLSOPs and the optimization over the solution set of a related linear MOP. This direction of research is successfully continued by Glackin [169] and Glackin et al. [170] who propose an algorithm for solving linear BLSOPs using linear MOPs.

Fliege and Vicente [148] reformulate an original, unconstrained nonlinear BLSOP problem into a four-objective MOP and outline a reformulation for constrained problems. Ivanenko and Plyasunov [209, 210] continue the reformulation efforts in more general directions using a parametric perturbation function of the lower-level problem and the Karush-Kuhn-Tucker conditions for optimality for the convex lower-level problem. Independently of [170], Pieume et al. [304] extend the result in [155] to BLSOPs with nonlinear upper-level objective functions and constraints. For unconstrained nonlinear BLSOPs, they modify the result in [148] which reduces solving the overall BLSOP to solving two four-objective MOPs and taking the intersection of their solution sets.

Sakawa and Nishizaki [319] argue that a cooperative BLSOP is equivalent to the biobjective program whose vector-valued objective function involves the objective function of the leader and that of the follower, and propose solution methods based on fuzzy programming.

18.7.4.2 Theory of Bilevel Multiobjective Programming

Introducing multiple objective functions to a bilevel problem poses not only technical but also conceptual challenges. With multiple objective functions at the lower level, one can no longer assume that there exists a unique solution to the lower-level problem. A common approach is to assume that the efficient set of the lower-level MOP becomes the set of optimal solutions for the lower level. The different formulations resulting from the leader's anticipations of the follower's actions are analyzed by Nishizaki and Sakawa [289], Nie [288], and Sakawa and Nishizaki [319].

Let $X_u \subset \mathbb{R}^{n_u}$ and $X_l(x^u) \subset \mathbb{R}^{n_l}$ for all $x^u \in X_u$, and let $X = \{x = (x^u, x^l) : x^u \in X_u, x^l \in X_l(x^u)\} \subset \mathbb{R}^n$, where $n = n_u + n_l$. An optimistic bilevel multiobjective program (BLMOP) has the form

$$\begin{aligned} \min_{x^u, x^l} \quad & f^u(x^u, x^l) \\ \text{subject to } & x_l \in E(X_l(x^u), f^l(x^u, \cdot), \mathbb{R}^{p_l}_{\geqq}) \\ & x^u \in X_u, \end{aligned} \tag{18.34}$$

where $f^u : X \to \mathbb{R}^{p_u}$ is the vector of upper-level objective functions and $f^l : X \to \mathbb{R}^{p_l}$ is the vector of lower-level objective functions. The notation $X_l(x^u)$ reflects that the feasible set X_l of the lower-level problem depends on the upper-level decision x^u. One may consider a more general formulation in which the upper-level feasible set X_u depends on the lower-level decision x^l. If the functions f^u and f^l are scalar-valued, then problem (18.34) reduces to a BLSOP.

The derivation of new optimality conditions for different classes of BLMOPs has been an important research direction. Ye and Zhu [403] study optimality conditions for MOPs with variational inequality constraints and apply their results to BLMOPs. Zhang et al. [418] formulate a class of fuzzy linear BLMOPs and derive optimality conditions for them. Problems with a single objective at the upper level and multiple objective functions at the lower level are studied by Nie [288] who proposes new solution concepts for the overall problem including a risk solution, a conservative solution, and mean-optimal solution. For the convex case on the lower level, that level's problem is scalarized with the weighted-sum method and then replaced by the KKT conditions of the scalarized problem. Necessary and sufficient optimality conditions for the three types of solutions are derived. Dell'Aere [100] derives optimality conditions for BLMOPs with a convex, equality-constrained lower-level problem and also performs a sensitivity analysis for this class. Jahn and Schaller [213] present two types of optimality conditions for a general class of BLMOPs in infinite dimensions with objective spaces partially ordered by pointed convex cones and under differentiability conditions: one using the Lagrange multiplier rule generalized for this class and the other using the contingent cone. However, the conditions give only partial characterizations of the solutions and cannot be used for the development of solution methods. Necessary conditions for efficiency for bilevel problems with multiple objectives only at the upper level are derived by Ye [402] who, under certain assumptions, replaces the lower-level SOP with its KKT optimality conditions. Zhang et al. [419] present necessary conditions for efficiency for multistage bilevel problems with multiple objectives at the upper level and a stage-dependent lower-level SOP. The solution of each SOP at the current stage depends on the solutions of the SOP and the upper-level MOP in the previous stage.

Some researchers link BLMOPs with SOPs or MOPs. Gadhi and Dember [156] establish an equivalence between a bilevel problem with multiple objectives only at the upper level and a single-level SOP, and derive necessary conditions for efficiency for the original problem. Eichfelder [127] shows that the feasible set of the upper-level problem of a general BLMOP can be expressed as a set of efficient solutions of an MOP. Ruuska et al. [316] provide sufficient conditions to reduce a BLMOP to an MOP.

A result on the nonemptiness of the solution set is established by Calvete and Galé [64] for linear bilevel problems with multiple objectives only at the upper level.

18.7.4.3 Methodology for Bilevel Multiobjective Programming

Methods for solving BLMOPs depend on the class of considered problems and the solution goal. The class may include linear or convex problems while the goal results from the paradigm resolving the multiobjective nature of the problem. In multiobjective programming, one is typically interested in finding the entire solution set of the MOP while in MCDM often only a particular element of this set, which optimizes decision maker's preferences, is sought. The latter often makes use of interactive MCDM methods that interchangeably navigate through the solution set and elicit preferences from the decision maker in order to arrive at a preferred solution. Following this distinction, methods for solving BLMOPs can generally be classified as methods aiming at the computation of the entire (or part of) solution set and methods supporting MCDM. Due to the scope of this chapter interactive methods requiring active involvement of a decision-maker are not discussed.

Some authors have proposed solution approaches to specially structured problems. The class of problems that had first been considered is related to performing the decision stage of MCDM during which a decision maker's utility function is optimized over the efficient set for the purpose of finding a preferred solution. In particular, optimization over the solution set for linear MOPs is extensively studied by Benson and his co-authors in a series of papers [32–34, 38–40] and also by Bolintinéanu [53], Ecker and Song [110], Dauer and Fosnaugh [95], Sayin [324], Thach [360], Horst and Thoai [201], Thi et al. [361–363], Jorge [218], Horst et al. [202], Thoai [365], and others, with a review provided by Yamamoto [398]. Optimization over the efficient set of nonlinear MOPs is studied by An et al. [11], Horst and Thoai [200], Thoai [364], Tuy and Hoai-Phuong [368], and others. More recently, Leyffer [246] and Faulkenberg and Wiecek [143] present bilevel methods to optimize a quality measure of a discrete representation of the solution set of single-level convex MOPs. Reasoning in the opposite direction, Liu [251] starts with a nonlinear BLSOP and transforms it into the problem of maximizing a function over the solution set of a related parametric linear MOP.

Based on the presented optimality conditions for BLMOPs with convex lower-level problems, Dell'Aere [100] develops solution algorithms of both subdivision and recovering type. An algorithm for bilevel problems with indefinite quadratic objective functions and polyhedral feasible sets is proposed by Arora and Arora [16].

Calvete and Gale [64] and Alves et al. [7] independently work on the same class of bilevel linear problems with multiple objectives only at the upper level. The former propose a solution approach using scalarizations of the upper-level problem such as the weighted-sum method, the ε-constraint method, and Benson's method, which results in solving a BLSOP, and show that applying the weighted-sum method does not guarantee finding the entire solution set to the overall problem. The latter reformulate the original problem into a linear MOP with mixed $0-1$ variables and offer a characterization of the Pareto set in the biobjective case. Calvete and Gale

[65] later work on bilevel problems with multiple linear objectives only at the lower level and polyhedral feasible sets, and reformulate this problem to an SOP with a nonconvex feasible set.

For noncooperative and cooperative linear BLMOPs, Sakawa and Nishizaki [319] propose solution methods based on scalarizations with a utility function or an achievement function, and based on fuzzy programming.

Krüger et al. [240] solve more general bilevel problems in the context of applications for mechatronic systems. Their bilevel problem has multiple MOPs on the lower level and an MOP on the upper level. Each lower-level MOP has only its own variables while the upper-level MOP has its own variable but parametrically depends on the lower-level efficient sets.

Algorithms proposed by Jahn and Schaller [213] and Eichfelder [127] seem to be the only global solvers for general nonconvex BLMOPs. In [213], the lower-level problem is parametrized by a scalar parameter varying in an interval. For an arbitrary value of the parameter, a search algorithm equipped with the subdivision technique MOSAST is applied to find a representation of the lower-level solution set. The Graef-Younes method is then used to collect solutions on the upper level and determine the solution set of the overall problem. Eichfelder's algorithm [127] is suitable for solving bilevel problems with two objectives at each level and a scalar decision variable at the upper-level problem. The Pascoletti-Serafini scalarization approach (18.18) and an adaptive parameter control based on a sensitivity analysis are used to generate solution points of the lower-level problem. For several discretizations of the upper-level variable, lower-level solution sets are approximated and unified into a set being a representation of the feasible set of the upper-level problem over which this problem is solved.

18.8 Current and Future Research Directions

The rapid development of optimization techniques and computational power over the last decades has made it possible to solve many MOPs of practically relevant size in reasonable time. At the same time we observe an increasing awareness of decision makers and analysts that it is necessary to incorporate multiple objectives in decision processes. Thus in the future we expect to see a growing number of new real-world applications of multiobjective programming.

In the last ten years, many application areas have already adopted a multi-objective modeling and solution paradigm. Küfer et al. [241] describe a linear MOP formulation of the radiation therapy planning problem. These models can have thousands of variables and tens of thousands of constraints; nevertheless an approximation of (a part of) the efficient set can be computed effectively. Ehrgott and Ryan [119] solve bicriteria set partitioning problems with a few hundred constraints and many thousands of variables for an application in airline crew scheduling. As a reaction to the recent economic and financial crisis, Gaganis et al. [157] reevaluate previous decision aid techniques and investigate the use of new criteria addressing the macroeconomic, institutional and regulatory environment in addition to basic characteristics of the banking and financial sector. Bruggemann and Patil [61] attempt to remedy the complexity of today's data-centric world by bridging between multiobjective programming and environmental and ecological statistics with applications to watersheds, biomanipulation, and engineering systems. Interestingly, Köhn [229] highlights the emerging use of multiobjective programming with modern applications in quantitative psychology, while Shoval et al. [339] associate Pareto optimality with evolution.

From this perspective we believe that the following are valuable directions of future research.

18.8.1 Research on Set-Oriented Methods

The advent of set-oriented methods rooted in stochastic principles or global optimization complements the earlier development of metaheuristics that are also set-oriented but based on different principles. Metaheuristics have provided the multiobjective programming community with algorithmic schemes having two important properties: (1) the schemes are very effective for problems on which exact algorithms fail; (2) the schemes are relatively easily adaptable to many special problems. At present, evolutionary techniques constitute probably the most successful approach to solving MOPs in practice and we expect this trend to continue. In particular, hybrid algorithms combining evolutionary approaches with principles of exact algorithms give promise of computational improvements [333]. However, the set-oriented methods have emerged as a viable alternative and we expect that these new methods will better establish themselves as a non-traditional optimization tool or working in concert with metaheuristics.

18.8.2 Theoretical and Methodological Studies Motivated by Mathematical and Real-Life Applications

Areas such as bilevel programming or robust optimization give opportunities to develop MOPs whose solutions are associated with the solutions of the optimization problems in those areas. While the synergy between bilevel programming and MOPs is addressed in this study, connections between robust optimization and MOPs have already been developed in the literature but are yet to be reviewed. We believe that other mathematical applications will come to light in the future. For real-life applications, there is no "one size fits all" methodology for MOPs. A method that works well in theory can fail in practice and one that works well on some problem may not be suitable for another one. So MOP methodology will increasingly be studied in problem contexts.

18.8.3 Applications in New Areas

Disciplines and research areas such as sustainable development, financial engineering, quality control, engineering design, processing and manufacturing of new materials, astronomy and medicine will continue to provide new opportunities for challenging applications of multiobjective programming.

18.8.4 Integration of Multiobjective Programming with Multicriteria Decision Analysis (MCDA)

In the current multicriteria decision-making (MCDM) methodology, multiobjective programming methods and MCDA methods are often seen as two ends of a spectrum. However, current applications indicate that both paradigms are needed in order for MCDM to succeed. In a majority of applications there is an objective stage, where multiobjective programming techniques are appropriate, and there is a subjective stage, where human judgment and preferences modeled within MCDA come into play. At this stage the formal mathematical approach is likely to be less effective and human factors-oriented strategies are needed to guide the decision maker. In any case, human participation will never be eliminated from the decision process but will be given stronger support by MCDM methodologies.

18.9 Conclusion

In this chapter we summarized the state of the art in continuous multiobjective programming. Our main attention has been devoted to optimality concepts, optimality conditions, solution techniques and approximations of the solution sets for general and specially structured MOPs. We recognize that the content of this chapter is subjective, as we excluded many facets of the subject such as duality and sensitivity, other stability results, variational inequalities, generalized convexity, nonsmoothness, Arrow-Barankin-Blackwell theorems, results for more general problems in vector spaces, other special classes of problems, etc. The topics of this chapter as well as other related topics have been discussed in other sources such as Ehrgott and Gandibleux [117], Engau [132], Gal et al. [159], and some other chapters in this book.

References

1. Abhyankar, S.S., Morin, T.L., Trafalis, T.: Efficient faces of polytopes: interior point algorithms, parametrization of algebraic varieties, and multiple objective optimization. Contemp. Math. **114**, 319–341 (1990)
2. Achary, K.K.: A direct simplex type method for determining all efficient solution pairs of time-cost transportation problem. Econ. Comput. Econ. Cybern. Stud. Res. **24**(2/3), 139–150 (1989)
3. Aghezzaf, B.: Efficiency and duality in multiobjective fractional programming. Opsearch **32**(1), 31–43 (1995)
4. Aghezzaf, B.: Second-order necessary conditions of the Kuhn-Tucker type in multiobjective programming problems. Control. Cybern. **28**(2), 213–224 (1999)
5. Aghezzaf, B., Hachimi, M.: Second order optimality conditions in multiobjective optimization problems. J. Optim. Theory Appl. **102**(1), 37–50 (1999)
6. Aghezzaf, B., Ouaderhman, T.: An interactive interior point algorithm for multiobjective linear programming problems. Oper. Res. Lett. **29**(4), 163–170 (2001)
7. Alves, M.J., Dempe, S., Júdice, J.J.: Computing the Pareto frontier of a bi-objective bi-linear problems using a multiobjective mixed-integer programming algorithm. Optimization **61**(3), 335–358 (2012)
8. Ammar, E.E.: On solutions of fuzzy random multiobjective quadratic programming with applications in portfolio problem. Inf. Sci. **178**(2), 468–484 (2008)
9. Ammar, E.E.: On fuzzy random multiobjective quadratic programming. Eur. J. Oper. Res. **193**(2), 329–341 (2009)
10. Ammar, E.E.: On solutions of generalized multiobjective quadratic with fuzzy random environments. J. Fuzzy Math. **19**(3), 695–710 (2011)
11. An, L.T.H., Tao, P.D., Muu, L.D.: Numerical solution for optimization over the efficient set by D.C. optimization algorithms. Oper. Res. Lett. **19**, 117–128 (1996)
12. Arbel, A.: An interior multiobjective primal-dual linear programming algorithm based on approximated gradients and efficient anchoring points. Comput. Oper. Res. **24**(4), 353–365 (1997)
13. Armand, P.: Finding all maximal efficient faces in multiobjective linear programming. Math. Program. **61**, 357–375 (1993)
14. Armand, P., Malivert, C.: Determination of the efficient set in multiobjective linear programming. J. Optim. Theory Appl. **70**(3), 467–489 (1991)
15. Armann, R.: Solving multiobjective programming problems by discrete representation. Optimization **20**, 483–492 (1989)
16. Arora, S.R., Arora, R.: Indefinite quadratic bilevel programming problem with multiple objectives at both levels. Int. J. Optim.: Theory Methods Appl. **1**(3), 318–327 (2009)
17. Baatar, D., Wiecek, M.M.: Advancing equitability in multiobjective programming. Comput. Math. Appl. **52**(1–2), 225–234 (2006)
18. Badra, N.M., Maaty, M.A.: Stability of multicriteria quadratic programming. Inst. Math. Comput. Sci. **11**(2), 133–140 (2000)
19. Balbás, A., Galperin, E., Jiménez Guerra, P.: Sensitivity of Pareto solutions in multiobjective optimization. J. Optim. Theory Appl. **126**(2), 247–264 (2005)
20. Bard, J.F.: An efficient point algorithm for a linear two-stage optimization problem. Oper. Res. **31**, 670–684 (1983)
21. Bard, J.F.: Practical Bilevel Optimization: Algorithms and Applications. Nonconvex Optimization and Its Applications. Kluwer, Dordrecht (1998)
22. Bárdossy, A., Bogárdi, I., Duckstein, L.: Composite programming as an extension of compromise programming. In: Serafini, P. (ed.) Mathematics of Multi Objective Optimization, pp. 375–408. Springer, Wien (1985)

23. Beato-Moreno, A., Luque-Calvo, P., Osuna-Gómez, R., Rufián-Lizana, A.: Finding efficient points in multiobjective quadratic programming with strictly convex objective functions. In: Caballero, R., Ruiz, F., Steuer, R.E. (eds.) Advances in Multiple Objective and Goal Programming. Lecture Notes in Economics and Mathematical Systems, vol. 455, pp. 365–373. Springer, Berlin (1997)
24. Beato-Moreno, A., Ruiz-Canales, P., Luque-Calvo, P.-L., Blanquero-Bravo, R.: Multiobjective quadratic problem: characterization of the efficient points. In: Crouzeix, J.P., Martinez-Legaz, J.E., Volle, M. (eds.) Generalized Convexity, Generalized Monotonicity: Recent Results, pp. 425–438. Kluwer, Dordrecht (1998)
25. Behringer, F.A.: Lexicographic quasiconcave multiobjective programming. Z. Oper. Res. **21**, 103–116 (1977)
26. Ben-Tal, A., El Ghaoui, L., Nemirovski, A.: Robust Optimization. Princeton Series in Applied Mathematics. Princeton University Press, Princeton (2009)
27. Benson, H.P.: Existence of efficient solutions for vector maximization problems. J. Optim. Theory Appl. **26**(4), 569–580 (1978)
28. Benson, H.P.: An improved definition of proper efficiency for vector maximization with respect to cones. J. Optim. Theory Appl. **71**, 232–241 (1979)
29. Benson, H.P.: Finding an initial efficient extreme point for a linear multiple objective program. J. Oper. Res. Soc. **32**, 495–498 (1981)
30. Benson, H.P.: Finding certain weakly-efficient vertices in multiple objective linear fractional programming. Manag. Sci. **31**(2), 240–245 (1985)
31. Benson, H.P.: Multiple-objective linear programming with parametric criteria coefficients. Manag. Sci. **31**, 461–474 (1985)
32. Benson, H.P.: An all linear programming relaxation algorithm for optimizing over the efficient set. J. Glob. Optim. **1**, 83–104 (1991)
33. Benson, H.P.: A finite, nonadjacent extreme-point search algorithm for optimization over the efficient set. J. Optim. Theory Appl. **73**(1), 47–64 (1992)
34. Benson, H.P.: A bisection-extreme point search algorithm for optimizing over the efficient set in the linear dependence case. J. Glob. Optim. **3**, 95–111 (1993)
35. Benson, H.P.: Further analysis of an outcome set-based algorithm for multiple-objective linear programming. J. Optim. Theory Appl. **97**(1), 1–10 (1998)
36. Benson, H.P.: Hybrid approach for solving multiple-objective linear programs in outcome space. J. Optim. Theory Appl. **98**(1), 17–35 (1998)
37. Benson, H.P.: An outer approximation algorithm for generating all efficient extreme points in the outcome set of a multiple objective linear programming problem. J. Glob. Optim. **13**, 1–24 (1998)
38. Benson, H.P., Lee, D.: Outcome-based algorithm for optimizing over the efficient set of a bicriteria linear programming problem. J. Optim. Theory Appl. **88**(1), 77–105 (1996)
39. Benson, H.P., Sayin, S.: Optimization over the efficient set: four special cases. J. Optim. Theory Appl. **80**(1), 3–18 (1992)
40. Benson, H.P., Sayin, S.: A face search heuristic algorithm for optimizing over the efficient set. Nav. Res. Logist. **40**, 103–116 (1993)
41. Benson, H.P., Sayin, S.: Towards finding global representations of the efficient set in multiple objective mathematical programming. Nav. Res. Logist. **44**, 47–67 (1997)
42. Benson, H.P., Sun, E.: Outcome space partition of the weight set in multiobjective linear programming. J. Optim. Theory Appl. **105**(1), 17–36 (2000)
43. Bento, G.C., Ferreira, O.P., Oliveira, P.R.: Unconstrained steepest descent method for multicriteria optimization on Riemannian manifolds. J. Optim. Theory Appl. **154**(1), 88–107 (2012)
44. Berezkin, V.E., Kamenev, G.K., Lotov, A.V.: Hybrid adaptive methods for approximating a nonconvex multidimensional Pareto frontier. Comput. Math. Math. Phys. **46**(11), 1918–1931 (2006)
45. Bergstresser, K., Yu, P.-L.: Domination structures and multicriteria problems in N-person games. Theor. Decis. **8**(1), 5–48 (1977)

46. Bergstresser, K., Charnes, A., Yu, P.-L.: Generalization of domination structures and nondominated solutions in multicriteria decision making. J. Optim. Theory Appl. **18**(1), 3–13 (1976)
47. Bernau, H.: Use of exact penalty functions to determine efficient decisions. Eur. J. Oper. Res. **49**, 348–355 (1990)
48. Bhatia, D., Gupta, P.: Generalized linear complementarity problem and multiobjective programming problem. Optimization **46**(2), 199–214 (1999)
49. Bigi, G., Castellani, M.: Second order optimality conditions for differentiable multiobjective programming. RAIRO Oper. Res. **34**, 411–426 (2000)
50. Bitran, G.R.: Linear multi-objective programs with interval coefficients. Manag. Sci. **26**, 694–706 (1980)
51. Blanco, V., Puerto, J., Ben Ali, S.E.H.: A semidefinite programming approach for solving multiobjective linear programming. J. Glob. Optim. **58**(3), 465–480 (2014)
52. Bokrantz, R., Forsgren, A.: An algorithm for approximating convex Pareto surfaces based on dual techniques. INFORMS J. Comput. **25**(2), 377–393 (2013)
53. Bolintinéanu, S.: Minimizing of a quasi-concave function over an efficient set. Nav. Res. Logist. **40**, 103–116 (1993)
54. Bolintinéanu, S., El Maghri, M.: Second-order efficiency conditions and sensitivity of efficient points. J. Optim. Theory Appl. **98**(3), 569–592 (1998)
55. Borwein, J.: Proper efficient points for maximization with respect to cones. SIAM J. Control. Optim. **15**(1), 57–63 (1977)
56. Borwein, J.M.: On the existence of Pareto efficient points. Math. Oper. Res. **8**(1), 64–73 (1983)
57. Borwein, J.M., Zhuang, D.M.: Super efficiency in convex vector optimization. Z. Oper. Res. **35**(3), 175–184 (1991)
58. Borwein, J.M., Zhuang, D.M.: Super efficiency in vector optimization. Trans. Am. Math. Soc. **338**(1), 105–122 (1993)
59. Bowman, V.J.: On the relationship of the Tchebycheff norm and the efficient frontier of multiple-criteria objectives. In: Thieriez, H. (ed.) Multiple Criteria Decision Making. Lecture Notes in Economics and Mathematical Systems, vol. 130. Springer, Berlin (1976)
60. Bozkurt, B., Fowler, J.W., Gel, E.S., Kim, B., Köksalan, M., Wallenius, J.: Quantitative comparison of approximate solution sets for multicriteria optimization problems with weighted Tchebycheff preference function. Oper. Res. **58**(3), 650–659 (2010)
61. Bruggemann, R., Patil, G.P.: Multicriteria prioritization and partial order in environmental and ecological statistics. J. Stat. Theory Pract. **4**(4), 631–641 (2010)
62. Buchanan, J., Gardiner, L.: A comparison of two reference point methods in multiple objective mathematical programming. Eur. J. Oper. Res. **149**, 17–34 (2003)
63. Buryak, Y.I., Insarov, V.V.: Control in many-object organizational systems. III. Parametric optimizationion a system with several objective functions. J. Comput. Syst. Sci. Int. **46**(5), 756–765 (2007)
64. Calvete, H.I., Galé, C.: Linear bilevel programs with multiple objectives at the upper level. J. Comput. Appl. Math. **234**, 950–959 (2010)
65. Calvete, H.I., Galé, C.: On linear bilevel problems with multiple objectives at the lower level. Omega **39**, 33–40 (2011)
66. Cambini, R.: Second order optimality conditions in multiobjective programming. Optimization **44**(2), 139–160 (1998)
67. Cambini, A., Martein, L., Cambini,. R.: A new approach to second order optimality conditions in vector optimization. In: Caballero, R., Ruiz, F., Steuer, R.E. (eds.) Advances in Multiple Objective and Goal Programming. Lecture Notes in Economics and Mathematical Systems, vol. 455, pp. 219–227. Springer, Berlin (1997)
68. Cambini, A., Martein, L., Stancu-Minasian, I.M.: A survey on bicriteria fractional problems. Math. Rep. (Romanian Academy, Bucureşti) **2**(2), 127–162 (2000)
69. Cambini, A., Luc, D.T., Martein, L.: Order-preserving transformations and applications. J. Optim. Theory Appl. **118**(2), 275–293 (2003)

70. Candler, W.: A linear bilevel programming algorithm: a comment. Comput. Oper. Res. **15**, 297–298 (1988)
71. Ceng, L.-C., Huang, S.: Existence theorems for generalized vector variational inequalities with a variable ordering relation. J. Glob. Optim. **46**(4), 521–535 (2010)
72. Chankong, V., Haimes, Y.Y.: Multiobjective Decision Making Theory and Methodology. Elsevier Science, New York (1983)
73. Charnes, A., Cooper, W.: Management Models and Industrial Applications of Linear Programming. Wiley, New York (1961)
74. Chen, G.Y., Yang, X.Q.: Characterizations of variable domination structures via nonlinear scalarization. J. Optim. Theory Appl. **112**(1), 97–110 (2002)
75. Chen, G.Y., Yang, X.Q., Yu, H.: A nonlinear scalarization function and generalized quasi-vector equilibrium problems. J. Glob. Optim. **32**(4), 451–466 (2005)
76. Chernykh, O.: Approximating the Pareto-hull of a convex set by polyhedral sets. Comput. Math. Math. Phys. **35**, 1033–1039 (1995)
77. Chew, K.L.: Domination structures in abstract spaces. In: Proceedings of the First Franco-Southeast Asian Mathematical Conference (Singapore, 1979), Vol. II, Special Issue b, pp. 190–204 (1979)
78. Chinchuluun, A., Yuan, D., Pardalos, P.M.: Optimality conditions and duality for nondifferentiable multiobjective fractional programming with generalized convexity. Ann. Oper. Res. **154**, 133–147 (2007)
79. Choo, E.U., Atkins, A.D.: Connectedness in multiple linear fractional programming. Manag. Sci. **29**(2), 250–255 (1983)
80. Chuong, T.D., Yao, J.-Ch.: Steepest descent methods for critical points in vector optimization problems. Appl. Anal. **91**(10), 1811–1829 (2012)
81. Churkina, S.Yu.: Search for efficient solutions of multi-criterion problems by target-level method. Comput. Math. Model. **13**(2), 208–214 (2002)
82. Clarke, P.A., Westerberg, A.W.: A note on the optimality conditions for the bilevel programming problem. Nav. Res. Logist. **35**(5), 413–418 (1988)
83. Cohon, J.L.: Multiobjective Programming and Planning. Academic, New York (1978)
84. Cohon, J.L., Church, R.L., Sheer, D.P.: Generating multiobjective tradeoffs: an algorithm for bicriterion problems. Water Resour. Res. **15**, 1001–1010 (1979)
85. Collette, Y., Siarry, P.: Three new metrics to measure the convergence of metaheuristics towards the Pareto frontier and the aesthetic of a set of solution in biobjective optimization. Comput. Oper. Res. **32**, 773–792 (2005)
86. Corley, H.W.: An existence result for maximization with respect to cones. J. Optim. Theory Appl. **31**(2), 277–281 (1980)
87. Costa, J.P.: An interative method for multiple objective linear fractional programming problems. OR Spectr. **27**(4), 633–652 (2005)
88. Craft, D.L., Halabi, T.F., Shih, H.A., Bortfeld, T.R.: Approximating convex Pareto surfaces in multiobjective radiotherapy planning. Med. Phys. **33**(9), 3399–3407 (2006)
89. Czyżak, P., Jaszkiewicz, A.: Pareto simulated annealing - a metaheuristic technique for multipleobjective combinatorial optimization. J. Multi-Criteria Decis. Anal. **7**, 34–47 (1998)
90. Da Cuhna, N.O., Polak, E.: Constrained minimization under vector-valued criteria in finite dimensional spaces. J. Math. Anal. Appl. **19**(1), 103–124 (1967)
91. Das, I.: An improved technique for choosing parameters for Pareto surface generation using normal-boundary intersection. Short Paper Proceedings of the Third World Congress of Structural and Multidisciplinary Optimization, vol. 2, pp. 411–413 (1999)
92. Das, I., Dennis, J.E.: Normal-boundary intersection: a new method for generating the Pareto surface in nonlinear multicriteria optimization problems. SIAM J. Optim. **8**(3), 631–657 (1998)
93. Daskilewicz, M.J., German, B.J.: Nondomination-level coordinate system for parametrizing and exploring Pareto frontiers. AIAA **51**(8), 1946–1959 (2013)
94. Dauer, J.P.: On degeneracy and collapsing in the construction of the set of objective values in a multiple objective linear program. Ann. Oper. Res. **46/47**(1–4), 279–292 (1993)

95. Dauer, J.P., Fosnaugh, T.A.: Optimization over the efficient set. J. Glob. Optim. **7**(3), 261–277 (1995)
96. Dauer, J.P., Gallagher, R.J.: A combined constraint-space, objective-space approach for determining high-dimensional maximal efficient faces of multiple objective linear programs. Eur. J. Oper. Res. **88**, 368–381 (1990)
97. Dauer, J.P., Liu, Y.H.: Solving multiple objective linear programs in objective space. Eur. J. Oper. Res. **46**, 350–357 (1990)
98. Dauer, J.P., Saleh, O.A.: Constructing the set of efficient objective values in multiple objective linear programs. Eur. J. Oper. Res. **46**, 358–365 (1990)
99. Deb, K., Pratap, A., Agarwal, S., Meyarivan, T.: A fast and elitist multiobjective genetic algorithm: Nsga-ii. IEEE Trans. Evol. Comput. **6**(2), 182–197 (2002)
100. Dell'Aere, A.: Numerical Methods for the Solution of Bi-Level Multi-Objective Optimization Problems. Ph.D. thesis, Universität Paderborn, Germany (2008)
101. Dellnitz, M., Witting, K.: Computation of robust pareto points. Int. J. Comput. Sci. Math. **2**(3), 243–266 (2009)
102. Dellnitz, M., Schütze, O., Hestermeyer, T.: Covering Pareto sets by multilevel subdivision techniques. J. Optim. Theory Appl. **124**(1), 113–136 (2005)
103. Dempe, S.: Foundations of Bilevel Programming. Nonconvex Optimization and Its Applications. Kluwer, New York (2006)
104. Désidéri, J.-A.: Multiple-gradient descent algorithm (MGDA) for multiobjective optimization. C. R. Acad. Bulg. Sci. - Ser. I **350**, 313–318 (2012)
105. Graña Drummond, L.M., Iusem, A.N.: A projected gradient method for vector optimization problems. Comput. Optim. Appl. **28**, 5–29 (2004)
106. Graña Drummond, L.M., Svaiter, B.F.: A steepest descent method for vector optimization. J. Comput. Appl. Math. **175**, 395–414 (2005)
107. Ebiefung, A.: Nonlinear mappings associated with the generalized linear complementarity problem. Math. Program. **69**(2), 255–268 (1995)
108. Ecker, J.G., Kouada, I.A.: Finding efficient points for linear multiple objective programs. Math. Program. **8**, 375–377 (1975)
109. Ecker, J.G., Kouada, I.A.: Finding all efficient extreme points for multiple objective linear programs. Math. Program. **14**(2), 249–261 (1978)
110. Ecker, J., Song, J.: Optimizing a linear function over the efficient set. J. Optim. Theory Appl. **83**, 541–563 (1994)
111. Ecker, J.G., Hegner, N.S., Kouada, I.A.: Generating all maximal efficient faces for multiple objective linear programs. J. Optim. Theory Appl. **30**(3), 353–381 (1980)
112. Efremov, R.V., Kamenev, G.K.: Properties of a method for polyhedral approximation of the feasible criterion set in convex multiobjective problems. Ann. Oper. Res. **166**, 271–279 (2009)
113. Ehrgott, M.: Discrete decision problems, multiple criteria optimization classes and lexicographic max-ordering. In: Stewart, T.J., van den Honert, R.C. (eds.) Trends in Multicriteria Decision Making. Lecture Notes in Economics and Mathematical Systems, vol. 465, pp. 31–44. Springer, Berlin (1998)
114. Ehrgott, M.: Multicriteria Optimization. Lecture Notes in Economics and Mathematical Systems, vol. 491. Springer, Berlin (2000)
115. Ehrgott, M.: The balance space approach to multicriteria decision making - involving the decision maker. Comput. Math. Appl. **44**(7), 909–923 (2002)
116. Ehrgott, M., Galperin, E.A.: Min-max formulation of the balance number in multiobjective global optimization. Comput. Math. Appl. **44**(7), 899–907 (2002)
117. Ehrgott, M., Gandibleux, X., (eds.): Multiple Criteria Optimization – State of the Art Annotated Bibliographic Surveys. Kluwer's International Series in Operations Research and Management Science, vol. 52. Kluwer, Boston (2002)
118. Ehrgott, M., Ruzika, S.: Improved ε-constraint method for multiobjective programming. J. Optim. Theory Appl. **138**, 375–396 (2008)
119. Ehrgott, M., Ryan, D.M.: Constructing robust crew schedules with bicriteria optimization. J. Multi-Criteria Decis. Anal. **11**, 139–150 (2002)

120. Ehrgott, M., Skriver, A.J.V.: Solving biobjective combinatorial max-ordering problems by ranking methods and a two-phases approach. Eur. J. Oper. Res. **147**, 657–664 (2003)
121. Ehrgott, M., Hamacher, H.W., Klamroth, K., Nickel, S., Schöbel, A., Wiecek, M.M.: A note on the equivalence of balance points and Pareto solutions in multiple-objective programming. J. Optim. Theory Appl. **92**(1), 209–212 (1997)
122. Ehrgott, M., Nickel, S., Hamacher, H.W.: Geometric methods to solve max-ordering location problems. Discret. Appl. Math. **93**, 3–20 (1999)
123. Ehrgott, M., Puerto, J., Rodríguez-Chía, A.M.: Primal-dual simplex method for multiobjective linear programming. J. Optim. Theory Appl. **134**(3), 483–497 (2007)
124. Ehrgott, M., Shao, L., Schöbel, A.: An approximation algorithm for convex multi-objective programming problems. J. Glob. Optim. **50**(3), 397–416 (2011)
125. Ehrgott, M., Löhne, A., Shao, L.: A dual variant of Benson's "outer approximation algorithm" for multiple objective linear programming. J. Glob. Optim. **52**(4), 757–778 (2012)
126. Eichfelder, G.: An adaptive scalarization method in multiobjective optimization. SIAM J. Optim. 19(4), 1694–1718 (2009)
127. Eichfelder, G.: Multiobjective bilevel optimization. Math. Program. **123**(2, Ser. A), 419–449 (2010)
128. Eichfelder, G.: Optimal elements in vector optimization with a variable ordering structure. J. Optim. Theory Appl. **151**(2), 217–240 (2011)
129. Eichfelder, G.: Variable ordering structures in vector optimization. In: Recent Developments in Vector Optimization. Vector Optimization, pp. 95–126. Springer, Berlin (2012)
130. Eichfelder, G., Ha, T.X.D.: Optimality conditions for vector optimization problems with variable ordering structures. Optimization **62**(5), 597–627 (2013)
131. Engau, A.: Variable preference modeling with ideal-symmetric convex cones. J. Glob. Optim. **42**(2), 295–311 (2008)
132. Engau, A.: Nonlinear multiobjective programming. In: Wiley Encyclopedia of Operations Research and Management Science. Wiley, New York (2010)
133. Engau, A., Wiecek, M.M.: Cone characterizations of approximate solutions in real vector optimization. J. Optim. Theory Appl. **134**(3), 499–513 (2007)
134. Engau, A., Wiecek, M.M.: Introducing nonpolyhedral cones to multiobjective programming. In: Barichard, K. (ed.) Multiobjective Programming and Goal Programming: Theoretical Results and Practical Applications. Lecture Notes in Economics and Mathematical Systems, vol. 618, pp. 35–45. Springer, Berlin (2009)
135. Enkhbat, R., Guddat, J., Chinchuluun, A.: Parametric multiobjective optimization. In: Chinchuluun, A., Pardalos, P.M., Migdalas, A., Pitsoulis, L. (eds.) Pareto Optimality, Game Theory and Equilibria. Optimization and Its Applications, vol. 17, pp. 529–538. Springer, New York (2008)
136. Ester, J., Tröltzsch, F.: On generalized notions of efficiency in multicriteria decision making. Syst. Anal. Model. Simul. **3**, 153–161 (1986)
137. Eusébio, A., Figueira, J.R., Ehrgott, M.: A primal-dual simplex algorithm for bi-objective network flow problems. 4OR **7**(3), 255–273 (2009)
138. Evans, J.P., Steuer, R.E.: A revised simplex method for linear multiple objective programs. Math. Program. **5**, 375–377 (1973)
139. Fang, Y.P., Yang, X.Q.: Smooth representations of optimal solution sets of piecewise linear parametric multiobjective programs. In: Variational Analysis and Generalized Differentiation in Optimization and Control. Springer Optimization and Its Applications , vol. 47, pp. 163–176. Springer, New York (2010)
140. Fang, Y.P., Meng, K., Yang, X.Q.: Piecewise linear multicriteria programs: the continuous case and its discontinuous generalization. Oper. Res. **60**(2), 398–409 (2012)
141. Farhang-Mehr, A., Azarm, S.: An information-theoretic entropy metric for assessing multi-objective optimization solution set quality. J. Mech. Des. **125**, 655–663 (2003)
142. Faulkenberg, S., Wiecek, M.M.: On the quality of discrete representations in multiple objective programming. Optim. Eng. **11**(3), 423–440 (2010)

143. Faulkenberg, S.L., Wiecek, M.M.: Generating equidistant representations in biobjective programming. Comput. Optim. Appl. **51**(3), 1173–1210 (2012)
144. Fernández, J., Tóth, B.: Obtaining an outer approximation of the efficient set of nonlinear biobjective problems. J. Glob. Optim. **38**, 315–331 (2007)
145. Fliege, J.: Gap-free computation of Pareto-points by quadratic scalarizations. Math. Methods Oper. Res. **59**(1), 69–89 (2004)
146. Fliege, J., Heseler, A.: Constructing approximations to the efficient set of convex quadratic multiobjective problems. Technical Report Ergebnisberichte Angewandte Mathematik No. 211, Universität Dortmund, Fachbereich Mathematik (2002)
147. Fliege, J., Svaiter, B.F.: Steepest descent methods for multicriteria optimization. Math. Methods Oper. Res. **51**, 479–494 (2000)
148. Fliege, J., Vicente, L.N.: Multicriteria approach to bilevel optimization. J. Optim. Theory Appl. **131**(2), 209–225 (2006)
149. Fliege, J., Graña Drummond, L.M., Svaiter, B.F.: Newton's method for multiobjective optimization. SIAM J. Optim. **20**(2), 602–626 (2009)
150. Fruhwirth, B., Mekelburg, K.: On the efficient point set of tricriteria linear programs. Eur. J. Oper. Res. **72**, 192–199 (1994)
151. Fruhwirth, B., Burkard, R.E., Rote, G.: Approximation of convex curves with application to the bicriterial minimum cost flow problem. Eur. J. Oper. Res. **42**, 326–388 (1989)
152. Fu, Y., Diwekar, U.M.: An efficient sampling approach to multiobjective optimization. Ann. Oper. Res. **132**, 109–134 (2004)
153. Fukuda, E.H., Graña Drummond, L.M.: On the convergence of the projected gradient method for vector optimization. Optimization **60**(8–9), 1009–1021 (2011)
154. Fukuda, E.H., Graña Drummond, L.M.: Inexact projected gradient method for vector optimization. Comput. Optim. Appl. **54**, 473–493 (2013)
155. Fülop, J.: On the equivalence between a linear bilevel programming problem and linear optimization over the efficient set. Technical Report WP93-1, Laboratory of Operations Research and Decision Systems, Computer and Automation Institute, Hungarian Academy of Sciences (1993)
156. Gadhi, N., Dempe, S.: Necessary optimality conditions and a new approach to multiobjective bilevel optimization problems. J. Optim. Theory Appl. **155**(1), 100–114 (2012)
157. Gaganis, C., Pasiouras, F., Doumpos, M., Zopounidis, C.: Modelling banking sector stability with multicriteria approaches. Optim. Lett. **4**(4), 543–558 (2010)
158. Gal, T.: A general method for determining the set of all efficient solutions to a linear vectormaximum problem. Eur. J. Oper. Res. **1**, 307–322 (1977)
159. Gal, T., Stewart, T., Hanne, T. (eds.): Multicriteria Decision Making – Advances in MCDM Models, Algorithms, Theory, and Applications. Kluwer's International Series in Operations Research and Management Science, vol. 21. Kluwer, Norwell (2002)
160. Galperin, E.A.: Nonscalarized multiobjective global optimization. J. Optim. Theory Appl. **75**(1), 69–85 (1992)
161. Galperin, E.A., Wiecek, M.M.: Retrieval and use of the balance set in multiobjective global optimization. Comput. Math. Appl. **37**(4/5), 111–123 (1999)
162. Gao, Y., Rong, W.: Optimality conditions and duality in multiobjective generalized fractional programming with generalized convexity. Pac. J. Optim. **5**(3), 403–413 (2009)
163. Gardenghi, M., Gómez, T., Miquel, F., Wiecek, M.M.: Algebra of efficient sets for complex systems. J. Optim. Theory Appl. **149**, 385–410 (2011)
164. Gardenghi, M., Wiecek, M.M.: Efficiency for multiobjective multidisciplinary optimization problems with quasiseparable subproblems. Optim. Eng. **13**(2), 293–318 (2012)
165. Gearhart, W.B.: Compromise solutions and estimation of the noninferior set. J. Optim. Theory Appl. **28**, 29–47 (1979)
166. Gembicki, F.W., Haimes, Y.Y.: Approach to performance and multiobjective sensitivity optimization: the goal attainment method. IEEE Trans. Autom. Control **20**, 769–771 (1975)
167. Geoffrion, A.M.: Proper efficiency and the theory of vector maximization. J. Math. Anal. Appl. **22**, 618–630 (1968)

168. Gert, C., Weidner, P.: Nonconvex separation theorems and some applications in vector optimization. J. Optim. Theory Appl. **67**, 297–320 (1990)
169. Glackin, J.: Solving Bilevel Binear Programs Using Multiple Objective Linear Programming. Ph.D. thesis, Rensselaer Polytechnic Institute, Troy (2006)
170. Glackin, J., Ecker, J.G., Kupferschmid, M.: Solving bilevel linear programs uisng multiple objective linear programming. J. Optim. Theory Appl. **140**, 197–212 (2009)
171. Goh, C.J., Yang, X.Q.: Analytic efficient solution set for multi-criteria quadratic programs. Eur. J. Oper. Res. **92**(1), 166–181 (1996)
172. Gorrisen, B.L., den Hertog, D.: Approximating the Pareto set of multiobjective linear programs via robust optimization. Oper. Res. Lett. **40**(5), 319–324 (2012)
173. Gourion, D., Luc, D.T.: Generating the weakly efficient set of nonconvex multiobjective problems. J. Glob. Optim. **41**(4), 517–538 (2008)
174. Gourion, D., Luc, D.T.: Finding efficient solutions by free disposal outer approximation. SIAM J. Optim. **20**(6), 2939–2958 (2010)
175. Guddat, J., Vasquez, F.G., Tammer, K., Wendler, K.: Multiobjective and Stocahstic Optimization Based on Parametric Optimization. Mathematical Research, vol. 26. Akademie, Berlin (1985)
176. Guddat, J., Vazquez, F.G., Jongen, H.Th.: Parametric Optimization: Singularities, Pathfollowing and Jumps. Wiley, Chichester (1990)
177. Gulati, T.R., Islam, M.A.: Proper efficiency in a linear fractional vector maximum problem with generalized convex constraints. Eur. J. Oper. Res. **36**(3), 339–345 (1988)
178. Gulati, T.R., Islam, M.A.: Efficiency in linear fractional vector maximization problem with nonlinear constraints. Optimization **20**(4), 477–482 (1989)
179. Gupta, B.: Finding the set of all efficient solutions for the linear fractional multiobjective program with zero-one variables. Opsearch **18**(4), 204–214 (1982)
180. Gupta, B.: Programming with multi-objective linear fractional functionals. Acta Ciencia Indica Math. **9**(1–4), 195–201 (1983)
181. Gupta, R., Malhotra, R.: Multi-criteria integer linear fractional programming problem. Optimization **35**(4), 373–389 (1995)
182. Gutiérrez, C., Jiménez, B., Novo, V.: On approximate efficiency in multiobjective programming. Math. Methods Oper. Res. **64**(1), 165–185 (2006)
183. Gutiérrez, C., Jiménez, B., Novo, V.: On second-order Fritz John type optimality conditions in nonsmooth multiobjective programming. Math. Program. **123**(1, Ser. B), 199–223 (2010)
184. Gutiérrez, C., Jiménez, B., Novo,V.: Equivalent ϵ-efficiency notions in vector optimization. TOP **20**(2), 437–455 (2012)
185. Haanpää, T.: Approximation Method for Computationally Expensive Nonconvex Multiobjective Optimization Problems. Jyväskylä Studies in Computing, vol. 157. Jyväskylän Yliopisto, Jyväskylä (2012)
186. Hartikainen, M., Miettinen, K., Wiecek, M.M.: Constructing a Pareto front approximation for decision making. Math. Methods Oper. Res. **73**, 209–234 (2011)
187. Hartikainen, M., Miettinen, K., Wiecek, M.M.: Decision making on Pareto front approximations with inherent nondominance. In: New State of MCDM in the 21st Century. Lecture Notes in Economics and Mathematical Systems, vol. 648, pp. 35–45. Springer, Heidelberg (2011)
188. Hartikainen, M., Miettinen, K., Wiecek, M.M.: PAINT: Pareto front interpolation for nonlinear multiobjective optimization. Comput. Optim. Appl. **52**, 845–867 (2012)
189. Hartley, R.: On cone-efficiency, cone-convexity and cone-compactness. SIAM J. Appl. Math. **34**(2), 211–222 (1978)
190. Haurie, A., Savard, G., White, D.: A note on an efficient point algorithm for a linear two-stage optimization problem. Oper. Res. **38**, 553–555 (1990)
191. Hazen, G.B.: Differential characterizations of nonconical dominance in multiple objective decision making. Math. Oper. Res. **13**(1), 174–189 (1988)
192. Hazen, G.B., Morin, T.L.: Nonconical optimality conditions: some additional results. J. Optim. Theory Appl. **41**(4), 619–623 (1983)

193. Hazen, G.B., Morin, T.L.: Optimality conditions in nonconical multiple-objective programming. J. Optim. Theory Appl. **40**(1), 25–60 (1983)
194. Helbig, S.: Approximation of the efficient point set by perturbation of the ordering cone. ZOR: Methods Models Oper. Res. **35**, 197–220 (1991)
195. Helbig, S.: On a constructive approximation of the efficient outcomes in bicriterion vector optimization. J. Glob. Optim. **5**, 35–48 (1994)
196. Henig, M.: Proper efficiency with respect to cones. J. Optim. Theory Appl. **36**(3), 387–407 (1982)
197. Heyde, F., Löhne, F.: Geometric duality in multiple objective linear programming. SIAM J. Comput. **19**(2), 836–845 (2008)
198. Hillermeier, C.: Nonlinear Multiobjective Optimization - A Generalized Homotopy Approach. Birkhäuser, Basel (2001)
199. Hirschberger, M.: Computation of efficient compromise arcs in convex quadratic multicriteria optimization. J. Glob. Optim. **31**(3), 535–546 (2005)
200. Horst, R., Thoai, N.V.: Utility function programs and optimization over the efficient set in multiple-objective decision making. J. Optim. Theory Appl. **92**(3), 605–631 (1997)
201. Horst, R., Thoai, N.V.: Maximizing a convex function over the efficient or weakly efficient set. Eur. J. Oper. Res. **117**, 239–252 (1999)
202. Horst, R., Thoai, N.V., Yamamoto, Y., Zenke, D.: On optimization over the efficient set in linear multicriteria programming. J. Optim. Thoery Appl. **134**, 433–443 (2007)
203. Huang, N.J., Yang, X.Q., Chan, W.K.: Vector complementarity problems with a variable ordering relation. Eur. J. Oper. Res. **176**(1), 15–26 (2007)
204. Hunt, B.J., Wiecek, M.M.: Cones to aid decision making in multicriteria programming. In: Tanino, T., Tanaka, T., Inuiguchi, M. (eds.) Multi-Objective Programming and Goal Programming, pp. 153–158. Springer, Berlin (2003)
205. Isac, G., Kostreva, M.M., Wiecek, M.M.: Multiple-objective approximation of feasible but unsolvable linear complementarity problems. J. Optim. Theory Appl. **86**(2), 389–405 (1995)
206. Isermann, H.: The enumeration of the set of all efficient solutions for a linear multiple objective program. Oper. Res. Q. **28**(3), 711–725 (1977)
207. Isermann, H.: Linear lexicographic optimization. OR Spectr. **4**, 223–228 (1982)
208. Ismail-Yahaya, A., Messac, A.: Effective generation of the Pareto frontier: the normalized normal constraint method. In: 43rd AIAA/ASME/ASCE/AHS Structures, Structural Dynamics, and Materials Conference, Denver, CO, April 2002, vol. AIAA 2002-1232 (2002)
209. Ivanenko, D.S., Plyasunov, A.V.: On the reducibility of a bilevel programming problem to a vector optimization problems. In: Proceedings of the Institute of Computational Mathematics and Mathematical Geophysics, number 5 in Ser. Inform., pp. 132–143, Novosibirsk (2005)
210. Ivanenko, D.S., Plyasunov, A.V.: Reducibility of bilevel programming problems to vector optimization problems. J. Appl. Ind. Math. **2**(2), 179–195 (2008)
211. Jahn, J.: Multiobjective search algorithm with subdivision technique. Comput. Optim. Appl. **35**, 161–175 (2006)
212. Jahn, J., Merkel, A.: Reference point approximation method for the solution of bicriterial nonlinear optimization problems. J. Optim. Theory Appl. **74**, 87–103 (1992)
213. Jahn, J., Schaller, E.: Global solver for nonlinear bilevel vector optimization problems. Technical report, Institute of Applied Mathematics, Universität Erlangen-Nürnberg (2008)
214. Jeyakumar, V., Yang, X.Q.: Convex composite multi-objective nonsmooth programming. Math. Program. **59**(3), 325–343 (1993)
215. Jimenez, B., Novo, V.: First and second order sufficient conditions for strict minimality in multiobjective programming. Numer. Funct. Anal. Optim. **23**(3/4), 303–322 (2002)
216. Jin, Y., Sendhoff, B.: Applications of Evolutionary Computing. Lecture Notes in Computer Science, vol. 3005, pp. 525–536. Springer, New York (2004)
217. Jones, D.F., Tamiz, M.: Goal programming in the period 1900–2000. In: Ehrgott, M., Gandibleux, X. (eds.) Multiple Criteria Optimization: State of the Art Annotated Bibliographic Surveys, pp. 129–170. Kluwer, Boston (2002)

218. Jorge, J.: A bilinear algorithm for optimizing a linear function over the efficient set of a multiple objective linear programming problem. J. Glob. Optim. **31**, 1–16 (2005)
219. Kaliszewski, I.: A modified weighted Tchebycheff metric for multiple objective programming. Comput. Oper. Res. **14**(4), 315–323 (1987)
220. Kaliszewski, I.: Quantitative Pareto Analysis by Cone Separation Technique. Kluwer, Dordrecht (1994)
221. Kamenev, G.K.: Study of an adaptive single-phase method for approximating the multidimensional Pareto frontier in nonlinear systems. Comput. Math. Math. Phys. **49**(1), 2006–2016 (2009)
222. Karaivanova, J., Korhonen, P., Narula, S., Wallenius, J., Vassilev, V.: A reference direction approach to multiple objective integer linear programming. Eur. J. Oper. Res. **81**, 176–187 (1995)
223. Karasakal, E., Köksalan, M.: Generating a representative subset of the efficient frontier in multiple criteria decision making. Oper. Res. **57**(1), 187–199 (2009)
224. Kaul, R.N., Lyall, V.: A note on nonlinear fractional vector maximization. Opsearch **26**(2), 108–121 (1989)
225. Kim, I.Y., de Weck, O.L.: Adaptive weighted sum method for multiobjective optimization: a new method for Pareto front generation. Struct. Multidiscip. Optim. **31**, 105–116 (2006)
226. Kim, M.H., Kim, G.S., Lee, G.M.: On ϵ-optimality conditions for multiobjective fractional optimization problems. Fixed Point Theory Appl. **6**, 1–13 (2011)
227. Klamroth, K., Tind, J., Wiecek, M.M.: Unbiased approximation in multicriteria optimization. Math. Methods Oper. Res. **36**(3), 413–437 (2002)
228. Klinger, A.: Improper solutions of the vector maximum problem. Oper. Res. **15**(3), 570–572 (1967)
229. Köhn, H.-F.: A review of multiobjective programming and its application in quantitative psychology. J. Math. Psychol. **55**(5), 386–396 (2011)
230. Korhonen, P., Wallenius, J.: A Pareto race. Nav. Res. Logist. **35**, 615–623 (1988)
231. Korhonen, P., Yu, G.Y.: A reference direction approach to multiple objective quadratic-linear programming. Eur. J. Oper. Res. **102**(3), 601–610 (1997)
232. Kornbluth, J.S.H., Steuer, R.E.: On computing the set of all weakly efficient vertices in multiple objective linear fractional programming. In: Fandel, G., Gal, T. (eds.) Multiple Criteria Decision Making Theory and Application. Lecture Notes in Economics and Mathematical Systems, pp. 189–202. Springer, Berlin (1980)
233. Kostreva, M.M., Ogryczak, W.: Linear optimization with multiple equitable criteria. RAIRO Oper. Res. **33**, 275–297 (1999)
234. Kostreva, M.M., Wiecek, M.M.: Linear complementarity problems and multiple objective programming. Math. Program. **60**, 349–359 (1993)
235. Kostreva, M.M., Yang, X.Q.: Unified approaches for solvable and unsolvable linear complementarity problems. Eur. J. Oper. Res. **158**(2), 409–417 (2004)
236. Kostreva, M.M., Ordoyne, T.J., Wiecek, M.M.: Multiple objective programming with polynomial objectives and constraints. Eur. J. Oper. Res. **57**, 381–394 (1992)
237. Kostreva, M.M., Zheng, Q., Zhuang, D.: A method for approximating solutions of multicriteria nonlinear optimization problems. Optim. Methods Softw. **5**, 209–226 (1995)
238. Kostreva, M.M., Ogryczak, W., Wierzbicki, A.: Equitable aggregations and multiple criteria analysis. Eur. J. Oper. Res. **158**, 362–377 (2004)
239. Kouvelis, P., Yu, G.: Robust Discrete Optimization and Its Applications. Nonconvex Optimization and its Applications, vol. 14. Kluwer, Dordrecht (1997)
240. Krüger, M., Witting, K., Trächtler, A., Dellnitz, M.: Parametric model-order reduction in hierarchical multiobjective optimization of mechatronic systems. In: Preprints of the 18th IFAC World Congress, pp. 12611–12619, Milano (2011)
241. Küfer, K.H., Scherrer, A., Monz, M., Alonso, F., Trinkaus, H., Bortfeld, T., Thieke, C.: Intensity-modulated radiotherapy – a large scale multi-criteria programming problem. OR Spectr. **25**(2), 223–249 (2003)

242. Kuhn, H.W., Tucker, A.W.: Nonlinear programming. In: Neyman, J. (ed.) Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, pp. 481–492. University of California Press, Berkeley (1951)
243. Kutateladze, S.S.: Convex ϵ-programming. Sov. Math. Dokl. **20**, 391–393 (1979)
244. Lemaire, B.: Approximation in multiobjective optimization. J. Glob. Optim. **2**, 117–132 (1992)
245. Leung, Y.-W., Wang, Y.: U-measure: A quality measure for multiobjective programming. IEEE Trans. Syst. Man Cybern. Part A: Syst. Hum. **33**(3), 337–343 (2003)
246. Leyffer, S.: A complementarity constraint formulation of convex multiobjective optimization problems. INFORMS J. Comput. **21**(2), 257–267 (2009)
247. Li, D.: Convexification of a noninferior frontier. J. Optim. Theory Appl. **88**(1), 177–196 (1996)
248. Li, Y., Fadel, G.M., Wiecek, M.M.: Minimum effort approximation of the Pareto space of convex bi-criteria problems. Optim. Eng. **4**, 231–261 (2003)
249. Lin, J.G.: Multiple-objective problems: Pareto-optimal solutions by methods of proper equality constraints. IEEE Trans. Autom. Control **21**(5), 641–650 (1976)
250. Lin, S.A.Y.: A comparison of Pareto optimality and domination structure. Metroeconomica **28**(1–3), 62–74 (1976)
251. Liu, Y.: An approach for a class of bilevel multiobjective programming problem. In: 2009 International Conference on Computational Intelligence and Security, vol. 1, pp. 128–132, Beijing (2009)
252. Liu, H., Fan, N., Pardalos, P.M.: Generalized Lagrange function and generalized weak saddle points for a class of multiobjective fractional optimal control problems. J. Optim. Theory Appl. **154**(2), 370–381 (2012)
253. Long, X.J.: Optimality conditions and duality for nondifferentiable multiobjective fractional programming problems with (C, α, ρ, d)-convexity. J. Optim. Theory Appl. **148**(1), 197–208 (2011)
254. Lootsma, F.A., Athan, T.W., Papalambros, P.Y.: Controlling the search for a compromise solution in multi-objective optimization. Eng. Optim. **25**(1), 65–81 (1995)
255. Loridan, P.: ε-Solutions in vector minimization problems. J. Optim. Theory Appl. **43**, 265–276 (1984)
256. Lotov, A.V., Maiskaya, T.S.: Nonadaptive methods for polyhedral approximation of the Edgeworth-Pareto hull using suboptimal coverings on the direction sphere. Comput. Math. Math. Phys. **52**(1), 31–42 (2012)
257. Lotov, A.V., Kamenev, G.K., Berezkin, V.E.: Approximation and visualization of the Pareto frontier for nonconvex muilticriteria problems. Dokl. Akad. Nauk **386**, 738–741 (2002)
258. Lotov, A.V., Bushenkov, V.A., Kamenev, G.K.: Interactive Decision Maps: Approximation and Visualization of Pareto Frontier. Kluwer, Dordrecht (2004)
259. Lovison, A.: Singular continuation: generating piecewise linear approximations to Pareto sets via global analysis. SIAM J. Optim. **21**(2), 463–490 (2011)
260. Lovison, A.: Global search perspectives for multiobjective optimization. J. Glob. Optim. **57**(2), 385–398 (2012)
261. Luc, D.T., Phong, T.Q., Volle, M.: Scalarizing functions for generating the weakly efficient solution set in convex multiobjective problems. SIAM J. Optim. **15**(4), 987–1001 (2005)
262. Marchi, E., Oviedo, J.A.: Lexicographic optimality in the multiple objective linear programming: the nucleolar solution. Eur. J. Oper. Res. **57**, 355–359 (1992)
263. Marcotte, P., Savard, G.: A note on the Pareto optimality of solutions to the linear bilevel programming problem. Comput. Oper. Res. **18**(4), 355–359 (1991)
264. Marshall, A.W., Olkin, I.: Inequalities: Theory of Majorization and Its Applications. Academic, New York (1979)
265. Martin, J., Bielza, C., Insua, D.R.: Approximating nondominated sets in continuous multiobjective optimization problems. Nav. Res. Logist. **52**, 469–480 (2005)
266. Masin, M., Bukchin, Y.: Diversity maximization approach for multiobjective optimization. Oper. Res. **56**(2), 411–424 (2008)

267. Mattson, C.A., Mullur, A.A., Messac, A.: Smart pareto filter: obtaining a minimal representation of multiobjective design space. Eng. Optim. **36**(6), 721–740 (2004)
268. Meng, H.-Y., Zhang, X.-H., Liu, S.-Y.: New quality measures for multiobjective programming. In: Wang, L., Chen, K., Ong, Y.S. (eds.) Advances in Natural Computation First International Conference, ICNC 2005. Lecture Notes in Computer Science, vol. 3611, pp. 1044–1048 (1998)
269. Meng, Z., Shen, R., Jiang, M.: An objective penalty function algorithm for multiobjective optimization problem. Am. J. Oper. Res. **1**, 229–235 (2011)
270. Messac, A.: Physical programming effective optimization for computational design. AIAA J. **34**(1), 149–158 (1996)
271. Messac, A., Mattson, C.A.: Generating well-distributed sets of Pareto points for engineering design using physical programming. Optim. Eng. **3**, 431–450 (2002)
272. Messac, A., Mattson, C.A.: Normal constraint method with guarantee of even representation of complete Pareto frontier. AIAA J. **42**(10), 2101–2111 (2004)
273. Messac, A., Ismail-Yahaya, A., Mattson, C.A.: The normalized normal constraint method for generating the Pareto frontier. Struct. Multidiscip. Optim. **25**, 86–98 (2003)
274. Metev, B., Gueorguieva, D.: A simple method for obtaining weakly efficient points in multiobjective linear fractional programming problems. Eur. J. Oper. Res. **126**(2), 386–390 (2000)
275. Michalowski, W., Szapiro, T.: A bi-reference procedure for interactive multiple criteria programming. Oper. Res. **40**(2), 247–258 (1992)
276. Miettinen, K.: Nonlinear Multiobjective Optimization. International Series in Operations Research and Management Science, vol. 12. Kluwer, Dordrecht (1999)
277. Migdalas, A., Pardalos, P.M., Värbrand, P. (eds.): Multilevel Optimization: Algorithms and Applications. Nonconvex Optimization and Its Applications. Kluwer, Dordrecht (1998)
278. Miglierina, E., Mohlo, E., Recchioni, M.C.: Box-constrained mutli-objective optimization: a gradient like method without "a priori" scalarization. Eur. J. Oper. Res. **188**, 662–682 (2008)
279. Mishra, S.K., Jaiswal, M., Verma, R.U.: Optimality and duality for nonsmooth multiobjective fractional semi-infinite programming problem. Adv. Nonlinear Var. Inequal. **16**(1), 69–83 (2013)
280. Moosavian, S.A.A., Ghaffari, A., Salimi, A.: Sequential quadratic programming and analytic hierarchy process for nonlinear multiobjective optimization of a hydropower network. Optim. Control Appl. Methods **31**(4), 351–364 (2010)
281. Morse, J.N.: Reducing the size of the nondominated set: pruning by clustering. Comput. Oper. Res. **7**, 55–66 (1980)
282. Müller-Gritschneder, D., Graeb, H., Schlichtmann, U.: A successive approach to compute the bounded Pareto front of practical multiobjective optimization problems. SIAM J. Optim. **20**(2), 915–934 (2009)
283. Mut, M., Wiecek, M.M.: Generalized equitable preference in multiobjective programming. Eur. J. Oper. Res. **12**(3), 535–551 (2011)
284. Naccache, P.H.: Connectedness of the set of nondominated outcomes in multicriteria optimization. J. Optim. Theory Appl. **25**, 459–467 (1978)
285. Nefedov, V.N.: On the approximation of the Pareto set. USSR Comput. Math. Math. Phys. **24**(4), 19–28 (1984)
286. Nickel, S., Wiecek, M.M.: Multiple objective programming with piecewise linear functions. J. Multi-Criteria Decis. Anal. **8**, 322–332 (1999)
287. Niculescu, C.: Optimality and duality in multiobjective fractional programming involving ρ-semilocally type I-preinvex and related functions. J. Math. Anal. Appl. **335**(1), 7–19 (2007)
288. Nie, P.-Y.: A note on bilevel optimization problems. Int. J. Appl. Math. Sci. **2**(1), 31–38 (2005)
289. Nishizaki, I., Sakawa, M.: Stackelberg solutions to multiobjective two-level linear programming problems. J. Optim. Theory Appl. **103**(1), 161–182 (1999)
290. Ogryczak, W.: Inequality measures and equitable approaches to location problems. Eur. J. Oper. Res. **122**, 374–391 (2000)

291. Ogryczak, W.: Multiple criteria linear programming model for portfolio selection. Ann. Oper. Res. **97**, 143–162 (2000)
292. Ogryczak, W., Ruszczyński, A.: Dual stochastic dominance and related mean-risk models. SIAM J. Optim. **13**, 60–78 (2002)
293. Ogryczak, W., Wierzbicki, A., Milewski, M.: A multi-criteria approach to fair and efficient bandwidth allocation. Omega **36**, 451–463 (2008)
294. Ogryczak, W., Zawadzki, M.: Conditional median: a parametric solution concept for location problems. Ann. Oper. Res. **110**, 167–181 (2002)
295. Pascoletti, A., Serafini, P.: Scalarizing vector optimization problems. J. Optim. Theory Appl. **42**(4), 499–524 (1984)
296. Payne, A.N.: Efficient approximate representation of bi-objective tradeoff sets. J. Frankl. Inst. **330**, 1219–1233 (1993)
297. Payne, A.N., Polak, E.: An interactive rectangle elimination method for biobjective decision making. IEEE Trans. Autom. Control **25**(3), 421–432 (1980)
298. Payne, H.J., Polak, E., Collins, D.C., Meisel, W.S.: An algorithm for bicriteria optimization based on the sensitivity function. IEEE Trans. Autom. Control **20**(4), 546–648 (1975)
299. Penot, J.-P., Sterna-Karwat, A.: Parametrized multicriteria optimization: Continuity and closedness of optimal multifunctions. J. Math. Anal. Appl. **120**(1), 150–168 (1986)
300. Pereyra, V.: Fast computation of equispaced Pareto manifolds and Pareto fronts for multiobjective optimization problems. Math. Comput. Simul. **79**, 1935–1947 (2009)
301. Pereyra, V., Saunders, M., Castillo, J.: Equispaced Pareto front construction for constrained biobjective optimization. Math. Comput. Model. **57**(9), 2122–2131 (2013)
302. Philip, J.: Algorithms for the vector maximization problem. Math. Program. **2**, 207–229 (1972)
303. Philip, J.: Vector maximization at a degenerate vertex. Math. Program. **13**, 357–359 (1977)
304. Pieume, C.O., Fotso, L.P., Siarry, P.: Solving bilevel programming problems with multicriteria optimization techniques. Opsearch **46**(2), 169–183 (2009)
305. Polak, E.: On the approximation of solutions to multiple criteria decision making problems. In: Zeleny, M. (ed.) Multiple Criteria Decision Making. Lecture Notes in Economics and Mathematical Systems, vol. 123, pp. 271–282. Springer, New York (1976)
306. Poliščuk, L.I.: Piece-wise linear approximation of the Pareto boundary of convex two-criteria problems (Russian). In: Models and Methods of Investigating Economic Systems, pp. 108–116. Nauka Sibirsk. Otdel., Novosibirsk (1979)
307. Pourkarimi, L., Yaghoobi, M.A., Mashinchi, M.: Determining maximal efficient faces in multiobjective linear programming problem. J. Math. Anal. Appl. **354**(1), 234–248 (2009)
308. Rakowska, J., Haftka, R.T., Watson, L.T.: Tracing the efficient curve for multi-objective control-stucture optimization. Technical report 91-2, Computer Science, Virginia Polytechnic Institute and State University, Blacksburg (1991)
309. Rakowska, J., Haftka, R.T., Watson,L.T.: Multi-objective control-structure optimization via homotopy methods. SIAM J. Optim. **3**(3), 654–667 (1993)
310. Rennen, G., van Dam, E.R., den Hertog, D.: Enhancement of sandwich algorithms for approximating higher dimensional Pareto sets. INFORMS J. Comput. **23**(4), 493–517 (2011)
311. Rhode, R., Weber, R.: Multiple objective quadratic-linear programming. In: Brans, J.P. (ed.) Operational Research (Hamburg, 1981), pp. 405–420. North-Holland, Amsterdam (1981)
312. Romanko, O., Ghaffari-Hadigheh, A., Terlaky, T.: Multiobjective optimization via parametric optimization: models, algorithms and applications. In: Modeling and Optimization: Theory and Applications. Springer Proceedings in Mathematics and Statistics, vol. 21, pp. 77–119. Springer, New York (2012)
313. Romero, C.: A general structure of achievement function for a goal programming model. Eur. J. Oper. Res. **153**:675–686 (2004)
314. Ross, A.M., Rhodes, D.H., Hastings, D.E.: Using Pareto trace to determine system passive value robustness. In: IEEE SysCon 2009, 3rd Annual IEEE International Systems Conference, Vancouver (2009)

315. Roy, B.: Problems and methods with multiple objective functions. Math. Program. **1**:239–266 (1971)
316. Ruuska, S., Miettinen, K., Wiecek, M.M.: Connections between single-level and bilevel multiobjective optimization. J. Optim. Theory Appl. **153**(1), 60–74 (2012)
317. Ruzika, S., Wiecek, M.M.: Survey paper: Approximation methods in multiobjective programming. J. Optim. Theory Appl. **126**(3), 473–501 (2005)
318. Saad, O.M.: Stability on multiobjective quadratic programming problems with fuzzy parameters. Indian J. Pure Appl. Math. **35**(5), 639–653 (2004)
319. Sakawa, M., Nishizaki, I.: Cooperative and Noncooperative Multi-Level Programming. Springer, New York (2009)
320. Sankaran, J.K.: On a variant of lexicographic multi-objective programming. Eur. J. Oper. Res. **107**, 669–674 (1998)
321. Sawaragi, Y., Nakayama, H., Tanino, T.: Theory of Multiobjective Optimization. Academic, Orlando (1985)
322. Sayin, S.: An algorithm based on facial decomposition for finding the efficient set in multiple objective linear programming. Oper. Res. Lett. **19**, 87–94 (1996)
323. Sayin, S.: Measuring the quality of discrete representations of efficient sets in multiple objective mathematical programming. Math. Program. **87**, 543–560 (2000)
324. Sayin, S.: Optimizing over the efficient set using a top-down search of faces. Oper. Res. **48**, 65–72 (2000)
325. Sayin, S.: A procedure to find discrete representations of the efficient set with specified coverage errors. Oper. Res. **51**(3), 427–436 (2003)
326. Schäffler, S., Schultz, R., Weinzierl, K.: Stochastic method for the solution of unconstrained vector optimization problems. J. Optim. Theory Appl. **114**(1), 209–222 (2002)
327. Schandl, B., Klamroth, K., Wiecek, M.M.: Introducing oblique norms into multiple criteria programming. J. Glob. Optim. **23**, 81–97 (2002)
328. Schandl, B., Klamroth, K., Wiecek, M.M.: Norm-based approximation in multicriteria programming. Comput. Math. Appl. **44**(7), 925–942 (2002)
329. Schönfeld, K.P.: Effizienz und Dualität in der Aktivitätsanlyse. Ph.D. thesis, Freie Universität Berlin, Germany (1964) (in German)
330. Schott, J.R.: Fault Tolerant Design Using Single and Multicriteria Genetic Algorithm Optimization. Master's thesis, M.S. Thesis, Massachusetts Institute of Technology, Cambridge (1995)
331. Schütze, O., Dell'Aere, A., Dellnitz, M.: On continuation methods for the numerical treatment of multi-objective optimization problems. In: Branke, J., Deb, K., Miettinen, K., Steuer, R.E. (eds.) Dagstuhl Seminar Proceedings 04461 Practical Approaches to Multi-Objective Optimization, Dagstuhl (2005)
332. Schütze, O., Laumanns, M., Coello Coello, C.A., Dellnitz, M., Talbi, El-G.: Convergence of stochastic search algorithms to finite size Pareto set approximations. J. Glob. Optim. **41**, 559–577 (2008)
333. Schütze, O., Witting, K., Ober-Blöbaum, S., Dellnitz, M.: Set oriented methods for the numerical treatment of multiobjective optimization problems. In: Tantar, E.. Tantar, A.-A., Bouvry, P., Del Moral, P., Legrand, P., Coello Coello, C.A., Schütze, O. (eds.) EVOLVE - A Bridge Between Probability, Set Oriented Numerics and Evolutionary Computation. Studies in Computational Intelligence, vol. 447 (2013)
334. Shao, L., Ehrgott, M.: Finding representative nondominated points in multiobjective linear programming. In: Proceedings of the 2007 IEEE Symposium on Computational Intelligence in Multicriteria Decision Making (MCDM 2007), pp. 245–252 (2007)
335. Shao, L., Ehrgott, M.: Approximately solving multiobjective linear programmes in objective space and an application in radiotherapy treatment planning. Math. Methods Oper. Res. **68**(2), 257–276 (2008)
336. Sharma, V.: Multiobjective integer nonlinear fractional programming problem: a cutting plane approach. Opsearch **49**(2), 133–153 (2012)

337. Sharma, S., Ahmad, I.: Sufficiency in nonsmooth multiobjective fractional programming. J. Appl. Math. Inf. **31**(1–2), 189–197 (2013)
338. Shimizu, K., Ishizuka, Y., Bard, J.F.: Nondifferentiable and Two-Level Mathematical Programming. Kluwer, Norwell (1997)
339. Shoval, O., Sheftel, H., Shinar, G., Hart, Y., Ramote, O., Mayo, A., Dekel, E., Kavanagh, K., Alon, U.: Evolutionary trade-offs, Pareto optimality, and the geometry of phenotype space. Science **336**, 1157–1160 (2012)
340. Shukla, P.K.: On the normal boundary intersection method for generation of efficient front. In: Shi, Y., van Albada, G.D., Dongarra, J. (eds.) International Conference in Computer Science, Part I. Lecture Notes in Computer Science, vol. 4487, pp. 310–317, Springer, Berlin(2007)
341. Siem, A.Y.D., den Hertog, D., Hoffmann, A.L.: The effect of transformations on the approximation of univariate (convex) functions with applications to Pareto curves. Eur. J. Oper. Res. **189**(2), 347–362 (2008)
342. Skulimowski, A.M.: Decision Support Systems Based on Reference Sets. Dissertations Monographies, vol. 40. Wydawnictwa AGH, Krakow (1996)
343. Smale, S.: Global analysis and economics. I. Pareto optimum and a generalization of Morse theory. In:Peixoto, M. (ed.) Dynamical Systems (Proc. Sympos., Univ. Bahia, Salvador, 1971), pp. 531–544. Academic, New York (1973)
344. Soland, R.M.: Multicriteria optimization: a general characterization of efficient solutions. Decis. Sci. **10**, 26–38 (1979)
345. Solanki, R.S., Appino, P.A., Cohon, J.L.: Approximating the noninferior set in multiobjective linear programming problems. Eur. J. Oper. Res. **68**, 356–373 (1993)
346. Sonntag, Y., Zălinescu, C.: Comparison of existence results for efficient points. J. Optim. Theory Appl. **105**(1), 161–188 (2000)
347. Stanimirović, P.S., Stanimirović, I.P.: Implementation of polynomial multi-objective optimization in mathematica. Struct. Multidiscip. Optim. **36**(4), 411–428 (2008)
348. Steuer, R.E.: Multiple Criteria Optimization: Theory, Computation and Application. Wiley, New York (1985)
349. Steuer, R.: Random problem generation and the computation of efficient extreme points in multiple objective linear programming. Comput. Optim. Appl. **3**, 333–347 (1994)
350. Steuer, R.E.: ADBASE: A multiple objective linear programming solver. Technical report, Terry College of Business, University of Georgia, Athens (2000)
351. Steuer, R.E., Choo, E.U.: An interactive weighted Tchebycheff procedure for multiple objective programming. Math. Program. **26**, 326–344 (1983)
352. Steuer, R.E., Harris, F.W.: Intra-set point generation and filtering in decision and criterion space. Comput. Oper. Res. **7**, 41–53 (1980)
353. Strijbosch, L.W.G., VanDoorne, A.G.M., Selen, W.J.: A simplified MOLP algorithm: the MOLP-S procedure. Comput. Oper. Res. **18**(8), 709–716 (1991)
354. Suneja, S.K., Kohli, B.: Duality for multiobjective fractional programming problem using convexifactors. Math. Sci. **7**, 1–8 (2013)
355. Takeda, E., Nishida, T.: Multiple criteria decision problems with fuzzy domination structures. Fuzzy Sets Syst. **3**(2), 123–136 (1980)
356. Tamiz, M., Jones, D.: Goal programming and Pareto efficiency. J. Inf. Optim. Sci. **17**(2), 291–307 (1996)
357. Tanino, T., Kuk, H.: Nonlinear multiobjective programming. In: Ehrgott, M., Gandibleux, X. (eds.) Multiple Criteria Optimization: State of the Art Annotated Bibliographic Surveys, pp. 71–128. Kluwer, Boston (2002)
358. TenHuisen, M.L., Wiecek, M.M.: Vector optimization and generalized Lagrangian duality. Ann. Oper. Res. **51**, 15–32 (1994)
359. TenHuisen, M.L., Wiecek, M.M.: Efficiency and solution approaches to bicriteria nonconvex programs. J. Glob. Optim. **11**(3), 225–251 (1997)
360. Thach, P.T.: Dual approach to minimization on the set of pareto-optimal solutions. J. Optim. Theory Appl. **88**(3), 689–707 (1996)

361. Thi, H.A.L., Pham, D.T., Thoai, N.V.: Combination between global and local methods for solving an optimization problem over the sets. Eur. J. Oper. Res. **142**, 258–270 (2002)
362. Thi, H.A.L., Dinh, T.P., Muu, L.D.: Simplicially-constrained DC optimization over efficient and weakly efficient sets. J. Optim. Theory Appl. **117**(3), 503–531 (2003)
363. Thi, B.K.N., Thi, H.A.L., Tran, M.T.: Outcome-space polyblock approximation algorithm for optimizing over efficient sets. In: Thi, H.A.L., Pascal, B., Dinh, T.P. (eds.) Modelling, Computation and Optimization in Information Systems and Management Sciences. Communications in Computer and Information Science, vol. 17, pp. 234–243. Springer, Berlin (2008)
364. Thoai, N.V.: Conical algorithm in global optimization for optimizing over efficient sets. J. Glob. Optim. **18**, 321–336 (2000)
365. Thoai, N.V.: Reverse convex programming approach in the space of extreme criteria for optimization over efficient sets. J. Optim. Theory Appl. **147**, 263–277 (2010)
366. Thuan, L.V., Luc, D.T.: On sensitivity in linear multiobjective programming. J. Optim. Theory Appl. **107**(3), 615–626 (2000)
367. Tind, J., Wiecek, M.M.: Augmented Lagrangian and Tchebycheff approaches in multiple objective programming. J. Glob. Optim. **14**:251–266 (1999)
368. Tuy, N.T., Hoai-Phuong, H.: Optimization under composite monotonic constraints and constrained optimization over the efficient set. In: Global optimization. Nonconvex Optimization and Applications, vol. 84, pp. 3–31. Springer, New York (2006)
369. Ünlü, G.: A linear bilevel programming algorithm based on bicriteria programming. Comput. Oper. Res. **14**(2), 173–179 (1987)
370. Utyuzhnikov, S.V., Fantini, P., Guenov, M.D.: A method for generating a well-distributed Pareto set in nonlinear multiobjective optimization. J. Comput. Appl. Math. **223**, 820–841 (2009)
371. VanVeldhuizen, D.A.: Multiobjective Evolutionary Algorithms: Classifications, Analyses, and New Innovations. Ph.D. thesis, Air Force Institute of Technology Wright Patterson AFB (1999)
372. Verma, R.U.: Weak ϵ-efficiency conditions for multiobjective fractional programming. Appl. Math. Comput. **219**(12), 6819–6827 (2013)
373. Vieira, D.A.G., Takahashi, R.H.C., Saldanha, R.R.: Multicriteria optimization with a multiobjective golden section line search. Math. Program. **131**(1–2, Ser. A), 131–161 (2012)
374. Voinalovich, V.: External approximation to the Pareto set in criterion space for multicriterion linear programming tasks. Cybern. Comput. Technol. **1**, 135–142 (1984)
375. Wang, S.: Second-order necessary and sufficient conditions in multiobjective programming. Numer. Funct. Anal. Optim. **12**(1–2), 237–252 (1991)
376. Warburton, A.R.: Quasiconcave vector maximization: connectedness of the sets of Pareto-optimal and weak Pareto-optimal alternatives. J. Optim. Theory Appl. **40**, 537–557 (1983)
377. Weidner, P.: Domination sets and optimality conditions in vector optimization theory. Wissenschaftliche Zeitschrift der Technischen Hochschule Ilmenau **31**(2), 133–146 (1985) (in German)
378. Weidner, P.: On interdependencies between objective functions and dominance sets in vector optimization. In: 19.Jahrestagung "Mathematische Optimierung" (Sellin, 1987). Seminarberichte, vol. 90, pp. 136–145. Humboldt University of Berlin, Berlin (1987)
379. Weidner, P.: Complete efficiency and interdependencies between objective functions in vector optimization. Z. Oper. Res. **34**, 91–115 (1990)
380. Weidner, P.: Problems in scalarizing multicriteria approaches. In: Zionts, S., Köksalan, M. (eds.) In Multiple Criteria Decision Making in the New Millennium. Advances in Soft Computing, pp. 199–209. Springer, Berlin (2001)
381. Weidner, P.: Tradeoff directions and dominance sets. In: Tanino, T. (ed.) Multi-Objective Programming and Goal Programming. Advances in Soft Computing, pp. 275–280. Springer, Berlin (2003)
382. Wen, U.-P., Hsu, Sh.-T.: A note on a linear bilevel programming algorithm based on bicriteria programming. Comput. Oper. Res. **16**(1), 79–83 (1989)

383. Wendell, R.E., Lee, D.N.: Efficiency in multiple objective optimization problems. Math. Program. **12**(3), 406–414 (1977)
384. White, D.J.: Epsilon efficiency. J. Optim. Theory Appl. **49**(2), 319–337 (1986)
385. White, D.J.: Weighting factor extensions for finite multiple objective vector minimization problems. Eur. J. Oper. Res. **36**, 256–265 (1988)
386. White, D.J.: A characterisation of the feasible set of objective function vectors in linear multiple objective problems. Eur. J. Oper. Res. **52**, 361–366 (1991)
387. Wiecek, M.M.: Advances in cone-based preference modeling for decision making with multiple criteria. Decis. Mak. Manuf. Serv. **1**(1–2), 153–173 (2007)
388. Wiecek, M.M., Zhang, H.: A parallel algorithm for multiple objective linear programs. Comput. Optim. Appl. **8**(1), 41–56 (1997)
389. Wiecek, M.M., Chen, W., Zhang, J.: Piecewise quadratic approximation of the non-dominated set for bi-criteria programs. J. Multi-Criteria Decis. Anal. **10**, 35–47 (2001)
390. Wierzbicki, A.P.: A methodological guide to multiobjective optimization. In: Iracki, K., Malanowski, K., Walukiewicz, S. (eds.) Optimization Techniques, Part 1. Lecture Notes in Control and Information Sciences, vol. 22, pp. 99–123. Springer, Berlin (1980)
391. Wierzbicki, A.P.: A methodological approach to comparing parametric characterizations of efficient solutions. In: Fandel, G., Grauer, M., Kurzhanski, A., Wierzbicki, A.P. (eds.) Large-Scale Modelling and Interactive Decision Analysis. Lecture Notes in Economics and Mathematical Systems, vol. 273, pp. 27–45. Springer, Berlin (1986)
392. Wierzbicki, A.P.: On the completeness and constructiveness of parametric characterizations to vector optimization problems. OR Spectr. **8**(2), 73–87 (1986)
393. Witting, K.: Numerical Algorithms for the Treatment of Parametric Multiobjective Optimization Problems and Applications. Ph.D. Thesis, University of Paderborn, Paderborn (2011)
394. Witting, K., Schulz, B., Dellnitz, M., Böcker, J., Fröhleke, N.: A new approach for online multiobjective optimization of mechatronic systems. Int. J. Softw. Tools Technol. Transfer **10**, 223–231 (2008)
395. Witting, K., Ober-Blöbaum, S., Dellnitz, M.: A variational approach to define robustness for parametric multiobjective optimization problems. J. Glob. Optim. **57**(2), 331–345 (2012)
396. Wu, H.C.: A solution concept for fuzzy multiobjective programming problems based on convex cones. J. Optim. Theory Appl. **121**(2), 397–417 (2004)
397. Wu, J., Azarm, S.: Metrics for quality assessment of a multiobjective design optimization solution set. Trans. ASME **123**, 18–25 (2001)
398. Yamamoto, Y.: Optimization over the efficient set: overview. J. Glob. Optim. **22**, 285–317 (2002)
399. Yan, H., Wei, Q., Wang, J.: Constructing efficient solutions structure of multiobjective linear programming. J. Math. Anal. Appl. **307**(2), 504–523 (2005)
400. Yang, X.Q., Goh, C.J.: A method for convex curve approximation. Eur. J. Oper. Res. **97**, 205–212 (1997)
401. Yang, X.Q., Yen, N.D.: Structure and weak sharp minimum of the Pareto solution set for piecewise linear multiobjective optimization. J. Optim. Theory Appl. **147**(1), 113–124 (2010)
402. Ye, J.J.: Necessary optimality conditions for multiobjective bilevel programs. Math. Oper. Res. **36**(1), 165–184 (2011)
403. Ye, J.J., Zhu, Q.J.: Multiobjective optimization problem with variational inequality constraints. Math. Program. **96**(1), 139–160 (2003)
404. Ying, G.: Higher-order symmetric duality for a class of multiobjective fractional programming problems. J. Inequal. Appl. **27**, 142 (2012)
405. Yu, P.L.: A class of solutions for group decision making. Manag. Sci. **19**(8), 936–946 (1973)
406. Yu, P.-L.: Introduction to domination structures in multicriteria decision problems. In: Cochrane, J.L., Zeleny, M. (eds.) Multiple Criteria Decision Making, pp. 249–261. University of South Carolina Press, Columbia (1973)
407. Yu, P.L.: Cone convexity, cone extreme points and nondominated solutions in decision problems with multiobjectives. J. Optim. Theory Appl. **14**, 319–377 (1974)

408. Yu, P.-L.: Domination structures and nondominated solutions. In: Multicriteria Decision Making. International Centre for Mechanical Sciences (CISM) Courses and Lectures, No. 211, pp. 227–280. Springer, Vienna (1975)
409. Yu, P.-L.: Multiple-criteria Decision Making. Concepts, Techniques, and Extensions. Mathematical Concepts and Methods in Science and Engineering, vol. 30. Plenum, New York/London (1985)
410. Yu, P.-L., Leitmann, G.: Compromise solutions, domination structures, and Salukvadze's solution. J. Optim. Theory Appl. **13**, 362–378 (1974)
411. Yu, P.L., Zeleny, M.: The set of all nondominated solutions in linear cases and a multicriteria simplex method. J. Math. Anal. Appl. **49**, 430–468 (1975)
412. Yu, P.L., Zeleny, M.: Linear multiparametric programming by multicriteria simplex method. Manag. Sci. **23**(2), 159–170 (1976)
413. Zalmai, G.J., Zhang, Q.: Semiinfinite multiobjective fractional programming, part I: sufficient efficiency conditions. J. Appl. Anal. **16**(2), 199–224 (2010)
414. Zalmai, G.J., Zhang, Q.: Semiinfinite multiobjective fractional programming, part II: duality models. J. Appl. Anal. **17**(1), 1–35 (2011)
415. Zeleny, M.: Compromise programming. In: Cochrane, J.L., Zeleny, M. (eds.) Multiple Criteria Decision Making, pp. 262–301. University of South Carolina Press, Columbia (1973)
416. Zeleny, M.: Linear Multiobjective Programming. Lecture Notes in Economics and Mathematical Systems, vol. 95. Springer, Berlin (1974)
417. Zhang, W.H., Gao, T.: A min-max method with adaptive weightings for uniformly spaced Pareto optimum points. Comput. Struct. **84**, 1760–1769 (2006)
418. Zhang, G., Lu, J., Dillon, Th.: Solution concepts and an approximation Kuhn-Tucker approach for fuzzy multiobjective linear bilevel programming. In: Chinchuluun, A., Pardalos, P.M., Migdalas, A., Pitsoulis, L. (eds.) Pareto Optimality, Game Theory and Equilibria. Optimization and Its Applications, vol. 17, pp. 457–480. Springer, New York (2008)
419. Zhang, R., Truong, B., Zhang, Q.: Multistage hierarchical optimization problems with multicriterion objectives. J. Ind. Manag. Optim. **7**(1), 103–115 (2011)
420. Zheng, X.Y., Yang, X.Q.: The structure of weak Pareto solution sets in piecewise linear multiobjective optimization in normed spaces. Sci. China (Ser. A: Math.) **51**(7), 1243–1256 (2008)
421. Zheng, X.Y., Yang, X.Q.: Weak sharp minima for piecewise linear multiobjective optimization in normed spaces. Nonlinear Anal. Ser. A: Theory Methods **68**(12), 3771–3779 (2008)
422. Zionts, S., Wallenius, J.: Identifying efficient vectors: some theory and computational results. Oper. Res. **28**(3), 785–793 (1980)
423. Zitzler, E.: Evolutionary Algorithms for Multiobjective Optimization: Methods and Applications. Ph.D. thesis, Swiss Federal Institute of Technology, Zurich (1999)
424. Zitzler, E., Thiele, L.: Multiobjective optimization using evolutionary algorithms - a comparative case study. In: Eiben, A.E., Beck, Th., Schoenauer, M., Schwefel, H.-P. (eds.) Parallel Problem Solving from Nature - PPSN V: 5th International Conference, Amsterdam, 27–30 September 1998. Lecture Notes in Computer Science, pp. 292–301 (1998)
425. Zitzler, E., Knowles, J., Thiele, L.: Quality assessment of Pareto set approximations. In: Branke, J., Deb, K., Miettinen, K., Slowinski, R. (eds.) Multiobjective Optimization: Interactive and Evolutionary Approaches. Lecture Notes in Computer Science, vol. 5252, pp. 373–404. Springer, Berlin (1998)

Chapter 19
Exact Methods for Multi-Objective Combinatorial Optimisation

Matthias Ehrgott, Xavier Gandibleux, and Anthony Przybylski

Abstract In this chapter we consider multi-objective optimisation problems with a combinatorial structure. Such problems have a discrete feasible set and can be formulated as integer (usually binary) optimisation problems with multiple (integer valued) objectives. We focus on a review of exact methods to solve such problems. First, we provide definitions of the most important classes of solutions and explore properties of such problems and their solution sets. Then we discuss the most common approaches to solve multi-objective combinatorial optimisation problems. These approaches include extensions of single objective algorithms, scalarisation methods, the two-phase method and multi-objective branch and bound. For each of the approaches we provide references to specific algorithms found in the literature. We end the chapter with a description of some other algorithmic approaches for MOCO problems and conclusions suggesting directions for future research.

Keywords Multi-objective optimisation • Combinatorial optimisation • Exact methods • Scalarisation • Branch and bound • Two-phase method

19.1 Introduction

In this section we provide basic definitions and notations for multi-objective combinatorial optimisation (MOCO) problems, including definitions of efficient solutions and non-dominated points. We discuss the theoretical background of multi-objective combinatorial optimisation. We recall results from computational complexity, highlighting that MOCO problems are almost always NP-hard and

---

M. Ehrgott (✉)
Department of Management Science, Lancaster University, Bailrigg, Lancaster LA1 4YX, UK
e-mail: m.ehrgott@lancaster.ac.uk

X. Gandibleux • A. Przybylski
Faculté des Sciences et des Techniques, Université de Nantes, 44322 Nantes Cedex 03, France
e-mail: Xavier.Gandibleux@univ-nantes.fr; Anthony.Przybylski@univ-nantes.fr

S. Greco et al. (eds.), *Multiple Criteria Decision Analysis*, International Series in Operations Research & Management Science 233,
DOI 10.1007/978-1-4939-3094-4_19

#P-hard. Moreover, for many problems, instances with an exponential number of non-dominated points exist. We mention results on the (non)-connectedness of efficient solutions. We also give an outlook on the rest of the chapter.

19.1.1 Definitions

Formally, a multi-objective combinatorial optimisation problem can be written as a linear integer programme with multiple objectives

$$\min\{z(x) = Cx : Ax = b, x \in \{0,1\}^n\}, \tag{19.1}$$

where $x \in \{0,1\}^n$ is a (column) vector of n binary variables $x_j, j = 1, \ldots, n$; $C \in \mathbb{Z}^{p \times n}$ contains the rows c^k of coefficients of p linear objective functions $z_k(x) = c^k x, k = 1, \ldots, p$ and $A \in \mathbb{Z}^{m \times n}, b \in \mathbb{Z}^m$ describe m constraints $a^i x = b_i, i = 1, \ldots, m$, where $a^i, i = 1, \ldots, m$ are m row vectors from $\mathbb{Z}^n$. The constraints define combinatorial structures such as paths, trees, or cycles in a network or partitions of a set, etc. and it will be convenient to assume that all coefficients, i.e. all entries of A, b, and C, are integers.

In order to define what solving MOCO problem (19.1) means, we define orders of vectors in $\mathbb{R}^p$. We use the notations $\leqq, \leq$, and $<$ to define componentwise orders in Definition 1.

Definition 1. Let $y^1, y^2 \in \mathbb{R}^p$. We write

- $y^1 \leqq y^2$ if $y^1_k \leqq y^2_k$ for $k = 1, \ldots, p$;
- $y^1 \leq y^2$ if $y^1 \leqq y^2$ but $y^1 \neq y^2$ and
- $y^1 < y^2$ if $y^1_k < y^2_k$ for $k = 1, \ldots, p$.

According to the componentwise orders, we define $\mathbb{R}^p_{\geqq} := \{y \in \mathbb{R}^p : y \geqq 0\}$ as the non-negative orthant in $\mathbb{R}^p$, and analogously $\mathbb{R}^p_{\geq}$ and $\mathbb{R}^p_{>}$.

The set $X = \{x \in \{0,1\}^n : Ax = b\}$ is called feasible set in decision space $\mathbb{R}^n$ and $Y := z(X) = \{Cx : x \in X\}$ is the feasible set in objective space $\mathbb{R}^p$. The set of points $\text{conv}(Y) + \mathbb{R}^p_{\geqq}$, sometimes called the Edgeworth-Pareto hull of Y, is very important. Figure 19.1 illustrates the feasible set Y of a MOCO problem with two objectives as a finite set of circles and its Edgeworth-Pareto hull $\text{conv}(Y) + \mathbb{R}^p_{\geqq}$ as a shaded area.

Individual minimisers of the p objective functions $z_k(x)$, for $k = 1, \ldots, p$, i.e. feasible solutions $\hat{x} \in X$ such that $z_k(\hat{x}) \leqq z_k(x)$ for all $x \in X$ for some $k \in \{1, \ldots, p\}$ do only minimise a single objective function and do not provide any control over the values of the other $p - 1$ objectives if they are not unique, see Fig. 19.2a. In that case their objective function vectors may differ in values $z_i(\hat{x})$ for $i \neq k$. It is clear (see Fig. 19.2a) that a point combining the minimal values for all objectives, called the ideal point and denoted y^I, i.e. $(1,1)^T$ in Fig. 19.2a, does usually not belong to Y or even $\text{conv}(Y) + \mathbb{R}^p_{\geqq}$. Hence an ideal feasible solution $x^I \in X$ such that $z_k(x^I) \leqq z_k(x)$ for all $x \in X$ and all $k = 1, \ldots, p$ does in general not exist.

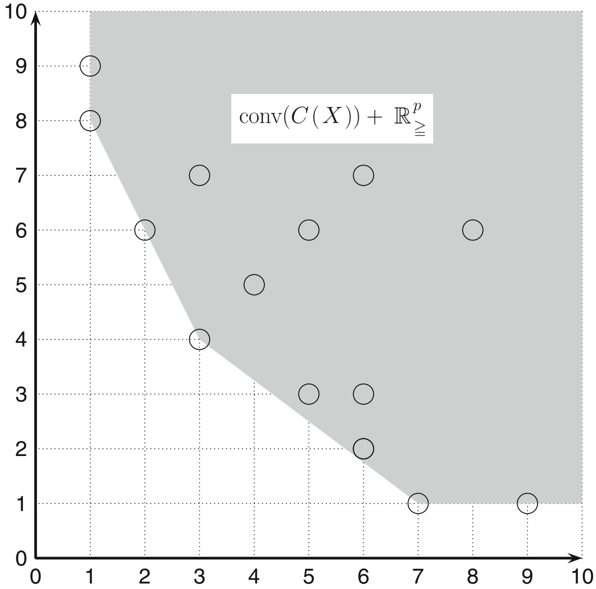


Fig. 19.1 Feasible set and Edgeworth-Pareto hull

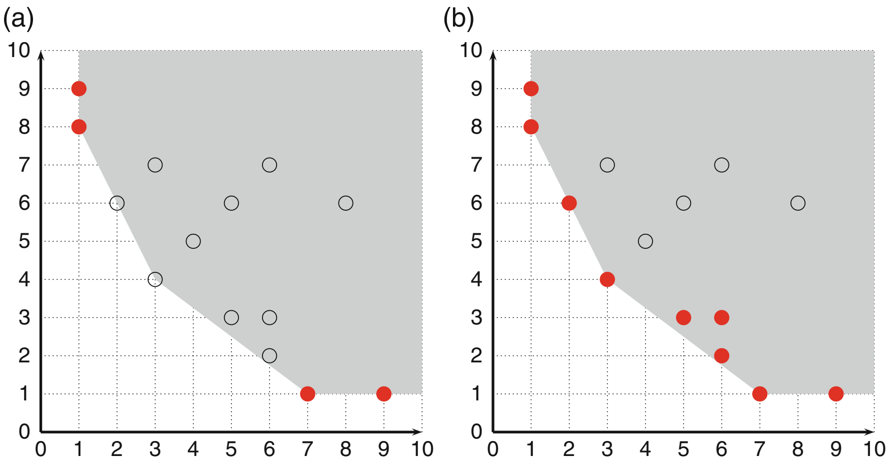


Fig. 19.2 (**a**) Individual and lexicographic minima. (**b**) (Weakly) non-dominated points

The ambiguity in considering individual optima is avoided by turning to lexicographic optima, i.e. feasible solutions $\hat{x}$ such that $z(\hat{x}) \leqq_{lex} z(x)$ for all $x \in X$, and more generally, $z^{\pi}(\hat{x}) \leqq_{lex} z^{\pi}(x)$ for all $x \in X$ and some permutation $z^{\pi} = (z_{\pi(1)}, \dots, z_{\pi(p)})$ of the components of the vector valued function $z = (z_1, \dots, z_p)$. The lexicographic order is defined in Definition 2.

Definition 2. Let $y^1, y^2 \in \mathbb{R}^p$. We write $z^\pi(\hat{x}) <_{lex} z^\pi(x)$ (y^1 is lexicographically smaller than y^2 with respect to permutation π) if there is some $k \in \{1, \ldots, p\}$ such that $y^1_{\pi(i)} = y^2_{\pi(i)}$ for $i = 1, \ldots k-1$ and $y^1_{\pi(k)} < y^2_{\pi(k)}$. Hence $z(\hat{x}) \leqq_{lex} z(x)$ if either $z^\pi(\hat{x}) <_{lex} z^\pi(x)$ or $z(\hat{x}) = z(x)$

Figure 19.2a shows non-unique individual minima for z_1 and z_2 (filled circles). Among those, $\hat{y}^1 = (1, 8)^T$ and $\hat{y}^2 = (7, 1)^T$ are lexicographically minimal for the permutation of objectives (z_1, z_2) and (z_2, z_1), respectively.

While individual and lexicographic optima refer to total (pre)orders on $\mathbb{R}^p$, multi-objective optimisation is based on the concept of efficiency or Pareto optimality. It is defined using partial (pre)orders based on the componentwise comparison of vectors in Definition 1.

Definition 3. A feasible solution $\hat{x} \in X$ belongs to the set of weakly efficient solutions X_{wE} if there is no $x \in X$ with $z(x) < z(\hat{x})$. In that case, $z(\hat{x})$ is called weakly non-dominated. We denote $Y_{wN} := z(X_{wN})$ the set of all weakly non-dominated points. Feasible solution $\hat{x} \in X_E$, the set of efficient solutions, if there is no feasible x with $z(x) \leq z(\hat{x})$. The objective vector $z(\hat{x})$ of an efficient solution is called non-dominated point and $Y_N := z(X_E)$ is the set of all non-dominated points.

Definition 4 provides two further definitions of efficiency, that only play a limited role in multi-objective combinatorial optimisation.

Definition 4. Feasible solution $\hat{x} \in X$ is called strictly efficient if there is no $x \in X$ such that $z(x) \leqq z(\hat{x})$. It is called properly efficient if it is efficient and there exists a real number $M > 0$ such that for every $i \in \{1, \ldots, p\}$ and $x \in X$ such that $z_i(x) < z_i(\hat{x})$ there exists some $j \in \{1, \ldots, p\} \setminus \{i\}$ such that $z_j(\hat{x}) < z_j(x)$ and $(z_i(\hat{x}) - z_i(x))/(z_j(x) - z_j(\hat{x})) \leqq M$.

Strict efficiency of $\hat{x}$ implies that the pre-image $z^{-1}(z(\hat{x}))$ is a singleton. In single objective combinatorial optimisation, this corresponds to unique optimisers and is rarely considered. Proper efficiency on the other hand relates to bounded trade-offs between the objectives. While this is an important issue in continuous non-linear multi-objective optimisation (see Chap. 18), in our setting we assume that $Y \subset \mathbb{Z}^p$ and differences between objective values are therefore integers, implying that all efficient solutions are properly efficient. The following distinction between supported and non-supported efficient solutions is, on the other hand, crucial in multi-objective combinatorial optimisation.

Definition 5. An efficient solution is supported if there is some $\lambda \in \mathbb{R}^p_>$ such that $\lambda^T C\hat{x} \leqq \lambda^T Cx$ for all $x \in X$. In case $C\hat{x}$ is an extreme point of $\text{conv}(Y) + \mathbb{R}^p_{\geqq}$ it is an extreme efficient solution and $C\hat{x}$ is an extreme non-dominated point. We let X_{SE1} be the set of extreme efficient solutions and let X_{SE2} denote the set of supported efficient solutions such that $C\hat{x}$ is in the relative interior of a face of $\text{conv}(Y) + \mathbb{R}^p_{\geqq}$. All supported efficient solutions are $X_{SE} = X_{SE1} \cup X_{SE2}$. Finally, $X_{NE} = X_E \setminus X_{SE}$ is

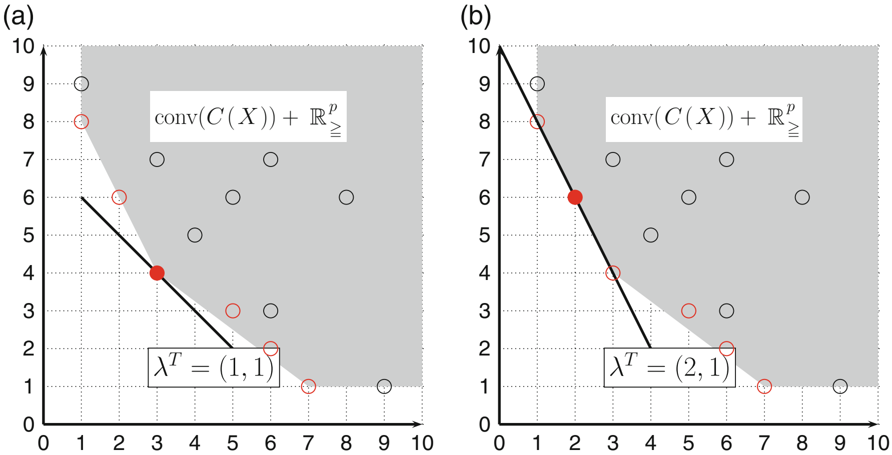


Fig. 19.3 (**a**) Extreme non-dominated point for $\lambda^T = (1, 1)$. (**b**) Supported non-dominated point in the relative interior of a face for $\lambda^T = (2, 1)$

the set of non-supported efficient solutions, i.e. efficient solutions $\hat{x}$ such that $C\hat{x}$ is in interior of $\text{conv}(Y) + \mathbb{R}^p_{\geqq}$. The counterparts of X_{SE} and X_{NE} in objective space are called the sets of (non-)supported non-dominated points Y_{SN} and Y_{NN}, respectively. Analogously, Y_{SN1} and Y_{SN2} denote the sets of non-dominated extreme points of Y and supported non-dominated points that are not extreme points of Y, respectively.

It is very easy to construct small examples that show that even the bi-objective shortest path, spanning tree, and assignment problems have non-supported efficient solutions. In our small example, Fig. 19.2b shows (weakly) non-dominated points as filled circles. Note that $(1, 9)^T$, $(6, 3)^T$ and $(9, 1)^T$ are weakly non-dominated but not non-dominated. Moreover, $(5, 3)^T$ and $(6, 2)^T$ are non-supported non-dominated points. Figures 19.3a,b illustrate supported non-dominated points. Figure 19.3a shows that $(3, 4)^T$ is an extreme non-dominated point and Fig. 19.3b that $(2, 6)^T$ is a supported non-dominated point in the relative interior of a face of $\text{conv}(Y) + \mathbb{R}^p_{\geqq}$.

Following [38] we call $x^1, x^2 \in X_E$ equivalent if $Cx^1 = Cx^2$. A complete set of efficient solution is a set $\hat{X} \subset X_E$ such that for all $y \in Y_N$ there is some $x \in \hat{X}$ with $z(x) = y$. A minimal complete set contains no equivalent solutions, the maximal complete set X_E contains all equivalent solutions. We can now speak about, e.g. a minimal complete set of extreme efficient solutions. This classification allows to precisely describe what is to be understood by statements that some algorithm "solves" a certain MOCO problem. We shall always understand solving a MOCO problem as finding all non-dominated points and for each $y \in Y_{ND}$ one x such that $Cx = y$, i.e. finding a minimal complete set of efficient solutions.

19.1.2 Computational Complexity

The most striking feature concerning computational complexity in the sense of worst case performance of algorithms is that MOCO problems are hard in the conventional notions of computational complexity. We formally define the decision problem related to optimisation problem (19.1) as: "Given $d \in \mathbb{Z}^p$, does there exist $x \in X$ such that $Cx \leqq d$?" and the associated counting problem as: "Given $d \in \mathbb{Z}^p$, how many $x \in X$ satisfy $Cx \leqq b$?" Apart from these questions we are also interested in knowing how many efficient solutions (non-dominated points) may exist in the worst case. General upper bounds on the cardinality of Y_N (for MOCO problems in the presence of both sum and bottleneck objectives) are given by Stanojević [91] and earlier, lower bounds on $|Y_N|$ have been derived in [32].

Obviously, the multi-objective version of any NP-hard single objective combinatorial optimisation problem is also NP-hard. Another source of NP-hardness of MOCO problem (19.1) derives from the NP-hardness of so-called resource-constrained single objective combinatorial optimisation problems. These are problems that ask for the minimisation of a single linear function c^1x over feasible set X subject to the additional constraint $c^2x \leqq d$. Since the decision and counting problems associated with a resource-constrained combinatorial optimisation problem are identical to those of bi-objective combinatorial optimisation problems, hardness results for the resource constrained problems imply hardness results for the bi-objective problems.

It is well known that if A is a totally unimodular matrix, then the polyhedron conv(X) has only integer extreme points. Hence, in the single objective case, (19.1) can be solved by linear programming. The presence of non-supported non-dominated points makes the total unimodularity property much less useful in the multi-objective case. However, Kouvelis and Carlson [48] show for the bi-objective case, that if the variables of (19.1) are separable, i.e. $x = (x^1, x^2)$ such that $c^kx = c^kx^k$ for $k = 1, 2$ and A is totally unimodular, then the set of non-supported efficient solutions is empty. Hence, in this case, (19.1) can be solved by (parametric) linear programming methods.

Specifically addressing MOCO problems, we first consider a binary optimisation problem without any constraints. Ehrgott [25] uses a parsimonious transformation to the binary knapsack problem, which is known to be NP-complete and #P-complete [42, 102], to show that the most basic MOCO problem, called the unconstrained MOCO problem, is hard.

Theorem 1. *The unconstrained bi-objective combinatorial optimisation problem*

$$\min \left\{ \sum_{i=1}^{n} c_i^k x_i \; for \; k = 1, 2 : x_i \in \{0, 1\} \; for \; i = 1, \dots, n \right\} \tag{19.2}$$

is NP-hard and #P-hard. Moreover, there is an instance of (19.2) *that has an exponential number of non-dominated points.*

For the latter part of this theorem, it is sufficient to set $c_i^k := (-1)^k 2^{i-1}$ to see that $Y = Y_N$, because the cost coefficients are simply binary representations of the positive and negative values of integers $1, \ldots, 2^{n-1}$. This is a common feature of many of the instances constructed to show that the number of non-dominated points can be exponential in the worst case. As a consequence, all the non-dominated points of such instances lie on a straight line in objective space defined by a constant sum of both objectives. The more interesting question of whether the number of extreme non-dominated points, i.e. $|Y_{SN1}|$, can be exponential in the size of an instance has not been investigated widely. The existence of instances with an exponential number of non-dominated point is called intractability of the MOCO problem.

Although intractability results imply that even Y_{SN} can be exponential in the size of a problem instance, numerical tests reveal that the number of non-dominated points is often "small", in particular in real world applications. This has been observed by Raith and Ehrgott [75] and Müller-Hannemann and Weihe [60] for randomly generated and real world instances of the shortest path problem and by Przybylski et al. [71] for randomly generated instances of the bi-objective assignment problem. This evidence suggests that the numerical values of objective function coefficients in C play an important role just as does the combinatorial structure of the instance.

We summarise other complexity results in Table 19.1.

Despite these mainly negative results, Blanco and Puerto [8] have shown that encoding the entire set of non-dominated solutions of a multi-objective integer programming problem in a short sum of rational functions is polynomially doable, when the dimension of the decision space is fixed.

Table 19.1 Complexity results for MOCO problems

MOCO problem	Result	Reference
Bi-objective shortest path	NP-hard	[89]
	$\lvert Y_{SN}\rvert$ is exponential	[38]
Bi-objective integer minimum cost flow	$\lvert Y_{SN1}\rvert$ is exponential	[81]
Bi-objective minimum spanning tree	NP-hard	[10]
	$\lvert Y_{SN}\rvert$ is exponential	[37]
	$\lvert Y_{SN1}\rvert = O(\lvert\mathscr{E}\rvert^2)$	[83]
	$\lvert Y_{NN}\rvert$ is exponential	[82]
Bi-objective global minimum cut	$\lvert Y_N\rvert = O(\lvert\mathscr{V}\rvert^7)$	[1]
Bi-objective assignment	NP-hard	[89]
	#P-hard	[61]
Bi-objective search problem on a line	NP-hard	[66]
	$\lvert Y_N\rvert$ is exponential	[66]
Bi-objective unform matroid	NP-complete	[23]

19.1.3 Connectedness of Efficient Solutions

One approach to design algorithms to solve MOCO problems is the idea of local search. In local search, a neighbourhood of a feasible solution is defined, i.e. for $x \in X$, $N(x)$ is a subset of X, so that each solution in $N(x)$ can be obtained from x by a certain "move". Such a move can be, e.g. the exchange of an edge of a spanning tree by another edge of the underlying graph or the basis exchange of a pivot operation in the simplex algorithm. It is then possible to define a graph $\mathscr{E}\mathscr{G} = (\mathscr{V}, \mathscr{E})$, called the efficiency graph in [28] and the adjacency graph in [82], the vertices of which are the efficient solutions of a MOCO problem and the edges of which are defined by $[x^1, x^2] \in E$ if and only if $x^2 \in N(x^1)$ and $x^1 \in N(x^2)$. If the adjacency graph of an instance of a MOCO problem is connected, it is then possible to find all efficient solutions of the MOCO problem by a local search procedure. The set X_E is called connected, if the corresponding efficiency graph is connected. Ehrgott and Klamroth [28] show the first result concerning the (non)connectedness of the set of efficient solutions of a MOCO problem.

Theorem 2. *The adjacency graph of the set X_E of an instance of the bi-objective shortest path and bi-objective minimum spanning tree problem is not connected in general.*

Ruzika [82] notes that this result also holds for the adjacency graph of weakly efficient solutions X_{wE}. Furthermore, Ruzika [82] demonstrates that the number and cardinality of the connected components of the adjacency graph can be exponential. Przybylski et al. [69] modify the example used in [28] to show that the efficiency graph of the bi-objective integer minimum cost flow problem need not be connected. Ruzika [82] also proves non-connectedness results for the bi-objective binary knapsack problem with uniform weights and cardinality constraint, the binary multiple choice knapsack problem with equal weights, the unconstrained MOCO problem (19.2) and the bi-objective assignment problem.

Very few positive results concerning the connectedness of efficient solutions are known. They are generally obtained for very specific instances. For example it is clear that the adjacency graph of efficient solutions of a multi-objective minimum spanning tree problem on a graph that contains exactly one cycle is connected. Gorski et al. [36] prove the connectedness of the efficient solutions for a class of two-dimensional knapsack problems with binary weights.

19.1.4 Bounds and Bound Sets

The success of exact algorithms to solve combinatorial optimisation problems often depends on the availability of good lower and upper bounds, i.e. scalars l and u such that $l \leqq z^* \leqq u$, where z^* is the optimal value of the optimisation problem.

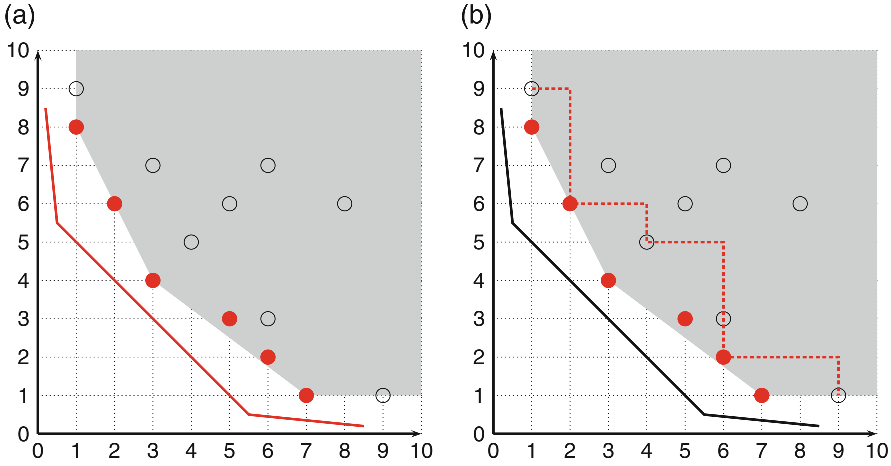


Fig. 19.4 (**a**) A lower bound set. (**b**) An upper bound set defined by feasible points

Straightforward generalisations of lower and upper bounds are given by the ideal and nadir points y^I and y^N. They are defined in Eqs. (19.3) and (19.4),

$$y_k^I := \min\{y_k : y \in Y\}; k = 1, \ldots, p; \tag{19.3}$$

$$y_k^N := \max\{y_k : y \in Y_N\}; k = 1, \ldots, p. \tag{19.4}$$

The ideal and nadir point are tight lower and upper bounds on the values of any non-dominated point, i.e. there is no $y \in Y_N$ that dominates y^I nor that is dominated by y^N, yet for each $k = 1, \ldots, p$ there exists some $y \in Y_N$ such that $y_k = y_k^I$ and some $y \in Y_N$ such that $y_k = y_k^N$. However, in general neither the ideal nor the nadir point are feasible and can hence be "far away" from the non-dominated set. Note that in Fig. 19.4a the ideal point $y^I = (1, 1)^T$ and the nadir point $y^N = (7, 8)^T$.

To overcome this drawback it is necessary to combine the notion of a set of non-dominated points with the idea of bounds, leading to the definition of bound sets. A first definition of lower and upper bound sets L and U is given in [99]. A (different) definition for the bi-objective case is due to [26]. Using only part of the conditions of [26], Delort and Spanjaard [20] propose a third definition. Here, we formally give the definition of [27]. Note that a set $S \subset \mathbb{R}^p$ is called $\mathbb{R}^p_{\geqq}$-closed, respectively $\mathbb{R}^p_{\geqq}$-bounded if $S + \mathbb{R}^p_{\geqq}$ is closed, respectively if there exists some $\hat{s} \in \mathbb{R}^p$ such that $S \subset \hat{s} + \mathbb{R}^p_{\geqq}$, see [27].

Definition 6. • A lower bound set L is a $\mathbb{R}^p_{\geqq}$-closed, $\mathbb{R}^p_{\geqq}$-bounded set such that $Y_N \subset L + \mathbb{R}^p_{\geqq}$ and $L \subset \left(L + \mathbb{R}^p_{\geqq}\right)_N$.

- An upper bound set U is a $\mathbb{R}^p_{\geqq}$-closed, $\mathbb{R}^p_{\geqq}$-bounded set such that Y_N is contained in $\operatorname{cl}\left[\left(U + \mathbb{R}^p_{\geqq}\right)^c\right]$ and $U \subset \left(U + \mathbb{R}^p_{\geqq}\right)_N$, where ^{c} indicates the complement of a set in $\mathbb{R}^p$.

Definition 6 implies that neither L nor U contain points dominated by other points from the same set. Moreover, this definition says that the objective vectors of any set of feasible solutions, filtered by dominance, define an upper bound set. In the single objective case the definitions reduce to the usual definitions of lower and upper bound, and in particular encompass the idea of the incumbent best known solution providing an upper bound. Moreover, it preserves the condition that $L = U$ implies $L = U = Y_N$.

An important lower bound set is defined by the non-dominated set of the convex hull of (supported) non-dominated points, $L = (\operatorname{conv}(Y_{SN}))_N$, a set that is sometimes called the non-dominated frontier of MOCO problem (19.1). This is in fact the tightest possible $\mathbb{R}^p_{\geqq}$-convex lower bound set. In Fig. 19.4a this will be the union of the two line segments connecting $(1, 8)^T$, $(3, 4)^T$ and $(7, 1)^T$.

Figure 19.4a shows a lower bound set consisting of three line segments that could represent the non-dominated set of the (multi-objective) linear programming relaxation of a MOCO problem (19.1). Figure 19.4b illustrates how six feasible points $U = \{(1, 9)^T, (3, 6)^T, (4, 5)^T, (6, 2)^T, (9, 1)^T\}$ define an upper bound set. In this way, any multi-objective relaxation of MOCO problem (19.1) defines a lower bound set and any set of feasible points (filtered by dominance) defines an upper bound set, preserving this important property from single objective branch and bound. The arguably simplest and most often used lower bound set is $L = \{y^I\}$, i.e the set consisting of the ideal point y^I defined as in (19.3). Similarly, an upper bound set can be obtained by the anti-ideal point y^{AI} with $y^{AI}_k := \max\{c^k x : x \in X\}$ or the nadir point y^N (see Eq. (19.4)) consisting of the worst objective values over the efficient set. Note that the nadir point is hard to compute for $p \geqq 3$, even for linear problems, see [31].

19.1.5 Outlook

In what follows, we discuss exact solution approaches proposed in the literature to solve multi-objective versions of combinatorial optimisation problems. This material is presented in four sections. In Sect. 19.2 we consider algorithms for single objective combinatorial problems that can be extended to solve their multi-objective counterparts. Two prime examples are labelling algorithms for the shortest path problem and the greedy algorithm. In Sect. 19.3 the topic of scalarisation is discussed, which plays a role in Sect. 19.4, where we discuss approaches that rely heavily on the repeated solution of single objective combinatorial optimisation problems. Such approaches are particularly beneficial if polynomial time algorithms are available for the single objective problem. We present an outline of the two-phase method [97] and review two-phase algorithms from the literature. More general approaches, that do not revert to repeatedly solve single objective versions of the problem under consideration have been proposed for general MOCO problems. In Sect. 19.5 we address the principle of branch and bound algorithms and we conclude the survey with a review of other exact solution methods. The chapter concludes with a brief discussion and suggestions for further research.

19.2 Extending Single Objective Algorithms

In this section, we shall discuss algorithms to solve multi-objective combinatorial optimisation problems for which efficient polynomial time algorithms exist to solve their single objective versions, and for which it is possible to extend these algorithms to deal with the multi-objective versions. The prime examples are the multi-objective shortest path and spanning tree problems.

19.2.1 Labelling Algorithms

Let $\mathscr{G} = (\mathscr{V}, \mathscr{A})$ be a directed graph defined by node set $\mathscr{V}$ and arc set $\mathscr{A}$ with p arc costs $c_{ij}^k, k = 1, \ldots, p$ on arcs $(i,j) \in \mathscr{A}$. The single-source single-sink multi-objective shortest path problem is to find a minimal complete set of efficient paths from an origin node s to a destination node t. The single-source variant finds a complete set of efficient paths from origin s to all other nodes of $\mathscr{G}$, and the all-pairs version efficient paths between all pairs of origins and destinations.

In the single objective case $p = 1$, label setting algorithms, such as Dijkstra's algorithm [22], or label correcting algorithms, such as Bellman's algorithm [5], are well known polynomial time algorithms.

Multi-objective labelling algorithms rely on the following fact. Assuming that all $c_{ij}^k \geqq 0$, let P_{st} be an efficient path from s to t. Then any subpath P_{uv} from u to v, where u and v are nodes on P_{st}, is an efficient path from u to v. Notice that, on the other hand, concatenations of efficient paths need not be efficient. This principle of optimality shows that the multi-objective shortest path problem is an example of multi-objective dynamic programming and implies that generalisations of both Dijkstra's and Bellman's algorithms are possible. Such algorithms have vector valued labels and therefore, due to the partial orders used, need to maintain sets of non-dominated labels at each node rather than single labels.

For a label setting algorithm, lists of permanent and temporary labels are stored and it is necessary to ensure that a permanent label defines an efficient path from s to the labelled node. This can be done by selecting the lexicographically smallest label from the temporary list to become permanent. A label setting algorithm then follows the same steps as in the single objective case, only that newly created labels need to be compared with label sets. New labels dominating existing labels lead to the deletion of those dominated labels. An existing label dominating a new label results in the new label being discarded, otherwise the new label is added to the label list. An interesting feature of the multi-objective shortest path problem is that, in contrast to the single objective one, it is not possible to terminate a multi-objective label setting algorithm once the destination node t has been reached, since further unprocessed temporary labels at nodes in $\mathscr{V} \setminus \{s,t\}$ may lead to the detection of more efficient paths from s to t, another effect of the partial order. Nevertheless, any label at t can serve as an upper bound and temporary labels at any intermediate node dominated by such an upper bound can be eliminated.

Just as in the single objective case, label setting algorithms fail if negative arc lengths are permitted. In the multi-objective case the following situations may occur: If there is a cycle C with $\sum_{(i,j)\in C} c_{ij} \leq 0$ there is no efficient path; if there is a cycle C with $\sum_{(i,j)\in C} c^k_{ij} < 0$ and $\sum_{(i,j)\in C} c^l_{ij} > 0$ for some k and some $l \neq k$ there are infinitely many efficient paths as every pass of the cycle reduces one objective and increases another thereby creating one more efficient path every time. In the presence of negative arc lengths label correcting algorithms are necessary. Once again, one proceeds with processing the labels as in the single objective case, keeping in mind that all newly created labels need to be compared with existing label sets so that all dominated labels can be eliminated. In each iteration either a single label or a set of labels is processed and extended along an arc (arcs) out of the node where the label(s) reside. Of course, no label is permanent until termination of the algorithm. For details on a variety of multi-objective shortest path problems including pseudocode and numerical results the reader is referred to [75]. Another review of multi-objective shortest path algorithms is provided in [95]. Paixão and Santos [65] report on a computational study of a variety of labelling algorithms for the multi-objective shortest path problem. Newer labelling algorithms can be found for example in [39, 84, 87, 88].

More generally than algorithms for shortest path problems, dynamic programming can be generalised to multiple objective problems along the same lines as labelling algorithms, see for example [17, 99]. Rong and Figueira [80] use dynamic programming to solve bi-objective binary knapsack problems.

19.2.2 Greedy Algorithms

Another problem that we discuss is the multi-objective spanning tree problem. Given a graph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ with edge set $\mathscr{E}$ and edge costs $c^k_{ij}, k = 1, \ldots, p$ for all edges $[i,j] \in \mathscr{E}$ its aim is to find efficient spanning trees of $\mathscr{G}$, i.e. spanning trees the cost vectors of which are non-dominated. The following theorem is the foundation for greedy algorithms that generalise Prim's [68] and Kruskal's [49] algorithms.

Theorem 3 ([37]). *Let T be an efficient spanning tree of $\mathscr{G}$. The following assertions hold.*

1. *Let* $e \in \mathscr{E}(T)$ *be an edge of* T*. Let* $(\mathscr{V}(T_1), \mathscr{E}(T_1))$ *and* $(\mathscr{V}(T_2), \mathscr{E}(T_2))$ *be the two connected components of* $T \setminus \{e\}$*. Let* $C(e) := \{f = [i,j] \in \mathscr{E} : i \in \mathscr{V}(T_1), j \in \mathscr{V}(T_2)\}$ *be the cut defined by deleting e. Then* $c(e) \in \min\{c(f) : f \in C(e)\}$*.*
2. *Let* $f \in \mathscr{E} \setminus \mathscr{E}(T)$ *and let* $P(f)$ *be the unique path in* T *connecting the end nodes of* f*. Then* $c(f) \leq c(e)$ *does not hold for any* $e \in P(f)$*.*

The first statement of Theorem 3 shows that starting from a single node and adding efficient edges between nodes already included and nodes not yet included in the tree will eventually construct all efficient trees. Notice that several such edges might exist, and hence in every iteration there will be a set of efficient partial trees. As in the multi-objective shortest path problem adding edges to an efficient partial tree does not necessarily lead to another efficient partial tree, but also adding efficient edges to a dominated partial tree may yield an efficient partial tree. It is therefore necessary to filter out dominated trees at termination of the algorithm. In a similar way, the second statement provides a justification for a Kruskal-like algorithm efficient spanning tree problem, see [14] for an adaptation to the multi-objective case.

Since spanning trees are examples of matroid bases, the above extends to multi-objective matroid optimisation and the greedy algorithm. A result by Serafini [89] shows that efficient matroid bases can be found by the greedy algorithm working with a topological order of the cost vectors of the elements. Let $\mathscr{M} = (\mathscr{E}, \mathscr{I})$ be a matroid and $c_j \in \mathbb{R}^p$ be the cost vectors of elements $e_j \in \mathscr{E}$. Recall that the componentwise order $\leqq$ is a partial order. A topological order of the elements of $\mathscr{E}$ is a total order $\preceq$ such that $c_j \leqq c_{j'}$ implies $e_j \preceq e_{j'}$.

Theorem 4 ([89]). *Let B be an efficient matroid base. Then there exists a topological order of the elements of* $\mathscr{E}$ *such that the greedy algorithm applied to this order yields B.*

Notice that Theorem 4 provides a necessary, but not a sufficient condition.

Greedy algorithms have also been developed in [36] to solve three-objective unconstrained combinatorial optimisation problems in polynomial time.

19.3 Scalarisation

The idea of scalarisation is to convert a multi-objective optimisation problem to a (parameterised) single objective problem that is solved repeatedly with different parameter values. We are interested in the following desirable properties of scalarisations [103]. Correctness requires that an optimal solution of the scalarised problem is (at least weakly) efficient. A scalarisation method is complete, if all efficient solutions can be found by solving a scalarised problem with appropriately chosen parameters. For computability it is important that the scalarisation is not harder than the single objective version of the MOCO problem. This relates to theoretical computational complexity as well as computation time in practice. Furthermore,

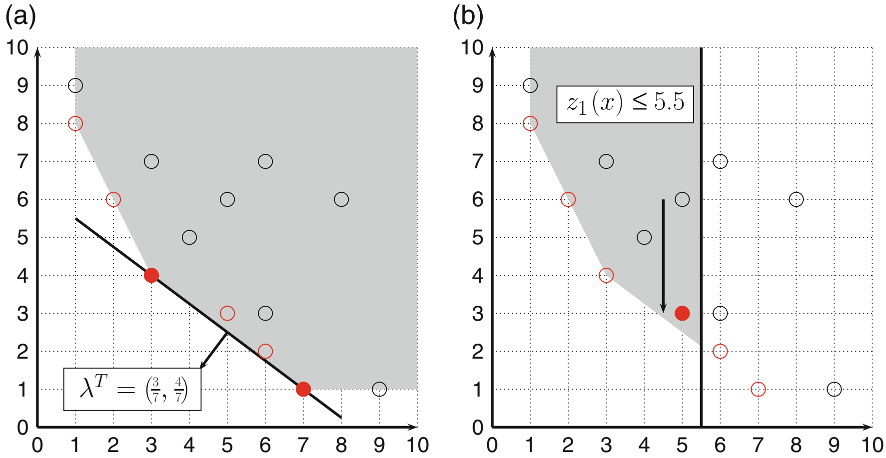
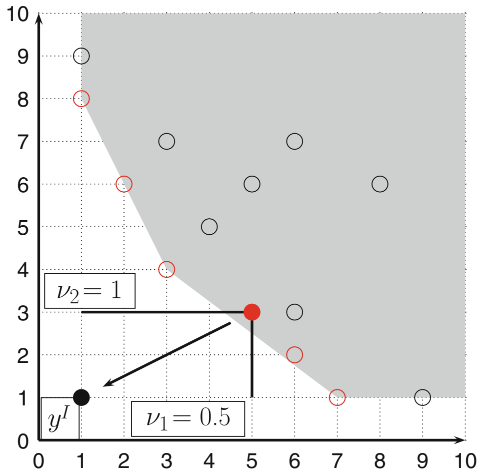


Fig. 19.5 (**a**) The weighted sum scalarisation. (**b**) The ε-constraint scalarisation

Fig. 19.6 The Chebychev scalarisation



since MOCO problem (19.1) has linear constraints and objectives, the scalarisation should have a linear formulation.

The scalarisation techniques that are most often applied in MOCO, illustrated in Figs. 19.5a,b and 19.6, are the weighted sum method (Fig. 19.5a)

$$\min\left\{\lambda^T z(x) : x \in X\right\}, \tag{19.5}$$

the ε-constraint method (Fig. 19.5b)

$$\min\left\{z_l(x) : z_k(x) \leq \varepsilon_k, k \neq l, x \in X\right\}, \tag{19.6}$$

Table 19.2 Properties of popular scalarisation methods

Scalarisation	Correct	Complete	Computable	Linear
Weighted sum	+	–	+	+
ε-Constraint	+	+	–	+
Chebychev	+	+	–	+

and the weighted Chebychev method (Fig. 19.6)

$$\min \left\{ \max_{k=1,\ldots,p} \nu_k (z_k(x) - y_k^I) : x \in X \right\} . \tag{19.7}$$

In Table 19.2 we summarise the properties of the methods listed in Eqs. (19.5)–(19.7).

All three scalarised problems clearly have a linear formulation (with integer variables, of course). They do also all compute (weakly) efficient solutions, see, e.g. [25] for proofs. As for completeness, it follows from the definition of non-supported non-dominated points, that the weighted sum scalarisation cannot compute any non-supported efficient solution. On the other hand, both the ε-constraint scalarisation and the Chebychev scalarisation can be tuned to find all efficient solutions. For proofs we do again refer the reader to, e.g. [25]. In terms of computability, we have seen before that the weighted sum scalarisation (19.5) does maintain the structure of the single objective variant of MOCO problem (19.1), it is therefore solvable with the same computational effort. On the other hand, the ε-constraint method (19.6) and the (linearised version of the) Chebychev scalarisation (19.7) both include bounds on objective function values. Such additional constraints usually make single objective versions of MOCO problem (19.1) harder to solve, because they destroy the structure of the problem which is exploited in efficient algorithms for their solution by adding knapsack type constraints.

In [24] it has been shown that all three scalarisations (and several others) are special cases of the more general formulation

$$\min_{x \in X} \left\{ \max_{k=1}^{p} [\nu_k (c_k x - \rho_k)] + \sum_{k=1}^{p} [\lambda_k (c_k x - \rho_k)] : c_k x \leqq \varepsilon_k, k = 1, \ldots, p \right\} , \tag{19.8}$$

where ν and λ denote (non-negative) weight vectors in $\mathbb{R}^p$, $\rho \in \mathbb{R}^p$ is a reference point and scalars ε_k represent bounds on objective function values. To see this, set $\nu_k = 0, \rho_k = 0$ and $\varepsilon_k = M$ for all $k = 1, \ldots, p$ and sufficiently large M to obtain the weighted sum problem (19.5); $\nu_k = 0$ for $k = 1, \ldots, p$, $\lambda_l = 1, \lambda_k = 0$ for all $k \neq l$, $\varepsilon_l = M$ and $\rho_k = 0$ for all $k = 1, \ldots, p$ for the ε-constraint scalarisation (19.6); and finally $\rho = y^I, \lambda = 0, \varepsilon_k = M$ for all $k = 1, \ldots, p$ for the Chebychev scalarisation (19.7). With regard to the general scalarisation in Eq. (19.8) we cite the following result.

Theorem 5 ([24]). *1. The general scalarisation* (19.8) *is correct, complete, and NP-hard.*
2. An optimal solution of the Lagrangean dual of the linearised general scalarisation is a supported efficient solution of the MOCO problem (19.1)*.*

Theorem 5 shows that the general scalarisation will be difficult to solve and that solving it by Lagrangian relaxation is not useful to obtain non-supported non-dominated points. Moreover, Table 19.2 indicates that complete scalarisations are not computable, whereas computable ones are not complete. To resolve this dilemma, one may ask whether it is possible to come up with a compromise, that is a scalarisation that falls somewhere between the weighted sum scalarisation and the ε-constraint/Chebychev scalarisation, combining their strengths and eliminating their weaknesses. This is indeed possible with the elastic constraint scalarisation. This scalarisation is derived from the ε-constraint scalarisation, but allows the constraints on objective function values to be violated, with the violation penalised in the objective function. Formally, the elastic constraint scalarisation is defined as in Eq. (19.9),

$$\min\left\{c_l x + \sum_{k\neq l} \mu_k w_k : c_k x + v_k - w_k = \varepsilon_k, k \neq l; \quad x \in X; \quad v_k, w_k \geq 0, k \neq l\right\}. \tag{19.9}$$

The constraints on objective values in the ε-constraint scalarisation are turned into equality constraints by means of slack and surplus variables v_k and w_k. Positive values of w_k indicate constraint violations in the ε-constraint scalarisation and are penalised with a contribution of penalty parameter μ_k in the objective function. Figure 19.7a,b compare these two scalarisations. Figure 19.7a repeats Fig. 19.5b, where the vertical line indicates a hard constraint on $z_1(x) \leqq 5.5$ and the arrow indicates minimisation of z_2. The optimal point is indicated by a filled circle. Figure 19.7b shows that points to the right of the vertical line are feasible in the elastic constraint scalarisation, with lighter shading indicating greater violation of the limit of ε on z_2. To the right of the vertical line, for every feasible point of the MOCO problem, an arrow indicates the objective value of the same point in Eq. (19.9). Note that the optimal point for Eq. (19.9) in this example is now to the right of the vertical line. Theorem 6 summarises the properties of the elastic constraint scalarisation.

Theorem 6 ([30]). *The method of elastic constraints is correct and complete. It contains the weighted sum and ε-constraint method as special cases.*

For a proof of the first part we refer to [30]. The second part follows by setting first $\varepsilon_k < min_{x\in X} c^k x$ for $k \neq l$. Then $v_k = 0$ for all feasible solutions $x \in X$ and hence $w_k = c^k x - \varepsilon_k$ and the problem reduces to the weighted sum problem. Secondly, we can set $\mu_k = M$ for a sufficiently large number M. Then any feasible point with $y_k > \varepsilon_k$ will contribute so much penalty to the objective function of elastic

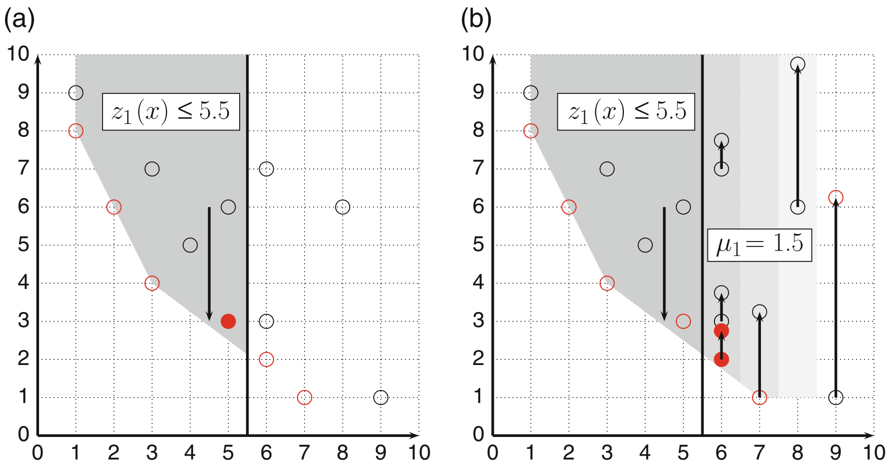


Fig. 19.7 (**a**) The ε-constraint scalarisation. (**b**) The elastic constraint scalarisation

constraint scalarisation (19.9) that it is not optimal. Hence scalarisations (19.9) and (19.6) will have the same set of optimal solutions.

Although the elastic constraint scalarisation (19.9) is *NP*-hard, it is often solvable in reasonable time in practice because it respects problem structure better than the ε-constraints of Eq. (19.6). It limits the damage done by adding hard ε-constraints to the model. A successful implementation of the method for an application to bi-objective set partitioning problems has been reported in [29].

When solving a MOCO problem using scalarisation, bearing in mind that the scalarised combinatorial optimisation problems may be hard to solve, it is of interest to know how many scalarised problems need to be solved in order to find a minimal complete set of efficient solutions. Theorem 7 summarises the results in this regard.

Theorem 7. *1. In the case $p = 2$, the number of scalarised single objective problems to be solved in order to determine Y_N is bounded by $2|Y_N| - 1$ [11, 77]. In case the ε-constraint scalarisation is used, this bound is $|Y_N| + 1$ [51].*
2. In the case $p = 3$ the bound is $3|Y_N| - 2$ and $2|Y_N| - 1$ for the ε-constraint scalarisation [15]. For $p > 3$ the general bound is $O\left(|Y_N|^{\lfloor \frac{p}{2} \rfloor}\right)$, [46].

We note that general scalarisation algorithms to solve MOCO problems often make use of scalarisations that use resource constrained single objective problems of the form

$$\min\{g(f(x)) : f(x) \leqq u, x \in X\}, \tag{19.10}$$

where $g : \mathbb{R} \to \mathbb{R}$ is a strongly increasing function, see [46] for a general scheme and the references in Sect. 19.3.1 for specific examples of such algorithms. These

Table 19.3 Algorithms based on scalarisation

MOCO problem	Scalarisation	Reference
BO binary LP	Weighted sum with ε-constraints	[11]
BO knapsack, capacitated network routing	Weighted Chebychev	[77]
TO multidimensional knapsack	Lexicographic ε-constraint	[51]
Generic	Weighted sum with ε-constraints	[46]
TO three-dimensional knapsack	General scalarisation	[15]
BO integer minimum cost flow (*)	ε-constraint	[33]
BO knapsack	Lexicographic weighted Chebychev	[85]
BO multidimensional knapsack	Weighted sum with constraints	[93]
TO three-dimensional knapsack	Lexicographic ε-constraint	[62]
MO knapsack, shortest path, spanning tree	Lexicographic ε-constraint	[53]
MO three-dimensional knapsack, assignment	Lexicographic ε-constraint	[63]
MO TSP TO knapsack, assignment	Lexicographic ε-constraint	[44]
BO knapsack	Augmented weighted Chebychev	[16]
MO integer LP	Single objective with constraints	[47]
BO, TO multidimensional knapsack	Augmented ε-constraint	[56]
BO set partitioning (*)	Elastic constraint	[29], [94]
BO TSP with profits (*)	ε-constraint	[6]

algorithms work with an upper bound set that is updated throughout the algorithm. They end as soon as it is guaranteed that all non-dominated points have been found.

19.3.1 Scalarisation Algorithms from the Literature

In this section we present a tabular overview of scalarisation algorithms for MOCO problems. Most of these algorithms are generic, in the sense that they can be applied to any multi-objective integer programming problem. Hence Table 19.3 presents the type of scalarisation applied, for which MOCO problem the algorithm has been tested, and the references. If the algorithm has been specifically developed for a particular MOCO problem, this is denoted by (*) appearing behind the problem.

19.4 The Two-Phase Method

If it is not possible to adapt a single objective algorithm to the multi-objective case directly, then it is desirable to use the single objective algorithm to solve single objective instances of the multi-objective problem (repeatedly) to obtain some efficient solutions. This is in particular true for problems for which the single objective counterpart is solvable in polynomial time. In this section we consider algorithms for such multi-objective optimisation problems. Considering the hardness of even the "simplest" bi-objective combinatorial optimisation problems as discussed in Sect. 19.1.1, this indicates that there is benefit in solving the single objective version as a subproblem even if that has to be done often, so that the polynomial time algorithms can be exploited as much as possible. We shall describe the two phase method, originally proposed by Ulungu and Teghem [97], as a general algorithmic framework to solve multi-objective combinatorial optimisation problems in a way that relies heavily on the use of fast algorithms for single objective optimisation. A detailed exposition of the method can be found in [74].

19.4.1 The Two Phase Method for Two Objectives

We first explain the two phase method for MOCO problems with two objectives. In Phase 1 at least a (minimal) complete set of extreme efficient solutions X_{SE1} is found. This is typically done starting from two lexicographically optimal solutions. The dichotomic method then recursively calculates a weight vector $\lambda^T = (\lambda_1, \lambda_2) \in \mathbb{R}^2_>$ as a normal to the line connecting two currently known non-dominated points y^l and y^r with $y^l_1 < y^r_1$ (and therefore $y^l_2 > y^r_2$ according to Eq. (19.11) and solving a weighted sum problem $\min\{\lambda^T Cx : x \in X\}$, where

$$\lambda := (y^l_2 - y^r_2, y^r_1 - y^l_1). \tag{19.11}$$

In Fig. 19.8a, lexicographic minima $(1, 8)^T$ and $(7, 1)^T$ are identified and define $\lambda^T = (8 - 1, 7 - 1) = (7, 6)$. The corresponding weighted sum problem yields

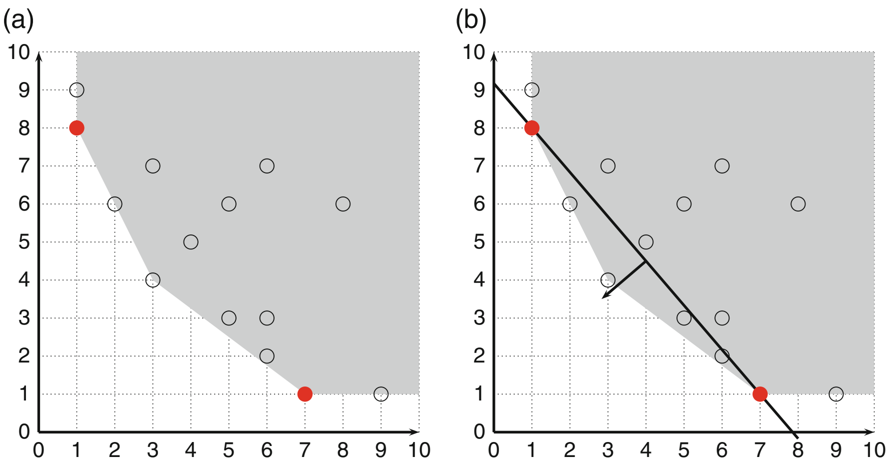


Fig. 19.8 (**a**) Lexicographically optimal points. (**b**) The first weighted sum problem

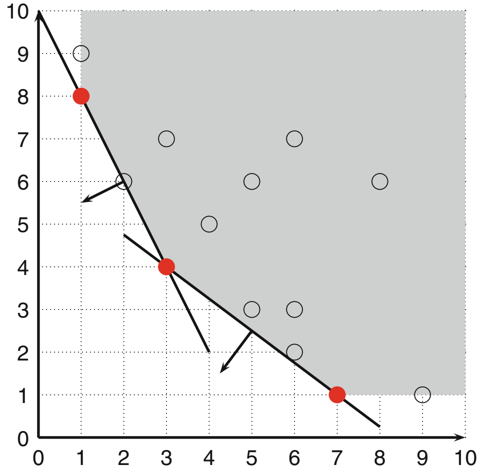


Fig. 19.9 Phase 1 of the two phase method

non-dominated point $(3,4)^T$ (Fig. 19.8b) and allows dividing the problem in two: The first weighted sum problem with $\lambda^T = (8-4, 3-1) = (4,2)$ looks for new non-dominated extreme points between $(1,8)^T$ and $(3,4)^T$. The second with $\lambda^T = (4-1, 7-3) = (3,4)$ is used to explore the area between $(3,4)^T$ and $(7,1)^T$. Neither finds any further non-dominated extreme points in the example. The first, however, may or may not find supported (but non-extreme) non-dominated point $(2,6)^T$, depending on the solver (Fig. 19.9).

This dichotomic scheme has been independently described by Cohon [13], Aneja and Nair [2] and Dial [21]. Alternatively, it is possible to apply a parametric scheme, which starts with one of the lexicographically optimal points, say for permutation (z^1, z^2) of the objective functions, and then moves to further extreme non-dominated points by systematically increasing the weight of the second objective function in the weighted sum problem. This scheme is usually applied, if the single objective version of MOCO problem (19.1) can be solved by linear programming. In this case a parametric version of the simplex algorithm to solve bi-objective linear programmes is applied, see, e.g. [75] and references therein for further details. Since we assume that all objective function values are integer, lexicographically optimal solutions can be found by solving two single objective problems, as Przybylski et al. [71] have observed. First, weighted sum problem $\min\{\lambda^T Cx : x \in X\}$ is solved with $\lambda^T = (1,0)$. Since an optimal solution of this problem may be weakly efficient, resulting in weakly non-dominated point y^1 the weighted sum problem is then resolved with $\lambda^T = (y_2^1 + 1, 1)$ to either confirm that y^1 is the lexicographically optimal point or to replace it by the true lexicographically optimal point.

In Phase 2 any other non-dominated points are determined, in particular non-supported non-dominated points Y_{NN}. After Phase 1 it is possible to restrict the search for non-dominated points to triangles defined by consecutive non-dominated extreme points, see Fig. 19.10a. In the literature, several methods have been proposed to achieve that. Neighbourhood search as suggested, e.g. in [52, 86] is

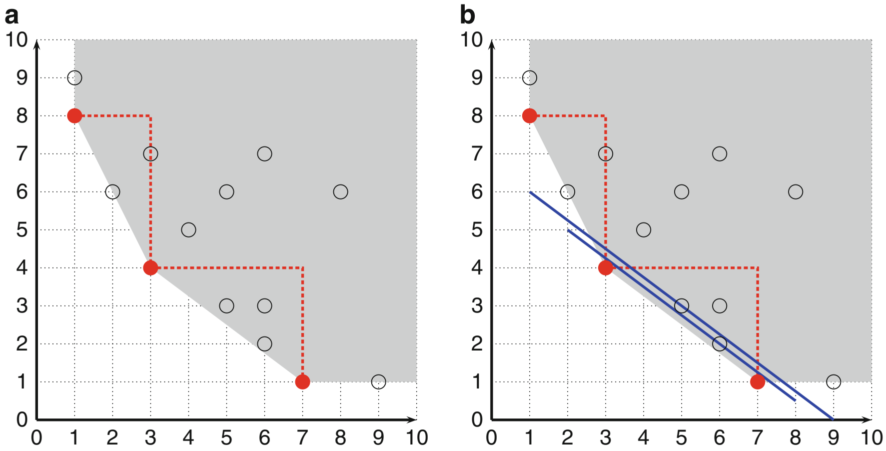


Fig. 19.10 (**a**) The triangles where non-supported non-dominated points may be located. (**b**) Ranking non-supported non-dominated points

in general wrong, because efficient solutions are in general not connected via a neighbourhood structure, see Sect. 19.1.3. One can apply constraints to restrict objective values to triangles and modify those constraints as further points are discovered. However, the computational effort may be high, in particular because problems tend to get harder when adding further (knapsack type) constraints, see Sect. 19.1.2. This idea is therefore contrary to the spirit of the two phase method. Variable fixing strategies as suggested by Ulungu and Teghem [97] can be a reasonable alternative. However, currently the best performing two phase algorithms are those that exploit a ranking algorithm that generates solutions of single objective (weighted sum) problems in order of their objective value (Fig. 19.10b). The ranking method has been successfully applied for a number of bi-objective problems, namely bi-objective assignment [71], multi-modal assignment [67], spanning tree [92], shortest path [75], integer network flow [76], and binary knapsack [40]. It is important to note that for all these problems efficient ranking algorithms to find r-best solutions of the single objective version of the problem are available, references to which can be found in the original papers.

The ranking algorithm is applied to weighted sum problems corresponding to each triangle (see Fig. 19.10b). Let $\lambda \in \mathbb{R}^2_>$ be a weight vector defined as the normal to the line connecting two consecutive non-dominated extreme points, e.g. $\lambda^T = (4, 2)$ (left triangle) and $\lambda^T = (3, 4)$ (right triangle) in Fig. 19.10b. The ranking algorithm then finds second, third, etc. best solutions for the problem of minimising $\lambda^T Cx$, in increasing order of objective values for $\lambda^T Cx$ as shown in Fig. 19.10b. The two non-dominated extreme points defining the triangle (and λ according to Eq. (19.11)) define optimal values for this problem.

This process stops at the latest once a solution, the (weighted sum) objective function value of which is worse than that of the third corner of the triangle, is found.

In Fig. 19.10b, with $\lambda^T = (3, 4)$ the algorithm finds $(6, 2)^T$ with objective 26, then $(5, 3)^T$ with objective value 27. Both points are non-dominated. The 4th best point is $(6, 3)^T$, which is dominated by both previous points. The algorithm can now be stopped, because any further point would also be dominated. This indicates that the enumeration of solutions can be stopped even before the ranking algorithm has identified a feasible solution outside the triangle.

In order to facilitate this, upper bounds on the weighted sum objective value of any efficient solution in the triangle can be derived. Let $\{x^i : 0 \leqq i \leqq q\}$ be feasible solutions with $z(x^i) \in \Delta(x^0, x^q)$, the triangle defined by $z(x^0)$ and $z(x^q)$, sorted by increasing value of z_1, where $z(x^0)$ and $z(x^q)$ are two consecutive non-dominated extreme points of conv(Y). Let

$$\beta_0 := \max_{i=0}^{q-1}\{\lambda_1 z_1(x^{i+1}) + \lambda_2 z_2(x^i)\}, \tag{19.12}$$

$$\beta_1 := \max\left\{\max_{i=1}^{q-1}\{\lambda^1 z^1(x^i) + \lambda^2 z^2(x^i)\},\right. \\ \left.\max_{i=0}^{q}\{\lambda^1(z^1(x^i) - 1) + \lambda^2(z^2(x^{i-1}) - 1)\}\right\}, \tag{19.13}$$

$$\beta_2 := \max_{i=0}^{q}\{\lambda^1(z^1(x^i) - 1) + \lambda^2(z^2(x^{i-1}) - 1)\}. \tag{19.14}$$

Then $\beta_0 \geqq \beta_1 \geqq \beta_2$ are upper bounds for the weighted sum objective value of any non-dominated point in $\Delta(x^0, x^q)$. The ranking process can be stopped as soon as bound β_j is reached. The use of β_0 and β_1 from Eqs. (19.12) and (19.13) allows the computation of the maximal complete set, whereas β_2 guarantees a minimal complete set only. Notice that these bounds will be improved every time a new efficient solution is found. In Fig. 19.10b, the initial bound β_2 using Eq. (19.14) with $\{y^0 = (3, 4)^T, y^1 = (7, 1)^T\}$ is 30, but once $(6, 2)^T$ is found this improves to 27 and with $(5, 3)^T$ also discovered to 24. Clearly no additional feasible point in the triangle can be better than that bound.

The two phase method involves the solution of enumeration problems. In order to find a maximal complete set one must, of course, enumerate all optimal solutions of $\min_{x\in X} \lambda^T Cx$ for all weighted sum problems solved in Phase 1 and enumerate all $x \in X_{NE}$ with $Cx = y \in Y_{ND}$ for all non-supported non-dominated points y. But even in order to compute a minimal complete set enumeration is necessary to find X_{SE2}. There can indeed be many optimal solutions of $\min_{x\in X} \lambda^T Cx$ that are non-extreme and not equivalent to one another (see, e.g. points $(1, 8)^T$, $(2, 6)^T$, $(3, 4)^T$ in Fig. 19.8a).

As an example of an effective implementation of a two phase method, Przybylski et al.[71] have developed a two phase algorithm for the bi-objective assignment problem using the Hungarian method [50] to solve weighted sum assignment problems, an enumeration algorithm by Fukuda and Matsui [35] to enumerate all optimal solutions of these problems, and a ranking algorithm for (non-optimal) solutions of $\min_{x\in X} \lambda^T Cx$ by Chegireddy and Hamacher [12]. The algorithm outperformed a two phase method using variable fixing, a two phase method using a heuristic to find good feasible solution before Phase 2, general exact MOCO algorithms, and CPLEX using constraints on objectives. An explanation for the good performance of the method is given by the distribution of objective function values for randomly generated instances.

19.4.2 The Two Phase Method for Three Objectives

Extending the two phase method to three objectives is not trivial. In Phase 1, a weight vector defines the normal to a plane. While three non-dominated points are sufficient to calculate such a weight vector, there may be up to six different lexicographically optimal points, so it is unclear which points to choose for calculating weights to start with. Moreover, even if the minimisers of the three objective functions are unique (and therefore there are only three lexcicographically optimal solutions), the normal to the plane defined by three non-dominated points may not be positive. In this case no further non-dominated points would be calculated, see an example presented in [72].

Hence, a direct generalisation of the two phase method already fails with the initialisation. Two generalisations of Phase 1 have been proposed by Przybylski et al. [72] and Özpeynirci and Köksalan [64]. We present the ideas of [72], where Phase 1 relies on decomposition of the simplex of all non-negative normalised weights

$$W^0 := \left\{ \lambda \in \mathbb{R}^p_> : \lambda_p = 1 - \sum_{k=1}^{p-1} \lambda_k \right\} \tag{19.15}$$

into subsets

$$W^0(y) := \{\lambda \in W^0 : \lambda^T y \leqq \lambda^t Y' \text{ for all } y' \in Y\} \tag{19.16}$$

of W^0 consisting of all weight vectors λ such that y is a point minimising $\lambda^T y$ over Y. It turns out that y is a non-dominated extreme point if and only if $W^0(y)$ has dimension $p - 1$. This allows us to define adjacency of non-dominated extreme points as follows. Non-dominated extreme points y^1 and y^2 are adjacent if and only if $W^0(y^1) \cap W^0(y^2)$ is a polytope of dimension $p - 2$, which then makes it possible to derive the optimality condition of Theorem 8.

Theorem 8 ([72]). *If S is a set of supported non-dominated points then*

$$Y_{SN1} \subseteq S \iff W^0 = \bigcup_{y \in S} W^0(y). \tag{19.17}$$

The results above lead to a new Phase 1 algorithm. Let S be a set of supported non-dominated points. Let $W_p^0(y) = \{\lambda \in W^0 : \lambda^T y \leqq \lambda^T y^* \text{ for all } y^* \in S\}$. Then $W^0(y) \subseteq W_p^0(y)$ for all $y \in S$ and $W^0 = \bigcup_{y \in S} W_p^0(y)$. The algorithm initialises S with the lexicographically optimal points. While S is not empty it chooses $\hat{y} \in S$, computes $W_p^0(\hat{y})$ and investigates all facets F of $W_p^0(\hat{y})$ defined by $\lambda^T \hat{y} = \lambda^T y'$ for $y' \in S$ to determine whether F is also a facet of $W^0(\hat{y})$. If $\hat{y}$ minimises $\lambda^T y$ for all $\lambda \in F$ then $\hat{y}$ and y' are adjacent and F is the common face of $W^0(\hat{y})$ and $W^0(y')$. If there are $y^* \in Y$ and $\lambda \in F$ such that $\lambda^T y^* < \lambda^T y$ then $W^0(\hat{y})$ is a proper subset of $W_p^0(\hat{y})$, y^* is added to S and $W_p^0(\hat{y})$ is updated.

At the end of Phase 1, all non-dominated extreme points of Y and a complete set of extreme efficient solutions are known. Any other supported efficient solutions must be optimal solutions to weighted sum problems with λ belonging to 0- and 1-dimensional faces of some $W^0(y)$ of some non-dominated extreme point y. To find these, for each $y \in Y_{SN1}$, we find all optimal solutions of weighted sum problems (19.5) defined firstly by weight vectors λ that are extreme points of $W^0(y)$ which are not located on the boundary of W^0, and secondly by weight vectors λ located in the interior of edges of $W^0(y)$ the extreme points of which belong to the boundary of W^0.

To find non-supported non-dominated points one can once again employ a ranking algorithm for weighted sum problems, where λ is chosen to be a normal to a facet defining hyperplane of $\text{conv}(Y) + \mathbb{R}^p_{\geqq}$. Note that this is analogous to the bi-objective case. The difficult part in the completion of the Phase 2 algorithm is the computation of good upper bounds and the selection of weights in a way that keeps the ranking of solutions as limited as possible. Here the difficulty arises from the fact that unlike in the case of two objectives, the area where non-supported non-dominated points can be found does not decompose into disjoint subsets, see also Sect. 19.1.4. Details about Phase 2 for multi-objective combinatorial optimisation problems can be found in [73], where its generalisation to more than three objectives via a recursive scheme is also discussed.

Numerical results in [70] show that the method outperforms three general methods to solve MOCO problems by a factor of up to 1000 on the three objective assignment problem. The biggest advantage of the two phase method is that it respects problem structure, thereby enabling to use efficient algorithms for single objective problems as much as possible.

19.4.3 Two-Phase Algorithms from the Literature

Two-phase algorithms from the literature are summarised in Table 19.4. We list the MOCO problem for which the algorithm has been designed, the methods used for Phases 1 and 2, and the reference.

Table 19.4 Two-phase algorithms

MOCO problem	Phase 1 approach	Phase 2 approach	Reference
Bi-objective integer network flow	Parametric	Local search	[52]
Bi-objective integer network flow	Parametric	Local search	[86]
Bi-objective integer network flow	Parametric	Ranking	[76]
Bi-objective assignment	Dichotomic	Variable fixing	[97]
Bi-objective assignment	Dichotomic	Variable fixing	[96]
Bi-objective assignment	Dichotomic	Ranking	[71]
Three-objective assignment	Dichotomic	Ranking	[72, 73]
Bi-objective multimodal assignment	Dichotomic	Ranking	[67]
Bi-objective spanning tree	Dichotomic	Ranking, branch and bound	[92]
Bi-objective shortest path	Parametric	Label correcting	[59]
Bi-objective shortest path	Dichotomic Parametric	Label correcting, label setting Ranking	[75]
Bi-objective knapsack	Dichotomic	Branch and bound	[101]
Bi-objective knapsack	Dichotomic	Ranking	[40]
Three-objective knapsack	Dichotomic	Ranking	[40]

19.5 Multi-Objective Branch and Bound

Branch and bound is a standard method to solve single objective combinatorial optimisation problems and is contained in any textbook on combinatorial optimisation. In order to apply it to multi-objective combinatorial optimisation problems, we need to define the branching and the bounding parts of the algorithm. Branching refers to the method used to partition the feasible set of a combinatorial optimisation problem into two or more disjoint subsets that define new subproblems. Bounding refers to the computation of lower and upper bounds (bound sets), see Sect. 19.1.4, and their use to eliminate subproblems from further consideration. Apart from branching and bounding, a strategy to select the next subproblem for evaluation is required. We address these questions based on contributions to the MOCO literature in turn.

19.5.1 Branching and Node Selection

The branching strategy concerns only the feasible set of the problem, hence there is no difference in partitioning the feasible set of a subproblem between the single and multiple objective cases. Nevertheless, the rules used to decide the branching may involve the objective function coefficients. Several authors propose branching strategies for the multi-objective binary knapsack problem

$$\max\left\{\left(\sum_{j=1}^{n} c_j^1 x_j, \ldots, \sum_{j=1}^{n} c_j^p x_j\right) : \sum_{j=1}^{n} w_j x_j \leqq \omega, x \in \{0,1\}^p\right\}, \tag{19.18}$$

where c_j^k and w_j are non-negative integer numbers. For $p = 1$, algorithms often use the rank of a variable according to decreasing value-to-weight ratio c_j/w_j. They define the branching strategy by setting variables with small rank equal to one and those with high rank equal to 0, see [43].

In the multi-objective case, Ulungu and Teghem [98] propose to consider the rank r_j^k of each item j according to ratio c_j^k/w_j for $k \in \{1,\ldots,p\}$, and to order the items by decreasing value of the sum of the ranks $\sum_{k=1,\ldots,p} r_j^k$. Bazgan et al. [3] use an order obtained from the decreasing sum of ratios $\sum_{k=1,\ldots,p} c_j^k/w_j$, increasing worst rank $\max_{k=1,\ldots,p} r_j^k$ and best rank $\min_{k=1,\ldots,p} r_j^k$. Jorge [40] proposes orders respecting the dominance relation between vectors of ratios $(c_j^1/w_j,\ldots,c_j^p/w_j)$. Jorge [40] also proposes to count for each item j the number of items $|dom(j)|$ dominating it, ordering items by increasing value of $|dom(j)|$. Finally, Jorge [40] suggests to order items according increasing rank following a principle from evolutionary multi-objective algorithms, see [18]. The choice of order has a considerable impact on solution times, as [40] demonstrates.

Other branching strategies that are popular in single objective branch and bound methods require the evaluation of (relaxed) solutions of subproblems (such as computing the fractionality of variables in the LP relaxation of a binary optimisation problem) and have not yet been modified for the use in multi-objective branch and bound algorithms.

Typical node selection strategies follow a depth-first or breadth-first principle and can be adapted to multi-objective branch and bound without changes. In fact, most published algorithms follow a depth-first strategy. More elaborate methods, such as best-first strategies on the other hand require adaptations such as the comparison of bound sets, and the difficulties this raises have so far prevented researchers to implement such strategies.

19.5.2 Bounding and Fathoming Nodes

Multi-objective branch and bound algorithms from the literature maintain a set of (feasible) points that defines an upper bound set U and the algorithms terminate as soon as $U = Y_N$. U can be initialised as $\{(\infty,\ldots,\infty)\}$ or by a set of feasible solutions that are obtained by heuristic methods. The upper bound set is updated any time a new feasible solution is found. For each subproblem, or node of the branch and bound tree, a lower bound set L for the set Y_N of non-dominated points of the subproblem is computed.

A node of the branch and bound tree can be fathomed if

1. The subproblem has an empty feasible set (infeasibility);
2. The non-dominated set Y_N of the subproblem belongs to L (optimality);
3. For every $l \in L$ there exists some $u \in U$ such that $u \leqq l$ (dominance).

Notice that fathoming a subproblem by optimality is often only practically possible if L consists of a single feasible point. The general condition $Y_N \subseteq L$ is, however, very difficult to verify. Indeed, because L is generally a continuous set, whereas U is a discrete set of points, fathoming a node by dominance (or even optimality) is not trivial. Several researchers have proposed methods for the practical application of the dominance test. Przybylski et al. [73] describe a method which, for a given (feasible) upper bound set U computes a set of points $D(U)$ such that the condition that the non-dominated set lies between the lower and upper bound set can be written in a way that is easier to use algorithmically, see Fig. 19.11a,b for illustrations of $D(U)$.

Proposition 1. *Let L and Ube upper bound sets for Y_N. Then*

$$\left(L + \mathbb{R}^p_{\geqq}\right) \setminus \left(U + \mathbb{R}^p_{>}\right) = \left(L + \mathbb{R}^p_{\geqq}\right) \cap \cup_{u \in D(U)} \left(u - \mathbb{R}^p_{\geqq}\right). \tag{19.19}$$

Then, all points in L are weakly dominated by points in U if and only if all points in $D(U)$ are not dominated by any points in L. This rule for fathoming by dominance is illustrated in Fig. 19.11a. In the case that all $y \in Y$ are integer vectors, the fathoming rule can be improved, as shown in Fig. 19.11b, for details we have to refer to [73, 90].

Sourd and Spanjaard [90] reformulate the dominance condition via strictly monotone functions $h : \mathbb{R}^p \to \mathbb{R}$, i.e. $y^1 < y^2$ implies $h(y^1) < h(y^2)$. If a strictly

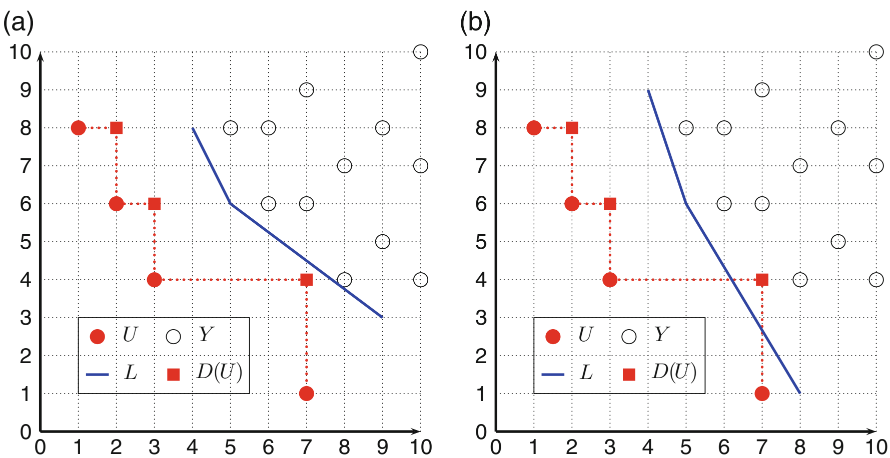


Fig. 19.11 (**a**) The node can be fathomed by dominance. (**b**) The node can be fathomed by dominance assuming $Y \subset \mathbb{Z}^p$

monotone function h is such that $h(l) \geqq 0$ for all $l \in L$ and $h(u) \leqq 0$ for all $u \in U$ then $h(y) \geqq 0$ for all $y \in Y_N$. Sourd and Spanjaard [90] provide classes of functions h for the case that $L + \mathbb{R}^p_{\geqq}$ is a polyhedron and $p = 2$.

19.5.3 Multi-Objective Branch and Bound Algorithms from the Literature

In this section, we provide a tabular summary of multi-objective branch and bound algorithm from the literature. For each algorithm in Table 19.5, we first mention the MOCO problem to which the algorithm is applied, where BO stands for bi-objective, TO for three-objective and MO for multi-objective. Then we provide specifications for the algorithm, namely lower and upper bound sets used, the branching and node selection strategies and the method used to find new feasible solutions. Finally, we list the reference. Note that the column headings refer to minimisation problems, so the interpretation of lower and upper bound set is reversed for problems that are formulated with maximisation objectives.

19.6 Conclusion

In this chapter we have summarised exact algorithmic approaches for solving multi-objective combinatorial optimisation problems. We have focused on general algorithmic schemes of increasing complexity. Starting from dynamic programming and greedy schemes, we also considered scalarisation and general algorithms based on scalarisation and the two-phase method as a specific technique for multi-objective optimisation. Finally, we covered multi-objective branch and bound algorithms. We do not claim that the surveys in any of these sections are exhaustive, since new algorithms appear all the time, and researchers in diverse fields are today involved in the development of new algorithms. A lot of algorithms are also being developed specifically for particular MOCO problems. We just mention [57] for an adaptation of the concept of the core of a knapsack problem to the bi-objective case and [79] for non-additive multi-objective shortest path problems. Furthermore, new general algorithmic ideas begin to make an appearance in multi-objective combinatorial optimisation. Jozefowiez et al. [41] combine an ε-constraint method with branch-and-cut to solve the bi-objective covering tour problem.

Table 19.5 Multi-objective branch and bound algorithms

MOCO problem	Upper bound	Lower bound	Branching	Node selection	Feasible solutions	Reference
MO binary	Incumbent set	Utopia point	Variable fixing	Depth-first	Variable fixing	[45]
BO knapsack	Incumbent set	Utopia point	Variable fixing	Depth-first	Variable fixing	[98]
BO knapsack	Adaptation of [98] restricted to triangles					[101]
BO spanning tree	Incumbent set	Utopia point	Edge fixing	Depth-first	At leaves	[78]
BO spanning tree	Incumbent set	Convex relaxation	Edge fixing	Depth-first	Convex relaxation at nodes	[90]
MO knapsack	Incumbent set	Ideal point LP relaxation	Variable fixing	Depth-first	At leaves	[34]
TO knapsack	Incumbent set	Utopia point	Variable fixing	Depth-first	At leaves	[40]
TO knapsack	Incumbent set	Convex relaxation	Variable fixing	Lexicographic order of 5 criteria	Part of UB	[40]
BO assignment	Adaptation of [90] restricted to triangles					[19]
BO flow shop	No details	No details	No details	Depth-first	No details	[58]
BO mixed integer	Incumbent set	Ideal point of LP relaxation	Variable fixing	Depth-first	LP at leaves	[54, 55]
BO mixed integer	Extended incumbent set	LP relaxation	Variable fixing	Depth-first	LP at leaves	[100]

Recently, researchers are developing algorithms that use (single objective) integer programming solvers as a black box. These approaches exploit the tremendous advantages that have been made in single objective combinatorial optimisation algorithms and solvers, see [9] for one such method. Also, new methodologies using tools from algebraic geometry emerge, see [7]. We have not touched at all on the vast field of heuristic and metaheuristic algorithms for MOCO problems, or the increasing interest that approximation algorithms received, see e.g. [4] and references therein. All of these established research directions and newly emerging areas offer plenty open questions for future research.

References

1. Aissi, H., Mahjoub, R., McCormick, S.T., Queyranne, M.: A strongly polynomial time algorithm for multicriteria global minimum cuts. In: Lee, J., Vygen, J. (eds.) Proceedings of 17th International Conference on Integer Programming and Combinatorial Optimization, IPCO 2014, Bonn, June 23–25, 2014. Lecture Notes in Computer Science, vol. 8494, pp. 25–36. Springer, Berlin (2014)
2. Aneja, Y.P., Nair, K.P.K.: Bicriteria transportation problem. Manag. Sci. **25**, 73–78 (1979)
3. Bazgan, C., Hugot, H., Vanderpooten, D.: Solving efficiently the 0–1 multi-objective knapsack problem. Comput. Oper. Res. **36**, 260–279 (2009)
4. Bazgan, C., Jamain, F., Vanderpooten, D.: Approximate Pareto sets of minimal size for multiobjective optimization problems. Oper. Res. Lett. **43**, 1–6 (2015)
5. Bellman, R.: On a routing problem. Q. Appl. Math. **16**, 87–90 (1958)
6. Bérubé, J.-F., Gendreau, F., Potvin, J.-Y.: An exact ε-constraint method for bi-objective combinatorial optimization problems: application to he travelling slaesman problem with profits. Eur. J. Oper. Res. **194**, 39–50 (2009)
7. Blanco, V., Puerto, J.: Partial Gröbner bases for multiobjective integer linear ptimization. SIAM J. Discret. Math. **23**, 571–595 (2009)
8. Blanco, V., Puerto, J.: A new complexity result on multiobjective linear integer programming using short rational generating functions. Optim. Lett. **6**, 537–543 (2012)
9. Boland, N., Charkhgard, H., Savelsbergh, M.: The triangle splitting method for biobjective mixed integer programming. In: Lee, J., Vygen, J. (eds.) Proceedings of 17th International Conference on Integer Programming and Combinatorial Optimization, IPCO 2014, Bonn, June 23–25, 2014. Lecture Notes in Computer Science, vol. 8494, pp. 162–173. Springer, New York (2014)
10. Camerini, P.M., Galbiati, G., Maffioli, F.: The complexity of multi-constrained spanning tree problems. In: Lovasz, L. (ed.) Theory of Algorithms, pp. 53–101. North-Holland, Amsterdam (1984)
11. Chalmet, L.G., Lemonidis, L., Elzinga, D.J.: An algorithm for the bi-criterion integer programming problem. Eur. J. Oper. Res. **25**, 292–300 (1986)
12. Chegireddy, C.R., Hamacher, H.W.: Algorithms for finding k-best perfect matchings. Discret. Appl. Math. **18**, 155–165 (1987)
13. Cohon, J.L.: Multiobjective Programming and Planning. Academic Press, New York (1978)
14. Corley, H.W.: Efficient spanning trees. J. Optim. Theory Appl. **45**, 481–485 (1985)
15. Dächert, K., Klamroth, K.: A linear bound on the number of scalarizations needed to solve discrete tricriteria optimization problems. J. Glob. Optim. **61**, 643–676 (2015)
16. Dächert, K., Gorski, J., Klamroth, K.: An augmented weighted Tchebycheff method with adaptively chosen parameters for discrete bicriteria optimization problems. Comput. Oper. Res. **39**, 2929–2943 (2012)
17. Daellenbach, H.G., De Kluyver, C.A.: Note on multiple objective dynamic programming. J. Oper. Res. Soc. **31**, 591–594 (1980)
18. Deb, K.: Multi-Objective Optimization Using Evolutionary Algorithms. Wiley, Chichester (2001)
19. Delort, C.: Algorithmes d'énumération implicite pour l'optimisation multi-objectifs: Exploitation d'ensembles bornant et application aux problèmes de sac à dos et d'affectation. Ph.D. thesis, Université Pierre et Marie Curie (Paris VI) (2011)
20. Delort, C., Spanjaard, O.: Using bound sets in multiobjective optimization: Application to the biobjective binary knapsack problem. In: Desta, P. (ed.) Proceedings of 9th International Symposium on Experimental Algorithms, SEA 2010, Ischia Island, Naples, March 20–22, 2010. Lecture Notes in Computer Science, vol. 6049, pp. 253–265. Springer, Heidelberg (2010)
21. Dial, R.: A model and algorithm for multicriteria route-mode choice. Transp. Res. B **13**, 311–316 (1979)

22. Dijkstra, E.W.: A note on two problems in connexion with graphs. Numer. Math. **1**, 269–271 (1959)
23. Ehrgott, M.: On matroids with multiple objectives. Optimization **38**, 73–84 (1996)
24. Ehrgott, M.: A discussion of scalarization techniques for multiobjective integer programming. Ann. Oper. Res. **147**, 343–360 (2005)
25. Ehrgott, M.: Multicriteria Optimization, 2nd edn. Springer, Berlin (2005)
26. Ehrgott, M., Gandibleux, X.: Bounds and bound sets for biobjective combinatorial optimization problems. In: Köksalan, M., Zionts, St. (eds.) Multiple Criteria Decision Making in the New Millennium. Lectures Notes in Economics and Mathematical Systems, vol. 507, pp. 241–253. Springer, Berlin (2001)
27. Ehrgott, M., Gandibleux, X.: Bound sets for biobjective combinatorial optimization problems. Comput. Oper. Res. **34**, 2674–2694 (2007)
28. Ehrgott, M., Klamroth, K.: Connectedness of efficient solutions in multiple criteria combinatorial optimization. Eur. J. Oper. Res. **97**(1), 159–166 (1997)
29. Ehrgott, M., Ryan, D.M.: Constructing robust crew schedules with bicriteria optimization. J. Multi-Criteria Decis. Anal. **11**, 139–150 (2002)
30. Ehrgott, M., Ryan, D.M.: The method of elastic constraints for multiobjective combinatorial optimization and its application in airline crew scheduling. In: Tanino, T., Tanaka, T., Inuiguchi, M. (eds.) Multi-Objective Programming and Goal-Programming: Theory and Applications. Advances in Soft Computing, vol. 21, pp. 117–122. Springer, Berlin (2003)
31. Ehrgott, M., Tenfelde-Podehl, D.: Computation of ideal and nadir values and implications for their use in MCDM methods. Eur. J. Oper. Res. **151**, 119–131 (2003)
32. Emelichev, V.A., Perepelitsa, V.A.: On cardinality of the set of alternatives in discrete many-criterion problems. Discret. Math. Appl. **2**, 461–471 (1992)
33. Eusébio, A., Figuiera, J.R.: Finding non-dominated solutions in bi-objective integer network flow problems. Comput. Oper. Res. **36**, 2554–2564 (2009)
34. Florios, K., Mavrotas, G., Diakoulaki, D.: Solving multiobjective, multiconstraint knapsack problems using mathematical programming and evolutionary algorithms. Eur. J. Oper. Res. **203**, 14–21 (2010)
35. Fukuda, K., Matsui, T.: Finding all the perfect matchings in bipartite graphs. Networks **22**, 461–468 (1992)
36. Gorski, J., Paquete, L., Pedrosa, F.: Greedy algorithms for a class of knapsack problems with binary weights. Comput. Oper. Res. **39**, 498–511 (2012)
37. Hamacher, H.W., Ruhe, G.: On spanning tree problems with multiple objectives. Ann. Oper. Res. **52**, 209–230 (1994)
38. Hansen, P.: Bicriterion path problems. In: Fandel, G., Gal, T. (eds.) Multiple Criteria Decision Making Theory and Application. Lecture Notes in Economics and Mathematical Systems, vol. 177, pp. 109–127. Springer, Berlin (1979)
39. Iori, M., Martello, S., Pretolani, D.:An aggregate label setting policy for the multi-objective shortest path problem. Eur. J. Oper. Res. **207**, 1489–1496 (2010)
40. Jorge, J.: Nouvelles propositions pour la résolution exacte du sac à dos multi-objectif unidimensionnel en variables binaires. Ph.D. thesis, Université de Nantes (2010)
41. Jozefowiez, N., Semet, F., Talbi, E.-G.: The bi-objective covering tour problem. Comput. Oper. Res. **34**, 1929–1942 (2007)
42. Karp, R.M.: Reducibility among combinatorial problems. In: Miller, R.E., Thatcher, J.W. (eds.) Complexity of Computer Computations, pp. 85–103. Plenum Press, New York (1972)
43. Kellerer, H., Pferschy, U., Pisinger, D.: Knapsack Problems. Springer, Berlin (2004)
44. Kirlik, G., Sayin, S.: A new algorithm for generating all nondominated solutions of multiobjective discrete optimization problems. Eur. J. Oper. Res. **232**, 479–488 (2014)
45. Kiziltan, G., Yucaoglu, E.: An algorithm for multiobjective zero-one linear programming. Manag. Sci. **29**, 1444–1453 (1983)
46. Klamroth, K., Lacour, R., Vanderpooten, D.: On the representation of the search region in multi-objective optimization. Eur. J. Oper. Res. **245**(3), 767–778 (2015)

47. Klein, D., Hannan, E.: An algorithm for the multiple objective integer linear programming problem. Eur. J. Oper. Res. **9**, 378–385 (1982)
48. Kouvelis, P., Carlson, R.C.: Total unimodularity applications in bi-objective discrete optimization. Oper. Res. Lett. **11**, 61–65 (1992)
49. Kruskal, J.B.: On the shortest spanning subtree of a graph and the travelling salesman problem. Proc. Am. Math. Soc. **7**, 48–50 (1956)
50. Kuhn, H.W.: The Hungarian method for the assignment problem. Nav. Res. Logist. **2**, 83–97 (1955)
51. Laumanns, M., Thiele, L., Zitzler, E.: An efficient, adaptive parameter variation scheme for metaheuristics based on the epsilon-constraint method. Eur. J. Oper. Res. **169**, 932–942 (2006)
52. Lee, H., Pulat, P.S.: Bicriteria network flow problems: Integer case. Eur. J. Oper. Res. **66**, 148–157 (1993)
53. Lokman, B., Köksalan, M.: Finding all nondominated points of multi-objective integer programs. J. Glob. Optim. **57**, 347–365 (2013)
54. Mavrotas, G., Diakoulaki, D.: A branch and bound algorithm for mixed zero-one multiple objective linear programming. Eur. J. Oper. Res. **107**(3), 530–541 (1998)
55. Mavrotas, G., Diakoulaki, D.: Multi-criteria branch and bound: a vector maximization algorithm for mixed 0–1 multiple objective linear programming. Appl. Math. Comput. **171**(3), 53–71 (2005)
56. Mavrotas, G., Florios, K.: An improved version of the augmented ε-constraint method (augmecon2) for finding the exact pareto set in multi-objective integer programming problems. Appl. Math. Comput. **219**, 9652–9669 (2013)
57. Mavrotas, G., Figueira, J.R., Florios, K.: Solving the bi-objective multi-dimensional knapsack problem exploiting the concept of core. Appl. Math. Comput. **215**, 2502–2514 (2009)
58. Mezmaz, M., Melab, N., Talbi, E.G.: A grid-based parallel approach of the multi-objective branch and bound. In: 15th EUROMICRO International Conference on Parallel, Distributed and Network-Based Processing (PDP07), Napoli, pp. 23–27 (2007). IEEE, New York (2007). doi:10.1109/PDP.2007.7
59. Mote, J., Murthy, I., Olson, D.L.: A parametric approach to solving bicriterion shortest path problems. Eur. J. Oper. Res. **53**, 81–92 (1991)
60. Müller-Hannemann, M., Weihe, K.: On the cardinality of the Pareto set in bicriteria shortest path problems. Ann. Oper. Res. **147**, 269–286 (2006)
61. Neumayer, P.: Complexity of optimization on vectorweighted graphs. In: Bachem, A., Derigs, U., Jünger, M., Schrader, R. (eds.) Operations Research, vol. 93, pp. 359–361. Physica Verlag, Heidelberg (1994)
62. Özlen, M., Azizoglu, M.: Multi-objective integer programming: a general approach for generating all non-dominated solutions. Eur. J. Oper. Res. **199**, 25–35 (2009)
63. Ozlen, M., Burton, B.A., MacRae, C.A.G.: Multi-objective integer programming: An improved recursive algorithm. J. Optim. Theory Appl. **160**, 470–482 (2014)
64. Özpeynirci, Ö., Köksalan, M.: An exact algorithm for finding extreme supported nondominated points of multiobjective mixed integer programs. Manag. Sci. **56**, 2302–2315 (2010)
65. Paixão, J.M., Santos, J.L.: Labeling methods for the general case of the multi-objective shortest path problem—a computational study. In: Madureira, A., Reis, C., Marques, V. (eds.) Computational Intelligence and Decision Making: Trends and Applications. Intelligent Systems, Control and Automation: Science and Engineering, vol. 61, pp. 489–502. Springer, New York (2013)
66. Paquete, L., Jaschob, M., Klamroth, K., Gorski, J.: On a biobjective search problem in a line: formulations and algorithms. Theor. Comput. Sci. **507**, 61–71 (2013)
67. Pedersen, C.R., Nielsen, L.R., Andersen, K.A.: The bicriterion multi modal assignment problem: introduction, analysis, and experimental results. INFORMS J. Comput. **20**, 400–411 (2008)
68. Prim, J.C.: Shortest connection networks and some generalizations. Bell Syst. Tech. J. **36**, 1389–1401 (1957)
69. Przybylski, A., Gandibleux, X., Ehrgott, M.: The biobjective integer minimum cost flow problem—Incorrectness of Sedeõ-Noda and Gonzàlez-Martin's algorithm. Comput. Oper. Res. **33**, 1459–1463 (2006)

70. Przybylski, A., Gandibleux, X., Ehrgott, M.: Computational results for four exact methods to solve the three-objective assignment problem. In: Barichard, V., Ehrgott, M., Gandibleux, X., T'Kindt, V. (eds.) Multiple Objective Programming and Goal Programming: Theoretical Results and Practical Applications. Lecture Notes in Economics and Mathematical Systems, vol. 618, pp. 79–88. Springer, Berlin (2008)
71. Przybylski, A., Gandibleux, X., Ehrgott, M.: Two phase algorithms for the bi-objective assignment problem. Eur. J. Oper. Res. **185**, 509–533 (2008)
72. Przybylski, A., Gandibleux, X., Ehrgott, M.: A recursive algorithms for finding all nondominated extreme points in the outcome set of a multiobjective integer program. INFORMS J. Comput. **22**, 371–386 (2010)
73. Przybylski, A., Gandibleux, X., Ehrgott, M.: A two phase method for multi-objective integer programming and its application to the assignment problem with three objectives. Discret. Optim. **7**, 149–165 (2010)
74. Przybylski, A., Gandibleux, X., Ehrgott, M.: The two-phase method for multiobjective combinatorial optimization problem. In: Majoub, R. (ed.) Progress in Combinatorial Optimization, pp. 559–596. ISTE Wiley, London (2011)
75. Raith, A., Ehrgott, M.: A comparison of solution strategies for biobjective shortest path problems. Comput. Oper. Res. **36**, 1299–1331 (2009)
76. Raith, A., Ehrgott, M.: A two-phase algorithm for the biobjective integer minimum cost flow problem. Comput. Oper. Res. **36**, 1945–1954 (2009)
77. Ralphs, T.K., Saltzmann, M.J., Wiecek, M.M.: An improved algorithm for solving biobjective integer programs. Ann. Oper. Res. **147**, 43–70 (2006)
78. Ramos, R.M., Alonso, S., Sicilia, J., González, C.: The problem of the optimal biobjective spanning tree. Eur. J. Oper. Res. **111**, 617–628 (1998)
79. Reinhardt, L.B., Pisinger, D.: Multi-objective and multi-constrained non-additive shortest path problems. Comput. Oper. Res. **38**, 605–616 (2011)
80. Rong, A., Figueira, J.R.: Computational performance of basic state reduction based dynamic programming algorithms for bi-objective 0–1 knapsack problems. Comput. Math. Appl. **63**, 1462–1480 (2012)
81. Ruhe, G.: Complexity results for multicriteria and parametric network flows using a pathological graph of Zadeh. Z. Oper. Res. **32**,9–27 (1988)
82. Ruzika, S.: On multipe objective combinatorial optimization. Ph.D. thesis, Department of Mathematics, Technical University of Kaiserslautern (2007)
83. Ruzika, S., Hamacher, H.W.: A survey on multiple objective minimum spanning tree problems. In: Lerner, D., Wagner, D., Zweig, K.A. (eds.) Algorithmics. Lecture Notes in Computer Science, vol. 5515, pp. 104–116. Springer, Heidelberg (2009)
84. Sanders, P., Mandow, L.: Parallel label-setting multi-objective shortest path search. In: 2013 IEEE 27th International Symposium on Parallel & Distributed Processing (IPDPS), pp. 215–224. IEEE, New York (2013)
85. Sayin, S., Kouvelis, P.: The multiobjective discrete optimization problem: a weighted min-max two-stage optimization approach and a bicriteria algorithm. Manag. Sci. **51**, 1572–1581 (2005)
86. Sedeño-Noda, A., González-Martín, C.: An algorithm for the biobjective integer minimum cost flow problem. Comput. Oper. Res. **28**, 139–156 (2001)
87. Sedeño-Noda, A., González-Martín, C.: An efficient label setting/correcting shortest path algorithm. Comput. Optim. Appl. **51**, 437–455 (2012)
88. Sedeño-Noda, A., Raith, A.: A Dijkstra-like method computing all extreme supported non-dominated solutions of the biobjective sortest path problem. Comput. Oper. Res. **57**, 83–94 (2015)
89. Serafini, P.: Some considerations about computational complexity for multi objective combinatorial problems. In: Jahn, J., Krabs, W. (eds.) Recent Advances and Historical Development of Vector Optimization. Lecture Notes in Economics and Mathematical Systems, vol. 294, pp. 222–232. Springer, Berlin (1986)

90. Sourd, F., Spanjaard, O.: A multiobjective branch-and-bound framework: application to the biobjective spanning tree problem. INFORMS J. Comput. **20**, 472–484 (2008)
91. Stanojević, M., Vujosevic, M., Stanojević, B.: On the cardinality of the nondominated set of multi-objective combinatorial optimization problems. Oper. Res. Lett. **41**, 197–200 (2013)
92. Steiner, S., Radzik, T.: Computing all efficient solutions of the biobjective minimum spanning tree problem. Comput. Oper. Res. **35**, 198–211 (2008)
93. Sylva, J., Crema, A.: A method for finding the set of non-dominated vectors for multiple objective integer linear programs. Eur. J. Oper. Res. **158**, 46–55 (2004)
94. Tam, B., Ryan, D., Ehrgott, M.: Multi-objective approaches to the unit crewing problem in airline crew scheduling. J. Multi-Criteria Decis. Anal. **21**, 257–277 (2014)
95. Tarapata, Z.: Selected multicriteria shortest path problems: an analysis of complexity, models and adaptation of standrad algorithms. Int. J. Appl. Math. Comput. Sci. **1**, 269–287 (2007)
96. Tuyttens, D., Teghem, J., Fortemps, P., Van Nieuwenhuyse, K.: Performance of the MOSA method for the bicriteria assignment problem. J. Heuristics **6**, 295–310 (2000)
97. Ulungu, E.L., Teghem, J.: The two phases method: an efficient procedure to solve bi-objective combinatorial optimization problems. Found. Comput. Decis. Sci. **20**, 149–165 (1995)
98. Ulungu, E.L., Teghem, J.: Solving multi-objective knapsack problem by a branch-and-bound procedure. In: Climaco, J. (ed.) Multicriteria Analysis, pp. 269–278. Springer, Berlin (1997)
99. Villarreal, B., Karwan, M.H.: Multicriteria integer programming: a (hybrid) dynamic programming recursive approach. Math. Program. **21**, 204–223 (1981)
100. Vincent, T., Seipp, F., Ruzika, S., Przybylski, A., Gandibleux, X.: Multiple objective branch and bound for mixed 0–1 linear programming: corrections and improvements for the biobjective case. Comput. Oper. Res. **40**, 498–509 (2013)
101. Visée, M., Teghem, J., Pirlot, M., Ulungu, E.L.: Two-phases method and branch and bound procedures to solve the bi-obective knapsack problem. J. Glob. Optim. **12**, 139–155 (1998)
102. Welsh, D.J.A.: Complexity: Knots, Colourings and Counting. Cambridge University Press, Cambridge (1993)
103. Wierzbicki, A.P.: On the completeness and constructiveness of parametric characterizations to vector optimization problems. OR-Spektrum **8**, 73–87 (1986)

Chapter 20
Fuzzy Multi-Criteria Optimization: Possibilistic and Fuzzy/Stochastic Approaches

Masahiro Inuiguchi, Kosuke Kato, and Hideki Katagiri

Abstract In this chapter, we review fuzzy multi-criteria optimization focusing on possibilistic treatments of objective functions with fuzzy coefficients and on interactive fuzzy stochastic multiple objective programming approaches. In the first part, treatments of objective functions with fuzzy coefficients dividing into single objective function case and multiple objective function case. In single objective function case, multi-criteria treatments, possibly and necessarily optimal solutions, and minimax regret solutions are described showing the relations to multi-criteria optimization. In multiple objective function case, possibly and necessarily efficient solutions are investigated. Their properties and possible and necessary efficiency tests are shown. As one of interactive fuzzy stochastic programming approaches, multiple objective programming problems with fuzzy random parameters are discussed. Possibilistic expectation and variance models are proposed through incorporation of possibilistic and stochastic programming approaches. Interactive algorithms for deriving a satisficing solution of a decision maker are shown.

Keywords Fuzzy programming • Possibility measure • Necessity measure • Minimax regret • Possible efficiency • Necessary efficiency • Fuzzy random variable • Random fuzzy variable • Interactive algorithm

---

M. Inuiguchi (✉)
Graduate School of Engineering Science, Osaka University,
Toyonaka, Osaka 560-8531, Japan
e-mail: inuiguti@sys.es.osaka-u.ac.jp

K. Kato
Faculty of Applied Information Science, Hiroshima Institute of Technology,
Saeki-ku, Hiroshima 731-5193, Japan
e-mail: k.katoh.me@it-hiroshima.ac.jp

H. Katagiri
Graduate School of Engineering, Hiroshima University,
Higashi-Hiroshima 739-8527, Japan
e-mail: katagiri-h@hiroshima-u.ac.jp

S. Greco et al. (eds.), *Multiple Criteria Decision Analysis*, International Series in Operations Research & Management Science 233,
DOI 10.1007/978-1-4939-3094-4_20

20.1 Introduction

In mathematical programming problems, parameters such as coefficients and right-hand side values of constraints have been assumed to be given as real numbers. However, in real world problems, there are cases that those parameters cannot be given precisely by lack of knowledge or by uncertain nature of coefficients. For example, rate of return of investment, demands for products, and so on are known as uncertain parameters. Moreover, time for manual assembly operation and the cost of a taxi ride can also be ambiguous and depend on the worker's skill and the degree of traffic congestion, respectively. Those uncertain parameters have been treated as random variables so that the mathematical programming problems become stochastic programming problems [85, 86].

To formulate a stochastic programming problem, we should estimate a proper probability distribution which parameters obey. However, the estimation is not always a simple task because of the following reasons: (1) historical data of some parameters cannot be obtained easily especially when we face a new uncertain variable, and (2) subjective probabilities cannot be specified easily when many parameters exist. Moreover, even if we succeeded to estimate the probability distribution from historical data, there is no guarantee that the current parameters obey the distribution actually.

We may often come across that we can estimate the possible ranges of the uncertain parameters. For example, we may find out a possible range of cost of taxi ride through experience if we almost know the distance and the traffic quantity. Then, it is conceivable that we represent the possible ranges by fuzzy sets and formulate the mathematical programming problems as fuzzy programming problems [10, 25, 31, 59, 66, 78, 80, 83, 85].

In this paper, we introduce approaches to mathematical programming problems with fuzzy parameters dividing into two parts. In the first part, we review treatments of objective functions with fuzzy coefficients dividing into single objective function case and multiple objective function case. In both cases, the solutions are studied first in problems with interval coefficients and then in the problems with fuzzy coefficients.

In the single objective function case, we show that multi-criteria treatments of an objective function with coefficients using lower and upper bounds do not always produce good solutions. Then possibly and necessarily optimal solutions are introduced. The relations of those solution concepts with solution concepts in multi-criteria optimization are described. A necessarily optimal solution is the most reasonable solution but it does not exist in many cases while a possibly optimal solution always exists when the feasible region is bounded and nonempty but it is only one of least reasonable solutions. Then minimax regret and maximin achievement solutions are introduced as a possibly optimal solution minimizing the deviation from the necessary optimality. Those solutions can be seen as robust suboptimal solutions.

In the multiple objective function case, possibly and necessarily efficient solutions are introduced as the extensions of possibly and necessarily optimal solutions.

Because many efficient solutions exist usually in the conventional multiple objective programming problem, it is highly possible that necessarily optimal solutions exist. The properties of possibly and necessarily efficient solutions are investigated. Moreover the possible and necessary efficient tests are described.

In the second part, we consider a case where a part of uncertain parameters can be expressed by random variables but the other part can be expressed by fuzzy numbers. In order to take into consideration not only fuzziness but also randomness of the coefficients in objective functions, multiple objective programming problems with fuzzy random coefficients are discussed. By incorporating possibilistic and stochastic programming approaches, possibilistic expectation and variance models are proposed. It is shown that multiple objective programming problems with fuzzy random coefficients can be deterministic linear or nonlinear multiple objective fractional programming problems. Interactive algorithms for deriving a satisficing solution of a decision maker are provided.

20.2 Problem Statement and Preliminaries

Multiple objective linear programming (MOLP) problems can be written as

$$\begin{aligned} &\text{maximize } (\boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{x}, \boldsymbol{c}_2^{\mathrm{T}}\boldsymbol{x}, \ldots, \boldsymbol{c}_p^{\mathrm{T}}\boldsymbol{x})^{\mathrm{T}}, \\ &\text{subject to } \boldsymbol{a}_i^{\mathrm{T}}\boldsymbol{x} = b_i,\ i = 1, 2, \ldots, m, \\ &\qquad\qquad \boldsymbol{x} \geq \boldsymbol{0}, \end{aligned} \tag{20.1}$$

where $\boldsymbol{c}_k = (c_{k1}, c_{k2}, \ldots, c_{kn})^{\mathrm{T}}$, $k = 1, 2, \ldots, p$ and $\boldsymbol{a}_i = (a_{i1}, a_{i2}, \ldots, a_{in})^{\mathrm{T}}$, $i = 1, 2, \ldots, m$ are constant vectors and b_i, $i = 1, 2, \ldots, m$ constants. $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$ is the decision variable vector.

In MOLP problems, many solution concepts are considered (see [13]). In this chapter, we describe only the following three solution concepts:

Complete optimality: A feasible solution $\hat{\boldsymbol{x}}$ is said to be completely optimal if and only if we have $\boldsymbol{c}_k^{\mathrm{T}}\hat{\boldsymbol{x}} \geq \boldsymbol{c}_k^{\mathrm{T}}\boldsymbol{x}$, $k = 1, 2, \ldots, p$ for all feasible solution $\boldsymbol{x}$.

Efficiency: A feasible solution $\hat{\boldsymbol{x}}$ is said to be efficient if and only if there is no feasible solution $\boldsymbol{x}$ such that $\boldsymbol{c}_k^{\mathrm{T}}\boldsymbol{x} \geq \boldsymbol{c}_k^{\mathrm{T}}\hat{\boldsymbol{x}}$, $k = 1, 2, \ldots, p$ with at least one strict inequality.

Weak efficiency: A feasible solution $\hat{\boldsymbol{x}}$ is said to be weakly efficient if and only if there is no feasible solution $\boldsymbol{x}$ such that $\boldsymbol{c}_k^{\mathrm{T}}\boldsymbol{x} > \boldsymbol{c}_k^{\mathrm{T}}\hat{\boldsymbol{x}}$, $k = 1, 2, \ldots, p$.

In the conventional MOLP problem (20.1), the coefficients and right-hand side values are assumed to be specified as real numbers. However, in the real world applications, we may face situations where coefficients and right-hand side values cannot be specified as real numbers by the lack of exact knowledge or by their fluctuations. Even in those situations there are cases when ranges of possible values for coefficients and right-hand side values can be specified by experts' vague knowledge. In the first part of this paper, we assume that those ranges are expressed by fuzzy sets and consider the MOLP problem with fuzzy coefficients. Because we focus on the treatments of objective functions with fuzzy coefficients, we assume the constraints are crisp so that they do not include any fuzzy parameters. However, the constraints with fuzzy parameters are often reduced to the crisp constraints in fuzzy/possibilistic programming approaches [10, 25, 31].

MOLP problems with fuzzy numbers treated in the first part of this chapter can be represented as

$$\begin{aligned} &\text{maximize } (\tilde{\boldsymbol{c}}_1^{\mathrm{T}}\boldsymbol{x}, \tilde{\boldsymbol{c}}_2^{\mathrm{T}}\boldsymbol{x}, \ldots, \tilde{\boldsymbol{c}}_p^{\mathrm{T}}\boldsymbol{x})^{\mathrm{T}}, \\ &\text{subject to } \boldsymbol{x} \in X, \end{aligned} \tag{20.2}$$

where we define

$$X = \{\boldsymbol{x} \in \mathbf{R}^n \mid \boldsymbol{a}_i^{\mathrm{T}}\boldsymbol{x} = b_i,\ i = 1, 2, \ldots, m,\ \boldsymbol{x} \geq \boldsymbol{0}\}. \tag{20.3}$$

$\tilde{\boldsymbol{c}}_k = (\tilde{c}_{k1}, \tilde{c}_{k2}, \ldots, \tilde{c}_{kn})^{\mathrm{T}}$, $k = 1, 2, \ldots, p$ is a vector of fuzzy coefficients. $\tilde{c}_{kj}$, $j = 1, 2, \ldots, n$, $k = 1, 2, \ldots, p$ are fuzzy numbers. A fuzzy number $\tilde{c}$ is a fuzzy set on a real line whose membership function $\mu_{\tilde{c}} : \mathbf{R} \to [0, 1]$ satisfies (see, for example, [11])

(i) $\tilde{c}$ is normal, i.e., there exists $r \in \mathbf{R}$ such that $\mu_{\tilde{c}}(r) = 1$.
(ii) $\mu_{\tilde{c}}$ is upper semi-continuous, i.e., the h-level set $[\tilde{c}]_h = \{r \in \mathbf{R} \mid \mu_{\tilde{c}}(r) \geq h\}$ is a closed set for any $h \in (0, 1]$.
(iii) $\tilde{c}$ is a convex fuzzy set. Namely, $\mu_{\tilde{c}}$ is a quasi-concave function, i.e., for any r_1, $r_2 \in \mathbf{R}$, for any $\lambda \in [0, 1]$, $\mu_{\tilde{c}}(\lambda r_1 + (1-\lambda) r_2) \geq \min(\mu_{\tilde{c}}(r_1), \mu_{\tilde{c}}(r_2))$. In other words, h-level set $[\tilde{c}]_h$ is a convex set for any $h \in (0, 1]$.
(iv) $\tilde{c}$ is bounded, i.e., $\lim_{r \to +\infty} \mu_{\tilde{c}}(r) = \lim_{r \to -\infty} \mu_{\tilde{c}}(r) = 0$. In other words, the h-level set $[\tilde{c}]_h$ is bounded for any $h \in (0, 1]$.

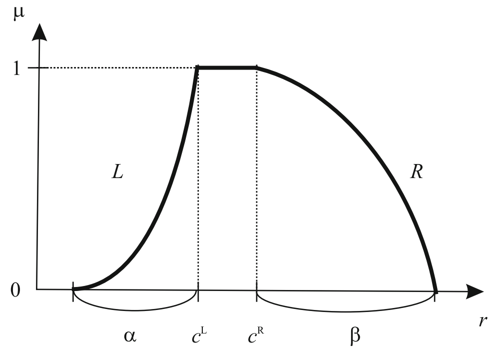
From (ii) to (iv), an h-level set $[\tilde{c}]_h$ is a bounded closed interval for any $h \in (0, 1]$ when $\tilde{c}$ is a fuzzy number. L-R fuzzy numbers are often used in literature. An L-R fuzzy number $\tilde{c}$ is a fuzzy number defined by the following membership function:

$$\mu_{\tilde{c}}(r) = \begin{cases} L\left(\dfrac{c^{\mathrm{L}} - r}{\alpha}\right), & \text{if } r \leq c^{\mathrm{L}} \text{ and } \alpha > 0, \\ 1, & \text{if } r \in [c^{\mathrm{L}}, c^{\mathrm{R}}], \\ R\left(\dfrac{r - c^{\mathrm{R}}}{\beta}\right), & \text{if } r \geq c^{\mathrm{R}} \text{ and } \beta > 0, \\ 0, & \text{otherwise}, \end{cases} \tag{20.4}$$

where L and $R : [0, +\infty) \to [0, 1]$ are reference functions, i.e., $L(0) = R(0) = 1$, $\lim_{r \to +\infty} L(r) = \lim_{r \to +\infty} R(r) = 0$ and L and R are upper semi-continuous non-increasing functions. α and β are assumed to be non-negative.

An example of L-R fuzzy number $\tilde{c}$ is illustrated in Fig. 20.1. As shown in Fig. 20.1, c^{L} and c^{R} are lower and upper bounds of the core of $\tilde{c}$, i.e., $\mathrm{Core}(\tilde{c}) = \{r \mid \mu_{\tilde{c}}(r) = 1\}$. α and β show the left and right spreads of $\tilde{c}$. Functions L and R specify the left and right shapes. Using those parameters and functions, fuzzy number $\tilde{c}$ is

Fig. 20.1 L-R fuzzy number $\tilde{c} = (c^{\mathrm{L}}, c^{\mathrm{R}}, \alpha, \beta)_{LR}$



represented as $\tilde{c} = (c^{\mathrm{L}}, c^{\mathrm{R}}, \alpha, \beta)_{LR}$. A membership degree $\mu_{\tilde{c}}(r)$ of fuzzy coefficient $\tilde{c}$ shows the possibility degree of an event 'the coefficient value is r'.

Problem (20.2) has fuzzy coefficients only in the objective functions. In Problem (20.2), we should calculate fuzzy linear function values $\tilde{\boldsymbol{c}}_k^{\mathrm{T}}\boldsymbol{x}$, $k = 1, 2, \ldots, p$. Those function values can be fuzzy quantities since the coefficients are fuzzy numbers. The extension principle [11] defines the fuzzy quantity of function values of fuzzy numbers. Let $g : \mathbf{R}^q \to \mathbf{R}$ be a function. A function value of $(\tilde{c}_1, \tilde{c}_2, \ldots, \tilde{c}_q)$, i.e., $g(\tilde{c}_1, \tilde{c}_2, \ldots, \tilde{c}_q)$ is a fuzzy quantity $\tilde{Y}$ with the following membership function:

$$\mu_{\tilde{Y}}(y) = \begin{cases} \sup\limits_{\boldsymbol{r}:g(\boldsymbol{r})=y} \min\left(\mu_{\tilde{c}_1}(r_1), \mu_{\tilde{c}_2}(r_2), \ldots, \mu_{\tilde{c}_q}(r_q)\right), \\ \qquad \text{if } \exists \boldsymbol{r} = (r_1, r_2, \ldots, r_q); g(\boldsymbol{r}) = y, \\ 0, \text{ otherwise.} \end{cases} \tag{20.5}$$

Since $\tilde{\boldsymbol{c}}$ is a vector of fuzzy numbers $\tilde{c}_i$ whose h-level set is a bounded closed interval for any $h \in (0, 1]$, we have the following equation (see [11]) when g is a continuous function;

$$[\tilde{Y}]_h = g([\tilde{\boldsymbol{a}}]_h), \ \forall h \in (0, 1], \tag{20.6}$$

where $[\tilde{\boldsymbol{c}}]_h = ([\tilde{c}_1]_h, [\tilde{c}_2]_h, \ldots, [\tilde{c}_q]_h)$. Note that $[\tilde{c}_j]_h$ is a closed interval since $\tilde{c}_j$ is a fuzzy number. Equation (20.6) implies that h-level set of function value $\tilde{Y}$ can be obtained by interval calculations. Moreover, since g is continuous, from (20.6), we know that $[\tilde{Y}]_h$ is also a closed interval and $[\tilde{Y}]_1 \neq \emptyset$. Therefore, $\tilde{Y}$ is also a fuzzy number.

Let $g(\boldsymbol{r}) = \boldsymbol{r}^{\mathrm{T}}\boldsymbol{x}$, where we define $\boldsymbol{r} = (r_1, r_2, \ldots, r_n)^{\mathrm{T}}$. We obtain the fuzzy linear function value $\tilde{\boldsymbol{c}}_k^{\mathrm{T}}\boldsymbol{x}$ as a fuzzy number $g(\tilde{\boldsymbol{c}}_k)$. For $\boldsymbol{x} \geq \boldsymbol{0}$, we have

$$[\tilde{\boldsymbol{c}}_k^{\mathrm{T}}\boldsymbol{x}]_h = \left[\sum_{j=1}^{n} c_{kj}^{\mathrm{L}}(h)x_j, \sum_{j=1}^{n} c_{kj}^{\mathrm{R}}(h)x_j\right], \ \forall h \in (0, 1], \tag{20.7}$$

where $c_{kj}^{\mathrm{L}}(h)$ and $c_{kj}^{\mathrm{R}}(h)$ are lower and upper bounds of h-level set $[\tilde{c}_{kj}]_h$, i.e., $c_{kj}^{\mathrm{L}}(h) = \inf[\tilde{c}_{kj}]_h$ and $c_{kj}^{\mathrm{R}}(h) = \sup[\tilde{c}_{kj}]_h$. Note that when $\tilde{c}_{kj}$ is an L-R fuzzy number $(c_{kj}^{\mathrm{L}}, c_{kj}^{\mathrm{R}}, \gamma_{kj}^{\mathrm{L}}, \gamma_{kj}^{\mathrm{R}})_{L_{kj}R_{kj}}$, we have

$$c_{kj}^{\mathrm{L}}(h) = c_{kj}^{\mathrm{L}} - \gamma_{kj}^{\mathrm{L}} L_{kj}^{(-1)}(h), \quad c_{kj}^{\mathrm{R}}(h) = c_{kj}^{\mathrm{R}} + \gamma_{kj}^{\mathrm{R}} R_{kj}^{(-1)}(h), \tag{20.8}$$

where $L_{kj}^{(-1)}$ and $R_{kj}^{(-1)}$ are pseudo-inverse functions of L_{kj} and R_{kj} defined by $L_{kj}^{(-1)}(h) = \sup\{r \mid L_{kj}(r) \geq h\}$ and $R_{kj}^{(-1)}(h) = \sup\{r \mid R_{kj}(r) \geq h\}$.

In Problem (20.2), each objective function value $\tilde{\boldsymbol{c}}_k^{\mathrm{T}}\boldsymbol{x}$ is obtained as a fuzzy number. Minimizing a fuzzy number $\tilde{\boldsymbol{c}}_k^{\mathrm{T}}\boldsymbol{x}$ cannot be clearly understood. Therefore, Problem (20.2) is an ill-posed problem. We should introduce an interpretation of Problem (20.2) so that we can transform the problem to a well-posed problem. Many interpretations have been proposed. In the first part of this paper, we describe the interpretations from the viewpoint of optimization. For the other interpretations from viewpoint of satisficing, see, for example, [10, 25].

Possibility and necessity measures of a fuzzy set $\tilde{B}$ under a fuzzy set $\tilde{A}$ are defined as follows (see [12]):

$$\Pi_{\tilde{A}}(\tilde{B}) = \sup_r \min(\mu_{\tilde{A}}(r), \mu_{\tilde{B}}(r)), \tag{20.9}$$

$$N_{\tilde{A}}(\tilde{B}) = \inf_r \max(1 - \mu_{\tilde{A}}(r), \mu_{\tilde{B}}(r)). \tag{20.10}$$

Those possibility and necessity measures are depicted in Fig. 20.2.

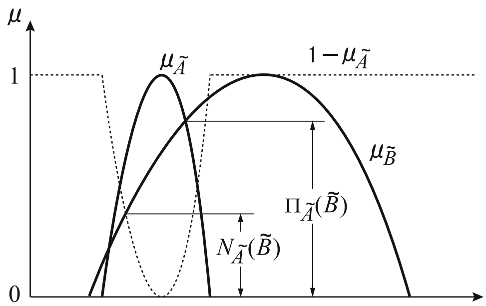
When fuzzy sets $\tilde{A}$ and $\tilde{B} \subseteq \mathbf{R}^q$ have upper semi-continuous membership functions and $\tilde{A}$ is bounded, we have, for any $h \in (0, 1]$,

$$\Pi_{\tilde{A}}(\tilde{B}) \geq h \Leftrightarrow [\tilde{A}]_h \cap [\tilde{B}]_h \neq \emptyset, \tag{20.11}$$

$$N_{\tilde{A}}(\tilde{B}) \geq h \Leftrightarrow (\tilde{A})_{1-h} \subseteq [\tilde{B}]_h \;\Leftrightarrow\; \mathrm{cl}(\tilde{A})_{1-h} \subseteq [\tilde{B}]_h, \tag{20.12}$$

where $\tilde{A}$ is said to be bounded when $[\tilde{A}]_h$ is bounded for any $h \in (0, 1]$. $(\tilde{A})_{1-h}$ is a strong $(1-h)$-level set of $\tilde{A}$ defined by $(\tilde{A})_{1-h} = \{r \mid \mu_{\tilde{A}}(r) > 1 - h\}$. In (20.11) and (20.12), we may understand that the possibility measure shows to what extent $\tilde{A}$ intersects with $\tilde{B}$ while the necessity measure shows to what extent $\tilde{A}$ is included in $\tilde{B}$. This interpretation is true even for other conjunction and implication functions T and I.

Fig. 20.2 Possibility and necessity measures



20.3 Single Objective Function Case

In this section, we treat Problem (20.2) with $p = 1$, i.e., single objective function case. In the single objective function case, there are many approaches (see for example, [15, 32, 40, 79, 90]). These approaches can be divided into two classes: satisficing approach and optimizing approach. The satisficing approach use a goal, the objective function value with which the decision maker is satisfied, while the optimizing approach does not use such a goal but generalizes the optimality concept to the case with uncertain coefficients. We describe the optimizing approaches to Problem (20.2) with $p = 1$ in this section. We demonstrate that even in the single objective function case, Problem (20.2) with $p = 1$ has a deep connection to multi-criteria optimization.

When $p = 1$, Problem (20.2) is reduced to

$$\begin{aligned} &\text{maximize } \tilde{\boldsymbol{c}}_1^{\mathrm{T}}\boldsymbol{x}, \\ &\text{subject to } \boldsymbol{x} \in X. \end{aligned} \tag{20.13}$$

20.3.1 Optimization of Upper and Lower Bounds

When fuzzy coefficients $\tilde{c}_{1j}$, $j = 1, 2, \ldots, n$ degenerate to intervals $[c_{1j}^{\mathrm{L}}, c_{1j}^{\mathrm{R}}]$, $j = 1, 2, \ldots, n$, Problem (20.13) becomes an interval programming problem. In this case, Problem (20.13) is formulated as the following bi-objective linear programming problem in many papers [15, 40, 79, 91]:

$$\begin{aligned} &\text{maximize } \left(\boldsymbol{c}_1^{\mathrm{L}\,\mathrm{T}}\boldsymbol{x}, \boldsymbol{c}_1^{\mathrm{R}\,\mathrm{T}}\boldsymbol{x}\right)^{\mathrm{T}}, \\ &\text{subject to } \boldsymbol{x} \in X, \end{aligned} \tag{20.14}$$

where $\boldsymbol{c}_1^{\mathrm{L}} = (c_{11}^{\mathrm{L}}, c_{12}^{\mathrm{L}}, \ldots, c_{1n}^{\mathrm{L}})^{\mathrm{T}}$ and $\boldsymbol{c}_1^{\mathrm{R}} = (c_{11}^{\mathrm{R}}, c_{12}^{\mathrm{R}}, \ldots, c_{1n}^{\mathrm{R}})^{\mathrm{T}}$.

The inequality relation between two interval $A = [a^{\mathrm{L}}, a^{\mathrm{R}}]$ and $B = [b^{\mathrm{L}}, b^{\mathrm{R}}]$ is frequently defined by

$$A \geq B \Leftrightarrow a^{\mathrm{L}} \geq b^{\mathrm{L}} \text{ and } a^{\mathrm{R}} \geq b^{\mathrm{R}}. \tag{20.15}$$

Problem (20.14) would be understood as a problem inspired from this inequality relation. Moreover, because Problem (20.14) maximizes the lower and upper bounds of objective function value simultaneously, it can be also seen as a problem maximizing the worst objective function value and the best objective function value. Namely, it is a model applied simultaneously the maximin criterion and the maximax criterion proposed for decision making under strict uncertainty. An efficient solution to Problem (20.14) is considered as a reasonable solution in this approach.

Extending this idea to general fuzzy coefficient case, we may have the following linear programming (LP) problem with infinitely many objective functions [91]:

$$\begin{aligned}&\text{maximize} \begin{pmatrix} \boldsymbol{c}_1^{\mathrm{L}}(h)^{\mathrm{T}}\boldsymbol{x}, \ \forall h \in (0,1] \\ \boldsymbol{c}_1^{\mathrm{R}}(h)^{\mathrm{T}}\boldsymbol{x}, \ \forall h \in (0,1] \end{pmatrix}, \\ &\text{subject to } \boldsymbol{x} \in X,\end{aligned} \tag{20.16}$$

where $\boldsymbol{c}_1^{\mathrm{L}}(h) = (c_{11}^{\mathrm{L}}(h), c_{12}^{\mathrm{L}}(h), \ldots, c_{1n}^{\mathrm{L}}(h))^{\mathrm{T}}$ and $\boldsymbol{c}_1^{\mathrm{R}}(h) = (c_{11}^{\mathrm{R}}(h), c_{12}^{\mathrm{R}}(h), \ldots, c_{1n}^{\mathrm{R}}(h))^{\mathrm{T}}$.

This formulation is also related to the following inequality relation between fuzzy numbers $\tilde{A}$ and $\tilde{B}$ (see [75, 91]):

$$\tilde{A} \succeq \tilde{B} \Leftrightarrow a^{\mathrm{L}}(h) \geq b^{\mathrm{L}}(h) \text{ and } a^{\mathrm{R}}(h) \geq b^{\mathrm{R}}(h), \ \forall h \in (0,1], \tag{20.17}$$

where we define for $h \in (0,1]$, $a^{\mathrm{L}}(h) = \inf[\tilde{A}]_h$, $b^{\mathrm{L}}(h) = \inf[\tilde{B}]_h$, $a^{\mathrm{R}}(h) = \sup[\tilde{A}]_h$ and $b^{\mathrm{R}}(h) = \sup[\tilde{B}]_h$.

When all fuzzy coefficients $\tilde{c}_{1j}$ are assumed to be L-R fuzzy numbers $(c_{1j}^{\mathrm{L}}, c_{1j}^{\mathrm{R}}, \gamma_{1j}^{\mathrm{L}}, \gamma_{1j}^{\mathrm{R}})_{LR}$ with same left and right reference functions L and R such that $L(1) = R(1) = 0$ and $\forall r \in [0,1)$, $L(r) > 0$, $R(r) > 0$, the following LP problem with four objective functions are considered:

$$\begin{aligned}&\text{maximize} \left(\boldsymbol{c}_1^{\mathrm{L}\,\mathrm{T}}\boldsymbol{x},\ \boldsymbol{c}_1^{\mathrm{R}\,\mathrm{T}}\boldsymbol{x},\ (\boldsymbol{c}_1^{\mathrm{L}} - \boldsymbol{\gamma}_1^{\mathrm{L}})^{\mathrm{T}}\boldsymbol{x},\ (\boldsymbol{c}_1^{\mathrm{R}} + \boldsymbol{\gamma}_1^{\mathrm{R}})^{\mathrm{T}}\boldsymbol{x}\right)^{\mathrm{T}}, \\ &\text{subject to } \boldsymbol{x} \in X,\end{aligned} \tag{20.18}$$

where $\boldsymbol{c}_1^{\mathrm{L}} = (c_{11}^{\mathrm{L}}, c_{12}^{\mathrm{L}}, \ldots, c_{1n}^{\mathrm{L}})^{\mathrm{T}}$, $\boldsymbol{c}_1^{\mathrm{R}} = (c_{11}^{\mathrm{R}}, c_{12}^{\mathrm{R}}, \ldots, c_{1n}^{\mathrm{R}})^{\mathrm{T}}$, $\boldsymbol{\gamma}_1^{\mathrm{L}} = (\gamma_{11}^{\mathrm{L}}, \gamma_{12}^{\mathrm{L}}, \ldots, \gamma_{1n}^{\mathrm{L}})^{\mathrm{T}}$, $\boldsymbol{\gamma}_1^{\mathrm{R}} = (\gamma_{11}^{\mathrm{R}}, \gamma_{12}^{\mathrm{R}}, \ldots, \gamma_{1n}^{\mathrm{R}})^{\mathrm{T}}$.

The following theorem shows the equivalence between Problems (20.16) and (20.18).

Theorem 1. *The efficient solution set $\mathit{Eff}_{\mathrm{many}}$ of Problem (20.16) coincides with the efficient solution set $\mathit{Eff}_{\mathrm{four}}$ of Problem (20.18).*

Proof. Let $\boldsymbol{x} \notin \mathit{Eff}_{\mathrm{four}}$. Then there exists $\bar{\boldsymbol{x}} \in X$ such that

$$\begin{aligned}&\boldsymbol{c}_1^{\mathrm{L}\,\mathrm{T}}\bar{\boldsymbol{x}} \geq \boldsymbol{c}_1^{\mathrm{L}\,\mathrm{T}}\boldsymbol{x},\ \ \boldsymbol{c}_1^{\mathrm{R}\,\mathrm{T}}\bar{\boldsymbol{x}} \geq \boldsymbol{c}_1^{\mathrm{R}\,\mathrm{T}}\boldsymbol{x}, \\ &(\boldsymbol{c}_1^{\mathrm{L}} - \boldsymbol{\gamma}_1^{\mathrm{L}})^{\mathrm{T}}\bar{\boldsymbol{x}} \geq (\boldsymbol{c}_1^{\mathrm{L}} - \boldsymbol{\gamma}_1^{\mathrm{L}})^{\mathrm{T}}\boldsymbol{x}, \\ &(\boldsymbol{c}_1^{\mathrm{R}} + \boldsymbol{\gamma}_1^{\mathrm{R}})^{\mathrm{T}}\bar{\boldsymbol{x}} \geq (\boldsymbol{c}_1^{\mathrm{R}} + \boldsymbol{\gamma}_1^{\mathrm{R}})^{\mathrm{T}}\boldsymbol{x}\end{aligned} \tag{$*$}$$

with at least one strict inequality. For L-R fuzzy numbers $(c_{1j}^{\mathrm{L}}, c_{1j}^{\mathrm{R}}, \gamma_{1j}^{\mathrm{L}}, \gamma_{1j}^{\mathrm{R}})_{LR}$, as in (20.8), we have

$$c_{1j}^{\mathrm{L}}(h) = c_{1j}^{\mathrm{L}} - \gamma_{1j}^{\mathrm{L}}L^{(-1)}(h), \quad c_{1j}^{\mathrm{R}}(h) = c_{1j}^{\mathrm{R}} + \gamma_{1j}^{\mathrm{R}}R^{(-1)}(h),$$

for any $h \in (0,1]$. From $L(1) = R(1) = 0$ and $\forall r \in (0,1], L(r) > 1$ and $R(r) > 1$, we have $\forall h \in (0,1], L^{(-1)}(h) \in [0,1]$ and $R^{(-1)}(h) \in [0,1]$. From $(*)$, we obtain

$$\boldsymbol{c}_1^{\mathrm{L}}(h)^{\mathrm{T}}\bar{\boldsymbol{x}} \geq \boldsymbol{c}_1^{\mathrm{L}}(h)^{\mathrm{T}}\boldsymbol{x},\ \ \boldsymbol{c}_1^{\mathrm{R}}(h)^{\mathrm{T}}\bar{\boldsymbol{x}} \geq \boldsymbol{c}_1^{\mathrm{R}}(h)^{\mathrm{T}}\boldsymbol{x},\ \forall h \in (0,1] \tag{$**$}$$

with at least one strict inequality. Then we obtain $\boldsymbol{x} \notin \mathit{Eff}_{\mathrm{many}}$.

On the contrary, let $\boldsymbol{x} \notin \mathit{Eff}_{\mathrm{many}}$. Then there exists $\bar{\boldsymbol{x}} \in X$ such that $(**)$ with at least one strict inequality holds. Because $(**)$ holds, we have $(*)$. Then we shall show that $(*)$ holds with at least one strict inequality. We can prove this dividing into two cases: (a) $\exists \bar{h} \in (0,1]$, $\boldsymbol{c}_1^{\mathrm{L}}(h)^{\mathrm{T}}\bar{\boldsymbol{x}} > \boldsymbol{c}_1^{\mathrm{L}}(h)^{\mathrm{T}}\boldsymbol{x}$ and (b) $\exists \bar{h} \in (0,1]$, $\boldsymbol{c}_1^{\mathrm{R}}(h)^{\mathrm{T}}\bar{\boldsymbol{x}} > \boldsymbol{c}_1^{\mathrm{R}}(h)^{\mathrm{T}}\boldsymbol{x}$. In case (a), if $L^{(-1)}(\bar{h}) = 0$, we obtain $\boldsymbol{c}_1^{\mathrm{L}^{\mathrm{T}}}\bar{\boldsymbol{x}} > \boldsymbol{c}_1^{\mathrm{L}^{\mathrm{T}}}\boldsymbol{x}$ and this directly implies that $(*)$ holds with at least one strict inequality. Then we assume $L^{(-1)}(\bar{h}) \neq 0$ and $\boldsymbol{c}_1^{\mathrm{L}^{\mathrm{T}}}\bar{\boldsymbol{x}} = \boldsymbol{c}_1^{\mathrm{L}^{\mathrm{T}}}\boldsymbol{x}$. This and condition for (a) imply $-L^{(-1)}(\bar{h})(\boldsymbol{\gamma}_1^{\mathrm{L}^{\mathrm{T}}}\bar{\boldsymbol{x}}) > -L^{(-1)}(\bar{h})(\boldsymbol{\gamma}_1^{\mathrm{L}^{\mathrm{T}}}\boldsymbol{x})$. Because we have $L^{(-1)}(\bar{h}) \neq 0$ and $L^{(-1)}(\bar{h}) \in [0,1]$, we obtain $(\boldsymbol{c}_1^{\mathrm{L}} - \boldsymbol{\gamma}_1^{\mathrm{L}})^{\mathrm{T}}\bar{\boldsymbol{x}} > (\boldsymbol{c}_1^{\mathrm{L}} - \boldsymbol{\gamma}_1^{\mathrm{L}})^{\mathrm{T}}\boldsymbol{x}$, i.e., $(*)$ holds with at least one strict inequality. In case (b), we can prove in the same way. Then $(*)$ holds with at least one strict inequality, i.e., $\boldsymbol{x} \notin \mathit{Eff}_{\mathrm{four}}$. □

In this approach, an efficient solution to the reduced multiple objective programming problems is considered as a reasonable solution [15, 40, 79]. If the complete optimal solution exists, it is considered as the best solution. Furukawa [15] proposed an efficient enumeration method of efficient solutions of Problem (20.16).

The following example given in [33] shows the limitation of this approach.

Example 1. Consider the following LP problem with interval objective function:

$$\begin{aligned} \text{maximize } & [1,3]x_1 + [1,3]x_2, \\ \text{subject to } & 45x_1 + 50x_2 \leq 530, \\ & 50x_1 + 45x_2 \leq 515, \\ & 0 \leq x_1 \leq 8,\ 0 \leq x_2 \leq 8. \end{aligned} \tag{20.19}$$

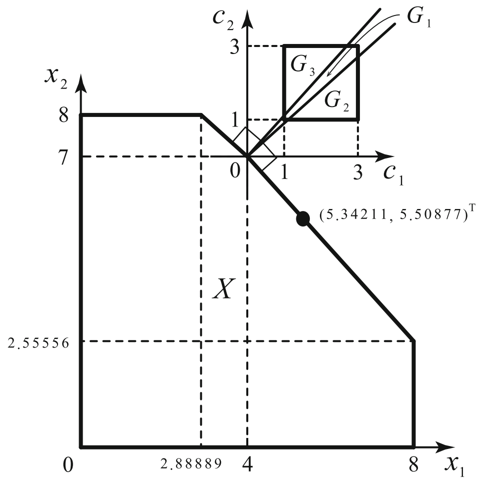
In this case, the following bi-objective problem corresponds to Problem (20.16):

$$\begin{aligned} \text{maximize } & (x_1 + x_2,\ 3x_1 + 3x_2)^{\mathrm{T}}, \\ \text{subject to } & 45x_1 + 50x_2 \leq 530, \\ & 50x_1 + 45x_2 \leq 515, \\ & 0 \leq x_1 \leq 8,\ 0 \leq x_2 \leq 8. \end{aligned} \tag{20.20}$$

The efficient optimal solution to Problem (20.20) is unique it is $(x_1, x_2)^{\mathrm{T}} = (4,7)^{\mathrm{T}}$. In other words, $(x_1, x_2)^{\mathrm{T}} = (4,7)^{\mathrm{T}}$ is the complete optimal solution. This solution on the feasible region is depicted in Fig. 20.3.

In Fig. 20.3, a box on c_1-c_2 coordinate shows all possible realizations of the objective function coefficient vector. Area G_1 shows the possible realizations of the objective function coefficient vector to which solution $(x_1, x_2)^{\mathrm{T}} = (4,7)^{\mathrm{T}}$ is optimal.

Fig. 20.3 Example 1



Similarly, Area G_2 and G_3 show the possible realizations of the objective function coefficient vector to which solutions $(x_1, x_2)^{\mathrm{T}} = (8, 2.55556)^{\mathrm{T}}$ and $(x_1, x_2)^{\mathrm{T}} = (2.8889, 8)^{\mathrm{T}}$ are optimal, respectively. Although solution $(x_1, x_2)^{\mathrm{T}} = (4, 7)^{\mathrm{T}}$ is the unique efficient solution to Problem (20.20), Area G_1 is much smaller than Areas G_2 and G_3. If all possible realizations of the objective function coefficient vector are equally probable, the probability that $(x_1, x_2)^{\mathrm{T}} = (4, 7)^{\mathrm{T}}$ is not the optimal solution is rather high. From this point of view, the validity of selecting solution $(x_1, x_2)^{\mathrm{T}} = (4, 7)^{\mathrm{T}}$ may be controversial.

20.3.2 Possibly and Necessarily Optimal Solutions

Let $S(\boldsymbol{c})$ be a set of optimal solutions to an LP problem with objective function $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$,

$$\begin{array}{l}\text{maximize } \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x},\\ \text{subject to } \boldsymbol{x} \in X.\end{array} \tag{20.21}$$

Consider Problem (20.13) when $\tilde{c}_{1j}, j = 1, 2, \ldots, n$ degenerate to intervals $[c_{1j}^{\mathrm{L}}, c_{1j}^{\mathrm{R}}]$, $j = 1, 2, \ldots, n$ and define $\Gamma = \prod_{j=1}^{n}[c_{1j}^{\mathrm{L}}, c_{1j}^{\mathrm{R}}] = \{(c_1, c_2, \ldots, c_n)^{\mathrm{T}} \mid c_{1j}^{\mathrm{L}} \le c_j \le c_{1j}^{\mathrm{R}},\ j = 1, 2, \ldots, n\}$. Then we define the following two optimal solution sets:

$$\Pi S = \bigcup \{S(\boldsymbol{c}) \mid \boldsymbol{c} \in \Gamma\}, \tag{20.22}$$

$$NS = \bigcap \{S(\boldsymbol{c}) \mid \boldsymbol{c} \in \Gamma\}. \tag{20.23}$$

An element of ΠS is a solution optimal for at least one $\boldsymbol{c} = (c_1, c_2, \ldots, c_n)^{\mathrm{T}}$ such that $c_{1j}^{\mathrm{L}} \leq c_j \leq c_{1j}^{\mathrm{R}}$, $j = 1, 2, \ldots, n$. Since $[c_{1j}^{\mathrm{L}}, c_{1j}^{\mathrm{R}}]$, $j = 1, 2, \ldots, n$ show the possible ranges of objective function coefficients c_{1j}, $j = 1, 2, \ldots, n$, an element of ΠS is called a "possibly optimal solution". On the other hand, an element of NS is a solution optimal for all $\boldsymbol{c} = (c_1, c_2, \ldots, c_n)^{\mathrm{T}}$ such that $c_{1j}^{\mathrm{L}} \leq c_j \leq c_{1j}^{\mathrm{R}}$, $j = 1, 2, \ldots, n$ and called a "necessarily optimal solution". Solution set ΠS was originally considered by Steuer [88] for a little different purpose while the concept of solution set NS was proposed by Bitran [4] in the setting of MOLP problems. Luhandjula [65] introduced those concepts into MOLP problem with fuzzy objective function coefficients. However, his definition was a little different from the one we described in what follows. Inuiguchi and Kume [30] and Inuiguchi and Sakawa [34] connected those concepts to possibility theory [12, 98] and termed ΠS and NS 'possibly optimal solution set' and 'necessarily optimal solution set'.

Consider the following MOLP problem:

$$\begin{array}{l} \text{maximize } \left(\bar{\boldsymbol{c}}_1^{\mathrm{T}}\boldsymbol{x}, \bar{\boldsymbol{c}}_2^{\mathrm{T}}\boldsymbol{x}, \ldots, \bar{\boldsymbol{c}}_q^{\mathrm{T}}\boldsymbol{x}\right)^{\mathrm{T}}, \\ \text{subject to } \boldsymbol{x} \in X, \end{array} \tag{20.24}$$

where $\bar{\boldsymbol{c}}_j$, $j = 1, 2, \ldots, q$ are all extreme points of box set $\Gamma = \prod_{j=1}^{n}[c_{1j}^{\mathrm{L}}, c_{1j}^{\mathrm{R}}]$. Accordingly, we have $q = 2^n$ when $c_{1j}^{\mathrm{L}} < c_{1j}^{\mathrm{R}}$, $j = 1, 2, \ldots, n$ and $q < 2^n$ when there exists at least one $j \in \{1, 2, \ldots, n\}$ such that $c_{1j}^{\mathrm{L}} = c_{1j}^{\mathrm{R}}$. We have $NS \subseteq \Pi S$, i.e., a necessarily optimal solution is a possibly optimal solution. The following theorem given by Inuiguchi and Kume [30] connects possibly and necessarily optimal solutions to weakly efficient and completely optimal solutions, respectively.

Theorem 2. *A solution is possibly optimal to Problem (20.13) with* $\tilde{c}_{1j} = [c_{1j}^{\mathrm{L}}, c_{1j}^{\mathrm{R}}]$, $j = 1, 2, \ldots, n$ *if and only if it is weakly efficient to Problem (20.24). A solution is necessarily optimal to Problem (20.13) when* $\tilde{c}_{1j} = [c_{1j}^{\mathrm{L}}, c_{1j}^{\mathrm{R}}]$, $j = 1, 2, \ldots, n$ *if and only if it is completely optimal to Problem (20.24).*

Proof. Suppose $\hat{\boldsymbol{x}}$ is a weakly efficient solution to Problem (20.24). There are no feasible solutions such that $\bar{\boldsymbol{c}}_j^{\mathrm{T}}\boldsymbol{x} > \bar{\boldsymbol{c}}_j^{\mathrm{T}}\hat{\boldsymbol{x}}$, $j = 1, 2, \ldots, q$. As is well known in the literature, there is a vector $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_q)^{\mathrm{T}}$ such that $\sum_{j=1}^{q} \lambda_j = 1$, $\lambda_j \geq 0$, $j = 1, 2, \ldots, q$ and $\hat{\boldsymbol{x}}$ is an optimal solution to the following LP problem:

$$\max_{\boldsymbol{x} \in X} \sum_{j=1}^{q} \lambda_j \bar{\boldsymbol{c}}_j^{\mathrm{T}}\boldsymbol{x}. \tag{$*$}$$

Thus, we have $\hat{\boldsymbol{x}} \in S\left(\sum_{j=1}^{q} \lambda_j \bar{\boldsymbol{c}}_j\right)$. By the definition of $\bar{\boldsymbol{c}}_j$'s, $\sum_{j=1}^{q} \lambda_j \bar{\boldsymbol{c}}_j \in \Gamma$. Hence, $\hat{\boldsymbol{x}}$ is a possibly optimal solution to Problem (20.13) with $\tilde{c}_{1j} = [c_{1j}^{\mathrm{L}}, c_{1j}^{\mathrm{R}}]$, $j = 1, 2, \ldots, n$. Conversely, suppose $\hat{\boldsymbol{x}}$ is a possibly optimal solution to Problem (20.13) with $\tilde{c}_{1j} = [c_{1j}^{\mathrm{L}}, c_{1j}^{\mathrm{R}}]$, $j = 1, 2, \ldots, n$, there is a vector $\boldsymbol{c} \in \Gamma$ such that $\hat{\boldsymbol{x}} \in S(\boldsymbol{c})$. By the definition of $\boldsymbol{c}^j$'s, there is a vector $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_q)^{\mathrm{T}}$ such that $\sum_{j=1}^{q} \lambda_j = 1$, $\lambda_j \geq 0$,

$j = 1, 2, \ldots, q$ and $\boldsymbol{c} = \sum_{j=1}^{q} \lambda_j \bar{\boldsymbol{c}}_j$. Thus, $\hat{\boldsymbol{x}}$ is an optimal solution to the problem $(*)$ and from this fact, it is a weakly efficient solution to problem (20.24). Hence, the first assertion is proved.

The second assertion can be proved similarly. □

Possibly and necessarily optimal solutions are exemplified in the following example.

Example 2. Let us consider the following LP problems with interval objective function:

$$\begin{aligned} \text{maximize } & [2,3]x_1 + [1.5, 2.5]x_2, \\ \text{subject to } & 3x_1 + 4x_2 \le 42, \\ & 3x_1 + x_2 \le 24, \\ & x_1 \ge 0,\ 0 \le x_2 \le 9. \end{aligned} \tag{20.25}$$

For this problem, we obtain $\Gamma = \{(c_1, c_2)^{\mathrm{T}} \mid 2 \le c_1 \le 3,\ 1.5 \le c_2 \le 2.5\}$ and $X = \{(x_1, x_2)^{\mathrm{T}} \mid 3x_1 + 4x_2 \le 42,\ 3x_1 + x_2 \le 24,\ x_1 \ge 0,\ 0 \le x_2 \le 9\}$ at $(6,6)^{\mathrm{T}}$. Consider solution $(x_1, x_2)^{\mathrm{T}} = (6,6)^{\mathrm{T}}$ and the normal cone to X at this solution, i.e., a set of vectors $(c_1, c_2)^{\mathrm{T}}$ such that $(x_1 - 6, x_2 - 6)^{\mathrm{T}}(c_1, c_2) \le 0$. The normal cone to X at $(6,6)^{\mathrm{T}}$ is obtained as

$$P((6,6)^{\mathrm{T}}) = \left\{(c_1, c_2)^{\mathrm{T}} \mid c_1 - 3c_2 \le 0,\ 4c_1 - 3c_2 \ge 0\right\}. \tag{20.26}$$

As shown in Fig. 20.4, we obtain $\Gamma \subseteq P((6,6)^{\mathrm{T}})$. This implies that solution $(6,6)^{\mathrm{T}}$ is optimal for all $(c_1, c_2)^{\mathrm{T}} \in \Gamma$. Therefore, solution $(6,6)^{\mathrm{T}}$ is a necessarily optimal solution.

On the other hand, when the objective function of Problem (20.25) is changed to

$$[1,3]x_1 + [1.5, 3]x_2, \tag{20.27}$$

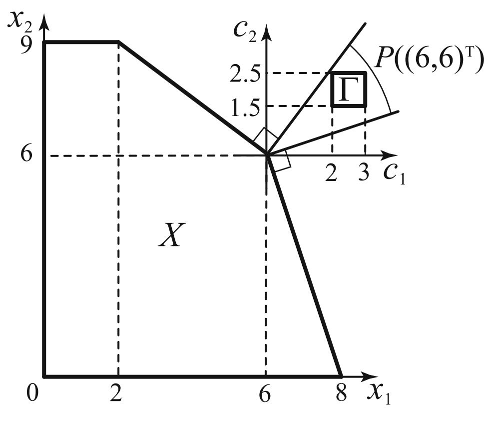


Fig. 20.4 Problem (20.25)

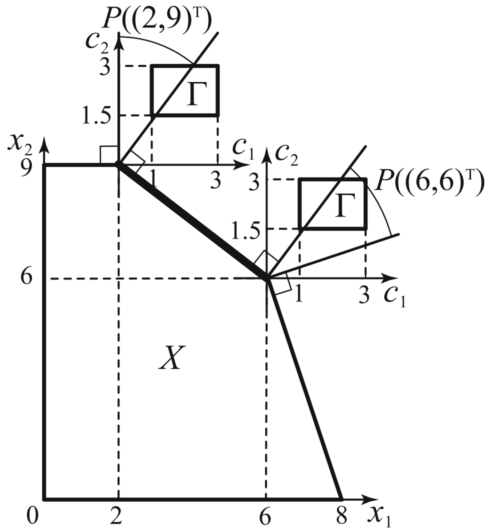


Fig. 20.5 Problem (20.25) with the updated objective function

Γ is updated to $\Gamma = \{(c_1, c_2)^{\mathrm{T}} \mid 1 \leq c_1 \leq 3,\ 1.5 \leq c_2 \leq 3\}$. As shown in Fig. 20.5, $\Gamma \subseteq P((6,6)^{\mathrm{T}})$ is no longer valid. In this case, $\Gamma \subseteq P((2,9)^{\mathrm{T}}) \cup P((6,6)^{\mathrm{T}})$ is obtained and solutions on the line segment between $(2,9)^{\mathrm{T}}$ and $(6,6)^{\mathrm{T}}$ are all possibly optimal solutions. As shown in this example, there are infinitely many possibly optimal solutions. However, the number of possibly optimal basic solutions (extreme points) is finite.

As shown in this example, a necessarily optimal solution does not exist in many cases but if it exists it is the most reasonable solution. On the other hand, a possibly optimal solution always exist whenever X is bounded and nonempty but it is often non-unique. If a possibly optimal solution is unique, it is a necessarily optimal solution. Moreover, as is conjectured from this example, we can prove the following equivalences for a given $\boldsymbol{x} \in X$:

$$\boldsymbol{x} \in NS \Leftrightarrow \Gamma \subseteq P(\boldsymbol{x}), \tag{20.28}$$

$$\boldsymbol{x} \in \Pi S \Leftrightarrow \Gamma \cap P(\boldsymbol{x}) \neq \emptyset, \tag{20.29}$$

where $P(\boldsymbol{x})$ is the normal cone to X at solution $\boldsymbol{x}$.

The possibly and necessarily optimal solutions are extended to the case where $\tilde{c}_{1j}$, $j = 1, 2, \ldots, n$ are fuzzy numbers. In this case, the possible range Γ_1 of coefficient vectors $\boldsymbol{c}$ becomes a fuzzy set defined by the following membership function:

$$\mu_{\Gamma_1}(\boldsymbol{c}) = \min_{j=1,2,\ldots,n} \mu_{\tilde{c}_{1j}}(c_j), \tag{20.30}$$

where $\boldsymbol{c} = (c_1, c_2, \ldots, c_n)^{\mathrm{T}}$ and $\mu_{\tilde{c}_{1j}}$ is the membership function of $\tilde{c}_{1j}$. Accordingly the possibility optimal solution set ΠS and the necessarily optimal solution set NS become fuzzy sets defined by the following membership functions:

$$\mu_{\Pi S}(\boldsymbol{x}) = \sup_{\boldsymbol{c}:\, \boldsymbol{x} \in S(\boldsymbol{c})} \mu_{\tilde{c}_{1j}}(\boldsymbol{c}), \tag{20.31}$$

$$\mu_{NS}(\boldsymbol{x}) = \inf_{\boldsymbol{c}:\, \boldsymbol{x} \notin S(\boldsymbol{c})} 1 - \mu_{\tilde{c}_{1j}}(\boldsymbol{c}), \tag{20.32}$$

where $\mu_{\Pi S}$ and μ_{NS} are membership functions of the possibility optimal solution set ΠS and the necessarily optimal solution set NS. Because $\tilde{c}_{1j}$ has membership function, each solution $\boldsymbol{x} \in X$ has possible optimality degree $\mu_{\Pi S}(\boldsymbol{x})$ and necessary optimality degree $\mu_{NS}(\boldsymbol{x})$. Because $\tilde{c}_{1j}$, $j = 1, 2, \ldots, n$ are fuzzy numbers, from (20.11) and (20.12), we have the following properties for any $h \in (0, 1]$:

$$\mu_{\Pi S}(\boldsymbol{x}) \geq h \Leftrightarrow \exists \boldsymbol{c} \in [\Gamma]_h,\ \boldsymbol{x} \in S(\boldsymbol{c}), \tag{20.33}$$

$$\mu_{NS}(\boldsymbol{x}) \geq h \Leftrightarrow \forall \boldsymbol{c} \in (\Gamma)_{1-h},\ \boldsymbol{x} \in S(\boldsymbol{c}). \tag{20.34}$$

As shown in those properties, the chance that a necessarily optimal solution exists increases by defining $[\Gamma]_1$ smaller. Especially, if we define Γ with a continuous membership function such that $[\Gamma]_1$ is a singleton composed of the most plausible objective function coefficient vector and $(\Gamma)_0$ shows the largest possible range, we can analyze the degree of robust optimality of a solution $\boldsymbol{x}$ by $\mu_{NS}(\boldsymbol{x})$.

Computation methods for the degree of possible optimality and the degree of necessary optimality of a given feasible solution are investigated by Inuiguchi and Sakawa [32]. They showed that the former can be done by solving an LP problem while the latter by solving many LP problems. On the other hand, Steuer [88] investigated enumeration methods of all possibly optimal basic solutions of Problem (20.13) when $\tilde{c}_j$'s are closed intervals. Inuiguchi and Tanino [38] proposed an enumeration method of all possibly optimal basic solutions of Problem (20.13) with possible optimality degree $\mu_{\Pi S}(\boldsymbol{x})$.

Remark 1. Consider a MOLP problem,

$$\begin{array}{l} \text{maximize } \left(\boldsymbol{c}_1{}^{\mathrm{T}}\boldsymbol{x}, \boldsymbol{c}_2{}^{\mathrm{T}}\boldsymbol{x}, \ldots, \boldsymbol{c}_p{}^{\mathrm{T}}\boldsymbol{x}\right)^{\mathrm{T}}, \\ \text{subject to } \boldsymbol{x} \in X, \end{array} \tag{20.35}$$

and solve it by weighting method. If the weight $\boldsymbol{w} \geq \boldsymbol{0}$ cannot be specified uniquely but by a fuzzy set $\tilde{\boldsymbol{w}}$, the possible and necessary optimalities are useful to find candidate solutions.

20.3.3 Minimax Regret Solutions and the Related Solution Concepts

As seen in the previous subsection, a necessarily optimal solution is the most reasonable solution to Problem (20.13) but its existence is not guaranteed. On the other hand, possibly optimal solutions are the least reasonable solutions to Problem (20.13) but there are usually many possibly optimal solutions. Therefore, these solution concepts are two extremes.

In this subsection, we consider intermediate solution concepts such that

1. the solution is a possibly optimal solution,
2. it coincides with the necessarily optimal solution when the necessarily optimal solution exists, and
3. it minimizes the deviation from the necessary optimality, or it maximizes the proximity to the necessity optimality.

For the sake of ease, we first consider cases where Γ is a crisp set. To measure the deviation from the necessary optimality and the proximity to the necessity optimality, the following two functions $R : X \to [0, \infty)$ and $WA : X \to (-\infty, 1]$ have been considered so far (see Inuiguchi and Kume [30], Inuiguchi and Sakawa [33, 35]):

$$R(\boldsymbol{x}) = \max_{\boldsymbol{c} \in \Gamma} \max_{\boldsymbol{y} \in X} \boldsymbol{c}^{\mathrm{T}} (\boldsymbol{y} - \boldsymbol{x}), \qquad WA(\boldsymbol{x}) = \min_{\boldsymbol{c} \in \Gamma} \frac{\boldsymbol{c}^{\mathrm{T}} \boldsymbol{x}}{\max_{\boldsymbol{y} \in X} \boldsymbol{c}^{\mathrm{T}} \boldsymbol{y}}, \tag{20.36}$$

where $R(\boldsymbol{x})$ is known as the maximum regret. $R(\boldsymbol{x})$ takes its minimum value zero if and only if $\boldsymbol{x}$ is a necessarily optimal solution. On the other hand, $WA(\boldsymbol{x})$ shows the worst achievement rate and is defined only when $\max_{\boldsymbol{y} \in X} \boldsymbol{c}^{\mathrm{T}} \boldsymbol{y} > 0$. $WA(\boldsymbol{x})$ takes its maximum value one if and only if $\boldsymbol{x}$ is a necessarily optimal solution.

Hence, we obtain the following programming problems:

$$\underset{\boldsymbol{x} \in X}{\text{minimize}}\, R(\boldsymbol{x}), \qquad \underset{\boldsymbol{x} \in X}{\text{maximize}}\, WA(\boldsymbol{x}). \tag{20.37}$$

The former problem is the minimax regret problem and the latter problem is the maximin achievement rate problem. Optimal solutions to those problems are called 'a minimax regret solution' and 'a maximin achievement rate solution', respectively. The possible optimalities of minimax regret solutions and maximin achievement rate solutions are proved by using Theorem 2 as shown in the following theorem (Inuiguchi and Kume [30] and Inuiguchi and Sakawa [35]).

Theorem 3. *Minimax regret solutions as well as maximin achievement rate solutions are possibly optimal solutions to the problem (20.13).*

Proof. We prove the possible optimality of a minimax regret solution because that of a maximin achievement rate solution can be proved in the same way. Let $\hat{\boldsymbol{x}}$ be a minimax regret solution. Assume it does not a possibly optimal solution. Then it does not a weakly efficient solution to MOLP problem (20.24) from Theorem 2. Thus, there exists a feasible solution $\boldsymbol{x}$ such that $\bar{\boldsymbol{c}}_j^{\mathrm{T}}\boldsymbol{x} > \bar{\boldsymbol{c}}_j^{\mathrm{T}}\hat{\boldsymbol{x}}, j = 1, 2, \ldots, q$. Namely,

$$\sum_{j=1}^{q} \lambda_j \bar{\boldsymbol{c}}_j^{\mathrm{T}} \boldsymbol{x} > \sum_{j=1}^{q} \lambda_j \bar{\boldsymbol{c}}_j^{\mathrm{T}} \hat{\boldsymbol{x}}$$

holds for all $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_p)$ such that $\sum_{j=1}^{q} \lambda_j = 1$ and $\lambda_j \geq 0$, $j = 1, 2, \ldots, q$. Since $\bar{\boldsymbol{c}}_j$, $j = 1, 2, \ldots, q$ are all extreme points of Γ, this inequality can be rewritten as

$$\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} > \boldsymbol{c}^{\mathrm{T}}\hat{\boldsymbol{x}}, \text{ for all } \boldsymbol{c} \in \Gamma.$$

Thus we have

$$R(\boldsymbol{x}) = \max_{\boldsymbol{c}\in\Gamma} \max_{\boldsymbol{y}\in X} \boldsymbol{c}^{\mathrm{T}} (\boldsymbol{y} - \boldsymbol{x}) < \max_{\boldsymbol{c}\in\Gamma} \max_{\boldsymbol{y}\in X} \boldsymbol{c}^{\mathrm{T}} (\boldsymbol{y} - \hat{\boldsymbol{x}}) = R(\hat{\boldsymbol{x}}).$$

This contradicts the fact that $\hat{\boldsymbol{x}}$ is a minimax regret solution. Hence, a minimax regret solution is a possibly optimal solution. □

Example 3. Consider Problem (20.19) again. The minimax regret solution is obtained as point $(5.34211, 5.50877)^{\mathrm{T}}$ in Fig. 20.3. As shown in Fig. 20.3, this solution is on the polygonal line segment composed of $(2.8889, 8)^{\mathrm{T}}$, $(4, 7)^{\mathrm{T}}$ and $(8, 2.55556)^{\mathrm{T}}$. The polygonal line segment shows the possibly optimal solution set. Then we know that the minimax regret solution is a possibly optimal solution. Moreover, from Fig. 20.3, we observe the solution $(5.34211, 5.50877)^{\mathrm{T}}$ is located at a well-balanced place on the polygonal line segment.

Now we consider cases where Γ is a fuzzy set. In this case, by the extension principle, we define fuzzy regret $\tilde{r}(\boldsymbol{x})$ and fuzzy achievement rate $\widetilde{ac}(\boldsymbol{x})$ for a feasible solution $\boldsymbol{x} \in X$ by the following membership functions:

$$\mu_{\tilde{r}(\boldsymbol{x})}(r) = \sup \left\{ \mu_{\Gamma}(\boldsymbol{c}) \;\middle|\; r = \max_{\boldsymbol{y}\in X} \boldsymbol{c}^{\mathrm{T}}(\boldsymbol{y} - \boldsymbol{x}) \right\}, \tag{20.38}$$

$$\mu_{\widetilde{ac}(\boldsymbol{x})}(r) = \sup \left\{ \mu_{\Gamma}(\boldsymbol{c}) \;\middle|\; r \cdot \max_{\boldsymbol{y}\in X} \boldsymbol{c}^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \right\}. \tag{20.39}$$

Moreover, we specify fuzzy goal G_r having an upper semi-continuous non-increasing membership function $\mu_{G_r} : [0, +\infty) \to [0, 1]$ such that $\mu_{G_r}(0) = 1$ on the regret, and fuzzy goal G_{ac} having an upper semi-continuous non-decreasing membership function $\mu_{G_{ac}} : (-\infty, 1] \to [0, 1]$ such that $\mu_{G_{ac}}(1) = 1$ on the achievement rate. Then, using necessity measure, the problem is formulated as

$$\operatorname*{maximize}_{\boldsymbol{x}\in X} N_{\tilde{r}(\boldsymbol{x})}(G_r), \qquad \operatorname*{maximize}_{\boldsymbol{x}\in X} N_{\widetilde{ac}(\boldsymbol{x})}(G_{ac}). \tag{20.40}$$

We note that optimal solutions to these problem can be seen as relaxations of necessarily optimal solutions. Let us define two kinds of suboptimal solution sets to LP problem (20.13) with objective function $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$ as fuzzy sets $S_{dif}(\boldsymbol{c})$ and $S_{rat}(\boldsymbol{c})$ by the following membership functions:

$$\mu_{S_{dif}(\boldsymbol{c})}(\boldsymbol{x}) = \min\left(\chi_X(\boldsymbol{x}), \mu_{G_r}\left(\max_{\boldsymbol{y}\in X} \boldsymbol{c}^{\mathrm{T}}\boldsymbol{y} - \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}\right)\right), \tag{20.41}$$

$$\mu_{S_{rat}(\boldsymbol{c})}(\boldsymbol{x}) = \min\left(\chi_X(\boldsymbol{x}), \mu_{G_{ac}}\left(\frac{\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}}{\max_{\boldsymbol{y}\in X} \boldsymbol{c}^{\mathrm{T}}\boldsymbol{y}}\right)\right), \tag{20.42}$$

where χ_X is the characteristic function of feasible region X, i.e., $\chi_X(\boldsymbol{x}) = 1$ for $\boldsymbol{x} \in X$ and $\chi_X(\boldsymbol{x}) = 0$ for $\boldsymbol{x} \notin X$.

Based on these, we define two kinds of necessarily suboptimal solution sets NS_{dif} and NS_{rat} by the following membership functions:

$$\mu_{NS_{dif}}(\boldsymbol{x}) = \inf_{\boldsymbol{c}} \max\left(1 - \mu_{\Gamma}(\boldsymbol{c}), \mu_{S_{dif}(\boldsymbol{c})}(\boldsymbol{x})\right), \tag{20.43}$$

$$\mu_{NS_{rat}}(\boldsymbol{x}) = \inf_{\boldsymbol{c}} \max\left(1 - \mu_{\Gamma}(\boldsymbol{c}), \mu_{S_{rat}(\boldsymbol{c})}(\boldsymbol{x})\right). \tag{20.44}$$

We obtain $\mu_{NS_{dif}}(\boldsymbol{x}) = N_{\tilde{r}(\boldsymbol{x})}(G_r)$ and $\mu_{NS_{rat}}(\boldsymbol{x}) = N_{\widetilde{ac}(\boldsymbol{x})}(G_r)$ for $\boldsymbol{x} \in X$. Therefore, problems in (20.40) are understood optimization problems of necessary suboptimality degrees.

The minimax regret problem was considered by Inuiguchi and Kume [30] and Inuiguchi and Sakawa [33]. Inuiguchi and Sakawa [33] proposed a solution method based on the relaxation procedure when all possibly optimal basic solutions are known. Mausser and Laguna [71] proposed a mixed integer programming approach to the minimax regret problem. Inuiguchi and Tanino [37] proposed a solution approach based on outer approximation and cutting hyperplane. The maximin achievement rate approach was proposed by Inuiguchi and Sakawa [35] and a relaxation procedure for a maximin achievement rate solution was proposed when all possibly optimal basic solutions are known. The necessarily suboptimal solution set is originally proposed by Inuiguchi and Sakawa [36]. They treated the regret case and proposed a solution algorithm based on the relaxation procedure and the bisection method. Inuiguchi et al. [39] further investigated a solution algorithm for both problems in (20.40). In those solution algorithms, the relaxation procedure and bisection method converges at the same time. The reduced problems described in this subsection are non-convex optimization problems. The recent global optimization techniques [21] would work well for those problems. The minimax regret solution concept is applied to discrete optimization problems [41] and MOLP problems [76]. The minimax regret solution to a MOLP problem minimizes the deviation from the complete optimality. The computational complexity of minimax regret solution is investigated in [1].

20.4 Multiple Objective Function Case

Now we describe the approaches to Problem (20.2) with $p > 1$, i.e., multiple objective function case.

20.4.1 Possibly and Necessarily Efficient Solutions

The concepts of possibly and necessarily optimal solutions can be extended to the case of multiple objective functions. In this case, the corresponding solution concepts are possibly and necessarily efficient solutions.

Before giving the definitions of possibly and necessarily efficient solutions, we define a set of efficient solutions, $E(C)$ to the following MOLP problem:

$$\begin{aligned} &\text{maximize } (\boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{x}, \boldsymbol{c}_2^{\mathrm{T}}\boldsymbol{x}, \ldots, \boldsymbol{c}_p^{\mathrm{T}}\boldsymbol{x})^{\mathrm{T}}, \\ &\text{subject to } \boldsymbol{x} \in X, \end{aligned} \tag{20.45}$$

where we define $p \times n$ matrix C by $C = (\boldsymbol{c}_1\ \boldsymbol{c}_2\ \cdots\ \boldsymbol{c}_p)^{\mathrm{T}}$.

First, we describe the case where $\tilde{c}_{kj}$, $k = 1, 2, \ldots, p$, $j = 1, 2, \ldots, n$ degenerate to intervals $[c_{kj}^{\mathrm{L}}, c_{kj}^{\mathrm{R}}]$, $k = 1, 2, \ldots, p$, $j = 1, 2, \ldots, n$ and define $\Theta = \prod_{k=1}^{p} \Gamma_k$ and $\Gamma_k = \prod_{j=1}^{n} [c_{kj}^{\mathrm{L}}, c_{kj}^{\mathrm{R}}] = \{(c_1, c_2, \ldots, c_n)^{\mathrm{T}} \mid c_{kj}^{\mathrm{L}} \le c_j \le c_{kj}^{\mathrm{R}},\ j = 1, 2, \ldots, n\}$, $k = 1, 2, \ldots, p$. Namely, Θ is a box set of $p \times n$ matrices. Then, in the analogy, we obtain the possibly efficient solution set ΠE and the necessarily efficient solution set NE by

$$\Pi E = \bigcup \{E(C) \mid C \in \Theta\}, \tag{20.46}$$

$$NE = \bigcap \{E(C) \mid C \in \Theta\}. \tag{20.47}$$

Elements of ΠE and NE are interpreted in the same way as those of ΠS and NS, respectively. Namely, an element of ΠE is a solution efficient for at least one $C \in \Theta$. Because Θ shows the possible range of objective function coefficient matrix, an element of ΠE is called a "possibly efficient solution". On the other hand, an element of NE is a solution efficient for all $C \in \Theta$ and called a "necessarily efficient solution".

Let $K(C) = \{\boldsymbol{s} \mid C\boldsymbol{s} \ge \boldsymbol{0} \text{ and } C\boldsymbol{s} \ne \boldsymbol{0}\}$ and $R(C) = \{C^{\mathrm{T}}\boldsymbol{z} \mid \boldsymbol{z} > \boldsymbol{0}\}$. Namely, $K(C)$ shows the set of improving directions while $R(C)$ is the set of positively weighted sum of objective coefficient vectors. Using $K(C)$ and $R(C)$, we define the following sets:

$$K\Pi(\Theta) = \bigcap \{K(C) \mid C \in \Theta\}, \tag{20.48}$$

$$KN(\Theta) = \bigcup \{K(C) \mid C \in \Theta\}, \tag{20.49}$$

$$\begin{aligned} R\Pi(\Theta) &= \{\boldsymbol{c} \mid \exists \boldsymbol{z} > \boldsymbol{0}\ \exists C \in \Theta, \boldsymbol{c} = C^{\mathrm{T}}\boldsymbol{z}\} \\ &= \bigcup \{R(C) \mid C \in \Theta\}, \end{aligned} \tag{20.50}$$

$$\begin{aligned} RN(\Theta) &= \{\boldsymbol{c} \mid \forall C \in \Theta\ \exists \boldsymbol{z} > \boldsymbol{0}, \boldsymbol{c} = C^{\mathrm{T}}\boldsymbol{z}\} \\ &= \bigcap \{R(C) \mid C \in \Theta\}. \end{aligned} \tag{20.51}$$

Let $T(\boldsymbol{x})$ be the tangent cone of feasible region X at point $\boldsymbol{x} \in X$, i.e., $T(\boldsymbol{x}) = \mathrm{cl}\{r(\boldsymbol{y} - \boldsymbol{x}) \mid \boldsymbol{y} \in X,\ r \geq 0\}$, where $\mathrm{cl}K$ is the closure of a set K. Let $P(\boldsymbol{x})$ be the normal cone of X at $\boldsymbol{x} \in X$, in other words, $P(\boldsymbol{x}) = \{\boldsymbol{c} \mid \boldsymbol{c}^{\mathrm{T}}(\boldsymbol{y} - \boldsymbol{x}) \leq 0,\ \forall \boldsymbol{y} \in X\} = \left\{\boldsymbol{c} \,\middle|\, \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} = \max_{\boldsymbol{y} \in X} \boldsymbol{c}^{\mathrm{T}}\boldsymbol{y}\right\}$.

Because the following equivalence for $\boldsymbol{x} \in X$ is known in MOLP Problem (see, for example, Steuer [89]):

$$\boldsymbol{x} \in E(C) \Leftrightarrow (K(C) \cup \{\boldsymbol{0}\}) \cap T(\boldsymbol{x}) = \{\boldsymbol{0}\}, \tag{20.52}$$

$$\boldsymbol{x} \in E(C) \Leftrightarrow R(C) \cap P(\boldsymbol{x}) \neq \emptyset, \tag{20.53}$$

Then, for $\boldsymbol{x} \in X$, we have

$$\boldsymbol{x} \in \Pi E \Leftrightarrow (K\Pi(\Theta) \cup \{\boldsymbol{0}\}) \cap T(\boldsymbol{x}) = \{\boldsymbol{0}\}, \tag{20.54}$$

$$\boldsymbol{x} \in NE \Leftrightarrow (KN(\Theta) \cup \{\boldsymbol{0}\}) \cap T(\boldsymbol{x}) = \{\boldsymbol{0}\}, \tag{20.55}$$

$$\boldsymbol{x} \in \Pi E \Leftrightarrow R\Pi(\Theta) \cap P(\boldsymbol{x}) \neq \emptyset, \tag{20.56}$$

$$RN(\Theta) \cap P(\boldsymbol{x}) \neq \emptyset \Rightarrow \boldsymbol{x} \in NE. \tag{20.57}$$

Let Φ be the subset of matrices of Θ having all elements of each column at the upper bound or at the lower bound. Namely, $C \in \Phi$ implies $C_{\cdot j} = L_{\cdot j}$ or $C_{\cdot j} = U_{\cdot j}$ for $j = 1, 2, \ldots, p$, where $L = (c_{ij}^{\mathrm{L}})$, $U = (c_{ij}^{\mathrm{R}})$ and $C_{\cdot j}$ is the j-th column of matrix C. We have the following proposition (see Bitran [4]).

Proposition 1. *We have the following equations:*

$$KN(\Theta) = KN(\Phi), \tag{20.58}$$

$$NE = \bigcap \{E(C) \mid C \in \Phi\}, \tag{20.59}$$

$$RN(\Theta) = RN(\Phi). \tag{20.60}$$

Proof. We prove (20.58) and (20.59). Equation (20.60) is obtained from (20.59) in a straightforward manner.

$KN(\Phi) \subseteq KN(\Theta)$ is obvious. Then we prove the reverse inclusion relation. Assume $\boldsymbol{s} \in KN(\Theta)$, Then there exists $C \in \Theta$ such that $C\boldsymbol{s} \geq \boldsymbol{0}$ and $C\boldsymbol{s} \neq \boldsymbol{0}$. Consider $\bar{C}$ defined by $\bar{C}_{\cdot j} = L_{\cdot j}$ if $s_j < 0$ and $\bar{C}_{\cdot j} = U_{\cdot j}$ otherwise, for $j = 1, 2, \ldots, p$. Then $\bar{C} \in \Phi$. We have $\bar{C}\boldsymbol{s} \geq C\boldsymbol{s} \geq \boldsymbol{0}$ and $\bar{C}\boldsymbol{s} \neq \boldsymbol{0}$. This implies $\boldsymbol{s} \in K(\bar{C}) \subseteq KN(\Phi)$. Hence, $KN(\Phi) \supseteq KN(\Theta)$.

Now let use prove (20.59). By definition, we have $NE = \bigcap\{E(C) \mid C \in \Theta\} \subseteq \bigcap\{E(C) \mid C \in \Phi\}$. Then we prove the reverse inclusion relation. Assume $\boldsymbol{x} \notin NE$. Thus, from (20.55), we have $(KN(\Theta) \cup \{\boldsymbol{0}\}) \cap T(\boldsymbol{x}) \neq \{\boldsymbol{0}\}$. From (20.58), we obtain $(KN(\Phi) \cup \{\boldsymbol{0}\}) \cap T(\boldsymbol{x}) \neq \{\boldsymbol{0}\}$. Namely, $\boldsymbol{x} \notin E(C)$ for some $C \in \Phi$. Consequently, $\boldsymbol{x} \notin \bigcap\{E(C) \mid C \in \Phi\}$. Hence, $NE = \bigcap\{E(C) \mid C \in \Theta\} \supseteq \bigcap\{E(C) \mid C \in \Phi\}$. □

As is known in the literature, we have

$$P(\boldsymbol{x}) = T(\boldsymbol{x})^* \text{ and } T(\boldsymbol{x}) = P(\boldsymbol{x})^*, \tag{20.61}$$

where D^* stands for the polar cone of a set D, i.e., $D^* = \{\boldsymbol{y} \mid \boldsymbol{x}^{\mathrm{T}}\boldsymbol{y} \leq 0, \forall \boldsymbol{x} \in D\}$. We obtain the following proposition.

Lemma 1. *The following are true:*

$$\forall \boldsymbol{s} \in K(C),\ \forall \boldsymbol{y} \in R(C); \boldsymbol{s}^{\mathrm{T}}\boldsymbol{y} > 0, \tag{20.62}$$

$$-R(C) \subseteq K(C)^*, \tag{20.63}$$

$$-R(C) \subseteq \mathrm{int}\{\boldsymbol{s}\}^*,\ \forall \boldsymbol{s} \in K(C). \tag{20.64}$$

where int$D$ *is the interior of set* $D \subset \mathbf{R}^n$.

Proof. From definition, we obtain (20.62). Equations (20.63) and (20.64) are obtained from (20.62) in a straight forward manner. □

We obtain the following theorem (Inuiguchi [27]).

Theorem 4. *If* $RN(\Theta)$ *is not empty, we have*

$$\boldsymbol{x} \in NE \Leftrightarrow RN(\Theta) \cap P(\boldsymbol{x}) \neq \emptyset. \tag{20.65}$$

Proof. We prove that $RN(\Theta) \cap P(\boldsymbol{x}) \neq \emptyset$ implies $\boldsymbol{x} \in NE$ because the reverse implication is obtained from (20.57).

Assume $\boldsymbol{x} \notin NE$. Then, from (20.55) and $\boldsymbol{0} \in T(\boldsymbol{x})$, $KN(\Theta) \cap T(\boldsymbol{x}) \neq \emptyset$. Let $\hat{\boldsymbol{s}} \in KN(\Theta) \cap T(\boldsymbol{x})$. There exists $C \in \Theta$ such that $\hat{\boldsymbol{s}} \in K(C) \cap T(\boldsymbol{x})$. Considering $\{\hat{\boldsymbol{s}}\}^* = \{\boldsymbol{c} \in \mathbf{R}^n \mid \boldsymbol{c}^{\mathrm{T}}\hat{\boldsymbol{s}} \leq 0\}$, we have

$$K(C)^* \subseteq \{\hat{\boldsymbol{s}}\}^* \text{ and } T(\boldsymbol{x})^* \subseteq \{\hat{\boldsymbol{s}}\}^*.$$

From (20.61), the second inclusion relation implies $P(\boldsymbol{x}) \subseteq \{\hat{\boldsymbol{s}}\}^*$, i.e.,

$$\forall \boldsymbol{y} \in P(\boldsymbol{x});\ \hat{\boldsymbol{s}}^{\mathrm{T}}\boldsymbol{y} \leq 0. \tag{$*$}$$

On the other hand, from (20.63), we have $-R(C) \subseteq \{\hat{\boldsymbol{s}}\}^*$. From $\hat{\boldsymbol{s}} \in K(C)$ and (20.64), we obtain $-R(C) \subseteq \mathrm{int}\{\hat{\boldsymbol{s}}\}^*$. This means

$$\forall \boldsymbol{y} \in R(C);\ \hat{\boldsymbol{s}}^{\mathrm{T}}\boldsymbol{y} > 0. \tag{$**$}$$

From $(*)$ and $(**)$, we find that $\{\boldsymbol{y} \mid \hat{\boldsymbol{s}}^{\mathrm{T}}\boldsymbol{y} = 0\}$ is a separating hyperplane of $P(\boldsymbol{x})$ and $R(C)$. Therefore, we obtain $R(C) \cap P(\boldsymbol{x}) = \emptyset$. By definition of $RN(\Theta)$, this implies

$$RN(\Theta) \cap P(\boldsymbol{x}) = \emptyset.$$

Hence, we have $RN(\Theta) \cap P(\boldsymbol{x}) \neq \emptyset \Rightarrow \boldsymbol{x} \in NE$. □

Let us consider the following set of objective function coefficients:

$$\Lambda(\Theta) = \bigcap \{\Lambda \mid \Lambda \text{ is a convex cone, and } \forall C \in \Theta,\ R(C) \cap \Lambda \neq \emptyset\} \tag{20.66}$$

When $\Lambda(\Theta)$ is not empty, from the definition, we have

$$\boldsymbol{x} \in NE \Leftrightarrow \Lambda(\Theta) \subseteq P(\boldsymbol{x}). \tag{20.67}$$

Let $Uni = \{\boldsymbol{c} = (c_1, c_2, \ldots, c_n)^{\mathrm{T}} \mid \sum_{j=1}^{n} |c_j| = 1\}$. We find the following strong relations between $RN(\Theta)$ and $\Lambda(\Theta)$:

- $RN(\Theta)$ is empty if and only if $\Lambda(\Theta) \cap Uni$ is neither an empty set nor a singleton.
- $RN(\Theta) \cap Uni$ is neither an empty set nor a singleton if and only if $\Lambda(\Theta) = \emptyset$.
- $RN(\Theta) \cap Uni$ is a singleton if and only if $\Lambda(\Theta) \cap Uni$ is a singleton, and moreover we have $\Lambda(\Theta) = RN(\Theta)$.

We note $RN(\Theta) \subseteq R\Pi(\Theta)$ and $\Lambda(\Theta) \subseteq R\Pi(\Theta)$.

Moreover, comparing (20.65) and (20.67) with (20.29) and (20.28), respectively, we found the following relations:

$$\boldsymbol{x} \in NE \Leftrightarrow \begin{cases} \boldsymbol{x} \in \Pi S \text{ with } \Gamma = RN(\Theta), & \text{if } RN(\Theta) \neq \emptyset, \\ \boldsymbol{x} \in NS \text{ with } \Gamma = \Lambda(\Theta), & \text{otherwise,} \end{cases} \tag{20.68}$$

where we note that we apply possible and necessary optimality concepts even when Γ is not a box set. Namely, when $RN(\Theta)$ is not empty, the necessary efficiency can be tested by the possible optimality with objective coefficient vector set $RN(\Theta)$. On the contrary, when $RN(\Theta)$ is empty, the necessary efficiency can be tested by the necessary optimality with objective coefficient vector set $\Lambda(\Theta)$. Moreover, cones $RN(\Theta)$ and $\Lambda(\Theta)$ can be replaced with bounded sets $RN(\Theta) \cap Uni$ and $\Lambda(\Theta) \cap Uni$, respectively. We may apply the techniques in single objective function case including minimax regret solution concepts to multiple objective function case if we obtain $RN(\Theta)$ and $\Lambda(\Theta)$.

When $\tilde{c}_{kj}$, $k = 1, 2, \ldots, p$, $j = 1, 2, \ldots, n$ degenerate to intervals $[c_{kj}^{\mathrm{L}}, c_{kj}^{\mathrm{R}}]$, $k = 1, 2, \ldots, p$, $j = 1, 2, \ldots, n$, possibility efficient solutions and necessarily efficient solutions are illustrated in the following example.

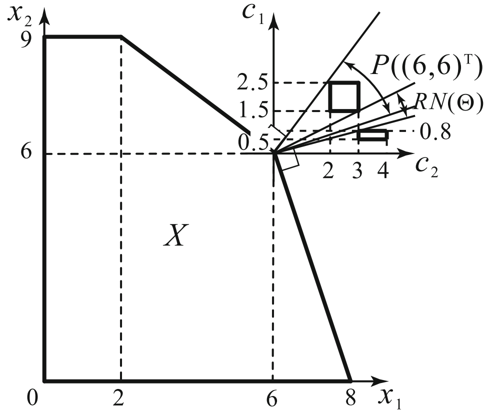


Fig. 20.6 An example of a necessarily efficient solution

Example 4. Let us consider the following LP problem with multiple interval objective functions (Inuiguchi and Sakawa [34]):

$$\begin{aligned} &\text{maximize } ([2,3]x_1 + [1.5,2.5]x_2, [3,4]x_1 + [0.5,0.8]x_2)^{\mathrm{T}}, \\ &\text{subject to } 3x_1 + 4x_2 \le 42, \\ &\qquad 3x_1 + x_2 \le 24, \\ &\qquad x_1 \ge 0,\ 0 \le x_2 \le 9. \end{aligned}$$

To this problem, from Fig. 20.6, we obtain

$$RN(\Theta) = \{\boldsymbol{c} \mid \boldsymbol{c} = r_1(3, 0.8) + r_2(3, 1.5),\ r_1 > 0,\ r_2 > 0\},$$

while $\Lambda(\Theta) = \emptyset$. Consider a solution $\boldsymbol{x} = (x_1, x_2)^{\mathrm{T}} = (6,6)^{\mathrm{T}}$. The normal cone of the feasible region at $(6,6)^{\mathrm{T}}$ is obtained as

$$P\left((6,6)^{\mathrm{T}}\right) = \{\boldsymbol{c} \mid \boldsymbol{c} = r_1(2, 2.5) + r_2(3, 1),\ r_1 \ge 0,\ r_2 \ge 0\}.$$

We obtain $RN(\Theta) \cap P((6,6)^{\mathrm{T}}) \neq \emptyset$. From Theorem 4, this implies that $(6,6)^{\mathrm{T}}$ is a necessarily efficient solution. Moreover, any solution $(x_1, x_2)^{\mathrm{T}}$ on the line segment from $(6,6)^{\mathrm{T}}$ to $(8,0)^{\mathrm{T}}$ includes $\{k(3, 0.8) \mid k \ge 0\} \subset RN(\Theta)$ in its normal cone $P((x_1, x_2)^{\mathrm{T}})$, and therefore, it is also a necessarily efficient solution. Thus there are many necessarily efficient solutions.

On the other hand, we obtain

$$R\Pi(\Theta) = \{\boldsymbol{c} \mid \boldsymbol{c} = r_1(2, 2.5) + r_2(4, 0.5),\ r_1 > 0,\ r_2 > 0\},$$

and $R\Pi(\Theta) \cap P((6,6)^{\mathrm{T}}) \neq \emptyset$. Thus, $(6,6)^{\mathrm{T}}$ is also a possibly efficient solution. Moreover solutions on the line segment from $(6,6)^{\mathrm{T}}$ to $(8,0)^{\mathrm{T}}$ are all possibly efficient solutions because we have $R\Pi(\Theta) \cap P((x_1, x_2)^{\mathrm{T}}) \neq \emptyset$. There are no other possibly efficient solutions because other feasible solutions $(x_1, x_2)^{\mathrm{T}}$ satisfy $R\Pi(\Theta) \cap P((x_1, x_2)^{\mathrm{T}}) = \emptyset$. Thus, in this example, we have $\Pi E = NE$.

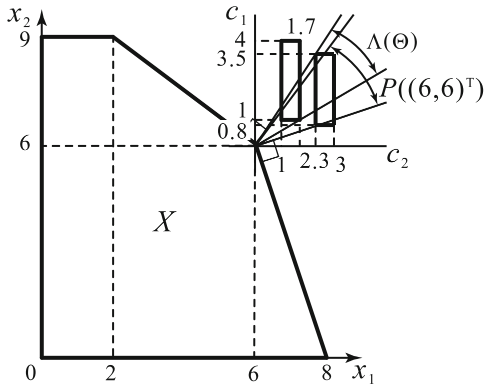


Fig. 20.7 An example of a non-necessarily efficient solution

Next, let us consider the following LP problem with multiple interval objective functions:

$$\begin{aligned}&\text{maximize } ([1,1.7]x_1 + [1,4]x_2, [2.3,3]x_1 + [0.8,3.5]x_2)^{\mathrm{T}},\\ &\text{subject to } 3x_1 + 4x_2 \le 42,\\ &\qquad 3x_1 + x_2 \le 24,\\ &\qquad x_1 \ge 0,\ 0 \le x_2 \le 9.\end{aligned}$$

For this problem, we obtain $RN(\Theta) = \emptyset$ while

$$\Lambda(\Theta) = \{\boldsymbol{c} \mid \boldsymbol{c} = r_1(2,1) + r_2(2.3,3.5),\ r_1 \ge 0,\ r_2 \ge 0\}.$$

Because the constraints are same as the previous problem, the normal cone of the feasible region at $(6,6)^{\mathrm{T}}$ is same as $P((6,6)^{\mathrm{T}})$. As shown in Fig. 20.7, we have $\Lambda(\Theta) \not\subseteq P((6,6)^{\mathrm{T}})$. Then $(6,6)^{\mathrm{T}}$ is not a necessarily efficient solution. However, as shown in Fig. 20.7, we have $\Lambda(\Theta) \cap P((6,6)^{\mathrm{T}}) \neq \emptyset$ and this implies $R\Pi(\Theta) \cap P((6,6)^{\mathrm{T}}) \neq \emptyset$. Namely, $(6,6)^{\mathrm{T}}$ is a possibly optimal solution. In this case, we obtain

$$R\Pi(\Theta) = \{\boldsymbol{c} \mid \boldsymbol{c} = r_1(1,4) + r_2(3,0.8),\ r_1 > 0,\ r_2 > 0\}.$$

Then solutions (x_1, x_2) on the polygon passing $(2,9)^{\mathrm{T}}$, $(6,6)^{\mathrm{T}}$ and $(8,0)^{\mathrm{T}}$ are all possibly optimal solutions because they satisfy $R\Pi(\Theta) \cap P((x_1,x_2)^{\mathrm{T}}) \neq \emptyset$. No other solutions are possibly optimal.

Finally, let us consider the following LP problem with multiple interval objective functions:

$$\begin{aligned}&\text{maximize } ([2.5,3.5]x_1 + [-1,0.5]x_2, [-2,-1]x_1 + [-0.5,1]x_2)^{\mathrm{T}},\\ &\text{subject to } 3x_1 + 4x_2 \le 42,\\ &\qquad 3x_1 + x_2 \le 24,\\ &\qquad x_1 \ge 0,\ 0 \le x_2 \le 9.\end{aligned}$$

For this problem, we obtain $RN(\Theta) = \emptyset$ and $\Lambda(\Theta) = \mathbf{R}^2$. Because $P((x_1, x_2)^{\mathrm{T}}) \subset \mathbf{R}^2$ for any $(x_1, x_2)^{\mathrm{T}} \in X$, there is no necessarily optimal solution. Moreover, $R\Pi(\Theta) = \mathbf{R}^2$ and thus, all feasible solutions are possibly efficient.

The possibly efficient solution set ΠE and the necessarily efficient solution set NE are extended to the case where Θ is fuzzy set. Namely, they are defined by the following membership functions:

$$\mu_{\Pi E}(\boldsymbol{x}) = \sup_{C:\boldsymbol{x} \in E(C)} \mu_{\Theta}(C), \tag{20.69}$$

$$\mu_{NE}(\boldsymbol{x}) = \inf_{C:\boldsymbol{x} \notin E(C)} 1 - \mu_{\Theta}(C). \tag{20.70}$$

Similar to possibly and necessarily optimal solution sets, we have

$$\mu_{\Pi E}(\boldsymbol{x}) \geq h \Leftrightarrow \exists C \in [\Theta]_h,\ \boldsymbol{x} \in E(C), \tag{20.71}$$

$$\mu_{NE}(\boldsymbol{x}) \geq h \Leftrightarrow \forall C \in (\Theta)_{1-h},\ \boldsymbol{x} \in E(C), \tag{20.72}$$

where $[\Theta]_h$ and $(\Theta)_{1-h}$ are h-level set and strong $(1-h)$-level set of Θ. From those we have

$$[\Pi E]_h = \bigcup \{E(C) \mid C \in [\Theta]_h\}, \tag{20.73}$$

$$[NE]_h = \bigcap \{E(C) \mid C \in (\Theta)_{1-h}\}. \tag{20.74}$$

Therefore, the h-level sets of possibly and necessarily efficient solution sets with fuzzy objective function coefficients are treated almost in the same way as possibly and necessarily efficient solution sets with interval objective function coefficients.

The examples of possibly and necessarily efficient solutions in fuzzy coefficient case can be found in Inuiguchi and Sakawa [34].

Remark 2. By taking a positively weighted sum of objective functions of Problem (20.2), we obtain an LP problem with a single objective function. To this single objective LP problem, we obtain possibly and necessarily optimal solution sets. Let $\Pi S(\boldsymbol{w})$ and $NS(\boldsymbol{w})$ be possibly and necessarily optimal solution sets of the single objective LP problem with weight vector $\boldsymbol{w}$, respectively. We have the following relations to possibly and necessarily efficient solution sets of Problem (20.2) (see Inuiguchi [27]):

$$\Pi E = \bigcup_{\boldsymbol{w} > \boldsymbol{0}} \Pi S(\boldsymbol{w}), \quad NE \supseteq \bigcup_{\boldsymbol{w} > \boldsymbol{0}} NS(\boldsymbol{w}). \tag{20.75}$$

Remark 3. Luhandjula [64] and Sakawa and Yano [81, 82] earlier defined similar but different optimal and efficient solutions to Problems (20.13) and (20.2), respectively. Those pioneering definitions are based on the inequality relations between objective function values of solutions. However the interactions between objective function values are discarded. The omission of the interaction between fuzzy objective function values are not always reasonable as shown by Inuiguchi [26]. On the other hand, Inuiguchi and Kume [29] proposed several extensions of efficient solutions based on the extended dominance relations between solutions. They showed the relations of the proposed extensions of efficient solutions including possibly and necessarily efficient solutions.

20.4.2 Efficiency Test and Possible Efficiency Test

In this subsection, we describe a method to confirm the possible and necessary efficiency of a given feasible solution. To confirm this, we solve mathematical programming problems called Possible and necessary efficiency test problems. The test problems are often investigated for given basic feasible solutions while Inuiguchi and Sakawa [34] investigated the possible efficiency test problem of any feasible solution.

First let us consider a basic feasible solution $\boldsymbol{x}^0 \in X$. Let C_{B} and C_{N} be the submatrices of objective function coefficient matrix C corresponding to the basic matrix B and the non-basic matrix N which are submatrices of $A = (\boldsymbol{a}_1\ \boldsymbol{a}_2\ \dots\ \boldsymbol{a}_m)^{\mathrm{T}}$. We define a vector function $V : \mathbf{R}^{p\times n} \to \mathbf{R}^{p\times(n-m)}$ by

$$V(C) = V((C_{\mathrm{B}}\ C_{\mathrm{N}})) = C_{\mathrm{N}} - C_{\mathrm{B}}B^{-1}N. \tag{20.76}$$

Let J_{B} and J_{N} be the index sets of basic and non-basic variables, respectively, i.e., $J_{\mathrm{B}} = \{j \mid x_j \text{ is a basic variable}\}$ and $J_{\mathrm{N}} = \{j \mid x_j \text{ is a non-basic variable}\}$. A solution $\boldsymbol{s}$ satisfying the following system of linear inequalities shows an improvement direction of objective function without violation of constraints from $\boldsymbol{x}^0$:

$$\begin{aligned}
&V(C)\boldsymbol{s} \geq \boldsymbol{0},\ V(C)\boldsymbol{s} \neq \boldsymbol{0},\\
&B_{i\cdot}^{-1}N\boldsymbol{s} \leq \boldsymbol{0},\ i \in D = \{i \mid x_i^0 = 0,\ i \in J_{\mathrm{B}}\},\\
&\boldsymbol{s} \geq \boldsymbol{0},
\end{aligned} \tag{20.77}$$

where we note that D is an index set of basic variables which degenerate at $\boldsymbol{x}^0$. Then D is empty if $\boldsymbol{x}^0$ is nondegenerate. Then the necessary and sufficient condition that $\boldsymbol{x}^0$ is efficient solution with respect to objective function coefficient matrix C is given by the inconsistency of (20.77) (see Evans and Steuer [14]).

Using Tucker' theorem of alternatives [70], the inconsistency of (20.77) is equivalent to the consistency of

$$\begin{aligned}
&V(C)^{\mathrm{T}}\boldsymbol{t}_0 - \sum_{i\in D} N^{\mathrm{T}}(B_{i\cdot}^{-1})^{\mathrm{T}} t_{1i} \leq 0,\\
&\boldsymbol{t}_0 > \boldsymbol{0},\ t_{1i} \geq 0,\ i \in D,
\end{aligned} \tag{20.78}$$

or equivalently,

$$
\begin{aligned}
& C_{\mathrm{N}}^{\mathrm{T}}\boldsymbol{t}_0 - N^{\mathrm{T}}B^{-\mathrm{T}}C_{\mathrm{B}}^{\mathrm{T}}\boldsymbol{t}_0 - \sum_{i\in D} N^{\mathrm{T}}(B_{i\cdot}^{-1})^{\mathrm{T}}t_{1i} \leq 0, \\
& \boldsymbol{t}_0 \geq \mathbf{1},\ t_{1i} \geq 0,\ i \in D,
\end{aligned}
\tag{20.79}
$$

where $\mathbf{1} = (1, 1, \ldots, 1)^{\mathrm{T}}$.

The necessary and sufficient conditions described above are applicable to basic solutions. Now let us consider a feasible solution $\boldsymbol{x}^0$ which is not always a basic solution. Because an efficient solution is a proper efficient solution [16], an optimal solution to an LP problem with objective function $\boldsymbol{u}_2^{\mathrm{T}}C\boldsymbol{x}$ for some $\boldsymbol{u}_2 > \mathbf{0}$ is an efficient solution of Problem (20.45) and vice versa. Then, the necessary and sufficient condition that $\boldsymbol{x}^0$ is efficient solution with respect to objective function coefficient matrix C is given by the consistency of the following system of linear inequalities [34]:

$$
A^{\mathrm{T}}\boldsymbol{u}_0 - \boldsymbol{u}_1 = C^{\mathrm{T}}\boldsymbol{u}_2,\ \boldsymbol{x}^{0\mathrm{T}}\boldsymbol{u}_1 = 0,\ \boldsymbol{u}_1 \geq \mathbf{0},\ \boldsymbol{u}_2 \geq \mathbf{1}. \tag{20.80}
$$

In Problem (20.2), the objective function coefficient matrix is not clearly given by a matrix but by a set of matrices, $\Theta = \{C \mid L \leq C \leq U\}$. The necessary and sufficient condition that $\boldsymbol{x}^0$ is possibly efficient solution to Problem (20.2) is given by the consistency of the following system of linear inequalities [34]:

$$
L^{\mathrm{T}}\boldsymbol{u}_2 \leq A^{\mathrm{T}}\boldsymbol{u}_0 - \boldsymbol{u}_1 \leq U^{\mathrm{T}}\boldsymbol{u}_2,\ \boldsymbol{x}^{0\mathrm{T}}\boldsymbol{u}_1 = 0,\ \boldsymbol{u}_1 \geq \mathbf{0},\ \boldsymbol{u}_2 \geq \mathbf{1}. \tag{20.81}
$$

Moreover, if $\boldsymbol{x}^0$ is a basic solution, the necessary and sufficient condition that $\boldsymbol{x}^0$ is a possibly efficient solution to Problem (20.2) is given also by the consistency of the following system of linear inequalities [28]:

$$
\begin{aligned}
& L_{\mathrm{N}}^{\mathrm{T}}\boldsymbol{t}_0 - N^{\mathrm{T}}B^{-\mathrm{T}}\boldsymbol{t}_2 - \sum_{i\in D} N^{\mathrm{T}}(B_{i\cdot}^{-1})^{\mathrm{T}}t_{1i} \leq 0, \\
& L_{\mathrm{B}}\boldsymbol{t}_0 \leq \boldsymbol{t}_2 \leq U_{\mathrm{B}}\boldsymbol{t}_0,\ \boldsymbol{t}_0 \geq \mathbf{1},\ t_{1i} \geq 0,\ i \in D.
\end{aligned}
\tag{20.82}
$$

Inuiguchi and Kume [28] showed that, for $i \in D$, the i-th row of C_{B} can be fixed at the i-th row of L_{B} as we fixed C_{N} at L_{N} by the consideration of (20.77).

As shown above the possible efficiency of a given feasible solution can be checked easily by the consistency of a system of linear inequalities.

Let us consider a case where the (k, j)-component c_{kj} of C is given by L-R fuzzy number $\tilde{c}_{kj} = (c_{kj}^{\mathrm{L}}, c_{kj}^{\mathrm{R}}, \gamma_{kj}^{\mathrm{L}}, \gamma_{kj}^{\mathrm{R}})_{L_{kj}R_{kj}}$. Define matrices with parameter h, $\Delta^{\mathrm{L}}(h) = (\gamma_{kj}^{\mathrm{L}}L_{kj}^{(-1)}(h))$ and $\Delta^{\mathrm{R}}(h) = (\gamma_{kj}^{\mathrm{R}}R_{kj}^{(-1)}(h))$. Then $[\Theta]_h$ is obtained by

$$
[\Theta]_h = \{C \mid L - \Delta^{\mathrm{L}}(h) \leq C \leq U + \Delta^{\mathrm{R}}(h)\}. \tag{20.83}
$$

Then, from (20.81), the degree of possible optimality of a given feasible solution $\boldsymbol{x}^0$ is obtained as (see Inuiguchi and Sakawa [32])

$$\begin{aligned}\mu_{\Pi E}(\boldsymbol{x}^0) = \sup\{h \in [0,1] \mid \exists \boldsymbol{u}_0, \exists \boldsymbol{u}_1 \geq \boldsymbol{0}, \exists \boldsymbol{u}_2 \geq \boldsymbol{1};\ {\boldsymbol{x}^0}^{\mathrm{T}}\boldsymbol{u}_1 = 0,\\ (L - \Delta^{\mathrm{L}}(h))^{\mathrm{T}}\boldsymbol{u}_2 \leq A^{\mathrm{T}}\boldsymbol{u}_0 - \boldsymbol{u}_1 \leq (U + \Delta^{\mathrm{U}}(h))^{\mathrm{T}}\boldsymbol{u}_2\}\end{aligned} \tag{20.84}$$

For a fixed h, the conditions in the set of the right-hand side in (20.84) become a system of linear inequalities. Then the supremum can be obtained approximately by a bisection method of $h \in [0,1]$ and LP for finding a solution satisfying the system of linear inequalities.

Moreover, when $\boldsymbol{x}^0$ is a basic solution, from (20.82), we obtain

$$\begin{aligned}\mu_{\Pi E}(\boldsymbol{x}^0) = \sup\{h \in [0,1] \mid \exists \boldsymbol{t}_0 \geq \boldsymbol{1}, \exists t_{1i} \geq 0,\ i \in D, \exists \boldsymbol{t}_2;\\ (L_{\mathrm{N}}^{\mathrm{T}} - \Delta_{\mathrm{N}}^{\mathrm{L}}(h))\boldsymbol{t}_0 - N^{\mathrm{T}}B^{-\mathrm{T}}\boldsymbol{t}_2 - \sum_{i \in D} N^{\mathrm{T}}(B_{i\cdot}^{-1})^{\mathrm{T}}t_{1i} \leq 0,\\ (L_{\mathrm{B}}^{\mathrm{T}} - \Delta_{\mathrm{B}}^{\mathrm{L}}(h))\boldsymbol{t}_0 \leq \boldsymbol{t}_2 \leq (U_{\mathrm{B}}^{\mathrm{T}} + \Delta_{\mathrm{B}}^{\mathrm{R}}(h))\boldsymbol{t}_0\},\end{aligned} \tag{20.85}$$

where $\Delta_{\mathrm{B}}^{\mathrm{L}}$ and $\Delta_{\mathrm{B}}^{\mathrm{R}}$ are submatrices of Δ^{L} and Δ^{R} corresponding to basic variables while $\Delta_{\mathrm{N}}^{\mathrm{L}}$ and $\Delta_{\mathrm{N}}^{\mathrm{R}}$ are submatrices of Δ^{L} and Δ^{R} corresponding to non-basic variables. Similar to (20.84), for a fixed h, the conditions in the set of the right-hand side in (20.85) become a system of linear inequalities. Then the supremum can be obtained approximately by a bisection method of $h \in [0,1]$ and LP for finding a solution satisfying the system of linear inequalities.

As shown above, even in fuzzy coefficient case, the possible efficiency degree of a given feasible solution can be calculated rather easily by a bisection method and an LP technique.

20.4.3 Necessary Efficiency Test

The necessary efficiency test is much more difficult than the possible efficiency test. Bitran [4] proposed an enumeration procedure for the necessary efficiency test of a non-degenerate basic solution when Θ is a crisp set. In this paper, we describe the implicit enumeration algorithm for the necessary efficiency test of a basic solution based on (20.77) when Θ is a crisp set. The difference from the Bitran's approach is only that we have additional constraints $B_{i\cdot}^{-1}N\boldsymbol{s} \leq \boldsymbol{0}$, $i \in D = \{i \mid x_i^0 = 0, i \in J_{\mathrm{B}}\}$.

Because the necessary and sufficient condition for a basic feasible solution $\boldsymbol{x}^0$ to be an efficient solution with respect to objective coefficient matrix C is given as the inconsistency of (20.77), from Proposition 1, we check the inconsistency of (20.77) for all $C \in \Phi \subseteq \Theta$. To do this, we consider the following non-linear programming problem:

$$
\begin{aligned}
&\text{maximize } \mathbf{1}^{\mathrm{T}}\boldsymbol{y},\\
&\text{subject to } (C_{\mathrm{N}} - C_{\mathrm{B}}B^{-1}N)\boldsymbol{s} - \boldsymbol{y} = \mathbf{0},\\
&\qquad B_{i\cdot}^{-1}N\boldsymbol{s} \le \mathbf{0},\ i \in D = \{i \mid x_i^0 = 0, i \in J_{\mathrm{B}}\},\\
&\qquad C_{\mathrm{N}} \in \{L_{\mathrm{N}},\ U_{\mathrm{N}}\},\ C_{\mathrm{B}} \in \{L_{\mathrm{B}},\ U_{\mathrm{B}}\},\\
&\qquad \boldsymbol{s} \ge \mathbf{0},\ \boldsymbol{y} \ge \mathbf{0}.
\end{aligned}
\tag{20.86}
$$

If the optimal value of Problem (20.86) is zero, the given basic solution $\boldsymbol{x}^0$ is a necessarily optimal solution. Otherwise, $\boldsymbol{x}^0$ is not a necessarily optimal solution.

We obtain the following Proposition.

Proposition 2. *In Problem (20.86), there is always an optimal solution* $(C_{\mathrm{B}}^*, C_{\mathrm{N}}^*, \boldsymbol{s}^*, \boldsymbol{y}^*)$ *with* $C_{\mathrm{N}}^* = U_{\mathrm{N}}$ *and* $C_{\mathrm{B}\cdot i}^* = U_{\mathrm{B}\cdot i}$, $i \in D$.

Proof. It is trivial from $\boldsymbol{s} \ge \mathbf{0}$ and $B_{i\cdot}^{-1}N\boldsymbol{s} \le \mathbf{0}$, $i \in D$. □

From Proposition 2, some part of $C = (C_{\mathrm{B}}\ \ C_{\mathrm{N}})$ can be fixed to solve Problem (20.86). For each non-basic variable x_j, let $C_{\mathrm{B}}(j)$ be the $p \times n$ matrix with columns $C_{\mathrm{B}}(j)_{\cdot k}$ defined by

$$
C_{\mathrm{B}}(j)_{\cdot k} = \begin{cases} L_{\mathrm{B}\cdot k} \text{ if } k \notin D \text{ and } B_{k\cdot}^{-1}N_{\cdot j} \ge 0, \\ U_{\mathrm{B}\cdot k} \text{ if } k \in D \text{ or } B_{k\cdot}^{-1}N_{\cdot j} < 0, \end{cases} \quad k \in J_{\mathrm{B}}.
\tag{20.87}
$$

We obviously have $C_{\mathrm{B}}(j)B^{-1}N_{\cdot j}\boldsymbol{s} \le C_{\mathrm{B}}B^{-1}N_{\cdot j}\boldsymbol{s}$ for $L_{\mathrm{B}} \le C_{\mathrm{B}} \le U_{\mathrm{B}}$ and $\boldsymbol{s} \ge \mathbf{0}$. We have the following proposition.

Proposition 3. *If Problem (20.86) has a feasible solution* $(C_{\mathrm{B}}^*, C_{\mathrm{N}}^*, \boldsymbol{s}^*, \boldsymbol{y}^*)$ *such that* $\mathbf{1}^{\mathrm{T}}\boldsymbol{s}^* > 0$ *then the following problem with an arbitrary index set* $M_1 \subseteq J_{\mathrm{B}} \setminus D$ *has a feasible solution* $\mathbf{1}^{\mathrm{T}}\boldsymbol{s} > 0$.

$$
\begin{aligned}
&\textit{maximize } \mathbf{1}^{\mathrm{T}}\boldsymbol{y},\\
&\textit{subject to } \sum_{j \in J_{\mathrm{N}}} \Bigg(U_{\mathrm{N}\cdot j} - \sum_{k \in D} U_{\mathrm{B}\cdot k}B_{k\cdot}^{-1}N_{\cdot j} - \sum_{k \in M_1} C_{\mathrm{B}\cdot k}B_{k\cdot}^{-1}N_{\cdot j}\\
&\qquad\qquad - \sum_{k \in J_{\mathrm{B}} \setminus (M_1 \cup D)} C_{\mathrm{B}}(j)_{\cdot k}B_{k\cdot}^{-1}N_{\cdot j} \Bigg) s_j - \boldsymbol{y} = \mathbf{0},\\
&\qquad B_{i\cdot}^{-1}N_{\cdot j}s_j \le 0,\ j \in J_{\mathrm{N}},\ i \in D,\\
&\qquad C_{\mathrm{B}\cdot k} \in \{L_{\mathrm{B}\cdot k},\ U_{\mathrm{B}\cdot k}\},\ k = 1, 2, \ldots, m_1 \textit{ such that } k \notin D,\\
&\qquad \boldsymbol{y} \ge \mathbf{0},\ s_j \ge 0,\ j \in J_{\mathrm{N}}.
\end{aligned}
\tag{20.88}
$$

Proof. We have $U_{\mathrm{N}\cdot j} \ge C_{\mathrm{N}\cdot j}$, $j \in J_{\mathrm{N}}$ and $C_{\mathrm{B}}(j)_{\cdot k}B_{\cdot k}^{-1}N_{\cdot j} \le C_{\mathrm{B}\cdot k}B_{\cdot k}^{-1}N_{\cdot j}$, $j \in J_{\mathrm{N}}$, $C_{\mathrm{B}\cdot k} \in \{L_{\mathrm{B}\cdot k},\ U_{\mathrm{B}\cdot k}\}$. Moreover, for $s_j \ge 0$ such that $B_{i\cdot}^{-1}N_{\cdot j}s_j \le 0$, $j \in J_{\mathrm{N}}$, $i \in D$, we have $U_{\mathrm{B}\cdot k}B_{\cdot k}^{-1}N_{\cdot j}s_j \le C_{\mathrm{B}\cdot k}B_{\cdot k}^{-1}N_{\cdot j}s_j$. Then the constraints of Problem (20.88) is a relaxation of those of Problem (20.86). Hence, we obtain this proposition. □

This proposition enables us to apply an implicit enumeration algorithm. We explain the procedure following Bitran's explanation [4]. However, the description in this paper is different from Bitran's because Bitran proposed the method when the basic solution is not degenerate.

Let $w = |J_{\mathrm{B}} \setminus D|$ and $J_{\mathrm{B}} \setminus D = \{k_1, k_2, \ldots, k_w\}$. If $w = 0$, the necessary efficiency can be checked by solving Problem (20.88) with $M_1 = \emptyset$. Then, we assume $w \neq 0$ in what follows. We consider $M_1 = \{k_1, k_2, \ldots, k_{m_1}\}$ with $m_1 \leq k_w$. For convenience, let $P(\boldsymbol{x}^0, m_1 = 0)$ be Problem (20.88) with $M_1 = \emptyset$. Then the implicit enumeration algorithm is described as follows.

Implicit Enumeration Algorithm [4]

Start by solving $P(\boldsymbol{x}^0, m_1 = 0)$. If the optimal value is zero, terminate the algorithm and $\boldsymbol{x}^0$ is necessarily efficient. Otherwise, let $m_1 = 1$ and generate the following two problems:

$$\begin{aligned}
&P(\boldsymbol{x}^0, m_1 = 1, 1) : \text{maximize } \ \mathbf{1}^{\mathrm{T}}\boldsymbol{y},\\
&\text{subject to} \sum_{j\in J_{\mathrm{N}}} \Bigg(U_{\mathrm{N}\cdot j} - \sum_{k\in D} U_{\mathrm{B}\cdot k} B^{-1}_{k\cdot} N_{\cdot j} - U_{\mathrm{B}\cdot k_1} B^{-1}_{k_1\cdot} N_{\cdot j}\\
&\qquad\qquad - \sum_{l=2}^{w} C_{\mathrm{B}}(j)_{\cdot k_l} B^{-1}_{k_l\cdot} N_{\cdot j} \Bigg) s_j - \boldsymbol{y} = \mathbf{0},\\
&\qquad B^{-1}_{i\cdot} N_{\cdot j} s_j \leq 0,\ j \in J_{\mathrm{N}},\ i \in D,\\
&\qquad \boldsymbol{y} \geq \mathbf{0},\ s_j \geq 0,\ j \in J_{\mathrm{N}}.
\end{aligned}$$

and

$$\begin{aligned}
&P(\boldsymbol{x}^0, m_1 = 1, 0) : \text{maximize } \ \mathbf{1}^{\mathrm{T}}\boldsymbol{y},\\
&\text{subject to} \sum_{j\in J_{\mathrm{N}}} \Bigg(U_{\mathrm{N}\cdot j} - \sum_{k\in D} U_{\mathrm{B}\cdot k} B^{-1}_{k\cdot} N_{\cdot j} - L_{\mathrm{B}\cdot k_1} B^{-1}_{k_1\cdot} N_{\cdot j}\\
&\qquad\qquad - \sum_{l=2}^{w} C_{\mathrm{B}}(j)_{\cdot k_l} B^{-1}_{k_l\cdot} N_{\cdot j} \Bigg) s_j - \boldsymbol{y} = \mathbf{0},\\
&\qquad B^{-1}_{i\cdot} N_{\cdot j} s_j \leq 0,\ j \in J_{\mathrm{N}},\ i \in D,\\
&\qquad \boldsymbol{y} \geq \mathbf{0},\ s_j \geq 0,\ j \in J_{\mathrm{N}}.
\end{aligned}$$

Where in the notation $P(\boldsymbol{x}^0, m_1 = 1, z)$, $z = 1$ ($z = 0$) indicates that the column, in C_{B} corresponding to $m_1 = 1$ has all its elements at the upper (lower) bound. If the optimal value of $P(\boldsymbol{x}^0, m_1 = 1, 1)$ is zero, by Proposition 3, there is no optimal matrix C_{B} in Problem (20.86), with $\mathbf{1}^{\mathrm{T}}\boldsymbol{y} > 0$ and having $C_{\mathrm{B}\cdot k_1} = U_{\mathrm{B}\cdot k_1}$. In this case we do not need to consider any descendent of $P(\boldsymbol{x}^0, m_1 = 1, 1)$ and the branch is fathomed. If the optimal value of $P(\boldsymbol{x}^0, m_1 = 1, 1)$ is positive we generate two new problems $P(\boldsymbol{x}^0, m_1 = 2, 1, 1)$ and $P(\boldsymbol{x}^0, m_1 = 2, 1, 0)$. These two problems

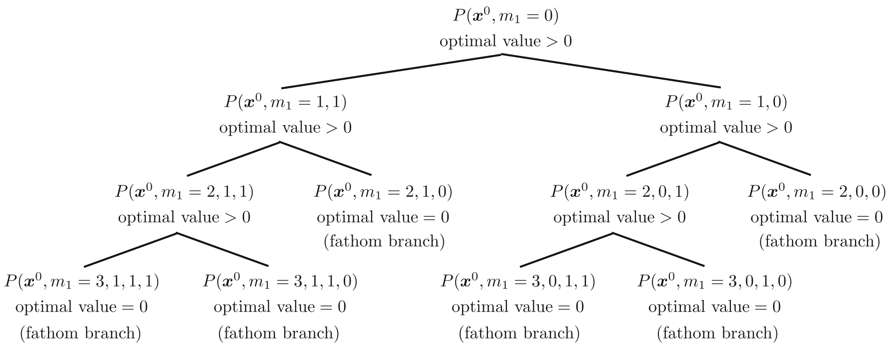


Fig. 20.8 Example of a tree generated by the implicit enumeration algorithm

are obtained by substituting $U_{\mathrm{B}\cdot k_2}$ and $L_{\mathrm{B}\cdot k_2}$, respectively, for $C_{\mathrm{B}}(j)_{\cdot k_2}$ in $P(\boldsymbol{x}^0, m_1 = 1, 1)$. We proceed in the same way, i.e., branching on problems with optimal value positive and fathoming those with optimal value zero until, we either conclude that $\boldsymbol{x}^0$ is necessarily efficient or obtain a C_{B} such that $L_{\mathrm{B}} \le C_{\mathrm{B}} \le U_{\mathrm{B}}$ and the optimal value of Problem (20.86) is positive. An example of a tree generated by the implicit enumeration algorithm is given in Fig. 20.8. In this figure, $P(\boldsymbol{x}^0, m_1 = 2, 0, 1, 0)$ is the problem,

$$
\begin{aligned}
P(\boldsymbol{x}^0, m_1 = 2, 0, 1, 0) :\ & \text{maximize}\ \ \mathbf{1}^{\mathrm{T}}\boldsymbol{y}, \\
\text{subject to} \sum_{j \in J_{\mathrm{N}}} \Bigg(& U_{\mathrm{N}\cdot j} - \sum_{k \in D} U_{\mathrm{B}\cdot k} B_{k\cdot}^{-1} N_{\cdot j} - L_{\mathrm{B}\cdot k_1} B_{k_1\cdot}^{-1} N_{\cdot j} \\
& - U_{\mathrm{B}\cdot k_2} B_{k_2\cdot}^{-1} N_{\cdot j} - L_{\mathrm{B}\cdot k_3} B_{k_3\cdot}^{-1} N_{\cdot j} \\
& \quad - \sum_{l=4}^{w} C_{\mathrm{B}}(j)_{\cdot k_l} B_{k_l\cdot}^{-1} N_{\cdot j} \Bigg) s_j - \boldsymbol{y} = \mathbf{0}, \\
& B_{i\cdot}^{-1} N_{\cdot j} s_j \le 0,\ j \in J_{\mathrm{N}},\ i \in D, \\
& \boldsymbol{y} \ge \mathbf{0},\ s_j \ge 0,\ j \in J_{\mathrm{N}}.
\end{aligned}
$$

The convergence of the algorithm, after solving a finite number of LP problems, follows from Proposition 3 and the fact that the number of matrices C_{B} that can possibly be enumerated is finite.

The implicit enumeration algorithm may be terminated earlier when $\boldsymbol{x}^0$ is necessarily efficient because we can fathom the branches only when the optimal value is zero. As Ida [22] pointed out, we may build the implicit enumeration algorithm which may be terminated earlier when $\boldsymbol{x}^0$ is not necessarily optimal. To this end, we define

$$
\check{C}_{\mathrm{B}}(j)_{\cdot k} = \begin{cases} L_{\mathrm{B}\cdot k} & \text{if } k \notin D \text{ and } B_{k\cdot}^{-1} N_{\cdot j} < 0, \\ U_{\mathrm{B}\cdot k} & \text{if } k \in D \text{ or } B_{k\cdot}^{-1} N_{\cdot j} \ge 0, \end{cases} \quad k \in J_{\mathrm{B}} \tag{20.89}
$$

and problem $\check{P}(\boldsymbol{x}^0, m_1 = l, z_1, z_2, \ldots, z_l)$ as the problem $P(\boldsymbol{x}^0, m_1 = l, z_1, z_2, \ldots, z_l)$ with substitution of $\check{C}_{\rm B}(j)_{\cdot k}$ for $C_{\rm B}(j)_{\cdot k}$, where $0 \le l \le w$ and $z_i \in \{0, 1\}$, $i = 1, 2, \ldots, l$. We obtain the following proposition.

Proposition 4. *If the optimal value of* $\check{P}(\boldsymbol{x}^0, m_1 = l, \bar{z}_1, \bar{z}_2, \ldots, \bar{z}_l)$ *is positive for some* $\bar{z}_i \in \{0, 1\}$*,* $i = 1, 2, \ldots, l$*, so is the optimal value of* $Q(\boldsymbol{x}^0, m_1 = l + 1, \bar{z}_1, \bar{z}_2, \ldots, \bar{z}_l, z_{l+1})$.

Proof. The proposition can be obtained easily. □

From Proposition 4, at the each node of the tree generated by the implicit enumeration, we solve $Q(\boldsymbol{x}^0, m_1 = l, \bar{z}_1, \bar{z}_2, \ldots, \bar{z}_l)$ as well as $P(\boldsymbol{x}^0, m_1 = l, \bar{z}_1, \bar{z}_2, \ldots, \bar{z}_l)$. If the optimal value of $Q(\boldsymbol{x}^0, m_1 = l, \bar{z}_1, \bar{z}_2, \ldots, \bar{z}_l)$ is positive, from the applications of Proposition 4, we know the optimal value of $Q(\boldsymbol{x}^0, m_1 = w, \bar{z}_1, \bar{z}_2, \ldots, \bar{z}_l, z_{l+1}, \ldots, z_w)$ is positive for any $z_i \in \{0, 1\}$, $i = l + 1, l + 2, \ldots, w$. This implies that (20.77) is consistent with $C \in \Phi \subseteq \Theta$ specified by $(\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_l, z_{l+1}, \ldots, z_w)$. Namely, we know that $\boldsymbol{x}^0$ is not necessarily efficient. Therefore, if the optimal value of $Q(\boldsymbol{x}^0, m_1 = l, \bar{z}_1, \bar{z}_2, \ldots, \bar{z}_l)$ is positive, we terminate the algorithm with telling that $\boldsymbol{x}^0$ is not necessarily efficient.

An example of a tree generated by this extended enumeration algorithm is shown in Fig. 20.9. While Fig. 20.8 illustrates a tree generated by the original enumeration algorithm when $\boldsymbol{x}^0$ is necessarily efficient, Fig. 20.9 illustrates a tree generated by the extended enumeration algorithm when $\boldsymbol{x}^0$ is not necessarily efficient. Even if the optimal value of $P(\boldsymbol{x}^0, m_1 = l, \bar{z}_1, \bar{z}_2, \ldots, \bar{z}_l)$ is zero, we do not terminate the

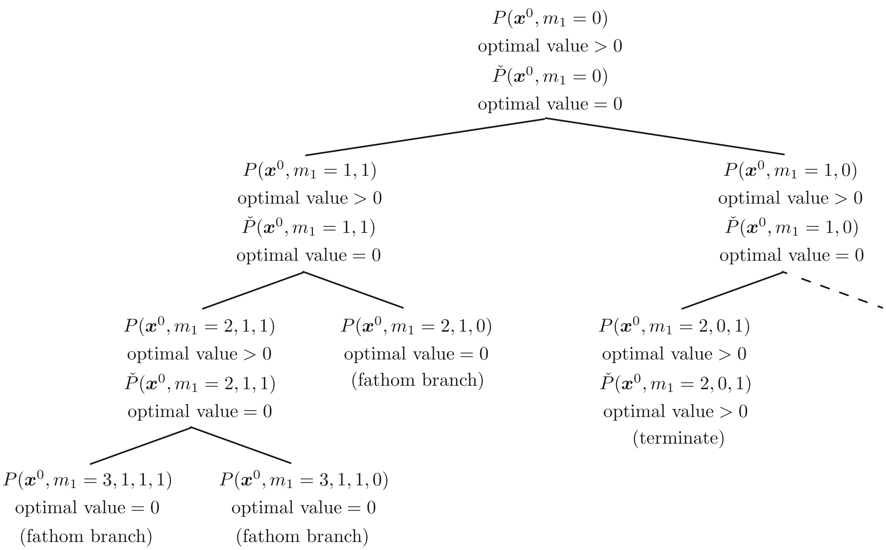


Fig. 20.9 Example of a tree generated by the extended implicit enumeration

algorithm but fathom the subproblem. On the contrary, if the optimal value of $Q(\boldsymbol{x}^0, m_1 = l, \bar{z}_1, \bar{z}_2, \ldots, \bar{z}_l)$ is positive, we know that $\boldsymbol{x}^0$ is not necessarily efficient and terminate the algorithm.

As Ida [22] proposed, we may build the implicit enumeration algorithm only with solving $Q(\boldsymbol{x}^0, m_1 = l, \bar{z}_1, \bar{z}_2, \ldots, \bar{z}_l)$. Moreover, Ida [22, 23] proposed a modification of extreme ray generation method [7] suitable for the problem. In either way, the necessary efficiency test requires a lot of computational cost. Recently, Hladík [20] showed that the necessary efficiency test problem is co-NP-complete even for the case of only one objective and Hladík [19] gives a necessary condition for necessary efficiency which can solve easily. An overview of MOLP models with interval coefficients is also done by Oliveira and Antunes [72]. The necessity efficiency test of a given non-basic feasible solution can be done based on the consistency of (20.80) for all $C \in \Phi \subseteq \Theta$. However, it is not easy to build an implicit enumeration algorithm as we described above in basic feasible solution case because we cannot easily obtain a proposition corresponding to Proposition 3. The necessity efficiency test of a given basic feasible solution in fuzzy coefficient case can be done by the introduction of a bisection method to the implicit enumeration method. However, this becomes a complex algorithm. The studies on effective methods for necessity efficiency tests in non-basic solution case as well as necessity efficiency tests in fuzzy coefficient case are a part of future topics.

20.5 Interactive Fuzzy Stochastic Multiple Objective Programming

One of the traditional tools for taking into consideration uncertainty of parameters involved in mathematical programming problems is stochastic programming [3, 9, 24], in which the coefficients in objective functions and/or constraints are represented with random variables. Stochastic programming with multiple objective functions were first introduced by Contini [8] as a goal programming approach to multiobjective stochastic programming, and further studied by Stancu-Minasian [86]. For deriving a compromise or satisficing solution for the DM in multiobjective stochastic decision making situations, an interactive programming method for multiobjective stochastic programming with Gaussian random variables were first presented by Goicoecha et al. [18] as a natural extension of the so-called STEP method [2] which is an interactive method for deterministic problems. An interactive method for multiobjective stochastic programming with discrete random variables, called STRANGE, was proposed by Teghem et al. [92] and Słowiński and Teghem [84]. The subsequent works on interactive multiobjective stochastic programming have been accumulated [57, 93, 94]. There seems to be no explicit definitions of the extended Pareto optimality concepts for multiobjective stochastic programming, until White [97] defined the Pareto optimal solutions for the expectation optimization model and the variance minimization model. More comprehensive discussions were provided by Stancu-Minasian [86] and Caballero et al. [6] through the introduction of extended Pareto optimal solution concepts for the probability maximization model and the fractile criterion optimization model. An overview of models and solution techniques for multiobjective stochastic programming problems were summarized in the context of Stancu-Minasian [87].

When decision makers formulate stochastic programming problems as representations of decision making situations, it is implicitly assumed that uncertain parameters or coefficients involved in multiobjective programming problems can be expressed as *random variables*. This means that the realized values of random parameters under the occurrence of some event are assumed to be definitely represented with real values. However, it is natural to consider that the possible realized values of these random parameters are often only ambiguously known to the experts. In this case, it may be more appropriate to interpret the experts' ambiguous understanding of the realized values of random parameters as fuzzy numbers. From such a practical point of view, this subsection introduces multiobjective linear programming problems where the coefficients of the objective function are expressed as *fuzzy random variables*.

20.5.1 Fuzzy Random Variable

A fuzzy random variable was first introduced by Kwakernaak [58], and its mathematical basis was constructed by Puri and Ralescu [73]. An overview of the developments of fuzzy random variables was found in the recent article of Gil et al. [17].

In general, fuzzy random variables can be defined in an n dimensional Euclidian space $\mathbb{R}^n$ [73]. From a practical viewpoint, as a special case of the definition by Puri and Ralescu, following the definition by Wang and Zhang [96], we present the definition of a fuzzy random variable in a single dimensional Euclidian space $\mathbb{R}$.

Definition 1 (Fuzzy Random Variable). Let $(\Omega, \mathfrak{A}, P)$ be a probability space, where Ω is a sample space, $\mathfrak{A}$ is a σ-field and P is a probability measure. Let F_N be the set of all fuzzy numbers and $\mathfrak{B}$ a Borel σ-field of $\mathbb{R}$. Then, a map $\tilde{\bar{C}} : \Omega \to F_N$ is called a fuzzy random variable if it holds that

$$\left\{(\omega, \tau) \in \Omega \times \mathbb{R} \;\middle|\; \tau \in \tilde{\bar{C}}_\alpha(\omega)\right\} \in \mathfrak{A} \times \mathfrak{B}, \;\forall \alpha \in [0, 1], \tag{20.90}$$

where $\tilde{\bar{C}}_\alpha(\omega) = \left[\tilde{\bar{C}}_\alpha^-(\omega), \tilde{\bar{C}}_\alpha^+(\omega)\right] = \left\{\tau \in \mathbb{R} \;\middle|\; \mu_{\tilde{\bar{C}}(\omega)}(\tau) \geq \alpha\right\}$ is an α-level set of the fuzzy number $\tilde{\bar{C}}(\omega)$ for $\omega \in \Omega$.

Intuitively, fuzzy random variables are considered to be random variables whose realized values are not real values but fuzzy numbers or fuzzy sets.

In Definition 1, $\tilde{\bar{C}}(\omega)$ is a fuzzy number corresponding to the realized value of fuzzy random variable $\tilde{\bar{C}}$ under the occurrence of each elementary event ω in the sample space Ω. For each elementary event ω, $\tilde{\tilde{C}}_\alpha^-(\omega)$ and $\tilde{\tilde{C}}_\alpha^+(\omega)$ are the left and right end-points of the closed interval $\left[\tilde{\tilde{C}}_\alpha^-(\omega), \tilde{\tilde{C}}_\alpha^+(\omega)\right]$ which is an α-level set of the fuzzy number $\tilde{\tilde{C}}(\omega)$ characterized by the membership function $\mu_{\tilde{\tilde{C}}(\omega)}(\tau)$. Observe that the values of $\tilde{\tilde{C}}_\alpha^-(\omega)$ and $\tilde{\tilde{C}}_\alpha^+(\omega)$ are real values which vary randomly due to the random occurrence of elementary events ω. With this observation in mind, realizing that $\tilde{\tilde{C}}_\alpha^-$ and $\tilde{\tilde{C}}_\alpha^+$ can be regarded as random variables, it is evident that fuzzy random variables can be viewed as an extension of ordinary random variables.

In general, if the sample space Ω is uncountable, positive probabilities cannot be always assigned to all the sets of events in the sample space due to the limitation that the sum of the probabilities is equal to one. Realizing such situations, it is significant to introduce the concept of σ-field which is a set of subsets of the sample space.

To understand the concept of fuzzy random variables, consider discrete fuzzy random variables. To be more specific, when a sample space Ω is countable, the discrete fuzzy random variable can be defined by setting the σ-field $\mathfrak{A}$ as the power set 2^Ω or some other smaller set, together with the probability measure P associated with the probability mass function p satisfying

$$P(A) = \sum_{\omega \in A} p(\omega), \ \forall A \in \mathfrak{A}.$$

Consider a simple example: Let a sample space be $\Omega = \{\omega_1, \omega_2, \omega_3\}$, a σ-field $\mathfrak{A} = 2^\Omega$, and a probability measure $P(A) = \sum_{\omega \in A} p(\omega)$ for all $A \in \mathfrak{A}$. Then, Fig. 20.10 illustrates a discrete fuzzy random variable where fuzzy numbers $\tilde{\tilde{C}}(\omega_1)$, $\tilde{\tilde{C}}(\omega_2)$ and $\tilde{\tilde{C}}(\omega_3)$ are randomly realized at probabilities $p(\omega_1)$, $p(\omega_2)$ and $p(\omega_3)$, respectively, satisfying $\sum_{j=1}^3 p(\omega_j) = 1$.

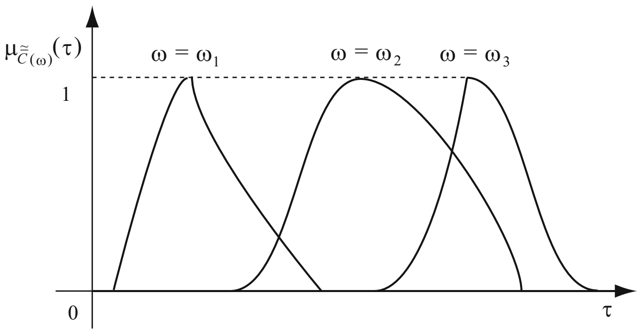


Fig. 20.10 Example of discrete fuzzy random variables

20.5.2 Brief Survey of Fuzzy Random Multiple Objective Programming

Studies on linear programming problems with fuzzy random variable coefficients, called fuzzy random linear programming problems, were initiated by Wang and Qiao [95] and Qiao et al. [74] as a so-called distribution problem of which goal is to seek the probability distribution of the optimal solution and optimal value. Optimization models of fuzzy random linear programming were first developed by Luhandjula et al. [67, 69], and further studied by Liu [60, 61] and Rommelfanger [77]. A brief survey of major fuzzy stochastic programming models including fuzzy random programming was found in the paper by Luhandjula [68].

On the basis of possibility theory, Katagiri et al. firstly introduced possibilistic programming approaches to fuzzy random linear programming problems [42, 44] where only the right-hand side of an equality constraint involves a fuzzy random variable, and considered more general cases where both sides of inequality constraints involve fuzzy random variables [45]. They also tackled the problem where the coefficients of the objective functions are fuzzy random variables [43]. Through the combination of a stochastic programming model and a possibilistic programming model, Katagiri et al. introduced a possibilistic programming approach to fuzzy random programming model [50] and proposed several multiobjective fuzzy random programming models using different optimization criteria such as possibility expectation optimization [46], possibility variance minimization [48], possibility-based probability maximization [53] and possibility-based fractile optimization [51].

Extensions to multiobjective 0-1 programming problems with fuzzy random variables were provided by incorporating the branch-and-bound method into the interactive methods [49].

Along this line, this section devotes to discussing the optimization models for multiobjective fuzzy random programming problems where each of coefficients in the objective functions are represented with fuzzy random variables.

20.5.3 Problem Formulation

Assuming that the coefficients of the objective functions are expressed as fuzzy random variables, we consider a multiobjective fuzzy random programming problem

$$\left.\begin{array}{l} \text{minimize } z_1(\boldsymbol{x}) = \tilde{\bar{\boldsymbol{C}}}_1\boldsymbol{x} \\ \qquad\qquad \cdots\cdots \\ \text{minimize } z_k(\boldsymbol{x}) = \tilde{\bar{\boldsymbol{C}}}_k\boldsymbol{x} \\ \text{subject to } A\boldsymbol{x} \leq \boldsymbol{b},\ \boldsymbol{x} \geq \boldsymbol{0}, \end{array}\right\} \tag{20.91}$$

where $\boldsymbol{x}$ is an n dimensional decision variable column vector, A is an $m \times n$ coefficient matrix, $\boldsymbol{b}$ is an m dimensional column vector and $\tilde{\tilde{\boldsymbol{C}}}_l = (\tilde{\tilde{C}}_{l1}, \ldots, \tilde{\tilde{C}}_{ln}),\ l = 1, \ldots, k$ are n dimensional coefficient row vectors of fuzzy random variables.

For notational convenience, let F denote the feasible region of (20.91), namely

$$F \triangleq \{\boldsymbol{x} \in \mathbb{R}^n \mid A\boldsymbol{x} \leq \boldsymbol{b},\ \boldsymbol{x} \geq \boldsymbol{0}\}.$$

Considering a simple but practical fuzzy random variables satisfying the conditions in Definition 1, suppose that each element $\tilde{\tilde{C}}_{lj}$ of the vector $\tilde{\tilde{\boldsymbol{C}}}_l = (\tilde{\tilde{C}}_{l1}, \ldots, \tilde{\tilde{C}}_{ln})$ is a fuzzy random variable whose realized value is a fuzzy number $\tilde{C}_{ljs_l}$ depending on a scenario $s_l \in \{1, \ldots, S_l\}$ which occurs with a probability p_{ls_l}, where $\sum_{s_l=1}^{S_l} p_{ls_l} = 1$.

The sample space is defined as $\Omega = \{1, \ldots, S_l\}$, and the corresponding σ-field is $\mathfrak{A} = 2^{\Omega}$. Unfortunately, however, if the shapes of $\tilde{C}_{ljs_l},\ s_l = 1, \ldots, S_l$ are not the same as shown in Fig. 20.10, it is quite difficult to calculate the fuzzy random variable representing the objective function involving fuzzy random variables in problem (20.91). Realizing such difficulty, Katagiri et al. [43, 46, 48, 49] considered a discrete fuzzy random variable as an extended concept of the discrete random variable. Along this line, in this section, we restrict ourselves to considering the case where the realized values $\tilde{C}_{ljs_l},\ s_l = 1, \ldots, S_l$ are triangular fuzzy numbers with the membership function defined as

$$\mu_{\tilde{C}_{ljs_l}}(\tau) = \begin{cases} \max\left\{1 - \dfrac{d_{ljs_l} - \tau}{\beta_{lj}}, 0\right\} & \text{if } \tau \leq d_{ljs_l} \\ \max\left\{1 - \dfrac{\tau - d_{ljs_l}}{\gamma_{lj}}, 0\right\} & \text{if } \tau > d_{ljs_l}, \end{cases} \tag{20.92}$$

where the value of d_{ljs_l} varies depending on which scenario $s_l \in \{1, \ldots, S_l\}$ occurs, and β_{lj} and γ_{lj} are not random parameters but constants. Figure 20.11 illustrates an example of the membership function $\mu_{\tilde{C}_{ljs_l}}(\tau)$. Formally, the membership function of the fuzzy random variable $\tilde{\tilde{C}}_{lj}$ is represented by

$$\mu_{\tilde{\tilde{C}}_{lj}}(\tau) = \begin{cases} \max\left\{1 - \dfrac{\bar{d}_{lj} - \tau}{\beta_{lj}}, 0\right\} & \text{if } \tau \leq \bar{d}_{lj} \\ \max\left\{1 - \dfrac{\tau - \bar{d}_{lj}}{\gamma_{lj}}, 0\right\} & \text{if } \tau > \bar{d}_{lj}. \end{cases} \tag{20.93}$$

Through the Zadeh's extension principle, each objective function $\tilde{\tilde{\boldsymbol{C}}}_l\boldsymbol{x}$ is represented by a single fuzzy random variable of which realized value for scenario s_l is a triangular fuzzy number $\tilde{\boldsymbol{C}}_{ls_l}\boldsymbol{x}$ characterized by the membership function

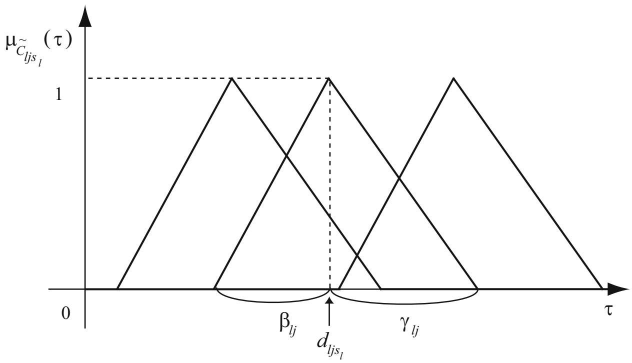
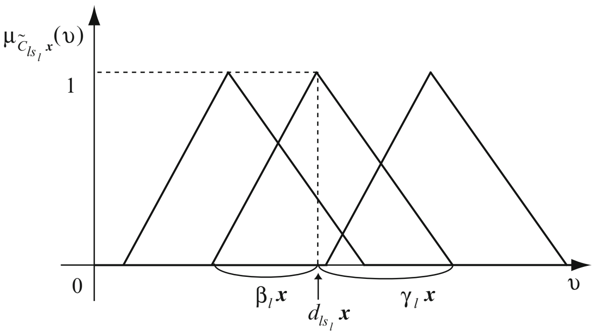


Fig. 20.11 Example of the membership function $\mu_{\tilde{C}_{ljs_l}}$

Fig. 20.12 Example of the membership function $\mu_{\tilde{\boldsymbol{C}}_{ls_l}\boldsymbol{x}}$



$$\mu_{\tilde{\boldsymbol{C}}_{ls_l}\boldsymbol{x}}(\upsilon) = \begin{cases} \max\left\{1 - \dfrac{\boldsymbol{d}_{ls_l}\boldsymbol{x} - \upsilon}{\boldsymbol{\beta}_l\boldsymbol{x}}, 0\right\} & \text{if } \upsilon \le \boldsymbol{d}_{ls_l}\boldsymbol{x} \\ \max\left\{1 - \dfrac{\upsilon - \boldsymbol{d}_{ls_l}\boldsymbol{x}}{\boldsymbol{\gamma}_l\boldsymbol{x}}, 0\right\} & \text{if } \upsilon > \boldsymbol{d}_{ls_l}\boldsymbol{x}, \end{cases} \tag{20.94}$$

where $\boldsymbol{d}_{ls_l}$ is an n dimensional column vector which is different from the other $\boldsymbol{d}_{l\hat{s}_l}$, $\hat{s}_l \in \{1,\ldots,S_l\}$, $\hat{s}_l \neq s_l$, and $\boldsymbol{\beta}_l$ and $\boldsymbol{\gamma}_l$ are n dimensional constant column vectors. Figure 20.12 illustrates an example of the membership function $\mu_{\tilde{\boldsymbol{C}}_{ls_l}\boldsymbol{x}}(\upsilon)$. Also for the lth objective function $\tilde{\tilde{\boldsymbol{C}}}_l\boldsymbol{x}$, its membership function is formally expressed as

$$\mu_{\tilde{\tilde{\boldsymbol{C}}}_l\boldsymbol{x}}(\upsilon) = \begin{cases} \max\left\{1 - \dfrac{\bar{\boldsymbol{d}}_l\boldsymbol{x} - \upsilon}{\boldsymbol{\beta}_l\boldsymbol{x}}, 0\right\} & \text{if } \upsilon \le \bar{\boldsymbol{d}}_l\boldsymbol{x} \\ \max\left\{1 - \dfrac{\upsilon - \bar{\boldsymbol{d}}_l\boldsymbol{x}}{\boldsymbol{\gamma}_l\boldsymbol{x}}, 0\right\} & \text{if } \upsilon > \bar{\boldsymbol{d}}_l\boldsymbol{x}. \end{cases} \tag{20.95}$$

Considering the imprecise nature of human judgments, it is quite natural to assume that the decision maker (DM) may have a fuzzy goal for each of the objective functions $z_l(\boldsymbol{x}) = \tilde{\bar{\boldsymbol{C}}}_l\boldsymbol{x}$, and in a minimization problem, the DM specifies the fuzzy goal such that "the objective function value should be substantially less than or equal to some value." Such a fuzzy goal can be quantified by eliciting the corresponding membership functions through some interaction process from the DM. In this subsection, for simplicity, the linear membership function expressed as the following is assumed:

$$\mu_{\tilde{G}_l}(y) = \begin{cases} 0 & \text{if } y > z_l^0 \\ \dfrac{y - z_l^0}{z_l^1 - z_l^0} & \text{if } z_l^1 \le y \le z_l^0 \\ 1 & \text{if } y < z_l^1, \end{cases} \tag{20.96}$$

where z_l^0 and z_l^1 are parameters determined by decision makers so as to represent the DM's degree of satisfaction of the objective function values

$$\left.\begin{aligned} z_l^0 &= \max_{s_l \in \{1,\ldots,S_l\}} \max_{\boldsymbol{x} \in F} \sum_{j=1}^n d_{ljs_l} x_j, \; l = 1, \ldots, k, \\ z_l^1 &= \min_{s_l \in \{1,\ldots,S_l\}} \min_{\boldsymbol{x} \in F} \sum_{j=1}^n d_{ljs_l} x_j, \; l = 1, \ldots, k. \end{aligned}\right\} \tag{20.97}$$

It should be noted here that z_l^0 and z_l^1 are obtained by solving linear programming problems. Figure 20.13 illustrates an example of the membership function $\mu_{\tilde{G}_l}$ of a fuzzy goal $\tilde{G}_l$.

Recalling that the membership function is regarded as a possibility distribution, the degree of possibility that the objective function value $\tilde{\bar{\boldsymbol{C}}}_{ls_l}\boldsymbol{x}$ for a given scenario $s_l \in \{1, \ldots, S_l\}$ attains the fuzzy goal $\tilde{G}_l$ is expressed as

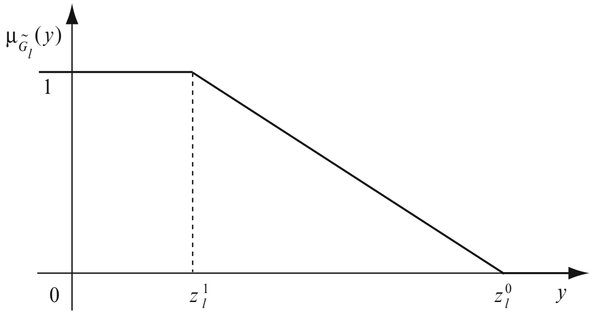


Fig. 20.13 Example of the membership function of a fuzzy goal

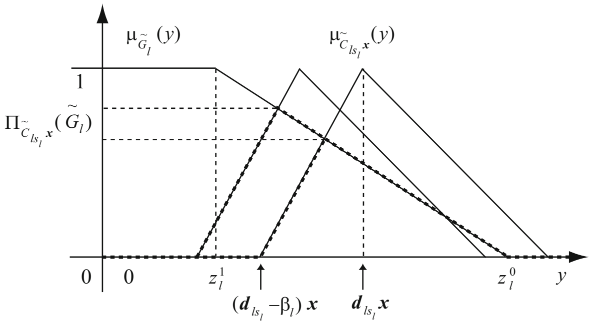


Fig. 20.14 Degree of possibility $\Pi_{\tilde{\boldsymbol{C}}_{ls_l}\boldsymbol{x}}(\tilde{G}_l)$

$$\Pi_{\tilde{\boldsymbol{C}}_{ls_l}\boldsymbol{x}}(\tilde{G}_l) = \sup_y \min\left\{\mu_{\tilde{\boldsymbol{C}}_{ls_l}\boldsymbol{x}}(y),\ \mu_{\tilde{G}_l}(y)\right\},\ l = 1,\ldots,k. \tag{20.98}$$

Figure 20.14 illustrates the degree of possibility that the fuzzy goal $\tilde{G}_l$ is fulfilled under the possibility distribution $\mu_{\tilde{C}_{ls_l}\boldsymbol{x}}$.

A possibility measure is useful to a decision maker who observes decision making situations from an optimistic point of view. However, when a decision maker is pessimistic about the situation, it is reasonable to use a necessity measure rather than a possibility measure. Then, the degree of necessity that the objective function value $\tilde{\bar{\boldsymbol{C}}}_{ls_l}\boldsymbol{x}$ for a given scenario $s_l \in \{1,\ldots,S_l\}$ attains the fuzzy goal $\tilde{G}_l$ is expressed as

$$N_{\tilde{\boldsymbol{C}}_{ls_l}\boldsymbol{x}}(\tilde{G}_l) = \inf_y \max\left\{\mu_{\tilde{\boldsymbol{C}}_{ls_l}\boldsymbol{x}}(y),\ 1-\mu_{\tilde{G}_l}(y)\right\},\ l = 1,\ldots,k. \tag{20.99}$$

Observing that the degrees of possibility vary randomly depending on which scenario occurs, it should be noted here that conventional possibilistic programming approaches cannot be directly applied to (20.91). With this observation in mind, realizing that (20.91) involves not only fuzziness but also randomness, Katagiri et al. considered fuzzy random decision making models such as possibilistic expectation model [43, 46, 49] and possibilistic variance model [48] by incorporating the possibility theory into stochastic programming models.

20.5.4 Possibilistic Expectation Model

One of the natural solution approaches to such decision making situations as discussed in the previous subsection is to maximize expectation of the degree of possibility and/or necessity. Katagiri et al. [43, 46, 49] introduced possibilistic expectation models under the assumption that a DM intends to maximize the expected degree of possibility and/or necessity that each of the original objective functions involving fuzzy random variable coefficients attains the fuzzy goals. On the basis of possibilistic expectation models the original multiobjective fuzzy random programming problem (20.91) can be reformulated as the following problem:

$$\left.\begin{array}{l}\text{maximize } E\left[\Pi_{\tilde{\tilde{\boldsymbol{C}}}_l\boldsymbol{x}}(\tilde{G}_l)\right],\ l\in\mathscr{L}_{pos}\\ \text{maximize } E\left[N_{\tilde{\tilde{\boldsymbol{C}}}_l\boldsymbol{x}}(\tilde{G}_l)\right],\ l\in\mathscr{L}_{nec}\\ \text{subject to } \boldsymbol{x}\in F,\end{array}\right\} \tag{20.100}$$

where $E[\cdot]$ denotes the expectation operator. $\mathscr{L}_{pos}$ and $\mathscr{L}_{nec}$ are index sets satisfying $\mathscr{L}_{pos}\cup\mathscr{L}_{nec}=\{1,2,\ldots,k\}$ and $\mathscr{L}_{pos}\cap\mathscr{L}_{nec}=\emptyset$.

When the triangular fuzzy random variable (20.94) and the linear fuzzy goal (20.96) are given, the degree of possibility (20.98) is explicitly represented by

$$\Pi_{\tilde{\boldsymbol{C}}_{ls_l}\boldsymbol{x}}(\tilde{G}_l)=\frac{\displaystyle\sum_{j=1}^{n}(\beta_{lj}-d_{ljs_l})x_j+z_l^0}{\displaystyle\sum_{j=1}^{n}\beta_{lj}x_j-z_l^1+z_l^0}. \tag{20.101}$$

On the other hand, the degree of necessity (20.99) is explicitly expressed as

$$N_{\tilde{\boldsymbol{C}}_{ls_l}\boldsymbol{x}}(\tilde{G}_l)=\frac{\displaystyle -\sum_{j=1}^{n}d_{ljs_l}x_j+z_l^0}{\displaystyle\sum_{j=1}^{n}\gamma_{lj}x_j-z_l^1+z_l^0}. \tag{20.102}$$

Recalling that the occurrence probability of scenario s_l is p_{ls_l}, the expectation of the degree of possibility or necessity is calculated as

$$E\left[\Pi_{\tilde{\tilde{\boldsymbol{C}}}_l\boldsymbol{x}}(\tilde{G}_l)\right]\triangleq\sum_{s_l=1}^{S_l}p_{ls_l}\Pi_{\tilde{\boldsymbol{C}}_{ls_l}\boldsymbol{x}}(\tilde{G}_l)=\frac{\displaystyle\sum_{j=1}^{n}\left(\beta_{lj}-\sum_{s_l=1}^{S_l}p_{ls_l}d_{ljs_l}\right)x_j+z_l^0}{\displaystyle\sum_{j=1}^{n}\beta_{lj}x_j-z_l^1+z_l^0}. \tag{20.103}$$

$$E\left[N_{\tilde{\tilde{\boldsymbol{C}}}_l\boldsymbol{x}}(\tilde{G}_l)\right]\triangleq\sum_{s_l=1}^{S_l}p_{ls_l}N_{\tilde{\boldsymbol{C}}_{ls_l}\boldsymbol{x}}(\tilde{G}_l)=\frac{\displaystyle -\sum_{j=1}^{n}\sum_{s_l=1}^{S_l}p_{ls_l}d_{ljs_l}x_j+z_l^0}{\displaystyle\sum_{j=1}^{n}\gamma_{lj}x_j-z_l^1+z_l^0}. \tag{20.104}$$

Let $Z_l^E(\boldsymbol{x})$ denote

$$Z_l^E(\boldsymbol{x}) = \begin{cases} E\left[\Pi_{\tilde{\tilde{\boldsymbol{C}}}_l\boldsymbol{x}}(\tilde{G}_l)\right] \text{ if } l \in \mathscr{L}_{pos} \\ E\left[N_{\tilde{\tilde{\boldsymbol{C}}}_l\boldsymbol{x}}(\tilde{G}_l)\right] \text{ if } l \in \mathscr{L}_{nec} \end{cases} \tag{20.105}$$

To calculate a candidate for the satisficing solution which is also Pareto optimal, in interactive multiobjective programming, the DM is asked to specify reference levels $\hat{z}_l^E$, $l = 1, \ldots, k$ of the objective function values of (20.105), and it is called the reference (expected possibility) levels. For the DM's reference levels $\hat{z}_l^E$, $l = 1, \ldots, k$, an Pareto optimal solution, which is the nearest to a vector of the reference levels or better than it if the reference levels are attainable in a sense of minimax, is obtained by solving the minimax problem

$$\left.\begin{array}{l} \text{minimize } \max\limits_{1 \le l \le k} \left\{\hat{z}_l^E - Z_l^E(\boldsymbol{x})\right\} \\ \text{subject to } \boldsymbol{x} \in F. \end{array}\right\} \tag{20.106}$$

Following the preceding discussion, we can now present an interactive algorithm for deriving a satisficing solution for the DM from among the Pareto optimal solution set.

20.5.4.1 Interactive Satisficing Method for the Possibilistic Expectation Model

Step 1: Determine the linear membership functions $\mu_{\tilde{G}_l}$, $l = 1, \ldots, k$ defined as (20.96) by calculating z_l^0 and z_l^1, $l = 1, \ldots, k$.

Step 2: Set the initial reference levels at 1s, which can be viewed as the ideal values, i.e., $\hat{z}_l^E = 1$, $l = 1, \ldots, k$.

Step 3: For the current reference levels $\hat{z}_l^E$, $l = 1, \ldots, k$, solve the minimax problem (20.106).

Step 4: The DM is supplied with the corresponding Pareto optimal solution $\boldsymbol{x}^*$. If the DM is satisfied with the current objective function values $Z_l^E(\boldsymbol{x}^*)$, $l = 1, \ldots, k$, then stop the algorithm. Otherwise, ask the DM to update the reference levels $\hat{z}_l^E$, $l = 1, \ldots, k$ by considering the current objective function values, and return to step 3.

Here it should be stressed for the DM that any improvement of one expectation of the degree of possibility can be achieved only at the expense of at least one of other expected possibilities or expected necessities.

20.5.5 Possibilistic Variance Model

As discussed in the previous subsection, the possibilistic expectation model would be appropriate if the DM intends to simply maximize the expected degrees of possibility without concerning about those fluctuations.

However, when the DM prefers to decrease the fluctuation of the objective function values, the possibilistic expectation model is not relevant because some scenario yielding a very low possibility of good performance may occur even with a small probability.

To avoid such risk, from the risk-averse point of view, by minimizing the variance of the degree of possibility under the constraints of feasibility together with the conditions for the expected degrees of possibility, Katagiri et al. [48] considered a possibilistic variance model for fuzzy random multiobjective programming problems. Along this line, in this section, we consider the following problem as a risk-aversion approach to the original problem (20.91):

$$\left.\begin{array}{ll}\text{minimize} & Var\left[\Pi_{\tilde{\tilde{\boldsymbol{C}}}_l\boldsymbol{x}}(\tilde{G}_l)\right],\ l\in\mathscr{L}_{pos}\\ \text{minimize} & Var\left[N_{\tilde{\tilde{\boldsymbol{C}}}_l\boldsymbol{x}}(\tilde{G}_l)\right],\ l\in\mathscr{L}_{nec}\\ \text{subject to} & E\left[\Pi_{\tilde{\tilde{\boldsymbol{C}}}_l\boldsymbol{x}}(\tilde{G}_l)\right]\geq\xi_l,\ l\in\mathscr{L}_{pos}\\ & E\left[N_{\tilde{\tilde{\boldsymbol{C}}}_l\boldsymbol{x}}(\tilde{G}_l)\right]\geq\xi_l,\ l\in\mathscr{L}_{nec}\\ & \boldsymbol{x}\in F,\end{array}\right\} \tag{20.107}$$

where $Var[\cdot]$ denotes the variance operator, and $\xi_l,\ l=1,\dots,k$ are permissible expectation levels for the expected degrees of possibility specified by the DM.

For notational convenience, let $F(\boldsymbol{\xi})$ be the feasible region of (20.107), namely

$$F(\boldsymbol{\xi})\triangleq\left\{\boldsymbol{x}\in F\ \middle|\ E\left[\Pi_{\tilde{\tilde{\boldsymbol{C}}}_l\boldsymbol{x}}(\tilde{G}_l)\right]\geq\xi_l,\ l\in\mathscr{L}_{pos},\ \ E\left[N_{\tilde{\tilde{\boldsymbol{C}}}_l\boldsymbol{x}}(\tilde{G}_l)\right]\geq\xi_l,\ l\in\mathscr{L}_{nec}\right\}.$$

Recalling (20.98) and (20.99), each of the objective functions in (20.107) is calculated as

$$Var\left[\Pi_{\tilde{\tilde{\boldsymbol{C}}}_l\boldsymbol{x}}(\tilde{G}_l)\right]=\frac{1}{\left(\displaystyle\sum_{j=1}^{n}\beta_{lj}x_j-z_l^1+z_l^0\right)^2}\,Var\left[\sum_{j=1}^{n}\bar{d}_{lj}x_j\right]$$

$$= \frac{1}{\left(\sum_{j=1}^{n} \beta_{lj} x_j - z_l^1 + z_l^0\right)^2} \boldsymbol{x}^T V_l \boldsymbol{x}, \tag{20.108}$$

$$Var\left[N_{\tilde{\bar{\boldsymbol{C}}}_l\boldsymbol{x}}(\tilde{G}_l)\right] = \frac{1}{\left(\sum_{j=1}^{n} \gamma_{lj} x_j - z_l^1 + z_l^0\right)^2} Var\left[\sum_{j=1}^{n} \bar{d}_{lj} x_j\right]$$

$$= \frac{1}{\left(\sum_{j=1}^{n} \gamma_{lj} x_j - z_l^1 + z_l^0\right)^2} \boldsymbol{x}^T V_l \boldsymbol{x}, \tag{20.109}$$

where V_l is the variance-covariance matrix of $\bar{\boldsymbol{d}}_l$ expressed by

$$V_l = \begin{bmatrix} v_{11}^l & v_{12}^l & \cdots & v_{1n}^l \\ v_{21}^l & v_{22}^l & \cdots & v_{2n}^l \\ \vdots & \vdots & \ddots & \vdots \\ v_{n1}^l & v_{n2}^l & \cdots & v_{nn}^l \end{bmatrix}, \quad l = 1, \ldots, k,$$

and

$$v_{jj}^l = Var[\bar{d}_{lj}] = \sum_{s_l=1}^{S_l} p_{ls_l} \{d_{ljs_l}\}^2 - \left\{\sum_{s_l=1}^{S_l} p_{ls_l} d_{ljs_l}\right\}^2, \ j = 1, \ldots, n,$$

$$v_{jr}^l = Cov[\bar{d}_{lj}, \bar{d}_{lr}] = E[\bar{d}_{lj}, \bar{d}_{lr}] - E[\bar{d}_{lj}]E[\bar{d}_{lr}]$$

$$= \sum_{s_l=1}^{S_l} p_{ls_l} d_{ljs_l} d_{lrs_l} - \sum_{s_l=1}^{S_l} p_{ls_l} d_{ljs_l} \sum_{s_l=1}^{S_l} p_{ls_l} d_{lrs_l}, \ j \neq r, \ r = 1, \ldots, n.$$

Furthermore, from (20.103) and (20.104), the constraint of the expected degree of possibility $E\left[\Pi_{\tilde{\bar{\boldsymbol{C}}}_l\boldsymbol{x}}(\tilde{G}_l)\right] \geq \xi_l$ and that of necessity $E\left[N_{\tilde{\bar{\boldsymbol{C}}}_l\boldsymbol{x}}(\tilde{G}_l)\right] \geq \xi_l$ are explicitly represented as

$$\sum_{j=1}^{n}\left\{\sum_{s_l=1}^{S_l} p_{ls_l} d_{ljs_l} - (1-\xi_l)\beta_{lj}\right\} x_j \le z_l^0 - \xi_l(z_l^0 - z_l^1). \tag{20.110}$$

and

$$\sum_{j=1}^{n}\left\{\sum_{s_l=1}^{S_l} p_{ls_l} d_{ljs_l} + \xi_l\gamma_{lj}\right\} x_j \le z_l^0 - \xi_l(z_l^0 - z_l^1). \tag{20.111}$$

By substituting (20.109), (20.110) and (20.111) into (20.107), (20.107) is equivalently transformed as

$$\left.\begin{array}{ll}
\text{minimize} & \dfrac{1}{\left(\displaystyle\sum_{j=1}^{n}\beta_{lj}x_j - z_l^1 + z_l^0\right)^2}\, \boldsymbol{x}^T V_l \boldsymbol{x},\ l \in \mathscr{L}_{pos} \\
\text{minimize} & \dfrac{1}{\left(\displaystyle\sum_{j=1}^{n}\gamma_{lj}x_j - z_l^1 + z_l^0\right)^2}\, \boldsymbol{x}^T V_l \boldsymbol{x},\ l \in \mathscr{L}_{nec} \\
\text{subject to} & \displaystyle\sum_{j=1}^{n}\left\{\sum_{s_l=1}^{S_l} p_{ls_l} d_{ljs_l} - (1-\xi_l)\beta_{lj}\right\} x_j \le z_l^0 - \xi_l(z_l^0 - z_l^1),\ l \in \mathscr{L}_{pos} \\
& \displaystyle\sum_{j=1}^{n}\left\{\sum_{s_l=1}^{S_l} p_{ls_l} d_{ljs_l} + \xi_l\gamma_{lj}\right\} x_j \le z_l^0 - \xi_l(z_l^0 - z_l^1),\ l \in \mathscr{L}_{nec} \\
& \boldsymbol{x} \in F.
\end{array}\right\} \tag{20.112}$$

From the fact that it holds

$$\sum_{j=1}^{n}\beta_{lj}x_j - z_l^1 + z_l^0 > 0,\ \ \sum_{j=1}^{n}\gamma_{lj}x_j - z_l^1 + z_l^0 > 0$$

and $\boldsymbol{x}^T V_l \boldsymbol{x} \ge 0$ due to the positive-semidefinite property of V_l, by computing the square root of the objective functions of (20.112), (20.112) is equivalently rewritten as

$$
\left.\begin{array}{ll}
\text{minimize} & \dfrac{\sqrt{\boldsymbol{x}^T V_l \boldsymbol{x}}}{\sum_{j=1}^{n} \beta_{lj} x_j - z_l^1 + z_l^0}, \ l \in \mathscr{L}_{pos} \\
\text{minimize} & \dfrac{\sqrt{\boldsymbol{x}^T V_l \boldsymbol{x}}}{\sum_{j=1}^{n} \gamma_{lj} x_j - z_l^1 + z_l^0}, \ l \in \mathscr{L}_{nec} \\
\text{subject to} & \sum_{j=1}^{n} \left\{ \sum_{s_l=1}^{S_l} p_{ls_l} d_{ljs_l} - (1-\xi_l)\beta_{lj} \right\} x_j \leq z_l^0 - \xi_l (z_l^0 - z_l^1), \ l \in \mathscr{L}_{pos} \\
& \sum_{j=1}^{n} \left\{ \sum_{s_l=1}^{S_l} p_{ls_l} d_{ljs_l} + \xi_l \gamma_{lj} \right\} x_j \leq z_l^0 - \xi_l (z_l^0 - z_l^1), \ l \in \mathscr{L}_{nec} \\
& \boldsymbol{x} \in F,
\end{array}\right\} \tag{20.113}
$$

where each of the objective functions represents the standard deviation of the degree of possibility or necessity.

It should be noted here that the minimization of the variance is equivalent to the minimization of the standard deviation.

To calculate a candidate for the satisficing solution, the DM is asked to specify the reference levels $\hat{z}_l^D$, $i = 1, \ldots, k$ of the objective function values of (20.113).

Let $Z_l^D(\boldsymbol{x})$ denote

$$
Z_l^D(\boldsymbol{x}) = \begin{cases} \dfrac{\sqrt{\boldsymbol{x}^T V_l \boldsymbol{x}}}{\sum_{j=1}^{n} \beta_{lj} x_j - z_l^1 + z_l^0}, \ l \in \mathscr{L}_{pos} \\ \dfrac{\sqrt{\boldsymbol{x}^T V_l \boldsymbol{x}}}{\sum_{j=1}^{n} \gamma_{lj} x_j - z_l^1 + z_l^0}, \ l \in \mathscr{L}_{nec} \end{cases}
$$

Then, for the DM's reference levels $\hat{z}_l^D$, $i = 1, \ldots, k$, a (weakly) Pareto optimal solution is obtained by solving the minimax problem

$$
\left.\begin{array}{ll}
\text{minimize} & \max_{1 \leq l \leq k} \left\{ Z_l^D(\boldsymbol{x}) - \hat{z}_l^D \right\} \\
\text{subject to} & \sum_{j=1}^{n} \left\{ \sum_{s_l=1}^{S_l} p_{ls_l} d_{ljs_l} - (1-\xi_l)\beta_{lj} \right\} x_j \leq z_l^0 - \xi_l (z_l^0 - z_l^1), \ l \in \mathscr{L}_{pos} \\
& \sum_{j=1}^{n} \left\{ \sum_{s_l=1}^{S_l} p_{ls_l} d_{ljs_l} + \xi_l \gamma_{lj} \right\} x_j \leq z_l^0 - \xi_l (z_l^0 - z_l^1), \ l \in \mathscr{L}_{nec} \\
& \boldsymbol{x} \in F.
\end{array}\right\} \tag{20.114}
$$

For notational convenience, we introduce $N_l(\boldsymbol{x})$ and $D_l(\boldsymbol{x})$ such that

$$Z_l^D(\boldsymbol{x}) - \hat{z}_l^D \triangleq \frac{N_l(\boldsymbol{x})}{D_l(\boldsymbol{x})}, \tag{20.115}$$

where

$$N_l(\boldsymbol{x}) = \begin{cases} \sqrt{\boldsymbol{x}^T V_l \boldsymbol{x}} - \hat{z}_l^D \left(\sum_{j=1}^n \beta_{lj} x_j - z_l^1 + z_l^0 \right), \ \forall l \in \mathscr{L}_{pos} \\ \sqrt{\boldsymbol{x}^T V_l \boldsymbol{x}} - \hat{z}_l^D \left(\sum_{j=1}^n \gamma_{lj} x_j - z_l^1 + z_l^0 \right), \ \forall l \in \mathscr{L}_{nec} \end{cases}$$

$$D_l(\boldsymbol{x}) = \begin{cases} \sum_{j=1}^n \beta_{lj} x_j - z_l^1 + z_l^0, \ \forall l \in \mathscr{L}_{pos} \\ \sum_{j=1}^n \gamma_{lj} x_j - z_l^1 + z_l^0, \ \forall l \in \mathscr{L}_{nec}. \end{cases}$$

Since the numerator $N_l(\boldsymbol{x})$ is a convex function and the denominator $D_l(\boldsymbol{x})$ is an affine function, it follows that $N_l(\boldsymbol{x})/D_l(\boldsymbol{x})$ is a quasi-convex function. Using this property, we can solve (20.114) by using the following extended Dinkelbach-type algorithm [5]:

20.5.5.1 Extended Dinkelbach-Type Algorithm for Solving (20.114)

Step 1: Set $r := 0$ and find a feasible solution $\boldsymbol{x}^r \in F(\boldsymbol{\xi})$.
Step 2: For a q^r calculated by

$$q^r = \max_{1 \le l \le k} \left\{ \frac{N_l(\boldsymbol{x}^r)}{D_l(\boldsymbol{x}^r)} \right\},$$

find an optimal solution $\boldsymbol{x}^c$ to the convex programming problem

$$\left.\begin{aligned} &\text{minimize } v \\ &\text{subject to } \frac{1}{D_l(\boldsymbol{x}^r)} \{D_l(\boldsymbol{x}) - q^r N_l(\boldsymbol{x})\} \le v, \ l = 1, \ldots, k \\ &\qquad \sum_{j=1}^n \left\{ \sum_{s_l=1}^{S_l} p_{ls_l} d_{ljs_l} - (1-\xi_l)\beta_{lj} \right\} x_j \le z_l^0 - \xi_l (z_l^0 - z_l^1), \ l \in \mathscr{L}_{pos} \\ &\qquad \sum_{j=1}^n \left\{ \sum_{s_l=1}^{S_l} p_{ls_l} d_{ljs_l} + \xi_l \gamma_{lj} \right\} x_j \le z_l^0 - \xi_l (z_l^0 - z_l^1), \ l \in \mathscr{L}_{nec} \\ &\qquad \boldsymbol{x} \in F. \end{aligned}\right\} \tag{20.116}$$

Step 3: For a sufficiently small positive number ε, if $v < \varepsilon$, stop the algorithm. Otherwise, set $\boldsymbol{x}^r := \boldsymbol{x}^c$, $r := r + 1$, and return to step 2.

Now we are ready to summarize an interactive algorithm for deriving a satisficing solution for the DM from among the Pareto optimal solution set.

20.5.5.2 Interactive Satisficing Method for the Possibilistic Variance Model

Step 1: Determine the linear membership functions $\mu_{\tilde{G}_l}$, $l = 1, \ldots, k$ with z_l^0 and z_l^1, $l = 1, \ldots, k$ obtained by solving linear programming problems (20.97).
Step 2: Calculate the individual minima and maxima of $Z_l^E(\boldsymbol{x})$, $l = 1, \ldots, k$.
Step 3: Ask the DM to specify the permissible levels ξ_l, $l = 1, \ldots, k$ taking into account the individual minima and maxima obtained in step 2.
Step 4: Set the initial reference levels at 0s, which can be viewed as the ideal values, i.e., $\hat{z}_l^D = 0$, $l = 1, \ldots, k$.
Step 5: For the current reference levels $\hat{z}_l^D$, $l = 1, \ldots, k$, solve the minimax problem (20.114) by using the extended Dinkelbach-type algorithm.
Step 6: The DM is supplied with the obtained Pareto optimal solution $\boldsymbol{x}^*$. If the DM is satisfied with the current objective function values $Z_l^D(\boldsymbol{x}^*)$, $l = 1, \ldots, k$, then stop. Otherwise, ask the DM to update the reference levels $\hat{z}_l^D$, $l = 1, \ldots, k$, and return to step 5.

20.5.6 Recent Topics: Random Fuzzy Multiple Objective Programming

When a random variable is used to express an uncertain parameter related to a stochastic factor of real systems, it is implicitly assumed that there exists a single random variable as a proper representation of the uncertain parameter. However, in some cases, experts may consider that it is suitable to employ a set of random variables, rather than a single one, in order to more precisely express the uncertain parameter. In this case, depending on the degree to which experts convince that each element (random variable) in the set is compatible with the uncertain parameter, it would be quite natural to assign different values (different degrees of possibility) to the elements in the set. For handling such real-world decision making situations, a random fuzzy variable was introduced by Liu [62] and explicitly defined [63] as a function from a possibility space to a collection of random variables.

Recently, by considering the experts' ambiguous understanding of mean and variance of random variables, Katagiri et al. [47] introduced a linear programming problem where an objective function contains random fuzzy parameters and discussed the problem in the framework of random fuzzy variables. They focused on the case where the mean of each random variable is represented with a fuzzy number and constructed a novel decision making model on the basis of possibility theory. Their model was extended to a multiobjective case [52] where only the coefficients of the objective functions are given as random fuzzy variables. A more general type of random fuzzy programming problems, in which not only objective functions but also constraints involve random fuzzy variables, was developed [56]. In these models, it is shown that the original problems can be transformed into deterministic nonlinear programming problems, and that the obtained deterministic problems can be exactly solved using conventional nonlinear programming techniques under some assumptions. These random fuzzy programming models are extended to other decision making problems such as two-level (bilevel) programming problems [54] and minimum spanning tree problems [55].

References

1. Averbakh, I., Lebedev, V.: On the complexity of minmax regret linear programming. Eur. J. Oper. Res. **160**, 227–231 (2005)
2. Benayoun, R., de Montgolfier, J., Tergny, J., Larichev, O.I.: Linear programming with multiple objective functions: STEP Method (STEM). Math. Program. **1**, 366–375 (1971)
3. Birge, J.R., Louveaux, F.: Introduction to Stochastic Programming, 2nd edn. Springer, New York/Dordrecht/Heidelberg/London (2011)
4. Bitran, G.R.: Linear multiple objective problems with interval coefficients. Manag. Sci. **26**, 694–706 (1980)
5. Borde, J., Crouzeix, J.P.: Convergence of a Dinkelbach-type algorithm in generalized fractional programming. Z. Oper. Res. **31**, 31–54 (1987)
6. Caballero, R., Cerdá, E., Muñoz, M.M., Rey, L., Stancu-Minasian, I.M.: Efficient solution concepts and their relations in stochastic multiobjective programming. J. Optim. Theory Appl. **110**, 53–74 (2001)
7. Chernikova, N.V.: Algorithm for finding a general formula for the nonnegative solutions of a system of linear inequalities. USSR Comput. Math. Math. Phys. **5**, 228–233 (1965)
8. Contini, B.: A stochastic approach to goal programming. Oper. Res. **16**, 576–586 (1968)
9. Dantzig, G.B.: Linear programming under uncertainty. Manag. Sci. **1**, 197–206 (1955)
10. Dubois, D.: Linear programming with fuzzy data. In: Bezdek, J.C. (ed.) Analysis of Fuzzy Information. Application in Engineering and Science, vol. III, pp. 241–263. CRC Press, Boca Raton (1987)
11. Dubois, D., Prade, H.: Fuzzy numbers: an overview. In: Bezdek, J.C. (ed.) Analysis of Fuzzy Information. Mathematics and Logic, vol. I, pp. 3–39. CRC Press, Boca Raton (1987)
12. Dubois, D., Prade, H.: Possibility Theory: An Approach to Computerized Processing of Uncertainty. Plenum Press, New York (1988)
13. Ehrgott, M., Gandibleux, X. (eds.): Multiple Criteria Optimization: State of the Art Annotated Bibliographic Surveys. Kluwer Academic Publishers, Boston (2002)
14. Evans, J.P., Steuer, R.E.: A revised simplex method for linear multiple objective programs. Math. Program. **5**, 54–72 (1973)
15. Furukawa, N.: Parametric orders on fuzzy numbers and their roles in fuzzy optimization problems. Optimization **40**, 171–192 (1997)
16. Geoffrion, A.M.: Proper efficiency and the theory of vector maximization. J. Math. Anal. Appl. **22**, 618–630 (1968)
17. Gil, M.A., Lopez-Diaz, M., Ralescu, D.A.: Overview on the development of fuzzy random variables. Fuzzy Sets Syst. **157**, 2546–2557 (2006)

18. Goicoecha, A., Hansen, D.R., Duckstein, L.: Multiobjective Decision Analysis with Engineering and Business Applications. Wiley, New York (1982)
19. Hladík, M.: On necessary efficient solutions in interval multiobjective linear programming. In: CD-ROM Proceedings of the 25th Mini-EURO Conference Uncertainty and Robustness in Planning and Decision Making URPDM 2010, pp. 1–10 (2010)
20. Hladík, M.: Complexity of necessary efficiency in interval linear programming and multiobjective linear programming. Optim. Lett. **6**, 893–899 (2012)
21. Horst, R., Tuy, H.: Global Optimization: Deterministic Approaches, 3rd Revised and Enlarged Edition. Springer, Berlin (1995)
22. Ida, M.: Necessary efficient test in interval multiobjective linear programming. In: Proceedings of the Eighth International Fuzzy Systems Association World Congress, pp. 500–504 (1999)
23. Ida, M.: Efficient solution generation for multiple objective linear programming and uncertain coefficients. In: Proceedings of the 8th Bellman Continuum, pp. 132–136 (2000)
24. Infanger, G. (eds.): Stochastic Programming. Springer, New York/Dorderecht/Heidelberg/London (2011)
25. Inuiguchi, M.: Multiple objective linear programming with fuzzy coefficients. In: Figueira, J., Greco, S., Ehrgott, M. (eds.) Multiple Criteria Decision Analysis: State of the Art Surveys, pp.723–760. Kluwer Academic Publishers, Boston (2002)
26. Inuiguchi, M.: On possibilistic/fuzzy optimization. In: Melin, P., Castillo, O., Aguilar, L.T., Kacprzyk, J., Pedrycz, W. (eds.) Foundations of Fuzzy Logic and Soft Computing. Proceedings of 12th International Fuzzy Systems Association World Congress, IFSA 2007, Cancun, June 2007. Lecture Notes in Artificial Intelligence, vol. 4529, pp. 351–360. Springer, Berlin (2007)
27. Inuiguchi, M.: Necessary efficiency is partitioned into possible and necessary optimalities. In: Proceedings of 15th International Fuzzy Systems Association World Congress, IFSA-NAFIPS 2013 (2013)
28. Inuiguchi, M., Kume, Y.: A discrimination method of possibly efficient extreme points for interval multiobjective linear programming problems (in Japanese). Trans. SICE **25**, 824–826 (1989)
29. Inuiguchi, M., Kume, Y.: Solution concepts for fuzzy multiple objective programming problems. Jpn. J. Fuzzy Theory Syst. **2**, 1–22 (1990)
30. Inuiguchi, M., Kume, Y.: Minimax regret in linear programming problems with an interval objective function. In: Tzeng, G.H., Wang, H.F., Wen, U.P., Yu, P.L. (eds.) Multiple Criteria Decision Making, pp. 65–74. Springer, New York (1994)
31. Inuiguchi, M., Ramík, J.: Possibilistic linear programming: a brief review of fuzzy mathematical programming and a comparison with stochastic programming in portfolio selection problem. Fuzzy Sets Syst. **111**, 3–28 (2000)
32. Inuiguchi, M., Sakawa, M.: Possible and necessary optimality tests in possibilistic linear programming problems. Fuzzy Sets Syst. **67**, 29–46 (1994)
33. Inuiguchi, M., Sakawa, M.: Minimax regret solutions to linear programming problems with an interval objective function. Eur. J. Oper. Res. **86**, 526–536 (1995)
34. Inuiguchi, M., Sakawa, M.: Possible and necessary efficiency in possibilistic multiobjective linear programming problems and possible efficiency test. Fuzzy Sets Syst. **78**, 231–241 (1996)
35. Inuiguchi, M., Sakawa, M.: An achievement rate approach to linear programming problems with an interval objective function. J. Oper. Res. Soc. **48**, 25–33 (1997)
36. Inuiguchi, M., Sakawa, M.: Robust optimization under softness in a fuzzy linear programming problem. Int. J. Approx. Reason. **18**, 21–34 (1998)
37. Inuiguchi, M., Tanino, T.: On computation methods for a minimax regret solution based on outer approximation and cutting hyperplanes. Int. J. Fuzzy Syst. **3**, 548–557 (2001)
38. Inuiguchi, M., Tanino, T.: Enumeration of all possibly optimal vertices with possible optimality degrees in linear programming problems with a possibilistic objective function. Fuzzy Optim. Decis. Making **3**, 311–326 (2004)

39. Inuiguchi, M., Tanino, T., Tanaka, H.: Optimization approaches to possibilistic linear programming problems. In: Proceedings of Joint 9th IFSA World Congress and 20th NAFIPS International Conference, pp. 2724–2729 (2001)
40. Ishibuchi, H., Tanaka, H.: Multiobjective programming in optimization of the interval objective function. Eur. J. Oper. Res. **48**, 219–225 (1990)
41. Kasperski, A.: Discrete Optimization with Interval Data: Minimax Regret and Fuzzy Approach. Springer, Berlin (2008)
42. Katagiri, H., Ishii, H.: Linear programming problem with fuzzy random constraint. Math. Jpn. **52**, 123–129 (2000)
43. Katagiri, H., Sakawa, M.: A study on fuzzy random linear programming problems based on possibility and necessity. In: Bilgic, T., DeBaets, B., Kaynak, O. (eds.) Fuzzy Sets and Systems, IFSA 2003: 10th International Fuzzy Systems Association World Congress, Istanbul. Lecture Notes in Computer Science, pp. 725–732. Springer, Berlin (2003)
44. Katagiri, H., Ishii, H., Itoh, T.: Fuzzy random linear programming problem. In: Proceedings of Second European Workshop on Fuzzy Decision Analysis and Neural Networks for Management, Planning and Optimization, Dortmund, pp. 107–115 (1997)
45. Katagiri, H., Ishii, H., Sakawa, M.: Linear programming problem under fuzziness and randomness. In: Proceedings of International Conference on Applied Stochastic System Modeling, Kyoto, pp. 97–106 (2000)
46. Katagiri, H., Sakawa, M., Ishii, H.: Multiobjective fuzzy random linear programming using E-model and possibility measure. In: Joint 9th IFSA World Congress and 20th NAFIPS International Conference, Vancouver, pp. 2295–2300 (2001)
47. Katagiri, H., Ishii, H., Sakawa, M.: Linear programming problems with random fuzzy variable coefficients. In: Proceedings of 5th Czech-Japan Seminar on Data Analysis and Decision Making under Uncertainty, Koyasan, vol. 1, pp. 55–58 (2002)
48. Katagiri, H., Sakawa, M., Ohsaki, S.: An interactive satisficing method through the variance minimization model for fuzzy random linear programming problems. In: Tanino, T., Tamaki, T., Inuiguchi, M. (eds.) Multi-Objective Programming and Goal-Programming: Theory and Applications. Advances in Soft Computing, pp.171–176. Springer, Berlin (2003)
49. Katagiri, H., Sakawa, M., Kato, K., Nishizaki, I.: A fuzzy random multiobjective 0-1 programming based on the expectation optimization model using possibility and necessity measures. Math. Comput. Model. **40**, 411–421 (2004)
50. Katagiri, H., Sakawa, M., Ishii, H.: Studies of stochastic programming models using possibility and necessity measures for linear programming problems with fuzzy random variable coefficients. Electron. Commun. Jpn. Part III-Fundam. Electron. Sci. **88**, 68–75 (2005)
51. Katagiri, H., Sakawa, M., Kato, K., Ohsaki, S.: An interactive fuzzy satisficing method based on the fractile optimization model using possibility and necessity measures for a fuzzy random multiobjective linear programming problem. Electron. Commun. Jpn. Part III-Fundam. Electron. Sci. **88**, 20–28 (2005)
52. Katagiri, H., Hasuike, T., Ishii, H., Nishizaki, I.: Interactive multiobjective programming under random fuzzy environments. In: Proceedings of the 11th Czech-Japan Seminar on Data Analysis and Decision Making Under Uncertainty, Sendai, pp. 33–38 (2008)
53. Katagiri, H., Sakawa, M., Kato, K., Nishizaki, I.: Interactive multiobjective fuzzy random linear programming: maximization of possibility and probability. Eur. J. Oper. Res. **188**, 530–539 (2008)
54. Katagiri, H., Niwa, K., Kubo, D., Hasuike, T.: Interactive random fuzzy two-level programming through possibility-based fractile criterion optimality. In: Proceedings of The International MultiConference of Engineers and Computer Scientists 2010, vol. 3, pp. 2113–2118 (2010)
55. Katagiri, H., Kato, K., Hasuike, T.: A random fuzzy minimum spanning tree problem through a possibility-based value at risk model. Expert Syst. Appl. **39**, 205–220 (2012)

56. Katagiri, H., Uno, T., Kato, K., Tsuda, H., Tsubaki, H.: Random fuzzy multiobjective programming: optimization of possibilistic value at risk (pVaR). Expert Syst. Appl. **40**, 563–574 (2013)
57. Klein, G., Moskowitz, H., Ravindran, A.: Interactive multiobjective optimization under uncertainty. Manag. Sci. **36**, 58–75 (1990)
58. Kwakernaak, H.: Fuzzy random variable-1. Inf. Sci. **15**, 1–29 (1978)
59. Lai, Y.-J., Hwang, C.L.: Fuzzy Mathematical Programming. Springer, Berlin (1992)
60. Liu, B.: Fuzzy random chance-constrained programming. IEEE Trans. Fuzzy Syst. **9**, 713–720 (2001)
61. Liu, B.: Fuzzy random dependent-chance programming. IEEE Trans. Fuzzy Syst. **9**, 721–726 (2001)
62. Liu, B.: Random fuzzy dependent-chance programming and its hybrid intelligent algorithm. Inf. Sci. **141**, 259–271 (2002)
63. Liu, B.: Uncertainty Theory. Springer, Berlin/Heidelberg/New York (2004)
64. Luhandjula, M.K.: On possibilistic linear programming. Fuzzy Sets Syst. **18**, 15–30 (1986)
65. Luhandjula, M.K.: Multiple objective programming problems with possibilistic coefficients. Fuzzy Sets Syst. **21**, 135–145 (1987)
66. Luhandjula, M.K.: Fuzzy optimization: an appraisal. Fuzzy Sets Syst. **30**, 257–282 (1989)
67. Luhandjula, M.K.: Fuzziness and randomness in an optimization framework. Fuzzy Sets Syst. **77**, 291–297 (1996)
68. Luhandjula, M.K.: Fuzzy stochastic linear programming: survey and future research directions. Eur. J. Oper. Res. **174**, 1353–1367 (2006)
69. Luhandjula, M.K., Gupta, M.M.: On fuzzy stochastic optimization. Fuzzy Sets Syst. **81**, 47–55 (1996)
70. Mangasarian, O.L.: Nonlinear Programming. McGraw-Hill, New York (1969)
71. Mausser, H.E., Laguna, M.: A new mixed integer formulation for the maximum regret problem. Int. Trans. Oper. Res. **5**, 389–403 (1998)
72. Oliveira, C., Antunes, C.H.: Multiple objective linear programming models with interval coefficients—an illustrated overview. Eur. J. Oper. Res. **181**, 1434–1463 (2007)
73. Puri, M.L., Ralescu, D.A.: Fuzzy random variables. J. Math. Anal. Appl. **14**, 409–422 (1986)
74. Qiao, Z., Zhang, Y., Wang, G.-Y.: On fuzzy random linear programming. Fuzzy Sets Syst. **65**, 31–49 (1994)
75. Ramík, J., Římánek, J.: Inequality relation between fuzzy numbers and its use in fuzzy optimization. Fuzzy Sets Syst. **16**, 123–138 (1985)
76. Rivaz, S., Yaghoobi, M.A.: Minimax regret solution to multiobjective linear programming problems with interval objective functions coefficients. Cent. Eur. J. Oper. Res. **21**, 625–649 (2013)
77. Rommelfanger, H.: A general concept for solving linear multicriteria programming problems with crisp, fuzzy or stochastic variables. Fuzzy Sets Syst. **158**, 1892–1904 (2007)
78. Rommelfanger, H., Słowiński, R.: Fuzzy linear programming with single or multiple objective functions. In: Słowiński, R. (ed.) Fuzzy Sets in Decision Analysis. Operations Research and Statistics. Kluwer Academic Publishers, Boston (1998)
79. Rommelfanger, H., Hanuscheck, R., Wolf, J.: Linear programming with fuzzy objective. Fuzzy Sets Syst. **29**, 31–48 (1989)
80. Sakawa, M.: Fuzzy Sets and Interactive Multiobjective Optimization. Plenum Press, New York (1993)
81. Sakawa, M., Yano, H.: Feasibility and Pareto optimality for multiobjective nonlinear programming problems with fuzzy parameters. Fuzzy Sets Syst. **43**, 1–15 (1991)
82. Sakawa, M., Takahashi, J., Yano, H.: Extended Pareto optimality concept for multiobjective linear programming problems with fuzzy parameters and its properties. Electron. Commun. Jpn. Part III-Fundam. Electron. Sci. **73**, 1–9 (1990)
83. Słowiński, R.: Fuzzy multi-objective linear programming. Belg. J. Oper. Res. Stat. Comput. Sci. **37**, 83–98 (1997)

84. Słowiński, R., Teghem, J.: Fuzzy vs. stochastic approaches to multicriteria linear programming under uncertainty. Nav. Res. Logist. **35**, 673–695 (1988)
85. Słowiński, R., Teghem, J. (eds.): Stochastic Versus Fuzzy Approaches to Multiobjective Mathematical Programming Under Uncertainty. Kluwer Academic Publishers, Dordrecht (1990)
86. Stancu-Minasian, I.M.: Stochastic Programming with Multiple Objective Functions. D. Reidel Publishing Company, Dordrecht (1984)
87. Stancu-Minasian, I.M.: Overview of different approaches for solving stochastic programming problems with multiple objective functions. In: Słowiński, R., Teghem, J. (eds.) Stochastic Versus Fuzzy Approaches to Multiobjective Mathematical Programming Under Uncertainty, pp.71–101. Kluwer Academic Publishers, Dordrecht/Boston/London (1990)
88. Steuer, R.E.: Algorithms for linear programming problems with interval objective function coefficients. Math. Oper. Res. **6**, 333–348 (1981)
89. Steuer, R.E.: Multiple Criteria Optimization: Theory, Computation, and Application. Wiley, New York (1986)
90. Tanaka, H., Asai, K.: Fuzzy linear programming problems with fuzzy numbers. Fuzzy Sets Syst. **13**, 1–10 (1984)
91. Tanaka, H., Ichihashi, H., Asai, K.: A formulation of fuzzy linear programming problem based on comparison of fuzzy numbers. Control Cybern. **13**, 185–194 (1984)
92. Teghem, J., Dufrane, D., Thauvoye, M., Kunsch, P.: STRANGE: an interactive method for multi-objective linear programming under uncertainty. Eur. J. Oper. Res. **26**, 65–82 (1986)
93. Urli, B., Nadeau, R.: Stochastic MOLP with incomplete information: an interactive approach with recourse. J. Oper. Res. Soc. **41**, 1143–1152 (1990)
94. Urli, B., Nadeau, R.: PROMISE/scenarios: an interactive method for multiobjective stochastic linear programming under partial uncertainty. Eur. J. Oper. Res. **155**, 361–372 (2004)
95. Wang, G.-Y., Qiao, Z.: Linear programming with fuzzy random variable coefficients. Fuzzy Sets Syst. **57**, 295–311 (1993)
96. Wang, G., Zhang, Y.: The theory of fuzzy stochastic processes. Fuzzy Sets Syst. **51**, 161–178 (1992)
97. White, D.J.: Optimality and Efficiency. Wiley, New York (1982)
98. Zadeh, L.A.: Fuzzy sets as a basis for a theory of possibility. Fuzzy Sets Syst. **1**, 3–28 (1978)

Chapter 21
A Review of Goal Programming

Dylan Jones and Mehrdad Tamiz

Abstract The field of goal programming is continuing to develop at a rapid pace. New variants of the goal programming model are being introduced into the literature and existing variants combined together to form a more comprehensive and flexible modelling structure. Goal programming is also being applied to wide a range of modern applications and is increasingly being used in combination with other techniques from operations research and artificial intelligence to enhance its modelling flexibility. This paper presents a review of the field of goal programming focussing on recent developments. The current range of goal programming variants is described. The range of techniques that goal programming has been combined or integrated with is discussed. A range of modern applications of goal programming are given and conclusions are drawn.

Keywords Goal programming • Bibliography

21.1 Introduction

Goal programming is a technique within the field of multi-criteria decision making (MCDM) primarily based around the Simon [118] philosophy of reaching a set of multiple goals as closely as possible. The earliest goal programming formulation was introduced by Charnes et al. [30], in the context of executive compensation. At this point the term 'goal programming' was not used and the model was seen as an adaptation of linear programming. A more formal theory of goal programming is given by Charnes and Cooper [27]. Books by Lee [81], Ignizio [63], Romero [108] and Ignizio and Cavalier [65], have helped shape the direction and development of goal programming. A recent book by Jones and Tamiz [75] gives a detailed

D. Jones (✉)
Department of Mathematics, University of Portsmouth, Lion Terrace, Portsmouth PO1 3HF, UK
e-mail: dylan.jones@port.ac.uk

M. Tamiz
College of Business Administration, Kuwait University, P.O. Box 5486, Safat, Kuwait
e-mail: Mehrdad@cba.edu.kw

S. Greco et al. (eds.), *Multiple Criteria Decision Analysis*, International Series in Operations Research & Management Science 233,
DOI 10.1007/978-1-4939-3094-4_21

explanation of goal programming from a practical modelling perspective. This paper gives an overview of the field of goal programming focussing on recent developments and applications.

The remainder of this paper is divided into four sections. Section 21.2 details the goal programming variants and gives key references for each variant. Section 21.3 examines goal programming as part of a mixed modelling framework and gives references for each technique that goal programming has been combined or integrated with. Section 21.4 gives references for some recent applications of goal programming. Finally, Sect. 21.5 draws conclusions.

21.2 Goal Programming Variants

The field of goal programming can be categorised into a number of variants. Jones and Tamiz [75] draw the distinction between 'Distance-metric based variants' who differ in the way they compare deviations from amongst the set of goals and 'Decision Variable and Goal based variants' which differ in the way they define the decision variables or calculate individual deviations from goals. Sections 21.2.1–21.2.6 detail the distance-metric based-variants and Sects. 21.2.7–21.2.10 detail the decision-variable and goal based variants. It is important to note that a goal programme can be a combination of many types of variants. For example, an non-linear integer lexicographic goal programme [15], or a fuzzy weighted goal programme [135].

21.2.1 Lexicographic Goal Programming

Lexicographic goal programming is a key variant used when the decision maker has a natural priority ordering of the goals. It enjoyed prominence in the early goal programming literature, with [125] recording 75 % of goal programming models using the lexicographic form. Books by Lee [81], Ignizio [63], Ignizio and Cavalier [65], and Schniederjans [114], all have a strong emphasis on the lexicographic variant. The lexicographic variant is used when the decision maker has a natural ordering of goals into a number of priority levels in mind or when they do not wish to make direct trade-offs between goals for political or ethical reasons.

The algebraic formulation of a lexicographic goal programme with the number of priority levels defined as L with corresponding index $l = 1, \ldots, L$ is given below. Each priority level is a function of a subset of unwanted deviational variables which we define as $h_l\left(\underline{n}, \underline{p}\right)$. This leads to the following formulation:

$$Lex\ Min\ \boldsymbol{a} = \left[h_1\left(\underline{n}, \underline{p}\right), h_2\left(\underline{n}, \underline{p}\right), \ldots, h_L\left(\underline{n}, \underline{p}\right)\right]$$

Subject to:

$$f_q\left(\underline{\boldsymbol{x}}\right) + n_q - p_q = b_q \qquad q = 1, \ldots Q$$

$$\underline{x} \in F$$

$$n_q, p_q \geq 0 \qquad q = 1, \ldots, Q$$

Where each $h_l(\underline{n}, \underline{p})$ contains a number of unwanted deviational variables. The exact nature of $h_l(\underline{n}, \underline{p})$ depends on the nature of the goal programme to be formulated, but if we assume that it is linear and separable then it will assume the form:

$$h_l(\underline{n}, \underline{p}) = \sum_{q=1}^{Q} \left(\frac{u_q^l n_q}{k_q} + \frac{v_q^l p_q}{k_q} \right)$$

Where u_q^l is the preferential weight associated with the minimisation of n_q in the l'th priority level. v_q^l is the preferential weight associated with the minimisation of p_q in the l'th priority level. k_q is the normalisation constant of the q' th goal, further explained in Sect. 21.2.2 below.

The lexicographic structure has been criticised by some authors on the grounds of its incompatibility with utility function theory [96]. However Romero [108] points out its practicality in modelling situations where the decision maker has a pre-defined ordering of the goals in mind. Furthermore, Tamiz et al. [126, 127] explore the relationship between lexicographic ordering and utility functions and conclude that the non-compensatory lexicographic model can be appropriate when modelling some real-life decisions. Therefore, whilst it is true that lexicographic goal programming will not be appropriate for every multi-objective situation, it can be seen that there is a class of situations in which it proves to be an effective an appropriate decision aiding tool.

21.2.2 Weighted Goal Programming

The weighted goal programme variant is covered in details in books by Romero [108] and Jones and Tamiz [75]. It is recorded by Jones and Tamiz [74] as accounting for 41 % of goal programming articles in the period 1990–2000. Assuming linearity of the achievement function, the linear weighted goal programme can be represented by the following formulation:

$$Min\ \boldsymbol{a} = \sum_{q=1}^{Q} \left(\frac{u_q n_q}{k_q} + \frac{v_q p_q}{k_q} \right)$$

Subject to:

$$f_q\left(\underline{\boldsymbol{x}}\right) + n_q - p_q = b_q \qquad q = 1,\ldots Q$$

$$\underline{x} \in F$$

$$n_q, p_q \geq 0 \qquad q = 1,\ldots,Q$$

With the variable definitions identical to those introduced for the lexicographic goal programming variant in Sect. 21.2.1 above, except that the preference weights u_q and v_q are no longer indexed by priority level. This model allows for trade-offs between goals to be investigated by varying the preference weights. Major issues in using the weighted variant have included the choice of normalisation weight k_q. Romero [108] and Jones and Tamiz [75] both give descriptions of possible choices of normalising weights. The setting the preferential weights is another issue with [70] suggesting a weight sensitivity algorithm to overcome difficulties associated with weight setting.

21.2.3 Chebyshev Goal Programming

The Chebyshev (or MinMax) goal programming variant is proposed by Flavell [49] and explained by Jones and Tamiz [75]. It differs from the lexicographic and weighted variants in that it has an underlying Chebyshev ($L\infty$) distance metric rather than a Manhattan (L1) one. This has the effect of ensuring a balance between the satisfaction of the goals rather than just concentrating on optimisation. This means that the Chebyshev variant should be relevant to a large number of application areas, especially those with multiple decision makers each of whom has a preference to their own subset of goals that they regard as most important. However, surveys of the literature (e.g. Jones and Tamiz [74]) find little practical use of the Chebyshev goal programming variant.

If we let λ be the maximal deviation from amongst the set of goals then the Chebyshev goal programming has the following algebraic format:

$$Min\ \boldsymbol{a} = \lambda$$

Subject to:

$$f_q\left(\underline{\boldsymbol{x}}\right) + n_q - p_q = b_q \qquad q = 1,\ldots Q$$

$$\frac{u_q n_q}{k_q} + \frac{v_q p_q}{k_q} \leq \lambda \qquad q = 1,\ldots Q$$

$$\underline{x} \in F$$

$$n_q, p_q \geq 0 \qquad q = 1, \ldots, Q, \qquad \lambda \geq 0$$

A case study of the use of the Chebyshev variant is given by Ignizio [64] in the context of maintenance planning.

21.2.4 Extended Goal Programming

The extended Lexicographic goal programming (ELGP) is introduced by Romero [109] with the purpose of providing a general framework which covers and allows the combination of the most common goal programming variants. It is also encompasses several other distance-based MCDM techniques. This work is further extended by Romero [110] who provides a more generalised form of the achievement function and by Arenas et al. [5] who extend the framework to include fuzzy models (detailed in Sect. 21.2.7).

The ELGP model is given in its general algebraic form as:

$$Min\ a = \begin{bmatrix} \left(\alpha_1 \lambda_1 + (1 - \alpha_1) \left\{ \sum_{i=1}^{q} \left(u_i^1 n_i^1 + v_i^1 p_i^1 \right) \right\} \right), .. \\ ., \left(\alpha_l \lambda_l + (1 - \alpha_l) \left\{ \sum_{i=1}^{q} \left(u_i^l n_i^l + v_i^l p_i^l \right) \right\} \right), \ldots, \\ \left(\alpha_L \lambda_L + (1 - \alpha_L) \left\{ \sum_{i=1}^{q} \left(u_i^L n_i^L + v_i^L p_i^L \right) \right\} \right) \end{bmatrix}$$

Subject to:

$$\begin{array}{ll} \alpha_l \left(u_i^l n_i^l + v_i^l p_i^l \right) \leq \lambda_l & l = 1, \ldots, L \\ f_i(x) + n_i - p_i = b_i & i = 1, \ldots, q \\ n_i,\ p_i \geq 0 & i = 1, \ldots, q \end{array}$$

Where, in common with the notation introduced in Sects. 21.2.1–21.2.3, unwanted deviations are given a positive weight and deviations which are not desired to be minimised in a given priority level are given a zero weight in the achievement function. The parameter notation u_i^l, v_i^l, λ_l now contains an extra superscript l which refers to the priority level.

The ELGP formulation allows for the inclusion and combination of the lexicographic, weighted, and Chebyshev goal programming as detailed in Sects. 21.2.1–21.2.3 above. The balance between optimisation (efficiency) and balance (equity) can be controlled at each priority level through the parameter α_l which can be varied between complete emphasis on optimisation ($\alpha = 0$) and complete emphasis on balance ($\alpha = 1$). The ELGP framework is therefore a comprehensive tool for the inclusion of all relevant types of underlying philosophies into the goal programming framework.

21.2.5 Meta Goal Programming

The Meta goal programming framework is proposed by Rodriguez et al. [107] as a means of allowing the decision maker more flexibility in expressing their preferences by means of setting meta-goals. These can be thought of as secondary goals derived from the original set of goals. They can be achieved by means of a lexicographic or a weighted structure as deemed appropriate by the decision maker. The three types of meta-goals proposed are:

Type 1: A meta-goal relating to the percentage sum of unwanted deviations
Type 2: A meta-goal relating to the maximum percentage deviation
Type 3: A meta-goal relating to the percentage of unachieved goals.

Considering these three types of meta-goals from the perspective of underlying distance metrics allows for an understanding of the philosophy of this method. The type 1 meta-goal has an L_1 underlying metric whereas the type 2 meta goal has an L_∞ underlying metric. The type 3 meta-goal has an L_0 underlying metric of the type sometimes found in classification models, with a binary value of 1 if the goal is satisfied and 0 otherwise. Thus the meta-goal programming framework is valuable in allowing decision maker(s) to explore their preference structure without having to commit beforehand to a specific distance metric or philosophy. The meta-goal programming framework is extended to include an interactive methodology by Caballero et al. [18] and is used in a collaborative manufacturing context by Lin et al. [89].

21.2.6 Multi-Choice Goal Programming

A recently proposed variant is multi-choice goal programming [23]. A modification is given by Chang [24] which results in a more computationally efficient model termed revised multi-choice goal programming. Multi-choice goal programming has since been applied to problems arising in the fields of supplier selection [84, 88]; product planning [83]; and supply chain management marketing [104]. Multi-Choice goal programming allows the decision maker to specify a set of multiple target values for each goal from which they wish to see deviations minimised.

21.2.7 Fuzzy Goal Programming

The fuzzy goal programming variant is becoming increasingly popular within the field of goal programming. This variant utilises fuzzy set theory [138] to deal with a level of imprecision in the goal programming model. This imprecision normally relates to the goal target values (b_q) but could also relate to other aspects of the goal programme such as the priority structure. The early fuzzy goal programming models used both Chebyshev [61, 100] and weighted [61, 128] distance metrics. Yaghoobi et al. [135] give a recent framework for the weighted variant. There are various possibilities for measuring the fuzziness around the target goals, each of which leads to a different fuzzy membership function. These functions model the drop in dissatisfaction from a state of total satisfaction (where the membership function takes the value 1) to a state of total dissatisfaction (where the membership function takes the value 0). There are many possible fuzzy membership functions, with the most common forms being 'left-sided', 'right sided', 'triangular', and 'trapezoidal' which are algebraically defined in Jones and Tamiz [75]. Some recent applications of fuzzy goal programming are given by Amiri and Salehi-Sadaghiani [4] to optimise multi-response problems and [78] for assembly line balancing.

21.2.8 Goal Programming with Non-standard Preferences

The standard goal programming model assumes a per unit penalty (termed u_q or v_q in Sects. 21.2.2 and 21.2.3) for every unit of unwanted deviation from the target value. These lead to a linear relationship between the magnitude of the unwanted deviation and the penalty imposed. However, a linear relationship is not sufficient to adequately represent the decision maker's preference structure in many real-life applications. Hence a number of 'non-standard' penalty structures have been proposed in order to improve the flexibility and scope of goal programming. The first proposed method is termed interval goal programming [26]. This method keeps a linear per-unit penalty but relaxes the condition that a single goal target value should be specified. Instead the decision maker chooses an interval which is satisfactory and penalise deviations from either end of this interval. The next development is that of a penalty functions approach that allows an increasing per-unit penalty at distances further out from the goal. This is first proposed by Charnes et al. [29] in the context of resource allocation for a marine environmental problem. Further developments on penalty function modelling are found in [19] the context of water resource planning and [108] who gives description of the underlying theory.

The next developments allowed a complete range of preference functions to be modelled. Martel and Aouni [94] provide the first global preference change framework with their adaptation of the discrete multi-criteria method [13] to the goal programming format. This framework is valuable in providing linkages between the discrete Multi-Criteria approaches and goal programming. Jones and Tamiz [73] provide a more computationally efficient framework that breaks the non-standard preferences down to a series of four basic preference changes. Chang [22] proposes a model that is efficient at handling 'penalty function' type changes and is similar to the [19] model.

21.2.9 Integer and Binary Goal Programming

A (mixed) integer goal programme occurs when one or more of the decision or deviational variables are restricted to take an integer value. A (mixed) binary goal programme is a special case of integer goal programming where all the integer variables are restricted to take the vales of zero or one. Any other variant of goal programming listed in Sect. 21.2 can also be an integer or a binary goal programme, e.g. an integer lexicographic goal programme.

Most of the modelling methodologies and solution challenges associated with single objective integer programming (see Williams [134]) also apply to integer goal programming. Integer and binary goal programmes are of particular use when formulating many practical problems that have both logical conditions and multiple, conflicting goals. Typical application areas include multi-objective shortest path, assignment, logistics, network flow, spanning tree, travelling salesperson, knapsack, scheduling, location, and set covering problems [45]. Recent applications of integer and binary goal programming are given by Oddoye et al. [102] in the context of healthcare planning; Caballero et al. [15] in the context of sawmill planning and Chen and Su [33] in the context of logistics planning.

21.2.10 Non-linear and Fractional Goal Programming

A goal programme where one or more constraints, goals, or the achievement function is non-linear is termed a non-linear goal programme. Any of the other goal programming variants listed in Sect. 21.2 could be non-linear, e.g. a non-linear extended goal programme. Non-linear goal programmes can be solved in similar ways to single objective non-linear programmes. Solution methods employed include exact methods and modern heuristic methods. Recent applications involving exact methods include [90] who present a mixed integer non-linear weighted goal programme for supermarket planning and [43] who present a non-linear goal programming model for an oil blending problem.

Recent application using heuristic methods as a solution tool include [85] who presents a weighted non-linear model for transportation planning that is solved using genetic algorithms and [15] who present a non-linear integer lexicographic model for sawmill planning that is solved using a scatter-based tabu search method.

A fractional goal programming is a special case of a non-linear goal programme with one or more goals of the form:

$$\frac{f_q(\underline{\boldsymbol{x}})}{g_q(\underline{\boldsymbol{x}})} + n_q - p_q = b_q$$

Where $g_q(\underline{\boldsymbol{x}})$ is a generic function of the decision variables. This type of goal programme is mentioned by Romero [108] as arising in the fields of financial planning, production planning, and engineering. This variant also occurs in some goal programming based methods for deriving weighting vectors from pairwise comparison matrices [41]. Fasakhodi et al. [48] give a recent use of fractional goal programming in water resource planning.

21.3 Goal Programming as Part of a Mixed-Modelling Framework

The complexity of modern Operational Research studies is leading to the requirement to combine techniques in order to gain a more effective means of modelling the real-life situation. This has lead to an increase in goal programming being used as part of mixed-modelling strategies. This section looks at some of the most common cases of the integration and combination of goal programming with techniques from the wider fields of Multi-Criteria Decision Making, Operational Research, and Artificial Intelligence.

21.3.1 Goal Programming as a Statistical Tool

The most frequent use of goal programming is as a decision making tool where the goals correspond to a range of diverse, conflicting criteria. However, the situation where the target levels correspond to a measured value on a single criterion presents a different type of goal programming model. The x_j values now represent coefficients of an equation corresponding to the other criteria and the (weighted) achievement function aims to provide a best possible fit to the set of observed target values by minimising the sum of deviations from the target values. The technological (a_{ij}) coefficients now represent the score of the i'th observation on the j'th criteria. This leads to the base model:

$$Min\ a = \sum_{i=1}^{q} (u_i n_i + v_i p_i)$$

Subject to

$$\sum_{j=1}^{m} a_{ij}x_j + n_i - p_i = b_i \quad i = 1,\ldots,q$$

$$x \in F$$

$$x_j\ free \;\; j = 1,\ldots,m \quad n_i, p_i \geq 0 \quad i = 1,\ldots,q$$

This model leads to goal programming becoming a useful tool for the fields of statistics and of data mining. The above equation is in fact a least-absolute value (LAV) regression model, which is based on the L_1 distance metric. This is used as an alternative to the standard least-squares regression model which has an underlying L_2 distance metric. This has the advantage of being less influenced by outlying observations. The other advantage gained by using this type of mathematical programming based model is that it is more flexible in that it requires fewer assumptions than standard least-squares regression. It is also more flexible in that it allows extra constraints on the combinations of the x_{ij} values and weighting of the observations according to their importance to the decision maker. The original goal programming model of Charnes et al. [30] was this type of model for calculating executive compensation packages, and took advantage of the ability to add extra constraints to ensure implementable results. A further significant application of goal programming as a regression tool is given by Charnes et al. [32] in the context of corporate policy planning. The theory and statistical properties of the goal programming based L_1 regression are detailed in Sueyoshi [122].

Recent applications of goal programming as a regression tool are given by Bhattacharya [12] who takes advantage of the fewer assumptions required in order to provide improved pre-harvest forecasts in an agricultural planning model. Da Silva et al. [37] use the technique for forestry management. They conclude that goal programming produces similar results to least squares regression.

Other recent uses of goal programming in relation to statistical methods include its use to model user preferences in small area statistics [115]; the use of fuzzy goal programming to optimise multi-response problems [4]; and a weighted goal programming model to optimize multiresponse-multivariate surfaces in complex experiments [62].

21.3.2 Goal Programming and Other Distance-Metric Based Approaches

The distance metric based approaches in multi-criteria decision making are defined as those techniques that minimise a function of the distance between the desired and achieved solutions. The desired solution could be a decision maker set of

Table 21.1 Distance metrics used in MCDM distance-based techniques

Technique	Distance metrics used	Nature of b_q
Weighted goal programming	L_1	Decision maker set target values
Chebyshev goal programming	L_∞	Decision maker set target values
Compromise programming	Set of $L_1, \ldots, L_p, \ldots, L_\infty$	Ideal values
Reference point method	L_∞ followed by L_1	Decision maker set reference values

goals for each objective or it could be the ideal point. The principal distance-metric based approaches are goal programming, compromise programming [139], and the reference-point method [133].

The key difference between the distance-metric techniques is mainly that of underlying philosophy and distance metric used. The measure of distance can be summarised by the L_ρ set of distance metrics which have the following algebraic form:

$$Min\ L_\rho = \left[\sum_{q=1}^{Q} \left(\frac{|f_q(x) - b_q|}{k_q} \right)^\rho \right]^{1/\rho}$$

Table 21.1 gives the distance metrics used by the major distance-metric based techniques:

Romero et al. [111] investigate linkages between the distance metric based techniques. They show that the compromise programming with the L_1 distance metric is equivalent to a weighted goal programme with the target values set at ideal levels. Also, the compromise programming with the L_∞ distance metric is equivalent to a Chebyshev goal programme with the target values set at ideal levels. It is noted that the original formulation of compromise programming is intended to use a (compromise) set of distance metrics between $\rho = 1$ and $\rho = \infty$ to give a scale of solutions between ruthless optimisation and balance. In fact, many applications just concentrate in practice on these two end-points of the compromise set, in which case there is a strong connection between the process of compromise programme and the process of solving weighted and Chebyshev goal programmes with optimistic target values. The other distance metric sometime used is $\rho = 2$ which corresponds to a non-linear (quadratic) goal programme with optimistic target values. These theoretical linkages led to the development of the extended goal programming framework [110] as described in Sect. 21.4.3, which encompasses both techniques.

A further linkage is given between the reference point method and goal programming. The reference point method is shown to be able to set into a goal programming framework as an initial Chebyshev goal programme followed by an L_1 Pareto restoration phase by Romero et al. [111] and Ogryczak [103].

21.3.3 Goal Programming and Pairwise Comparison Techniques

The most well-known pairwise comparison technique is the Analytical Hierarchy Process (AHP) [112]. The synergy between goal programming and the AHP can take two forms, which are outlined separately below.

21.3.3.1 Using the AHP to Determine Goal Programming Preferential Weights

There exists a natural combination whereby the AHP can be used to elicit a set of preferential weights for a weighted or Chebyshev goal programme. This methodology was first used by Gass [55] in the context of military planning. It has since been applied to variety of application areas including recently in project selection [76], transportation resource allocation [132], healthcare [86], and energy planning [36]. Mahmoud et al. [92] combine multi-choice goal programming and the AHP in the context of quality management. Jones [70] gives an algorithm for weight sensitivity in goal programming that can incorporate pairwise comparison information in order to define the limits of the sensitivity analysis.

21.3.3.2 Using Goal Programming as a Technique to Derive the Weighting Vector in AHP

The other major combination of AHP and goal programming can be viewed of as a reverse of the first combination. In this case, goal programming is used as an alternative method for deriving the AHP weights rather than the other way around. The standard means of deriving AHP weights is the Eigenvector method, as detailed by Saaty [112]. However there has been a lot of discussion in the literature about the advantages and disadvantages of the eigenvalue methods and other methods have been proposed for deriving the weighting vector from the pairwise comparison matrix [113]. These include the logarithmic weighted goal programming approach of Bryson [14] and the Chebyshev goal programming approach of Despotis [41]. Jones and Mardle [72] give a generic distance-metric framework that encompasses the works of Bryson and Despotis. A methodology for producing interval weighting vectors from interval pairwise comparisons is given by Wang [130]. Gong et al. [58] present a methodology that encompasses intuitionistic fuzzy preference relations.

21.3.4 Goal Programming and Other Multi-Criteria Decision Analysis Techniques

Goal programming is traditionally regarded as an a priori multi-criteria decision making technique. That is, all the preference information is specified by the decision making prior to solving the model. However, there are studies either combine the goal programme with one of the other two major classes of Multi-Criteria Decision Analysis techniques—interactive and a posteriori methods—or modify the goal programme into one of these two classes. These are detailed below.

21.3.4.1 Goal Programming and Interactive Methods

There have been several works that look at the formal intersection of interactive techniques with goal programming dating back to Dyer [44]. Gardiner and Steuer [54] classify the principal interactive methods from the wider field of MCDM into a unified framework. Tamiz and Jones [123] present an interactive method for weighted and lexicographic goal programmes and discuss the design of interactive methods for goal programming.

There are also specialist formal interactive goal programming methods for several variants. Caballero et al. [18] give an interactive meta-goal programming approach. Interactive methods are particularly used in the fuzzy goal programming variant, with [6] in the context of bi-level programming; [87, 116] in the context of supply chain management; and [1] in the context of transportation being recent examples. In addition, there are many articles which use goal programming in an iterative or repeated manner, sometimes within a larger multi-technique modelling and solution system. This leads to a more informal, but nevertheless effective, form of interactive use of goal programming. Recent examples of this type of interaction include within a land use planning decision and support system [66] and within a environmental management tool to identify best available techniques [95].

21.3.4.2 Goal Programming and A Posteriori Techniques

The a posteriori techniques in MCDM are concerned with generating Pareto Efficient solutions before eliciting the decision maker(s) preferences. This could either be by classical methods, as detailed by Steuer [119] or by evolutionary methods [40]. The major use of these techniques for goal programming relate to the topic of Pareto Efficiency detection and restoration. These allow points on, or a portion of, the Efficient set that dominates a goal programming solution that is Pareto Inefficient to be generated. The first algorithms for this purpose are given by Hannan [60] and Romero [108] which can be applied to the lexicographic and weighted variants. Tamiz and Jones [124] develop a model that detects and restore Pareto Efficiency in accordance with the decision makers preferences.

This is extended to the integer and binary goal programming variants by Tamiz et al. [128]. Romero et al. [111] and Ogryczak [103] provide detection and restoration methods for the Chebyshev variant. Caballero and Hernandez [16] deal with the detection and restoration of fractional goal programmes. Most recently, Larbani and Aouni [80] develop a general purpose detection and restoration methodology for goal programming.

Goal programming and efficient set generation solutions to a model can also be generated by the same computer package and compared. An example of this is given by the MOPEN package [17].

21.3.4.3 Goal Programming and Discrete Choice/Outranking Methods

The discrete choice and outranking methods are used to choose between or provide a ranking of a discrete number of alternative solutions in the presence of multiple criteria. Although there is potential for combination with goal programming there are not as yet many examples in the literature. One possible combination is to use an outranking method to provide preference information for a goal programme. Martel and Aouni [94] use the Promethee method for this purpose. Another possibility is to use the discrete choice method to choose between or the outranking method to rank a number of goal programme solutions produced by different weighting schemes or variants. Perez Gladish et al. [106] apply this methodology to a mutual fund portfolio selection problem using the ELECTRE I method as the outranking technique. Most recently Yilmaz and Dagdeviren [137] combine the fuzzy Promethee method with binary lexicographic goal programming in order to model an equipment selection decision process.

21.3.5 Goal Programming and Computing/Artificial Intelligence Techniques

21.3.5.1 Goal Programming and Pattern Recognition

Pattern recognition models classify a set of observations into a number of groups based on their characteristics. The most common situation and easily analyzable is the two-group classification problem where items have to be classified into one of two distinct groups by consideration of a number of attributes of the item. Two group classification models arise in fields such as finance (creditworthy of non-creditworthy), medicine (benign or malignant), and defence (friendly or hostile). A related field is that of discriminant analysis, which concentrates on the values and nature of the factors used to discriminate the different classes. Normally, a training set of observations whose class and attributes are known is available in order to train or form the model in its classification. An overview of different methods used to solve pattern classification models is given by Zhai et al. [140].

The use of mathematical programming techniques for pattern classification is described by Baek and Ignizio [9]. Freed and Glover [51, 52] describe and analyze the use of linear and goal programming methods for pattern classification and discriminant analysis [98] give a goal programming model that uses a piecewise linear discriminant line for the purposes of classification. Nakayama et al. [99] examine the use of goal programming to assist the support vector machine (SVM) approach to pattern classification [71] give a framework for modelling two group pattern classification and discriminant analysis using different goal programming variants. Most recently, Pendharkar and Troutt [105] propose a weighted goal programming model to assist in DEA-based dimensionality reduction for a class of classification problems.

21.3.5.2 Goal Programming and Fuzzy Logic

The major combination between the [138] theory of fuzzy numbers and goal programming is that of the fuzzy goal programming variant, which is detailed in Sect. 21.2.7. However, there is also benefit to be gained from combining goal programming with the wider field of fuzzy logic in order to configure decision maker systems that can both process fuzzy information and make satisficing decisions based on the outcome of the fuzzy logic process. A recent example of this combination is found in Famuyiwa et al. [47] who use fuzzy logic to identify and classify potential suppliers in the supplier selection problem and goal programming to subsequently choose suppliers according to the manufacturer's preferences. Similarly, Nepal et al. [101] use a combination of fuzzy logic and the Chebyshev goal programming variant for a quality design problem.

21.3.5.3 Goal Programming and Meta Heuristic Methods

Meta-Heuristic methods allow for solution of goal programming models which are too complex to be solved by conventional exact methods. A selection of uses of popular meta-heuristics used to solve goal programmes is given below.

- **Genetic Algorithms**: Ghoseiri and Ghannadpour [56] in the context of vehicle routing); Wang and Chang [131] in the context of network topology design; Leung [85] in the context of transportation planning; and Stewart et al. [121] in the context of land use planning. Mishra et al. [97] combine concepts from genetic algorithms and simulated annealing to solve a fuzzy goal programming model relating to machine-tool selection and operation allocation.
- **Simulated Annealing**: Baykasoglu [11] presents a general method for solving lexicographic goal programmes using simulated annealing. Mishra et al. [97] combine concepts from genetic algorithms and simulated annealing to solve a

fuzzy goal programming model relating to machine-tool selection and operation allocation. Zolfaghari et al. [141] use simulated annealing to solve a goal programme for to the multi-period task assignment model.

- **Tabu Search**: Yang and Feng [136] present a tabu search approach to solve a stochastic goal programme related to solid transportation.
- **Ant Colony Optimisation**: Chan and Swarnkar [21] use ant colony optimisation to solve a fuzzy goal programme for machine-tool selection and operation allocation.
- **Artificial Immune Systems**: Chan et al. [20] use a combination of artificial immune systems and goal programming for a e-procurement design model.
- **Scatter Search**: Caballero et al. [15] use a multiple-objective adaptation of a scatter search heuristic to solve a non-linear integer goal programme related to sawmill planning.

21.3.6 Goal Programming and Data Envelopment Analysis

The technique of data-envelopment analysis originated from two of the three authors of goal programming. In 1978 Charnes et al. [31] gave the first DEA formulation, with the purpose of measuring the efficiency of decision making units. Glover and Sueyoshi [57] describe DEA as having its roots in the goal programming [27, 30] and fractional programming [28] work of Charnes and Cooper.

The similarities between DEA and goal programming are well documented. The technical details of the mathematical similarities between the weighted goal programming and additive DEA techniques are given in Cooper [34] and the historical context of the two techniques given by Glover and Sueyoshi [57]. The philosophical differences between the two techniques are also emphasised by Cooper [34], who states the goal programming is primarily a planning technique whereas DEA is primarily a control and evaluation technique.

Given the different purposes of the two techniques, there is potential benefit in their combination to form integrated planning and control systems. A recent example is given by Dharmapala et al. [42] in the context of academic salary planning. Goal programming is used as a planning tool to set the average salaries and then DEA is used as a measurement tool to set the merit payments for individual staff. An integrated GP/DEA approach with both planning and performance measurement aspects is given in the context of transportation planning by Sheth et al. [117]. Another possibility is to use DEA to first eliminate some inefficient units before using goal programming as a selection tool amongst the efficient units. This approach is applied by Ekinci [46] in the context of supplier selection. Lam and Bai [79] use a goal programming model in order to minimise deviations from the means of both inputs and outputs in the DEA model. Stewart [120] develops a goal programming model that extends the DEA analysis to include managerial goals.

21.4 Application of Goal Programming

Goal programming has been used extensively to solve classical models arising in Operational Research and a range of applications from a wide variety of fields. A non-exhaustive list includes recent goal programming versions, applications, or combinations from the following fields:

Agricultural Planning Babic and Peric [8] use weighted and lexicographic goal programming models to determine optimal livestock food blends. Manzano-Agugliaro and Canero-Leon [93] present a weighted goal programme to model economic and environmental effects for intensive agriculture. Fleskens and de Graaff [50] compare a number of goal programming variants for olive orchard management.

Business Management Garcia et al. [53] compare weighted, Chebyshev, and extended goal programming models for ranking firms. Stewart [120] uses goal programming to provide a benchmarking analysis that includes managerial goals.

Defence Lee et al. [82] use a combinations of the AHP and weighted goal programming to solve a weapons selection problem.

Energy Planning Cowan et al. [36] combine the AHP and goal programming in order to explore the impact of technology development and adoption for sustainable hydroelectric power and storage technologies.

Engineering de Oliveira and Saramago [39] contrast goal programming with other multi-criteria methods for two engineering problems.

Facility Location Kanoun et al. [77] present a weighted goal programming with non-standard preference functions for emergency services location.

Finance Abdelaziz et al. [2] present a stochastic goal programming model for portfolio selection. Ballestero et al. [10] present a stochastic goal programming model to select portfolios with fuzzy beta values.

Forecasting Coshall and Charlesworth [35] present a lexicographic goal programme for tourism forecasting.

Forestry Lundstrom et al. [91] present a quadratic goal programming model for Boreal forest management. Gonzalez-Pachon and Romero [59] present an extended goal programming model that assists in achieving a consensus in a forestry management problem.

HealthCare Adan et al. [3] present a weighted goal programming model for surgery scheduling. Oddoye et al. [102] present a combine of simulation and weighted binary goal programming to plan resources on a medical assessment unit. Jerbi and Kamoun [67] combine simulation and weighted goal programming to devise a schedule for outpatient appointments at a hospital.

Human Resource Management de Andres et al. [38] present an extended goal programming model for performance appraisal.

Manufacturing Kara et al. [78] present a fuzzy binary goal programming for assembly line balancing. Arunraj and Maiti [7] present a lexicographic goal programme for risk-based maintenance with a case study in a chemical plant.

Marketing Jha et al. [68] use a lexicographic goal programme for multi-project media planning.

Supply chain management Liao and Kao [88] combine a fuzzy TOPSIS method and multi-choice goal programming to model a supplier selection model. Lotfi et al. [90] use goal programming to plan supermarket space allocation and inventory policy. Jolai et al. [69] combine TOPSIS and weighted goal programming for a multi-product, multi-supplier selection problem.

Tourism Tsai et al. [129] use a combination of binary goal programming, the analytical network process (ANP), and activity based costing (ABC) to aid decision making in the hotel industry.

Transportation Ghoseiri and Ghannadpour [56] combine goal programming and genetic algorithms to solve a multiple objective vehicle routing problem. Chang and Lee [25] apply goal programming to an airport selection process.

21.5 Conclusions

It can be seen from the review presented in this paper that goal programming is a active and growing technique. The number of variants of goal programming continues to increase and there is a growing awareness that variants can be combined in order to increase modelling flexibility. There is also a growing body of literature concerning the combination of integration of goal programming with other techniques from multi-criteria decision making, operational research, computing and artificial intelligence techniques. The purpose of these combinations could be to increase the modelling power of goal programming; to allow solution of ill structured or difficult to solve goal programmes; or to encompass a new application area that requires a mix of techniques. Goal programming remains a very applied technique with a range of modelling application papers detailed in Sect. 21.4 giving evidence of this fact.

References

1. Abd El-Wahed, W.F., Lee, S.M.: Interactive fuzzy goal programming for multi-objective transportation problems. Omega-Int. J. Manag. Sci. **34**, 158–166 (2006)
2. Abdelaziz, F.B., El Fayedh, R., Rao, A.: A discrete stochastic goal program for portfolio selection: the case of United Arab Emirates Equity Market. INFOR **47**, 5–13 (2009)

3. Adan, I., Bekkers, J., Dellaert, N., Jeunet, J., Vissers, J.: Improving operational effectiveness of tactical master plans for emergency and elective patients under stochastic demand and capacitated resources. Eur. J. Oper. Res. **213**, 290–308 (2011)
4. Amiri, M., Salehi-Sadaghiani, J.: A methodology for optimizing statistical multi-response problems using fuzzy goal programming. Scientia Iranica **15**, 389–397 (2008)
5. Arenas, M., Bilbao, A., Perez, B., Rodriguez, M.V.: Fuzzy extended lexicographic goal programming. In: LopezDiaz, M., Gil, M.A., Grzegorzewski, P., Hryniewicz, O., Lawry, J. (eds.) Soft Methodology and Random Information Systems, Advances in Soft Computing, pp. 543–550. Springer, Berlin (2004)
6. Arora, S.R., Gupta, R.: Interactive fuzzy goal programming approach for bilevel programming problem. Eur. J. Oper. Res. **194**, 368–376 (2008)
7. Arunraj, N.S., Maiti, J.: Risk-based maintenance policy selection using AHP and goal programming. Saf. Sci. **48**, 238–247 (2010)
8. Babic, Z., Peric, T.: Optimization of livestock feed blend by use of goal programming. Int. J. Prod. Econ. **130**, 218–223 (2011)
9. Baek, W., Ignizio, J.P.: Pattern-classification via linear-programming. Comput. Ind. Eng. **25**, 393–396 (1993)
10. Ballestero, E., Perez-Gladish, B., Arenas-Parra, M., Bilbao-Terol, A.: Selecting portfolios given multiple Eurostoxx-based uncertainty scenarios: a stochastic goal programming approach from fuzzy betas. INFOR **47**, 59–70 (2009)
11. Baykasoglu, A.: Preemptive goal programming using simulated annealing. Eng. Optim. **37**, 49–63 (2005)
12. Bhattacharya, A.: A goal programming approach for developing pre-harvest forecasts of crop yield. J. Oper. Res. Soc. **57**, 1014–1017 (2006)
13. Brans, J.P., Vincke, P., Mareschal, B.: A preference ranking organization method. Manag. Sci. **31**, 647–656 (1985)
14. Bryson, N.: A goal programming method for generating priority vectors. J. Oper. Res. Soc. **46**, 641–648 (1995)
15. Caballero, R., Gomez, T., Molina, J., Fosado, O., Leon, M.A., Garofal, M., Saavedra, B.: Sawing planning using a multicriteria approach. J. Ind. Manag. Optim. **8,** 303**–**317 (2009)
16. Caballero, R., Hernandez, M.: Restoration of efficiency in a goal programming problem with linear fractional criteria. Eur. J. Oper. Res. **172**, 31–39 (2006)
17. Caballero, R., Luque, M., Molina, J., Ruiz, F.: MOPEN: a computational package for linear multiobjective and goal programming problems. Decis. Support Syst. **41**, 160–175 (2005)
18. Caballero, R., Ruiz, F., Uria, M.V., Romero, C.: Interactive meta-goal programming. Eur. J. Oper. Res. **175**, 135–154 (2006)
19. Can, E.K., Houck, M.H.: Real time reservoir operations by goal programming. J. Water Resour. Plan. Manag. **110**, 297–309 (1984)
20. Chan, F.T.S., Shukla, M., Tiwari, M.K., Shankar, R., Choy, K.L.: B2B multi-attribute e-procurement: an artificial immune system based goal programming approach. Int. J. Prod. Res. **49**, 321–341 (2011)
21. Chan, F.T.S., Swarnkar, R.: Ant colony optimization approach to a fuzzy goal programming model for a machine tool selection and operation allocation problem in an FMS. Robot. Comput-Integr. Manuf. **22**, 353–362 (2006)
22. Chang, C.T.: Mixed binary interval goal programming. J. Oper. Res. Soc. **57**, 469–473 (2006)
23. Chang, C.T.: Multi-choice goal programming. Omega-Int. J. Manag. Sci. **35**, 389–396 (2007)
24. Chang, C.T.: Revised multi-choice goal programming. Appl. Math. Model. **32**, 2587–2595 (2008)
25. Chang, Y.C., Lee, N.: A multi-objective goal programming airport selection model for low-cost carriers' networks. Transp. Res. E-Logist. Transp. Rev. **46**, 709–718 (2010)
26. Charnes, A., Collomb, B.: Optimal economic stabilization policy: linear goal-interval programming models. Socioecon. Plann. Sci. **6**, 431–435 (1972)
27. Charnes, A., Cooper, W.W.: Management Models and Industrial Applications of Linear Programming. Wiley, New York (1961)

28. Charnes, A., Cooper, W.W.: Programming with linear fractional functionals. Nav. Res. Logist. Q. **9**, 181–186 (1961)
29. Charnes, A., Cooper, W.W., Harrald, J., Karwan, K., Wallace, W.: A goal interval programming model for resource allocation in a marine environmental protection problem. J. Environ. Econ. Manag. **3**, 347–362 (1976)
30. Charnes, A., Cooper, W.W., Ferguson, R.: Optimal estimation of executive compensation by linear programming. Manag. Sci. **1**, 138–151 (1955)
31. Charnes, A., Cooper, W.W., Rhodes, E.: Measuring the efficiency of decision making units. Eur. J. Oper. Res. **2**, 429–444 (1978)
32. Charnes, A., Cooper, W.W., Sueyoshi, T.: A goal programming constrained regression review of the Bell system breakup. Manag. Sci. **34**, 1–26 (1988)
33. Chen, K.H., Su, C.T.: Activity assigning of fourth party logistics by particle swarm optimization-based preemptive fuzzy integer goal programming. Expert Syst. Appl. **37**, 3630–3637 (2010)
34. Cooper, W.W.: Origins, uses of, and relations between goal programming and data envelopment analysis. J. Multi-Criteria Decis. Anal. **15**, 3–11 (2005)
35. Coshall, J.T., Charlesworth, R.: A management orientated approach to combination forecasting of tourism demand. Tour. Manage. **32**, 759–776 (2010)
36. Cowan, K., Daim, T., Anderson, T.: Exploring the impact of technology development and adoption for sustainable hydroelectric power and storage technologies in the Pacific Northwest United States. Energy **35**, 4771–4779 (2010)
37. Da Silva, L.M.S., Rodriguez, L.C.E., Caixeta, J.V., Bauch, S.C.: Fitting a taper function to minimize the sum of absolute deviations. Scientia Agricola **63**, 460–470 (2006)
38. de Andres, R., Garcia-Lapresta, J.L., Gonzalez-Pachon, J.: Performance appraisal based on distance function methods. Eur. J. Oper. Res. **207**, 1599–1607 (2010)
39. de Oliveira, L.S., Saramago, S.F.P.: Multiobjective optimization techniques applied to engineering problems. J. Brazilian Soc. Mech. Sci. Eng. **32**, 94–105 (2010)
40. Deb, K.: Multi-Objective Optimization Using Evolutionary Algorithms. Wiley, New York (2001)
41. Despotis, D.K.: Fractional minmax goal programming: a unified approach to priority estimation and preference analysis in MCDM. J. Oper. Res. Soc. **47**, 989–999 (1996)
42. Dharmapala, P.S., Ghosh, J.B., Saber, H.M.: Market- and merit-based adjustment of faculty salaries. Asia-Pacific J. Oper. Res. **24**, 1–19 (2007)
43. Dua, R., Bhandari, N., Kumar, V.: Multi-criteria optimization for obtaining efficiently blended transformer oils. IEEE Trans. Dielectr. Electr. Insul. **15**, 879–887 (2008)
44. Dyer, J.S.: Interactive goal programming. Manag. Sci. **19**, 62–70 (1973)
45. Ehrgott, M., Gandibleux, X.: Multiobjective combinatorial optimization—theory, methodology, and applications. In: Ehrgott, M., Gandibleux, X. (eds.) Multi-Criteria Optimization—State of the Art Annotated Bibliographic Surveys, pp. 369–444. Kluwer Academic Publishers, Boston (2002)
46. Ekinci, Y.: Demand assignment: a DEA and goal programming approach. In: Demiralp, M., Udriste, C., Bognar, G., Soni, R., Nassar, H. (eds.) Applied Mathematics for Science and Engineering, pp. 394–397. World Scientific and Engineering Acad and Soc, Athens (2007)
47. Famuyiwa, O., Monplaisir, L., Nepal, B.: An integrated fuzzy-goal-programming-based framework for selecting suppliers in strategic alliance formation. Int. J. Prod. Econ. **113**, 862–875 (2008)
48. Fasakhodi, A.A., Nouri, S.H., Amini, M.: Water resources sustainability and optimal cropping pattern in farming systems; a multi-objective fractional goal programming approach. Water Resour. Manag. **24**, 4639–4657 (2010)
49. Flavell, R.B.: A new goal programming formulation. Omega **4**, 731–732 (1976)
50. Fleskens, L., de Graaff, J.: Conserving natural resources in olive orchards on sloping land: alternative goal programming approaches towards effective design of cross-compliance and agri-environmental measures. Agr. Syst. **103**, 521–534 (2010)

51. Freed, N., Glover, F.: Simple but powerful goal programming-models for discriminant problems. Eur. J. Oper. Res. **7**, 44–60 (1981)
52. Freed, N., Glover, F.: Resolving certain difficulties and improving the classification power of LP discriminant-analysis formulations. Decis. Sci. **17**, 589–595 (1986)
53. Garcia, F., Guijarro, F., Moya, I.: A goal programming approach to estimating performance weights for ranking firms. Comput. Oper. Res. **37**, 1597–1609 (2010)
54. Gardiner, L., Steuer, R.: Unified interactive multiple objective programming. Eur. J. Oper. Res. **74**, 391–406 (1994)
55. Gass, S.I.: The setting of weights in linear goal-programming problems. Comput. Oper. Res. **14**, 227–229 (1987)
56. Ghoseiri, K., Ghannadpour, S.F.: Multi-objective vehicle routing problem with time windows using goal programming and genetic algorithm. Appl. Soft Comput. **10**, 1096–1107 (2010)
57. Glover, F., Sueyoshi, T.: Contributions of Professor William W Cooper in operations research and management science. Eur. J. Oper. Res. **197**, 1–16 (2009)
58. Gong, Z.W., Li, L.S., Forrest, J., Zhao, Y.: The optimal priority models of the intuitionistic fuzzy preference relation and their application in selecting industries with higher meteorological sensitivity. Expert Syst. Appl. **38**, 4394–4402 (2011)
59. Gonzalez-Pachon, J., Romero, C.: The design of socially optimal decisions in a consensus scenario. Omega-Int. J. Manag. Sci. **39**, 179–185 (2011)
60. Hannan, E.L.: Nondominance in goal programming. INFOR **18**, 300–309 (1980)
61. Hannan, E.L.: On fuzzy goal programming. Decis. Sci. **12**, 522–531 (1981)
62. Hejazi, T.H., Bashiri, M., Noghondarian, K., Atkinson, C.: Multiresponse optimization with consideration of probabilistic covariates. Qual. Reliab. Eng. Int. **27**, 437–449 (2010)
63. Ignizio, J.P.: Goal Programming and Extensions. Lexington Books, Lexington (1976)
64. Ignizio, J.P.: Optimal maintenance headcount allocation: an application of Chebyshev goal programming. Int. J. Prod. Res. **42**, 201–210 (2004)
65. Ignizio, J.P., Cavalier, T.: Linear Programming. Prentice-Hall, Upper Saddle River (1994)
66. Janssen, R., van Herwijnen, M., Stewart, T.J., Aerts, J.C.J.H.: Multiobjective decision support for land-use planning. Environ. Plann. B-Plann. Des. **35**, 740–756 (2008)
67. Jerbi, B., Kamoun, H.: Multiobjective study to implement outpatient appointment system at Hedi Chaker Hospital. Simul. Model. Pract. Theory **19**, 1363–1370 (2011)
68. Jha, P.C., Aggarwal, R., Gupta, A.: Optimal media planning for multi-products in segmented market. Appl. Math. Comput. **217**, 6802–6818 (2011)
69. Jolai, F., Yazdian, S.A., Shahanaghi, K., Khojasteh, M.A.: Integrating fuzzy TOPSIS and multi-period goal programming for purchasing multiple products from multiple suppliers. J. Purch. Supply Manag. **17**, 42–53 (2011)
70. Jones, D.F.: A practical weight sensitivity algorithm for goal and multiple objective programming. Eur. J. Oper. Res. **213**, 238–245 (2011)
71. Jones, D.F., Collins, A., Hand, C.: A classification model based on goal programming with non-standard preference functions with application to prediction of cinema-going behaviour. Eur. J. Oper. Res. **177**, 515–524 (2007)
72. Jones, D.F., Mardle, S.J.: A distance-metric methodology for the derivation of weights from a pairwise comparison matrix. J. Oper. Res. Soc. **55**, 869–875 (2004)
73. Jones, D.F., Tamiz, M.: Improving the flexibility of goal programming via preference modelling techniques. Omega **23**, 41–48 (1995)
74. Jones, D.F., Tamiz, M.: Goal Programming in the period 1990–2000. In: Ehrgott, M., Gandibleux, X. (eds.) Multi-Criteria Optimization: State of the art Annotated Bibliographic Surveys, pp. 129–170. Kluwer, Dordrecht (2002)
75. Jones, D.F., Tamiz, M.: Practical Goal Programming. Springer Books, New York (2010)
76. Kahraman, C., Buyukozkan, G.: A combined fuzzy AHP and fuzzy goal programming approach for effective six-sigma project selection. J. Multiple-Valued Log. Soft Comput. **14**, 599–615 (2008)
77. Kanoun, I., Chabchoub, H., Aouni, B.: Goal programming model for fire and emergency service facilities site selection. INFOR **48**, 143–153 (2010)

78. Kara, Y., Gokcen, H., Atasagun, Y.: Balancing parallel assembly lines with precise and fuzzy goals. Int. J. Prod. Res. **48**, 1685–1703 (2010)
79. Lam, K.F., Bai, F.: Minimizing deviations of input and output weights from their means in data envelopment analysis. Comput. Ind. Eng. **60**, 527–533 (2011)
80. Larbani, M., Aouni, B.: A new approach for generating efficient solutions within the goal programming model. J. Oper. Res. Soc. **62**, 173–181 (2011)
81. Lee, S.M.: Goal Programming for Decision Analysis. Auerbach, Philadelphia (1972)
82. Lee, J., Kang, S.H., Rosenberger, J., Kim, S.B.: A hybrid approach of goal programming for weapon systems selection. Comput. Ind. Eng. **58**, 521–527 (2010)
83. Lee, A.H.I., Kang, H.Y., Yang, C.Y., Lin, C.Y.: An evaluation framework for product planning using FANP, QFD and multi-choice goal programming. Int. J. Prod. Res. **48**, 3977–3997 (2010)
84. Lee, A.H.I., Kang, H.Y., Chang, C.T.: Fuzzy multiple goal programming applied to TFT-LCD supplier selection by downstream manufacturers. Expert Syst. Appl. **36**, 6318–6325 (2009)
85. Leung, S.C.H.: A non-linear goal programming model and solution method for the multi-objective trip distribution problem in transportation engineering. Optim. Eng. **8**, 277–298 (2007)
86. Li, X., Beullens, P., Jones, D., Tamiz, M.: An integrated queuing and multi-objective bed allocation model with application to a hospital in China. J. Oper. Res. Soc. **60**, 330–338 (2009)
87. Liang, T.F.: Applying fuzzy goal programming to production/transportation planning decisions in a supply chain. Int. J. Syst. Sci. **38**, 293–304 (2007)
88. Liao, C.N., Kao, H.P.: An integrated fuzzy TOPSIS and MCGP approach to supplier selection in supply chain management. Expert Syst. Appl. **38**, 10803–10811 (2011)
89. Lin, H.W., Nagalingam, S.V., Lin, G.C.I.: An interactive meta-goal programming-based decision analysis methodology to support collaborative manufacturing. Robot. Comput-Integr. Manuf. **25**, 135–154 (2009)
90. Lotfi, M.M., Rabbani, M., Ghaderi, S.F.: A weighted goal programming approach for replenishment planning and space allocation in a supermarket. J. Oper. Res. Soc. **62**, 1128–1137 (2011)
91. Lundstrom, J., Ohman, K., Perhans, K., Ronnqvist, M., Gustafsson, L.: Cost-effective age structure and geographical distribution of boreal forest reserves. J. Appl. Ecol. **48**, 133–142 (2011)
92. Mahmoud, H.B., Ketata, R., Ben Romdhane, T., Ben Ahmed, S.: A multiobjective-optimization approach for a piloted quality-management system: a comparison of two approaches for a case study. Comput. Ind. **62**, 460–466 (2011)
93. Manzano-Agugliaro, F., Canero-Leon, R.: Economics and environmental analysis of Mediterranean greenhouse crops. African J. Agric. Res. **5**, 3009–3016 (2010)
94. Martel, J.M., Aouni, B.: Incorporating the decision-makers preferences in the goal-programming model. J. Oper. Res. Soc. **41**, 1121–1132 (1990)
95. Mavrotas, G., Georgopoulou, E., Mirasgedis, S., Sarafidis, Y., Lalas, D., Hontou, V., Gakis, N.: Multi-objective combinatorial optimisation for selecting best available techniques (Bat) in the industrial sector: the COMBAT tool. J. Oper. Res. Soc. **60**, 906–920 (2009)
96. Min, H., Storbeck, J.: On the origin and persistence of misconceptions in goal programming. J. Oper. Res. Soc. **42**, 301–312 (1991)
97. Mishra, S., Prakash, N., Tiwari, M.K., Lashkari, R.S.: A fuzzy goal-programming model of machine-tool selection and operation allocation problem in FMS: a quick converging simulated annealing-based approach. Int. J. Prod. Res. **44**, 43–76 (2006)
98. Nakayama, H., Kagaku, N.: Pattern classification by linear goal programming and its extensions. J. Glob. Optim. **12**, 111–126 (1998)
99. Nakayama, H., Yun, Y.B., Asada, T., Yoon, M.: MOP/GP models for machine learning. Eur. J. Oper. Res. **166**, 756–768 (2005)
100. Narasimhan, R.: Goal programming in a fuzzy environment. Decis. Sci. **11**, 325–336 (1980)

101. Nepal, B., Monplaisir, L., Singh, N.: A methodology for integrating design for quality in modular product design. J. Eng. Des. **17**, 387–409 (2006)
102. Oddoye, J.P., Tamiz, M., Jones, D.F., Schmidt, P.: Combining simulation and goal programming for healthcare planning in a medical assessment unit. Eur. J. Oper. Res. **193**, 250–261 (2009)
103. Ogryczak, W.: On goal programming formulations of the reference point method. J. Oper. Res. Soc. **52**, 691–698 (2001)
104. Paksoy, T., Chang, C.T.: Revised multi-choice goal programming for multi-period, multi-stage inventory controlled supply chain model with popup stores in Guerrilla marketing. Appl. Math. Model. **34**, 3586–3598 (2010)
105. Pendharkar, P.C., Troutt, M.D.: DEA based dimensionality reduction for classification problems satisfying strict non-satiety assumption. Eur. J. Oper. Res. **212**, 155–163 (2011)
106. Perez Gladish, B., Jones, D.F., Tamiz, M., Bilbao Terol, A.: An interactive three stage model for mutual fund portfolio selection. Omega-Int. J. Manag. Sci. **35**, 75–88 (2007)
107. Rodriguez, M.V., Caballero, R., Ruiz, F., Romero, C.: Meta-goal Programming. Eur. J. Oper. Res. **136**, 422–429 (2002)
108. Romero, C.: A Handbook of Critical Issues in Goal Programming. Pergamon Press, Oxford (1991)
109. Romero, C.: Extended lexicographic goal programming: a unifying approach. Omega **29**, 63–71 (2001)
110. Romero, C.: A general structure of achievement function for a goal programming model. Eur. J. Oper. Res. **153**, 675–686 (2004)
111. Romero, C., Tamiz, M., Jones, D.F.: Goal programming, compromise programming and reference point method formulations: linkages and utility interpretations. J. Oper. Res. Soc. **49**(9), 986–991 (1998)
112. Saaty, T.L.: The Analytical Hierarchy Process. McGraw-Hill, New York (1981)
113. Saaty, T.L., Hu, G.: Ranking by eigenvector versus other methods in the analytic hierarchy process. Appl. Math. Lett. **11**, 121–125 (1998)
114. Schniederjans, M.J.: Goal Programming, Methodology and Applications. Kluwer, Boston (1995)
115. Sedeno-Noda, A., Gonzalez-Davila, E., Gonzalez-Martin, C., Gonzalez-Yanes, A.: Preemptive benchmarking problem: an approach for official statistics in small areas. Eur. J. Oper. Res. **196**, 360–369 (2009)
116. Selim, H., Ozkarahan, I.: A supply chain distribution network design model: an interactive fuzzy goal programming-based solution approach. Int. J. Adv. Manuf. Technol. **36**, 401–418 (2008)
117. Sheth, C., Triantis, K., Teodorovic, D.: Performance evaluation of bus routes: a provider and passenger perspective. Transp. Res. E-Logist. Transp. Rev. **43**, 453–478 (2007)
118. Simon, H.A.: Models of Man. Wiley, New York (1955)
119. Steuer, R.E.: Multiple Criteria Optimization: Theory Computation and Application. Wiley, New York (1986)
120. Stewart, T.J.: Goal directed benchmarking for organizational efficiency. Omega-Int. J. Manag. Sci. **38**, 534–539 (2010)
121. Stewart, T.J., Janssen, R., van Herwijnen, M.: A genetic algorithm approach to multiobjective land use planning. Comput. Oper. Res. **31**, 2293–2313 (2004)
122. Sueyoshi, T.: Least absolute value estimation. J. Oper. Res. Soc. Japan **40**, 261–275 (1997)
123. Tamiz, M., Jones, D.F.: Interactive frameworks for investigation of goal programming models: theory and practice. J. Multicriteria Decis. Anal. **6**, 52–60 (1997)
124. Tamiz, M., Jones, D.F.: Goal programming and Pareto efficiency. J. Inf. Optim. Sci. **17**, 291–307 (1996)
125. Tamiz, M., Jones, D.F, El-Darzi, E.: A review of goal programming and its applications. Ann. Oper. Res. **58**, 39–53 (1995)
126. Tamiz, M., Mirrazavi, S.K., Jones, D.F.: Extensions of Pareto efficiency analysis to integer goal programming. Omega-Int. J. Manag. Sci. **27**, 179–188 (1999)

127. Tamiz, M., Jones, D.F., Romero, C.: Goal programming for decision making: an overview of the current state-of-the-art. Eur. J. Oper. Res. **111**, 569–581 (1998)
128. Tiwari, R.N., Dhahmar, S., Rao, J.R.: Fuzzy goal programming: an additive model. Fuzzy Set. Syst. **24**, 27–34 (1987)
129. Tsai, W.H., Hsu, J.L., Chen, C.H., Lin, W.R., Chen, S.P.: An integrated approach for selecting corporate social responsibility programs and costs evaluation in the international tourist hotel. Int. J. Hosp. Manag. **29**, 385–396 (2010)
130. Wang, Y.M.: A goal programming method for obtaining interval weights from an interval comparison matrix. Eur. J. Oper. Res. **177**, 458–471 (2007)
131. Wang, C.S., Chang, C.T.: Integrated genetic algorithm and goal programming for network topology design problem with multiple objectives and multiple criteria. IEEE-ACM Trans. Netw. **16**, 680–690 (2008)
132. Wey, W.M., Wu, K.Y.: Using ANP priorities with goal programming in resource allocation in transportation. Math. Comput. Model. **46**, 985–1000 (2007)
133. Wierzbicki, A.P.: A mathematical basis for satisficing decision making. Math. Model. **3**, 391–405 (1982)
134. Williams, H.P.: Logic and Integer Programming. Springer, New York (2009)
135. Yaghoobi, M.A., Jones, D.F., Tamiz, M.: Weighted additive models for solving fuzzy goal programming problems. Asia-Pacific J. Oper. Res. **25**, 715–733 (2008)
136. Yang, L.X., Feng, Y.: A bicriteria solid transportation problem with fixed charge under stochastic environment. Appl. Math. Model. **31**, 2668–2683 (2007)
137. Yilmaz, B., Dagdeviren, M.: A combined approach for equipment selection: F-PROMETHEE method and zero–one goal programming. Expert Syst. Appl. **38**, 11641–11650 (2011)
138. Zadeh, L.A.: Fuzzy sets. Inf. Control. **8**, 338–353 (1965)
139. Zeleny, M.: Multi Criteria Decision Making. McGraw Hill, New York (1982)
140. Zhai, J.H., Zhang, S.F., Wang, X.Z.: An overview of pattern classification methodologies. Proc. 2006 Int. Conf. Mach. Learn. Cybern. **1–7**, 3222–3227 (2006)
141. Zolfaghari, S., Jaber, M.Y., Hamoudi, H.: A goal programming approach for a multi-period task assignment problem. INFOR **42**, 299–309 (2004)

Chapter 22
Interactive Nonlinear Multiobjective Optimization Methods

Kaisa Miettinen, Jussi Hakanen, and Dmitry Podkopaev

Abstract An overview of interactive methods for solving nonlinear multiobjective optimization problems is given. In interactive methods, the decision maker progressively provides preference information so that the her or his most satisfactory Pareto optimal solution can be found. The basic features of several methods are introduced and some theoretical results are provided. In addition, references to modifications and applications as well as to other methods are indicated. As the role of the decision maker is very important in interactive methods, methods presented are classified according to the type of preference information that the decision maker is assumed to provide.

Keywords Multiple criteria decision making • Multiple objectives • Nonlinear optimization • Interactive methods • Pareto optimality

22.1 Introduction

Nonlinear multiobjective optimization means multiple criteria decision making involving nonlinear functions of (continuous) decision variables. In these problems, the best possible compromise, that is, Pareto optimal solution, is to be found from an infinite number of alternatives represented by decision variables restricted by constraint functions. Thus, enumerating the solutions is impossible.

K. Miettinen • J. Hakanen
University of Jyvaskyla, Department of Mathematical Information Technology,
P.O. Box 35 (Agora), FI-40014 University of Jyvaskyla, Finland
e-mail: kaisa.miettinen@jyu.fi; jussi.hakanen@jyu.fi

D. Podkopaev (✉)
University of Jyvaskyla, Department of Biological and Environmental Sciences,
P.O. Box 35 (YA), FI-40014 University of Jyvaskyla, Finland
e-mail: dmitry.podkopaev@gmail.com

S. Greco et al. (eds.), *Multiple Criteria Decision Analysis*, International Series in Operations Research & Management Science 233,
DOI 10.1007/978-1-4939-3094-4_22

Solving multiobjective optimization problems usually requires the participation of a human decision maker who is supposed to have insight into the problem and who can express preference relations between alternative solutions or objective functions. Multiobjective optimization methods can be divided into four classes according to the role of the decision maker in the solution process. If the decision maker is not involved, we use methods where no articulation of preference information is used, in other words, *no-preference methods*. If the decision maker expresses preference information after the solution process, we speak about *a posteriori methods* whereas *a priori methods* require articulation of preference information before the solution process. The most extensive class is *interactive methods*, where the decision maker specifies preference information progressively during the solution process. Here we concentrate on this last-mentioned class and introduce several examples of interactive methods.

In the literature, interactive methods have proven useful for various reasons. They have been found efficient from both computational and cognitive points of view. Because the decision maker directs the solution process with one's preferences, only those Pareto optimal solutions that are interesting to her or him need to be calculated. This means savings in computational cost when compared to a situation where a big set of Pareto optimal solutions should be calculated. On the other hand, the amount of new information generated per iteration is limited and, in this way, the decision maker does not need to compare too many solutions at a time. An important advantage of interactive methods is learning. Once the decision maker has provided preferences, (s)he can see from the Pareto optimal solutions generated, how attainable or feasible the preferences were. In this way, the decision maker gains insight about the problem. (S)he learns about the interdependencies between the objective functions and also about one's own preferences. The decision maker can also change her or his mind after the learning, if so desired.

Many real-world phenomena behave in a nonlinear way. Besides, linear problems can always be solved using methods created for nonlinear problems but not vice versa. For these reasons, we here devote ourselves to nonlinear problems. We assume that all the information involved is deterministic and that we have a single decision maker.

In this presentation we concentrate on general-purpose interactive methods and, thus, methods tailored for some particular problem type are not included. In recent years, interactive approaches have been developed in the field of evolutionary multiobjective optimization (see, for example, [14]), but we do not consider them here. The literature survey of years since 2000 has been limited to journal articles. We describe in more detail methods that have published applications.

22.2 Concepts

Let us begin by introducing several concepts and definitions. We study *multiobjective optimization problems* of the form

$$\begin{array}{ll} \text{minimize} & \{f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_k(\mathbf{x})\} \\ \text{subject to} & \mathbf{x} \in S \end{array} \tag{22.1}$$

involving k ($\geq$ 2) *objective functions* or objectives $f_i : \mathbf{R}^n \rightarrow \mathbf{R}$ that we want to minimize simultaneously. The *decision (variable) vectors* $\mathbf{x}$ belong to the (nonempty) *feasible region* $S \subseteq \mathbf{R}^n$. The feasible region is formed by *constraint functions* but we do not fix them here.

We denote the image of the feasible region by $Z \subset \mathbf{R}^k$ and call it a *feasible objective region*. *Objective (function) values* form *objective vectors* $\mathbf{z} = \mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_k(\mathbf{x}))^T$. Note that if f_i is to be maximized, it is equivalent to minimize $-f_i$.

We call a multiobjective optimization problem *convex* if all the objective functions and the feasible region are convex. On the other hand, the problem is *nondifferentiable* if at least one of the objective or the constraint functions is nondifferentiable. (Here nondifferentiability means that the function is not necessarily continuously differentiable but that it is locally Lipschitz continuous.)

We assume that the objective functions are at least partly conflicting and possibly incommensurable. This means that it is not possible to find a single solution that would optimize all the objectives simultaneously. As the definition of optimality we employ Pareto optimality. An objective vector is Pareto optimal (or noninferior or efficient or nondominated) if none of its components can be improved without deterioration to at least one of the other components. More formally, we have the following definition.

Definition 1. A decision vector $\mathbf{x}^* \in S$ is (globally) *Pareto optimal* if there does not exist another decision vector $\mathbf{x} \in S$ such that $f_i(\mathbf{x}) \leq f_i(\mathbf{x}^*)$ for all $i = 1, \ldots, k$ and $f_j(\mathbf{x}) < f_j(\mathbf{x}^*)$ for at least one index j.

An objective vector $\mathbf{z}^* \in Z$ is Pareto optimal if there does not exist another vector $\mathbf{z} \in Z$ such that $z_i \leq z_i^*$ for all $i = 1, \ldots, k$ and $z_j < z_j^*$ for at least one index j; or equivalently, $\mathbf{z}^*$ is Pareto optimal if the decision vector corresponding to it is Pareto optimal.

Local Pareto optimality is defined in a small neighborhood of $\mathbf{x}^* \in S$. Naturally, any globally Pareto optimal solution is locally Pareto optimal. The converse is valid, for example, for convex multiobjective optimization problems; see [21, 139], among others.

For the sake of brevity, we usually speak about Pareto optimality in the sequel. In practice, however, we only have locally Pareto optimal solutions computationally available, unless some additional requirement, such as convexity, is fulfilled or unless we have global solvers available.

A *Pareto optimal set* consists of (an infinite number of) Pareto optimal solutions. In interactive methods, we usually move around the Pareto optimal set and forget the other solutions. However, one should remember that this limitation may have weaknesses. Namely, the real Pareto optimal set may remain unknown. This may be the case if an objective function is only an approximation of an unknown function or if not all the objective functions involved are explicitly expressed.

Moving from one Pareto optimal solution to another necessitates trading off. To be more specific, a *trade-off* reflects the ratio of change in the values of the objective functions concerning the increment of one objective function that occurs when the value of some other objective function decreases (see, for example, [23, 139]).

For any two solutions equally preferable to the decision maker there is a trade-off involving a certain increment in the value of one objective function that the decision maker is willing to tolerate in exchange for a certain amount of decrement in some other objective function while the preferences of the two solutions remain the same. This is called the *marginal rate of substitution* (see, for example, [139] for further details and properties).

Usually, one of the objective functions is selected as a *reference function* when trade-offs and marginal rates of substitution are treated. The pairwise trade-offs and the marginal rates of substitution are generated with respect to it.

Sometimes Pareto optimal sets are not enough but we need wider or smaller sets: weakly and properly Pareto optimal sets, respectively. An objective vector is *weakly Pareto optimal* if there does not exist any other objective vector for which all the components are smaller. Weakly Pareto optimal solutions are sometimes computationally easier to generate than Pareto optimal solutions. Thus, they have relevance from a technical point of view. On the other hand, a vector is *properly Pareto optimal* if unbounded trade-offs are not allowed. For a collection of different definitions of proper Pareto optimality, see, for example, [139].

Multiobjective optimization problems are usually solved by *scalarization* which means that the problem is converted into one or a family of single (scalar) objective optimization problems. This produces a new scalarized problem with a real-valued objective function, possibly depending on some parameters. The resulting new problem must be solved with a single objective optimization method which is appropriate to the characteristics of the problem in question (taking into account, for example, differentiability and convexity). When scalarization is done properly, it can be guaranteed that the solution obtained is Pareto optimal to the original multiobjective optimization problems. For further details see, for example, [139, 210].

Interactive methods differ from each other by the way the problem is transformed into a single objective optimization problem, by the form in which information is provided by the decision maker and by the form in which information is given to the decision maker at each iteration of the solution process.

One way of inquiring the decision maker's opinions is to ask for satisfactory or desirable objective function values. They are called *aspiration levels* and denoted by $\bar{z}_i$, $i = 1, \ldots, k$. They form a vector $\bar{\mathbf{z}} \in \mathbf{R}^k$ to be called a *reference point*.

The ranges of the objective functions in the set of Pareto optimal solutions give valuable information to the decision maker about the possibilities and restrictions of the problem (assuming the objective functions are bounded over S). The components of the *ideal objective vector* $\mathbf{z}^\star \in \mathbf{R}^k$ are the individual optima of the objective functions. This vector represents the lower bounds of the Pareto optimal set. (In nonconvex problems, we need a global solver for minimizing the k functions.) Note that we sometimes need a vector that its strictly better than the ideal objective vector. This vector is called a *utopian objective vector* and denoted by $\mathbf{z}^{\star\star}$.

The upper bounds of the Pareto optimal set, that is, the components of a *nadir objective vector* $\mathbf{z}^{\mathrm{nad}}$, are much more difficult to obtain. Actually, there is no constructive method for calculating the nadir objective vector for nonlinear problems. However, a rough estimate can be obtained by keeping in mind the solutions where each objective function attains its lowest value and calculating the values of the other objectives. The highest value obtained for each objective can be selected as the estimated component of $\mathbf{z}^{\mathrm{nad}}$. This approach was originally proposed in [9] and later named as a *pay-off table* method. Some approaches for estimating the nadir objective vector for nonlinear multiobjective optimization are summarized in [139]. Examples of latest approaches include [37, 38].

It is sometimes assumed that the decision maker makes decisions on the basis of an underlying *value function* $U : \mathbf{R}^k \to \mathbf{R}$ representing her or his preferences among the objective vectors [93]. Even though value functions are seldom explicitly known, they have been important in the development of multiobjective optimization methods and as a theoretical background. Thus, the value function is sometimes presumed to be known implicitly.

The value function is usually assumed to be *strongly decreasing*. In other words, the preferences of the decision maker are assumed to increase if the value of one objective function decreases while all the other objective values remain unchanged. In brief, we can say that *less is preferred to more*. In that case, the maximal solution of U is assured to be Pareto optimal. Note that regardless of the existence of a value function, in what follows, we shall assume that lower objective function values are preferred to higher, that is, less is preferred to more by the decision maker.

An alternative to the idea of maximizing some value function is *satisficing decision making* [210]. In this approach, the decision maker tries to achieve certain aspirations. If the aspirations are achieved, the solution is called a *satisficing solution*.

22.3 Introduction to Interactive Methods

A large variety of methods has been developed for solving multiobjective optimization problems. We can say that none of them is generally superior to all the others. As mentioned earlier, we apply here the classification of the methods into four classes according to the participation of the decision maker in the solution process. This classification was originally suggested in [79] and it was followed later, for example, in [139].

While we discuss interactive methods, we divide them into ad hoc and non ad hoc methods (based on value functions) as suggested in [228]. Even if one knew the decision maker's value function, one would not exactly know how to respond to the questions posed by an *ad hoc* method. On the other hand, in *non ad hoc* methods, the responses can be determined or at least confidently simulated based on a value function.

Before describing the methods, we mention several references for further information. This presentation is mainly based on [139]. Concepts and methods for multiobjective optimization are also treated in [16, 23, 43, 44, 79, 132, 200, 210, 223, 227, 232, 246, 250, 275].

Interactive multiobjective optimization methods, in particular, are collected in [155, 181, 211, 247, 259]. Furthermore, methods with applications to large-scale systems and industry are presented in [65, 220, 236].

We shall not discuss non-interactive methods here. However, we mention some of such methods by name and give references for further information. Examples of no-preference methods are the method of the global criterion [274, 277] and the multiobjective proximal bundle method [145]. From among a posteriori methods we mention the weighting method [56, 276], the ε-constraint method [64] and the hybrid method [31, 258] as well as the method of weighted metrics [277] and the achievement scalarizing function approach [261–263, 265]. Multiobjective evolutionary algorithms are also a posteriori in nature, see, for example, [14, 36] and references therein. A priori methods include the value function method [93], the lexicographic ordering [52] and the goal programming [24, 25, 81, 198, 199].

In what follows, we concentrate on interactive methods. In interactive methods, a solution pattern is formed and repeated several times. After every iteration, some information is given to the decision maker and (s)he is asked to answer some questions or to provide some other type of information. In this way, only a part of the Pareto optimal solutions has to be generated and evaluated, and the decision maker can specify and correct her or his preferences and selections during the solution process when (s)he gets to know the problem better. Thus, the decision maker does not need to have any global preference structure. Further information about the topics treated here can be found in [139, 155].

An interactive method typically contains the following main steps: (1) initialize (for example, calculate ideal and nadir objective vectors and show them to the decision maker), (2) generate a Pareto optimal starting point (some neutral compromise solution or solution given by the decision maker) and show it to the decision maker, (3) ask for preference information from the decision maker (for example, aspiration levels or number of new solutions to be generated, depending on the method in question), (4) generate new Pareto optimal solution(s) according to the preferences and show it/them and possibly some other information about the problem to the decision maker, (5) if several solutions were generated, ask the decision maker to select the best solution so far, and (6) stop, if the decision maker wants to. Otherwise, go to step (3).

Three main stopping criteria can be identified in interactive methods. In the best situation, the decision maker finds a desirable solution and wants to stop. Alternatively, the decision maker gets tired and stops or some algorithmic stopping rule is fulfilled. In the last-mentioned case, one must check that the decision maker agrees to stop.

As a matter of fact, as stated in [155], solving a multiobjective optimization problem with an interactive method can be regarded as a constructive process where, while learning, the decision maker builds a conviction of what is possible (that is, what kind of solutions are available and attainable) and confronting this knowledge with her or his preferences that also evolve. Based on this understanding, in interactive methods we should pay attention to psychological convergence, rather than to mathematical convergence (like, for example, optimizing some value function).

Sometimes, two different phases can be identified in interactive solution processes: *learning phase* and *decision phase* [155]. In the learning phase, the decision maker learns about the problem and gains understanding of what kind of solutions are attainable whereas the most preferred solution is found in the decision phase in the region identified in the first phase. Naturally, the two phases can also be used iteratively.

In what follows, we present several interactive methods. The idea is to describe a collection of methods based on different approaches. In addition, plenty of references are included. Note that although all the calculations take place in the decision variable space, we mostly speak about the corresponding objective vectors and refer to both as solutions since the space is apparent from the context.

When presenting the methods we apply the classification given in [129, 205] according to the type of preference information that the methods utilize. This is an important aspect because a reliable and an understandable way of extracting preference information from the decision maker is essential for the success of applying interactive methods. The decision maker must feel being in control and must understand the questions posed. Otherwise, the answers cannot be relied on in the solution process. It is also important to pay attention to the cognitive load set on the decision maker, as discussed in [115]. Applying the method should not set too much cognitive load on the decision maker.

In the first class, the decision maker specifies aspiration levels (in other words, a reference point) representing desirable objective function values. In the second class, the decision maker provides a classification indicating which of the objective function values should be improved, maintained at the current value or allowed to impair. One should note that providing aspiration levels and a classification are closely related as justified in [150]. From classification information one can derive a reference point but not vice versa. The third class is devoted to methods where the decision maker compares different solutions and chooses a solution among several ones. The fourth class involves marginal rates of substitution referring to the amount of decrement in the value of one objective function that compensates to the decision maker an infinitesimal increment in the value of another objective function while the values of other objective functions remain unaltered. In addition to the four classes given in [129, 205], we consider a fifth class devoted to navigation based methods where the decision maker moves around in the set of Pareto optimal solutions in real time and controls the direction of movement in different ways.

22.4 Methods Using Aspiration Levels

What is common to the methods in this section is a reference point consisting of desirable aspiration levels. With a reference point, the decision maker can conveniently express one's desires without any cognitive mapping as (s)he gives objective function values and obtains objective function values generated by the method. Some of the methods in this section utilize other types of preference information as well but the reference point is an integral element of each method.

22.4.1 Reference Point Method

The reference point method [260, 261, 263] is based on vectors formed of reasonable or desirable aspiration levels. These reference points are used to derive scalarizing functions having minimal values at weakly, properly or Pareto optimal solutions.

No specific assumptions are set in this method. The idea is to direct the search by changing the reference point $\bar{\mathbf{z}}^h$ (at iteration h) in the spirit of satisficing decision making rather than optimizing any value function. It is important that reference points are intuitive and easy for the decision maker to specify and their consistency is not an essential requirement.

Note that specifying a reference point can be considered as a way of classifying the objective functions. If the aspiration level is lower than the current objective value, that objective function is currently unacceptable, and if the aspiration level is equal to or higher than the current objective value, that function is acceptable. The difference here is that the reference point can be infeasible in every component. Naturally, trading off is unavoidable in moving from one Pareto optimal solution to another and it is impossible to get a solution where all objective values are better than in the previous Pareto optimal solution but different solutions can be obtained with different approaches.

Scalarizing functions used in the reference point method are so-called achievement (scalarizing) functions and the method relies on their properties. We can define so-called order representing and order approximating achievement functions.

An example of a scalarized problem with an order representing achievement function is

$$\begin{aligned} \text{minimize} \quad & \max_{i=1,\ldots,k} [\, w_i(f_i(\mathbf{x}) - \bar{z}_i^h) \,] \\ \text{subject to} \quad & \mathbf{x} \in S, \end{aligned} \tag{22.2}$$

where **w** is some fixed positive weighting vector. An example of a scalarized problem with an order-approximating achievement function is

$$\begin{aligned} &\text{minimize} \quad \max_{i=1,\ldots,k} [\, w_i(f_i(\mathbf{x}) - \bar{z}_i^h)\,] + \rho \sum_{i=1}^{k} w_i(f_i(\mathbf{x}) - \bar{z}_i^h) \\ &\text{subject to} \quad \mathbf{x} \in S, \end{aligned} \tag{22.3}$$

where **w** is as above and $\rho > 0$.

Theorem 1. *If the achievement function is order-representing, then its solution is weakly Pareto optimal. If the function is order-approximating, then its solution is Pareto optimal and the solution is properly Pareto optimal if the function is also strongly increasing. Any (weakly) Pareto optimal solution can be found if the achievement function is order representing. Finally, any properly Pareto optimal solution can be found if the function is order-approximating.*

The reference point method is very simple. Before the solution process starts, some information is given to the decision maker about the problem. If possible, the ideal objective vector and the (approximated) nadir objective vector are presented. Another possibility is to minimize and maximize the objective functions individually in the feasible region (if it is bounded). Naturally, the maximized objective function values do not typically represent components of the nadir objective vector but they can give some information to the decision maker in any case.

The basic steps of the reference point algorithm are the following:

1. Select the achievement function. Present information about the problem to the decision maker. Set $h = 1$.
2. Ask the decision maker to specify a reference point $\bar{\mathbf{z}}^h \in \mathbf{R}^k$.
3. Minimize the achievement function and obtain a (weakly, properly or) Pareto optimal solution $\mathbf{z}^h$. Present it to the decision maker.
4. Calculate a number of k other (weakly, properly or) Pareto optimal solutions with perturbed reference points $\bar{\mathbf{z}}(i) = \bar{\mathbf{z}}^h + d^h \mathbf{e}^i$, where $d^h = \|\bar{\mathbf{z}}^h - \mathbf{z}^h\|$ and $\mathbf{e}^i$ is the ith unit vector for $i = 1, \ldots, k$.
5. Present the alternatives to the decision maker. If (s)he finds one of the $k + 1$ solutions satisfactory, stop. Otherwise, ask the decision maker to specify a new reference point $\bar{\mathbf{z}}^{h+1}$. Set $h = h + 1$ and go to step 3.

The idea in perturbing the reference point in step 4 is that the decision maker gets a better conception of the possible solutions around the current solution. If the reference point is far from the Pareto optimal set, the decision maker gets a wider description of the Pareto optimal set and if the reference point is near the Pareto optimal set, then a finer description of the Pareto optimal set is given.

In this method, the decision maker has to specify aspiration levels and compare objective vectors. The decision maker is free to change her or his mind during the process and can direct the solution process without being forced to understand complicated concepts and their meaning. On the other hand, the method does not necessarily help the decision maker to find more satisfactory solutions.

The reference point method is an ad hoc method because a reference point cannot directly be defined based on a value function. On the other hand, alternatives are easy to compare whenever a value function is known.

Let us mention that a software family called DIDAS (Dynamic Interactive Decision Analysis and Support) has been developed on the basis of the reference point ideas of Wierzbicki. It is described, for example, in [267].

Applications and modifications of the reference point method are provided in [11, 62, 159, 186, 187, 215, 217, 219, 231, 245, 248, 249, 264, 266].

22.4.2 GUESS Method

The GUESS method is also called a *naïve method* [18]. The method is related to the reference point method.

It is assumed that a global ideal objective vector $\mathbf{z}^{\star}$ and a global nadir objective vector $\mathbf{z}^{\text{nad}}$ are available. The structure of the method is very simple: the decision maker specifies a reference point (or a guess) $\bar{\mathbf{z}}^h$ and a Pareto optimal solution is generated which is somehow closest to the reference point. Then the decision maker specifies a new reference point and so on.

The general idea is to maximize the minimum weighted deviation from the nadir objective vector. The problem to be solved is

$$\begin{aligned} \text{maximize} \quad & \min_{i=1,\dots,k} \left[\frac{z_i^{\text{nad}} - f_i(\mathbf{x})}{z_i^{\text{nad}} - \bar{z}_i^h} \right] \\ \text{subject to} \quad & \mathbf{x} \in S. \end{aligned} \tag{22.4}$$

Notice that the aspiration levels have to be strictly lower than the components of the nadir objective vector.

Theorem 2. *The solution of (22.4) is weakly Pareto optimal and any Pareto optimal solution can be found.*

The GUESS algorithm has five basic steps.

1. Calculate $\mathbf{z}^{\star}$ and $\mathbf{z}^{\text{nad}}$ and present them to the decision maker. Set $h = 1$.
2. Let the decision maker specify upper or lower bounds to the objective functions if (s)he so desires. Update the problem, if necessary.
3. Ask the decision maker to specify a reference point $\bar{\mathbf{z}}^h$ between $\mathbf{z}^{\star}$ and $\mathbf{z}^{\text{nad}}$.
4. Solve (22.4) and present the solution to the decision maker.
5. If the decision maker is satisfied, stop. Otherwise, set $h = h + 1$ and go to step 2.

In step 2, upper or lower bounds mean adding constraints to the problem (22.4), but the ideal or the nadir objective vectors are not affected. The only stopping rule is the satisfaction of the decision maker. No guidance is given to the decision maker in setting new aspiration levels. This is typical of many reference point based methods.

The GUESS method is simple to use and no consistency of the preference information provided is required. The only information required from the decision maker is a reference point and possible upper and lower bounds, which are optional. Note that inappropriate lower bounds may lead to solutions that are not weakly Pareto optimal. Unfortunately, the GUESS method relies heavily on the availability of the nadir objective vector, which is usually only an estimation.

The GUESS method is an ad hoc method. The existence of a value function would not help in specifying reference points or bounds for the objective functions. The method has been compared to several other interactive methods in [17, 20, 33] and it has performed surprisingly well. The reasons may be its simplicity and flexibility. One can say that decision makers seem to prefer solution methods where they can feel that they are in control.

22.4.3 Light Beam Search

The light beam search [83, 84] employs tools of multiattribute decision analysis (see, for example, [250]) together with reference point ideas. The basic setting is identical to the reference point method. The problem to be solved is

$$\begin{aligned} &\text{minimize} \quad \max_{i=1,\dots,k} [\, w_i(f_i(\mathbf{x}) - \bar{z}_i^h)\,] + \rho \sum_{i=1}^{k} (f_i(\mathbf{x}) - \bar{z}_i^h) \\ &\text{subject to} \quad \mathbf{x} \in S, \end{aligned} \tag{22.5}$$

where $\mathbf{w}$ is a weighting vector, $\bar{\mathbf{z}}^h$ is the current reference point and $\rho > 0$.

Theorem 3. *The solution of (22.5) is properly Pareto optimal and any properly Pareto optimal solution can be found.*

The reference point is here assumed to be infeasible, that is, unattainable. It is also assumed that the objective and the constraint functions are continuously differentiable and that the objective functions are bounded over S. Furthermore, none of the objective functions is allowed to be more important than all the others together.

In the light beam search, the decision maker directs the search by specifying reference points. In addition, other solutions in the neighbourhood of the current solution are displayed. Thus, the idea is identical to that of the reference point method. The main difference is in the way the alternatives are generated. The motivation is to avoid comparing too similar alternatives or alternatives that are indifferent to the decision maker. To achieve this goal, concepts of ELECTRE methods (developed for handling with discrete problems in multiattribute decision analysis) are utilized (see, for example, [202]).

It is not always possible for the decision maker to distinguish between different alternatives. This means that there is an interval where indifference prevails. For this reason, the decision maker is asked to provide *indifference thresholds* for each objective function. The line between indifference and preference does not have to be sharp, either. The hesitation between indifference and preference can be expressed by *preference thresholds*. Finally, a *veto threshold* prevents a good performance in some objectives from compensating for poor values on some other objectives.

In the light beam search, *outranking relations* are established between alternatives. An objective vector $\mathbf{z}^1$ is said to outrank $\mathbf{z}^2$ if $\mathbf{z}^1$ is at least as good as $\mathbf{z}^2$. The idea is to generate k new alternative objective vectors such that they outrank the current solution. In particular, incomparable or indifferent alternatives are not shown to the decision maker. The alternatives to be shown are called *characteristic neighbours*. The neighbours are determined by projecting the gradient of one objective function at a time onto the linear approximation of those constraints that are active in the current solution.

We can now outline the light beam algorithm.

1. If the decision maker can specify the best and the worst values for each objective function, denote them by $\mathbf{z}^{\star}$ and $\mathbf{z}^{\text{nad}}$, respectively. Alternatively, calculate $\mathbf{z}^{\star}$ and $\mathbf{z}^{\text{nad}}$. Set $h = 1$ and $\bar{\mathbf{z}}^h = \mathbf{z}^{\star}$. Initialize the set of saved solutions as $B = \emptyset$. Ask the decision maker to specify an indifference threshold for each objective. If desired, (s)he can also specify preference and veto thresholds.
2. Calculate current Pareto optimal solution $\mathbf{z}^h$ by solving (22.5).
3. Present $\mathbf{z}^h$ to the decision maker. Calculate k Pareto optimal characteristic neighbours of $\mathbf{z}^h$ and present them as well to the decision maker. If the decision maker wants to see alternatives between any two of the $k + 1$ alternatives displayed, set their difference as a search direction, take different steps in this direction and project them onto the Pareto optimal set before showing them to the decision maker. If the decision maker wants to save $\mathbf{z}^h$, set $B = B \cup \{\mathbf{z}^h\}$.
4. If the decision maker wants to revise the thresholds, save them, set $\mathbf{z}^h = \mathbf{z}^{h+1}$, $h = h + 1$ and go then to step 3. If the decision maker wants to give another reference point, denote it by $\bar{\mathbf{z}}^{h+1}$, set $h = h + 1$ and go to step 2. If the decision maker wants to select one of the alternatives or one solution in B as a current solution, set it as $\mathbf{z}^{h+1}$, set $h = h + 1$ and go to step 3. If one of the alternatives is satisfactory, stop.

The option of saving desirable solutions in the set B increases the flexibility of the method. A similar option could be added to many other methods as well.

The name of the method comes from the idea of projecting a focused beam of light from the reference point onto the Pareto optimal set. The lighted part of the Pareto optimal set changes if the location of the spotlight, that is, the reference point or the point of interest in the Pareto optimal set are changed.

In the light beam search, the decision maker specifies reference points, compares alternatives and affects the set of alternatives in different ways. Specifying different thresholds may be demanding for the decision maker. Note, however, that the thresholds are not constant but can be altered at any time. The developers of the method point out that it may be computationally rather demanding to find the exact characteristic neighbours in a general case. It is, however, noteworthy that the neighbours can be generated in parallel.

The light beam search is an ad hoc method because a value function could not directly determine new reference points. It could, however, be used in comparing alternatives. Remember that the thresholds are important here and they must come from the decision maker.

A modification of the method is described in [264].

22.4.4 Other Methods Using Aspiration Levels

Many interactive methods of the class of methods using aspiration levels originate from the goal programming approach because the interpretation of a goal and a reference point are closely related. Examples of such methods include [133, 162, 185, 218, 237, 255]. Methods adopting a fuzzy approach to setting aspiration levels have been proposed in [75, 77, 160, 209]. Some other aspiration level based interactive methods can be found in [13, 34, 61, 70, 97, 124, 182, 235, 238, 254, 256, 257].

22.5 Methods Using Classification

With a classification, the decision maker can express what kind of changes should be made to the current Pareto optimal solution to get a more desirable solution. Classification reminds the decision maker of the fact that it is not possible to improve all objective values of a Pareto optimal objective vector but impairment in some objective(s) must be allowed. The methods presented in this section utilize different numbers of classes. Some of the methods involve preference information other than classification but classification is the core element in all of them.

22.5.1 Step Method

The step method (STEM) [9] is one of the first interactive methods developed for multiobjective optimization problems. Here we describe an extension for nonlinear problems according to [46] and [210], pp. 268–269.

STEM is based on the classification of the objective functions at the current iteration at $\mathbf{z}^h = \mathbf{f}(\mathbf{x}^h)$. It is assumed that the decision maker can indicate both functions that have acceptable values and those whose values are too high, that is, functions that are unacceptable. Then the decision maker is supposed to give up a little in the value(s) of some acceptable objective function(s) f_i (denoted by $i \in I^>$) in order to improve the values of some unacceptable objective functions f_i (denoted by $i \in I^<$) (here $I^> \cup I^< = \{1, \ldots, k\}$). To be more specific, the decision maker is asked to specify upper bounds $\varepsilon_i^h > f_i(\mathbf{x}^h)$ for the functions in $I^>$.

The only requirement in the method is that the objective functions are bounded over S because distances are measured to the (global) ideal objective vector. The first problem to be solved is

$$\begin{aligned} \text{minimize} \quad & \max_{i=1,\ldots,k} \left[\frac{e_i}{\sum_{j=1}^k e_j} (f_i(\mathbf{x}) - z_i^\star) \right] \\ \text{subject to} \quad & \mathbf{x} \in S, \end{aligned} \tag{22.6}$$

where $e_i = \frac{1}{z_i^\star} \frac{z_i^{\text{nad}} - z_i^\star}{z_i^{\text{nad}}}$ as suggested in [46], or $e_i = \frac{z_i^{\text{nad}} - z_i^\star}{\max\left[|z_i^{\text{nad}}|, |z_i^\star|\right]}$ as suggested in [247].

Theorem 4. *The solution of (22.6) is weakly Pareto optimal. The problem has at least one Pareto optimal solution.*

After the decision maker has classified the objective functions, the feasible region is restricted according to the information of the decision maker. The weights of the relaxed objective functions are set equal to zero, that is $e_i = 0$ for $i \in I^>$. Then a new distance minimization problem

$$\begin{aligned} \text{minimize} \quad & \max_{i=1,\ldots,k} \left[\frac{e_i}{\sum_{j=1}^k e_j} (f_i(\mathbf{x}) - z_i^\star) \right] \\ \text{subject to} \quad & f_i(\mathbf{x}) \le \varepsilon_i^h \ \text{ for all } \ i \in I^>, \\ & f_i(\mathbf{x}) \le f_i(\mathbf{x}^h) \ \text{ for all } \ i \in I^<, \\ & \mathbf{x} \in S \end{aligned} \tag{22.7}$$

is solved.

The basic phases of the STEM algorithm are the following:

1. Calculate $\mathbf{z}^\star$ and $\mathbf{z}^{\text{nad}}$ and the weighting coefficients. Set $h = 1$. Solve (22.6). Denote the solution by $\mathbf{z}^h \in Z$.
2. Ask the decision maker to classify the objective functions at $\mathbf{z}^h$ into $I^>$ and $I^<$. If the latter class is empty, stop. Otherwise, ask the decision maker to specify relaxed upper bounds ε_i^h for $i \in I^>$.
3. Solve (22.7) and denote the solution by $\mathbf{z}^{h+1} \in Z$. Set $h = h + 1$ and go to step 2.

The solution process continues until the decision maker does not want to change any component of the current objective vector. If the decision maker is not satisfied with any of the components, then the procedure must also be stopped.

In STEM, the decision maker is moving from one weakly Pareto optimal solution to another. The idea of classification is quite simple for her or him. However, it may be difficult to estimate appropriate amounts of increment that would allow the desired amount of improvement in those functions whose values should be decreased.

STEM is an ad hoc method because the existence of a value function would not help in the classification process.

Applications and modifications of STEM are given in [6, 23, 35, 79, 86].

22.5.2 Satisficing Trade-Off Method

The satisficing trade-off method (STOM) [174, 176] utilizes classification and reference points. As its name suggests, STOM is based on satisficing decision making. The decision maker is asked to classify the objective functions at the current solution $\mathbf{z}^h = \mathbf{f}(\mathbf{x}^h)$ into three classes: the unacceptable objective functions whose values should be improved ($I^<$), the acceptable objective functions whose values may increase ($I^>$) and the acceptable objective functions whose values are acceptable as they are (denoted by $I^=$) (such that $I^< \cup I^> \cup I^= = \{1, \ldots, k\}$).

The decision maker only has to specify aspiration levels for the functions in $I^<$. The aspiration levels (that is, upper bounds) for the functions in $I^>$ can be derived using so-called automatic trade-off. In addition, the aspiration levels for the functions in $I^=$ are set equal to $f_i(\mathbf{x}^h)$. All the three kinds of aspiration levels form a reference point $\bar{\mathbf{z}}^h$.

Different scalarizing functions can be used in STOM. One alternative is to solve the scalarized problem

$$\begin{aligned} &\text{minimize} \quad \max_{i=1,\ldots,k} \left[\frac{f_i(\mathbf{x}) - z_i^{\star\star}}{\bar{z}_i^h - z_i^{\star\star}} \right] \\ &\text{subject to} \quad \mathbf{x} \in S, \end{aligned} \tag{22.8}$$

where the reference point must be strictly worse in each component than the utopian objective vector.

Theorem 5. *The solution of (22.8) is weakly Pareto optimal and any Pareto optimal solution can be found.*

If weakly Pareto optimal solutions are to be avoided, the scalarized problem to be solved is

$$\begin{aligned} &\text{minimize} \quad \max_{i=1,\ldots,k} \left[\frac{f_i(\mathbf{x}) - z_i^{\star\star}}{\bar{z}_i^h - z_i^{\star\star}} \right] + \rho \sum_{i=1}^{k} \frac{f_i(\mathbf{x})}{\bar{z}_i^h - z_i^{\star\star}} \\ &\text{subject to} \quad \mathbf{x} \in S, \end{aligned} \tag{22.9}$$

where $\rho > 0$ is some sufficiently small scalar.

Theorem 6. *The solution of (22.9) is properly Pareto optimal and any properly Pareto optimal solution can be found.*

Here the utopian objective vector must be known globally. However, if some objective function f_j is not bounded from below on S, then some small scalar value can be used as $z_j^{\star\star}$.

Assuming all the functions involved are differentiable, the scalarizing functions can be written in a differentiable form by introducing a scalar variable α to be optimized and setting it as an upper bound for each function in the max-term. Under certain assumptions, trade-off rate information can be obtained from the Karush-Kuhn-Tucker multipliers connected to the solution of this formulation. In *automatic trade-off*, upper bounds for the functions in $I^>$ are derived with the help of this trade-off information.

Let us now describe the STOM algorithm.

1. Select the scalarizing function. Calculate $\mathbf{z}^{\star\star}$. Set $h = 1$.
2. Ask the decision maker to specify a reference point $\bar{\mathbf{z}}^h \in \mathbf{R}^k$ such that $\bar{z}_i^h > z_i^{\star\star}$ for every $i = 1, \ldots, k$.
3. Minimize the scalarizing function used. Denote the solution by $\mathbf{z}^h$. Present it to the decision maker.
4. Ask the decision maker to classify the objective functions. If $I^< = \emptyset$, stop. Otherwise, ask the decision maker to specify new aspiration levels $\bar{z}_i^{h+1}$ for $I \in I^<$. Set $\bar{z}_i^{h+1} = z_i^h$ for $i \in I^=$.
5. Use automatic trade-off to obtain new levels (upper bounds) $\bar{z}_i^{h+1}$ for the functions in $I^>$. Set $h = h + 1$ and go to step 3.

The decision maker can modify the levels calculated based on trade-off rate information if they are not agreeable. On the other hand, the decision maker can specify those upper bounds herself or himself, if so desired. If trade-off rate information is not available, for example, in a case when the functions are nondifferentiable, STOM is almost the same as the GUESS method. The only difference is the scalarizing function used.

There is no need to repeat comments mentioned in connection with STEM and the GUESS method. In all of them, the role of the decision maker is easy to understand. STOM requires even less input from the decision maker if automatic trade-off is used.

As said before, in practice, classifying the objective functions into three classes and specifying the amounts of increment and decrement for their values is a subset of specifying a new reference point. A new reference point is implicitly formed.

STOM is an ad hoc method like all the other classification based methods. However, one must remember that the aim of the method is particularly in satisficing rather than optimizing some value function.

Modifications and applications of STOM are described in [96, 158, 168–176, 178–180, 189, 253].

22.5.3 Reference Direction Method

In the classification based reference direction (RD) method [183, 184], a current objective vector $\mathbf{z}^h$ is presented to the decision maker at iteration h, and (s)he is asked to specify a reference point $\bar{\mathbf{z}}^h$ consisting of desired levels for the objective functions. However, as the idea is to move around the weakly Pareto optimal set, some objective functions must be allowed to increase in order to attain lower values for some other objectives.

As mentioned earlier, specifying a reference point is equivalent to an implicit classification indicating those objective functions whose values should be decreased till they reach some acceptable aspiration level, those whose values are satisfactory at the moment, and those whose values are allowed to increase to some upper bound. We denote again these three classes by $I^<$, $I^=$ and $I^>$, respectively. Furthermore, we denote the components of the reference point corresponding to $I^>$ by ε_i^h (at iteration h) because they represent upper bounds.

Here, steps are taken in the reference direction $\bar{\mathbf{z}}^h - \mathbf{z}^h$ and the decision maker specifies a priori the number of steps to be taken, that is, the number of solutions to be generated. The idea is to move step by step as long as the decision maker wants to. In this way, extra computation is avoided when only those alternatives are calculated that the decision maker wants to see.

Alternatives are generated along the reference direction by solving the problem

$$\begin{aligned} &\text{minimize} \quad \max_{i \in I^<} \left[\frac{f_i(\mathbf{x}) - z_i^h}{z_i^h - \bar{z}_i^h} \right] \\ &\text{subject to} \quad f_i(\mathbf{x}) \le \varepsilon_i^h + \alpha(z_i^h - \varepsilon_i^h) \ \text{ for all } \ i \in I^>, \\ &\qquad\qquad\quad f_i(\mathbf{x}) \le z_i^h \ \text{ for all } \ i \in I^=, \\ &\qquad\qquad\quad \mathbf{x} \in S, \end{aligned} \tag{22.10}$$

where $0 \le \alpha < 1$ is the step-size in the reference direction, $\bar{z}_i^h < z_i^h$ for $i \in I^<$ and $\varepsilon_i^h > z_i^h$ for $i \in I^>$.

Theorem 7. *The solution of (22.10) is weakly Pareto optimal for every* $0 \le \alpha < 1$ *and any Pareto optimal solution can be found.*

The steps of the RD algorithm are the following:

1. Find a starting solution $\mathbf{z}^1$ and show it to the decision maker. Set $h = 1$.
2. If the decision maker does not want to decrease any component of $\mathbf{z}^h$, stop. Otherwise, ask the decision maker to specify $\bar{\mathbf{z}}^h$, where some of the components are lower and some higher or equal when compared to those of $\mathbf{z}^h$. If there are no higher values, set $P = r = 1$ and go to step 3. Otherwise, ask the decision maker to specify the maximum number of alternatives P (s)he wants to see. Set $r = 1$.
3. Set $\alpha = 1 - r/P$. Solve (22.10) and get $\mathbf{z}^h(r)$. Set $r = r + 1$.
4. Show $\mathbf{z}^h(r)$ to the decision maker. If (s)he is satisfied, stop. If $r \le P$ and the decision maker wants to see another solution, go to step 3. Otherwise, if $r > P$ or the decision maker wants to change the reference point, set $\mathbf{z}^{h+1} = \mathbf{z}^h(r)$, $h = h + 1$ and go to step 2.

The RD method does not require artificial or complicated information from the decision maker; only reference points and the number of intermediate solutions are used. Some decision makers may appreciate the fact that they are not asked to compare several alternatives but only to decide whether another alternative is to be generated or not.

The decision maker must a priori determine the number of steps to be taken, and then intermediate solutions are calculated one by one as long as the decision maker wants to. This has both positive and negative sides. On one hand, it is computationally efficient since it may be unnecessary to calculate all the intermediate solutions. On the other hand, the number of steps to be taken cannot be changed.

The RD method is an ad hoc method because a value function would not help in specifying reference points or the numbers of steps to be taken. It could not even help in selecting the most preferred alternative. Here one must decide for one solution at a time whether to calculate new alternative solutions or not. If the new alternative happens to be less preferred than its predecessor, one cannot return to the previous solution.

Applications and modifications of the RD method are described in [60, 148].

22.5.4 NIMBUS Method

The NIMBUS method was originally presented in [139, 145, 148] but here we describe the so-called synchronous version introduced in [151]. Originally, NIMBUS was particularly directed for nondifferentiable problems but nowadays it is a general interactive multiobjective optimization method for nonlinear problems.

NIMBUS offers flexible ways of performing interactive consideration of the problem and determining the preferences of the decision maker during the solution process. Classification is used as the means of interaction between the decision maker and the algorithm. In addition, the decision maker can ask for intermediate Pareto optimal solutions to be generated between any two Pareto optimal solutions.

In the classification, the decision maker can easily indicate what kind of improvements are desirable and what kind of impairments are tolerable. Opposedite to the classification based methods introduced so far, NIMBUS has five classes available. The decision maker examines at every iteration h the current objective vector $\mathbf{z}^h$ and divides the objective functions into up to five classes according to how the current solution should be changed to get a more desirable solution. The classes are functions f_i whose values

- should be decreased ($i \in I^<$),
- should be decreased till an aspiration level $\bar{z}_i^h < z_i^h$ ($i \in I^\leq$),
- are satisfactory at the moment ($i \in I^=$),
- are allowed to increase till an upper bound $\varepsilon_i^h > z_i^h$ ($i \in I^>$), and
- are allowed to change freely ($i \in I^\diamond$),

where $I^< \cup I^\leq \neq \emptyset$ and $I^> \cup I^\diamond \neq \emptyset$.

In addition to the classification, the decision maker is asked to specify the aspiration levels and the upper bounds if the second and the fourth class, respectively, are used. The difference between the classes $I^{<}$ and $I^{\leq}$ is that the functions in $I^{<}$ are to be minimized as far as possible but the functions in $I^{\leq}$ only as far as the aspiration level.

As mentioned, NIMBUS has more classes than STEM, STOM or the RD method. This means that the decision maker has more freedom and flexibility in specifying the desired changes in the objective values. Note that not all of the classes have to be used. The availability of the class $I^{\diamond}$ means that some functions can be left unclassified for a while to be able to follow how their values change while the others are classified.

After the classification information has been obtained, a scalarized problem is solved and the Pareto optimal solution $\hat{x}^h$ obtained reflects the desires of the decision maker as well as possible. In this way, the decision maker can learn about the attainability of her or his preferences. In the synchronous version of NIMBUS [151], the idea is to provide to the decision maker up to four slightly different Pareto optimal solutions based on the same preference information. The decision maker can decide how many solutions (s)he wants to see and compare. In this way, the decision maker can learn more about what kind of solutions are available in the area of the Pareto optimal set that (s)he is interested in.

After the classification, up to four scalarized problems are solved. The one that follows the classification information closest is

$$\begin{aligned}
\text{minimize} \quad & \max_{\substack{i\in I^{<}\\ j\in I^{\leq}}} \left[\frac{f_i(\mathbf{x}) - z_i^{\star}}{z_i^{\text{nad}} - z_i^{\star\star}}, \frac{f_j(\mathbf{x}) - \bar{z}_j}{z_j^{\text{nad}} - z_j^{\star\star}} \right] + \rho \sum_{i=1}^{k} \frac{f_i(\mathbf{x})}{z_i^{\text{nad}} - z_i^{\star\star}} \\
\text{subject to} \quad & f_i(\mathbf{x}) \leq f_i(\mathbf{x}^h) \;\; \text{for all} \;\; i \in I^{<} \cup I^{\leq} \cup I^{=}, \\
& f_i(\mathbf{x}) \leq \varepsilon_i \;\; \text{for all} \;\; i \in I^{\geq}, \\
& \mathbf{x} \in S,
\end{aligned} \tag{22.11}$$

where a so-called augmentation coefficient $\rho > 0$ is a relatively small scalar, and $z_i^{\star}$ for $i \in I^{<}$ are components of the ideal objective vector. The weighting coefficients $1/(z_j^{\text{nad}} - z_j^{\star\star})$ involving components of the nadir and the utopian objective vectors, respectively, have proven to facilitate capturing the preferences of the decision maker well. They also increase computational efficiency [154].

The other three problems are based on a reference point. As mentioned in Sect. 22.1, one can derive a reference point from classification information. If the decision maker has provided aspiration levels and upper bounds, they are directly used as components of the reference point. Similarly it is straightforward to use the current objective function value of the class $I^{=}$. In the class $I^{<}$, the component of the ideal objective vector is used in the reference point and in the class $I^{\diamond}$, the component of the nadir objective vector is used. In this way, we can get a k-dimensional reference point and can solve reference point based scalarized problems. In the synchronous NIMBUS, the problems (22.4) of GUESS, (22.3) of the reference point method and (22.8) of the STOM method are used.

Theorem 8. *The solution of (22.11) is Pareto optimal.*

The decision maker can also ask for intermediate solutions between any two Pareto optimal solutions $\mathbf{x}^h$ and $\hat{\mathbf{x}}^h$ to be generated. This means that we calculate a search direction $\mathbf{d}^h = \hat{\mathbf{x}}^h - \mathbf{x}^h$ and provide more solutions by taking steps of different sizes in this direction. In other words, we generate $P - 1$ new vectors $\mathbf{f}(\mathbf{x}^h + t_j \mathbf{d}^h)$, $j = 2, \ldots, P - 1$, where $t_j = \frac{j-1}{P-1}$. Their Pareto optimal counterparts [by setting each of the new vectors at a time as a reference point for (22.3)] are presented to the decision maker, who then selects the most satisfying solution among the alternatives.

The NIMBUS algorithm is given below. The solution process stops if the decision maker does not want to improve any objective function value or is not willing to impair any objective function value.

We denote the set of saved solutions by A. At the beginning, we set $A = \emptyset$. The starting point of the solution process can come from the decision maker or it can be some neutral compromise [265] between the objectives. The nadir and utopian objective vectors must be calculated or estimated before starting the solution process.

The main steps of the synchronous NIMBUS algorithm are the following.

1. Generate a Pareto optimal starting point.
2. Ask the decision maker to classify the objective functions at the current solution and to specify the aspiration levels and upper bounds if they are needed.
3. Ask the decision maker to select the maximum number of different solutions to be generated (between one and four) and solve as many problems (listed above).
4. Present the different new solutions obtained to the decision maker.
5. If the decision maker wants to save one or more of the new solutions to A, include it/them to A.
6. If the decision maker does not want to see intermediate solutions between any two solutions, go to step 8. Otherwise, ask the decision maker to select the two solutions from among the new solutions or the solutions in A. Ask the number of the intermediate solutions from the decision maker.
7. Generate the desired number of intermediate solutions and project them to the Pareto optimal set. Go to step 4.
8. Ask the decision maker to choose the most preferred one among the new and/or the intermediate solutions or the solutions in A. Denote it as the current solution. If the decision maker wants to continue, go to step 2. Otherwise, stop.

In NIMBUS, the decision maker is free to explore the Pareto optimal set, to learn and also to change her or his mind if necessary. The selection of the most preferred alternative from a given set is also possible but not necessary. The decision maker can also eliminate undesirable solutions from further consideration. Unlike some other classification based methods, NIMBUS does not depend entirely on how well the decision maker manages in the classification. It is important that the classification is not irreversible. If the solution obtained is not satisfactory, the decision maker can go back to the previous solution or explore intermediate solutions. The method aims at being flexible and the decision maker can select to what extent (s)he exploits the versatile possibilities available. The method does not introduce too massive calculations, either.

Being a classification based method, NIMBUS is ad hoc in nature. A value function could only be used to compare different alternatives.

An implementation of NIMBUS is available on the Internet. This WWW-NIMBUS system is at the disposal of every academic Internet user at http://nimbus.mit.jyu.fi/. Positive sides of a WWW implementation are that the latest version of the system is always available and the user saves the trouble of installing the software. The operating system used or compilers available set no restrictions because all that is needed is a WWW browser. Furthermore, WWW provides a graphical user interface with possibilities for visualizing the classification phase, alternative solutions etc. The system contains both a nondifferentiable local solver and a global solver (genetic algorithm). For details, see [147, 149, 151]. The first version of WWW-NIMBUS was implemented in 1995. Then, it was a pioneering interactive optimization system on the Internet.

There is also an implementation of NIMBUS in the Windows/Linux operating systems called IND-NIMBUS [82, 140]. It can be connected to different simulation and modelling tools like Matlab and GAMS. Several local and global single objective optimization methods and their hybrids are available. It is also possible to utilize, for example, the optimization methods of GAMS. IND-NIMBUS has different tools for supporting graphical comparison of selected solutions and it also contains implementations of the Pareto Navigator method and the NAUTILUS method (see Sects. 22.8.2 and 22.6.2, respectively).

Applications and modifications of the NIMBUS method can be found in [47, 66–68, 71, 72, 117, 118, 141, 145, 146, 148, 150, 152, 153, 157, 206, 214, 222].

22.5.5 Other Methods Using Classification

Interactive physical programming is an interactive method developed for trade-off analysis and decision making in multidisciplinary optimization [240]. It is based on a physical programming approach to produce Pareto optimal solutions [136]. A second order approximation of the Pareto optimal set at the current Pareto optimal solution is produced and the decision maker is able to generate solutions in the approximation obeying her or his classification. However, this necessitates differentiability assumptions. A modification can be found in [76].

Some other classification based methods can be found in [7, 92, 138].

22.6 Methods Where Solutions Are Compared

In this section we present some methods where the decision maker is assumed to compare Pareto optimal solutions and select one of them. Thus, the decision maker is not assumed to provide much information but the cognitive load related to the comparison naturally depends on the number of alternatives to be considered.

22.6.1 Chebyshev Method

The Chebyshev method was originally called the Tchebycheff method. It was proposed in [223], pp. 419–450 and [226] and refined in [224] and it is also known by the name *interactive weighted Tchebycheff procedure*. The idea in this weighting vector set reduction method is to develop a sequence of progressively smaller subsets of the Pareto optimal set until a final solution is located.

This method does not have too many assumptions. All that is assumed is that the objective functions are bounded (from below) over S. To start with, a (global) utopian objective vector $\mathbf{z}^{\star\star}$ is established. Then the distance from the utopian objective vector to the feasible objective region is minimized by solving the scalarized problem

$$\begin{aligned} &\text{lex minimize} \quad \max_{i=1,\dots,k} \left[w_i^h (f_i(\mathbf{x}) - z_i^{\star\star}) \right], \sum_{i=1}^{k} (f_i(\mathbf{x}) - z_i^{\star\star}) \\ &\text{subject to} \quad \mathbf{x} \in S. \end{aligned} \tag{22.12}$$

The notation above means that if the min-max problem does not have a unique solution, the sum term is minimized subject to the obtained solutions.

Theorem 9. *The solution of (22.12) is Pareto optimal and any Pareto optimal solution can be found.*

In the Chebyshev method, different Pareto optimal solutions are generated by altering the weighting vector $\mathbf{w}^h$. At each iteration h, the weighting vector set $W^h = \{\mathbf{w}^h \in \mathbf{R}^k \mid l_i^h < w_i^h < u_i^h,\ \sum_{i=1}^k w_i^h = 1\}$ is reduced to W^{h+1}, where $W^{h+1} \subset W^h$. At the first iteration, a sample of the whole Pareto optimal set is generated by solving (22.12) with well dispersed weighting vectors from $W = W^1$ (with $l_i^1 = 0$ and $u_i^1 = 1$). The space W^h is reduced by tightening the upper and the lower bounds for the weights.

Let $\mathbf{z}^h$ be the objective vector that the decision maker chooses from the sample at the iteration h and let $\mathbf{w}^h$ be the corresponding weighting vector in the problem. Now a concentrated group of weighting vectors centred around $\mathbf{w}^h$ is formed. In this way, a sample of Pareto optimal solutions centred around $\mathbf{z}^h$ is obtained.

Before the solution process starts, the decision maker must set the number of alternative solutions P to be compared at each iteration and the number of iterations to be taken *itn*. We can now present the main features of the Chebyshev algorithm.

1. Set the set size P and a tentative number of iterations *itn*. Set $l_i^1 = 0$ and $u_i^1 = 1$ for all $i = 1, \ldots, k$. Construct $\mathbf{z}^{\star\star}$. Set $h = 1$.
2. Form the weighting vector set W^h and generate $2P$ dispersed weighting vectors $\mathbf{w}^h \in W^h$.
3. Solve (22.12) for each of the $2P$ weighting vectors.
4. Present the P most different of the resulting objective vectors to the decision maker and let her or him choose the most preferred among them.
5. If $h = itn$, stop.
6. Reduce W^h to get for W^{h+1}, set $h = h + 1$ and go to step 2.

The problem (22.12) is solved more that P times so that solutions very close to each other do not have to be presented to the decision maker. On the other hand, the predetermined number of iterations is not necessarily conclusive. The decision maker can stop iterating when (s)he obtains a satisfactory solution or continue the solution process longer if necessary.

In this method, the decision maker is only asked to compare Pareto optimal objective vectors. The number of these alternatives and the number of objective functions affect the easiness of the comparison. The personal capabilities of the decision maker are also important. Note that some consistency is required from the decision maker because the discarded parts of the weighting vector space cannot be restored.

It must be mentioned that a great deal of calculation is needed in the method. That is why it may not be applicable for large and complex problems. However, parallel computing can be utilized when generating the alternatives.

The Chebyshev method is a non ad hoc method. It is easy to compare the alternative solutions with the help of a value function.

Applications and modifications of the Chebyshev method are given in [1, 88, 95, 128, 189, 196, 212, 225, 234, 269].

22.6.2 NAUTILUS Method

The NAUTILUS method, introduced in [156], has a different philosophy from many other interactive methods. It is based on the assumptions that past experiences affect the hopes of decision makers and that people do not react symmetrically to gains and losses. This is derived from the prospect theory of [87]. Typically, interactive multiobjective optimization methods move around the set of Pareto optimal solutions according to the preference of the decision maker and (s)he must trade-off, that is, give up in some objective functions in order to enable improvement in some others to get from one Pareto optimal solution to another. But according to the prospect theory, the decision makers may have difficulties in allowing impairment: the decision maker may get anchored in the vicinity of the starting point and the solution process may even be prematurely terminated.

The NAUTILUS method is different from most interactive methods because it does not generate Pareto optimal solutions at every iteration. Instead, the solution process starts from the nadir objective vector representing bad values for all objective functions. In this way, the decision maker can attain improvement in each objective function without any trading-off and can simply indicate how much each of the objectives should be improved. It has also been observed that the decision maker may be more satisfied with a given solution if the previous one was very undesirable, and this lays the foundation of the NAUTILUS method.

The method utilizes the scalarized problem (22.3) of the reference point method but unlike other methods utilizing this problem where weights are kept unaltered during the whole process and their purpose is mainly to normalize different ranges of objectives, in NAUTILUS the weights have a different role as proposed in [127]. In NAUTILUS, the weights are varied to get different Pareto optimal solutions and some preference information is included in the weights. As mentioned earlier, the optimal solution of problem (22.3) is assured to be Pareto optimal for any reference point (see, for example, [139]).

As said, the NAUTILUS method starts from the nadir objective vector and at every iteration the decision maker gets a solution where all objective function values improve from the previous iteration. Thus, only the solution of the last iteration is Pareto optimal. To get started, the decision maker is asked to give the number of iterations (s)he plans to carry out, denoted by *itn*. This is an initial estimate and can be changed at any time.

As before, we denote by $\mathbf{z}^h$ the objective vector corresponding to the iteration h. We set $\mathbf{z}^0 = \mathbf{z}^{\text{nad}}$. Therefore, $\mathbf{z}^0$ (except in trivial problems) is not Pareto optimal. Furthermore, we denote by it^h the number of iterations left (including iteration h). Thus, $it^1 = itn$. At each iteration, the range of reachable values that each objective function can have without impairment in any other objective function (in this and further iterations) will shrink. Lower and upper bounds on these reachable values will be calculated when possible. For iteration h, we denote by $\mathbf{z}^{h,lo} = (z_1^{h,lo}, \ldots, z_k^{h,lo})^T$ and $\mathbf{z}^{h,up} = (z_1^{h,up}, \ldots, z_k^{h,up})^T$ these lower and upper bounds, respectively. Initially, $\mathbf{z}^{1,lo} = \mathbf{z}^\star$ and $\mathbf{z}^{1,up} = \mathbf{z}^{nad}$. This information can be regarded as an actualization of the pay-off table (see, for example, [139]) (indicating new ideal and nadir values) at each iteration, thus informing the decision maker of what values are achievable for each objective function.

For iteration $h-1$, the objective vector $\mathbf{z}^{h-1} = (z_1^{h-1}, \ldots, z_k^{h-1})^T$ is shown to the decision maker, who has two possibilities to provide her or his preference information:

1. Ranking the objective functions according to the relative importance of improving current objective function values. Here the decision maker is not asked to give any global preference ranking of the objectives, but the local importance of improving each of the current objective function values. (S)he is asked to assign objective functions to classes in an increasing order of importance for improving

the corresponding objective value z_i^{h-1}. With this information the k objective functions can be allocated into index sets J_r which represent the importance levels $r = 1, \ldots, s$, where $1 \leq s \leq k$. If $r < t$, then improving objective function values in J_r is less important than improving objective function values in J_t. Each objective function can belong to only one index set, but several objectives can be in the same index set J_r. We then set

$$w_i^h = \frac{1}{r(z_i^{\text{nad}} - z_i^{\star\star})} \quad \text{for all } i \in J_r, \quad r = 1, \ldots, s. \tag{22.13}$$

2. Answering the question: *Assuming you have one hundred points available, how would you distribute them among the current objective values so that the more points you allocate, the more improvement on the corresponding current objective value is desired?* If the decision maker gives p_i^h points to the objective function f_i, we set $\Delta q_i^h = p_i^h/100$ and

$$w_i^h = \frac{1}{\Delta q_i^h(z_i^{\text{nad}} - z_i^{\star\star})} \quad \text{for all } i = 1, \ldots, k. \tag{22.14}$$

We set $\bar{\mathbf{z}}^h = \mathbf{z}^{h-1}$, and $w_i = w_i^h$ $(i = 1, \ldots, k)$, as defined in (22.13) or (22.14), depending on the way the decision maker specifies the preference information and solve the scalarized problem (22.3). Let us denote by $\mathbf{x}^h$ the Pareto optimal decision vector obtained and set $\mathbf{f}^h = \mathbf{f}(\mathbf{x}^h)$. Then, at the next iteration we take a step from the current solution towards $\mathbf{f}^h$ and show to the decision maker

$$\mathbf{z}^h = \frac{it^h - 1}{it^h}\mathbf{z}^{h-1} + \frac{1}{it^h}\mathbf{f}^h. \tag{22.15}$$

As mentioned, if h is the last iteration, then $it^h = 1$ and $\mathbf{z}^h = \mathbf{f}^h$ is the final Pareto optimal objective vector while $\mathbf{x}^h$ is the final solution in the decision space. But if h is not the last iteration, then $\mathbf{z}^h$ can even be an infeasible vector in the objective space. Nevertheless, it has the following properties:

Theorem 10. *At any iteration h, components of $\mathbf{z}^h$ are all better than the corresponding components of $\mathbf{z}^{h-1}$.*

It is important to point out that although $\mathbf{z}^h$ is not a Pareto optimal objective vector of problem (22.1) (if h is not the last iteration), and it may even be infeasible for this problem, it is assured to either be in the feasible objective set Z for problem (22.1) or there is some Pareto optimal objective vector where each objective function has a better value. On the other hand, each objective vector $\mathbf{z}^h$ produced has better objective function values than the corresponding values in all previous iterations. In addition, at each iteration, a part of the Pareto optimal set is eliminated from consideration in the sense that it is not reachable unless a step backwards is taken.

Vectors $\mathbf{z}^{h,lo}$ providing bounds for the objective values that can be attained at the next iteration can be calculated by solving k problems of the ε-constraint method so that each objective function is optimized in turn and the upper bounds for the other objective functions are taken from the corresponding components of $\mathbf{z}^{h-1}$.

Thus, the attainable values of $\mathbf{z}^h$ are bound in the following way:

$$z_i^h \in [z_i^{h,lo}, z_i^{h-1}] \quad (i = 1, \ldots, k).$$

By denoting $\mathbf{z}^{h,up} = \mathbf{z}^{h-1}$, we have

$$z_i^h \in [z_i^{h,lo}, z_i^{h,up}] \quad (i = 1, \ldots, k). \tag{22.16}$$

Depending on the computational cost of solving the k problems of the ε-constraint method, it must be evaluated whether these bounds are worth to be calculated at each iteration. If this is regarded to be too time-consuming, calculating the bounds can be skipped.

In addition, a measure of the closeness of the current vector to the Pareto optimal set can be shown to the decision maker. This allows the decision maker to determine whether the approach rhythm to the Pareto optimal set is appropriate or whether it is too fast or too slow. The decision maker can affect this by adjusting the number of iterations still to be taken. Given the information available, the decision maker may take a step backwards if (s)he does not like the new solution generated or the bounds and/or change the number of remaining iterations. In the latter case, we assign a new value to it^h. In the former case, the decision maker can either:

- *continue with old preference information*. A new solution is obtained by considering a smaller step size starting from the previous solution (for example, a half of the former step size), or
- *provide new preference information*. Then a new iteration step is taken, starting from $\mathbf{z}^{h-1}$.

To get started, the ideal and the nadir objective vectors must be calculated or estimated. Then, an overview of the NAUTILUS algorithm can be summarized as follows.

1. Ask the decision maker to give the number of iterations, itn. Set $h = 1$, $\mathbf{z}^0 = \mathbf{f}^{1,up} = \mathbf{z}^{\text{nad}}$, $\mathbf{f}^{1,lo} = \mathbf{z}^\star$ and $it^1 = itn$.
2. Ask the decision maker to provide preference information in either of the two ways and calculate weights w_i^h $(i = 1, \ldots, k)$.
3. Set the reference point and the weights and solve problem (22.3) to get $\mathbf{x}^h$ and the corresponding $\mathbf{f}^h$.
4. Calculate $\mathbf{z}^h$ according to (22.15).
5. Given $\mathbf{z}^h$, find $\mathbf{f}^{h+1,lo}$ by solving k ε-constraint problems. Furthermore, set $\mathbf{f}^{h+1,up} = \mathbf{z}^h$. Calculate the distance to the Pareto optimal set.
6. Show the current objective values z_i^h $(i = 1, \ldots, k)$, together with the additional information $[f_i^{h+1,lo}, f_i^{h+1,up}]$ $(i = 1, \ldots, k)$ and the distance to the decision maker.

7. Set a new value for it^h if the decision maker wants to change the number of remaining iterations.
8. Ask the decision maker whether (s)he wants to take a step backwards. If so, go to step 10. Otherwise, continue.
9. If $it^h = 1$, stop with the last solution $\mathbf{x}^h$ and $\mathbf{f}^h$ as the final solution. Otherwise, set $it^{h+1} = it^h - 1$ and $h = h + 1$. If the decision maker wants to give new preference information, go to step 1. Alternatively, the decision maker can take a new step in the same direction (using the preference information of the previous iteration). Then, set $\mathbf{f}^h = \mathbf{f}^{h-1}$, and go to step 4.
10. Ask the decision maker whether (s)he would like to provide new preference information starting from $\mathbf{z}^{h-1}$. If so, go to step 2. Alternatively, the decision maker can take a shorter step with the same preference information given in step 2. Then, set $\mathbf{z}^h = \frac{1}{2}\mathbf{z}^h + \frac{1}{2}\mathbf{z}^{h-1}$ and go to step 5.

The algorithm looks more complicated than it actually is. There are many steps to provide to the decision maker different options of how to continue the solution process. A good user interface plays an important role in making the options available intuitive.

The NAUTILUS method has been located in this class of methods because the decision maker must compare at each iteration the solution generated to the solution of the previous iteration and decide whether to proceed or to go backwards. Naturally, preference information indicating how important it is to improve each of the objective functions from their current levels is also needed.

NAUTILUS is ad hoc in nature because all preference information needed cannot be obtained from a value function.

A modification of the NAUTILUS method is presented in [213].

22.6.3 *Other Methods Where Solutions Are Compared*

Methods where the decision maker is asked to compare different solutions have been developed rather recently. Such methods targeted at nonlinear problems can be found in [26, 89, 91, 102, 121, 130, 131].

22.7 Methods Using Marginal Rates of Substitution

In this section we present methods that utilize preference information in the form of marginal rates of substitution or desirability of trade-off information provided. These methods are included here because they have played a role in the history of developing interactive methods. They aim at some sort of mathematical convergence in optimizing an estimated value function rather than psychological convergence. It is important that the decision maker understands well the concepts used in these methods to be able to apply them.

22.7.1 Interactive Surrogate Worth Trade-Off Method

The interactive surrogate worth trade-off (ISWT) method is introduced in [22] and [23], pp. 371–379. The ISWT method utilizes the scalarized ε-*constraint problem* where one of the objective functions is minimized subject to upper bounds on all the other objectives:

$$\begin{aligned} &\text{minimize} && f_\ell(\mathbf{x}) \\ &\text{subject to} && f_j(\mathbf{x}) \leq \varepsilon_j \ \text{ for all } \ j = 1, \dots, k, \ \ j \neq \ell, \\ & && \mathbf{x} \in S, \end{aligned} \tag{22.17}$$

where $\ell \in \{1, \dots, k\}$ and ε_j are upper bounds for the other objectives.

Theorem 11. *The solution of (22.17) is weakly Pareto optimal. The decision vector* $\mathbf{x}^* \in S$ *is Pareto optimal if and only if it solves (22.17) for every* $\ell = 1, \dots, k$, *where* $\varepsilon_j = f_j(\mathbf{x}^*)$ *for* $j = 1, \dots, k$, $j \neq \ell$. *A unique solution is Pareto optimal for any upper bounds.*

The idea of the ISWT method is to maximize an approximation of an underlying value function. A search direction is determined based on the opinions of the decision maker concerning trade-off rates at the current solution. The step-size to be taken in the search direction is determined by solving several ε-constraint problems and asking the decision maker to select the most satisfactory solution.

It is assumed that the underlying value function exists and is implicitly known to the decision maker. In addition, it must be continuously differentiable and strongly decreasing. Furthermore, the objective and the constraint functions must be twice continuously differentiable and the feasible region has to be compact. Finally, it is assumed that the Pareto optimality of the solutions of the ε-constraint problem is guaranteed and that trade-off rate information is available in the Karush-Kuhn-Tucker (KKT) multipliers related to the ε-constraint problem.

Changes in objective function values between a reference function f_ℓ and all the other objectives are compared. For each $i = 1, \dots, k$, $i \neq \ell$, the decision maker must answer the following question: Let an objective vector $\mathbf{z}^h$ be given. If the value of f_ℓ is decreased by λ_i^h units, then the value of f_i is increased by one unit (or vice versa) and the other objective values remain unaltered. How desirable do you find this trade-off?

The response of the decision maker indicating the degree of preference is called a *surrogate worth* value. According to [22, 23] the response must be an integer between 10 and -10 whereas it is suggested in [242] to use integers from 2 to -2.

The gradient of the underlying value function is then estimated with the help of the surrogate worth values. This gives a search direction with a steepest ascent for the value function. Several different steps are taken in the search direction and the decision maker must select the most satisfactory of them. In practice, the upper bounds of the ε-constraint problem are revised based on surrogate worth values with different step-sizes.

The main features of the ISWT algorithm can be presented with four steps.

1. Select f_ℓ to be minimized and give upper bounds to the other objective functions. Set $h = 1$.
2. Solve (22.17) to get a solution $\mathbf{z}^h$. Trade-off rate information is obtained from the KKT multipliers.
3. Ask the decision maker for the surrogate worth values at $\mathbf{z}^h$.
4. If some stopping criterion is satisfied, stop. Otherwise, update the upper bounds with the help of the answers obtained in step 3 and solve several ε-constraint problems. Let the decision maker choose the most preferred alternative $\mathbf{z}^{h+1}$ and set $h = h + 1$. Go to step 3.

As far as stopping criteria are concerned, one can always stop when the decision maker wants to do so. A common stopping criterion is the situation where all the surrogate worth values equal zero. One more criterion is the case when the decision maker wants to proceed only in an infeasible direction.

In the ISWT method, the decision maker is asked to specify surrogate worth values and compare Pareto optimal alternatives. It may be difficult for the decision maker to provide consistent surrogate worth values throughout the decision process. In addition, if there is a large number of objective functions, the decision maker has to specify a lot of surrogate worth values at each iteration. On the other hand, the easiness of the comparison of alternatives depends on the number of objectives and on the personal abilities of the decision maker.

The ISWT method can be regarded as a non ad hoc method. The sign of the surrogate worth values can be judged by comparing trade-off rates with marginal rates of substitution (obtainable from the value function). Furthermore, when comparing alternatives, it is easy to select the one with the highest value function value.

Modification of the ISWT method are presented in [23, 27, 49, 63, 69].

22.7.2 Geoffrion-Dyer-Feinberg Method

In the Geoffrion-Dyer-Feinberg (GDF) method proposed in [57], the basic idea is related to that of the ISWT method. In both the methods, the underlying (implicitly known) value function is approximated and maximized. In the GDF method, the approximation is based on marginal rates of substitution.

It is assumed that an underlying value function exists, is implicitly known to the decision maker and is strongly decreasing with respect to the reference function f_ℓ. In addition, the corresponding value function with decision variables as variables must be continuously differentiable and concave on S. Furthermore, the objective functions have to be continuously differentiable and the feasible region S must be compact and convex.

Let $\mathbf{x}^h$ be the current solution. We can obtain a local linear approximation for the gradient of the value function with the help of marginal rates of substitution m_i^h involving a reference function f_ℓ and the other functions f_i. Based on this information we solve the problem

$$\begin{aligned} &\text{maximize} \quad \left(\sum_{i=1}^{k} -m_i^h \nabla_x f_i(\mathbf{x}^h) \right)^T \mathbf{y} \\ &\text{subject to} \quad \mathbf{y} \in S, \end{aligned} \tag{22.18}$$

where $\mathbf{y} \in \mathbf{R}^n$ is the variable. Let us denote the solution by $\mathbf{y}^h$. Then, the search direction is $\mathbf{d}^h = \mathbf{y}^h - \mathbf{x}^h$.

The following problem is to find a step-size. The decision maker can be offered objective vectors where steps of different sizes are taken in the search direction starting from the current solution. Unfortunately, these alternatives are not necessarily Pareto optimal.

Now we can present the GDF algorithm.

1. Ask the decision maker to select f_ℓ. Set $h = 1$.
2. Ask the decision maker to specify marginal rates of substitution between f_ℓ and the other objectives at the current solution $\mathbf{z}^h$.
3. Solve (22.18). Set the search direction $\mathbf{d}^h$. If $\mathbf{d}^h = \mathbf{0}$, stop.
4. Determine with the help of the decision maker the appropriate step-size t^h to be taken in $\mathbf{d}^h$. Denote the corresponding solution by $\mathbf{z}^{h+1} = \mathbf{f}(\mathbf{x}^h + t^h\mathbf{d}^h)$.
5. Set $h = h + 1$. If the decision maker wants to continue, go to step 2. Otherwise, stop.

In the GDF method, the decision maker has to specify marginal rates of substitution and select the most preferred solution from a set of alternatives. The theoretical foundation of the method is convincing but the practical side is not as promising. At each iteration the decision maker has to determine $k - 1$ marginal rates of substitution in a consistent and correct way. On the other hand, it is obvious that in practice the task of selection becomes more difficult for the decision maker as the number of objective functions increases. Another drawback is that not all the solutions presented to the decision maker are necessarily Pareto optimal. They can naturally be projected onto the Pareto optimal set but this necessitates extra effort.

The GDF method is a non ad hoc method. The marginal rates of substitution and selections can be done with the help of value function information. Note that if the underlying value function is linear, the marginal rates of substitution are constant and only one iteration is needed.

Applications and modifications of the GDF method are described in [3, 40, 42, 51, 53, 73, 79, 85, 143, 144, 167, 190, 201, 207, 223, 268].

22.7.3 Other Methods Using Marginal Rates of Substitution

Although preference information about relative importance of different objectives in one form or another is utilized in many interactive methods, there are very few methods where the desirable marginal rates of substitute are the main preference information. Such methods are presented in [123, 134, 271].

22.8 Navigation Methods

By navigation we refer to methods where new Pareto optimal solution candidates are generated in a real-time imitating fashion along directions that are derived from the information the decision maker has specified. In this way, the decision maker can learn about the interdependencies among the objective functions. The decision maker can either continue the movement along the current direction or change the direction, that is, one's preferences. Increased interest has been devoted to navigation based methods in the literature in recent years. In these methods, the user interface plays a very important role in enabling the navigation.

22.8.1 Reference Direction Approach

The reference direction approach [104, 109] is also known by the name *visual interactive approach*. It contains ideas from, for example, the GDF method and the reference point method. However, more information is provided to the decision maker.

In reference point based methods, a reference point is projected onto the Pareto optimal set by optimizing an achievement function. Here a whole so-called *reference direction* is projected onto the Pareto optimal set. It is a vector from the current solution $\mathbf{z}^h$ to the reference point $\bar{\mathbf{z}}^h$. In practice, steps of different sizes are taken along the reference direction and projected. The idea is to plot the objective function values on a computer screen as value paths. The decision maker can move the cursor back and forth and see the corresponding numerical values at each solution.

Solutions along the reference direction are generated by solving the scalarized problem

$$\begin{aligned} \text{minimize} \quad & \max_{i \in I} \left[\frac{f_i(\mathbf{x}) - \bar{z}_i^h}{w_i} \right] \\ \text{subject to} \quad & \bar{\mathbf{z}}^h = \mathbf{z}^h + t\mathbf{d}^{h+1}, \\ & \mathbf{x} \in S, \end{aligned} \tag{22.19}$$

where $I = \{i \mid w_i > 0\} \subset \{1, \ldots, k\}$ and t has different discrete nonnegative values. The weighting vector can be, for example, the reference point specified by the decision maker.

Theorem 12. *The solution of (22.19) is weakly Pareto optimal.*

The algorithm of the reference direction approach is as follows.

1. Find an arbitrary objective vector $\mathbf{z}^1$. Set $h = 1$.
2. Ask the decision maker to specify a reference point $\bar{\mathbf{z}}^h \in \mathbf{R}^k$ and set $\mathbf{d}^{h+1} = \bar{\mathbf{z}}^h - \mathbf{z}^h$.
3. Find the set Z^{h+1} of weakly Pareto optimal solutions with different values of t in (22.19).
4. Ask the decision maker to select the most preferred solution $\mathbf{z}^{h+1}$ in Z^{h+1}.
5. If $\mathbf{z}^h \neq \mathbf{z}^{h+1}$, set $h = h + 1$ and go to step 2. Otherwise, check the optimality conditions. If the conditions are satisfied, stop. Otherwise, set $h = h + 1$ and set $\mathbf{d}^{h+1}$ to be a search direction identified by the optimality checking procedure. Go to step 3.

Checking the optimality conditions in step 5 is the most complicated part of the algorithm. Thus far, no specific assumptions have been set on the value function. However, we can check the optimality of $\mathbf{z}^{h+1}$ if the cone containing all the feasible directions has a finite number of generators. We must then assume that an underlying value function exists and is pseudoconcave on Z. In addition, S must be convex and compact and the constraint functions must be differentiable.

The role of the decision maker is similar in the reference point method and in the reference direction approach: specifying reference points and selecting the most preferred alternative. But by providing similar reference point information, in the reference direction approach, the decision maker can explore a wider part of the weakly Pareto optimal set. This possibility brings the task of comparing the alternatives.

The performance of the method depends greatly on how well the decision maker manages to specify the reference directions that lead to more satisfactory solutions. The consistency of the decision maker's answers is not important and it is not checked in the algorithm.

The reference direction approach can be characterized as an ad hoc method as the other reference point based methods. The aim is to support the decision maker in getting to know the problem better.

A dynamic user interface to the reference direction approach and its adaptation to generalized goal programming is introduced in [111]. This method for linear multiobjective optimization problems is called the *Pareto race* and the software system implementing the Pareto race is called VIG (Visual Interactive Goal programming) [113, 114].

Applications and modifications of the reference direction approach are described in [10, 103–108, 110].

22.8.2 Pareto Navigator Method

Pareto Navigator is an interactive method utilizing a polyhedral approximation of the Pareto optimal set for convex problems [48]. Pareto Navigator consists of two phases, namely an initialization phase, where the decision maker is not involved and a navigation phase. In a initialization phase, a relatively small set of Pareto optimal objective vectors is assumed to be available to form a polyhedral approximation of the Pareto optimal set in the objective space. These objective vectors can be computed, for example, by using some a posteriori approach.

Pareto Navigator has been developed especially for the learning phase of interactive solution processes introduced in Sect. 22.3 and for computationally expensive problems where objective function and/or constraint function value evaluations may be time-consuming because the problem is, for example, simulation-based. In these problems, computing Pareto optimal solutions can take a lot of time. For this reason, besides the original (computationally expensive) problem, an approximation is used to enable fast computations so that the decision maker does not need to wait for new solutions being generated based on her or his preferences.

In Pareto Navigator the decision maker is not involved in the part of the solution process where the set of objective vectors representing the Pareto optimal set is generated. Once the approximation has been created based on the objective vectors available, the original problem is not solved (in the navigation phase). When the navigation phase starts, the decision maker can navigate dynamically in the approximated Pareto optimal set in real time since approximated Pareto optimal solutions can be produced by solving linear programming problems that are computationally inexpensive.

Whenever the decision maker has found an interesting approximated Pareto optimal solution, the corresponding solution to the original problem can be generated by solving problem (22.3) with the approximated solution as a reference point. This can be seen as projecting the approximated solution to the Pareto optimal set of the original problem. However, this step may take time as the original problem is computationally expensive.

As mentioned, the multiobjective optimization problem is assumed to be convex, that is, the objective functions and the feasible region must be convex. The algorithm of Pareto Navigator is as follows.

1. Compute first a polyhedral approximation of the Pareto optimal set in the objective space based on a small set of Pareto optimal objective vectors. Use the extreme values present in this set to approximate the ideal and nadir objective vectors. Ask the decision maker to select a starting point for navigation (for example, one of the Pareto optimal objective vectors available).
2. Show the objective values of the current solution to the decision maker and ask her or him whether a preferred solution has been found. If yes, go to step 6. Otherwise, continue.

3. Ask the decision maker whether (s)he would like to proceed to some other direction. If the decision maker does not want to change the direction, go to step 5.
4. Ask the decision maker to specify how the current solution should be improved by giving aspiration levels for the objectives. To aid her or him, show the ideal and the nadir objective vectors. Based on the resulting reference point $\bar{\mathbf{z}}$ and the current solution $\mathbf{z}^c$, set a search direction.
5. Ask the decision maker to indicate a speed of movement, that is, a step size $\alpha > 0$ to the direction specified. Generate approximated Pareto optimal solutions in the direction specified by using a reference point based approach for each step in the direction starting from the current solution $\mathbf{z}^c$. Once an approximated solution is produced, it is instantly shown to the decision maker. New approximated solutions are produced to the direction specified until the decision maker stops the movement. Then go to step 2.
6. Once the decision maker has found a satisfactory solution, stop. Project the approximated Pareto optimal solution to the actual Pareto optimal set and show the resulting solution to the decision maker.

The search direction is based on decision maker's preferences and there are different ways of defining a direction where to move on the approximation. In Pareto Navigator, the direction is specified by $\mathbf{d} = \bar{\mathbf{z}} - \mathbf{z}^c$. The approximated Pareto optimal solutions are then computed by solving problems of the form

$$\begin{array}{ll} \text{minimize} & \max\limits_{i=1,\ldots,k} w_i\,(z_i - \bar{z}_i(\alpha)) \\ \text{subject to} & \mathbf{Az} \leq \mathbf{b}, \end{array} \tag{22.20}$$

where $\bar{\mathbf{z}}(\alpha) = \mathbf{z}^c + \alpha\mathbf{d}$ is the reference point depending on the step parameter $\alpha > 0$ (being varied) to the direction $\mathbf{d}$ and w_i, $i = 1, \ldots, k$, are the scaling coefficients. The scaling coefficient can be set as one divided by the difference of the estimated nadir and ideal objective values. The linear constraints of problem (22.20) form a convex hull for a set of Pareto optimal solutions used to form the polyhedral approximation and, in practice, the reference point $\bar{\mathbf{z}}(\alpha)$ is projected to the nondominated facets of the convex hull.

The objective function of problem (22.20) is nonlinear with respect to $\mathbf{z}$ but can be linearized by adding a new real variable $\xi \in \mathbf{R}$ replacing the max term. The resulting problem is then linear with respect to a new variable $\mathbf{z}' = (\xi, \mathbf{z})^T$. Due to linearity, approximated Pareto optimal solutions can be produced and shown to the decision maker in real time by shifting the reference point along the direction $\mathbf{d}$ by increasing the value of α. At any point, the decision maker is able to find the closest actual Pareto optimal solution for any approximated Pareto optimal solution. However, as said, this can be time consuming.

Because the decision maker must specify desirable objective function values, this method is ad hoc by nature.

During the navigation, the approximated solutions are shown to the decision maker by presenting the approximated values as a continuous path (value path) for each objective function separately (bar charts can be used as well). Pareto Navigator is implemented in the IND-NIMBUS system [140] and the graphical user interface development is described in [241].

22.8.3 Pareto Navigation Method

The Pareto Navigation method developed in [163] assumes the convexity of all objective functions and a convex feasible region. Similar to the Pareto Navigator method, the idea is to enable a fast generation of new solutions in the navigation phase. Thus, the method starts with formulating a surrogate problem based on a set of pre-computed Pareto optimal decision vectors $\{\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(m)}\}$. The most preferred solution is sought among their convex hull

$$\mathcal{X} = \left\{ \sum_{j=1}^{m} v_j \mathbf{x}^{(j)} : \sum_{j=1}^{m} v_j = 1,\ v_j \geq 0 \text{ for all } j = 1, \ldots, m \right\}.$$

This allows replacing the feasible region of the original problem with the set of convex combination coefficients $v_1, \ldots, v_m$ in the definition of $\mathcal{X}$.

The current state of the navigation process is represented by the current Pareto optimal solution $\mathbf{x}^h$ and the vector of current upper bounds $\mathbf{b} \in \mathbf{R}^k$ on objective function values. Using the surrogate problem with these bounds as additional constraints, the ideal objective vector is calculated and the nadir objective vector is estimated via a pay-off table. They define ranges of objective function values for Pareto optimal solutions. These ranges together with the current solution are displayed in a radar chart also known as a spider-web chart.

By moving sliders on the radar chart with the mouse, the decision maker can provide two types of preference information: upper bounds on objective values and a desired value (aspiration level) of any objective function. Changes made by the decision maker are immediately reflected in the current state of the navigation process and shown in the radar chart. Setting the upper bounds influences the objective function ranges as described above. Setting the value of any objective function f_{i^*} to a desired value τ yields updating the current solution with the solution of the following problem

$$
\begin{aligned}
\text{minimize} \quad & \max_{\substack{i=1,\ldots,k,\\ i \neq i^*}} y_i - f_i(x^h) \\
\text{subject to} \quad & \mathbf{y} = f\left(\sum_{j=1}^{m} v_j \mathbf{x}^{(j)}\right) + \mathbf{s}, \\
& y_i \leq b_i, \; i = 1, \ldots, k, \\
& y_{i^*} = \tau, \\
& \sum_{j=1}^{m} v_j = 1, \\
& \mathbf{v} \text{ and } \mathbf{s} \text{ are non-negative.}
\end{aligned}
$$

By using the two above-described mechanisms of expressing preferences the decision maker explores the set of Pareto optimal solutions of the surrogate problem until a most preferred or satisfactory solution is found. Because the decision maker must provide upper bounds and aspiration levels, the method is ad hoc by nature.

The method has been developed and implemented for intensity modulated radiation therapy treatment planning. Therefore, in addition to the radar chart, some application-specific information about the current solution (treatment plan) is displayed. Nevertheless, there are no obstacles of adapting the method elsewhere when the multiobjective optimization problem is convex and the convex hull of some finite set of pre-calculated Pareto optimal solutions may serve as a good enough approximation of the Pareto optimal set.

22.8.4 Other Navigation Methods

Other navigation based methods developed for nonlinear multiobjective optimization problems and implemented as software tools include [125, 127]. A collection of methods and software for solving multiobjective linear optimization problems [4, 5] can also be mentioned for they can be partly extended to nonlinear problems.

22.9 Other Interactive Methods

The number of interactive methods developed for multiobjective optimization is large. So far, we have given several examples of them. Let us next mention references to some more methods based on miscellaneous ideas: [8, 28–30, 39, 45, 50, 54, 55, 78, 90, 94, 98, 101, 119, 120, 122, 135, 161, 165, 166, 177, 191–194, 203, 204, 208, 216, 220, 221, 227, 229, 230, 233, 239, 252, 270, 272, 273].

22.10 Comparing the Methods

None of the many multiobjective optimization methods can be claimed to be superior to the others in every aspect. One can say that selecting a multiobjective optimization method is a problem with multiple objectives itself. The properties of the problem and the capabilities and the desires of the decision maker have to be charted before a solution method can be chosen. Some methods may suit some problems and some decision makers better than some others.

A decision tree is provided in [139] for easing the method selection. The tree is based on theoretical facts concerning the assumptions on the problem to be solved and the preferences of the decision maker. Further aspects to be taken into account when evaluating and selecting methods are collected, for example, in [12, 58, 74, 80, 139, 232, 243, 244].

In addition to theoretical properties, practical applicability, in particular, plays an important role in the selection of an appropriate method. The difficulty is that practical applicability is hard to determine without experience.

Some comparisons of the methods have been reported in the literature. They have been carried out with respect to a variety of criteria and under varied circumstances. Instead of a human decision maker one can sometimes employ value functions in the comparisons. Unfortunately, replacing the decision maker with a value function does not fully reflect the real usefulness of the methods. One of the problems is that value functions cannot really help in testing ad hoc methods.

Tests with human decision makers are described in [15, 17, 19, 20, 32, 33, 41, 112, 137, 188, 251] while tests with value functions are reported in [2, 59, 164, 195]. Finally, comparisons based on intuition are provided in [46, 99, 100, 116, 134, 138, 189, 197, 211, 247, 250].

22.11 Conclusions

We have outlined several interactive methods for solving nonlinear multiobjective optimization problems and indicated references to many more. One of the challenges in this area is spreading the word about the existing methods to those who solve real-world problems. Another challenge is to develop methods that support the decision maker even better. User-friendliness cannot be overestimated because interactive methods must be able to correspond to the characteristics of the decision maker. Specific methods for different areas of application that take into account the characteristics of the problems are also important.

An alternative to creating new methods is to use different methods in different phases of the solution process. This hybridization means that the positive features of various methods can be exploited to their best advantage in appropriate phases. In this way, it may also be possible to overcome some of the weaknesses of the methods. Ways to enable changing the type of preference information specified, that is, the method used during the solution process are presented in [129, 205].

The decision maker can be supported by using visual illustrations and further development of such tools is essential. For instance, one may visualize (parts of) the Pareto optimal set and, for example, use 3D slices of the feasible objective region (see [125, 126], among others) and other tools. On the other hand, one can illustrate sets of alternatives by means of bar charts, value paths, spider-web charts and petal diagrams etc. For more details see, for example, [139] and references therein as well as [142] for a more detailed survey.

References

1. Agrell, P.J., Lence, B.J., Stam, A.: An interactive multicriteria decision model for multipurpose reservoir management: the Shellmouth reservoir. J. Multi-Criteria Decis. Anal. **7**, 61–86 (1998)
2. Aksoy, Y., Butler, T.W., Minor III, E.D.: Comparative studies in interactive multiple objective mathematical programming. Eur. J. Oper. Res. **89**, 408–422 (1996)
3. Al-alvani, J.E., Hobbs, B.F., Malakooti, B.: An interactive integrated multiobjective optimization approach for quasiconcave/quasiconvex utility functions. In: Goicoechea, A., Duckstein, L., Zionts, S. (eds.) Multiple Criteria Decision Making: Proceedings of the Ninth International Conference: Theory and Applications in Business, Industry, and Government, pp. 45–60. Springer, New York (1992)
4. Antunes, C.H., Alves, M.J., Silva, A.L., Clímaco, J.N.: An integrated MOLP method base package – a guided tour of TOMMIX. Comput. Oper. Res. **19**, 609–625 (1992)
5. Antunes, C.H., Melo, M.P., Clímaco, J.N.: On the integration of an interactive MOLP procedure base and expert system technique. Eur. J. Oper. Res. **61**, 135–144 (1992)
6. Aubin, J.-P., Näslund, B.: An exterior branching algorithm. Working Paper 72–42, European Institute for Advanced Studies in Management, Brussels (1972)
7. Baba, N., Takeda, H., Miyake, T.: Interactive multi-objective programming technique using random optimization method. Int. J. Syst. Sci. **19**, 151–159 (1988)
8. Bard, J.F.: A multiobjective methodology for selecting subsystem automation options. Manag. Sci. **32**, 1628–1641 (1986)
9. Benayoun, R., de Montgolfier, J., Tergny, J., Laritchev, O.: Linear programming with multiple objective functions: step method (STEM). Math. Program. **1**, 366–375 (1971)
10. Benson, H.P., Aksoy,Y.: Using efficient feasible directions in interactive multiple objective linear programming. Oper. Res. Lett. **10**, 203–209 (1991)
11. Bischoff, E.: Two empirical tests with approaches to multiple-criteria decision making. In: Grauer, M., Thompson, M., Wierzbicki, A.P. (eds.) Plural Rationality and Interactive Decision Processes, pp. 344–347. Springer, Berlin, Heidelberg (1985)
12. Bischoff, E.: Multi-objective decision analysis – the right objectives? In: Fandel, G., Grauer, M., Kurzhanski, A., Wierzbicki, A.P. (eds.) Large-Scale Modelling and Interactive Decision Analysis, pp. 155–160. Springer, Heidelberg (1986)
13. Bogetoft, P., Hallefjord, Å., Kok, M.: On the convergence of reference point methods in multiobjective programming. Eur. J. Oper. Res. **34**, 56–68 (1988)
14. Branke, J., Deb, K., Miettinen, K., Slowinski, R. (eds.): Multiobjective Optimization: Interactive and Evolutionary Approaches. Springer, Berlin, Heidelberg (2008)
15. Brockhoff, K.: Experimental test of MCDM algorithms in a modular approach. Eur. J. Oper. Res. **22**, 159–166 (1985)
16. Buchanan, J.T.: Multiple objective mathematical programming: a review. N. Z. Oper. Res. **14**, 1–27 (1986)

17. Buchanan, J.T.: An experimental evaluation of interactive MCDM methods and the decision making process. J. Oper. Res. Soc. **45**, 1050–1059 (1994)
18. Buchanan, J.T.: A naïve approach for solving MCDM problems: the GUESS method. J. Oper. Res. Soc. **48**, 202–206 (1997)
19. Buchanan, J.T., Corner, J.L.: The effects of anchoring in interactive MCDM solution methods. Comput. Oper. Res. **24**, 907–918 (1997)
20. Buchanan, J.T., Daellenbach, H.G.: A comparative evaluation of interactive solution methods for multiple objective decision models. Eur. J. Oper. Res. **29**, 353–359 (1987)
21. Censor, Y.: Pareto optimality in multiobjective problems. Appl. Math. Optim. **4**, 41–59 (1977)
22. Chankong, V., Haimes, Y.Y.: The interactive surrogate worth trade-off (ISWT) method for multiobjective decision-making. In: Zionts, S. (ed.) Multiple Criteria Problem Solving, pp. 42–67. Springer, Berlin, Heidelberg (1978)
23. Chankong, V., Haimes, Y.Y.: Multiobjective Decision Making Theory and Methodology. Elsevier, New York (1983)
24. Charnes, A., Cooper, W.W.: Management Models and Industrial Applications of Linear Programming, vol. 1. Wiley, New York (1961)
25. Charnes, A., Cooper, W.W.: Goal programming and multiple objective optimization; part 1. Eur. J. Oper. Res. **1**, 39–54 (1977)
26. Chen, J., Lin, S.: An interactive neural network-based approach for solving multiple criteria decision-making problems. Decis. Support Syst. **36**, 137–146 (2003)
27. Chen, T., Wang, B.-I.: An interactive method for multiobjective decision making. In: Straszak, A. (ed.) Large Scale Systems: Theory and Applications 1983, pp. 277–282. Pergamon, Oxford (1984)
28. Clímaco, J.N., Antunes, C.H.: Flexible method bases and man-machine interfaces as key features in interactive MOLP approaches. In: Korhonen, P., Lewandowski, A., Wallenius, J. (eds.), Multiple Criteria Decision Support, pp. 207–216. Springer, Berlin (1991)
29. Clímaco, J.N., Antunes, C.H.: Man-machine interfacing in MCDA. In: Tzeng, G.H., Wand, H.F., Wen, U.P., Yu, P.L. (eds.) Multiple Criteria Decision Making – Proceedings of the Tenth International Conference: Expand and Enrich the Domains of Thinking and Application, pp. 239–253. Springer, New York (1994)
30. Clímaco, J.N., Antunes, C.H., Alves, M.J.: From TRIMAP to SOMMIX – building effective interactive MOLP computational tools. In: Fandel, G., Gal, T. (eds.) Multiple Criteria Decision Making: Proceedings of the Twelfth International Conference, pp. 285–296. Springer, Berlin, Heidelberg (1997)
31. Corley, H.W.: A new scalar equivalence for Pareto optimization. IEEE Trans. Automat. Control **25**, 829–830 (1980)
32. Corner, J.L., Buchanan, J.T.: Experimental consideration of preference in decision making under certainty. J. Multi-Criteria Decis. Anal. **4**, 107–121 (1995)
33. Corner, J.L., Buchanan, J.T.: Capturing decision maker preference: experimental comparison of decision analysis and MCDM techniques. Eur. J. Oper. Res. **98**, 85–97 (1997)
34. Costa, J.P., Clímaco, J.N.: A multiple reference point parallel approach in MCDM. In: Tzeng, G.H., Wand, H.F., Wen, U.P., Yu, P.L. (eds.) Multiple Criteria Decision Making – Proceedings of the Tenth International Conference: Expand and Enrich the Domains of Thinking and Application, pp. 255–263. Springer, New York (1994)
35. Crama, Y.: Analysis of STEM-like solutions to multi-objective programming problems. In: French, S., Hartley, R., Thomas, L.C., White, D.J. (eds.) Multi-Objective Decision Making, pp. 208–213. Academic, New York (1983)
36. Deb, K.: Multi-Objective Optimization using Evolutionary Algorithms. Wiley, Chichester (2001)
37. Deb, K., Miettinen, K.: Nadir point estimation using evolutionary approaches: better accuracy and computational speed through focused search. In: Ehrgott, M., Naujoks, B., Stewart, T.J., Wallenius, J. (eds.) Multiple Criteria Decision Making for Sustainable Energy and Transportation Systems, Proceedings, pp. 339–354. Springer, Berlin, Heidelberg (2010)

38. Deb, K., Miettinen, K., Chaudhuri, S.: Towards an estimation of nadir objective vector using a hybrid of evolutionary and local search approaches. IEEE Trans. Evol. Comput. **14**, 821–841 (2010)
39. Diaz, A.: Interactive solution to multiobjective optimization problems. Int. J. Numer. Methods Eng. **24**, 1865–1877 (1987)
40. Dyer, J.S.: Interactive goal programming. Manag. Sci. **19**, 62–70 (1972)
41. Dyer, J.S.: An empirical investigation of a man-machine interactive approach to the solution of the multiple criteria problem. In: Cochrane, J.L., Zeleny, M. (eds.) Multiple Criteria Decision Making, pp. 202–216. University of South Carolina Press, Columbia (1973)
42. Dyer, J.S.: A time-sharing computer program for the solution of the multiple criteria problem. Manag. Sci. **19**, 1379–1383 (1973)
43. Dyer, J.S., Sarin, R.K.: Multicriteria decision making. In: Holzman, A.G. (ed.) Mathematical Programming for Operations Researchers and Computer Scientists, pp. 123–148. Marcel Dekker, New York (1981)
44. Ehrgott, M.: Multicriteria Optimization. Springer, Berlin, Heidelberg (2000)
45. Eschenauer, H.A., Schäfer, E., Bernau, H.: Application of interactive vector optimization methods with regard to problems in structural mechanics. In: Eschenauer, H.A., Thierauf, G. (eds.) Discretization Methods and Structural Optimization – Procedures and Applications, pp. 95–101. Springer, Berlin, Heidelberg (1989)
46. Eschenauer, H.A., Osyczka, A., Schäfer, E.: Interactive multicriteria optimization in design process. In: Eschenauer, H., Koski, J., Osyczka, A. (eds.) Multicriteria Design Optimization Procedures and Applications, pp. 71–114. Springer, Berlin, Heidelberg (1990)
47. Eskelinen, P., Miettinen, K.: Trade-off analysis approach for interactive nonlinear multiobjective optimization. OR Spectr. **34**, 803–816 (2012)
48. Eskelinen, P., Miettinen, K., Klamroth, K., Hakanen, J.: Pareto Navigator for interactive nonlinear multiobjective optimization. OR Spectr. **32**, 211–227 (2010)
49. Feng, J.: Dual worth trade-off method and its application for solving multiple criteria decision making problems. J. Syst. Eng. Electron. **17**, 554–558 (2006)
50. Ferreira, P.A.V., Geromel, J.C.: An interactive projection method for multicriteria optimization problems. IEEE Trans. Syst. Man Cybern. **20**, 596–605 (1990)
51. Ferreira, P.A.V., Machado, M.E.S.: Solving multiple-objective problems in the objective space. J. Optim. Theory Appl. **89**, 659–680 (1996)
52. Fishburn, P.C.: Lexicographic orders, utilities and decision rules: a survey. Manag. Sci. **20**, 1442–1471 (1974)
53. Friesz, T.L.: Multiobjective optimization in transportation: the case of equilibrium network design. In: Morse, N.N. (ed.) Organizations: Multiple Agents with Multiple Criteria, pp. 116–127. Springer, Berlin, Heidelberg (1981)
54. Gardiner, L.R., Steuer, R.E.: Unified interactive multiple objective programming. Eur. J. Oper. Res. **74**, 391–406 (1994)
55. Gardiner, L.R., Steuer, R.E.: Unified interactive multiple objective programming: an open architecture for accommodating new procedures. J. Oper. Res. Soc. **45**, 1456–1466 (1994)
56. Gass, S., Saaty, T.: The computational algorithm for the parametric objective function. Nav. Res. Logist. Q. **2**, 39–45 (1955)
57. Geoffrion, A.M., Dyer, J.S., Feinberg, A.: An interactive approach for multi-criterion optimization, with an application to the operation of an academic department. Manag. Sci. **19**, 357–368 (1972)
58. Gershon, M., Duckstein, L.: An algorithm for choosing of a multiobjective technique. In: Hansen, P. (ed.) Essays and Surveys on Multiple Criteria Decision Making, pp. 53–62. Springer, Berlin, Heidelberg (1983)
59. Gibson, M., Bernardo, J.J., Chung, C., Badinelli, R.: A comparison of interactive multiple-objective decision making procedures. Comput. Oper. Res. **14**, 97–105 (1987)
60. Gouljashki, V.G., Kirilov, L.M., Narula, S.C., Vassilev, V.S.: A reference direction interactive algorithm of the multiple objective nonlinear integer programming. In: Fandel, G., Gal, T. (eds.) Multiple Criteria Decision Making: Proceedings of the Twelfth International Conference, pp. 308–317. Springer, Berlin, Heidelberg (1997)

61. Granat, J., Makowski, M.: Interactive specification and analysis of aspiration-based preferences. Eur. J. Oper. Res. **122**, 469–485 (2000)
62. Grauer, M., Lewandowski, A., Wierzbicki, A.: DIDASS – theory, implementation and experiences. In: Grauer, M., Wierzbicki, A.P. (eds.) Interactive Decision Analysis, pp. 22–30. Springer, Heidelberg (1984)
63. Haimes, Y.Y.: The surrogate worth trade-off (SWT) method and its extensions. In: Fandel, G., Gal, T. (eds.) Multiple Criteria Decision Making Theory and Application, pp. 85–108. Springer, Berlin, Heidelberg (1980)
64. Haimes, Y.Y., Lasdon, L.S., Wismer, D.A.: On a bicriterion formulation of the problems of integrated system identification and system optimization. IEEE Trans. Syst. Man Cybern. **1**, 296–297 (1971)
65. Haimes, Y.Y., Tarvainen, K., Shima, T., Thadathil, J.: Hierarchical Multiobjective Analysis of Large-Scale Systems. Hemisphere, New York (1990)
66. Hakanen, J., Kawajiri, Y., Miettinen, K., Biegler, L.T.: Interactive multi-objective optimization for simulated moving bed processes. Control Cybern. **36**, 282–320 (2007)
67. Hakanen, J., Miettinen, K., Sahlstedt, K.: Wastewater treatment: new insight provided by interactive multiobjective optimization. Decis. Support Syst. **51**, 328–337 (2011)
68. Hakanen, J., Sahlstedt, K., Miettinen, K.: Wastewater treatment plant design and operation under multiple conflicting objective functions. Environ. Model. Softw. **46**, 240–249 (2013)
69. Hall, W.A., Haimes, Y.Y.: The surrogate worth trade-off method with multiple decision-makers. In: Zeleny, M. (ed.) Multiple Criteria Decision Making Kyoto 1975, pp. 207–233. Springer, Berlin, Heidelberg (1976)
70. Hallefjord, Å., Jörnsten, K.: An entropy target-point approach to multiobjective programming. Int. J. Syst. Sci. **17**, 639–653 (1986)
71. Hämäläinen, J.P., Miettinen, K., Tarvainen, P., Toivanen, J.: Interactive solution approach to a multiobjective optimization problem in a paper machine headbox design. J. Optim. Theory Appl. **116**, 265–281 (2003)
72. Heikkola, E., Miettinen, K., Nieminen, P.: Multiobjective optimization of an ultrasonic transducer using NIMBUS. Ultrasonics **44**, 368–380 (2006)
73. Hemming, T.: Some modifications of a large step gradient method for interactive multicriterion optimization. In: Morse, N.N. (ed.) Organizations: Multiple Agents with Multiple Criteria, pp. 128–139. Springer, Berlin, Heidelberg (1981)
74. Hobbs, B.F.: What can we learn from experiments in multiobjective decision analysis? IEEE Trans. Syst. Man. Cybern. **16**, 384–394 (1986)
75. Hu, C., Shen, Y., Li, S.: An interactive satisficing method based on alternative tolerance for fuzzy multiple objective optimization. Appl. Math. Model. **33**, 1886–1893 (2009)
76. Huang, H., Tian, Z.: Application of neural network to interactive physical programming. In: Wang, J., Liao, X., Yi, Z. (eds.) Advances in Neural Networks, pp. 725–730. Springer, Berlin, Heidelberg (2005)
77. Huang, H.-Z., Gu, Y.-K., Du, X.: An interactive fuzzy multi-objective optimization method for engineering design. Eng. Appl. Artif. Intell. **19**, 451–460 (2006)
78. Hussein, M.L., El-Ghaffar, F.S.A.: An interactive approach for vector optimization problems. Eur. J. Oper. Res. **89**, 185–192 (1996)
79. Hwang, C.-L., Masud, A.S.M.: Multiple Objective Decision Making – Methods and Applications: A State-of-the-Art Survey. Springer, Berlin, Heidelberg (1979)
80. Hwang, C.-L., Yoon, K.: Multiple Attribute Decision Making Methods and Applications: A State of the Art Survey. Springer, Berlin, Heidelberg (1981)
81. Ignizio, J.P.: Goal Programming and Extensions. Lexington Books, Lexington (1976)
82. IND-NIMBUS website. http://ind-nimbus.it.jyu.fi/
83. Jaszkiewicz, A., Słowiński, R.: The light beam search – outranking based interactive procedure for multiple-objective mathematical programming. In: Pardalos, P.M., Siskos, Y., Zopounidis, C. (eds.) Advances in Multicriteria Analysis, pp. 129–146. Kluwer, Dordrecht (1995)

84. Jaszkiewicz, A., Słowiński, R.: The 'light beam search' approach – an overview of methodology and applications. Eur. J. Oper. Res. **113**, 300–314 (1999)
85. Jedrzejowicz, P., Rosicka, L.: Multicriterial reliability optimization problem. Found. Control Eng. **8**, 165–173 (1983)
86. Jeong, I.-J., Kim, K.-J.: D-STEM: a modified Step method with desirability function concept. Comput. Oper. Res. **32**, 3175–3190 (2005)
87. Kahneman, D., Tversky, A.: Prospect theory: an analysis of decision under risk. Econometrica **47**, 263–291 (1979)
88. Kaliszewski, I.: A modified weighted Tchebycheff metric for multiple objective programming. Comput. Oper. Res. **14**, 315–323 (1987)
89. Kaliszewski, I.: Using trade-off information in decision-making algorithms. Comput. Oper. Res. **27**, 161–182 (2000)
90. Kaliszewski, I., Michalowski, W.: Searching for psychologically stable solutions of multiple criteria decision problems. Eur. J. Oper. Res. **118**, 549–562 (1999)
91. Kaliszewski, I., Zionts, S.: Generalization of the Zionts-Wallenius algorithm. Control Cybern. **33**, 477–500 (2004)
92. Kaliszewski, I., Michalowski, W., Kersten, G.: A hybrid interactive technique for the MCDM problems. In: Karwan, M.H., Spronk, J., Wallenius, J. (eds.) Essays in Decision Making: A Volume in Honour of Stanley Zionts, pp. 48–59. Springer, Berlin, Heidelberg (1997)
93. Keeney, R.L., Raiffa, H.: Decisions with Multiple Objectives: Preferences and Value Tradeoffs. Wiley, New York (1976)
94. Kim, S.H., Gal, T.: A new interactive algorithm for multi-objective linear programming using maximally changeable dominance cone. Eur. J. Oper. Res. **64**, 126–137 (1993)
95. Kim, J., Kim, S.-K.: A CHIM-based interactive Tchebycheff procedure for multiple objective decision making. Comput. Oper. Res. **33**, 1557–1574 (2006)
96. Kitayama, S., Yamazaki, K.: Compromise point incorporating trade-off ratio in multiobjective optimization. Appl. Soft Comput. **12**, 1959–1964 (2012)
97. Klamroth, K., Miettinen, K.: Integrating approximation and interactive decision making in multicriteria optimization. Oper. Res. **56**, 222–234 (2008)
98. Klepper, L.Y.: An interactive method for optimization of irradiation plans in radiation therapy of malignant tumors. Biomed. Eng. **40**, 291–297 (2006)
99. Kok, M.: Scalarization and the interface with decision makers in interactive multi objective linear programming. In: Serafini, P. (ed.) Mathematics of Multi Objective Optimization, pp. 433–438. Springer, New York (1985)
100. Kok, M.: The interface with decision makers and some experimental results in interactive multiple objective programming method. Eur. J. Oper. Res. **26**, 96–107 (1986)
101. Kok, M., Lootsma, F.A.: Pairwise-comparison methods in multiple objective programming, with applications in a long-term energy-planning model. Eur. J. Oper. Res. **22**, 44–55 (1985)
102. Köksalan, M.M., Moskowitz, H.: Solving the multiobjective decision making problem using a distance function. In: Tzeng, G.H., Wand, H.F., Wen, U.P., Yu, P.L. (eds.) Multiple Criteria Decision Making – Proceedings of the Tenth International Conference: Expand and Enrich the Domains of Thinking and Application, pp. 101–107. Springer, New York (1994)
103. Köksalan, M., Plante, R.D.: Interactive multicriteria optimization for multiple-response product and process design. Manuf. Serv. Oper. Manag. **5**, 334–347 (2003)
104. Korhonen, P.: Reference direction approach to multiple objective linear programming: historical overview. In: Karwan, M.H., Spronk, J., Wallenius, J. (eds.) Essays in Decision Making: A Volume in Honour of Stanley Zionts, pp. 74–92. Springer, Berlin, Heidelberg (1997)
105. Korhonen, P., Halme, M.: Using lexicographic parametric programming for searching a nondominated set in multiple objective linear programming. J. Multi-Criteria Decis. Anal. **5**, 291–300 (1996)
106. Korhonen, P., Laakso, J.: A visual interactive method for solving the multiple-criteria problem. In: Grauer, M., Wierzbicki, A.P. (eds.) Interactive Decision Analysis, pp. 146–153. Springer, Heidelberg (1984)

107. Korhonen, P., Laakso, J.: On developing a visual interactive multiple criteria method – an outline. In: Haimes, Y.Y., Chankong, V. (eds.) Decision Making with Multiple Objectives, pp. 272–281. Springer, Berlin, Heidelberg (1985)
108. Korhonen, P., Laakso, J.: Solving generalized goal programming problems using a visual interactive approach. Eur. J. Oper. Res. **26**, 355–363 (1986)
109. Korhonen, P., Laakso, J.: A visual interactive method for solving the multiple criteria problem. Eur. J. Oper. Res. **24**, 277–287 (1986)
110. Korhonen, P., Narula, S.C.: An evolutionary approach to support decision making with linear decision models. J. Multi-Criteria Decis. Anal. **2**, 111–119 (1993)
111. Korhonen, P., Wallenius, J.: A Pareto race. Nav. Res. Logist. **35**, 615–623 (1988)
112. Korhonen, P., Wallenius, J.: Observations regarding choice behaviour in interactive multiple criteria decision-making environments: an experimental investigation. In: Lewandowski, A., Stanchev, I. (eds.), Methodology and Software for Interactive Decision Support, pp. 163–170. Springer, Heidelberg (1989)
113. Korhonen, P., Wallenius, J.: VIG – a visual and dynamic decision support system for multiple objective linear programming. In: Karpak, B., Zionts, S. (eds.) Multiple Criteria Decision Making and Risk Analysis Using Microcomputers, pp. 251–281. Springer, Berlin, Heidelberg (1989)
114. Korhonen, P., Wallenius, J.: A multiple objective linear programming decision support system. Decis. Support Syst. **6**, 243–251 (1990)
115. Larichev, O.: Cognitive validity in design of decision-aiding techniques. J. Multi-Criteria Decis. Anal. **1**, 127–138 (1992)
116. Larichev, O.I., Polyakov, O.A., Nikiforov, A.O.: Multicriterion linear programming problems. J. Econ. Psychol. **8**, 389–407 (1987)
117. Laukkanen, T., Tveit, T.-M., Ojalehto, V., Miettinen, K., Fogelholm, C.-J.: An interactive multi-objective approach to heat exchanger network synthesis. Comput. Chem. Eng. **34**, 943–952 (2010)
118. Laukkanen, T., Tveit, T.-M., Ojalehto, V., Miettinen, K., Fogelholm, C.-J.: Bilevel heat exchanger network synthesis with an interactive multi-objective optimization method. Appl. Therm. Eng. **48**, 301–316 (2012)
119. Lazimy, R.: Interactive relaxation method for a broad class of integer and continuous nonlinear multiple criteria problems. J. Math. Anal. Appl. **116**, 553–573 (1986)
120. Lazimy, R.: Solving multiple criteria problems by interactive decomposition. Math. Program. **35**, 334–361 (1986)
121. Lazimy, R.: Interactive polyhedral outer approximation (IPOA) strategy for general multiobjective optimization problems. Ann. Oper. Res. **210**, 73–99 (2012)
122. Lieberman, E.R.: Soviet multi-objective mathematical programming methods: an overview. Manag. Sci. **37**, 1147–1165 (1991)
123. Loganathan, G.V., Sherali, H.D.: A convergent interactive cutting-plane algorithm for multiobjective optimization. Oper. Res. **35**, 365–377 (1987)
124. Lotov, A.V.: Computer-based support for planning and negotiation on environmental rehabilitation of water resource systems. In: Loucks, D.P. (ed.) Restoration of Degraded Rivers: Challenges, Issues and Experiences, pp. 417–445. Kluwer, Boston (1998)
125. Lotov, A.V., Bushenkov, V.A., Chernykh, O.L.: Multicriteria DSS for river water-quality planning. Microcomput. Civil Eng. **12**, 57–67 (1997)
126. Lotov, A.V., Bushenkov, V.A., Kamenev, G.K.: Interactive Decision Maps: Approximation and Visualization of Pareto Frontier. Kluwer, Norwell (2004)
127. Luque, M., Miettinen, K., Eskelinen, P., Ruiz, F.: Incorporating preference information in interactive reference point methods for multiobjective optimization. Omega **37**, 450–462 (2009)
128. Luque, M., Ruiz, F., Steuer, R.E.: Modified interactive Chebyshev algorithm (MICA) for convex multiobjective programming. Eur. J. Oper. Res. **204**, 557–564 (2010)
129. Luque, M., Ruiz, F., Miettinen, K.: Global formulation for interactive multiobjective optimization. OR Spectr. **33**, 27–48 (2011)

130. Mackin, P., Roy, A., Wallenius, J.: An interactive weight space reduction procedure for nonlinear multiple objective mathematical programming. Math. Program. **127**, 425–444 (2011)
131. Malakooti, B., Raman, V.: An interactive multi-objective artificial neural network approach for machine setup optimization. J. Intell. Manuf. **11**, 41–50 (2000)
132. Marler, R.T., Arora, J.S.: Survey of multi-objective optimization methods for engineering. Struct. Multidiscip. Optim. **26**, 369–395 (2004)
133. Masud, A.S.M., Hwang, C.L.: Interactive sequential goal programming. J. Oper. Res. Soc. **32**, 391–400 (1981)
134. Masud, A.S.M., Zheng, X.: An algorithm for multiple-objective non-linear programming. J. Oper. Res. Soc. **40**, 895–906 (1989)
135. Meng, Z., Shen, R., Jiang, M.: An objective penalty functions algorithm for multiobjective optimization problem. Am. J. Oper. Res. **1**, 229–235 (2011)
136. Messac, A.: Physical programming: effective optimization for computational design. AIAA J. **34**, 149–158 (1996)
137. Michalowski, W.: Evaluation of a multiple criteria interactive programming approach: an experiment. INFOR Inf. Syst. Oper. Res. **25**, 165–173 (1987)
138. Michalowski, W., Szapiro, T.: A bi-reference procedure for interactive multiple criteria programming. Oper. Res. **40**, 247–258 (1992)
139. Miettinen, K.: Nonlinear Multiobjective Optimization. Kluwer, Boston (1999)
140. Miettinen, K.: IND-NIMBUS for demanding interactive multiobjective optimization. In: Trzaskalik, T. (ed.) Multiple Criteria Decision Making '05, pp. 137–150. The Karol Adamiecki University of Economics in Katowice, Katowice (2006)
141. Miettinen, K.: Using interactive multiobjective optimization in continuous casting of steel. Mater. Manuf. Process. **22**, 585–593 (2007)
142. Miettinen, K.: Survey of methods to visualize alternatives in multiple criteria decision making problems. OR Spectr. **36**, 3–37 (2014)
143. Miettinen, K., Mäkelä, M.M.: An interactive method for nonsmooth multiobjective optimization with an application to optimal control. Optim. Methods Softw. **2**, 31–44 (1993)
144. Miettinen, K., Mäkelä, M.M.: A nondifferentiable multiple criteria optimization method applied to continuous casting process. In: Fasano, A., Primicerio, M. (eds.) Proceedings of the Seventh European Conference on Mathematics in Industry, pp. 255–262. B. G. Teubner, Stuttgart (1994)
145. Miettinen, K., Mäkelä, M.M.: Interactive bundle-based method for nondifferentiable multiobjective optimization: NIMBUS. Optimization **34**, 231–246 (1995)
146. Miettinen, K., Mäkelä, M.M.: Interactive method NIMBUS for nondifferentiable multiobjective optimization problems. In: Clímaco, J. (ed.) Multicriteria Analysis, pp. 310–319. Springer, Berlin, Heidelberg (1997)
147. Miettinen, K., Mäkelä, M.M.: Interactive MCDM support system in the Internet. In: Stewart, T., van den Honert, R. (eds.) Trends in Multicriteria Decision Making: Proceedings of the 13th International Conference on Multiple Criteria Decision Making, pp. 419–428. Springer, Berlin, Heidelberg(1998)
148. Miettinen, K., Mäkelä, M.M.: Comparative evaluation of some interactive reference point-based methods for multi-objective optimisation. J. Oper. Res. Soc. **50**, 949–959 (1999)
149. Miettinen, K., Mäkelä, M.M.: Interactive multiobjective optimization system WWW-NIMBUS on the Internet. Comput. Oper. Res. **27**, 709–723 (2000)
150. Miettinen, K., Mäkelä, M.M.: On scalarizing functions in multiobjective optimization. OR Spectr. **24**, 193–213 (2002)
151. Miettinen, K., Mäkelä, M.M.: Synchronous approach in interactive multiobjective optimization. Eur. J. Oper. Res. **170**, 909–922 (2006)
152. Miettinen, K., Mäkelä, M.M., Männikkö, T.: Optimal control of continuous casting by nondifferentiable multiobjective optimization. Comput. Optim. Appl. **11**, 177–194 (1998)
153. Miettinen, K., Lotov, A.V., Kamenev, G.K., Berezkin, V.E.: Integration of two multiobjective optimization methods for nonlinear problems. Optim. Methods Softw. **18**, 63–80 (2003)

154. Miettinen, K., Mäkelä, M.M., Kaario, K.: Experiments with classification-based scalarizing functions in interactive multiobjective optimization. Eur. J. Oper. Res. **175**, 931–947 (2006)
155. Miettinen, K., Ruiz, F., Wierzbicki, A.P.: Introduction to multiobjective optimization: interactive approaches. In: Branke, J., Deb, K., Miettinen, K., Slowinski, R. (eds.) Multiobjective Optimization: Interactive and Evolutionary Approaches, pp. 27–57. Springer, Berlin, Heidelberg (2008)
156. Miettinen, K., Eskelinen, P., Ruiz, F., Luque, M.: NAUTILUS method: an interactive technique in multiobjective optimization based on the nadir point. Eur. J. Oper. Res. **206**, 426–434 (2010)
157. Miettinen, K., Mustajoki, J., Stewart, T.J.: Interactive multiobjective optimization with NIMBUS for decision making under uncertainty. OR Spectr. **36**, 39–56 (2014)
158. Mitani, K., Nakayama, H.: A multiobjective diet planning support system using the satisficing trade-off method. J. Multi-Criteria Decis. Anal. **6**, 131–139 (1997)
159. Mocci, U., Primicerio, L.: Ring network design: an MCDM approach. In: Fandel, G., Gal, T. (eds.) Multiple Criteria Decision Making: Proceedings of the Twelfth International Conference, pp. 491–500. Springer, Berlin, Heidelberg (1997)
160. Mohan, C., Nguyen, H.T.: Reference direction interactive method for solving multiobjective fuzzy programming problems. Eur. J. Oper. Res. **107**, 599–613 (1998)
161. Moldavskiy, M.A.: Singling out a set of undominated solutions in continuous vector optimization problems. Sov. Autom. Control **14**, 47–53 (1981)
162. Monarchi, D.E., Kisiel, C.C., Duckstein, L.: Interactive multiobjective programming in water resources: a case study. Water Res. Res. **9**, 837–850 (1973)
163. Monz, M., Kufer, K.H., Bortfeld, T.R., Thieke, C.: Pareto navigation – algorithmic foundation of interactive multi-criteria IMRT planning. Phys. Med. Biol. **53**, 985–998 (2008)
164. Mote, J., Olson, D.L., Venkataramanan, M.A.: A comparative multiobjective programming study. Math. Comput. Model. **10**, 719–729 (1988)
165. M'silti, A., Tolla, P.: An interactive multiobjective nonlinear programming procedure. Eur. J. Oper. Res. **64**, 115–125 (1993)
166. Mukai, H.: Algorithms for multicriterion optimization. IEEE Trans. Autom. Control **25**, 177–186 (1980)
167. Musselman, K., Talavage, J.: A tradeoff cut approach to multiple objective optimization. Oper. Res. **28**, 1424–1435 (1980)
168. Nakayama, H.: On the components in interactive multiobjective programming methods. In: Grauer, M., Thompson, M., Wierzbicki, A.P. (eds.) Plural Rationality and Interactive Decision Processes, pp. 234–247. Springer, Berlin, Heidelberg (1985)
169. Nakayama, H.: Sensitivity and trade-off analysis in multiobjective programming. In: Lewandowski, A., Stanchev, I. (eds.) Methodology and Software for Interactive Decision Support, pp. 86–93. Springer, Heidelberg (1989)
170. Nakayama, H.: Satisficing trade-off method for problems with multiple linear fractional objectives and its applications. In: Lewandowski, A., Volkovich, V. (eds.) Multiobjective Problems of Mathematical Programming, pp. 42–50. Springer, Heidelberg (1991)
171. Nakayama, H.: Trade-off analysis based upon parametric optimization. In: Korhonen, P., Lewandowski, A., Wallenius, J. (eds.) Multiple Criteria Decision Support, pp. 42–52. Springer, Heidelberg (1991)
172. Nakayama, H.: Trade-off analysis using parametric optimization techniques. Eur. J. Oper. Res. **60**, 87–98 (1992)
173. Nakayama, H.: Engineering applications of multi-objective programming: recent results. In: Tzeng, G.H., Wand, H.F., Wen, U.P., Yu, P.L. (eds.) Multiple Criteria Decision Making – Proceedings of the Tenth International Conference: Expand and Enrich the Domains of Thinking and Application, pp. 369–378. Springer, New York (1994)
174. Nakayama, H.: Aspiration level approach to interactive multi-objective programming and its applications. In: Pardalos, P.M., Siskos, Y., Zopounidis, C. (eds.) Advances in Multicriteria Analysis, pp. 147–174. Kluwer, Dordrecht (1995)

175. Nakayama, H., Furukawa, K.: Satisficing trade-off method with an application to multiobjective structural design. Large Scale Syst. **8**, 47–57 (1985)
176. Nakayama, H., Sawaragi, Y.: Satisficing trade-off method for multiobjective programming. In: Grauer, M., Wierzbicki, A.P. (eds.) Interactive Decision Analysis, pp. 113–122. Springer, Heidelberg (1984)
177. Nakayama, H., Tanino, T., Sawaragi, Y.: An interactive optimization method in multicriteria decisionmaking. IEEE Trans. Syst. Man Cybern. **10**, 163–169 (1980)
178. Nakayama, H., Nomura, J., Sawada, K., Nakajima, R.: An application of satisficing trade-off method to a blending problem of industrial materials. In: Fandel, G., Grauer, M., Kurzhanski, A., Wierzbicki, A.P. (eds.) Large-Scale Modelling and Interactive Decision Analysis, pp. 303–313. Springer, Heidelberg (1986)
179. Nakayama, H., Kaneshige, K., Takemoto, S., Watada, Y.: An application of a multi-objective programming technique to construction accuracy control of cable-stayed bridges. Eur. J. Oper. Res. **87**, 731–738 (1995)
180. Nakayama, H., Kaneshige, K., Takemoto, S., Watada, Y.: A construction accuracy control system of cable stayed bridge using a multi-objective programming technique. In: Fandel, G., Gal, T. (eds.) Multiple Criteria Decision Making: Proceedings of the Twelfth International Conference, pp. 501–509. Springer, Berlin, Heidelberg (1997)
181. Narula, S.C., Weistroffer, H.R.: Algorithms for multi-objective nonlinear programming problems: an overview. In: Lockett, A.G., Islei, G. (eds.) Improving Decision Making in Organisations, pp. 434–443. Springer, Berlin, Heidelberg (1989)
182. Narula, S.C., Weistroffer, H.R.: A flexible method for nonlinear multicriteria decisionmaking problems. IEEE Trans. Syst. Man Cybern. **19**, 883–887 (1989)
183. Narula, S.C., Kirilov, L., Vassilev, V.: An interactive algorithm for solving multiple objective nonlinear programming problems. In: Tzeng, G.H., Wand, H.F., Wen, U.P., Yu, P.L. (eds.) Multiple Criteria Decision Making – Proceedings of the Tenth International Conference: Expand and Enrich the Domains of Thinking and Application, pp. 119–127. Springer, New York (1994)
184. Narula, S.C., Kirilov, L., Vassilev, V.: Reference direction approach for solving multiple objective nonlinear programming problems. IEEE Trans. Syst. Man Cybern. **24**, 804–806 (1994)
185. Nijkamp, P., Spronk, J.: Interactive multiple goal programming: an evaluation and some results. In: Fandel, G., Gal, T. (eds.) Multiple Criteria Decision Making Theory and Applications, pp. 278–293. Springer, Berlin, Heidelberg (1980)
186. Ogryczak, W.: Preemptive reference point method. In: Clímaco, J. (ed.) Multicriteria Analysis, pp. 156–167. Springer, Berlin, Heidelberg (1997)
187. Olbrisch, M.: The interactive reference point approach as a solution concept for econometric decision models. In: Beckmann, M.J., Gaede, K.-W., Ritter, K., Schneeweiss, H. (eds.) X. Symposium on Operations Research, Part 1, Sections 1–5, pp. 611–619. Verlag Anton Hain Meisenheim GmbH, Königstein (1986)
188. Olson, D.L.: Review of empirical studies in multiobjective mathematical programming: subject reflection of nonlinear utility and learning. Decis. Sci. **23**, 1–20 (1992)
189. Olson, D.L.: Tchebycheff norms in multi-objective linear programming. Math. Comput. Model. **17**, 113–124 (1993)
190. Oppenheimer, K.R.: A proxy approach to multi-attribute decision making. Manag. Sci. **24**, 675–689 (1978)
191. Osyczka, A., Kundu, S.: A new method to solve generalized multicriteria optimization problems using the simple genetic algorithm. Struct. Optim. **10**, 94–99 (1995)
192. Ramesh, R., Karwan, M.H., Zionts, S.: Theory of convex cones in multicriteria decision making. Ann. Oper. Res. **16**, 131–147 (1988)
193. Ramesh, R., Karwan, M.H., Zionts, S.: Interactive multicriteria linear programming: an extension of the method of Zionts and Wallenius. Nav. Res. Logist. **36**, 321–335 (1989)
194. Ramesh, R., Karwan, M.H., Zionts, S.: Preference structure representation using convex cones in multicriteria integer programming. Manag. Sci. **35**, 1092–1105 (1989)

195. Reeves, G.R., Gonzalez, J.J.: A comparison of two interactive MCDM procedures. Eur. J. Oper. Res. **41**, 203–209 (1989)
196. Reeves, G.R., MacLeod, K.R.: Robustness of the interactive weighted Tchebycheff procedure to inaccurate preference information. J. Multi-Criteria Decis. Anal. **8**, 128–132 (1999)
197. Rietveld, P.: Multiple Objective Decision Methods and Regional Planning. North-Holland, Amsterdam (1980)
198. Romero, C.: Handbook of Critical Issues in Goal Programming. Pergamon, Oxford (1991)
199. Romero, C.: Extended lexicographic goal programming: a unifying approach. Omega **29**, 63–71 (2001)
200. Rosenthal, R.E.: Principles of multiobjective optimization. Decis. Sci. **16**, 133–152 (1985)
201. Rosinger, E.E.: Interactive algorithm for multiobjective optimization. J. Optim. Theory Appl. **35**, 339–365 (1981). Errata Corrige in J. Optim. Theory Appl. **38**, 147–148 (1982)
202. Roy, B.: The outranking approach and the foundations of ELECTRE methods. In: Bana e Costa, C.A. (ed.) Readings in Multiple Criteria Decision Aid, pp. 155–183. Springer, Berlin, Heidelberg (1990)
203. Roy, A., Mackin, P.: Multicriteria optimization (linear and nonlinear) using proxy value functions. In: Korhonen, P., Lewandowski, A., Wallenius, J. (eds.) Multiple Criteria Decision Support, pp. 128–134. Springer, Heidelberg (1991)
204. Roy, A., Wallenius, J.: Nonlinear multiobjective optimization: an algorithm and some theory. Math. Program. **55**, 235–249 (1992)
205. Ruiz, F., Luque, M., Miettinen, K.: Improving the computational efficiency of a global formulation (GLIDE) for interactive multiobjective optimization. Ann. Oper. Res. **197**, 47–70 (2012)
206. Ruotsalainen, H., Miettinen, K., Palmgren, J.-E., Lahtinen, T.: Interactive multiobjective optimization for anatomy based three-dimensional HDR brachytherapy. Phys. Med. Biol. **55**, 4703–4719 (2010)
207. Sadagopan, S., Ravindran, A.: Interactive algorithms for multiple criteria nonlinear programming problems. Eur. J. Oper. Res. **25**, 247–257 (1986)
208. Sakawa, M.: Interactive multiobjective decision making by the sequential proxy optimization technique: SPOT. Eur. J. Oper. Res. **9**, 386–396 (1982)
209. Sakawa, M., Yauchi, K.: An interactive fuzzy satisficing method for multiobjective nonconvex programming problems through floating point genetic algorithms. Eur. J. Oper. Res. **117**, 113–124 (1999)
210. Sawaragi, Y., Nakayama, H., Tanino, T.: Theory of Multiobjective Optimization. Academic, Orlando (1985)
211. Shin, W.S., Ravindran, A.: Interactive multiple objective optimization: survey I – continuous case. Comput. Oper. Res. **18**, 97–114 (1991)
212. Silverman, J., Steuer, R.E., Whisman, A.W.: A multi-period, multiple criteria optimization system for manpower planning. Eur. J. Oper. Res. **34**, 160–170 (1988)
213. Sindhya, K., Ruiz, A.B., Miettinen, K.: A preference based interactive evolutionary algorithm for multiobjective optimization: PIE. In: Takahashi, R.H.C., Deb, K., Wanner, E.F., Greco, S. (eds.) Evolutionary Multi-Criterion Optimization: 6th International Conference, Proceedings, pp. 212–225. Springer, Berlin, Heidelberg (2011)
214. Sindhya, K., Ojalehto, V., Savolainen, J., Niemistö, H., Hakanen, J., Miettinen, K.: Coupling dynamic simulation and interactive multiobjective optimization for complex problems: an APROS-NIMBUS case study. Expert Syst. Appl. **41**, 2546–2558 (2014)
215. Skulimowski, A.M.J.: Decision Support Systems Based on Reference Sets. Wydawnictwa AGH, Kraków (1996)
216. Slowinski, R.: Interactive multiobjective optimization based on ordinal regression. In: Lewandowski, A., Volkovich, V. (eds.) Multiobjective Problems of Mathematical Programming, pp. 93–100. Springer, Heidelberg (1991)
217. Song, A., Cheng, W.-M.: A method for multihuman and multi-criteria decision making. In: Sydow, A., Tzafestas, S.G., Vichnevetsky, R. (eds.) Systems Analysis and Simulation 1988 I: Theory and Foundations, pp. 213–216. Akademie, Berlin (1988)

218. Spronk, J.: Interactive multifactorial planning: state of the art. In: Bana e Costa, C.A. (ed.) Readings in Multiple Criteria Decision Aid, pp. 512–534. Springer, Berlin, Heidelberg (1990)
219. Stam, A., Kuula, M., Cesar, H.: Transboundary air pollution in Europe: an interactive multicriteria tradeoff analysis. Eur. J. Oper. Res. **56**, 263–277 (1992)
220. Statnikov, R.B.: Multicriteria Design: Optimization and Identification. Kluwer, Dordrecht (1999)
221. Statnikov, R.B., Matusov, J.: Use of p_τ-nets for the approximation of the Edgeworth-Pareto set in multicriteria optimization. J. Optim. Theory Appl. **91**, 543–560 (1996)
222. Steponavice, I., Ruuska, S., Miettinen, K.: A solution process for simulation-based multiobjective design optimization with an application in paper industry. Comput. Aided Des. **47**, 45–58 (2014)
223. Steuer, R.E.: Multiple Criteria Optimization: Theory, Computation, and Applications. Wiley, New York (1986)
224. Steuer, R.E.: The Tchebycheff procedure of interactive multiple objective programming. In: Karpak, B., Zionts, S. (eds.) Multiple Criteria Decision Making and Risk Analysis Using Microcomputers, pp. 235–249. Springer, Berlin, Heidelberg (1989)
225. Steuer, R.E.: Implementing the Tchebycheff method in a spreadsheet. In: Karwan, M.H., Spronk, J., Wallenius, J. (eds.) Essays in Decision Making: A Volume in Honour of Stanley Zionts, pp. 93–103. Springer, Berlin, Heidelberg (1997)
226. Steuer, R.E., Choo, E.-U.: An interactive weighted Tchebycheff procedure for multiple objective programming. Math. Program. **26**, 326–344 (1983)
227. Steuer, R.E., Gardiner, L.R.: Interactive multiple objective programming: concepts, current status, and future directions. In: Bana e Costa, C.A. (ed.) Readings in Multiple Criteria Decision Aid, pp. 413–444. Springer, Berlin, Heidelberg, (1990)
228. Steuer, R.E., Gardiner, L.R.: On the computational testing of procedures for interactive multiple objective linear programming. In: Fandel, G., Gehring, H. (eds.) Operations Research, pp. 121–131. Springer, Berlin, Heidelberg (1991)
229. Steuer, R.E., Sun, M.: The parameter space investigation method of multiple objective nonlinear programming: a computational investigation. Oper. Res. **43**, 641–648 (1995)
230. Steuer, R.E., Whisman, A.W.: Toward the consolidation of interactive multiple objective programming procedures. In: Fandel, G., Grauer, M., Kurzhanski, A., Wierzbicki, A.P. (eds.) Large-Scale Modelling and Interactive Decision Analysis, pp. 232–241. Springer, Heidelberg (1986)
231. Steuer, R.E., Silverman, J., Whisman, A.W.: A combined Tchebycheff/aspiration criterion vector interactive multiobjective programming procedure. Manag. Sci. **39**, 1255–1260 (1993)
232. Stewart, T.J.: A critical survey on the status of multiple criteria decision making theory and practice. Omega **20**, 569–586 (1992)
233. Sun, M., Stam, A., Steuer, R.E.: Solving multiple objective programming problems using feed-forward artificial neural networks: the interactive FFANN procedure. Manag. Sci. **42**, 835–849 (1996)
234. Sun, M., Stam, A., Steuer, R.E.: Interactive multiple objective programming using Tchebycheff programs and artificial neural networks. Comput. Oper. Res. **27**, 601–620 (2000)
235. Sunaga, T., Mazeed, M.A., Kondo, E.: A penalty function formulation for interactive multiobjective programming problems. In: Iri, M., Yajima, K. (eds.) System Modelling and Optimization, pp. 221–230. Springer, Heidelberg (1988)
236. Tabucanon, M.T.: Multiple Criteria Decision Making in Industry. Elsevier, Amsterdam (1988)
237. Tamiz, M., Jones, D.F.: A general interactive goal programming algorithm. In: Fandel, G., Gal, T. (eds.) Multiple Criteria Decision Making: Proceedings of the Twelfth International Conference, pp. 433–444. Springer, Berlin, Heidelberg (1997)
238. Tapia, C.G., Murtagh, B.A.: The use of preference criteria in interactive multiobjective mathematical programming. Asia Pac. J. Oper. Res. **6**, 131–147 (1989)
239. Tapia, C.G., Murtagh, B.A.: A Markovian process in interactive multiobjective decision-making. Eur. J. Oper. Res. **57**, 421–428 (1992)

240. Tappeta, R.V., Renaud, J.E., Messac, A., Sundararaj, G.J.: Interactive physical programming: tradeoff analysis and decision making in multidisciplinary optimization. AIAA J. **38**, 917–926 (2000)
241. Tarkkanen, S., Miettinen, K., Hakanen, J., Isomäki, H.: Incremental user-interface development for interactive multiobjective optimization. Expert Syst. Appl. **40**, 3220–3232 (2013)
242. Tarvainen, K.: On the implementation of the interactive surrogate worth trade-off (ISWT) method. In: Grauer, M., Wierzbicki, A.P. (eds.) Interactive Decision Analysis, pp. 154–161. Springer, Berlin, Heidelberg (1984)
243. Tecle, A., Duckstein, L.: A procedure for selecting MCDM techniques for forest resources management. In: Goicoechea, A., Duckstein, L., Zionts, S. (eds.) Multiple Criteria Decision Making: Proceedings of the Ninth International Conference: Theory and Applications in Business, Industry, and Government, pp. 19–32. Springer, New York (1992)
244. Teghem J. Jr., Delhaye, C., Kunsch, P.L.: An interactive decision support system (IDSS) for multicriteria decision aid. Math. Comput. Model. **12**, 1311–1320 (1989)
245. Udink ten Cate, A.: On the determination of the optimal temperature for the growth of an early cucumber crop in a greenhouse. In: Grauer, M., Thompson, M., Wierzbicki, A.P. (eds.) Plural Rationality and Interactive Decision Processes, pp. 311–318. Springer, Berlin, Heidelberg (1985)
246. Vanderpooten, D.: Multiobjective programming: basic concepts and approaches. In: Slowinski, R., Teghem, J. (eds.) Stochastic Versus Fuzzy Approaches to Multiobjective Mathematical Programming under Uncertainty, pp. 7–22. Kluwer, Dordrecht (1990)
247. Vanderpooten, D., Vincke, P.: Description and analysis of some representative interactive multicriteria procedures. Math. Comput. Model. **12**, 1221–1238 (1989)
248. Vetschera, R.: Feedback-oriented group decision support in a reference point framework. In: Lewandowski, A., Volkovich, V. (eds.) Multiobjective Problems of Mathematical Programming, pp. 309–314. Springer, Heidelberg (1991)
249. Vetschera, R.: A note on scalarizing functions under changing sets of criteria. Eur. J. Oper. Res. **52**, 113–118 (1991)
250. Vincke, P.: Multicriteria Decision-Aid. Wiley, Chichester (1992)
251. Wallenius, J.: Comparative evaluation of some interactive approaches to multicriterion optimization. Manag. Sci. **21**, 1387–1396 (1975)
252. Wang, S.: Algorithms for multiobjective and nonsmooth optimization. In: Kleinschmidt, P., Radermacher, F.J., Schweitzer, W., Wildermann, H. (eds.) Methods of Operations Research 58, pp. 131–142. Athenäum Verlag, Frankfurt am Main (1989)
253. Wang, S.: An interactive method for multicriteria decision making. In: Phua, K.H., Wang, C.M., Yeong, W.Y., Leong, T.Y., Loh, H.T., Tan, K.C., Chou, F.S. (eds.), Optimization: Techniques and Applications. Proceedings of the International Conference (ICOTA), vol. 1, pp. 307–316. World Scientific, Singapore (1992)
254. Weistroffer, H.R.: Multiple criteria decision making with interactive over-achievement programming. Oper. Res. Lett. **1**, 241–245 (1982)
255. Weistroffer, H.R.: An interactive goal-programming method for non-linear multiple criteria decision-making problems. Comput. Oper. Res. **10**, 311–320 (1983)
256. Weistroffer, H.R.: A combined over- and under-achievement programming approach to multiple objective decision-making. Large Scale Syst. **7**, 47–58 (1984)
257. Weistroffer, H.R.: A flexible model for multi-objective optimization. In: Jahn, J., Krabs, W. (eds.) Recent Advances and Historical Developments of Vector Optimization, pp. 311–316. Springer, Berlin, Heidelberg, (1987)
258. Wendell, R.E., Lee, D.N.: Efficiency in multiple objective optimization problems. Math. Program. **12**, 406–414 (1977)
259. White, D.J.: A selection of multi-objective interactive programming methods. In: French, S., Hartley, R., Thomas, L.C., White, D.J. (eds.) Multi-Objective Decision Making, pp. 99–126. Academic, London (1983)
260. Wierzbicki, A.P.: The use of reference objectives in multiobjective optimization. In: Fandel, G., Gal, T. (eds.) Multiple Criteria Decision Making Theory and Applications, pp. 468–486. Springer, Berlin, Heidelberg (1980)

261. Wierzbicki, A.P.: A mathematical basis for satisficing decision making. Math. Model. **3**, 391–405 (1982)
262. Wierzbicki, A.P.: A methodological approach to comparing parametric characterizations of efficient solutions. In: Fandel, G., Grauer, M., Kurzhanski, A., Wierzbicki, A.P. (eds.) Large-Scale Modelling and Interactive Decision Analysis, pp. 27–45. Springer, Heidelberg (1986)
263. Wierzbicki, A.P.: On the completeness and constructiveness of parametric characterizations to vector optimization problems. OR Spectr. **8**, 73–87 (1986)
264. Wierzbicki, A.P.: Convergence of interactive procedures of multiobjective optimization and decision support. In: Karwan, M.H., Spronk, J., Wallenius, J. (eds.) Essays in Decision Making: A Volume in Honour of Stanley Zionts, pp. 19–47. Springer, Berlin, Heidelberg (1997)
265. Wierzbicki, A.P.: Reference point approaches. In: Gal, T., Stewart, T.J., Hanne, T. (eds.) Multicriteria Decision Making: Advances in MCDM Models, Algorithms, Theory, and Applications, pp. 9-1–9-39. Kluwer, Boston (1999)
266. Wierzbicki, A.P., Granat, J.: Multi-objective modeling for engineering application in decision support. In: Fandel, G., Gal, T. (eds.) Multiple Criteria Decision Making: Proceedings of the Twelfth International Conference, pp. 529–540. Springer, Berlin, Heidelberg (1997)
267. Wierzbicki, A.P., Granat, J.: Multi-objective modeling for engineering applications: DIDASN++ system. Eur. J. Oper. Res. **113**, 372–389 (1999)
268. Winkels, H.-M., Meika, M.: An integration of efficiency projections into the Geoffrion approach for multiobjective linear programming. Eur. J. Oper. Res. **16**, 113–127 (1984)
269. Wood, E., Greis, N.P., Steuer, R.E.: Linear and nonlinear applications of the Tchebycheff metric to the multi criteria water allocation problem. In: Rinaldi, S. (ed.) Environmental Systems Analysis and Management, pp. 363–376. North-Holland, Amsterdam (1982)
270. Yang, J.-B.: Gradient projection and local region search for multiobjective optimization. Eur. J. Oper. Res. **112**, 432–459 (1999)
271. Yang, J.-B., Li, D.: Normal vector identification and interactive tradeoff analysis using minimax formulation in multiobjective optimization. IEEE Trans. Syst. Man Cybern. **32**, 305–319 (2002)
272. Yang, J.-B., Sen, P.: Preference modelling by estimating local utility functions for multiobjective optimization. Eur. J. Oper. Res. **95**, 115–138 (1996)
273. Yang, J.-B., Chen, C., Zhang, Z.-J.: The interactive step trade-off method (ISTM) for multiobjective optimization. IEEE Trans. Syst. Man Cybern. **20**, 688–695 (1990)
274. Yu, P.L.: A class of solutions for group decision problems. Manag. Sci. **19**, 936–946 (1973)
275. Yu, P.L.: Multiple-Criteria Decision Making Concepts, Techniques, and Extensions. Plenum, New York (1985)
276. Zadeh, L.: Optimality and non-scalar-valued performance criteria. IEEE Trans. Autom. Control **8**, 59–60 (1963)
277. Zeleny, M.: Compromise programming. In: Cochrane, J.L., Zeleny, M. (eds.) Multiple Criteria Decision Making, pp. 262–301. University of South Carolina Press, Columbia (1973)

Chapter 23
MCDA and Multiobjective Evolutionary Algorithms

Juergen Branke

Abstract Evolutionary multiobjective optimization promises to efficiently generate a representative set of Pareto optimal solutions in a single optimization run. This allows the decision maker to select the most preferred solution from the generated set, rather than having to specify preferences a priori. In recent years, there has been a growing interest in combining the ideas of evolutionary multiobjective optimization and MCDA. MCDA can be used before optimization, to specify partial user preferences, after optimization, to help select the most preferred solution from the set generated by the evolutionary algorithm, or be tightly integrated with the evolutionary algorithm to guide the optimization towards the most preferred solution. This chapter surveys the state of the art of using preference information within evolutionary multiobjective optimization.

Keywords Evolutionary algorithms • Interactive multiobjective optimization

23.1 Introduction

Single objective Evolutionary Algorithms (EAs) are general purpose optimization heuristics inspired by natural evolution. Because they make very few assumptions about the optimization problem (for example, they do not require that the objective function is differentiable and can work with almost arbitrary constraints), they are recognized as very versatile and powerful tool for complex optimization problems that can not be solved with exact methods. They are successfully used in industry on a wide variety of complex optimization problems including, for example, scheduling, transportation, or engineering design.

For multi-objective problems, they have an additional appeal. Because they maintain a population of candidate solutions throughout the optimization process, they are able to simultaneously search for a set of solutions in a single run. In other words, they are able to search for a representative set of Pareto-optimal solutions,

J. Branke (✉)
Warwick Business School, University of Warwick, Coventry CV4 7AL, UK
e-mail: juergen.branke@wbs.ac.uk

S. Greco et al. (eds.), *Multiple Criteria Decision Analysis*, International Series in Operations Research & Management Science 233,
DOI 10.1007/978-1-4939-3094-4_23

approximating the true Pareto set, in a single run. As multi-objective evolutionary algorithms (MOEAs) usually don't require preference information from the user, they are often called "a posteriori" methods: The user reveals his/her preferences only after optimization, by picking a solution from the set. Being presented with a set of Pareto-optimal solutions to choose from is very appealing for many decision makers, and MOEAs have become one of the most active research areas in evolutionary computation.

In the beginning, the MOEA community has developed more or less independently from the "classical" MCDM community. Only in recent years, most notably with the initiation of regular Dagstuhl workshops,[1] has it been recognized that MOEAs and MCDM have a lot to offer to each other, and subsequently the communities have grown together. Promising possibilities for combining MOEA and MCDM techniques include:

1. Use an MOEA to generate an approximation of the Pareto frontier, but then use an MCDM technique to help the decision maker (DM) to select the best solution from this approximation set. While the latter step may be almost trivial in the case of two objectives (which was the focus of the MOEA community in the early years), an MCDM support may be very useful in case of more objectives.
2. Start by eliciting partial or approximate user preferences, and use this information to narrow down the search of the MOEA. That is, rather than searching for an approximation of the entire Pareto frontier, the search is focused on what is believed to be the most interesting region for the DM, consistent with the provided preference information.
3. Interleaving the use of MOEA and MCDM techniques. The MOEA is run for a few generations, then MCDA techniques are used to elicit some user preferences, which can then in turn be used to guide the next few generations of the MOEA, before the next preference information is elicited.

In this chapter, after an introduction to MOEAs, the different possible combinations between MOEA and MCDA will be discussed. Throughout this chapter, unless specified otherwise we will assume minimization of objectives.

23.2 Multiobjective Evolutionary Algorithms

Evolutionary algorithms (EAs) are general purpose optimization heuristics inspired by natural evolution, in particular Darwin's principle of "survival of the fittest". Due to the inspiration from biology, many biological metaphors are used to describe EAs.

Starting with a set of candidate solutions (*population*), in each iteration (*generation*), the better solutions are selected (*parents*) and used to generate new solutions (*offspring*). For generating offspring, two operators are usually used: *Crossover*

[1] http://www.dagstuhl.de.

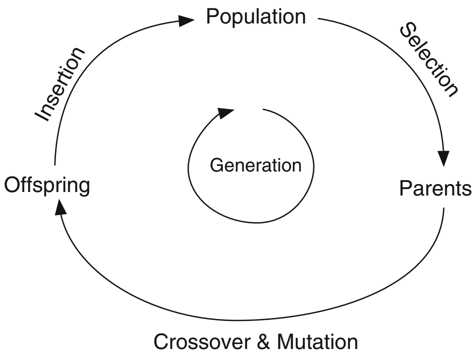


Fig. 23.1 Basic loop of an evolutionary algorithm

recombines the information of two parents in a new way, while *mutation* introduces small random modifications to a solution. These offspring are then inserted into the population, replacing some of the weaker solutions (*individuals*) so that the overall population size remains the same. By iteratively selecting the better solutions and using them to create new candidates, the population evolves, and the solutions become better and better adapted to the optimization problem at hand, just like in nature, where the individuals become better and better adapted to their environment through evolution. Darwin's principle of survival of the fittest is used twice: Good solutions are preferred when selecting the parents used to generate offspring, and only good solutions survive from one generation to the next. For an illustration, see Fig. 23.1.

To design an evolutionary algorithm for a particular optimization problem, not much is needed. One has to define the search space, i.e., the description of a solution and the constraints; an objective function that evaluates the quality of a solution (usually called *fitness function* in EA parlance), and the search operators crossover and mutation. Note that there are almost no restrictions in defining these components. In particular, the objective function can be an arbitrary black box and does not have to be continuous or differentiable. This makes EAs, as many other metaheuristics such as tabu search or simulated annealing, a very versatile tool that can be successfully used in domains where exact optimization methods are not applicable. Different to many other metaheuristics, EAs work with a population of solutions, rather than moving from one solution to the next. This not only helps avoiding getting stuck in local optima, it also introduces a new neighborhood structure defined by the crossover. For a good introduction to evolutionary algorithms, the reader is referred to [32].

From the point of view of multi-objective optimization, the most important aspect is that EAs work with a population of solutions. This makes it possible to use them to generate a set of solutions in one run, such as an approximation to the Pareto optimal set. This is much more efficient than generating an approximation of the Pareto set by running an algorithm multiple times, with different weights for the objectives or different constraint settings. This ability to search for a representative set of Pareto optimal solutions is appealing to many researchers and practitioners, and has made MOEAs one of the most active research areas in evolutionary computation. The number of publications in this area has soared over the past 10–15 years, and a comprehensive repository of MOEA publications maintained by Carlos Coello-Coello listed more than 6800 papers by April 2012 (http://delta.cs.cinvestav.mx/~ccoello/EMOO).

All that needs to be changed when moving from a single objective EA to a multi objective EA is the selection process and how individuals in the population are ranked. If there is only one objective, individuals are naturally ranked according to this objective, and it is clear which individuals are the best and should be selected as parents or survive to the next generation. In case of multiple objectives, it is still necessary to rank the individuals, but it is no longer obvious how to do this, and many different ranking schemes have been developed. In the remainder of this chapter, we will first describe the most widely used MOEA, the Non-dominated Sorting GA (NSGA-II), and then briefly describe two more recent developments that work on different principles. The section concludes with a brief discussion of interactive evolutionary algorithms which bear many similarities to interactive multi-objective evolutionary algorithms.

23.2.1 Non-dominated Sorting Genetic Algorithm (NSGA-II)

The non-dominated sorting genetic algorithm (NSGA-II) [29] is probably one of the most popular and widely used MOEA. It is based on the idea that a good approximation to the Pareto front is characterized by

1. a small distance of the solutions to the true Pareto front,
2. a wide range of solutions, i.e., an approximation of the extreme values, and
3. a good distribution of solutions, i.e., an even spread along the Pareto frontier.

So, NSGA-II tries to rank individuals according to how much they contribute to the above goals, ordering the goals in the above sequence. Because the true Pareto front is generally not known (otherwise, optimization would not be needed), the distance to the true Pareto frontier can not be measured. Instead NSGA-II computes a proxy measure by a method called the *non-dominance ranking*. For this ranking, the procedure first determines all non-dominated solutions and assigns them to the first (best) class. Then, it iteratively removes these solutions from the population, again determines all non-dominated solutions, and assigns them to the next best class, until the population is empty. An example of this classification can be seen in Fig. 23.2.

Within a class, the algorithm gives the highest rank to the extreme solutions in any objective in order to maintain a wide range of solutions according to the second of the above goals. Finally, the aim of an even distribution of solutions is followed by using the so-called *crowding distance* to generate a full order of the individuals. The crowding distance is the sum of differences between an individual's left and right neighbor, in each objective, where large distances are preferred.

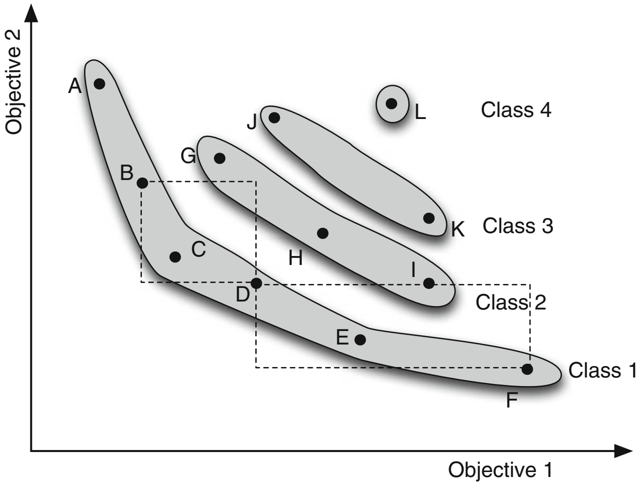


Fig. 23.2 Non-dominated sorting of solutions as in NSGA-II

In the example in Fig. 23.2, individual E is preferred over individual C because of a higher crowding distance. The overall order of individuals in Fig. 23.2 is thus $(A, F), E, D, B, C, (G, I), H, (J, K), L$, with parentheses indicating equivalence classes which may be ordered randomly.

Another popular MOEA that follows a similar idea of using a combination of proxy criteria to rank individuals is the Strength Pareto Evolutionary Algorithm (SPEA) [75].

23.2.2 Indicator-Based MOEAs

NSGA-II and SPEA-II are using several proxy criteria to determine the quality of an approximation to the Pareto frontier, and combine them in lexicographic order. More recently, researchers have defined single (unary) criteria to determine the quality of a set of solutions, and use an individual's marginal contribution to this criterion for ranking. That is, an individual is evaluated by the loss of performance in this criterion if the individual would be removed.

The most widely accepted criterion for the quality of an approximation set is the *hypervolume*. It measures the volume of the dominated portion of the objective space, bounded by a reference point, see Fig. 23.3. Individuals can then be ranked according to their marginal contribution to the hypervolume. If $HV(P)$ is the hypervolume of population P, the marginal HV of individual i would be calculated as $MHV(i) = HV(P) - HV(P \setminus \{i\})$, where individuals with larger MHV are preferred. In the example of Fig. 23.3, solution B has the largest marginal

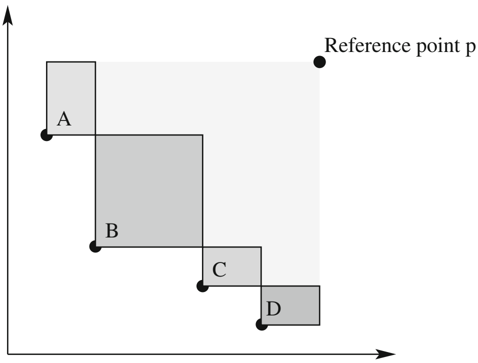


Fig. 23.3 Example for (marginal) Hypervolume

hypervolume and would be ranked first. This is the idea behind the S-MOEA [33] or IBEA [74]. Note that the HV depends on the location of the reference point which is not uniquely defined, and that it is expensive to calculate in higher-dimensional objective spaces [13]. Other unary criteria that have been used are epsilon-dominance [74] and marginal utility [8].

23.2.3 Multiobjective Evolutionary Algorithm Based on Decomposition (MOEA/D)

The multiobjective evolutionary algorithm based on decomposition (MOEA/D) [73] decomposes a multiobjective optimization problem into a number of single objective optimization sub-problems equal to the population size, e.g., by defining different weight vectors for a linear combination of objectives or different achievement scalarizing functions. But rather than solving these sub-problems independently, the idea is to solve them simultaneously, and allow the different search processes to influence each other. In short, the population comprises of the best solution found so far for each of the sub-problems. In every generation, a new offspring is created for each sub-problem by randomly selecting two parents from the sub-problem's neighbourhood, performing crossover and mutation, and re-inserting the individual into the population. The new individual replaces all individuals in the population for which it is better with respect to the corresponding sub-problem. In effect, this means mating is restricted to among individuals from the same region of the non-dominated frontier, and diversity in the population is maintained implicitly by the definition of the different sub-problems. It may be challenging to define appropriate sub-problems without any knowledge of the Pareto frontier.

23.2.4 Interactive Evolutionary Algorithms

Interactive EAs are single-objective EAs where the evaluation is entirely based on human judgement, for example when the aesthetics of a design is optimized [53]. It is up to the user to evaluate and rank solutions during the run, and this ranking is then used for selection. Human fatigue is a crucial factor in such algorithms, as the number of solutions usually looked at by EAs may become very large. Thus, various approaches based on approximate modeling (e.g., with a function learned from evaluation examples) of the DM's preferences have been proposed in this field, see, e.g., [64]. The evolutionary algorithm tries to predict a DM's answers using this model, and asks the DM to evaluate only some of the new solutions. There are apparent similarities of this field to interactive multi-objective optimization. In both cases we are looking for solutions being the best from the point of view of subjective preferences. Thus, an interactive evolutionary algorithm could be directly applied to a multi-objective problem, simply asking the DM to evaluate presented solutions. However, in multi-objective optimization, we assume to at least know the criteria that form the basis for the evaluation of a solution, and that these can be computed. Only how these objectives are combined to the overall utility of a solution is subjective. In other words, in a multi-objective problem, user evaluation is only necessary to compare mutually non-dominated solutions.

23.3 MCDM to Support the Selection from a Set of Solutions Generated by an MOEA

An MOEA is usually designed to find a representative set of the entire Pareto-optimal frontier. In the case of more than two objectives, however, there may be very many Pareto optimal solutions, and it may not be easy for a DM to identify the most preferred one. Thus it is natural to use MCDA techniques to help the DM select a solution from the set generated by the MOEA (see, e.g., [45]). This is just a straightforward application of one method after the other, so methodologically there is not much to say about this. Compared to a truly interactive optimization, the advantage is that the representative set of the efficient frontier is pre-computed, so the interaction can be very fast. Also, over the entire process, the DM is only shown solutions that are mutually non-dominating and hopefully close to the true Pareto front. On the other hand, the user can not go beyond the solutions generated by the EA in the first place, e.g., asking for a refined resolution in the most interesting area.

23.4 Integrating User Preferences in MOEA

While it may be impractical for a DM to completely specify his or her preferences before any alternatives are known (and turn the multi-objective problem into an single-objective problem), it makes sense to assume that the DMs have at least a rough idea about their preferences. The methods discussed in this section aim at integrating such imprecise knowledge into the EMO approach, biasing the search towards solutions that are considered as relevant by the DM. The goal is no longer to generate a good approximation to all Pareto optimal solutions, but a small set of solutions that contains the DM's preferred solution with the highest probability. This may yield three important advantages:

1. Instead of a diverse set of solutions, many of them clearly irrelevant to the DM, a search bias based on the DM's partial preferences will provide a more suitable sample of all Pareto optimal alternatives. It could either be a smaller set of only the most relevant solutions, or offer a more fine-grained resolution of the relevant parts of the Pareto frontier.
2. By focusing the search onto the relevant part of the search space, we expect the optimization algorithm to find these solutions more quickly.
3. As the number of objectives increases, it becomes more and more difficult to identify the complete Pareto optimal frontier. This is partly because of the increasing number of Pareto optimal solutions, but also because with an increasing number of objectives, almost all solutions become non-dominated, rendering dominance as selection criterion useless. Partial user preferences re-introduce the necessary selection pressure.

The literature contains quite a few techniques to incorporate full or partial preference information into MOEAs, and previous surveys on this topic include [15, 17, 61]. This section is partially based on [5]. In the following, we classify the different approaches based on the type of partial preference information they ask from the DM, namely objective scaling (Sect. 23.4.1), constraints (Sect. 23.4.2), a goal or reference point (Sect. 23.4.3), trade-off information (Sect. 23.4.4), a weight function over the objective space (Sect. 23.4.5), a distribution of possible utility functions (Sect. 23.4.6), outranking relations (Sect. 23.4.7) and direct solution rankings (Sect. 23.4.8).

Note that many of the approaches in this section were originally designed as two-step procedure: The DM is asked to reveal some information on their preferences, then the MOEA is run to identify an appropriate subset of solutions. But from there, it is only a small step to a truly interactive procedure that alternates between the elicitation of user preferences and MOEA steps. So, in this chapter we will not distinguish between MOEAs that take into account partial user preferences and truly interactive approaches, but will point out when the algorithm was described as interactive in the paper. One advantage of interactive methods may be that when the DM controls the search process, he/she gets more involved in the process, learns about potential alternatives, and is eventually more confident about the final choice.

23.4.1 Scaling

One of the often claimed advantages of MOEAs is that they do not require an a priori specification of user preferences because they generate a good approximation of the whole Pareto front, allowing the DM to pick his/her preferred solution afterwards. However, the whole Pareto optimal front may contain very many alternatives, in which case MOEAs can only hope to find a representative subset of all Pareto optimal solutions. Therefore, most basic EMO approaches attempt to generate a uniform distribution of representatives along the Pareto front. For this goal, they rely on distance information in the objective space, be it in the crowding distance of NSGA-II, the calculation of the hypervolume in IBEA or the definition of a neighborhood in MOEA/D. Thus, what is considered uniform depends on the scaling of the objectives. This is illustrated in Fig. 23.4. The left panel (a) shows an evenly distributed set of solutions along the Pareto front. Scaling the second objective by a factor of 100 (e.g., using centimeters instead of meters as unit), leads to a bias of the distribution and more solutions along the front parallel to the axis of the second objective (right panel). Note that depending on the shape of the front, this means that there is a bias towards objective 1 (as in the convex front in Fig. 23.4), or objective 2 (if the front is concave). So, the user-defined scaling is actually a usually ignored form of user preference specification necessary also for MOEAs.

Many current implementations of MOEAs (e.g., NSGA-II and SPEA) scale objectives based on the solutions currently in the population (see, e.g., [23, p. 248]). While this results in nice visualizations if the front is plotted with a 1:1 ratio, and relieves the DM from specifying a scaling, it assumes that ranges of values covered

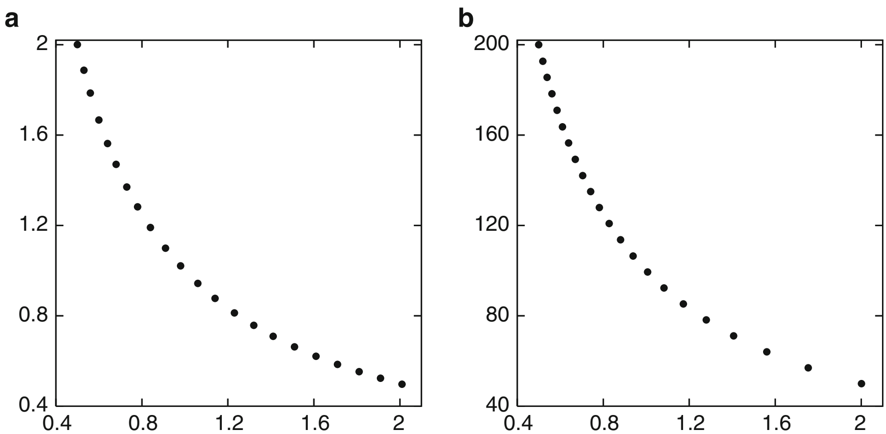


Fig. 23.4 Influence of scaling on the distribution of solutions along the Pareto front as generated by MOEAs. On the left figure (**a**), the front is plotted with a 1:1 ratio. On the right figure (**b**), the y-axis has been scaled by a factor of 100

by the Pareto front in each objective are equally important. Whether this assumption is justified certainly depends strongly on the application and the DM's preferences.

The sensitivity of most MOEAs to the scaling of the objectives also allows to introduce preferences explicitly by scaling objectives. Deb [24] was the first to make use of this idea by allowing to scale objectives linearly. As a result, distances in an objective that is scaled up appear greater, and most MOEAs would then rank individuals higher which are in regions of the Pareto frontier that are parallel to the scaled-up objective. However, a linear scaling of the objectives does not allow to focus on a compromise region (for equal scaling of the objectives, the effect of scaling cancels out).

Branke and Deb [6] refined the mechanism of [24] with a better control of the region based on trade-offs rather than objectives. Basically, scaling is done relative to a hyperplane, rather than to individual objectives. The DM defines the hyperplane in objective space, and the distribution of solutions is biased towards areas of the Pareto front that are parallel of the hyperplane. The extent of the bias can be controlled by a separate parameter.

Trautmann and Mehnen [68] suggest to use non-linear *desirability functions* for scaling. A desirability function maps the values of each objective to the interval [0, 1], describing the desirability of an objective function value, independently for each objective. If the desirability function is monotonic, Trautmann and Mehnen [68] suggests sigmoid functions. The MOEA is then applied to the space of desirability values. The non-dominance relations are not changed by a monotonic transformation, meaning the algorithm will still converge to the Pareto frontier, only the distribution of solutions along the frontier will change. The MOEA will focus on areas along the Pareto frontier where the desirability function has the largest gradient. This idea is studied in more detail in [69], where it is shown that the approach also produces sensible results when the regions of largest gradient lie completely outside or inside the Pareto frontier.

23.4.2 Constraints

Often, the DM can formulate preferences in the form of constraints, for example "Criterion 1 should be less than β". Handling constraints is a well-researched topic in evolutionary algorithms in general, and most of the techniques carry over to MOEAs in a straightforward manner. One of the simplest and most common techniques is probably to rank infeasible solutions according to their degree of infeasibility, and inferior to all feasible solutions [22, 51]. A detailed discussion of constraint handling techniques is out of the scope of this chapter. Instead, the interested reader is referred to [16] for a general survey on constraint handling techniques, and [23], Chapter 7, for a survey with focus on MOEA techniques.

23.4.3 Providing a Reference Point

Perhaps the most widely used way to provide preference information is a reference point, a technique that has a long tradition in multi-criteria decision making, see, e.g., [71, 72] and also Chap. 22 in this book. A reference point consists of aspiration levels reflecting desirable values for the objective function, i.e., a target the user is hoping for. Such an information can then be used in different ways to focus the search.

The use of a reference point to guide the MOEA has first been proposed by Fonseca and Fleming [38]. The basic idea there is to give a higher priority to objectives in which the goal is not fulfilled. Thus, when deciding whether a solution $\mathbf{x}$ is preferred over a solution $\mathbf{y}$ or not, first, only the objectives in which solution $\mathbf{x}$ does not satisfy the goal are considered, and $\mathbf{x}$ is preferred to $\mathbf{y}$ if it dominates $\mathbf{y}$ on these objectives. If $\mathbf{x}$ is equal to $\mathbf{y}$ in all these objectives, or if $\mathbf{x}$ satisfies the goal in all objectives, $\mathbf{x}$ is preferred over $\mathbf{y}$ either if $\mathbf{y}$ does not fulfill some of the objectives fulfilled by $\mathbf{x}$, or if $\mathbf{x}$ dominates $\mathbf{y}$ on the objectives fulfilled by $\mathbf{x}$. More formally, this can be stated as follows. Let $\mathbf{r}$ denote the reference point, and let there be m objectives without loss of generality sorted such that $\mathbf{x}$ fulfills objectives $k+1\dots m$ but not objectives $1\dots k$, i.e.

$$f_i(\mathbf{x}) > r_i \quad \forall i = 1\dots k \tag{23.1}$$

$$f_i(\mathbf{x}) \le r_i \quad \forall i = k+1\dots m. \tag{23.2}$$

Then, $\mathbf{x}$ is preferred to $\mathbf{y}$ if and only if

$$\begin{aligned} &\mathbf{x} \succ_{1\dots k} \mathbf{y} \vee \\ &\quad \mathbf{x} =_{1\dots k} \mathbf{y} \wedge [(\exists l \in [k+1\dots n] : f_l(\mathbf{y}) > r_k) \vee (\mathbf{x} \succ_{k+1\dots n} \mathbf{y})] \end{aligned} \tag{23.3}$$

with $\mathbf{x} \succ_{i..j} \mathbf{y}$ meaning that solution $\mathbf{x}$ dominates solution $\mathbf{y}$ on objectives i to j (i.e., for minimization problems as considered here, $f_k(\mathbf{x}) \le f_k(\mathbf{y}) \forall k = i\dots j$ with at least one strict inequality). A slightly extended version that allows the decision maker to additionally assign priorities to objectives has been published in [39]. This publication also contains the proof that the proposed preference relation is transitive. Figure 23.5 visualizes what part of the Pareto front remains preferred depending on whether the reference point is reachable (a) or not (b). If the goal has been set so ambitious that there is no solution which can reach the goal in even a single objective, the goal has no effect on search, and simply the whole Pareto front is approximated.

Deb [21] proposed a simpler variant that just ignores improvements over a goal value by replacing a solution's objective value $f_i(\mathbf{x})$ with $\max\{f_i(\mathbf{x}), r_i\}$. If the goal vector $\mathbf{r}$ is outside the feasible range, the method is almost identical to the definition in [38]. However, if the goal can be reached, the approach from [21] will lose its selection pressure and basically stop search as soon as the reference point has been

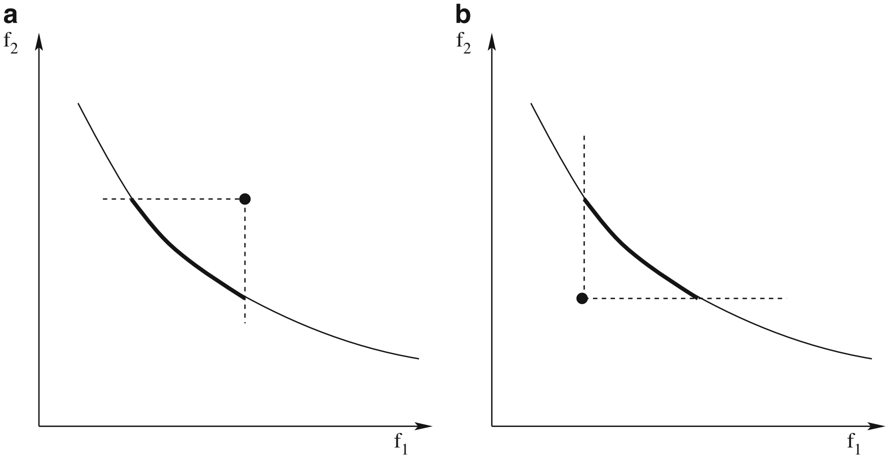


Fig. 23.5 Part of the Pareto optimal front that remains optimal with a given reference point **r** and the preference relation from [38]. The left panel (**a**) shows a reachable reference point, while the right panel (**b**) shows an unreachable one

found, i.e., return a solution which is not Pareto optimal. On the other hand, the approach by Fonseca and Fleming [38] keeps improving beyond the reference point. The goal-programming idea has been extended in [23] to allow for reference regions in addition to reference points.

Tan et al. [65] proposed another ranking scheme which in a first stage prefers individuals fulfilling all criteria, and ranks those individuals according to standard non-dominance sorting. Among the remaining solutions, solution $\mathbf{x}$ dominates solution $\mathbf{y}$ if and only if $\mathbf{x}$ dominates $\mathbf{y}$ with respect to the objectives in which $\mathbf{x}$ does not fulfill the goal (as in [38]), or if $|\mathbf{x}-\mathbf{r}| \succ |\mathbf{y}-\mathbf{r}|$. The latter corresponds to a "mirroring" of the objective vector along the axis of the fulfilled criteria. This may lead to some strange effects, such as non-transitivity of the preference relation (x is preferred to y, and y to z, but x and z are considered equal). Also, it seems odd to "penalize" solutions for largely exceeding a goal. What is more interesting in [65] is the suggestion on how to account for multiple reference points, connected with AND and OR operations. The idea here is to rank the solutions independently with respect to all reference points. Then, rankings are combined as follows. If two reference points are connected by an AND operator, the rank of the solution is the maximum of the ranks according to the individual reference points. If the operator is an OR, the rank of the solution is the minimum of the ranks according to the individual reference points. This idea of combining the information of several reference points can naturally be combined with other preference relations using a reference point. The paper also presents a way to prioritize objectives by introducing additional goals. In effect, however, the prioritization is equivalent to the one proposed in [39].

Deb and Sundar [28, 30] replace the crowding distance calculation in NSGA-II by the distance to the reference point, where solutions with a smaller distance are preferred. More specifically, solutions with the same non-dominated rank are sorted with respect to their distance to the reference point. To control the extent of obtained solutions, all solutions having a distance of ϵ or less between them are grouped. Only one randomly picked solution from each group is retained, while all other group members are assigned a large rank to discourage their use. As [39, 65], this approach is able to improve beyond a reference point within the feasible region, because the non-dominated sorting keeps driving the population to the Pareto optimal front. Also, as [65], it can handle multiple reference points simultaneously. With the parameter ϵ, it is possible to explicitly influence the diversity of solutions returned. Whether this extra parameter is an advantage or a burden may depend on the application.

A reference-point based modification of dominance where solutions fulfilling all goals and solutions fulfilling none of the goals are preferred over solutions fulfilling only some of the goals has been proposed in [58]. This, again, drives the search beyond the reference point if it is feasible, but it can obviously lead to situations where a solution which is dominated (fulfilling none of the goals) is actually preferred over the solution that dominates it (fulfilling some of the goals). More interesting in this paper is perhaps the way of interaction with the user. While the DM is not satisfied, in each round he/she can either set a completely new reference point, or select one of the evolved solutions and the new reference point is automatically determined as a linear combination of the old reference point and the selected solution. The representative set of non-dominated solutions shown to the user are the extreme solutions in each objective plus some representative solutions resulting from a clustering procedure.

Another modification of dominance based on a reference point is the *r-dominance* proposed by Said et al. [63]. Similar to [28, 30], the Euclidean distance to the reference point is taken into account as criterion in addition to the normal dominance. But rather than modifying the crowding distance calculation as in [28], the dominance relation is modified. Consider a pair of solutions $\mathbf{x}$ and $\mathbf{y}$ incomparable in the sense of Pareto dominance. Let $d(\mathbf{x}, \mathbf{y})$ define the Euclidean distance between two solutions $\mathbf{x}$ and $\mathbf{y}$, and d_{max} and d_{min} be the maximum and minimum distance of any solution in the population to the reference point $\mathbf{g}$. If the normalized difference between their distances to the reference point, $\frac{d(\mathbf{y},\mathbf{g})-d(\mathbf{x},\mathbf{g})}{d_{max}-d_{min}}$, is larger than some threshold δ, then the solution $\mathbf{x}$ closer to the reference point is said to r-dominate the solution $\mathbf{y}$ further away. The idea is integrated into NSGA-II simply by using r-dominance instead of the normal Pareto dominance in the non-dominated ranking procedure. The parameter δ allows the DM to influence the spread of the solutions along the Pareto frontier, with smaller δ generally leading to a smaller area covered. The authors note that focusing the search towards a reference point may lead to a loss of diversity and premature convergence. Said et al. [63] thus propose to linearly decrease δ over the run. An empirical comparison with g-dominance [58], R-NSGA-II [28] and PBEA [66] shows that r-dominance converges better and/or allows a better influence on the spread of solutions obtained. As many others, the approach allows the simultaneous consideration of multiple reference points.

Yet another modification of the dominance relation is proposed by Jaimes et al. [47], where the normal Pareto dominance is used if the two solutions to be compared are close to the reference point (in terms of the notation above, $d(\mathbf{x}, \mathbf{g}) < (d_{min} + \delta) \wedge d(\mathbf{y}, \mathbf{g}) < (d_{min} + \delta)$). If one of the two solutions has a larger distance to the reference point, they are simply compared based on their distance to the reference point, with shorter distance preferred. As [63], [47] makes the observation that a very strong focus on the preference information (small δ) may lead to premature convergence.

Thiele et al. [66] integrate reference point information into the Indicator-Based Evolutionary Algorithm. It relies on the ϵ-indicator that measures the minimal distance by which an individual needs to be improved in each objective to become non-dominated (or can be worsened before it becomes dominated). This indicator is weighted by means of an achievement scalarizing function based on a user specified reference point, giving solutions closer to the reference point a higher weight. This idea is somewhat related to the use of weight distributions in the objective space discussed in Sect. 23.4.5. The paper demonstrates that this allows to focus the search on the area around the specified reference point, and find interesting solutions faster.

The classical MCDM literature also includes some approaches where, in addition to a reference point, some further indicators are used to generate a set of alternative solutions. These include the reference direction method [56] and light beam search [48]. Recently, these methods have also been adopted into MOEAs.

In brief, the reference direction method allows the user to specify a starting point and a reference point, with the difference of the two defining the reference direction. Then, several points on this vector are used to define a set of achievement scalarizing functions, and each of these is used to search for a point on the Pareto optimal frontier. In [26], an MOEA is used to search for all these points simultaneously. For this purpose, the NSGA-II ranking mechanism has been modified to focus the search accordingly.

The light beam search also uses a reference direction, and additionally asks the user for some thresholds which are then used so find some possibly interesting neighboring solutions around the (according to the reference direction) most preferred solution. Deb and Kumar [27] use an MOEA to simultaneously search for a number of solutions in the neighborhood of the solution defined by the reference direction. This is achieved by first identifying the “most preferred” or “middle” solution using an achievement scalarizing function based on the reference point. Then, a modified crowding distance calculation is used to focus the search on those solutions which are not worse by more than the allowed threshold in all the objectives.

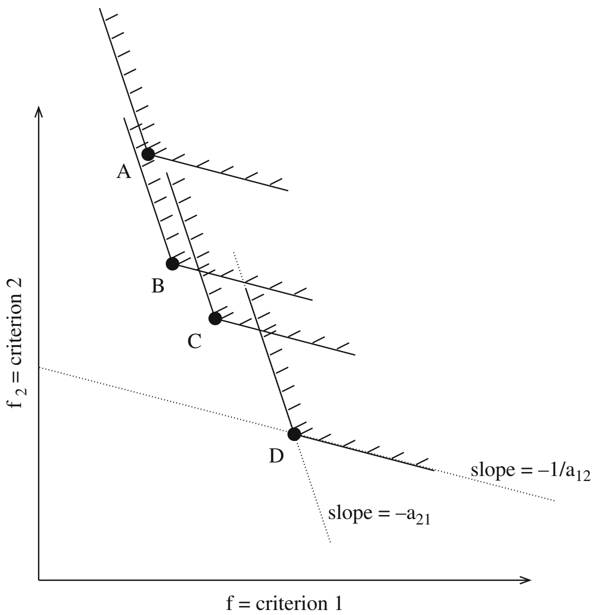
Summarizing, the first approach proposed in [38] still seems to be a good way to include reference point information. While in most approaches the part of the Pareto optimal front considered as relevant depends on the reference point and the shape and location of the Pareto optimal front, in [28] the desired spread of solutions in the vicinity of the Pareto optimal solution closest to the reference point is specified explicitly. A number of approaches such as [28, 63, 65] allow to consider several reference points simultaneously. In [63], the focus on the reference point is gradually increased over the run in order to maintain diversity and avoid getting stuck in local optima. The MOEAs based on the reference direction and light beam search [26, 27] allow the user to specify additional information that influences the focus of the search and the set of solutions returned.

23.4.4 Limiting Possible Trade-Offs

If the user has no idea about what kind of solutions may be reachable, it may be easier to specify suitable trade-offs, i.e., how much gain in one objective is necessary to balance the loss in the other.

In the guided MOEA proposed by Branke et al. [7], the user is allowed to specify preferences in the form of maximally acceptable trade-offs like "one unit improvement in objective *i* is worth at most a_{ji} units in objective *j*". The basic idea is to modify the dominance criterion accordingly, so that it reflects the specified maximally acceptable trade-offs. A solution **x** is now preferred to a non-dominated solution **y** if the gain in the objective where **y** is better does not outweigh the loss in the other objective, see Fig. 23.6 for an example. The region dominated by a solution is adjusted by changing the slope of the boundaries according to the specified maximal and minimal trade-offs. In this example, solution A is now dominated by solution B, because the loss in objective 2 is too big to justify the improvement in objective 1. On the other hand, solutions D and C are still mutually non-dominated.

Fig. 23.6 Effect of the modified dominance scheme used by G-MOEA



This idea can be implemented by a simple transformation of the objectives: It is sufficient to replace the original objectives with two auxiliary objectives Ω_1 and Ω_2 and use these together with the standard dominance principle, where

$$\Omega_1(x) = f_1(x) + a_{12}f_2(x)$$
$$\Omega_2(x) = a_{21}f_1(x) + f_2(x)$$

Because the transformation is so simple, the guided dominance scheme can be easily incorporated into standard MOEAs based on dominance, and it does not change the complexity nor the inner workings of the algorithm. However, an extension of this simple idea to more than two dimensions is not straightforward.

A very similar effect to the above guided MOEA is achieved in [44], where maximal and minimal trade-offs are not elicited explicitly, but derived from pairwise comparisons of solutions. This approach leads to identical dominance regions, but requires the solution of up to two linear programs whenever the dominance relation between two solutions is determined. On the other hand, it naturally extends to many dimensions. Because it uses a different way to elicit user preferences, it will be discussed in more detail in Sect. 23.4.8.

Another approach trying to restrict the possible trade-offs by using a different notion of Pareto dominance ("proper" Pareto dominance in this case), has been proposed in [12].

The idea proposed by Yin and Sendhoff [52] is to aggregate the different objectives into one objective via weighted summation, but to vary the weights gradually over time during the optimization. For two objectives, it is suggested to set $w_1(t) = |\sin(2\pi t/F)|$ and $w_2(t) = 1 - w_1(t)$, where t is the generation counter and F is a parameter to influence the oscillation period. The range of weights used in this process can be easily restricted to reflect the preferences of the DM by specifying a maximal and minimal weight $w_1^{\max}$ and $w_1^{\min}$, setting $w_1(t) = w_1^{\min} + (w_1^{\max} - w_1^{\min}) \cdot (sin(2\pi t/F) + 1)/2$ and adjusting w_2 accordingly. The effect is a population moving along the Pareto front, covering the part of the front which is optimal with respect to the range of possible weight values. Because the population will not converge but keep oscillating along the front, it is necessary to collect all non-dominated solutions found in an external archive. Note also the slight difference in effect to restricting the maximal and minimal trade-off as do the other approaches in this section. While the other approaches enforce these trade-offs locally, on a one-to-one comparison, the dynamic weighting modifies the global fitness function. Therefore, the approach runs into problems if the Pareto front is concave, because a small weight change would require the population to make a big "jump".

23.4.5 Weighting the Objective Space

Zitzler et al. [76] allow the DM to define a weight distribution over the objective space, giving a higher weight to more preferred regions. This information can then be incorporated in the MOEA by biasing selection towards individuals in areas of the objective space which have been assigned a higher weight. In [76], Zitzler et al. integrate this sort of information into the indicator-based EAs (cf. Sect. 23.2.2), with the aim to find a set of solutions that optimize the weighted hypervolume. Friedrich et al. [41] integrates the weight distribution into other MOEAs such as NSGA-II by modifying the crowding distance and SPEA-2.

The challenge is probably to facilitate the specification of a weighting of the entire objective space. While this may be relatively simple in 2D, it seems not clear whether a DM can specify such a weighting easily in more than two dimensions. In [76], three different weighting schemes are proposed that could serve as prototypes: a weight distribution which favors extremal solutions, a weight distribution which favors one objective over the other (but still keeping the best solution with respect to the less important objective), and a weight distribution based on a reference point, which generates a ridge-like function through the reference point parallel to the diagonal. A way to derive a weighting function based on a reference front and a desired density of solutions on that front has been proposed in [2]. Auger et al. [1] look into efficient sampling techniques to estimate the weighted hypervolume.

23.4.6 Specifying a Distribution over Utility Functions

Branke et al. [8] use the "expected utility" as indicator in an indicator-based EA (cf. Sect. 23.2.2), i.e., a solution is evaluated by the expected loss in utility if this solution would be absent from the population. To calculate the expected utility, it is assumed that the DM has a linear utility function of the form $u(\mathbf{x}) = \lambda f_1(\mathbf{x}) + (1 - \lambda)f_2(\mathbf{x})$, and λ is unknown but follows a uniform distribution over $[0, 1]$. The expected marginal utility (EMU) of a solution $\mathbf{x}$ is then the utility difference between the best and second best solution, integrated over all utility functions where solution $\mathbf{x}$ is best:

$$\text{EMU}(\mathbf{x}) = \int_{\lambda=0}^{1} \max\{0, \min_{\mathbf{y}}\{u(\mathbf{y}) - u(\mathbf{x})\}\} d\lambda \tag{23.4}$$

While the expected marginal utility can be calculated exactly in the case of two objectives, numerical integration is required for more objectives. Without preference information, the result of using this indicator is a natural focus of the search on so-called "knees", i.e., convex regions with strong curvature. In these regions,

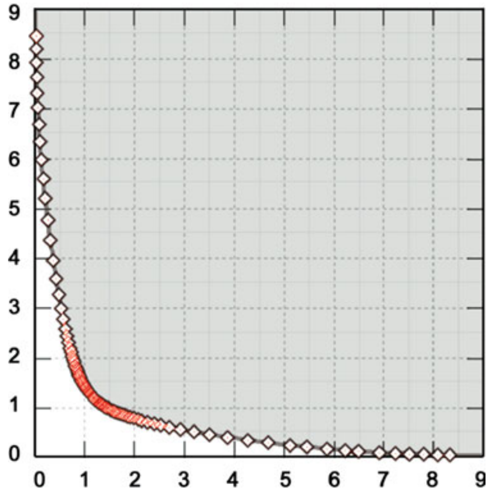


Fig. 23.7 Marginal contribution calculated according to expected utility result in a concentration of the individuals in knee areas

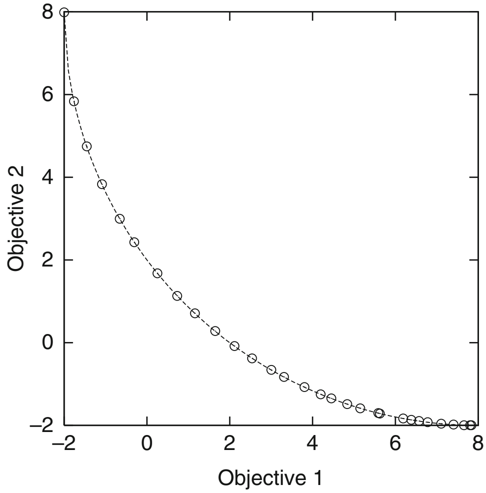


Fig. 23.8 Resulting distribution of individuals with the marginal expected utility approach and a linearly decreasing probability distribution for λ

an improvement in either objectives requires a significant worsening of the other objective, and such solutions are often preferred by DMs [20]. An example of the resulting distribution of individuals along a Pareto front with a single knee is shown in Fig. 23.7. Additional explicit user preferences can be taken into account by allowing the user to specify the probability distribution for λ [5], an example for the resulting biased front is provided in Fig. 23.8. Obviously, any distribution over any space of utility functions could be considered with this approach.

23.4.7 Approaches Based on Outranking Relations

The method by Cvetkovic and Parmee [19] assigns each criterion a weight w_i, and additionally requires a minimum level for dominance τ, which corresponds to the concordance criterion of the ELECTRE method [36]. Accordingly, the following weighted dominance criterion is used as dominance relation in the MOEA.

$$\mathbf{x} \succ_w \mathbf{y} \Leftrightarrow \sum_{i:f_i(\mathbf{x}) \leq f_i(\mathbf{y})} w_i \geq \tau.$$

To facilitate specification of the required weights, they suggest a method to turn fuzzy preferences into specific quantitative weights. However, since for every criterion the dominance scheme only considers whether one solution is better than another solution, and not by how much it is better, this approach allows only a very coarse guidance and is difficult to control.

Rekiek et al. [62], Coelho et al. [18], and Parreiras et al. [59] use preference flow according to PROMETHEE II [11] to rank solutions. NOSGA, proposed in [34] and further developed as NOSGA-II in [35], uses a similar mechanism to the non-dominance sorting in NSGA-II, but rather than identifying the non-dominated solutions in each step, it places all the solutions that are not strictly outranked into the same rank. Among the solutions with the same rank, for each solution i, the number W_i of solutions weakly outranking i and the number F_i of solutions with a better preference flow than i are determined, and solutions with a low sum $W_i + F_i$ are preferred.

The challenge with all methods based on outranking is to set appropriate parameters. MCDA techniques such as ELECTRE [36] and PROMETHEE [11] may help set those values.

23.4.8 Approaches Based on Solution Comparison

Perhaps the easiest form of providing preference information is to specify which of two solutions would be preferable. In this case, the DM can compare all aspects of the two solutions and make a holistic judgement. Of course, the number of such comparisons a DM can make is limited, while optimization usually needs to rank very many solutions. Thus, such information is only useful if it can somehow be generalized to compare solutions other than the ones for which the DM provided explicit preference information. The approaches in this subsection all ask the DM to rank solutions. Sometimes this means comparing just two solutions, sometimes more. Sometimes a full ranking is required, sometimes only the best or worst solution have to be identified. The way this information is used differs quite substantially. This section is structured according to the underlying algorithmic principle, whether it calculates a most representative value function, works with a set of compatible value functions, or is based on some other principle.

23.4.8.1 Determining a Representative Value Function

The approaches in this subsection use the elicited preference information to derive a single value function to approximate user preferences. Value functions can have different complexity, ranging from simple linear functions to non-parametric approaches such as artificial neural networks or support vector machines. Most approaches simply use the derived value function for ranking individuals, sometimes as secondary criterion after non-dominance, but other uses can also be found.

Phelps and Köksalan [60] proposed an interactive evolutionary algorithm that periodically asks the DM to rank pairs of solutions. Assuming **linear** value functions (actually, the objectives are modified before the optimization to the squared distance to a reference value, which effectively results in ellipsoidal iso-utility curves), these preferences are turned into constraints for possible weights. For example, if solution **x** is preferred over **y**, it is clear that

$$\sum_{k=1}^{n} w_k(f_k(\mathbf{x}) - f_k(\mathbf{y})) < 0. \tag{23.5}$$

The method determines the most discriminative weight vector compatible with the preference information. Most discriminative here means the weight vector that maximizes the minimum value difference over all pairs of solutions ranked by the DM.

Denote with A the set that contains all the pairs of solutions $(\mathbf{a}, \mathbf{b})$ which have been ranked by the DM as **a** is preferred over **b**. Then, the following linear program (LP) identifies the most discriminative value function.

$$\begin{aligned} &\max \epsilon \\ &\text{s.t.} \\ &\sum_{k=1}^{n} w_k(f_k(\mathbf{y}) - f_k(\mathbf{x})) > \epsilon \qquad \forall(\mathbf{x}, \mathbf{y}) \in A \\ &\sum_{k=1}^{n} w_k = 1, \quad w_k \geq 0. \end{aligned} \tag{23.6}$$

The resulting weight vector is then used for ranking individuals in the evolutionary algorithm that works as a single objective evolutionary algorithm between user interactions. If the LP is overconstrained and no feasible solution is found, the oldest preference information is discarded.

A very similar idea is used in [3], but instead of an LP, a second evolutionary algorithm is used to determine a compatible linear value function. The two EAs are run in alternating fashion: first, both populations (solutions and weights) are initialized, then the DM is asked to rank the solutions. After that, the population of weights is evolved for some generations to produce a weighting which is most compatible with the user ranking. Then, this weighting is used to evolve the solutions for some generations, and the process repeats.

Deb et al. [31] derive a **polynomial** value function model. The user is shown a set of (five in the paper) solutions and asked to (at least partially) rank them. Then, similar to the approach by Phelps and Köksalan [60], the most discriminative value function is determined. However, where [60] uses a linear value function model [31] uses a polynomial value function model of the form

$$V(\mathbf{x}) = \prod_{i=1}^{n} S_i(\mathbf{x}) = \prod_{i=1}^{n} \left(\sum_{j=1}^{n} [k_{ij} f_j(\mathbf{x}) + k_{i(n+1)}] \right), \tag{23.7}$$

with n being the number of objectives and k_{ij} the parameters that need to be chosen appropriately. Fitting the value function model to the specified preferences involves solving the following optimization problem (assuming maximization of objectives):

$$\max \epsilon$$

$$s.t. \quad S(\mathbf{x}) \geq 0 \quad \forall \mathbf{x} \tag{23.8}$$

$$V(\mathbf{x}) - V(\mathbf{y}) \geq \epsilon \quad \forall (\mathbf{x}, \mathbf{y}) \in A \tag{23.9}$$

$$k_{ij} \geq 0 \tag{23.10}$$

where A is again the set that contains all the pairs of solutions $(\mathbf{a}, \mathbf{b})$ which have been ranked by the DM as $\mathbf{a}$ is preferred over $\mathbf{b}$. Inequalities (23.8) and (23.10) aim to ensure the value function is monotonically increasing, Inequality (23.9) ensures compatibility with preference information. With a slight modification, also equivalence relationships could be modeled. Note that the above is a non-linear optimization problem, and the authors propose to use sequential quadratic programming to solve it.

Once a most discriminative value function has been identified, this information is used in the MOEA's ranking of individuals. Basically, the objective space is separated into two areas: All individuals with an estimated value (according to the approximated value function) better than the solution ranked *second* by the DM are assumed to dominate all the solutions with an estimated value worse than the solution ranked second. This is visualized in Fig. 23.9. If B was the solution ranked second in the last interaction with the DM and the curve represents the iso-utility line of all solutions with equivalent value according to the most discriminative value function, then all solutions above this curve are assumed to dominate all solutions below this curve. If both solutions lie either above or below the curve, they are compared based on the usual Pareto dominance.

The authors additionally use the approximated value function to perform a local single-objective optimization starting with the solution ranked best by the DM. If this local improvement step is not able to improve the solution's value by at least a certain margin, it is concluded that the algorithm has found the most preferred solution and the optimization is stopped.

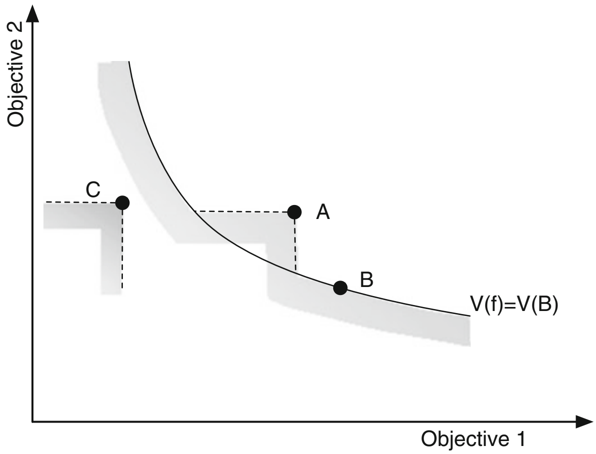


Fig. 23.9 Example for dominated region in the approach from [31]. Maximization of objectives is assumed. The *curve* represents all solutions equivalent to *B* according to the approximated value function. All solutions with an estimated value better than *B* (above the *curve*) dominate all solutions with an estimated value worse than *B* (below the *curve*). The *grey areas* indicate the areas dominated by solutions *A* and *C*, respectively

Todd and Sen [67] use **artificial neural networks** to represent the DM's value function. Periodically, they present the DM with a set of solutions and ask for a score between 0 and 1. The set of solutions is chosen such that they represent a broad variety regarding the approximated value function, in particular, the estimated best and worst individual of the population are always included. Information from several user interactions is accumulated after normalizing preference scores.

Another model that allows to represent complex value functions are **support vector machines** (SVM). SVMs have the additional advantage of being able to trade-off model accuracy and model complexity. Battiti and Passerini [4] use SVMs in the setting of an interactive MOEA, more specifically NSGA-II. Periodically, the DM is presented with a set of solutions and asked to (at least partially) rank them. This information is then used to train the SVM, with cross-validation employed to select an appropriate kernel. The derived approximate value function is then used to replace the crowding distance in NSGA-II by sorting individuals in the same non-dominance rank based on their value according to the learned value function.

The solutions shown to the DM during interaction are the best according to the approximated value function, or randomly selected non-dominated solutions in the first step. The paper examines the influence of the number of solutions shown to the DM (assuming full ranking) and the number of interactions with the DM. The results suggest that a relatively large number of solutions need to be ranked for the SVM to learn a useful value function (around 10–20), but only two interactions with the DM seem sufficient to come very close to results that would have been obtained had the DM's true value function been known from the beginning. The authors recommend to not start interaction until the MOEA has found a reasonable coverage of the entire Pareto frontier, which somewhat defeats the purpose of narrowing down the search early on. In [14], the approach's robustness to incorrect (noisy) DM preferences is examined and it is shown that the algorithm can cope well with noise, in particular if the number of solutions ranked by the DM is large.

23.4.8.2 Determining a Set of Compatible Value Functions

Rather than deriving a single value function, Jaszkiewicz [50] notes that there may be several value function compatible with the specified user preferences and samples the preference function used in each generation from the set of preference functions (in this case **linear** weightings are assumed). The proposed approach is based on the Pareto memetic algorithm (PMA) [49] and uses the value function also for local search. In the interactive version, preference information from pairwise comparisons of solutions is used to reduce the set of possible weight vectors.

Greenwood et al. [44] suggested an imprecise value function approach which considers all compatible **linear** value functions *simultaneously*. The procedure asks the user to rank a few alternatives, and from this derives constraints for the weightings of the objectives consistent with the given ordering. Then, these are used to check whether there is a feasible linear weighting such that solution **x** would be preferred to solution **y**.

Let A denote the set of all pairs of solutions $(\mathbf{x}, \mathbf{y})$ ranked by the DM, and **x** preferred to **y**. Then, to compare any two solutions **u** and **v**, simultaneously all linearly weighted additive utility functions are considered which are consistent with the ordering on the initially ranked solutions. A preference of **u** over **v** is inferred if **u** is preferred to **v** for all such utility functions. A linear program (LP) is used to search for a utility function where **u** is not preferred to **v**.

$$\min Z = \sum_{k=1}^{n} w_k(f_k(\mathbf{u}) - f_k(\mathbf{v})) \tag{23.11}$$

$$\sum_{k=1}^{n} w_k(f_k(\mathbf{x}) - f_k(\mathbf{y})) < 0 \qquad \forall (\mathbf{x}, \mathbf{y}) \in A \tag{23.12}$$

$$\sum_{k=1}^{n} w_k = 1, \quad w_k \geq 0.$$

If the LP returns a solution value $Z > 0$, it can be concluded that there is no linear combination of objectives consistent with inequality (23.12) such that **u** would be preferable, and therefore **v** is preferred over **u** independent of the weight vector. If the LP can find a linear combination with $Z < 0$, it only means that **v** is not necessarily preferred to **u**. To test whether **u** is preferred to **v**, one has to solve another LP and fail to find a linear combination of objectives such that **v** would be preferable. Overall, the method requires to solve 1 or 2 LPs for each pair of solutions in the population. Also, it needs special mechanisms to make sure that the allowed weight space does not become empty, i.e., that the user ranking is consistent with at least one possible linear weight assignment. The authors suggest to use a mechanism from [70] which removes a minimal set of the DM's preference statements to make the weight space non-empty.

Although developed independently and with a different motivation, the guided MOEA discussed in Sect. 23.4.4 leads to the same preference relation as the imprecise value function approach above. The differences are in the way the maximally acceptable trade-offs are derived (specified directly by the DM in the guided MOEA, and inferred from a ranking of solutions in [44]), and in the different implementation (a simple transformation of objectives in guided MOEA, and the solving of many LPs in the imprecise value function approach). While the guided MOEA is more elegant and computationally efficient for two objectives, the imprecise value function approach works independent of the number of objectives.

In [57], the value function model is only implicit. Under the assumption of **quasi-concave** value functions, specified preferences between solutions can be generalized to preference cones [57]. For a simple 2D example, see Fig. 23.10. If the DM specified that solutions B and C are both preferable over A, it can be concluded that all solutions in the cone's polyhedron (light grey area, i.e., solutions B, C, F) would be preferred over the vertex of the cone (solution A) which in turn would be preferred over all solutions under the cone (dark grey area, solution D). Thus, of the 10 possible pairwise relationships between the 5 solutions, the information about 2 pairwise relationships in this example allowed to derive another five ($A \succ D$, $B \succ D, C \succ D, F \succ D, F \succ A$). Identifying a solution's location relative to the cone requires solving two linear programming problems.

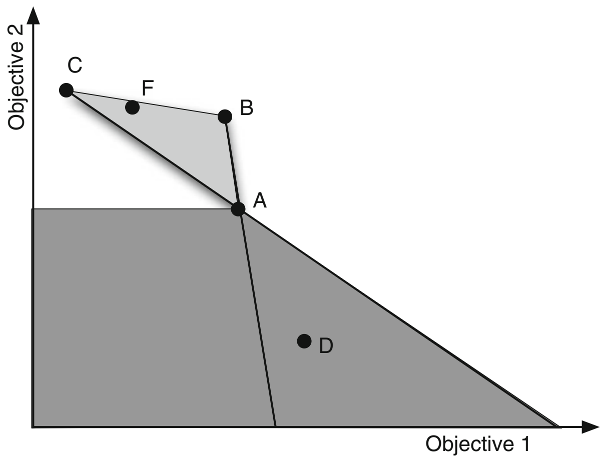


Fig. 23.10 Visualization of the preference cone in 2D, assuming quasi concave utility function and maximization of objectives

This idea is used by Fowler et al. [40] to partially rank the non-dominated solutions in an MOEA. The DM is asked to consider a set of six solutions and specify the best and worst. From this information, six preference cones are derived (five 2-point cones involving the best and any of the other solutions, and one 6-point preference cone specifying that five solutions are better than the worst). All generated cones are kept throughout the optimization run, even if the solutions defining the cone are deleted from the population. The solutions shown to the DM are selected from the set of non-dominated solutions that can not already be ranked with the existing cones.

Another approach that uses the entire set of compatible linear utility functions is the one based on expected marginal utility [8]. But since it asks the user for a probability distribution over weights rather than deriving the compatible weights from pairwise comparisons, it is described in Sect. 23.4.6.

Branke et al. [9, 10] proposed a framework called NEMO (Necessary preference enhanced Evolutionary Multiobjective Optimizer). It also uses pairwise comparisons of solutions to learn user preferences. Similar to the imprecise value function approach by Greenwood et al. [44], it simultaneously considers the set of *all* value functions compatible with the elicited preference information. But rather than being restricted to linear value functions, it allows for **piecewise-linear** [10] or **general monotonic additive** [9, 10] value functions. This is possible because it is based on robust ordinal regression [43], a method that has recently been introduced into multi-criteria ranking [37]. It can take into account a preference ranking of solutions, such as "x is preferred over y", but also intensities of preferences, such as "x is preferred over y more than w over z".

A solution x is necessarily preferred over solution y, if it is preferred according to *all* value functions compatible with the elicited preference information. As in [44], whether one solution is necessarily preferred over another can be detected by solving two linear programs. The main difference is that the variables are not weights, but values of characteristic points of marginal value functions. NEMO replaces the use of the dominance relation in the non-dominance sorting step of NSGA-II by the necessary preference relation. Additionally, it computes a most representative utility function that it uses for scaling in the crowding distance calculation. Note that when no preference information is provided, NEMO reduces to the standard NSGA-II.

23.4.8.3 Other Algorithmic Principles

Karahan and Köksalan [54] propose the Territory Defining Evolutionary Algorithm (TDEA), where new individuals are only allowed to enter the archive or survive, if they do not fall into the "territory" of already existing solutions. The territory is an incomparable region around an individual defined by the maximum absolute distance over all objectives, see Fig. 23.11. The territory mechanism replaces other crowding or diversity preservation mechanisms in MOEA. By choosing τ small in regions of high relevance for the DM, the MOEA will keep and subsequently

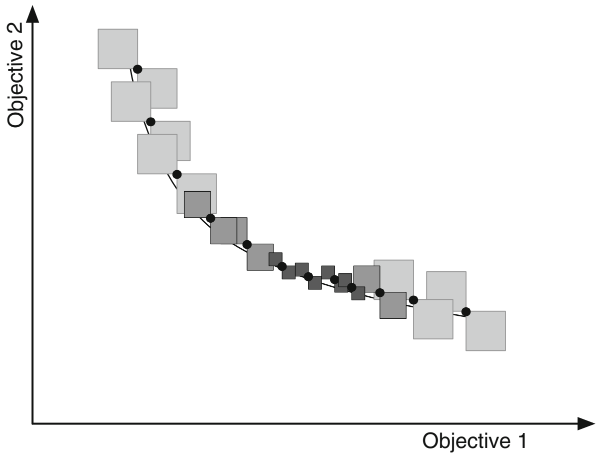


Fig. 23.11 Solutions (*black points*) and territories (*squares*) with different sizes as used in [54]. Regions with smaller territories will maintain a higher density of solutions

also generate more Pareto-optimal solutions in those regions. The mechanism thus effectively biases the distribution of solutions along the Pareto front. In [55], an interactive preference elicitation based on TDEA is proposed. The approach iteratively narrows down a "preferred region", defined as a range of weight vectors for calculating the Tchebycheff distance to an ideal point. It starts with a pre-specified territory size τ_0 and considers the entire weight range as equally preferred. Then, in each iteration, the user is asked to identify the most preferred solution from a representative sample of solutions found so far. The weight vector minimizing the Tchebycheff distance of this solution to the ideal point is then taken as the center of the new preferred weight range, the interval width and τ used in this interval are decreased exponentially over time. As a result, the resolution becomes increasingly fine-grained in the areas around solutions identified as most interesting by the DM. The method assumes knowledge of the ideal point, it is not clear how sensitive the method is to imprecise knowledge of the ideal point.

Gong et al. [42] also ask the DM to identify the best out of a representative set of non-dominated solutions (they use 10 candidate solutions in their paper). This information is then integrated into MOEA/D. Recall that MOEA/D works by simultaneously solving a diverse set of single-objective problems with different weight vectors (cf. Sect. 23.2.3). Once the user has specified their most preferred solution, a region of interest is defined around this solution, and some weight vectors outside that region are re-positioned inside the region of interest. That way, the search increasingly focuses around the solutions preferred by the DM.

23.5 Summary and Open Research Questions

If a single solution is to be selected in a multi-objective optimization problem, at some point during the process, the DM has to reveal his/her preferences. Specifying these preferences a priori, i.e., before alternatives are known, often means to ask too much of the DM. On the other hand, searching for all non-dominated solutions, as most MOEA do, may result in a waste of optimization efforts to find solutions that are clearly unacceptable to the DM.

This chapter overviewed intermediate approaches, that ask for partial preference information from the DM a priori or interactively, and then focus the search to those regions of the Pareto optimal front that seem most interesting to the DM. That way, it is possible to provide a larger number of relevant solutions more quickly.

Table 23.1 summarizes some aspects of some of the most prominent approaches. It lists the information required from the DM (Information), the part of the MOEA modified (Modification), and whether the result is a bounded region of the Pareto

Table 23.1 Comparison of some selected approaches to incorporate partial user preferences

Name	Information	Modification	Influence
Constraints [16]	Constraint	Miscellaneous	Region
Preference-based EA [66]	Reference point	Quality indicator	Distribution
Preference relation [38]	Reference point	Dominance	Region
Reference point based EMO [30]	Reference point	Crowding dist.	Region
r-Dominance [63]	Reference point, threshold	Dominance	Region
Light beam search based EMO, [27]	Reference direction thresholds	Crowding dist.	Region
Guided MOEA [7]	Maximal/minimal trade-off	Objectives	Region
Weighted integration [76]	Weighting of objective space	Quality indicator	Distribution
Marginal expected utility [8]	Value fct. probability distribution	Crowding dist.	Distribution
Desirability Functions [69]	Scaling function	Objectives	Distribution
Biased crowding [6]	Desired trade-off	Crowding dist.	Distribution
Territory defining EA [54]	Convergence Schedule	Replacement	Distribution
Interactive MOEA/D [42]	Best of set	Weight vectors	Distribution
Cone dominance [40]	Best and worst from set	Dominance	Region
Imprecise value function [44]	Pairwise comparisons	Dominance	Region
NEMO [10]	Pairwise comparisons	Dominance and crowding distance	Region and distribution
Progressively interactive EMO [31]	Ranking of set	Dominance	Region
Brain-computer EMO [4]	Ranking of set	Crowding distance	Distribution

optimal front or a biased distribution (Influence). What method is most appropriate certainly depends on the application (e.g., whether the Pareto front is convex or concave, or whether the DM has a good conception of what is reachable) and on the kind of information the DM feels comfortable to provide.

Integrating preference information with MOEAs is still a relatively young research area with ample opportunities for future work. Many of the ideas can be combined, allowing the DM to provide preference information in different ways. For example, it would be straightforward to combine a reference point based approach which leads to sharp boundaries of the area in objective space considered as interesting with a marginal contribution approach which alters the distribution within this area. As a first step in this direction, Deb and Chaudhuri [25] proposed an interactive decision support system called I-MODE that implements an interactive procedure built over a number of existing EMO and classical decision making methods. The main idea of the procedure is to allow the DM to interactively focus on interesting region(s) of the Pareto front. The DM has options to use several tools for generation of potentially Pareto optimal solutions concentrated in the desired regions. For example, he/she may use weighted sum approach, utility function based approach, Tchebycheff function approach or trade-off information. Note that the preference information may be used to define a number of interesting regions. For example, the DM may define a number of reference (aspiration) points defining different regions. The preference information is then used by an EMO to generate new solutions in (hopefully) interesting regions.

A big obstacle for progress is the current lack of comparative studies. The problem of course is that when DMs are involved, a comparison is much more difficult than if it were a fully automated system. What is needed is an automated DM that behaves similar to a human DM, which would allow reproducible and very extensive empirical comparisons.

The handling of noise, be it noisy objective evaluations or noisy preference information, is another topic that requires more attention. Hughes [46] is an early paper specifically addressing noisy objective function values. The main idea to cope with the noise is to rank individuals by the sum of probabilities of being dominated by any other individual. To take preferences into account, the paper proposes a kind of weighting of the domination probabilities.

Finally, while almost all approaches surveyed in this chapter assume a single DM, the interaction with groups of DMs would be another worthwhile research direction.

References

1. Auger, A., Bader, J., Brockhoff, D., Zitzler, E.: Articulating user preference in many-objective problems by sampling the weighted hypervolume. In: Raidl, G., et al. (eds.) Genetic and Evolutionary Computation Conference, pp. 555–562. ACM, New York (2009)
2. Auger, A., Bader, J., Brockhoff, D., Zitzler, E.: Investigating and exploiting the bias of weighter hypervolume to articulate user preferences. In: Raidl, G., et al. (eds.) Genetic and Evolutionary Computation Conference, pp. 563–570. ACM, New York (2009)

3. Barbosa, H.J.C., Barreto, A.M.S.: An interactive genetic algorithm with co-evolution of weights for multiobjective problems. In: Spector, L., et al. (eds.) Genetic and Evolutionary Computation Conference, pp. 203–210. Morgan Kaufmann, Los Altos, CA (2001)
4. Battiti, R., Passerini, A.: Brain-computer evolutionary multiobjective optimization: a genetic algorithm adapting to the decision maker. IEEE Trans. Evol. Comput. **14**(5), 671–687 (2010)
5. Branke, J.: Consideration of user preferences in evolutionary multi-objective optimization. In: Branke, J., Deb, K., Miettinen, K., Slowinski, R. (eds.) Multiobjective Optimization—Interactive and Evolutionary Approaches. Lecture Notes in Computer Science, vol. 5252, pp. 157–178. Springer, Berlin (2008)
6. Branke, J., Deb, K.: Integrating user preference into evolutionary multi-objective optimization. In: Jin, Y. (ed.) Knowledge Incorporation in Evolutionary Computation, pp. 461–478. Springer, Berlin (2005)
7. Branke, J., Kaußler, T., Schmeck, H.: Guidance in evolutionary multi-objective optimization. Adv. Eng. Softw. **32**, 499–507 (2001)
8. Branke, J., Deb, K., Dierolf, H., Osswald, M.: Finding knees in multi-objective optimization. In: Parallel Problem Solving from Nature. Lecture Notes in Computer Science, vol. 3242, pp. 722–731. Springer, Berlin (2004)
9. Branke, J., Greco, S., Słowiński, R., Zielniewicz, P.: Interactive evolutionary multiobjective optimization using robust ordinal regression. In: Ehrgott, M., et al. (eds.) International Conference on Evolutionary Multi-Criterion Optimization. Lecture Notes in Computer Science, vol. 5467, pp. 554–568. Springer, Berlin (2009)
10. Branke, J., Greco, S., Słowiński, R., Zielniewicz, P.: Interactive evolutionary multiobjective optimization driven by robust ordinal regression. Bull. Pol. Acad. Sci. Tech. Sci. **58**(3), 347–358 (2010)
11. Brans, J.P., Mareschal, B.: PROMETHEE methods. In: Figueira, J., et al. (eds.) Multiple Criteria Decision Analysis, pp. 163–196. Springer, Berlin (2005)
12. Braun, M.A., Shukla, P.K., Schmeck, H.: Preference ranking schemes in multi-objective evolutionary algorithms. In: Takahashi, R.H.C., et al. (eds.) Evolutionary Multi-Criterion Optimization. Lecture Notes in Computer Science, vol. 6576, pp. 226–240. Springer, Berlin (2011)
13. Bringmann, K., Friedrich, T.: An efficient algorithm for computing hypervolume contributions. Evol. Comput. J. **18**(3), 383–402 (2010)
14. Campigotto, P., Passerini, A.: Adapting to a realistic decision maker: experiments towards a reactive multi-objective optimizer. In: International Conference on Learning and Intelligent Optimization. Lecture Notes in Computer Science, vol. 6073, pp. 338–341. Springer, Berlin (2010)
15. Coello, C.A.C.: Handling preferences in evolutionary multiobjective optimization: a survey. In: Congress on Evolutionary Computation, vol. 1, pp. 30–37. IEEE (2000)
16. Coello, C.A.C.: Theoretical and numerical constraint-handling techniques used with evolutionary algorithms: a survey of the state of the art. Comput. Methods Appl. Mech. Eng. **191**(11–12), 1245–1287 (2002)
17. Coello, C.A.C., Veldhuizen, D.A.V., Lamont, G.B.: Evolutionary Algorithms for Solving Multi-Objective Problems. Kluwer, Dordrecht (2002)
18. Coelho, R.F., Bersini, H., Bouillard, P.: Parametrical mechanical design with constraints and preferences: application to a purge valve. Comput. Methods Appl. Mech. Eng. **192**, 4355–4378 (2003)
19. Cvetkovic, D., Parmee, I.C.: Preferences and their application in evolutionary multiobjective optimisation. IEEE Trans. Evol. Comput. **6**(1), 42–57 (2002)
20. Das, I.: On characterizing the 'knee' of the pareto curve based on normal-boundary intersection. Struct. Optim. **18**(2/3), 107–115 (1999)
21. Deb, K.: Solving goal programming problems using multi-objective genetic algorithms. In: Proceedings of Congress on Evolutionary Computation, pp. 77–84 (1999)
22. Deb, K.: An efficient constraint handling method for genetic algorithms. Comput. Methods Appl. Mech. Eng. **186**(2–4), 311–338 (2000)

23. Deb, K.: Multi-Objective Optimization using Evolutionary Algorithms. Wiley, New York (2001)
24. Deb, K.: Multi-objective evolutionary algorithms: introducing bias among Pareto-optimal solutions. In: Ghosh, A., Tsutsui, S. (eds.) Advances in Evolutionary Computing: Theory and Applications, pp. 263–292. Springer, London (2003)
25. Deb, K., Chaudhuri, S.: I-MODE: an interactive multi-objective optimization and decision-making using evolutionary methods. Technical Report KanGAL Report No. 2007003, Indian Institute of Technology Kanpur (2007)
26. Deb, K., Kumar, A.: Interactive evolutionary multi-objective optimization and decision-making using reference direction method. In: Genetic and Evolutionary Computation Conference, pp. 781–788. ACM, New York (2007)
27. Deb, K., Kumar, A.: Light beam search based multi-objective optimization using evolutionary algorithms. In: Congress on Evolutionary Computation, pp. 2125–2132. IEEE, Los Alamitos (2007)
28. Deb, K., Sundar, J.: Reference point based multi-objective optimization using evolutionary algorithms. In: Genetic and Evolutionary Computation Conference, pp. 635–642. ACM, New York (2006)
29. Deb, K., Agrawal, S., Pratap, A., Meyarivan, T.: A fast and elitist multi-objective genetic algorithm: NSGA-II. IEEE Trans. Evol. Comput. **6**(2), 182–197 (2002)
30. Deb, K., Sundar, J., Udaya Bhaskara Rao, N., Chaudhuri, S.: Reference point based multi-objective optimization using evolutionary algorithms. Int. J. Comput. Intell. Res. **2**(3), 273–286 (2006)
31. Deb, K., Sinha, A., Korhonen, P., Wallenius, J.: An interactive evolutionary multiobjective optimization method based on progressively approximated value functions. IEEE Trans. Evol. Comput. **14**(5), 723–739 (2010)
32. Eiben, A.E., Smith, J.E.: Introduction to Evolutionary Computing. Springer, Berlin (2003)
33. Emmerich, M., Beume, N., Naujoks, B.: An EMO algorithm using the hypervolume measure as selection criterion. In: Evolutionary Multi-Criterion Optimization. Lecture Notes in Computer Science, vol. 3410. Springer, Berlin (2005)
34. Fernandez, E., Lopez, E., Bernal, S., Coello, C.A.C., Navarro, J.: Evolutionary multiobjective optimization using an outranking-based dominance generalization. Comput. Oper. Res. **37**(2), 390–395 (2010)
35. Fernandez, E., Lopez, E., Lopez, F., Coello, C.A.C.: Increasing selective pressure towards the best compromise in evolutionary multiobjective optimization: the extended NOSGA method. Inf. Sci. **181**, 44–56 (2011)
36. Figueira, J., Mousseau, V., Roy, B.: ELECTRE methods. In: Figueia, J., Greco, S., Ehrgott, M. (eds.) Multiple Criteria Decision Analysis: State of the Art Surveys, pp. 134–162. Springer, Berlin (2005)
37. Figueira, J., Greco, S., Słowiński, R.: Building a set of additive value functions representing a reference preorder and intensities of preference: Grip method. Eur. J. Oper. Res. **195**(2), 460–486 (2009)
38. Fonseca, C.M., Fleming, P.J.: Genetic algorithms for multiobjective optimization: formulation, discussion, and generalization. In: Proceedings of the Fifth International Conference on Genetic Algorithms, pp. 416–423 (1993)
39. Fonseca, C.M., Fleming, P.J.: Multiobjective optimization and multiple constraint handling with evolutionary algorithms—part I: a unified formulation. IEEE Trans. Syst. Man Cybern. Part A **28**(1), 26–37 (1998)
40. Fowler, J.W., Gel, E.S., Köksalan, M.M., Korhonen, P., Marquis, J.L., Wallenius, J.: Interactive evolutionary multi-objective optimization for quasi-concave preference functions. Eur. J. Oper. Res. **206**, 417–425 (2010)
41. Friedrich, T., Kroeger, T., Neumann, F.: Weighted preferences in evolutionary multi-objective optimization. Int. J. Mach. Learn. Cybern. (to appear) **4**(2), 139–148, Springer (2013)

42. Gong, M., Liu, F., Zhang, W., Jiao, L., Zhang, Q.: Interactive MOEA/D for multi-objective decision making. In: Genetic and Evolutionary Computation Conference, pp. 721–728. ACM, New York (2011)
43. Greco, S., Słowiński, R., Figueira, J., Mousseau, V.: Robust ordinal regression. In: Ehrgott, M., et al. (eds.) New Advances in Multiple Criteria Decision Analysis, pp. 273–320. Springer, Berlin (2010)
44. Greenwood, G.W., Hu, X.S., D'Ambrosio, J.G.: Fitness functions for multiple objective optimization problems: combining preferences with Pareto rankings. In: Belew, R.K., Vose, M.D. (eds.) Foundations of Genetic Algorithms, pp. 437–455. Morgan Kaufmann, Los Altos, CA (1997)
45. Hapke, M., Jaszkiewicz, A., Słowiński, R.: Interactive analysis of multiple-criteria project scheduling problems. Eur. J. Oper. Res. **107**, 315–324 (1998)
46. Hughes, E.J.: Constraint handling with uncertain and noisy multi-objective evolution. In: Congress on Evolutionary Computation, pp. 963–970. IEEE, Los Alamitos (2001)
47. Jaimes, A.L., Montano, A.A., Coello, C.A.C.: Preference incorporation to solve many-ojective airfoil design problems. In: Congress on Evolutionary Computation, pp. 1605–1612. IEEE, New Orleans (2011)
48. Jaszkiewicz, A., Slowinski, R.: The light beam search over a non-dominated surface of a multiple-objective programming problem. Eur. J. Oper. Res. **113**(2), 300–314 (1999)
49. Jaszkiewicz, A.: A comparative study of multiple-objective metaheuristics on the bi-objective set covering problem and the pareto memetic algorithm. Ann. Oper. Res. **131**(1–4), 135–158 (2004)
50. Jaszkiewicz, A.: Interactive multiobjective optimization with the Pareto memetic algorithm. Found. Comput. Decis. Sci. **32**(1), 15–32 (2007)
51. Jiménez, F., Verdegay, J.L.: Evolutionary techniques for constrained optimization problems. In: Zimmermann, H.J. (ed.) European Congress on Intelligent Techniques and Soft Computing. Verlag Mainz, Germany (1999)
52. Jin, Y., Sendhoff, B.: Incorporation of fuzzy preferences into evolutionary multiobjective optimization. In: Wang, L., et al. (eds.) Asia-Pacific Conference on Simulated Evolution and Learning, pp. 26–30. Nanyang Technical University, Singapore (2002)
53. Kamalian, R., Takagi, H., Agogino, A.M.: Optimized design of mems by evolutionary multi-objective optimization with interactive evolutionary computation. In: Genetic and Evolutionary Computation Conference, pp. 1030–1041 (2004)
54. Karahan, I., Köksalan, M.: A territory defining multiobjective evolutionary algorithmsw and preference incorporation. IEEE Trans. Evol. Comput. **14**(4), 636–664 (2010)
55. Köksalan, M., Karahan, I.: An interactive territory defining evolutionary algorithm: iTDEA. IEEE Trans. Evol. Comput. **14**(5), 702–722 (2010)
56. Korhonen, P., Laakso, J.: A visual interactive method for solving the multiple criteria problem. Eur. J. Oper. Res. **24**, 277–287 (1986)
57. Korhonen, P., Wallenius, J., Zionts, S.: Solving the discrete multiple criteria problem using convex cones. Manage. Sci. **30**, 1336–1345 (1984)
58. Luque, J.M., Santana-Quintero, L.V., Hernandez-Diaz, A.G., Coello, C.A.C., Caballero, R.: g-dominance: reference point based dominance for multiobjective metaheuristics. Eur. J. Oper. Res. **197**(2), 685–692 (2009)
59. Parreiras, R.O., Vasconcelos, J.A.: Decision making in multiobjective optimization problems. In: Nedjah, N., de Macedo Mourelle, L. (eds.) Real-World Multi-Objective System Engineering, pp. 29–52. Nova Science Publishers, New York (2005)
60. Phelps, S., Köksalan, M.: An interactive evolutionary metaheuristic for multiobjective combinatorial optimization. Manage. Sci. **49**(12), 1726–1738 (2003)
61. Rachmawati, L., Srinivasan, D.: Preference incorporation in multi-objective evolutionary algorithms: a survey. In: Congress on Evolutionary Computation, pp. 3385–3391. IEEE, Los Alamitos (2006)

62. Rekiek, B., Lit, P.D., Fabrice, P., L'Eglise, T., Emanuel, F., Delchambre, A.: Dealing with users's preferences in hybrid assembly lines design. In: Binder, Z., et al. (eds.) Management and Control of Production and Logistics Conference, pp. 989–994. Pergamon, New York (2000)
63. Said, L.B., Bechikh, S., Ghedira, K.: The r-dominance: a new dominance relation for interactive evolutionary multicriteria decision making. IEEE Trans. Evol. Comput. **14**(5), 801–818 (2010)
64. Takagi, H.: Interactive evolutionary computation: fusion of the capabilities of ec optimization and human evaluation. In: Proceedings of the IEEE, vol. 89, pp. 1275–1296 (2001)
65. Tan, K.C., Lee, T.H., Khor, E.F.: Evolutionary algorithms with goal and priority information for multi-objective optimization. In: Congress on Evolutionary Computation, pp. 106–113. IEEE, Los Alamitos (1999)
66. Thiele, L., Miettinen, K., Korhonen, P.J., Molina, J.: A preference-based interactive evolutionary algorithm for multiobjective optimization. Evol. Comput. J. **17**(3), 411–436 (2009)
67. Todd, D.S., Sen, P.: Directed multiple objective search of design spaces using genetic algorithms and neural networks. In: Banzhaf, W., et al. (eds.) Genetic and Evolutionary Computation Conference, pp. 1738–1743. Morgan Kaufmann, San Francisco, CA (1999)
68. Trautmann, H., Mehnen, J.: A method for including a-priori-preference in multicriteria optimization. Technical Report 49/2005, SFG 475, University of Dortmund, Germany (2005)
69. Wagner, T., Trautmann, H.: Integration of preferences in hypervolume-based multiobjective evolutionary algorithms by means of desirability functions. IEEE Trans. Evol. Comput. **14**(5), 688–701 (2010)
70. White, C., Sage, A., Dozono, S.: A model of multiattribute decision-making and tradeoff weight determination under uncertainty. IEEE Trans. Syst. Man Cybern. **14**, 223–229 (1984)
71. Wierzbicki, A.P.: Basic properties of scalarizing functions for multiobjective optimization. Optimization **8**(1), 55–60 (1977)
72. Wierzbicki, A.P.: On the completeness and constructiveness of parametric characterizations to vector optimization problems. OR Spektrum **8**(2), 73–87 (1986)
73. Zhang, Q., Li, H.: MOEA/D: a multi-objective evolutionary algorithm based on decomposition. IEEE Trans. Evol. Comput. Evol. Comput. **11**(6), 712–731 (2007)
74. Zitzler, E., Künzli, S.: Indicator-based selection in multiobjective search. In: Parallel Problem Solving from Nature. Lecture Notes in Computer Science, vol. 3242, pp. 832–842. Springer, Berlin (2004)
75. Zitzler, E., Laumanns, M., Thiele, L.: SPEA2: Improving the strength Pareto evolutionary algorithm for multiobjective optimization. In: Giannakoglou, K., et al. (eds.) Evolutionary Methods for Design, Optimisation and Control with Application to Industrial Problems (EUROGEN 2001), pp. 95–100. International Center for Numerical Methods in Engineering, Spain (2002)
76. Zitzler, E., Brockhoff, D., Thiele, L.: The hypervolume indicator revisited: on the design of Pareto-compliant indicators via weighted integration. In: Obayashi, S., et al. (eds.) Evolutionary Multi-Criterion Optimization. Lecture Notes in Computer Science, vol. 4403, pp. 862–876. Springer, Berlin (2007)

Part VII
Applications

Chapter 24
Multicriteria Decision Aid/Analysis in Finance

Jaap Spronk, Ralph E. Steuer, and Constantin Zopounidis

Abstract Over the past decades the complexity of financial decisions has increased rapidly, thus highlighting the importance of developing and implementing sophisticated and efficient quantitative analysis techniques for supporting and aiding financial decision making. Multicriteria decision aid (MCDA), an advanced branch of operations research, provides financial decision makers and analysts with a wide range of methodologies well-suited for the complexity of modern financial decision making. The aim of this chapter is to provide an in-depth presentation of the contributions of MCDA in finance focusing on the methods used, applications, computation, and directions for future research.

Keywords Multicriteria decision aid • Finance • Portfolio theory • Multiple criteria optimization • Outranking relations • Preference disaggregation analysis

24.1 Introduction

Over the past decades, the globalization of financial markets, the intensification of competition among organizations, and the rapid social and technological changes that have taken place have only led to increasing uncertainty and instability in the business and financial environment. Within this more recent context, both the

J. Spronk
Rotterdam School of Management, Erasmus University Rotterdam, P.O.Box 1738, 3000DR Rotterdam, The Netherlands
e-mail: jspronk@rsm.nl

R.E. Steuer
Department of Finance, University of Georgia, Athens, GA 30602-6253, USA
e-mail: rsteuer@uga.edu

C. Zopounidis (✉)
Financial Engineering Laboratory, Department of Production Engineering and Management, Technical University of Crete, University Campus, 73100 Chania, Greece

Audencia Nantes School of Management, 8 route de la Jonelire, 44312 Nantes Cedex 3, France
e-mail: kostas@dpem.tuc.gr

S. Greco et al. (eds.), *Multiple Criteria Decision Analysis*, International Series in Operations Research & Management Science 233,
DOI 10.1007/978-1-4939-3094-4_24

importance of financial decision making and the complexity of the process by which financial decision making is carried out have increased. This is clearly evident by the variety and volume of new financial products and services that have appeared on the scene.

In this new era of financial reality, researchers and practitioners acknowledge the requirement to address financial decision-making problems through integrated and realistic approaches utilizing sophisticated analytical techniques. In this way, the connections between financial theory, the tools of operations research, and mathematical modelling have become more entwined. Techniques from the fields of optimization, forecasting, decision support systems, MCDA, fuzzy logic, stochastic processes, simulation, etc. are now commonly considered valuable tools for financial decision making.

The use of mathematics and operations research in finance got its start in the 1950s with the introduction of Markowitz's portfolio theory [131, 133]. Since then, in addition to portfolio selection and management, operations research has contributed to financial decision making problems in other areas including venture capital investments, bankruptcy prediction, financial planning, corporate mergers and acquisitions, country risk assessment, etc. These contributions are not limited to academic research; they are now often found in daily practice.

Within the field of operations research, MCDA has evolved over the last three decades into one of its pillar disciplines. The development of MCDA is based upon the common finding that a sole objective, goal, criterion, or point of view is rarely used to make real-world decisions. In response, MCDA is devoted to the development of appropriate methodologies to support and aid decision makers across ranges of situations in which multiple conflicting decision factors (objectives, goals, criteria, etc.) are to be considered simultaneously.

The methodological framework of MCDA is well-suited to the growing complexities encountered in financial decision making. While there have been in finance MCDA stirrings going back 20–30 years, the topic of MCDA, as can be seen from the bulk of the references, really hasn't come into its own until recently (see [215] for a recent survey of the literature). As for early stirrings, we have, for example, Bhaskar [22] in which microeconomic theory was criticized for largely pursuing a single criterion approach arguing that things like profit maximization are too naive to meet the evolving decision-making demands in many financial areas. Also, in another paper [23], the unavoidable presence of multiple objectives in capital budgeting was noted and the necessity for developing ways to deal with the unique challenges posed by multiple criteria was stressed. It is upon what has taken place since these early roots, and on what are today promising directions in MCDA in finance, that this contribution is focused.

Such observations and findings have motivated researchers to explore the potentials of MCDA in addressing financial decision-making problems. The objective of this chapter is to provide a state-of-the-art comprehensive review of the research made up to date on this issue. Section 24.2 presents discussions to justify the presence of MCDA in financial decision making. Section 24.3, focuses on MCDA in resource allocation problems (continuous problems) as in the field of portfolio management. Section 24.4, presents the contribution of MCDA methodologies in supporting financial decisions that require the evaluation of a discrete set of alternatives (firms, countries, stocks, investment projects, etc.). Finally, Sect. 24.5 concludes the chapter and discusses possible future research directions on the implementation of multicriteria analysis in financial institutions and firms.

24.2 Financial Decision Making

Financial-economic decision problems come in great variety. Individuals are involved in decisions concerning their future pensions, the financing of their homes, and investments in mutual funds. Firms, financial institutions, and advisors are involved in cross-country mergers, complicated swap contracts, and mortgage-backed securities, to name just a few.

Despite the variety, such decisions have much in common. Maybe "money" comes first to mind, but there are typically other factors that suggest that financial-economic problems should most appropriately be treated as multiple criteria decision problems in general: multiple actors, multiple policy constraints, and multiple sources of risk (see e.g., Spronk and Hallerbach [177], and Hallerbach and Spronk [79, 80], Martel and Zopounidis [134], Zopounidis [202], and Steuer and Na [182]).

Two other common elements in financial decisions are that their outcomes are distributed over time and uncertainty, and thus involve risk. A further factor is that most decisions are made consciously, with a clear and constant drive to make "good", "better" or even "optimal" decisions. In this drive to improve on financial decisions, we stumble across an area of tension between decision making in practice on the one hand and the potential contributions of finance theory and decision tools on the other. Although the bulk of financial theory is of a descriptive nature, thus focusing on the "average" or "representative" decision maker, we observe a large willingness to apply financial theory in actual decision-making. At the same time, knowledge about decision tools that can be applied in a specific decision situation, is limited. Clearly, there is need of a framework that can provide guidance in applying financial theory, decision tools, and common sense to solving financial problems.

24.2.1 Issues, Concepts, and Principles

> Finance is a sub field of economics distinguished by both its focus and its methodology. The primary focus of finance is the workings of the capital markets and the supply and the pricing of capital assets. The methodology of finance is the use of close substitutes to price financial contracts and instruments. This methodology is applied to value instruments whose characteristics extend across time and whose payoffs depend upon the resolution of uncertainty. (Ross [158], p. 1)

The field of finance is concerned with decisions with respect to the efficient allocation of scarce capital resources over competing alternatives. The allocation is efficient when the alternative with the highest value is chosen. Current value is viewed as the (present) value of claims on future cash flows. Hence we can say that financial decisions involve the valuation of future, and hence uncertain or "risky," cash flow streams. Cash flow stream X is valued by comparing it with cash flow streams {A,...,Z} that are traded on financial markets. When a traded cash flow stream Y has been identified that is a substitute for X, then their values must be the same. After all, when introducing X to the market, it cannot be distinguished value-wise from Y. Accepting the efficient market hypothesis (stipulating that all available information is fully and immediately incorporated in market prices), the market price of Y equals the value of Y, and hence the value of X. This explains the crucial role of financial markets.

The valuation of future cash flow streams is a key issue in finance. The process of valuation must be preceded by evaluation: without analyzing the characteristics of a cash flow stream, no potential substitute can be identified. Since it is uncertain what the future will bring, the analysis of the risk characteristics will be predominant. Moreover, as time passes, the current value must be protected against influences that may erode its value. This in turn implies the need for risk management. There are basically three areas of financial decisions:

1. **Capital budgeting**: to what portfolio of real investment projects should a firm commit its capital? The central issues here are how to evaluate investment opportunities, how to distinguish profitable from non-profitable projects and how to choose between competing projects.
2. **Corporate financing**: this encompasses the capital structure policy and dividend policy and addresses questions as: how should the firm finance its activities? What securities should the firm issue or what financial contracts should the firm engage in? What part of the firm's earnings should be paid as cash dividends and what part reinvested in the firm? How should the firm's solvency and liquidity be maintained?
3. **Financial investment**: this is the mirror image of the previous decision area and involves choosing a portfolio of financial securities with the objective to change the consumption pattern over time.

In each of these decision areas the financial key issues of valuation, risk analysis and risk management, and performance evaluation can be recognized, and from the above several financial concepts emerge: financial markets, efficient allocation and market value. In approaching the financial decision areas, some financial principles or maxims are formulated. The first is self-interested behavior: economic subjects are driven by *non-satiation* ("greed"). This ensures the goal of value maximization. Prices are based on financial markets, and under the efficient market hypothesis, prices of securities coincide with their value. Value has time and risk dimensions. With regard to the former, *time preference* is assumed (a dollar today is preferred to a dollar tomorrow). With respect to the latter, *risk aversion* is assumed (a safe dollar is preferred to a risky dollar). Overall risk may be reduced by *diversification*:

combining risky assets or cash flow streams may be beneficial. In one way or another, the trade-off between expected return and risk that is imposed by market participants on the evaluation of risky ventures will translate into a *risk-return trade-off* that is offered by investment opportunities in the market.

Since value has time and risk aspects, the question arises about what mechanisms can be invoked to incorporate these dimensions in the valuation process. There are basically two mechanisms. The first is the arbitrage mechanism. Value is derived from the presumption that there do not exist arbitrage opportunities. This no-arbitrage condition excludes sure profits at no cost and implies that perfect substitutes have the same value. This is the *law of one price*, one of the very few laws in financial economics. It is a strong mechanism, requiring very few assumptions on market subjects, only non-satiation. Examples of valuation models built on no-arbitrage are the Arbitrage Pricing Theory for primary financial assets and the Option Pricing Theory for derivative securities. The second is the equilibrium mechanism. In this case value is derived from the market clearing condition that demand equals supply. The latter mechanism is much weaker than the former: the exclusion of arbitrage opportunities is a necessary but by no means a sufficient condition for market equilibrium. In addition to non-satiation also assumptions must be made regarding the risk attitudes of all market participants. Examples of equilibrium-based models are the Capital Asset Pricing Model and its variants. Below we discuss the differences between the two valuation approaches in more detail. It suffices to remark that it is still a big step from the principles to solving actual decision problems.

24.2.2 Focus of Financial Research

An alternative, albeit almost circular, definition of finance is provided by Jarrow [101, p. 1].

> Finance theory (. . .) includes those models most often associated with financial economics. (. . .) [*A*] practical definition of financial economics is found in those topics that appear with some regularity in such publications as *Journal of Finance, Journal of Financial and Quantitative Analysis, Journal of Financial Economics*, and *Journal of Banking and Finance*.

Browsing through back volumes of these journals and comparing them to the more recent ones reveals a blatant development in nature and focus. In early days of finance, the papers were descriptive in a narrative way and in the main focused on financial instruments and institutions. Finance as a decision science emerged in the early 1950s, when Markowitz [130, 131] studied the portfolio selection decision and launched what now is known as "modern portfolio theory." In the 1960s and the early 1970s, many financial economic decision problems were approached by operational research techniques; see for example Ashford et al. [8] and McInnes and Carleton [138] for an overview. However, since then, this type of research has became more and more absorbed by the operations research community and in their journals.

But what direction did finance take? Over the last 25 years mathematical models have replaced the verbal models and finance has founded itself firmly in a neo-classical micro-economic tradition. Over this period we observe a shift to research that is descriptive in a sophisticated econometrical way and that focuses on the statistical characteristics of (mainly well-developed) financial markets where a host of financial instruments is traded. Bollerslev [27, p. 41], aptly describes this shift as follows.

> A cursory look at the traditional econometrics journals (...) severely underestimates the scope of the field [of financial econometrics], as many of the important econometric advances are now also published in the premier finance journals - the *Journal of Finance*, the *Journal of Financial Economics*, and the *Review of Financial Studies* – as well as a host of other empirically oriented finance journals.

The host of reported research addresses the behavior of financial market prices. The study of the pricing of primary securities is interesting for its own right, but it is also relevant for the pricing of derivative securities. Indeed, the description of the pricing of primary assets and the development of tools for pricing derivative assets mark the success story of modern finance.

The body of descriptive finance theory has grown enormously. According to modern definitions of the field of finance, the descriptive nature is even predominant.

> The core of finance theory is the study of the behavior of economic agents in allocating and deploying their resources, both spatially and across time, in an uncertain environment. (Merton [140], p. 7)

Compared to Ross' [158] definition cited earlier, the focus is purely positive. The question arises to what extent the insights gained from descriptive finance—how sophisticated they may be from a mathematical, statistical or econometric point of view—can serve as guidelines for financial decisions in practice. Almost 30 years ago, in the preface of their book *The Theory of Finance*, Eugene Fama and Merton Miller defended their omission of detailed examples, purporting to show how to apply the theory to real-world decision problems, as

> (...) a reflection of our belief that the potential contribution of the theory of finance to the decision-making process, although substantial, is still essentially indirect. The theory can often help expose the inconsistencies in existing procedures; it can help keep the really critical questions from getting lost in the inevitable maze of technical detail; and it can help prevent the too easy, unthinking acceptance of either the old clichs or new fads. But the theory of finance has not yet been brought, and perhaps never will be, to the cookbook stage. (Fama and Miller [65], p. viii)

Careful inspection of current finance texts reveals that in this respect not much has changed. However, pure finance theory and foolproof financial recipes are two extremes of a continuum. The latter cookbook stage will never be achieved, of course, and in all realism and wisdom this alchemic goal should not be sought for. But what we dearly miss is an extensive body of research that bridges the apparent gap between the extremes: research that shows how to solve real-world financial decision problems without violating insights offered by pure finance theory on the one hand and without neglecting the peculiarities of the specific decision problem on the other.

On another matter, the role of assumptions in modelling is to simplify the real world in order to make it tractable. In this respect the art of modelling is to make assumptions where they most contribute to the model's tractability and at the same time detract from the realism of the model as little as possible. The considerations in this trade-off are fundamentally different for positive (descriptive) models on the one hand and conditional-normative models on the other. In the next section we elaborate further on the distinctions between the two types of modelling as concerns the role of assumptions.

24.2.3 Descriptive vs. Conditional-Normative Modelling

In a positive or descriptive model simplified assumptions are made in order to obtain a testable implication of the model. The validity of the model is evaluated according to the inability to reject the model's implications at some level of significance. So validity is of an empirical nature, solely judged by the implications of the model. Consider the example of an equilibrium asset-pricing model. As a starting point, assumptions are made with respect to the preferences of an imaginary investor and the risk-return characteristics of the investment opportunities. These assumptions are sufficiently strong to allow solving the portfolio optimization problem. Next a homogeneity condition is imposed: all investors in the market possess the same information and share the same expectations. This allows focusing on "a representative investor". Finally the equilibrium market clearing condition is imposed: all available assets (supply) must be incorporated in the portfolio of the representative investor (demand). The first order conditions of portfolio optimality then stipulate the trade-off between risk and expected return that is required by the investor. Because of the market clearing, the assets offer the same trade-off. Hence a market-wide relationship between risk and return is established and this relationship is the object of empirical testing. As long as the pricing relationship is not falsified the model is accepted, irrespective of whether the necessary assumptions are realistic or not. When the model is falsified, deduction may help to amend the assumptions where after the same procedure is followed. This hypothetic-deductive cycle ends when the model is no longer falsified by the empirical data at hand.

In a conditional-normative model, simplifying assumptions are also made in order to obtain a tractable model. These assumptions relate to the preferences of the decision maker and to the representation of the set of choice alternatives. The object of the conditional-normative modelling is not to infer a testable implication but to obtain a decision rule. This derived decision rule is valid and can normatively be applied conditional on the fact that the decision maker satisfies the underlying assumptions; cf. Keynes [110].

In order to support decisions in finance, obviously both the preferences of the decision maker and the characteristics of the choice alternatives should be understood and related to each other. Unfortunately, the host of financial-economic modelling is of a positive nature and focuses on the "average" decision maker instead of addressing the particular (typically non-average) decision maker. The assumptions underlying financial theory at best describe "average individuals" and "average decision situations" and hence are not suited to describe specific individual decision problems. The assumptions made to simplify the decision situation often completely redefine the particular problem at hand. The real world is replaced by an over-simplified model-world. As a consequence, not the initial problem is solved but a synthesized and redefined problem that is not even recognized by the decision maker himself. The over-simplified model becomes a Procrustes bed for the financial decision maker who seeks advice.

For example, it is assumed that a decision maker has complete information and that this information can be molded into easily manipulated probability distributions. Even worse, positive knowledge and descriptive theories that by definition reflect the outcomes of decisions made by some representative decision maker are used to prescribe what actions to take in a specific decision situation. For example, equilibrium asset pricing theories predict the effects of decisions and actions of many individuals on the formation of prices in financial markets. Under the homogeneity condition the collection of investors is reduced to the representative investor. When the pricing implications of the model are simply used to guide actual investment behavior, then the decision maker is forced into the straitjacket of this representative investor.

Unfortunately we observe that conditional-normative financial modelling is only regarded as a starting point for descriptive modelling and is not pursued for its own sake. After almost 20 years, Hastie's [83] lament has not lost its poignancy.

> In American business today, particularly in the field of finance, what is needed are approximate answers to the precise problem rather than precise answers to the approximate problem.

Apart from the positive modelling of financial markets as described above, there is one other field in finance in which the achievements of applied modelling are apparent: option pricing theory, the set of models that enable the pricing of derivative securities and all kinds of contingent claims. Indeed, the option pricing formulas developed by Black and Scholes [25] and Merton [139] mark a huge success in the history of financial modelling. Contingent claims analysis made a flying start, and

> .... when judged by its ability to explain the empirical data, option pricing theory is the most successful theory not only in finance, but in all of economics. (Ross [158], p. 24)
> Given a theory that works so well, the best empirical work will be to use it as a tool rather than to test it. (Ross [158], p. 23)

Indeed, modern-day derivatives trading would be unthinkable without the decision support of an impressive coherent toolbox for analyzing the risk characteristics of derivatives and for pricing them in a consistent way. Compared to this framework, the models and theories developed and tested for primary assets look pale. What is the reason for the success of derivatives research?

For an explanation we turn to the principal tool used in option pricing theory: no-arbitrage valuation. By definition derivative securities derive their value from primary underlying assets. Under some mild assumptions, a dynamic trading strategy can be designed in which the derivative security is exactly replicated with a portfolio of the primary security and risk-free bonds. Under the no-arbitrage condition, the current value of the derivative security and the replicating portfolio should be identical. Looking from another perspective, a suitably chosen hedge combination of the derivative and the underlying security produces a risk-free position. On this position the risk-free rate must be earned, otherwise there exist arbitrage opportunities. Since the position is risk free, risk attitudes and risk aversion do not enter the story. Therefore a derivative security will have the same value in a market environment with risk neutral investors as in a market with risk averse investors. This in turn implies that a derivative can be priced under the assumption that investors are risk neutral. As a consequence, no assumptions are required on preferences (other than non-satiation), utility functions, the degree of risk aversion, and risk premia. Thus, option pricing theory can escape from the burden of modelling of preference structures. Instead, research attention shifts to analyzing price dynamics on financial markets. An additional reason for the success in derivatives research is that the analytical and mathematical techniques are similar to those used in the physical sciences (see for example Derman [44]).

Of course, even in derivatives modelling some assumptions are required. This introduces model risk. When the functional relationships stipulated in the model are wrong, or when relevant input parameters of the model are incorrectly estimated, the model produces the wrong value and the wrong risk profile of the derivative. To an increasing degree, financial institutions are aware that great losses can be incurred because of model risk. Especially in risk management and derivatives trading model risk is a hot item (see Derman [43]). This spurred Merton to ventilate this warning.

> At times, the mathematics of the models become too interesting and we lose sight of the models' ultimate purpose. The mathematics of the models is precise, but the models are not, being only approximations to the complex, real world. Their accuracy as a useful approximation to that world varies considerably across time and place. The practitioner should therefore apply the models only tentatively, assessing their limitations carefully in each application. (Merton [140], p. 14)

Ironically this quote was taken just after the very successful launch of Long Term Capital Management (LTCM), the hedge fund of which Merton and Myron Scholes were the founding partners. In 1998, LTCM collapsed and model risk played a very important role in this debacle.

Summarizing we draw the conclusion that successful applied financial modelling does exist, and blossoms in the field of derivatives. Here also the validity of the assumptions is crucial, this in contrast to positive modelling. However, in the field of derivatives with replicating strategies and arbitrage-based valuation, the concept of "absence of risk" is well defined and no preference assumptions are needed in the modelling process. For modelling decisions regarding the underlying primary assets, in contrast, assumptions on the decision maker's preferences and on the "risk" attached to the outcomes of the choice alternatives are indispensable. For these types of financial problems, the host of simplifying assumptions that are made in the descriptive modelling framework invalidate the use of the model in a specific decision situation. Thus we face the following challenge: how can we retain the conceptual foundation of the financial-economic framework and still provide sound advice that can be applied in multifarious practice? As a first step we will sketch the relationship between decision sciences and financial decision-making.

24.2.4 Decision Support for Financial Decisions

Over the last 50 years or so, the financial discipline has shown continuously rapid and profound changes, both in theory and in practice. Many disciplines have been affected by globalization, deregulation, privatization, computerization, and communication technologies. Hardly any field has been influenced as much as finance. After the mainly institutional and even somewhat *ad hoc* approaches before the 1950s, Markowitz [130, 131] has opened new avenues by formalizing and quantifying the concept of "risk". In the decades that followed, a lot of attention was paid to the functioning of financial markets and the pricing of financial assets including options. The year 1973 gave birth to the first official market in options (CBOE) and to crucial option pricing formulas that have become famous quite fast (Black-Scholes and Cox-Ross-Rubinstein, see Hull [90]), both in theory and practice. At that time, financial decision problems were structured by (a) listing a number of mutually exclusive decision alternatives, (b) describing them by their (estimated) future cash flows, including an estimation of their stochastic variation and later on including the effect of optional decisions, and (c) valuing them by using the market models describing financial markets.

In the 1970s, 1980s and 1990s, the financial world saw enormous growth in derivative products, both in terms of variety and in terms of market volumes. Financial institutions have learned to work with complex financial products. Academia has contributed by developing many pricing models, notably for derivatives. Also, one can say that financial theory has been rewritten in the light of contingent claims ("optional decisions") and will soon be further reshaped by giving more attention to game elements in financial decisions. The rapid development of the use of complex financial products has certainly not been without accidents. This has led regulators to demand more precise evaluations and the reporting of financial positions (cf. e.g., the emergence of the Value-at-Risk concept, see Jorion [107]).

In addition to the analysis of financial risk, the structured management of financial risk has come to the forefront. In their textbook, Bodie and Merton [26] describe the threefold tasks of the financial discipline as Valuation, Risk Management, and Optimization. We would like to amend the threefold tasks of financial management to Valuation, Risk Management, and Decision Making. The reason is that financial decision problems often have to be solved in dynamic environments where information is not always complete, different stakeholders with possibly conflicting goals and constraints play a role and clear-cut optimization problems cannot always be obtained (and solved).

At the same time, many efforts from the decision-making disciplines are misdirected. For instance, some approaches fail to give room for the inherent complexity of the decision procedure given the decision maker's specific context. Other approaches concentrate on the beauties of a particular decision method without doing full justice to the peculiarities of the decision context. Aside from being partial in this respect, useful principles and insights offered by financial-economic theory are often not integrated in the decision modelling. It is therefore no surprise that one can observe in practice unstructured *ad hoc* approaches as well as complex approaches that severely restrict the decision process.

24.2.5 Relevance of MCDA for Financial Decisions

The central issue in financial economics is the efficient allocation of scarce capital and resources over alternative uses. The allocation (and redistribution) of capital takes place on financial markets and is termed "efficient" when market value is maximized. Just as water will flow to the lowest point, capital will flow to uses that offer the highest return. Therefore it seems that the criterion for guiding financial decisions is one-dimensional: maximize market value or maximize future return.

From a financial-economic perspective, the goal of the firm, for example, is very much single objective. Management should maximize the firm's contribution to the financial wealth of its shareholders. Also the shareholders are considered to be myopic. Their only objective is to maximize their single-dimensional financial wealth. The link between the shareholders and the firm is footed in law. Shareholders are the owners of the firm. They possess the property rights of the firm and are thus entitled to decide what the firm should aim for, which according to homogeneity is supposed to be the same for all shareholders, i.e., maximize the firm's contribution to the financial wealth of the shareholders. The firm can accomplish this by engaging in investment projects with positive net present value. This is the neo-classical view on the role of the firm and on the relationship between the firm and its shareholders in a capitalist society. Figure 24.1 depicts a simplified graphical representation of this line of thought.

It is important to note that this position is embedded in a much larger framework of stylized thinking in among others economics (general equilibrium framework) and law (property rights theory and limited liability of shareholders). Until today, this view is seen as an ideal by many; see for example Jensen [102]. Presently, however, the societal impact of the firm and its governance structure is a growing topic of debate. Here we will show that also in finance there are many roads leading to Rome, or rather to the designation MCDA. Whether one belongs to the camp of Jensen or to the camp of those advocating socially responsible entrepreneurship, one has to deal with multiple criteria.

There is a series of situations in which the firm chooses (or has to take account of) a multiplicity of objectives and (policy) constraints. An overview of these situations is depicted in Fig. 24.2. One issue is who decides on the objective(s) of the firm. If there is a multiplicity of parties who may decide what the firm is aiming for, one generally encounters a multitude of goals, constraints and considerations that—more often than not—will be at least partially conflictive. A clear example is the

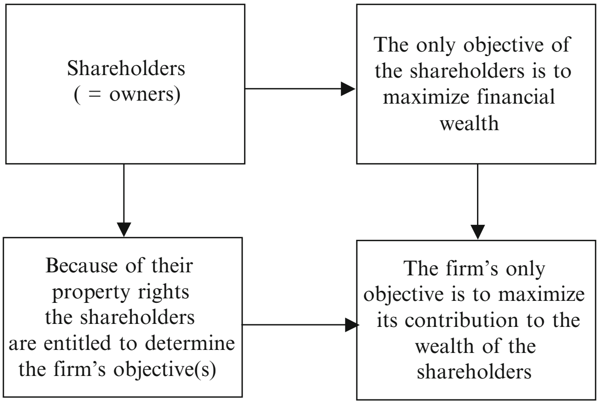


Fig. 24.1 The neo-classical view on the objective of the firm

conflicting objectives arising from agency problems (Jensen and Meckling [103]). This means that many decision problems include multiple criteria and multiple actors (viz. group decision making, negotiation theory, see Box 3 in Fig. 24.2). Sometimes, all those who decide on what the firm should aim for agree upon exactly the same objective(s). In fact, this is what neo-classical financial theory assumes when adopting shareholder value maximization (Box 1 in Fig. 24.2). In practice, there are many firms that explicitly strive for a multiplicity of goals, which naturally leads to decision problems with multiple criteria (Box 2 in Fig. 24.2).

However, although these firms do explicitly state to take account of multiple objectives, there are still very few of these firms that make use of tools provided by the MCDA literature. In most cases firms maximize one objective subject to (policy) constraints on the other objectives. As such there is nothing wrong with such a procedure as long as the location of these policy constraints is chosen correctly. In practice, however, one often observes that there is no discussion at all about the location of the policy constraints. Moreover, there is often no idea about the trade-offs between the location of the various constraints and the objective function that is maximized. In our opinion, multiple criteria decision methodologies may help decision makers to gain better insights in the trade-offs they are confronted with.

Now let us get back to the case in which the owner(s)/shareholders do have only one objective in mind: wealth maximization. Although this is by definition the most prominent candidate for single criteria decision-making, we will argue that even in this case there are many circumstances in which the formulation as a multiple criteria decision problem is opportune.

In order to contribute maximally to the wealth of its shareholders, an individual firm should maximize the value of its shares. The value of these shares is determined on the financial markets by the forces of demand and supply. Shares represent claims on the future residual cash flows of the firm (and also on a usually very limited right on corporate control). In the view of the financial markets, the value

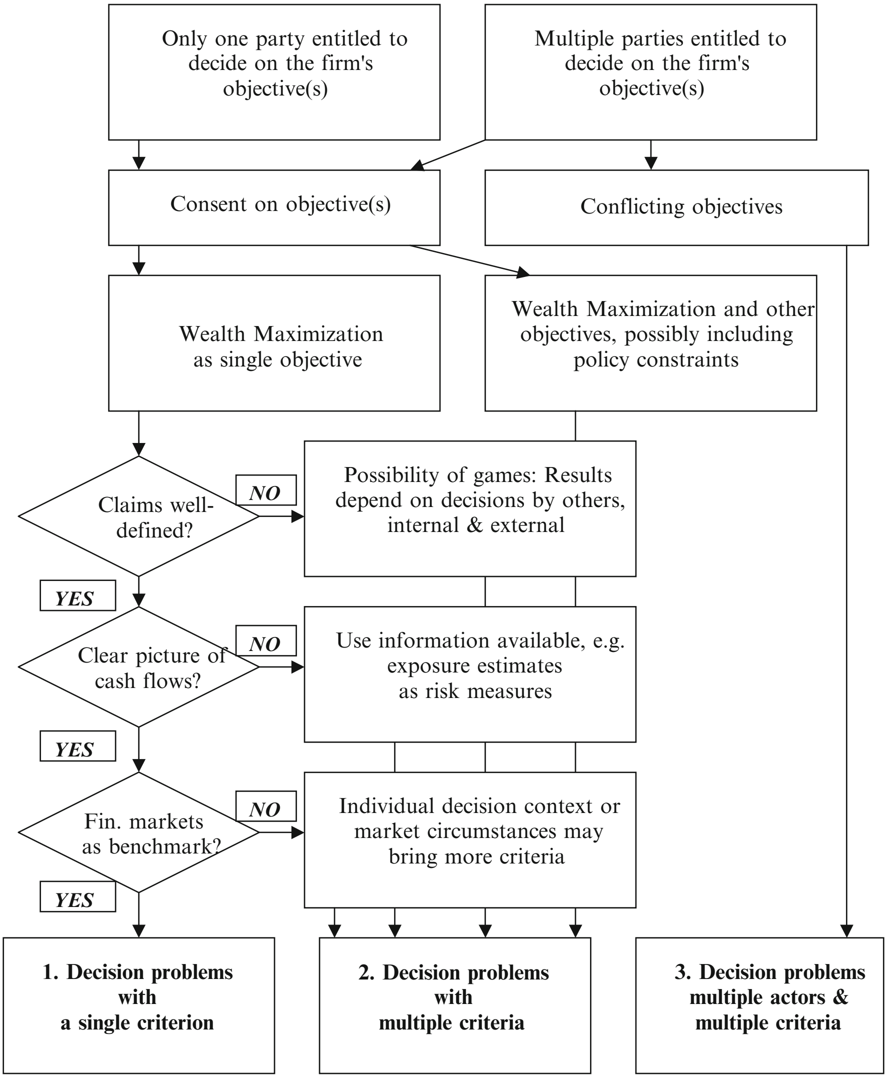


Fig. 24.2 Situations leading to MCDA in the firm

of such a claim is determined relative to the claims of other firms that are traded on these markets. The financial markets' perception of the quality of these cash flow claims is crucial for the valuation of the shares. Translated to the management of the individual firm, the aim is not only to maximize the quality of the future residual cash flows of the firm but also to properly communicate all news about these cash flows to the financial markets. Only by the disclosure of such information can informational asymmetries be resolved and the fair market value of a cash flow claim be determined. In evaluating the possible consequences of its decision alternatives, management should estimate the effects on the uncertain (future) cash flows followed by an estimation of the financial markets' valuation of these effects. Then (and only then) the decision rule of management is very simple: choose the decision alternative that generates the highest estimated market value.

The first problem that might arise while following the above prescription is that residual claims cannot always be defined because of "gaming effects" (see Fig. 24.2, Box 2). In other words, the future cash flows of the firm do not only depend on the present and future decisions of the firm's management, but also on the present and future decisions of other parties. An obvious example is the situation of oligopolistic markets in which the decisions of the competitors may strongly influence each other. Similar situations may arise with other external stakeholders such as powerful clients, powerful suppliers, and powerful financiers. Games may also arise within the firm, for instance between management and certain key production factors. The problem with game situations is that their effect on a firm's future cash flows caused by other parties involved cannot be treated in the form of simple constraints or as cost factors in cash flow calculations. MCDA may help to solve this problem by formulating multi-dimensional profiles of the consequences of the firm's decision alternatives. In these profiles, the effects on parties other than the firm are also included. These multi-dimensional profiles are the keys to open the complete MCDA toolbox.

A second problem in dealing with the single-objective wealth maximization problems is that the quality of information concerning the firm's future cash flows under different decision alternatives is far from complete. In addition, the available information may be biased or flawed. One way to approach the incomplete information problem is suggested by Spronk and Hallerbach [177]. In their multi-factorial approach, different sources of uncertainty should be identified after which the exposures of the cash flows to these risk sources are estimated. The estimated exposures can next be included in a multi-criteria decision method. In the case that the available information is not conclusive, different "views" on the future cash flows may develop. Next each of these views can be adopted as representing a different dimension of the decision problem. The resulting multi-dimensional decision problem can then be handled by using MCDA (see Fig. 24.2, Box 2).

A third potential problem in wealth maximization is that the financial markets do not always provide relevant pricing signals to evaluate the wealth effects of the firm's decisions, for example, because of market inefficiencies. This means that the firm may want to include attributes in addition to the market's signals in order to measure the riskiness and wealth effects of its decisions.

24.2.6 A Multicriteria Framework for Financial Decisions

In our view it, is the role of financial modelling to support financial decision making, as described in Hallerbach and Spronk [81], to build pointed models that take into account the peculiarities of the precise problem. The goal here is to bridge the gaps between decision-making disciplines, the discipline of financial economics, and the need for adequate decision support.

24.2.6.1 Principles

This framework is built on the principle that assumptions should be made where they help the modelling process the most and hurt the particular decision problem the least.[1] We call this the *Principle of Low Fat Modelling*. When addressing a decision situation, make use of all available information, but do not make unrealistic assumptions with respect to the availability of information. Do not make unrealistic assumptions that disqualify the decision context at hand. There should be ample room to incorporate idiosyncrasies of the decision context within the problem formulation, thus recognizing that the actual (non-average) decision maker is often very different from the "representative" decision maker. The preferences of the decision maker may not be explicitly available and may not even be known in detail by the decision maker himself. The uncertainty a decision maker faces with respect to the potential outcomes of his decisions may not be readily represented by means of a tractable statistical distribution. In many real-life cases, uncertainty can only be described in imprecise terms and available information is far from complete. And when the preferences of the decision maker are confronted with the characteristics of the decision alternatives, the conditional-normative nature of derived decision rules and advice should be accepted.

A second principle underlying our framework is the *Principle of Eclecticism*. One should borrow all the concepts and insights from modern financial theory that help to make better financial decisions. Financial theory can provide rich descriptions of uncertainty and risk. Examples are the multi-factor representation of risk in which the risk attached to the choice alternatives is conditioned on underlying factors such as the contingent claims approach in which the decision outcomes are conditioned on the opportunity to adjust or revise decisions in the future or game theory in which the outcomes are also conditioned on potential (conflicting) decisions made by other parties. But it is not the availability of theoretical insights that determines their application; it depends on the specific decision context at hand.

By restricting one thinking to a prechoosen set of problem characteristics, there is obviously more "to be seen" but at the same time it is possible to make observation errors, and maybe more worrisome, the problem and its context may be changing over time. This calls for the *Principle of Permanent Learning*, which stresses the process nature of decision making in which both the representation of the problem and the problem itself can change over time. Therefore, there is a permanent

[1]The underlying assumptions must be validated and the effectiveness and efficiency of the actions taken must be evaluated systematically. The latter calls for a sophisticated performance evaluation process that explicitly acknowledges the role of learning.

need to critically evaluate the problem formulation, the decisions made and their performance. Obviously, decision making and performance evaluation are two key elements in the decision-making process. As argued in Spronk and Vermeulen [178], performance evaluation of decisions should be structured such that the original idiosyncrasies of the problem (i.e., at the time the decision is made) are fully taken into account at the moment of evaluation, (i.e., *ex post*). By doing so, one increases the chance of learning from errors and misspecifications in the past.

24.2.6.2 Allocation Decisions

Financial decisions are allocation decisions, in which both time and uncertainty (and thus risk) play a crucial role. In order to support decisions in finance, both the preferences of the decision maker and the characteristics of the choice alternatives should be adequately understood and related to each other. A distinction can be made between "pure" financial decisions in which cash flows and market values steer the decision and "mixed" financial decisions in which other criteria are also considered. In financial theory, financial decisions are considered to be pure. In practice, most decisions are mixed. Hallerbach and Spronk [80] show that many financial decisions are mixed and thus should be treated as multiple criteria decision problems.

The solution of pure financial decisions requires the analysis, valuation, and management of risky cash flow streams and risky assets. The solution of mixed financial problems involves, in addition, the analysis of other effects. This implies that, in order to describe the effects of mixed decisions, multi-dimensional impact profiles should be used (cf. Spronk and Hallerbach [177]). The use of multi-dimensional impact profiles naturally opens the door to MCDA. Another distinction that can be made is between the financial decisions of individuals on one hand the financial decisions of companies and institutions on the other. The reason for the distinction results from the different ways in which decision makers steer the solutions. Individual decisions are guided by individual preferences (e.g., as described by utility functions), whereas the decisions of corporations and institutions are often guided by some aggregate objective (e.g., maximization of market value).

24.2.6.3 Uncertainty and Risk[2]

In each of the types of financial decisions just described, the effects are distributed over future time periods and are uncertain. In order to evaluate these possible effects, available information should be used to develop a "picture" of these effects and their likelihood. In some settings there is complete information but more often information is incomplete. In our framework, we use multi-dimensional risk profiles

[2]This section draws heavily on a part of Hallerbach and Spronk [79].

for modelling uncertainty and risk. This is another reason why multicriteria decision analysis is opportune when solving financial decision problems. *Two questions* play a crucial role:

1. Where does the uncertainty stem from or, in other words, what are the sources of risk?
2. When and how can this uncertainty be changed?

The answer to the first question leads to the decomposition of uncertainty. This involves attributing the inherent risk (potential variability in the outcomes) to the variability in several underlying state variables or factors. We can thus view the outcomes as being *generated* by the factors. Conversely, the stochastic outcomes are *conditioned* on these factors. The degree in which fluctuations in the factors propagate into fluctuations in the outcomes can be measured by response coefficients. These sensitivity coefficients can then be interpreted as exposures to the underlying risk factors and together they constitute the multi-dimensional risk profile of a decision alternative.

The answer to the second question leads to three prototypes of decision problems:

(1) The decision maker makes and implements a final decision and waits for its outcome. This outcome will depend on the evolution of external factors, beyond the decision maker's control.
(2) The decision maker makes and implements a decision and observes the evolution of external factors (which are still beyond the decision maker's control). However, depending on the value of these factors, the decision maker may make and implement additional decisions. For example, a decision maker may decide to produce some amount of a new and spectacular software package and then, depending on market reaction, he may decide to stop, decrease, or increase production.
(3) As in (2), but the decision maker is not the sole player and thus has to take account of the potential impact of decisions made by others sometime in the future (where the other(s) are of course confronted with a similar type of decision problem). The interaction between the various players in the field gives rise to dynamic game situations.

24.2.6.4 A Bird's-Eye View of the Framework

In Fig. 24.3, a bird's-eye view of the framework is presented. The framework integrates several elements in a process-oriented approach towards financial decisions. The left side of Fig. 24.3 represents the elements that lead to decisions, represented by the Resolution/Conclusion box at the lower left hand side. As mentioned above, performance evaluation (shown at the lower right hand side of the figure) is an integral part of the decision-making process. However, in this article we do not pay further attention to performance evaluation or to the feedback leading from performance evaluation to other elements of the decision-making process.

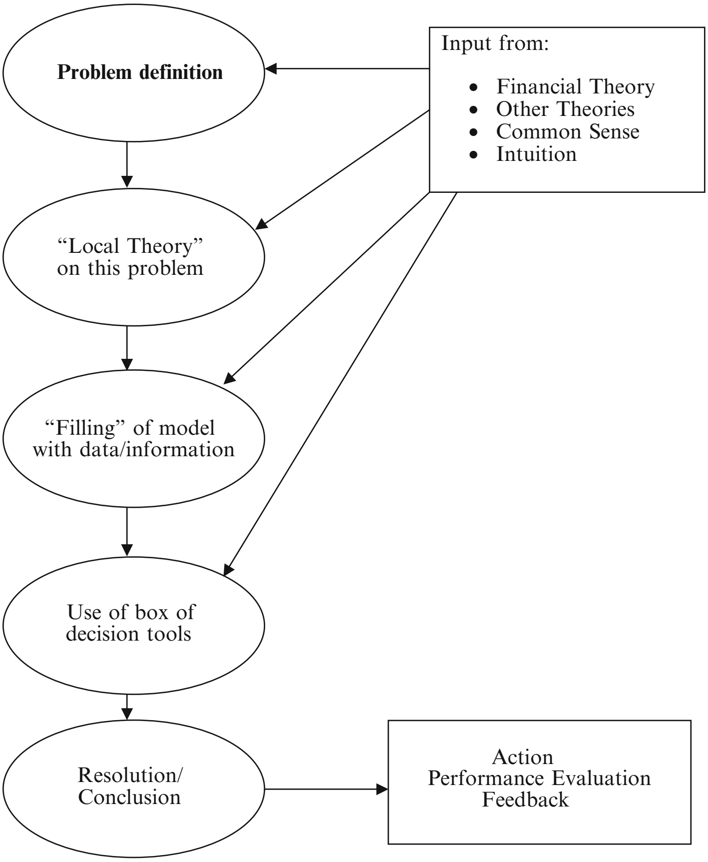


Fig. 24.3 A bird's-eye view of the framework

Financial decision problems will often be put as allocation problems. At this stage, it is important to determine whether the problem is a mixed or pure financial problem. Also, one should know who decides and which objectives are to be served by the decisions.

In the next step, the problem is defined more precisely. Many factors play a role here. For instance, the degree of upfront structure in the problem definition, the similarity with other problems, time and commitment from the decision makers, availability of time, similarity to problems known in theory and so on. In this stage, the insights from financial theory often have to be supplemented (or even amended) by insights from other disciplines and by the discipline of common sense. The problem formulation can thus be seen as a theoretical description (we use the label "local theory") of the problem.

After the problem formulation, data have to be collected, evaluated and sometimes transformed into estimates. These data are then used as inputs for the formalization of the problem description. The structure of the problem, together with the quality and availability of the data determines what tools can be used and in which way. As explained above, the use of multi-dimensional impact profiles almost naturally leads to the use of multicriteria decision analysis.

24.2.6.5 The Framework and Modern Financial Theory

In our framework we try to borrow all concepts and insights from modern financial theory that help to make better financial decisions. Financial theory provides rich and powerful tools for describing uncertainty and risk. Examples are the multi-factor representation of risk, which leads to multi-dimensional impact profiles that can be integrated within multicriteria decision analysis. A very important contribution of financial theory is the contingent claims approach in which the decision outcomes are conditioned on the opportunity to adjust or revise decisions in the future. This comes together with financial markets where contingent claims are being traded in volume. This brings us to the role of financial markets as instruments to trade risks, to redistribute risks, and even to decrease or eliminate risk. We believe and hope that contingent claims thinking will also be used in other domains than finance. In the first place because of what it adds when describing decision problems. Secondly, new markets may emerge in which also non-financial risks can be handled in a better way.

In addition to helping to better describe decision problems, financial theory provides a number of crucial insights. The most obvious (which is clearly not limited to financial economics) is probably the concept of "best alternative opportunity" thinking. Whenever making an evaluation of decision alternatives, one should take into account that the decision maker may have alternative opportunities (often but not exclusively provided by markets), the best of which sets a benchmark for the evaluation of the decision alternatives considered.

Other concepts are the efficient market hypothesis and the no-arbitrage condition. These point both to the fact that in competitive environments, it is not obvious that one can outsmart all the others. So if you find ways to make easy money, you should at least try to answer the question why you have been so lucky and how the environment will react.

24.3 MCDA in Portfolio Decision-Making Theory

We now turn our attention to the area of finance known as portfolio theory. In portfolio theory, we study the attributes of collections of securities called portfolios and how investors make judgements based upon these attributes. The problem that characterizes this area is the problem of portfolio selection.

Formulated as an optimization problem, this problem has been studied extensively. Thousands of papers have been written on it. A feel for many of these papers can be gained by scanning the references contained in Elton et al. [63]. As far as mainstream finance is concerned, the problem is only two-dimensional, able to address only tradeoffs between risk (typically measured by standard deviation) and return. To more realistically model the problem and be better prepared for a future which will only be more complicated, we now discuss issues involved in generalizing portfolio selection to include criteria beyond standard deviation and return, such as liquidity, dividend yield, sustainability, and so forth. See, for example, Lo et al. [123], Ehrgott et al. [62], Ben Abdelaziz et al. [1], Ballestero et al. [13], and Xidonas at al. [192]. In this way, MCDA in the form of multiple criteria optimization enters the picture. While the word “multiple” includes two, we will generally use it for more than two. We now explore the possibilities of multiple objectives in portfolio selection and discuss the effects of recognizing multiple criteria on the traditional assumptions and practice of portfolio selection in finance.

For this, we are organized as follows. In Sect. 24.3.1 we introduce the risk-return problem of portfolio selection, and in Sect. 24.3.2 we demonstrate the problem in a multiple criteria optimization framework. In Sect. 24.3.3 we discuss two variants of the portfolio selection model, and in Sect. 24.3.4 we discuss the bullet-shaped feasible regions that so often accompany portfolio selection problems. In the context of some key assumptions, in Sect. 24.3.5 we discuss the sensitivity of the nondominated set to changes in various factors, and in Sect. 24.3.6 we update the assumptions in accordance with the indicated presence of additional criteria. In Sect. 24.3.7 we talk about how to deal with resulting nondominated surfaces, and in Sect. 24.3.8 we report on the idea that the “modern portfolio analysis” of today can be viewed as the projection onto the risk-return plane of the real multiple criteria portfolio selection problem in higher dimensional space. In Sect. 24.3.9 we comment on future directions.

24.3.1 Portfolio Selection Problem

In finance, due to Markowitz [131–133], we have the canonical problem of portfolio selection as follows. Assume

(a) *n* securities
(b) a sum of money to be invested
(c) beginning of a holding period
(d) end of a holding period.

Let $\mathbf{x} = (x_1, \ldots, x_n)$ be a *portfolio* where the x_i are weights that specify the proportions of the sum to be invested in the different securities at the beginning of the holding period. For security i, let r_i be the random variable for the percent return realized on security i between the beginning of the holding period and the end of the holding period. Then for r_p, the random variable for the percent return realized on a portfolio between the beginning of the holding period and the end, we have

$$r_p = \sum_{i=1}^{n} r_i x_i$$

Unfortunately, it is not possible to know at the beginning of the holding period the value to be achieved by r_p at the end of the holding period. However, it is assumed that at the beginning of the holding period we have in our possession all expected values $E\{r_i\}$, variances σ_{ii}, and covariances σ_{ij} for the n securities.

Since r_p is not deterministic and an investor would presumably wish to protect against low values of r_p from turning out to be the case, the approach considered prudent in portfolio selection is to seek a portfolio solution that produces a high expected value of r_p and a low standard deviation of r_p. Using the $E\{r_i\}$, σ_{ii} and σ_{ij}, the expected value of r_p is given by

$$E\{r_p\} = \sum_{i=1}^{n} E\{r_i\} x_i \tag{24.1}$$

and the standard deviation of r_p is given by

$$\sigma\{r_p\} = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} x_i \sigma_{ij} x_j} \tag{24.2}$$

As for constraints, there is always the full investment constraint

$$\sum_{i=1}^{n} x_i = 1 \tag{24.3}$$

Depending on the version of the problem, there may be additional constraints such as

$$\ell_i \leq x_i \leq \mu_i \text{ for all } i \tag{24.4}$$

which are very common.

The way (24.1)–(24.4) is solved is as follows. First compute the set of all of the model's "nondominated" combinations of expected return and standard deviation. Then, after examining the set, which portrays as a non-negatively sloped concave curve, the investor selects the nondominated combination that he or she feels strikes the best balance between expected return and standard deviation.

With $E\{r_p\}$ to be maximized and $\sigma\{r_p\}$ to be minimized, (24.1)–(24.4) is a multiple objective program. Although the power of multiple criteria optimization is generally not necessary with two-objective programs (they can often be addressed with single criterion techniques), the theory of multiple criteria optimization, however, is necessary when wishing to generalize portfolio selection, as we do, to take into account additional criteria.

24.3.2 Background on Multicriteria Optimization

In multiple criteria optimization, to handle both maximization and minimization objectives, we have

$$
\begin{aligned}
&\max \textit{ or } \min \{f_1(\mathbf{x}) = z_1\} \\
&\qquad \vdots \\
&\max \textit{ or } \min \{f_k(\mathbf{x}) = z_k\} \\
&\qquad \text{s.t.} \qquad \mathbf{x} \in S
\end{aligned}
\tag{MC}
$$

where $\mathbf{x} \in \mathbb{R}^n$, k is the number of objectives, and z_i is a *criterion value*. In multiple criteria optimization, we have the feasible region in two different spaces. One is $S \subset \mathbb{R}^n$ in *decision space* and the other is $Z \subset \mathbb{R}^k$ in *criterion space*. Let $\mathbf{z} \in \mathbb{R}^k$. Then *criterion vector* $\mathbf{z} \in Z$ if and only if there exists an $\mathbf{x} \in S$ such that $\mathbf{z} = (f_1(\mathbf{x}), \ldots, f_k(\mathbf{x}))$. In this way, Z is the set of all *images* of the $\mathbf{x} \in S$.

Criterion vectors in Z are either *nondominated* or *dominated*, and points in S are either *efficient* or *inefficient*. Let $J^+ = \{ i \mid f_i(\mathbf{x}) \text{ is to be maximized}\}$ and $J^- = \{j \mid f_j(\mathbf{x}) \text{ is to be minimized}\}$. Then we have

Definition 1. Assume (MC). Then $\bar{\mathbf{z}} \in Z$ is a *nondominated* criterion vector if and only if there does not exist another $\mathbf{z} \in Z$ such that (i) $z_i \geq \bar{z}_i$ for all $i \in J^+$, and $z_j \leq \bar{z}_j$ for all $j \in J^-$, and (ii) $z_i > \bar{z}_i$ or $z_j < \bar{z}_j$ for at least one $i \in J^+$ or $j \in J^-$. Otherwise, $\bar{\mathbf{z}} \in Z$ is *dominated*.

The set of all nondominated criterion vectors is designated N and is called the *nondominated set*.

Definition 2. Let $\bar{\mathbf{x}} \in S$. Then $\bar{\mathbf{x}}$ is *efficient* in (MC) if and only if its criterion vector $\bar{\mathbf{z}} = (f_1(\bar{\mathbf{x}}), \ldots, f_k(\bar{\mathbf{x}}))$ is nondominated, that is, if and only if $\bar{\mathbf{z}} \in N$. Otherwise, $\bar{\mathbf{x}}$ is *inefficient*.

The set of all efficient points is designated E and is called the *efficient set*. Note the distinction with regard to terminology. While nondominance is a criterion space concept, in multiple criteria optimization, efficiency is only a decision space concept.

To define optimality in a multiple criteria optimization problem, let $U\colon \mathbb{R}^k \to \mathbb{R}$ be the decision maker's utility function. Then, any $\mathbf{z}^o \in Z$ that maximizes U over Z is an *optimal criterion vector*, and any $\mathbf{x}^o \in S$ such that $(f_1(\mathbf{x}^o), \ldots, f_k(\mathbf{x}^o)) = \mathbf{z}^o$ is an *optimal solution*. We are interested in the efficient and nondominated sets because if U is such that *more-is-better-than-less* for each z_i, $i \in J^+$, and *less-is-better-than-more* for each z_j, $j \in J^-$, then any $\mathbf{z}^o$ optimal criterion vector is such that $\mathbf{z}^o \in N$, and any feasible *inverse image* $\mathbf{x}^o$ is such that $\mathbf{x}^o \in E$. The significance of this is that to find an optimal criterion vector $\mathbf{z}^o$, it is only necessary to find a best point in N. After a $\mathbf{z}^o$ has been found, it is only necessary to obtain an $\mathbf{x}^o \in S$ inverse image to know what to implement to achieve the k simultaneous performances specified by the values in $\mathbf{z}^o$.

Although N in portfolio selection is a portion of the surface of $Z \in \mathbb{R}^k$, locating the best solution in N, when $k > 2$, is generally a non-trivial task because of the size of N. As a result, a large part of the field of multiple criteria optimization is concerned with procedures for computing or sampling N to locate an optimal or *near-optimal* solution, where a near-optimal solution is close enough to being optimal to terminate the decision process.

Within this framework, (24.1)–(24.4) can now be expressed in the form of a bi-objective multiple criteria optimization problem

$$\min\{\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} x_i \sigma_{ij} x_j} = z_1\} \tag{MC-O}$$

$$\max\{\sum_{i=1}^{n} E\{r_i\} x_i = z_2\}$$

$$\text{s.t.} \quad \sum_{i=1}^{n} x_i = 1$$

$$\ell_i \leq x_i \leq \mu_i \text{ for all } i$$

24.3.3 Two Model Variants

Two model variants of (24.1)–(24.4) have evolved as classics. One is the *unrestricted* model

$$\min\{\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} x_i \sigma_{ij} x_j} = z_1\} \tag{MC-U}$$

$$\max\{\sum_{i=1}^{n} E\{r_i\} x_i = z_2\}$$

$$\text{s.t.} \quad \sum_{i=1}^{n} x_i = 1$$

$$\text{all } x_i \text{ unrestricted}$$

meaning that there are no constraints beyond the full investment constraint in the model. The other is the *variable-restricted* model

$$\min\{\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} x_i\sigma_{ij}x_j} = z_1\} \tag{MC-B}$$

$$\max\{\sum_{i=1}^{n} E\{r_i\}x_i = z_2\}$$

$$\text{s.t.} \quad \sum_{i=1}^{n} x_i = 1$$

$$\ell_i \leq x_i \leq \mu_i \text{ for all } i$$

in which lower and upper bounds exist on the x_i. In the unrestricted model there are no lower limits on the weights, meaning that *unlimited* short selling is permitted. To illustrate, let $x_3 = -0.2$. This says the following to an investor. Borrow a position in security 3 to the extent of 20 % of the initial sum to be invested and then sell. With the extra 20 % and the initial sum, invest it in accordance with the other x_i.

The unrestricted model is a favorite in teaching because of its elegant mathematical properties. For example, as long as the covariance matrix

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & & \\ \vdots & & & \vdots \\ \sigma_{n1} & & \cdots & \sigma_{nn} \end{bmatrix}$$

is nonsingular, every imaginable piece of information about the model appears to be analytically derivable in closed form (for instance see Roll [157]).

The variable-restricted model, despite requiring mathematical programming (typically some form of quadratic programming) because of the extra constraints, is the favorite in practice. For instance, in the US, short selling is prohibited by law in the $11 trillion mutual fund business. It is also prohibited in the management of pension assets. And even in hedge funds where short selling is almost standard, it is all but impossible to imagine any situation in which there wouldn't be limits. A question is, when trying to locate an optimal solution, how much difference might there be between the two models?

24.3.4 Bullet-Shaped Feasible Regions

When looking through the portfolio chapters of almost any university investments text, it is hard to miss seeing graphs of bullet-shaped regions, often with dots in them, with standard deviation on the horizontal axis and expected return on the

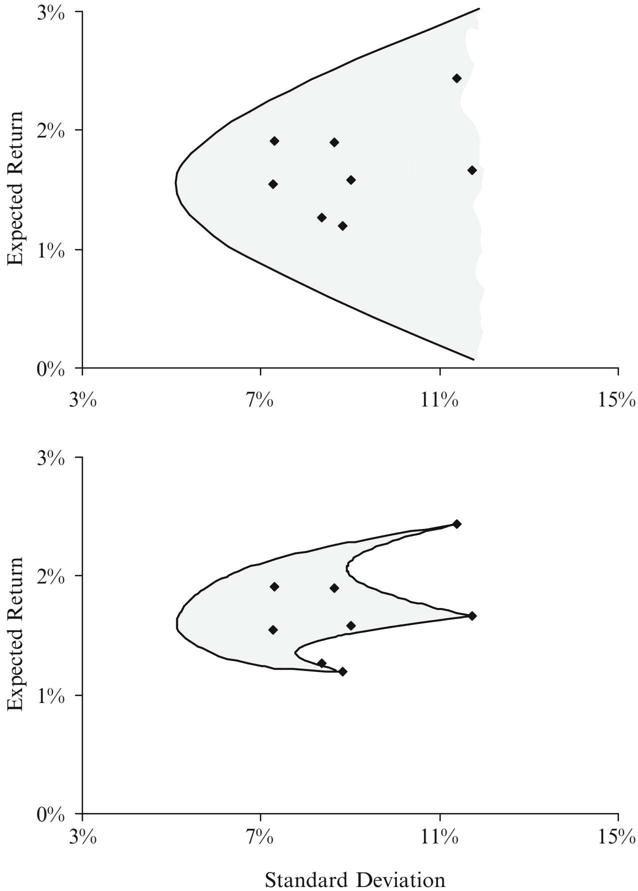


Fig. 24.4 Feasible regions Z of (MC-U) and (MC-B) for the same eight securities

vertical. When unbounded as in Fig. 24.4 (*top*), these are graphs of the feasible region Z of (MC-U) in criterion space. When bounded as in Fig. 24.4 (*bottom*), these are graphs of the feasible region Z of (MC-B) in criterion space. The dots are typically the criterion vectors $(\sigma\{r_i\}, E\{r_i\})$ of individual securities.

To see why a feasible region Z of (MC-U) is bullet-shaped and unbounded, consider securities A and B in Fig. 24.5. The unbounded line sweeping through A and B, which is a hyperbola, is the set of criterion vectors of all two-stock portfolios resulting from all linear combinations of A and B whose weights sum to one. In detail, all points on the hyperbola strictly between A and B correspond to weights $x_a > 0$ and $x_b > 0$; all points on the hyperbola above and to the right of A correspond to weights $x_a > 1$ and $x_b < 0$; and all points on the hyperbola below and to the right of B correspond to weights $x_a < 0$ and $x_b > 1$. The degree of "bow" toward the vertical axis of the hyperbola is a function of the correlation coefficient ρ_{ab} between

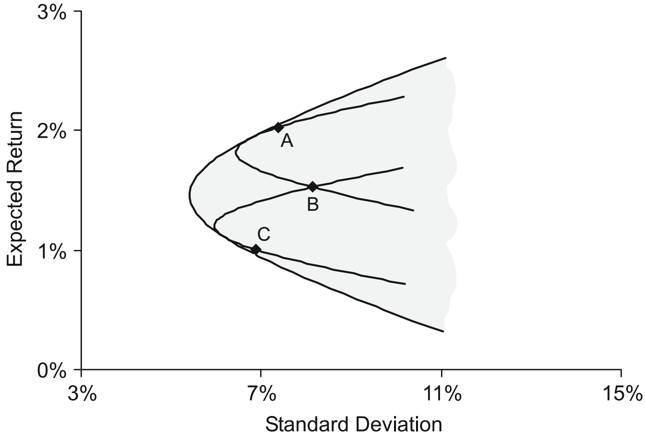


Fig. 24.5 Unbounded bullet-shaped feasible region Z created by securities A, B and C

A and B. This is seen by looking at the components of the $(\sigma\{r_{ab}\}, E\{r_{ab}\})$ criterion vector of any two-stock portfolio which are given by

$$\sigma\{r_{ab}\} = \sqrt{\sigma_{aa}x_a^2 + 2\rho_{ab}\sigma_a\sigma_b x_a x_b + \sigma_{bb}x_b^2}$$

and

$$E\{r_{ab}\} = E\{r_a\}x_a + E\{r_b\}x_b$$

in which $\sigma\{r_a\} = \sqrt{\sigma_{aa}}$ and $\sigma\{r_b\} = \sqrt{\sigma_{bb}}$.

Through B and C in Fig. 24.5 there is another hyperbola. Since through any point on the hyperbola through A and B and any point on the hyperbola through B and C there is yet another hyperbola, feasible region Z fills in and takes on its bullet shape whose leftmost boundary is, in the case of (MC-U), a single hyperbola.

With regard to the feasible region Z of (MC-B), the hyperbolic lines through the criterion vectors of any two financial products are not unbounded. In every case, they end at some point because of the bounds on the variables. While still filling in to create a bullet-shaped Z, the leftmost boundary, instead of being formed by a single hyperbola, is in general formed by segments from several hyperbolas. The rightmost boundary, instead of being unbounded, takes on a “scalloped” effect as in Fig. 24.4 (*bottom*).

Because standard deviation is to be minimized and expected return is to be maximized, we look to the “northwest” of Z for the nondominated set. This causes the nondominated set to be the upper portion of the leftmost boundary (the portion that is non-negatively sloped). In finance, it is called the "efficient frontier." Here, because of our interests in portfolio analysis with multiple criteria, we prefer to call it the "nondominated frontier."

24.3.5 Assumptions and Nondominated Sensitivities

The assumptions surrounding the use of (MC-U) and (MC-B) and theories based upon them in finance are largely as follows.

(a) There are many investors, each small, none of which can affect prices.
(b) There are no taxes.
(c) There are no transactions costs.
(d) Each investor's asset universe is all publicly traded securities.
(e) All investors are rational mean-variance optimizers.
(f) All investors have utility functions whose indifference curves are convex-to-the-origin.
(g) All investors share the same expected returns, variances, and covariances about the future. This is called *homogeneous expectations*.
(h) All investors have the same single holding period.
(i) Each security is infinitely divisible.

We now discuss the sensitivity of the nondominated frontier to factors that have implications about the appropriateness of this set of the assumptions. Sensitivity is measured by noting what happens to the nondominated frontier as the parameter associated with a given factor changes. We start by looking at the sensitivity of the nondominated frontier to changes in an upper bound common to all investment proportion weights. Then we discuss the likely sensitivities of the nondominated frontier to changes in other things such as dividend yield, a liquidity measure, a social responsibility attribute, and so forth. The computer work required for testing such sensitivities is outlined in the following procedure.

1. Start the construction of what is recognized in multiple criteria optimization as an ϵ-constraint program by converting the expected return objective in (MC-U) and (MC-B) to a $\geq$ constraint with right-hand side ϵ.
2. Set the factor parameter to its starting value.
3. Set ϵ to its starting value.
4. Solve the ϵ-constraint program and take the square root of the outputted variance to form the nondominated point $(\sigma\{r_p\}, E\{r_p\})$.
5. If ϵ has reached its ending value, go to Step 6. Otherwise, increment ϵ and go to Step 4.
6. Connect on a graph all of the nondominated points obtained from the current value of the factor parameter to achieve a portrayal of the nondominated frontier of this factor parameter value. If the factor parameter has reached its ending value, stop. Otherwise increment the factor parameter and go to Step 3.

For the procedure, the ϵ-constraint program is

$$\min\{\sum_{i=1}^{n}\sum_{j=1}^{n} x_i\sigma_{ij}x_j = \sigma^2\{r_p\}\} \tag{Eps-1}$$

$$\text{s.t.} \quad \sum_{i=1}^{n} E\{r_i\}x_i \geq \epsilon$$

$$\sum_{i=1}^{n} x_i = 1$$

$$\ell_i \leq x_i \leq \mu \text{ for all } i$$

and to obtain results for the sensitivity of the nondominated frontier due to changes in the upper bound μ, let us consider a problem in which $n = 20$; $\ell = -0.05$ to permit mild short selling; and μ is set in turn to $1.00, 0.15, 0.10$ to generate three frontiers. Running 25 different ϵ values (experimenter's choice) for each μ-value, the three nondominated frontiers of Fig. 24.6 result. The topmost frontier is for $\mu = 1.00$, the middle frontier is for $\mu = 0.15$, and the bottommost frontier is for $\mu = 0.10$.

As seen in Fig. 24.6, the nondominated frontier undergoes major changes as we step through the three values of μ. Hence there is considerable sensitivity to the value of μ. Since, in the spirit of diversification, investors would presumably prefer smaller values of μ to larger values as long as portfolio performance is not seriously deteriorated in other respects, we can see that an examination of the tradeoffs among risk, return, and μ are involved before a final decision can be made. Since an investor would probably have no way of knowing in advance his or her optimal value of μ without reference to its effects on risk and return, μ conceivably could be a criterion to be optimized, too.

Using the same procedure, other experiments (results not shown) could be conducted. For example, if we wished to test the sensitivity of the nondominated frontier to changes in expected dividend yield, we would then work with the following ϵ-constraint program

$$\min\{\sum_{i=1}^{n}\sum_{j=1}^{n} x_i\sigma_{ij}x_j = \sigma^2\{r_p\}\} \tag{Eps-2}$$

$$\text{s.t.} \quad \sum_{i=1}^{n} E\{r_i\}x_i \geq \epsilon$$

$$\sum_{i=1}^{n} E\{d_i\}x_i \geq \delta$$

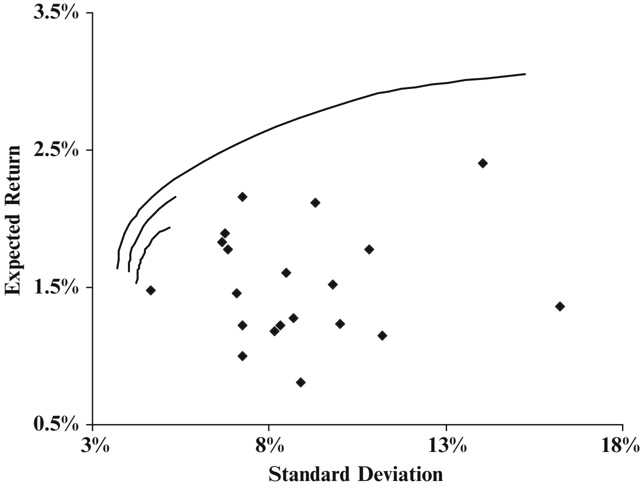


Fig. 24.6 Nondominated frontiers as a function of changes in the value of upper bound parameter μ

$$\sum_{i=1}^{n} x_i = 1$$

$$\ell_i \leq x_i \leq \mu_i \text{ for all } i$$

where d_i is the random variable for the dividend yield realized on security i between the beginning and end of the holding period and δ is the minimum dividend yield requirement parameter value to be changed in turn as μ was in (Eps-1) to test for different nondominated frontiers. A similar formulation could be set up for social responsibility.

For both dividend yield and social responsibility we can probably expect to see nondominated frontier sensitivities along the lines of that for μ. If this is indeed the case, this would signal that dividends and social responsibility could also be criteria. With μ, we now see how it is easy to have more criteria than two in investing. Whereas the assumptions at the beginning of this section assume a two-criterion world, we are led to see new things by virtue of these experiments. One is that the assumption about risk and return being the only criteria is certainly under seige. Another is that, in the company of μ, dividends, and social responsibility, the last of which can be highly subjective, *individualism* should be given more play. By individualism, no investor's criteria, opinions, or assessments need conform to those of another. In conflict with the assumption about homogeneous expectations, individualism allows an investor to have differing opinions about any security's expected return, risk profile, liquidity, dividend outlook, social responsibility quotient, and so forth. At the portfolio level, for example, individualism allows investors to possess different lists of criteria, have differing objective functions for even the same criteria, work from different asset universes, and enforce different attitudes about the nature of short selling. Therefore, with different lists of criteria, different objective functions, and different sets of constraints, all investors would not face the same feasible region with the same nondominated set. Each would have his or her own portfolio problem with its own optimal solution. The benefit of this enlarged outlook would be that portfolio theory would then not only have to focus on explaining equilibrium solutions, but on customized solutions as well.

24.3.6 Expanded Formulations and New Assumptions

Generally, in multiple criteria, we distinguish a constraint from an objective as follows. If when modelling we realize that we can not easily fix a right-hand side value without knowing how other output measures turn out, then we are probably looking at an objective. With this in mind, a list of possible extra objectives in portfolio selection could be

$$\max\ \{f_3(\mathbf{x}) = \text{dividend yield}\}$$
$$\min\ \{f_4(\mathbf{x}) = \text{maximum investment proportion weight}\}$$
$$\max\ \{f_5(\mathbf{x}) = \text{social responsibility}\}$$
$$\max\ \{f_6(\mathbf{x}) = \text{liquidity}\}$$
$$\max\ \{f_7(\mathbf{x}) = \text{momentum}\}$$
$$\max\ \{f_8(\mathbf{x}) = \text{investment in R\&D}\}$$

While one can imagine more exotic criteria, all of the above at least have the simplicity that they can be modelled linearly.

Updating to take a new look at portfolio selection, the following is proposed as a more appropriate set of assumptions with which to now approach the study of portfolio theory.

(a) There are many investors, each small, none of which can affect prices.
(b) There are no taxes.
(c) There are no transactions costs.
(d) An investor's asset universe can be any subset of all publicly traded securities.
(e) Investors may possess any mix of three or more objectives.
(f) All investors have utility functions whose indifference curves are convex-to-the-origin.

(g) Heterogeneity of expectations is the rule. That is, investors can have widely different forecasts and assessments about any security attribute including expected returns, variances, covariances, expected dividends, and so forth.
(h) Short selling is allowed but to only some limited extent.

The first three assumptions remain the same as they are nice to retain in that they establish benchmarks against which some of the world's imperfections can be measured. The assumption about convex-to-the-origin utility function contours is also retained as we see no compelling difficulty with it at the present time, but all the rest have either been modified or deleted.

24.3.7 Nondominated Surfaces

Let k be the number of criteria in a given portfolio selection model. Then the nondominated set of current-day finance that exists as a frontier in $\mathbb{R}^2$ is a *surface* in $\mathbb{R}^k$. The simplest case with a surface is with three criteria. The question is, how to solve? This is not a trivial question. Perhaps, to get a feel for the nondominated surface, the method that might first come to mind would be to solve repetitively the following ϵ-constraint program

$$\begin{aligned}
\min \{ \sum_{i=1}^{n} \sum_{j=1}^{n} x_i \sigma_{ij} x_j &= \sigma^2\{r_p\}\} \\
\text{s.t.} \qquad \sum_{i=1}^{n} E\{r_i\} x_i &\geq \epsilon_e \\
\sum_{i=1}^{n} E\{l_i\} x_i &\geq \epsilon_l \\
\sum_{i=1}^{n} x_i &= 1 \\
\ell_i \leq x_i \leq \mu_i & \text{ for all } i
\end{aligned}$$

where for sake of variety liquidity is the third criterion. We use ϵ_e and ϵ_l to distinguish between the ϵ's for expected return and liquidity. However, this approach involves many optimizations. If one might normally characterize a nondominated frontier with 50 points, up to a thousand points might be needed with a non-dominated surface to achieve about the same degree of representation density. Some references to help appreciate this might include Qi et al. [153], Şakar and Köksalan[164], and Mavrotas [137].

Instead of looking at the problem in ϵ-constraint terms, another approach is to look at it (since it contains three objectives) in tri-criterion form as follows

$$\min \{ \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} x_i \sigma_{ij} x_j} = z_1 \}$$

$$\max \{ \sum_{i=1}^{n} E\{r_i\} x_i = z_2 \}$$

$$\max \{ \sum_{i=1}^{n} E\{l_i\} x_i = z_3 \}$$

$$\text{s.t.} \quad \sum_{i=1}^{n} x_i = 1$$

$$\ell_i \leq x_i \leq \mu_i \text{ for all } i$$

and then try to compute the whole nondominated surface exactly by multi-parametric quadratic programming. Whereas a nondominated frontier was shown earlier to be piecewise hyperbolic, a nondominated surface is platelet-wise or patch-wise hyperboloidic. One can think of the back of a turtle. This is new material and, as of this writing, the first paper on this is by Hirschberger et al. [84].

24.3.8 Idea of a Projection

In traditional risk-return finance there is the "market portfolio". By theory, the market portfolio contains every security in proportion to its market capitalization, is anticipated to be somewhere in the midst of the nondominated frontier, and is supposed to be everyone's optimal portfolio when not including the risk-free asset. Since the market portfolio is impractical, indices like the S&P 500 are used as surrogates. But empirically, the surrogates, which should be essentially as desirable as the market portfolio, have always been found to be quite below the nondominated frontier, in fact so below that this cannot be explained by chance variation. Whereas this is an anomaly in conventional risk-return finance, this is exactly what we would expect in multiple criteria finance.

To take a glimpse at the logic why, consider the following. In a risk-return portfolio problem, let us assume that the feasible region Z is the ellipse in Fig. 24.7. Here, the nondominated frontier is the portion of the boundary of the ellipse in the second quadrant emanating from the center of the ellipse. Similarly, in a k-criterion portfolio problem (with $k-2$ objectives beyond risk and return), let us assume that the feasible region is an ellipsoid in k-space. Here, the nondominated surface is the portion of the surface of the ellipsoid in a similar orthant emanating from the center of the ellipsoid.

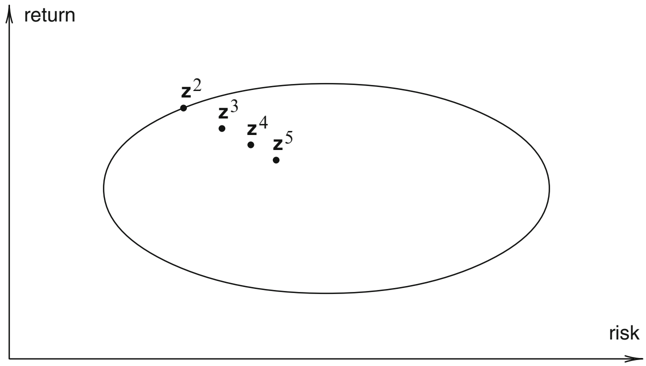


Fig. 24.7 An ellipsoidal feasible region projected onto two-dimensional risk-return space

Now assume that the market portfolio, which by theory is nondominated, is in the middle of the nondominated set. Then, when $k = 2$, the market portfolio would be at $\mathbf{z}^2$ on the ellipse. However, if (1) there is a third objective, (2) the feasible region is ellipsoidal in three-space, and (3) the market portfolio is in the middle of the nondominated surface in $\mathbb{R}^3$, then the market portfolio would project onto risk-return space at $\mathbf{z}^3$. If (1) there is a fourth objective, (2) the feasible region is ellipsoidal in four-space, and (3) the market portfolio is in the middle of the nondominated surface in $\mathbb{R}^4$, then the market portfolio would project onto risk-return space at $\mathbf{z}^4$. With five objectives under the same conditions, the market portfolio would project onto risk-return space at $\mathbf{z}^5$, and so forth, becoming deeper and deeper.

Consequently, it can be viewed that the "modern portfolio theory" of today is only a first-order approximation—a projection onto the risk-return plane of the real multiple criteria problem from higher dimensional criterion space.

24.3.9 Further Research in MCDA in Portfolio Analysis

So far we have only talked about extending the canonical model in the direction of multiple criteria. In addition to multiple criteria, we also find intriguing for future research the areas of special variable treatments and alternative risk measures. By special variable treatments, we mean conditions on the variables such as the following:

(a) No fewer than a given number of securities, and no more than a given number of securities, can be in a portfolio (either long or short).
(b) No more than a given number of securities can be sold short.
(c) If a stock is in a portfolio, then its weight must be in market cap proportion to the weights of all other stocks in the portfolio.
(d) No more than a given proportion of a portfolio can be involved in stocks sold short.

(e) Some or all of the x_i are semi-continuous. When an x_i is semi-continuous, x_i is either zero or in a given interval $[a, b], a > 0$.
(f) No more than a given number of stocks may have a given upper bound. For instance, at most one stock (but which one is not known beforehand) may constitute as much as 10 % of a portfolio, with all other stocks having an upper bound of 5 %.

While some of these can be modelled with auxiliary 0–1 variables as in Xidonas and Mavrotas [190], others may be best approached by evolutionary-style procedures as in Anagnostopoulos and Mamanis [4]. Having at one's disposal well-researched methods for dealing with special variable treatments would extend the power of our new look at portfolio analysis.

By alternative risk measures, we are thinking of measures like mean absolute deviation (MAD) as broached by Konno and Yamazaki [116] and conditional value at risk (CVaR) as integrated into a financial study such as by Şakar and Köksalan [164]. Finally, it may be that multiple criteria and behavioral finance (see Shefrin [168]) reinforce one another as both areas see more going on in investing than the traditional.

24.4 MCDA in Discrete Financial Decision-Making Problems

Several financial decision-making problems require the evaluation of a finite set of alternatives $A = \{a_1, a_2, \ldots, a_m\}$, which may involve firms, investment projects, stocks, credit applications, etc. These types of problems are referred to as "discrete" problems. The outcome of the evaluation process may have different forms, which are referred to as "problematics" [162]: (1) problematic α: Choosing one or more alternatives, (2) problematic β: Sorting the alternatives in pre-defined ordered categories, (3) problematic γ: Ranking the alternatives from the best to the worst ones, and (4) problematic δ: Describing the alternatives in terms of their performance on the criteria. The selection of an investment project is a typical example of a financial decision-making problem where problematic α (choice) is applicable. The prediction of business failure is an example of problematic β (classification of firms as healthy or failed), the comparative evaluation and ranking of stocks according to their financial and stock market performance is an example of problematic γ, whereas the description of the financial characteristics of a set of firms is a good example of problematic δ.

In all cases, the evaluation process involves the aggregation of all decision criteria $F = \{g_1, g_2, \ldots, g_n\}$. The aggregation process can be performed in many different ways depending on the form of the criteria aggregation model. Three main forms of aggregation models can be distinguished: (1) outranking relations (relational form), (2) utility functions (functional form), (3) decision rules (symbolic form). In order to make sure that the aggregation model is developed in accordance to the decision maker's judgment policy, some preferential information must be specified, such as the relative importance of the criteria. This information can be obtained either through direct procedures in which a decision analyst elicits it directly from the decision maker, or through indirect procedures in which the decision maker provides representative decision examples, which are used to infer the preferential parameters consistent with the decision maker's global evaluations. The latter approach is known in the MCDA field as "preference disaggregation analysis" [99, 100].

The subsequent subsections in this part of the chapter present several MCDA discrete evaluation approaches which are suitable for addressing financial decision-making problems. The presentation is organized in terms of the criteria aggregation model employed by each approach (outranking relations, utility functions, decision rules).

24.4.1 Outranking Relations

The foundations of the outranking relations theory have been set by Bernard Roy during the late 1960s through the development of the ELECTRE family of methods (**EL**imination **E**t **C**hoix **T**raduisant la **RE**alité; [160]). Since then, they have been widely used by MCDA researchers in several problem contexts.

An outranking relation is a binary relation that enables the decision maker to assess the strength of the outranking character of an alternative a_i over an alternative a_j. This strength increases if there are enough arguments (coalition of the criteria) to confirm that a_i is at least as good as a_j, while there is no strong evidence to refuse this statement.

Outranking relations techniques operate into two stages. The first stage involves the development of an outranking relation among the considered alternatives, while the second stage involves the exploitation of the developed outranking relation to choose the best alternatives (problematic α), to sort them into homogenous groups (problematic β), or to rank them from the most to the least preferred ones (problematic γ).

Some of the most widely known outranking relations methods include the family of the ELECTRE methods [161] and the family of the PROMETHEE methods [28]. These methods are briefly discussed below. A detailed presentation of all outranking methods can be found in the books of Roy and Bouyssou [163] and Vincke [186].

ELECTRE Methods The family of ELECTRE methods was initially introduced by Roy [160], through the development of the ELECTRE I method, the first method to employ the outranking relation concept. Since then, several extensions have been proposed, including ELECTRE II, III, IV, IS and TRI [161]. These methods address different types of problems, including choice (ELECTRE I, IS), ranking (ELECTRE II, III, IV) and sorting/classification (ELECTRE TRI).

Given a set of alternatives $A = \{a_1, a_2, \ldots a_m\}$ any of the above ELECTRE methods can be employed depending on the objective of the analysis (choice, ranking, sorting/classification). Despite their differences, all the ELECTRE methods are based on the identification of the strength of affirmations of the form Q ="alternative a_i is at least as good as alternative a_j". The specification of this strength requires the consideration of the arguments that support Q as well as the consideration of the arguments that are against it. The strength of the arguments that support Q is analyzed through the "concordance test". The measure used to assess this strength is the global concordance index $C(a_i, a_j) \in [0, 1]$. The closer is C to unity, the higher is the strength of the arguments that support the affirmation Q. The concordance index is estimated as the weighted average of partial concordance indices defined for each criterion:

$$C(a_i, a_j) = \sum_{k=1}^{n} w_k c_k(g_{ik} - g_{jk})$$

where w_k is the weight of criterion g_k ($\sum w_k = 1, w_k \geq 0$) and $c_k(g_{ik} - g_{jk})$ is the partial concordance index defined as a function of the difference $g_{ik} - g_{jk}$ between the performance of a_i and a_j on criterion g_k. The partial concordance index measures the strength of the affirmation Q_k ="a_i is at least as good as a_j on the basis of criterion g_k". The partial index is normalized in the interval [0, 1], with values close to 1 indicating that Q_k is true and values close to 0 indicating that Q_k is false.

Except for assessing the strength of the arguments that support the affirmation Q, the strength of the arguments against Q is also assessed. This is performed through the "discordance test", which leads to the calculation of the discordance index $D_k(g_{ik} - g_{jk})$ for each criterion g_k. The higher is the discordance index the more significant is the opposition of a criterion on the validity of Q.

The concordance C and the discordance indices D_k are combined to construct the final outranking relation. The way that this combination is performed, as well as the way that the results are employed to choose, rank, or sort the alternatives depends on the specific ELECTRE method that is used. Details on these issues can be found in the works of Roy [161, 162] as well as in the book of Roy and Bouyssou [163].

PROMETHEE Methods The development of the PROMETHEE family of methods (**P**reference **R**anking **O**rganization **METH**hod of **E**nrichment **E**valuations) began in the mid 1980s with the work of Brans and Vincke [28] on the PROMETHEE I and II methods.

The PROMETHEE method leads to the development of an outranking relation that can be used to choose the best alternatives (PROMETHEE I) or to rank the alternatives from the most preferred to the least preferred ones (PROMETHEE II). For a given set of alternatives A, the evaluation process in PROMETHEE involves pairwise comparisons (a_i, a_j) to determine the preference index $\pi(a_i, a_j)$ measuring the degree of preference for a_i over a_j, as follows:

$$\pi(a_i, a_j) = \sum_{k=1}^{n} w_k P_k(g_{ik} - g_{jk}) \in [0, 1]$$

The higher is the preference index (closer to unity) the higher is the strength of the preference for a_i over a_j. The calculation of the preference index depends on the specification of the criteria weights w_k ($\sum w_k = 1$, $w_k \geq 0$) and the criteria preference function $P_1, \ldots, P_n$. The criteria preference functions are increasing functions of the difference $g_{ik} - g_{jk}$ between the performances of a_i and a_j on criterion g_k. The preference functions are normalized between 0 and 1, with higher values indicating stronger preference for a_i over a_j in terms of criterion g_k. Brans and Vincke [28] proposed six specific types of criteria preference functions (generalized criteria) which seem sufficient in practice.

On the basis of all pairwise comparisons for m alternatives, two overall performance measures can be defined. The first is the leaving flow $\phi^+(a_i) = \frac{1}{m-1}\sum_j \pi(a_i, a_j)$ which indicates the strength of preference for a_i over all other alternatives in A. In a similar way, the entering flow $\phi^-(a_i) = \frac{1}{m-1}\sum_j \pi(a_j, a_i)$ is also defined to measure the weaknesses of a_i compared to all other alternatives.

On the basis of these measures the procedures of PROMETHEE I and II are employed to rank the alternatives [28]. PROMETHEE I builds a partial ranking (with incomparabilities) through the combination of the rankings defined from the leaving and entering flows. On the other hand, PROMETHEE II provides a complete ranking on the basis of the net flow index $\phi(a_i) = \phi^+(a_i) - \phi^-(a_i)$, which constitutes an overall index of the performance of the alternatives.

24.4.2 Utility Functions-Based Approaches

Multiattribute utility theory (MAUT; [109]) extends the traditional utility theory to the multi-dimensional case. The objective of MAUT is to model and represent the decision maker's preferential system into a utility/value function $U(a_i)$. The utility function is defined on the criteria space, such that:

$$U(a_i) > U(a_j) \Leftrightarrow a_i \succ a_j \quad (a_i \text{ is preferred to } a_j) \tag{24.5}$$

$$U(a_i) = U(a_j) \Leftrightarrow a_i \sim a_j \quad (a_i \text{ is indifferent to } a_j) \tag{24.6}$$

The most commonly used form of utility function is the additive one:

$$U(a_i) = w_1 u_1(g_{i1}) + w_2 u_2(g_{i2}) + \ldots + w_n u_n(g_{in}) \tag{24.7}$$

where, $u_1, u_2, \ldots, u_n$ are the marginal utility functions corresponding the evaluation criteria. Each marginal utility function $u_k(g_k)$ defines the utility/value of the alternatives for each individual criterion g_k. The constants $w_1, w_2, \ldots, w_n$ represent the criteria trade-offs that the decision maker is willing to take.

A detailed description of the methodological framework underlying MAUT and its applications is presented in the book of Keeney and Raiffa [109].

Generally, the process for developing an additive utility function is based on the cooperation between the decision analyst and the decision maker. This process involves the specification of the criteria trade-offs and the form of the marginal utility functions. The specification of these parameters is performed through interactive procedures, such as the midpoint value technique [109]. The realization of such interactive procedures is often facilitated by the use of multicriteria decision support systems, such as the MACBETH system [16].

However, the implementation of such interactive procedures in practice can be cumbersome, mainly because it is rather time consuming and it depends on the willingness of the decision maker to provide the required information and the ability of the decision analyst to elicit it efficiently. The preference disaggregation approach of MCDA (PDA; [99, 100]) provides a methodological framework for coping with this problem. PDA refers to the analysis (disaggregation) of the global preferences (judgement policy) of the decision maker in order to identify the criteria aggregation model that underlies the preference result (ranking or classification/sorting). In PDA, the parameters of the decision model are estimated through the analysis of the decision maker's overall preference on some reference alternatives A', which may include either examples of past decisions or a small subset of the alternatives under consideration. The decision maker is asked to provide some examples regarding the evaluation of the reference alternatives according to his decision policy (global preferences). Then, using regression-based techniques the global preference model is estimated so that the decision maker's global evaluation is reproduced as consistently as possible by the model. A comprehensive bibliography on preference disaggregation methods can be found in Jacquet-Lagrèze and Siskos [99, 100], whereas some recent trends are discussed in [170].

PDA methods are particularly useful in addressing financial decision-making problems [203]. The repetitive character of financial decisions and the requirement for real-time decision support are two features of financial decisions which are consistent with the PDA framework. Thus, several PDA methods have been extensively used in addressing financial decision problems, mainly in cases where a ranking or sorting/classification of the alternatives is required. The following subsections provide a brief description of some representative PDA methods which have been used in financial problems.

UTA Method The UTA method (**UT**ilités **A**dditives; [98]) is an ordinal regression method developed to address ranking problems. The objective of the method is to develop an additive utility function which is as consistent as possible with the decision maker's judgment policy. The input to the method involves a pre-order of a set of reference alternatives A' . The developed utility model is assumed to be consistent with the decision maker's judgment policy if it is able to reproduce the given pre-order of the reference alternatives as consistently as possible.

In developing the utility model to meet this requirement, there are two types of possible errors which may occur [171]): (1) the under-estimation error when the developed model assigns a reference alternative to a lower (better) rank than the one specified in the given pre-order (the alternative is under-estimated by the decision maker), and (2) the over-estimation error when the developed model assigns a reference alternative to a higher (worse) rank than the one specified in the given pre-order (the alternative is over-estimated by the decision maker). The objective of the model development process is to minimize the sum of these errors. This is performed through linear programming techniques [98].

UTADIS Method The UTADIS method (**UT**ilités **A**dditives **DIS**criminantes; [54, 97]) is a variant of the UTA method, developed for classification problems. Similarly to the UTA method, the decision maker is asked to provide a classification of a set of reference alternatives A' into ordered categories $C_1, C_2, \ldots, C_q$ defined such that $C_1 \succ C_2 \succ \cdots \succ C_q$ (i.e., group C_1 includes the most preferred alternatives, whereas group C_q includes the least preferred ones). Within this context, the developed additive utility model will be consistent with the decision maker's global judgment if $t_k < U(a_i) < t_{k-1}$ for any alternative a_i that belongs in category C_k, where $t_0 = 1 > t_1 > t_2 > \ldots > t_{q-1} > 0 = t_q$ are thresholds that discriminate the groups. Similarly, to the UTA method, the under-estimation and over-estimation errors are also used in the UTADIS method to measure the differences between the model's results and the predefined classification of the reference alternatives. In this case, the two types of errors are defined as follows: (1) the under-estimation error $\sigma_i^+ = \max\{0, t_\ell - U(a_i)\}$, $\forall\, a_i \in C_\ell$, $\ell = 1, 2, \ldots, q-1$, (2) the over-estimation error $\sigma_i^- = \max\{0, U(a_i) - t_{\ell-1}\}$, $\forall\, a_i \in C_\ell$, $\ell = 2, 3, \ldots, q$. The additive utility model is developed to minimize these errors using a linear programming formulation [54].

Several variants of the original UTADIS method have been proposed (UTADIS I, II, III) to consider different optimality criteria during the development of the additive utility classification model [54, 205]. Other recent extensions can be found in [50, 55, 73, 74, 115].

MHDIS Method The MHDIS method (**M**ulti-group **H**ie-rarchical **DIS**crimination [209]) extends the PDA framework of the UTADIS method in complex sorting/classification problems involving multiple groups. MHDIS addresses sorting problems through a hierarchical procedure, in which the groups are distinguished progressively, starting by discriminating group C_1 (most preferred alternatives) from all the other groups $\{C_2, C_3, \ldots, C_q\}$, and then proceeding to the discrimination between the alternatives belonging to the other groups. At each stage of this sequential/hierarchical process two additive utility functions are developed for the classification of the alternatives. Assuming that the classification of the alternatives should be made into q ordered classes $C_1 \succ C_2 \succ \cdots \succ C_q$, $2(q-1)$ additive utility functions U_ℓ and $U_{\sim\ell}$ are developed. The function U_ℓ measures the utility for the decision maker of a decision to assign an alternative into group C_ℓ, whereas the second function $U_{\sim\ell}$ corresponds to the classification into the set of groups

$C_{\sim\ell} = \{C_{\ell+1}, C_{\ell+2}, \ldots, C_q\}$. The rules used to perform the classification of the alternatives are the following:

$$\left.\begin{array}{l}\text{If } U_1(a_i) > U_{\sim 1}(a_i) \text{ then } a_i \in C_1\\ \quad \text{Else if } U_2(a_i) > U_{\sim 2}(a_i) \text{ then } a_i \in C_2\\ \quad\quad \ldots\ldots\ldots\ldots\ldots\ldots\\ \quad\quad\quad \text{Else if } U_{q-1}(a_i) > U_{\sim(q-1)}(a_i) \text{ then } a_i \in C_{q-1}\\ \quad\quad\quad\quad \text{Else } a_i \in C_q\end{array}\right\} \tag{24.8}$$

The fitting of the decision model on the reference data is performed through a combination of linear and mix-integer programming formulation, which take into account the number of classification errors introduced by the model, as well as the robustness of the model's recommendations. A detailed description of the model optimization process in the MHDIS method can be found in Zopounidis and Doumpos [209].

24.4.3 Decision Rule Models: Rough Set Theory

Pawlak [151] introduced rough set theory as a tool to describe dependencies between attributes, to evaluate the significance of attributes and to deal with inconsistent data. The rough set approach assumes that every alternative is described by two types of attributes: condition and decision attributes. Condition attributes are those used to describe the characteristics of the alternatives (e.g., criteria), whereas the decision attributes define a one or multiple decision recommendations (usually expressed in a classification scheme). Alternatives that have the same description in terms of the condition attributes are considered to be indiscernible. The indiscernibility relation constitutes the main basis of the rough set theory. Any set of alternatives, which can be obtained through a union of some indiscernible alternatives is considered to be crisp otherwise it is a rough set. The existence of rough sets in a decision problem is due to imprecise, vague or inconsistent data. The rough set approach enables the identification of such cases, without requiring their elimination, which may actually lead to loss of useful information. Furthermore, it enables the discovery of important subsets of attributes as well as attributes that can be ignored without affecting the quality of the model's recommendations.

The rough set approach assumes a symbolic decision model expressed in the form of a set of "IF ... THEN ..." rules. Decision rules can be consistent if they include only one recommendation in their conclusion part, or approximate if their conclusion involves a disjunction of elementary decisions that describe rough sets.

This traditional framework of the rough set theory, has been extended towards the development of a new preference modelling framework within MCDA [71, 72]. The main novelty of the new rough set approach concerns the possibility of handling criteria, i.e., attributes with preference ordered domains, and preference ordered groups. Within this context the rough approximations are defined according to the dominance relation, instead of the indiscernibility relation used. The decision rules derived from these approximations constitute a preference model.

24.4.4 Applications in Financial Decisions

MCDA discrete evaluation methods are well suited for the study of several financial decision-making problems. The diversified nature of the factors (evaluation criteria) that affect financial decisions, the complexity of the financial, business and economic environments, the subjective nature of many financial decisions, are only some of the features of financial decisions which are in accordance with the MCDA modelling framework. This section reviews the up-to-date applications of MCDA discrete evaluation methods in some typical financial decision making contexts.

Bankruptcy and Credit Risk Assessment The assessment of bankruptcy and credit risk have been major research fields in finance for the last decades. The recent credit crisis that started from USA has highlighted once again the importance of these issues in a worldwide economic and business context. Bankruptcy risk is derived by the failure of a firm to meet its debt obligations to its creditors, thus leading the firm either to liquidation (discontinuity of the firm's operations) or to a reorganization program [204]. The concept of credit risk is similar to that of bankruptcy risk, in the sense that in both cases the likelihood that a debtor (firm, organization or individual) will not be able to meet its debt obligations to its creditors, is a key issue in the analysis. However, while bankruptcy is generally associated with legislative procedures, credit risk is a more general concept that takes into account any failure of a debtor to meet his/her debt obligations on the basis of a pre-specified payment schedule. In both bankruptcy and credit risk assessment, decision models are developed to classify firms or individuals into predefined groups (problematic β), e.g., classification of firms as bankrupt/non-bankrupt, or as high credit risk firms/low credit risk firms. Such models are widely used by financial institutions for credit granting decisions, loan pricing, credit portfolio risk analysis, and investment planning.

Statistical and econometric techniques (discriminant analysis, logit and probit analysis, etc.) have been widely used for developing bankruptcy prediction and credit risk models. Over the past couple of decades, however, new methodologies have attracted the interest of researchers and practitioners, including MCDA techniques [47, 146, 204].

Bankruptcy prediction and credit scoring models are fitted on historical default data. In that sense, the model construction process is mostly involved with the identification of powerful (statistical) patterns that explain past defaults and bankruptcies, which can also be used for handling future cases. However, there are a number of features that make MCDA methods particularly useful. First, every bankruptcy

Table 24.1 Applications of MCDA approaches in bankruptcy and credit risk assessment

Approaches	Methods	Studies
Multiattribute utility theory	AHP	[96, 179–181]
	MACBETH	[17]
Outranking relations	ELECTRE	[20, 46, 57, 88, 111]
	Other	[6, 89, 197]
Preference disaggregation	UTA	[198, 201]
	UTADIS	[51, 206–208]
	MHDIS	[52, 61, 148]
	Other	[31, 68, 76, 117, 121, 122, 180, 194]
Rough set theory		[32, 48, 70, 174, 176]

prediction and credit scoring model provides a risk rating, which is purely ordinal (e.g., the ratings of major rating agencies such as Moody's, Standard & Poor's, and Fitch). This is in accordance with the standard ordinal classification setting in MCDA. Furthermore, the attributes describing the performance and viability of corporate entities, organization, or individual clients (e.g., financial ratios) are not some arbitrary statistical predictor variables. Instead, their use in a prediction/decision model should be made in way that has clear economic and business relevance, not only in a general context, but also in the specific application setting of a particular country, region, business sector, or financial institution. Incorporating expert knowledge of senior credit risk analysts and policy makers into statistical models is not a straightforward process. On the other hand, MCDA methods provide this possibility, thus enhancing the model calibration process with information that is crucial for the successful use of the model in practice.

A representative list of the MCDA evaluation approaches applied in bankruptcy and credit risk assessment is presented in Table 24.1.

Portfolio Selection and Management Portfolio selection and management involves the construction of a portfolio of securities (stocks, bonds, treasury bills, mutual funds, etc.) that maximizes the investor's utility. This problem can be realized as a two stage process [92, 93, 192]: (1) the evaluation of the available assets to select the ones that best meet the investor's preferences, (2) specification of the amount of capital to be invested in each of the assets selected in the first stage. The implementation of these two stages in the traditional portfolio theory is based on the mean-variance approach introduced by Markowitz [131, 133].

Nevertheless, numerous studies have emphasized the multi-dimensional aspects of portfolio selection and management [29, 177, 183, 195]. Section 24.3 discussed this issue in a comprehensive manner in the context of portfolio optimization under multiple objectives. Except for the optimization phase, one could also consider the asset selection phase or even the process of selecting the most suitable capital allocation strategy among multiple Pareto efficient portfolios. The asset selection phase is most useful for large-scale portfolio problems with too many assets. In such

Table 24.2 Applications of MCDA approaches in portfolio selection and management

Approaches	Methods	Studies
Multiattribute utility theory	AHP	[77, 120, 165, 167]
	MACBETH	[14, 15, 124]
	Other	[3, 12, 36, 49, 64, 105, 155]
Outranking relations	ELECTRE	[91–93, 113, 135, 136, 184, 191]
	PROMETHEE	[2, 78, 112, 136]
	Other	[69, 85]
Preference disaggregation	UTA	[92, 93, 166, 200, 212]
	UTADIS	[10, 210, 214]
	MHDIS	[58]
Rough set theory		[106]

cases, investors often employ screening rules to select the assets that best suit their investment policy and have the best future growth prospects. Such rules are usually based on technical analysis and a careful examination of fundamental variables and factors. MCDA is well-suited in this context enabling the investor to combine multiple criteria related to the prospects of each investment option and its suitability to the investor's policy. The portfolio optimization process is then performed on a limited number of assets selected through a multicriteria evaluation and screening process. However, as demonstrated in Sect. 24.4, the optimization phase leads to a set of suitable portfolios (Pareto efficient portfolios), among which an investor must select the most suitable one. This can be achieved directly through the multiobjective optimization process, which may lead to a single efficient portfolio (the one that best meets the investor's policy), or through a multicriteria portfolio evaluation process implemented after a small number of representative efficient portfolios has been constructed. In the latter case, discrete MCDA methods can be employed to evaluate the performance of the selected efficient portfolios under multiple investment criteria.

Table 24.2 summarizes several studies involving the application of MCDA evaluations methods in portfolio selection and management, covering both the asset selection and the portfolio selection stages.

Corporate Performance Evaluation The evaluation of the performance of corporate entities and organizations is an important activity for their management and shareholders as well as for investors and policy makers. Such an evaluation provides the management and the shareholders with a tool to assess the strength and weakness of the firm as well as its competitive advantages over its competitors, thus providing guidance on the choice of the measures that need to be taken to overcome the existing problems. Investors (institutional and individual) are interested in the assessment of corporate performance for guidance to their investment decisions, while policy makers may use such an assessment to identify the existing problems in the business environment and take measures that will ensure a sustainable economic

Table 24.3 Applications of MCDA approaches in the assessment of corporate performance

Approaches	Methods	Studies
Multiattribute utility theory	AHP	[9, 11, 119, 142]
	Other methods	[45, 67, 193]
Outranking relations	ELECTRE	[24, 37, 66, 94]
	PROMETHEE	[11, 18, 37, 56, 108, 126–128]
		[129, 147, 196]
Preference disaggregation	UTA	[75, 173, 211, 213]
	UTADIS	[34, 66, 95, 141, 187]

growth and social stability. The performance of a firm or an organization is clearly multi-dimensional, since it is affected by a variety of factors of different nature, such as: (1) financial factors indicating the financial position of the firm/organization, (2) strategic factors of qualitative nature that define the internal operation of the firm and its relation to its customers and the market (organization, management, market trend, etc. [198], (3) economic factors that define the economic and business environment. The aggregation of all these factors into a global evaluation index is a subjective process that depends on the decision maker's values and judgment policy. This is in accordance with the MCDA paradigm, thus leading several operational researchers to the investigation of the capabilities that MCDA methods provide in supporting decision maker's in making decisions regarding the evaluation of corporate performance. An indicative list of studies on this topic is given in Table 24.3.

Investment Appraisal In most cases the choice of investment projects is an important strategic decision for every firm, public or private, large or small. Therefore, the process of an investment decision should be conveniently modelled. In general, the investment decision process consists of four main stages: perception, formulation, evaluation, and choice. The financial theory intervenes only in the stages of evaluation and choice based on traditional financial criteria such as the payback period, the accounting rate of return, the net present value, the internal rate of return, the discounted payback method, etc. [35]. This approach, however, entails some shortcomings such as the difficulty in aggregating the conflicting results of each criterion and the elimination of important qualitative variables from the analysis [202]. MCDA, on the other hand, contributes in a very original way to the investment decision process, supporting all stages of the investment process. Concerning the stages of perception and formulation, MCDA contributes to the identification of possible actions (investment opportunities) and to the definition of a set of potential actions (possible variants, each variant constituting an investment project in competition with others). Concerning the stages of evaluation and choice, MCDA supports the introduction in the analysis of both quantitative and qualitative criteria. Criteria such as the urgency of the project, the coherence of the objectives of the projects with those of the general policy of the firm, the social and environmental aspects should be taken into careful consideration. Therefore, MCDA contributes through the identification of the best investment projects according to the

Table 24.4 Applications of MCDA approaches in investment appraisal

Approaches	Methods	Studies
Multiattribute utility theory	AHP	[7, 114, 159]
	Other	[33, 40, 64, 82, 104, 152]
Outranking relations	ELECTRE	[30, 41]
	PROMETHEE	[118, 125, 154, 188]
	ORESTE	[41]
Preference disaggregation	UTA	[21, 169]
	UTADIS	[97]

Table 24.5 Applications of MCDA approaches in other financial decision-making problems

Topic	Methodology	Studies
Venture capital	Conjoint analysis	[143, 156]
	UTA	[172, 199]
	MAUT	[19, 87]
Country risk	MAUT	[86, 185]
	UTA	[5, 39]
	UTADIS	[5, 59, 205]
	MHDIS	[53, 59, 60]
	Other	[38, 42, 144, 145]
Mergers and acquisitions	UTADIS, MHDIS	[149, 150]
	Rough sets	[175]
	Other	[189]

problematic chosen, the satisfactory resolution of the conflicts between the criteria, the determination of the relative importance of the criteria in the decision-making process, and the revealing of the investors' preferences and system of values. These attractive features have been the main motivation for the use of MCDA methods in investment appraisal in several real-world cases. A representative list of studies in presented in Table 24.4.

Other Financial Decision Problems Except for the above financial decision-making problems, discrete MCDA evaluation methods are also applicable in several other fields of finance. Table 24.5 list some additional applications of MCDA methods in other financial problems, including venture capital, country risk assessment and the prediction of corporate mergers and acquisitions. In venture capital investment decisions, MCDA methods are used to evaluate firms that seek venture capital financing, and identify the factors that drive such financing decisions. In country risk assessment, MCDA methods are used to developed models that aggregate the appropriate economic, financial and socio-political factors, to support the evaluation of the creditworthiness and the future prospects of the countries. Finally, in corporate mergers and acquisitions MCDA methods are used to assess the likelihood that a firm will be merged or acquired on the basis of financial information (financial ratios) and strategic factors.

24.5 Conclusions and Future Perspectives

This chapter discussed the contribution of MCDA in financial decision-making problems, focusing on the justification of the multi-dimensional character of financial decisions and the use of different MCDA methodologies to support them.

Overall, the main advantages that the MCDA paradigm provides in financial decision making, could be summarized in the following aspects [202]: (1) the possibility of structuring complex evaluation problems, (2) the introduction of both quantitative and qualitative criteria in the evaluation process, (3) the transparency in the evaluation, allowing good argumentation in financial decisions, and (4) the introduction of sophisticated, flexible and realistic scientific methods in the financial decision-making process.

In conclusion, MCDA methods seem to have a promising future in the field of financial management, because they offer a highly methodological and realistic framework to decision problems. Nevertheless, their success in practice depends heavily on the development of computerized multicriteria decision support systems. Financial institutions as well as firms acknowledge the multi-dimensional nature of financial decision problems [23]. Nevertheless, they often use optimization or statistical approaches to address their financial problems, since optimization and statistical software packages are easily available in relatively low cost, even though many of these software packages are not specifically designed for financial decision-making problems. Consequently, the use of MCDA methods to support real time financial decision making, calls upon the development of integrated user-friendly multicriteria decision support systems that will be specifically designed to address financial problems. Examples of such systems are the CGX system [181], the BANKS system [128], the BANKADVISER system [126], the INVEX system [188], the FINEVA system [213], the FINCLAS system [206], the INVESTOR system [210], etc. The development and promotion of such systems is a key issue in the successful application of MCDA methods in finance.

Acknowledgements The authors wish to thank Winfried Hallerbach by allowing to draw in this text on work co-authored by him (see our list of references).

References

1. Abdelaziz, F.B., Aouni, B., El-Fayedh, R.: Multi-objective stochastic programming for portfolio selection. Eur. J. Oper. Res. **177**(3), 1811–1823 (2007)
2. Albadvi, A., Chaharsooghi, S.K., Esfahanipour, A.: Decision making in stock trading: an application of PROMETHEE. Eur. J. Oper. Res. **177**(2), 673–683 (2007)
3. Amiri, M., Zandieh, M., Vahdani, B., Soltani, R., Roshanaei, V.: An integrated eigenvector-DEA-TOPSIS methodology for portfolio risk evaluation in the FOREX spot market. Expert Syst. Appl. **37**(1), 509–516 (2010)
4. Anagnostopoulos, K.P., Mamanis, G.: A portfolio optimization model with three objectives and discrete variables. Comput. Oper. Res. **37**, 1285–1297 (2010)

5. Anastassiou, Th., Zopounidis, C.: Country risk assessment: a multicriteria analysis approach. J. Euro-Asian Manag. **3**(1), 51–73 (1997)
6. Andenmatten, A.: Evaluation du Risque de Défaillance des Emetteurs d'Obligations: Une Approche par l'Aide Multicritère à la Décision. Presses Polytechniques et Universitaires Romandes (1995)
7. Angelou, G.N., Economides, A.A.: A compound real option and AHP methodology for evaluating ICT business alternatives. Telematics Inform. **26**(4), 353–374 (2009)
8. Ashford, R.W., Berry, R.H., Dyson, R.G.: Operational research and financial management. Eur. J. Oper. Res. **36**(2), 143–152 (1988)
9. Aznar, J., Guijarro, F., Moreno-Jimenez, J.: Mixed valuation methods: a combined AHP-GP procedure for individual and group multicriteria agricultural valuation. Ann. Oper. Res. **190**, 221–238 (2011)
10. Babalos, V., Philippas, N., Doumpos, M., Zopounidis, C.: Mutual funds performance appraisal using stochastic multicriteria acceptability analysis. Appl. Math. Comput. **218**(9), 5693–5703 (2012)
11. Babic, Z., Plazibat, N.: Ranking of enterprises based on multicriteria analysis. Int. J. Prod. Econ. **56–57**, 29–35 (1998)
12. Ballestero, E., Günther, E., Pla-Santamaria, D., Stummer, C.: Portfolio selection under strict uncertainty: a multi-criteria methodology and its application to the Frankfurt and Vienna stock exchanges. Eur. J. Oper. Res. **181**(3), 1476–1487 (2007)
13. Ballestero, E., Bravo, M., Perez-Gladish, B., Arenas-Parra, M., Plà-Santamaría, D.: Socially responsible investment: a multicriteria approach to portfolio selection combining ethical and financial objectives. Eur. J. Oper. Res. **216**(2), 497–494 (2012)
14. Bana e Costa, C.A., Soares, J.O.: Multicriteria approaches for portfolio selection: an overview. Rev. Financ. Mark. **4**(1), 19–26 (2001)
15. Bana e Costa, C.A., Soares, J.O.: A multicriteria model for portfolio management. Eur. J. Financ. **10**(3), 198–211 (2004)
16. Bana e Costa, C.A., Vansnick, J.C.: MACBETH: an interactive path towards the construction of cardinal value functions. Int. Trans. Oper. Res. **1**, 489–500 (1994)
17. Bana e Costa, C.A., Barroso, L., Soares, J.O.: Qualitative modelling of credit scoring: a case study in banking. J. Eur. Res. Stud **5**(1–2), 37–51 (2002)
18. Baourakis, G., Doumpos, M., Kalogeras, N., Zopounidis, C.: Multicriteria analysis and assessment of financial viability of agri-businesses: the case of marketing co-operatives and juice producing companies. Agribusiness **18**(4), 543–558 (2002)
19. Beim, G., Lévesque, M.: Country selection for new business venturing: a multiple criteria decision analysis. Long Range Plan. **39**(3), 265–293 (2006)
20. Bergeron, M., Martel, J.M., Twarabimenye, P.: The evaluation of corporate loan applications based on the MCDA. J. Euro-Asian Manag. **2**(2), 16–46 (1996)
21. Beuthe, M., Eeckhoudt, L., Scannella, G.: A practical multicriteria methodology for assessing risk public investments. Soc. Econ. Plan. Sci. **34**, 121–139 (2000)
22. Bhaskar, K.: A multiple objective approach to capital budgeting. Account. Bus. Res. **10**, 25–46 (1979)
23. Bhaskar, K., McNamee, P.: Multiple objectives in accounting and finance. J. Bus. Financ. Account. **10**(4), 595–621 (1983)
24. Biggiero, L., Iazzolino, G., Laise, D.: Financial performance measurement: a multicriteria methodology. J. Financ. Decis. Mak. **1**(2), 37–55 (2005)
25. Black, F., Scholes, M.: The pricing of options and corporate liabilities. J. Polit. Econ. **81**(3), 637–659 (1973)
26. Bodie, Z., Merton, R.C.: Finance. Prentice-Hall, Upper Saddle River (2000)
27. Bollerslev, T.: Financial econometrics: past developments and future challenges. J. Econom. **100**, 41–51 (2001)
28. Brans, J.P., Vincke, Ph.: A preference ranking organization method. Manag. Sci. **31**(6), 647–656 (1985)

29. Briec, W., Kerstens, K.: Portfolio selection in multidimensional general and partial moment space. J. Econ. Dyn. Control. **34**(4), 636–656 (2010)
30. Buchanan, J., Sheppard, Ph., Vanderpooten, D.: Project ranking using Electre III. Department of Management Systems, Research Report Series 1999–01 (1999)
31. Bugera, V., Konno, H., Uryasev, S.: Credit cards scoring with quadratic utility functions. J. Multi-Criteria Decis. Anal. **11**(4–5), 197–211 (2002)
32. Capotorti, A., Barbanera, E.: Credit scoring analysis using a fuzzy probabilistic rough set model. Comput. Stat. Data Anal. **56**(4), 981–994 (2012)
33. Chan, Y., DiSalvo, J.P., Garrambone, M.W.: A goal-seeking approach to capital budgeting. Soc. Econ. Plan. Sci. **39**(2), 165–182 (2005)
34. Cohen, S., Doumpos, M., Neophytou, E., Zopounidis, C.: Assessing financial distress where bankruptcy is not an option: an alternative approach for local municipalities. Eur. J. Oper. Res. **218**(1), 270–279 (2012)
35. Colasse, B.: Gestion Financière de l'Entreprise. Presses Universitaires de France, Paris (1993)
36. Colson, G., DeBruyn, Ch.: An integrated multiobjective portfolio management system. Math. Comput. Model. **12**(10–11), 1359–1381 (1989)
37. Colson, G., Mbangala, M.: Evaluation multicritère d'entreprises publiques du rail. FINECO **8**(1), 45–72 (1998)
38. Cook, W.D., Hebner, K.J.: A multicriteria approach to country risk evaluation: with an example employing Japanese data. Int. Rev. Econ. Financ. **2**(4), 327–438 (1993)
39. Cosset, J.C., Siskos, Y., Zopounidis, C.: Evaluating country risk: a decision support approach. Glob. Financ. J. **3**(1), 79–95 (1992)
40. Cundrič, A., Kern, T., Rajkovič, V.: A qualitative model for road investment appraisal. Transp. Policy **15**(4), 225–231 (2008)
41. Danila, N.: Méthodologie d' Aide à la Décision Dans le Cas d' Investissements Fort Dépendants. Thèse de Doctorat de 3e Cycle, UER Sciences des Organisations, Université de Paris–Dauphine (1980)
42. De Smet, Y., Guzmán, L.M.: Towards multicriteria clustering: an extension of the k-means algorithm. Eur. J. Oper. Res. **58**(2), 390–398 (2004)
43. Derman, E.: Model risk. Risk **9**(5), 34–37 (1996)
44. Derman, E.: Valuing models and modelling value. J. Portf. Manag. **22**(Spring), 106–114 (1996)
45. Diakoulaki, D., Mavrotas, G., Papagyannakis, L.: A multicriteria approach for evaluating the performance of industrial firms. Omega **20**(4), 467–474 (1992)
46. Dimitras, A.I., Zopounidis, C., Hurson, C.: A multicriteria decision aid method for the assessment of business failure risk. Found. Comput. Decis. Sci. **20**(2), 99–112 (1995)
47. Dimitras, A.I., Zanakis, S.H., Zopounidis, C.: A survey of business failures with an emphasis on prediction methods and industrial applications. Eur. J. Oper. Res. **90**(3), 487–513 (1996)
48. Dimitras, A.I., Slowinski, R., Susmaga, R., Zopounidis, C.: Business failure prediction using rough sets. Eur. J. Oper. Res. **114**(2), 263–280 (1999)
49. Dominiak, C.: Portfolio selection using the idea of reference solution. In: Fandel, G., Gal, T. (eds.) Lecture Notes in Economics and Mathematical Systems, vol. 448, pp. 593–602. Springer, Berlin (1997)
50. Doumpos, M.: Learning non-monotonic additive value functions for multicriteria decision making. OR Spectr. **34**(1), 89–106 (2012)
51. Doumpos, M., Pasiouras, F.: Developing and testing models for replicating credit ratings: a multicriteria approach. Comput. Econ. **25**(4), 327–341 (2005)
52. Doumpos, M., Zopounidis, C.: A multicriteria discrimination method for the prediction of financial distress: the case of Greece. Multinational Financ. J. **3**(2), 71–101 (1999)
53. Doumpos, M., Zopounidis, C.: Assessing financial risks using a multicriteria sorting procedure: the case of country risk assessment. Omega **29**(1), 97–109 (2000)
54. Doumpos, M., Zopounidis, C.: Multicriteria Decision Aid Classification Methods. Kluwer, Dordrecht (2002)

55. Doumpos, M., Zopounidis, C.: Regularized estimation for preference disaggregation in multiple criteria decision making. Comput. Optim. Appl. **38**, 61–80 (2007)
56. Doumpos, M., Zopounidis, C.: A multicriteria decision support system for bank rating. Decis. Support. Syst. **50**(1), 55–63 (2010)
57. Doumpos, M., Zopounidis, C.: A multicriteria outranking modeling approach for credit rating. Decis. Sci. **42**(3), 721–742 (2011)
58. Doumpos, M., Zopounidis, C., Pardalos, P.M.: Multicriteria sorting methodology: application to financial decision problems. Parallel Algorithm. Appl. **15**(1–2), 113–129 (2000)
59. Doumpos, M., Zanakis, S.H., Zopounidis, C.: Multicriteria preference disaggregation for classification problems with an application to global investing risk. Decis. Sci. **32**(2), 333–385 (2001)
60. Doumpos, M., Pentaraki, K., Zopounidis, C., Agorastos, C.: Assessing country risk using a multi-group discrimination method: a comparative analysis. Manag. Financ. **27**, 16–34 (2001)
61. Doumpos, M., Kosmidou, K., Baourakis, G., Zopounidis, C.: Credit risk assessment using a multicriteria hierarchical discrimination approach: a comparative analysis. Eur. J. Oper. Res. **138**(2), 392–412 (2002)
62. Ehrgott, M., Klamroth, K., Schwehm, C.: An MCDM approach to portfolio optimization. Eur. J. Oper. Res. **155**(3), 752–770 (2004)
63. Elton, E.J., Gruber, M.J., Brown, S.J., Goetzmann, W.N.: Modern Portfolio Theory and Investment Analysis, 8th edn. Wiley, New York (2010)
64. Evrard, Y., Zisswiller, R.: Une analyse des décisions d'investissement fondée sur les modèles de choix multi-attributs. Finance **3**(1), 51–68 (1982)
65. Fama, E.F., Miller, M.H.: The Theory of Finance. Dryden Press, Hinsdale (1972)
66. Gaganis, C., Pasiouras, F., Doumpos, M., Zopounidis, C.: Modelling banking sector stability with multicriteria approaches. Optim. Lett. **4**(4), 543–558 (2010)
67. Garcìa, F., Guijarro, F., Moya, I.: Ranking Spanish savings banks: A multicriteria approach. Math. Comput. Model. **52**(7–8), 1058–1065 (2010)
68. García, F., Giménez, V., Guijarro, F.: Credit risk management: A multicriteria approach to assess creditworthiness. Math. Comput. Model. **57**(7–8), 2009–2015 (2013)
69. Gladish, B.P., Jones, D.F., Tamiz, M., Terol, A.B.: An interactive three-stage model for mutual funds portfolio selection. Omega **35**(1), 75–88 (2007)
70. Greco, S., Matarazzo, B., Slowinski, R.: A new rough set approach to evaluation of bankruptcy risk. In: Zopounidis, C. (ed.) Operational Tools in the Management of Financial Risks, pp. 121–136. Kluwer, Dordrecht (1998)
71. Greco, S., Matarazzo, B., Slowinski, R.: Extension of the rough set approach to multicriteria decision support. Inf. Syst. Oper. Res. **38**(3), 161–196 (2000)
72. Greco, S., Matarazzo, B., Slowinski, R.: Rough sets theory for multicriteria decision analysis. Eur. J. Oper. Res. **129**, 1–47 (2001)
73. Greco, S., Mousseau, V., Slowinski, R.: Multiple criteria sorting with a set of additive value functions. Eur. J. Oper. Res. **207**(3), 1455–1470 (2010)
74. Greco, G., Kadzinski, M., Slowinski, R.: Selection of a representative value function in robust multiple criteria sorting. Comput. Oper. Res. **38**(11), 1620–1637 (2011)
75. Grigoroudis, E., Orfanoudaki, E., Zopounidis, C.: Strategic performance measurement in a healthcare organisation: a multiple criteria approach based on balanced scorecard. Omega **40**(1), 104–119 (2012)
76. Gupta, Y.P., Rao, R.P., Bagghi, P.K.: Linear goal programming as an alternative to multivariate discriminant analysis: a note. J. Bus. Financ. Account. **17**(4), 593–598 (1990)
77. Gupta, P., Mehlawat, M.K., Saxena, A.: A hybrid approach to asset allocation with simultaneous consideration of suitability and optimality. Inf. Sci. **180**(11), 2264–2285 (2010)
78. Hababou, M., Martel, J.M.: A multicriteria approach for selecting a portfolio manager. Inf. Syst. Oper. Res. **36**(3), 161–176 (1998)
79. Hallerbach, H.W., Spronk, J.: A multicriteria framework for risk analysis. In: Haimes, Y.Y., Steuer, R.E. (eds.) Lecture Notes in Economics and Mathematical Systems, vol. 487, pp. 272–283. Springer, Berlin (2000)

80. Hallerbach, W.G., Spronk, J.: A multidimensional framework for financial-economic decisions. J. Multi-Criteria Decis. Anal. **11**(3), 111–124 (2002)
81. Hallerbach, W.G., Spronk, J.: The relevance of MCDM for financial decisions. J. Multi-Criteria Decis. Anal. **11**(4–5), 187–195 (2002)
82. Han, S., Diekmann, J., Lee, Y., Ock, J.: Multicriteria financial portfolio risk management for international projects. J. Construct. Eng. Manag. **130**(3), 346–356 (2004)
83. Hastie, K.L.: A perspective on management education. Financ. Manag. **11**(4), 55–62 (1982)
84. Hirschberger, M., Steuer, R.E., Utz, S., Wimmer, M., Qi, Y.: Computing the nondominated set in tri-criterion portfolio selection. Oper. Res. **61**(1), 169–183 (2013)
85. Ho, W.R.J., Tsai, C.-L., Tzeng, G.-H., Fang, S.-K.: Combined DEMATEL technique with a novel MCDM model for exploring portfolio selection based on CAPM. Expert Syst. Appl. **38**(1), 16–25 (2011)
86. Hogan, K.M., Lipton, A.F., Olson, G.T.: An evaluation of country risk in assessing direct foreign investment. In: Applications of Management Science, pp. 251–276. Emerald, Bingley (2012)
87. Hsu, Y.-G., Shyu, J.Z., Tzeng, G.-H.: Policy tools on the formation of new biotechnology firms in taiwan. Technovation **25**(3), 281–292 (2005)
88. Hu, Y.-C.: Bankruptcy prediction using ELECTRE-based single-layer perceptron. Neurocomputing **72**, 3150–3157 (2009)
89. Hu, Y.-C., Chen, C.-J.: A PROMETHEE-based classification method using concordance and discordance relations and its application to bankruptcy prediction. Inf. Sci. **181**(22), 4959–4968 (2011)
90. Hull, J.C.: Options, Futures and Other Derivatives. Prentice-Hall, Upper Saddle River (2003)
91. Hurson, Ch., Ricci, N.: Multicriteria decision making and portfolio management with arbitrage pricing theory. In: Zopounidis, C. (ed.) Operational Tools in The Management of Financial Risks, pp. 31–55. Kluwer, Dordrecht (1998)
92. Hurson, Ch., Zopounidis, C.: On the use of multi-criteria decision aid methods to portfolio selection. J. Euro-Asian Manag. **1**(2), 69–94 (1995)
93. Hurson, Ch., Zopounidis, C.: Gestion de Portefeuille et Analyse Multicritère. Economica, Paris (1997)
94. Iazzolino, G., Laise, D., Marraro, L.: Business multicriteria performance analysis: a tutorial. Benchmark. Int. J. **19**(3), 395–411 (2012)
95. Ioannidis, C., Pasiouras, F., Zopounidis, C.: Assessing bank soundness with classification techniques. Omega **38**(5), 345–357 (2010)
96. Jablonsky, J.: Multicriteria evaluation of clients in financial houses. Cent. Eur. J. Oper. Res. Econ. **3**(2), 257–264 (1993)
97. Jacquet-Lagrèze, E.: An application of the UTA discriminant model for the evaluation of R & D projects. In: Pardalos, P.M., Siskos, Y., Zopounidis, C. (eds.) Advances in Multicriteria Analysis, pp. 203–211. Kluwer, Dordrecht (1995)
98. Jacquet-Lagrèze, E., Siskos, Y.: Assessing a set of additive utility functions for multicriteria decision making: the UTA method. Eur. J. Oper. Res. **10**(2), 151–164 (1982)
99. Jacquet-Lagrèze, E., Siskos, Y.: Méthodes de Décision Multicritère. Editions Hommes et Techniques, Paris (1983)
100. Jacquet-Lagrèze, E., Siskos, Y.: Preference disaggregation: twenty years of MCDA experience. Eur. J. Oper. Res. **130**(2), 233–245 (2001)
101. Jarrow, R.A.: Finance Theory. Prentice-Hall, Upper Saddle River (1988)
102. Jensen, M.C.: Value maximization, stakeholder theory, and the corporate objective function. J. Appl. Corp. Financ. **14**(3), 8–21 (2001)
103. Jensen, M.C., Meckling, W.H.: Theory of the firm: managerial behavior, agency costs and ownership structure. J. Financ. Econ. **3**, 305–360 (1976)
104. Jiménez, L.G., Pascual, L.B.: Multicriteria cash-flow modeling and project value-multiples for two-stage project valuation. Int. J. Proj. Manag. **26**(2), 185–194 (2008)
105. Jog, V., Kaliszewski, I., Michalowski, W.: Using attribute trade-off information in investment. J. Multi-Criteria Decis. Anal. **8**(4), 189–199 (1999)

106. Jog, V., Michalowski, W., Slowinski, R., Susmaga, R.: The rough sets analysis and the neural networks classifier: a hybrid approach to predicting stocks' performance. In: Despotis, D.K., Zopounidis, C. (eds.) Proceedings of the 5th International Meeting of the Decision Sciences Institute. Integrating Technology and Human Decisions: Bridging into the 21st Century, vol. II, pp. 1386–1388. New Technologies Editions, Athens (1999)
107. Jorion, Ph.: Value at Risk: The Benchmark for Controlling Market Risk. McGraw-Hill, Chicago (2001)
108. Kalogeras, N., Baourakis, G., Zopounidis, C., van Dijk, G.: Evaluating the financial performance of agri-food firms: a multicriteria decision-aid approach. J. Food Eng. **70**(3), 365–371 (2005)
109. Keeney, R.L., Raiffa, H.: Decisions with Multiple Objectives: Preferences and Value Tradeoffs. Cambridge University Press, Cambridge (1993)
110. Keynes, J.N.: The Scope and Method of Political Economy. Macmillan, London (1891)
111. Khalil, J., Martel, J.M., Jutras, P.: A multicriterion system for credit risk rating. Gestion 2000: Belg. Manag. Mag. **15**(1), 125–146 (2000)
112. Khoury, N.T., Martel, J.M.: The relationship between risk-return characteristics of mutual funds and their size. Finance **11**(2), 67–82 (1990)
113. Khoury, N.T., Martel, J.M., Veilleux, M.: Méthode multicritère de sélection de portefeuilles indiciels internationaux. L'Actualité Economique, Revue d'Analyse Economique **69**(1), 171–190 (1993)
114. Kivijärvi, H., Tuominen, M.: A decision support system for semistructured strategic decisions: a multi-tool method for evaluating intangible investments. Revue des Systèmes de Décision **1**, 353–376 (1992)
115. Köksalan, M., Özpeynirci, B.S.: An interactive sorting method for additive utility functions. Comput. Oper. Res. **36**(9), 2565–2572 (2009)
116. Konno, H., Yamazaki, H.: Mean-absolute deviation portfolio optimization model and its application to Tokyo stock market. Manag. Sci. **37**(1), 519–531 (1991)
117. Kou, G., Peng, Y., Shi, Y., Wise, M., Xu, W.: Discovering credit cardholders' behavior by multiple criteria linear programming. Ann. Oper. Res. **135**, 261–274 (2005)
118. Kunsch, P.L.: Advanced operations research techniques in capital budgeting. In: Zopounidis, C., Doumpos, M., Pardalos, P.M. (eds.) Handbook of Financial Engineering, pp. 301–342. Springer, New York (2008)
119. Lee, H., Kwak, W., Han, I.: Developing a business performance evaluation system: an analytic hierarchical model. Eng. Econ. **30**(4), 343–357 (1995)
120. Lee, W.-S., Tzeng, G.-H., Guan, J.-L., Chien, K.-T., Huang, J.-M.: Combined MCDM techniques for exploring stock selection based on Gordon model. Expert Syst. Appl. **36**(3), 6421–6430 (2009)
121. Li, A., Shi, Y., He, J.: Mclp-based methods for improving "bad" catching rate in credit cardholder behavior analysis. Appl. Soft Comput. **8**(3), 1259–1265 (2008)
122. Li, J., Wei, L., Li, G., Xu, W.: An evolution strategy-based multiple kernels multi-criteria programming approach: the case of credit decision making. Decis. Support. Syst. **51**(2), 292–298 (2011)
123. Lo, A.W., Petrov, C., Wierzbicki, M.: It's 11pm – Do you know where your liquidity is? The mean-variance-liquidity frontier. J. Invest. Manag. **1**(1), 55–93 (2003)
124. Lourenco, J.C., Morton, A., Bana e Costa, C.A.: Probe-a multicriteria decision support system for portfolio robustness evaluation. Decis. Support. Syst. **54**(1), 534–550 (2012)
125. Macharis, C., de Witte, A., Ampe, J.: The multi-actor, multi-criteria analysis methodology (MAMCA) for the evaluation of transport projects: theory and practice. J. Adv. Transp. **43**(2), 183–202 (2009)
126. Mareschal, B., Brans, J.P.: BANKADVISER: an industrial evaluation system. Eur. J. Oper. Res. **54**, 318–324 (1991)
127. Mareschal, B., Mertens, D.: Evaluation financière par la méthode multicritère GAIA: Application au Secteur Bancaire Belge. Revue de la Banque **6**, 317–329 (1990)

128. Mareschal, B., Mertens, D.: BANKS: a multicriteria PROMETHEE-based decision support system for the evaluation of the international banking sector. Revue des Systèmes de Décision **1**(2–3), 175–189 (1992)
129. Mareschal, B., Mertens, D.: Evaluation multicritère par la méthode multicritère GAIA: Application au secteur de l'assurance en belgique. L'Actualité Economique, Revue d'Analyse Economique **69**(1), 206 (1993)
130. Markowitz, H.: Theories of uncertainty and financial behavior. Econometrica **19**, 325–326 (1950)
131. Markowitz, H.: Portfolio selection. J. Financ. **7**, 77–91 (1952)
132. Markowitz, H.: The optimization of a quadratic function subject to linear constraints. Nav. Res. Logist. Q. **3**, 111–133 (1956)
133. Markowitz, H.: Portfolio Selection: Efficient Diversification in Investments. Wiley, New York (1959)
134. Martel, J.M., Zopounidis, C.: Critères multiples et décisions financières. FINECO **8**, 1–12 (1998)
135. Martel, J.M., Khoury, N.T., Bergeron, M.: An application of a multicriteria approach to portfolio comparisons. J. Oper. Res. Soc. **39**(7), 617–628 (1988)
136. Martel, J.M., Khoury, N.T., M'Zali, B.: Comparaison performancetaille des fonds mutuels par une analyse multicritère. L' Actualité Economique, Revue d'Analyse Economique **67**(3), 306–324 (1991)
137. Mavrotas, G.: Effective implementation of the ϵ-constraint method in multiobjective programming problems. Appl. Math. Comput. **213**(2), 455–465 (2009)
138. McInnes, J.M., Carleton, W.J.: Theory, models and implementation in financial management. Manag. Sci. **28**(9), 957–978 (1982)
139. Merton, R.C.: Theory of rational option pricing. Bell J. Econ. **4**(Spring), 141–183 (1973)
140. Merton, R.C.: Influence of mathematical models in finance on practice: past, present, and future. Financ. Pract. Educ. **5**(1), 7–15 (1995)
141. Michalopoulos, M., Zopounidis, C., Doumpos, M.: Evaluation des succursales bancaires è l'aide d'une méthode multicritère. FINECO **8**(2), 123–136 (1998)
142. Mikkilä, M., Kolehmainen, T., Pukkala, O.: Multi-attribute assessment of acceptability of operations in the pulp and paper industries. For. Policy Econ. **7**(2), 227–243 (2005)
143. Muzyka, D., Birley, S., Leleux, B.: Trade-offs in the investment decisions of European venture capitalists. J. Bus. Ventur. **11**(4), 273–287 (1996)
144. Oral, M., Chabchoub, H.: An estimation model for replicating the rankings of the world competitiveness report. Int. J. Forecast. **13**, 527–537 (1997)
145. Oral, M., Kettani, O., Cosset, J.C., Daouas, M.: An estimation model for country risk rating. Int. J. Forecast. **8**, 583–593 (1992)
146. Papageorgiou, D., Doumpos, M., Zopounidis, C.: Credit rating systems: regulatory framework and comparative evaluation of existing methods. In: Zopounidis, C., Doumpos, M., Pardalos, P.M. (eds.) Handbook of Financial Engineering, pp. 457–488. Springer, New York (2008)
147. Pardalos, P.M., Michalopoulos, M., Zopounidis, C.: On the use of multicriteria methods for the evaluation of insurance companies in Greece. In: Zopounidis, C. (ed.) New Operational Approaches for Financial Modelling, pp. 271–283. Physica-Verlag, Heidelberg (1997)
148. Pasiouras, F., Gaganis, C., Doumpos, M.: A multicriteria discrimination approach for the credit rating of Asian banks. Ann. Financ. **3**, 351–367 (2007)
149. Pasiouras, F., Tanna, S., Zopounidis, C.: The identification of acquisition targets in the eu banking industry: an application of multicriteria approaches. Int. Rev. Financ. Anal. **16**(3), 262–281 (2007)
150. Pasiouras, C., Gaganis, F., Zopounidis, C.: Multicriteria classification models for the identification of targets and acquirers in the Asian banking sector. Eur. J. Oper. Res. **204**(2), 328–335 (2010)
151. Pawlak, Z.: Rough sets. Int. J. Inf. Comput. Sci. **11**, 341–356 (1982)

152. Pearman, A.D., Mackie, P.J., May, A.D., Simon, D.: The use of multi-criteria techniques to rank highway investment proposals. In: Lockett, A.G., Islei, G. (eds.) Lecture Notes in Economics and Mathematical Systems, vol. 335, pp. 157–165. Springer, Berlin (1989)
153. Qi, Y., Hirschberger, M., Steuer, R.E.: Dotted representations of mean-variance efficient frontiers and their computation. Inf. Syst. Oper. Res. **1**(3), 15–21 (2009)
154. Ribarovic, Z., Mladineo, N.: Application of multicriterional analysis to the ranking and evaluation of the investment programmes in the ready mixed concrete industry. Eng. Costs Prod. Econ. **12**, 367–374 (1987)
155. Rios-Garcia, S., Rios-Insua, S.: The portfolio problem with multiattributes and multiple criteria. In: Hansen, P. (ed.) Lecture Notes in Economics and Mathematical Systems, vol. 209, pp. 317–325. Springer, Berlin (1983)
156. Riquelme, H., Rickards, T.: Hybrid conjoint analysis: an estimation probe in new venture decisions. J. Bus. Ventur. **7**, 505–518 (1992)
157. Roll, R.: A critique of the asset pricing theory's tests. J. Financ. Econ. **4**, 129–176 (1977)
158. Ross, S.A.: Finance. The New Palgrave Finance, pp. 1–34. MacMillan, Hong Kong (1989)
159. Rousos, E.-P., Lee, B.S.: Multicriteria analysis in shipping investment evaluation. Marit. Policy Manag. **39**(4), 423–442 (2012)
160. Roy, B.: Classement et choix en présence de points de vue multiples: La méthode ELECTRE. Revue Francaise d'Informatique et de Recherche Operationnelle **8**, 57–75 (1968)
161. Roy, B.: The outranking approach and the foundations of ELECTRE methods. Theory Decis. **31**(1), 49–73 (1991)
162. Roy, B.: Multicriteria Methodology for Decision Aiding. Kluwer, Dordrecht (1996)
163. Roy, B., Bouyssou, D.: Aide Multicritère à la Décision: Méthodes et Cas. Economica, Paris (1993)
164. Şakar, C., Köksalan, M.: A stochastic programming approach to multicriteria portfolio optimization. J. Glob. Optim. **57**(2), 299–314 (2013)
165. Saaty, T.L., Rogers, P.C., Pell, R.: Portfolio selection through hierarchies. J. Portf. Manag. **6**(Spring), 16–21 (1980)
166. Samaras, G.D., Matsatsinis, N.F., Zopounidis, C.: A multicriteria DSS for stock evaluation using fundamental analysis. Eur. J. Oper. Res. **187**(3), 1380–1401 (2008)
167. Sevastjanov, P., Dymova, L.: Stock screening with use of multiple criteria decision making and optimization. Omega **37**, 659–671 (2009)
168. Shefrin, H.: Beyond Greed and Fear: Understanding Behavioral Finance and the Psychology of Investing. Oxford University Press, New York (2002)
169. Siskos, J., Assimakopoulos, N.: Multicriteria highway planning: a case study. Math. Comput. Model. **12**(10–11), 1401–1410 (1989)
170. Siskos, Y., Grigoroudis, E.: New trends in aggregation-disaggregation approaches. In: Zopounidis, C., Pardalos, P.M. (eds.) Handbook of Multicriteria Analysis, pp. 189–214. Springer, Berlin/Heidelberg (2010)
171. Siskos, Y., Yannacopoulos, D.: UTASTAR: an ordinal regression method for building additive value functions. Investigação Operacional **5**(1), 39–53 (1985)
172. Siskos, J., Zopounidis, C.: The evaluation criteria of the venture capital investment activity: an interactive assessment. Eur. J. Oper. Res. **31**(3), 304–313 (1987)
173. Siskos, Y., Zopounidis, C., Pouliezos, A.: An integrated DSS for financing firms by an industrial development bank in Greece. Decis. Support. Syst. **12**, 151–168 (1994)
174. Slowinski, R., Zopounidis, C.: Application of the rough set approach to evaluation of bankruptcy risk. Int. J. Intell. Syst. Account. Financ. Manag. **4**, 27–41 (1995)
175. Slowinski, R., Zopounidis, C., Dimitras, A.I.: Prediction of company acquisition in Greece by means of the rough set approach. Eur. J. Oper. Res. **100**, 1–15 (1997)
176. Slowinski, R., Zopounidis, C., Dimitras, A.I., Susmaga, R.: Rough set predictor of business failure. In: Ribeiro, R.R., Yager, R.R., Zimmermann, H.J., Kacprzyk, J. (eds.) Soft Computing in Financial Engineering, pp. 402–424. Physica-Verlag, Heidelberg (1999)
177. Spronk, J., Hallerbach, W.G.: Financial modelling: where to go? With an illustration for portfolio management. Eur. J. Oper. Res. **99**(1), 113–125 (1997)

178. Spronk, J., Vermeulen, E.M.: Comparative performance evaluation under uncertainty. Eur. J. Oper. Res. **150**(3), 482–495 (2003)
179. Srinivasan, V., Bolster, P.J.: An industrial bond rating model based on the analytic hierarchy process. Eur. J. Oper. Res. **48**(1), 105–119 (1990)
180. Srinivasan, V., Kim, Y.H.: Credit granting: a comparative analysis of classification procedures. J. Financ. **XLII**(3), 665–683 (1987)
181. Srinivasan, V., Ruparel, B.: CGX: an expert support system for credit granting. Eur. J. Oper. Res. **45**, 293–308 (1990)
182. Steuer, R.E., Na, P.: Multiple criteria decision making combined with finance: a categorized bibliography. Eur. J. Oper. Res. **150**(3), 496–515 (2003)
183. Steuer, R., Qi, Y., Hirschberger, M.: Suitable-portfolio investors, nondominated frontier sensitivity, and the effect of multiple objectives on standard portfolio selection. Ann. Oper. Res. **152**, 297–317 (2007)
184. Szala, A.: L' aide à la décision en gestion de portefeuille. Diplôme Supérieur de Recherches Appliquées, Université de Paris–Dauphine (1990)
185. Tang, J.C.S., Espinal, C.G.: A model to assess country risk. Omega **17**(4), 363–367 (1989)
186. Vincke, Ph.: Multicriteria Decision Aid. Wiley, New York (1992)
187. Voulgaris, F., Doumpos, M., Zopounidis, C.: On the evaluation of Greek industrial SMEs' performance via multicriteria analysis of financial ratios. Small Bus. Econ. **15**, 127–136 (2000)
188. Vranes, S., Stanojevic, M., Stevanovic, V., Lucin, M.: INVEX: investment advisory expert system. Expert Syst. **13**(2), 105–119 (1996)
189. Wang, T.-C., Lin, Y.-L.: Applying the consistent fuzzy preference relations to select merger strategy for commercial banks in new financial environments. Expert Syst. Appl. **36**(3), 7019–7026 (2009)
190. Xidonas, P., Mavrotas, G.: Multiobjective portfolio optimization with non-convex constraints: evidence from the Eurostoxx 50. Eur. J. Financ. **20**(11), 957–977 (2014)
191. Xidonas, P., Mavrotas, G., Psarras, J.: A multicriteria methodology for equity selection using financial analysis. Comput. Oper. Res. **36**(12), 3187–3203 (2009)
192. Xidonas, P., Mavrotas, G., Krintas, T., Psarras, J., Zopounidis, C.: Multicriteria Portfolio Management. Springer, New York (2012)
193. Yeh, C.H., Deng, H., Chang, Y.H.: Fuzzy multicriteria analysis for performance evaluation of bus companies. Eur. J. Oper. Res. **126**(3), 459–473 (2000)
194. Yu, L., Wang, S., Lai, K.K.: An intelligent-agent-based fuzzy group decision making model for financial multicriteria decision support: the case of credit scoring. Eur. J. Oper. Res. **195**(3), 942–959 (2009)
195. Zeleny, M.: Multidimensional measure of risk: the prospect ranking vector. In: Zionts, S. (ed.) Lecture Notes in Economics and Mathematical Systems, vol. 155, pp. 529–548. Springer, Heidelberg (1977)
196. Zmitri, R., Martel, J.M., Dumas, Y.: Un indice multicritère de santé financière pour les succursales bancaires. FINECO **8**(2), 107–121 (1998)
197. Zollinger, M.: L'analyse multicritère et le risque de crédit aux entreprises. Revue Française de Gestion, pp. 56–66. Janvier-Février (1982)
198. Zopounidis, C.: A multicriteria decision making methodology for the evaluation of the risk of failure and an application. Found. Control Eng. **12**(1), 45–67 (1987)
199. Zopounidis, C.: La Gestion du Capital-Risque. Economica, Paris (1990)
200. Zopounidis, C.: On the use of the MINORA decision aiding system to portfolio selection and management. J. Inf. Sci. Technol. **2**(2), 150–156 (1993)
201. Zopounidis, C.: Evaluation du Risque de Défaillance de l' Entreprise: Méthodes et Cas d'Application. Economica, Paris (1995)
202. Zopounidis, C.: Multicriteria decision aid in financial management. Eur. J. Oper. Res. **119**(2), 404–415 (1999)

203. Zopounidis, C.: Preference disaggregation approach: basic features, examples from financial decision making. In: Floudas, C.A., Pardalos, P.M. (eds.) Encyclopedia of Optimization, vol. IV, pp. 344–357. Kluwer, Dordrecht (2001)
204. Zopounidis, C., Dimitras, A.I.: Multicriteria Decision Aid Methods for the Prediction of Business Failure. Kluwer, Dordrecht (1998)
205. Zopounidis, C., Doumpos, M.: A multicriteria decision aid methodology for the assessment of country risk. Eur. Res. Manag. Bus. Econ. **3**(3), 13–33 (1997)
206. Zopounidis, C., Doumpos, M.: Developing a multicriteria decision support system for financial classification problems: the FINCLAS system. Optim. Methods Softw. **8**, 277–304 (1998)
207. Zopounidis, C., Doumpos, M.: Business failure prediction using UTADIS multicriteria analysis. J. Oper. Res. Soc. **50**(11), 1138–1148 (1999)
208. Zopounidis, C., Doumpos, M.: A multicriteria decision aid methodology for sorting decision problems: the case of financial distress. Comput. Econ. **14**(3), 197–218 (1999)
209. Zopounidis, C., Doumpos, M.: Building additive utilities for multi-group hierarchical discrimination: the MHDIS method. Optim. Methods Softw. **14**(3), 219–240 (2000)
210. Zopounidis, C., Doumpos, M.: INVESTOR: a decision support system based on multiple criteria for portfolio selection and composition. In: Colorni, A., Paruccini, M., Roy, B. (eds.) A-MCD-A (Aide Multi Critere a la Decision - Multiple Criteria Decision Aiding), pp. 371–381. European Commission Joint Research Centre, Brussels (2000)
211. Zopounidis, C., Despotis, D.K., Stavropoulou, E.: Multiattribute evaluation of Greek banking performance. Appl. Stoch. Model. Data Anal. **11**(1), 97–107 (1995)
212. Zopounidis, C., Godefroid, M., Hurson, Ch.: Designing a multicriteria decision support system for portfolio selection and management. In: Janssen, J., Skiadas, C.H., Zopounidis, C. (eds.) Advances in Stochastic Modelling and Data Analysis, pp. 261–292. Kluwer, Dordrecht (1995)
213. Zopounidis, C., Matsatsinis, N.F., Doumpos M.: Developing a multicriteria knowledge-based decision support system for the assessment of corporate performance and viability: the FINEVA system. Fuzzy Econ. Rev. **1**(2), 35–53 (1996)
214. Zopounidis, C., Doumpos, M., Zanakis, S.H.: Stock evaluation using a preference disaggregation methodology. Decis. Sci. **30**(2), 313–336 (1999)
215. Zopounidis, C., Galariotis, E., Doumpos, M., Sarri, S., Andriosopoulos, K.: Multiple criteria decision aiding for finance: An updated bibliographic survey, European Journal of Operational Research, **247**(2), 339–348 (2015)

Chapter 25
Multi-Objective Optimization and Multi-Criteria Analysis Models and Methods for Problems in the Energy Sector

Carlos Henggeler Antunes and Carla Oliveira Henriques

Abstract The energy sector has been a fertile ground for the application of operational research (OR) models and methods (Antunes and Martins, OR Models for Energy Policy, Planning and Management, Annals of Operational Research, vols. 120/121, 2003). Even though different concerns have been present in OR models to assess the merit of potential solutions for a broad range of problems arising in the energy sector, the use of multi-objective optimization (MOO) and multi-criteria analysis (MCA) approaches is more recent, dating back from mid-late 1970s. The need to consider explicitly multiple uses of water resource systems or environmental aspects in energy planning provided the main motivation for the use of MOO and MCA models and methods with a special evidence in scientific literature since the 1980s. The increasing need to account for sustainability issues, which is inherently a multi-criteria concept, in planning and operational decisions, the changes in the organization of energy markets, the conflicting views of several stakeholders, the prevalent uncertainty associated with energy models, have made MOO and MCA approaches indispensable to deal with complex and challenging problems in the energy sector. This paper aims at providing an overview of MOO and MCA models and methods in a vast range of energy problems, namely in the electricity sector, which updates and extends the one in Diakoulaki et al. (In J. Figueira, S. Greco, M. Ehrgott (Eds.). Multiple Criteria Decision Analysis – State of the Art Surveys. International Series in Operations Research and

C.H. Antunes (✉)
Department of Electrical and Computer Engineering, Faculty of Sciences and Technology, Polo II, University of Coimbra, 3030-290 Coimbra, Portugal

INESC Coimbra, Rua Antero de Quental 199, 3000-033 Coimbra, Portugal
e-mail: ch@deec.uc.pt

C.O. Henriques
ISCAC, Polytechnic Institute of Coimbra, Quinto Agrícola, Bencanto, 3040-316 Coimbra, Portugal

INESC Coímbra, Rua Antero Quental 199, 3000-033 Coímbra, Portugal
e-mail: coliv@inescc.pt

S. Greco et al. (eds.), *Multiple Criteria Decision Analysis*, International Series in Operations Research & Management Science 233,
DOI 10.1007/978-1-4939-3094-4_25

Management Science, vol. 78, pp. 859–897, Springer, New York, 2005). Broadly, models and methods dealing with multi-objective mathematical programming and a priori explicitly known discrete alternatives are distinguished and some of the main types of problems are stated. The main conclusion is that MOO and MCA approaches are essential for a thorough analysis of energy problems at different decision levels, from strategic to operational, and with different timeframes.

Keywords Multi-objective optimization • Multi-criteria decision analysis • Energy sector

25.1 Introduction

The capability of reliable provision of energy to meet a vast range of needs and requirements in residential, services/commerce, agriculture, industrial and transportation sectors, is one of the most distinctive features of modern developed societies. From supplying power and heat to production systems to satisfying heating, cooling, lighting, and mobility needs, energy is pervasive in everyday life. Until mid 1970s, when an energy crisis occurred caused by the peaking of oil demand in major industrial nations and embargoes from producer countries, energy planning was almost exclusively driven by cost minimization models subject to demand satisfaction and technology constraints. This paradigm, in which per capita energy consumption was an index of a nation's prosperity, started to change due to the energy crises in the 1970s (in the aftermath of the Yom Kippur war and the Iranian revolution) and also the growing concerns regarding environmental impacts associated with the energy life-cycle from extraction, including the depletion of fossil resources, to end-use. Therefore, the merits of energy plans and policies could not be judged by considering just economic costs, but other evaluation aspects such as reliability of supply, environmental impacts, source diversification, etc., should be explicitly taken into account to address energy problems in a societal perspective. Although issues other than economic costs were often present at the outset of some studies, usually those concerns were then amalgamated into an overall cost dimension by monetizing, for instance, environmental impacts and energy losses, rather than operationalizing those multiple, incommensurate and conflicting evaluation axes as expressing distinct perspectives of the merits of courses of action.

In this context, multi-objective optimization (MOO) and multi-criteria analysis (MCA) models and methods naturally gained an increasing relevance and acceptance in the appraisal of energy technologies and policies in a vast range of energy planning problems at different decision levels (strategic, tactical, operational) and timeframes (from long-term planning to near real-time control). The recognized need and advantages of explicitly using multiple objectives/criteria not just provided a value-added in exploring a larger range of possible decisions embodying different trade-offs between the competing axes of evaluation but also enabled a richer critical analysis of potential solutions. Furthermore, this modeling and methodological framework made possible to include in a coherent manner in the decision process the preferences and interests of multiple stakeholders, in order to increase solution acceptance, and the several sources of uncertainty at stake, in order to obtain more robust recommendations.

Two major trends may be identified, which have a strong impact of MOO/MCA research and practice on the energy sector: the increasing awareness of the need to ensure sustainable development, in which energy provision plays a key role, and the trend for liberalization and market deregulation, at least in some industry segments. The concern of sustainable provision of energy meeting the present needs without compromising the ability of future generations to meet their needs is inescapable in the development of decision support models in the energy sector. Sustainability is inherently a multi-criteria concept, which makes MOO/MCA approaches indispensable to deal with the complex and challenging problems arising in the energy sector. The exploitation of energy resources must be balanced with the threats of climate change, mitigation of impacts on human health and natural ecosystems, assessment of geo-political risks, etc., also recognizing the uncertainty, the long-term and possible large-scale effects of today's energy decisions. Technologies that promote sustainable energy include renewable energy sources, such as hydroelectricity, solar energy, wind energy, wave power, geothermal energy, and tidal power, and also those designed to improve energy efficiency. Besides the important investments at stake in several energy decisions, these embody also complex and controversial issues related to global (for instance, pollution knows no border) and inter-generational (for instance, a power plant will operate during the next 40 years) effects for which the MOO/MCA tool bag offers the methodological instruments to reach balanced decisions due to their ability to combine powerful models and methods with subjective judgments and perspectives of reality.

For many years companies in the energy industry were generally vertically integrated, although at different degrees, i.e. owning generation plants, transmission and distribution networks, and customer access equipment including billing services. Most of these companies were, and still are in several cases, state-owned with the aim of protecting public interests in face of the essential nature of provision of energy, namely gas and electricity. Energy markets began to be shaped with the privatization of electric power systems in the early 1980s in some South-American countries and in UK in the early 1990s with the privatization of the electricity supply industry. The main aim of deregulation, whether involving or not privatization, was encouraging competition in many areas to curb economic inefficiencies associated with the operation of energy monopolies. In general, in the (more or less concentrated) wholesale electricity markets competing generators offer their electricity production to retailing companies, which then sell it to clients. In the retail markets end-use clients are able to select their supplier from competing retailing companies (although annual customer switching rates among European

Union member states are, in general, well below 20 % by volume). A very important issue here is that end-users do not see the possibly highly variable wholesale prices and therefore do not have incentives to reduce their consumption at peaking prices by shifting it to periods of lower prices. Models and methods to address end-users' demand response to reduce peak demand and energy bill are currently a challenging research area in which MOO/MCA models and methods are being used. The technological improvements enabling small-scale production of electricity is also expected to introduce further changes in the industry, since the *prosumer* (i.e., simultaneously producer and consumer) will expectedly be able to manage demand, have a local micro/mini generation facility (e.g., a small wind turbine or photovoltaic panels), store energy in static batteries or in an electric vehicle, and buy from or sell electricity back to the grid. Therefore, new and challenging decision contexts emerge at different energy industry levels, which should balance economic efficiency, environmental concerns, social interests, and technological issues.

The first historical applications of MOO/MCA in energy planning date back to the late 1970s, namely concerning power generation expansion planning or the choice of sites for nuclear and fossil-fired generation plants. As the relevance of MOO/MCA models was recognized, a vast amount of literature reported new models, algorithmic approaches and real-world applications to several problems, also witnessing the need to take duly into account problem structuring techniques to shape problems to be tackled and dealing with uncertainty with the aim to obtain robust conclusions [81, 107, 124]. Greening and Bernow [107] even advocated the implementation of several multi-criteria methods in an integrated assessment framework for the design of coordinated energy and environmental policies.

This paper aims at providing an overview of MOO/MCA approaches in a vast range of energy problems, which updates and extends the one in Diakoulaki et al. [81]. Broadly, models and methods dealing with multi-objective optimization models and multi-criteria decision analysis are distinguished. In MOO, mathematical programming models are developed consisting of multiple objective functions to be optimized in a feasible region defined by a set of constraints, with different types of decision variables (binary, integer, continuous, etc.). In MCA a limited number of courses of action (alternatives) are, in general, explicitly known a priori to be evaluated according to multiple evaluation criteria, possibly organized as a hierarchical criterion tree, and the performances of the alternatives may be qualitatively and/or quantitatively expressed using different types of scales (ratio, interval, etc.) thus leading to a bi-dimensional impact matrix (alternatives vs. criteria). The most frequent types of problems reported in the literature are mentioned, briefly stating the structure of models and methods to tackle them. Due to the vastness of literature on this topic, which cannot be exhaustively reviewed, this chapter makes a selection of papers appearing in international journals in the twenty-first century, mostly in the areas of operational research and energy, with the main goal of providing selected references for a variety of problems, modeling and methodological approaches and also displaying general trends as perceived by the domains being covered and methods being used. The main conclusion is that MOO/MCA approaches are essential for a thorough analysis of a vast range of problems in the energy sector at different decision levels and with different timeframes in order to generate usable recommendations that balance multiple, conflicting and incommensurate evaluation aspects.

25.2 Multi-Objective Optimization Models and Methods for Energy Planning

This section addresses MOO models and methods to deal with a large variety of energy problems at different organizational levels and timeframes. Problems are briefly described, the main characteristics of mathematical models are pointed out, and the methods to compute non-dominated solutions are mentioned. It would be a value-added to ascertain the real-world nature of applications. However, papers almost exclusively focus on model formulation and algorithmic approaches. Therefore, these contributions do not generally convey the significance, subtleties and richness of important details that contribute for the success or failure of true real-world studies, from which lessons could be learned for similar decision situations. Nevertheless, these papers play an important role as experimentation frameworks of models and algorithms in realistic (if not real) settings thus contributing to illustrate the potential advantages offered by MOO approaches in hard problems (due to the dimension, non-linear and often combinatorial nature of the search space). In this context, it is worthwhile to mention that MOO has been considered a relevant topic to be included in the undergraduate electric power engineering curriculum [120].

The focus is herein placed on power systems planning problems, which may be broadly categorized according to the analysis timeframe and the type of decisions to be made. A common distinction is made between long-term/strategic, operational and short-term problems (see Table 25.1).

Table 25.1 Categories of planning problems in power systems according to the organizational level and timeframe

Planning	Typical timeframe	Examples of decisions to be made
Long-term/strategic	Several years-decades	Generation expansion planning Transmission facility expansion Siting of new power plants Energy-environment-economy models Market design
Operational	Months-years	Generation scheduling Transmission scheduling Reactive power planning
Short-term	Hours-days-weeks	Unit commitment Power flow Demand-side management

25.2.1 Power Generation Expansion Planning and Operation Planning

Power generation capacity expansion planning was one of the first problems to be addressed using MOO, initially as an extension of single objective cost minimization models. As environmental issues gained an increasing attention, models began to include them as explicit objective functions rather than encompassing them in an overall cost function by using, for instance, monetized pollutant emissions associated with power generation. In these problems the aim is, in general, identifying the amount of power to be installed (number and type of generating units, that is primary energy source and energy conversion technology, sometimes also involving siting decisions) and output (energy to be produced by new and already installed units) throughout a planning period, in general of a few decades. With the development of renewable energy resources, technologies for power generation expansion involve coal units, large scale and small hydro units, conventional and combined cycle natural gas units, nuclear plants, wind farms, geothermal units, photovoltaic units, etc.

The objective functions include the minimization of the total expansion cost (or just production costs), the minimization of pollutant emissions (SO_2, CO_2, NO_x, etc.), the minimization of a surrogate for environmental impacts (an economic indicator obtained by monetizing the pollutant emissions, a ton-equivalent indicator or an aggregate dimensionless indicator), the maximization of the system reliability/safety, the minimization of outage cost, the minimization of radioactive wastes produced, the minimization of the external energy dependence of the country, the minimization of a potential technical risk/damage indicator, the minimization of option portfolio investment risk, the minimization of fuel price risks, and the maximization of employment at national or regional level.

The constraints mainly express generation capacity lower/upper bounds, minimum load requirements, satisfaction of forecasted demand including a reserve margin, resource availability, technology restrictions due to technical or political reasons (e.g., the amount of nuclear power allowed to be installed), domestic fuel quotas, energy security (as a surrogate for diversification of the energy supply), committed power limits, budgetary limitations, operational availability of generating units, rate of growth of the addition of new capacity, transmission constraints due to generation units placement, coal/gas production and transportation capacities, need to account for multiple water uses and capacity in hydro reservoirs, pumping capacities.

Multiple-use hydroelectric systems, and in particular multi-reservoir cascaded systems, impose additional issues to be considered, either as objective functions or constraints, such as competition of different operators on the same basin (scheduling of reservoirs) and balancing energy and non-energy uses, including dam safety, discharges and spills, flood protection and control, agriculture irrigation, industrial and domestic water supply, navigation, dilution of pollutants and heated effluents, recreation, ecological sustainability and protection of species, etc.

Pollutant emissions materialize either expressed as constraints in physical quantities (tons), in general reflecting (national or international) legislation, or (surrogate) environmental objective functions consisting in aggregate indicators penalizing the installed capacity and the energy output.

Besides considering conventional and renewable supply-side options, some models adopt a broader perspective of integrated resource planning by also including demand-side options in the planning process. Demand-side options, resulting from demand-side management programs aimed at operating over end-use loads to shape the load diagram in such a way that peaks are flattened and valleys are filled, such as direct load control activities, may be modeled as an equivalent-generating group with associated operational constraints of capacity and time of operation (in general, just allowed to operate during demand peaks).

The first models proposed in literature were multiple objective linear programming (MOLP) models, thus disregarding the discrete nature of generation units used for power expansion. In some cases, the continuous solutions were then subject to a "discretization" phase using typical capacities of units available for expansion. This issue was then taken into account at the outset by using multiple objective mixed-integer linear programming (MOMILP) models. Non-linear relationships, such as reliability metrics, were often converted into linear expressions by using some kind of linearization technique.

The algorithmic approaches to tackle generation capacity expansion planning models are very diversified and in some way denote the trend from "classical" MOO approaches, both generating methods to characterize as exhaustively as possible the non-dominated solution set and interactive methods using the preference information expressed by decision makers/planners to guide and reduce the scope of the search, to multi-objective meta-heuristics and, in particular, multi-objective genetic/evolutionary algorithms. The use of MOMILP models led to the development of MOO algorithms based on branch-and-bound or cutting planes, in general aimed at characterizing the whole non-dominated solution set. In general, in "classical" approaches a single non-dominated solution is generated through the optimization of a surrogate scalar(izing) function that temporarily aggregates the original multiple objectives also including preference information parameters, in such a way that the optimal solution to this function is non-dominated to the MOO model. Population-based meta-heuristics (genetic/evolutionary algorithms—GA/EA, particle swarm optimization—PSO, differential evolution—DE, etc.) are often justified, besides the combinatorial complexity of the problems, on the grounds that using a population of solutions that expectedly converge to the true non-dominated front (which is generally unknown) is more efficient than resorting to the optimization of scalarizing functions.

Having in mind the exploitation of results in practice, interactive methods are well suited to support decision makers (DM) in selecting a final recommendation, or even a set of non-dominated solutions for additional scrutiny. This is accomplished through a feedback process consisting in the computation of a compromise solution between the competing objectives and a dialogue stage in which the DM's input on this solution is used to modify the scalarizing function to be used in the next computation phase or to guide the search through an adequate change of the meta-heuristics parameters according to that preference information.

Several sources of uncertainty are at stake in these power generation expansion planning models, namely concerning demand growth, primary energy prices, inflows to hydro reservoirs, etc., and even regulations. The uncertainty associated with the model coefficients is usually modeled through stochastic coefficients or, in fewer cases, fuzzy sets, and models are then tackled by means of stochastic or fuzzy programming. Also, scenarios to which a probability distribution is assigned are sometimes used, those embodying sets of plausible instantiations of uncertain model elements, such as the ones mentioned above. The paradigm of robust solutions is also used, in the sense that the variation in objective functions, and even constraints, is within acceptable ranges for uncertain model coefficients and parameters, thus displaying a certain degree of "immunity" of solutions to "moderate" changes in the inputs. Since these types of prior incorporation of uncertainty in the mathematical model leads, in general, to a significant additional computational burden in obtaining solutions, uncertainty may be also tackled by performing (a posteriori) sensitivity analysis of selected compromise solutions.

Generation capacity expansion planning models should take into account the deregulation/liberalization and privatization trends underway, namely in the electricity and gas industries. Therefore, models developed for centrally planned contexts are not adequate whenever the generation segment is operating under a market setting, although this may assume very distinct configurations even keeping some characteristics of the traditional vertical organization. In some way the problem shifts from cost minimization to profit maximization (market revenues minus operational costs) in the perspective of the private generation company competing with similar companies for market share, although very dependent on forms of market or contracts. Therefore, some approaches based on market equilibrium have been proposed, in which each player attempts to maximize its own profits. Some form of central planning authority generally assesses the individual company proposals accounting for keeping overall market efficiency, system reliability levels, and transmission network requirements. It should be noticed that the conjecture of competition among generation companies driving prices down did not materialize in many cases.

Some works are briefly reviewed below, highlighting the characteristics of the model and the method used to compute non-dominated solutions.

Soloveitchick et al. [253] present an MOLP model for setting up the marginal abatement cost in the long run, considering the minimization of generation costs and emissions. Constraints refer to available capacity, including at peak conditions, and satisfaction of instantaneous power demand.

Antunes et al. [21] present an MOMILP model considering as objective functions the total expansion cost, the environmental impact associated with the installed power capacity and the environmental impact associated with the energy output. It takes into account the modular expansion capacity values of supply-side options as well as Demand Side Management (DSM) as an option in the planning process. Constraints are related to the reliability of the supply system, the availability of the generating units, the capacity of equivalent DSM generating group, the total capacity installed throughout the planning period, the pollutant emissions and the available capacity modules for expansion for each generating technology. Decision variables refer to the power to be installed and energy output of generating technologies considered for additions (gas, simple and combined cycle, coal and DSM unit) and those existing at the beginning of the planning period (coal and oil). An interactive MOMILP approach to compute supported and unsupported non-dominated solutions is proposed.

Meza et al. [184] present a long-term MOLP model considering the minimization of total (investment and operation) costs, environmental impact, imported fuel and fuel price risks to decide the location of the planned generation units in a multi-period planning horizon. The Analytic Hierarchy Process (AHP) is used after the non-dominated solutions to the MOLP model are computed.

Kannan et al. [149] use the well-known elitist non-dominated sorting genetic algorithm (NSGA-II) to deal with MOO models to minimize cost and a sum of normalized constraint violations, and to minimize investment cost and outage cost. A 6-year planning horizon and five types of candidate generation units are considered. Murugan et al. [193] use NSGA-II in transmission constrained generation expansion planning, in which objective functions are the same as in Kannan et al. [149] but transmission related constraints are treated as hard constraints.

Meza et al. [185] propose an MOO mixed-integer bilinear model to determine the number and capacity of new generating units (conventional steam units, coal units, combined cycle modules, nuclear plants, gas turbines, wind farms, and geothermal and hydro units), number of new circuits required in the network to accommodate new generation, the voltage phase angle at each node, and the amount of required imported fuel for a single-period generation expansion plan. An EA is used to obtain the non-dominated front and AHP is then used to select the best alternative according to the DM's preferences. A limited sensitivity analysis phase is performed to account for fuel price scenarios.

In several models the problems of power generation and transmission network planning are coordinated. Unsihuay-Vila et al. [273] describe an MOO multi-area and multistage model to long-term expansion planning of integrated generation and transmission corridors. The objective functions are total cost (investments and operational costs, investment costs in DSM programs, investment and operation of carbon capture technologies), life-cycle Greenhouse Gas (GHG) emissions and electricity generation mix diversification. Constraints include supply/demand balance of electricity, bounds on energy supply, DSM programs, transmission lines, lifetime of the infrastructures, carbon capture technology projects, and energy diversification. The carbon abatement policy under the Clean Development Mechanism (CDM) within the European Union Greenhouse Gas Emission Trading Scheme is considered. A compromise approach based on weights and Manhattan and Tchebycheff metrics is used to obtain non-dominated solutions.

Aghaei et al. [9] present a multi-period MOMILP model for generation expansion planning including renewable energy sources (RES). The objective functions are the minimization of overall costs, emissions, energy consumption and portfolio investment risk, and the maximization of system reliability (which is converted into a set of linear expressions). The method to obtain non-dominated solutions is based on hybrid augmented-weighted ε-constraint and lexicographic optimization. In a final stage, fuzzy decision techniques are used to select the most preferred solution based on the DM's goals. A 6-year planning horizon and seven types of candidate generation units are considered.

Planning of distributed energy resources (DER)/dispersed generation (DG) is closely coupled with distribution network planning whenever power injection points are in this network. DER play a key role in addressing energy and environmental challenges, but also require important investments and may bring about technical and reliability issues. DER located near consumption points contribute to reduce power flow in lines, which is also dependent on time coincidence of generation with demand, leading to better voltage profile (quality of service) and lower losses. The possibility of working in an islanded mode is also relevant for the Distribution System Operator (DSO). The DER developer aims at maximizing the energy traded, within technical operation limits. From a societal standpoint renewable DER provide a cleaner energy supply than fossil-fuel energy generation and contribute to mitigate foreign energy dependency. Alarcon-Rodriguez et al. [15] offer a review of MO planning of DER.

Carpinelli et al. [50] present a methodology aimed at finding the best development plan for the system, which considers the management of risks and uncertainties. The problem of optimal sizing and siting DG is formulated as a constrained non-differentiable MOO model to maximize some power quality indicators (including voltage quality and harmonic distortion) and minimize costs (including investment and energy losses). A so-called double trade-off procedure is used, which consists in first obtaining a wide range of DG siting and sizing solutions for the scenarios considered (for instance, associated with different wind speed at possible locations) by means of an ε-constraint method and then identifying the most robust solutions.

Celli et al. [60] present an MOO model for the siting and sizing of DG resources into distribution networks. The methodology offers the exploitation of trade-offs between network upgrading cost, power losses cost, energy not supplied cost, and cost of energy required by the customers served. Non-dominated solutions are obtained using a GA and an ε-constrained method.

Ochoa et al. [207] present an approach to locate and sizing DG in distribution networks and then computing a multi-dimensional performance index for each configuration considering a wide range of operational and security issues. This index may be used to shape the nature of the contract between the utility and the DG. A similar work is described in Singh and Verma [250] to size and locate DG in distribution systems with different load models based on a performance index and using a GA to derive solutions. The main issue is demonstrating that load models can significantly affect the optimal location and sizing of DG resources in distribution systems.

Wang and Singh [284] use an improved PSO algorithm to compute non-dominated solutions to a multi-source hybrid power generation system including wind turbines, photovoltaic panels, and storage batteries, considering cost, reliability, and emissions objective functions. Different sources of uncertainty are taken into account by means of probabilistic methods, which are associated with equipment failures, wind speed, solar insolation, and stochastic generation/load variations.

Katsigiannis et al. [150] study a small autonomous hybrid power system including renewable and conventional power sources, as well as energy storage (lead-acid batteries and hydrogen storage), for which an MOO model is developed considering as an economic objective the minimization of energy costs and as an environmental objective the minimization of the total GHG emissions during the system lifetime. The computation of GHG emissions is based on life cycle analysis of each system component. NSGA-II coupled with a local search procedure is used to obtain non-dominated solutions.

El-Zonkoly [90] describes an MO index-based approach to optimally determine the size and location of DG units in distribution system considering different load models, including the representation of protection device requirements. A range of technical issues such as active and reactive power losses of the system, voltage profile, line loading and the power injected into the grid are accounted for in the objective functions.

Niknam et al. [204] present an MOO model for placement and sizing of RES (photovoltaic, wind turbines and fuel cell units) electricity generators. The objective functions are total costs, deviation of the bus voltage, power losses and emissions. Constraints consider voltage limits, number and size of renewable electricity generators due to budgetary restrictions. Solutions are computed using a honey bee mating optimization algorithm.

Zangeneh et al. [297] develop a fuzzy MOO model to determine the optimal size, location and technology of DG units in distribution systems. The objective functions are the profit of a distribution company selling the DG output power to its customers (including cost, revenue and marginal revenue terms), a weighted sum of technical violation risk (of over/under node voltage, line and transformer overloading, and short circuit capacity), and the amount of pollutant emissions (accounting for CO_2, NO_x, SO_2, CO and PM10). Constraints refer to maximum installed DG capacity at each node and power flow. The forecasted load, electricity market price and parameters related to the DG technologies are uncertain and modeled using fuzzy sets.

Arnette and Zobel [24] develop an MOLP model to determine the optimal mix of RES (wind, photovoltaic, biomass) and existing fossil fuel facilities on a regional basis. The objective functions are total (capital, fixed O&M, variable O&M) cost and emissions. Constraints include biomass availability, biomass as a percentage of total fuel generation, electricity demand, and budgetary restrictions. Solutions are obtained by means of weighting techniques and a min-max approach.

Soroudi and Afrasiab [254] propose a stochastic dynamic MOO model for integration of DG in distribution networks, considering the minimization of technical constraint dissatisfaction, costs and environmental emissions, to determine the optimal location, size and timing of investment for both DG units and network elements. The uncertainties of load, electricity price and wind power generation are taken into account using scenarios. A binary PSO algorithm is used to compute non-dominated solutions and a fuzzy satisfying method is applied to select the best solution considering the planner's preferences.

Hybrid (i.e. combining several sources) renewable energy systems have been increasingly used as a sustainable and reliable power supply option for stand-alone applications, especially in remote areas. A review of MOO models using evolutionary algorithms devoted to stand-alone systems is presented in Fadaee and Radzi [93], including placement, sizing, operation, design, planning and control decisions.

Dufo-López et al. [87] apply the Strength Pareto Evolutionary Algorithm (SPEA) to the MOO model of a stand-alone photovoltaic-wind-diesel system with battery storage. The objective functions are energy costs and the equivalent CO_2 life cycle emissions, subject to load satisfaction. Solutions display different combinations of energy conversion technologies and operating schedule during the year.

Operational problems arising at generation level are also tackled using MOO models. A method to design the power-pressure mapping at fossil fuel power plants is presented by Garduno-Ramirez and Lee [99] by defining an MOO problem that is developed as a supervisory set-point scheduler. A nonlinear goal programming method is used to compute a single solution from the set of non-dominated solutions based on the assignment of relative preference values to the objective functions, encompassing a diversity of operating scenarios. Heo et al. [118] then addressed this problem of establishing the set points for controllers in fossil fuel plants by using variations of PSO, including an evolutionary PSO. Later, Heo and Lee [117] present a multi-agent methodology to derive an MOO model that is then tackled by a PSO algorithm to define the set points (for steam pressure and reheater/superheater steam temperatures) in control loops for a large oil-fired power plant, which should be mapped with the changing load demand and satisfy the conflicting requirements in plant operation.

Another important topic in power systems for which MOO models are relevant deals with maintenance scheduling problems. Yare et al. [295] use a modified discrete PSO algorithm to derive preventive maintenance schedules of generating units for economical and reliable operation of a power system, while satisfying system load demand and crew constraints. Jin et al. [139] propose an MOO model to design and operate a wind DG system, involving the determination of equipment sizing, siting, and maintenance schedules in order to minimize system cost (consisting in capital, operations, maintenance, downtime losses, and environmental penalty costs) and maximize turbine reliability. Power intermittency (due to wind speed) and load uncertainty are taken into account and an MO GA is used to compute non-dominated solutions for the equipment siting, sizing, and maintenance intervals.

The economic and environmental challenges associated with nuclear energy are also addressed. Zhang et al. [300] carry out an economic/environmental analysis of power generation expansion in Japan taking into account the Fukushima nuclear accident, which obliged the Japanese government to review its nuclear power policy. The objective functions are total net present value cost and CO_2 emissions, subject to supply–demand balance, fuel, installed capacity, budgetary, and environmental constraints, and considering nuclear power scenarios (actively anti-nuclear, passively negative towards nuclear, conservative towards nuclear, and active nuclear expansion).

25.2.2 Transmission and Distribution Network Planning

The network infrastructure (both transmission and distribution) plays a critical role in providing energy to consumers. When utilities were vertically integrated, thus owning transmission network and generation assets, the planning process was generally integrated. Being the sole provider of services along the whole industry chain, utilities had complete data and forecasting capability about demand and its evolution. The complete knowledge about decisions concerning the installation of new generation units or the retirement of existing ones enabled also a more controlled planning of the transmission network. Since nowadays generation and transmission are usually separated by means of functional unbundling or company split, and due to competition in electricity generation, transmission network planning is a more complex task. This may lead to sub-optimal decisions from a societal perspective since, for instance, a lower rate of transmission expansion can impair investments in new generation to serve increasing load. Moreover, since the "owner of power" is likely to change between generation and delivering to loads, transmission flows in liberalized markets impose further transmission network planning challenges. A reliable and efficient network infrastructure is essential to ensure competitive wholesale and retail segments of the market, which requires adequate planning models and methods. These (namely gas and electricity) network infrastructures have natural monopoly characteristics, which in turn imposes suitable access mechanisms and, in general, regulatory frameworks able to take duly into account the overall societal perspective.

Focusing on power networks, the transmission network has a central position in system operations and wholesale markets. This network is generally managed by a Transmission Network Operator (TNO) that is responsible for planning, namely regarding infrastructure expansion and technological modernization, and operation to offer efficient, reliable, and nondiscriminatory service. In several countries, namely due to the network extension, regional operators exist, thus requiring further coordination. Transmission network planning models are aimed at determining the location, the size and the time frame of the installation of new circuit additions to supply the forecasted load throughout the planning period, considering economic, environmental, technical and quality of service objectives subject to operating constraints given existing network configuration and generation units. Aspects generally contemplated either as objective functions or constraints are: economic—construction/reinforcement costs, equipment (transformer stations, protection devices, etc.) upgrade costs, congestion costs, energy losses costs, regional or national economic growth induced by projects, facilitating competitive wholesale markets; environmental—impacts of line corridors, effects on location of power plants, need to account for remote disperse renewable generation; technical—network topology, inter-control area flows, reliability standards associated with thermal, voltage and stability requirements; quality of service indicators—system/customer average interruption frequency/duration indices, momentary average interruption frequency index; public health—population exposure to electromagnetic fields.

Distribution networks carry electric energy from transmission networks to customers. Distribution Network Operators (DNOs) are generally organized on a geographical basis and should provide a reliable operation complying with technical and quality of service parameters, taking into account the dynamics of end-use loads at different time frames. The network distribution planning should also support the operation of electricity market by enabling non-discriminatory access to the network. The introduction of dispersed renewable generation, sometimes at the distribution network level, is changing the distribution network planning process since this now needs to accommodate not just traditional and new loads (for instance, the electric vehicle) but also micro- and mini-generation facilities. The ongoing evolution to smart grids, offering the technological basis using sophisticated Information and Communication Technologies (ICT) techniques to accommodate responsive demand, storage, and local generation, creates new challenges regarding distribution network planning in a more dynamic stance taking into account the integrated management of supply and demand resources.

A set of representative works is briefly reviewed below, underlining the characteristics of the model and the methods used to compute non-dominated solutions.

Bhowmick et al. [39] consider the minimization of the substation and feeder costs and the interruption costs as linear objective functions. Constraints refer to network radial characteristics, load satisfaction, power flow and interruption duration.

Ramirez-Rosado and Bernal-Agustin [230] present an MOO approach based on an EA to maximize network reliability and minimize the distribution system expansion costs. A non-linear mixed integer model provides the sizing and location of future (reserve and operation) feeders and substations.

Chung et al. [72] consider investment cost, reliability and environmental impact as objective functions. A GA is developed to compute possible planning schemes. This step is followed by a fuzzy decision analysis method to select a final solution.

Carvalho and Ferreira [55] study the trade-off of backup-circuit investment decisions and the cost of energy not supplied in distribution network planning, taking into account reliability issues.

Ramirez-Rosado and Dominguez-Navarro [231, 232] present an MO Tabu Search approach to solve a fuzzy model for optimal planning of distribution networks considering three objective functions: minimization of fuzzy economic cost, maximization of fuzzy reliability, and maximization of solution robustness. The size and location of reserve feeders and substations for maximizing the level of reliability at the lowest economic cost, for a given level of robustness, is determined.

Braga and Saraiva [42] present a multi-year dynamic transmission expansion planning MOO model, considering as objective functions the investment costs, operation costs, and expected energy not supplied. An interactive approach is used starting from a non-dominated solution that is computed by specifying aspiration levels for two of those objectives and using simulated annealing to deal with the integer nature of investment decisions. The DM can then change the aspiration levels and obtain new solutions. Once an expansion plan is accepted, the algorithm computes long-term marginal costs, reflecting both investment and operation costs, which are more stable than short-term ones and inherently address the revenue reconciliation problem in short-term approaches.

Carrano et al. [52] present an MOO approach for providing decision support to electrical distribution network evolution planning, considering two objective functions to be minimized: an aggregate (installation and energy losses) cost and a system failure index. An MO GA is used with a problem-specific variable encoding scheme and mutation and crossover operators.

Mendoza et al. [181] apply NSGA and SPEA (with a fuzzy c-means clustering algorithm) approaches to an MOO model for designing power distribution networks. The objective functions consist in minimizing the total costs and maximizing the reliability, subject to technical constraints.

Carrano et al. [51] develop an immune-based EA for the electric distribution network expansion problem under uncertainty in the evolution of node loads. A Monte-Carlo simulation of the future load conditions is performed to evaluate solutions within a set of possible scenarios. A dominance analysis is then carried out to compare the candidate solutions, considering as objectives the infeasibility rate, the nominal cost, the mean cost and the fault cost. The design outcome is a network that has a satisfactory behavior under the considered scenarios, thus leading to networks displaying more robust performances under load evolution uncertainties.

Harrison et al. [113] present an MO optimal power flow model to simulate how (the UK scheme) incentives to DG developers and DNO affect their choice of DG capacity within the limits of the existing network. Costs, benefits and tradeoffs associated with DG in terms of connection, losses and network deferral are explored to assess whether incentives encourage both parties to make DG connections.

Hazra and Sinha [115] present a non-linear model for congestion management in transmission networks by generation rescheduling and/or load shedding of participating generators and loads considering two objective functions: minimization of overload and cost of operation. An MO PSO approach is used to derive the non-dominated front.

Maghouli et al. [174] develop a multi-stage transmission expansion methodology using a mixed integer MOO framework with internal scenario analysis. The objective functions are total social cost, maximum regret (robustness criterion), and maximum adjustment cost (flexibility criterion). Uncertainties are considered by defining a number of scenarios. NSGA II is used to obtain the non-dominated front and fuzzy sets are applied to obtain the most preferred solution.

Soroudi et al. [255] present a long-term dynamic MOO model for distribution network planning involving determining the optimal sizing, placement and timing of investments on DG units and network reinforcements over the planning period. The objective functions are the benefits of DNO and DG operators. Uncertainty of loads, electricity prices and wind turbine power generation is dealt with using the point estimation method. A two-stage heuristic method is used to solve the model.

Zhao et al. [301] develop a market simulation-based method to assess the economical attractiveness of different generation technologies to shape future scenarios of generation expansion. An MOO model for transmission expansion planning is proposed to select transmission expansion plans that are flexible given the uncertainties of generation expansion, system load, and other market variables, namely concerning the impacts of distributed generation.

Ganguly et al. [98] develop an MOO model for electrical distribution systems planning to determine the optimal feeder routes and branch conductor sizes. The objective functions are the total cost associated with network failure (non-delivered energy, cost of repair, customer damage cost due to interruptions) and economic costs (installation of new facilities, capacity expansion, maintenance, cost of energy losses). Constraints include power demand and supply balance, limits on power flows in substation and feeder branches, upper and lower limits for the node voltages, and network radiality. An MO dynamic programming algorithm based on weighted aggregation is used to compute non-dominated solutions.

Gitizadeh et al. [102] present an MOO model for multistage distribution network expansion planning model considering DG. The objective functions are investment and operations costs and energy not supplied (reliability index). Constraints include power flow equations, distribution transformers capacities, feeders and branches capacities, distributed generation resources capacities, voltage limits, and radial structure of the network. A hybrid PSO and shuffled frog-leaping algorithm is used to compute solutions.

Tant et al. [261] propose an MOO method to assess the trade-offs between three objective functions—voltage regulation, peak power reduction, and annual cost—to study the potential of using battery energy storage systems in the public low-voltage distribution network with the aim of deferring upgrades needed to increase the penetration of photovoltaic generation systems. Results are related with dimensioning decisions of the battery (considering different technologies, such as lithium-ion and lead-acid) and the inverter.

The installation of flexible ac transmission systems (FACTS) in existing transmission networks can be used to improve the transmission system load margin and reduce the network expansion cost. Ara et al. [22] present a mixed-integer nonlinear MOO model to determine the location of FACTS shunt-series controllers (phase-shifting transformer, hybrid flow controller, and unified power-flow controller). The objective functions are the total fuel cost, power losses, and system loadability with and without minimum cost of FACTS installation. The ε-constraint technique is used to obtain non-dominated solutions. Chang [63] develop an MOO model to determine which buses need static var compensators (SVC), considering the maximization of load margin and the minimization of SVC installation cost. The model is then solved using a fitness sharing MO PSO algorithm to obtain the non-dominated front, under each contingency with high risk index.

The growth in the penetration of (especially large scale) wind farms in power systems leads to the need of considering its impacts on transmission network expansion planning. For this purpose Moeini-Aghtaie et al. [188] propose an MOO model considering as objective functions the investment cost, risk cost and congestion cost. A combination of Monte Carlo simulation and Point Estimation Method is used to capture the effects of network uncertainties. NSGA II is used to compute the non-dominated front and a fuzzy decision making approach based on the DM's preferences is used for selecting the final solution.

Fuel cells are environmentally clean, can operate with low noise levels, and can provide energy in a controlled way with high efficiency. Niknam et al. [204] present an MO fuzzy self-adaptive PSO-EA to solve an operational management problem considering fuel cell power plants in the distribution network. The objective functions to be minimized are total electrical energy losses, total electrical energy cost, total pollutant emissions, and deviation of bus voltages.

The reconfiguration of distribution feeders, which are generally operated in a radial structure, is done by the DSO during normal or emergency operational planning. Network reconfiguration is carried out by changing the status of sectionalizing (normally closed) switches and tie line (normally open) switches, thus leading to combinatorial problems. Decisions about the status of those switches should be evaluated by means of objective functions such as line losses, load balancing, voltage drop and number of switching actions.

Hsiao and Chien [130] develop a constrained non-differentiable MOO model for the feeder reconfiguration problem to reduce power loss, increase system security and improve power quality, subject to operational constraints. The objective functions are formulated as fuzzy sets to capture their imprecise nature and an EA approach is then used.

Huang [133] proposes a fuzzy network for MO service restoration of distribution systems in which the multiple objective functions are combined into a weighted-sum function with weights derived using AHP. Fuzzy cause-effect networks are built to represent the knowledge and the inference scheme elicited from operators' needs, as well as heuristic rules expressed in imprecise linguistic terms.

Lin et al. [168] present an immune algorithm for determining switching operations to achieve loss minimization and loading balance among feeders and main transformers. An interactive best-compromise method is applied to solve the distribution-feeder reconfiguration providing quantitative measures to aid the decision-making process.

Hsiao [129] uses an MO EA approach for distribution feeder reconfiguration considering the minimization of power losses and switching operations, and the maximization of voltage quality and service reliability. An interactive fuzzy algorithm is used to obtain a compromise solution based on the operator's judgments.

Hong and Ho [125] present a fuzzy MO model dealt with a genetic algorithm to determine the configuration of a radial distribution system, taking normal condition and contingencies (faults) into account. Minimization of the active losses is central when the system operates in a normal condition, while voltage drop should be minimized when a fault occurs.

Prasad et al. [228] propose a fuzzy mutated GA for reconfiguration of radial distribution systems considering the minimization of power loss and a voltage deviation index. The algorithm guarantees the radial property of the network without islanding any load point.

Das [79] presents an algorithm for radial network reconfiguration based on heuristic rules and a fuzzy MO approach. The objective functions to be optimized involve load balancing among the feeders, real power loss, deviation of nodes voltage, and branch current constraint violation. Fuzzy sets are used to deal with the imprecise nature of these objectives and heuristic rules are incorporated for minimizing the number of tie-switch operations.

Ahuja et al. [13] propose an MOO model for distribution system reconfiguration, which is solved using a hybrid algorithm based on artificial immune systems and ant colony optimization. The search space is explored by means of the hyper-mutation operator that perturbs existing antibodies to produce new ones to obtain solutions to restore the distribution system under contingency situations. The objective functions are real losses, transformer load balancing, and voltage deviation.

Savier and Das [242] present a model to allocate power losses to consumers connected to radial distribution networks before and after network reconfiguration in a deregulated environment. The network reconfiguration algorithm is based on a fuzzy MO approach using the max-min principle. The objective functions are related with real power loss reduction, node voltage deviation, and absolute value of branch currents.

Falaghi et al. [94] present a method for sectionalizing switches placement in distribution networks with DG sources. The objective functions are the maximization of reliability and the minimization of sectionalizing switches cost. A fuzzy membership function is defined for each term in the objective functions. The relocation of existing switches and operational constraints concerning distribution networks and DG during post-fault service restoration are considered. The fuzzy MOO model is dealt with an ant colony optimization approach.

Mendoza et al. [182] propose an MO GA approach for power distribution network reconfiguration considering as objective functions power system losses and reliability indices.

Yin and Lu [296] present a distribution network feeder operation MOO model considering network efficiency balance, switching and reliability costs. The annual feeder load curve is divided into multi-periods of load levels and the feeder configurations for different load levels in annual operation planning are optimized. A binary PSO search is used to determine the feeder switching schedule.

Bernardon et al. [36] propose a fuzzy MOO model to minimize power losses and maximize reliability (number of interrupted customers) in distribution network reconfiguration, considering sub-transmission systems. Constraints refer to the radial characteristics of the network, and limits for the current magnitude in network elements and voltage magnitude in network nodes. A heuristic search procedure is used based on the branch-exchange strategy to design configurations that are then evaluated by the Bellman-Zadeh approach.

Gupta et al. [111] present an MOO model for the reconfiguration of radial distribution systems in a fuzzy framework, which is dealt with an adaptive GA. The genetic operators are adapted with the help of graph theory to generate feasible individuals. The objective functions are the minimization of real power loss, node voltage constraint violation, branch current constraint violation, and number of switching operations, subject to the network radial structure with all nodes energized.

Santos et al. [240] present a formulation for system reconfiguration in large-scale distribution networks. Problems are modeled using a node-depth encoding, a technique for tree encoding instead of a graph chain representation to network design, for which operators are easier to implement and adapt to different problems. An MO EA based on sub-population tables is then used to explore the constrained search space, considering objective functions related to power losses, number of switching operations, network loading, substation loading, voltage ratio and aggregation function.

Singh and Misra [249] present an MO feeder reconfiguration model usable in different tariff structures to minimize the overall cost of MW, MVAr and MVA intakes of an in-house distribution system. Different load types and tariff structures are simulated to conclude that load type is a major factor in reconfiguration and cannot be represented by constant load models.

Tsai and Hsu [268] use Gray Correlation Analysis to integrate the objective functions and provide a relative measure to a particular switching plan associated with a chromosome in an EA framework without any prior knowledge of the system under reconfiguration. The objective functions are the system real power losses, the estimated maximum percentage voltage drop in the system, the load balancing index, and the total number of switching actions during feeder reconfiguration.

Niknam et al. [206] propose a stochastic MOO model for distribution feeder reconfiguration, considering as objectives functions total power losses, voltage deviation and total cost. Uncertainties associated with wind power generation and active and reactive load are explicitly considered. The methodology consists, in the first stage, of a roulette wheel mechanism in combination with Weibull/Gaussian probability distribution function of wind/load forecast variations for random scenario generation in which the stochastic problem is converted into deterministic scenarios. In the second stage, a modified MOO PSO is implemented for each deterministic scenario.

Sardou et al. [241] propose a modified shuffled frog-leaping algorithm to derive the optimal placement of manual and automatic switches in distribution systems. The objective functions are the minimization of the customer interruption cost and switches' investment and maintenance cost. Customer types, customers load patterns, and different time-varying loading rates are considered besides network branch failure rates, restoration time and repair time.

Gu and Zhong [108] develop a network reconfiguration approach based on a two-layer unit-restarting framework for the power system restoration process, i.e. network-layer unit restarting and plant-layer unit restarting. An MOO model is based on this two-layer framework, in which the network-layer unit restarting, the plant-layer unit restarting and the restoration of important loads are separately considered with their own models and solving algorithms. A lexicographic method is then used to solve the MOO model by coordinating the solution search processes of the three sub-problems to determine the restarting sequence of all units and restoration of the important loads.

Guedes et al. [109] develop a heuristic based on a branch-and-bound implicit enumeration scheme to minimize the total power loss and the maximum current of electrical radial networks, in the reconfiguration of an electrical radial network. Pareto dominance is used for pruning the search tree.

The likelihood of fault occurrence and the size of the area affected by a fault in the network have augmented due to increasing demand, the size and complexity of power distribution systems. Therefore, the fast restoration of the power supply to the unaffected out-of-service areas is an important issue to sustain customer satisfaction and revenue level. Mao and Miu [177] develop a non-differentiable MOO model for switch placement to form self-supported areas after fault isolation aimed at improving system reliability for radial distribution systems with DG under fault conditions, also considering customer priority. Graph-based algorithms incorporating direct load control are developed to locate switches.

Kumar et al. [158] present an NSGA-II based approach for solving the service restoration problem in an electric power distribution system. The objective functions are the minimization of out-of-service area, the number of manually controlled switch operations, the number of remotely controlled switch operations, and losses. Constraints are related to radial network structure, bus voltage limits, feeder line current limits, and customer priority.

Cossi et al. [74] develop an MO mixed-integer non-linear programming model (MINLP) for primary distribution network planning problem. The objective functions are the expansion and operation costs and the system reliability costs in contingency events. Reliability costs result from the non-supplied energy due to repairing and switching operations in the distribution network to isolate and to redistribute loads in the affected sections by permanent faults. An MO reactive Tabu Search algorithm is used based on dominance to obtain the Pareto optimal frontier. The problem of placement of sectionalizing (automatic or manual) switches to restore the distribution network and to reduce non-supplied energy costs when permanent faults occur in the network is solved simultaneously with the network expansion planning problem using a GA.

Competitive electricity markets require the consideration of additional technical issues in network expansion planning. Louie and Strunz [170] develop a hierarchical MOO approach to address the economic market objective of an independent system operator in competitive electricity markets while considering secondary objectives, which are locally optimized, through coordinated control of network devices such as phase shifting transformers and series FACTS. The secondary objective is a wide-area impact index assessing the effects of parallel flow over multiple lines in a region.

Xu et al. [292] present a model to expand the transmission network in open-access schemes. The method starts with a candidate pool of feasible expansion plans and the selection of the best candidates is carried out through MOO integrating the market operation and planning in a deregulated system. The objective functions are the total expansion investment, the generation cost and the profit prospect of the expanded transmission lines, which is based on the MW-mile pricing method to allocate costs based on actual system usage. Human intervention is required in both stages to take into account practical engineering and management concerns. Reliability criteria intervene before an expansion plan is adopted.

The evolution to smart grids imposes several challenging problems regarding the integration of significant levels of distributed renewable generation (with their characteristics of higher variability and, in some cases, non-dispatchability in comparison with convention thermal plants), storage options (also taking into account the dissemination of electric vehicles) and demand responsive energy management systems. Brown et al. [46] propose a model to determine potential locations for adding interties between feeders in a radial distribution system to improve the reliability in the islanded mode of operation. The objective functions are cost and a reliability measure associated with feeder addition. Constraints are related to budget allowed, improving reliability beyond the base case, voltage lower and upper bounds at buses, and loading on the lines. An empirical equation incorporating the capacity factors of renewable generation is used to model the power output of the distributed sources. An MO GA is used to compute non-dominated solutions.

The guarantee of asset performance is a leading goal for electric power network managers. Ascertaining the optimal balance between preventive and corrective maintenance is of utmost importance for achieving that goal, being necessary pondering life-cycle and maintenance costs as well as constraints imposed by demand and regulators. Hilber et al. [122] develop an MOO model with this aim, including customer interruption cost and network operator's (preventive and corrective) maintenance costs. An evolutionary PSO approach is proposed to compute solutions.

Yang et al. [294] present an MO EA to minimize the overall substation cost and maximizing reliability in electric power distribution networks. The scheduling of substation preventive maintenance provides different trade-offs between these objective functions. Decision-varying Markov models relating the deterioration process with maintenance operations are developed to predict the availability of individual components then enabling to identify critical components and evaluate the overall substation reliability.

Yang and Chang [293] present an integrated methodology to compute preventive maintenance schedules to optimize overall cost and reliability consisting of three functional modules to model the stochastic deterioration process of individual component with a continuous-time Markov model, evaluate the reliability of a composite power system taking into account the configuration and failure dependence of the system, and compute compromise solutions using a Pareto-based MO EA.

Power quality in distribution systems generally decline due to an increase in nonlinear loads. Hong and Huang [126] present an interactive MO nonlinear programming approach based on GA to passive filter planning. Short-circuit capacity of the point of common coupling and the individual bus loads are modeled with fuzzy sets. The objective functions to be minimized are the harmonic voltages and the filter cost, subject to satisfying the harmonic standard and harmonic power flow equations.

25.2.3 Reactive Power Planning and Voltage Regulation

Consumer loads impose active and reactive power demand. Active (real) power is converted into "useful" energy, such as light or heat. Inductive reactive power that is imposed, for instance, to generate the magnetic field of asynchronous electric machines, must be compensated. This can be achieved by installing capacitor banks in order to guarantee an efficient delivery of active power to loads, releasing system capacity, reducing system losses, and improving bus voltage profile, thus promoting economic and operational/quality of service benefits. The reactive power compensation planning (also referred to as VAr planning) problem involves determining the location and size of capacitors, which provide locally reactive power, to be installed in electrical distribution networks, which are generally operated in a radial structure. This is a non-linear problem with binary and continuous variables. Objective functions generally express investment, installation, and operation and maintenance costs, power losses, economical operating conditions, system security margin (line overloads due to excessive power flow), voltage deviation from the ideal voltage at buses and quality of service indicators. More recent methods to deal with this problem are based on meta-heuristics to cope with its combinatorial nature, namely population based approaches devoted to MOO models.

Hsiao and Chien [131] deal with the optimal capacitor allocation problem considering as objective functions investment cost, operating efficiency, system security and service quality. An interactive trade-off algorithm is used based on the ε-constraint technique and the DM's preferences on system operating policies to obtain a compromise non-dominated solution.

Augugliaro et al. [26] deal with the problem of optimal control of shunt capacitor banks under load tap changers located at HV/MV substations coupled with optimal control of tie-switches and capacitor banks on the feeders of a large radially operated meshed distribution system. The objective functions are the minimization of power losses and the flattening of the voltage profile. An MO heuristic strategy based on fuzzy sets is used.

Pires et al. [223] present a non-linear mixed integer MOO model considering capacitor installation cost and active power losses as objective functions to evaluate the quality of solutions the capacitor location problem in radial distribution networks. Solutions are obtained using a Tabu Search approach. A similar model is dealt with in Antunes et al. [20] using an MO simulated annealing approach based on the non-dominance relation and some form of aggregation of the objective functions to compute the acceptance probability function.

Li et al. [166] propose an integer-coded MO GA using non-dominance for reactive power compensation planning, considering both intact and contingent operating states, to solve the siting problem of the installation of new devices and the operational problem of preventive transformer taps and the controller characteristics of dynamic compensation devices. The objective functions are the voltage deviation from the ideal setting and the cost associated with the installation and use of reactive power compensation devices.

Ma et al. [171] deal with a real-time power voltage control problem. Control devices such as capacitors and on-load tap changers, as well as load shedding are modeled as discrete control variables. The objective functions are the deviation of instant voltage at buses and number of operations of control devices and load shedding. An MO jump gene EA is used to obtain widespread control solutions that are then analyzed using the Simple Multi-attribute Rating Technique (SMART).

Malekpour and Niknam [175] present an MOO model for the Volt/VAr control problem in distribution systems with high wind power penetration. A probabilistic load flow approach using the point estimate method is employed to model the uncertainty in load demands and electrical power generation of wind farms. The objective functions are electrical energy costs generated by fuel cell power plants, wind farms and distribution companies, total electrical energy losses, and emissions. Constraints refer to balanced distribution power flow equations, renewable sources active and reactive power, distribution line active power flow limits, transformer taps, capacitors reactive power, power factor, and bus voltage magnitude at each load level. A modified Frog Leaping Algorithm is used to achieve the optimal values for active and reactive power of wind farms and fuel cell power plants, reactive power of capacitors and transformers tap positions for the next day ahead. The objective functions are fuzzified and a max-min approach is used to compute solutions.

Segura et al. [244] formulate the capacitor placement problem as an MOO model including economic and technical aspects. The quadratic minimization of the voltage harmonic distortion produced by the harmonic currents drawn by non-linear loads is also considered, besides cost and losses functions. The real capacitor lifetime issue is dealt with using a resonance index and an aging model of capacitor dielectric insulation under a non-sinusoidal waveform scenario.

Alonso et al. [16] present an MOO model for reactive power planning in networks with wind power generation. The aim is aiding power system operators to determine the optimal placement to locate wind farms and FACTS devices as well as the amount of reactive power to be injected into the network. The objective functions are voltage stability, active power losses, and investment costs of the VAr injection sources. Constraints refer to power flow equations, power generation from fixed speed and variable speed wind turbines, voltage limits at buses, limits of the loadability factor, limits of variable speed generators, limits of static VAr compensators, and physical restrictions in the wind farm connection point. An MO GA is used to obtain non-dominated solutions.

Niknam et al. [200] propose a stochastic nonlinear mixed-integer MOO model for daily voltage/VAr control including hydro turbine, fuel cell, wind turbine, and photovoltaic power plants. The objective functions to be minimized are electrical energy losses, voltage deviations, total electrical energy costs, and total emissions of renewable energy sources and grid. The uncertainty associated with hourly load, wind power, and solar irradiance forecasts are modeled in a scenario-based stochastic framework to convert the stochastic MOO model into a series of equivalent deterministic models. An EA is used to solve these models, including a mutation scheme to enhance the global searching ability and mitigate the premature convergence.

Especially in lengthy rural electric power distribution systems, automatic voltage regulators (AVR), which are auto-transformers with individual taps on windings, may be installed at a substation or along distribution lines aimed at providing customers with steady voltage independent of how much power is drawn from the line. AVR compensate voltage drops through distribution lines to reduce energy losses thus improving energy quality. Chang and Yang [64] develop an MOO model for planning series compensation devices, series voltage restorer and fault current limiter, for power quality improvement in distribution systems, involving a composite set of fuzzy performance indices comprising the cost expenditure of installed series compensation devices, voltage boosts across sensitive loads and overall voltage improvement. A Tabu Search approach is used to obtain solutions and robustness is taken into account considering preselected internal faults, external faults and simultaneous disturbances.

Mendoza et al. [183] present an MOO approach to define the optimal location of AVR in electric distribution networks, considering the total power losses and the voltage drop in the system as the objectives to be optimized. A micro GA is used to compute Pareto optimal solutions and offering the DM a set of possible (trade-off) solutions.

Souza and de Almeida [256] use SPEA-2 to determine the installation of AVR banks and capacitors in radial distribution feeders to reduce losses and improve the network voltage profile. Expert knowledge is taken into account via fuzzy logic in order to reduce the search space.

Wang et al. [285] use a PSO approach for improving the dynamic voltage security of a power system, involving continuous and discrete control variables. The aim of the optimal coordinated preventive control is formulated as a non-linear MOO model to optimize the terminal voltage, the output power of each generator and the tap position of each on-load tap changer so as to keep the voltage secure along the trajectory of a quasi-steady-state time-domain simulation when contingencies in generation facilities or transmission line occur.

Niknam et al. [202] propose an MOO model to determine the location of AVR in distribution systems with DG. The objective functions are electricity generation costs, electrical losses in the distribution system and the voltage deviations. Constraints refer to active power constraints of DG units, AVR's tap position, and bus voltage magnitude. An algorithm based on a modified teaching-learning EA is used to obtain Pareto optimal solutions.

A vast range of operational problems is at stake in power systems, such as load frequency control and harmonic distortion, which should be addressed considering multiple objectives to assess the merits of solutions. Darvishi et al. [78] use a fuzzy MO model to optimize the power factor and total harmonic distortion, while limiting selective harmonic distortion, in the framework of an active filter based power quality scheme. A differential evolution approach is used to obtain solutions.

25.2.4 Unit Commitment and Dispatch Problems

Broadly, the unit commitment problem consists in scheduling generating power plants to be on, off, or in stand-by mode, within a planning period to meet demand load. When the power system is vertically integrated, unit commitment is carried out by the utility in a centralized manner and the objective function is minimizing overall costs (the generation cost function is generally approximated as a quadratic function of the power output) subject to meeting demand and reserve margins. When generation is under competition, a generation company must decide locally the unit commitment plan in order to maximize its profit taking into account established power contracts and the energy it estimates it may sell in a competitive (spot) market according to price forecasts. Technical constraints such as capacity constraints, stable operating levels, minimum time period the unit is up and/or down, or maximum rate of ramping up or down should be included in mathematical models. Economic dispatch problems consist in determining the optimal combination of power output of online generating power plants to minimize the total fuel cost while satisfying load demand and operational constraints. Since load demand can vary swiftly, dispatch should be able to react and adapt while guaranteeing adequate cost or profit levels, considering technical issues such as voltage control, congestion, transmission losses, line overloading, voltage profile, deviations of technical indicators from standard values. Also, particular market structures should be taken into account. While a generation company in a competitive environment intends to maximize profits, entities such as an independent system operator aims at maximizing social welfare, and these perspectives should be reconciled in decision aid models. Economic-environmental dispatch generally leads to MOO models in which cost minimization, or profit maximization, and environmental impact minimization (namely harmful emissions originated at fossil-fuel power plants) are explicitly considered. Hobbs et al. [123] investigate the implications of a deregulated market on unit commitment models.

Some works are briefly outlined below, including the main characteristics of the model and the methods used to obtain non-dominated solutions.

Dhillon et al. [80] present a fuzzy MOO model to determine the generation schedule of a short-range hydrothermal problem, considering the minimization of cost, NO_x emissions, SO_2 emissions, CO_2 emissions, and variance of generation mismatch. Statistical uncertainties in the thermal generation cost, NO_x, SO_2 and CO_2 emission curves and power demand, are captured as random variables. A weighting approach is used to exploit the trade-offs between these objectives. Fuzzy sets are then used to assist the system operator to choose the weights to obtain the operating point that maximizes the overall satisfaction.

Abido [1] formulates the environmental/economic power dispatch problem as a nonlinear constrained model. An NSGA approach is used to compute the non-dominated frontier and fuzzy sets then extract a “best compromise” solution from the trade-off curve.

Brar et al. [44] consider cost and emission objectives in a thermal power dispatch problem to allocate the electricity demand among the committed generating units, subject to physical and technological constraints. A "best compromise" solution is obtained by searching for the optimal weighting pattern (the one that attains the maximum satisfaction level of the objective membership functions) using a GA.

Centeno et al. [61] develop a weighted mixed integer goal programming model to deal with the problem of converting an energy schedule into a power schedule, respecting the reserve schedule as well as technical constraints. Goals are associated with the unit's energy schedule, total energy scheduled, unit's positive reserve schedule, total cost for the company, and smoothness of power changes. The constraints are related to the linearization of energy costs, limits for power generation and reserve, reserve limits intervals, ramp rates, and security limits.

An EA is presented in Tsay [269] to solve the economical operation of a cogeneration system under emission constraints. The objective functions are the minimization of cost and several types of emissions. The cost model includes fuel cost and tie-line energy. The emissions considered are CO_2, SOx, and NO_x, which are derived as a function of fuel enthalpy. The constraints include fuel mix, operational constraints, and emissions. The steam output, fuel mix, and power generation are computed considering the time-of-use dispatch between cogeneration systems and utility companies.

Bath et al. [29] present an interactive fuzzy satisfying weighting method to decide the generation schedule considering explicitly statistical uncertainties in the system production cost data, pollutant emission data and load demand. The objectives are the operating cost, NO_x emissions and risk due to the variance of active and reactive power generation mismatch. The Hooke-Jeeves' algorithm and evolutionary search techniques are used to generate the "best" solution in the framework of an interactive approach.

Abido [2] discusses the potential and effectiveness of different Pareto-based MO EA for solving a constrained non-linear MOO environmental-economic electric power dispatch problem. A hierarchical clustering algorithm is used to provide the power system operator with a representative and manageable Pareto optimal set. A fuzzy set based approach is developed to identify one of those solutions as the best compromise one.

Bath et al. [30] present a mixed stochastic-fuzzy MOO approach to decide the generation schedule of committed thermal stations, considering the minimization of fuel cost, gaseous pollutant emission, variance of active and reactive generation mismatch, and a voltage deviation to avoid violation of active power-line flow limits. Statistical uncertainties are taken into account in the thermal generation cost, gaseous emission curves, active and reactive power demand, and voltage magnitude at each bus. The Hooke-Jeeve method is used to generate non-dominated solutions within minimum and maximum limits of power generation and then a min-max technique is used to select the optimal solution interactively.

Agrawal et al. [12] present a fuzzy clustering-based PSO approach to solve the constrained environmental-economic dispatch problem with conflicting objectives. The algorithm is endowed with several features to preserve non-dominated solutions found along the search process, to direct the particles towards less explored regions of the Pareto front, and to avoid entrapment into local optima. It also incorporates a fuzzy feedback mechanism to determine a compromise solution. The objective functions are the minimization of fuel cost and emissions, subject to power balance (total power generated must satisfy total demand and account for transmission losses), generation capacity and transmission line loading limits.

Borghetti et al. [41] present a short-term scheduling procedure in two stages: a day-ahead scheduler for the optimization of DG production during the following day and an intra-day scheduler that every 15 min adjusts the scheduling in order to take into account the operation requirements and constraints of the distribution network. The intra-day scheduler solves a non-linear mixed integer MOO problem, in which the objective functions are the minimization of the voltage deviations with respect to the rated value, dispatchable DG production deviations with respect to the set points calculated by the day-ahead scheduler, and network losses.

Pourmousavi et al. [227] present a PSO approach to find real-time optimal energy management solutions for a stand-alone hybrid wind-microturbine (MT) energy system, requiring the capability to make rapid and robust plans regarding the dispatch of electrical power produced by generation assets. The objective functions are the cost of generated electricity, the MT operational efficiency, and environmental emissions.

Vahidinasab and Jadid [274] develop an MOO model for joint economic/environmental dispatch in energy markets. The objective functions are the (quadratic) cost of generators and pollutant emissions. Constraints include power (generation, demand and losses) balance, power generation limits, and line flow upper bounds. An ε-constraint technique is used to compute the non-dominated front and fuzzy sets identify a compromise solution. Vahidinasab and Jadid [276] present a stochastic MOO model for self-scheduling of a power producer participating in the day-ahead joint energy and reserves markets. The objective functions are expected profit and (SO_2 and NO_x) emissions when committing its generation of thermal units. Constraints consider supplying energy and ancillary services in the spot market, ramp rate restrictions, and minimum up/down time. Uncertainties associated with price forecasting and forced outage of generating units are modeled as scenarios using a combined fuzzy c-mean/Monte-Carlo simulation approach to convert the problem into a deterministic mixed-integer optimization problem. Each deterministic scenario is then tackled using an ε-constraint method. Schedules may be used by the power producers to decide on emission quota arbitrage opportunities and strategic bidding to the energy and reserve market.

Zhihuan et al. [302] incorporate the concept of robust solution into an MOO reactive power dispatch model to take into account uncertain load perturbations during system operation. The aim is searching for solutions that are immune to parameter drifts and load changes using information of load increase directions to promote the stability of optimal solutions in face of load perturbations. NSGA-II is used to search for robust non-dominated solutions regarding perturbations, which are more practical in reactive power optimization of real-time operation systems.

Catalão and Mendes [56] propose an MOO approach to solve the profit-based short-term thermal scheduling problem with environmental concerns under competitive and environmentally constrained market conditions. The objective functions are the minimization of emissions and the maximization of profits.

Bayon et al. [31] develop an analytical solution for the environmental-economic dispatch optimization problem obtaining the Pareto optimal set under different loading conditions. The objective functions are the minimization of fuel costs and NO_x emissions. Constraints refer to power balance (total power generated must supply total load demand and transmission losses) and unit capacity constraints.

Chandrasekaran and Simon [62] propose a fuzzy artificial bee colony (ABC) algorithm for solving the MO unit commitment problem. The objective functions are fuel cost, emissions and system reliability level. The binary coded ABC algorithm finds the on/off status of the generating units whereas the economic dispatch is solved using the real coded ABC. The fuzzy membership design variables are tuned using the real coded ABC, thus not requiring expert information.

Guo et al. [110] present an MOO dispatch model considering the integration of wind power. The objective functions are generation (wind turbines and coal-fired) cost, reserve capacity and the environmental emissions. Constraints refer to power output and load demand balance, upper and lower limits of power output, maximum climbing rate and back-up capacity supported by coal-fired generators. An approach based on a coordination degree combined with a satisfaction degree is used to transform the problem into a single-objective one (optimal generation dispatch), which is then solved using PSO.

Niknam and Doagou-Mojarrad [201] introduce an adaptive θ-PSO algorithm for the MO economic-emission dispatch. The algorithm is based on the phase angle vector to generate solutions faster than in the original PSO and evolutionary methods. New mutation and inertia weight factors adjustment techniques are used.

Niknam et al. [202] develop a stochastic MOO model for operation and management of electrical energy, hydrogen production and thermal load supplement by fuel cell power plants in distribution systems, taking into account uncertainty associated with load demand, price of natural gas, fuel cost for residential loads, electricity purchasing/selling tariffs, hydrogen selling price, operation, and maintenance costs. The objective functions are active power losses, emissions and total (fuel cell power plant and grid) cost. Constraints refer to power flow equations, line limits, active power generation, bus voltage, and ramp rate. A so-called teacher-learning algorithm is proposed to integrate the operation management of fuel cell power plants and the configuration of the system.

Aghaei et al. [11] present a nonlinear MOO dynamic economic emission dispatch model including wind turbines. The objective functions are expected total electrical energy costs and emissions. Constraints represent power balance equation, transmission network losses, up and down ramp rate limits, generation limits, spinning reserve requirements and wind power generation. A scenario-based stochastic programming framework is used to model the random nature of load demand and wind forecast errors. The stochastic problem is transformed into an equivalent deterministic scenario-based nonlinear, non-smooth, and non-differentiable problem, which is tackled using a PSO approach with a self-adaptive probabilistic mutation strategy. A similar model is presented by Bahmani-Firouzi et al. [27] using a PSO algorithm with a fuzzy adaptive technique and self-adaptive learning strategy for velocity updating.

Fazlollahi and Maréchal [96] propose an MOO model for process design and energy integration for sizing and operation optimization of poly-generation technologies, including biomass resources. The objective functions are investment cost, operating cost including incomes, and CO_2 emissions. Constraints consider heat balance, CO_2 emissions and electricity balance. MO EA and mixed integer LP approaches are used to obtain solutions.

Micro-grids encompass generation, loads and power flows as a sub-system generally operating in a grid-connected mode (i.e., power can be imported or exported from and to the main grid) allowing for local control of DG and thereby reducing or eliminating the need for central dispatch. In disturbance or fault conditions, the micro-grid can be isolated (islanded) from the distribution system to keep quality of service locally. Chaouachi et al. [65] develop a generalized formulation for energy management of a micro-grid using artificial intelligence techniques coupled with an MOLP model. The objective functions are the minimization of operation costs and the environmental impact, taking into account the future availability of renewable energy and load demand that is predicted using an artificial neural network ensemble (24 h ahead photovoltaic generation and 1 h ahead wind power generation, and load demand). Uncertainties regarding the overall micro-grid operation and the forecasted parameters are considered.

Moghaddam et al. [189] present an MOO model for operation management in micro-grids. The objective functions are operating costs and pollutant emissions, including CO_2, SO_2 and NO_x. Constraints deal with power balance, active power generation capacity, and battery limits. The micro-grid includes DG sources such as micro-turbines, fuel cells, photovoltaic units, wind turbine and batteries. An MO adaptive PSO is used to compute solutions to a nonlinear MOO model, including a chaotic local search mechanism and a fuzzy self-adaptive structure.

Sanseverino et al. [239] use the output of data mining tools, as loads and generation production forecasting models, to determine the generator scheduling by identifying optimal real and reactive power dispatch among distributed energy resources in micro-grids, including storage units. The objective functions consist in the minimization of the energy losses, production (fuel) costs, and CO_2 emissions. Constraints refer to DG power output taking into account the required power reserves and voltage drop at network buses. Solutions are obtained using NSGA-II within an execution monitoring and replanning approach to capture uncertainty associated with weather conditions and loads profiles. The controller allows monitoring the execution of the scheduling plan, interrupting the monitoring to input new information and repairing the plan under execution every time interval.

Niknam et al. [203] develop a stochastic mixed integer nonlinear and non-differentiable MOO model for micro-grid operation, considering cost and emissions as the objective functions. Constraints refer to power balance, active power limits of units, spinning reserve requirements, and charge/discharge rate of storage devices. Uncertainties associated with load demand, power output of wind and photovoltaic units and market prices are modeled using a scenario-based stochastic programming. An MO teaching–learning EA is used to characterize the Pareto optimal front.

Optimal power flow is an important problem in (steady-state) power system analysis due to operational security concerns and savings potential. A power flow analysis provides the magnitude and phase angle of the voltage at each bus, and the real and reactive power flowing in each line of the network. Therefore, power flow analysis is instrumental for several problems in power systems. The determination of the optimal power flow (OPF) may be based on nonlinear MOO models to establish the optimal settings of control variables for minimizing the cost of generation, emissions, transmission losses and voltage and power flow deviations. Nangia et al. [195] present an OPF model with three objectives: cost of generation, system transmission loss and pollution. Solutions to the power system operation problem are obtained by minimizing the Euclidean distance to the ideal point.

Rosehart et al. [236] develop OPF techniques based on MOO to optimize active and reactive power dispatch while maximizing voltage security in power systems. Interior point methods coupled with goal programming and linearly combined objective functions are used to obtain non-dominated solutions. The effects of minimizing operating costs, minimizing reactive power generation, and/or maximizing loading margins are compared to suggest possible ways of costing voltage security in power systems.

Amorim et al. [19] present an MO EA for OPF to deal with a large-scale non-convex constrained nonlinear model with continuous and discrete variables. The objective functions are associated with the violated inequality constraints associated with physical and operational aspects. The MO EA is based on Pareto optimality and uses a diversity-preserving mechanism to overcome premature convergence. Fuzzy set theory is then used to identify the best compromise solutions.

Bhattacharya and Roy [37] present a heuristic technique called gravitational search algorithm, which is inspired by swarm behavior and based on the Newton's law of gravity and mass interactions, to solve the MO OPF problem. The objective functions are minimization of fuel cost, active power losses, and voltage deviation.

Niknam et al. [205] present a PSO approach for the MO OPF problem, considering cost, losses, voltage stability, and emission impacts as the objective functions. A fuzzy decision-based mechanism is used to select the best compromise solution from the Pareto optimal set. The algorithm uses chaos queues and self-adaptive concepts to adjust the PSO parameters.

25.2.5 Load Management

Demand-side resources have been used by utilities with the main goals of achieving cost reduction and operational benefits (such as reducing peak power demand, improving reliability, increasing load factor, or reducing losses), which maintain their potential attractiveness even in an unbundled electricity industry. Appropriate power curtailment actions impose changes on the regular working cycles of loads to reduce peak demand without compromising the quality of the energy services provided, either by interrupting loads through direct load control or voluntary load shedding, shifting their operation cycles to other time periods or changing operating settings (such a thermostats). These types of demand-side actions have attracted further attention mainly due to the volatility and spikes of wholesale electricity prices and reliability concerns (transmission congestion and generation shortfalls). Loads that provide energy services whose quality is not substantially affected by short duration supply interruptions (for instance, thermostatic loads such as electric water heaters and air conditioners in the residential sector) are adequate targets for these actions. The goal is to design and select adequate load management actions, considering a comprehensive set of objectives of different nature (economic, technical, comfort, quality of services) and different players in the power industry. In a progressively deregulated market, these actions are an opportunity for a retailer facing volatile wholesale prices and fixed, over a certain time period, retail prices. A distribution utility (which owns and manages the distribution network) is generally interested in decreasing peak demand at primary substation and transformer stations levels due to capacity constraints, reliability concerns, or efficiency improvement through loss reduction. Impacts on spinning reserve and reliability may also be taken into account. The reduction of power demand may also be appropriate due to costs associated with a specific demand level, where profits may substantially decrease because average wholesale prices are much higher than retail prices in a certain period. Peak reduction enables both the distribution utility and the retailer to have a better capability of continuously exploring the differences between purchasing and selling prices in order to increase profits. However, since the energy service provided by loads under control is changed, possibly postponed or even not provided at all, when load management actions are implemented, attention should be paid to discomfort caused to customers so that those actions become also interesting for them, namely due to the ensuing reduction in their electricity bill. Therefore, multiple incommensurate and conflicting objectives of economical, technical and quality of service nature are at stake in the design and selection of load management actions. Some of the aspects mentioned above are modeled as hard or soft constraints (by establishing thresholds whose violation is included into a penalty function).

Some works are briefly reviewed below, highlighting the characteristics of the model and the methods proposed.

Jorge et al. [140] compute non-dominated control strategies for load management that minimize peak demand, maximize the utility profit associated with the energy services delivered by the controlled loads, and maximize quality of service subject to constraints of maximum number of loads allowed to violate a comfort threshold. An interactive procedure based on the STEM method is used.

Gomes et al. [104] consider an MO model for minimizing peak power demand at different load aggregation levels (sub-station and power transformers, thus enabling the model to be used in different scenarios of power systems structure and by different entities), maximizing profits (which depend on the amount of electricity sold and the time of day/season/year), minimizing the loss factor, and minimizing discomfort caused to consumers (maximum continuous time interval and a state variable controlled by loads is over or under a pre-specified threshold). An MO EA is used to derive load shedding patterns to be applied to groups of loads. This model has been then dealt with an interactive MO EA for the identification and selection of direct load control actions (population individuals), including a progressive articulation of the DM's preferences by changing aspiration or reservation levels used in fitness assessment, as well as adaptive operators [105]. The demand imposed by loads when subject to control actions is assessed using physically-based load models.

Manjure and Makram [176] consider the minimization of costs, which are associated with re-dispatching generation or curtailing interruptible loads, and the maximization of the security margin in the event of a generation shortage.

Pedrasa et al. [218] use a binary PSO algorithm to schedule diverse interruptible loads over a given period. The objective functions are minimizing the total payment and the frequency of interruptions imposed upon the loads, while satisfying a system requirement of total hourly curtailments and operational constraints of interruptible loads.

Hong et al. [127] investigate the demand response achieved by an energy management system in a smart home environment to obtain the optimal temperature scheduling for air-conditioning according to the day-ahead electricity price and outdoor temperature forecasts. Since these are predicted 24 h in advance, the predicted retail electricity prices and temperatures are modeled using fuzzy sets. An immune clonal selection approach is used to determine the day-ahead 24 h temperature schedule for air-conditioning to minimize electricity costs and maximize comfort.

Shahnia et al. [246] present a peak load management system in low-voltage distribution networks. An MOO model is used to select the loads to be controlled in order to minimize peak load and maximize customer satisfaction. Low-cost controllers with low-bandwidth two-way communication are installed in costumers' premises and at distribution transformers to implement solutions online.

Hong and Wei [128] determine the percentage of load allowed to be shedded and parameters including the number of stages and delayed time for the under-frequency relay. A hierarchical GA is employed to minimize the amount of load shedded and maximize the lowest swing frequency caused by a disturbance. An autonomous system with diesel generators and wind-power generators is employed to illustrate the method.

Recently, MOO models have been used for the design of energy management systems for (pure or hybrid) electric vehicles (EV), also encompassing energy and power management problems in multi-source EV. Zhang and Liu [299] present an energy control strategy for parallel hybrid electric vehicles using fuzzy multi-objective optimization. The objective functions are the overall vehicle fuel economy and emissions, by converting the electric energy consumed by the electric motor into equivalent fuel consumption. A minimum average weighted deviation method is used to compute the non-dominated solution set, considering variations in emission requirement in different districts, accounting for the battery state of charge within its operation range.

Chen et al. [69] develop a fuzzy logic controller to manage the energy distribution for a hybrid EV, in which an MO EA is used to optimize the fuzzy membership function according to known fuzzy control rules.

Buildings account for about 40 % of overall energy consumption and several countries and supra-national institutions, e.g. the European Union, have produced specific regulations for improving energy efficiency in buildings, regarding the building envelope and equipment (e.g., solar thermal for hot water). Evins et al. [92] develop an MOO framework that is applied to the outputs of Standard Assessment Procedure, involving all energy calculations for building regulations compliance and Code for Sustainable Homes ratings for domestic buildings in the UK, considering as objective functions carbon emissions (which is based on the percentage improvement of the Dwelling Emission Rate over the Target Emission Rate) and costs (construction costs plus energy costs). Constraints refer to limits on over-heating and roof area. NSGA-II is used to compute non-dominated solutions.

Asadi et al. [25] present a mixed-integer MOO to optimize the retrofit cost, energy savings and the thermal comfort of a residential building. The decision space is defined by the combination of alternative materials for the external walls insulation and roof insulation, different window types, and installation of solar collectors in an existing building. The objective functions are retrofit cost, energy savings and thermal comfort (using the predicted mean vote metric). A weighted Tchebycheff metric is used to compute non-dominated solutions. Combined cooling heating and power (CCHP) systems have revealed to be economical, energy-efficient and environmental friendly, even more than conventional cogeneration plants, enabling the utilization of waste heat (in cooling, space heating and hot water). These systems may range from large-scale applications such as in industry and commercial buildings to small-scale systems. Wu et al. [289] deal with a mixed-integer non-linear programming MOO model to optimize the operation of a micro CCHP system (gas engine and adsorption chiller, and auxiliary devices such as gas boiler, heat pump and electric chiller) under different load conditions. The objective functions are the energy saving ratio and cost saving ratio, subject to equipment and energy balance constraints.

25.2.6 Energy-Economy Planning Models

The study of the interactions between the economy (at national or regional levels), the energy sector and the corresponding impacts on the environment inherently involves multiple axes of evaluation of distinct policies. In general, MOO models for this purpose are developed based on input–output analysis (IOA) or general equilibrium models (GEM). The analytical framework of IOA enables to model the interactions between the whole economy and the energy sector, thus identifying the energy required for the provision of goods and services in an economy and also quantifying the corresponding pollutant emissions. GEM include interrelated markets and represent the (sub-)systems (energy, environment, economy) and the dynamic mechanisms of agent's behavior to compute the competitive market equilibrium and determine the optimal balance for energy demand/supply and emissions/abatement.

Hsu and Chou [132] suggest an MOLP approach integrated with IOA to evaluate the impact of energy conservation policy on the cost of reducing CO_2 emissions and undertaking industrial adjustment in Taiwan. An inter-temporal CO_2 reduction model, consisting of two objective equations (maximization of the Gross Domestic Product (GDP) and the minimization of CO_2 emissions) and 1340 constraint equations, is constructed to simulate alternative scenarios consisting of Case I (no constraint on CO_2 emissions), Case II (per capita CO_2 emissions at Taiwan year 2000 levels), Case III (Case II emission levels with energy conservation), and Case IV (Case II emission levels with energy conservation plus improved electricity efficiency). Constraints include inter-temporal inter-industry constraints, water resource constraints, labor constraints for each industry and industrial expansion constraints.

Chen [68] employs an MOLP model combined with an IOA model to determine the trade-off between GDP growth and CO_2 emissions on Taiwan's economy. The author derives non-inferior solutions by using the 'center-point' method.

Oliveira and Antunes [209] also propose an economy-energy-environment planning model based on IOA whose objective functions are private consumption, employment level, CO_2 emissions and the self-production of electricity. Constraints refer to balance of payments, gross-added value, production capacity, bounds on exports and imports, public deficit, storage capacity and security stocks for hydrocarbons. Solutions to this MOLP model were obtained using the STEM method. An interactive approach to tackle uncertainty and imprecision associated with the coefficients of this type of models is presented in Borges and Antunes [40], where some of the coefficients are triangular fuzzy numbers. Interactive techniques are used to perform the decomposition of the parametric (weight) diagram into indifference solutions corresponding to basic non-dominated solutions. Three objective functions are considered which enable to graphically display the decomposition of the parametric diagram: energy imports, self-production of electricity and CO_2 emissions. Oliveira and Antunes [210] develop an MOLP model based on an IOA considering the minimization of acidification potential and energy imports, and the maximization of GDP, employment, and self-power generation. Constraints refer to a large set of economic indicators, gross fixed capital formation, trade balance, production capacity, stock changes, public deficit, storage capacity and security stocks, and several pollutant emissions. Non-dominated solutions are obtained using an interactive procedure based on a min-max scalarizing function associated with reference points that are displaced according the DM's preferences expressed through average annual growth rates. The structure of this model has been then updated in several directions including capturing the MOLP model coefficients through intervals, considering as objective functions the GDP, employment, global warming potential and energy imports [211, 212]. This model enables to provide information regarding the robustness of non-dominated solutions (that is, solutions that attain desired levels for the objective functions across a set of plausible scenarios) and also a more optimistic or pessimistic stance by the DM. With the introduction of (direct and indirect) employment multipliers, this IOA structure has been used to extend the interval MOLP to assess the trade-offs between economic growth (GDP), social welfare (employment), and electricity generation based on renewable energy sources [213].

Cristóbal [76] suggest an IOA MOLP model combined with goal programming to assess the economic goals—the level of output must be as close as possible to the level of the year 2005; Social goals—labour requirements must be as close as possible to the level of the year 2005; energy goals—coal requirements must be reduced by 5 %; environmental goals—and total emissions of GHGs and wastes emissions must be reduced by 10 %. Solutions are obtained by considering the minimization of the total deviations from the goals.

Wu and Xu [290] propose a system dynamics and fuzzy MOO integrated support model to predict energy consumption and CO_2 emissions for a world heritage area. The objective functions are the increase of GDP per capita, energy consumption, and CO_2 emissions, subject to minimum GDP growth rate, investment for energy savings and CO_2 emission reduction, energy intensity, and carbon intensity. A simple weighted sum scheme is used to obtain solutions (policy suggestions).

Pérez-Fortes et al. [219] develop an MO MILP model for design and operation of bio-based supply chains that use locally available biomass to generate electricity (through a gasification technology). The objective functions express economic, environmental and social concerns. Decisions are related to location and capacity of technologies, connectivity between the supply entities, biomass storage periods, matter transportation and biomass utilization.

Aki et al. [14] study the introduction of an integrated energy service system in an urban area, which supplies electricity, gas, cooling, and heating to consumers. CO_2 emissions, energy pricing and economic impact on the consumers are considered under several scenarios using linear programming models

25.2.7 Energy Markets

The liberalization of energy markets is aimed at increasing overall efficiency through the introduction of competition in some of the industry branches, namely generation and retailing. The underlying idea is that by enhancing efficiency and productivity gains, lower energy (namely electricity) prices and lower production costs are achieved. This trend of energy markets should go in line with security of supply (by minimizing risks and overall impacts of supply disruptions, diversifying energy sources including renewables and energy efficiency), competitive energy systems (to minimize energy costs for consumers and industry thus contributing to social policies and economic competitiveness), and environmental protection (thus minimizing the impacts of energy generation and use on populations and ecosystems). Issues such as the internalization of external costs to the environment into energy prices, in accordance with the polluter pays principle, are also at stake in designing market-based mechanisms balancing multiple objectives, such as taxes or tradable emission permits.

Niimura and Nakashima [199] analyze the trade-offs between different objectives of power system operation and the influence of policies such as environmental impact minimization on deregulated electricity trade, using a fuzzy interactive MOO procedure to reach a coordinated solution.

Kaleta et al. [147] present a stochastic short-term planning model for supporting decisions of small energy suppliers (price takers). The objective functions modeling the generator attitude towards risk are the mean return, the mean loss, the mean semi-deviation below the mean return, the worst return realization and the conditional value-at-risk. The technical constraints lead to a mixed integer LP model. The uncertainty associated with market prices is modeled using a set of scenarios with assigned probabilities. Solutions are computed using an interactive approach based on aspiration/reservation levels and achievement scalarizing functions.

Milano et al. [186] propose a technique for representing system security in the operation of decentralized electricity markets, with special emphasis on voltage stability. An MOO model considers the maximization of the social benefit and the distance to maximum loading conditions, which is dealt with an interior point method to solve the optimal power flow problem. Elastic and inelastic demand conditions are considered. It is shown that system security can be improved yielding better market conditions through increased transaction levels and improved locational marginal prices throughout the system.

Amjady et al. [18] develop an MOO model for day-ahead joint market clearing. The objective functions include augmented generation offer cost and security indices (overload index, voltage drop index, and voltage stability margin). System uncertainties including generating units and branches contingencies and load uncertainty are explicitly considered in the stochastic market clearing scheme. The solution methodology consists of two stages: a roulette wheel mechanism and Monte Carlo simulation for random adaptive 24 h scenario generation wherein the stochastic MO market clearing procedure is converted into deterministic scenarios. For each scenario, an MO method based on the ε-constraint technique is used for provision of spinning and non-spinning reserve as well as energy.

Dukpa et al. [88] propose a strategy for wind electric generators employing an energy storage device for participating in the day-ahead unit commitment process. The objective functions are the maximization of returns from the market considering the best forecast and the minimization of risks considering the forecast uncertainties. Risk in the participation strategy is quantified by computing expected energy not served. The MO mixed integer LP model is transformed into a fuzzy optimization model. Energy storage enables to shift wind energy produced during hours with low marginal prices to hours with higher marginal prices by appropriately storing and releasing it and maintain an energy reserve similar to spinning reserve to minimize the risk of the optimal participation schedule.

Farahani et al. [95] propose an MOO model for reactive power market clearing with the presence of plug-in hybrid EVs, considering as objective functions the total payment function to the vehicles and generators for their reactive power compensation and total grid losses. An MO PSO approach is used and the "best" compromise solution is chosen based on preferences revealed in face of non-dominated solutions using a fuzzy approach.

Aghaei et al. [10] present an MOO model for electricity market clearing, considering both voltage and dynamic security aspects. The objective functions are offer cost of energy and reserves, corrected transient energy margin, and voltage stability margin. Constraints are related to AC power flow constraints and operation limits of units, security and reserve requirements. Solutions are computed combining a lexicographic approach and an augmented ε-constraint technique.

Khazali et al. [155] use a fuzzy MOO approach for clearing the reactive power market. The objective functions are the total payment function, voltage stability and the voltage deviation of the network buses, for which membership functions are specified and a single goal attainment function is tackled using a fuzzy adaptive PSO to determine the amount of reactive power provided by each generator and the reactive compensation devices including adjustment tap settings of transformers. The reactive power compensation devices are assumed to compete in an integrated market with the generators and then a separate reactive power market is proposed.

Reddy et al. [234] also address the reactive power price clearing considering voltage stability using an MOO approach. The objective functions are the minimization of total payment and transmission losses, and the maximization of a voltage enhancement index and the load served. Constraints refer to nodal power balance, generator reactive power restrictions, determination of market clearing prices, reactive power capability limits of generators, and security (voltage, thermal limits, transformer tap settings). The SPEA and an MO PSO are used to obtain results in base and stressed cases with constant and voltage dependent load modeling.

Vahidinasab and Jadid [275] present a bilevel model, in which the upper-level sub-problem maximizes the individual supplier pay-off and the lower-level sub-problem solves the Independent System Operator's market clearing problem. The objective functions are social welfare and pollutant emissions, subject to power flow equations as well as generator, security and power transfer limits. The algorithm used to solve the ISO's MO optimal power flow is based on the ε-constraint technique.

25.3 Energy Planning Decisions with *Discrete Alternatives*

MCDA methods become increasingly popular in energy decision-making due to their capability to deal with complex decision processes, in face of multiple and conflicting evaluation criteria, different stakeholders with different views and preferences, several sources of uncertainty and distinct time frames. Literature reviews with specific focuses on the use of MCDA in energy problems have been reported. Hobbs and Meier [124] provide a wide review of MCDM methods and energy-environment applications. Keefer et al. [153] offer a perspective on trends and developments regarding decision analysis applications. Greening and Bernow [107] describe a modeling framework incorporating developments in integrated assessment of energy and environmental issues, and suggest a strategy for developing a set of coordinated policies from varying levels of information about policy attributes and DM's preferences. Pohekar and Ramachandran [224] review the application of various MCDM methods in the framework of sustainable energy planning. Kiker et al. [156] suggest recommendations for applying MCDA techniques in environmental projects. Polatidis et al. [226] develop a methodological framework to provide insights regarding the suitability of multi-criteria techniques in the context of renewable energy planning. Jebaraj and Iniyan [138] review emerging issues related to energy modeling. Wei et al. [287] analyze energy models developed by various international organizations, focusing on modeling approaches and structures, as well as their typical applications. Zhou et al. [303] update a previous study on decision analysis in energy and environmental modeling. Løken [169] provide an overview of some of the most relevant MCDA methods proposed in the literature. Higgs et al. [121] outline alternative methodologies that involve the use of information technology methods in enabling a possible consensus to be reached between participatory groups on decisions that may affect their local environment. Wang et al. [283] review methods in different stages of MCDM processes for sustainable energy planning. Kowalski et al. [158] combine the use of scenario building and participatory MCDA in the context of renewable energy from a methodological point of view. Behzadian et al. [34] suggest a classification scheme and provide a comprehensive literature review to uncover, classify, and interpret research studies based on PROMETHEE methodologies and applications. Carrera and Mack [53] review the process of sustainability assessment of energy technologies using expert judgments to rate energy technologies on a set of social indicators that were generated in a discursive procedure. Huang et al. [134] review environmental applications of MCDA. Abu-Taha [3] presents a review of MCDA in the area of renewable energy, revealing that AHP has been the most used of all MCDM methodologies. Bhattacharyya [38] reviews methodologies for off grid electrification projects. Scott et al. [243] review works dealing with problems arising in the bioenergy sector. Behzadian et al. [35] conduct a survey to offer a taxonomy of the research on TOPSIS applications and methodologies in energy management, having concluded that most concentrate on evaluating and selecting energy generation technologies as well as assessing energy system performance. Doukas [84] explores different linguistic representation and computational MCDA models that are or can be applied to energy policy support, concluding that MCDA methodologies with direct computation on linguistic variables can aid the design of energy policy frameworks, by bridging the gap between energy policy makers thinking, reasoning, representation and computing. Herva and Roca [119] review the advantages of combining complementary environmental evaluation tools and the applicability of multi-criteria analysis in decision support, explicitly considering energy decision-making applications. Mirakyan and Guio [187] present a review of methods and tools for integrated energy planning in cities and territories, concluding that the purpose of MCDA methodologies is not just required to define the "right" energy plan but rather to support the understanding of the multi-criteria complex situation that supports interactive planning and learning, helping people to systematically consider, articulate and apply value judgments.

This section is devoted to review MCDA models and methods dealing with energy decision-making with the aim of analyzing the main trends of methodological approaches and specific domains of application. The articles reviewed address different energy supply systems, such as renewable energy systems (e.g. photovoltaic, wind, hydrogen, biomass, biogas, geothermal and biofuels) and conventional technologies (e.g. natural gas, coal, fuel, large hydro and nuclear power), and different sorts of energy decision-making applications: comparison of power generation technologies, evaluation of energy plans and policies, selection of energy projects, siting decisions, evaluation of energy efficiency measures either in technology replacement or in building refurbishment, etc. Criteria usually considered to evaluate the merit of different alternatives in energy decision-making problems are included in the main broad categories: technical, economic, environmental and social.

25.3.1 Comparison of Power Generation Technologies

The aim of these problems is mainly focused on the appraisal of available primary energy source and technological options, for conventional technologies and/or renewable energy technologies. The main energy focus of these studies remains on electricity generation using conventional sources but with renewable electricity generation and hydrogen gaining increasing attention (see Table 25.2).

25.3.2 Energy Plans and Policies

The energy decision-making problems framed in this category are concerned with the choices faced by energy planners or regulators at the national, regional or local level seeking to identify the most desired one among alternative scenarios, energy policies and strategies for the future. The main purpose is to guide the formulation and development of energy policies taking into account the public debate on energy policy, energy conservation strategies and energy resource allocation issues, involving concerns with renewable sources, hydrogen and bioenergy (see Table 25.3).

Table 25.2 Studies grouped in power generation comparison problems

Energy focus	Scope	Alternatives	Stakeholders	MCDA method	Uncertainty	Application-origin	References
RES-E wind	Assessment of new and renewable energy power plants	Technologies	n.a.	SAW—general index of sustainability	Stochastic approach	Not specified—Portugal	Afgan and Carvalho [4]
RES-E wind	Choice of wind farms as a form of renewable energy in comparison to other potential sources	Technologies	CREW members—engineers, university professors, students, entrepreneurs and employees of municipal local government, among others	AHP	n.a.	Local—South Ontario, Canada	Nigim et al. [198]
RES-E biogas	Assessment of forty one agricultural biogas plants	DMU—representative set of energy crop digestion plants	n.a.	ELECTRE TRI	n.a.	Not specified—Austria	Madlener et al. [172]
RES-E	Assessment of small-scale or large-scale approaches to renewable energy provision	Technologies	Five professionals with experience in the energy sector; three of these were from Kirklees Council, a further two were from other organisations	MACBETH	Sensitivity analysis	Local—Metropolitan Borough of Kirklees in Yorkshire, UK	Burton and Hubacek [49]
RES-E	Characterization of electricity generation technologies	Future scenarios	Official authorities	PROMETHEE	Sensitivity analysis	National—Greece	Diakoulaki and Karangelis [82]

RES-E wind	Evaluation of off-shore wind energy technologies	Technologies	Research group and three experts	'Adjacency matrix' and 'Sub-matrix algorithm'. Synergic relationships are combined with independent weights so that decision maker can select the optimal combination based on the final priorities calculated by the integration of the elements in the hierarchy	n.a.	Local—US Pacific Northwest	Daima et al. [77]
RES	Selection among renewable energy alternatives	Energy sources	Experts	Fuzzy AHP	Sensitivity analysis and fuzzy aggregation	National—Turkey	Kahraman et al. [146]
Electricity generation	Analysis on electric power system (EPS) expansion	Options of the UEPS expansion	n.a.	Use of a hierarchy analysis technique different from the traditional one in the fact that there is no need to introduce special scales to compare elements	Probability of each scenario	National—Russia	Voropai and Ivanova [280]
Electricity generation	Evaluation of energy resources that will enable the selection of a suitable electricity generation alternative	Technologies	n.a.	PROMETHEE	Robustness analysis and Weight stability intervals	National—Turkey	Topcu and Ulengin [265]

(continued)

Table 25.2 (continued)

Energy focus	Scope	Alternatives	Stakeholders	MCDA method	Uncertainty	Application-origin	References
Electricity generation	Evaluation of sustainable technologies for electricity generation	Technologies	Programme participants, stakeholders, etc.	PROMETHEE II	n.a.	National—Greece	Doukas et al. [86]
Electricity generation	Sustainability assessment of CHP systems	Technologies	n.a.	SAW—general index of sustainability	Stochastic approach	Not specified—Greece, Belgium and Portugal	Pilavachi et al. [221]
Electricity generation—natural gas	Evaluation of the potential natural gas utilization in energy sector	Technologies	n.a.	SAW—general index of sustainability	Stochastic approach	Not specified—Portugal and Greece	Afgan et al. [8]
Electricity generation	Assessment of various options of the energy power system	Options	n.a.	SAW—index at information deficiency	Stochastic approach and non-numeric (ordinal), nonexact (interval) and non-complete information	National—Bosnia and Herzegovina	Begić and Afgan [33]
Electricity generation	Formulation of sustainable technological energy priorities	Technologies	Public utility, independent power producers, financing organizations, researchers and academics, governmental managers, regulatory authority, transmission system operator, Center for Renewable Energy Sources	OWA	Fuzzy—linguistic approach	National—Greece	Doukas et al. [85]

Electricity generation	Ranking of power expansion alternatives	Expansion alternatives	n.a.	MAVT	Robustness, sensitivity analysis and scenarios	National—South Africa	Heinrich et al. [116]
Electricity generation	Evaluation of alternative fuels for electricity generation	Energy sources	n.a.	ANP	Scenario based	National—Turkey	Köne and Büke [157].
Electricity generation	Evaluation of hybrid energy systems	Technologies	n.a.	SAW—general index of sustainability	Stochastic approach	Not specified—Belgium and Portugal	Afgan and Carvalho [5]
Electricity generation	Technological, economic and sustainability evaluation of power plants	Technologies	n.a.	AHP	Sensitivity analysis	Not specified—Greece	Chatzimouratidis and Pilavachi [67]
Electricity generation	Evaluation of the sustainability of current and future electricity supply options	Technologies	85 employees of the Axpo Group	SAW—“sustainability index”	Sensitivity analysis and scenarios	Local utility company—Switzerland	Roth et al. [237]
Electricity generation—hydrogen and natural gas	Evaluation of nine types of electrical energy generation options	Technologies	n.a.	AHP	Scenario based	n.a.	Pilavachi et al. [222]
Electricity generation	Analysis to prioritize investment portfolios in capacity expansion and energy security	Technologies	n.a.	MAVT	Sensitivity analysis and scenarios	National—Mexico	Martinez et al. [178]

(continued)

Table 25.2 (continued)

Energy focus	Scope	Alternatives	Stakeholders	MCDA method	Uncertainty	Application-origin	References
Electricity generation	Choosing the most sustainable electricity production technologies	Technologies	n.a.	TOPSIS	Sensitivity analysis and scenarios	Not specified—Lituania	Streimikiene et al. [258]
Electricity generation	Evaluation of different electricity production scenarios	Scenarios	A group of experts and academics with background in economics, engineering and environment	SAW	Sensitivity analysis	National—Portugal	Ribeiro et al. [235]
Hydrogen	Exploring the commercialization of future hydrogen fuel processor technologies	Technologies	Pairwise comparisons were based on literature studies and the expert opinion of these authors	AHP	Sensitivity analysis and scenarios	National—USA	Winebrakea and Creswick [288]
Hydrogen	Assessment of hydrogen systems	Technologies	n.a.	SAW—sustainability index	Scenario based	n.a.	Afgan et al. [7]
Hydrogen	Establishing a strategic hydrogen energy technology roadmap	Technologies	n.a.	Fuzzy AHP	Fuzzy techniques	National—Korea	Lee et al. [163]
Hydrogen	Establishing a strategic long-term strategic energy technology roadmap for hydrogen energy technologies	Technologies	Eight experts who have been carrying out the development of energy technologies and energy policy over 10 and 15 years	Fuzzy AHP and DEA	Fuzzy techniques	National—Korea	Lee et al. [164]

Energy in general	Evaluation of alternative energy sources for the country	Energy sources	Private enterprise, universities, associations, government and research associations, other countries, the public	ANP	n.a.	National—Turkey	Ulutas [272]
Energy in general	Evaluation and selection of current energy resources in a selected industry	Energy sources	n.a.	ANP	Sensitivity analysis	National—Turkey	Önüt et al. [214]
Bioenergy	Assessment of bioenergy systems	Technologies	Biomass feedstock producers and suppliers; heat, electricity and biofuel project developers, utilities and transport fuel suppliers, and end-users; the financial community; technology providers	Multi-criterion decision analysis framework and decision-conferencing approach	Sensitivity analysis and NUSAP	National—UK	Elghali et al. [89]
Biomass	Ranking different biomass feedstock-based pellets	Scenarios	Experts in this field, and pellet manufacturers and users	PROMETHEE	Sensitivity analysis and scenarios	National—Canada	Sultana and Kumar [259]

Table 25.3 Studies grouped in energy plans and policies problems

Energy focus	Scope	Alternatives	Stakeholders	MCDA method	Uncertainty	Application-origin	References
RES-E hydropower and geothermal energy	Framework plan for the use of hydropower and geothermal energy	Energy sources	National Energy Authority, Institute of Natural History and the Nature Conservation Agency, Environmental Association and the Icelandic Touring Association, as well as the four workgroup chairpersons	AHP	n.a.	National—Iceland	Thórhallsdóttir [264]
RES-E	Determining the achievable penetration of renewable energy sources into an insular system for the purpose of electricity generation	Scenarios	Local authorities and private actors that were part of the project	ELECTRE III	Sensitivity analysis	Local—Karpathos and Kassos, Greece	Papadopoulos and Karagiannidis [216]
RES	Assessment of an action plan for the diffusion of renewable energy technologies at regional scale	Technologies/actions	n.a.	ELECTRE III	Scenario based	Local—Sardinia, Italy	Beccali et al. [32]
RES	Design of renewable energy promotion policies	Renewable energy technology	Renewable energy technology	PROMETHEE	Fuzzy techniques	National—Austria	Madlener and Stagl [173]

RES	Assessing the renewable energy producers' operational environment	EU—accession member states	n.a.	OWA	n.a.	Transnational—fourteen EU—accession Member States	Patlitzianas et al. [217]
RES	Combined use of scenario building and participatory multi-criteria analysis in the context of renewable energy from a methodological point of view	Scenarios	National (Governmental bodies, private firms, Power distributers, NGOS, Research Institutes) and Local (local energy experts, regional and national energy experts, mayors and deputy mayors, citizens)	PROMETHEE and PMCA	Scenario based	National and Local—Austria	Kowalski et al. [158]
RES	Establishment of strategies needed to reach, in the long term, an energy system more sustainable	Strategies	Nine experts from University, Energy Administration, Provincial Energy Agency, Electrical distribution Company, Andalusian Energy Agency, Andalusian Institute of Renewables, Andalusian Development Institute and Ecologist Associations	PROMETHEE	n.a.	Local—Jaén Province, Spain	Terrados et al. [263]

(continued)

Table 25.3 (continued)

Energy focus	Scope	Alternatives	Stakeholders	MCDA method	Uncertainty	Application-origin	References
RES	Sustainable energy planning on an island	Energy policy alternatives	Local authorities; potential investors; local communities; academic institutions; environmental groups; governments and European Union	PROMETHEE I/II	n.a.	Local—Crete	Tsoutsos et al. [270]
RES	Determining the best renewable energy alternative and selection among alternative energy production sites	Technology and possible locations	Three RES experts	VIKOR-AHP	Fuzzy techniques	Local—Istanbul	Kaya and Kahraman [151]
RES	Assessment of renewable energy sources with policy and technology concerns	Energy sources	Experts in Taiwan who are familiar with the status quo development of renewable energy technologies, the market conditions, and the renewable energy policy	Fuzzy AHP	Fuzzy techniques	National—Taiwan	Shen et al. [247]
RES	An assessment of the EU renewable energy targets and supporting policies	Scenarios	n.a.	Energy and Climate Policy Interactions (ECPI) Decision Support Tool	n.a.	Transnational-EU	Oikonomou et al. [208]

Electricity generation	Select, define and apply a set of sustainability indicators for the energy system assessment	Energy sources	n.a.	SAW—general index of sustainability	Standard deviations which are measuring uncertainty of weight-estimation are taken into a consideration	Local—a small island not specified	Afgan et al. [6]
Electricity generation	Assessment of various energy policies for power alternatives	Energy sources	Expert's group mostly composed of academicians	Fuzzy-AHP Chang's Model	Fuzzy techniques	National—Turkey	Talinli et al. [260]
Electricity generation	Energy planning decision-making problem	Energy sources	Energy planning experts	AHP/TOPSIS	Sensitivity analysis and fuzzy synthetic extent values	National—Turkey	Kaya and Kahraman [152]
Electricity generation	Recommend future energy sources, taking into consideration water consumption and the possibility of desalination of sea water	Energy sources	n.a.	ELECTRE III	Sensitivity analysis	Local—Yorkshire and the Humber region in the UK	Hunt et al. [135]

(continued)

Table 25.3 (continued)

Energy focus	Scope	Alternatives	Stakeholders	MCDA method	Uncertainty	Application-origin	References
Hydrogen	Assessment of the environmental, social and economic sustainability of six possible future hydrogen energy systems	Scenarios	UK hydrogen-related organisations (H2Net, London Hydrogen Partnership and the Low Carbon Vehicle Partnership), ensuring representation from hydrogen production, distribution and end-use industries as well as relevant policy and civil society stakeholders	MCM	Sensitivity analysis	National—UK	McDowall and Eames [180]
Hydrogen	Analysis of the potential of Korea to be competitive in development of hydrogen energy technology	Nations	n.a.	AHP	Scenario based	National—Korea	Lee et al. [161]
Hydrogen	Assessment of national competitiveness in the hydrogen technology sector	Nations	Korea's MEST and MKE	Fuzzy AHP	Scenario based and fuzzy and interval programming techniques	National—Korea	Lee et al. [165]
Hydrogen	Measuring the relative efficiency of the R&D performance in the national hydrogen energy technology development	Nations	51 experts from the academic, government, industrial, and research sectors	Fuzzy AHP and DEA	Fuzzy techniques	National—Korea	Lee et al. [162]

Biomass	Assessment of a range of possibilities for perennial energy crops conversion	Energy crops	Experts	Capability index—weighted sum	Sensitivity analysis, Monte Carlo simulations and jack-knifing techniques	Local—Yorkshire and the Humber Region in Northern UK	Tenerelli, and Carver [262]
Energy in general	Sustainable development of rural energy and its appraising system in China	Different areas and periods	n.a.	AHP	n.a.	Local—Jinhu, Sheyang, Taixing, Suining, Wujin and Wuxian Counties in Jiangsu Province of China	Xiaohua and Zhenmin [291]
Energy in general	Defining national priorities for greenhouse gases emissions reduction in the energy sector	CO_2 reduction measures	Ministry for Development, Ministry for Environment, Physical Planning and Public Works, Ministry for Transport, Ministry for Agriculture, Ministry for National Economy, Ministry of Finance, Public Power Corporation, Public Gas Enterprise, Oil Distilleries of Greece, Organisation of Urban Transport for Athens, Corporation of Manufacturers of Solar Systems, Centre for Renewable Energy Sources	ELECTRE TRI	Sensitivity analysis	National—Greece	Georgopoulou et al. [100]

(continued)

Table 25.3 (continued)

Energy focus	Scope	Alternatives	Stakeholders	MCDA method	Uncertainty	Application-origin	References
Energy in general	Prioritization process of Policy instruments for promoting energy conservation	Policy instruments	Manager of the RUE Division at the NERC	AHP	n.a.	National—Jordan	Kablan [144]
Energy in general	Evaluation of sustainability scenarios	Scenarios	n.a.	SAW	Fuzzy sets synthesis technique	Local—Belgrade	Jovanovic et al. [141]
Energy in general	Sustainability assessment at the macro scale	Years	UN expert; National experts—statisticians, researchers, members of national governments, and representatives from European Commission services; OECD Working Group on Environmental Information and Outlooks; Stakeholders in Austria—federal ministries, representatives from the state (Länder) and district levels, social partners, different interest groups and NGOs	NAIADE	Sensitivity analysis and fuzzy or stochastic techniques	National—Austria	Shmelev and Rodríguez-Labajos [248]

Energy in general	Measurement of the sustainability of an urban energy system	Scenarios	Experts	Additive synthesising function	Fuzzy sets of synthesis technique	Local—Belgrade, Serbia	Jovanovic et al. [142]
Energy in general	Assessment of six policy measures or scenarios relating to residential heating energy and domestic electricity consumption	Scenarios	n.a.	NAIADE	Sensitivity analysis and scenarios	Local—Irish city region	Browne et al. [47]
Energy in general	Selection among energy policies	Energy sources	n.a.	Fuzzy AHP	Fuzzy techniques	National—Turkey	Kahraman and Kaya [145]
Energy in general	This paper presents a study on the options for energy and carbon development for the city of Bangkok	Energy policies and interventions	Team members	MAVT and AHP	n.a.	Local—Bangkok, Thailand	Phdungsilp [220]
Energy in general	A transdisciplinary process to address the future energy system	Scenarios	Nine households that represented the different combinations of building type and heating system, a small local industry and a representative from the local business association	AHP, stakeholder-based MCA	Scenario based	Local—Urnäsch, Switzerland	Trutnevyte et al. [266]

(continued)

Table 25.3 (continued)

Energy focus	Scope	Alternatives	Stakeholders	MCDA method	Uncertainty	Application-origin	References
Energy in general	Environmental performance evaluation of Beijing's energy use planning	Scenarios	A group of decision-makers composed of scientists, urban planning engineers, government officials and other specialists	AHP and Fuzzy extent analysis	Fuzzy techniques	Local—Beijing, China	[285, 286]
Energy in general	Energy resource planning activities	Energy sources	Industry representatives, environmentalists, local residents, academicians, public authorities	AHP	n.a.	Local—Aydin, Turkey	Erol and Kılkıs [91]
Energy in general	Linking visions with quantitative resource allocation scenarios which show different options in implementing the visions	Scenarios	Energy consumers, experts, representatives from academia and the energy industry	MAVT	Scenario based	Local—Urnäsch, Switzerland	Trutnevyte et al. [267]
Natural gas	Selecting optimal energetic scenario	Scenarios	n.a.	NAIADE	Sensitivity analysis	National—Romania	Dinca et al. [83]

25.3.3 Selection of Energy Projects

Project selection is a typical multi-criteria decision situation: in face of a considerable number of projects the DM is asked to identify the most attractive subset of alternatives. The majority of these projects is focused on renewable energy investments for electricity generation (see Table 25.4).

25.3.4 Siting Decisions

Another common situation in energy decision-making studies refers to location problems. Most applications concerning the location of facilities focus on the siting of new wind farms and hydro and thermal power plants, in some cases also complemented with choices regarding operational parameters (see Table 25.5).

25.3.5 Energy Efficiency

Energy efficiency studies mainly consider the evaluation and sorting of energy efficiency measures and programs either in technology replacement or building refurbishment. Also, attention is paid to the identification of the relevant barriers to energy efficiency and their importance in several contexts (domestic, industry clusters, etc.) (see Table 25.6).

25.3.6 Miscellaneous

This miscellaneous category includes rather unique and specialized areas which could not be included in any of the above alternative classifications: carbon capture and storage (CCS), cooking technologies, transportation and heating systems, trigeneration systems, thermal technologies, heat pumps, natural gas pipeline, production processes, domestic hot water production, DSM, and micro grids (see Table 25.7).

25.3.7 The Choice of Criteria

A wide range of criteria is considered for the design of optimal energy systems configurations [215]. The evaluation criteria in energy decision-making problems can act as a driving force for the discussion on sustainable energy systems

Table 25.4 Studies grouped in energy project problems

Energy focus	Scope	Alternatives	Stakeholders	MCDA method	Uncertainty	Application-origin	References
RES—geothermal energy	Evaluation and ranking of alternative energy exploitation schemes of a low temperature geothermal field	Alternative geothermal energy exploitation schemes	n.a.	PROMETHEE II and F—PROMETHEE	Scenarios and Fuzzy techniques	Local—Nea Kessani, Greece	Goumas and Lygerou [106]
RES—geothermal energy	Assisting with multi-criteria analysis in a renewable energy project	Alternative geothermal energy exploitation schemes	Local authorities, potential investors, central government, and public pressure groups (NGOs and local media)	PROMETHEE II	Sensitivity analysis and scenarios	Local—island of Chios, Greece	Haralambopoulos and Polatidis [112]
RES-E wind	Selection of wind energy projects	Projects	Regulatory Authority of Energy	ELECTRE TRI	Sensitivity analysis and scenarios	Local—Greece	Mavrotas et al. [179]
RES-E	Selection of a renewable energy investment project	Energy sources	The government, the banks and the development companies	VIKOR-AHP	n.a.	National—Spain	Cristóbal [75]
RES-E PV	Selection among 16 projects that have been proposed on farming fields in Haute Corse	Projects	The President of CA2B and the President's councillors	ELECTRE IS	Sensitivity analysis	Local—Haute-Corse, France	Haurant et al. [114]

Electricity generation	Selection of optimal power generation projects and attracting private investors	Projects	Relevant experts	Fuzzy AHP	Sensitivity analysis and Fuzzy techniques	National—China	Liang et al. [167]
Bioenergy	Facilitation of the design and implementation of sustainable bioenergy projects	Scenarios	Nine key stakeholders of Kasonga were invited to two workshops, and eight of them took part in the MCA evaluation. Participants represented the local and national government, NGOs, and gender groups	AHP, PROMETHEE, NAIADE	Scenario based	Local-Kasonga (Uganda)	Buchholz et al. [48]
Energy in general	Prioritization of project proposals in the energy sector	Projects	Ministry of Energy, energy experts, humanitarian aid and financing international organizations	ELECTRE III and PROMETHEE	Definition of preferences and uncertainties through the thresholds for all the criteria	National—Armenia	Goletsis et al. [103]
RES-E-wind	Selection of a suitable wind farm project	Projects	A committee of experts in the industry	AHP	n.a.	Local—anonymous province in China	Lee et al. [160]

Table 25.5 Studies grouped in siting decision problems

Energy focus	Scope	Alternatives	Stakeholders	MCDA method	Uncertainty	Application-origin	References
RES-E-wind	Determine the most convenient location for a wind observation station to be built on the campus of a university	Locations	Expert who had supervised several other projects of establishing WOS	AHP	n.a.	Local—Turkey	Aras et al. [23]
RES-E-wind	Assessment regarding the feasibility of installing some wind energy turbines in a site	Wind turbine configurations	n.a.	NAIADE	n.a.	Local—Island of Salina, Italy	Cavallaro and Ciraolo [59]
RES-E-wind	Social multicriteria evaluation is proposed as a general framework for dealing with the problem of wind park location	Locations	Scientists and social actors	SAW	Sensitivity and robustness analysis	Local—Catalonian—Urgell and Conca de Barberà, Spain	Gamboa and Munda [97]
RES-E-PV	Selecting optimal sites for grid-connected photovoltaic power plants	Locations	n.a.	AHP	Sensitivity analysis	Local—Andalusia, Spain	Carrión et al. [54]
RES-E-wind	Siting new wind farms	Locations	n.a.	SAW	n.a.	Local—La Rioja, Spain	Ramirez-Rosado et al. [233]

RES-E-wind	Derive wind farm land suitability index and classification under Geographical Information System environment	Locations	n.a.	AHP-OWA (ordered weight averaging aggregation function)	n.a.	National—Oman	Al-Yahyai et al. [17]
Electricity generation—thermal power	Evaluation and selection of optimal locations for thermal power plant	Locations	The committee has experts from key Ministries like Coal, Transport, Environment and Forest, Water Resource etc., power corporation and various State Electricity Boards	STEEP-fuzzy AHP-TOPSIS	Sensitivity analysis and fuzzy techniques	Local—Central part of India	Choudhary and Shankar [71]
Electricity generation—hydropower	Decisions on site selection and plant technical and operational parameters	Location and plant technical and operational parameters	Elements from CH2OICE project	VIKOR	n.a.	National—Bosnia and Herzegovina	Vučijak et al. [281]

Table 25.6 Studies grouped in energy efficiency problems

Target	Energy focus	Scope	Alternatives	Stakeholders	MCDA method	Uncertainty	Application-origin	References
Identify barriers	Energy in general	Identify relevant barriers to energy efficiency and their dimensions in small scale industry clusters	Barriers	The entrepreneurs	AHP	n.a.	National—India	Nagesha and Balachandra [194]
Initiatives	Energy in general	Sorting energy-efficiency initiatives, promoted by electric utilities, with or without public funds authorized by a regulator, or promoted by an independent energy agency	Energy efficiency initiatives	The Energy Agency, the Regulator, the companies	ELECTRE-TRI	Robustness analysis	National—Portugal	Neves et al. [197]
Technology	Energy in general	Ranking alternatives for induction motors replacement	Technologies	Management, Financial, Production and Maintenance Professionals	PROMETHEE II	Sensitivity analysis	National—Brazil	Sola et al. [252]
Buildings	Energy in general	Proposing a comprehensive design process—'Integrated Energy-Efficient Building Design Process'	Energy efficient strategy	One owner, four LEED accredited professionals and two end users	AHP	Sensitivity analysis	Local—New Delhi, India	Kanagaraj and Mahalingam [148]
Technology	Energy in general	Replace technologies in industrial energy systems, regarding organizational barriers for energy efficiency	Portfolio	The participation of all sectors in the company, as well as specialists in decision-making and the energy area	MAUT	Sensitivity analysis	Local—southern Brazil	Sola and Mota [251]

Table 25.7 Studies grouped in energy miscellaneous problems

Target	Energy focus	Scope	Alternatives	Stakeholders	MCDA method	Uncertainty	Application-origin	References
Cooking energy sources	Energy in general	Ranking alternative cooking energy sources	Cooking energy alternatives	Thirty experts—educators, policy makers, researchers and actual users covering a variety of population	PROMETHEE	Sensitivity analysis	National—India	Pohekar and Ramachandran [225]
Transportation	Energy in general—fuels	Analysis of alternative-fuel buses for public transportation	Alternative-fuel mode	Experts from different decision-making groups, such as bus users, the social community, and the operators	AHP, TOPSIS and VIKOR	n.a.	Local—Taipei City, Taiwan	Tzeng et al. [271]
CCS	Electricity generation	Examine the stakeholders' reactions to the mitigation option of capturing CO_2 from power stations and storing it in suitable off-shore geological reservoirs	Scenarios	The stakeholders were selected to represent key interests and expertise from the energy business, government and NGOs. A specialist on the environmental assessment and construction of long pipeline routes was also interviewed	SAW	n.a.	Local—North West region of England	Shackley and McLachlan [245]

(continued)

Table 25.7 (continued)

Target	Energy focus	Scope	Alternatives	Stakeholders	MCDA method	Uncertainty	Application-origin	References
DSM	Electricity generation	Evaluation of DSM implementation strategies	Strategies	Top management officials of Rajasthan Electricity Regulatory Commission, Transmission Company, Distribution companies and some independent consultants in the state	AHP	n.a.	National—India	Vashishtha and Ramachandran [277]
Heating system	Energy in general	Evaluation of conventional and renewable energy sources for space heating in the household sector	Heating system	Based on most recent energy survey of the household sector	AHP and Fuzzy sets	Fuzzy techniques	National—Jordan	Jaber et al. [137]
Heating system	Energy in general	Planning of community heating systems modernization and development	Heating system	n.a.	ELECTRE III	Scenarios	Local—Poland	Mroz [191]
Trigeneration systems	Gas	Selection and evaluation of trigeneration systems	Technologies	Experts	Fuzzy AHP	Fuzzy techniques	Local building in Shanghai—China	Wang et al. [282]

Thermal technology	RES—Solar	Assessment of concentrated solar thermal technologies	Concentrated solar parabolic technologies	n.a.	PROMETHEE	Use of weight stability intervals	Not specified—Italy	Cavallaro [57]
Transportation	Biofuels	Rank different road transportation fuel-based vehicles (both renewable and non-renewable)	Fuel alternatives	n.a.	PROMETHEE	Sensitivity analysis and scenarios	Not specified—Canada	Mohamadabadi et al. [190]
Heat pumps	Energy in general	Use of heat pumps for the integration of wind power	Installed heat pump capacity	n.a.	MAVT	n.a.	Local—Western Denmark	Østergaard [215]
Natural gas pipelines	Natural Gas	Assessing risk in natural gas pipelines and classifying sections of pipeline into risk categories	Pipeline section	Indirectly influenced by the perception of several stakeholders, including government authorities and a regulatory agency	ELECTRE TRI	Sensitivity analysis	Not specified—Brazil	Brito et al. [45]
Heating system	Energy in general	Evaluation and ranking energy sources available for a case of district heating system	Energy sources	Developer, Environmental group, Community representative group	PROMETHEE II	Scenario based	Local—Vancouver, Canada	Ghafghazi et al. [101]
Production processes	RES—E—PV	Assessment of thin-film photovoltaic production processes	Processes	Expert's opinion	ELECTRE III	Sensitivity analysis	Not specified—Italy	Cavallaro [58]

(continued)

Table 25.7 (continued)

Target	Energy focus	Scope	Alternatives	Stakeholders	MCDA method	Uncertainty	Application-origin	References
Heating system	Energy in general	Selection of space heating systems for an industrial building	Heating system	The company manager and a number of potential suppliers	AHP	Sensitivity analysis	Local—Treviso (North Eastern Italy)	Chinese et al. [70]
Domestic hot water production	Energy in general	An evaluation of the sustainability of different energy options for obtaining thermal energy	Energy options	n.a.	SAW—general index of sustainability	Fuzzy techniques	Local—Belgrade, Serbia	Jovanovic et al. [143]
Pollution abatement	Energy in general	Obtaining a priority list of abatement options, achieving consensus and securing the adoption of the resulting optimal solution	Control air pollution alternatives	Sixteen experts, representing local authorities, universities, research institutes, industries and public bodies actively involved in the corresponding field	ELECTRE III	Sensitivity analysis, Scenarios and Fuzzy techniques	Local—Thessaloniki, Greece	Vlachokostas et al. [278]
Micro grids	Electricity generation	Assessment and quantification of the sustainability and reliability of different power production scenarios in a regional system, focusing on the interaction of microgrids with the existing transmission/distribution grid	Scenarios	n.a.	SAW—composite sustainability index	Sensitivity analysis	Transnational—Northwestern European electricity market (Belgium, France, Germany and the Netherlands)	Prete et al. [229]

development [283]. A holistic approach is required for the development of energy systems that can help solving broader problems associated with the essential linkages between the energy systems, the environment and the socio-economic development. Thus, the consideration of evaluation criteria and methods that can perform a thorough assessment of the energy decisions at stake is a prerequisite for selecting the best course of action (or a subset of alternatives for further screening), ranking the alternatives or assigning them to categories of merit, also informing DMs about their integrated performance and monitoring their impacts on the environment and the socio-economic context. The diversity of problems, evaluation criteria, stakeholders and methodological approaches considered in this fast-growing field highlights the importance of problem structuring methods, which are aimed at rising out a set of interests, preferences and concerns of the relevant stakeholders and their relations of power (Neves et al. [196, 197]; Coelho et al. [73]).

The main criteria used in energy decision-making studies are briefly reviewed and classified into technical, economic, environmental and social aspects, which are summarized in Fig. 25.1.

25.3.8 Technical Criteria

Adaptability—represents the technology's potential to be adapted to the country's conditions for energy production. Depending on the approach considered it can be measured in qualitative or quantitative units [17, 70, 86, 158, 160, 214, 272].

Availability—evaluates whether the energy resource is readily available [17, 67, 71, 75, 91, 137, 160, 198, 214, 247, 258, 260, 272]. An average availability factor is based on typical load factors. Availability can also indicate the amount of time a unit can be used for electricity and/or steam production [83, 114]. Due to the intermittent nature of renewable energy, some studies imply that the stability of electricity output is critical for the development of renewable energy. Depending on the indicator used, it can be measured in qualitative or quantitative units.

Continuity and predictability—reflects the technology's ability to maintain stable the energy generated without being affected by external factors. This criterion is important to know if the technology operates continuously and confidently. It is usually measured in qualitative units [32, 86, 145, 146, 214, 225, 263, 265].

Diversity—this criterion is understood as diversity of installed power, calculated according to the Shannone-Wiener Index [235], diversity of energy production mix [248, 266], diversity of technologies [158, 180] or diversity of supply [70, 135, 173]. Depending on the indicator it can be measured in quantitative or qualitative units.

Efficiency—refers to how much useful energy can be obtained from an energy source. The efficiency coefficient is the ratio of the output energy to the input energy, which is used to evaluate energy systems. It is usually measured in quantitative units [4–6, 8, 33, 48, 67, 77, 86, 86, 135, 137, 141–143, 148, 151, 152, 172, 178, 194, 221, 222, 259, 260, 263, 271, 285].

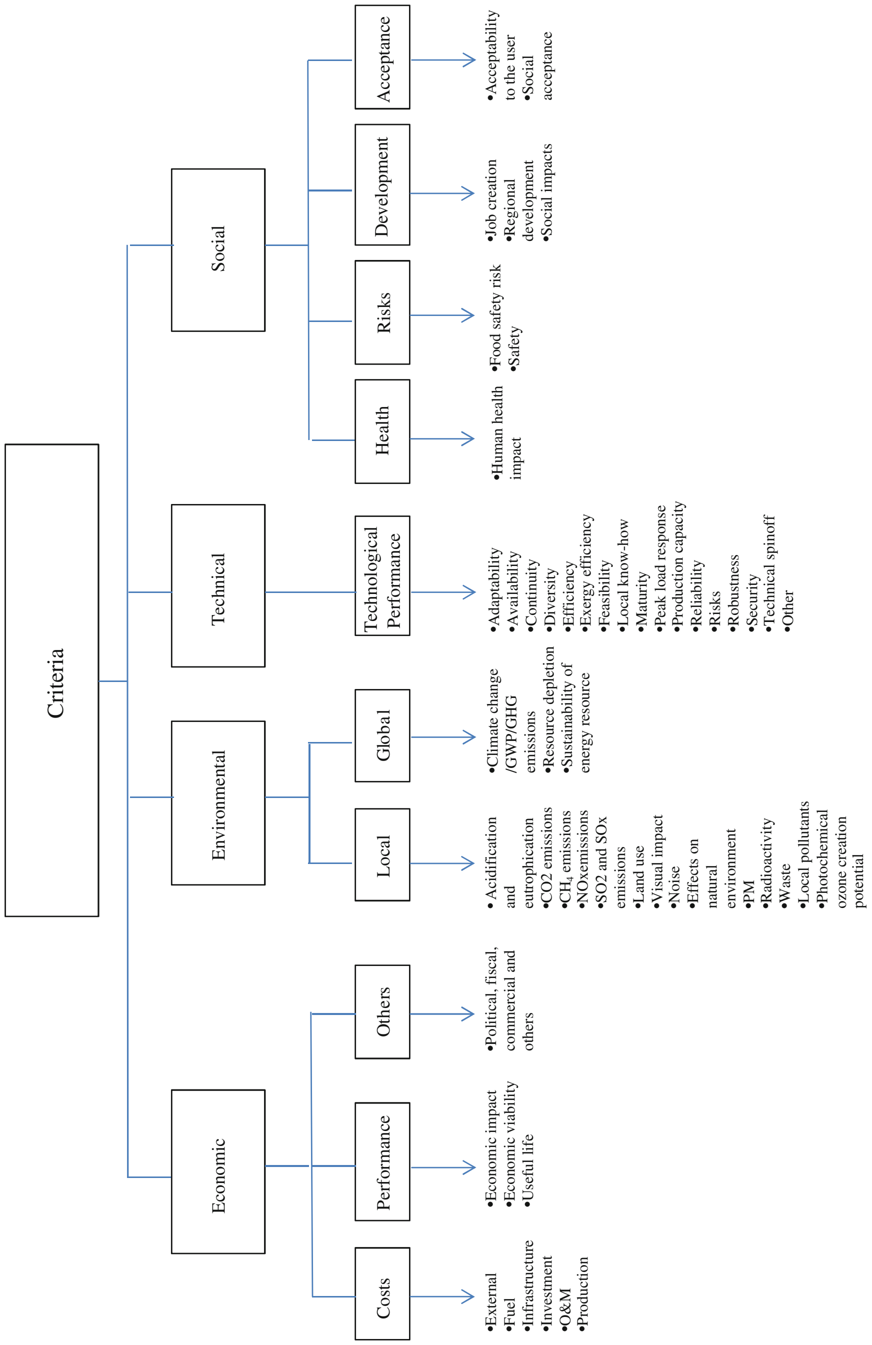


Fig. 25.1 Criteria considered in energy decision-making studies

Exergy efficiency—also known as the second-law efficiency or rational efficiency, computes the efficiency of a process taking the second law of thermodynamics into account. Exergy is the energy that is available to be used. The evaluation of CHP (CCHP) systems is often performed using this criterion [83, 151, 152, 229, 282]. It is usually measured in quantitative units.

Feasibility—measures the confidence of the implementation of the energy policy. The number of times the policy has been tested successfully can be taken into account as a decision parameter. This criterion can be considered in diverse categories and, depending on the indicators considered, it can be measured in qualitative or quantitative units [103, 145, 146, 180, 198].

Local technical know-how—includes an evaluation which is based on a qualitative comparison between the complexity of the technology considered and the capacity of local actors to ensure an appropriate operating support for its installation and maintenance [32, 48, 71, 103, 145, 146, 263].

Maturity of the technology—represents the technology's maturity rate as well as its penetration percentage in the international market [32, 57–59, 86, 86, 91, 103, 160, 178, 225, 247, 259, 260, 263, 282]. It may be also perceived as technological advantage [158]. The following stages can be considered: (1) technologies that are only tested in laboratory; (2) technologies that are only performed in pilot plants, where the demonstrative goal is linked to the experimental one, referring to the operating and technical conditions; (3) technologies that could be still improved; and (4) consolidated technologies, which are close to reaching the theoretical efficiency limits. It is usually measured in qualitative units. In Mavrotas et al. [179] the maturity criterion refers to the maturity of the certification procedure.

Peak load response—reflects the technology-specific ability to respond promptly to large temporal variations in demand. This capability is particularly attractive in view of market liberalization. Base-load technologies and those renewables that strongly depend on climatic conditions are not suitable in this context and have very low score. It is usually measured in qualitative units [17, 82, 135, 216, 258].

Production Capacity—refers to the availability of a fuel as a feedstock for a given alternative to the installed or new generation capacity. Alternatives that have large feedstock reserves and greater generation capacity are better [49, 59, 67, 75, 97, 160, 163, 164, 191, 260, 271, 288]. Depending on the indicator it can be measured in qualitative or quantitative units.

Reliability—evaluates the capacity of a device or system to perform as designed, the resistance to failure, the ability to perform a required function under stated conditions for a specified period of time, or the ability of failure without catastrophic consequences. It may be applied to technologies or even to energy policies. Technology may have been only tested in laboratory or only performed in pilot plants, it can be still improved or it is consolidated. It is usually measured in qualitative units [32, 70, 83, 91, 137, 145, 146, 160, 179, 197, 225, 245, 252, 271].

Risk—evaluates the safety of the implementation of an energy policy or the risks of a major disaster [103, 145, 146, 160, 245, 260]. Depending on the indicator it can be measured in qualitative or quantitative units.

Robustness/Durability—measures robustness based on the materials' fatigue life and reliability. It is usually measured in qualitative units [23, 77, 91, 180, 225, 259].

Security—evaluates the security of the supply system, the reduction of energy dependence or fuel imports, reflecting mostly geopolitical factors that may affect the continuous availability of non-renewable energy carriers from their origin. Secure energy supplies are essential to maintaining economic activity and to providing reliable energy services to the society [47, 48, 67, 82, 86, 157, 158, 173, 180, 197, 208, 235, 245, 247, 258, 266, 267, 270]. Depending on the indicator it can be measured in qualitative or quantitative units. It is also considered as an economic criterion.

Technical spin-off—refers to the development of other analogous technology [163, 164]. It was used in the hydrogen context and it is measured in qualitative units.

Other specific technical criteria—these are criteria directly related to the type of energy source under analysis and specifically considered in a particular study, for instance**:** deficit of electric power in a problem of electric power system expansion planning [280]; multiplicative effects on the local technology for the prioritization of energy projects [103]; peak power, range, vehicle operation and performance start-up time and transient response for the assessment of the future hydrogen fueling systems for transportation [288]; topography in the choice of location of a wind observation station [23]; the satisfaction of energy demand for decision support in energy conservation promotion [144]; nutrition value of food for the evaluation of cooking energy alternatives [225]; the suitability of a potential site for the energy resource [265]; lock-in and deliverability for assessment of the role of CCS [245]; orography and climate conditions for a siting problem [54]; the technological status of hydrogen energy measured by the number of SCI papers, the number of patents and the number of proceedings [161, 165]; the technological infrastructure of the hydrogen technology [161, 162, 165]; the co-generation ratio for the appraisal of a heating system modernization [191]; reduction of capacity costs for sorting actions for energy efficiency promotion [197]; the ratio of consumption of primary energy to the demand, and the control and regulation property for the assessment of the trigeneration systems [282]; planning and monitoring needs for bioenergy [48]; temperature and solar capacity factor for the assessment of concentrated solar thermal technologies [57]; energy intensity for the appraisal of an energy system [141, 142]; micro-siting of WEGs and WEG functions for the strategic selection of wind farms [160]; reserve capacity requirement, island mode, connected mode or connected island mode, and condensing mode operation for energy systems analysis of renewable integration [215]; conventional fuel savings for sustainable energy planning [270]; utilization rate of material and thickness of active material for the comparative assessment of thin-film photovoltaic production processes [58]; time to repair for the selection of space heating systems in an industrial building [70]; deployment of renewable energy systems in case of the assessment of EU renewable energy targets [208]; service conditions and operation factor for improving energy efficiency in an industrial motor system [252]; effectiveness for the assessment of pollution abatement measures [278]; structure of energy use and industry for environmental performance of energy use planning [285, 286]; wind characteristics for a problem of wind farm land suitability indexing [17]; accessibility for the selection of a thermal power plant location [71]; expected loss of energy and loss of load probability in the case of sustainability and reliability assessment of micro-grids [229]; bulk density, deposit formation, lower heating value and storage time before degradation in the case of biomass pellets assessment [259]; growing degree days, slope, soil wetness, annual rainfall, soil texture, soil pH, soil depth and slope for agro-energy spatial modeling [262]; proximity to sea and to heat demand for an application to the UK energy sector [135]; the rate of dispatchable power, considering the ratio between the sum of installed power of coal, CCGT, dam hydro power plants, and all the installed power for the evaluation of future scenarios for the power generation sector [235].

It may be concluded that regarding the technical criteria, the efficiency criterion is the most used in the energy applications herein reviewed (being present in 32 papers out of the 104 papers considered). The highest concern with efficiency is attained in the framework of power generation studies, followed by energy efficiency and energy plans (see Fig. 25.2). Technology maturity and security are the next most important criteria: security concerns have a higher participation in studies in energy plans and policies, followed by power generation technologies, siting decisions and energy projects; technology maturity aspects have a higher presence in energy project decisions, energy studies, and energy plans and policies studies. Availability criteria are taken into account mostly in energy project decisions and siting decisions. Reliability has its highest expression in miscellaneous energy studies, energy project decisions and energy efficiency studies. Production capacity is more relevant in siting and energy project decisions.

25.3.9 Economic Criteria

25.3.9.1 Costs

Externality costs—are costs imposed on society and environment but not accounted for by the producers and consumers of energy, and therefore generally not included in market prices. This criterion is measured in quantitative units [66, 67, 180, 215, 229, 258, 265].

Fuel cost—refers to the provision of raw materials necessary (e.g. coal or natural gas for conventional thermal plants or uranium for nuclear power plants) for the operation of the energy supply system. Fuel costs may include extraction or mining, transportation and possible fuel processing to be used in a power plant. Fuel costs may vary considerably in different time periods and geographies as a result of several reasons, including demand, production and political matters. Fuel cost is excluded from operation and maintenance cost when both are selected to perform the evaluations [7, 67, 70, 82, 83, 86, 86, 135, 137, 141–143, 157, 190, 214, 221, 222, 229, 235, 258, 260, 265–267, 272, 280, 288].

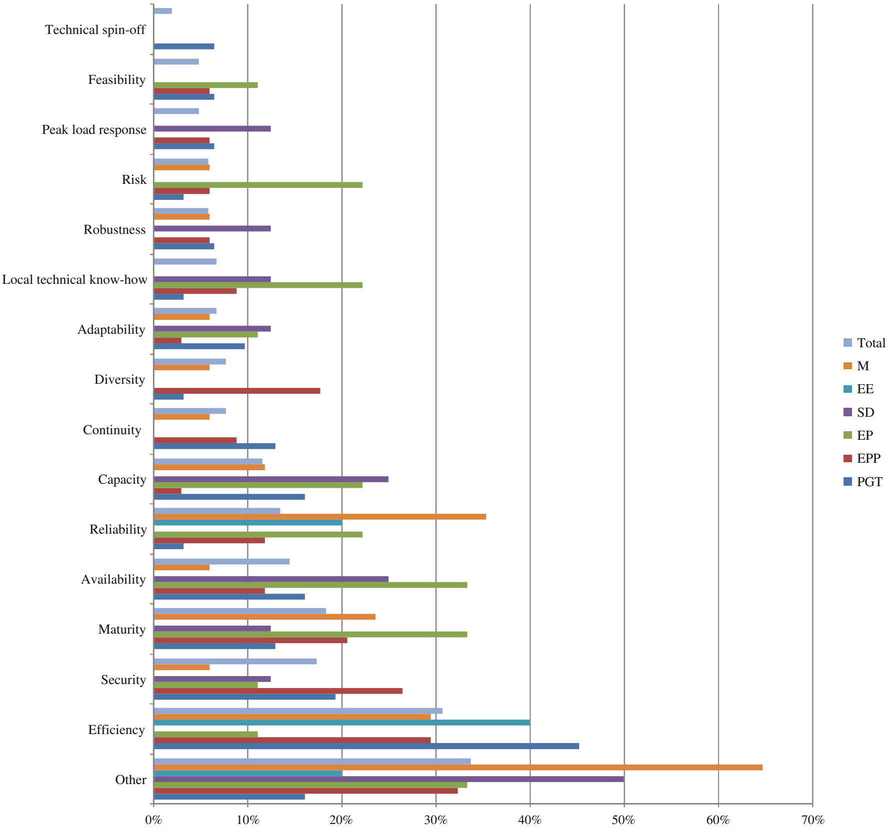


Fig. 25.2 Technical criteria. *M* miscellaneous, *EE* energy efficiency, *SD* siting decisions, *EP* energy projects, *EPP* energy plans and policies, *PGT* power generation technologies

Infrastructure costs—may include additional investments in transmission network required by each scenario [235], investment in grid connections [258] or investment in infrastructures as a whole [71, 135, 235, 245, 288].

Investment cost—may include the purchase of mechanical equipment, technological installations, construction of roads and connections to the national grid, engineering services, drilling and other incidental construction work. Nuclear and coal-fired units are characterized by high investment costs and low operating costs while gas-fired generation is characterized by lower capital costs and higher operating costs. Investment cost is the most used economic criterion to evaluate energy systems [4–6, 23, 33, 46, 49, 57–59, 67, 70, 71, 75, 77, 82, 83, 86, 86, 91, 100, 135, 137, 141–143, 145, 146, 151, 152, 157, 158, 160, 164, 167, 190, 214, 216, 220–222, 225, 229, 235, 245, 247, 251, 252, 258, 260, 265–267, 270, 272, 280–282, 288].

Operation and maintenance (O&M) costs—include wages and the funds spent for energy, products and services, and preventive and corrective maintenance works. The operation and maintenance costs may be divided into fixed and variable costs [7, 46, 49, 57, 59, 67, 70, 75, 77, 82, 83, 100, 137, 141, 143, 151, 152, 157, 158, 160, 214, 216, 220, 222, 225, 229, 235, 245, 258, 260, 265–267, 270–272, 281]. This criterion is usually measured in quantitative units.

Production costs—This criterion is important and useful for assessing how commercially competitive the system is compared with other production technologies [4, 5, 7, 8, 33, 57, 197, 215, 221, 259, 271, 278, 282].

25.3.9.2 Economic Performance

Economic impact—refers to the capacity of the energy project or policy of promoting local/regional/national economic development [48, 71, 114, 137, 144, 148, 158, 163, 164, 173, 198, 247], or the impact on the dynamics of the national industry and local income [97, 235], or even the impact on GDP or GNP [6, 237, 248]. Depending on the indicators used, it can be measured in a qualitative scale or in quantitative units, respectively.

Economic viability—this criterion evaluates the proposed energy policy/project namely using the following economic appraisal techniques [46, 89, 103, 173, 180, 194, 233]:

- Net Present Value (NPV)—defined as the total present value of a time series of cash flows. It is a well-known method for the appraisal of long-term energy projects and it measures the excess or shortfall of cash flows, in present value terms, once financing charges are met. NPV is often used to assess the feasibility of an energy project by an investor [106, 145, 146, 191, 216, 251, 252, 282].
- Internal Rate of Return (IRR)—is the discount rate that makes the net present value of all cash flows from a particular project equal to zero. The higher a project's IRR, the more desirable it is to undertake the project [112, 145, 146, 179, 252, 282].
- Cost-Benefit Analysis (CBA)—prior to erecting a new power plant or taking on a new energy project, a CBA should be conducted as a means of evaluating all of the potential costs and revenues that may be generated if the project is completed. The outcome of this analysis will determine whether the project is financially viable or if another project should be pursued 145, 146].
- Payback period—refers to the period of time required for the return on an investment to "repay" the sum of the original investment. It intuitively measures how long a project takes to "pay for itself", shorter payback periods being obviously preferable to longer payback periods to investors. Although primarily a financial term, the concept of a payback period is occasionally extended to energy payback period, i.e. the period of time over which the energy savings of a project equal the amount of energy expended since project inception [86, 86, 251, 252].

Useful life—refers to the number of years the power plant can operate before the equipment needs to be replaced. Generally, the energy system timelife follows the "bathtub curve" [49, 75, 197, 222].

Other specific economic criteria—Other political, fiscal, legal, and commercial criteria include: availability of funds, compatibility with the national energy policy objectives, political acceptance, geopolitical issues, legal framework, commercial aspects, market size, energy price stability, duration of preparation and implementation phases for the evaluation of different energy policy implementations, energy saved [17, 32, 46, 47, 70, 71, 75, 89, 100, 103, 106, 112, 143, 145, 146, 158, 160–165, 167, 173, 178, 180, 191, 194, 197, 208, 214, 217, 220, 225, 237, 247, 248, 251, 252, 260, 263, 271, 272, 277, 288].

The investment cost criterion is the most used regarding all types of criteria considered (see Fig. 25.3). When taken into account, the presence of this criterion in each energy decision problem ranges from 76 % in miscellaneous energy studies

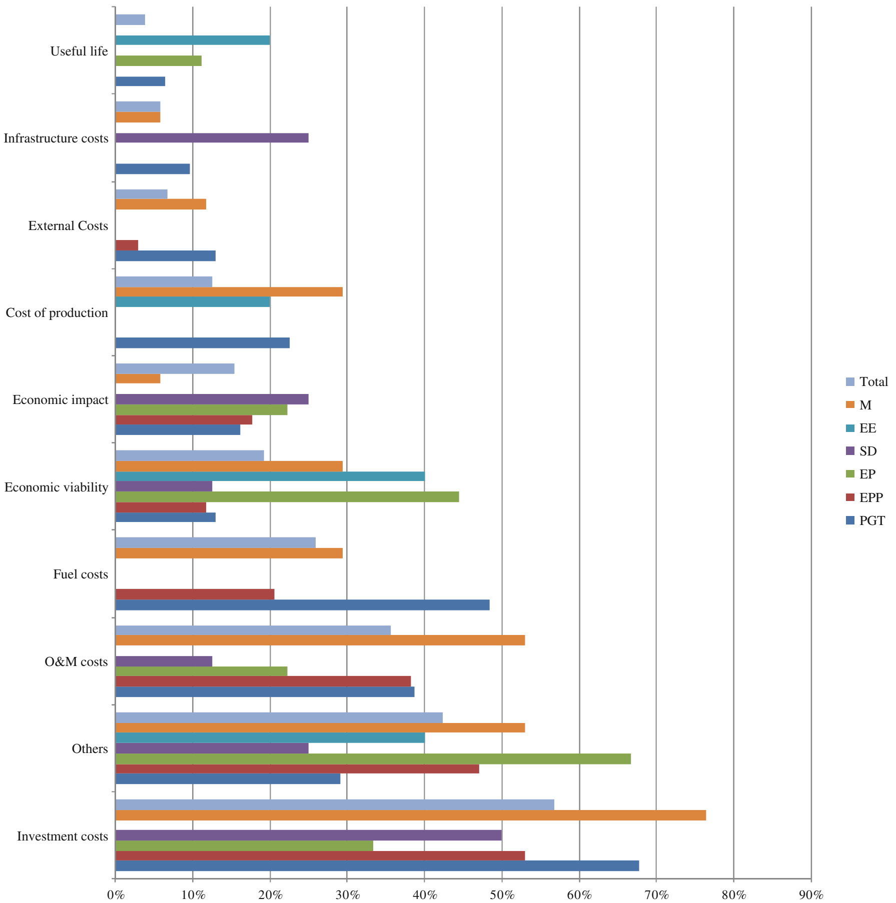


Fig. 25.3 Economic criteria

to 33 % in energy project studies. O&M costs are the next most important economic criterion, followed by fuel cost. While O&M costs have a highest participation in studies framed in miscellaneous energy applications (with focus on heating systems), followed by studies grouped in power generation technologies and energy plans and policies, the fuel cost criterion has a highest presence in power generation technology comparisons. Economic viability concerns are taken into account mainly in energy projects and energy efficiency decisions. Economic impact criteria have its highest expression in energy siting decisions and in energy projects. Production costs are present in miscellaneous energy applications, appraisal of power generation technologies and energy efficiency problems.

25.3.10 Environmental Criteria

25.3.10.1 Local Impacts

Acidification and eutrophication—refers to the acidification potential and to the acid rain (sulfuric and nitric acid created from the binding of the sulfur dioxide and nitrogen with water in the atmosphere) and the contribution of the deposition of nitrogen-containing compounds to eutrophication (excess nutrient enrichment) of terrestrial and marine ecosystems [83, 214, 272].

CH_4 emissions—methane is another important organic compound released during biomass burning because of its carbon content. Methane emissions largely depend on the burning method, decreasing with increasing combustion efficiency [157, 259].

CO_2 emissions—CO_2 is mainly released through conventional energy systems and contributes to the GHG effect leading to global warming. Indicators consider either a direct contribution for the increase (in case of fuel combustion) or the reduction (in case of RES generation) of this pollutant. The computation of emissions may be based on lifecycle assessment techniques. This criterion is usually measured in quantitative units [4–8, 33, 49, 59, 66, 75, 82, 89, 97, 100, 135, 141–143, 151, 152, 157, 158, 180, 191, 197, 208, 215, 216, 220–222, 229, 235, 247, 248, 263, 270, 282, 285, 286, 291].

Effects on natural environment—mainly refers to the potential risk to ecosystems caused by energy policies and strategies [17, 46, 49, 57, 59, 71, 83, 86, 86, 91, 112, 114, 144, 148, 151, 173, 178, 191, 197, 198, 214, 221, 233, 237, 245, 248, 260, 264, 271, 272, 288, 291]. It may include impacts on air, water and soil quality [158, 281], reflect human toxicity, fresh water eco-toxicity potential, marine eco-toxicity potential (MAETP) and terrestrial eco-toxicity potential [77], as well as ecological footprint [47]. Depending on the indicators considered, it can be either evaluated in qualitative or quantitative terms.

Land use—the land required by power plants is a matter of concern for their evaluation, since different energy systems may have different land use impacts for the same output level [4, 32, 48, 54, 66, 71, 97, 114, 135, 145, 146, 151, 152, 173, 247, 248, 288, 291].

Local pollutants—it includes the emissions of pollutants with local and regional impacts not previously mentioned (e.g. non-methane volatile organic compounds, ash emission and carbon monoxide, smell) [32, 48, 57, 66, 103, 135, 137, 145, 146, 158, 173, 225, 280, 282, 288].

Noise—refers to the noise impact caused in neighbor areas by new infrastructures, such as wind parks. It is usually measured in a qualitative scale [17, 49, 57, 59, 77, 97, 103, 151, 158, 173, 233, 235, 263, 271, 282].

NO_x emissions—contribute to air pollution, acid deposition and climate change. Reacting with organic chemicals, NO_x can form a wide variety of toxic products that may damage health. This criterion is usually measured in quantitative units [5, 6, 33, 47, 66, 82, 100, 141–143, 151, 152, 157, 197, 220, 222, 229, 247, 259, 263, 282, 285, 286, 288].

Particulate matters (PM)—are the primary cause for the rise of mortality and morbidity in the vicinity of power plants and it is mainly released by coal/lignite and oil as well as biomass and photovoltaic power plants (during cell manufacturing, which is accounted for in lifecycle assessment studies). The risk for human health depends on size, distribution, microstructure and chemical composition of particulates released into the atmosphere [66, 135, 158, 197, 266, 267, 285, 286, 288].

Photochemical ozone creation potential—Photochemical pollution is formed from emissions of nitrogen oxides and volatile organic compounds and carbon monoxide in the presence of sunlight [83, 135, 158].

Radioactivity—small amounts of radioactivity are released to the atmosphere from both coal-fired and nuclear power stations. In the case of coal combustion, small quantities of uranium, radium and thorium present in the coal produce various levels of radioactive fly ash. Nuclear power stations and reprocessing plants also release small quantities of radioactive gases [66].

SO_2 and SO_x emissions—sulfur emissions into the atmosphere in the form of SO_2 and SO_3 derive from the burning of fossil fuels or even wood, straw, alfalfa, switchgrass or poultry litter. These emissions are main contributors to acidification and are responsible for serious damage to human health and ecosystems [6, 33, 47, 66, 82, 100, 135, 157, 158, 197, 220, 229, 259, 263, 285, 286].

Visual impact—reflects the visual nuisance that may be created by the establishment of a wind turbine in a specific area or by the construction of new power plants upon the sightseeing. The landscape of the different sites, the distance from the nearest observer, the type and size of plants to be installed and the possibility to integrate them with their surroundings are considered when evaluating the alternatives. It is usually measured in a qualitative scale [54, 59, 77, 97, 112, 114, 151, 158, 233, 235, 237, 263, 264, 266, 267].

Wastes—relate to damages on the quality of the environment and it may include wastes that are related to secondary products by fumes treatment or water processing, and solid wastes. The evaluation of this criterion may include the type and quantity of emissions, and costs associated with waste treatments [6, 103, 112, 145, 146, 173, 237, 248, 260].

25.3.10.2 Global Impacts

Climate change/GWP/GHG emissions—refers to the GHG emissions or carbon footprint [32, 47, 71, 83, 86, 86, 91, 172, 190, 214, 237, 258, 266, 267, 272, 288].

Resource depletion—relates to the cumulated energy input and material input throughout the project or energy policy lifetime [6, 33, 83, 91, 148, 158, 173, 191, 215, 220, 237, 248].

Sustainability of energy resource—is a measure of feedstock supply availability over a long period of time (40–70 years), in amounts sufficient for supplying large market penetration of a given technology/energy source alternative. Alternatives associated with renewable resources are better scored since they allow savings of finite energy resources [32, 59, 71, 91, 144, 178, 215, 247, 248, 265, 288, 291].

The most used environmental criterion is CO_2 emissions. The highest concern with this criterion is attained in the framework of the studies grouped in energy plans and policies followed by power generation studies (see Fig. 25.4). Effects on natural environment are the next most important environmental criteria, having a high presence in energy siting decisions, energy efficiency, power generation technologies and energy plans and policies. NO_x emissions are broadly taken into account, namely in energy plans and policies studies, power generation technologies, and energy efficiency studies. Land use is also an important evaluation concern, having its highest expression in siting decisions, energy plans and policies and energy projects studies. Climate change is a global assessment criterion considered in power generation technologies and energy plans and policies studies. While SO_2 emissions are mostly present in energy plans and policies studies, local pollutants, noise and visual impacts criteria appear mostly in siting decision studies. Resource depletion and sustainability of energy resources are considered in about 10 % of studies.

25.3.11 Social Criteria

25.3.11.1 Health Impacts

Human health—refers to health hazards caused by the different energy sources. Relative hazards to human health due to various sources can be compared using the concept of expected years-of-life lost. Depending on the indicator considered, this criterion can be measured in qualitative or quantitative units [46, 77, 157, 173, 220, 235, 237, 288].

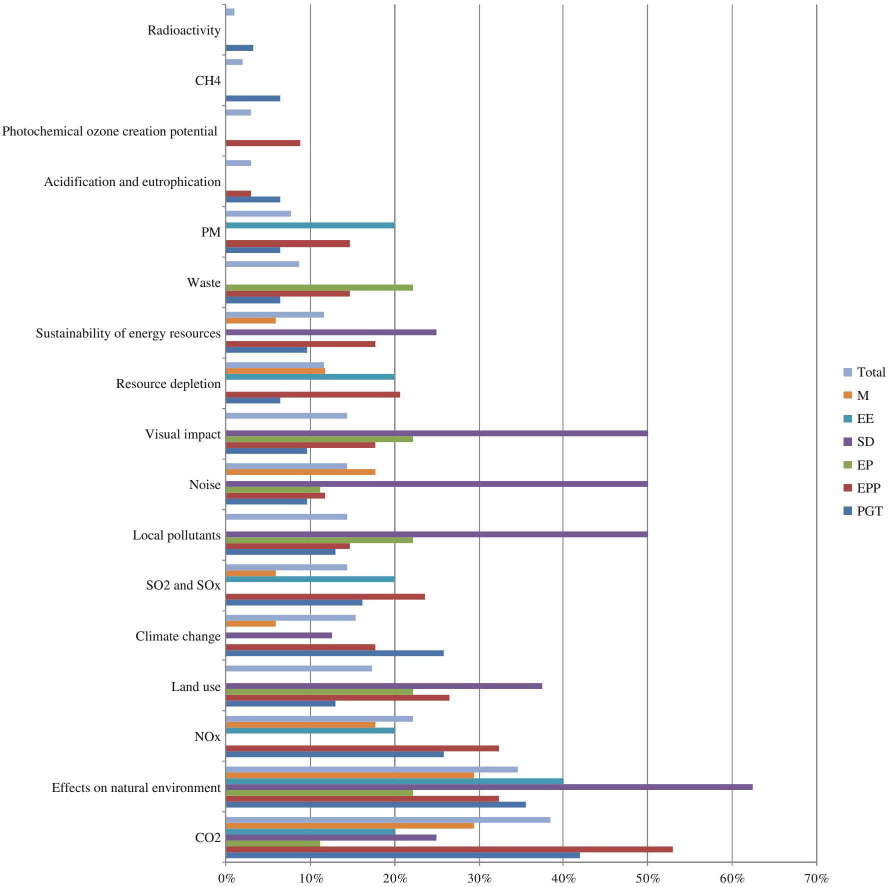


Fig. 25.4 Environmental criteria

25.3.11.2 Risks

Food safety—is a qualitative indicator generally used for assessing the risks of utilizing bio-fuels on food supply safety and food prices. This criterion is very relevant as the increased use of bio-fuels in the transportation sector causes important problems related with the increase of food prices [258].

Safety—represents the risk of fatal accidents or injuries according to the frequency of occurrence of an accident in the past and the number of fatalities or injuries involved. Severe accidents perceived in the future represent a qualitative assessment of risk [66, 137, 141–143, 225, 237, 258, 260, 282, 288].

25.3.11.3 Development

Job creation—evaluates energy policies by taking into account their impacts on direct, indirect or induced employment, either with local or national implications. It can be measured in quantitative or qualitative units [6–8, 32, 33, 47, 48, 66, 71, 86, 86, 100, 106, 112, 135, 145, 146, 151, 152, 158, 173, 197, 217, 220, 235, 247, 258, 263, 266, 267, 285, 286].

Regional development—expresses the progress induced in the less developed regions of a country by the deployment of new technologies or energy plans [71, 86, 86, 217, 270].

Social impacts—refers to the (either positive or negative) effects produced by the implementation of the energy project to the community [114, 135, 137, 173, 197, 198, 214, 215, 220, 237, 245, 248, 270, 272, 285, 286].

25.3.11.4 Acceptability

Acceptability to the user—depends on attributes such as usability, reliability, efficiency of use and comfort, among others [23, 70, 103, 148, 225, 259, 271, 282, 285, 286, 291]. It can also represent the refueling convenience, a measure of consumer access to a given fuel through the development of adequate refueling stations. Alternatives that have an existing network of fueling options are better [190, 288]. It is usually measured in a qualitative scale.

Social acceptability—refers to enhancement of consensus among social partners. It takes into account opinions related to the energy systems by the local population. It is extremely important since the opinion of the population and pressure groups may heavily influence the amount of time needed to complete an energy project. It is measured in a qualitative scale [59, 66, 77, 89, 91, 103, 145, 146, 151, 152, 158, 160, 180, 217, 220, 245, 258, 260, 270, 278].

Other social criteria—are generally directly related to the kind of energy source under analysis and specifically considered in a particular study, for instance: human resources dedicated to research and development activities [161, 162, 165]; social components of risks [237]; educational potential [198]; share of household income spent on fuel and electricity; working hours per energy produced [141–143]; cultural heritage [264]; environmental protection level [167]; educational supportive actions for RES in order to increase the energy market competitiveness and the energy environmental awareness [217]; amount of capital per kWh produced in the lifetime and the number of entities created per kWh produced in lifetime for assessing a diversity and vitality indicator [6].

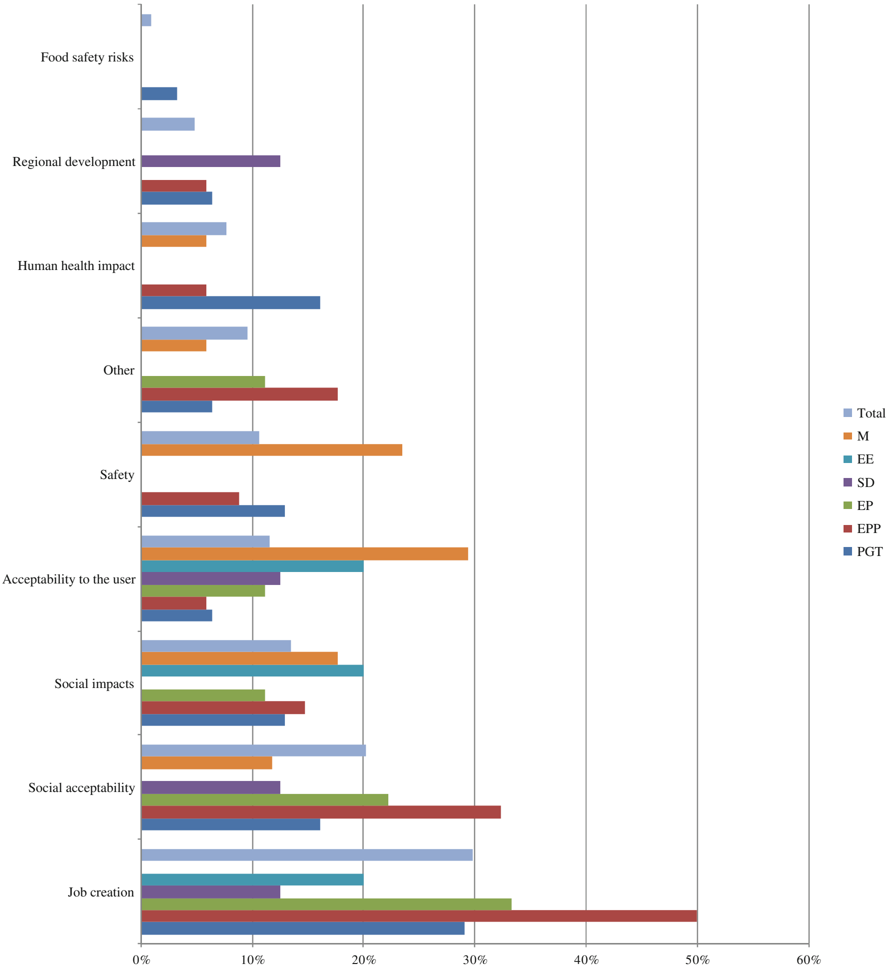


Fig. 25.5 Social criteria

Job creation is the most used social criteria. The highest concern with this criterion is attained in the framework of the studies grouped in energy plans and policies, followed by energy project studies (see Fig. 25.5). Social acceptability is the next most important social criteria with a high participation in energy plans and policies studies and energy project studies. Social impacts are mostly considered in energy efficiency studies. Acceptability to the users is an assessment criterion considered in miscellaneous energy studies and energy efficiency studies. Safety concerns are also important being taken into account in more than 10 % of the papers reviewed.

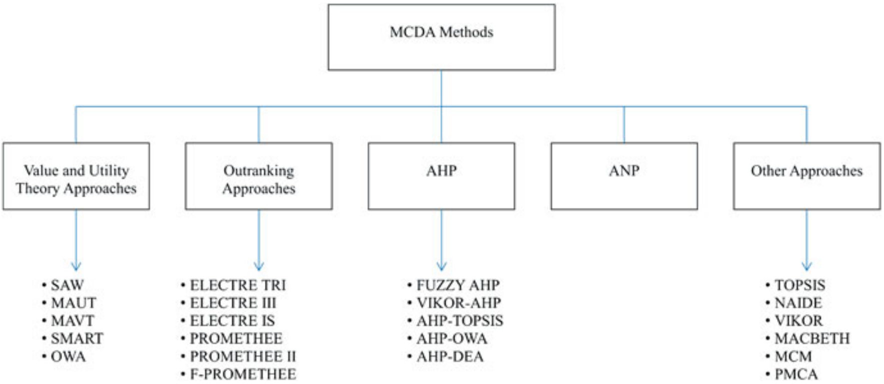


Fig. 25.6 MCDA methods used in energy decision-making studies

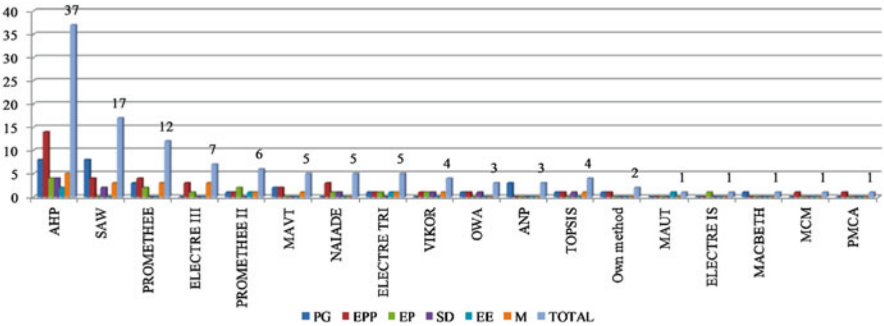


Fig. 25.7 MCDA methods used in each type of energy application (number of papers)

25.3.12 MCDA Methods

MCDA methods are classified into five main groups as shown in Fig. 25.6: Value and utility theory approaches, outranking approaches, AHP/ANP approaches and other approaches not fitting into the previous groups. The most popular MCDA methods used in energy decision-making studies herein reviewed are also illustrated in Fig. 25.6.

The value and utility theory approaches or function-based methods that are more frequently used include:

- Simple Additive Weighting (SAW), in which the global value of each alternative is equal to the sum of the products of the criterion weight and attribute data. This method is the second most used one, being mainly considered in power generation technology comparisons (see Fig. 25.7), and typically refers to a general index of sustainability—Afgan and Carvalho [4, 5]; Pilavachi et al. [221]; Afgan et al. [7, 8]; Begić and Afgan [33]; Roth et al. [237].

- Multiple Attribute Utility Theory (MAUT) [154], which allows DMs to consider their preferences in the form of multiple attribute utility functions, is mostly used to capture the uncertainty relating to the outcomes of alternatives rather than the uncertainty relating to attribute values. In our review it is only used in energy efficiency studies [251].
- The Multiple Attribute Value Theory (MAVT), a special case of MAUT where there is no uncertainty in the consequences of the alternatives, is mainly used in power generation technology comparisons, in particular to rank power expansion alternatives [116] or to prioritize investment portfolios in capacity expansion and energy security [178]. It is also applied in energy plans and policies, more specifically in the study of the options for energy and carbon development for a city [220] and to reconcile visions in quantitative resource allocation scenarios [267] (see Fig. 25.7).
- The Simple Multi-Attribute Rated Technique (SMART) [279] is mentioned as an appropriate MCDA technique for renewable energy planning in Polatidis et al. [226], although no reports were found in scientific literature of the use of this methodology in the framework of energy decision-making.
- The Ordered Weighted Average (OWA), which can combine non-weighted and weighted linguistic information, has been used in the formulation of sustainable technological energy priorities [86], the assessment of renewable energy producers' operational environment [217], and to derive a wind farm land suitability index and classification using Geographical Information Systems [17].

The outranking approaches include:

- The ELECTRE (elimination and choice translating reality) family of methods [238], namely ELECTRE III, ELECTRE TRI and ELECTRE IS methods are frequently used in energy decision-making. ELECTRE III has been mainly used in energy plans and policies studies for the ranking of renewable energy resources [32, 216] and recommending future energy sources [135], and also in miscellaneous energy studies in the context of the appraisal of heating systems [191], production processes [58] and pollution control [278]. ELECTRE TRI, which assigns alternatives to predefined ordered categories of merit has been used for the assessment of agricultural biogas plants [172], the definition of national priorities for GHG emissions reduction in the energy sector [100], the selection of wind energy projects [179], the appraisal of energy-efficiency initiatives [197], and the assessment of risk in natural gas pipelines [46]. ELECTRE IS, which is devoted to choice problems including a robustness analysis of results, has been used to select among energy projects in farming fields in a region [114].
- The several versions of the PROMETHEE, Preference Ranking Organization Method for Enrichment Evaluation [43], has been mainly used in energy plans and policies concerning the design of renewable energy policies [158, 173] and sustainable energy planning [263, 270]. It has also been used in the comparison of power generation technologies [82, 265] and ranking the performance of different biomass feedstock-based pellets [259], and in miscellaneous energy studies, such as in the appraisal of cooking energy sources [225], thermal technologies [57]

and public transportation vehicles [190]. PROMETHEE II for performing a complete ranking of all actions has been used in the evaluation of sustainable technologies for electricity generation [86], sustainable energy planning on an island [270], energy exploitation schemes of a low temperature geothermal field [106], assessment of renewable energy projects [112], determining induction motors replacement schemes [252], and evaluation of heating systems [101]. In the fuzzy PROMETHEE method the performance of each scenario in each criterion is introduced as a fuzzy number [106]. PROMETHEE is the third most used method considered in the papers herein analyzed (see Fig. 25.7).

ANP has been applied on power generation technology studies, in particular in the evaluation of alternative fuels for electricity generation [157], and energy sources for a country [272] and a particular industry [214].

Several applications of AHP in combination with other methods have been reported in the literature. Tzeng et al. [271] apply AHP to determine the relative weighting of the evaluation criteria and use TOPSIS to determine the best compromise alternative fuel mode for public transportation. Jaber et al. [137] use the fuzzy sets and AHP to perform the evaluation of conventional and renewable energy sources for space heating in the household sector. Wang et al. [282] combine fuzzy sets with AHP to assess trigeneration systems. Kahraman et al. [146] and Kahraman and Kaya [145] used the modified Fuzzy AHP method developed by Zeng et al. [298] by including simplified fuzzy operations and similar steps to classical AHP for selecting renewable energy alternatives and energy policy options. Kaya and Kahraman [151] and Cristóbal [75] propose the integrated VIKOR-AHP methodology for the selection of energy policies and production sites and to select a renewable energy project, in which the criterion weights are determined by pairwise comparison matrices of AHP. Shen et al. [247] combine AHP and fuzzy set theory to aid in measuring the ambiguity and uncertainty in the DM's judgments to assess the 3E goals and renewable energy sources regulated by the Renewable Energy Development Bill. Talinli et al. [260] and Choudhary and Shankar [71] consider priority weights derived from linguistic comparison terms and their equivalent triangular fuzzy numbers for assessing the energy policies and selecting optimal locations for thermal power plants. Wang et al. [285, 286] use AHP and fuzzy analysis to determine the environmental performance of urban energy use plans. Al-Yahyai et al. [17] use an AHP-OWA within a GIS environment.

TOPSIS—Technique for Order Preference by Similarity to Ideal Solution [136] applications have mostly focused on the evaluation and selection of energy generation methods and technologies as well as energy systems performance. Tzeng et al. [271] apply TOPSIS to determine the best compromise alternative fuel mode for public transportation. Kaya and Kahraman [152] propose a modified fuzzy TOPSIS for the selection of the best energy technology alternative. Choudhary and Shankar [71] use it to rank the alternative locations of thermal power plants. Streimikiene et al. [258] apply TOPSIS for choosing the most sustainable electricity production technologies.

NAIADE—Novel Approach to Imprecise Assessment and Decision Environment [192] applications to energy decision-making studies are focused on the assessment of scenarios. Cavallaro and Ciraolo [59] use it to make a preliminary assessment of the feasibility of installing wind energy turbines on an island site. Dinca et al. [83] apply NAIADE to select the best energy scenario. Buchholz et al. [48] review four multi-criteria tools, including NAIADE, for analyzing their suitability to assess sustainability of bioenergy systems with a special focus on multiple stakeholders. Shmelev and Rodríguez-Labajos [248] make a dynamic analysis of sustainability at the macro-scale considering long-term, medium-term and short-term plans. Browne et al. [47] use NAIADE to assess policy measures or scenarios relating to residential heating energy and domestic electricity consumption.

VIKOR [281] has been used in applications ranging from the assessment of energy sources to energy projects and siting decision evaluations. Tzeng et al. [271] use it to determine the best compromise alternative fuel mode for public transportation. Kaya and Kahraman [151] suggest a modified fuzzy VIKOR methodology to make a selection among alternative renewable energy options and production sites. Cristóbal [75] use VIKOR in the selection of a renewable energy project within a Renewable Energy Plan launched by a government. Vučijak et al. [281] apply it to site selection and plant technical and operational parameters decisions, based on the effects of the hydro power plants on the indicators defining the ecological status of the affected water body.

Burton and Hubacek [49] apply MACBETH—Measuring Attractiveness by a Categorical Based Evaluation TecHnique [28] to assess whether small scale or large scale approaches to renewable energy provision are best placed to help meet the targets set in the Energy White Paper for the UK at the lowest social, economic and environmental costs.

McDowall and Eames [180] describe the application of Multi-criteria Mapping (MCM) [257], combining participatory scenario development, to provide an integrated and transparent assessment of the environmental, social and economic sustainability of possible future hydrogen energy systems for the UK.

Participatory multi-criteria analysis (PMCA) is used by Kowalski et al. [158] to overcome some of the problems of monetary valuation and account for the multiple dimensions and long-term nature of sustainable development, in a participatory process for appraising future energy systems, considering the complexity of socio-economic and biophysical systems featuring high uncertainty, conflicting objectives, different forms of data and information, multiple interests and perspectives.

25.3.13 Uncertainty Treatment

Due to the lack, inconsistency or imprecision of data and the subjectivity or vagueness of human judgments, the processing of all the different types and sources of uncertainty is required to provide results in which the DM can have confidence. The difficulty of providing exact numerical values for the criteria, making precise evaluations and translating human reasoning into a qualitative/quantitative scale has been largely recognized [84]. Therefore, most of the input data and parameters required by the methods cannot be given precisely. Data associated with the performance of the alternatives according to multiple criteria, namely those of a more subjective nature, and parameters, such as weights whether or not understood as criterion importance coefficients, may be expressed in some methods by linguistic terms. Human judgments on qualitative attributes are always subjective and thus inherently imprecise.

Sensitivity analysis is the most popular uncertainty handling technique found in the studies herein reviewed. Sensitivity analysis investigates the model response to different types of variation in the input information, including raw data, technical parameters conveying preferences, and additional assumptions. Several flavors of sensitivity analysis are combined with the MCDA methods previously mentioned. In most applications sensitivity analysis is based on a "one at time" approach by considering changes in the results due to variations in a single parameter. Sensitivity analysis may be conducted by considering:

- the computation of the intervals of the weights of the fundamental criteria with the NAIADE method [59, 83, 248];
- whether there are other preferences or weights affecting the overall ordering of the options [89];
- the variation between optimistic and pessimistic scores [180];
- the range of weights within which the dominant alternative remains stable in the framework of the MACBETH method [49];
- the change of weights regarding the costs, benefits and risks dimensions with the ANP method [214];
- the assignment of various criterion weighting schemes to accommodate a range of perspectives combined with the PROMETHEE method [82, 190, 224, 259], MAVT [116, 178]; the MAUT method [251]; AHP [54, 66, 67, 70, 146, 148, 167, 288]; AHP and TOPSIS [71, 151, 152]; PROMETHEE II [112, 252]; SAW [97, 229, 235, 237, 262]; and TOPSIS [258];
- the modification of the (indifference, preference) thresholds in ELECTRE III [58, 103, 135, 216, 278] and ELECTRE TRI [46, 100];
- a study of the degree of robustness for each outranking situation in ELECTRE IS [114].

In methods based on value and utility theory approaches, the treatment of uncertainty is mainly held by using sensitivity analysis and/or stochastic distributions. Moreover, sensitivity analysis is the most used technique in outranking methods and the second most used technique in AHP.

Fuzzy sets and fuzzy logic are also used to address uncertainty in MCDA. The use of fuzzy logic techniques has allowed developing a quantitative approach using a qualitative representation, so it has been capable of simultaneously handling numerical data and linguistic knowledge expression. Fuzzy techniques are mostly used with the AHP method (see Fig. 25.8).

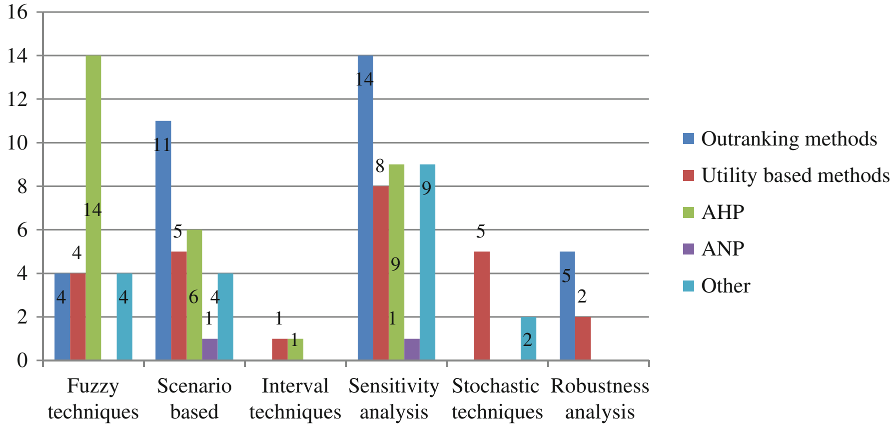


Fig. 25.8 Uncertainty handling techniques used with different MCDA methods

Scenario based analysis is the second most frequent uncertainty handling technique, being mainly used in outranking methods. In scenario planning a limited number of scenarios is constructed to analyze likely or relevant projections of the future.

Robustness analysis, which is generally defined as the determination of the degree to which a solution is affected, in terms of any attribute, by unknown parameters or changing assumptions, has gained an increased attention in MCDA. Flavors of robustness analysis are mainly used in outranking methods, mainly with PROMETHEE.

25.4 Conclusions

The energy sector is of outstanding importance for the satisfaction of societal needs, providing directly or indirectly the fundamental requirements for almost all activities in modern societies. The application of models and methods of operational research has contributed to effective decision support in several problems arising in the energy sector. Until mid 1970s models for energy planning were mainly based on an overall cost minimization perspective subject to demand satisfaction and technology constraints. Nowadays, energy planning models, from strategic long-term to operational short-term ones, are mostly based on multi-objective optimization and multi-criteria decision analysis approaches, thus recognizing the need to encompass multiple, conflicting and incommensurate aspects of evaluation of the merits of distinct courses of action pertaining to economy, environment, reliability, quality of service, etc. These models not just capture in a more realistic manner the complexity of these problems, but also provide a value-added in exploring a variety of possible decisions representing different trade-offs between the competing objectives/criteria thus enabling a more comprehensive analysis of potential solutions also including in the decision process the preferences and interests of stakeholders.

The quest for sustainability, namely concerning renewable energy resources, technological advancements, new market designs, the significance of investments, etc., make problems in the energy sector important challenges, for which MOO and MCDA possess the right tools to be offered to planners and decision makers (governments, regulators, utilities, consumers, interest groups) for a thorough analysis and balanced recommendations. Therefore, it is expected that the energy sector will remain one of the most active and exciting areas of application of MOO/MCDA models and methods, with an enriching cross-fertilization between challenging problems and innovative models and methods.

Acknowledgments This work has been framed under the Energy for Sustainability Initiative of the University of Coimbra and supported by the Portuguese Foundation for Science and Technology under project grants PEst-C/EEI/UI0308/2011 and MIT/SET/0018/2009, and R&D Project EMSURE—Energy and Mobility for Sustainable Regions (CENTRO 07 0224 FEDER 002004).

References

1. Abido, M.: A novel multiobjective evolutionary algorithm for environmental/economic power dispatch. Electr. Power Syst. Res. **65**(1), 71–81 (2003)
2. Abido, M.A.: Multiobjective evolutionary algorithms for electric power dispatch problem. IEEE Trans. Evol. Comput. **10**(3), 315–329 (2006)
3. Abu-Taha, R. Multi-criteria Applications in Renewable Energy Analysis: A Literature Review, Technology Management in the Energy Smart World (PICMET), 2011 Proceedings of PICMET '11, IEEE, 1–8 (2011)
4. Afgan, N.H., Carvalho, M.G.: Multi-criteria assessment of new and renewable energy power plants. Energy **27**(8), 739–755 (2002)
5. Afgan, N.H., Carvalho, M.G.: Sustainability assessment of a hybrid energy system. Energy Policy **36**(8), 2903–2910 (2008)
6. Afgan, N.H., Carvalho, M.G., Hovanov, N.V.: Energy system assessment with sustainability indicators. Energy Policy **28**(9), 603–612 (2000)
7. Afgan, N.H., Pilavachi, P.A., Carvalho, M.G.: Multi-criteria evaluation of natural gas resources. Energy Policy **35**(1), 704–713 (2007)
8. Afgan, N.H., Veziroglu, A., Carvalho, M.G.: Multi-criteria evaluation of hydrogen system options. Int. J. Hydrog. Energy **32**(15), 3183–3193 (2007)
9. Aghaei, J., Akbari, M.A., Roosta, A., Gitizadeh, M., Niknam, T.: Integrated renewable – conventional generation expansion planning using multiobjective framework. Gener. Transm. Distrib. IET **6**(8), 773–784 (2012)
10. Aghaei, J., Amjady, N., Shayanfar, H.: Multi-objective electricity market clearing considering dynamic security by lexicographic optimization and augmented epsilon constraint method. Appl. Soft Comput. **11**(4), 3846–3858 (2011)
11. Aghaei, J., Niknam, T., Azizipanah-Abarghooee, R., Arroyo, J.: Scenario-based dynamic economic emission dispatch considering load and wind power uncertainties. Electr. Power Energy Syst. **47**, 351–367 (2013)

12. Agrawal, S., Panigrahi, B.K., Tiwari, M.K.: Multiobjective particle swarm algorithm with fuzzy clustering for electrical power dispatch. IEEE Trans. Evol. Comput. **12**(5), 529–541 (2008)
13. Ahuja, A., Das, S., Pahwa, A.: An AIS-ACO hybrid approach for multi-objective distribution system reconfiguration. IEEE Trans. Power Syst. **22**(3), 1101–1111 (2007)
14. Aki, H., Oyama, T., Tsuji, K.: Analysis of energy pricing in urban energy service systems considering a multiobjective problem of environmental and economic impact. IEEE Trans. Power Syst. **18**(4), 1275–1282 (2003)
15. Alarcon-Rodriguez, A., Ault, G., Galloway, S.: Multi-objective planning of distributed energy resources: a review of the state-of-the-art. Renew. Sustain. Energy Rev. **14**(5), 1353–1366 (2010)
16. Alonso, M., Amaris, H., Alvarez-Ortega, C.: A multiobjective approach for reactive power planning in networks with wind power generation. Renew. Energy **37**(1), 180–191 (2012)
17. Al-Yahyai, S., Charabi, Y., Gastli, A., Al-Badi, A.: Wind farm land suitability indexing using multi-criteria analysis. Renew. Energy **44**, 80–87 (2012)
18. Amjady, N., Aghaei, J., Shayanfar, H.A.: Stochastic multiobjective market clearing of joint energy and reserves auctions ensuring power system security. IEEE Trans. Power Syst. **24**(4), 1841–1854 (2009)
19. Amorim, E.A., Hashimoto, S.H.M., Lima, F.G.M., Mantovani, J.R.S.: Multi objective evolutionary algorithm applied to the optimal power flow problem. Latin America Trans. IEEE **8**(3), 236–244 (2010)
20. Antunes, C.H., Lima, P., Oliveira, E., Pires, D.F.: A multi-objective simulated annealing approach to reactive power compensation. Eng. Optim. **43**(10), 1063–1077 (2011)
21. Antunes, C.H., Martins, A.G., Brito, I.: A multiple objective mixed integer linear programming model for power generation expansion planning. Energy **29**(4), 613–627 (2004)
22. Ara, A.L., Kazemi, A., Nabavi Niaki, S.A.: Multiobjective optimal location of FACTS shunt-series controllers for power system operation planning. IEEE Trans. Power Delivery **27**(2), 481–490 (2012)
23. Aras, H., Erdogmus, S., Koc, E.: Multi-criteria selection for a wind observation station location using analytic hierarchy process. Renew. Energy **29**(8), 1383–1392 (2004)
24. Arnette, A., Zobel, C.W.: An optimization model for regional renewable energy development. Renew. Sustain. Energy Rev. **16**(7), 4606–4615 (2012)
25. Asadi, E., Silva, M.G., Antunes, C.H., Dias, L.: A multi-objective optimization model for building retrofit strategies using TRNSYS simulations, GenOpt and MATLAB. Build. Environ. **56**, 370–378 (2012)
26. Augugliaro, A., Dusonchet, L., Favuzza, S., Sanseverino, E.R.: Voltage regulation and power losses minimization in automated distribution networks by an evolutionary multiobjective approach. IEEE Trans. Power Syst. **19**(3), 1516–1527 (2004)
27. Bahmani-Firouzi, B., Farjah, E., Azizipanah-Abarghooee, R.: An efficient scenario-based and fuzzy self-adaptive learning particle swarm optimization approach for dynamic economic emission dispatch considering load and wind power uncertainties. Energy **50**, 232–244 (2013)
28. Bana e Costa, C.A., De Corte, J.-M., Vansnick, J.-C.: MACBETH. Int. J Inf. Technol. Decis. Mak. **11**(2), 359–387 (2012)
29. Bath, S.K., Dhillon, J.S., Kothari, D.P.: Fuzzy satisfying stochastic multi-objective generation scheduling by weightage pattern search methods. Electr. Power Syst. Res. **69**(2–3), 311–320 (2004)
30. Bath, S.K., Dhillon, J.S., Kothari, D.P.: Stochastic multiobjective generation allocation using pattern-search method. Gener. Transm. Distrib. IEE Proc. **153**(4), 476–484 (2006)
31. Bayon, L., Grau, J.M., Ruiz, M.M., Suarez, P.M.: The exact solution of the environmental/economic dispatch problem. IEEE Trans. Power Syst. **27**(2), 723–731 (2012)
32. Beccali, M., Cellura, M., Mistretta, M.: Decision-making in energy planning. Application of the Electre method at regional level for the diffusion of renewable energy technology. Renew. Energy **28**(13), 2063–2087 (2003)

33. Begić, F., Afgan, N.H.: Sustainability assessment tool for the decision making in selection of energy system-Bosnian case. Energy **32**(10), 1979–1985 (2007)
34. Behzadian, M., Kazemzadeh, R.B., Albadvi, A., Aghdasi, M.: PROMETHEE: a comprehensive literature review on methodologies and applications. Eur. J. Oper. Res. **200**(1), 198–215 (2010)
35. Behzadian, M., Otaghsara, S.K., Yazdani, M., Ignatius, J.: A state-of the-art survey of TOPSIS applications. Expert Syst. Appl. **39**(17), 13051–13069 (2012)
36. Bernardon, D.P., Garcia, V.J., Ferreira, A.S.Q., Canha, L.N.: Multicriteria distribution network reconfiguration considering subtransmission analysis. IEEE Trans. Power Delivery **25**(4), 2684–2691 (2010)
37. Bhattacharya, A., Roy, P.K.: Solution of multi-objective optimal power flow using gravitational search algorithm. Gener. Transm. Distrib. IET **6**(8), 751–763 (2012)
38. Bhattacharyya, S.C.: Review of alternative methodologies for analysing off-grid electricity supply. Renew. Sustain. Energy Rev. **16**(1), 677–694 (2012)
39. Bhowmick, S., Goswami, S.K., Bhattacherjee, P.K.: An LP-based multiobjective distribution system planning using goal attainment method. Electr. Power Compon. Syst. **29**(5), 479–490 (2001)
40. Borges, A., Antunes, C.H.: A fuzzy multiple objective decision support model for energy-economy planning. Eur. J. Oper. Res. **45**(2), 304–316 (2003)
41. Borghetti, A., Bosetti, M., Grillo, S., Massucco, S., Nucci, C.A., Paolone, M., Silvestro, F.: Short-term scheduling and control of active distribution systems with high penetration of renewable resources. Syst. J. IEEE **4**(3), 313–322 (2010)
42. Braga, A.S.D., Saraiva, J.T.: A multiyear dynamic approach for transmission expansion planning and long-term marginal costs computation. IEEE Trans. Power Syst. **20**(3), 1631–1639 (2005)
43. Brans, J.P., Vincke, P.: A preference ranging organization method. The PROMETHEE method for MCDM. Manag. Sci. **31**, 647–656 (1985)
44. Brar, Y.S., Dhillon, J., Kothari, D.P.: Genetic-fuzzy logic based weightage pattern search for multiobjective load dispatch problem. Asian J. Inf. Technol. **2**(4), 364–372 (2003)
45. Brito, A.J., Almeida, A.T., Mota, C.M.M.: A multicriteria model for risk sorting of natural gas pipelines based on ELECTRE TRI integrating utility theory. Eur. J. Oper. Res. **200**(1), 812–821 (2010)
46. Brown, H.E., Suryanarayanan, S., Natarajan, S.A., Rajopadhye, S.: Improving reliability of islanded distribution systems with distributed renewable energy resources. IEEE Trans. Smart Grid **3**(4), 2028–2038 (2012)
47. Browne, D., O'Regan, B., Moles, R.: Use of multi-criteria decision analysis to explore alternative domestic energy and electricity policy scenarios in an Irish city-region. Energy **35**(2), 518–528 (2010)
48. Buchholz, T., Rametsteiner, E., Volk, T., Luzadis, V.A.: Multi criteria analysis for bioenergy systems assessments. Energy Policy **37**(2), 484–495 (2009)
49. Burton, J., Hubacek, K.: Is small beautiful? A multicriteria assessment of small-scale energy technology applications in local governments. Energy Policy **35**(12), 6402–6412 (2007)
50. Carpinelli, G., Celli, G., Mocci, S., Pilo, F., Russo, A.: Optimisation of embedded generation sizing and siting by using a double trade-off method. Gener. Transm. Distrib. IEE Proc. **152**(4), 503–513 (2005)
51. Carrano, E.G., Guimarães, F.G., Takahashi, R.H.C., Neto, O.M., Campelo, F.: Electric distribution network expansion under load-evolution uncertainty using an immune system inspired algorithm. IEEE Trans. Power Syst. **22**(2), 851–861 (2007)
52. Carrano, E.G., Soares, L.A.E., Takahashi, R.H.C., Saldanha, R.R., Neto, O.M.: Electric distribution multiobjective network design using a problem-specific genetic algorithm. IEEE Trans. Power Delivery **21**(2), 995–1005 (2006)
53. Carrera, D., Mack, A.: Sustainability assessment of energy technologies via social indicators: results of a survey among European energy experts. Energy Policy **38**(2), 1030–1039 (2010)

54. Carrión, J., Estrella, A., Dolsa, F., Toro, M., Rodrríguez, M., Ridao, A.: Environmental decision-support systems for evaluating the carrying capacity of land areas: optimal site selection for grid-connected photovoltaic power plants. Renew. Sustain. Energy Rev. **12**(9), 2358–2380 (2008)
55. Carvalho, P.M.S., Ferreira, L.A.F.M.: Urban distribution network investment criteria for reliability adequacy. IEEE Trans. Power Syst. **19**(2), 1216–1222 (2004)
56. Catalão, J.P.S., Mendes, V.M.F.: Influence of environmental constraints on profit-based short-term thermal scheduling. IEEE Trans. Sustain. Energy **2**(2), 131–138 (2011)
57. Cavallaro, F.: Multi-criteria decision aid to assess concentrated solar thermal technologies. Renew. Energy **34**(7), 1678–1685 (2009)
58. Cavallaro, F.: A comparative assessment of thin-film photovoltaic production processes using the ELECTRE III method. Energy Policy **38**(1), 463–474 (2010)
59. Cavallaro, F., Ciraolo, L.: A multicriteria approach to evaluate wind energy plants on an Italian island. Energy Policy **33**(2), 235–244 (2005)
60. Celli, G., Ghiani, E., Mocci, S., Pilo, F.: A multiobjective evolutionary algorithm for the sizing and siting of distributed generation. IEEE Trans. Power Syst. **20**(2), 750–757 (2005)
61. Centeno, E., Vitoriano, B., Campos, A., Muñoz, A., Villar, J., Sánchez, E.: A goal programming model for rescheduling of generation power in deregulated markets. Ann. Oper. Res. **120**, 45–57 (2003)
62. Chandrasekaran, K., Simon, S.P.: Multi-objective unit commitment problem with reliability function using fuzzified binary real coded artificial bee colony algorithm. Gener. Transm. Distrib. IET **6**(10), 1060–1073 (2012)
63. Chang, Y.-C.: Multi-objective optimal SVC installation for power system loading margin improvement. IEEE Trans. Power Syst. **27**(2), 984–992 (2012)
64. Chang, C.S., Yang, S.W.: Optimal multiobjective planning of dynamic series compensation devices for power quality improvement. Gener. Transm. Distrib. IEE Proc. **148**(4), 361–370 (2001)
65. Chaouachi, A., Kamel, R.M., Andoulsi, R., Nagasaka, K.: Multiobjective intelligent energy management for a microgrid. IEEE Trans. Ind. Electron. **60**(4), 1688–1699 (2013)
66. Chatzimouratidis, A.I., Pilavachi, P.A.: Multicriteria evaluation of power plants impact on the living standard using the analytic hierarchy process. Energy Policy **36**(3), 1074–1089 (2008)
67. Chatzimouratidis, A.I., Pilavachi, P.A.: Sensitivity analysis of technological, economic and sustainability evaluation of power plants using the analytic hierarchy process. Energy Policy **37**(3), 788–798 (2009)
68. Chen, T.: The impact of mitigating CO emissions on Taiwan's economy. Energy Econ. **23**(2), 141–151 (2001)
69. Chen, W., Zhang, Y., Liang, Y., Chen, L., Yang, J.: Multi-objective optimisation of parallel hybrid electric vehicles based on fuzzy logic control. Int. J. Veh. Auton. Syst. **6**(3–4), 236–250 (2009)
70. Chinese, D., Nardin, G., Saro, O.: Multi-criteria analysis for the selection of space heating systems in an industrial building. Energy **36**(1), 556–565 (2011)
71. Choudhary, D., Shankar, R.: An STEEP-fuzzy AHP-TOPSIS framework for evaluation and selection of thermal power plant location: a case study from India. Energy **42**(1), 510–521 (2012)
72. Chung, T.S., Li, K.K., Chen, G.J., Xie, J.D., Tang, G.Q.: Multiobjective transmission network planning by a hybrid GA approach with fuzzy decision analysis. Int. J. Electr. Power Energy Syst. **25**(3), 187–192 (2003)
73. Coelho, D., Antunes, C.H., Martins, A.G.: Using SSM for structuring decision support in urban energy planning. Technol. Econ. Dev. Econ. **16**(4), 641–653 (2010)
74. Cossi, A.M., da Silva, L.G.W., Lázaro, R.A.R., Mantovani, J.R.S.: Primary power distribution systems planning taking into account reliability, operation and expansion costs. Gener. Transm. Distrib. IET **6**(3), 274–284 (2012)
75. Cristóbal, J.R.: Multi-criteria decision-making in the selection of a renewable energy project in Spain: the Vikor method. Renew. Energy **36**(2), 498–502 (2011)

76. Cristóbal, J.R.: A goal programming model for environmental policy analysis: application to Spain. Energy Policy **43**, 303–307 (2012)
77. Daima, T., Bayraktaroglu, E., Estep, J., Lima, D., Upadhyay, J., Yang, J.: Optimizing the NW off-shore wind turbine design. Math. Comput. Model. **55**(3–4), 396–404 (2012)
78. Darvishi, A., Alimardani, A., Hosseinian, S.H.: Gener. Transm. Distrib. IET **5**(9), 921–929 (2011)
79. Das, D.: A fuzzy multiobjective approach for network reconfiguration of distribution systems. IEEE Trans. Power Delivery **21**(1), 202–209 (2006)
80. Dhillon, J.S., Parti, S.C., Kothari, D.P.: Fuzzy decision-making in stochastic multiobjective short-term hydrothermal scheduling. Gener. Transm. Distrib. IEE Proc. **149**(2), 191–200 (2002)
81. Diakoulaki, D., Antunes, C.H., Martins, A.G.: In: Figueira, J., Greco, S., Ehrgott, M. (eds.) Multiple Criteria Decision Analysis – State of the Art Surveys. International Series in Operations Research and Management Science, vol. 78, pp. 859–897. Springer, New York (2005)
82. Diakoulaki, D., Karangelis, F.: Multi-criteria decision analysis and cost-benefit analysis of alternative scenarios for the power generation sector in Greece. Renew. Sustain. Energy Rev. **11**(4), 716–727 (2007)
83. Dinca, C., Badea, A., Rousseaux, P., Apostol, T.: A multi-criteria approach to evaluate the natural gas energy systems. Energy Policy **35**(11), 5754–5765 (2007)
84. Doukas, H.: Modelling of linguistic variables in multicriteria energy policy support. Eur. J. Oper. Res. (2013). doi:10.1016/j.ejor.2012.11.026
85. Doukas, H., Andreas, B., Psarras, J.: Multi-criteria decision aid for the formulation of sustainable technological energy priorities using linguistic variables. Eur. J. Oper. Res. **182**(2), 844–855 (2007)
86. Doukas, H., Patlitzianas, K., Psarras, J.: Supporting sustainable electricity technologies in Greece using MCDM. Resour. Policy **31**(2), 129–136 (2006)
87. Dufo-López, R., Bernal-Agustí, J., Yusta-Loyo, J., Domínguez-Navarro, J., Ramírez-Rosado, I., Lujano, J., Aso, I.: Multi-objective optimization minimizing cost and life cycle emissions of stand-alone PV–wind–diesel systems with batteries storage. Appl. Energy **88**(11), 4033–4041 (2011)
88. Dukpa, A., Duggal, I., Venkatesh, B., Chang, L.: Optimal participation and risk mitigation of wind generators in an electricity market. Renew. Power Gener. IET **4**(2), 165–175 (2010)
89. Elghali, L., Clift, R., Sinclair, P., Panoutsou, C., Bauen, A.: Developing a sustainability framework for the assessment of bioenergy systems. Energy Policy **35**(12), 6075–6083 (2007)
90. El-Zonkoly, A.M.: Optimal placement of multi-distributed generation units including different load models using particle swarm optimization. Gener. Transm. Distrib. IET **5**(7), 760–771 (2011)
91. Erol, Ö., Kılkıs, B.: An energy source policy assessment using analytical hierarchy process. Energy Convers. Manag. **63**, 245–252 (2012)
92. Evins, R., Pointer, P., Vaidyanathan, R., Burgess, S.: A case study exploring regulated energy use in domestic buildings using design-of-experiments and multi-objective optimization. Build. Environ. **54**, 126–136 (2012)
93. Fadaee, M., Radzi, M.A.M.: Multi-objective optimization of a stand-alone hybrid renewable energy system by using evolutionary algorithms: a review. Renew. Sustain. Energy Rev. **16**(5), 3364–3369 (2012)
94. Falaghi, H., Haghifam, M.-R., Singh, C.: Ant colony optimization-based method for placement of sectionalizing switches in distribution networks using a fuzzy multiobjective approach. IEEE Trans. Power Delivery **24**(1), 268–276 (2009)

95. Farahani, H.F., Shayanfar, H.A., Ghazizadeh, M.S.: Multi-objective clearing of reactive power market including plug-in hybrid electric vehicle. Electr. Power Compon. Syst. **41**(2), 197–220 (2013)
96. Fazlollahi, S., Maréchal, F.: Multi-objective, multi-period optimization of biomass conversion technologies using evolutionary algorithms and mixed integer linear programming (MILP). Appl. Thermal Eng. **50**(2), 1504–1513 (2013)
97. Gamboa, G., Munda, G.: The problem of windfarm location: a social multi-criteria evaluation framework. Energy Policy **35**(3), 1564–1583 (2007)
98. Ganguly, S., Sahoo, N., Das, D.: Multi-objective planning of electrical distribution systems using dynamic programming. Electr. Power Energy Syst. **46**, 65–78 (2013)
99. Garduno-Ramirez, R., Lee, K.Y.: Multiobjective optimal power plant operation through coordinate control with pressure set point scheduling. IEEE Trans. Energy Convers. **16**(2), 115–122 (2001)
100. Georgopoulou, E., Sarafidis, Y., Mirasgedis, S., Zaimi, S., Lalas, D.P.: A multiple criteria decision-aid approach in defining national priorities for greenhouse gases emissions reduction in the energy sector. Eur. J. Oper. Res. **146**(1), 199–215 (2003)
101. Ghafghazi, S., Sowlati, T., Sokhansanj, S., Melin, S.: A multicriteria approach to evaluate district heating system options. Appl. Energy **87**(4), 1134–1140 (2010)
102. Gitizadeh, M., Vahed, A., Aghaei, J.: Multistage distribution system expansion planning considering distributed generation using hybrid evolutionary algorithms. Appl. Energy **101**, 655–666 (2013)
103. Goletsis, Y., Psarras, J., Samouilidis, J.E.: Project ranking in the Armenian energy sector using a multicriteria method for groups. Ann. Oper. Res. **120**(1–4), 135–157 (2003)
104. Gomes, A., Antunes, C.H., Martins, A.: A multiple objective evolutionary approach for the design and selection of load control strategies. IEEE Trans. Power Syst. **19**(2), 1173–1180 (2004)
105. Gomes, A., Antunes, C.H., Martins, A.: A multiple objective approach to electric load management using an interactive evolutionary algorithm. IEEE Trans. Power Syst. **22**(3), 1004–1011 (2007)
106. Goumas, M., Lygerou, V.: An extension of the PROMETHEE method for decision making in fuzzy environment: ranking of alternative energy exploitation projects. Eur. J. Oper. Res. **23**(3), 606–613 (2000)
107. Greening, L., Bernow, S.: Design of coordinated energy and environmental policies: use of multi-criteria decision-making. Energy Policy **32**(6), 721–735 (2004)
108. Gu, X., Zhong, H.: Optimisation of network reconfiguration based on a two-layer unit-restarting framework for power system restoration. Gener. Transm. Distrib. IET **6**(7), 693–700 (2012)
109. Guedes, L.S.M., Lisboa, A.C., Vieira, D.A.G., Saldanha, R.R.: A multiobjective heuristic for reconfiguration of the electrical radial network. IEEE Trans. Power Delivery **28**(1), 311–319 (2013)
110. Guo, C., Bai, Y., Zheng, X., Zhan, J., Wu, Q.: Optimal generation dispatch with renewable energy embedded using multiple objectives. Electr. Power Energy Syst. **42**(1), 440–447 (2012)
111. Gupta, N., Swarnkar, A., Niazi, K.R., Bansal, R.C.: Multi-objective reconfiguration of distribution systems using adaptive genetic algorithm in fuzzy framework. Gener. Transm. Distrib. IET **4**(12), 1288–1298 (2010)
112. Haralambopoulos, D., Polatidis, H.: Renewable energy projects: structuring a multi-criteria group decision-making framework. Renew. Energy **28**(6), 961–973 (2003)
113. Harrison, G.P., Piccolo, A., Siano, P., Wallace, A.R.: Exploring the trade-offs between incentives for distributed generation developers and DNOs. IEEE Trans. Power Syst. **22**(2), 821–828 (2007)
114. Haurant, P., Oberti, P., Muselli, M.: Multicriteria selection aiding related to photovoltaic plants on farming fields on Corsica Island: a real case study using the ELECTRE outranking framework. Energy Policy **39**(2), 676–688 (2011)

115. Hazra, J., Sinha, A.K.: Congestion management using multiobjective particle swarm optimization. IEEE Trans. Power Syst. **22**(4), 1726–1734 (2007)
116. Heinrich, G., Basson, L., Cohen, B., Howells, M., Petrie, J.: Ranking and selection of power expansion alternatives for multiple objectives under uncertainty. Energy **32**(12), 2350–2369 (2007)
117. Heo, J.S., Lee, K.Y.: A multiagent-system-based intelligent reference governor for multiobjective optimal power plant operation. IEEE Trans. Energy Convers. **23**(4), 1082–1092 (2008)
118. Heo, J.S., Lee, K.Y., Garduno-Ramirez, R.: Multiobjective control of power plants using particle swarm optimization techniques. IEEE Trans. Energy Convers. **21**(2), 552–561 (2006)
119. Herva, M., Roca, E.: Review of combined approaches and multi-criteria analysis for corporate environmental evaluation. J. Clean. Prod. **39**, 355–371 (2013)
120. Heydt, G.T.: Multiobjective optimization: an educational opportunity in power engineering. IEEE Trans. Power Syst. **2**(3), 1631–1632 (2009)
121. Higgs, G., Berry, R., Kidner, D., Langford, M.: Using IT approaches to promote public participation in renewable energy planning: prospects and challenges. Land Use Policy **25**(4), 596–607 (2008)
122. Hilber, P., Miranda, V., Matos, M.A., Bertling, L.: Multiobjective optimization applied to maintenance policy for electrical networks. IEEE Trans. Power Syst. **22**(4), 1675–1682 (2007)
123. Hobbs, B., Hobbs, B.F., Rothkopf, M.H., O'Neill, R.P., Chao, H.-p. (eds.): The Next Generation of Electric Power Unit Commitment Models. International Series in Operations Research & Management Science, vol. 36. Kluwer Academic Publishers, Dordrecht (2001). VII 319 pp
124. Hobbs, B., Meier, P.: Energy Decisions and the Environment: A Guide to the Use of Multicriteria Methods. International Series in Operations Research and Management Science. Kluwer Academic Publishers, Dordrecht (2000)
125. Hong, Y.-Y., Ho, S.-Y.: Determination of network configuration considering multiobjective in distribution systems using genetic algorithms. IEEE Trans. Power Syst. **20**(2), 1062–1069 (2005)
126. Hong, Y.-Y., Huang, W.-F.: Interactive multiobjective passive filter planning with fuzzy parameters in distribution systems using genetic algorithms. IEEE Trans. Power Delivery **18**(3), 1043–1050 (2003)
127. Hong, Y.-Y., Lin, J.-K., Wu, C.-P., Chuang, C.-C.: Multi-objective air-conditioning control considering fuzzy parameters using immune clonal selection programming. IEEE Trans. Smart Grid **3**(4), 1603–1610 (2012)
128. Hong, Y.-Y., Wei, S.-F.: Multiobjective underfrequency load shedding in an autonomous system using hierarchical genetic algorithms. IEEE Trans. Power Delivery **25**(3), 1355–1362 (2010)
129. Hsiao, Y.-T.: Multiobjective evolution programming method for feeder reconfiguration. IEEE Trans. Power Syst. **19**(1), 594–599 (2004)
130. Hsiao, Y.-T., Chien, C.-Y.: Multiobjective optimal feeder reconfiguration. Gener. Transm. Distrib. IEE Proc. **148**(4), 333–336 (2001)
131. Hsiao, Y.-T., Chien, C.-Y.: Optimisation of capacitor allocation using an interactive trade-off method. Gener. Transm. Distrib. IEE Proc. **148**(4), 371–374 (2001)
132. Hsu, G., Chou, F.: Integrated planning for mitigating CO_2 emissions in Taiwan: a multi-objective programming approach. Energy Policy **28**(8), 519–523 (2000)
133. Huang, C.-M.: Multiobjective service restoration of distribution systems using fuzzy cause-effect networks. IEEE Trans. Power Syst. **18**(2), 867–874 (2003)
134. Huang, I., Keisler, J., Linkov, I.: Multi-criteria decision analysis in environmental sciences: ten years of applications and trends. Sci. Total Environ. **409**(19), 3578–3594 (2011)
135. Hunt, J.D., Bañares-Alcántara, R., Hanbury, D.: A new integrated tool for complex decision making: application to the UK energy sector. Decis. Support Syst. **54**(3), 1427–1441 (2013)
136. Hwang, C.L., Yoon, K.: Multiple Attribute Decision Making: Methods and Applications. Springer, New York (1981)

137. Jaber, J., Jaber, Q., Sawalha, S., Mohsen, M.: Evaluation of conventional and renewable energy sources for space heating in the household sector. Renew. Energy Sustain. Rev. **12**(1), 278–289 (2008)
138. Jebaraj, S., Iniyan, S.: A review of energy models. Renew. Sustain. Energy Rev. **10**(4), 281–311 (2006)
139. Jin, T., Tian, Y., Zhang, C.W., Coit, D.W.: Multicriteria planning for distributed wind generation under strategic maintenance. IEEE Trans. Power Delivery **28**(1), 357–367 (2013)
140. Jorge, H., Antunes, C.H., Martins, A.: A multiple objective decision support model for the selection of remote load control strategies. IEEE Trans. Power Syst. **15**(2), 865–872 (2000)
141. Jovanovic, M., Afgan, N., Radovanovi, P., Stevanovic, V.: Sustainable development of the Belgrade energy system. Energy **34**(5), 532–539 (2009)
142. Jovanovic, M., Afgan, N., Bakic, V.: An analytical method for the measurement of energy system sustainability in urban areas. Energy **35**(9), 3909–3920 (2010)
143. Jovanovic, M., Turanjanin, V., Bakic, V., Pezo, M., Vucicevic, B.: Sustainability estimation of energy system options that use gas and renewable resources for domestic hot water production. Energy **36**(4), 2169–2175 (2011)
144. Kablan, M.M.: Decision support for energy conservation promotion: an analytic hierarchy process approach. Energy Policy **32**(10), 1151–1158 (2004)
145. Kahraman, C., Kaya, I.: A fuzzy multicriteria methodology for selection among energy alternatives. Expert Syst. Appl. **37**(9), 6270–6281 (2010)
146. Kahraman, C., Kaya, I., Cebi, S.: A comparative analysis for multiattribute selection among renewable energy alternatives using fuzzy axiomatic design and fuzzy analytic hierarchy process. Energy **34**(10), 1603–1616 (2009)
147. Kaleta, M., Ogryczak, W., Toczylowski, E., Zoltowska, I.: On multiple criteria decision support for suppliers on the competitive electric power market. Ann. Oper. Res. **121**(1–4), 79–104 (2003)
148. Kanagaraj, G., Mahalingam, A.: Designing energy efficient commercial buildings – a systems framework. Energy Build. **43**(9), 232–2343 (2011)
149. Kannan, S., Baskar, S., McCalley, J.D., Murugan, P.: Application of NSGA-II algorithm to generation expansion planning. IEEE Trans. Power Syst. **24**(1), 454–461 (2009)
150. Katsigiannis, Y.A., Georgilakis, P.S., Karapidakis, E.S.: Multiobjective genetic algorithm solution to the optimum economic and environmental performance problem of small autonomous hybrid power systems with renewables. IET Renew. Power Gener. **4**(5), 404–419 (2010)
151. Kaya, T., Kahraman, C.: Multicriteria renewable energy planning using an integrated fuzzy VIKOR & AHP methodology: the case of Istanbul. Energy **35**(6), 2517–2527 (2010)
152. Kaya, T., Kahraman, C.: Multicriteria decision making in energy planning using a modified fuzzy TOPSIS. Expert Syst. Appl. **38**(6), 6577–6585 (2011)
153. Keefer, D., Kirkwood, C.W., Corner, J.L.: Perspective on decision analysis applications, 1990–2001. Decis. Anal. **1**, 4–22 (2004)
154. Keeney, R.L., Raiffa, H.: Decisions with Multiple Objectives: Preference and Value Tradeoffs. Wiley, New York (1976)
155. Khazali, A.H., Kalantar, M., Khazali, A.: Fuzzy multi-objective reactive power clearing considering reactive compensation sources. Energy **36**(5), 3319–3327 (2011)
156. Kiker, G.A., Bridges, T.S., Varghese, A., Seager, T.P., Linkov, I.: Application of multicriteria decision analysis in environmental decision making. Integr. Environ. Assess. Manag. **1**(2), 95–108 (2005)
157. Köne, A., Büke, T.: An Analytical Network Process (ANP) evaluation of alternative fuels for electricity generation in Turkey. Energy Policy **35**(10), 5220–5228 (2007)
158. Kowalski, K., Stagl, S., Madlener, R., Omann, I.: Sustainable energy futures: methodological challenges in combining scenarios and participatory multi-criteria analysis. Eur. J. Oper. Res. **197**(3), 1063–1074 (2009)
159. Kumar, Y., Das, B., Sharma, J.: Multiobjective, multiconstraint service restoration of electric power distribution system with priority customers. IEEE Trans. Power Delivery **23**(1), 261–270 (2008)

160. Lee, A.H.I., Chen, H.H., Kang, H.Y.: Multi-criteria decision making on strategic selection of wind farms. Renew. Energy **34**(1), 120–126 (2009)
161. Lee, S.K., Mogi, G., Kim, J.W., Jin Gim, B.: A fuzzy analytic hierarchy process approach for assessing national competitiveness in the hydrogen technology sector. Int. J. Hydrog. Energy **33**(23), 6840–6848 (2008)
162. Lee, S., Mogi, G., Kon Lee, S., Hui, K.S., Wook Kim, J.: Econometric analysis of the R&D performance in the national hydrogen energy technology development for measuring relative efficiency: the fuzzy AHP/DEA integrated model approach. Int. J. Hydrog. Energy **35**(6), 2236–2246 (2010)
163. Lee, S., Mogi, G., Lee, S., Kim, J.: Prioritizing the weights of hydrogen energy technologies in the sector of the hydrogen economy by using a fuzzy AHP approach. Int. J. Hydrog. Energy **36**(2), 1897–1902 (2011)
164. Lee, S., Mogi, G., Li, Z., Hui, K., Kon Lee, S., Hui, K.N., Yong Park, S., Jin Ha, Y., Wook Kim, J.: Measuring the relative efficiency of hydrogen energy technologies for implementing the hydrogen economy: an integrated fuzzy AHP/DEA approach. Int. J. Hydrog. Energy **36**(20), 12655–12663 (2011)
165. Lee, S.K., Mogi, G., Kim, J.W.: The competitiveness of Korea as a developer of hydrogen energy technology: the AHP approach. Energy Policy **36**(4), 1284–1291 (2008)
166. Li, F., Pilgrim, J.D., Dabeedin, C., Chebbo, A., Aggarwal, R.K.: Genetic algorithms for optimal reactive power compensation on the national grid system. IEEE Trans. Power Syst. **20**(1), 493–500 (2005)
167. Liang, Z., Yang, K., Sun, Y., Yuan, J., Zhang, H., Zhang, Z.: Decision support for choice optimal power generation projects: fuzzy comprehensive evaluation model based on the electricity market. Energy Policy **34**(17), 3359–3364 (2006)
168. Lin, C.H., Chen, C.S., Wu, C.J., Kang, M.S.: Application of immune algorithm to optimal switching operation for distribution-loss minimisation and loading balance. Gener. Transm. Distrib. IEE Proc. **150**(2), 183–189 (2003)
169. Løken, E.: Use of multicriteria decision analysis methods for energy planning problems. Renew. Sustain. Energy Rev. **11**(7), 1584–1595 (2007)
170. Louie, H., Strunz, K.: Hierarchical multiobjective optimization for independent system operators (ISOs) in electricity markets. IEEE Trans. Power Syst. **21**(4), 1583–1591 (2006)
171. Ma, H.M., Ng, K.-T., Man, K.F.: A multiple criteria decision-making knowledge-based scheme for real-time power voltage control. IEEE Trans. Ind. Inf. **4**(1), 58–66 (2008)
172. Madlener, R., Antunes, C.H., Dias, L.C.: Assessing the performance of biogas plants with multi-criteria and data envelopment analysis. Eur. J. Oper. Res. **197**(3), 1084–1094 (2009)
173. Madlener, R., Stagl, S.: Sustainability-guided promotion of renewable electricity generation. Ecol. Econ. **53**(2), 147–167 (2005)
174. Maghouli, P., Hosseini, S.H., Buygi, M.O., Shahidehpour, M.: A scenario-based multi-objective model for multi-stage transmission expansion planning. IEEE Trans. Power Syst. **26**(1), 470–478 (2011)
175. Malekpour, A., Niknam, T.: A probabilistic multi-objective daily Volt/Var control at distribution networks including renewable energy sources. Energy **36**(5), 3477–3488 (2011)
176. Manjure, D., Makram, E.: Optimal load curtailment as a bi-criteria program. Electr. Power Syst. Res. **66**(2), 155–161 (2003)
177. Mao, Y., Miu, K.N.: Switch placement to improve system reliability for radial distribution systems with distributed generation. IEEE Trans. Power Syst. **18**(4), 1346–1352 (2003)
178. Martinez, L., Lambert, J., Karvetski, C.: Scenario-informed multiple criteria analysis for prioritizing investments in electricity capacity expansion. Reliab. Eng. Syst. Saf. **96**(8), 883–891 (2011)
179. Mavrotas, G., Diakoulaki, D., Capros, P.: Combined MCDA-IP approach for project selection in the electricity market. Ann. Oper. Res. **120**(1–4), 159–170 (2003)
180. McDowall, W., Eames, M.: Towards a sustainable hydrogen economy: a multicriteria sustainability appraisal of competing hydrogen futures. Int. J. Hydrog. Energy **32**(18), 4611–4626 (2007)

181. Mendoza, F., Bernal-Agustin, J.L., Dominguez-Navarro, J.A.: NSGA and SPEA applied to multiobjective design of power distribution systems. IEEE Trans. Power Syst. **21**(4), 1938–1945 (2006)
182. Mendoza, J.E., Lopez, M.E., Coello, C.A.C., Lopez, E.A.: Microgenetic multiobjective reconfiguration algorithm considering power losses and reliability indices for medium voltage distribution network. Gener. Transm. Distrib. IET **3**(9), 825–840 (2009)
183. Mendoza, J.E., Morales, D.A., Lopez, R.A., Lopez, E.A., Vannier, J.-C., Coello, C.A.C.: Multiobjective location of automatic voltage regulators in a radial distribution network using a micro genetic algorithm. IEEE Trans. Power Syst. **22**(1), 404–412 (2007)
184. Meza, J.L.C., Yildirim, M.B., Masud, A.S.M.: A model for the multiperiod multiobjective power generation expansion problem. IEEE Trans. Power Syst. **22**(2), 871–878 (2007)
185. Meza, J.L.C., Yildirim, M.B., Masud, A.S.M.: A multiobjective evolutionary programming algorithm and its applications to power generation expansion planning. IEEE Trans. Syst. Man Cybern. A Syst. Hum. **39**(5), 1086–1096 (2009)
186. Milano, F., Canizares, C.A., Invernizzi, M.: Multiobjective optimization for pricing system security in electricity markets. IEEE Trans. Power Syst. **18**(2), 596–604 (2003)
187. Mirakyan, A., Guio, R.: Integrated energy planning in cities and territories: a review of methods and tools. Renew. Sustain. Energy Rev. **22**, 289–297 (2013)
188. Moeini-Aghtaie, M., Abbaspour, A., Fotuhi-Firuzabad, M.: Incorporating large-scale distant wind farms in probabilistic transmission expansion planning—part I: theory and algorithm. IEEE Trans. Power Syst. **27**(3), 1585–1593 (2012)
189. Moghaddam, A., Seifi, A., Niknam, T., Pahlavani, M.: Multi-objective operation management of a renewable MG (micro-grid) with back-up micro-turbine/fuel cell/battery hybrid power source. Energy **36**(11), 6490–6507 (2011)
190. Mohamadabadi, H., Tichkowsky, G., Kumar, A.: Development of a multi-criteria assessment model for ranking of renewable and non-renewable transportation fuel vehicles. Energy **34**(1), 112–125 (2009)
191. Mroz, T.: Planning of community heating systems modernization and development. Appl. Thermal Eng. **28**(14–15), 1844–1852 (2008)
192. Munda, G.: Multicriteria Evaluation in a Fuzzy Environment. Contributions to Economics Series. Physica-Verlag, Heidelberg (1995)
193. Murugan, P., Kannan, S., Baskar, S.: Application of NSGA-II algorithm to single-objective transmission constrained generation expansion planning. IEEE Trans. Power Syst. **24**(4), 1790–1797 (2009)
194. Nagesha, N., Balachandra, P.: Barriers to energy efficiency in small industry clusters: multicriteria-based prioritization using the analytic hierarchy process. Energy **31**(12), 1633–1647 (2006)
195. Nangia, U., Jain, N.K., Wadhwa, C.L.: Multiobjective optimal load flow based on ideal distance minimization in 3D space. Int. J. Electr. Power Energy Syst. **23**(8), 847–855 (2001)
196. Neves, L.P., Dias, L.C., Antunes, C.H., Martins, A.G.: Structuring an MCDA model using SSM: a case study in energy efficiency. Eur. J. Oper. Res. **199**(3), 834–845 (2009)
197. Neves, L., Martins, A.G., Antunes, C.H., Dias, L.: A multi-criteria decision approach to sorting actions for promoting energy efficiency. Energy Policy **36**(7), 2351–2363 (2008)
198. Nigim, K., Munier, N., Green, J.: Pre-feasibility MCDM tools to aid communities in prioritising local viable renewable energy sources. Renew. Energy **29**(11), 1775–1791 (2004)
199. Niimura, T., Nakashima, T.: Multiobjective tradeoff analysis of deregulated electricity transactions. Int. J. Electr. Power Energy Syst. **25**(3), 179–185 (2003)
200. Niknam, T., Azizipanah-Abarghooee, R., Narimani, M.: An efficient scenario-based stochastic programming framework for multi-objective optimal micro-grid operation. Appl. Energy **99**, 455–470 (2012)
201. Niknam, T., Doagou-Mojarrad, H.: Multiobjective economic/emission dispatch by multiobjective θ-particle swarm optimization. IET Gener. Transm. Distrib. **6**(5), 363–377 (2012)
202. Niknam, T., Fard, A., Baziar, A.: Multi-objective stochastic distribution feeder reconfiguration problem considering hydrogen and thermal energy production by fuel cell power plants. Energy **42**(1), 563–573 (2012)

203. Niknam, T., Kavousifard, A., Aghaei, J.: Scenario-based multiobjective distribution feeder reconfiguration considering wind power using adaptive modified particle swarm optimization. IET Renew. Power Gener. **6**(4), 236–247 (2012)
204. Niknam, T., Meymand, H.Z., Mojarrad, H.D., Aghaei, J.: Multi-objective daily operation management of distribution network considering fuel cell power plants. IET Renew. Power Gener. **5**(5), 356–367 (2011)
205. Niknam, T., Narimani, M.R., Aghaei, J., Azizipanah-Abarghooee, R.: Improved particle swarm optimisation for multi-objective optimal power flow considering the cost, loss, emission and voltage stability index. IET Gener. Transm. Distrib. **6**(6), 515–527 (2012)
206. Niknam, T., Zare, M., Aghaei, J.: Scenario-based multiobjective Volt/Var control in distribution networks including renewable energy sources. IEEE Trans. Power Delivery **27**(4), 2004–2019 (2012)
207. Ochoa, L.F., Padilha-Feltrin, A., Harrison, G.P.: Evaluating distributed generation impacts with a multiobjective index. IEEE Trans. Power Delivery **21**(3), 1452–1458 (2006)
208. Oikonomou, V., Flamos, A., Gargiulo, M., Giannakidis, G., Kanudia, A., Spijker, E., Grafakos, S.: Linking least-cost energy system costs models with MCA: an assessment of the EU renewable energy targets and supporting policies. Energy Policy **39**(5), 2786–2799 (2011)
209. Oliveira, C., Antunes, C.H.: An input–output model for decision support in energy-economy planning – a multiobjective interactive approach. J. Syst. Anal. Model. Simul. **42**(5), 769–790 (2002)
210. Oliveira, C., Antunes, C.H.: A multiple objective model to deal with economy-energy-environment interactions. Eur. J. Oper. Res. **153**(2), 370–375 (2004)
211. Oliveira, C., Antunes, C.H.: A multi-objective multi-sectoral economy–energy–environment model: application to Portugal. Energy **36**(5), 2856–2866 (2011)
212. Oliveira, C., Antunes, C.H.: Interactions of economic growth, energy consumption and the environment in the context of the crisis – a study with uncertain data. Energy **48**(1), 415–422 (2012)
213. Oliveira, C., Coelho, D., Silva, P., Antunes, C.H.: How many jobs can the RES-E sectors generate in the Portuguese context? Renew. Sustain. Energy Rev. **21**, 444–455 (2013)
214. Önüt, S., Tuzkaya, U., Saadet, N.: Multiple criteria evaluation of current energy resources for Turkish manufacturing industry. Energy Convers. Manag. **49**(6), 1480–1492 (2008)
215. Østergaard, P.: Reviewing optimisation criteria for energy systems analyses of renewable energy integration. Energy **34**(9), 1236–1245 (2009)
216. Papadopoulos, A., Karagiannidis, A.: Application of the multi-criteria analysis method Electre III for the optimisation of decentralised energy systems. Omega **36**(5), 766–776 (2008)
217. Patlitzianas, K.D., Ntotas, K., Doukas, H., Psarras, J.: Assessing the renewable energy producers' environment in EU accession member states. Energy Convers. Manag. **48**(3), 890–897 (2007)
218. Pedrasa, M., Spooner, T.D., MacGill, I.F.: Scheduling of demand side resources using binary particle swarm optimization. IEEE Trans. Power Syst. **24**(3), 1173–1181 (2009)
219. Pérez-Fortes, M., Laínez-Aguirre, J., Arranz-Piera, P., Velo, E., Puigjaner, L.: Design of regional and sustainable bio-based networks for electricity generation using a multi-objective MILP approach. Energy **44**(1), 79–95 (2012)
220. Phdungsilp, A.: Integrated energy and carbon modeling with a decision support system: policy scenarios for low-carbon city development in Bangkok. Energy Policy **38**(9), 4808–4817 (2010)
221. Pilavachi, P.A., Roumpeas, C.P., Minett, S., Afgan, N.H.: Multi-criteria evaluation for CHP system options. Energy Convers. Manag. **47**(20), 3519–3529 (2006)
222. Pilavachi, P.A., Stephanidis, S.D., Pappas, V.A., Afgan, N.H.: Multi-criteria evaluation of hydrogen and natural gas fuelled power plant technologies. Appl. Thermal Eng. **29**(11–12), 2228–2234 (2009)

223. Pires, D.F., Martins, A.G., Antunes, C.H.: A multiobjective model for VAR planning in radial distribution networks based on tabu search. IEEE Trans. Power Syst. **20**(2), 1089–1094 (2005)
224. Pohekar, S., Ramachandran, M.: Application of multi-criteria decision making to sustainable energy planning – a review. Renew. Sustain. Energy Rev. **8**(4), 365–381 (2004)
225. Pohekar, S., Ramachandran, M.: Multi-criteria evaluation of cooking energy alternatives for promoting parabolic solar cooker in India. Renew. Energy **29**(9), 1449–1460 (2004)
226. Polatidis, H., Harambopoulos, D.A., Munda, G., Vreeker, R.: Selecting an appropriate multi-criteria decision analysis technique for renewable energy planning. Energy Sources **1**(2), 181–193 (2006)
227. Pourmousavi, S.A., Nehrir, M.H., Colson, C.M., Wang, C.: Real-time energy management of a stand-alone hybrid wind-microturbine energy system using particle swarm optimization. IEEE Trans. Sustain. Energy **1**(3), 193–201 (2010)
228. Prasad, K., Ranjan, R., Sahoo, N.C., Chaturvedi, A.: Optimal reconfiguration of radial distribution systems using a fuzzy mutated genetic algorithm. IEEE Trans. Power Delivery **20**(2), 1211–1213 (2005)
229. Prete, C.L., Hobbs, B.F., Norman, C.S., Cano-Andrade, S., Fuentes, A., von Spakovsky, M.R., Mili, L.: Sustainability and reliability assessment of microgrids in a regional electricity market. Energy **41**(1), 192–202 (2012)
230. Ramirez-Rosado, I.J., Bernal-Agustin, J.L.: Reliability and costs optimization for distribution networks expansion using an evolutionary algorithm. IEEE Trans. Power Syst. **16**(1), 111–118 (2001)
231. Ramirez-Rosado, I.J., Dominguez-Navarro, J.A.: Possibilistic model based on fuzzy sets for the multiobjective optimal planning of electric power distribution networks. IEEE Trans. Power Syst. **19**(4), 1801–1810 (2004)
232. Ramirez-Rosado, I.J., Dominguez-Navarro, J.A.: New multiobjective tabu search algorithm for fuzzy optimal planning of power distribution systems. IEEE Trans. Power Syst. **21**(1), 224–233 (2006)
233. Ramirez-Rosado, I.J., Garcia-Garrido, E., Fernandez-Jimenez, A., Zorzano-Santamaria, P., Monteiro, C., Miranda, V.: Promotion of new wind farms based on a decision-support system. Renew. Energy **33**(4), 558–566 (2008)
234. Reddy, S., Abhyankar, A., Bijwe, P.: Reactive power price clearing using multi-objective optimization. Energy **36**(5), 3579–3589 (2011)
235. Ribeiro, F., Ferreira, P., Araújo, M.: Evaluating future scenarios for the power generation sector using a multi-criteria decision analysis (MCDA) tool: the Portuguese case. Energy **52**, 126–136 (2013). doi:10.1016/j.energy.2012.12.036
236. Rosehart, W.D., Canizares, C.A., Quintana, V.H.: Multiobjective optimal power flows to evaluate voltage security costs in power networks. IEEE Trans. Power Syst. **18**(2), 578–587 (2003)
237. Roth, S., Hirschberg, S., Bauer, C., Burgherr, P., Dones, R., Heck, T., Schenler, W.: Sustainability of electricity supply technology portfolio. Ann. Nucl. Energy **36**(3), 409–416 (2009)
238. Roy, B.: Multicriteria Methodology for Decision Aiding. Springer, New York (1996)
239. Sanseverino, E., Silvestre, M.L., Ippolito, M., De Paola, A., Re, G.L.: An execution, monitoring and replanning approach for optimal energy management in microgrids. Energy **36**(5), 3429–3436 (2011)
240. Santos, A.C., Delbem, A.C.B., London, J.B.A., Bretas, N.G.: Node-depth encoding and multiobjective evolutionary algorithm applied to large-scale distribution system reconfiguration. IEEE Trans. Power Syst. **25**(3), 1254–1265 (2010)
241. Sardou, I.G., Banejad, M., Hooshmand, R., Dastfan, A.: Modified shuffled frog leaping algorithm for optimal switch placement in distribution automation system using a multi-objective fuzzy approach. Gener. Transm. Distrib. IET **6**(6), 493–502 (2012)
242. Savier, J.S., Das, D.: Impact of network reconfiguration on loss allocation of radial distribution systems. IEEE Trans. Power Delivery **22**(4), 2473–2480 (2007)

243. Scott, J.A., Ho, W., Dey, P.K.: A review of multi-criteria decision-making methods for bioenergy systems. Energy **42**(1), 146–156 (2012)
244. Segura, S., da Silva, L.C.P., Romero, R., Salles, D.: Gener. Transm. Distrib. IET **6**(7), 646–656 (2012)
245. Shackley, S., McLachlan, C.: Trade-offs in assessing different energy futures: a regional multi-criteria assessment of the role of carbon dioxide capture and storage. Environ. Sci. Policy **9**(4), 376–391 (2006)
246. Shahnia, F., Wishart, M.T., Ghosh, A., Ledwich, G., Zare, F.: Smart demand side management of low-voltage distribution networks using multi-objective decision making. Gener. Transm. Distrib. IET **6**(10), 968–1000 (2012)
247. Shen, Y.-C., Lin, G.T.R., Li, K.-P., Yuan, B.J.C.: An assessment of exploiting renewable energy sources with concerns of policy and technology. Energy Policy **38**(8), 4604–4616 (2010)
248. Shmelev, S., Rodríguez-Labajos, B.: Dynamic multidimensional assessment of sustainability at the macro level: the case of Austria. Ecol. Econ. **68**(10), 2560–2573 (2009)
249. Singh, D., Misra, R.K.: Multi-objective feeder reconfiguration in different tariff structures. Gener. Transm. Distrib. IET **4**(8), 974–988 (2010)
250. Singh, D., Verma, K.S.: Multiobjective optimization for DG planning with load models. IEEE Trans. Power Syst. **24**(1), 427–436 (2009)
251. Sola, A., Mota, C.: A multi-attribute decision model for portfolio selection aiming to replace technologies in industrial motor systems. Energy Convers. Manag. **57**, 97–106 (2012)
252. Sola, A., Mota, C., Kovaleski, J.: A model for improving energy efficiency in industrial motor system using multicriteria analysis. Energy Policy **39**(6), 3645–3654 (2011)
253. Soloveitchick, D., Ben-Aderet, N., Grimman, M., Lotov, A.: Multiobjective optimization and marginal pollution abatement cost in the electricity sector – an Israeli case study. Eur. J. Oper. Res. **140**(3), 571–583 (2002)
254. Soroudi, A., Afrasiab, M.: Binary PSO-based dynamic multi-objective model for distributed generation planning under uncertainty. Renew. Power Gener. IET **6**(2), 67–68 (2012)
255. Soroudi, A., Ehsan, M., Caire, R., Hadjsaid, N.: Hybrid immune-genetic algorithm method for benefit maximisation of distribution network operators and distributed generation owners in a deregulated environment. Gener. Transm. Distrib. IET **5**(9), 961–972 (2011)
256. Souza, B.A., de Almeida, A.M.F.: Multiobjective optimization and fuzzy logic applied to planning of the Volt/Var problem in distributions systems. IEEE Trans. Power Syst. **25**(3), 1274–1281 (2010)
257. Stirling, A.: Multi-Criteria Mapping: A Detailed Analysis Manual, Version 2.0. Mimeo. SPRU, University of Sussex, Brighton (2005)
258. Streimikiene, D., Balezentis, T., Krisciukaitiene, I., Balezentisa, A.: Prioritizing sustainable electricity production technologies: MCDM approach. Renew. Sustain. Energy Rev. **16**(5), 3302–3311 (2012)
259. Sultana, A., Kumar, A.: Ranking of biomass pellets by integration of economic, environmental and technical factors. Biomass Bioenergy **39**, 344–355 (2012)
260. Talinli, I., Topuz, E., Akbay, M.U.: Comparative analysis for energy production processes (EPPs): sustainable energy futures for Turkey. Energy Policy **38**(8), 4479–4488 (2010)
261. Tant, J., Geth, F., Six, D., Tant, P., Driesen, J.: Multiobjective battery storage to improve PV integration in residential distribution grids. IEEE Trans. Sustain. Energy **4**(1), 182–191 (2013)
262. Tenerelli, P., Carver, S.: Multi-criteria, multi-objective and uncertainty analysis for agro-energy spatial modelling. Appl. Geogr. **32**(2), 724–736 (2012)
263. Terrados, J., Almonacid, G., Pérez-Higueras, P.: Proposal for a combined methodology for renewable energy planning. Application to a Spanish region. Renew. Sustain. Energy Rev. **13**(8), 2022–2030 (2009)
264. Thórhallsdóttir, T.E.: Strategic planning at the national level: evaluating and ranking energy projects by environmental impact. Environ. Impact Assess. Rev. **27**(6), 545–568 (2007)

265. Topcu, Y.I., Ulengin, F.: Energy for the future: an integrated decision aid for the case of Turkey. Energy **29**(1), 137–154 (2004)
266. Trutnevyte, E., Stauffacher, M., Scholz, R.: Supporting energy initiatives in small communities by linking visions with energy scenarios and multi-criteria assessment. Energy Policy **39**(12), 7884–7895 (2011)
267. Trutnevyte, E., Stauffacher, M., Scholz, R.: Linking stakeholder visions with resource allocation scenarios and multi-criteria assessment. Eur. J. Oper. Res. **219**(3), 762–772 (2012)
268. Tsai, M.-S., Hsu, F.-Y.: Application of grey correlation analysis in evolutionary programming for distribution system feeder reconfiguration. IEEE Trans. Power Syst. **25**(2), 1126–1133 (2010)
269. Tsay, M.-T.: Applying the multiobjective approach for operation strategy of cogeneration systems under environmental constraints. Int. J. Electr. Power Energy Syst. **25**(3), 219–226 (2003)
270. Tsoutsos, T., Drandaki, M., Frantzeskaki, N., Iosifidis, E., Kiosses, I.: Sustainable energy planning by using multi-criteria analysis application in the island of Crete. Energy Policy **37**(5), 1587–1600 (2009)
271. Tzeng, G.-H., Lin, C.-W., Opricovic, S.: Multi-criteria analysis of alternative-fuel buses for public transportation. Energy Policy **33**(11), 1373–1383 (2005)
272. Ulutas, B.H.: Determination of the appropriate energy policy for Turkey. Energy **30**(7), 1146–1161 (2005)
273. Unsihuay-Vila, C., Marangon-Lima, J.W., Souza, A., Perez-Arriaga, I.J.: Multistage expansion planning of generation and interconnections with sustainable energy development criteria: a multiobjective model. Electr. Power Energy Syst. **33**(2), 258–270 (2011)
274. Vahidinasab, V., Jadid, S.: Multiobjective environmental/techno-economic approach for strategic bidding in energy markets. Appl. Energy **86**(4), 496–504 (2009)
275. Vahidinasab, V., Jadid, S.: Stochastic multiobjective self-scheduling of a power producer in joint energy and reserves markets. Electr. Power Syst. Res. **80**(7), 760–769 (2010)
276. Vahidinasab, V., Jadid, S.: Joint economic and emission dispatch in energy markets: a multiobjective mathematical programming approach. Energy **35**(3), 1497–1504 (2010)
277. Vashishtha, S., Ramachandran, M.: Multicriteria evaluation of demand side management (DSM) implementation strategies in the Indian power sector. Energy **31**(12), 2210–2225 (2006)
278. Vlachokostas, C., Achillas, C., Moussiopoulos, N., Banias, G.: Multicriteria methodological approach to manage urban air pollution. Atmos. Environ. **45**(25), 4160–4169 (2011)
279. von Winterfeldt, D., Edwards, W.: Decision Analysis and Behavioral Research. Cambridge University Press, Cambridge, MA (1986)
280. Voropai, N.I., Ivanova, E.Y.: Multicriteria decision analysis techniques in electric power system expansion planning. Electr. Power Energy Syst. **24**(1), 71–78 (2002)
281. Vučijak, B., Kupusović, T., Midžić-Kurtagić, S., Ćerić, A.: Applicability of multicriteria decision aid to sustainable hydropower. Appl. Energy **101**, 261–267 (2013)
282. Wang, J.-J., Jing, Y.-Y., Zhang, C.-F., Shi, G.-H., Zhang, X.-T.: A fuzzy multi-criteria decision making model for trigeneration system. Energy Policy **36**(10), 3823–3832 (2008)
283. Wang, J.-J., Jing, Y.-Y., Zhang, C.-F., Zhao, J.-H.: Review on multi-criteria decision analysis aid in sustainable energy decision-making. Renew. Sustain. Energy Rev. **13**(9), 2263–2278 (2009)
284. Wang, L., Singh, C.: Multicriteria design of hybrid power generation systems based on a modified particle swarm optimization algorithm. IEEE Trans. Energy Convers. **24**(1), 163–172 (2009)
285. Wang, Y.H., Wang, Y.Y., Chan, K.W., Pong, T.Y.G.: Dynamic voltage security constrained optimal coordinated voltage control using enhanced particle swarm optimization. Gener. Transm. Distrib. IET **5**(2), 239–248 (2011)
286. Wang, L., Xu, L., Song, H.: Environmental performance evaluation of Beijing's energy use planning. Energy Policy **39**(6), 3483–3495 (2011)

287. Wei, Y., Wu, G., Fan, Y., Liu, L.: Progress in energy complex system modelling and analysis. Int. J. Global Energy Issues **25**(1–2), 109–128 (2006)
288. Winebrakea, J.J., Creswick, B.P.: The future of hydrogen fueling systems for transportation: an application of perspective-based scenario analysis using the analytic hierarchy process. Technol. Forecast. Soc. Chang. **70**(4), 359–384 (2003)
289. Wu, J., Wang, J., Li, S.: Multi-objective optimal operation strategy study of micro-CCHP system. Energy **48**(1), 472–483 (2012)
290. Wu, Z., Xu, J.: Predicting and optimization of energy consumption using system dynamics-fuzzy multiple objective programming in world heritage areas. Energy **49**, 19–31 (2013)
291. Xiaohua, W., Zhenmin, F.: Sustainable development of rural energy and its appraising system in China. Renew. Energy Sustain. Rev. **6**(4), 395–404 (2002)
292. Xu, Z., Dong, Z.Y., Wong, K.P.: A hybrid planning method for transmission networks in a deregulated environment. IEEE Trans. Power Syst. **21**(2), 925–932 (2006)
293. Yang, F., Chang, C.S.: Optimisation of maintenance schedules and extents for composite power systems using multi-objective evolutionary algorithm. Gener. Transm. Distrib. IET **3**(10), 930–940 (2009)
294. Yang, F., Kwan, C.M., Chang, C.S.: Multiobjective evolutionary optimization of substation maintenance using decision-varying Markov model. IEEE Trans. Power Syst. **23**(3), 1328–1335 (2008)
295. Yare, Y., Venayagamoorthy, G.K., Aliyu, U.O.: Optimal generator maintenance scheduling using a modified discrete PSO. Gener. Transm. Distrib. IET **2**(6), 834–846 (2008)
296. Yin, S.-A., Lu, C.-N.: Distribution feeder scheduling considering variable load profile and outage costs. IEEE Trans. Power Syst. **24**(2), 652–660 (2009)
297. Zangeneh, A., Jadid, S., Rahimi-Kian, A.: A fuzzy environmental-technical-economic model for distributed generation planning. Energy **36**(5), 3437–3445 (2011)
298. Zeng, J., An, M., Smith, J.N.: Application of a fuzzy based decision making methodology to construction project risk assessment. Int. J. Project Manag. **25**(6), 589–600 (2007)
299. Zhang, Y., Liu, H.-P.: Fuzzy multi-objective control strategy for parallel hybrid electric vehicle. Electr. Syst. Transportation IET **2**(2), 39–50 (2012)
300. Zhang, Q., Mclellan, B., Tezuka, T., Ishihara, K.: Economic and environmental analysis of power generation expansion in Japan considering Fukushima nuclear accident using a multi-objective optimization model. Energy **44**(1), 986–995 (2012)
301. Zhao, J.H., Foster, J., Dong, Z.Y., Wong, K.P.: Flexible transmission network planning considering distributed generation impacts. IEEE Trans. Power Syst. **26**(3), 1434–1443 (2011)
302. Zhihuan, L., Yinhong, L., Xianzhong, D.: Non-dominated sorting genetic algorithm-II for robust multi-objective optimal reactive power dispatch. Gener. Transm. Distrib. IET **4**(9), 1000–1008 (2010)
303. Zhou, P., Ang, B.W., Poh, K.L.: Decision analysis in energy and environmental modeling: an update. Energy **31**(14), 2604–2622 (2006)

Chapter 26
Multicriteria Analysis in Telecommunication Network Planning and Design: A Survey

João Clímaco, José Craveirinha, and Rita Girão-Silva

Abstract The interaction between a complex socio-economic environment and the extremely fast pace of development of new telecommunication technologies and services justifies the interest of multicriteria evaluation in decision making processes associated with several phases of network planning and design. Based on an overview of current and foreseen evolutions in telecommunication network technologies and services we begin by identifying and discussing challenges and issues concerning the use of multicriteria analysis in telecommunication network planning and design problems. Next we present a review of contributions on these areas, with particular emphasis on routing and network design models. We will also outline an agenda of current and future research trends and issues in this application area of multicriteria modelling.

Keywords Telecommunication planning and design • Multicriteria analysis

26.1 Motivation

Telecommunication systems and network technologies and the associated services have been and are in a process of very rapid evolution. Major changes in telecommunication system technologies and service offerings are currently underway. The evolution of telecommunication networks is a process of paramount importance not only because of the large investments required but also due to its significant impacts on the economic activities and on the society as a whole. The development of these networks gives rise to a variety of complex multidimensional problems.

J. Clímaco • J. Craveirinha (✉) • R. Girão-Silva
Institute of Computers and Systems Engineering of Coimbra (INESC-Coimbra), R. Antero de Quental, 199, 3000-033 Coimbra, Portugal
e-mail: jclimaco@inescc.pt

J. Craveirinha • R. Girão-Silva
Department of Electrical and Computer Engineering of the Faculty of Science and Technology of the University of Coimbra, Pólo II, Pinhal de Marrocos, 3030-290 Coimbra, Portugal
e-mail: jcrav@deec.uc.pt; rita@deec.uc.pt

S. Greco et al. (eds.), *Multiple Criteria Decision Analysis*, International Series in Operations Research & Management Science 233,
DOI 10.1007/978-1-4939-3094-4_26

Therefore, the interaction between a complex socio-economic environment and the extremely fast pace of development of new telecommunication technologies and services justify the interest of multicriteria evaluation in decision making processes associated with several phases of network planning and design. In the present work a state of the art review on this subject is done.

In the second section of this study an overview of major historical evolutions, current and foreseen major developments in telecommunication network technologies and services, is presented. Section 26.3 discusses general issues concerning the use of multicriteria analysis (MA) in telecommunication network planning and design. Section 26.4 is dedicated to a comprehensive discussion of applications of multicriteria analysis to telecommunication network planning and design problems. The first part (Sect. 26.4.1) is dedicated to a survey on routing models, an area where there has been a very great increase in contributions using some form of multicriteria based modelling. Section 26.4.2 deals with network planning and design models and Sect. 26.4.3 is focused on studies that present multicriteria evaluation approaches focusing on socio-economic evolutions associated with specific telecommunication issues. Of course it should be noted that there is no sharp frontier between Sects. 26.4.1 and 26.4.2 and that network design includes implicitly or explicitly some routing sub-model.

It must be remarked that we decided, in Sect. 26.4.1, to describe in more detail, as compared to other models, a recent multicriteria routing model dedicated to optical WDM networks, developed by our research group, as an example of the potentialities of multicriteria decision aiding approaches, to deal with new challenges in telecommunication routing problems. This is in fact an area in which there has been a very significant increase in terms of interest both from the Operations Research (OR) and from the engineering community as reflected in an increasing number of contributions. Also some aspects of two bicriteria shortest path Quality of Service (QoS) routing models will be analysed in more detail to draw attention to some methodological issues related to the calculation and selection of non-dominated solutions in an automated decision environment. In our opinion these models typify well cases in which the use of multicriteria (in this chapter used as synonymous with multiple objective) mathematical programming models is justified. Applications of multiattribute approaches mostly used in socio-economic evolution models associated with specific telecommunication issues, are just outlined because, in technical terms, they are not much different from their application in other areas and for lack of space for a thorough discussion.

26.2 Overview of Current Evolutions in Telecommunication Networks and Services

26.2.1 Major Technological Evolutions

For better understanding the decisive impact of network evolutions in the emergence of a significant number of new sets of problems of network planning and design, involving multiple objectives and constraints, we now present an overview of the major trends and factors underlying past, current and future evolutions.

Firstly, from an historical perspective, it can be said that major telecommunication network evolutions have been centred on and around two major modes of information transfer: circuit switching (typical of classical telephone networks) and packet switching (typical of Internet). In circuit switching when a call is generated the network routing mechanisms seek to find an available path (with the required bandwidth) from origin to destination and when that path (usually designated as route) is found then it is seized (in terms of the corresponding resources needed for each call) for the duration of the call; if no path is found in the required conditions the call is lost. In packet switching, the information to be transmitted is divided into packets (carrying the information about their origin and destination) of variable size that are routed through an available path and may suffer delays in the intermediate nodes. The development of the TCP/IP (Transmission Control Protocol/Internet Protocol) protocol suite enabled the very rapid expansion of the Internet in the 1980s, strongly accelerated in the 1990s through the release by the European Laboratory CERN (European Organization for Nuclear Research), in 1993, of the basic Web technologies. As for the public telephone networks they rapidly evolved from the 1980s through the development of ISDNs (Integrated Services Digital Networks) enabling the convergence on the same network of different types of services. The extremely rapid expansion of the demand for data services and for new and more bandwidth "greedy" services, soon required the development of technologies enabling to implement the concept of broadband ISDNs (B-ISDNs), mainly based during the 1990s, on the information transfer technology, ATM (Asynchronous Transfer Mode). ATM sought to take advantage of the inherent merits of packet-switching (namely the flexibility in terms of management of available bandwidth, when transporting large amounts of data) and circuit-switching (in terms of the provision of interactive real-time connection-oriented services with guaranteed QoS requirements). Since the early 2000 ATM has been rapidly abandoned in parallel with the emergence of multiservice Internet based technologies enabling the implementation of connection-oriented services and advanced QoS routing mechanisms. Among such technologies we should refer to IntServ (Integrated Services), DiffServ (Differentiated Services) and especially to MPLS (Multiprotocol Label Switching) and GMPLS (Generalised MPLS). A fundamental reason for the increasing success of Internet as a basic communication transfer platform is the fact that it enables a high percentage of the capabilities of an "ideal network" at a very low relative cost, as analysed in [125].

At the level of the transport infrastructure (underlying transmission networks) these trends were supported and have stimulated the development of optical networks capable of making the most of the large bandwidths associated with the very low wavelengths that may be carried by optical fibres. In particular WDM (Wavelength Division Multiplexing), enabling the simultaneous transport of several high capacity signals in each fiber, and DWDM (Dense WDM), using tens of wavelengths on each fiber, with extremely large information rates and increased traffic carrying flexibility, especially when associated with the introduction of wavelength conversion in the network nodes, permitted to take further advantage of the very large economies of scale provided by optical networks. All these developments led to the concept of *intelligent optical network*. As an example of the new services provided by these networks we could mention [89]: intelligent ultra-high bandwidth (lines at 2.5, 10 and 40 Gb/s provisioned from the customer equipment or from a centralised centre, including service protection mechanisms for failure situations), dynamic trunking (allowing to establish optical channels between equipments with the desired bandwidth and during the required time) and Gigabit Ethernet. Regarding transmission technologies (also known as carrier technologies) based on optical fibres besides the currently dominant SONET/SDH (Synchronous Optical Network/Synchronous Digital Hierarchy) systems a new carrier technology, Carrier Ethernet, is spreading as a cost-effective and functionally advanced alternative (see e.g. [228]).

Also rapid evolutions in digital radio communication technologies enabled an extremely rapid expansion in wireless and mobile networks with new or enhanced service functionalities. This was mainly driven by an increasing demand for mobile data services including Internet access, so that the total number of mobile subscribers tends to overtake the number of fixed lines on a world level. Related developments in hardware/microelectronics made the coexistence of multiple and heterogeneous access techniques possible, leading to the development of heterogeneous wireless networks (see e.g. [23]). As for the next generation wireless networks, a recent report by Wu et al. [279] describes a vision on the future networks over the next decade, focusing on the potential for new business opportunities and on the need for an increase in the network performance, so as to handle the current growth expectations in terms of energy efficiency, capacity, throughput and deployment.

More recently the development of 4G (fourth generation) systems compliant with IMT-Advanced (International Mobile Telecommunications Advanced) standards should provide mobile broadband solutions to different mobile devices in a comprehensive and secure IP-based network. 4G networks should be interoperable with existing wireless standards and offer global roaming across multiple networks. A wide range of services is expected, including bandwidth-consuming applications like HD (High Definition) broadcast, video calls and mobile TV. 4G should also provide a higher reliability and a significant improvement in performance and QoS. These advanced mobile services should be supported by networks that are increasingly packet-based.

In the last decade telecommunication networks have been subject to an extremely rapid evolution that is the result of the combination of two major forces: *traffic growth* and a very fast pace of *technological advances*.

As for traffic growth this is both quantitative and qualitative that is involving increases in traffic volumes in response to broad socio-economic developments and also related to the demand for new services, more bandwidth demanding, as these become available through technological evolution or simply perceived as desirable by groups of customers. In this respect it should be stressed the extremely rapid increase in Internet traffic that has occurred in very recent years (for instance in early 2000, the annual increase rates were 60–80 %—*apud* [89]). At the same time the increase in the number of subscribers of broadband services and wireless networks has grown at average rates of 60 % and 25 %, respectively. More recent figures of the United States (US) wireless subscriber connections show an increase by almost 3 times of the number of subscribers in the last decade, with a current wireless penetration value of 102.4 % and a total of 29.7 % of US households having wireless-only systems [74].

The strong interactions between those two driving forces and socio-economic factors should be stressed. A relevant example is the fact that the explosive growth of Internet enabled the rapid development of the so-called electronic commerce, as an increasingly important business practice, with very strong impact on economy and society as a whole. The impacts of telecommunication network developments in the structure, management and organisational culture of the companies in association with the present days globalisation, are also obvious. Overall it can be said that there is a strong correlation between the technological development and expansion of telecommunication networks and economic and social evolutions. Secondly, at the market level, the steady transition from regulatory monopolies to liberalisation led to fierce competition among operators and service providers both at the level of national networks and local access networks. All these evolutions are multifaceted and prone to conflicts and contradictions, an example being the tensions between the recent drive for big mergers and acquisitions between operators and the antitrust policies of the regulatory bodies (Federal Trade Commission and Federal Communications Commission in the US and the EU Competition Directorate). Needless to say that there are strong social interactions associated with the development of new network technologies simultaneously in terms of strictly human interactions, in terms of the relationships between humans and all types of organisations and in terms of the intra and inter-organisational relations. The detailed analysis of these trends and interactions is naturally out of the scope of this study.

In a simplified manner it can be said that the factors mentioned above, favoured the development of technologies and network architectures capable of satisfying increasing traffic volumes and more sophisticated services, at the lowest possible cost (per basic information unit that can be carried with a certain QoS satisfaction degree).

From the previous analysis the following great trends in future telecommunication network technological evolutions can be put forward:

- the convergence of Internet wired transport infrastructure towards an intelligent optical network;
- the evolution of 4G wireless network in the direction of an all IP multiservice converged network;
- the increasing relevance of multidimensional QoS issues in the new technological platforms.

Since the last trend plays a decisive role in the interest in developing multicriteria decision analysis (MCDA) models we will analyse it in more detail.

26.2.2 Increasing Relevance of QoS Issues in the New Technological Platforms

The simplicity of the IP that provides the basic end-to-end data delivery service in the existing Internet, based on a “best effort” (BE) service concept, lacks a mechanism capable of guaranteeing the multiple QoS requirements of the new type of applications, namely multimedia applications. This led to the introduction of new functionalities in the next generation Internet, namely the IntServ and the DiffServ mechanisms, providing certain QoS guarantees concerning the transport of traffic of different types (for an overview see [185]). Also the MPLS technology contains QoS mechanisms that enable different QoS parameter levels to be guaranteed on separate LSPs—Label Switched Paths (or network “tunnels”) as well as functions of network load balancing (through traffic engineering operations) and fast rerouting under failure. All these developments pave the way to a new, high performance multiservice Internet corresponding to the concept of *QoS based packet network* proposed in [89].

On the other hand the Universal Mobile Telecommunications System (UMTS) platform provides mechanisms of QoS support for 3G (third generation) and 4G wireless networks. These mechanisms are based on a QoS architecture that uses several traffic classes intended for different types of applications where each class corresponds to applications with similar statistical behaviour and similar QoS requirements. This and other developments are creating the technical conditions for the full interoperability of these networks and its convergence towards an all IP network, as discussed above.

All these innovations and technological trends put in evidence the increasing relevance of the issues related to the definition and assessment of multidimensional QoS parameters and the associated network control mechanisms. These issues have important reflexes in the type and nature of many new problems of network planning and design, namely concerning routing methods and the choice of alternative network architectures. The natural inclusion in the OR models associated with such problems of multiple, eventually conflicting objectives and various types of constraints, technical and socio-economic, lays the ground for the potential advantage of the introduction of MA methods. In fact, and concerning the *type of problems* that need to be addressed, the demand for new services, the rapid traffic growth and the extremely rapid technological evolution have led to the multiplication of new types of problems of routing, network planning and design (as it will become clear in the next sections) in many of which there is a potential advantage in considering explicitly several criteria. Also, and with respect to the *nature* of many of such new problems, it is important to address the multidimensional character of the problems, together with the consideration of relevant technical and socio-economic constraints. This necessity becomes more apparent if one takes into account the increasing importance of the QoS issues (of a multidimensional nature) related to the development of new services and the rapid evolution of the technological platforms. Finally, the importance, in various decision problems, of the inclusion of negotiation processes involving various decision agents (in complex cases the customer, the end service provider and the network operator) and the uncertainty associated with many objective functions (OFs, in short) and constraint parameters, makes it clear the interest in considering MA approaches in this context.

26.3 Multicriteria Analysis in Telecommunication Network Planning and Design

From the last section, it is clear that decision making processes related to telecommunication networks take place in an ever increasing complex and turbulent environment characterised by a fast pace of technological evolution, drastic changes in available services, market structures and societal expectations, involving multiple and potentially conflicting options. This is obviously an area where different socio-economic decisions involving communication issues have to be made, but it is also a case where technological issues are of paramount importance as it is recognised, for instance, by Nurminen [205]: "(. . .) The network engineering process starts with a set of requirements or planning goals. Typical requirements deal with issues like functionality, cost, reliability, maintainability, and expandability. Often there are case specific additional requirements such as location of the maintenance personnel, access to the sites, company policies, etc. In practice the requirements are often obscure. (. . .)". Nurminen, who has collaborated in the development of network planning mathematical models for Nokia, recognises the limitations of single criterion models. However he emphasises the difficulties in the tuning of parameters in mathematical programming models and also draws attention to the fact that this aspect becomes more difficult to tackle when multiple objective formulations are used, since the procedures of preference aggregation by the decision maker(s), or DM(s), imply, in general, the definition of specific parameters, such as, for example, the fixation of some kind of "weights". This difficulty does not justify less interest in multicriteria modelling but must be taken into account.

In many situations the mathematical models for decision support, in this area, become more realistic if different evaluation aspects are explicitly considered by building a consistent set of criteria (or objectives) rather than aggregating them a priori in a single economic indicator. In fact, multicriteria models explicitly address different concerns that are at stake so that DMs may grasp the conflicting nature of the criteria and the compromises to be made in order to identify satisfactory solutions. In a context involving multiple and conflicting criteria, the concept of optimal solution gives place to the concept of non-dominated solutions set, that is feasible solutions for which no improvement in any criterion is possible without sacrificing on at least one of the other criteria. In general, multicriteria approaches look for the identification of one or more non-dominated, or approximately non-dominated, satisfactory solutions. Of course, the choice of the approach or method to aggregate the preferences is also multicriteria in nature. Beyond the problem mentioned above concerning the fixation of parameters, it must be taken into account whether or not there is a possibility of using interactive procedures especially in relation to the speed of calculation. In fact, the procedure can not be interactive if the calculations in each interaction are too slow. Also, in many telecommunication network decision problems, no more than a few seconds (sometimes less) are available for finding the solution to be implemented, equally situations in which interactive procedures cannot be applied. As we will see later on, when presenting a concrete example, the simplicity of the questions the DM has to answer in the phase of preference aggregation, is crucial. Cognitive as well as technical aspects are at stake that may compromise, in many cases, the quality of the selected solutions.

Concerning methodological aspects it is clear that many routing models are based on multicriteria shortest path models, as it will be analysed in Sect. 26.4.1. In this respect we would like to refer to the paper by Tarapata [259], which presents a review on selected multicriteria (multiobjective) shortest path (MOSP) problems and resolution methods. It also provides an analysis of the complexity of the resolution algorithms, ways of adapting classical algorithms, in particular using fast implementations of Dijkstra's algorithm and a presentation of properties of MOSP problems formulated as mathematical programming problems, including comparative performance analysis experiments with different resolution approaches. A recent review of the literature on multicriteria path and tree problems, namely multicriteria minimum spanning tree problems, including an in-depth discussion on exact algorithms and their foundations and an outline of important applications, is presented in [53]. Also examples of applications of multiobjective combinatorial optimisation illustrating how this type of approach provides more realistic modelling and potential benefits, including reference to examples in telecommunication routing are in [87].

Another aspect in which there are compromises to be made, concerns the type of implementations to be executed with respect to single criterion problems that have to be solved in each step of a multicriteria approach. This question is not exclusive of multicriteria models but it is more critical in this case than in single criterion models, since the programs with the single criterion implementations are run several times. Let us examine what is at stake: in many situations the mathematical programming models to be used have a network structure. In many of these cases there are very efficient specific algorithms for solving them, sometimes exact resolution procedures, other times heuristics. In this respect, the remarkable development of metaheuristics in recent years has to be noted.

Many optimisation problems in the field of telecommunications, are characterised by their large size and the presence of multiple, conflicting objectives. Solving these problems exactly may be difficult or infeasible either as a result of their combinatorial complexity or due to computational limitations in the context of a given application. Therefore, heuristics and metaheuristics are often required for their solution in an acceptable time, especially when on-line methods (and in particular, real-time methods) are at stake. A state-of-the-art in multiobjective metaheuristics at the beginning of the last decade is in [135]. More recent reviews can be found in [88, 256, 257]. In [88], an overview over approximation methods in multiobjective combinatorial optimisation is provided. A summary of "classical" metaheuristics and a focus on recent approaches, where metaheuristics are hybridised and/or combined with exact methods, are presented in that reference. In [256], a unified view of metaheuristics is provided. A background on metaheuristics and the implementation of algorithms to solve complex optimisation problems across a diverse range of applications, including routing, are presented in that reference. In [257], the focus is on open research lines related to non-evolutionary metaheuristics, hybrid multiobjective metaheuristics, parallel multiobjective optimisation, and multiobjective optimisation under uncertainty.

A particular case of metaheuristics are the evolutionary algorithms, which have been extensively used to solve multiobjective optimisation problems in the field of telecommunications. A state-of-the-art in multiobjective evolutionary algorithms (MOEA) at the beginning of the last decade is in [268]. More recent reviews can be found in [59, 60], where the features of MOEAs are presented, the aim being to find good solutions for high-dimensional real-world design applications. State-of-the-art research results and applications to different practical problems are presented.

The question is that the very rapid development of modern telecommunication networks makes it advisable, in many situations, to use generic algorithms, often less efficient but more robust concerning its applicability when there are technological shifts, in order to avoid heavy implementation overheads for each specific new case.

It is also important to discuss in broad terms which multicriteria model is more adequate to each situation. Up till now we have talked about mathematical programming models that may be linear, non-linear and additionally may have, or not, a special structure. In counterpoint other type of models that we will designate as multiattribute models have been developed. While multicriteria mathematical programming models assume the set of feasible alternatives is defined implicitly through the introduction of constraints, in multiattribute models a finite and small set of alternatives is defined explicitly. These alternatives are analysed taking into account multiple criteria. This type of models allows a more detailed evaluation of the considered alternatives, without computational explosion, but in most situations it implies a very reductive point of view when considering telecommunication planning and design. In fact, the explicit definition of a small set of alternatives is a hard task and not realistic in many cases. As we will see later on, in some circumstances the complementary utilisation of both types of models can be advisable. It is out of the scope of this paper to enter in details on the approaches that are available to analyse multicriteria models since it is a matter of study in other chapters of this book. In a few words concerning multiattribute models there is the so-called American School where, to support the evaluation of a discrete set of alternatives, a multiattribute utility function, linear or not (depending on the approaches) is built [138]. The Analytical Hierarchy Process (AHP) can be viewed as a special branch of the American School where a hierarchy of interrelated decision levels is identified [239–241]. On the other hand, the so-called French School is based on the introduction of partial orders, i.e. outranking relations are considered. No more complete comparability of alternatives and transitivity are obtained. As an example of the French School approaches we can refer to

ELECTRE methods [235]. Depending on the situation the purpose is to select the most preferred alternative, to rank the alternatives or to classify the alternatives in groups. They are less demanding than the American School approaches, namely in terms of fixing parameters, however the results are less conclusive regarding the aggregation of the DM preferences.

Concerning the approaches dedicated to multicriteria mathematical programming models attention should be paid to the dimension of the real problems to deal with and, many times, the necessity of a rapid execution, for the reasons discussed above. In this respect one should put in relief, from the bibliography concerning telecommunication applications: the use of interactive approaches dedicated to multicriteria linear programming models; the use of methaheuristics for analysing integer and mixed-integer programming models; and the use of approaches based on the resolution of shortest and *k*-shortest path problems in the context of routing models. It should be noted that network multicriteria shortest path models are the only ones for which sufficiently rapid exact algorithms are available, either to generate the whole or part of the non-dominated solution set, or to study the problem in an interactive manner.

Last but not least, the uncertainties in various instances of the models, are also a key issue in telecommunication planning and design. The uncertainty associated with the representation of traffic flows offered to the network is of major importance in many models. Such representation is a twofold task: the use of adequate stochastic models (these are often mere approximations) for representing the traffic flows as required by the model and the obtainment of estimates of the probabilistic parameters that are needed in the stochastic sub-models. Also the uncertainties and/or imprecisions associated with other parameters of the OR model of different origins, from data collection to preference aggregation modelling (see [39]) are a relevant issue in this context.

As it is well known, multicriteria approaches allow in these circumstances to identify the set of criteria related to the stable part of the DMs' preferences, leaving to later analysis further aggregation of their preferences. In many situations, the output of the MA is not a solution but some satisfactory solutions according to the model used. So, an a posteriori analysis studying in more detail (namely, taking into account characteristics not included in the model) those solutions may be advisable. Furthermore, in some situations (as, for instance, in strategic planning) the analysis may not lead to a prescription but just to a clarification of the decision situation. This attitude dealing with the problems may help to reduce the gap between models and real world problems.

A generic analysis of the reasons why there are potential advantages in considering explicitly multiple points of view in the evaluation of telecommunication planning and management problems and of major factors to be taken into account in the use of multicriteria approaches is in [50].

Wierzbicki [275] shows how MA in telecommunications can be seen from the point of view of knowledge creation theory. After reviewing a method called creative space used for integrating various approaches to knowledge creation, a discussion on how this type of approach can be useful in supporting new technology creation by constructing specialised "creative environments" is presented.

In the next sections of this chapter a schematic review and discussion of works using multicriteria models, published in the context of planning and design of telecommunication networks, is presented. Also a discussion of future trends in these areas will be outlined. Special attention will be paid to the section concerning routing models (in Sect. 26.4.1) since, as it was seen in the previous section, this is an area that raises great challenges having in mind the introduction of new technologies and services, of a multidimensional character, since beyond costs various dimensions associated with QoS are involved. An historical perspective about the way in which various dimensions were treated in different models and proposals to consider explicitly more than one criterion in situations of static routing and of dynamic routing, will be presented. In this context, and from a methodological point of view, exact algorithms for the calculation of shortest paths in single criterion and multicriteria situations as well as heuristics, will be mentioned. Secondly, a reference to studies on strategic planning of the evolution of telecommunication networks, using multicriteria linear programming models and interactive methodologies of analysis (Sect. 26.4.2), will be made. A model that intends to evaluate the introduction of new basic services in the local access network, in face of some of the remarkable technological developments previously discussed, is reviewed. It is also briefly outlined an expansion planning model, concerning the cellular phone system in a Brasilian state, based on a multiattribute approach. Next, a reference to several studies focusing on problems which may be grouped in the area of operational planning (in the context of Sect. 26.4.2), will be made. In particular: a link frequency assignment problem, a power management policy problem in wireless communication, an Internet caches placement problem, an hub location problem dedicated to rural area telecommunication networks taking advantage of new technologies and a frequency allocation problem in mobile telephone networks. Very different models were used in these applications, however all of them belong to the category of multicriteria mathematical programming.

Finally, some socio-economic application models related to telecommunication issues are reported. Namely: several strategic studies concerning electronic commerce decisions and a study of quality concerning the provided telecommunication services. In all these situations multiattribute models were used. Furthermore, some studies concerning the complementarity/substitution between travelling and telecommuting are referred to, namely studies where multicriteria network equilibrium modelling is proposed in several situations.

26.4 Review and Discussion of Applications of MA to Telecommunication Network Planning

26.4.1 Routing Models

26.4.1.1 Background Concepts

Routing is a key functionality in any communication network and has a decisive impact on network performance (in terms of traffic carried and supplied grade of service (GoS) for end-to-end connections) and cost. Routing is essentially concerned with the definition and selection of a path or set of paths from an originating to a terminating node (assuming the network functional topology is represented by a graph), seeking to optimise certain objective(s) and satisfy certain technical-economic constraint(s). The routing problems have different nature and multiple formulations, depending fundamentally on the mode of information transfer, the type of service(s) associated with the routed "calls", the level of representation of the network (typically two levels are considered: the physical or transmission network and the logical or functional network), and the features of the routing paradigm (for example whether it is static or time varying according to traffic fluctuations or network conditions). The term "call" is taken here in its broadest sense, as an end-to-end service request with certain requirements that must be met by the path (or route) along which that call is routed. Examples are a telephone call, a video call, a data packet stream or an end-to-end wavelength assignment (in an optical network). In the broader context of the planning and design activities routing is a fundamental network functionality that may be considered as an integral part of the network operational planning decision process, strongly related to other planning instances, namely network structure design (involving topological design and capacity facility calculation) and traffic network management. At a lower level of the network functionalities routing is intimately related to the entities that actually implement its working in a real network, entities usually designated as routing protocols, critically interrelated with the technological requirements. Two examples, for the Internet, are the OSPF (Open Shortest Path First) protocol and the BGP (Border Gateway Protocol). These aspects and interdependencies are a decisive factor in the formulation of the routing problems from the OR perspective. An overview of some of these issues and possible modelling and resolution approaches can be seen in [180].

When formulating routing problems it is useful to model networks as *teletraffic networks* the specification of which includes the following elements: (1) a graph $(\mathscr{V}, \mathscr{L})$ defining the network topology where the nodes (in $\mathscr{V}$) may represent switches, exchanges (groups of switches interconnected in a certain manner) or routers, and the edges (or links in $\mathscr{L}$) represent transmission facilities with a certain capacity; (2) the capacities of the arcs, that are usually expressed in terms of bandwidth (in bit/s) or equivalent number of certain basic transmission channels (for example in multiples of 64 kb/s channels); (3) the node-to-node traffic flows that may be modelled in general as marked point processes (e.g. a marked Poisson process, see [62]), which enable a representation of the instants of arrivals, call durations and associated bandwidth requirements in the links; (4) the used routing principle(s), that is, the basic features of the network routing function (for example, whether it is static or dynamic or whether a maximal number of links per path was prescribed). Here we consider the term *routing method* as a particular specification of certain routing principle(s), including the algorithm or set of rules which are used to perform the path computation and path selection for every traffic flow or connection request, at a given time, having in mind to get the objective function value(s) as good as possible and satisfy certain constraint(s)—associated with the underlying routing principle(s) and possible additional constraint(s) reflecting bounds on other objective(s) or requirement(s) inherent to the method. It must be emphasised that the specification of the objective(s) and constraint(s) depends strongly on the nature of the network and services (in various technical instances) and on the rationale of the routing method. The procedure/algorithm of path computation and route selection is normally designated as *routing algorithm* and is a key element of the routing method. At a lower level of specification routing is described through what we designate as *routing technique*, a technical entity that actually enables the implementation of a routing method in a given real network with a given technology and architecture and assuming a particular structure of available information. A routing technique is typically implemented through standardised *routing protocols*, critically dependent on the features of the concrete technological platform, such as the OSPF for Internet.

A detailed analysis on background concepts useful to the development of OR based routing models is in [58] in the context of a review on multicriteria routing models. An extensive analysis of basic routing models and a presentation of key OR approaches in this area, namely network flow-programming approaches can be seen in [217]. Wang et al. present an overview on routing optimisation procedures for Internet, analysed from a traffic engineering perspective in [271].

26.4.1.2 Review of Multiple Criteria Routing Approaches

QoS Routing Models

The extremely rapid pace of technological evolution and the increase in the demand for new communication services lead to the necessity of multiservice network functionalities dealing with multiple, heterogeneous QoS dimensions. This trend (discussed in Sect. 26.2) led to a new routing paradigm in telecommunication networks designated as *QoS routing*. This type of routing involves the selection of a chain of network resources along a feasible path satisfying certain requirements (dependent on traffic features associated with service types) and seeking to optimise some relevant metric(s) such as delay, cost, number of edges of a path or loss probability. Therefore, in this context, routing algorithms need to consider distinct metrics [160].

In commonly used approaches the path calculation problem is formulated as a shortest path problem with a single OF, corresponding either to a single metric or to a function encompassing different metrics, while QoS requirements are incorporated into these models by means of additional constraints. This is the usually proposed approach for QoS routing problems, generally designated as *constrained-based QoS routing*. This type of routing problems is particularly relevant in the new Internet technologies, namely MPLS, as explained in Sect. 26.2, and in some ATM routing protocols.

A well known approach in multicriteria model analysis consists of transforming the OFs into constraints, except one of them which is then optimised. In adequate conditions the obtained solution will necessarily be non-dominated, concerning the original multicriteria model; furthermore by varying the second member of the constraints it is possible to obtain different non-dominated solutions (see [253]). In this sense constrained-based QoS routing models can be envisaged as a first tentative of MA. On the other hand, the necessity of determining the solution to be implemented in the network in a very short time (usually a few seconds or even less, depending on several factors) makes that the most common approach is to develop heuristics that include classical algorithms for shortest path computations.

In [45] an overview of the majority of QoS routing procedures up to 1998, is presented. A report on the state of art on QoS routing up to 1999 is presented in [264]. A comprehensive review on constrained-based routing is provided in [152, 153]. The latter authors recognise that QoS routing requires that multiple parameters, related to current network state, have to be frequently updated and the corresponding information has to be distributed throughout the network. Hence the creation of routing protocols capable of efficiently computing the required paths and processing and distributing that dynamically varying information, is still an open issue that needs further investigation. In these circumstances they opted for a review of methods dedicated to this type of problem where the network state is temporarily static. In the same study several exact algorithms and heuristics dedicated to the multiple-constrained path (MCP), to the multiple-constrained optimal path (MCOP) and to the restricted shortest path (RSP) problems, are discussed. In the MCP problem, the aim is to obtain path(s) which satisfy constraints on all metrics while in MCOP and RSP (this is a particular case of the former with one constraint alone) problems there is an OF to be optimised.

Kant et al. [136] describe an extensive study on the impact of various QoS routing heuristics typically MCOP or shortest path tree procedures, considering various metrics as path calculation objectives/constraints and compares their results in terms of various network metrics, through an integrated analytic toolset.

The reference list includes several models on variants of QoS routing problems and various resolution procedures. Namely: [122, 127] (focusing on the RSP problem); [78] (dealing with the MCP problem through an heuristic with tunable accuracy, based on a k-shortest path algorithm) and [267] (proposing a procedure for dealing with the MCP and the MCOP problems, also based on a k-shortest path algorithm); a similar type of problems is tackled in [30], using a heuristic based on a k-shortest path algorithm; Reeves and Salama [227] propose a distributed implementation of a heuristic for a delay-constrained least cost path problem (a specific MCOP problem); [224] (presenting and comparing several algorithms for the MCP problem); [46] (proposing two heuristics for the MCP problem, based on Dijkstra and Bellman-Ford algorithms); [147, 148] (proposing a heuristic for the MCOP problem based on modified versions of Dijkstra algorithm); [172] (developing an exact algorithm for finding *k*-shortest paths satisfying multiple constraints); [8] (dealing with the CSP—Constrained Shortest Path problem); [37, 124] (proposing a dual algorithm for the CSP problem); [283, 285] (both dealing with heuristics for the MCP problems).

An in depth analysis of complexity issues of the MCP problem, stating that the problem is NP-complete but not strong NP-complete and presenting reasons explaining why in most practical instances of the problem, concerning communication network applications, exact solutions can be achieved is in [150]. The former related work in [154] presents a study on performance evaluation of MCP and RSP algorithms based on complexity analysis and simulation results. Also for the MCP problema, Kuipers and van Mieghem [151] apply the concept of dominance as an efficient search-space reduction technique and evaluate the advantages of using this technique via a simulation study. A comparison study, focused on exact algorithms and heuristics of specific type, for the MCOP routing problem is shown in [155]. Avallone et al. [21] presents a comparison of algorithms for the MCOP problem, based on simulations. In this reference the authors propose a routing scheme that tries to maximise the throughput (or minimise the blocking), which are typical goals in traffic engineering algorithms, while trying to satisfy the users' QoS requirements. This combination of objectives considering both the perspective of the service providers and the perspective of the service users in an integrated manner, is also studied in [20], where simulation results are presented.

Cui et al. [76] propose a simulated annealing metaheuristic using Dijkstra's algorithm, for solving MCP routing problems, and analyse its computational complexity and scalability in simulated networks.

Various papers focusing on particular applications of QoS routing models are also included in the references.

In this category we included a QoS inter-domain routing model for a high speed wide area network (WAN) given in [144], solved with a heuristic approach. Applications of QoS routing approaches to classical integrated service networks are in [113, 120, 177, 178]. Models of this type for Internet are quite numerous, particularly having in mind their interest in the DiffServ, IntServ and MPLS technological platforms as shown in [93, 99, 219].

An application of a QoS routing model in an ATM network, focused on a problem with multiple constraints, is in [222]. Applications to MPLS networks are in [27] where an overview in this specific application area is presented.

Rocha et al. [232] describe an evolutionary algorithm seeking to optimise the link costs (in the context of an OSPF routing protocol) from the point of view of network performance, assuming a MCOP base formulation.

A QoS routing procedure involving multiple constraints (MCP type problem) and using a fuzzy system based routing technique is described in [288].

Also other papers dealing with specific application models involving problems of this type are included in the bibliography, namely: [27] (presenting an overview of application models for MPLS networks); [77] (focusing on an application to ATM networks); [78, 93] (dealing with applications to Internet routing); [113, 120, 177, 178] (showing applications to routing protocols for traffic with bandwidth guarantees in integrated services networks); [219, 269] (making an analysis of various formulations and mathematical properties of the MCP problem with respect to the metrics more relevant to QoS routing).

Several routing methods require the calculation of several paths simultaneously, leading to a class of routing problems designated as *multipath routing* problems. Examples arise in routing models with reliability requirements or resilient routing in which an active path and a back-up path, to be used in the event of failure, are to be computed simultaneously for each pair of origin-destination nodes (which may be referred to as *point-to-point multipath routing*) and in *multicast routing* where a set of paths has to be calculated from an originating node to a set of destination nodes—point-to-multipoint routing (for example in the case of distributional services supplied by a certain service provider) or interconnecting a sub-set of the network nodes—multipoint-to-multipoint routing (for example in teleconferencing services in Internet). If all the network nodes have to be interconnected the associated multicast routing problem, which may be designated as *broadcast routing*, can be formulated as a *spanning tree problem*. If the set of destination nodes is a proper subset of the set of network nodes, the multicast routing problem can be formulated as a *Steiner tree problem* where the destination nodes and the originating node are the terminal nodes.

In [149] a QoS routing procedure for a constrained multicast routing problem is presented. It involves the simultaneous selection of paths from a source node to multiple destination nodes dealing with applications to routing problems in integrated services packet networks. In [250] an algorithm for a multicast constrained problem is described and its performance is analysed through simulation. A multicast routing procedure involving the calculation of multiple trees is in [223]. In [5], a multicast QoS routing model for wireless networks is described. It is based on the calculation of trees satisfying multiple constraints, using a heuristic combining features of genetic algorithms and competitive learning. A multipath routing model where the load of a traffic flow between an originating node and a terminating node can be divided by a set of feasible paths—a routing procedure usually designated as *traffic splitting*—considering several criteria namely concerning the required bandwidth, for application to Internet, is in [100].

Wu et al. [278] propose a multiple-constraint multicast routing heuristic based on a genetic algorithm and compare its application with three single-constrained Steiner tree heuristics. Other QoS-constrained multicast routing heuristics are presented in [168] and in [47] which takes available bandwidth as the prime metric, considering the constraints of the surplus energy of the node, delay and delay jitter. Hou et al. [129] describes a QoS multicast routing model with multiple constraints using a genetic algorithm.

Seok et al. [244] address a dynamic version of this problem with a hop-count constraint where the routing requests of traffic arrive one-by-one; it is formulated as a mixed-integer programming problem and a new heuristic for its resolution is proposed. A similar problem is treated in [42] by using a modified Bellman-Ford algorithm, so that the proposed routing method builds a multicast tree, where a node is added to the existing multicast tree without re-routing and satisfying QoS constraints. Other QoS-constrained multicast routing heuristics for dynamic variants of this problem are presented in [169].

Zhu et al. [290] present a heuristic for constructing minimum-cost multicast trees with delay constraints. The same type of problem is addressed in [214] with a formulation that handles two variants of the network cost optimisation goal: minimising the total bandwidth utilisation of the tree and another minimising the maximal link cost. The problem is solved by a heuristic.

Alrabiah and Znati [4] describe three heuristics based on shortest path calculations, for dealing with this type of problem and analyse their complexity and compare their results in terms of the tree costs. Another heuristic for the same problem is proposed in [98].

The paper [271] describes a heuristic for a multipath constrained routing problem involving the calculation of two shortest link-disjoint paths, for protection purposes. Another multipath constrained routing problem is addressed in [48] for application to MPLS.

In [2], a specific QoS routing model for robust routing design in MPLS networks is described, considering a point-to-point two-path calculation problem and two network performance metrics obtained with and without failures in the links; a mixed-integer linear programming formulation is used.

In [171] a multipath QoS routing model for ad-hoc wireless networks considering four criteria and presenting a resolution procedure based on fuzzy set theory and evolutionary computing, is described.

Special attention should be drawn to some cases where the concerns which lead to this type of approaches, are relevant to MA. In [273] an exact RSP algorithm, designated as constrained Bellman-Ford (CBF), is proposed. This enables, for example, to obtain successive shortest paths between a pair of nodes for different values of the right hand-side constraint on the delay, hence obtaining non-dominated solutions. That author proposes an exact algorithm dedicated to the RSP problem. The bicriteria nature of this proposal is clear and we could put in evidence that the bicriteria shortest path problem approach in [52] could perform a similar study in a much more efficient manner.

We would also like to draw attention to the fact that the principles underlying the bicriteria routing approach described in [16] based on a specific k-shortest path algorithm and on the introduction of preference thresholds in the OF space have clear relations with the principles underlying the algorithm in [134] focused on the MCP problem and with other algorithms intended to improve some aspects of that algorithm.

Consider now approaches based on Lagrangian decomposition, where, for example, one intends to calculate the minimal cost path subject to a delay constraint.

The costs and delays on the links are combined linearly and hence the shortest path, regarding the obtained OF, is calculated. Kuipers et al. [153] recognise that a key issue in such approaches is the way in which the appropriate multipliers are determined when delay and cost are combined, since this obviously conditions the solution that is obtained. It is a question of the same type that arises in the definition of weights when in MA one intends to optimise a weighted sum of OFs. Note that in bicriteria shortest path problems, there may exist unsupported non-dominated solutions. In the example above nothing guarantees that the obtained solution is optimal for the original RSP problem. Approaches where one seeks to close the gap between the optimal solution and the solution obtained from a linear combination by using k-shortest path algorithms are referred to. Also approaches for calculating unsupported non-dominated solutions based on k-shortest path algorithms can be developed.

Other multidimensional approaches, where there is an a priori articulation of preferences in the path selection, taking as basis bandwidth, delay and hop count, could be mentioned [153]. Relevant examples are the widest-shortest and the shortest-widest path approaches. Examples of such approaches can be seen in [177, 178, 210], [269] (in this case the purpose is to calculate the shortest path in terms of delay, with maximal minimal arc bandwidth; note that the minimal bandwidth of all arcs of the path is usually known as *bottleneck bandwidth*), [212] and [267] (presenting a heuristic approach based on an utility function, as an alternative to the widest-shortest path model for routing "elastic[1] traffic flows" in Internet).

Sobrinho [249] seeks to treat in an unified form several QoS routing related path computation problems (including connectivity, shortest path, widest path, most-reliable path, widest-shortest path and most-reliable shortest path problems) by using an algebra of weights (hence treating in an articulated manner the aggregation of preferences) and also enabling to take into account a specific requirement of the routing procedure implementation in the Internet. As an application of this approach a variant of the Dijkstra algorithm which guarantees the satisfaction of that requirement, is constructed.

Riedl [230] describes a genetic algorithm for calculating paths seeking to minimise the sum of the delays in the links and the inverse of the bottleneck bandwidth, for application in MPLS networks. Riedl and Schupke [231] describe a routing optimisation approach considering one or two metrics, enabling to include concave metrics and multipath routing through equal cost paths. The model seeks to minimise the maximum link utilisation (MLU) in the network by adjusting delays and capacities of the links. A linear combination of the two link metrics is considered in the two metric case and a heuristic solution is proposed for application to large networks as an alternative to a mixed-integer programming formulation. Also note that this type of approach is a particular form of the "weight setting

[1] An elastic traffic flow has a rate that can adapt to the available network capacity.

problem", a NP hard problem originally addressed in [99] for application to OSPF routing protocols, if we identify the weights as the individual link metrics.

Mitra et al. [203] develop a network revenue maximisation model in which pricing and routing (with traffic splitting) are jointly calculated and show that link shadow costs provide the basis for selecting optimal prices and routing policies. The resulting concave programming formulation with constraints is solved by a Lagrangian method and it enables selecting the solution which optimises the revenue while guaranteeing the minimum traffic splitting so that it may be very useful in a bicriteria optimisation context.

Explicitly Multicriteria Routing Models

Let us now consider the cases in which the modelling is *more explicitly multicriteria*. We think there are potential advantages in considering many routing problems in modern telecommunication networks explicitly as multiple criteria problems. This type of modelling is potentially advantageous although one cannot ignore that, in many situations, the solution to be implemented has to be obtained in a limited time that may range from a very small fraction of a second (typically tens of ms) to a few seconds. This practical limitation implies the impossibility of using interactive methods in many cases, hence leading to the necessity of implementing automatic path calculation procedures. The exception is in static routing problems, in transport networks where transmission routes are maintained for large time periods or in some form of periodic dynamic routing models where the input parameters are estimated in advance (for example, node-to-node traffic intensities in different time periods), cases in which an interactive procedure could be used to select the routes (for every node pair) to be stored in routing tables assigned to every node. This explains the predominance of methods where there is an a priori articulation of preferences. It should be noted that, even in these cases, there are advantages in considering explicit multicriteria modelling hence rendering the mechanisms of preference aggregation transparent. In this manner, several aspects, namely cost and QoS parameters such as blocking probability, delay or bandwidth, can be addressed explicitly by the mathematical models, part as OFs and the remainder as constraints, seeking to reflect in a more realistic manner the underlying engineering problem.

There is still a different type of multicriteria model, which deserves a reference. In many types of telecommunication networks there is a mechanism, closely associated with the routing function that is usually designated as *admission control*. This mechanism involves a decision on whether or not each call/packet is accepted, as a function of certain call characteristics (e.g. associated type of service, tariff system and QoS requirements) and, possibly, network working conditions (this is typical of dynamic routing methods that include admission control mechanisms). The underlying objective of this mechanism is to maximise the operator revenue while satisfying the QoS guarantees for every customer class.

Very early papers on multiobjective models in telecommunication networks focused on flow control models were [82, 83, 86]. Douligeris [82, 83] describes a multiobjective flow control model in a multiclass packet traffic network with queueing mechanisms where each class has a performance objective. The used formulation is a non-linear multiobjective optimisation problem, involving as criteria throughput and average delay, that is transformed in a linear multicriteria program, solved by standard techniques. Also in [86], a problem of optimisation of a multiserver two-class packet queueing model is dealt with by considering a bi-objective Nash game formulation. In this context each class of packets, in competition with the other class, seeks to minimise its own cost function (average delay or blocking probability).

The game theoretic approach to routing in Internet in [22] considers that users may determine individually the routes for flows they control, each with several objectives, giving rise to a non-cooperative multicriteria game. The existence and uniqueness of Nash equilibria for this type of problem are analysed in various conditions and the model is applied to pricing.

In classical B-ISDN and in broadband multimedia networks in general, admission control is also a relevant issue, since the supplied QoS guarantees are directly related to the obtained revenue, via the tariff (or "charging") system (a comprehensive analysis and discussion on charging models for multiservice networks is in [251]). Brown et al. [41] address an admission control problem in broadband multiservice networks, modelled as a specific semi-Markov decision process that might be considered as a first tentative stochastic multicriteria approach. In this approach the objective is to maximise the total revenue rate of ongoing calls while satisfying the QoS guarantees of all carried calls. The resolution approach is based on a reinforcement learning technique. The solutions are compared with simple heuristic admission control solutions, by using a simulation model for a test communication system with two types of traffic sources.

Concerning multicriteria approaches for flow control calculation, Hassanein et al. [126] describe a multipoint-to-point flow control multiobjective optimisation model with three criteria: overall network throughput, fairness amongst sources in terms of carried rates and fairness amongst groups of sources. The OFs are of quadratic type and the minimisation of a weighted sum of the three functions is used for finding non-dominated solutions.

Bezruk et al. [36] give a highlight of a generic multicriteria optimisation formulation as a network design tool and of the type of compromise solutions it may supply in the case of flow control calculation in a packet switched network.

On the other hand, as it will be seen in the cases that we are describing next in more detail, it is possible to conciliate the automatic path calculation and selection with some flexibility in the form of preference aggregation. This enables the grasping of the compromises among different objectives, taking into account certain QoS requirements, by treating in a consistent manner the comparison among distinct routing possibilities in the context of a certain routing principle.

Following this methodological framework an explicitly multiobjective routing model for telecommunication networks was firstly (as far as we know) presented by Antunes et al. [16]. In this approach a static routing problem (that is a routing problem where the OF(s) coefficients are constant values) is formulated as a bi-objective shortest path problem. The model can be adapted to different metrics associated with different types of services. An algorithmic approach was developed to deal with this problem which computes non-dominated paths based on the optimisation of weighted sums of the multiple OFs, based on a very efficient *k*-shortest path algorithm in [188]. QoS requirements are represented in the model through 'soft constraints' (that is constraints not directly incorporated in the mathematical formulation) in terms of 'acceptable' and 'requested' values for each of those metrics. Note that since the routing problem is modelled as a multiple objective shortest path problem without side constraints, no metrics other than the ones considered as objectives are represented in this model. This limitation could be surpassed, but in such case it should be necessary to check whether each new calculated path, respects the side constraints. The resolution approach proposed in [16] is inspired by the one presented in [233], in the framework of a procedure enabling to search interactively non-dominated supported and unsupported shortest paths in the bicriteria case. It should be stressed that the node-to-node routing plans are supposed to run in an automatic manner, in the framework of a routing control network mechanism. The procedure satisfies this requirement integrating the use of a *k*-shortest path algorithm [188] (likewise in [233]) together with new devices designated by soft constraints. To sum up, in this approach a specialised automatic algorithm was developed to obtain non-dominated solutions, which takes into account the specific aspects of a routing problem in a multiservice environment. Note that updating the thresholds regarding the soft constraints related to QoS requirements, according to the evolution of the network state, is a very simple and clear procedure in operational terms.

To understand the main features of this type of approach, used in various forms in a number of contributions referred to later, an illustrative example of its working is presented, based on Fig. 26.1, considering as metrics cost and delay.

Firstly the vertex solutions, which optimise each OF separately, are computed, by solving two shortest path problems using Dijkstra's algorithm. This yields information regarding the value range of each OF over the non-dominated solution set. QoS requirements for each of those metrics are specified by means of the thresholds concerning a requested value (aspiration level) and an acceptable value (reservation level). The addition of this type of soft constraints (that is, constraints not directly incorporated into the mathematical formulation) defines priority regions, in which non-dominated solutions are searched. Region A is a first priority region where both requested values are satisfied. Regions B1 and B2 are second priority regions where only one of the requested values is met and the acceptable value for the other metric is also guaranteed. A further distinction can be made between these second priority regions by establishing a preference order on the OFs. For instance, stating that cost is more important than delay, would give preference to region B1. Region C is a third priority (or fourth if B1 and B2 have different priorities) region in which only acceptable values for both metrics are fulfilled. For the example in Fig. 26.1 the first solution found within (first priority) region A (solution 3) is selected. Note that any solution in the first priority region dominates any solution in region C. Of course, in other situations, solutions within second priority regions B1 and B2 could be found

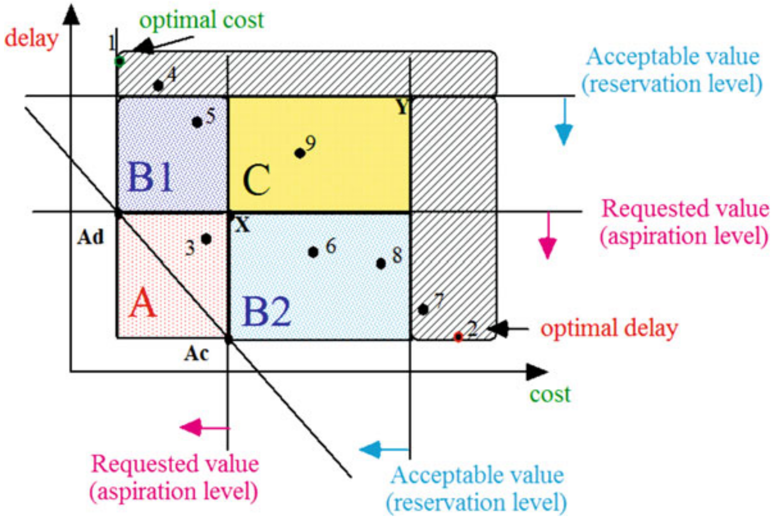


Fig. 26.1 Priority regions and example in [16]

first. These solutions should be stored but not reported until the first priority region is entirely searched (i.e. when the constant cost line of the OF used in the *k*-shortest path problem passes through point X). If there are no non-dominated solutions within region A, the search proceeds to second priority regions. The previously computed solutions in regions B1 and B2, if any, are now reconsidered. In the example, solutions 5, 6 and 8 are found within second priority regions. In general, it is (again) possible to obtain solutions in the third priority region (C) before all second regions (B1, B2) are searched. Again these solutions are stored and reported only when regions B are completely searched without finding any non-dominated solutions within them. If the algorithm proceeds to this point it means that no paths exist satisfying at least one of the requested QoS values (aspiration levels) and only acceptable values (reservation levels) can be met. Beyond point Y even acceptable values for QoS requirements cannot be met. In this case a possible relaxation in the acceptable value thresholds would have to be considered. In fact non-dominated solutions may possibly exist outside the priority regions (such as solutions 4 and 7), which could be used as "last chance" routes.

The use of the capability of this type of model incorporating preference thresholds is strongly dependent on the application environment, in terms of network technological constraints (with repercussion on the teletraffic network model) and capabilities, as well as on QoS requirements, types of traffic flows and characteristics of provided services. For example, in conventional NB-ISDN (Narrow Band ISDN), only constraints concerning "acceptable" levels of GoS need to be considered, which should follow standard ITU recommendations. On the other hand, in ATM and multiservice IP networks, namely based on IntServ, DiffServ and especially MPLS platforms, where traffic sources of quite different nature and a multiplicity of requirements may occur, the connection-oriented services allow the user to indicate the communication needs during the connection set-up phase and the network may tailor the transfer properties of the connection to specific user needs. This gives rise, in particular, to the concept of traffic contract or SLA (Service Level Agreement) with its inherent flexibility in terms of resource management. In this framework both types of (soft) constraints, concerning "acceptable" and "requested" values become significant. In this context, it must be noted that the (possible) occurrence of non-dominated paths which lead to a better value than the one "requested" by the user raises questions regarding their admissibility as outcomes of the algorithm, since they correspond to an overutilisation (albeit temporary) of network resources. This type of questions, which does not bring any further algorithmic or computational complexity to the proposed approach, nevertheless requires further analysis, which will be necessarily dependent on the network features. So, an important point put forward in this paper is to draw attention to the potential advantages in the application of MA to routing problems in multiservice networks and to provide an efficient algorithmic approach for resolving the problem with the consideration of relevant 'soft' constraints/preference thresholds, in addition to normal 'hard' constraints.

Many other multicriteria routing models have been developed since [16] that are based either directly or in terms of auxiliary resolution procedures in multicriteria shortest path algorithms. Having in mind the importance of this type of procedure we would like to refer to reviews in this area or papers of methodological nature referring to applications in telecommunication network design.

Granat and Wierzbicki [119] present an overview of MA techniques applied to the design of telecommunication networks and outline how a special model for interactive MA can be adapted for this purpose.

The work [58] presents an overview on papers concerning multicriteria routing models and describes a bi-level optimisation multicriteria routing model that may be applied to ATM or to IP/MPLS with QoS constraints. In this model the first level OFs are path cost and number of arcs and the second level OFs are bottleneck bandwidth and path average delay. The model is solved exactly by calculating the non-dominated solutions concerning the first level functions, which are "filtered" by using bounds defined through the second level OFs.

The paper [276] presents a conceptual framework for the development of multicriteria QoS routing approaches in IP networks. After a critical analysis of the use of the weighted sum aggregation of criteria as a basic method for multiple criteria decision making in a general context and in the context of routing in communication networks, the authors analyse, in depth, the features of routing approaches that may be considered consistently multicriteria and describe an illustrative routing model recurring to achievement functions in the context of a reference point approach.

Another important type of networks where multicriteria routing models have been proposed is multiservice networks supporting multimedia applications. The utilisation of a QoS routing principle, as mentioned above, involves the selection of paths satisfying multiple constraints of a technical nature and which seek to optimise some relevant metric(s). This is naturally a sub-area of multicriteria routing where there is very great potential interest/advantage in the use of multicriteria shortest path based models.

A multiple objective flow-oriented optimisation model for this type of routing problem, intended for application to networks supporting multimedia applications, namely video services, was presented by Pornavalai et al. [220]. The OFs to be minimised are the number of links of the path (usually designated as hop-count), $z^{(1)}$, and a cost $z^{(2)}$ that is obtained by considering that the cost of a call using a link a_{ij} is the inverse of its available bandwidth, b_{ij}. The first OF is intended to minimise the number of resources used by a call while the second seeks to minimise the impact of the acceptance of a call by choosing 'least loaded' paths. As for the constraints, they are expressed by bounds on the minimum available bandwidth (bound BW_M), on the delay—sum of the delays d_{ij} on the links a_{ij} of the path—(bound DM_M), and on the delay jitter (bound J_M). This corresponds to the formulation of a bi-objective constrained shortest path problem ($P1_C$) obtained by adding to the classical bicriteria shortest path formulation $\min\{z^{(1)}, z^{(2)}\}$, the three constraints:

$$\begin{aligned} \min_{a_{ij}\in p} \{b_{ij}\} &\geq BW_M \\ \sum_{a_{ij}\in p} d_{ij} &\leq DM_M \\ \sum_{a_{ij}\in p} J_{ij} &\leq J_M \end{aligned} \tag{26.1}$$

where J_{ij} is the delay jitter on the link a_{ij} of path p. The OF coefficients are $c_{ij}^{(1)} = x_{ij}$ (with $x_{ij} = 1$ if the a_{ij} link is used in the path and $x_{ij} = 0$ otherwise) and $c_{ij}^{(2)} = \frac{1}{b_{ij}}$. The constraint coefficients d_{ij} and J_{ij} are calculated from stochastic models representing the queueing and jitter mechanisms associated with the link transmission functions, for each type of traffic flow. In some applications, such as video traffic in an ATM network using a specific queueing mechanism it is possible to transform the constraint (26.1) into a constraint on the number of links of the path. In [220] the resolution approach to this problem is a heuristic based on the Dijkstra shortest path (SP) algorithm. The heuristic is rule-based and has two phases: route metric selection (i.e. selection of the OF that it seeks to optimise in each iteration) and route composition rule (where SPs from the origin to intermediate nodes in terms of one metric are concatenated with SPs from those nodes to the destination). For each selected routing metric and composition rule if the SP or the composed path do not satisfy the constraints the heuristic will retry a new route metric and/or new composition rules until a feasible route is found or all routes are exhausted. In spite of its capability in supplying feasible solutions in short times (in networks with hundreds of nodes and average node degree of 4) it doesn't guarantee that the obtained solutions are non-dominated.

This type of routing problem was tackled in [55] by using an exact algorithmic approach for calculating the whole set of non-dominated paths of $P1_C$. This approach is based on the bi-objective shortest path algorithm by Clímaco and Martins [52] and on the MPS algorithm in [188], used for calculating k-shortest paths with respect to the convex combination of $z^{(1)}$ and $z^{(2)}$. In this approach it was necessary to adapt a ranking algorithm for generating the set of non-dominated paths. It might be expected that a labelling algorithm would be a better approach. However it was shown by the authors that the ranking algorithmic approach has better performance as a result of explicit consideration of the constraints in the bi-objective problem. This approach was applied to a problem of video traffic routing on ATM networks, by constructing random networks and networks based on the US inter-city spatial topology. In this particular application study it was shown that, although the used OFs were not strongly conflicting, there was a significant number of problems with 2, 3 and 4 non-dominated solutions. Also the algorithm, proposed in [55], enabled the calculation of the whole set of non-dominated solutions in networks with up to 3000 nodes and average degree of 4, in short processing times and modest memory requirements, up to certain bounds on the acceptable delay. This makes this algorithm attractive in many realistic problems, namely in more modern types of network platforms enabling the establishment of routes with guaranteed QoS levels in terms of bandwidth, delay or jitter.

An interactive procedure based on a reference point approach, for resolving a multiobjective routing problem is presented on [118]. The MA issue of selecting and ordering the solutions obtained in the context of a multicriteria shortest path routing model, taking into account that the route selection has to be performed in an automated manner, is addressed in [56]. A first approach to this problem based on the use of a k-shortest path algorithm and preference thresholds (leading to the specification of priority regions) defined in the OF space, is put forward. Later, in [57] an evolution of this method that is based on a reference point approach, using an algorithm to minimise a weighted Chebyshev distance to reference points defined in each priority region, is proposed and applied to a bi-objective video traffic routing problem with multiple constraints.

The authors in [34] propose a multiobjective shortest path model with constraints for calculating packet stream routes in an autonomous area of Internet considering OFs of min-sum and max-min type. In [35] a multiobjective routing model for the Internet using a multiobjective shortest path formulation is described; the model uses as path metrics total average delay, hop-count and residual bandwidth and it is solved by an exact algorithm that calculates the set of non-dominated solutions for connections from one node to all the other nodes and a selection procedure based on a weighted Chebyshev distance to the ideal point.

Yuan [284] describes a bi-objective optimisation model for multipath routing using traffic splitting, for application to the Internet. It assumes all paths are calculated from a single objective shortest path model (implemented in practice through an OSPF routing protocol) where the weights have to be optimised in order to obtain compromise solutions to a network optimisation bi-objective model. This type of routing method is known as ‘robust OSPF routing’. The two OFs of this model are traffic load balancing functions in full-operational and arc failure scenarios. The search for non-dominated solutions is a heuristic that uses a hash function and a diversification technique. Note that the associated single optimisation problem of calculation of the weights (cost coefficients of the arcs) such that the shortest paths obtained with those coefficients optimise some network cost function, assuming the traffic is split evenly among those shortest paths is known as ‘weight setting problem’—a NP hard problem originally addressed in [99].

Another paper focusing on a multiobjective formulation of the weight setting problem of link state protocols is [252]. An evolutionary metaheuristic is used for seeking approximate non-dominated solutions.

Komolafe and Sventek [146] address the problem of optimisation of the RSVP (Resource Reservation Protocol) routing protocol for Internet in terms of its timing parameters seeking to simultaneously optimise four network performance metrics, for various network traffic conditions. An evolutionary algorithm is used to seek approximate non-dominated solutions and an application study with a network simulator is described.

The authors in [3] present a study on the evolution of routing models used in the context of a sequence of releases of a network planning tool, including a bi-objective protection routing model. This model describes a trade-off between route length and disjointness of primary and back-up paths to be used in the event of failures.

Craveirinha et al. [68] describes a stochastic bicriteria approach for restorable QoS routing in MPLS networks, enabling to incorporate the most probable network failure states (including multiple failures) in the calculation of the active and protection paths by considering network “performability” measures. An exact resolution method is also put forward for the model assuming the link failure probabilities are known.

Concerning multipath routing models, Lee et al. [162] present a comparison on multipath routing algorithms for MPLS networks, with traffic splitting, that select candidate paths using multiple criteria.

In [216] an ant colony algorithmic approach, for dealing with a multiobjective multicast routing problem with four objectives, in a packet network, is presented. Donoso et al. [81] describe a multiobjective model for multicast routing the aim of which is to minimise the total delay and the total number of links. A heuristic combined with an evolutionary algorithm is proposed for obtaining approximate solutions.

A multiobjective multicast routing model in wireless networks is in [238] where a genetic algorithm, efficient in large networks, is used for obtaining approximate solutions.

A bicriteria multicast routing model for multipoint-to-multipoint virtual connections in transport networks where the metrics to be optimised are load cost and the number of arcs, is presented in [194] in the form of a bicriteria Steiner tree problem. A heuristic procedure based on a bicriteria spanning tree algorithm is proposed for solving the problem and its performance is analysed. An improvement of this bicriteria Steiner tree heuristic is shown in [187] together with extensive tests using the Steiner-Lib for comparison with the optimal values of each of the two OFs and for showing the effectiveness of the heuristic.

Talavera et al. [255] describe a multicriteria multicast routing model considering the formulation of a three criteria Steiner tree problem where the criteria to be optimised are MLU, tree cost and average link delay. The authors use as resolution tools four different MOEAs and compare their results in a network dynamic environment.

Cui et al. [75] describes a multiobjective optimisation model for multicast routing where an evolutionary algorithmic approach is proposed for calculating non-dominated solutions, and analyses its performance.

In [70] a procedure, based on the strength Pareto evolutionary algorithm (SPEA) aimed at solving a multiobjective multicast routing problem, is presented. The used formulation seeks to optimise simultaneously the cost of the tree, the maximal end-to-end delay, the average delay and the maximal link utilisation. The authors in [80, 197] present a multicast multiobjective model with traffic splitting, for application to MPLS networks which considers as OFs hop-count, total bandwidth consumption, maximal link utilisation, and the total end-to-end delay; the basis of the resolution approach uses a non-linear aggregated function of these four functions.

The authors in [94] present an overview on multiobjective multicast routing models and put forward a classification of publications in this area. The paper also describes an evolutionary algorithmic approach for obtaining the set of non-dominated solutions which is based on a SPEA procedure and shows experimental results for models with up to 11 objectives.

Meyerson et al. [198] analyses mathematical properties of a bicriteria Steiner tree problem the aim of which is to minimise the sum of the edge costs concerning one metric and the sum of source-sink distances in terms of an unrelated second metric. A specialised heuristic is proposed for obtaining solutions. Its application to multicast routing is also addressed.

Craveirinha et al. [69] presents a bicriteria minimum spanning tree routing model aimed at calculating and selecting non-dominated spanning trees for broadcasting messages or defining overlay networks over a MPLS network structure. The OFs of the model are the total load balancing cost and an average upper delay bound on the arcs of the spanning tree. An exact procedure is used for calculating supported non-dominated solutions and one of such solutions is selected by a method based on the approach in [57]. The network performance of the bicriteria model is also experimentally analysed.

The authors in [288] presents a bi-objective routing model in a context where the demand in the network is given by a set of offered traffic matrices and considering a weighted sum of the average and worst case network performance values under the given matrices. The trade-offs between these two criteria are analysed in case-study MPLS networks using OSPF based routing.

In [112], on-line algorithms for routing and bandwidth allocation which simultaneously try to maximise the throughput and assure fairness in the treatment of the communication sessions, are proposed.

Nace and Pioro [204] present a tutorial on the application of max-min fairness in routing optimisation in particular in relation to lexicographic optimisation models. Resolution algorithms for convex max-min optimisation are presented as well as their application in routing in communication networks.

Ogryczak et al. [206] describe lexicographic optimisation models using a max-min fairness principle for bandwidth resource allocation in a data network. Several algorithms for solving convex and non-convex models of this type are analysed. In a subsequent paper, [207] the authors tackle the problem of telecommunications network design with the objective of maximising service data flows and providing fair treatment of all services. Again a max-min fairness principle is used to formulate the resource allocation scheme. A lexicographic resolution approach is considered.

Pióro et al. [218] presents an analysis of the application of the max-min fairness principle to the design of telecommunication networks, using a lexicographic optimisation approach. An application of this type of formulation to a routing method for elastic traffic, is also presented.

Ogryczak et al. [208] develops a multicriteria model enabling equitable optimisation as an alternative to lexicographic optimisation, for dealing with bandwidth resource allocation in IP networks. This approach is applied to routing elastic traffic and a reference point procedure is developed as a resolution method.

A multicriteria routing model for wireless networks that seeks to obtain routes which minimise total energy consumption, latency and bit error rate simultaneously, is presented and its performance analysed in [182, 184]. The model uses a normalised weighted additive utility function to obtain non-dominated solutions and application results are presented. Marwaha et al. [195] presents a multiobjective routing approach for certain types of wireless networks, namely mobile ad-hoc networks. The model seeks to deal with the uncertainties of the routing model by using a fuzzy cost function of the different metrics and an evolutionary algorithm for tackling the corresponding routing optimisation problem.

A multiobjective routing model in wireless sensor networks (WSN) involving mobile agent routing, which also uses evolutionary algorithmic approaches, can be seen in [226]. In [181] a framework for routing in WSN that is adaptive, constraint-based, multiobjective and which is also solely dependent on localised knowledge, is proposed.

Malakooti et al. [183] propose a multicriteria routing method for satellite based Internet communications considering as criteria to be optimised the total packet latency to destination, the total processing time at a given node, the average delay jitter, and the packet loss failure. A weighted additive function of the criteria is used for seeking solutions.

Guerriero et al. [121] develop a bicriteria routing model for mobile ad-hoc networks where two OFs are to be minimised: energy consumption and link stability. A bicriteria CSP formulation is described and solutions are obtained through a convex combination of the two functions. A heuristic solution is developed taking into account the dynamic nature of the model and the distributed routing control.

Petrowski et al. [215] propose a method designated as “Russian doll method” based on the definition of a set of nested boxes in the criteria space and use a

Chebyshev metric to identify a so-called "most preferred" non-dominated vector. The method was applied to a multicriteria routing model for mobile ad-hoc networks considering two criteria: average packet delay and transmission error rate and it is tested by simulation.

Shafigh and Niyati [246] apply a learning automata procedure for finding a heuristic solution to a multicriteria routing problem in WSNs. A multicriteria dynamic routing model for data transport in WSNs, is shown in [171]. It considers three criteria associated with sensor-node properties and uses a heuristic procedure for adaptive tree reconfiguration.

Long et al. [173] describe a satellite routing model using as criteria average packet delay, residual link bandwidth and packet loss rate, and use a heuristic swarm intelligence type technique for obtaining solutions, namely a "beehive algorithm" inspired in the behaviour of bees. The results are compared with the ones from a basic heuristic designated as "prior order algorithm".

Roy and Das [236] present a multiobjective multicast routing model, for wireless networks, considering as criteria end-to-end delay, bandwidth guarantee and residual bandwidth utilisation. A genetic algorithm is developed as a resolution approach. This heuristic is used as the basis for a routing protocol in [237] and its performance is analysed through network simulation.

In [29] a bicriteria routing model for IP networks is described, aiming at optimising load balancing and average delay. A Non-Sorting Genetic Algorithm (NSGA) is used for seeking Pareto solutions.

Levin and Nuiriakhmetov [165] address a multiobjective multicast problem in Wi-Fi networks considering a multicriteria Steiner tree formulation where four criteria, namely total cost, total edge length, overall throughput (capacity) and estimate of QoS are to be optimised. A heuristic based on node clustering and a minimum spanning tree algorithm is used as resolution approach.

Xu and Qu [280] present a multiobjective multicast routing model, considering four objective functions in the associated multicriteria Steiner tree formulation (cost, maximal end-to-end delay, average delay in the tree, maximal link utilisation) and use a hybrid metaheuristic as a solution method: an Evolutionary Multiobjective Simulated Annealing procedure due to [170].

Minhas et al. [199] present a routing model seeking to optimise simultaneously lifetime and source-to-sink delay in WSNs by recurring to a fuzzy multiobjective optimisation approach.

Another specific type of routing models where multicriteria approaches have been proposed is *dynamic routing*. The advantages of using a dynamic routing principle, in telecommunication networks, are well known (see e.g. [18]). The essential feature of dynamic routing is the dependence of routing decisions on measurable network parameters such as number of channels occupied in a link, proportion of unaccepted connections (blocking probability), packet delays, estimated traffic offered, or events (e.g. whether a connection request is successful or not) hence reflecting, in one way or another, the network working conditions. This implies that selected end-to-end routes may vary in time, seeking to take advantage of the evolving network working conditions, with the aim of achieving, at any given time period, the best possible value(s) of some network performance criterion (or criteria). The impact of dynamic routing in network performance is particularly relevant in situations with highly variable traffic intensities, overload and failure conditions by enabling an effective response of the routing system to adverse network working states. The dynamic routing methods with this adaptive nature are usually designated as *adaptive routing* methods. A classical review and an overview on dynamic routing are in [17, 18, 61] where the advantages of dynamic routing methods concerning network performance and cost are clearly shown.

Some of the most challenging routing optimisation models may be designated as *network-wide optimisation* routing models and are characterised by the consideration of OF(s) formulated at a global network level and depending explicitly on all connection requests/flows present in the networks (a typical example is the expected total network revenue, expressed in terms of the means of all end-to-end flows). In contrast a more common type of models can be considered as *flow-oriented optimisation* models, in which the OF(s) are formulated at flow level, that is for any given node-to-node flow offered to the network (a typical example is the average delay experienced along the chosen route by the packets of a given flow).

Craveirinha et al. [65] describe a two-level hierarchical multicriteria routing model with traffic splitting for MPLS networks assuming that the required bandwidth is divided by two disjoint paths. It includes as first level OFs the sum of the "load balancing" costs of the two paths and the sum of the number of arcs of the two paths whereas the second level functions are the minimal bottleneck bandwidth and the maximal average delay of the two paths. An exact algorithm finds the non-dominated solutions of the first level OFs and the second level functions are used to "filter" one of those solutions according to acceptable bounds.

In [201] a bi-objective network-wide optimisation routing model for MPLS networks with two traffic classes (QoS and BE traffic), is proposed, using a lexicographic optimisation formulation. The problem is solved by a two-step heuristic approach based on multicommodity flow programming algorithms.

Having in mind to explore the potential advantages of a multiple objective routing principle of the type analysed in [16] and the inherent benefits of dynamic routing, Craveirinha et al. [63] propose and describe the essential features of a multiple objective dynamic routing method (designated as MODR) of periodic type where the selected node-to-node routes for all traffic flows change periodically as a function of estimates of certain network QoS related parameters, obtained from measurements in the network. In its initial formulation, for circuit-switched networks, it also uses a principle of alternative routing, that is any call of traffic flow f from node i to node j may attempt the routes (corresponding to loopless paths from i to j in the network graph): $r^1(f), r^2(f), \cdots, r^{\mathcal{O}}(f)$, in this order. The first of these paths with at least one free capacity unit (usually designated as channel or 'circuit', corresponding to the minimal arc capacity necessary to carry a call of flow f) in every arc and satisfying other possible requirements of the routing method, is the one which will be used by the call. If none of those $\mathcal{O}$ routes satisfies this condition, the call is lost, and the associated probability is designated as the (marginal) blocking probability for flow f or call congestion. The traffic flows were modelled as independent Poisson processes. In alternative dynamic routing methods the ordered route sets that may be used by calls of any traffic flow may vary in time in order to adapt the routing patterns to network conditions so that the 'best' possible network performance is obtained, under certain criterion (or criteria). In general these methods, when correctly designed, are the most efficient routing methods that may be used in this type of networks. MODR uses two metrics for path calculation purposes: blocking probability and implied costs, which define a specific form of a bi-objective shortest path problem. The implied cost of a link is an exact measure of the impact associated with the acceptance of a call in that link. This important mathematical concept was initially proposed by Kelly [139], for modelling routing problems in loss networks (that is in networks where calls are subject to a non-null blocking probability). It can be defined as the expected value (taking into account the revenue associated with the carried calls) of the increase in calls lost on all routes of all traffic flows which use a certain link, resulting from the acceptance of a call in that link. The method in [63] uses $\mathscr{O} = 2$: the first attempted route ($r^1(f)$) is the direct arc from i to j whenever it exists; the second choice route [alternative route, $r^2(f)$] has a maximum number D of links and is obtained from a modified version of the algorithmic approach in [16]. This new version of the algorithm (designated as MMRA—Modified Multiobjective Routing Algorithm), adapted to MODR, enables to select non-dominated paths, in the higher priority regions of the OF space. The priority region boundaries associated with soft constraints (required and acceptable values of the two metrics) are calculated as a function of periodic updates of the cost coefficients. In this model, in some situations, dominated solutions calculated in the first priority region(s) may be interesting for selection in some practical situations, leading to a change in the original procedure (for details see [63]). Examples in [63, 190], illustrative of the application of this bi-objective model to a fully-meshed circuit-switched network with telephone type traffic, show that path implied cost and blocking probability may be conflicting objectives in many practical network conditions, especially in cases of global or local traffic overload.

In [190] it is put in evidence an instability problem in the path calculation model presented in the previous paper [63] when that model is used directly to obtain the set of routes for every node-to-node traffic flow, in the context of a network-wide optimisation model. This instability is expressed by the fact that the paths calculated by the algorithm MMRA for all traffic flows, in each path updating period, tend to oscillate among a few sets of solutions. A preliminary analytical model showed that solution sets may be obtained by MMRA which lead to poor network performance from the point of view of two global network performance criteria: network mean blocking probability B_m (that is the mean blocking probability for a call offered to the network) and maximal node-to-node blocking probability, B_M. It is also shown experimentally that the minimisation of the implied cost of the paths ($z^{(1)}$) tends to minimise B_m while the minimisation of the blocking probabilities of the paths ($z^{(2)}$) tends to minimise B_M. That instability problem is a new "bi-objective" case of a known instability in single objective adaptive routing models, of particular relevance in packet-switched data networks (see e.g. [33]). To overcome this instability problem, associated with the great complexity of the routing model, the main requirement of a heuristic procedure is the capability of selecting "good" compromise solutions (set of routes for all traffic flows in every path updating period), from the point of view of the two mentioned global network performance criteria. Note that even a single objective formulation of the adaptive alternative routing problem is NP-complete in the strong sense (also in the degenerated case where $\mathscr{O} = 1$, i.e. no alternative route is provided), which is an indication of computational intractability even for near-optimal solutions. Martins [189] present a complete analytical model for the network routing problem in [190], enabling to make it explicit the mentioned instability problem and calculate, through the resolution of a system of non-linear teletraffic equations, the two global network performance values, for given traffic intensities and link capacities. This leads to a bi-objective dynamic alternative routing problem, formulated at the network level. A heuristic for resolving this problem was developed in the report [189], enabling to obtain good compromise solutions with respect to B_m and B_M, at every path updating period (heuristic for synchronous path selection), hence overcoming the mentioned instability problem. To show the effectiveness of the proposed approach, results from the MODR method (using this heuristic) are compared, for some test networks, with a reference dynamic routing method (RTNR or Real-Time Network Routing, developed by AT&T—see [17]), by recurring to a discrete event simulation platform.

Martins et al. [191] describe a heuristic based on a bi-objective shortest path model, for solving the multiobjective network-wide optimisation model in [190], and compare the resulting network performance with reference single objective dynamic routing methods. In [192] an extension of the previous model to multiservice networks (equivalent to multirate loss traffic networks) involving the specification of a bi-level hierarchical multiobjective routing optimisation model, is presented. This model includes OFs defined at network and service levels (including fairness objectives) and the performance of the developed heuristic is again compared with reference single objective dynamic routing, seeking to put in evidence potential advantages of multicriteria approaches in this area. A systematic analysis of the complexity and uncertainty issues involved in the multiobjective routing model is in [191] and the way they were dealt with in the proposed heuristic solution method, is presented in [64]. A modified and simplified version of the heuristic dynamic routing method for multiservice networks in [192]—very demanding in computational terms—is presented in [193], aimed at solving the same complex multiobjective network-wide optimisation model with much less computational resources (and similar performance) in the context of carrier IP/MPLS networks.

A different type of multiobjective network-wide optimisation model in terms of the nature of the used OFs is in [145] that proposes a routing model for multiservice networks with three OFs related to path cost, bandwidth utilisation in the arcs and a target arc utilisation, expressed in terms of bandwidth. The model is solved by an evolutionary algorithm. In [229] a bi-objective routing model for private circuit routing in the Internet, where the OFs are the packet delay and a traffic load balancing function, is proposed. The author in [90] presents a three-objective routing optimisation model for MPLS networks, considering multipath routing (corresponding to bandwidth splitting for each offered flow), proposing a mixed-integer formulation. Two of the OFs are analogous to the ones of the previous article and the third one aims at minimising the total number of used LSPs. The resolution approach uses an evolutionary algorithm and the results are compared with the results of another resolution procedure. Related papers, focusing on the same type of routing model are [91, 92]. In the paper [211] a genetic algorithm approach for dealing with multiobjective routing problems of generic type as well as a number of application results, are presented.

Another multiobjective network-wide optimisation routing model for MPLS networks considering multiple QoS traffic classes is proposed in [263]. This model is focused on the optimisation of admission control and routing performance and uses an auxiliary queueing model for estimating the average packet delay in the model. A lexicographic optimisation approach is used.

A discussion of key methodological issues raised by multiobjective routing models in MPLS networks is put forward in [66]. This reference also presents a proposal of a hierarchical multiobjective network-wide routing optimisation framework for networks with multiple service classes, including auxiliary approximate stochastic models for representing the traffic flows.

Girão-Silva et al. [107] describe a hierarchical multiobjective routing model in MPLS networks with two service classes, namely QoS and BE services. A bi-level network-wide optimisation model with fairness objectives for the different service classes is presented (in the framework of the approach in [66]) and a heuristic resolution method is proposed and tested in a reference network. The theoretical foundations of a resolution approach for this type of model, based on the use of a bicriteria shortest path sub-model using implied costs and blocking probabilities, are described in [67]. This is achieved through the proposal of a definition of 'marginal implied costs' in two-class service multirate loss networks, by extending earlier work on implied costs in [139, 200].

A meta-heuristic resolution approach for this model, namely a simulated annealing procedure and a tabu search procedure, are developed and tested in [108]. Girão-Silva et al. [109] describe a specialised heuristic based on a Pareto archive for solving that very complex routing model [107] and tests its performance by comparison with the previous heuristic approach. Also a dynamic version of the routing model is considered and its network performance tested via a discrete event stochastic simulator.

A multiple objective routing model for a stochastic network representing a large processing facility is approached in [141]. The nodes of the network correspond to finite capacity queues of different types (e.g. $M/G/1/m$, $GI/G/1/m$). The possibility of reattempts is considered and the arrival processes from the source nodes are renewal processes. The functions to be optimised are the average sojourn times for all customer types and the total routing costs and are often conflicting objectives. It should be noted, as mentioned by the authors, that this type of model, although having originally a formulation for manufacturing facilities, could be adapted to telecommunication networks, namely packed switched networks.

The proposed mathematical formulation is a multiple objective multicommodity integer programming problem with constraints. A heuristic is developed for solving the problem, based on the calculation of k-shortest paths, enabling to find an approximation to the non-dominated solution set.

The papers [128, 243] present a bi-objective routing model for MPLS networks that uses a lexicographic type formulation, considering as OFs the arc utilisation and the number of arcs per path. A two-step heuristic procedure based on a multicommodity flow approach is used for solving this problem.

A specific new routing problem in MPLS networks concerning "book ahead guaranteed services" (or BAG in short), modelled as a multicriteria decision problem, is approached in [260]. This problem is focused on the calculation ahead of time (with respect to the instant of generation of the actual call) of two paths, at the request of a user, with certain QoS guarantees. For example, the user may request the network administrator through a web-page sign-up of his/her access, at a future time, of the use of a supercomputer, with bandwidth and survivability guarantees in the event of failures. A pair of arc-disjoint paths (the first for the actual connection and the second to be activated in the event of failures) satisfying certain bandwidth constraints has to be calculated. The considered objectives are: to maximise the residual capacity in the network for other types of services (designated as "best effort services", such as e-mail or www), to minimise the routing costs of the BAG traffic, to minimise a penalty associated with the rejection of BAG service requests, and to maximise the revenue from accepted BAG demands. The proposed problem resolution is based on the aggregation of the four OFs and uses a heuristic to solve the resulting integer-linear programming problem.

A multiple objective routing model for B-ISDN (based on ATM), using a fuzzy optimisation approach, was presented by Aboelela and Douligeris [1]. The fuzzy programming model is focused on maximising the minimum membership function of all traffic class delays (corresponding to different service types) and the minimum membership function of the link utilisation factor of all network links. The efficiency and applicability of the approach are studied, under different network load conditions, by calculating several performance measures and comparing their values with the ones obtained from single objective models. The author discusses and recommends a hybrid resolution approach that combines the "generalised network model" that has been successfully applied to large zero-one integer programming problems [111] with the fuzzy programming technique.

The papers [174–176] deal with multicriteria routing models using heuristic approaches based on the concept of learning automata in fuzzy environments and present experimental results in communication networks by considering a model with two criteria concerning quality and price.

Anandalingam and Nam [6] propose a game theoretic approach to deal with a dynamic alternative routing problem in international circuit-switched networks, considering the cooperative and non-cooperative cases. In the non-cooperative case it is assumed that each player (corresponding to a given country involved in the network routing design) selects a routing strategy which optimises his/her payoff given the strategies chosen by the others and he/she equally assumes that the other players will attempt to use strategies which optimise their payoffs, where the payoff objectives of each player are expressed in terms of the minimisation of the cost of adding more links (with the required capacities) in his own part of the global network. This problem is modelled as a bi-level integer linear programming problem characterised by a player who works as “leader” and makes the initial decision (by minimising his/her own cost function) and then the other players, or “followers”, seek to minimise their own cost function given the leader’s decision. The leader has to pay a certain fraction of the link costs of a part of the network, jointly owned. Several application examples, where approximate solutions to the model are obtained from the branch-and-bound algorithm by Bard and Moore [28], are discussed. The major conclusions stress the great cost savings in global networks (an example is presented for a network interconnecting US, Japan and Hong-Kong), for all the involved players, obtained from the dynamic routing solutions, both in the cooperation and in the non-cooperation cases; this is a result of the distribution of the peak traffic loads of one country by the idle parts of the routes in other countries by making the most of the country different times.

An important type of routing problems in transport networks concerns routing in WDM optical networks. This type of problem, that may have multiple formulations, is usually designated in its most general form as the route and wavelength assignment problem (RWA in short). RWA refers to a type of routing problem that has become very important in optical networks, especially with the emergence of OXCs (Optical Cross-Connects), and is focused on the calculation of lightpaths (fixed bandwidth connection between two nodes via a succession of optical fibres). It can be decomposed in two inter-related sub-problems. Given an optical network, the arcs of which correspond to bundles of optical fibres each one with a number of available wavelengths, and the demand for node-to-node optical connections, the first sub-problem, or ‘routing problem’, involves the determination of the path (topological path) along which the connection should be established; the second sub-problem involves the assignment of wavelengths for every connection, on each arc of the selected path. RWA has multiple formulations depending on the nature of the traffic offered (optical connections), objectives (for example: to maximise the number of established connections for a fixed number of available wavelengths or to minimise the number of required wavelengths for a given set of requests) and technical constraints. An overview of basic concepts and formulations in this area of routing can be seen in [19]. A review of approaches for solving more common formulations of the RWA problem was presented in [286].

A multiobjective model for routing and provisioning in WDM networks is presented in [140]. The model considers a fixed budget, and a primary objective is the minimisation of a regret function concerning the amount of over and underprovisioning related to the uncertainty in the demand forecast. A secondary objective is the minimisation of the equipment cost. The proposed resolution approach of this lexicographic type formulation is a two-phase heuristic using mixed-integer linear programs.

Hua et al. [130] address the RWA problem considering multiple network optimisation objectives, namely profit and path length.

Huiban and Mateus [131] describe a model of virtual topology design and routing in optical WDM networks in terms of a multiobjective mixed-integer linear programming. The optimisation criteria are the number of used wavelengths and the maximum link load of lightpaths, and the resolution uses an ϵ-constraint method.

Crichigno et al. [71] develop a multiobjective routing model for WDM networks considering a multiobjective mixed-integer linear programming formulation. The optimisation criteria are the aggregated throughput, the resource consumption and the MLU, and the resolution procedure is an ϵ-constraint method. The network performance results are compared with those from single objective formulations. The same routing model is analysed in [72, 73].

A multiple-objective approach, based on genetic algorithms, is proposed in the report [289] for dealing with a specific routing problem in WDM optical networks. The problem is a particular version of the RWA problem and is modelled as a three objective integer linear programming problem and the resolution approach is a genetic algorithm using a Pareto ranking technique.

Leesutthipornchai et al. [163] describe a multiobjective RWA problem in WDM networks with static RWA where the OFs are the number of accepted communication requests and the number of required wavelength channels. It uses a genetic algorithm for solution calculation and compares the results with those from classical methods. The same type of problem was also tackled in [164] with a particular version of the SPEA.

Markovic and Acimovic-Raspopovic [186] address a routing problem in WDM networks modelled as a multicriteria shortest path problem where the criteria are the number of links, the number of free wavelengths and the blocking probability. Solutions are obtained with a shortest path algorithm applied to the weighted sum of the criteria.

To finalise this section we review in some detail a recent paper made by our research group, as an example of the potentialities of new multicriteria decision aiding approaches to deal with new challenges in telecommunication routing problems. The chosen paper is [117].

The routing problem in WDM networks involves multiple objectives and constraints. In this paper we propose a bicriteria routing model associated with the dynamic lightpath establishment (DLE) problem with incremental traffic in a WDM network. The model is intended for possible application in large WDM networks, with multiple wavelengths per fibre and multifibres per link. In order to enhance the range of application of the model, various types of nodes, with complete wavelength conversion capability, limited range conversion, or no wavelength capability, were considered. Furthermore, a mixture of bidirectional symmetric and unidirectional optical connections, the latter being in small percentage, characteristics often found in real optical networks, were considered.

Another important feature of the model is that the solution to the bicriteria routing optimisation problem should be calculated and selected in a short time and in an automated manner. Having in mind these factors and especially the incremental nature of the traffic offered, a flow-oriented optimisation formulation for the topological lightpath establishment (TLE) bicriteria problem was considered, that is, the bicriteria routing problem is formulated for each node-to-node connection request at a time and the wavelength assignment problem is solved separately, using an heuristic, after the TLE problem.

The candidate solutions to the topological RWA bicriteria model are topological paths which are non-dominated solutions to the following problem:

$$\begin{cases} \min_{p \in D_T} c(p) \\ \min_{p \in D_T} h(p) \end{cases}$$

where the set of admissible solutions, D_T, is composed of all the topological paths from the source to the destination node which correspond to viable lightpaths, that is, lightpaths with the same arcs as p and with a free and usable wavelength in every arc. The topological paths in these conditions (elements of D_T) are designated as viable topological paths, for the given origin-destination nodes. For obtaining D_T, the free wavelengths in each arc will have to be identified first taking into account the wavelength conservation specified capabilities, and then the set of viable paths for each pair of origin-destination nodes becomes implicitly defined.

The first objective function, $c(p)$ is related to the bandwidth usage in the links of the path p and is expressed in the inverse of the available bandwidth in the links:

$$c(p) = \sum_{l \in p} \frac{1}{b_l^T}, \quad p \in D$$

where D is the set of topological paths from the source to the destination node and b_l^T is the total available capacity in link l, in terms of available wavelengths. This criterion seeks a balanced distribution of traffic throughout the network, hence favouring the increase in the total traffic carried and in the associated expected revenue.

The second objective function is simply the number of arcs of the path, $h(p)$. The minimisation of $h(p)$ seeks to prevent the use of an excessive number of fibres in a connection, hence favouring global efficiency in the use of network resources as well as the reliability of optical connections.

The proposed resolution problem approach combines the use of a k-shortest path algorithm with preference thresholds identifying preference regions in the OFs space, in order to identify "good" non-dominated solutions. The final choice solution, if several non-dominated solutions belonging to the same preference region were calculated, is chosen automatically by minimising the Chebyshev distance to a reference point (which changes with the preference region under analysis), as proposed in [57].

Having obtained a non-dominated topological path, a heuristic procedure based on a path specific wavelength bottleneck bandwidth is then used to assign wavelengths to be used along the chosen topological lightpath.

The performance of the bicriteria model was analysed by comparing it with the resolution using separately the two single criterion approaches corresponding to each of the criteria used in the bicriteria approach (BC). The BC approach resulted in lower global blocking and less bandwidth than the solution using just the first criterion. The solution using just the second criterion uses less bandwidth than the BC approach because it leads to a significant lower number of successful connections. Although the BC approach uses more CPU time per request, its performance was nevertheless quite good, especially in the case of the more sparse network used in the experiences.

An extensive experimental study [247], using reference optical networks, analyses the network performance of the model in [117], by considering several important network performance metrics. The results of the bicriteria model are compared with the results from the associated single criterion optimisation models, in terms of network performance metrics. Gomes et al. [116] describe the extension of the model in [117] to protection routing in WDM networks, where two topological disjoint lightpaths are selected simultaneously (the active path and the protection path) to guarantee the continuity of the end-to-end optical connection in the event of failure in the active path. The network performance of the protection routing model in [116] was extensively tested and compared with single criterion models in [248] by performing a study similar to the one in [247].

26.4.2 Network Planning and Design

Telecommunication networks have been subject to continuing and extremely rapid technical innovations and to permanently evolving modes of communication. Also, in parallel, there is a significant increase in the demand for new services. It becomes more and more attractive for the telecommunication operating companies to offer the customers new ranges of new services, having in mind to take economic advantages of the new technology platforms and to respond to the customers' needs.

Operational planning designates a wide area of planning activities focused on the short term network design such as location, interconnection and dimensioning of transmission equipments and other facilities such as switching units, routers or traffic concentrators. In specific problems of this type there have been proposals of multicriteria modelling. Next we review papers in these areas.

In general, most network planning models try to express different aspects of the associated complex optimisation problems in currency units in order to encompass them in a unique economic OF. These telecommunication network planning models lack to capture explicitly the different and conflicting aspects arising in evaluating network modernisation policies. In fact, these problems are multidimensional in nature. Multicriteria models, taking explicitly into account economic, technological and social aspects (many times incommensurable) enable the DMs to grasp the conflicting nature of the objectives and the compromises to be made in order to select a satisfactory solution.

In particular, strategic planning is focused on the development and evaluation of scenarios of qualitative and quantitative network growth over a medium/long term period having in mind traffic increase, introduction of new technologies and services and the company economic objectives. This is a type of problem which involves a multiplicity of factors, some of which cannot be directly represented by an economic indicator. This particular area of network planning practically disappeared from the literature in more recent years because of the extremely fast pace of technological evolution on the one hand and the liberalisation in all areas of telecommunication operation, service provisioning and maintenance, on the other hand. These factors led to an environment where classic strategic planning tools don't have a role to play in practice except in the few cases where national or regional monopoly operators persist.

In the present socio-economic context the new telecommunication and information technologies are of paramount importance. Several trends are evident in recent rapid changes in telecommunication networks and services, which may be enlightened in terms of functional types of networks, in terms of the services offered and the underlying basic technologies. The evolution and growth of these networks and services pose difficult problems of forecasting planning and decision making. This stems from technological factors (namely the possibility of using alternative technologies for certain types of services and the difficulties in terms of standardisation) and socio-economic factors (the difficulty in foreseeing the associated economic constraints and potential benefits). In addition, the development of these networks gives rise to a variety of options and conflicts involving the government and the operators' policies. For example, policy makers must decide whether (and up to which extent) the potential economic and social benefits associated with these new networks justify the public support to their extensive capital costs. It is also clear that telecommunications, both at national and international levels have important impacts regarding the economic growth, the apparent reduction of geographical distances, social welfare and political options.

In [10, 13], the authors propose a multicriteria linear programming approach dedicated to the evaluation of the modernisation planning of telecommunication networks. These papers address an important strategic modernisation problem: the planning of the evolution of subscribers' lines in terms of classes of service offerings and basic technologies. An extension of this model was done in [12]. It concerns the possibility of evaluating the modernisation plans in terms of particular regional environments.

The original model is based on a state transition diagram the nodes of which characterise a subscriber line in terms of service offerings and supporting technologies, considering both the transition of lines to a more sophisticated state and the installation of new lines directly in any state. Five cash flows are defined concerning: (1) capital costs; (2) salvage value after dismantling a line; (3) annual operational and maintenance charges; (4) annual revenue of a line at a given year; (5) final value of a line at the end of the planning period. From these cash-flows an OF (to be maximised) quantifying the NPV (net present value) of network modernisation is defined. An external dependence function (to be minimised) quantifies the imported components associated with the investment costs and operational and maintenance charges. A "quality of service" function is defined in this model as the "degree of modernisation" associated with the "desirability" of new services and corresponds to the third OF (to be maximised). Finally, the model considers four main categories of constraints: upper bound on the cost and charges, degree of current satisfaction of the estimated demand, degree of penetration of the supporting technologies and continuity (line conservation) constraints. The policy of the telecommunication operator may also be reflected in the model through the inclusion of techno-economic constraints imposing upper bounds on the number of new lines of each technology to be installed at each year of the planning period.

Examples of application of this model using various sets of data may also be seen in [10, 13]. An extension of this model incorporating sensitivity analysis enabling to deal (in a systematic manner) with the inaccuracy of the OF coefficients, is in [14].

It must be stressed that since this was an outline seminal work in multiple objective modelling of strategic modernisation planning of telecommunication networks, the analysed model is naturally incomplete, subject to updates and modifications and its practical utilisation would certainly require additional information from telecommunication operators and major network equipment suppliers. This information—which we think is difficult to gather and has a high degree of uncertainty, having in mind the very rapid changes in technical, economic and social factors—would enable to tackle new challenges and opportunities associated with concrete scenarios of network evolution as perceived by network planners and managers. In fact, by modifying the state transition diagram (namely through the consideration of new nodes and arcs) or by including new objectives and/or constraints, or changing those in the model, other aspects, which might require consideration by the DMs, may be easily incorporated in the model without jeopardising its basic philosophy. So, this multicriteria model is sufficiently flexible, namely enabling to incorporate new evaluating criteria, which might become important in the assessment of network modernisation strategies in new contexts.

In [10] the interactive MA is based on the TRIMAP approach [49]. TRIMAP is an interactive calculation tool the aim of which is to aid the DM in the progressive and selective learning of the set of non-dominated solutions. It combines three main components: decomposition of the weighting space, introduction of constraints on the OF space and introduction of constraints on the weighting space. One important innovative feature of TRIMAP is that it enables the introduction of additional constraints on the OF values to be translated into the weighting space. The weighting space is used in TRIMAP mainly as a valuable means for collecting and presenting the information. In TRIMAP phases of computation alternate with phases of dialogue with the DM, this mainly in terms of the OF values, allowing a progressive and selective learning of the non-dominated solutions. In each computation phase a scalar problem consisting of a weighted sum of the OFs is solved with the main purpose of performing a progressive filling of the weighting space. In each step the DM will be called to decide whether or not the study of solutions corresponding to not yet searched regions of the weighting space is of interest. In this way it is intended to prevent the exhaustive search in regions with close OF values, situation found very often in real case studies. The underlying principle is to narrow progressively the scope of the search, using the knowledge accumulated in the previous interactions. The interactive process only ends when the DM considers to have gathered “sufficient knowledge” about the set of non-dominated solutions, which enables him/her to make a decision. This method uses an interface that offers the DM a flexible and user-friendly Human-computer interaction the use of which is easy and intuitive and enhances his/her capabilities of information processing and decision making.

The experience of the authors of Antunes et al. [10] with implementations and applications of different interactive multicriteria linear programming methods led to the conclusion that there is no simple method better than all the others in all circumstances [49]. This methodological posture led to the development of a flexible integrated computer package [11]: a method base, which seeks to take advantage of the combination of different types of interactive multicriteria linear programming methods. The basic principle of this integrated model is “to support interactively the DM in the progressive narrowing of the scope of the search, using the knowledge accumulated in the previous interactions. As more knowledge about the problem is gathered in each interaction, the preference system of the DM progressively evolves, thus making the DM to reflect upon his previously stated indications, or even to revise his preferences” [11, p. 343]. It is assumed that, in the process, the DM, beyond gathering knowledge, will gain new insights into the problem under analysis, which may be used for specifying new preferences and search directions. The method base main goal is therefore to support the DM in the task of exploring the problem and expressing his/her preferences by making the DM able, at each step, to reinforce or weaken his/her current convictions. The DM is considered a central and active element of this method base: the stopping criterion is the DM’s “satisfaction” and not the verification of a convergence condition on any implicit utility function. The main purpose was to create a flexible decision aid tool able to respect the underlying characteristics of the methods and facilitate their combination by guaranteeing a consistent transfer of usable information. This computer package is called TOMMIX [11] and integrates the STEM method, the Zionts-Wallenius method, the TRIMAP, the Interval Criterion Weights method, and the Pareto Race. In [13] the application of this package to the problem of modernisation planning of telecommunication networks, introduced above, is exemplified and discussed.

In [14], the flexibility of the proposed approach is enlarged by showing the way in which sensitivity analysis can be associated with the model. Interactive sensitivity analysis techniques concerning changes in the coefficients of the three OFs and the right hand side of the constraints, as well as the possibility of introducing new constraints, are proposed and discussed.

Finally, it must be mentioned the extension of TOMMIX to more than three OFs, leading to the development of SOMMIX [54]. This package can be of great interest in telecommunication strategic planning in those cases where the explicit consideration of more than three OFs is advisable.

Later, in [15], the authors extended the type of analysis mentioned above to other strategic telecommunication planning problems, namely regarding the evolution paths towards the deployment of technologies capable of providing broadband services in a residential and small business setting.

The emergence of new services based on broadband access technologies is recognised as an essential driver to generate additional revenues and support a long-term growth and financial strength of operators. Several factors are at stake, with many inter-related influences, such as the rapid pace of technical innovations, the development of multifaceted modes of communication and the changing market structures (even in local access networks). Therefore, the model described above has been extended as an attempt to exploit new avenues for studying the evolution policies towards broadband services [15]. The OFs considered in the extended model are: (1) the minimisation of the NPV of the total evolution cost; (2) the maximisation of the near-term service capability; (3) the maximisation of the compatibility with the embedded base of subscriber's equipment. Three main categories of constraints have been considered: (1) upper bound on cost and charges; (2) degree of satisfaction of the estimated demand; (3) degree of penetration of the supporting technologies.

As it is said in [15], the proposed approach required a great effort of data collection regarding the construction of the coefficients in the OFs and constraints. Hence the reliability of the analysis results is clearly questionable taking into account all types of uncertainties and imprecisions associated with estimates of the demand for services, investment, operational and maintenance cost and so on, as previously mentioned.

The study of approaches and methods suitable for tackling the inherent uncertainty and imprecision of the input information required by this and other types of planning models, such as interval programming, stochastic programming and fuzzy programming approaches, is a quite relevant research issue. A certainly difficult, but decisive question, is trying to identify which approaches are more suitable for a specific model, dedicated to a particular problem, in a given decision environment. Naturally these questions and challenges are common to most of the problematic areas discussed in this study.

In any case we think the discussed multiple objective mathematical programming approach is of interest to grasp certain compromises to be made and discover trends in this type of problem, which can be helpful to network operators to make decisions concerning the upgrade and expansion of access networks. The experiments displayed in the study [15], were carried out in the framework of an outline study more concerned with showing the usefulness of the multiple objective model rather than putting forward "prescriptive" conclusions. More experimentation with updated and more accurate data would be required, in particular involving sensitivity and robustness analysis on the model parameters and assumptions. Furthermore, in many cases, this type of studies could be complemented, at a lower level of analysis, with the screening of distinct alternatives to aid in making some "intermediate" decisions. Again multiple evaluation aspects are at stake. A possible approach to be developed would be to consider an impact matrix stating the level of performance of each potential course of action in terms of the evaluation criteria considered in this context, leading to a discrete alternative multiattribute decision model. This could be tackled by using several methods proposed in the scientific literature. An example of such approach is the possible consideration of the choice between Hybrid Fibre Coaxial (HFC) or Fibre To The Curb (FTTC) architectures using as evaluation criteria (among other significant possibilities): support for full service installation strategy, installed first cost, operations savings, fit to embedded plant, and evolutionary potential.

In the paper [25], the authors deal with a real world multicriteria decision aiding problem regarding the strategic study of the expansion of cellular telephony systems. The original problem concerns the determination of the municipalities of a Brasilian State in which a given mobile operator should expand its network. Economic factors (including budget limitations, costs, return of investments) as well as a significant number of technical factors (such as ease of installation and QoS parameters) are considered in the model attributes or criteria.

The authors pay particular attention to the phase of structuring the problem [234], i.e. the identification of the decision problem under study enabling to build a multiattribute model. Cognitive maps imported from psychology, were used in this task of organising and synthesising the points of view of the various actors. Although the integration of structuring methods with multicriteria evaluation approaches, following, for instance, the lines defended in [31], is an important practical issue it is beyond the scope of this paper. The analysis of the obtained multiattribute model is carried out using an additive value function approach to evaluate the alternatives. In order to build the criteria and to assess the scaling constants (weights), the methodology MACBETH [24] was used.

Flores and Cegla [96] present a bicriteria optimisation model for network topology design given the locations of the nodes, considering as objectives the cost of the links and the network reliability. It uses a SPEA as a metaheuristic resolution approach. The same problem is solved in [97] by a genetic algorithm "Non Dominated Sorting Genetic Algorithm" and the results compared with those obtained with the SPEA procedure.

Kumar [156] describes a bicriteria constrained network topological design problem where the OFs to be optimised are network delay and cost subject to satisfaction of reliability and flow constraints. The approach to the calculation of solutions is made through an evolutionary algorithm. A similar type of problem is addressed in [26] but using a self-similar stochastic modelling of the traffic. A MOEA solution approach is compared with a deterministic heuristic based on branch exchange.

A bicriteria model for topological, capacity and routing design of WANs seeking to minimise the total average delay per packet and the leasing cost of channels is in [114]. A branch and bound based method using bounds in one of the OFs is used to search for exact solutions. An extension of this model, considering external time-varying traffic arriving at network nodes, is described in [115].

Glaß et al. [110] develop a model for combined optimisation of topology and routing and use a MOEA for obtaining solutions to the formulated problem.

Wierzbicki [274] presents a multicriteria modelling approach for a problem concerning the placement of Internet caches. The underlying generic technical objective is to contribute to increase the network efficiency and the goal is to minimise the overall flow or the average packet delay. The problem of general cache location is formulated as a MILP (Multicriteria Integer Linear Programming Problem) and is reformulated using a reference point approach. Also the sensitivity of the model solutions to simplifications of the problem, is studied. Finally, simple greedy heuristic resolution approaches are tested for some medium size network topologies.

Setämaa-Kärkkäinen et al. [245] describe a bi-objective optimisation model in which the objective is to schedule data transmission to be as fast and as cheap as possible, in a wireless mobile network. A fast heuristic solution (having in mind the practical limitations of the application) is proposed and the results are compared with exact solutions from a weighted sum approach.

Papagianni et al. [213] discuss the application of particle swarm multiobjective optimisation heuristics to location and capacity design problems. The obtained results are compared with those from corresponding evolutionary algorithms.

Tiourine et al. [262] propose search algorithms for the problem of link frequency assignment, that has great relevance in its application to wireless networks, satellite communications, television and radio broadcast networks. The model proposed in this paper is in some sense a bi-objective combinatorial model. In fact it proposes a lexicographic sequence of two OFs. The principal objective consists in minimising interference and a secondary objective is the minimisation of the used radio spectrum. When optimising the latter objective it is assumed that a zero value of interference was obtained by solving the former optimisation problem. This study was included in CALMA (Combinatorial Algorithms for Military Applications) project, part of the long term European Cooperation Programme on Defence. Some local search approaches were considered such as tabu search, simulated annealing and variable-depth search, paying particular attention to the development of problem specific neighbourhood functions, as well as to the presentation and discussion of computational experiences.

A bicriteria optimisation model for an antenna arrangement problem where the OFs are the maximisation of the cover area and the minimisation of the cost of the antenna, with various transmission constraints is discussed in [272]. A parallel genetic algorithm is proposed and its results are compared with other genetic algorithms.

Levin and Petukhov [166, 167] address a multicriteria client assignment problem in wireless networks where the number of connected users, the reliability of the connection and three transmission quality metrics are to be maximised. A heuristic procedure is developed for finding solutions.

Brown [40] proposes the application of reinforcement learning methods to a packet wireless communication channel related problem. The addressed problem involves the search for a satisfactory power management policy considering simultaneously two criteria: trying to maximise the radio communication revenue and to decrease the battery usage. This problem is modelled as a Markov Decision Process, where the generated traffic is modelled as the traffic from an ON/OFF source and rewards are assigned to packets carried in each direction (between the mobile and the base station). Other technical elements of the communication system are also incorporated in the model in a simplified manner. This problem can be approached as a stochastic shortest path problem, introducing some simplifications that enable the reduction of the dimension and complexity of the state space.

Chan et al. [43] present a multicriteria model for transmission resource management in mobile transmission networks in which the transmission power and transmission rate control, are to be optimised. A multiobjective genetic algorithm is used and its solution is compared with those from a single optimisation model.

Charilas et al. [44] describe a multiattribute model for selection of the most efficient and suitable access networks to meet the QoS requirements in heterogeneous wireless networks. A fuzzy AHP and an ELECTRE method are used for the evaluation of alternatives.

26.4.3 Models Studying Interactions Between Telecommunication Evolution and Socio-Economic Issues

The use of multiattribute models in telecommunications planning and design, as far as we know, has been mainly proposed for application in models studying interactions between telecommunication evolution and socio-economic issues, as analysed next.

In [84], the authors show the way in which a customer can use the AHP (a survey on AHP can be seen in [143]) to choose a telecommunication company and/or particular services that are the best for satisfying his/her needs in terms of QoS or to decide between two telecommunication services providers. Raisinghani [225] studies multicriteria approaches for supporting strategic decisions on electronic commerce (e-commerce) based on AHP and ANP (Analytic Network Process). Remember that ANP is a generalisation of the AHP decision aiding methodology, where hierarchies are replaced by networks enabling the modelling of feedback loops (see [242]). The authors also discuss the possible advantages of this methodology, as a MCDA modelling approach, in the context of e-commerce.

Many more papers propose the use of AHP/ANP alone or combined with other approaches. Next we refer to some of these papers.

Andrew et al. [7] deal with the selection of communication technology for a rural telecommunication planning problem, considering uncertainty and multiple criteria. The AHP method is used. Also for rural telecommunication infrastructure technology selection [102, 103] propose the use of the ANP.

The prioritisation of a portfolio of Information and Communication Technologies (ICT) infrastructure projects is proposed in [9] by using the real options analysis together with AHP to evaluate ICT business alternatives and telecommunication investments analysis. Fialho et al. [95] propose a new level based approach, to prioritise telecommunications R&D projects, inspired in AHP principles. In [123], AHP is used in the prioritisation and selection of intellectual capital for the mobile telecommunication industry.

Giokas and Pentzaropoulos [106] propose the combined use of AHP and Data Envelopment Analysis (DEA) in order to rank the Organisation for Economic

Co-operation and Development (OECD) member states in the area of telecommunications.

Isiklar and Buyukozkan [133] evaluate mobile phone alternatives. AHP is applied to determine the relative weights of evaluation criteria and an extension of the Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) is applied to rank the mobile phone alternatives. Khademi-Zare et al. [142] also propose a combined use of TOPSIS and AHP. In this case, Fuzzy-QFD (Quality Function Deployment), TOPSIS and Fuzzy-QFD AHP are used. The model seeks to obtain a rank of strategic actions concerning mobile cellular technologies.

Kuo and Chen [157] deal with the selection of mobile value-added services based on a fuzzy synthetic evaluation and using AHP to evaluate the performance of mobile value-added services system operators.

Pramod and Banwet [221] evaluate the performance of an Indian telecommunication service supply chain using ANP analysis.

An application of AHP to socio-economic problems dealing with the vendor selection of a telecommunication system is reported in [258]. The developed model takes into account a double conflict related to multiple criteria and multiple DMs. The authors emphasise the feasibility of this application of AHP and its potential capability to reduce the time taken to select a vendor.

Thizy et al. [261] study how to support the decisions concerning investment in capital intensive telecommunication projects. A decision support system (DSS) combining a mathematical programming approach and AHP is proposed. The AHP is used for quantification of qualitative managerial judgement in regard to the relative value of projects through a two stage process.

Fuzzy versions of the AHP method are widely used: [44] deal with the selection of the most efficient and suitable access wireless network to meet the QoS requirements, using a fuzzy version of AHP and ELECTRE methods; [85] propose the use of a fuzzy version of the AHP for the evaluation of service quality on WAP (Wireless Application Protocol) service in wireless networks; [132] use a fuzzy AHP approach looking for "Always Best Connected" (ABC) users dedicated to WLAN (Wireless Local Area Network) and cellular networks; in [282] a fuzzy AHP methodology is used to evaluate four 3G licensing policies in Taiwan. In [101], PATTERN (Planning Assistance Through Technical Evaluation of Relevance Number) and fuzzy AHP are used in the context of Taiwan's virtual mobiles operators service planning.

On the other hand, different multiattribute approaches and some mathematical programming decision support tools have also been used in some applications. Namely: [79] study risk extreme events in investment plans for expansion of a telecommunications network using multiobjective decision trees; [104] propose a multicriteria approach that integrates technical concerns with perceptual considerations in the construction of tailor-made multimedia communication protocols; [105] propose a performance evaluation model for aiding a project manager of a telecommunication system operator in Brazil. It is a multicriteria constructivist decision support tool dedicated to the approval of outsourced service providers.

Keeney [137] discusses the issues concerning how to build a value model in the context of decision processes in telecommunication management. Special attention is paid to the identification and structuring of objectives both in qualitative and quantitative terms, including the use of utility functions.

Kyrylov and Bonanni [158] develop a strategy simulation game for analysing the telecommunications industry, using the agent-based technology. Also by using a simulation game, the same authors [159] propose a multicriteria optimisation of budgeting decisions by telecommunication service providers.

Lee et al. [161] develop a specific multicriteria decision support mathematical programming model for dealing with the definition of a "hub-structure", that is the selection of a number of "nucleus cities", in the context of a rural network planning process and present an application example for the State of Nebraska. The approach is a "compromise programming" technique [281, 287].

The aim of the study [179] is a DEA-based comparison of two hybrid multicriteria decision making (MCDM) approaches to evaluate the mobile phone options with respect to user's preferences.

Ondrus et al. [209] develop a simple multiactor, multicriteria approach to support mobile technology selection. Several multicriteria methods are considered, and the proposed approach is exemplified using the method ELECTRE I.

Sylla et al. [254] present a hybrid method for the multicriteria evaluation and selection of network technologies. It combines a web of system performance—WOSP (a cross-disciplinary information network systems performance tool based on general systems theory [32]) with a quantitative evaluation and selection multiattribute approach—QESM, based on the definition of problem requirements and the degree to which the alternatives meet those requirements. The method aims to choose among alternative ATM technologies.

Wojewnik and Szapiro [277] propose a new bi-reference procedure for interactive multicriteria optimisation with fuzzy coefficients and present an application on pricing for telecommunication services.

26.5 Future Trends

Now we will seek to give an outline of possible research trends in some areas of network planning and design, where challenges and opportunities for MA may arise. For simplifying this presentation, of a prospective nature, we will take as basis, application areas (or sub-areas) identified in the previous section, although one must be aware that new problematic areas are likely to emerge where MA may play a significant role in relation with some decision problems. The trends concerning the areas of network planning and design and models studying interactions between telecommunication evolution and socio-economic issues were aggregated in a common topic having in mind their strong interrelation.

26.5.1 Routing Models

From an OR methodological perspective major issues/challenges concerning routing models may be summarised in the next paragraphs in relation to the scope of our chapter. We may consider that 'QoS routing', viewed in its broadest sense as explained after, is the dominant paradigm concerning the development of routing models in IP-based networks. We would like to note here that, from our perspective, 'QoS routing' in its broadest sense includes not only the standard QoS routing formulations—as described in [152, 153, 160] and analysed from a teletraffic engineering or protocol implementation perspective in many RFCs (Request for Comments) reports and articles—but also all routing models that are explicitly multicriteria, as discussed in Sect. 26.4.1.

Firstly there is the need for investigating new implementations of exact algorithms for various MCOP problems in the context of QoS routing models having in mind to obtain better trade-offs in terms of exactness of the solution/computational efficiency for a given application model. Note that this issue is also very relevant in multicriteria shortest path based routing models the resolution of which involves shortest path or k-shortest path calculations. This is particularly important in cases in which there is no feasible optimal solution and the algorithm takes excessive time to detect such condition or if the memory requirements are a practical constraint. This is normally the case for networks of large dimension/connectivity and this type of limitations is critically related to the so called 'scalability' of the OR-based routing model, a concern usually found when we discuss a protocol implementation associated with a given algorithm. Concerning the complexity of exact algorithms, although classical NP-completeness analysis is naturally important, it must be stressed that this is a worst-case analysis and in some cases it may not be the key factor for choosing an algorithm in a certain application environment. As noted in [153], worst-case complexity and execution time can be quite different in different application environments and this is particularly relevant in classical QoS routing algorithms as well as in many multicriteria routing methods. An analysis of network features which lead to worst-case conditions in QoS routing procedures is addressed in [265]. Regarding the application of exact single criterion QoS routing algorithms, this work and also [266] conclude that worst-case conditions are very unlikely to occur. We would like to note that this conclusion agrees with our own experience in the development of exact resolution procedures, based on the utilisation of the extremely efficient (in practice in almost all the application experiments we considered) k-shortest path algorithm [188] in the context of multicriteria shortest path models. An example of this may be seen in [50].

Concerning the choice of metrics/cost functions adequate to a given application and other modelling aspects, this is an issue which requires a truly interdisciplinary work combining the analysis of traffic engineering and OR factors. In terms of OR this concern has obvious implications in the optimisation problem formulation, its complexity and the resolution approach.

The treatment of fairness issues also becomes increasingly relevant, namely when offered traffic flows with quite different intensities, different required bandwidths and/or belonging to different QoS classes of service compete for the network resources. In particular the optimisation of performance metrics at global network level or for QoS flows (i.e. those which have QoS guarantees) may deteriorate excessively the QoS of small intensity "best effort" flows. Examples of OR based approaches for IP/MPLS dealing with this issue were mentioned in Sect. 26.4.1: for instance [218] (considering max-min fairness principles) and [66, 207, 208], in the context of a multiobjective hierarchical optimisation model combined with max-min fairness at global network level and service class level.

The treatment of inaccuracy and uncertainty is an important and often difficult issue in the context of a routing model since these aspects may have a strong impact on the values of network performance metrics. Except in very simple static models (for example models based on the number of arcs alone) the arc costs usually reflect some link state dependent feature. This leads to an intrinsic inaccuracy taking into account that the related information is not, in practice, conveyed instantaneously to the nodes and/or is subject to estimation errors. Furthermore in a very common type of approach—dynamic flow-oriented optimisation models (for which routing calculation is performed for each node-to-node flow separately)—an important issue is the potential for routing instability since the successive updates of routing solutions for individual flows may originate network performance degradation, as studied in the context of packet networks in [33, Chap. 5] and also outlined in [196]. This instability phenomena also may arise in the solution generation process of certain network-wide optimisation models in which the OFs of the model depend explicitly on all network flows. Concerning uncertainty, this has to do with the intrinsic stochastic nature of the demand—and this issue should be tackled in some form in models that take explicitly into account the relation of traffic patterns in the network with the routing decisions. Examples of the way in which this issue may be tackled can be seen in the methodological frameworks for network-wide routing optimisation proposed in [66, 202]. In general we may consider that stochastic representations of traffic flows are more realistic but tend to introduce a heavy burden in terms of the numerical computation, which may lead to model intractability in networks with greater dimension.

Another relevant issue that deserves further investigation is the representation of the system of preferences, namely in automated routing procedures for which an interactive selection of solutions is not possible. Overviews with methodological discussions and case studies can be seen in [58, 66]. Also the paper [276] mentioned above analyses, in depth, this issue in the context of multicriteria routing in IP-based networks and describes an illustrative model tackling this issue by recurring to achievement functions in the context of a reference point approach.

Many types of routing methods require the calculation of several paths simultaneously, a general class of routing problems that may be designated as multipath routing problems. In particular multicast routing involves the calculation of a set of paths from an originating node to multiple destination nodes which involves in OR terms, the calculation of "minimum" (single criterion or multicriteria) Steiner trees.

Needless to say that these are research areas where many open problems, challenges and issues can be foreseen having in mind the great complexity of the associated combinatorial problems and taking into account the increasing multiplicity of new technological platforms, network architectures (two examples are MPLS over WDM and IP/MPLS over Carrier Ethernet) and service requirements.

Furthermore the development of heuristics and metaheuristics dedicated to the resolution of multicriteria routing models in IP-based networks is an area of increasing relevance and that is also expected to grow very quickly in a near future. This has to do with various factors, now briefly analysed. Firstly although many classic QoS routing problems which are NP-complete have exact resolution approaches these may easily become intractable in networks of greater dimension. Secondly, in many cases, the addition of constraints may significantly complicate the original formulations. Thirdly there are many other routing optimisation problems that are NP-hard in the strong sense for which there are no exact resolution methods with execution times compatible with the applications. This is especially relevant in dynamic routing with short routing update periods and in on-line (non real-time) routing. Finally this is an area where, in many routing models there is a confluence of one or several 'complicating factors' in the sense described by Jones et al. [135]: large number of variables (in particular in integer and mixed-integer formulations), non-linear OFs/constraints, the inclusion of stochasticity in the model formulation and non-standard utility functions, as in many multicriteria approaches (see [51]). These factors combined with the very rapid increase in computing power and the advances in metaheuristic techniques have fostered the increasing importance of these approaches in the solution of many routing models as noted in Sect. 26.4.

Regarding new application environments with great number of challenges and opportunities for the development of multicriteria routing approaches in the near future, we could point out:

- routing models for wireless and heterogeneous networks (these are networks where an end-to-end connection may use different technological solutions and has to transverse several networks or routing domains with distinct technical features);
- routing methods for overlay networks (a type of virtual transport network which interconnects a sub-set of nodes of a given underlying communication network) [270];
- routing models for Carrier Ethernet networks (a new, fast evolving, type of transport/switching networks based on high speed Ethernet frame transport which enable the implementation of advanced QoS routing schemes for unicast, multicast or broadcast connections) [228] or for MPLS—Transport Profile (MPLS-TP) networks.

We think that, from the interleaving between the aforementioned methodological aspects and the new technologic related application environments, a quite significant number of opportunities and challenges for the development of multicriteria approaches in this expanding and multifaceted area of application of OR methods and techniques, will arise.

26.5.2 Network Planning and Design and Models Studying Interactions Between Telecommunication Evolution and Socio-Economic Issues

Concerning these two quite interrelated topics the following points can be explored.

Firstly the study and development of new types of models (concerning new planning and design problems and different decision processes) and of new variants of models previously presented is a natural trend, having in mind the effects on the planning processes of the great turbulence of the socio-economic environment and the rapid market changes in interaction with an extremely fast technological evolution, as previously discussed. Regarding the problem and modelling frameworks it can be said, in general, that economic, social and technologic factors not only directly condition their form but also influence the perception of the DMs *vis a vis* the problems and the associated models, namely concerning the relative significance and importance of criteria or constraints.

In the particular case of modernisation planning of the access networks, the trend for the introduction of broadband services (requiring in many premises optical fibre directly to the customer) a type of problem in which different technological architectures can be used, a preliminary level of decision analysis for screening distinct alternatives, seems worth considering. This level of analysis might be concerned with the evaluation, under different performance criteria (for example, based on upgrade cost, operator revenue, response to estimated demand and user satisfaction in different technical instances) of various technologies and associated architectures available to the operator in a given market scenario.

Furthermore, mathematical programming approaches can be used to help the identification of more detailed multiattribute models, enabling a deeper analysis of the problem under study. It must be remarked that we believe in the complimentary use of both types of approaches. Last but not least, we emphasise a point regarding the modelling uncertainty, which requires particular attention in the future.

Other telecommunication applications with strong socio-economic implications deserve further investment in multicriteria modelling, in order to enable a more realistic evaluation of their impacts. As an example, we can refer to e-commerce and e-learning.

Operational planning involves certainly a vast number of problems some of which have already been treated, using multicriteria analysis models, as in the studies referred to in Sect. 26.4.2. It is expectable that, in the future, some other problems in this area will be prone to treatment in a multicriteria framework, especially having in mind the very rapid and multifaceted technological evolutions previously identified (in their major aspects) and their interactions with complex and fast changing economic and social trends. An example of such research challenges concerns cell partitioning and frequency allocation problems in the context of the very complex planning process of mobile cellular networks. Bourjolly et al. [38] present an overview of the application of OR-based decision support tools in this area. In particular the authors draw attention to the fact that cell partitioning (a decision process that has in mind to enable to use several times the available frequencies hence increasing the network capacity) addresses two conflicting issues, namely covered area and capacity (involving, in essence, a choice between a smaller number of larger cells versus a larger number of smaller cells). As for the frequency allocation problem, it involves the assignment of a certain number of radio frequencies to each cell, according to some "optimality" criteria and satisfying various technical constraints. In this type of problem several OFs can be considered such as discussed by those authors (namely the number of frequencies used, the frequency span and two types of signal interference, all to be minimised). It will also be expected that new and complex problems of transmission design have been and will continue to be fostered in the expanding area of sensor networks, WSNs and heterogeneous networks, an area in which multicriteria approaches have been drawing increasing attention.

Acknowledgements This work was financially supported by programme COMPETE of the EC Community Support Framework III and cosponsored by the EC fund FEDER and national funds (Portuguese Foundation for Science and Technology under project grants FCT–PTDC/EEA-TEL/101884/2008, FCT–PEst-C/EEI/UI0308/2011 and FCT–PEst-OE/EEI/UI308/2014).

References

1. Aboelela, E., Douligeris, C.: Fuzzy generalized network approach for solving an optimization model for routing in B-ISDN. Telecommun. Syst. **12**, 237–263 (1999)
2. Agrawal, G., Huang, D., Medhi, D.: Network protection design for MPLS networks. In: Proceedings of the 5th International Workshop on Design of Reliable Communication Networks (DRCN 2005), Naples (2005)
3. Akkanen, J., Nurminen, J.K.: Case study of the evolution of routing algorithms in a network planning tool. J. Syst. Softw. **58**(3), 181–198 (2001)
4. Alrabiah, T., Znati, T.: Delay-constrained, low-cost multicast routing in multimedia networks. J. Parallel Distrib. Comput. **61**(9), 1307–1336 (2001)
5. Al-Sharhan, S., Gueaieb, W.: A fast hybrid algorithm for multicast routing in wireless networks. Int. J. Artif. Intell. Tools **16**(1), 45–68 (2007)
6. Anandalingam, G., Nam, K.: Conflict and cooperation in designing international telecommunication networks. J. Oper. Res. Soc. **48**, 600–611 (1997)
7. Andrew, T.N., Rahoo, P., Nepal, T.: Enhancing the selection of communication technology for rural telecommunications: an analytic hierarchy process model. Int. J. Comput. Syst. Signals **6**(2), 26–34 (2005)
8. Aneja, Y.P., Nair, K.P.K.: The constrained shortest path problem. Nav. Res. Logist. Q. **25**, 549–555 (1978)
9. Angelou, G., Economides, A.A.: A compound real option and AHP methodology for evaluating ICT business alternatives. Telematics Inform. **26**(4), 353–374 (2009)
10. Antunes, C.H., Clímaco, J., Craveirinha, J.: A multiple objective linear programming approach to the modernization of telecommunication networks, in teletraffic and data-traffic: socioeconomic aspects. In: Jensen, A., Iversen, B. (eds.) Proceedings of the 13th International Teletraffic Congress (ITC13), Copenhagen. North-Holland Studies in Telecommunications, vol. 17, pp. 369–374. North-Holland, Amsterdam (1991)
11. Antunes, C.H., Alves, M.J., Silva, A.L., Clímaco, J.: An integrated MOLP method base package: a guided tour of TOMMIX. Comput. Oper. Res. **19**(7), 609–625 (1992)

12. Antunes, C.H., Clímaco, J., Craveirinha, J.: On multicriteria decision making approaches concerning strategic new telecommunication planning. In: Matsuda, T., Lesourne, J., Takahar, Y. (eds.) Proceedings of the International Conference on Economics Management and Information Technology, The Japanese Society of Management Information, pp. 95–98 (1992)
13. Antunes, C.H., Craveirinha, J., Clímaco, J.: A multiple criteria model for new telecommunication service planning. Eur. J. Oper. Res. **71**(3), 341–352 (1993)
14. Antunes, C.H., Craveirinha, J., Clímaco, J.: A flexible MOLP approach to the modernization of telecommunication networks incorporating sensitivity analysis. In: Labetoulle, J., Roberts, J.W. (eds.) Proceedings of the 14th International Teletraffic Congress on Advances in Teletraffic Science and Engineering (ITC14), pp. 1425–1434. Elsevier, Amsterdam (1994)
15. Antunes, C.H., Craveirinha, J., Clímaco, J.: Planning the evolution to broadband access networks: a multicriteria approach. Eur. J. Oper. Res. **109**(2), 530–540 (1998)
16. Antunes, C.H., Clímaco, J., Craveirinha, J., Barrico, C.: Multiple objective routing in integrated communication networks. In: Smith, D., Key, P. (eds.) Proceedings of the 16th International Teletraffic Congress (ITC16) – Teletraffic Engineering in a Competitive World, Edinburgh, pp. 1291–1300. Elsevier, Amsterdam (1999)
17. Ash, G.R.: Dynamic Routing in Telecommunications Networks, 1st edn. McGraw-Hill, New York (1998)
18. Ash, G.R., Chemouil, P.: 20 years of dynamic routing in circuit-switched networks: looking backward to the future. Glob. Commun. Newsl. **8**(10), 1–4 (2004). http://www.comsoc.org/gcn/back-issues
19. Assi, C., Shami, A., Ali, M.A., Kurtz, R., Guo, D.: Optical networking and real-time provisioning: an integrated vision for the next-generation Internet. IEEE Netw. **15**(4), 36–45 (2001)
20. Avallone, S., Ventre, G.: Q-BATE: a QoS constraint-based traffic engineering routing algorithm. In: Proceedings of the 2nd Conference on Next Generation Internet Design and Engineering (NGI'06), pp. 94–101 (2006)
21. Avallone, S., Kuipers, F., Ventre, G., van Mieghem, P.: Dynamic routing in QoS-aware traffic engineered networks. In: Kloos, C.D., Marin, A., Larrabeiti, D. (eds.) EUNICE 2005: Networks and Applications Towards a Ubiquitously Connected World, pp. 45–58. Springer, Berlin (2005)
22. Azouzi, R.E., Altman, E.: Constrained traffic equilibrium in routing. IEEE Trans. Autom. Control **48**(9), 1656–1660 (2003)
23. Badia, L., Miozzo, M., Rossi, M., Zorzi, M.: Routing schemes in heterogeneous wireless networks based on access advertisement and backward utilities for QoS support. IEEE Commun. Mag. **45**(2), 67–73 (2007)
24. Bana e Costa, C.A., Vansnick, J.C.: MACBETH: an interactive path towards the construction of cardinal value functions. Int. Trans. Oper. Res. **1**(4), 489–500 (1994)
25. Bana e Costa, C.A., Ensslin, L., Zanella, I.J.: A real-world MCDA application in cellular telephony systems. In: Stewart, T.J., van den Honert, R.C. (eds.) Trends in Multicriteria Decision Making. Lecture Notes in Economics and Mathematical Systems, vol. 465, pp. 412–423. Springer, Berlin (1998)
26. Banerjee, N., Kumar, R.: Multiobjective network design for realistic traffic models. In: Proceedings of the 9th Annual Genetic and Evolutionary Computation Conference (GECCO '07), London, England, pp. 1904–1911 (2007)
27. Banerjee, A., Drake, J., Lang, J.P., Turner, B., Kompella, K., Rekhter, Y.: Generalized multiprotocol label switching: an overview of routing and management enhancements. IEEE Commun. Mag. **39**(1), 144–150 (2001)
28. Bard, J.F., Moore, J.T.: A branch and bound algorithm for the bilevel programming problem. SIAM J. Sci. Stat. Comput. **11**(2), 281–292 (1990)
29. Baruani, J.A., Bagula, A.B., Muchanga, A.: On routing IP traffic using single- and multi-objective genetic optimization. In: Proceedings of the Southern African Telecommunications Networks and Applications Conference (SATNAC2006), Cape Town (2006)

30. Bellabas, A., Molnar, M., Lahoud, S.: Heuristics for the multicriteria routing problem. In: Proceedings of the Mosharaka International Conference on Communications Computers and Applications, Amman, Jordan (2009)
31. Belton, V., Ackermann, F., Shepherd, I.: Integrated support from problem structuring through to alternative evaluation using COPE and V.I.S.A. J. Multi-Criteria Decis. Anal. **6**(3), 115–130 (1997)
32. Bertalanffy, L.V.: General System Theory: Foundations, Development, Applications (Revised Edition). George Braziller, New York (1969)
33. Bertsekas, D., Gallager, R.: Data Networks, 2nd edn. Prentice-Hall, Englewood Cliffs, NJ (1992)
34. Beugnies, F., Gandibleux, X.: Multiobjective routing in IP networks. In: 7th PM2O Workshop (Programmation Mathématique Multi-Objectifs), Valenciennes (2003)
35. Beugnies, F., Gandibleux, X.: A multi-objective routing procedure for IP networks. In: Proceedings of the 18th International Conference on Multiple Criteria Decision Making (MCDM 2006), Chania (2006)
36. Bezruk, V., Svid, I., Korsun, I.: Methods of multicriteria optimization in telecommunication networks planning and controlling. In: Proceedings of the International Conference on Modern Problems of Radio Engineering, Telecommunications, and Computer Science (TCSET 2006), Lviv-Slavsko, pp. 381–383 (2006)
37. Blokh, D., Gutin, G.: An approximate algorithm for combinatorial optimization problems with two parameters. Aust. J. Comb. **14**, 157–164 (1996)
38. Bourjolly, J., Déjoie, L., Dioume, K., Lominy, M.: Frequency allocation in cellular phone networks: an OR success story. OR/MS Today **28**(2), 41–44 (2001)
39. Bouyssou, D.: Building criteria: a prerequisite for MCDA. In: Bana e Costa, C.A. (ed.) Readings in Multiple Criteria Decision Aid, pp. 58–80. Springer, Berlin (1990)
40. Brown, T.X.: Low power wireless communication via reinforcement learning. In: Solla, S.A., Leen, T.K., Müller, K.R. (eds.) Advances in Neural Information Processing Systems, vol. 12, pp. 893–899. MIT Press, Cambridge (2000)
41. Brown, T.X., Tong, H., Singh, S.: Optimizing admission control while ensuring quality of service in multimedia networks via reinforcement learning. In: Kearns, M.S., Cohn, D.A., Solla, S.A. (eds.) Advances in Neural Information Processing Systems, vol. 11, pp. 982–988. MIT Press, Cambridge (1999)
42. Chakraborty, D., Chakraborty, G., Shiratori, N.: A dynamic multicast routing satisfying multiple QoS constraints. Int. J. Netw. Manag. **13**, 321–335 (2003)
43. Chan, T.M., Kwong, S., Man, K.F.: Multiobjective resource optimization in mobile communication network. In: Proceedings of the 29th Annual Conference of the IEEE Industrial Electronics Society (IECON'03), vol. 3, pp. 2029–2034 (2003)
44. Charilas, D.E., Markaki, O.I., Psarras, J., Constantinou, P.: Application of fuzzy AHP and ELECTRE to network selection. In: Proceedings of MOBILIGHT 2009 – Mobile Lightweight Wireless Systems. Lecture Notes of the Institute for Computer Sciences, Social Informatics and Telecommunications Engineering, vol. 13, pp. 63–73. Springer, Berlin (2009)
45. Chen, S., Nahrstedt, K.: An overview of quality of service routing for next-generation high-speed networks: problems and solutions. IEEE Netw. **12**(6), 64–79 (1998)
46. Chen, S., Nahrstedt, K.: On finding multi-constrained paths. In: Proceedings of the IEEE International Conference on Communications (ICC'98), Atlanta, GA, vol. 2, pp. 874–879 (1998)
47. Chen, N., Li, L., Dong, W.: Multicast routing algorithm of multiple QoS based on widest-bandwidth. J. Syst. Eng. Electron. **17**(3), 642–647 (2006)
48. Cho, H.Y., Lee, J.Y., Kim, B.C.: Multi-path constraint-based routing algorithms for MPLS traffic engineering. In: Proceedings of the IEEE International Conference on Communications (ICC 2003), Anchorage, AK, pp. 1963–1967 (2003)
49. Clímaco, J., Antunes, C.H.: Implementation of a user friendly software package – a guided tour of TRIMAP. Math. Comput. Model. **12**(10/11), 1299–1309 (1989)

50. Clímaco, J.C.N., Craveirinha, J.M.F.: How and why to consider explicitly multiple points of view in the evaluation of telecommunication planning and management problems. In: Putnik, G., Gunasekaran, A. (eds.) Proceedings of Business Excellence I Conference – Performance Measures, Benchmarking and Best Practices in New Economy, Guimarães, pp. 73–78 (2003)
51. Clímaco, J.C.N., Craveirinha, J.M.F.: Multicriteria analysis in telecommunication network planning and design – problems and issues. In: Figueira, J., Greco, S., Ehrgott, M. (eds.) Multiple Criteria Decision Analysis: State of the Art Surveys. International Series in Operations Research & Management Science, vol. 78, pp. 899–951. Springer, New York (2005)
52. Clímaco, J.C.N., Martins, E.Q.V.: A bicriterion shortest path algorithm. Eur. J. Oper. Res. **11**(4), 399–404 (1982)
53. Clímaco, J.C.N., Pascoal, M.M.B.: Multicriteria path and tree problems: discussion on exact algorithms and applications. Int. Trans. Oper. Res. **19**(1–2), 63–98 (2012)
54. Clímaco, J., Antunes, C.H., Alves, M.J.: From TRIMAP to SOMMIX – building effective interactive MOLP computational tools. In: Fandel, G., Gal, T. (eds.) Proceedings of the XII International Conference on Multiple Criteria Decision Making. Lecture Notes in Economics and Mathematical Systems, vol. 448, pp. 285–296. Springer, Berlin (1997)
55. Clímaco, J.C.N., Craveirinha, J.M.F., Pascoal, M.M.B.: A bicriterion approach for routing problems in multimedia networks. Networks **41**(4), 206–220 (2003)
56. Clímaco, J.C.N., Craveirinha, J.M.F., Pascoal, M.M.B.: Routing calculation in multimedia: a procedure based on a bicriteria model. In: Neittaanmäki, P., Rossi, T., Korotov, S., Oñate, E., Périaux, J., Knörzer, D. (eds.) Proceedings of the European Congress on Computational Methods in Applied Sciences and Engineering (ECCOMAS 2004), Jyväskylä (2004)
57. Clímaco, J.C.N., Craveirinha, J.M.F., Pascoal, M.M.B.: An automated reference point-like approach for multicriteria shortest path problems. J. Syst. Sci. Syst. Eng. **15**(3), 314–329 (2006)
58. Clímaco, J.C.N., Craveirinha, J.M.F., Pascoal, M.M.B.: Multicriteria routing models in telecommunication networks – overview and a case study. In: Shi, Y., Olson, D.L., Stam, A. (eds.) Advances in Multiple Criteria Decision Making and Human Systems Management: Knowledge and Wisdom, pp. 17–46. IOS Press, Amsterdam (2007)
59. Coello Coello, C.A., Lamont, G.B.: Applications of Multi-Objective Evolutionary Algorithms. Advances in Natural Computation, vol. 1. World Scientific, Singapore (2004)
60. Coello Coello, C.A., Lamont, G.B., van Veldhuizen, D.A.: Evolutionary Algorithms for Solving Multi-Objective Problems, 2nd edn. Genetic and Evolutionary Computation. Springer, Berlin (2007)
61. Conte, M.: Dynamic Routing in Broadband Networks. Broadband Networks and Services. Kluwer Academic Publishers, Boston (2003)
62. Cox, D.R., Isham, V.: Point Processes. Monographs on Statistics and Applied Probability, vol. 12. Chapman and Hall/CRC, Boca Raton (1980)
63. Craveirinha, J., Martins, L., Gomes, T., Antunes, C.H., Clímaco, J.N.: A new multiple objective dynamic routing method using implied costs. J. Telecommun. Inf. Technol. **3**, 50–59 (2003)
64. Craveirinha, J., Martins, L., Clímaco, J.: Dealing with complexity in a multiobjective dynamic routing model for multiservice networks – a heuristic approach. In: Proceedings of the 15th Mini-EURO Conference on Managing Uncertainty in Decision Support Models (MUDSM 2004), Coimbra (2004)
65. Craveirinha, J.M.F., Clímaco, J.C.N., Pascoal, M.M.B., Martins, L.M.R.A.: Traffic splitting in MPLS networks – a hierarchical multicriteria approach. J. Telecommun. Inf. Technol. **4**, 3–10 (2007)
66. Craveirinha, J., Girão-Silva, R., Clímaco, J.: A meta-model for multiobjective routing in MPLS networks. Cent. Eur. J. Oper. Res. **16**(1), 79–105 (2008)

67. Craveirinha, J., Girão-Silva, R., Clímaco, J., Martins, L.: A hierarchical multiobjective routing model for MPLS networks with two service classes. In: Korytowski, A., Malanowski, K., Mitkowski, W., Szymkat, M. (eds.) Revised Selected Papers of the 23rd IFIP TC7 Conference on System Modeling and Optimization, Cracow, Poland, 23–27 July 2007. IFIP Advances in Information and Communication Technology, vol. 312, pp. 196–219. Springer, Berlin (2009)
68. Craveirinha, J., Gomes, T., Pascoal, M., Clímaco, J.: A stochastic bi-criteria approach for restorable QoS routing in MPLS. In: Proceedings of the 10th International Conference on Telecommunication Systems, Modeling and Analysis (ICTSM 2011), Prague (2011)
69. Craveirinha, J., Clímaco, J., Martins, L., Silva, C.G., Ferreira, N.: A bi-criteria minimum spanning tree routing model for MPLS/overlay networks. Telecommun. Syst. **52**(1), 203–215 (2013)
70. Crichigno, J., Barán, B.: Multiobjective multicast routing algorithm for traffic engineering. In: Proceedings of 13th IEEE International Conference on Computer and Communication Networks (ICCCN'04), Chicago, pp. 301–306 (2004)
71. Crichigno, J., Xie, C., Shu, W., Wu, M.Y., Ghani, N.: A multi-objective approach for throughput optimization and traffic engineering in WDM networks. In: Proceedings of the 43rd IEEE Asilomar Conference on Signals, Systems and Computers, Pacific Grove, CA, pp. 1043–1047 (2009)
72. Crichigno, J., Khoury, J., Shu, W., Wu, M.Y., Ghani, N.: Dynamic routing optimization in WDM networks. In: Proceedings of the IEEE Global Communications Conference (Globecom 2010), Miami, FL (2010)
73. Crichigno, J., Shu, W., Wu, M.Y.: Throughput optimization and traffic engineering in WDM networks considering multiple metrics. In: Proceedings of the IEEE International Communications Conference (ICC 2010) – Communications: Accelerating Growth and Development, Cape Town (2010)
74. CTIA – The Wireless Association: U.S. wireless quick facts. http://www.ctia.org/advocacy/research/index.cfm/aid/10323; http://www.ctia.org/your-wireless-life/how-wireless-works/wireless-quick-facts (2015). Accessed 11 Sept 2015
75. Cui, X., Lin, C., Wei, Y.: A multiobjective model for QoS multicast routing based on genetic algorithm. In: Proceedings of the 2003 International Conference on Computer Networks and Mobile Computing (ICCNMC'03), pp. 49–53 (2003)
76. Cui, Y., Xu, K., Wu, J., Yu, Z., Zhao, Y.: Multi-constrained routing based on simulated annealing. In: Proceedings of the IEEE International Conference on Communications (ICC'03), vol. 3, pp. 1718–1722 (2003)
77. de Neve, H., van Mieghem, P.: A multiple quality of service routing algorithm for PNNI. In: Proceedings of the IEEE ATM Workshop, Fairfax, VA, pp. 324–328 (1998)
78. de Neve, H., van Mieghem, P.: TAMCRA: a tunable accuracy multiple constraints routing algorithm. Comput. Commun. **23**(7), 667–679 (2000)
79. Dillon, R., Haimes, Y.Y.: Risk of extreme events via multiobjective decision trees: application to telecommunications. IEEE Transactions on Systems, Man IEEE Trans. Syst. Man Cybern. Syst. Hum. **26**(2), 262–271 (1996)
80. Donoso, Y., Fabregat, R., Marzo, J.L.: A multi-objective optimization scheme for multicast routing: a multitree approach. Telecommun. Syst. **27**(2–4), 229–251 (2004)
81. Donoso, Y., Pérez, A., Ardila, C., Fabregat, R.: Optimizing multiple objectives on multicast networks using memetic algorithms. GESTS Int. Trans. Comput. Sci. Eng. **20**(1), 192–204 (2005)
82. Douligeris, C.: Multiobjective telecommunication networks flow control. In: Proceedings of Southeastcon'91, Williamsburg, VA, vol. 2, pp. 647–651 (1991)
83. Douligeris, C.: Multiobjective flow control in telecommunication networks. In: Proceedings of IEEE INFOCOM'92, Florence, pp. 303–312 (1992)
84. Douligeris, C., Pereira, I.: A telecommunications quality study using the analytic hierarchy process. IEEE J. Sel. Areas Commun. [Special Issue on Quality of Telecommunications Services, Networks and Products, Bowick JW, Asatani K, Hoberg W, Malec H, Stockman S (eds.)] **12**(2), 241–250 (1994)

85. Du, Y., Zhou, W., Chen, B., Song, J.: A QoE based evaluation of service quality on WAP in wireless network. In: Proceedings of WiCom '09 – 5th International Conference on Wireless Communications, Networking and Mobile Computing, Beijing (2009)
86. Economides, A.A., Silvester, J.A.: Multi-objective routing in integrated services networks: A game theory approach. In: Proceedings of INFOCOM'91 - 10th Annual Joint Conference of the IEEE Computer and Communications Societies "Networking in the 90's", Bal Harbour, FL, vol. 3, pp. 1220–1227 (1991)
87. Ehrgott, M.: Multiobjective (combinatorial) optimisation – some thoughts on applications. In: Barichard, V., Ehrgott, M., Gandibleux, X., T'Kindt, V. (eds.) Multiobjective Programming and Goal Programming. Lecture Notes in Economics and Mathematical Systems, vol. 618, pp. 267–282. Springer, Berlin (2009)
88. Ehrgott, M., Gandibleux, X.: Hybrid metaheuristics for multi-objective combinatorial optimization. In: Blum, C., Blesa Aguilera, M.J., Roli, A., Sampels, M. (eds.) Hybrid Metaheuristics: An Emerging Approach to Optimization. Studies in Computational Intelligence, vol. 114, pp. 221–259. Springer, Berlin (2008)
89. El-Sayed, M., Jaffe, J.: A view of telecommunications network evolution. IEEE Commun. Mag. **40**(12), 74–81 (2002)
90. Erbas, S.C.: Utilizing evolutionary algorithms for multiobjective problems in traffic engineering. In: Ben-Ameur, W., Petrowski, A. (eds.) Proceedings of the International Networks Optimization Conference (INOC 2003), pp. 207–212. Institut National des Télécommunications, Evry/Paris (2003)
91. Erbas, S.C., Erbas, C.: A multiobjective off-line routing model for MPLS networks. In: Charzinski, J., Lehnert, R., Tran-Gia, P. (eds.) Proceedings of the 18th International Teletraffic Congress (ITC-18), pp. 471–480. Elsevier, Amsterdam, Berlin (2003)
92. Erbas, S.C., Mathar, R.: An off-line traffic engineering model for MPLS networks. In: Bakshi, B., Stiller, B. (eds.) Proceedings of the 27th Annual IEEE Conference on Local Computer Networks (27th LCN), Tampa, FL, pp. 166–174. IEEE Computer Society (2002)
93. Ergün, F., Sinha, R.K., Zhang, L.: QoS routing with performance-dependent costs. In: Sidi, M., Katzela, I., Shavitt, Y. (eds.) Proceedings of 19th Annual Joint Conference of the IEEE Computer and Communications Societies (INFOCOM 2000), Tel Aviv, vol. 1, pp. 137–146. IEEE Computer and Communications Societies (2000)
94. Fabregat, R., Donoso, Y., Baran, B., Solano, F., Marzo, J.L.: Multi-objective optimization scheme for multicast flows: a survey, a model and a MOEA solution. In: Proceedings of the 3rd International IFIP/ACM Latin American Conference on Networking (LANC'05), Cali, pp. 73–86 (2005)
95. Fialho, J., Godinho, P., Costa, J.P., Afonso, R., Regalado, J.G.: A level-based approach to prioritize telecommunications R&D. J. Telecommun. Inf. Technol. **4**, 40–46 (2008)
96. Flores, S.D., Cegla, B.B.: Multiobjective network design optimisation using parallel evolutionary algorithms. In: Proceedings of XXVII Conferencia Latinoamericana de Informática (CLEI'01), Mérida, Venezuela (2001)
97. Flores, S.D., Cegla, B.B., Cáceres, D.B.: Telecommunication network design with parallel multi-objective evolutionary algorithms. In: Proceedings of the 2003 IFIP/ACM Latin America Conference on "Towards a Latin American Agenda for Network Research", La Paz, pp. 1–11 (2003)
98. Forsati, R., Mahdavi, M., Haghighat, A.T., Ghariniyat, A.: An efficient algorithm for bandwidth-delay constrained least cost multicast routing. In: Proceedings of the 21st Canadian Conference on Electrical and Computer Engineering (CCECE 2008), Niagara Falls, ON, pp. 1641–1646 (2008)
99. Fortz, B., Thorup, M.: Internet traffic engineering by optimizing OSPF weights. In: Sidi, M., Katzela, I., Shavitt, Y. (eds.) Proceedings of the 19th Annual Joint Conference of the IEEE Computer and Communications Societies (INFOCOM 2000), Tel Aviv, vol. 2, pp. 519–528. IEEE Computer and Communications Societies (2000)
100. Fournié, L., Hong, D., Randriamasy, S.: Distributed multi-path and multi-objective routing for network operation and dimensioning. In: Proceedings of the 2nd EuroNGI Conference on Next Generation Internet Design and Engineering (NGI2006), Valencia (2006)

101. Fu, G.L., Yang, C., Tzeng, G.H.: A multicriteria analysis on the strategies to open Taiwan's mobile virtual network operators services. Int. J. Inf. Technol. Decis. Mak. **6**(1), 85–112 (2007)
102. Gasiea, Y., Emsley, M., Mikhailov, L.: On the applicability of the analytic network process to rural telecommunications infrastructure technology selection. In: Proceedings of the International Symposium on the Analytic Hierarchy Process (ISAHP) (2009)
103. Gasiea, Y., Emsley, M., Mikhailov, L.: Rural telecommunications infrastructure selection using the analytic network process. J. Telecommun. Inf. Technol. **2**, 28–42 (2010)
104. Ghinea, G., Magoulas, G.D., Siamitros, C.: Multicriteria decision making for enhanced perception-based multimedia communication. IEEE Transactions on IEEE Trans. Syst. Man Cybern. Syst. Hum. **35**(6), 855–866 (2005)
105. Giffhorn, E., Ensslin, L., Ensslin, S.R., Vianna, W.B.: Proposal of a multicriteria performance evaluation for outsourced project providers. In: PMA Conference – Theory and Practice in Performance Measurement, New Zealand (2009)
106. Giokas, D.I., Pentzaropoulos, G.C.: Efficiency ranking of the OECD member states in the area of telecommunications: a composite AHP/DEA study. Telecommun. Policy **32**(9–10), 672–685 (2008)
107. Girão-Silva, R., Craveirinha, J., Clímaco, J.: Hierarchical multiobjective routing in multiprotocol label switching networks with two service classes – a heuristic solution. Int. Trans. Oper. Res. **16**(3), 275–305 (2009)
108. Girão-Silva, R., Craveirinha, J., Clímaco, J.: Hierarchical multiobjective routing in MPLS networks with two service classes – a meta-heuristic solution. J. Telecommun. Inf. Technol. **3**, 20–37 (2009)
109. Girão-Silva, R., Craveirinha, J., Clímaco, J.: Hierarchical multiobjective routing model in multiprotocol label switching networks with two service classes – a Pareto archive strategy. Eng. Optim. **44**(5), 613–635 (2012)
110. Glaß, M., Lukasiewycz, M., Wanka, R., Haubelt, C., Teich, J.: Multi-objective routing and topology optimization in networked embedded systems. In: Najjar, W.A., Blume, H. (eds.) Proceedings of the International Conference on Embedded Computer Systems: Architectures, Modeling, and Simulation (IC-SAMOS 2008), Samos, pp. 74–81 (2008)
111. Glover, F., Mulvey, J.M.: Equivalence of the 0–1 integer programming problem to discrete generalized and pure networks. Oper. Res. **28**(3), 829–836 (1980)
112. Goel, A., Meyerson, A., Plotkin, S.: Combining fairness with throughput: online routing with multiple objectives. J. Comput. Syst. Sci. **63**(1), 62–79 (2001)
113. Goel, A., Ramakrishnan, K.G., Kataria, D., Logothetis, D.: Efficient computation of delay-sensitive routes from one source to all destinations. In: Sengupta, B., Bauer, F., Cavendish, D. (eds.) Proceedings of the 20th Annual Joint Conference of the IEEE Computer and Communications Societies (INFOCOM 2001), Anchorage, AK, vol. 2, pp. 854–858. IEEE Computer and Communications Societies (2001)
114. Gola, M., Kasprzak, A.: The two-criteria topological design problem in WAN with delay constraint: an algorithm and computational results. In: Kumar, V., Gavrilova, M.L., Tan, C.J.K., L'Ecuyer, P. (eds.) Proceedings of the International Conference on Computational Science and Its Applications (ICCSA 2003), Montreal. Lecture Notes in Computer Science, vol. 2667, pp. 971–972. Springer, New York (2003)
115. Gola, M., Kasprzak, A.: Topology design problem with combined cost criterion and time varying traffic in wide area networks. In: Proceedings of the 17th IMACS World Congress on Scientific Computation, Applied Mathematics and Simulation, Paris (2005)
116. Gomes, T., Craveirinha, J., Clímaco, J., Simões, C.: A bi-objective model for routing and wavelength assignment in resilient WDM networks. In: Martorell, S., Guedes Soares, C., Barnett, J. (eds.) Safety, Reliability and Risk Analysis: Theory, Methods and Applications: Proceedings of the European Safety and Reliability Conference (ESREL 2008), and 17th SRA-Europe, Valencia, vol. 4, pp. 2627–2634 (2008)
117. Gomes, T., Craveirinha, J., Clímaco, J., Simões, C.: A bicriteria routing model for multi-fibre WDM networks. Photon Netw. Commun. **18**(3), 287–299 (2009)

118. Granat, J., Guerriero, F.: The interactive analysis of the multicriteria shortest path problem by the reference point method. Eur. J. Oper. Res. **151**(1), 103–118 (2003)
119. Granat, J., Wierzbicki, A.P.: Multicriteria analysis in telecommunications. In: Proceedings of the 37th Hawaii International Conference on System Sciences (2004)
120. Guérin, R.A., Orda, A.: Networks with advance reservations: the routing perspective. In: Sidi, M., Katzela, I., Shavitt, Y. (eds.) Proceedings of the 19th Annual Joint Conference of the IEEE Computer and Communications Societies (INFOCOM 2000), Tel Aviv, vol. 1, pp. 118–127. IEEE Computer and Communications Societies (2000)
121. Guerriero, F., Rango, F.D., Marano, S., Bruno, E.: A biobjective optimization model for routing in mobile ad hoc networks. Appl. Math. Model. **33**(3), 1493–1512 (2009)
122. Guo, L., Matta, I.: Search space reduction in QoS routing. Comput. Netw. **41**(1), 73–88 (2003)
123. Han, D., Han, I.: Prioritization and selection of intellectual capital measurement indicators using analytic hierarchy process for the mobile telecommunications industry. Expert Syst. Appl. **26**(4), 519–527 (2004)
124. Handler, G.Y., Zang, I.: A dual algorithm for the constrained shortest path problem. Networks **10**(4), 293–309 (1980)
125. Handley, M.: Why the internet only just works. Br. Telecom Technol. J. **24**(3), 119–129 (2006)
126. Hassanein, H., Weng, X., Aboelfotoh, H.: Multi-objective optimization based flow control in multipoint-to-point communication. In: Proceedings of the IEEE Workshop on End-to-End Service Differentiation, EESD2003, Phoenix, AZ (2003)
127. Hassin, R.: Approximation schemes for the restricted shortest path problem. Math. Oper. Res. **17**(1), 36–42 (1992)
128. Haßlinger, G., Schnitter, S.: Optimized traffic load distribution in MPLS networks. In: Anandalingam, G., Raghavan, S. (eds.) Telecommunications Network Design and Management, pp. 125–141. Kluwer Academic Publishers, Boston (2003)
129. Hou, H.F., Li, F., Wang, H.Y.: QoS multicast routing algorithm with multiple constraints based on GA. In: Proceedings of the International Conference on Machine Learning and Cybernetics (ICMLC 2008), Kunming, pp. 1374–1378 (2008)
130. Hua, Q., Jihong, Z., Zengzhi, L.: Multi-objective routing and wavelength assignment method in WDM networks. In: Yoo, S.J.B., Cheung, K.W., Chung, Y.C., Li, G. (eds.) Proceedings of SPIE, Bellingham,WA. Network Architectures, Management, and Applications, vol. 5282, pp. 695–702 (2003)
131. Huiban, G., Mateus, G.R.: A multiobjective approach of the virtual topology design and routing problem in WDM networks. In: Proceedings of the 12th IEEE International Conference on Telecommunications (ICT), Cape Town (2005)
132. Isaksson, L., Fiedler, M.: Seamless connectivity in WLAN and cellular networks with multi criteria decision making. In: Proceedings of the 3rd EuroNGI Conference on Next Generation Internet Networks (NGI 2007), Trondheim, pp. 56–63 (2007)
133. Isiklar, G., Buyukozkan, G.: Using a multi-criteria decision making approach to evaluate mobile phone alternatives. Comput. Stand. Interfaces **29**(2), 265–274 (2007)
134. Jaffe, J.M.: Algorithms for finding paths with multiple constraints. Networks **14**(1), 95–116 (1984)
135. Jones, D.F., Mirrazavi, S.K., Tamiz, M.: Multi-objective meta-heuristics: an overview of the current state-of-the-art. Eur. J. Oper. Res. **137**(1), 1–9 (2002)
136. Kant, L., Manousakis, K., McAuley, A., Graff, C.: Network design approaches and tradeoffs in QoS route selection with diverse objectives and constraints. In: Proceedings of the IEEE Military Communications Conference (IEEE MILCOM 2008) (2008)
137. Keeney, R.L.: Modeling values for telecommunications management. IEEE Trans. Eng. Manag. **48**(3), 370–379 (2001)
138. Keeney, R.L., Raiffa, H.: Decisions with Multiple Objectives: Preferences and Value Tradeoffs. Cambridge University Press, Cambridge (1993)
139. Kelly, F.P.: Routing in circuit-switched networks: optimization, shadow prices and decentralization. Adv. Appl. Probab. **20**(1), 112–144 (1988)

140. Kennington, J., Lewis, K., Olinick, E., Ortynski, A., Spiride, G.: Robust solutions for the DWDM routing and provisioning problem: models and algorithms. Opt. Netw. Mag. **4**(2), 74–84 (2003)
141. Kerbache, L., Smith, J.M.: Multi-objective routing within large scale facilities using open finite queueing networks. Eur. J. Oper. Res. **121**(1), 105–123 (2000)
142. Khademi-Zare, H., Zarei, M., Sadeghieh, A., Owlia, M.S.: Ranking the strategic actions of Iran mobile cellular telecommunication using two models of fuzzy QFD. Telecommun. Policy **34**, 747–759 (2010)
143. Kim, J., Chang, K.K., Byunggon, Y.: A survey study on the relative importance of intranet functions: using the analytic hierarchy process. In: Fourth International Decision Sciences Institute Conference, Sydney (1997)
144. Kim, S.H., Lim, K., Kim, C.: A scalable QoS-based inter-domain routing scheme in a high speed wide area network. Comput. Commun. **21**, 390–399 (1998)
145. Knowles, J., Oates, M., Corne, D.: Advanced multi-objective evolutionary algorithms applied to two problems in telecommunications. BT Technol. J. **18**(4), 51–65 (2000)
146. Komolafe, O., Sventek, J.: RSVP performance evaluation using multi-objective evolutionary optimisation. In: Proceedings of the 24th Annual Joint Conference of the IEEE Computer and Communications Societies (IEEE Infocom 2005), Miami, FL, vol. 4, pp. 2447–2457 (2005)
147. Korkmaz, T., Krunz, M.: Multi-constrained optimal path selection. In: Sengupta, B., Bauer, F., Cavendish, D. (eds.) Proceedings of the 20th Annual Joint Conference of the IEEE Computer and Communications Societies (INFOCOM 2001), Anchorage, AK, vol. 2, pp. 834–843. IEEE Computer and Communications Societies, (2001)
148. Korkmaz, T., Krunz, M.: A randomized algorithm for finding a path subject to multiple QoS constraints. Comput. Netw. **36**(2–3), 251–268 (2001)
149. Kuipers, F., van Mieghem, P.: MAMCRA: a constrained-based multicast routing algorithm. Comput. Commun. **25**(8), 802–811 (2002)
150. Kuipers, F.A., van Mieghem, P.: Conditions that impact the complexity of QoS routing. IEEE/ACM Trans. Netw. **13**(4), 717–730 (2005)
151. Kuipers, F.A., van Mieghem, P.: Non-dominance in QoS routing: an implementational perspective. IEEE Commun. Lett. **9**(3), 267–269 (2005)
152. Kuipers, F., van Mieghem, P., Korkmaz, T., Krunz, M.: An overview of constraint-based path selection algorithms for QoS routing. IEEE Commun. Mag. **40**(12), 50–55 (2002)
153. Kuipers, F.A., Korkmaz, T., Krunz, M., van Mieghem, P.: A review of constraint-based routing algorithms. Technical Report, Technical University Delft (2002)
154. Kuipers, F.A., Korkmaz, T., Krunz, M., van Mieghem, P.: Performance evaluation of constraint-based path selection algorithms. IEEE Netw. **18**(5), 16–23 (2004)
155. Kuipers, F., Orda, A., Raz, D., van Mieghem, P.: A comparison of exact and ϵ-approximation algorithms for constrained routing. In: Proceedings of Networking 2006 - Fifth IFIP Networking Conference, Coimbra, Portugal (2006)
156. Kumar, R.: Multicriteria network design using distributed evolutionary algorithm. In: Pinkston, T.M., Prasanna, V.K. (eds.) Proceedings of the 10th International Conference on High Performance Computing (HiPC 2003), Hyderabad. Lecture Notes in Computer Science, vol. 2913/2003, pp. 343–352. Springer, New York (2003)
157. Kuo, Y.F., Chen, P.C.: Selection of mobile value-added services for system operators using fuzzy synthetic evaluation. Expert Syst. Appl. **30**(4), 612–620 (2006)
158. Kyrylov, V., Bonanni, C.: Modeling decision making by telecommunications services providers in a strategy market game. In: Proceedings of the 2004 Applied Telecommunications Symposium (ATS'04), Arlington, VA (2004)
159. Kyrylov, V., Bonanni, C.: Multi-criteria optimisation of budgeting decisions by telecommunication service providers in a simulation game. Int. J. Manag. Decis. Mak. **7**(2–3), 201–215 (2006)
160. Lee, W.C., Hluchyj, M.G., Humblet, P.A.: Routing subject to quality of service constraints in integrated communication networks. IEEE Netw. **9**(4), 46–55 (1995)

161. Lee, H., Shi, Y., Nazem, S.M., Kang, S.Y., Park, T.H., Sohn, M.H.: Multicriteria hub decision making for rural area telecommunication networks. Eur. J. Oper. Res. **133**, 483–495 (2001)
162. Lee, K., Toguyeni, A., Noce, A., Rahmani, A.: Comparison of multipath algorithms for load balancing in a MPLS network. In: Kim, C. (ed.) Proceedings of the International Conference on Information Networking – Convergence in Broadband and Mobile Networking (ICOIN2005), Jeju Island, Korea. Lecture Notes in Computer Science, vol. 3391/2005, pp. 463–470. Springer, Berlin (2005)
163. Leesutthipornchai, P., Wattanapongsakorn, N., Charnsripinyo, C.: Multi-objective design for routing wavelength assignment in WDM networks. In: Proceedings of the 3rd International Conference on New Trends in Information and Service Science (NISS 2009), Beijing, pp. 1315–1320 (2009)
164. Leesutthipornchai, P., Charnsripinyo, C., Wattanapongsakorn, N.: Solving multi-objective routing and wavelength assignment in WDM network using hybrid evolutionary computation approach. Comput. Commun. **33**(18), 2246–2259 (2010)
165. Levin, M.S., Nuiriakhmetov, R.I.: Multicriteria Steiner tree problem for communication network. Inf. Process. **9**(3), 199–209 (2009)
166. Levin, M.S., Petukhov, M.V.: Connection of users with a telecommunications network: multicriteria assignment problem. J. Commun. Technol. Electron. **55**(12), 1532–1541 (2010)
167. Levin, M.S., Petukhov, M.V.: Multicriteria assignment problem (Selection of access points). In: García-Pedrajas, N., Herrera, F., Fyfe, C., Benítez, J.M., Ali, M. (eds.) Trends in Applied Intelligent Systems. Lecture Notes in Computer Science, vol. 6097, pp. 277–287. Springer, Berlin (2010)
168. Li, L., Li, C.: A QoS-guaranteed multicast routing protocol. Comput. Commun. **27**(1), 59–69 (2004)
169. Li, L., Li, C.: A QoS multicast routing protocol for dynamic group. Inf. Sci. **169**, 113–130 (2005)
170. Li, H., Landa-Silva, D.: Evolutionary multi-objective simulated annealing with adaptive and competitive search direction. In: Proceedings of the 2008 IEEE Congress on Evolutionary Computation (CEC 2008), IEEE World Congress on Computational Intelligence, Hong Kong, pp. 3311–3318 (2008)
171. Li, Q., Beaver, J., Amer, A., Chrysanthis, P.K., Labrinidis, A., Santhanakrishnan, G.: Multi-criteria routing in wireless sensor-based pervasive environments. J. Pervasive Comput. Commun. **1**(4), 313–326 (2005)
172. Liu, G., Ramakrishnan, K.G.: A*Prune: an algorithm for finding k shortest paths subject to multiple constraints. In: Sengupta, B., Bauer, F., Cavendish, D. (eds.) Proceedings of the 20th Annual Joint Conference of the IEEE Computer and Communications Societies (INFOCOM 2001), Anchorage, AK, vol. 2, pp. 743–749. IEEE Computer and Communications Societies (2001)
173. Long, F., Sun, F., Yang, Z.: A novel routing algorithm based on multi-objective optimization for satellite networks. J. Netw. **6**(2), 238–246 (2011)
174. Lukač, K., Lukač, Z.: Multicriteria dynamic routing in communication networks based on F learning automata. In: Hamza, M.H. (ed.) Proceedings of Communications and Computer Networks (CCN 2002), Cambridge (2002)
175. Lukač, K., Lukač, Z., Tkalić, M.: Behaviour of F learning automata as multicriteria routing agents in connection oriented networks. In: Proceedings of the IEEE International Conference on Fuzzy Systems (FUZZ-IEEE 2003), St. Louis, MO, pp. 296–301 (2003)
176. Lukač, K., Lukač, Z., Tkalić, M.: On behaviour of fuzzy norms and F learning automata in distributed multicriteria network routing. In: Hamza, M.H. (ed.) Proceedings of the 21st IASTED International Conference on Applied Informatics, Innsbruck, pp. 686–691 (2003)
177. Ma, Q., Steenkiste, P.: On path selection for traffic with bandwidth guarantees. In: Proceedings of 5th IEEE International Conference on Network Protocols (ICNP'97), Atlanta, GA, pp. 191–202. IEEE Computer Society (1997)
178. Ma, Q., Steenkiste, P.: Routing traffic with quality-of-service guarantees in integrated services networks. In: Lesley, I., McAuley, D. (eds.) Proceedings of 8th International Workshop on Network and Operating System Support for Digital Audio and Video (NOSSDAV'98) (1998)

179. Mahdavi, I., Fazlollahtabar, H., Mozaffari, E., Heidari, M., Mahdavi-Amiri, N.: Data envelopment analysis based comparison of two hybrid multi-criteria decision-making approaches for mobile phone selection: a case study in Iranian telecommunication environment. Int. J. Inf. Decis. Sci. **1**(2), 194–220 (2008)
180. Mahey, P., Ribeiro, C.C.: Optimal routing for multi-service communication networks. OR/MS Today **29**(3), 40–43 (2002)
181. Mahjoub, D., El-Rewini, H.: Adaptive constraint-based multi-objective routing for wireless sensor networks. In: Proceedings of the IEEE International Conference on Pervasive Services (ICPS'07), Istanbul, pp. 72–75. IEEE Computer Society Press (2007)
182. Malakooti, B., Thomas, I.: A distributed composite multiple criteria routing using distance vector. In: Proceedings of the 2006 IEEE International Conference on Networking, Sensing and Control (ICNSC'06), Ft. Lauderdale, FL, pp. 42–47 (2006)
183. Malakooti, B., Griffith, M., Linsky, T., Liberatore, V., Malakooti, N., Bhasin, K.: Multi-criteria space based Internet routing. In: Proceedings of the 13th Industrial Engineering Research Conference (IERC), Houston, TX (2004)
184. Malakooti, B., Thomas, I., Tanguturi, S.K., Gajurel, S., Kim, H., Bhasin, K.: Multiple criteria network routing with simulation results. In: Proceedings of the 2006 Industrial Engineering Research Conference (IERC), Orlando, FL (2006)
185. Maniatis, S.I., Nikolouzou, E.G., Venieris, I.S.: QoS issues in the converged 3G wireless and wired networks. IEEE Commun. Mag. **40**(8), 44–53 (2002)
186. Markovic, G., Acimovic-Raspopovic, V.: An adaptive multi-criteria routing algorithm for wavelength routed optical networks. In: Proceedings of EUROCON 2005, the International Conference on Computer as a Tool, Belgrade, pp. 1353–1356 (2005)
187. Martins, L., Ferreira, N.G.: A bi-criteria approach for Steiner's tree problems in communication networks. In: International Workshop on Modeling, Analysis, and Control of Complex Networks (Cnet2011), 23rd International Teletraffic Congress (ITC 23), San Francisco (2011)
188. Martins, E.Q.V., Pascoal, M.M.B., Santos, J.L.E.: Deviation algorithms for ranking shortest paths. Int. J. Found. Comput. Sci. **10**(3), 247–263 (1999)
189. Martins, L., Craveirinha, J., Clímaco, J., Gomes, T.: Modeling and performance analysis of a new multiple objective dynamic routing method for multiexchange networks. Research Report ET-N8-5; 11/2002, INESC-Coimbra (`www.inescc.pt`) (2002)
190. Martins, L., Craveirinha, J., Clímaco, J.N., Gomes, T.: Implementation and performance of a new multiple objective dynamic routing method for multiexchange networks. J. Telecommun. Inf. Technol. **3**, 60–66 (2003)
191. Martins, L., Craveirinha, J., Clímaco, J., Gomes, T.: On a bi-dimensional dynamic alternative routing method. Eur. J. Oper. Res. [Special Issue on Advances in Complex Systems Modeling, Makowski, M., Nakamori, Y., Sebastian, H.-J. (eds.)] **166**(3), 828–842 (2005)
192. Martins, L., Craveirinha, J., Clímaco, J.: A new multiobjective dynamic routing method for multiservice networks: modelling and performance. Comput. Manag. Sci. **3**(3), 225–244 (2006)
193. Martins, L., Francisco, C., Redol, J., Craveirinha, J., Clímaco, J., Monteiro, P.: Evaluation of a multiobjective alternative routing method in carrier IP/MPLS networks. In: Fratta, L., Schulzrinne, H., Takahashi, Y., Spaniol, O. (eds.) Proceedings of NETWORKING 2009 – 8th International IFIP-TC 6 Networking Conference, Aachen. Lecture Notes in Computer Science, vol. 5550, pp. 195–206. Springer, Berlin (2009)
194. Martins, L., Ferreira, N., Craveirinha, J.: A bi-criteria algorithm for multipoint-to-multipoint virtual connections in transport networks. In: Proceedings of the 7th International Conference on Network and Service Management (CNSM 2011), Paris (2011)
195. Marwaha, S., Srinivasan, D., Chen, K.T., Vasilakos, A.: Evolutionary fuzzy multi-objective routing for wireless mobile ad hoc networks. In: Congress on Evolutionary Computation (CEC2004), Portland, OR, vol. 2, pp. 1964–1971 (2004)
196. Masip-Bruin, X., Yannuzzi, M., Domingo-Pascual, J., Fonte, A., Curado, M., Monteiro, E., Kuipers, F., van Mieghem, P., Avallone, S., Ventre, G., Aranda-Gutiérrez, P., Hollick, M., Steinmetz, R., Iannone, L., Salamatian, K.: Research challenges in QoS routing. Comput. Commun. **29**(5), 563–581 (2006)

197. Meisel, Y.D., Fabregat, R., Fàbrega, L.: Multi-objective scheme over multi-tree routing in multicast MPLS networks. In: Proceedings of the 2003 IFIP/ACM Latin America Networking Conference (LANC'03), La Paz (2003)
198. Meyerson, A., Munagala, K., Plotkin, S.: Cost distance: two metric network design. SIAM J. Comput. **38**(4), 1648–1659 (2008)
199. Minhas, M.R., Gopalakrishnan, S., Leung, V.C.M.: Multiobjective routing for simultaneously optimizing system lifetime and source-to-sink delay in wireless sensor networks. In: Proceedings of the 29th IEEE International Conference on Distributed Computing Systems Workshops (ICDCS Workshops '09), pp. 123–129 (2009)
200. Mitra, D., Ramakrishnan, K.G.: A case study of multiservice, multipriority traffic engineering design for data networks. In: Proceedings of Global Telecommunications Conference (IEEE GLOBECOM'99), vol. 1b, pp. 1077–1083 (1999)
201. Mitra, D., Ramakrishnan, K.G.: Techniques for traffic engineering of multiservice, multipriority networks. Bell Labs Tech. J. **6**(1), 139–151 (2001)
202. Mitra, D., Morrison, J.A., Ramakrishnan, K.G.: Optimization and design of network routing using refined asymptotic approximations. Perform. Eval. **36–37**, 267–288 (1999)
203. Mitra, D., Ramakrishnan, K.G., Wang, Q.: Combined economic modeling and traffic engineering: joint optimization of pricing and routing in multi-service networks. In: de Souza, J.M., da Fonseca, N.L.S., de Souza e Silva, E.A. (eds.) Proceedings of the 17th International Teletraffic Congress (ITC-17) – Traffic Engineering in the Internet Era, pp. 73–85. Elsevier, Salvador (2001)
204. Nace, D., Pioro, M.: Max-min fairness and its applications to routing and load-balancing in communication networks: a tutorial. IEEE Commun. Surv. Tutorials **10**(4), 5–17 (2008)
205. Nurminen, J.K.: Models and algorithms for network planning tools – practical experiences. Research Report E14, Systems Analysis Laboratory, Helsinki University of Technology (2003)
206. Ogryczak, W., Śliwiński, T., Wierzbicki, A.: Fair resource allocation schemes and network dimensioning problem. J. Telecommun. Inf. Technol. **3**, 34–42 (2003)
207. Ogryczak, W., Pióro, M., Tomaszewski, A.: Telecommunications network design and Max-min optimization problem. J. Telecommun. Inf. Technol. **3**, 43–56 (2005)
208. Ogryczak, W., Wierzbicki, A., Milewski, M.: A multi-criteria approach to fair and efficient bandwidth allocation. OMEGA Int. J. Manag. Sci. [Special Issue on Multiple Criteria Decision Making for Engineering, Wiecek, M.M., Ehrgott, M., Fadel, G., Figueira, J.R. (eds.)] **36**(3), 451–463 (2008)
209. Ondrus, J., Bui, T., Pigneur, Y.: A multi-actor, multi-criteria approach for technology selection when designing mobile information systems. In: Proceedings of IFIP TC8 Working Conference on Mobile Information Systems (MOBIS'05), Leeds (2005)
210. Orda, A.: Routing with end-to-end QoS guarantees in broadband networks. IEEE/ACM Trans. Netw. **7**(3), 365–374 (1999)
211. Osman, M.S., Abo-Sinna, M.A., Mousa, A.A.: An effective genetic algorithm approach to multiobjective routing problems (MORPs). Appl. Math. Comput. **163**(2), 769–781 (2005)
212. Oueslti-Boulahia, S., Oubagha, E.: An approach to routing elastic flows. In: Smith, D., Key, P. (eds.) Proceedings of the 16th International Teletraffic Congress (ITC16) – Teletraffic Engineering in a Competitive World, Edinburgh, pp. 1311–1320. Elsevier, Amsterdam (1999)
213. Papagianni, C., Papadopoulos, K., Pappas, C., Tselikas, N.D., Kaklamani, D.T., Venieris, I.S.: Communication network design using particle swarm optimization. In: Proceedings of the International Multiconference on Computer Science and Information Technology (IMCSIT 2008), Wisla, pp. 915–920 (2008)
214. Parsa, M., Zhu, Q., Garcia-Luna-Aceves, J.J.: An iterative algorithm for delay-constrained minimum-cost multicasting. IEEE/ACM Trans. Netw. **6**(4), 461–474 (1998)
215. Petrowski, A., Aissanou, F., Benyahia, I., Houcke, S.: Multicriteria reinforcement learning based on a Russian doll method for network routing. In: Proceedings of 5th IEEE International Conference on Intelligent Systems (IS'10), London (2010)

216. Pinto, D., Barán, B.: Solving multiobjective multicast routing problem with a new ant colony optimization approach. In: Proceedings of the 2nd IFIP/ACM Latin American Networking Conference (LANC'05), Cali (2005)
217. Pióro, M., Medhi, D.: Routing, Flow, and Capacity Design in Communication and Computer Networks, 1st edn. Morgan Kaufmann Publishers, San Francisco (2004)
218. Pióro, M., Dzida, M., Kubilinskas, E., Nilsson, P., Ogryczak, W., Tomaszewski, A., Zagożdżon, M.: Applications of the Max-min fairness principle in telecommunication network design. In: Proceedings of the Next Generation Internet Networks (EURONGI 2005), Rome, pp. 219–225 (2005)
219. Pornavalai, C., Chakraborty, G., Shiratori, N.: QoS based routing algorithm in integrated services packet networks. In: Proceedings of 5th IEEE International Conference on Network Protocols (ICNP'97), Atlanta, GA, pp. 167–174. IEEE Computer Society (1997)
220. Pornavalai, C., Chakraborty, G., Shiratori, N.: Routing with multiple QoS requirements for supporting multimedia applications. Telecommun. Syst. **9**(3,4), 357–373 (1998)
221. Pramod, V.R., Banwet, D.K.: Analytic network process analysis of an Indian telecommunication service supply chain: a case study. Serv. Sci. **2**(4), 281–293 (2010)
222. Prasithsangaree, P., Niehaus, D.: Multiple QoS routing in large PNNI ATM networks with heavy traffic. In: Proceedings of the 2002 International Symposium on Performance Evaluation of Computer and Telecommunication Systems (SPECTS'02), San Diego, CA, pp. 438–442 (2002)
223. Prieto, J., Barán, B., Crichigno, J.: Multitree-multiobjective multicast routing for traffic engineering. In: Bramer, M. (ed.) Proceedings of the 1st Artificial Intelligence in Theory and Practice, Santiago, Chile. IFIP International Federation for Information Processing, vol. 217, pp. 247–256. Springer, Berlin (2006)
224. Puri, A., Tripakis, S.: Algorithms for routing with multiple constraints. In: Proceedings of the 8th Scandinavian Workshop on Algorithm Theory (SWAT 2002), Turku, pp. 338–347 (2002)
225. Raisinghani, M.: A balanced analytic approach to strategic electronic commerce decisions: a framework of the evaluation method. In: van Grembergen, W. (ed.) Information Technology Evaluation Methods & Management, pp. 185–197. Idea Group Publishing, Hershey (2001)
226. Rajagopalan, R., Mohan, C.K., Varshney, P., Mehrotra, K.: Multi-objective mobile agent routing in wireless sensor networks. In: Proceedings of the 2005 IEEE Congress on Evolutionary Computation (CEC2005), Edinburgh, vol. 2, pp. 1730–1737 (2005)
227. Reeves, D.S., Salama, H.F.: A distributed algorithm for delay-constrained unicast routing. IEEE/ACM Trans. Netw. **8**(2), 239–250 (2000)
228. Reid, A., Willis, P., Hawkins, I., Bilton, C.: Carrier ethernet. IEEE Commun. Mag. **46**(9), 96–103 (2008)
229. Resende, M.G.C., Ribeiro, C.C.: A GRASP with path-relinking for private virtual circuit routing. Networks **41**(3), 104–114 (2003)
230. Riedl, A.: A hybrid genetic algorithm for routing optimization in IP networks utilizing bandwidth and delay metrics. In: Proceedings of the 2002 IEEE Workshop on IP Operations & Management (IPOM 2002), Dallas, TX (2002)
231. Riedl, A., Schupke, D.A.: Routing optimization in IP networks utilizing additive and concave link metrics. IEEE/ACM Trans. Netw. **15**(5), 1136–1148 (2007)
232. Rocha, M., Sousa, P., Rio, M., Cortez, P.: QoS constrained internet routing with evolutionary algorithms. In: Proceedings of the 2006 IEEE Congress on Evolutionary Computation, Vancouver, BC (2006)
233. Rodrigues, J.M., Clímaco, J.C., Current, J.R.: An interactive bi-objective shortest path approach: searching for unsupported nondominated solutions. Comput. Oper. Res. **26**, 789–798 (1999)
234. Rosenhead, J., Mingers, J.: Rational Analysis for a Problematic World Revisited: Problem Structuring Methods for Complexity, Uncertainty and Conflict, 2nd edn. Wiley, New York (2001)
235. Roy, B., Bouyssou, D.: Aide Multicritère à la Décision: Méthodes et Cas. Economica, Paris (1993)

236. Roy, A., Das, S.K.: Optimizing QoS-based multicast routing in wireless networks: a multi-objective genetic algorithmic approach. In: Gregori, E., Conti, M., Campbell, A.T., Omidyar, G., Zukerman, M. (eds.) Proceedings of the Second IFIP-TC6 Networking Conference, NETWORKING 2002, Pisa. Lecture Notes in Computer Science, vol. 2345, pp. 28–48. Springer, New York (2002)
237. Roy, A., Das, S.K.: QM^{2}RP: a QoS-based mobile multicast routing protocol using multi-objective genetic algorithm. Wirel. Netw. **10**(3), 271–286 (2004)
238. Roy, A., Banerjee, N., Das, S.K.: An efficient multi-objective QoS-routing algorithm for wireless multicasting. In: Proceedings of the IEEE 55th Vehicular Technology Conference (VTC 2002), Birmingham, AL, vol. 3, pp. 1160–1164 (2002)
239. Saaty, T.L.: The Analytic Hierarchy Process: Planning, Priority Setting, Resource Allocation. Decision Making. McGraw-Hill, New York (1980)
240. Saaty, T.L.: Highlights and critical points in the theory and application of the analytic hierarchy process. Eur. J. Oper. Res. **74**(3), 426–447 (1994)
241. Saaty, T.L.: How to make a decision: the analytic hierarchy process. Interfaces **24**(6), 19–43 (1994)
242. Saaty, T.L.: Decision Making with Dependence and Feedback – The Analytic Network Process: The Organization and Prioritization of Complexity, 2nd edn. RWS Publications, Pittsburgh, PA (2001)
243. Schnitter, S., Haßlinger, G.: Heuristic solutions to the LSP-design for MPLS traffic engineering. In: Proceedings of NETWORKS 2002 – 10th International Telecommunication Network Strategy and Planning Symposium, München, pp. 269–273 (2002)
244. Seok, Y., Lee, Y., Choi, Y., Kim, C.: Explicit multicast routing algorithms for constrained traffic engineering. In: Corradi, A., Daneshmand, M. (eds.) Proceedings of the 7th IEEE International Symposium on Computers and Communications (ISCC'02), Taormina (2002)
245. Setämaa-Kärkkäinen, A., Miettinen, K., Vuori, J.: Heuristic for a new multiobjective scheduling problem. Optim. Lett. **1**(3), 213–225 (2007)
246. Shafigh, A.S., Niyati, M.: A heuristic multi criteria routing protocol in wireless sensor networks. In: Unnikrishnan, S., Surve, S., Bhoir, D. (eds.) Advances in Computing, Communication and Control. Communications in Computer and Information Science Series, vol. 125, pp. 306–317. Springer, Berlin (2011)
247. Simões, C., Gomes, T., Craveirinha, J., Clímaco, J.: Performance analysis of a bi-objective model for routing and wavelength assignment in WDM networks. J. Telecommun. Inf. Technol. **3**, 13–24 (2010)
248. Simões, C., Gomes, T., Craveirinha, J., Clímaco, J.: Performance analysis of a bi-objective model for routing with protection in WDM networks. J. Telecommun. Inf. Technol. **3**, 25–35 (2010)
249. Sobrinho, J.L.: Algebra and algorithms for QoS path computation and hop-by-hop routing in the Internet. IEEE/ACM Trans. Netw. **10**(4), 541–550 (2002)
250. Song, J., Pung, H.K., Jacob, L.: A multi-constrained distributed QoS routing algorithm. In: Proceedings of 8th IEEE International Conference on Networks (ICON'00), pp. 165–171 (2000)
251. Songhurst, D.J.: Charging Communication Networks: From Theory to Practice. Elsevier Science B. V., Amsterdam (1999)
252. Sousa, P., Cortez, P., Rio, M., Rocha, M.: Traffic engineering approaches using multicriteria optimization techniques. In: Proceedings of Wired/Wireless Internet Communications. Lecture Notes in Computer Science, vol. 6649, pp. 104–115. Springer, New York (2011)
253. Steuer, R.E.: Multiple Criteria Optimization: Theory, Computation and Application. Probability and Mathematical Statistics. Wiley, New York (1986)
254. Sylla, C., Whitworth, B., Mahinda, E.: A hybrid method for the multicriteria evaluation of network technologies. In: Proceedings of ICT Africa (2008)
255. Talavera, F., Crichigno, J., Barán, B.: Polices for dynamical multiobjective environment of multicast traffic engineering. In: Proceedings of the 12th IEEE International Conference on Telecommunications (ICT), Cape Town (2005)

256. Talbi, E.-G.: Metaheuristics: From Design to Implementation. Wiley, New York (2009)
257. Talbi, E.-G., Basseur, M., Nebro, A.J., Alba, E.: Multi-objective optimization using metaheuristics: non-standard algorithms. Int. Trans. Oper. Res. **19**(1–2), 283–305 (2012)
258. Tam, M.C.Y., Tummala, V.M.R.: An application of the AHP in vendor selection of a telecommunications system. OMEGA Int. J. Manag. Sci. **29**(2), 171–182 (2001)
259. Tarapata, Z.: Selected multicriteria shortest path problems: an analysis of complexity, models and adaptation of standard algorithms. Int. J. Appl. Math. Comput. Sci. **17**(2), 269–287 (2007)
260. Thirumalasetty, S.R., Medhi, D.: MPLS traffic engineering for survivable book-ahead guaranteed services. Technical Report in Computer Science and Telecommunications. University of Missouri, Kansas City (2001)
261. Thizy, J.M., Lane, D.E., Pissarides, S., Rawat, S.: Interactive multiple criteria optimization for capital budgeting in a Canadian telecommunications company. In: Tamiz, M. (ed.) Multi-Objective Programming and Goal Programming: Theories and Applications. Lecture Notes in Economics and Mathematical Systems, vol. 432, pp. 128–147. Springer, Berlin (1996)
262. Tiourine, S.R., Hurkens, C.A.J., Lenstra, J.K.: Local search algorithms for the radio link frequency assignment problem. Telecommun. Syst. **13**(2–4), 293–314 (2000)
263. Tsai, W.K., Dai, W.: Joint routing and rate assignment in MPLS based networks. In: Proceedings of 9th IEEE International Conference on Networks (ICON'01), pp. 196–201 (2001)
264. van der Zee, M., Heijenk, G.: Quality of service routing: state of the art report. Open Report 1/0362-FCP NB 102 88 Uen, Ericsson / University of Twente (1999)
265. van Mieghem, P., Kuipers, F.A.: On the complexity of QoS routing. Comput. Commun. **26**(4), 376–387 (2003)
266. van Mieghem, P., Kuipers, F.A.: Concepts of exact QoS algorithms. IEEE/ACM Trans. Netw. **12**(5), 851–864 (2004)
267. van Mieghem, P., de Neve, H., Kuipers, F.A.: Hop-by-hop quality of service routing. Computer Netw. **37**(3–4), 407–423 (2001)
268. van Veldhuizen, D.A., Lamont, G.B.: Multiobjective evolutionary algorithms: analyzing the state-of-the-art. Evol. Comput. **8**(2), 125–147 (2000)
269. Wang, Z., Crowcroft, J.: Quality-of-service routing for supporting multimedia applications. IEEE J. Sel. Areas Commun. **14**(7), 1228–1234 (1996)
270. Wang, H., van Mieghem, P.: Constructing the overlay network by tuning link weights. In: Proceedings of the 2nd International Conference on Communications and Networking in China (CHINACOM '07), Shanghai, pp. 139–143; 324–328 (2007)
271. Wang, X., Wang, X., Xu, L.: A multi-constraint QoS routing algorithm with link-disjoint paths. Comput. Appl. Softw. 3 (2008). http://en.cnki.com.cn/Article_en/CJFDTOTAL-JYRJ200803011.htm
272. Watanabe, S., Hiroyasu, T., Miki, M.: Parallel evolutionary multi-criterion optimization for mobile telecommunication networks optimization. In: Giannakoglou, K.C., Tsahalis, D.T., Periaux, J., Papailiou, K.D., Fogarty, T. (eds.) Proceedings of the EUROGEN'01, Evolutionary Methods for Design, Optimization and Control with Applications to Industrial Problems, Barcelona, pp. 167–172 (2001)
273. Widyono, R.: The design and evaluation of routing algorithms for real-time channels. Technical Report TR-94-024, Tenet Group – University of California at Berkeley & International Computer Science Institute (1994)
274. Wierzbicki, A.: Multicriteria models for the placement of internet caches. In: Proceedings of the 16th MCDM World Conference – MCDM Winter Conference, Semmering (2002)
275. Wierzbicki, A.P.: Telecommunications, multiple criteria analysis and knowledge theory. J. Telecommun. Inf. Technol. **3**, 3–13 (2005)
276. Wierzbicki, A.P., Burakowski, W.: A conceptual framework for multiple-criteria routing in QoS IP networks. Int. Trans. Oper. Res. **18**(3), 377–399 (2011)
277. Wojewnik, P., Szapiro, T.: Bireference procedure fBIP for interactive multicriteria optimization with fuzzy coefficients. Cent. Eur. J. Econ. Model. Econ. **2**(3), 169–193 (2010)

278. Wu, J.J., Hwang, R.H., Lu, H.I.: Multicast routing with multiple QoS constraints in ATM networks. Inf. Sci. **124**, 29–57 (2000)
279. Wu, J., Rong, L., Wang, Y., Huang, H., Li, H., Pentikousis, K., Schramm, F., Schulz, E., Pampu, C., Chen, M., Lei, L., Womack, J., Periyalwar, S., Ramrekha, T.A., Politis, C., Usui, T., Wang, C.-X., Hunt, B., Kim, S.G.: Requirements and vision for NG-Wireless. Outlook 7 – Visions and research directions for the Wireless World. Wireless World Research Forum (2011)
280. Xu, Y., Qu, R.: Solving multi-objective multicast routing problems by evolutionary multi-objective simulated annealing algorithms with variable neighbourhoods. J. Oper. Res. Soc. [Special Issue on Heuristic Optimisation, Aickelin, U., Clark, A. (eds.)] **62**(2), 313–325 (2011)
281. Yu, P.L.: Multiple-Criteria Decision Making: Concepts, Techniques and Extensions. Mathematical Concepts and Methods in Science and Engineering. Plenum Press, New York (1985)
282. Yu, H.C., Lee, Z.Y., Chang, S.C.: Using a fuzzy multi-criteria decision making approach to evaluate alternative licensing mechanisms. Inf. Manag. **42**(4), 517–531 (2005)
283. Yuan, X.: Heuristic algorithms for multiconstrained quality-of-service routing. IEEE/ACM Trans. Netw. **10**(2), 244–256 (2002)
284. Yuan, D.: A bi-criteria optimization approach for robust OSPF routing. In: Proceedings of the 3rd IEEE Workshop on IP Operations and Management (IPOM 2003), Kansas City, MO, pp. 91–98 (2003)
285. Yuan, X., Liu, X.: Heuristic algorithms for multi-constrained quality of service routing. In: Sengupta, B., Bauer, F., Cavendish, D. (eds.) Proceedings of the 20th Annual Joint Conference of the IEEE Computer and Communications Societies (INFOCOM 2001), Anchorage, AK, vol. 2, pp. 844–853. IEEE Computer and Communications Societies (2001)
286. Zang, H., Jue, J.P., Mukherjee, B.: A review of routing and wavelength assignment approaches for wavelength-routed optical WDM networks. SPIE/Baltzer Opt. Netw. Mag. **1**(1), 47–60 (2000)
287. Zeleny, M.: A concept of compromise solutions and the method of the displaced ideal. Comput. Oper. Res. **1**(3–4), 479–496 (1974)
288. Zhang, R., Zhu, X.: Fuzzy routing in QoS networks. In: Wang, L., Jin, Y. (eds.) Proceedings of the Second International Conference on Fuzzy Systems and Knowledge Discovery (FSKD 2005), Changsha. Lecture Notes in Computer Science, vol. 3614, pp. 880–890. Springer, Berlin (2005)
289. Zhu, B.: Routing and wavelength assignment in all-optical networks by multi-objective genetic algorithms. Technical Report, Department of Mechanical Engineering, University of California at Berkeley (2000)
290. Zhu, Q., Parsa, M., Garcia-Luna-Aceves, J.J.: A source-based algorithm for delay-constrained minimum-cost multicasting. In: Proceedings of the 14th Annual Joint Conference of the IEEE Computer and Communication Societies (IEEE INFOCOM '95) – Bringing Information to People, Boston, MA, vol. 1, pp. 377–385. IEEE Computer Society (1995)

Chapter 27
Multiple Criteria Decision Analysis and Sustainable Development

Giuseppe Munda

Abstract Sustainable development is a multidimensional concept, including socio-economic, ecological, technical and ethical perspectives. In making sustainability policies operational, basic questions to be answered are sustainability of *what and whom*? As a consequence, sustainability issues are characterised by a high degree of conflict. The main objective of this chapter is to show that multiple-criteria decision analysis is an adequate approach for dealing with sustainability conflicts at both micro and macro levels of analysis. To achieve this objective, lessons, learned from both theoretical arguments and empirical experience, are reviewed. Guidelines of "*good practice*" are suggested too.

Keywords Sustainable development • Economics • Complex systems • Incommensurability • Social choice • Social multi-criteria evaluation

27.1 The Concept of Sustainable Development and the Incommensurability Principle

In the 1980s, the awareness of actual and potential conflicts between economic growth and the environment led to the concept of "*sustainable development*". Since then, all governments have declared, and still claim, their willingness to pursue economic growth under the flag of sustainable development although often development and sustainability are contradictory terms. The concept of sustainable development has wide appeal, partly because it does not set economic growth and environmental preservation in sharp opposition. Rather, sustainable development carries the ideal of a harmonisation or *simultaneous realisation* of economic growth and environmental concerns. For example, Barbier [8, p. 103] writes

G. Munda (✉)
European Commission, Joint Research Centre, Unit DDG.01 - Econometrics and Applied Statistics, TP 361 - Via E. Fermi 2749-I-21027, Ispra (Va), Italy
e-mail: giuseppe.munda@jrc.ec.europa.eu

S. Greco et al. (eds.), *Multiple Criteria Decision Analysis*, International Series in Operations Research & Management Science 233,
DOI 10.1007/978-1-4939-3094-4_27

that sustainable development implies: "*to maximise simultaneously*[1] *the biological system goals (genetic diversity, resilience, biological productivity), economic system goals (satisfaction of basic needs, enhancement of equity, increasing useful goods and services), and social system goals (cultural diversity, institutional sustainability, social justice, participation)*". This definition correctly points out that sustainable development is a *multidimensional* concept, but as our everyday life teaches us, it is generally impossible to maximise different objectives at the same time, and as formalised by multi-criteria decision analysis, compromise solutions must be found.

Let us try to clarify some fundamental points of the concept of "sustainable development". In economics by "development" is meant "*the set of changes in the economical, social, institutional and political structure needed to implement the transition from a pre-capitalistic economy based on agriculture, to an industrial capitalistic economy*" [17]. Such a definition of development has two main characteristics:

- The changes needed are not only quantitative (like the growth of gross domestic product), but qualitative too (social, institutional and political).
- There is only a possible model of development, i.e. the one of western industrialised countries. This implies that the concept of development is viewed as a process of cultural fusion toward the *best* knowledge, the *best* set of values, the *best* organisation and the *best* set of technologies.

Adding the term "sustainable" to the "set of changes" (the first point) means adding an ethical dimension to development. The issue of *distributional equity*, both within the same generation (intra-generational equity, e.g. the North–South conflict) and between different generations (inter-generational equity) becomes crucial [78]. Going further, a legitimate question could be raised: sustainable development of what and whom? [3]. Norgaard [92, p. 11] writes: "*consumers want consumption sustained, workers want jobs sustained. Capitalists and socialists have their "isms", while aristocrats and technocrats have their "cracies"*".

Martinez-Alier and O'Connor [71] have proposed the concept of ecological distribution to synthesise sustainability conflicts. The concept of *ecological distribution* refers to the social, spatial, and temporal asymmetries or inequalities in the use by humans of environmental resources and services. Thus, the territorial asymmetries between SO_2 emissions and the burdens of acid rain are an example of *spatial ecological distribution*. The inter-generational inequalities between the benefits of nuclear energy and the burdens of radioactive waste are an example of *temporal ecological distribution*. In the USA, "*environmental racism*", meaning locating polluting industries or toxic waste disposal sites in areas where poor people live, is an example of *social ecological distribution*. We can then conclude that sustainability management and planning is essentially a *conflict analysis*.

The second characteristic of the term "*development*" refers to the western industrialized production system as symbol of any successful development process.

[1] Emphasis added to the original.

However, serious environmental problem may stem from this vision. For example, according to actual social values in western countries, to have a car per two/three persons could be considered a reasonable objective in less developed countries. This would imply a number of cars ten times greater than the existent one, with possible consequences on global warming, reserves of petroleum, loss of agricultural land and noise. The contradiction between the terms "development" and "sustainable" may not be reconcilable unless other models of development are considered.

This is proposed by the so-called *co-evolutionary paradigm*. According to this view of social evolution, borrowed from biology [32], there is a constant and active *interaction* of the organisms with their environment. Organisms are not simply the results but they are also the causes of their own environments [50, 92]. Economic development can be viewed as a process of adaptation to a changing environment while itself being a source of environmental change. In real world societies, "*people survive to a large extent as members of groups. Group success depends on culture: the system of values, beliefs, artefacts, and art forms which sustain social organisation and rationalise action. Values and beliefs which fit the ecosystem survive and multiply; less fit ones eventually disappear. And thus cultural traits are selected much like genetic traits. At the same time, cultural values and beliefs influence how people interact with their ecosystem and apply selective pressure on species. Not only have people and their environment coevolved, but social systems and environmental systems have coevolved*" [92, p. 41]. From the co-evolutionary paradigm the following lessons can be learned:

(1) A priori, different models of co-evolution are possible, and then no unique optimal development path exists. The spatial dimension is a key feature of sustainable development.
(2) In environmental management local knowledge and expertise (being the result of a long co-evolutionary process) sometimes are more useful than experts' opinions. Social participation is then essential for successful sustainability policies.

Taking sustainability seriously into account creates a need for the inclusion of the physical appraisal of the environmental impacts on the socio-economic system too. As shown in Fig. 27.1, systemic approaches to sustainability issues consider the relationships between three systems: the economic system, the human system and the natural system [96]. The *economic system* includes the economic activities of humans, such as production, exchange and consumption. Given the scarcity phenomenon, such a system is efficiency oriented. The *human system* comprises all activities of human beings on our planet. It includes the spheres of biological human elements, of inspiration, of aesthetics, of social conflict, and of morality which constitute the frame of human life. Since it is clear that the economic system does not constitute the entire human system, one may assume that the economic system is a subsystem of the human system. Finally, the *natural system* includes both the human system and the economic system.

The previous discussion can be summarized by using the philosophical concept of *weak comparability* [70, 93]. From a philosophical perspective, it is possible

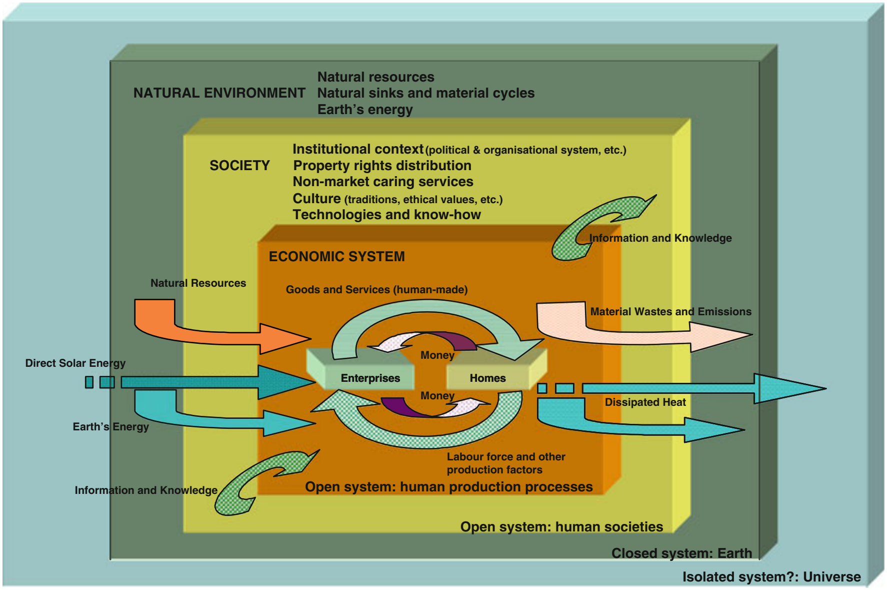


Fig. 27.1 A systemic vision of sustainability issues

to distinguish between the concepts of *strong comparability* (there exists a single comparative term by which all different actions can be ranked) implying *strong commensurability* (a common measure of the various consequences of an action based on a cardinal scale of measurement) or *weak commensurability* (a common measure based on an ordinal scale of measurement), and *weak comparability* (irreducible value conflict is unavoidable but compatible with rational choice employing, for example, multi-criteria evaluation).

In terms of formal logic, the difference between strong and weak comparability, and one defence of weak comparability, can be expressed in terms of Geach's distinction between *attributive and predicative adjectives* [46]. An adjective *A* is predicative if it passes the two following logical tests:

(1) *if x is AY, then x is A and x is Y;*
(2) *if x is AY and all Y's are Z's, then x is AZ.*

Adjectives that fail such tests are attributive. Geach claims that "*good*" is an attributive adjective. In many of its uses it clearly fails (2): "*X is a good economist, all economists are persons, and therefore X is a good person*" is an invalid argument. The fact that a comparative holds in one range of objects does not entail that it holds in the wider range. Given a claim that "*X is better than Y*" a proper response is "*X is better what than Y?*" Similar points can be made about the adjective "*valuable*" and "*is more valuable than*". If evaluative adjectives like "good" and "valuable" are attributive in standard uses, it follows that their comparative forms have a limited range. That does not however preclude the possibility of rational choices between objects that do not fall into the range of a single comparative. Weak comparability is compatible with the existence of such limited ranges.

It is in terms of such descriptions that evaluation takes place. A location is not evaluated as good or bad as such, but rather, as good, bad, beautiful or ugly *in relation to different descriptions.* It can be at one and the same time a "*good W*" and a "*bad X*", a "*beautiful Y*" and an "*ugly Z*". The use of these value terms in such contexts is attributive, not predicative. Evaluation of objects relative to different descriptions invokes not just different practices and perspectives, but also the different criteria and standards for evaluation associated with these. It presupposes value-pluralism. An appeal to different standards often results in conflicting appraisal of an object: as noted above, an object can have considerable worth as a *U, V, and W*, but little as an *X, Y and Z.*

In conclusion, weak comparability implies *incommensurability* i.e. there is an irreducible value conflict when deciding what common comparative term should be used to rank alternative actions. It is possible to further distinguish the concepts of social incommensurability and technical incommensurability [79, 81].

Social incommensurability refers to the existence of a multiplicity of legitimate values in society, and to deal with it, there is a need to consider the public participation issue. Any social decision problem is characterised by conflicts between competing values and interests and different groups and communities that represent them. In sustainability policies, biodiversity goals, landscape objectives, the direct services of different environments as resources and sinks, the historical and cultural meanings that places have for communities, the recreational options environments provide are a source of conflict [62]. Choosing any particular operational definition for *value* and its corresponding valuation technique involves making a decision about what is important and real. Distributional issues play a central role. Any policy option always implies winners and losers, thus it is important to check if a policy option seems preferable just because some dimensions (e.g. the environmental) or some social groups (e.g. the lower income groups) are not taken into account.

As a tool for conflict management, multi-criteria evaluation has demonstrated its usefulness in many sustainability policy and management problems in various geographical and cultural contexts (see e.g. [12, 13, 22, 42, 45, 57, 61, 73, 84, 90, 95, 101, 104, 117, 119, 123, 138]). The main point of force is the fact that the use of various evaluation criteria has a direct translation in terms of plurality of values used in the evaluation exercise. From this point of view, multiple-criteria decision analysis can be considered as a tool for implementing political democracy.

When dealing with sustainability issues neither an economic reductionism nor an ecological one is possible. Since in general, economic sustainability has an ecological cost and ecological sustainability has an economic cost, an integrative framework such as multi-criteria evaluation is needed for tackling sustainability issues properly. *Technical incommensurability* comes from the multidimensional nature of sustainability issues. One should note that the construction of a descriptive model of a real-world system depends on very strong assumptions about (1) the *purpose* of this construction, e.g. to evaluate the sustainability of a given city, (2) the *scale* of analysis, e.g. a block inside a city, the administrative unit constituting a Commune or the whole metropolitan area and (3) the set of dimensions, objectives and criteria used for the evaluation process. A reductionist approach for building a descriptive model can be defined as the use of just *one measurable indicator* (e.g. the monetary city product per person), *one dimension* (e.g. economic), *one scale of analysis* (e.g. the Commune), *one objective* (e.g. the maximisation of economic efficiency) and *one time horizon*. If one wants to avoid reductionism, there is a clear need to take into account incommensurable dimensions using different scientific languages coming from different legitimate representations of the same system [47–49]. This is what Neurath [89] called the need for an "*orchestration of sciences*".

The use of a multi-criteria framework is a very efficient tool to implement a multi/inter-disciplinary approach. When experts involved have various backgrounds in the beginning, the communication process is always very difficult; however it is astonishing to realize that when a multi-criterion framework is used, *immediately a common language is created*. This virtue of multi-criterion approaches has been corroborated in a great number of real-world case studies tackled by means of a variety of methods (see e.g., [11] who mainly uses MAUT approaches; [61] who builds on the DEFINITE software; [72] building on ELECTRE methods; [74] using AHP; [33] by means of NAIADE; [121] who use SMART). In terms of inter-disciplinarity, the issue is to find agreement on the set of criteria to be used; in terms of multi-disciplinarity, the issue is to propose and compute an appropriate criterion score. The efficiency of the interaction process can greatly increase and its effectiveness too.[2]

From this brief discussion the following conclusions can be drawn:

1. A proper evaluation of sustainability options needs to deal with a plurality of legitimate values and interests found in a society. From a societal point of view, economic optimization cannot be the only evaluation criterion. As is well known, not all goods have a market price, or this price is often too low (*market failures*). Environmental and distributional consequences (intra/inter-generational and for non-humans) must also be taken into account. In this framework multi-criteria evaluation is a very consistent approach.
2. If from a sustainability point of view, it is accepted that society as a whole has an indefinite lifespan, a much longer time horizon than is normally used on the market is required. A contradiction then arises: politicians usually have a very short time horizon (often 4–5 years depending on the electoral system) and this has the effect that sustainability is rarely among their priorities (thereby causing a *government failure* (for an overview of different perspectives on the role of governments in the economic sphere see e.g. [18]). For this reason evaluation of public projects should take into account the entire "*civil society*" (including

[2]Here I refer to the idea of orchestration of sciences as a combination of multi/inter-disciplinarity. Multi-disciplinarity: each expert takes her/his part. Inter-disciplinarity: methodological choices are discussed across the disciplines.

ethical concerns about *future generations*) and not only mythical benevolent policy-makers.

In the rest of this chapter, I will first analyse the role of multi-criteria decision analysis at a macroeconomic level, in particular with reference to the issue of construction and aggregation of sustainability assessment indicators and indexes. Then, I will discuss the use of multi-criteria techniques at a project level, for sustainability management and planning. At both levels, particular emphasis will be put on topics such as the role of problem structuring, the quality of the social process and the meaning of mathematical properties.

27.2 Measuring Sustainability: The Issue of Sustainability Assessment Indexes

From an economic point of view, traditionally Gross Domestic Product (GDP) has been considered as the best performance indicator for measuring national economy and welfare. But if resource depletion and degradation are factored into economic trends, what emerges is a radically different picture from the one depicted by conventional methods. In environmental terms, the GDP measure is plainly defective because:

1. no account is taken of environmental destruction or degradation;
2. natural resources as such are valued at zero;
3. repair and remedial expenditure such as pollution abatement measures, health care, etc., are counted as positive contribution to GDP inasmuch as they involve expenditures of economic goods and services.

In recent years, a growing stock of literature has been written on this topic and at the institutional level this debate has also invested the OECD and the European Commission, which devoted a number of recent conferences to the issue of well-being or happiness in the framework of "Measuring Progress". The purpose of "green accounting" is to provide information on the sustainability of the economy but there is no settled doctrine on how to combine different and sometimes contradictory indicators and indexes in a way immediately useful for policy (in the sense that GDP or other macroeconomic statistics have been useful for policy) [38]. The expression "*Taking nature into account*" (much used both in the UN system and in the European Union) hides the tension between money valuation, and appraisal through physical indicators and indexes (which themselves might show contradictory trends). So far, the elementary question of whether the European economy is moving towards sustainability or away from sustainability cannot be answered with consensus on the indicators and the integrative framework to be used (see e.g. [9, 23, 34, 59, 60, 78, 87, 98]).

A point of scientific controversy present in the contemporary debate is on the use of monetary or physical indexes. Examples of monetary indexes are Daly and

Cobb [26] ISEW (Index of Sustainable Economic Welfare), Pearce and Atkinson [97] Weak Sustainability Index, the so-called El Serafy approach [136]. Examples of physical indexes are HANPP (Human Appropriation of Net Primary Production) [131], the Ecological Footprint [133], MIPS (Material Input Per unit of Service) [116].

Although these approaches may look different, they all have some common characteristics:

1. The subcomponents needed for the building the aggregate index are *ad hoc*. No clear justification is given why e.g. diet enters in the computation of the ecological footprint and the generation of waste does not.
2. All the indexes are based on the assumptions that a common measurement rod needs to be established for aggregation purposes (money, energy, space, and so on). This creates the need of making very strong assumptions on conversion coefficients to be used and on compensability allowed (i.e. till which point better economical performances may cause environmental destruction or social exclusion?). The mathematical aggregation convention behind an index thus needs an explicit and well thought formulation.
3. The policy objective is often not clear. Inter-country or inter-city comparisons are a different policy objective than managing a particular country or city sustainability. Moreover, aggregate indexes are somewhat confusing, if one wishes to derive policy suggestions. For example, by looking at ISEW, we could know that indeed a country has a worst sustainability performance than the one pictured by standard GDP, but so what? ISEW being so aggregated does not supply any clear information of the cause of this bad performance and thus is useless for policy-making (while conventional GDP is at least giving clear information on the economic performance). The same applies to the ecological footprint, which sometimes can even give misleading policy suggestions (giving that diet is used, a more energy intensive agriculture might reduce the ecological footprint of e.g. a city, but in reality its environmental performance would be much worst!) or to the weak sustainability index (which is nothing but the classical golden rule of growth theory, where environmental physical destruction is never considered—above all if it is externalised outside the national borders).
4. All these approaches belong to the more general family of composite indicators and as a consequence, the assumptions used for their construction are common to them all.

Let's discuss this fourth point more in depth. Composite indicators[3] are very common in fields such as economic and business statistics and a variety of policy domains such as industrial competitiveness, sustainable development, globalisation and innovation. The proliferation of this kind of indicators is a clear symptom

[3]Composite indicators are indeed synthetic indexes, thus the two terms can be considered synonymous; here I use the term composite indicator since is the standard one in OECD/EC terminology [88].

of their political importance and operational relevance in decision-making; many international organizations propose their use in search of evidence based policy [88, 114]. From a formal point of view, a composite indicator is an aggregate of all dimensions, objectives, individual indicators and variables used for its construction. This implies that what defines a composite indicator is the set of properties underlying its aggregation convention. Although various functional forms for the underlying aggregation rules of a composite indicator have been developed in the literature, in the standard practice, a composite indicator is very often constructed by using a weighted linear aggregation rule applied to a set of variables. A typical composite indicator, I, is built as follows:

$$I = \sum_{i=1}^{N} w_i x_i \tag{27.1}$$

where x_i is a normalised variable and w_i a weight attached to x_i, with $\sum_{i=1}^{N} w_i = 1$ and $0 \le w_i \le 1$, $i = 1, 2, \ldots, N$. The main technical (i.e., without considering how variables have been selected) steps needed for its construction are two:

1. Standardisation of the variables to allow comparison without scale effect,
2. Weighted summation of these variables.

The standardisation step is a very delicate one. Main sources of a somewhat arbitrary assessment here are [88]:

- *Normalisation technique* used for the different measurement units dealt with.
- *Scale adjustment* used, for example population or GDP of each country considered.
- *Common measurement unit* used (money, energy, space and so on).

Let's first discuss the issue of linear aggregation of the variables chosen. As it is well known, the aggregation of several variables implies taking a position on the fundamental issue of compensability. The use of weights with intensity of preference originates compensatory aggregation conventions and gives the meaning of trade-offs to the weights. On the contrary, the use of weights with ordinal variable scores originates non-compensatory aggregation procedures and gives the weights the meaning of importance coefficients ([64, 99, 102, 129]; see also Chaps. 4 and 7 of this book).

Now the question arises: in their standard use weights in composite indicators are trade-offs or importance coefficients? "*Variables which are aggregated in a composite indicator have first to be weighted – all variables may be given equal weights or they may be given differing weights which reflect the significance, reliability or other characteristics of the underlying data. The weights given to different variables heavily influence the outcomes of the composite indicator. The rank of a country on a given scale can easily change with alternative weighting systems. … Greater weight should be given to components which are considered* *to be more significant in the context of the particular composite indicator*" [94, p. 10]. The concept of a weight used by OECD can be then classified as *symmetrical importance*, that is "... *if we have two non-equal numbers to construct a vector in R^2, then it is preferable to place the greatest number in the position corresponding to the most important criterion*." [99, p. 241].

Clearly, the mathematical convention underlying the additive aggregation model is a completely compensatory one. This means that in the weighted summation case, the substitution rates are equal to the weights of the variables up to a multiplicative coefficient. As a consequence, the estimation of weights is equivalent to that of substitution rates: the questions to be asked are in terms of "*gain with respect to one variable allowing to compensate loss with respect to another*" and NOT in terms of "*symmetrical importance*" of variables [16]. As a consequence in composite indicators, a theoretical inconsistency exists between the way weights are actually used and what their real theoretical meaning is.[4]

It is obvious that the aggregation convention used for composite indicators deal with the classical conflictual situation tackled in multi-criteria evaluation. Thus, the use of a multi-criterion framework for composite indicators in general and for sustainability indexes in particular is relevant and desirable [2, 35, 38, 80, 85, 128]. However, as made clear in this book, the so-called "*multi-criterion problem*" can be solved by means of a variety of mathematical approaches, all of them correct. This situation is due to Arrow's impossibility theorem [5], which proves that it is impossible to develop a "*perfect*" multi-criterion aggregation convention. This implies that it is desirable to have mathematical algorithms that may be recommended on some theoretical and empirical grounds. To deal with this problem, two main approaches can be distinguished.

1. The attempt of looking for a complete set of formal properties[5] that can be attributed to a specific method (e.g., [6, 130]).
2. The attempt to check under which specific circumstances each method could be more useful than others, i.e. the search of the right method for the right problem

[4]One should note that this inconsistency is present in the majority of the *environmental impact assessment studies* too. In fact it is a common practice to aggregate environmental impact indicators by means of a linear rule and to attach weights to them according to the relative importance idea. Moreover, the use of a linear aggregation procedure implies that among the different ecosystem aspects there are not phenomena of synergy or conflict. This appears to be quite an unrealistic assumption for environmental impact assessment studies [39]. For example, "*laboratory experiments made clear that the combined impact of the acidifying substances SO_2, NO_X, NH_3 and O_3 on plant growth is substantially more severe that the (linear) addition of the impacts of each of these substances alone would be.*" [31].

[5]Often this search for clear properties characterizing an algorithm is indicated as the axiomatic approach. However, one should note that properties or assumptions are NOT axioms. As perfectly synthesized by Saari [[110], p. 110] "*Many, if not most, results in this area are merely properties that happen to uniquely identify a particular procedure. But unless these properties can be used to construct, or be identified with <u>all</u> properties of the procedure (such as in the development of utility functions in the individual choice literature), they are not building blocks and they most surely are <u>not</u> axioms: they are properties that just happen to identify but not characterize, a procedure. As an example, the two properties (1) Finnish-American heritage (2) a particular DNA structure, uniquely identify me, but they most surely do not characterize me*".

(e.g., see [56] for a general approach, and [113] for a discussion in the context of environmental problems).

Next section gives an example of the first approach in the framework of sustainability composite indicators. Section 27.6 will deal with the second approach in the framework of multi-criteria evaluation of sustainability policies at a micro-level.

27.3 A Defensible Setting for Sustainability Composite Indicators

As discussed in the previous section, in the framework of sustainability composite indicators there is a need for a theoretical guarantee that weights are used with the meaning of "*symmetrical importance*". As a consequence, complete compensability should be avoided. This implies that *variables have to be used with an ordinal meaning*. This is not a problem since no loss of information is implied [6]. Moreover, given that often the measurement of variables is rough, it seems even desirable to use indicator scores with an ordinal meaning. Given that there is a consensus in the literature that the Condorcet' theory of voting is non-compensatory while Borda's one is fully compensatory, a first conclusion is that when one wishes to have weights as *importance coefficients*, there is a need for a Condorcet approach[6] while a Borda's one is desirable when weights are meaningful in the form of *trade-offs* [28, 75].

[6]Arrow and Raynaud [[6], pp. 77–78] arrive at the conclusion that a Condorcet aggregation algorithm has always to be preferred in a multi-criterion framework. On the complete opposite side one can find Saari [108–110]. His main criticism against Condorcet based approaches are based on two arguments: (1) if one wants to preserve relationships among pairs (e.g., to impose a side constraint to protect some relationship-balanced gender for candidates in a public concourse) then it is impossible to use pair-wise voting rules, a Borda count should be used necessarily. However, it is important to note that, although desirable in some cases, to preserve a relationship among pairs implies the loss of neutrality; this is not desirable on general grounds. (2) The individual rationality property (i.e. transitivity) has necessarily to be weakened if one wishes to adopt a Condorcet based voting rule. The underlying assumption of this definition is the identification of human rationality with consistency, and this can be criticized from many points of view. Simon [118] notes that humans have at their disposal neither the facts nor the consistent structure of values nor the reasoning power needed to apply the principles of utility theory. In microeconomics, where the assumption that an economic agent is always a utility maximize is a fundamental one, it is generally admitted that this behavioural assumption has a predictive meaning and not a descriptive one (see Friedman [37] for the most forceful defence of this non-descriptive meaning of the axioms of ordinal utility theory). As firstly noted by Luce [68], a down-to-earth preference modelling should imply the use of indifference and preference thresholds; this implies exactly the loss of the transitivity property of at least the indifference relation. A corroboration of this criticism in the framework of social choice can be found in Kelsey (1986), where it is stated that because of social choice problems, an individual with multiple objectives may find it impossible to construct a transitive ordering. Recent analyses of the concept of rational agent can also be found in Bykvist [19] and Sugden [124].

A problem inherent to the Condorcet consistent family of algorithms is the presence of cycles. The probability π (N, M) of obtain a cycle with N countries (regions, cities, etc.) and M individual indicators increases with N as well as the number of indicators. With many countries and individual indicators, cycles occur with an extremely high frequency. *As a consequence, the ranking procedure used has to deal with the cycle issue properly.*

Let's then discuss the cycle issue. A cycle breaking rule normally needs some arbitrary choice such as to delete the cycle with the lowest support. Now the question is: Is it possible to tackle the cycle issue in a more general way? Condorcet himself was aware of the problem of cycles in his approach; he built examples to explain it and he got close to find a consistent rule able to rank any number of alternatives when cycles are present. However, attempts to fully understand this part of Condorcet's voting theory came to a conclusions like "*... the general rules for the case of any number of candidates as given by Condorcet are stated so briefly as to be hardly intelligible ... and as no examples are given it is quite hopeless to find out what Condorcet meant*" (E.J. Nanson as quoted in [14, p. 175]). Or "*The obscurity and self-contradiction are without any parallel, so far as our experience of mathematical works extends ... no amount of examples can convey an adequate impression of the evils*" ([125, p. 352] as cited by Young [134, p. 1234]).

Attempts of clarifying, fully understanding and axiomatizing Condorcet's approach for solving cycles have been mainly done by Kemeny [65] who made the first intelligible description of the Condorcet approach, and by Young and Levenglick [135] who gave the clearest exposition and a complete axiomatisation. For this reason we can call this approach the Condorcet–Kemeny–Young–Levenglick (henceforth C-K-Y-L) ranking procedure.

Arrow and Raynaud [6, p. 77] also arrive at the conclusion that the highest feasible ambition for an aggregation algorithm building a multi-criterion ranking is to be Condorcet. These authors discard what they call the Kemeny's method, on the grounds that preference reversal phenomena may occur inside this approach [6, p. 96]. However, although the so-called Arrow-Raynaud's method does not present rank reversal, it is not applicable if cycles exist. Since in the context where composite indicators are built, cycles are very probable to occur, here the only solution is to choose the C-K-Y-L ranking procedure, thus accepting that rank reversals might appear.[7] The acceptance of rank reversals phenomena implies that the famous axiom of independence of irrelevant alternatives of Arrow's theorem is not respected. Anyway, Young [134, p. 1241] claims that the C-K-Y-L ranking procedure is the "*only plausible ranking procedure that is locally stable*". Where *local stability* means that the ranking of alternatives does not change if only an interval of the full ranking is considered.

[7] Anyway a Condorcet consistent rule always presents smaller probabilities of the occurrence of a rank reversal in comparison with any Borda consistent rule. This is again a strong argument in favour of a Condorcet's approach in this framework.

The adaptation of C-K-Y-L ranking procedure to the case of composite indicators is very simple. The maximum likelihood ranking of countries (regions, cities, etc.) is the ranking supported by the maximum number of individual indicators for each pair-wise comparison, summed over all pairs of countries considered.

Formally, a simple ranking algorithm of sustainability composite indicators, based on these concepts, can be the following [83].

Given a set of individual indicators $G = \{g_m\},\ m = 1,\ 2, \ldots,\ M,$ and a finite set $A = \{a_n\},\ n = 1,\ 2, \ldots,\ N$ of countries (cities or regions), let's assume that the evaluation of each country a_n with respect to an individual indicator g_m (i.e. the indicator score or variable) is based on an *interval or ratio* scale of measurement. For simplicity of exposition, let's assume that a higher value of an individual indicator is preferred to a lower one (the higher, the better), that is:

$$\begin{cases} a_j\, P\, a_k \iff g_m\ \left(a_j\right) > g_m\ \left(a_k\right) \\ a_j\, I\, a_k \iff g_m\ \left(a_j\right) = g_m\ \left(a_k\right) \end{cases} \tag{27.2}$$

Where, P and I indicate a preference and an indifference relation respectively, both fulfilling the transitive property.

Let's also assume the existence of a set of individual indicator weights derived as importance coefficients. The mathematical problem to be dealt with is then how to use this available information to rank in a complete pre-order (i.e. without any incomparability relation) all the countries from the best to the worst one.

The mathematical aggregation convention can be divided into two main steps:

1. Pair-wise comparison of countries according to the whole set of individual indicators used.
2. Ranking of countries in a complete pre-order.

For carrying out the pair-wise comparison of countries the following axiomatic system is needed (adapted from Arrow and Raynaud [6, pp. 81–82]).

Axiom 1: Diversity Each individual indicator is a total order on the finite set A of countries to be ranked, and there is no restriction on the individual indicators; they can be any total order on A.

Axiom 2: Symmetry Since individual indicators have incommensurable scales, the only preference information they provide is the ordinal pair-wise preferences they contain.[8]

[8]In our case, this axiom is needed since the intensity of preference of individual indicators is not considered to be useful preference information given that compensability has to be avoided and weights have to be symmetrical importance coefficients. Moreover, thanks to this axiom, a normalisation step is not needed. This reduces the sources of uncertainty and imprecise assessment.

Axiom 3: Positive Responsiveness The degree of preference between two countries *a* and *b* is a strictly increasing function of the number and weights of individual indicators that rank *a* before *b*.[9]

Thanks to these three axioms a $N \times N$ matrix, *E*, called *outranking matrix* [6, 106] can be built. Any generic element of E: $e_{jk}, j \neq k$ is the result of the pair-wise comparison, according to all the *M* individual indicators, between countries *j* and *k*. Such a global pair-wise comparison is obtained by means of Eq. (27.3).

$$e_{jk} = \sum_{m=1}^{M} \left(w_m \left(P_{jk} \right) + \frac{1}{2} w_m \left(I_{jk} \right) \right) \tag{27.3}$$

where $w_m(P_{jk})$ and $w_m(I_{jk})$ are the weights of individual indicators presenting a preference and an indifference relation respectively. It clearly holds

$$e_{jk} + e_{kj} = 1. \tag{27.4}$$

All the $N(N-1)$ pair-wise comparisons compose the outranking matrix *E*. Call *R* the set of all *N!* possible complete rankings of alternatives, $R = \{r_s\},\ s = 1,\ 2, \ldots,\ N!$. For each r_s, compute the corresponding score ϕ_s as the summation of e_{jk} over all the $\binom{N}{2}$ pairs *j, k* of alternatives, i.e.

$$\phi_s = \sum e_{jk}. \tag{27.5}$$

$$where\ j \neq k,\ \ s = 1,\ 2, \ldots N!\ \ and\ \ e_{jk} \in r_s$$

The final ranking (r_*) is the one which maximises Eq. (27.6), which is:

$$r_* \iff \phi_* = \max \sum e_{jk} \qquad where\ e_{jk} \in R. \tag{27.6}$$

Other formal properties of the C-K-Y-L ranking procedure are the following [135]:

- *Neutrality*: it does not depend on the name of any country, all countries are equally treated.

[9] In social choice terms then the *anonymity* property (i.e. equal treatment of all individual indicators) is broken. Indeed, given that full decisiveness yields to dictatorship, Arrow's impossibility theorem forces us to make a trade-off between *decisiveness* (an alternative has to be chosen or a ranking has to be made) and anonymity. In our case the loss of anonymity in favour of decisiveness is even a positive property. In general, it is essential that no individual indicator weight is more than 50 % of the total weight; otherwise the aggregation procedure would become lexicographic in nature, and the indicator would become a dictator in Arrow's term.

- *Unanimity* (sometimes called *Pareto Optimality*): if all individual indicators prefer country *a* to country *b* than *b* should not be chosen.
- *Monotonicity*: if country *a* is chosen in any pair-wise comparison and only the individual indicator scores (i.e. the variables) of *a* are improved, then *a* should be still the winning country.
- *Reinforcement*: if the set *A* of countries is ranked by two subsets G_1 and G_2 of the individual indicator set *G*, such that the ranking is the same for both G_1 and G_2, then $G_1 \cup G_2 = G$ should still supply the same ranking. This general consistency requirement is very important in the framework of composite indicators, since one may wish to apply the individual indicators belonging to each single dimension first and then pool them in the general model.

At this point a question arises: does the application of a formally correct mathematical aggregation procedure always guarantee the quality of the results obtained? This problem is tackled in the next section.

27.4 Warning! Not Always Rankings Have to Be Trusted . . .

Let's now take into consideration an illustrative example regarding four cities, two belonging to highly industrialized Countries (Amsterdam and New York) and two belonging to transitional economies (Budapest and Moscow) [80]. The indicators used are typical of the literature on urban sustainability (see e.g. [10, 20]). The profiles (i.e. the score of each city according to each indicator) of these four cities are the ones described in Table 27.1.

Several techniques can be used to standardise variables [94, 111]. However, although each normalisation technique entails different absolute values, the ranking

Table 27.1 Impact matrix for the four chosen cities according to the selected indicators

Matrix type	Impact	Case study	Alternatives			
Criteria			Budapest	Moscow	Amsterdam	New York
Houses owned (%)			50.5	40.2	2.2	10.3
Residential density (pers. /hectare)			123.3	225.2	152.1	72
Use of private car (%)			31.1	10	60	32.5
Mean travel time to work (minutes)			40	62	22	36.5
Solid waste generated per capita (t./year)			0.2	0.29	0.4	0.61
City product per person (US$/year)			4750	5100	28,251	30,952
Income disparity (Q5/Q!)			9.19	7.61	5.25	14.81
Households below poverty line (%)			36.6	15	20.5	16.3
Crime rate per 1000 (theft)			39.4	4.3	144.05	56.7

Table 27.2 Normalised impact matrix

100	78.674	0	16.770
66.515	0	47.72	100
57.8	100	0	55
55	0	100	63.75
100	78.05	51.22	0
0	1.335	89.691	100
58.787	75.314	100	0
0	100	74.538	93.982
74.884	100	0	62.505

Table 27.3 Outranking matrix of the four cities according to the nine indicators

	Budapest	Moscow	Amsterdam	New York
Budapest	0	4	4	5
Moscow	5	0	5	6
Amsterdam	5	4	0	3
New York	4	3	6	0

provided remains constant. In our example, the "*distance from the best and worst performers*" technique is applied, where positioning is in relation to the global maximum and minimum and the index takes values between 0 (laggard) and 100 (leader):

$$100\left(\frac{\text{actual value} - \text{minimum value}}{\text{maximum value} - \text{minimum value}}\right) \tag{27.7}$$

By applying Eq. (27.7) to the values contained in Table 27.2, the results presented in Table 27.3 are obtained. By applying Eq. (27.1) to the values contained in Table 27.3, the following results are obtained:

Budapest = 512.986
Moscow = 533.373
Amsterdam = 463.169
New York = 492.052

Thus the final ranking presents Amsterdam in the bottom position (worst than all the other cities considered), Moscow is in the top position, Budapest ranks second and New York ranks third. As a first reaction one might think that these somewhat surprising results are due to the use of the linear aggregation rule. Let's then apply the algorithm illustrated from Eqs. (27.2) to (27.6) to the impact matrix showed in Table 27.1.

The outranking matrix E is the one showed in Table 27.3.

The 24 possible rankings and the corresponding scores ϕ_s are the following:

B	**A**	**D**	**C**	**31**	C	B	D	A	27
B	**D**	**C**	**A**	**31**	D	B	A	C	27
A	B	D	C	30	D	C	B	A	27
B	D	A	C	30	A	C	B	D	26
B	C	A	D	29	A	D	C	B	26
B	A	C	D	28	D	A	B	C	26
B	C	D	A	28	D	C	A	B	26
C	B	A	D	28	D	A	C	B	25
D	B	C	A	28	C	A	D	B	24
A	B	C	D	27	C	D	B	A	24
A	D	B	C	27	A	C	D	B	23
C	A	B	D	27	C	D	A	B	23

Where A is Budapest, B is Moscow, C is Amsterdam and D is New York.

Also in this case Moscow is clearly in the top position. New York is surely better than Amsterdam. The position of Budapest with respect to both New York and Amsterdam is not well defined.

Let's look at Table 27.1 again. The nine indicators used seem reasonable; they indeed belong to three dimensions, i.e. economical, social and environmental, considered essential in any sustainability assessment. Let's then try to understand to which dimension each single indicator belongs. Roughly the following classification may be made:

Economic dimension

1. City product per person

Environmental dimension

2. Use of private car
3. Solid waste generated per capita

Social dimension

4. Houses owned
5. Residential density
6. Mean travel time to work
7. Income disparity
8. Households below poverty line
9. Crime rate

Clearly the social dimension is receiving implicitly a much bigger weight than any other dimension (considering that six indicators over nine belong to this dimension). A reasonable decision might be to consider the three dimensions equally important. This would imply to give the same weight to each dimension

Table 27.4 Weighted outranking matrix

	Budapest	Moscow	Amsterdam	New York
Budapest	0	0.3	0.4	0.4
Moscow	0.7	0	0.5	0.6
Amsterdam	0.6	0.5	0	0.3
New York	0.6	0.4	0.7	0

considered and finally to split this weight among the indicators. That is, each dimension has a weight of 0.333; then the economic indicator has a weight of 0.333, the two environmental indicators have a weight of 0.1666 each, and each one of the six social indicators receives a weight equal to 0.0555. As one can see, if dimensions are considered, weighting indicators by means of importance coefficients is crucial.

Let's now see if this weighting exercise provokes any change in the final ranking. The new outranking matrix is the one presented in Table 27.4.

The 24 possible rankings and the new corresponding scores ϕ_s are the following (where A is Budapest, B is Moscow, C is Amsterdam and D is New York):

B	**D**	**C**	**A**	**3, 6**	B	C	A	D	2, 9
D	B	C	A	3, 5	C	B	A	D	2, 9
D	C	B	A	3, 5	A	B	D	C	2, 9
B	D	A	C	3, 5	B	A	C	D	2, 8
D	B	A	C	3, 4	A	D	B	C	2, 8
B	A	D	C	3, 3	A	D	C	B	2, 8
B	C	D	A	3, 2	C	D	A	B	2, 7
C	B	D	A	3, 2	C	A	B	D	2, 6
D	C	A	B	3, 2	C	A	D	B	2, 5
C	D	B	A	3, 1	A	B	C	D	2, 5
D	A	B	C	3, 1	A	C	B	D	2, 5
D	A	C	B	3, 1	A	C	D	B	2, 4

As one can see, Moscow is still on the top position, but this time Budapest is on the bottom one. New York scores again better than Amsterdam.

Concluding, we can state that an advantage of this algorithm is to highlight the fact that rankings are not always robust, even if no parameter is changed. This type of lack of robustness is completely ignored by the linear aggregation rule. Moreover, the use of weights as importance coefficients can change the problem modelling significantly. However one has to note that the improvement of the mathematical aggregation procedure does not change the results spectacularly. The structuring process, and in this case above all, the input information used for the indicator scores determine clearly the ranking. *Garbage in, garbage out* phenomena are almost impossible to avoid.

At this point a general question needs to be answered: *From where are multi-criteria results coming from and what they mean?* The results obtained depend on:

1. *quality of the information available* (in our case for example the data concerning Amsterdam on the use of private cars and on criminality are suspiciously high, while criminality in Moscow or residential density in New York are suspiciously low),
2. *indicators chosen* (i.e. which representation of reality we are using, e.g. whose interests we are taken into account),
3. *Direction of each indicator* (i.e. the bigger the better or vice versa, e.g. in our example, it has been used the principle that house owners should be maximized, but this could be quite disputable and culturally dependent),
4. *relative importance of these indicators* (indicated by the weighting factor attached),
5. *ranking method used*.

All these uncertainties have to be taken into account when we state that an evaluation is made. Points from 1 to 4 clearly concern the way a given assessment exercise is structured; this implies that the quality of the aggregation convention is an important step to guarantee consistency between the assumptions used and the ranking obtained; but *the overall quality of a multi-criteria study depends crucially on the way this mathematical model is embedded in the social, political and technical structuring process.* This is the reason why in multi-criteria decision aid (MCDA) it is claimed that what is really important is the "*decision process*" and not the final solution [105, 106].

However, while it is clear what this means in terms of single-person decisions, how can we deal with the issue of a social process? To answer this question will be the aim of the next section.

27.5 The Issue of the "Quality of the Social Decision Processes"

In empirical evaluations of public projects and public provided goods, multi-criteria decision analysis seems to be an adequate policy tool since it allows taking into account a wide range of assessment criteria (e.g. environmental impact, distributional equity, and so on) and not simply profit maximisation, as a private economic agent would do. However, the management of a policy process involves many layers and kinds of decisions, and requires the construction of a *dialogue process* among many stakeholders, individual and collective, formal and informal, local and not.

In general, these concerns have not been considered very relevant by scientific research in the past (where the basic implicit assumption was that time was an infinite resource). On the other hand, the new nature of the policy problems faced in this third millennium (e.g., the mad cow, genetic modified organisms, ...), implies that very often when using science for policy-making, long term consequences may exist and scientists and policy-makers are confronting issues where, "*facts* *are uncertain, values in dispute, stakes high and decisions urgent*" [40, 41]. In this case, scientists cannot provide any useful input without interacting with the rest of society and the rest of the society cannot perform any sound decision making without interacting with the scientists. That is, the question on "*how to improve the quality of a social decision process*" must be put, quite quickly, on the agenda of "scientists", "decision makers" and indeed the whole society.

An outcome of this discussion is that the political and social framework must find a place in multi-criteria decision analysis. An effective policy exercise should consider not merely the measurable and contrastable dimensions of the simple parts of the system, that even if complicated may be technically simulated (technical incommensurability). To be realistic it should also deal with the higher dimensions of the system. Those dimensions in which power relations, hidden interests, social participation, cultural constraints, and other "soft" values, become relevant, and unavoidable variables that heavily, but not deterministically, affect the possible outcomes of the strategies to be adopted (social incommensurability).

At this point in the discussion, one question arises, who is making the decisions? Some critics of multi-criteria evaluation say that *in principle*, in cost-benefit analysis, votes expressed on the market by the whole population can be taken into account (of course with the condition that the distribution of income is accepted as a means to allocate votes).[10] On the contrary, multi-criteria evaluation can be based on the priorities and preferences of some decision-makers only (we could say that the way these decision-makers have reached their position is accepted as a way to allocate the right to express these priorities). This criticism may be correct if a "*technocratic approach*" is taken, where the analyst constructs the problem relying only upon experts' inputs (by experts meaning those who know the "technicalities" of a given problem).

For the formation of contemporary public policies, it is hard to imagine any viable alternative to *extended peer communities* [24, 30, 40, 41, 51, 53–55, 63]. They are already being created, in increasing numbers, either when the authorities cannot see a way forward, or know that without a broad base of consensus, no policies can succeed. They all have one important element in common: they assess the quality of policy proposals, including the scientific and technical component. And their verdicts all have some degree of moral force and hence political influence. Here the quality is not merely in the verification, but also in the *creation*; as local people can imagine solutions and reformulate problems in ways that the accredited experts, with the best will in the world, do not find natural [21].

This need of incorporating the general public into the policy processes has been more and more recognized by the multi-criteria community. Science for policy implies a responsibility of the scientists towards the whole society and not just towards a mythical decision-maker. The classical schematised relationship decision-maker/analyst is indeed embedded in a social framework, which is of a crucial

[10]One should note that indeed cost-benefit analysis can be easily criticised both from the distributive and environmental points of view (see e.g., [77, 120]). However I prefer not to deal with this issue here.

importance in the case of sustainability management and planning. Banville et al. [7] offers a very well structured and convincing argumentation on the need to extend Roy's concept of Multiple Criteria decision Aid by incorporating the notion of stakeholder [extension called "*Participative Multi-criteria Evaluation*" *(PMCE) or* "*Stakeholder Multi-Criteria Decision Aid*" *(SMCDA)*].

However, in my opinion, participation is a *necessary* condition but not a *sufficient* one, since the scientific team cannot simply accept uncritically the inputs of a participatory process. The main justifications of this statement are the following:

a) In a focus group, powerful stakeholders may influence deeply all the others.
b) Some stakeholders might not desire or be able to participate, but ethically the scientific team should not ignore them.
c) The notion of stakeholder only recognises relevant organised groups; this is the reason why the term "*social actor*" seems preferable to me.
d) Focus groups are never meant to be a representative sample of population. As a consequence, they can be a useful instrument to improve the knowledge of the scientific team of the institutional and social dimensions of the problem at hand, but never a way for deriving consistent conclusions on social preferences.
e) Since decision-makers search for legitimacy[11] of the decisions taken, it is extremely important that public participation or scientific studies do not become instruments of political de-responsibility. The deontological principles of the scientific team and policy-makers are essential for assuring the quality of the evaluation process. Social participation does not imply that scientists and decision-makers have no responsibility of policy actions defended and eventually taken.

Synthesising these arguments we can say that a participatory policy process can always be conditioned by heavy value judgements such as, have all the social actors the same importance (i.e. weight)? Should a socially desirable ranking be obtained on the grounds of the majority principle? Should some veto power be conceded to the minorities? Are income distribution effects important? And so on.

One of the most interesting research directions in the field of public economics is the attempt to introduce political constraints, interest groups and collusion effects explicitly (see e.g. [67]). In this context, *transparency* becomes an essential feature of public policy processes [122]. *Social Multi-Criteria Evaluation (SMCE)* has been explicitly designed to enhance transparency; the main idea being that results of an evaluation exercise depends on the way a given policy problem is *represented* and thus the assumptions used, the interests and values considered have to be made clear [79, 81].

A clear example of these considerations can be found in the determination of criterion weights. Can we have an elicitation of weights from all the social actors involved to be used in the evaluation process? As we know in society there are different legitimate values and points of view. This creates social pressure for taking

[11] On the issue of legitimacy see also Roy and Damart [107].

into account various policy dimensions, e.g. economic, social and environmental.[12] These dimensions are then translated by analysts into objectives and criteria. At this point a question arises who should attach criterion weights and how? To answer this question we have to accept a basic assumption: to attach weights to different criteria implies to give weights to different groups in society. This assumption has the following main consequences:

1. In social decision processes, weights cannot be derived as inputs coming from participatory techniques. This is *technically* very difficult (e.g., which elicitation method has to be used? Which statistical index is a good synthesis of the results obtained? Do average values of weights have meaning at all?), *pragmatically* not desirable (since strong conflicts among the various social actors are very probable to occur) and even *ethically* unacceptable (at least if a Kantian position is taken). *A plurality of ethical principles* seems the only consistent way to derive weights in a social framework.
2. Ethical judgements are unavoidable components of the evaluation exercise. These judgements always influence heavily the results. Let's imagine the extreme case where a development project in the Amazon forest will affect an indigenous community with little contact with other civilizations yet. Would it be ethically more correct to invite them in a focus group . . . or ethically compulsory to take into account the consequences of the project for their survival? As a consequence, transparency on the assumptions used is essential.
3. Weights in SMCE are clearly meaningful only as *importance coefficients* and not as trade-off (since different ethical positions leads to different ideas on criterion importance). This also implies that the aggregation conventions used should be non-compensatory mathematical algorithms. Non-compensability implies that minorities represented by criteria with smaller weights can still be very influent. This is for example clear in the use of the discordance index in the ELECTRE methods [105, 106].
4. *Sensitivity and robustness analysis* have a complete different meaning with respect to the case of single person and technical decisions [112, 115]. In fact in the case of SMCE, weights derive only from a few clear cut ethical positions. This means that sensitivity or robustness analysis have to check the consequences on the final ranking of only these positions and not of all the possible combinations of weights. Sensitivity and robustness analysis are then a way to improve transparency.[13]

[12]By *dimension*, here I mean the highest hierarchical level of analysis which indicates the scope of objectives and criteria.

[13]On this point I disagree with Kleijnen [66], who claims that "*modellers should try to develop robust models*", in the sense that models should not be very sensitive to modellers' assumptions. Some ethical positions might be very different and thus lead to different rankings of the policy options. What is essential in a social framework is then transparency on these assumptions.

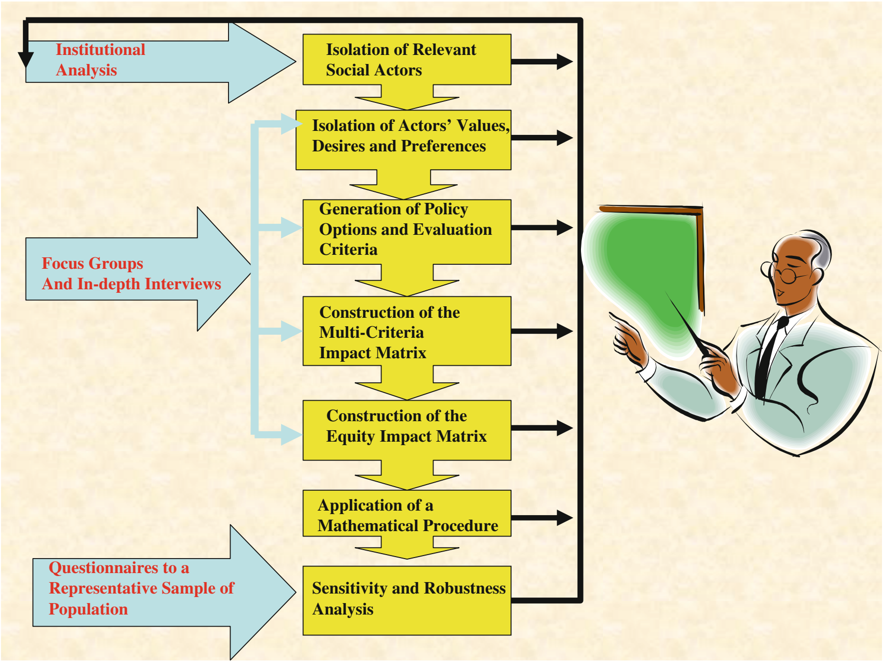


Fig. 27.2 The ideal problem structuring in SMCE

In a social multi-criteria evaluation framework, the pitfalls of the technocratic approach can be overtaken by applying different *methods of sociological research* (see Fig. 27.2). For example, "*institutional analysis*", performed mainly on historical, legislative and administrative documents, can provide a map of the relevant social actors. By means of focus groups it is possible to have an idea of people's desires and it is then possible to develop a set of policy options. Main limitations of the focus group technique are that they are not supposed to be a representative sample of the population and that sometimes people are not willing to participate or to state publicly what they really think (above all in small towns and villages). For this reason anonymous questionnaires and personal interviews are an essential part of the participatory process (for practical examples of participative and social multi-criteria evaluation see e.g., [4, 29, 43, 44, 52, 91, 103, 126]).

The selection of evaluation criteria has to be also based on what it is learned through the participation process. However, at this stage a problem generally arises: the evaluation criteria should come directly from the public participation process or they should be "translated" by the research team? I think that the rough material collected during interviews and focus groups could be used as a source of inspiration but the technical formulation of criteria having properties such as "non-redundancy", "legibility" and so on (see [15]) is a clear job of the researchers. Of course in this step, subjectivity is unavoidable, for this reason a widespread information campaign on the assumptions and conclusions of the study including local people, regional and national authorities, international scientists and even children at school is, in my opinion, highly recommendable.

Finally one has to note that policy evaluation is not a one-shot activity. On the contrary, it takes place as a *learning process* which is usually highly dynamic, so that judgements regarding the political relevance of items, alternatives or impacts may present sudden changes, hence requiring a policy analysis to be flexible and adaptive in nature. This is the reason why evaluation processes have a *cyclic nature*. By this is meant the possible adaptation of elements of the evaluation process due to continuous feedback loops among the various steps and consultations among the actors involved.

At this stage a question arises: which is the role of mathematical aggregation procedures in a social evaluation process of sustainability policies? In this framework, of course mathematical aggregation conventions play an important role, i.e. to assure that the rankings obtained are *consistent* with the information and the assumptions used along the structuring process. Next section then discusses the technical properties considered desirable for a multi-criteria algorithm to assure such a consistency.

27.6 The Issue of Consistency in Multi-Criteria Evaluation of Sustainability Policies

An issue, that makes multi-criterion aggregation conventions intrinsically complex, is the fact they are *formal, descriptive and normative* models simultaneously [76]. As a consequence, the properties of an approach have to be evaluated at least in the light of these three dimensions. Musgrave [86] in the framework of the debate on the maximisation assumption in microeconomics, made a very useful classification of the assumptions used in economic theory. He makes a distinction among *negligibility assumptions, domain assumptions and heuristic assumptions.* The first type is required to simplify and focus on the essence of the phenomena studied. The second type of assumptions is needed when applying a theory to specify the domain of applicability. The third type is needed either when a theory cannot be directly tested or when the essential assumptions give rise to such a complex model that successive approximation is required. One might see this last type of assumptions as the sake of learning about limits to the relationship between understandable implications and complexity.

In this section, by using these categories, I try to isolate some main properties that may be considered desirable for a discrete multi-criteria method in the framework of sustainability policies. Of course in another framework, e.g. stock exchange investments, these properties can easily be irrelevant or even undesirable.

When an economic/environmental integration has to be dealt with, a fundamental issue is the one of *compensability*. As we already saw, compensability refers to the existence of trade-offs, i.e. the possibility of offsetting a disadvantage on some criteria by a sufficiently large advantage on another criterion, whereas smaller advantages would not do the same. Thus a preference relation is non-compensatory if no trade-off occurs and is compensatory otherwise. The use of weights with intensity of preference originates compensatory multi-criteria methods and gives the meaning of trade-offs to the weights. On the contrary, the use of weights with ordinal criterion scores originates non-compensatory aggregation procedures and gives the weights the meaning of importance coefficients.

Mathematical compensability plays an important role in the implementation of the so-called "*weak and strong sustainability concepts*". Weak sustainability has been theorised mainly by those economists who have a quite optimistic view of *technological progress* and economic growth. They generally recognise that even if the production technologies of an economy can potentially yield increases in output commensurate with increases in inputs, overall output will be constrained by limited supplies of resources (growth theory with exhaustible resources). But these limits can be overcome by technological progress: if the rate of technological progress is high enough to offset the decline in the per capita quantity of natural resource services available, output per worker can rise indefinitely. A stronger statement is the following: *even in the absence of any technological progress exhaustible resources do not pose a fundamental problem if reproducible man-made capital is sufficiently* <u>*substitutable*</u> *for natural resources* [27]. Pearce and Atkinson [97] state that an economy is sustainable, if it saves more than the combined depreciation of natural and man-made capital. "*We can pass on less environment so long as we offset this loss by increasing the stock of roads and machinery, or other man-made (physical) capital. Alternatively, we can have fewer roads and factories so long as we compensate by having more wetlands or mixed woodlands or more education*" [127, p. 56].

From an ecological perspective, the expansion of the economic subsystem is limited by the size of the overall finite global ecosystem, by its dependence on the life support sustained by intricate ecological connections which are more easily disrupted as the scale of the economic subsystem grows relative to the overall system. This calls for a different concept of sustainability, that of *strong sustainability*, according to which certain sorts of natural capital are deemed critical and not readily substitutable by man-made capital [9]. Human expansion, with the associated exploitation and disposal of waste and pollutants, not only affects the natural environment as such, but also the level and composition of environmentally produced goods and services required to sustain society. Thus, the economic subsystem will be limited by the impacts of its own actions on the environment [36].

Unlimited growth cannot take place in a physically limited planet. Technology is, obviously, an important tool for a development truly sustainable but should not be mystified. The scale of human activities has a maximum expansion possibility defined either by the *regenerative or absorptive capacity of the ecosystem*. Strong sustainability implies that certain sorts of natural capital are deemed critical and not readily substitutable by man-made capital; it is clear that if one wants to operationalize strong sustainability, there is a clear need to use non-compensatory multi-criterion algorithms. Another argument in favour of non-compensatory algorithm is given by the desirability, in the framework of social decisions, that criterion weights can be attached in the form of importance coefficients and not as trade-offs. Clear examples of non-compensatory methods are the ELECTRE methods (see Chap. 4 of this book and [105, 106]) and the Condorcet type algorithm described in Sect. 27.3 of this chapter.

Another important desirable property is the possibility of dealing with mixed criterion scores. It has been argued that the presence of qualitative information in evaluation problems concerning socio-economic and physical planning is a rule, rather than an exception [90]. Thus, the idea of technical incommensurability implies that there is a clear need for methods that are able to take into account information of a "mixed" type (both qualitative and quantitative criterion scores). For simplicity, I refer to *qualitative information* as information measured on a nominal or ordinal scale, and to *quantitative information* as information measured on an interval or ratio scale. Examples of multi-criteria methods able to deal with mixed criterion scores are REGIME [58] and EVAMIX [132].

Moreover, ideally, this information should be precise, certain, exhaustive and unequivocal. But in reality, it is often necessary to use information which does not have those characteristics so that one has to face the uncertainty of a stochastic and/or fuzzy nature present in the data.

If it is impossible to establish exactly the future state of the system studied, a stochastic uncertainty exists, this type of uncertainty is well known in decision theory and economics, where it is called "*decisions under risk*". Applications of this concept in a multi-criteria framework can be found in D'Avignon and Vincke [25], Martel and Zaras [69], and Rietveld [100] among others.

Another framing of uncertainty, called *fuzzy uncertainty*, focuses on the ambiguity of information in the sense that the uncertainty does not concern the occurrence of an event but the event itself, which cannot be described unambiguously. This situation is very common in human systems. These systems are *complex systems* characterised by subjectivity, incompleteness and imprecision. Zadeh [137] writes: "*as the complexity of a system increases, our ability to make a precise and yet significant statement about its behaviour diminishes until a threshold is reached beyond which precision and significance (or relevance) become almost mutually exclusive characteristics*" (*incompatibility principle*). Fuzzy set theory is a mathematical theory for modelling situations, in which traditional modelling languages which are dichotomous in character and unambiguous in their description cannot be used. For a survey of multi-criteria approaches able to deal with fuzzy uncertainty see Part IV of this book and Munda [82]. In conclusion, *multi-criteria methods able to tackle consistently the widest types of mixed information and different sources of uncertainty should be considered as desirable ones*.

Another desirable property for mathematical aggregation procedures in the framework of sustainability decisions is *simplicity*, i.e. the use of a few parameters as possible. While in the context of multi-criteria decision aid, parameters helping the decision-maker to elicitate her/his preferences are desirable, in a social context there is the risk that their presence increases arbitrariness and reduces transparency. I think that in this second context the only exogenous parameters desirable are weights and, if absolutely necessary, indifference and preference thresholds.

Finally, in a policy framework, *to have a ranking of all the different courses of actions is better than to select just one alternative.* This mainly because in this way social compromises are easier (the second or the third alternative in the ranking may minimise opposition much more than the first one). Technically speaking this implies that multi-criteria methods able to deal with the γ decision problem formulation have to be preferred and that dominated alternatives cannot be excluded a priori.

Concluding, we can summarise a set of desirable properties for choosing an appropriate method for dealing with sustainability decision problems, as follows.

Descriptive domain assumptions:

- Mixed information on criterion scores should be tackled in the form of ordinal, crisp, stochastic and fuzzy criterion scores.

Normative domain assumptions:

- Simplicity is desirable and means the use of as less *ad hoc* parameters as possible.
- The most useful result for policy-making is a complete ranking of alternatives.
- Weights are meaningful only as importance coefficients and not as trade-offs.
- Complete compensability is not desirable.

Heuristic descriptive assumptions:

- When not all intensities of preference are meaningful, indifference and preference thresholds are useful exogenous parameters.
- Dominated alternatives have to be considered.

Finally one should note that these selection properties can be applied only to methods who achieve a set of minimum formal requirements, the main important being the following.

Formal domain assumptions:

- Unanimity.
- Monotonicity.
- Neutrality.

Negligibility formal assumptions:

- Anonymity.

27.7 Conclusion

When science is used for policy making, an appropriate management of decisions implies including the multiplicity of participants and perspectives. This also implies the impossibility of reducing all dimensions to a single unity of measure. "*The issue* *is not whether it is only the marketplace that can determine value, for economists have long debated other means of valuation; our concern is with the assumption that in any dialogue, all valuations or "numeraires" should be reducible to a single one-dimension standard*" [41, p. 198]. It is noteworthy that this call for citizen participation and transparency, when science is used for policy making, is more and more supported institutionally inside the European Union, where perhaps the most significant examples are the White Paper on Governance and the Directive on Strategic Environmental Impact Assessment.

Multi-criteria evaluation supplies a powerful framework for the implementation of the incommensurability principle. In fact it accomplishes the goals of being *inter/multi-disciplinary* (with respect to the research team), *participatory* (with respect to the local community) and *transparent* (since all criteria are presented in their original form without any transformations in money, energy or whatever common measurement rod). As a consequence multi-criteria evaluation looks as an adequate assessment framework for (micro and macro) sustainability policies.

However, one should remember that we are in a second best world. A useful analogy here is with Flatland, the classic Victorian science fiction and social parody [1]. There, the inhabitants of spaces with more dimensions had a richer awareness of themselves, and also could see beyond and through the consciousness of the simpler creatures inhabiting fewer dimensions. At this stage it is not unfair to reveal the dénouement of the story, namely that the Sphere of three-dimensional space showed himself to be just another Flatlander at heart, when he angrily refused to accept the reality of higher dimensions of being.

Acknowledgments This chapter builds on previous work carried out for the JRC's Centre for Research on Education and Lifelong Learning (CRELL), in the framework of the project KNOW (Human Capital for Prosperity and Sustainable Growth in Europe). The views expressed are purely those of the writer and may not in any circumstances be regarded as stating an official position of the European Commission.

References

1. Abbott, E.A.: Flatland: A Romance of Many Dimensions. Little, Brown & Co., Boston (1935)
2. Akgün, A.A., van Leeuwen, E., Nijkamp, P.: A multi-actor multi-criteria scenario analysis of regional sustainable resource policy. Ecol. Econ. **78**, 19–28 (2012)
3. Allen, T.F.H., Tainter, J.A., Hoekstra, T.W.: Supply-Side Sustainability. Columbia University Press, New York (2002)
4. Antunes, P., Karadzic, V., Santos, R., Beça, P., Osann, A.: Participatory multi-criteria analysis of irrigation management alternatives: the case of the Caia irrigation district, Portugal. Int. J. Agric. Sustain. **9**(2), 334–349 (2011)
5. Arrow, K.J.: Social Choice and Individual Values, 2nd edn. Wiley, New York (1963)
6. Arrow, K.J., Raynaud, H.: Social Choice and Multicriterion Decision Making. M.I.T Press, Cambridge (1986)
7. Banville, C., Landry, M., Martel, J.M., Boulaire, C.: A stakeholder approach to MCDA. Syst. Res. Behav. Sci. **15**, 15–32 (1998)

8. Barbier, E.B.: The concept of sustainable economic development. Environ. Conserv. **14**(2), 101–110 (1987)
9. Barbier, E.B., Markandya, A.: The conditions for achieving environmentally sustainable growth. Eur. Econ. Rev. **34**, 659–669 (1990)
10. Barbiroli, G.: Direct and indirect indicators of sustainable development. In: Proceedings International Conference on Environmental Pollution, Barcelona, pp. 489–496 (1993)
11. Beinat, E.: Value Functions for Environmental Management. Kluwer, Dordrecht (1997)
12. Beinat, E., Nijkamp, P. (eds.): Multicriteria Evaluation in Land-Use Management: Methodologies and Case Studies. Kluwer, Dordrecht (1998)
13. Bell, M.L., Hobbs, B., Elliott, E.M., Ellis, H., Robinson, Z.: An evaluation of multi-criteria methods in integrated assessment of climate policy. J. Multi-Criteria Decis. Anal. **10**(5), 229–256 (2001)
14. Black, D.: The Theory of Committees and Elections. Cambridge University Press, Cambridge (1958)
15. Bouyssou, D.: Building criteria: a prerequisite for MCDA. In: Bana e Costa, C.A. (ed.) Readings in Multiple Criteria Decision Aid, pp. 58–80. Springer, Berlin (1990)
16. Bouyssou, D., Marchant, T., Perny, P., Pirlot, M., Tsoukiàs, A., Vincke, P.: Evaluation and Decision Models: A Critical Perspective. Kluwer Academic Publishers, Dordrecht (2000)
17. Bresso, M.: Per un' economia ecologica. La Nuova Italia Scientifica, Rome (1993)
18. Buchanan, J.M., Musgrave, R.A.: Public Finance and Public Choice. The MIT Press, Cambridge (1999)
19. Bykvist, K.: Can unstable preferences provide a stable standard of well-being? Econ. Philos. **26**, 1–26 (2010)
20. Cerreta, M., Concilio, G., Monno, V. (eds.): Making Strategies in Spatial Planning. Springer, Dordrecht (2010)
21. Cerreta, M., De Toro, P.: Integrated spatial assessment for a creative decision-making process: a combined methodological approach to strategic environmental assessment. Int. J. Sustain. Dev. **13**(1–2), 17–30 (2010)
22. Cheng, F., Geertman, S., Kuffer, M., Zhan, Q.: An integrative methodology to improve Brownfield redevelopment planning in Chinese cities: a case study of Futian, Shenzhen. Comput. Environ. Urban. Syst. **35**(5), 388–398 (2011)
23. Chichilnisky, G.: An axiomatic approach to sustainable development. Soc. Choice Welf. **13**(2), 219–248 (1996)
24. Corral-Quintana, S., De Marchi, B., Funtowicz, S., Gallopín, G., Guimarães-Pereira, Â., Maltoni, B.: The Visions Project at the JRC. European Commission, Joint Research Centre, Institute for the Protection and Safety of the Citizens, EUR 19926 EN (2001). http://alba.jrc.it/visions
25. D'Avignon, G., Vincke, P.: An outranking method under uncertainty. Eur. J. Oper. Res. **36**, 311–321 (1988)
26. Daly, H.E., Cobb, J.J.: For the Common Good: Redirecting the Economy Toward Community, the Environment and a Sustainable Future. Beacon, Boston (1989)
27. Dasgupta, P., Heal, D.: Economic Theory and Exhaustible Resources. Cambridge University Press, Cambridge (1979)
28. de Condorcet, N.: Essai sur l'application de l'analyse à la probabilité des décisions rendues à la probabilité des voix. De l'Imprimerie Royale, Paris (1785)
29. De Marchi, B., Funtowicz, S.O., Lo Cascio, S., Munda, G.: Combining participative and institutional approaches with multi-criteria evaluation. An empirical study for water issue in Troina, Sicily. Ecol. Econ. **34**(2), 267–282 (2000)
30. De Marchi, B., Ravetz, J.: Participatory approaches to environmental policy. Concerted Action EVE, Policy Research Brief, No. 10 (2001)
31. Dietz, F.J., van der Straaten, J.: Rethinking environmental economics: missing links between economic theory and environmental policy. J. Econ. Issues **XXVI**(1), 27–51 (1992)
32. Ehrlich, P., Raven, P.: Butterflies and plants: a study in co-evolution. Evolution **18**, 586–608 (1964)

33. Espelta, J.M., Retana, J., Habrouk, A.: An economic and ecological multi-criteria evaluation of reforestation methods to recover burned *Pinus Negra* forests in NE Spain. For. Ecol. Manage. **180**, 185–198 (2003)
34. Faucheux, S., O'Connor, M. (eds.): Valuation for Sustainable Development: Methods and Policy Indicators. Edward Elgar, Cheltenham (1998)
35. Floridi, M., Pagni, S., Falorni, S., Luzzati, T.: An exercise in composite indicators construction: assessing the sustainability of Italian regions. Ecol. Econ. **70**(8), 1440–1447 (2011)
36. Folke, C.: Socio-economic dependence on the life-supporting environment. In: Folke, C., Kaberger, T. (eds.) Linking the Natural Environment and the Economy: Essays from the Eco-Eco Group, pp. 77–94. Kluwer, Dordrecht (1991)
37. Friedman, M.: On the methodology of positive economics. In: Essays in Positive Economics. University of Chicago Press, Chicago (1953)
38. Funtowicz, S., Martinez-Alier, J., Munda, G., Ravetz, J.: Information tools for environmental policy under conditions of complexity. European Environmental Agency, Experts' Corner, Environmental Issues Series, No. 9 (1999)
39. Funtowicz, S.O., Munda, G., Paruccini, M.: The aggregation of environmental data using multicriteria methods. Environmetrics **1**(4), 353–368 (1990)
40. Funtowicz, S.O., Ravetz, J.R.: A new scientific methodology for global environmental issues. In: Costanza, R. (ed.) Ecological Economics, pp. 137–152. Columbia University Press, New York/Columbia (1991)
41. Funtowicz, S.O., Ravetz, J.R.: The worth of a songbird: ecological economics as a post-normal science. Ecol. Econ. **10**, 197–207 (1994)
42. Gamboa, G.: Social multi-criteria evaluation of different development scenarios of the Aysén region, Chile. Ecol. Econ. **59**(1), 157–170 (2006)
43. Gamboa, G., Munda, G.: The problem of wind-park location: a social multi-criteria evaluation framework. Energy Policy **35**(3), 1564–1583 (2007)
44. Garmendia, E., Gamboa, G., Franco, J., Garmendia, J.M., Liria, P., Olazabal, M.: Social multi-criteria evaluation as a decision support tool for integrated coastal zone management. Ocean Coast. Manag. **53**(7), 385–403 (2010)
45. Garmendia, E., Stagl, S.: Public participation for sustainability and social learning: concepts and lessons from three case studies in Europe. Ecol. Econ. **69**(8), 1712–1722 (2010)
46. Geach, P.: Good and evil. In: Foot, P. (ed.) Theories of Ethics. Oxford University Press, Oxford (1967)
47. Giampietro, M., Mayumi, K.: Multiple-scale integrated assessment of societal metabolism: introducing the approach. Popul. Environ. **22**(2), 109–154 (2000)
48. Giampietro, M., Mayumi, K.: Multiple-scale integrated assessment of societal metabolism: integrating biophysical and economic representations across scales. Popul. Environ. **22**(2), 155–210 (2000)
49. Giampietro, M., Mayumi, K., Munda, G.: Integrated assessment and energy analysis: quality assurance in multi-criteria analyses of sustainability. Energy **31**(1), 59–86 (2006)
50. Gowdy, J.M.: Coevolutionary Economics: The Economy, Society, and the Environment. Kluwer, Boston (1994)
51. Gowdy, J.M., O'Hara, S.: Economic Theory for Environmentalists. Saint Lucie Press, New York (1996)
52. Grafakos, S., Flamos, A., Oikonomou, V., Zevgolis, D.: Multi-criteria analysis weighting methodology to incorporate stakeholders' preferences in energy and climate policy interactions. Int. J. Energy Sect. Manage. **4**(3), 434–461 (2010)
53. Guimarães-Pereira, A., Corral-Quintana, S.A., Funtowicz, S.: GOUVERNe: new trends in decision support for groundwater governance issues. Environ. Model. Software **20**, 111–118 (2005)
54. Guimarães-Pereira, A., Guedes, S., Tognetti, S. (eds.): Interfaces Between Science and Society. Greenleaf Publishing, Sheffield (2006)

55. Guimarães-Pereira, A., Rinaudo, J.D., Jeffrey, P., Blasuqes, J., Corral-Quintana, S.A., Courtois, N., Funtowicz, S., Petit, V.: ICT tools to support public participation in water resources governance and planning experiences from the design and testing of a multi-media platform. J. Environ. Assess. Policy Manag. **5**(3), 395–419 (2003)
56. Guitouni, A., Martel, J.M.: Tentative guidelines to help choosing an appropriate MCDA method. Eur. J. Oper. Res. **109**, 501–521 (1998)
57. Hayashi, K.: Multicriteria analysis for agriculture resource management: a critical survey and future perspectives. Eur. J. Oper. Res. **122**, 486–500 (2000)
58. Hinloopen, E., Nijkamp, P.: Qualitative multiple criteria choice analysis, the dominant regime method. Qual. Quant. **24**, 37–56 (1990)
59. Horwarth, R., Norgaard, R.B.: Intergenerational resource rights, efficiency and social optimality. Land Econ. **66**, 1–11 (1990)
60. Horwarth, R., Norgaard, R.B.: Environmental valuation under sustainable development. Am. Econ. Rev. Pap. Proc. **80**, 473–477 (1992)
61. Janssen, R.: Multiobjective Decision Support for Environmental Management. Kluwer, Dordrecht (1992)
62. Janssen, R., Munda, G.: Multi-criteria methods for quantitative, qualitative and fuzzy evaluation problems. In: van den Bergh, J. (ed.) Handbook of Environmental and Resource Economics, pp. 837–852. Edward Elgar, Cheltenham (1999)
63. Kasemir, B., Gardner, M., Jäger, J., Jaeger, C. (eds.): Public Participation in Sustainability Science. Cambridge University Press, Cambridge (2003)
64. Keeney, R., Raiffa, H.: Decision with Multiple Objectives: Preferences and Value Trade-Offs. Wiley, New York (1976)
65. Kemeny, J.: Mathematics without numbers. Daedalus **88**, 571–591 (1959)
66. Kleijnen, J.P.C.: Ethical issues in modelling: some reflections. Eur. J. Oper. Res. **130**, 223–230 (2001)
67. Laffont, J.J.: Incentives and Political Economy. Oxford University Press, Oxford (2000)
68. Luce, R.D.: Semiorders and a theory of utility discrimination. Econometrica **24**, 178–191 (1956)
69. Martel, J.M., Zaras, K.: Stochastic dominance in multicriteria analysis under risk. Theor. Decis. **35**(1), 31–49 (1995)
70. Martinez-Alier, J., Munda, G., O'Neill, J.: Weak comparability of values as a foundation for ecological economics. Ecol. Econ. **26**, 277–286 (1998)
71. Martinez-Alier, J., O'Connor, M.: Ecological and economic distribution conflicts. In: Costanza, R., Segura, O., Martinez-Alier, J. (eds.) Getting Down to Earth: Practical Applications of Ecological Economics. Island Press/ISEE, Washington, DC (1996)
72. Maystre, L., Pictet, J., Simos, J.: Méthodes multicritères ELECTRE – description, conseils pratiques et cas d'application à la gestion environnementale. Presses Polytechniques et Universitaires Romandes, Lausanne (1994)
73. Monterroso, I., Binimelis, R., Rodríguez-Labajos, B.: New methods for the analysis of invasion processes: multi-criteria evaluation of the invasion of Hydrilla verticillata in Guatemala. J. Environ. Manag. **92**(3), 494–507 (2011)
74. Moreno-Jiménez, J.M., Aguaron, J., Escobar, T., Turon, A.: Multicriteria procedural rationality on SISDEMA. Eur. J. Oper. Res. **119**(2), 388–403 (1999)
75. Moulin, H.: Axioms of Co-operative Decision Making, Econometric Society Monographs. Cambridge University Press, Cambridge (1988)
76. Munda, G.: Multiple-criteria decision aid: some epistemological considerations. J. Multi-Criteria Decis. Anal. **2**, 41–55 (1993)
77. Munda, G.: Cost-benefit analysis in integrated environmental assessment: some methodological issues. Ecol. Econ. **19**(2), 157–168 (1996)
78. Munda, G.: Environmental economics, ecological economics and the concept of sustainable development. Environ. Values **6**(2), 213–233 (1997)
79. Munda, G.: Social multi-criteria evaluation (SMCE): methodological foundations and operational consequences. Eur. J. Oper. Res. **158**(3), 662–677 (2004)

80. Munda, G.: "Measuring sustainability": a multi-criterion framework. Environ. Dev. Sustain. **7**(1), 117–134 (2005)
81. Munda, G.: Social Multi-Criteria Evaluation for a Sustainable Economy. Springer, Heidelberg (2008)
82. Munda, G.: Multicriteria Evaluation in a Fuzzy Environment. Theory and Applications in Ecological Economics. Physica-Verlag, Contributions to Economics Series, Heidelberg (1995)
83. Munda, G., Nardo, M.: Non-compensatory/non-linear composite indicators for ranking countries: a defensible setting. Appl. Econ. **41**, 1513–1523 (2009)
84. Munda, G., Russi, D.: Social multi-criteria evaluation of conflict over rural electrification and solar energy in Spain. Environ. Plan. C Gov. Policy **26**, 712–727 (2008)
85. Munda, G., Saisana, M.: Methodological considerations on regional sustainability assessment based on multicriteria and sensitivity analysis. Reg. Stud. **45**(2), 261–276 (2011)
86. Musgrave, A.: Unreal assumptions in economic theory: the F-twist untwisted. Kyklos **34**, 377–387 (1981)
87. Musu, I., Siniscalco, D.: National Accounts and the Environment. Kluwer Academic Publishers, Dordrecht (1996)
88. Nardo, M., Saisana, M., Saltelli, A., Tarantola, S., Hoffman, A., Giovannini, E.: OECD/JRC handbook on constructing composite indicators: methodology and user guide. OECD Statistics Working Paper, Paris, publication code: 302008251E1 (2008)
89. Neurath, O.: Empiricism and Sociology. Reidel, Dordrecht (1973)
90. Nijkamp, P., Rietveld, P., Voogd, H.: Multicriteria Evaluation in Physical Planning. North-Holland, Amsterdam (1990)
91. Nordström, E.-M., Eriksson, L.O., Öhman, K.: Integrating multiple criteria decision analysis in participatory forest planning: experience from a case study in northern Sweden. Forest Policy Econ. **12**(8), 562–574 (2010)
92. Norgaard, R.B.: Development Betrayed. Routledge, London (1994)
93. O'Neill, J.: Ecology, Policy and Politics. Routledge, London (1993)
94. OECD: Composite Indicators of Country Performance: A Critical Assessment. DST/IND(2003)5, Paris (2003)
95. Özkaynak, B.: Globalisation and local resistance: alternative city developmental scenarios on capital's global frontier-the case of Yalova, Turkey. Prog. Plan. **70**(2), 45–97 (2008)
96. Passet, R.: L'économique et le vivant. Payot, Paris (1979)
97. Pearce, D.W., Atkinson, G.D.: Capital theory and the measurement of sustainable development: an indicator of "weak" sustainability. Ecol. Econ. **8**, 103–108 (1993)
98. Pearce, D., Hamilton, G., Atkinson, G.D.: Measuring sustainable development: progress on indicators. Environ. Dev. Econ. **1**, 85–101 (1996)
99. Podinovskii, V.V.: Criteria importance theory. Math. Soc. Sci. **27**, 237–252 (1994)
100. Rietveld, P.: Using ordinal information in decision making under uncertainty. Syst. Anal. Model. Simul. **6**, 659–672 (1989)
101. Ringius, L., Asbjørn, T., Holtsmark, B.: Can multi-criteria rules fairly distribute climate burdens? OECD results from three burden sharing rules. Energy Policy **26**(10), 777–793 (1998)
102. Roberts, F.S.: Measurement Theory with Applications to Decision Making, Utility and the Social Sciences. Addison-Wesley, London (1979)
103. Roca, E., Gamboa, G., Tàbara, J.D.: Assessing the multidimensionality of coastal erosion risks: public participation and multicriteria analysis in a Mediterranean coastal system. Risk Anal. **28**(2), 399–412 (2008)
104. Romero, C., Rehman, T.: Multiple Criteria Analysis for Agricultural Decisions. Elsevier, Amsterdam (1989)
105. Roy, B.: Méthodologie multicritere d'aide à la decision. Economica, Paris (1985)
106. Roy, B.: Multicriteria Methodology for Decision Analysis. Kluwer, Dordrecht (1996)
107. Roy, B., Damart, S.: L'analyse Coûts-Avantages, outil de concertation et de légitimation? Metropolis **108/109**, 7–16 (2002)
108. Saari, D.G.: A dictionary for voting paradoxes. J. Econ. Theory **48**, 443–475 (1989)
109. Saari, D.G.: Mathematical structure of voting paradoxes. 1. Pairwise votes. Econ. Theory **15**, 1–53 (2000)

110. Saari, D.G.: Which is better: the Condorcet or Borda winner? Soc. Choice Welf. **26**, 107–129 (2006)
111. Saisana, M., Tarantola, S.: State-of-the-art report on current methodologies and practices for composite indicator development. EUR 20408 EN Report, European Commission, JRC, Ispra (2002)
112. Saisana, M., Tarantola, S., Saltelli, A.: Uncertainty and sensitivity techniques as tools for the analysis and validation of composite indicators. J. R. Stat. Soc. A **168**, 307–323 (2005)
113. Salminen, P., Hokkanen, J., Lahdelma, R.: Comparing multicriteria methods in the context of environmental problems. Eur. J. Oper. Res. **104**, 485–496 (1998)
114. Saltelli, A.: Composite indicators between analysis and advocacy. Soc. Indic. Res. **81**, 65–77 (2007)
115. Saltelli, A., Ratto, M., Andres, T., Campolongo, F., Cariboni, J., Gatelli, D., Saisana, M., Tarantola, S.: Global Sensitivity Analysis. The Primer. Wiley, England (2008)
116. Schmidt-Bleek, F.: Wieviel umwelt braucht der mensch? MIPS Das Mass für ökologisches wirtschaften. Birkh User, Berlin (1994)
117. Scolobig, A., Broto, V.C., Zabala, A.: Integrating multiple perspectives in social multicriteria evaluation of flood-mitigation alternatives: the case of Malborghetto-Valbruna. Environ. Plan. C Gov. Policy **26**(6), 1143–1161 (2008)
118. Simon, H.A.: Reason in Human Affairs. Stanford University Press, Stanford (1983)
119. Soma, K., Vatn, A.: Local democracy implications for coastal zone management-a case study in southern Norway. Land Use Policy **26**(3), 755–762 (2009)
120. Spash, C., Hanley, N.: Preferences, information, and biodiversity preservation. Ecol. Econ. **12**, 191–208 (1995)
121. Stewart, T.J., Joubert, A.: Conflicts between conservation goals and land use for exotic forest plantations in South Africa. In: Beinat, E., Nijkamp, P. (eds.) Multicriteria Evaluation in Land-Use Management: Methodologies and Case Studies, pp. 17–31. Kluwer, Dordrecht (1998)
122. Stiglitz, J.E.: New perspectives on public finance: recent achievements and future challenges. J. Public Econ. **86**, 341–360 (2002)
123. Straton, A.T., Jackson, S., Marinoni, O., Proctor, W., Woodward, E.: Exploring and evaluating scenarios for a river Catchment in Northern Australia using scenario development, multi-criteria analysis and a deliberative process as a tool for water planning. Water Resour. Manag. **25**(1), 141–164 (2010)
124. Sugden, R.: Opportunity as mutual advantage. Econ. Philos. **26**, 47–68 (2010)
125. Todhunter, I.: A History of Mathematical Theory of Probability. Chelsea, New York (1949)
126. Trutnevyte, E., Stauffacher, M., Scholz, R.: Linking stakeholder visions with resource allocation scenarios and multi-criteria assessment. Eur. J. Oper. Res. **219**(3), 762–772 (2012)
127. Turner, R.K., Pearce, D.W., Bateman, I.: Environmental Economics: An Elementary Introduction. Harvester Wheatsheaf, London (1994)
128. Ülengin, B., Ülengin, F., Güvenç, Ü.: A multidimensional approach to urban quality of life: the case of Istanbul. Eur. J. Oper. Res. **130**, 361–374 (2001)
129. Vincke, P.: Multicriteria Decision Aid. Wiley, New York (1992)
130. Vincke, P.: Recent progresses in multicriteria decision-aid. Rivista di Matematica per le scienze Economiche e Sociali **2**, 21–32 (1994)
131. Vitousek, P., Ehrlich, P., Ehrlich, A., Matson, P.: Human appropriation of the products of photosynthesis. Bioscience **34**(6), 368–373 (1986)
132. Voogd, H.: Multicriteria Evaluation for Urban and Regional Planning. Pion, London (1983)
133. Wackernagel, M., Rees, W.E.: Our Ecological Footprint: Reducing Human Impact on the Earth. New Society Publishers, Gabriola Island/Philadelphia (1995)
134. Young, H.P.: Condorcet's theory of voting. Am. Polit. Sci. Rev. **82**(4), 1231–1244 (1988)
135. Young, H.P., Levenglick, A.: A consistent extension of Condorcet's election principle. J. SIAM Appl. Math. **35**, 285–300 (1978)
136. Yusuf, J.A., El Serafy, S., Lutz, E.: Environmental Accounting for Sustainable Development. A UNEP World Bank Symposium, Washington, DC (1989)
137. Zadeh, L.A.: Fuzzy sets. Inf. Control **8**, 338–353 (1965)
138. Zendehdel, K., Rademaker, M., De Baets, B., Van Huylenbroeck, G.: Environmental decision making with conflicting social groups: a case study of the Lar rangeland in Iran. J. Arid Environ. **74**(3), 394–402 (2010)

Chapter 28
Multicriteria Portfolio Decision Analysis for Project Selection

Alec Morton, Jeffrey M. Keisler, and Ahti Salo

Abstract Multicriteria Portfolio Analysis spans several methods which typically build on MCDA to guide the selection of a subset (i.e., portfolio) of available objects, with the aim of maximising the performance of the resulting portfolio with regard to multiple criteria, subject to the requirement that the resources consumed by the portfolio does not exceed the availability of resources and, moreover, satisfies other relevant constraints as well. In this chapter, we present a formal model of this selection problem and describe how this model can present both challenges (e.g. portfolio value may, due to the interactions of elements, depend on project-level decisions in complex and non-additive ways) and opportunities (e.g. triage rules can be used to focus elicitation on projects which are critical for value assessment). We also survey the application of Portfolio Decision Analysis in several domains, such as allocation of R&D expenditure, military procurement, prioritisation of healthcare projects, and environment and energy planning, and conclude by outlining possible future research directions.

Keywords Project selection • Portfolio selection • Portfolio management • Multicriteria decision analysis

28.1 Introduction

Essentially all organizations are faced with the problem of choosing what activities to pursue. This is true of, for example, a high technology or pharmaceutical company, or public sector funder deciding what science to invest in; a Ministry

A. Morton (✉)
Department of Management Science, University of Strathclyde, Scotland, UK
e-mail: alec.morton@strath.ac.uk

J.M. Keisler
College of Management, University of Massachusetts, Boston, Massachusetts, USA
e-mail: jeff.keisler@umb.edu

A. Salo
Systems Analysis Laboratory, Aalto University, Finland
e-mail: ahti.salo@aalto.fi

S. Greco et al. (eds.), *Multiple Criteria Decision Analysis*, International Series in Operations Research & Management Science 233,
DOI 10.1007/978-1-4939-3094-4_28

of Defence deciding what equipment to procure; a hospital, insurance fund or health authority deciding what treatments to provide; a local council deciding on what services to provide, and how they are to be spacially distributed; a public authority seeking to distribute a budget for the maintenance of infrastructure; an IT department deciding what systems projects to initiate; or an international collaboration deciding what projects to pursue together.

Such problems are often multicriteria or multiobjective. This may be because the organisation's goals are themselves contested and the appropriate balance has to be negotiated between different stakeholders. Alternatively, it may be because while the organisation has, nominally, a single fundamental objective, such as profitability, the actions on the table are so far removed from this ultimate goal that tracing through the impacts of choices on this goal is not practical, and so decision makers (DMs) rely on assessments of proxies for that goal. Examples of such a situation might be upstream drug development, where detailed market modelling is not possible and so DMs rely on criteria such as unmet need, or allocation of a maintenance budget to roads, where typically the aim is to maintain the road network at a given quality level, rather than to minimise accidents or journey times per se.

These decision problems share a common structure. In all cases, the requirement is to choose a subset of items—a portfolio—from a choice set. This can be contrasted with typical situation in the textbook presentation of multicriteria decision analysis, which we call "single choice", where the DM has to choose a single item (action, option, alternative) from a set (such as a house, car or toaster to purchase). The distinction between portfolio and single choice has a long history and has been described by White [172] as the distinction between explicitly and implicitly defined alternatives, and by Roy [148] as the distinction between the globalised and fragmented concept of an action.

The relationship between these two problems can be conceptualised in various ways. One conceptualisation is to view portfolio choice as a generalisation of the single choice problem: in the single choice problem, the only available portfolios are those containing single items. In this sense, the portfolio choice problem is primary and the single choice problem secondary. From another point of view, the single choice problem is the more fundamental concept: portfolio choice can be seen as a single choice problem, subject to the interpretation that the set from which items are to be selected is a combinatorial set of portfolios. Indeed, some approaches deal with the portfolio nature of choice effectively by restructuring portfolio choice as a single choice problem and by screening the combinatorial set to identify a manageable subset of feasible or attractive portfolios which can then be considered explicitly. The Analysis of Interconnected Decision Areas (AIDA) method [46, 47] and the strategy table device [68, 158] are examples of this sort of approach.

The problem of portfolio choice also seems reminiscent of the "sorting problematique" [148, 177] where the DM classifies objects as belonging to a member of a set of ordered classes ("excellent", "good", "poor", etc), except that the classification is a binary one, into "accept" and "reject". Sorting differs, however, from portfolio choice in that in sorting there does not have to be a sense in which objects can be joined together or concatenated. Consequently from a sorting point of view, membership of the “accept” or “reject” class is typically determined by whether an object is better or worse than a reference object which lies on the boundary between two classes, rather than by the total cost of the accepted objects, which, from a sorting point of view, is not a meaningful concept.

The problem of portfolio choice can be approached by a common set of approaches. Elsewhere we have called these approaches which seek “to help DMs make informed selections from a discrete set of alternatives through mathematical modeling that accounts for relevant constraints, preferences, and uncertainties”, *Portfolio Decision Analysis* [151]. Although relatively neglected in the academic decision analysis literature (although see [91, 92]), Portfolio Decision Analysis accounts for a significant proportion of commercial decision analysis consulting [93]. Moreover, the label “Portfolio Decision Analysis” is a useful blanket term which serves to draw together different formal approaches to the management of portfolios of activity in different domains, underscoring key similarities.

The focus of the current chapter will be on approaches to Portfolio Decision Analysis in which there is explicit recognition of the multicriteria nature of the decision problem: we will call such approaches, *Multicriteria Portfolio Decision Analysis* or MCPDA. Although our own background is in Multiattribute Value and Utility Theory based methods, we aim, in keeping with the integrative spirit of the volume of which this chapter is a part, to cover and discuss approaches to MCPDA based on a broad range of methods. It should also be noted that many practitioner texts in this area also propose atheoretic scoring methods for project selection, although as this literature is not indexed, a systematic review does not seem possible. Readers are referred to other chapters in the current book for further technical details on the methods referred to. In this chapter, we have four main aims: firstly, to describe a framework for MCPDA; secondly, to draw attention to key modelling challenges and solutions; thirdly, to review practice in a number of different application domains; and fourthly, to conclude and point the way forward for further research in this area. We devote Sects. 28.2–28.5 to each of these aims respectively.

28.2 A Formal Framework for MCPDA

In this section we present a formal framework for MCPDA. The underlying theory of MCPDA is not very well developed and the main relevant reference we are aware of is [51], which the presentation of this chapter follows. However, we use ordinal rather than cardinal independence conditions (as these are easier to state and are more familiar), which give rise to a slightly different representation. We begin our formal development with a model of the portfolio space. It is normally most convenient to consider this space as a subset of $\{0, 1\}^m$, with the 0–1 entries representing m binary decisions to do a project or not. Normally that subset will be defined by a constraint set, and normally that subset will include a resource constraint of the form $c(x) \leq B$ which will have an especial significance as will become clear in the next section. In any case, we will denote the portfolio space as X, denote a typical member as $x = (x_1, \ldots, x_i, \ldots, x_m)$ and denote the index set $\{1, \ldots, m\}$ as M.

We suppose that there is a vector valued function $g(\cdot)$ which maps each choice of projects into a $m \times n$-dimensional space. The normal interpretation of g is that associated with each decision about each project there is a set of scores which depend on whether a project has been chosen or not: this interpretation requires that $g(x) = (g_1(x_1), \ldots, g_m(x_m))$ where $g_1(\cdot), \ldots, g_m(\cdot)$ are scoring functions associated with each decision which we will assume in the ensuing. Often in applications, the same scales will be used for all projects, and we will make that assumption here (although see [50] for an example where different scales are used for different types of projects). For example, a scoring system for scientific projects might include scales representing market size, innovativeness, and fit with company mission. We will denote the index set of criteria $N = \{1, \ldots, j, \ldots, n\}$. For a particular project i we will call the space of vectors of project scores $Y^i = \prod_{j \in N} Y_j$ with typical member as $y^i=(y_1, \ldots, y_j, \ldots, y_n)$. We will call the set of possible vectors of portfolio scores $Y = \prod_{i \in M} Y^i$, with typical member as $y=(y^1, \ldots, y^i, \ldots, y^m)$.

We introduce a preference relation $\succsim_Y$ over Y. Note that this preference relation is defined not just over possible portfolios of projects as they currently exist, but portfolios of counterfactual projects which do not in fact exist. For example, suppose that project 1 scores five on innovation and seven on strategic fit, and project 2 scores three on innovation and four on strategic fit. A counterfactual version of project 1 scores six on innovation and three on strategic fit. The preference model supposes that I can say how I feel about: the actual project 1 by itself, the actual project 2 by itself, the counterfactual project 1 by itself, the actual project 1 together with the actual project 2, and the counterfactual project 1 together with the actual project 2.

There are various forms which our preferences over this space might take but following [51], our approach will be to impose certain independence conditions on $\succsim_Y$. We shall think of independence in two parts: between-project independence and within-project independence. For a given set $I \subset N$ we write $Y^I = \prod_{i \in I} Y^i$ and $Y^{M/I} = \prod_{i \in M/I} Y^i$, denote typical elements y^I and $y^{M/I}$ respectively and use the notation $(y^I, y^{M/I})$ to denote the vector y which has corresponding entries equal to those of y^I for all $i \in I$ and $y^{M/I}$ for all $i \in M/I$. Our definition of between-project independence reads as follows.

Definition 1. If a preference ordering $\succsim_Y$ has the following property that for some $I \subset M$, for all $\dot{y}_I$ and $\ddot{y}_I \in Y_I$ and $\dot{y}_{M/I} \in Y_{M/I}$

$$(\dot{y}_I, \dot{y}_{M/I}) \succsim_Y (\ddot{y}_I, \dot{y}_{M/I}) \Longrightarrow (\dot{y}_I, y_{M/I}) \succsim_Y (\ddot{y}_I, y_{M/I}) \forall y_{M/I} \in Y_{M/I} \qquad (28.1)$$

then it is said to be **between-project independent for** I and if this condition holds for all $I \subset M$, it will be said to be **between-project independent over** M.

Subject to suitable auxiliary assumptions (weak ordering, restricted solvability, Archimedeanness and essentiality, as well as technical conditions which may be required in particular cases, for example when $m = 2$ or Z has uncountable cardinality), it is well-known that between-project independence over M allows us to write the value functions for portfolios of projects as $u(y) = \sum_{i=1}^{m} u^i(y^i)$. However, we also require some way to evaluate the projects. We can do this by defining partial preference orderings $\succsim_i$ by $\dot{y} \succsim_i \ddot{y}$ iff $(\dot{y}^i, y^{N/\{i\}}) \succsim_Y (\ddot{y}^i, y^{N/\{i\}})\ \forall y^{N/\{i\}} \in Y^{N/\{i\}}$. As should be evident, the partial preference ordering is represented by $u^i(y^i)$.

We now impose another condition on preferences (see [101, 130] for a use of this principle). To do this, we have to define an indifference relation: say that $\dot{y} \sim_Y \ddot{y}$ iff $\dot{y} \succsim_Y \ddot{y}$ and $\ddot{y} \succsim_Y \dot{y}$. This new assumption is an anonymity assumption: for any permutation σ on the set M, $(y^1, \ldots, y^m) \sim_Y (y^{\sigma(1)}, \ldots, y^{\sigma(m)})$. This condition embodies an idea that all that matters in the evaluation of the project is the scores: other attributes (names, sponsors, etc) do not influence preferences. It also follows that each $\succsim_i$ can be represented the same partial value functions so we can drop the index on $u^i(\cdot)$ and write them all as $u^*(\cdot)$.

Now for the final move, for a given set $J \subset N$ we write $Y_J = \prod_{j \in J} Y_j$ and $Y_{N/J} = \prod_{j \in N/J} Y_j$, denote typical elements y_J and $y_{N/J}$ respectively and use the notation $(y_J, y_{N/J})$ to denote the vector y which has corresponding entries equal to those of y_J for all $j \in J$ and $y_{N/J}$ for all $j \in N/J$. We now define a within-project independence condition:

Definition 2. If a partial preference ordering $\succsim_i$ has the following property that for some $J \subset N$, for all $\dot{y}_J$ and $\ddot{y}_J \in Y_J$ and $\dot{y}_{N/J} \in Y_{N/J}$

$$(\dot{y}_J, \dot{y}_{N/J}) \succsim_i (\ddot{y}_J, \dot{y}_{N/J}) \Longrightarrow (\dot{y}_J, y_{N/J}) \succsim_i (\ddot{y}_J, y_{N/J}) \forall y_{N/J} \in Y_{N/J} \tag{28.2}$$

then it is said to be **within-project independent for** J and if this condition holds for all $J \subset N$, it will be said to be **within-project independent over** N.

Again, subject to suitable auxiliary conditions, if within-project independence holds, the partial preference ordering $\succsim_i$ can be represented by a value function of the form $\sum_{j=1}^{n} u_j(y_j^i)$. Suppose within-project independence holds for all i. We know from the above that $u^*(\cdot)$ is also a value function representing the partial preference ordering $\succsim_i$ and so there must be a monotonically increasing transformation ϕ: $\phi\left(\sum_{j=1}^{n} u_j(y_j^i)\right) = u^*(y^i)$ and hence we have a value function for the portfolio $u(y) =$

$\sum_{i=1}^{m}\phi\left(\sum_{j=1}^{n}u_j(y_j^i)\right)$. Note that this result (in contrast to that of Golabi et al. [51]) does not imply that the value function is additive across both projects and criteria: for example, in the two criteria case, preferences represented by the value function $y_1^1y_2^1+y_1^2y_2^2$ with the $y_j^i s$ strictly positive, would respect both the between- and within-project independence axioms. It should also be noted that while if the auxiliary conditions hold at the between-project level there is no guarantee that they will hold at the within-project level (e.g. solvability may hold between projects but fail within projects). Indeed, if independence at the between-project level does not hold, the project-level partial preference ordering will not be complete and so not only can no additive value function exist, but no representation is possible at all.

A surprising feature of the literature is that other than [51, 101], essentially no authors seem to have taken on the task of axiomatising MCPDA models specifically for an exception see [101]. Thus, while the normative theory underpinning multi-criteria single choice has grown enormously since the early 1980s, the normative theory of MCPDA has been essentially stagnant. We hope that the remainder of this chapter will make clear some of the differences between the portfolio choice and single choice paradigm and will suggest to the interested reader possible directions for theoretic development.

28.3 Modelling Challenges

In this section, we discuss generic issues and process choices in the course of MCPDA modelling. We will organise the section under two headings: structuring the model, and exploring the portfolio space. In order to organise the discussion we will ask you to imagine that you (plus perhaps, families and/or partners) are confronted with the problem of furnishing a the living room of a new flat, where you will stay for a period of, say 18 months. The size of the budget for furnishing is not clear, but there is around £500. Borrowing money is not practically possible and the items you purchase have no salvage value after your lease runs out.

28.3.1 Structuring

The first stage in OR interventions is that of problem structuring. In the case of an MCDA model, we immediately face a dilemma, as there are two elements to be structured: the criteria and the alternatives (in the portfolio setting, the projects). Which does one do first? Keeney [83] has argued persuasively for “value-focussed thinking”—getting clear about values, in quite detailed and operational terms (for example, defining value scales and assessing tradeoffs) before thinking about the construction of alternatives to deliver these values. Thus in our example we might think about what you want to achieve through furnishing the flat (e.g. it is to be a place where you can work, relax, entertain guests, or store excess possessions?). This theme is echoed in the portfolio literature, where value-focussed thinking has been an influential and popular concept: e.g. Bordley [18] cites a situation where R&D scientists generate low value projects, reflecting "the fact that sometimes a scientist's main input about corporate priorities came from press releases". Nevertheless, there are cases where the projects may simply be given as part of the problem description, and so "alternative-focussed thinking" makes more sense. Examples which come to mind might be the allocation of fishing rights to applicants [160], or the choice of locations for army recruitment centres [10].

Without prejudice, we discuss structuring projects first of all. In general, an aim in MCPDA—as in decision analysis in general—is to come up with "creative, doable alternatives" [117]. One downside in encouraging idea generation, however, is that there may be too many alternatives generated to actually assess. A common prescription in the literature is to use some form of shortlisting—for example by using a screening model for a first pass and then more detailed economic or optimisation model to assess consequences in detail [13]. A second approach is to design some way to combine smaller projects into "package projects" (either making use of existing Problem Structuring Methods [11, 127] or customised approaches [9]) and assess these packages. In the flat furnishing example, for instance, you might choose to combine a dining table and chairs into a single item, although it would be possible to prioritise the table and chairs separately. This can have several advantages: the numbers of items which have to be evaluated is reduced, saving judgemental effort, packages can be constructed so that they are similar in size, thus avoiding scope insensitivity [92, 131] problems at the assessment phase, and it may be possible to construct packages such that the number of interactions between packages is minimised. The drawback is that good projects may be "hidden" within poor packages.

Montibeller et al. [128] observe that in situations where there may be natural groupings or "areas" for projects, analysts face a choice between establishing the areas first and using those areas to structure idea generation (how might one want to furnish the kitchen? how might one want to furnish the dining room?) on the one hand; and generating projects and then grouping them (as "kitchen projects", "dining room projects", etc), on the other. They observe that this distinction is similar to that between top-down and bottom-up structuring in criteria hierarchies, and between value- and alternative-focussed thinking in the generation of alternatives. This seems plausible: one would expect that these different methods would lead to different option sets (for example if one structures idea generation by asking for possible purchases in either the kitchen or the dining room, DMs may be less likely to generate a coat rack for the hallway which connects the two).

It is common in MCDA approaches to assume that the modelling plays no formal role in generating options. An interesting case where this assumption is relaxed is in the model of Souder and Mandakovic [113, 114], who propose a model for coordinating project selection in an organisational setting where subdepartments provide possible projects and the centre acts as DM. They observe that it is possible to think of this problem through mathematical programming decomposition, and propose a scheme whereby the centre passes information about preference to subdepartments and subdepartments respond by providing new, improved plans.

A complication which arises in MCPDA, but which is not present in single choice decision analysis applications, is that projects may interact [17, 43, 45, 152]. The standard classification of interactions is into:

- resource actions, where there are savings or additional costs from doing two (or more) projects simultaneously (e.g. it costs less to purchase a dining table and chairs as a set than to purchase them separately);
- technical interactions, where one project cannot be done without first doing another project, or alternatively, doing one project makes doing another impossible (for example, we may have space for the large refrigerator with icemaker or the smaller one without, but cannot accommodate both); and
- benefit interactions, where doing two (or more) projects together is worth more, or less than the sum of benefits of doing each project individually (we value the wide-screen television more when we have a sofa from which to watch it in comfort).

Modelling these interactions can require considerable ingenuity, e.g. [55, 56]. In some cases there may be some underlying model which can be used to define interactions automatically—for example [122] provide a model of road prioritisation in which the road degradation process is explicitly modelled through a Markov chain. In this case a difficulty is that the underlying model has to be incorporated into the PDA model, which may give rise to a problem which is computationally difficult to solve. On the other hand, direct judgemental assessment of interactions is also possible, perhaps through a device such as a cross-impact grid: an obvious drawback of this approach is asking a DM to explicitly assess whether an interaction exists and its sign for every pair of projects could represent a substantial judgemental burden. For this reason, [142] advocate deliberately not assessing interactions on the grounds that most of them will be not relevant to the decision, and those which are significant can be taken into account outside of the model.

In contrast to the formal model of the last section, some criteria may be at the level of the portfolio rather than at level of the individual project. A common example of such a portfolio-level criterion is “balance” [27, 28, 44, 78]. Often, it is felt that where there are groupings of projects such that it is desirable that a final allocation be balanced in the sense that there are not “too many” projects of one type or another. This feeling may come from a number of sources. It may, for instance, reflect an urge to diversify for the sake of robustness; on the other hand, it may by underpinned by a principle of “fairness” in resource allocation. Because it is often unclear exactly why balance is desirable, often it goes unmodelled, and is handled informally; alternatively, sometimes balance constraints are implemented within a model, in order to ensure that at least a certain number (or certain monetary value) of projects of a certain type are included in a portfolio [51]. Perhaps the most sophisticated approach to balance is that of Stewart [159], who models a concern for balance as a family of separate criteria which minimise deviation from some given distribution of manpower across project categories.

A related portfolio level concern is portfolio risk in situations where project success is probabilistic. In particular, if the success or failure of projects are correlated, it may be desirable to include a project in a portfolio if it is a good hedge against the risk associated with other projects. Seen through this lens, even in a setting where the aim is to maximise wealth, there may be multiple criteria in the sense that one cares both about the expected value and also about the risk. This idea features prominently in the theory of financial portfolios, most notably in the celebrated Markowitz mean-variance model. However, other measures of risk other than the variance are possible, and a rich class of models is available to capture and model this sort of concern [57, 59].

28.3.2 Exploring

Once the model elements have been defined, there follows a phase of exploration, where the DM and analyst work together to understand the DM's preferences and how they relate to the set of possible portfolios. The precise nature of this exploration will depend on the multicriteria method used. In the Multi-Attribute Value Theory modelling framework, for example, the analyst would take the DM through specific valuation and weighting questions designed construct partial value functions and aggregate them into a common value measure. Once this has been done, the problem has been effectively reduced to a single objective value maximisation problem—although of course one may want to go back to elicited weights and values subsequently in a sensitivity analysis phase. On the other hand, in a multiobjective programming setting, exploration may involve the generation of all non-dominated portfolios.

We do not intend in this section to describe the various MCDA methods which might be applied to the problem of portfolio choice, although this is a popular area of application and so it is probably true to say that every significant MCDA method has been applied to portfolio choice at some time or another (see the literature review in the following section). However, we do outline and comment on a few salient and generic ideas which seem to be popular and which can be used within the context of any MCDA method.

A first popular idea is that of *bubble charts* [17, 32, 95]. These methods are predominantly used in the context of models where there are at least two dimensions of value and a single cost dimension. The idea in these charts is to present the possible projects in the bicriteria space, representing a particular project by a circle, the size of which reflects the investment cost associated with the project—see Fig. 28.1 for a possible bubble chart for the flat furnishing example, with the various possible purchases scored on two dimensions, comfort and aesthetics. These displays seem to be found useful by DMs as a way of understanding more deeply the available alternatives. However, the displays themselves embed significant assumptions, most notably that there is a single criterion score associated with a project, which may not be the case if there are interactions or balance constraints.

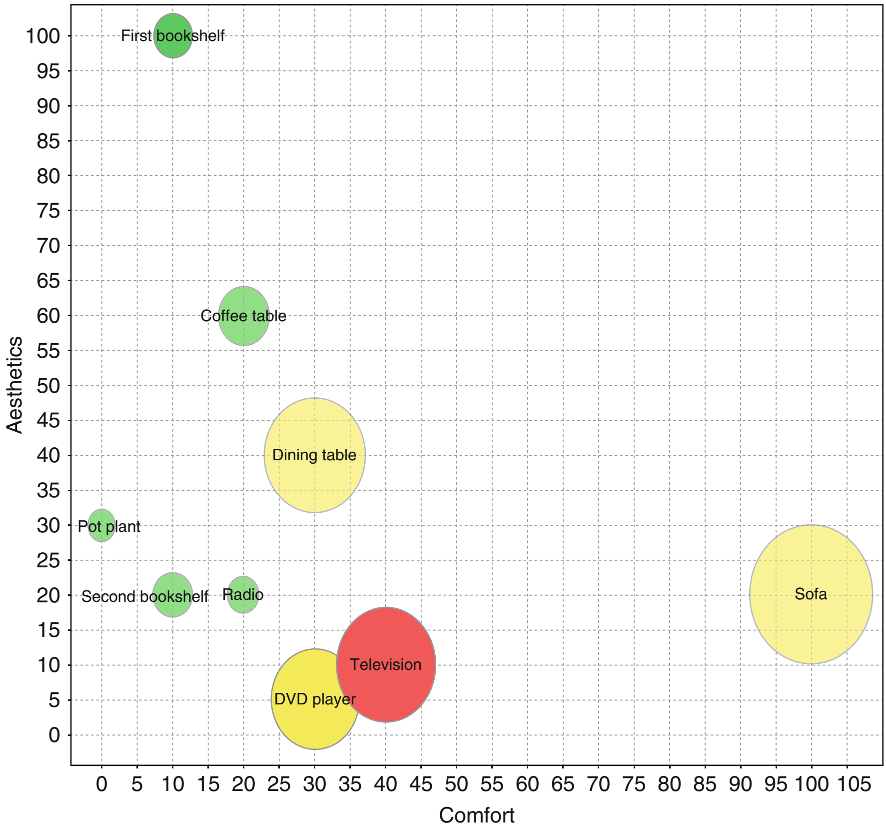


Fig. 28.1 Bubble chart for the flat furnishing example

A second popular idea is that of *triage*. This idea exploits the observation that even if one has incomplete information about the DM's preferences, it may be possible to "decide" a subset of projects. Projects which are contained in all members of the set of portfolios which are optimal with respect to some value function compatible with preference information expressed by the DM, can be decide positively; those which contained in no members of that set of portfolios can be decided negatively. Thus, with even limited, incomplete information about preference, it should be possible to narrow the space of investigation and concentrate attention on a small number of critical projects where analysis can really make a difference. This idea has been investigated in simulation studies by Keisler [84] (see also [85]) and forms the basis of the RPM-Decisions software described in [102, 103]. In Fig. 28.2, we show an example of the core index display from the RPM software for the flat furnishing problem. With no information about weights, it is not possible to say definitively what the optimal portfolio is, as there are two possibilities: however, both possible optimal portfolios contain the first and second

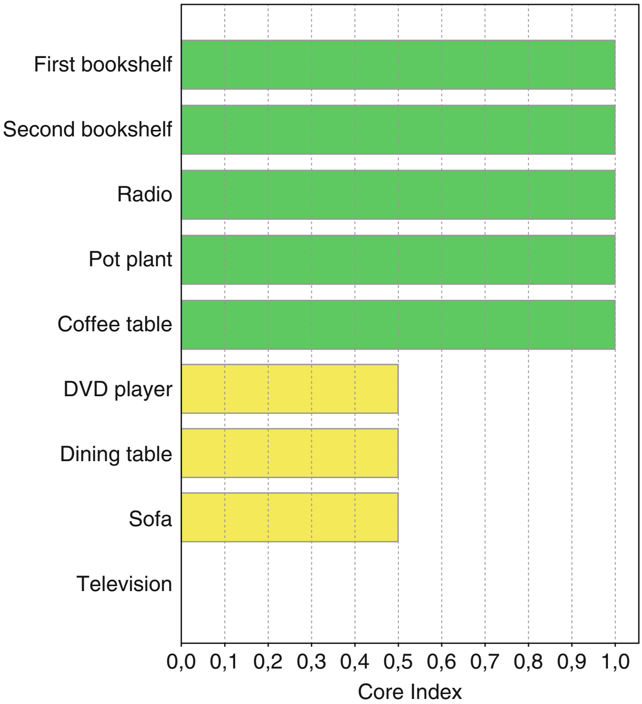


Fig. 28.2 Core index display for the flat furnishing example

bookshelves, the radio, the pot plant and the coffee table; neither of them contain the television; and the DVD player, dining table and sofa are each contained in one and one only of the two.

Another way to explore the portfolio space is to use a *cost-benefit display* of the type built into the Equity [142] and PROBE [110] software, which is applicable where there is single dimensional budget constraint $c(x) \leq B$, and cardinally measurable values have been assessed. From a mathematical standpoint, this display can be viewed as the Pareto front of a bicriterion problem which chooses non-dominated portfolios which maximise value and minimise cost; from a practical point of view it has the interpretation as the cumulative value obtained from implementing the optimal portfolio at some given level of spend, with linear interpolations between the discrete levels. In the simplest case, where criterion and value functions are linear, value increments associated with projects are unique and well-defined and the display can be generated by dividing benefit by cost for each project and proceeding down the resulting ranking, cumulating benefit and cost [91]. In more complicated environments, it may be necessary to solve a sequence of optimisation problems to generate this display. The Pareto front display for the flat furnishing example is shown in Fig. 28.3, assuming that the aesthetics and comfort scores can be combined with equal weights.

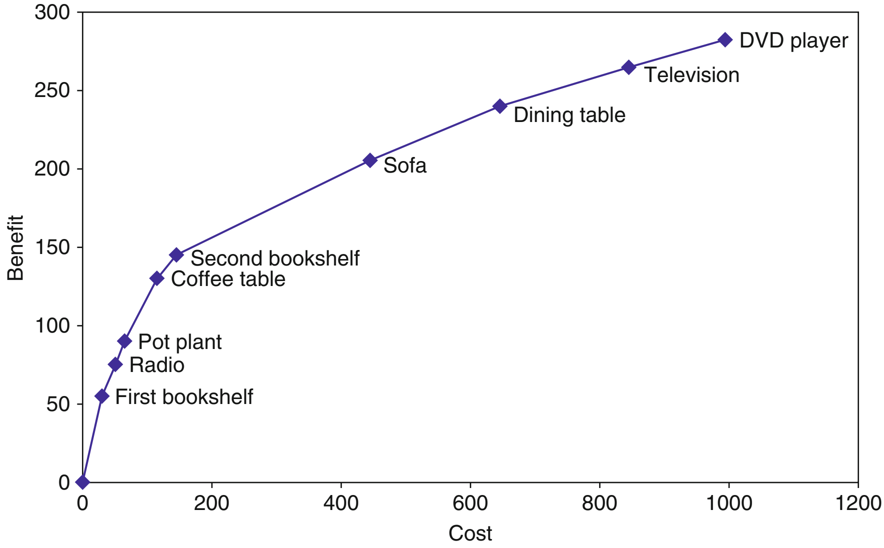


Fig. 28.3 Pareto front display for the flat furnishing example

28.4 Application Domains

In this section, we study four application domains: Research and Development (R&D) project selection; military planning and procurement; commissioning health services; and environment and energy planning. These application domains are not exhaustive. Nevertheless, they give a good general idea of the sorts of problems to which MCPDA has been applied, and the sorts of methods which have been used.

28.4.1 R&D Project Selection

Technical innovation is one of the engines of growth for both firms and nations in advanced economies. However, undertaking Research and Development (R&D) to support innovation is expensive and outcomes are hard to forecast, sometimes even hard to characterise. It should therefore come as no surprise that the OR community has been extremely active in developing solutions for R&D prioritisation. Indeed R&D management is easily the preeminent application area for portfolio and project selection models, and over several decades a vast literature dealing with this problem has accumulated (reviewed in [27, 32, 36, 63, 116, 153, 157]). Some of these models are mono-criterion in nature, typically in private sector settings where money provides a natural objective. Such models typically focus on the modelling of technical and production uncertainties about project delivery and market uncertainties about a project's ability to generate revenues. However, importantly for this chapter, there are several *multi*criteria models which have been proposed. It is important to realise that the term R&D covers a large number of different activities, and that the reasons for advocating multicriteria methods may be quite different depending on the purpose and context of the R&D in question. For example, in a government-sponsored programme of fundamental science, projects may have quite different outcomes (e.g. a Science and Engineering research council may fund projects which contribute to energy, healthcare, environmental improvement, transport. . .), and so (even if benefits could be accurately projected) there may be no single natural metric of value on which to compare projects. Further, the government sponsors of such a programme may have policy objectives, such as sponsoring interdisciplinarity, which reflect a philosophical view of the nature of the innovation process, rather than relating to specific benefits. Private sector R&D managers who have a nominal single financial objective, on the other hand, may be so far from the delivery to the customer that this objective does not meaningfully guide operational choices. This could be because the projects to be considered are in the very early phases of development and detailed market modelling is not possible, or it could be because projects are instrumental in nature, and are intended to contribute to, for example, internal operational objectives rather than generating revenues.

The literature on R&D prioritisation—even multicriteria R&D prioritisation—is vast and practically impossible to review completely. The simplest form of multicriteria approach is the scoring model [16, 25, 33, 38, 58, 65, 129], where projects are scored on a number of different dimensions using some form of attribute scale, then scores are weighted and combined, either additively or using some more complicated formula, to give an overall metric of value. Such models go back decades—for example as far back 1957, Rubinstein [149] discusses criteria for the evaluation and control of R&D projects, as well as then-current practice. However, despite (or perhaps more accurately, because of) their simplicity, scoring approaches are very much alive today [32]. While such models are likely to be useful as a tool for structuring reflection relative to holistic judgement, they are often developed and applied in apparent ignorance of the most basic decision analysis principles. As a result, it is very unclear what assumptions are being made about preferences and what the numbers (e.g. criteria weights) DMs are expected to provide mean. Such models may therefore produce a number which may guide decision making—but without DMs being forced to think as clearly as they could have been.

However, the use of multicriteria models is by no means limited to scoring models. Since the 1970s, practically all major multicriteria approaches have at some time between applied in connection with R&D prioritisation: indeed, typically a few years after a new multicriteria approach has been proposed it surfaces in the R&D project and portfolio selection context.

Considering first what one might think of as decision analysis or related methodologies, the early 80s were notable for several prominent applications using the then-new Multiattribute Utility Theory (MAUT) [50, 51], although certain scoring models proposed earlier do show awareness of decision theoretic principles and so might be considered proto-decision analyses [25, 58]. Of related interest are the applications of Keefer [80] and Keefer and Kirkwood [81] which seek to support resource allocation using a MAUT frame—although these do not constitute portfolio selection models, as utility functions are assessed directly on levels of investment with no intervening concept of a discrete project as a vehicle by which money is transformed into value. From the mid-1980s one starts to see the Analytic Hierarchy Process (AHP) and subsequently the Analytic Network Process (ANP) emerge as a competitor to MAUT [99, 118, 165] and a popular extension seems to be to combine the AHP with fuzzy numbers [69, 71]. Of particular interest in this area is the work by Lockett and Stratford [109] which uses both MAUT and AHP, and seeks to compare both approaches; also [135] describes an application of a technique called the Judgmental Analysis System (JAS), which like the AHP uses pairwise comparison data, but finds scores using geometric least squares rather than eigenvalue decomposition. Outranking approaches have been less prominently applied in this domain, but examples of applications using outranking principles do exist [36, 138], and it should also be noted that R&D prioritisation is a prominent running example in the book of Roy [148] which is the central text on outranking methods.

A difficulty with the use of decision analysis methods in the domain of R&D project prioritisation—and indeed of prioritisation generally—is that one has to capture the possibility of selecting more than one project and that projects may interact, at least through their consumption of a shared resource. A simple and popular way to model this shared resource consumption is to divide value scores by money for a "bang for the buck" index [32, 142]. A variant on this idea is to use some sort of efficiency analysis approach such as DEA [74, 107, 138], but careless use of DEA methods can be misleading for the reasons laid out in [19]. An alternative way to deal with shared resource consumption is to take the outputs from a MAUT or AHP model, and use these as coefficients in an optimisation model e.g [51, 99], in which the shared resource limit is modelled as a constraint. This has the advantage that other interactions can be modelled in the same framework. However, significant ingenuity may be required to incorporate the non-linear value functions which arise from the decision analysis modelling into an optimisation model [111, 119]. Using value scores within the context of an optimisation model seems a peculiar thing to do, however, partly for technical reasons—it is not clear whether solutions to the optimisation problem will be invariant with respect to permissible transformations of the utility scale [146]—but also because, since the decision recommendation will be based on a mono-criterion model, there is no natural way for the decision maker to contemplate the impact of value uncertainty on his decision.

In the light of the difficulty of articulating how the decision analysis part of the problem relates to the underlying optimisation problem, it should come as no surprise, therefore, that contemporaneously and to some extent in parallel, researchers have explored the application of various multiobjective optimisation based approaches in the R&D prioritisation context. Initially, one sees enthusiasm for (weighted) goal programming [87, 89, 120, 164], and latterly for exotic versions of goal programming, such as the stochastic version proposed in [12]. In one of the very few papers which bridges the divide between decision analysis and optimisation methods, [111] explore preemptive goal programming, compromise programming, and a minsum model which minimises the weighted sum of attribute distances from an ideal point. Ringuest and Graves [145] question the use of goal programming and advocate instead exact multiobjective optimisation methods, but they use a multiobjective linear programming framework which does not capture project indivisibility. Czajkowski and Jones [34] do capture indivisibilities, but the researchers restrict themselves to finding the supported efficient solutions of a rather small bicriteria problem. In general, modelling indivisible projects in an optimisation framework, raises difficulties with the combinatorial explosion in the number of efficient solutions. This can be dealt with through interactive methods [64, 159]; through the use of multiobjective metaheuristics [24, 39], or through some combination of both [161]. The flexibility of the metaheuristic approach can be seen in [61] where a multicriteria combined portfolio selection and manpower planning model is studied and solved, and in [60] which deals with a stochastic bicriteria version of the same problem. Moreover, recently the RPM approach [102], which blends decision analysis and optimisation methods and concepts, has been deployed in the R&D/innovation context, both at the level of prioritisation of specific projects [104], as well as at the level of prioritisation of research themes or topics [21, 170].

As noted above, the R&D prioritisation literature cannot be faulted for the absence of advanced analytic methods. However, a theme which recurs in the R&D prioritisation literature is—despite several detailed and published applications of implemented systems—that the advanced methods proposed in the literature are not finding widespread application in field settings [100, 155, 157]. The available empirical evidence on this point is rather out of date, and it could be that recent developments in organisational data and IT systems render these concerns obsolete. Nevertheless, if one believes such concerns have validity, a common prescription is that analytic methods should either be less technically complex (at least insofar as technical complexity imposes judgemental demands on DMs), and that there should be a renewed focus on the processual aspects of providing decision support, the institutional context of analysis, and the factors which drive successful institutionalisation [132, 142].

28.4.2 Military Planning and Procurement

Military planning and procurement decisions have been an important application area for MCPDA, in the light of the strategic importance of these decisions, the difficulty in characterising benefits, and the large sums of money involved. For this discussion, we draw heavily from the work of Burk and Parnell [23]. They describe and reference many military portfolio decision analyses, and note the following distinguishing characteristics as compared with other portfolio decisions: Legal constraints on the decisions and the decision process; strong political players; hostile adversaries; and complex systems. In addition, military portfolios involve non-financial resources and therefore may have multiple resource constraints. Furthermore, it may require considerable effort to define objectives that in essence make tradeoffs for unknown decision makers in hypothetical situations such as future battle conditions. Identifying and weighing the objectives in military portfolios is often a matter of discovery because they involve scenarios that have not occurred.

Burk and Parnell [23] identify military portfolio problems involving decisions about: "system acquisition, logistics, personnel management, training, maintenance, communications" and "weapon systems, types of forces, installations". Other portfolios in the literature involve forces, land use, infrastructure (e.g.[22]), system elements, arms transfers [156], and capabilities [37, 106]. Military portfolios may be defined in terms of assets to be procured or deployed, and portfolio elements may be distinguished in terms of location, asset type, and function.

Burk and Parnell [23] describe cases where stakeholders with different significant power bases have strong and sometimes conflicting preferences which makes the construction of value functions challenging. An extreme example might be one in which resources must be allocated across Navy, Army and Air Force assets, with senior leaders from each branch differing in their views of what is most important. Even within a single military service, there may be widely divergent views about the correct way forward: [141] describes how decision conferencing was used to develop value models for different stakeholders and to explore the implications of these value models for decisions about assets to be included in a major naval craft design.

Nevertheless, although stakeholders and their advisors may differ in their view of what drives value, unity of command means that there is one fundamental objective (force preparedness for national defence and the fulfilment of international obligations). However, because views about the best way to achieve this objective differ, and more importantly, because the military (at least in liberal democracies) is subject to strong political oversight, effective MCPDA processes have to include sensitivity analysis of results to assumptions about weights, and processes that are transparent and allow for clear explanation of the rationale behind recommendations.

Many of the objectives are derived from mission objectives, which are typically defined in formal documents and require operationalisation to a level where they can be applied to guide choice between particular types of equipment. Additional objectives include cost, safety, environmental impact, public acceptance (particularly in the case where decisions involve the use of specific sites for military purposes) and, higher level objectives than those of the mission, e.g., international relations [49].

Methods used for value modeling in military MCPDA include MultiAttribute Value and Utility theory, often within a Value-Focussed Thinking [83] framework; in some cases, the large scale of national efforts means that it is reasonable to assume that values are essentially linear, while in other cases, e.g., readiness of a particular unit, utility can easily be non-linear in some attributes—once force is overwhelming, there is limited additional benefit in investing further resources. Kangaspunta et al. [77] describe an approach to the selection of portfolios of weapons systems in a context where the underlying performances are generated by a combat simulator. The Analytic Hierarchy Process has also been used, e.g., for project selection [54]. When military portfolio decisions are meant to build preparedness for complex futures, scenario-based methods are often used [73, 79]. Some analyses are largely about determining project-level value measures for ranking and prioritisation of investments. However, many of the primary operations research methods originated in military applications, and so it is not surprising that other problems involve sophisticated mathematical optimisation, in settings where a very large number of decisions must be coordinated, e.g., personnel manpower assignment [26, 94], within a structured, stable and well-understood system, e.g. [48].

28.4.3 Commissioning Health Services

Broadly speaking, healthcare provision in developed countries takes one of two forms. Either healthcare is provided by insurance companies (e.g. Germany, the Netherlands), or it is provided by government, and funded out of general taxation (e.g. the UK, Italy, Canada). In both cases, there are opportunities for multicriteria analysis to help with the policy process (see [35] for two case studies of SMART to assist with health policy agenda-setting in the Netherlands), but in the latter case there is a particularly acute need as delivery organisations are effectively instruments of government, but their mandates are not articulated at a level of detail adequate to guide concrete choices about investment. It is on this latter case that we focus on in this section.

In order to provide support decision making in publicly funded healthcare systems, various multicriteria portfolio methods have been developed and proposed. Generally these proponents of these methods are reacting to a health economic orthodoxy in which prioritisation in health is considered to be a more-or-less a technical exercise in maximising health, with health measured through Quality-Adjusted Life Years or QALY [41, 52]. (A QALY is essentially a measure of time-integrated quality of life.) While the sophistication which has gone into refining these health metrics is considerable—health state measurement and valuation has become a small industry—for some, they miss the point. For one thing, for local decision makers, who are tasked with comparing alternative investments, building a full-blown health economic model may simply not be practical, and so multicriteria methods offer the opportunity to bring a greater degree of order to the prioritisation process without excessive cost [154]. For another, insofar as healthcare prioritisation depends critically on judgement, it has to involve deliberation by decision makers and dialogue with key stakeholders [35, 67]. As key value judgements are either hidden within the construction of the QALY, or ignored, in the view of these critics, the QALY is inadequate as a guide to priority setting, and hence the need for multicriteria methods.

As in the R&D management setting, many organisations have independently developed scoring rules which meet the need to provide some sort of orderly approach to prioritisation but which cannot be located in any particular theoretic tradition. As these scoring rules tend to be documented, if at all, in the grey literature and in non-peer reviewed publications, locating them can be quite challenging. Fortunately, [134, 154] review some of these approaches, going back to the last century (and [154] also provide a model which has been used by the Argyll and

Clyde Health Board in Scotland). More recently, however, there has been a greater interest in formal multicriteria approaches, particularly in the Program Budgeting and Marginal Analysis (PBMA) community. PBMA (e.g. [20, 124, 168]) is a structured approach to decision making about investment and disinvestment in healthcare, which involves identifying current patterns of resource use (program budgeting), and then identifying opportunities for investment and disinvestment (marginal analysis). Despite apparent success, indeed even popularity (a 2001 review [123] identified 78 applications of PBMA in 59 organisations), PBMA does not incorporate in its original form a formalised (or indeed even explicit) benefit metric, although some studies have used multiple criteria in an atheoretic way [125]. Recognising this limitation, [139] in Canada and [173, 174] and colleagues in England have conducted applications in the PBMA tradition but which also draw explicitly on MCDA literature (specifically the Multi-Attribute Value and Utility traditions)—for example in [139] a formal swing weighting approach was used. Airoldi and Morton [1] and Airoldi et al. [2] also describe case studies of a method for prioritisation which draws heavily on the decision conferencing approach of Phillips and Bana e Costa [142], which seems to be the first instance of a MCPA method, formally identified as such, being used in this setting. In a separate development, a team of researcher/practitioners associated with the World Health Organisation have also become advocates for multicriteria methods [6]. In their approach, policy makers are presented with pairs of multiattributed healthcare interventions and attribute weights are derived by fitting a logistic regression model. Case studies of this approach in Ghana and Nepal are presented in [7, 8, 75].

In terms of the methods used in this area, it should be noted that (certainly compared to R&D management and military) the methods seem to be relatively simple. The focus tends to be on valuing projects rather than valuing the portfolio as a whole ([1, 2] are exceptions). Although, there seems to be an emerging consensus that dividing value scores by costs and thus generating an efficiency ranking makes sense in an environment where one is concerned about resource use [175], the methods which are used to generate the value scores are predominantly single choice methods, which can give rise to misleading conclusions [31, 130]. There is no formal attention given to inter-project interdependencies, although there may well be interdependencies at the cost, benefit and value levels, or to balance concerns, for example between different diseases or subpopulations.

28.4.4 Environment and Energy Planning

In environmental planning and in energy planning (in which environmental considerations loom large, e.g.[53]), MCPDA has a natural place. MCDA is relevant because there are often non-financial impacts of importance to society in general and to different stakeholder groups, as demonstrated by the wealth of applications [5, 14, 70, 143, 176]. PDA is relevant in the environmental context because impacts of decisions have impact across numerous distinct portions of a larger ecosystem; and in energy because discrete units necessarily combine to form a larger energy delivery system.

There is such a variety of environmental applications (see [90]) that we do not aim here for any kind of comprehensive review. We instead give a flavor of the applications of MCPDA in this context, the commonly used attributes and resources as well as the range of stakeholders as they relate to these different portfolios, and the methods used and type of results generated.

Energy-related portfolio decisions most often involve discrete physical assets or types of assets to acquire, develop or deploy. These include: Acquisition of generation assets in a fleet or region, e.g., a mix of nuclear plants, coal plants, and wind turbines [147]; selection of generation sources for a particular user, e.g., a self-sufficient energy supply for a military installation [96]; selection of sources of a particular type such as hydropower [162] and technology investments as discussed elsewhere in Sect. 28.4.1. These are typically business decisions taken within regulatory constraints, or government funding decisions primarily (though not exclusively) viewed through an economic lens. Energy policy may also involve a set of decisions involving non-economic considerations, e.g. [76].

A common approach to project selection is to use MCDA to value essentially independent projects, while the portfolio aspect involves allocating funds across the projects. Applications include technology development or acquisition portfolios within for a single area (e.g., Solar energy, [51]) or across areas of renewable energy technologies [140] or other types of green technologies, e.g, [73]. Likewise, sets of possible remediation efforts [105] can treated as project portfolios. Another related problem is site selection, e.g., selecting a number of sites for the disposal or storage of industrial [136] or even nuclear [82] waste.

Some environmental portfolios have a spatial aspect. Land use problems (e.g., [29]), can be formulated as a portfolio of activities (uses) to which a set of regions (resources) is assigned; alternatively, the regions themselves can be the portfolio and their value can be a function of their dispositions (as suggested in [86]). Similarly, a region may contain a portfolio of habitats [163], ecosystems [108, 144] or species [40, 126] to be protected. Because areas on a map can be divided in arbitrarily many ways, environmental impacts of actions and policy can be viewed evaluated in terms of their effects on arrays of sub-regions, although such problems are not usually formulated explicitly as portfolio valuation or optimisation. Water management actions may also be considered at a portfolio level, with interactions between elements due to hydrology [30]. When geography is a constraint on portfolio formation, geographic information systems (GIS) are commonly used, and efforts to combine MCDA and GIS have been quite successful [112] in facilitating the decision process and valuing alternatives. A challenge for use of GIS in portfolio decisions is the incorporation of mathematical optimisation methods [98] with clearly defined decision variables that connect with the rich GIS representation of a situation.

Other applications involve management of environmental and resources, e.g., mining and agriculture [115, 121]. A sophisticated example of this class is described in [88] that considers a portfolio of forest areas to be planted, allowed to grow, and finally harvested.

Other portfolios are organized by levers controlled by different types of decision makers. Sets of laws and regulations can be treated as a portfolio, e.g., the fuel-economy portfolio [15]. Because laws are hard to describe as mathematical decision variables within a well-defined space, optimisation methods are difficult to apply, but evaluating a given set of portfolio alternatives using MCDA is practical. Business decision makers often consider portfolios of business units [72], portfolios of products [97], and portfolios of product design specifications [137]. These decision classes have been considered in the environmental context in some cases using project selection methods (e.g., choosing a point from the efficient frontier) along with MCDA, while in other cases analysis have used MCDA tools such as the balanced scorecard to evaluate alternative strategies [167].

In environmental MCPDA, there are many possible stakeholders [166]: nation states have an interest in levels of pollution; the whole of humanity (of today and of the future) and the natural ecosystem have a stake in whether there is global climate change. Communities have an interest in the local environmental effects, both positive and negative, of both industrial development and environmental remediation and protection; a special case of this is when the local residents are indigenous peoples and the activities are introduced by outside players. Regulators, governments and governmental organisations may represent these interests in their role as public representatives or bring interests of their own. Environmental groups may have concerns about particular ecosystems, species, regions or habitats, while individuals may be concerned about health, recreation, and even property values. Businesses have an interest in pursuing activities with economic benefits, as well as maintaining those benefits that arise from healthy environments (e.g., fishing), as well as in limiting the costs of compliance with environmental regulation. Energy portfolios involve similar sets of stakeholders, e.g., energy producers, energy consumers, and society at large.

In some cases, the criteria used in environmental decision models are hierarchical and first divided into health/safety, economic value, and ecological considerations, and then into more detailed considerations, e.g., types of emissions [171]. Criteria used in MCPDA for environmental applications tend to be similar to those used in non-portfolio MCDA. An exception is equity/fairness which naturally arises as a concern when a number of separate entities are affected, e.g.[169].

Energy criteria include cost and profit, of course, and also capacity, quality of power, local footprint and pollution from both generation (particularly CO2) and from obtaining fuel. Risk is often an issue (e.g. [3, 66]); in energy, some of the aggregate risk may be considered in a similar manner to that of financial portfolios (e.g., mean-variance models [4]). In environmental portfolios, outcomes of concern are mostly downsides to be avoided, either degradation or disaster, and cumulative risk and impact may be of concern.

Methods used in environmental problems tend to be quite participative. GIS tools, mentioned above, are prominent because of their usefulness in visualisation and their ability to make issues clear to varied audiences. Other stakeholder sensitive methods, e.g., PROMETHEE (for example as in [62]), and iterative methods, e.g., MCRID [133], are common in facilitating a decision process, while MAUT has also been used in studies supporting government bodies (e.g. [42]). While in energy portfolios, optimisation is common, in environmental portfolios, analysis is used either for simpler prioritisation and ranking, or for evaluation of a set of alternative strategies.

28.5 Conclusion and Directions for Future Research

In this chapter we have outlined what we see as some key themes in the use of multicriteria methods for the selection of portfolios of discrete projects. In Sect. 28.2, we presented a formal framework for MCPDA, based on the contribution of Golabi et al. [51]. In Sect. 28.3 we have described some of the main modelling challenges and opportunities which arise in applying multicriteria methods in the context of portfolio and project selection. In Sect. 28.4, we contextualised the discussion by surveying particular application domains.

We will conclude this chapter by discussing what we see as possible directions for future research, which draws on material we have presented in the introductory chapter of our book [150], and to which we refer the reader for more details. Our guiding philosophy is that as the selection of project portfolios is intrinsically a strategic issue, attention to technical modelling must go hand in hand with attention to social process. We group our discussion under three headings: *extending MCPDA theory, methods and tools; expanding the MCPDA knowledge base;* and *embedding MCPDA in organizational decision making*, which roughly echo the themes of Sects. 28.2–28.4 respectively.

- *Extending MCPDA theory, methods and tools.* As well as being a fruitful area for applications, MCPDA also offers a rich field for theoretic development. As we tried to lay out in Sect. 28.2, the fundamental axiomatics of which underlie MCPDA remains underdeveloped. Moreover, the design of software tools and algorithms also raises theoretic challenges. Insofar as the implied optimisation model underlying MCPDA is a knapsack problem, it is computationally hard in the deepest sense, and in the multicriteria environment, this is compounded by the difficulty of specifying completely the objective function. Moreover, in an environment where much of the analysis happens "on the fly" in workshops or in meetings with clients, algorithmic speed may be of critical importance. Therefore, there is a real role for mathematical and computational development in the mainstream of the OR tradition to support the advancement of MCPDA.
- *Expanding the MCPDA knowledge base*. In Sect. 28.3 we discussed the process of modelling in MCPDA, outlining some of the tools which have been proposed and found widespread use. Yet understanding of how best to structure and manage decision processes requires drawing on knowledge beyond the boundaries of what might be traditionally thought of as OR. One of the most

obvious linkages is to behavioural decision theory, which is profoundly relevant to questions of how best to elicit preference judgements and display information. And insofar as MCPDA is intended to support planning processes, many of the other organisational sciences (such as organisational development) have much to offer in terms of helping design better ways to structure and organise decision workshops.
- *Embedding MCPDA in organizational decision making.* One of the key lessons from Sect. 28.4, which dealt with application domains, is the extent to which practice varies and to which organisational context and sectoral matters. Indeed, contextualisation and customisation of MCPDA methods to particular settings can give rise to interesting modelling challenges: for example, some approaches to MCPDA may be purposely designed to reflect the information flows and hierarchical structure of the client organisation; others may reflect data limitations and preferred cognitive styles and professional backgrounds of individuals in a particular industry. Hence, the development of usable MCPDA tools can itself be seen as research into the characteristic features of portfolio decisions in a variety of different contexts.

Acknowledgements Thanks to Eeva Vilkkumaa and Juuso Liesiö for their help in generating Figs. 28.1 and 28.2.

References

1. Airoldi, M., Morton, A.: Portfolio decision analysis for population health. In: Salo, A., Keisler, J., Morton, A. (eds.) Portfolio decision analysis: improved methods for resource allocation. Springer, New York (2011)
2. Airoldi, M., Morton, A., Smith, J., Bevan, R.G.: STAR – people powered prioritization: a 21st century solution to allocation headaches. Med. Decis. Making **34**, 965–975 (2014)
3. Andrews, C.: Evaluating risk management strategies in resource planning. IEEE Trans. Power Syst. **10**(1), 420–426 (1995)
4. Awerbuch, S.: Portfolio-based electricity generation planning: policy implications for renewables and energy security. Mitig. Adapt. Strateg. Glob. Chang. **11**(3), 693–710 (2006)
5. Balasubramaniam, A., Voulvoulis, N.: The appropriateness of multicriteria analysis in environmental decision-making problems. Environ. Technol. **26**(9), 951–962 (2005)
6. Baltussen, R., Niessen, L.: Priority setting of health interventions: the need for multicriteria decision analysis. Cost Eff. Resour. Alloc. **4**(14) (2006)
7. Baltussen, R., Stolk, E., Chisholm, D., Aikins, M.: Towards a multi-criteria approach for priority setting: an application to Ghana. Health Econ. **15**(7), 689–696 (2006)
8. Baltussen, R., Ten Asbroek, A.H.A., Koolman, X., Shrestha, N., Bhattarai, P., Niessen, L.W.: Priority setting using multiple criteria: should a lung health programme be implemented in Nepal? Health Policy Plan. **22**(3), 178–185 (2007)
9. Bana e Costa, C.A., Oliveira, R.C.: Assigning priorities for maintenance, repair and refurbishment in managing a municipal housing stock. Eur. J. Oper. Res. **138**, 380–391 (2002)
10. Bana e Costa, C., Thomaz, J.: Locating centres of information and recruitment of volunteers for the Portugese armed forces: a decision analysis case study. In: Proceedings of the 42nd Annual Conference of the International Military Testing Association (IMTA), Edinburgh, pp. 173–180, 7-9 November 2000 (CD-ROM edition)

11. Bana e Costa, C., da Costa-Lobo, M.L., Ramos, I.A., Vansnick, J.-C.: Multicriteria approach for strategic planning: the case of Barcelos. In: Bouyssou, D., Jacquet-Lagreze, E., Perny, P., Slowinsky, R., Vanderpooten, D., Vincke, P. (eds.) Aiding Decisions with Multiple Criteria: Essays in Honour of Bernard Roy. Kluwer, Dordrecht (2002)
12. Bard, J.F.: Using multicriteria methods in the early stages of new product development. J. Oper. Res. Soc. **41**(8), 755–766 (1990)
13. Bard, J.F., Balachandra, R., Kaufmann, P.E.: An interactive approach to R-and-D project selection and termination. IEEE Trans. Eng. Manag. **35**(3), 139–146 (1988)
14. Bell, M.L., Hobbs, B.F., Elliott, E.M., Ellis, H., Robinson, Z.: An evaluation of multicriteria decision-making methods in integrated assessment of climate policy. In: Research and Practice in Multiple Criteria Decision Making: Proceedings of the XIVth International Conference on Multiple Criteria Decision Making, Charlottesville, 8–12 June 1998, p. 228. Springer, Berlin (2000)
15. Berry, T., Jaccard, M.: The renewable portfolio standard: design considerations and an implementation survey. Energy Policy **29**(4), 263–277 (2001)
16. Bitman, W.R., Sharif, N.: A conceptual framework for ranking R&D projects. IEEE Trans. Eng. Manag. **55**(2), 267–278 (2008)
17. Blau, G.E., Pekny, J.F., Varma, V.A., Bunch, P.R.: Managing a portfolio of interdependent new product candidates in the pharmaceutical industry. J. Prod. Innov. Manag. **21**(4), 227–245 (2004)
18. Bordley, R.F.: R&D project selection versus R&D project generation. IEEE Trans. Eng. Manag. **45**(4) (1998)
19. Bouyssou, D.: Using DEA as a tool for MCDM: some remarks. J. Oper. Res. Soc. **50**(9), 974–978 (1999)
20. Brambleby, P.: A survivors guide to program budgeting. Health Policy **33**(2), 127–145 (1995)
21. Brummer, V., Konnola, T., Salo, A.: Foresight within ERA-NETs: experiences from the preparation of an international research program. Technol. Forecast. Soc. Chang. **75**(4), 483–495 (2008)
22. Buckshaw, D.L., Parnell, G.S, Unkenholz, W.L., Parks, D.L., Wallner, J.L., Saydjari, O.S.: Mission oriented risk and design analysis of critical information systems. Mil. Oper. Res. Alexandria **10**(2), 19 (2005)
23. Burk, R.C., Parnell, G.S.: Portfolio decision analysis: lessons from military applications. In: Salo, A., Keisler, J., Morton, A. (eds.) Portfolio Decision Analysis: Methods for Improved Resource Allocation. Springer, New York (2011)
24. Carazo, A.F., Gomez, T., Molina, J., Hernandez-Diaz, A.G., Guerrero, F.M., Caballero, R.: Solving a comprehensive model for multiobjective project portfolio selection. Comput. Oper. Res. **37**(4), 630–639 (2010)
25. Cardus, D., Fuhrer, M.J., Martin, A.W., Thrall, R.M.: Use of benefit-cost-analysis in the peer-review of proposed research. Manag. Sci. **28**(4), 439–445 (1982)
26. Charnes, A., Cooper, W.W., Niehaus, R.J.: Dynamic multiattribute models for mixed man-power systems. Nav. Res. Logist. Q. **22**(2), 205–220 (1975)
27. Chien, C.F.: A portfolio-evaluation framework for selecting R&D projects. R&D Manag. **32**(4), 359–368 (2002)
28. Chien, C.F., Sainfort, F.: Evaluating the desirability of meals: an illustrative multiattribute decision analysis procedure to assess portfolios with interdependent items. J. Multi-Criteria Decis. Anal. **7**, 230–238 (1998)
29. Christou, M.D., Mattarelli, M.: Land-use planning in the vicinity of chemical sites: risk-informed decision making at a local community level. J. Hazard. Mater. **78**(1-3), 191–222 (2000)
30. Chung, E.S., Lee, K.S.: Prioritization of water management for sustainability using hydrologic simulation model and multicriteria decision making techniques. J. Environ. Manag. **90**(3), 1502–1511 (2009)
31. Clemen, R.T., Smith, J.E.: On the choice of baselines in multiattribute portfolio analysis: a cautionary note. Decis. Anal. **6**(4), 156–262 (2009)

32. Cooper, R.G., Edgett, S.J., Kleinschmidt, E.J.: Portfolio Management for New Products, 2nd edn. Basic Books, New York (2001)
33. Costello, D.: A practical approach to R-and-D project selection. Technol. Forecast. Soc. Chang. **23**(4), 353–368 (1983)
34. Czajkowski, A.F., Jones, S.: Selecting interrelated research-and-development projects in space technology planning. IEEE Trans. Eng. Manag. **33**(1), 17–24 (1986)
35. Daniels, N.: Accountability for reasonableness - establishing a fair process for priority setting is easier than agreeing on principles. Br. Med. J. **321**(7272), 1300–1301 (2000)
36. Danila, N.: Strategic evaluation and selection of R & D projects. R&D Manag. **19**(1), 47–62 (1989)
37. David, P.K., Shaver, R.D., Beck, J.: Portfolio analysis methods for assessing capability options. Rand Corporation, Santa Monica (2008)
38. Dean, B.V., Nishry, M.J.: Scoring and profitability models for evaluating and selecting research projects. Oper. Res. **13**(4), 550–569 (1965)
39. Doerner, K., Gutjahr, W.J., Hartl, R.F., Strauss, C., Stummer, C.: Pareto ant colony optimization: a metaheuristic approach to multiobjective portfolio selection. Ann. Oper. Res. **131**(1-4), 79–99 (2004)
40. Drechsler, M., Frank, K., Hanski, I., O'Hara, R.B., Wissel, C.: Ranking metapopulation extinction risk: from patterns in data to conservation management decisions. Ecol. Appl. **13**(4), 990–998 (2003)
41. Drummond, K.F., Sculpher, M.J., Torrance, G.W., O' Brien, B.J., Stoddart, G.L.: Methods for the Economic Evaluation of Health Care Programmes. Oxford University Press, Oxford (2005)
42. Dyer, J., Edmunds, T., Butler, J.C., Jia, J.: A multiattribute utility analysis of alternatives for the disposition of surplus weapons-grade plutonium. Oper. Res. **46**(6), 749–762 (1998)
43. Fahrni, P., Spatig, M.: An application-oriented guide to research-and-development project selection and evaluation methods. R&D Manag. **20**(2), 155–171 (1990)
44. Farquhar, P.H., Rao, V.R.: A balance model of revaluating subsets of multiattributed items. Manag. Sci. **22**(5), 528–539 (1976)
45. Fox, G.E., Baker, N.R., Bryant, J.L.: Economic-models for R-and-D project selection in the presence of project interactions. Manag. Sci. **30**(7), 890–902 (1984)
46. Friend, J., Hickling, A.: Planning Under Pressure: The Strategic Choice Approach, 2nd edn. Pergamon, Oxford (1997)
47. Friend, J.K., Jessop, W.N.: Local Government and Strategic Choice: An Operational Research Approach to the Processes of Public Planning. Tavistock, London (1969)
48. Gass, S.I.: Military manpower planning models. Comput. Oper. Res. **18**(1), 65–73 (1991)
49. Gass, N.: Conflict analysis in the politico-military environment of a new world order. J. Oper. Res. Soc. **45**(2), 133–142 (1994)
50. Golabi, K.: Selecting a portfolio of nonhomogeneous R and D proposals. Eur. J. Oper. Res. **21**(3), 347–357 (1985)
51. Golabi, K., Kirkwood, C.W., Sicherman, A.: Selecting a portfolio of solar energy projects using multiattribute preference theory. Manag. Sci. **27**, 174–189 (1981)
52. Gold, M.R., Siegel, J.E., Russell, L.B., Weinstein, M.C. (eds.): Cost-Effectiveness in Health and Medicine. Oxford University Press, Oxford (1996)
53. Greening, L.A., Bernow, S.: Design of coordinated energy and environmental policies: use of multi-criteria decision-making. Energy Policy **32**(6), 721–735 (2004)
54. Greiner, M.A., Fowler, J.W., Shunk, D.L., Carlyle, W.M., McNutt, R.T.: A hybrid approach using the analytic hierarchy process and integer programming to screen weapon systems projects. IEEE Trans. Eng. Manag. **50**(2), 192–203 (2003)
55. Grushka-Cockayne, Y., De Reyck, B.: Towards a single European sky. Interfaces **39**(5), 400–414 (2009)
56. Grushka-Cockayne, Y., De Reyck, B., Degraeve, Z.: An integrated decision-making approach for improving European air traffic management. Manag. Sci. **54**(8), 1395–1409 (2008)

57. Gustafsson, J., Salo, A.: Contingent portfolio programming for the management of risky projects. Oper. Res. **53**(6), 946–956 (2005)
58. Gustafson, D.H., Pai, G.K., Kramer, G.C.: A "weighted aggregate" approach to R & D project selection. AIIE Trans. **3**(1), 22–31 (1972)
59. Gustafsson, J., De Reyck, B., Degraeve, Z., Salo, A.: Valuation of risky projects and other illiquid investments using portfolio selection models. In: Salo, A., Keisler, J., Morton, A. (eds.) Portfolio Decision Analysis: Methods for Improved Resource Allocation. Springer, New York (2011)
60. Gutjahr, W.J., Reiter, P.: Bi-objective project portfolio selection and staff assignment under uncertainty. Optimization **59**(3), 417–445 (2010)
61. Gutjahr, W.J., Katzensteiner, S., Reiter, P., Stummer, C., Denk, M.: Multi-objective decision analysis for competence-oriented project portfolio selection. Eur. J. Oper. Res. **205**(3), 670–679 (2010)
62. Haralambopoulos, D.A., Polatidis, H.: Renewable energy projects: structuring a multi-criteria group decision-making framework. Renew. Energy **28**(6), 961–973 (2003)
63. Heidenberger, K., Stummer, C.: Research and development project selection and resource allocation: a review of quantitative modelling approaches. Int. J. Manag. Rev. **1**(2), 197–224 (1999)
64. Henig, M., Katz, H.: R and D project selection: a decision process approach. J. Multi-Criteria Decis. Anal. **5**, 169–171 (1996)
65. Henriksen, A.D., Traynor, A.J.: A practical R & D project-selection scoring tool. IEEE Trans. Eng. Manag. **46**(2), 158–170 (1999)
66. Hirst, E., Schweitzer, M.: Electric-utility resource planning and decision-making: the importance of uncertainty. Risk Anal. **10**(1), 137–146 (1990)
67. Holm, S.: Goodbye to the simple solutions: the second phase of priority setting in health care. Br. Med. J. **317**(7164), 1000–1002 (1998)
68. Howard, R.A.: Decision analysis: practice and promise. Manag. Sci. **34**(6), 679–695 (1988)
69. Hsu, Y.-G., Tzeng, W.-H., Shyu, J.Z.: Fuzzy multiple criteria selection of government-sponsored frontier technology projects. R&D Manag. **33**(5), 539–551 (2003)
70. Huang, J.P., Poh, K.L., Ang, B.W.: Decision analysis in energy and environmental modeling. Energy **20**(9), 843–855 (1995)
71. Huang, C.C., Chu, P.Y., Chiang, Y.H.: A fuzzy AHP application in government-sponsored R&D project selection. Omega Int. J. Manag. Sci. **36**(6), 1038–1052 (2008)
72. Ilinitch, A.Y., Schaltegger, S.C.: Developing a green business portfolio. Long Range Plan. **28**(2), 29–38 (1995)
73. Jackson, J., Kloeber Jr., J.M., Ralston, B.E., Deckro, R.F.: Selecting a portfolio of technologies: an application of decision analysis. Decis. Sci. **30**(1), 217–238 (1999)
74. Jayanthi, S., Witt, E.C., Singh, V.: Evaluation of potential of innovations: a DEA-based application to us photovoltaic industry. IEEE Trans. Eng. Manag. **56**(3), 478–493 (2009)
75. Jehu-Appiah, C., Baltussen, R., Acquah, C., Aikins, M., d'Almeida, S.A., Bosu, W.K., Koolman, X., Lauer, J., Osei, D., Adjei, S.: Balancing equity and efficiency in health priorities in Ghana: the use of multicriteria decision analysis. Value Health **11**(7), 1081–1087 (2008)
76. Jones, M., Hope, C., Hughes, R.: A multi-attribute value model for the study of UK energy policy. J. Oper. Res. Soc. **41**(10), 919–929 (1990)
77. Kangaspunta, J., Liesiö, J., Salo, A.: Cost-efficiency analysis of weapon system portfolios. Eur. J. Oper. Res. **223**(1), 264–275 (2012)
78. Karsu, Ö., Morton, A.: Incorporating balance concerns in resource allocation decisions: a bicriteria modelling approach. OMEGA – Int. J. Manag. Sci. **44**, 70–82 (2014)
79. Karvetski, C.W., Lambert, J.H., Linkov, I.: Scenario and multiple criteria decision analysis for energy and environmental security of military and industrial installations. Integr. Environ. Assess. Manag. **7**,228–236 (2010)
80. Keefer, D.L.: Allocation planning for R & D with uncertainty and multiple objectives. IEEE Trans. Eng. Manag. **EM-25**(1), 8–14 (1978)

81. Keefer, D.L., Kirkwood, C.W.: A multiobjective decision analysis: budget planning for product engineering. J. Oper. Res. Soc. **29**(5), 435–442 (1978)
82. Keeney, R.L.: An analysis of the portfolio of sites to characterize for selecting a nuclear repository. Risk Anal. **7**(2), 195–218 (1987)
83. Keeney, R.L.: Value focused Thinking: A Path to Creative Decision Making. Harvard University Press, Cambridge (1992)
84. Keisler, J.: Value of information in portfolio decision analysis. Decis. Anal. **1**(3), 177–189 (2004)
85. Keisler, J.: The value of assessing weights in multi-criteria portfolio decision analysis. J. Multi-Criteria Decis. Anal. **15**, 111–123 (2009)
86. Keisler, J.M., Sundell, R.C.: Combining multi-attribute utility and geographic information for boundary decisions: an application to park planning. J. Geogr. Inf. Decis. Anal. **1**(2), 101–118 (1997)
87. Keown, A.J., Taylor, B.W., Duncan, C.P.: Allocation of research and development funds - zero-one goal programming approach. Omega Int. J. Manag. Sci. **7**(4), 345–351 (1979)
88. Kettunen, J.: Applications of stochastic modelling for investment decision making under market uncertainties. Ph.D. thesis, Helsinki University of Technology (2009)
89. Khorramshahgol, R., Gousty, Y.: Delphic Goal Programming (DGP) - a multiobjective cost-benefit approach to research-and-development portfolio analysis. IEEE Trans. Eng. Manag. **33**(3), 172–175 (1986)
90. Kiker, G.A., Bridges, T.S., Varghese, A., Seager, T.P., Linkov, I.: Application of multicriteria decision analysis in environmental decision making. Integr. Environ. Assess. Manag. **1**(2), 95–108 (2005)
91. Kirkwood, C.W.: Strategic Decision Making: Multiobjective Decision Analysis with Spreadsheets. Duxbury, Belmont (1997)
92. Kleinmuntz, D.N.: Resource allocation decisions. In: Edwards, W., Miles, R.F., von Winterfeldt, D. (eds.) Advances in Decision Analysis, pp. 400–418. Cambridge University Press, Cambridge (2007)
93. Kleinmuntz, D.N.: Foreword. In: Salo, A., Keisler, J., Morton, A. (eds.) Advances in Portfolio Decision Analysis: Improved Methods for Resource Allocation. Springer, New York (2011)
94. Klingman, D., Phillips, N.V.: Topological and computational aspects of preemptive multicriteria military personnel assignment problems. Manag. Sci. **30**(11), 1362–1375 (1984)
95. Kloeber, J.: Current and cutting edge methods of portfolio decision analysis in pharmaceutical R&D. In: Salo, A., Keisler, J., Morton, A. (eds.) Portfolio Decision Analysis: Methods for Improved Resource Allocation. Springer, New York (2011)
96. Lambert, J.H., Karvetski, C.W., Ditmer, R.D., Abdallah, T., Johnson, M.D., Linkov, I.: Energy security for industrial and military installations: emergent conditions that influence the strategic selection of technologies. In: Gheorghe, A., Muresan, L. (eds.) Energy Security: International Local Issues: Theoretical Perspectives and Critical Energy Infrastructures. Springer, New York (2011)
97. Lee, B.W., Green, K.: Towards commercial and environmental excellence: a green portfolio matrix. Bus. Strateg. Environ. **3**(3), 1–9 (1994)
98. Li, X., Chen, Y., Liu, X., Li, D., He, J.: Concepts, methodologies, and tools of an integrated geographical simulation and optimization system. Int. J. Geogr. Inf. Sci. **25**(4), 633–655 (2011)
99. Liberatore, M.J.: An extension of the Analytic Hierarchy Process for industrial research-and-development project selection and resource-allocation. IEEE Trans. Eng. Manag. **34**(1), 12–18 (1987)
100. Liberatore, M.J., Titus, G.J.: The practice of management science in R-and-D project-management. Manag. Sci. **29**(8), 962–974 (1983)
101. Liesiö, J.: Measureable multiattribute value functions for portfolio decision analysis. Decis. Anal. **11**(1), 1–20 (2014)
102. Liesiö, J., Mild, P., Salo, A.: Preference programming for robust portfolio modeling and project selection. Eur. J. Oper. Res. **181**(3), 1488–1505 (2007)

103. Liesiö, J., Mild, P., Salo, A.: Robust portfolio modeling with incomplete cost information and project interdependencies. Eur. J. Oper. Res. **190**(3), 679–695 (2008)
104. Lindstedt, M., Liesiö, J., Salo, A.: Participatory development of a strategic product portfolio in a telecommunication company. Int. J. Technol. Manag. **42**(3), 250–266 (2008)
105. Linkov, I., Varghese, A., Jamil, S., Seager, T., Kiker, G., Bridges, T.: Multi-criteria decision analysis: a framework for structuring remedial decisions at contaminated sites. In: Linkov, I., Bakr Ramadan, A. (eds.) Comparative Risk Assessment and Environmental Decision Making. Kluwer, Dordrecht (2005)
106. Linkov, I., Satterstrom, F.K., Fenton, G.P.: Prioritization of capability gaps for joint small arms program using multi-criteria decision analysis. J. Multi-Criteria Decis. Anal. **16**(5-6), 179–185 (2009)
107. Linton, J.D., Walsh, S.T., Morabito, J.: Analysis, ranking and selection of R & D projects in a portfolio. R&D Manag. **32**(2), 139–148 (2002)
108. Liu, C., Frazier, P., Kumar, L., Macgregor, C., Blake, N.: Catchment-wide wetland assessment and prioritization using the multi-criteria decision-making method TOPSIS. Environ. Manag. **38**(2), 316–326 (2006)
109. Lockett, G., Stratford, M.: Ranking of research projects - experiments with 2 methods. Omega Int. J. Manag. Sci. **15**(5), 395–400 (1987)
110. Lourenço, J.C., Morton, A., Bana e Costa, C.A.B.: PROBE: a multicriteria decision support 1056 system for portfolio robustness evaluation. Decis. Support Syst. **54**(1), 534–550 (2012)
111. Madey, G.R., Dean, B.V.: Strategic planning in R&D using decision analysis and mathematical programming. IEEE Trans. Eng. Manag. **EM-32**(2), 84–90 (1985)
112. Malczewski, J.: GIS and Multicriteria Decision Analysis. Wiley, New York (1999)
113. Mandakovic, T., Souder, W.E.: A flexible hierarchical model for project selection and budget allocation. R&D Manag. **15**(1), 23–29 (1985)
114. Mandakovic, T., Souder, W.E.: An interactive decomposable heuristic for project selection. Manag. Sci. **31**(10), 1257–1271 (1985)
115. Mardle, S., Pascoe, S., Tamiz, M., Jones, D.: Resource allocation in the North sea demersal fisheries: a goal programming approach. Ann. Oper. Res. **94**(1), 321–342 (2000)
116. Martino, J.: R & D Project Selection. Wiley, Chichester (1995)
117. Matheson, D., Matheson, J.: The Smart Organisation: Creating Value Through Strategic R&D. Harvard Business School Press, Boston (1998)
118. Meade, L.M., Presley, A.: R&D project selection using the analytic networkwork process. IEEE Trans. Eng. Manag. **49**(1) (2002)
119. Mehrez, A., Sinuany-stern, Z.: Resource-allocation to interrelated risky projects using a multiattribute utility function. Manag. Sci. **29**(4), 430–439 (1983)
120. Mellichamp, J.M., Dixon, W.L., Mitchell, S.L.: Ballistic-missile defense technology management with goal programming. Interfaces **10**(5), 68–74 (1980)
121. Mendoza, G.A., Martins, H.: Multi-criteria decision analysis in natural resource management: a critical review of methods and new modelling paradigms. For. Ecol. Manag. **230**(1-3), 1–22 (2006)
122. Mild, P., Salo, A.: Combining a multiattribute value function with an optimization model: an application to dynamic resource allocation for infrastructure maintenance. Decis. Anal. **6**(3), 139–152 (2009).
123. Mitton, C., Donaldson, C.: Twenty-five years of programme budgeting and marginal analysis in the health sector, 1974-1999. J. Health Serv. Res. Policy **6**(4), 239–48 (2001)
124. Mitton, C.R., Donaldson, C.: Setting priorities and allocating resources in health regions: lessons from a project evaluating program budgeting and marginal analysis (PBMA). Health Policy **64**, 335–348 (2003)
125. Mitton, C.R., Patten, S., Waldner, H., Donaldson, C.: Priority setting in health authorities: a novel approach to a historical activity. Soc. Sci. Med. **57**, 1653–1663 (2003)
126. Moffett, A., Dyer, J.S., Sarkar, S.: Integrating biodiversity representation with multiple criteria in North-Central Namibia using non-dominated alternatives and a modified analytic hierarchy process. Biol. Conserv. **129**(2), 181–191 (2006)

127. Montibeller, G., Franco, L.A.: Resource allocation in local government with facilitated portfolio decision analysis. In: Salo, A., Keisler, J., Morton, A. (eds.) Portfolio Decision Analysis: Methods for Improved Resource Allocation. Springer, New York (2011)
128. Montibeller, G., Franco, L.A., Lord, E., Iglesias, A.: Structuring resource allocation decisions: a framework for building multi-criteria portfolio models with area-grouped options. Eur. J. Oper. Res. **199**(3), 846–856 (2009)
129. Moore, J.R., Baker, N.R.: Computational analysis of scoring models for R and D project selection. Manag. Sci. **16**(4), B212–232 (1969)
130. Morton, A.: Measurement issues in the evaluation of projects in a project portfolio. Eur. J. Oper. Res. **245**(16), 789–796 (2015)
131. Morton, A., Fasolo, B.: Behavioural decision theory for multi-criteria decision analysis: a guided tour. J. Oper. Res. Soc. **60**(2), 268–275 (2009)
132. Morton, A., Bird, D., Jones, A., White, M.: Decision conferencing for science prioritisation in the UK public sector: a dual case study. J. Oper. Res. Soc. **62**, 50–59 (2011)
133. Moskowitz, H., Preckel, P.V., Yang, A.: Multiple-criteria robust interactive decision analysis (MCRID) for optimizing public policies. Eur. J. Oper. Res. **56**(2), 219–236 (1992)
134. Mullen, P.M.: Quantifying priorities in healthcare: transparency or illusion. Health Serv. Manage. Res. **17**(1), 47–58 (2004)
135. Naudé, P., Lockett, G., Holmes, K.: A case study of strategic engineering decision making using judgmental modeling and psychological profiling. IEEE Trans. Eng. Manag. **44**(3), 237–247 (1997)
136. Ohman, K.V.H., Hettiaratchi, J.P.A., Ruwanpura, J., Balakrishnan, J., Achari, G.: Development of a landfill model to prioritize design and operating objectives. Environ. Monit. Assess. **135**(1), 85–97 (2007)
137. Olundh, G., Ritzén, S.: Making an ecodesign choice in project portfolio selection. In: Proceedings of the IEEE International Engineering Management Conference, vol. 3, pp. 913–917. IEEE (2004)
138. Oral, M., Kettani, O., Lang, P.: A methodology for collective evaluation and selection of industrial research-and-development projects. Manag. Sci. **37**(7), 871–885 (1991)
139. Peacock, S.J., Richardson, J.R.J., Carter, R., Edwards, D.: Priority setting in health care using multi-attribute utility theory and programme budgeting and marginal analysis (PBMA). Soc. Sci. Med. **64**(4), 897–910 (2007)
140. Peerenboom, J.P., Buehring, W.A., Joseph, T.W.: Selecting a portfolio of environmental programs for a synthetic fuels facility. Oper. Res. **37**(5), 689–699 (1989)
141. Phillips, L.D.: The royal navy's type 45 story: a case study. In: Salo, A., Keisler, J., Morton, A. (eds.) Portfolio Decision Analysis: Improved Methods for Resource Allocation. Springer, New York (2011)
142. Phillips, L.D., Bana e Costa, C.: Transparent prioritisation, budgeting and resource allocation with multi-criteria decision analysis and decision conferencing. Ann. Oper. Res. **154**(1), 51–68 (2007)
143. Pohekar, S.D., Ramachandran, M.: Application of multi-criteria decision making to sustainable energy planning–a review. Renew. Sust. Energ. Rev. **8**(4), 365–381 (2004)
144. Randhir, T., Shriver, D.M.: Deliberative valuation without prices: a multiattribute prioritization for watershed ecosystem management. Ecol. Econ. **68**(12), 3042–3051 (2009)
145. Ringuest, J.L., Graves, S.B.: The linear multi-objective R-and-D project selection problem. IEEE Trans. Eng. Manag. **36**(1), 54–57 (1989)
146. Roberts, F.S.: Meaningfulness of conclusions from combinatorial optimization. Discret. Appl. Math. **29**, 221–241 (1990)
147. Roth, S., Hirschberg, S., Bauer, C., Burgherr, P., Dones, R., Heck, T., Schenler, W.: Sustainability of electricity supply technology portfolio. Ann. Nucl. Energy **36**(3), 409–416 (2009)
148. Roy, B.: Méthodologie multicritère d'aide á la décision. Economica, Paris (1985)
149. Rubinstein, A.H.: Selecting criteria for R and D. Harv. Bus. Rev. **35**(1), 95–104 (1957)

150. Salo, A., Keisler, J., Morton, A.: Portfolio Decision Analysis: Improved Methods for Resource Allocation. Springer, New York (2011)
151. Salo, A., Keisler, J., Morton, A. (eds.): Portfolio Decision Analysis: Methods for Improved Resource Allocation. Springer, New York (2011)
152. Schmidt, R.L.: A model for R&D project selection with combined benefit, outcome and resource interactions. IEEE Trans. Eng. Manag. **40**(4), 403–410 (1993)
153. Schmidt, R.L., Freeland, J.R.: Recent progress in modeling R&D project-selection processes. IEEE Trans. Eng. Manag. **39**(2), 189–201 (1992)
154. Scott, S.N., Lees, A.: Developing a prioritisation framework: experiences from a Scottish health authority. Health Expect. **4**(1), 10–17 (2001)
155. Shane, S.A., Ulrich, K.T.: Technological innovation, product development, and entrepreneurship in management science. Manag. Sci. **50**(2), 133–144 (2004)
156. Sherwin, R.G., Laurance, E.J.: Arms transfers and military capability: measuring and evaluating conventional arms transfers. Int. Stud. Q. **23**(3), 360–389 (1979)
157. Souder, W.E., Mandakovic, T.: Research-and-development project selection models. Res. Manag. **29**(4), 36–42 (1986)
158. Spradlin, C.T., Kutoloski, D.M.: Action-oriented portfolio management. Res. Technol. Manag. **42**(2), 26–32 (1999)
159. Stewart, T.J.: A multicriteria decision support system for R-and-D project selection. J. Oper. Res. Soc. **42**(1), 17–26 (1991)
160. Stewart, T.J., Joubert, A., Janssen, R.: MCDA framework for fishing rights allocation in South Africa. Group Decis. Negot. **19**(3), 247–265 (2010)
161. Stummer, C., Heidenberger, K.: Interactive R&D portfolio analysis with project interdependencies and time profiles of multiple objectives. IEEE Trans. Eng. Manag. **50**(2), 175–183 (2003)
162. Supriyasilp, T., Pongput, K., Boonyasirikul, T.: Hydropower development priority using MCDM method. Energy Policy **37**(5), 1866–1875 (2009)
163. Svoray, T., Bar Kutiel, P., Bannet, T.: Urban land-use allocation in a mediterranean ecotone: habitat heterogeneity model incorporated in a GIS using a multi-criteria mechanism. Landsc. Urban Plan. **72**(4), 337–351 (2005)
164. Taylor, B.W., Moore, L.J., Clayton, E.R.: R-and-D project selection and manpower allocation with integer non-linear goal programming. Manag. Sci. **28**(10), 1149–1158 (1982)
165. Thurston, D.L., Tian, Y.Q.: Integration of the analytic hierarchy process and integer programming with linear constraints for long range production planning. Math. Comput. Model. **17**(4), 41–53 (1993)
166. Tonn, B., English, M., Travis, C.: A framework for understanding and improving environmental decision making. J. Environ. Plan. Manag. **43**(2), 163–183 (2000)
167. Tsai, W.H., Chou, W.C., Hsu, W.: The sustainability balanced scorecard as a framework for selecting socially responsible investment: an effective MCDM model. J. Oper. Res. Soc. **60**(10), 1396–1410 (2008)
168. Twaddle, S., Walker, A.: Program budgeting and marginal analysis - application within programs to assist purchasing in Greater Glasgow Health Board. Health Policy **33**(2), 91–105 (1995)
169. Vaillancourt, K., Waaub, J.: Equity in international greenhouse gases abatement scenarios: a multicriteria approach. Eur. J. Oper. Res. **153**(2), 489–505 (2004)
170. Vilkkumaa, E., Salo, A., Liesiö, J.: Multicriteria portfolio modeling for the development of shared action agendas. Group Decis. Negot. **23**, 772–783 (2014)
171. Wang, J.J., Jing, Y.Y., Zhang, C.F., Zhao, J.H.: Review on multi-criteria decision analysis aid in sustainable energy decision-making. Renew. Sust. Energ. Rev. **13**(9), 2263–2278 (2009)
172. White, D.J.: Decision Methodology: A Formalisation of the OR Process. Wiley, London (1975)
173. Wilson, E.C.F., Rees, S.J., Fordham, R.J.: Developing a prioritisation framework in an English Primary Care Trust. Cost Eff. Resour. Alloc. **4**(3) (2006)

174. Wilson, E., Sussex, J., MacLeod, C., Fordham, R.: Prioritizing health technologies in a Primary Care Trust. J. Health Serv. Res. Policy **12**(2), 80–85 (2007)
175. Wilson, E.C.F., Peacock, S.J., Ruta, D.: Priority setting in practice: what is the best way to compare costs and benefits? Health Econom. **18**(4), 467–478 (2009)
176. Zhou, P., Ang, B.W., Poh, K.L.: Decision analysis in energy and environmental modeling: an update. Energy **31**(14), 2604–2622 (2006)
177. Zopounidis, C., Doumpos, M.: Multicriteria classification and sorting methods: a literature review. Eur. J. Oper. Res. **138**(2), 229–246 (2002)

Part VIII
MCDM Software

Chapter 29
Multiple Criteria Decision Analysis Software

H. Roland Weistroffer and Yan Li

Abstract We provide an updated overview of the state of multiple criteria decision support software. Many methods and approaches have been proposed in the literature to handle multiple criteria decision analysis, and there is an abundance of software that implements or supports many of these approaches. Our review is structured around several decision considerations when searching for appropriate available software.

Keywords Multiple criteria decision analysis software • Decision support • Software package

29.1 Introduction

Multiple criteria decision models generally do not possess a mathematically well-defined optimum solution and thus the best the decision maker (DM) can do is to find a satisfactory compromise solution from among the efficient (non-dominated) solutions. Unless an explicit utility function representing the preferences of the DM is known a priori, interactive solution techniques are most appropriate to identify the preferred solution or perhaps a manageable set of desirable compromise solutions.

An abundance of multiple criteria decision analysis (MCDA) methods have been proposed in the literature, most of which require substantial amounts of computation. Many software packages have been developed to implement all or parts of these methods. This MCDA software covers various stages of the decision making process, from problem exploration and structuring to ascertaining the DM's preferences and identifying a most preferred compromise solution. Many business

H.R. Weistroffer (✉)
School of Business, Virginia Commonwealth University, Box 844000, Richmond, VA 23284-4000, USA
e-mail: hrweistr@vcu.edu

Y. Li (✉)
Center for Information Systems and Technology, Claremont Graduate University, 130 East 9th Street, Claremont, CA 91711, USA
e-mail: yan.li@cgu.edu

S. Greco et al. (eds.), *Multiple Criteria Decision Analysis*, International Series in Operations Research & Management Science 233,
DOI 10.1007/978-1-4939-3094-4_29

users, however, find it difficult to identify and choose an appropriate software package for their specific problem situation. The primary objective of this chapter is to provide an overview of commercially or otherwise readily available MCDA software and to offer users a practical guide on selecting the appropriate tools for their decision problems at hand.

In the following section, we summarize and categorize available MCDA software based on various decision-problem considerations. Such considerations include the type and characteristics of the decision problem to be resolved, the decision context, and the technology platform required by the software. In Sect. 29.3 of this chapter we then present more detailed reviews of each software package in alphabetical order. Finally, in Sect. 29.4, we offer some concluding observations.

29.2 General Overview of Available MCDA Software

Decision analysis software can assist DMs at various stages of the decision-making process, including problem exploration and formulation, identification of decision alternatives and solution constraints, structuring of preferences, and tradeoff judgments. Many commercially available, general decision analysis software packages have been included in biennial decision analysis software surveys in *OR/MS Today*, the first one published in 1993 [9]. The 2012 survey [53] included 47 decision analysis packages, some of which can be considered MCDA software and are also covered in our chapter. While specifically focusing on MCDA software, our review includes not only commercially marketed packages, but also software that has been developed at academic institutions for research purposes and is made available to the broader community, usually free of charge or for a nominal fee. Commercial packages may sell for hundreds or even thousands of dollars (though some vendors give educational discounts) and usually have dedicated websites and sophisticated marketing literature and may come with training courses and technical support. Software developed not-for-profit by academics usually comes without support and may have only limited documentation.

In order to provide some practical support for choosing the most appropriate software for a specific decision situation, we present a summary of MCDA software covered in this chapter, structured around the following considerations: the characteristics of the decision problem (viz. finite set of alternatives versus infinite options that can be defined by mathematical functions), the MCDA method(s) implemented by the software, the type of decision problem (viz. single DM versus group decision making), and the technology platform(s) supported by the software.

29.2.1 MADA Versus MOO Software

The first selection consideration is based on the characteristics of the decision problem formulation. MCDA normally involves the DM to choose a solution from the set of available alternatives, which can be finite or infinite [23]. Thus, MCDA problems can be roughly divided into two main types, viz. multiple attribute decision analysis (MADA) problems and multiple objective optimization (MOO) problems. In MADA problems, the DM must choose from among a finite number of *explicitly* identified alternatives, characterized by multiple attributes, where these attributes define the decision criteria. An example would be buying a new car and choosing among the various models available, characterized by attributes such as size, engine power, price, fuel consumption, etc. In contrast, MOO deals with problems where the alternatives are only *implicitly* known. In MOO problems, the decision criteria are expressed in the form of mathematical objective functions that are to be optimized. The argument vectors of the objective functions constitute the decision variables and can take on an infinite number of values within certain constraints. An example would be developing a new engine for an automobile manufacturer, where the decision criteria may include things like maximum power, fuel consumption, cost, etc., described by functions of the decision variables such as displacement capacity, compression rate, material used, etc. MOO models may involve linear or nonlinear objective functions and constraints, and may have continuous or integer decision variables. MOO software typically implements various optimization algorithms, such as linear programming, nonlinear programming, generic algorithms, meta-heuristics, etc. Table 29.1 categorizes the reviewed software packages according to these two types of problems.

29.2.2 MCDA Methods Implemented

The second selection consideration is the MCDA method implemented by the software. Corresponding to the two types of MCDA problem formulations, methods can be categorized into multiple criteria design methods and multiple criteria evaluation methods [13].

Multiple criteria design methods are intended to solve MOO problems, sometimes also referred to as multiple criteria design problems or continuous multiple criteria problems. A very large number of optimization methods of this type have been proposed, where each individual method is designed to solve a specific or a more generic type of MOO problem. Different MOO software generally implements different MOO methods.

Multiple criteria evaluation methods are intended to solve MADA problems, sometimes also called multi-criteria evaluation or selection problems. Brief descriptions of multiple criteria evaluation methods implemented by the software surveyed in this chapter are given in Table 29.2. More detailed descriptions of many of these methods can be found in earlier chapters in this book.

Table 29.3 shows which software packages implement methods from Table 29.2. Not all software packages explicitly state the method(s) employed, and often this information needs to be derived from their technical description. Some software packages implement multiple methods and are listed multiple times in Table 29.3.

Table 29.1 MADA and MOO software

Software for Multiple Attribute Decision Analysis (MADA)	1000Minds	4eMka2/jMAF	Accord
	CDP	DecideIT	Decision Explorer®
	Decision Desktop/Diviz	Decision Lab/Visual PROMETHEE	DPL 8
	D-Sight	ELCTRE III-IV	ELECTRE TRI
	ELECTRE IS	Equity	ESY
	Expert Choice	FuzzME	GeNIe-Smile
	HIPRE 3+	HIVIEW	IDS
	INPRE and ComPAIRS	IRIS (VIP)	JAMM/jRank
	Logical Decision	MakeIt Rational	Market Expert Markex
	MindDecider	MINORA	M-MACBETH/WISED
	MOIRA, MORIA Plus	NAIADE	OnBalance
	Prime Decisions	Priority Mapper	Prism decision System
	RICH Decision	Rubis	SANNA
	MC-SDSS for ArcGIS	TransparentChoice	Triptych
	UTA Plus	Very Good Choice	VIP Analysis
	VISA	Visual UTA	WINGDSS
	WINPRE		
Software for Multiple Objective Optimization (MOO)	ACADEA	Analytic Optimizer	APOGEE
	BENSOLVE	FGM	GUIMOO
	iMOLPe	IND-NIMBUS	interalg
	iSight	modeFrontier	Optimus
	ParadisEO-MOEO	Pareto Front Viewer	RGDB
	SOLVEX	TRIMAP	Visual Market

Table 29.2 Multiple criteria evaluation methods

AHP	AHP (*Analytic Hierarchy Process*) [61] provides a systematic procedure to model MADA problems as multilevel hierarchies, with the overall decision objective at the top of the hierarchy, the decision alternatives (options) at the bottom of the hierarchy, and the decision criteria (attributes) in the middle levels, possible with sub-criteria. AHP derives ratio scales from pair-wise comparisons of the elements at each level (with respect to elements at a higher level), which are then combined into overall preference weights for the alternatives. Preference consistency values are calculated.
Bayesian Analysis	Bayesian analysis is a statistical decision making approach for utilizing information in the form of possibilities [51]. Based on Bayesian decision theory, it prescribes the optimum alternative as one that maximizes the subjective expected utility. Bayesian analysis is useful in an environment of uncertainty and risk.
Cognitive Mapping	Cognitive mapping is based on *personal construct theory* [36] and has been used to explore and represent the DM's understanding of the relationships among interacting concepts [12]. Cognitive mapping can facilitate the DM in exploring and formulating MADA decision problems, as well as gathering and structuring qualitative data.
DELTA	The DELTA method [16] is based on classical statistical theory to evaluate decision problems using probability and utility intervals for alternatives and consequences when the decision information is imprecise.
DRSA	DRSA (*Dominance-based Rough Sets Approach*) [29, 30] extends original rough set theory to model and exploit the DM's preferences in the form of decision rules.
ELECTRE Family	The ELECTRE (*ELimination Et Choix Traduisant la REalité—Elimination and Choice Expressing Reality*) family of methods is based on the principles of outranking, a concept originated by Roy [57]. The outranking methods are based on multi-attribute utility theory principles, motivated by decision efficiency for pair-wise comparisons of all options. ELECTRE family methods have been applied to choosing, ranking and sorting problems, as well as fuzzy and non-fuzzy outranking relations.
Evidence Reasoning	The Evidence Reasoning approach adopts the principles of *utility theory* and *Dempster-Shafer theory of evidence* [18, 69]. It uses a belief decision matrix [77] to systematically model MADA decision problems under different types of uncertainties, such as objectivity, randomness, and incompleteness.
MACBETH	MACBETH (*Measuring Attractiveness by a Categorical Based Evaluation Technique*) is an interactive approach that requires only qualitative judgments about differences in values to help DMs to quantify the relative attractiveness of options [2]. It derives value scores and criteria weights using additive aggregation modeling.

(continued)

Table 29.2 (continued)

MAUT and MAVT	MAUT (*Multi-attribute Utility Theory*), developed by Keeney and Raiffa [35], uses utility functions to convert raw performance values of alternatives with respect to the decision criteria to a common scale. MAVT (*Multi-attribute value theory*) is a modified version of MAUT [24]. It differs from MAUT in that the aggregated utilities of alternatives are ranked and also recognizes the importance of criteria weights in decision making [13]. MAUT and MAVT constitute a family of methods that focus on the structure of multiple criteria, especially when risks or uncertainties play a significant role in defining and assessing alternatives [23].
PAPRIKA	PAPRIKA (*Potentially All Pairwise RanKings of all possible Alternatives*) [32] involves the DM performing pair-wise value ranking of *undominated* pairs of alternatives. The number of such rankings needed are kept to a minimum by identifying and eliminating implicitly ranked undominated pairs.
PROMETHEE	PROMETHEE (*Preference Ranking Organization METHod for Enrichment Evaluation*) is a popular family of outranking methods, developed by Brans [6]. The PROMETHEE methods provide flexible preference modeling capabilities through powerful interactive sensitivity analysis tools. PROMETHEE family methods have evolved to include partial ranking, complete ranking, ranking based on interval and continuous cases, etc. A visual interactive version (GAIA) was also introduced [5] to enable the DMs to better understand available choices and the impact of decision weights on the rankings.
SMART	SMART (*Simple Multi-attribute Rating Technique*) [73] is a simple form of a multi-attribute utility method. The alternatives are rated directly in the natural scales and the different scales of criteria are converted to a common scale mathematically using a simple value function. The weighted algebraic mean of the utility values associated with the alternative become its ranking value.
UTA	The UTA (*UTilités Additives*) method proposed by Jacquet-Lagrèze and Siskos [34] adopts the principles of MAUT but using a preference disaggregation rather than aggregation approach.

Table 29.3 Software by method implemented

Method	Software packages		
AHP	CDP Expert Choice HIPRE 3+ INPRE and ComPAIRS JAMM/jRank	Logical Decision MakeItRational MindDecider OnBalance Prime Decision	Priority Map TransparentChoice Triptych WINPRE
Bayesian Analysis	Accord		
Cognitive Mapping	Decision Explorer®	MindDecider	
DELTA	DecideIT		
DRSA	4eMka2/jMAF		
ELECTRE Family	ELECCALCELECTREIS	ELECTRE TRI IRIS (VIP)	SANNA Very Good Choice
Evidence Reasoning	IDS		
MACBETH	HIVIEW	M-MACBETH/WISED	
MAUT or MAVT	DPL 8 GeNie&Smile Prime Decision System NAIADE	Equity HiPriority RICH Decision VISA	ESYMOIRA, MORIA Plus MC-DSS for ArcGISWINGDSS
PAPRIKA	1000Minds		
PROMETHEE	Decision Lab/Visual PROMETHEE	D-Sight	SANNA
SMART	CDP HIPRE 3+ JAMM/jRANK	Logical decision Prime Decision	Priority Map WINPRE
UTA	Decision Desktop/Diviz Markex	VIP Analysis MINORA	UTA Plus

Table 29.4 Software with group decision support capabilities

Package	Specific GDSS	General and group	Specific version/add-in module
1000Minds		√	
Accord		√	
Decision Explorer®		√	
Decision Lab/Visual PROMETHEE		√	
D-Sight		√	Requires multi-actor plug-in
Equity3		√	Requires Catalyze Decision Conferencing Services
Expert Choice		√	Web-based Version
HIPRE 3+		√	Group-Link Version
HIVIEW3		√	Requires Catalyze Decision Conferencing Services
Logical Decision		√	LDW for Group Version
MindDecider		√	Group Version
OnBalance		√	
Prism GDSS	√		
TransparentChoice		√	
WINGDSS	√		

29.2.3 Group Decision Support

Group decision-making is a central concern in organizational settings since many important decisions are taken collectively by groups of people. The complexity of MCDA is greatly increased in the group setting. MCDA group decision support involves not only problem definition, criteria identification and prioritization, and individual preference elicitation, but also requires aggregating different individual preferences on a given set of alternatives into group judgments [38]. Table 29.4 lists software packages that provide group decision support capabilities. Some of the software packages are specific group decision support systems (GDSS), while others support both individual and group decision-making. Also, some of the packages provide group decision support only in specific versions or add-on modules.

29.2.4 Platform Supported

The computing environment supported by a software package is an important software selection criterion. If the desired software does not run on the user's currently available platform, extra updating costs may have to be taken into consideration. Also, some users may prefer a web-based application rather than a standalone package, while others may not want to host the data on a server and prefer a desktop version. One of the surveyed packages offers a software-as-a-service (SaaS) version. Some mobile-based MCDA applications are available, though they are not included in this survey, as currently these applications seem to be primarily intended for making personal decisions only. In the future, more mobile applications may be developed. Table 29.5 presents a summary of platforms supported by the surveyed software packages. Most MCDA packages were developed for Microsoft Windows based personal computers. Several software packages, mostly MOO software, are Microsoft Excel add-ons or Matlab solvers. There are some software packages exclusively implemented as web applications, and some with a web application version. There is also software implemented as plug-ins, or subroutine libraries. Two of the reviewed software packages are in fact subsystems of other packages. One software package requires a desktop client and a MySQL server. There is also an open source software package available.

29.3 Software Review

29.3.1 1000Minds

http://www.1000minds.com. 1000Minds implements the PAPRIKA (*Potentially All Pairwise RanKings of all possible Alternatives*) method [32], which involves the DM performing pair-wise value rankings of *undominated* pairs of alternatives. PAPRIKA keeps the number of such rankings needed to a minimum by identifying and eliminating implicitly ranked undominated pairs. 1000Minds prompts users, depending on what they want to do, to follow a simple six-step MCDM process: criteria selection (qualitative or quantitative), alternatives input (optional), pairwise ranking, preference values (derived by 1000Minds), ranked alternatives, and alternatives selection (including value-for-money analysis). Customized group decision-making processes involving potentially large numbers of participants can be created based on six decision activities provided by 1000Minds: decision surveys, online voting, alternatives entry, ranking surveys, categorization surveys, and ranking comparisons. The software supports an unlimited number of alternatives and a maximum of 30 decision criteria. 1000Minds is Internet based, with its servers housed in the USA and New Zealand. A 21-day trial use is available through the website and the software is available for free for unfunded research and study. A 1000Minds software development kit is also available as either a .Net class library or via web services.

Table 29.5 Software platforms

Platform	Software packages	
Windows	4eMka2/jMAF	JAMM/jRank
	Accord	Logical Decision
	Analytica Optimizer	MakeItRational
	CDP	Market Expert (Markex)
	DecideIT	M-MACBETH
	Decision Explorer®	MindDecider
	Decision Lab/Visual PROMETHEE	MINORA
	DPL 8	ModeFrontier
	D-Sight	MOIRA, and MORIA
	ELCTRE III-IV	Optimus
	ELECTRE TRI	OnBalance
	ELECTREIS	Pareto Front Viewer
	Equity	Plus
	Expert Choice	NAIADE
	FuzzME	Prime Decisions
	GUIMOO	Priority Mapper
	HIPRE 3+	TRIMAP
	HiPriority	UTA Plus
	HIVIEW	VIP Analysis
	IDS	VISA
	iMOLPe	Visual Market
	IND-NIMBUS	Visual UTA
	INPRE and ComPAIRS	WINGDSS
	interalg	WINPRE

Apple Mac	MakeItRational	interalg
Web-based	1000Minds D-Sight Expert Choice FGM GeNIe-Smile HIPRE 3+	IND-NIMBUS MakeItRational Prism Decision System RICH Decision RGDB TransparentChoice
Excel Add-on	SANNA 2009 Triptych Very Good Choice	ACADEA APOGEE
Subsystems	ESY	i-Sight
Plug-ins for d2	IRIS	Rubis
SaaS	Accord	WISED
Unix/Linux	ModeFrontier	interalg
Client–Server	Decision Desktop/Diviz	WISED
Fortran Library	SOLVEX	
Matlab Solver	BENSOLVE	
Open Source	ParadisEO-MOEO	
ArcGIS.net Extension	MC-SDSS for ArcGIS	

29.3.2 4eMka2/jMAF

http://idss.cs.put.poznan.pl/site/70.html. 4eMka2 is an implementation of the *dominance-based rough sets approach* (DRSA) [29, 30]. DRSA extends original rough set theory in the MCDM domain to model and exploit DMs preferences in terms of decision rules, with specific considerations of the characteristics of different types of multiple criteria problems. 4eMka2 system is specifically designed for solving multiple criteria sorting problems, by combining rough set theory with dominance relation to describe rough approximation of decision classes. Decision rules are extracted from a set of already classified examples (prepared by the user), and decision rules are represented in natural language as a set of "if … then … " statements. The system includes features like data validation, qualitative estimation of the ability of criteria and attributes to approximate the classification of objects, finding the core of criteria and attribute, inducing decision rules using the DOMLEM (minimal cover set of rules) and ALLRULES algorithms, and applying decision rules to reclassify objects with known decisions and to classify new objects. There is no a priori constraint imposed on the size of the decision problems. Rather, the size is said to depend on available memory and affordable computation time. 4eMka2 is Win32-based and free for download. 4eMKa2 is now outdated and has been replaced by jMAF, a Java application based on Eclipse Rich Client Platform UI.

29.3.3 ACADEA

ACADEA is a multi-objective optimization system for performance review of individual faculty in a university [1]. The system considers the aggregate performance of an academic department using the result of individual faculty member evaluations. Objectives are operationalized into criteria in the areas of research output, teaching output, external service, internal service and cost. Data envelopment analysis (DEA) approach is incorporated in the optimization model for efficiency measurement. Implemented as a spreadsheet add-on, the system can be used as an academic policy aid.

29.3.4 Accord

http://www.robustdecisions.com. Accord software is a decision support tool that helps individuals and groups make better decisions with uncertainty. The software integrates three main technologies: Taguchi's method of robust design, product design process, and Bayesian team support (BTS), among which BTS is a patented approach to decision support and the foundation for Accord. BTS is based on

Bayesian decision theory [51, 67]. Given a decision problem, the theory prescribes an optimal decision choice to select the alternative that maximizes the subjective expected utility. BTS extends Bayesian decision theory to integrate the "subjective expectances" from multiple DMs in a group decision-making situation. BTS also incorporates Bayesian methods with expert-based methods to support the decision-making process. BTS includes the following Bayesian analysis methods: subjective expected utility, marginal value of information, and probability of being best (combining preferences from multiple DMs' evaluations). The interface of the software includes four main features: (1) a belief map to provide belief modeling, (2) alternative comparison, (3) criteria used to compare alternatives, and (4) collaboration management of team members. Accord is offered in standalone, enterprise and SaaS versions. Thirty-days free trial is available.

29.3.5 Analytica Optimizer

http://www.lumina.com. Analytica is a family of decision support software that helps people visually create, analyze, and communicate decision models. Its underlying technologies are influence diagrams (visual representation of all essential elements of a decision problem in the form of decisions, uncertainties, and objectives) and Monte Carlo simulation (to evaluate risk and uncertainty). Analytica Optimizer, the highest edition level of Analytica, provides MOO support through its sublicensed solver engines from Frontline Systems.[1] It automatically distinguishes linear programming, quadratic programming or general non-linear programming optimization and seamlessly integrates optimization with all of other Analytica's core features. The optimization engines in Analytica Optimizer have various limits on the number of variables and constraints. For continuous linear programming and quadratic programming problems, there is a limit of 8000 variables and 8000 constraints. For integer or mixed-integer linear or quadratic programming, there is a limit of 2000 variables and 2000 constraints. For general non-linear problems, there is a limit of 500 variables and constraints. Add-on engines can be purchased to eliminate aforementioned limits on problem sizes. Anlytica optimizer is Windows based and is available for a free 30-day trial.

29.3.6 APOGEE

http://www.stat-design.com/Software/Apogee.html. Apogee is the statistical analysis, allocation and optimization engine for Triptych (see Sect. 29.3.59). Apogee works with mathematical functions $Y = f(X)$ created in Excel workbooks, where

[1] http://www.solver.com/about.htm.

X is a statistical variable (as a parameter) and Y is a mathematical function of the parameters (as a response). Apogee then provides statistical capabilities, including sensitivity analysis, Monte Carol Analysis, Allocation, and MOO, for assessing and improving the variation of the responses. Genetic algorithm approach is implemented to provide multi-objective, nonlinear, and global optimization. Unique to this tool is that the optimization approach allows X parameter uncertainty information to be included in the formulation and allows the Y response to be optimized for the mean, standard deviation, and/or probability of non-compliance (PNC) of multiple responses. A 10-day free trial is available.

29.3.7 BENSOLVE

http://ito.mathematik.uni-halle.de/~loehne/index_en_dl.php. BENSOLVE [39] is a free multi-objective linear programming (MOLP) solver in MatLab. It implements Benson's algorithm to solve linear vector optimization problems. The latest version, BENSOLVE-1.1, is available for free download.

29.3.8 Criterium Decision Plus (CDP)

http://www.infoharvest.com/ihroot/index.asp. CDP is a Windows-based visual multiple criteria decision support tool by InfoHarvest Inc. It supports both SMART [25] and AHP [61] methodology. Uncertainty is supported through graphical representation of uniform, triangular, normal, lognormal, and custom distributions for input attributes. CDP models can also be used directly in the freely available Ecosystem management Decision Support (EMDS) system for spatial MCDA decision-making, though CDP still has to be purchased separately. Version CDP 3.0 can support up to 200 alternatives and 500 blocks in total. To accommodate a greater number of alternatives, the *Weighted Decision Object* 3.0 (WDObj) that encapsulates the capability of CDP in an ActiveX (COM) object can be incorporated into the applications. A free CDP 3.0 student version, with all features but restricted model size, is downloadable from the vendor's website. CDP is compatible with Windows 95 to Windows 7, but Windows Vista is not supported.

29.3.9 DecideIT

http://www.preference.nu/site_en/decideit.php. DecideIT is marketed by Preference AB and is designed to integrate various procedures for handling vague and imprecise information in a complex decision situation and probabilistic decision analysis. Originated from MAUT, the tool utilizes the DELTA method [16, 17] to evaluate decision problems using weight, probability and utility intervals, and qualitative comparisons for criteria, alternatives and consequences. It provides decision trees and influence diagrams (transformed into a corresponding decision tree after evaluation) with criteria hierarchies to model users' decision architecture. Imprecise probabilities and utility value statements are captured through GUI, and results are graphically presented in various ways, e.g., as pair-wise comparison of alternatives. It also provides a graphical overview of the preference ordering among consequences and critical elements of a decision problem. The vendor also claims that "DecideIT provide means for analyzing decisions involving multiple and conflicting objectives and several stakeholders with differing views on the objectives." DecideIT supports 15 alternatives at the root level, 512 consequences per alternative, 1023 nodes per alternative, and 99 decision criteria. The software runs on Windows XP or Windows 7, with Java runtime environment and minimal 512 MB RAM. A trial version of DecideIT is available.

29.3.10 Decision Explorer®

http://www.banxia.com/dexplore. Decision Explorer® by Banxia Software is a Windows-based tool for managing qualitative information that surrounds complex or uncertain situations. The basic technique employed is cognitive mapping, a technique founded on the theory of personal constructs [36]. Decision Explorer® can facilitate group discussion and understanding by means of its visual development of problem issues. In addition to a number of tools to draw cognitive maps, the software provides a large number of analytical tools that assist in evaluating the similarities and differences of sets and in developing and analyzing clusters of information about the problem. The standard licenses are limited to 8000 concepts in its model sizes. The website provides a tutorial, case study, demonstration downloads, and a bibliography of material related to the software or the cognitive mapping method.

29.3.11 Decision Desktop Software (d2)/Diviz

http://www.decision-deck.org/d2/index.html. Decision Desktop Software, or d2, is a rich open source Java software containing several MCDA methods. It was the first software developed by the Decision Deck project, an effort to collaboratively develop open source multiple criteria decision aid software. The d2 allows decentralized evaluations from several experts, whose evaluation results can then be analyzed by a coordinator. Several MCDA methods and utilities plug-ins are bundled within the platform, including IRIS (see Sect. 29.3.31), Rubis, VIP (see Sect. 29.3.55), UTADISGMS and GRIP, and Weighted Sum. The d2 requires a local desktop installation of a client (Java 5 JRE is required) and uses a database to store application data on the server side (version 4.1.x or higher MySQL server is required). Decision Desktop is currently in a frozen development state due to a lack of developers. Another software under development by the same group is Diviz, currently used by students and researchers from around 15 universities in Europe. Diviz considers MCDA methods as sequences of more elementary algorithms, which can be rebuilt in the software as algorithmic workflows. Currently, there are about 100 algorithmic components available, ranging from outranking methods to value-based methods. The list of components can be viewed at http://www.decision-deck.org/diviz/webServices.html. A java-based Diviz client (which runs on Windows, Linux, or Mac) is required on the user's end. Calculations are done on servers located in France and Luxembourg.

29.3.12 Decision Lab 2000/Visual PROMETHEE

Decision Lab 2000 is an interactive decision support system [28] based on the outranking methods PROMETHEE [7, 8] and GAIA [5]. Sensitivity analyses are generated by using techniques of walking weights, intervals of stability, and the graphical axis of decision, displayed by the GAIA method. The software is also suitable for group decision support, providing profiles of actions and multi-scenario comparisons. The methodology used here requires fewer comparisons from the decision maker than the AHP method; it permits the user to define his own measurement scale. The original download link from its original developer and distributor, Visual Decision Inc. is no longer active. However, a new version of the software, Visual PROMETHEE beta is available for download (http://www.promethee-gaia.net/softwareF.html). Visual PROMETHEE is a Windows (XP, Vista, 7) application. Visual PROMETHEE also includes a PROMap GIS feature that is integrated with Google Maps. Internet connection is required to use the GIS PROMap feature.

29.3.13 DPL 8

http://www.syncopation.com. DPL 8 is a family of software products for decision and risk analysis. Decision modeling is provided through influence diagrams and decision trees. A typical decision tree includes a decision node to model decision alternatives, a chance node to model decision options, and a value node to model decision goals. After running the model, the decision analysis result is presented in the form of a policy tree. In case of a continuous chance node, a Monte Carlo simulation feature can be used to analyze a continuous model. The DPL 8 family includes Direct, Professional, Enterprise, and Portfolio, versions. The entry-level Direct version is a pure Excel add-in, while the other versions offer both an add-in interface and a standalone application interface. While there is no limit to the number of alternatives within a decision model, there is a limit of 1024 attributes for the decision criteria. DPL uses a standard Windows (XP or later) environment. Minimum storage requirement is 25MB of hard disk space, or 70MB for a full installation with all documentations. A demo is available for DPL 8 Direct and Professional. Discounted academic licenses are available for the Direct and Professional versions.

29.3.14 D-Sight

http://www.d-sight.com. D-Sight is relatively new MCDA software based on the PROMETHEE GAIA and utility-based methods. The evaluation criteria are organized through criteria hierarchy trees. The DM's preferences can be modeled through either pair-wise comparisons (PROMETHEE) or utility functions. After the specification of evaluation criteria and preferences, the software ranks and scores the alternatives. D-Sight software solutions are now using scoring scales between 0 and 100. However, D-Sight Desktop offers a PROMETHEE plug-in that displays scores using the -1 $+1$ PROMETHEE scale. A projection of alternatives and criteria (the GAIA plane) allows evaluation of how the alternatives perform with respect to the different criteria as well as how the criteria act as differentiators for alternatives. D-Sight is available as a desktop version or as a Web application. For the desktop version, additional functions can be obtained through D-Sight's plug-ins, such as Maps (free), multi-users plug-in (for group decision making), weights elicitation, and sub-set optimization. The D-Sight Web is a collaborative decision-making platform managing online projects in which people have specific roles, such as project managers, experts, etc. For the desktop version, a Windows-based Java Runtime version 6 or later, and 30MB free disk space are required. Special rates for academics are available, as well as a free 14-day trial version. A permanent free version of D-Sight Web is offered, which is thus not limited in time, but limited to one user account and one project at any time.

29.3.15 ELECTRE III-IV

http://www.lamsade.dauphine.fr/spip.php?article241. ELECTRE III aggregates partial preferences into a fuzzy outranking relation [27, 58]. ELECTRE IV builds several non-fuzzy outranking relations when criteria cannot be weighted. Two complete preorderings are obtained through a "distillation" procedure, either from the fuzzy outranking relation of ELECTRE III, or from the non-fuzzy outranking relations provided by ELECTRE IV. The intersection of these preorderings indicates the most reliable global preferences. A demo version of ELECTRE III-IV is available for download. ELECTRE III-IV runs on Windows.

29.3.16 ELECTRE IS

http://www.lamsade.dauphine.fr/spip.php?article238. ELECTRE IS represents an evolution of the ELECTRE I method [59] and enables the use of pseudo-criteria (criteria with thresholds). Given a finite set of alternatives evaluated on a consistent family of criteria, ELECTRE IS supports the user in the decision process of selecting one alternative or a subset of alternatives. The method consists of two parts: construction of one crisp outranking for modeling the DM's preferences, and exploitation of the graph corresponding to this relation. The subset searched is the kernel of the graph.

29.3.17 ELECTRE TRI

http://www.lamsade.dauphine.fr/spip.php?article244. ELECTRE TRI is a multiple criteria decision-aiding tool designed to deal with sorting problems. This software implements the ELECTRE TRI method that provides two different procedures (pessimistic or optimistic) to assign a finite set of actions to a set of categories corresponding to predefined guidelines [48]. ELECTRE TRI Assistant reduces the cognitive effort required from the DM to elicit the preference parameters by enabling weights to be inferred through a form of regression.

29.3.18 Equity3

http://www.catalyze.co.uk/?id=229. Equity3 is a PC-based MCDA tool originally developed by Catalyze Ltd in association with LSE Enterprise (London School of Economics and Political Science). It aims at helping DMs obtain better value-for-money from their portfolio decisions. Decision models in Equity3 are mostly built to aid the allocation of monetary resources to an investment portfolio. Building on the same methodological framework as HIVIEW3 (see Sect. 29.3.27), Equity3 includes five main model building stages, which are *model construction*, *scoring*, *setting preferences*, *analyzing models*, and *recommendations*. However, the model construction stage in Equity3 is quite different from HIVIEW3: it groups the portfolio of options into logical towers in the model structure. Detailed portfolio analysis in Equity 3 includes efficiency frontiers, affordability and trade-off analysis. Equity3 supports qualitative criteria and group decision making in the same manner as HIVIEW3. A 20 days free trial version is available for download. Educational licensing is also available, but support needs to be purchased separately.

29.3.19 ESY

ESY (*evaluation subsystem*) [52] employs the multi-attribute value theory model to help decision makers make more rational decisions and promote consistency in their decision making throughout all phases of a nuclear emergency. ESY provides decision support not only in the evaluation, but also in the formulation and appraisal of the decision strategies. It is one of the three distinct subsystems in RODOS (*real-time online decision support system*) architecture (http://www.rodos.fzk.de). Several other systems that evaluate strategies in nuclear emergencies are also provided, ranging from rule-based systems to those using multi-attribute value and utility theory.

29.3.20 Expert Choice

http://www.expertchoice.com. Expert Choice (EC) software employs AHP as its core methodology. EC products include Expert Choice Desktop, the web-based Comparion™ Suite for group decision-making, and Expert Choice Inside for application integration. EC desktop versions have been used for decision analysis for more than 20 years. In addition to hierarchies of alternatives, the desktop version also offers a rating template library of best practice ratings scales, portfolio scenarios to visualize different scenarios on the efficient frontier, 3D plotting to see results in more meaningful ways, and support for Microsoft project integration and Oracle database interfaces. Comparion™ is a collaborative application for DMs supporting five decision processes: (1) defining goals, (2) structuring decisions, (3) assigning roles, (4) collaborating, and (5) choosing among options. A 10-day free trial version of Comparion™ is available.

29.3.21 FGM

http://www.ccas.ru/mmes/mmeda/soft/first.htm. FGM is MCDM software for visualizing the Pareto frontier in decision problems with multiple objectives. FGM employs the ***F****easible* ***G****oals* ***M****ethod* to explore all possible results of all feasible decisions [42] and the *Interactive Decision Maps* technique to display various decision maps. It supports both linear optimization algorithms (mostly based on approximation of multi-dimensional convex bodies by polytopes) and non-linear optimization algorithms (based on stochastic covering of bodies by systems of simple figures). FGM 3.1 supports a maximum of 100 decision variables, 5 decision criteria, and 300 non-zero elements in a decision model. FGM-based applications can be coded in C language for PCs in the Windows environment and workstations in the Unix environment. Demo software is available for download, as well as a

Java-based web-application of the FGM demo. The same research group responsible for FGM also provides *reasonable-goal-method-based* (RGM-based) MCDA software, discussed in Sects. 29.3.53 and 29.3.65.

29.3.22 FuzzME

http://fuzzme.wz.cz. FuzzMe (**Fuzz**y Models of **M**ultiple-Criteria **E**valuation) is a tool for creating fuzzy models of multiple-criteria evaluation and decision-making. It was developed at the Faculty of Science at Palacký University Olomouc. Both quantitative and qualitative criteria are supported. For the aggregation of partial evaluations, different methods can be used, such as fuzzy weighted average, fuzzy ordered weighted average, or fuzzy Choquet integral [72]. FuzzME runs on Windows but requires the .NET framework. A demo version is available for download.

29.3.23 GeNIe & SMILE

http://genie.sis.pitt.edu. GeNIe & Smile is a decision-theoretic modeling system developed by the *Decision Systems Laboratory* at the University of Pittsburgh. The system provides a general-purpose modeling environment, SMILE (*Structural Modeling, Inference, and Learning Engine*), which is a fully portable library of C++ classes that implements decision-theoretic methods [22]. SMILE.NET is available with .NET framework, which can be used to create web-based applications. In addition, GeNIe, a Windows-based graphic click-and-drop interface for SMILE, is available to develop decision-theoretic models. The GeNIe & Smile system includes MADM-related modeling languages, such as multiple decision nodes, multiple utility nodes, and multiple attribute utility nodes. GeNIe, SMILE, and its wrappers, are available free of charge for any use.

29.3.24 GUIMOO

http://guimoo.gforge.inria.fr. GUIMOO (***G**raphical **U**ser **I**nterface for **M**ulti **O**bjective **O**ptimization*) is free software for analyzing results in MOO problems. It provides visualization of approximative Pareto frontiers and metrics for quantitative and qualitative performance evaluation, including S-metric, R-metric, size of the dominated space, coverage of the two sets and converge differences, etc. The latest release, GUIMOO-0.4-3 is developed in C++ in a Win32 desktop-based environment.

29.3.25 HIPRE 3+

http://www.sal.tkk.fi/en/resources/downloadables/hipre3. HIPRE 3+ is a software family that includes HIPRE 3+ (for desktop use), HIPRE 3+ Group Link (for group decision support), and Web-HIPRE. HIPRE 3+ is decision support software integrating AHP (*Analytic Hierarchy Process*) and SMART (*Simple Multiattribute Rating Technique*), which can be run separately or be combined in one. HIPRE 3+ provides a visual and customizable graphical interface for structuring, prioritization, and analysis of complex decision problems. HIPRE 3+ demo is restricted to run models with a maximum of three levels with three elements at each level. The full version of HIPRE 3+ can support up to 50 elements with up to 20 levels. HIPRE 3+ Group Link is group decision support software that combines individual prioritizations (through AHP) into an interval AHP model called preferences programming model [64]. HIPRE 3+ Group Link allows group members to combine AHP models, after individual AHP prioritizations are captured with HIPRE 3+. Web-HIPRE is a web-version of the HIPRE 3+ (http://www.hipre.hut.fi). It is a java-applet and provides a global platform for individual and group decision support.

29.3.26 HiPriority

http://www.quartzstar.com. HiPriority is designed to find best portfolio solutions, i.e. best subsets of alternatives subject to resource constraints. Weights are assigned to criteria and alternatives, and the software allows specifying dependencies between alternatives, as well as specifying mutually exclusive alternatives. HiPriority provides modeling of the consequences of interactions between options, such as multiple buffers to see the effects of forcing options in or out of a solution portfolio. To visualize benefit/cost ratios, the package creates simple value trees of cost elements together with their corresponding benefits, where cost is defined as any scarce resource. Miniature graphical views of the models are used as navigational tools. HiPriority is desktop-based and currently free to download as charity-ware.

29.3.27 HIVIEW3

http://www.catalyze.co.uk/?id=230. HIVIEW3 is a PC-based multiple criteria decision modeling tool original developed by Catalyze Ltd in association with LSE Enterprise (London School of Economics and Political Science). Hiview3 facilitates the building of decision models through choosing between mutually exclusive options. A complex decision modeling process is broken down into five simple management stages. In stage 1, the outline of a model is constructed as a value tree structure and the options are defined; in stage 2, each of the action options is scored against the criteria set up in the outlined model; in stage 3, DMs set preferences on the relative importance of different aspects of the model; in stage 4, the model is analyzed; and lastly, recommendations are presented in stage 5. One unique feature of HIVIEW3 is its support for both quantitative and qualitative criteria, and weight assessments. The support for qualitative criteria is implemented through the inclusion of MACBETH methodology, and is designed to work equally in a workshop or back office environment. In addition, HIVIEW3 also supports group decision-making through Catalyze decision conferencing services. A 20 days free trial version is available for download. Educational licensing is also available, but support needs to be purchased separately.

29.3.28 IDS Multicriteria Assessor (IDS Version 2.1)

http://www.e-ids.co.uk. IDS Multicriteria Assessor supports multi-attribute decision analysis based on the *Evidence Reasoning* (ER) approach, a decision method for dealing with uncertainties in multi-attribute decision analysis (MADA) problems of both quantitative and qualitative natures [76]. Based on *utility theory* and *Dempster-Shafer theory of evidence* [18, 69], the ER approach uses a belief decision matrix (a generalized decision matrix with attributes assessed using a belief structure) [77] to systematically model MADA decision problems under different types of uncertainties, such as objectivity, randomness, and incompleteness. A free demo version that supports ten attributes is available for download, as well as various price options for academic, professional and enterprise versions.

29.3.29 IND-NIMBUS

http://ind-nimbus.it.jyu.fi. IND-NIMBUS is an interactive multi-objective optimization system for solving continuous, nonlinear problems with conflicting objectives subject to equality and inequality constraints. It employs the NIMBUS [45] (***N****ondifferentiable* ***I****nteractive* ***M****ultiobjective* ***Bu****ndle-based Optimization* ***S****ystem*) method based on a classification of the objective functions. In NIMBUS, the user is asked to express preferences by classifying the objective functions at the current Pareto optimal solution into up to five classes according to how the current solution should be improved. The classes are *functions to be improved*, *to be improved till some aspiration level*, *satisfactory at the moment*, *allowed to impair till some bound*, and *allowed to change freely*. New Pareto optimal solutions are then generated by solving single-objective sub-problems created based on the preference information. Connections for using some commercial solvers have also been developed. While there is no theoretic restriction on problem size, IND-NIMBUS in practice can handle problems with less than ten objectives. IND-NIMBUS is desktop-based and can be connected with different simulator or modeling tools, such as Matlab. IND-NIMBUS can be used on the Windows and Linux platforms. It is commercial but free for academic testing purposes. Based on the same NIMBUS method, WWW-NIMBUS (http://nimbus.it.jyu.fi) is a free web-based version for academic teaching and research purposes.

29.3.30 INPRE and ComPAIRS

http://www.sal.tkk.fi/en/resources/downloadables/inpre. These two decision support tools are early implementations of techniques based on the imprecise preference statements in hierarchical weighting [63]. INPRE analyzes interval preference statements in the Analytic Hierarchy Process (AHP), while ComPAIRS works with similar statements in value tree analysis. The underlying methodology is similar to the one described in HIPRE 3+ (Sect. 29.3.25).

29.3.31 IRIS

http://www.uc.pt/feuc/ldias/software/iris. IRIS (*Interactive Robustness analysis and parameters' Inference for multicriteria Sorting problems*) is a DSS for sorting a set of actions (alternatives, projects, candidates) into predefined ordered categories, according to their evaluations (performances) on multiple criteria [21]. Application examples would be sorting funding requests according to merit categories, such as *very good*, *good*, *fair*, or *not eligible*, or sorting loan applicants into categories such as *accept*, *require more collateral*, or *reject*. IRIS uses a pessimistic concordance-only variant of the ELECTRE TRI method [19]. Rather than demanding precise values for the ELECTRE TRI parameters, IRIS allows one to enter constraints on these values. It adds a module to identify the source of inconsistency among the constraints when it is not possible to satisfy all of them at the same time, according to a method described by Mousseau et al. [47]. On the other hand, if the constraints are compatible with multiple assignments for the actions, IRIS allows drawing robust conclusions by indicating the range of assignments (for each action) that do not contradict any constraint. The software supports up to thousands of alternatives and up to 12 decision criteria. IRIS is windows-based and a demo version with limited problem sizes is available for download. IRIS is no longer actively supported, and an open source free alternative to IRIS is available as a plug-in for Decision Desktop (d2) software (see Sect. 29.3.11).

29.3.32 iMOLPe

http://www.uc.pt/en/org/inescc/products/molp_setup_limited. iMOLPe (Interactive Multi-Objective Linear Programming explorer) is an interactive software package to deal with linear programming problems with multiple objective functions, which includes scalarizing processes for computing efficient solutions based on weighted-sums, reference points and constraints on objective function values; distinct solution search strategies and visualization of results obtained with the TRIMAP method; and STEM, ICW and Pareto Race interactive methods. The downloadable version is limited to 6 objective functions, 100 decision variables and 100 functional constraints.

29.3.33 interalg

http://openopt.org/interalg. interalg (**inter**val **alg**orithm) is a free solver for multi-objective optimization with specifiable accuracy, possibly with categorical variables and general logical constraints. It uses an interval analysis based method and runs on Windows, Linux, or Mac. The software was initially released in March 2011, written in Python and NumPy. interalg includes a wide range of MOO functionalities, including searching for minima or maxima of non-linear problems, searching for global extrema of nonlinear problems with some discrete variables, searching full cover of Pareto front, and solution of non-linear equations. The software can handle some problems with hundreds of variables, though for some problems it may take too long to get a solution with the required accuracy.

29.3.34 iSight

http://www.3ds.com/products/simulia/portfolio/isight-simulia-execution-engine/isight-see-portfolio. Originally developed by Engineous Software, iSight is software for process integration and design optimization. It provides users with a suite of tools for creating simulation process flows to automatically exploit design alternatives and identify optimal performance parameters, taking advantage of its state-of-art multi-objective genetic algorithm approaches. In 2007, Engineous Software was acquired and iSight became a part of the Dassault Systèmes' SIMULIA brand product suite.

29.3.35 JAMM

http://idss.cs.put.poznan.pl/site/jamm.html. JAMM is designed to solve multi-criteria classification problems. Like 4eMka2 described in Sect. 29.3.2, JAMM is a family of software developed by the *Laboratory of Intelligent Decision Support Systems* (IDSS) at Poznań University of Technology to solve MCDM problems based on rough sets approach. The MCDM classification problem concerns the assignment of objects (alternatives) evaluated by a set of criteria to one of pre-defined and non-ordered decision classes, which is different from the sort problem in 4eMka2 where the decision classes are preference-ordered. The features in JAMM include: computation of rough approximations, induction of decision rules using DomLem and DomApriori (a complete set of rules), reduction of data table, classification of new examples, and data validation. It is Windows-based and available for free download. Based on communications with the software developers, JAMM is being replaced by jRank, a Java command-line application.

29.3.36 Logical Decisions

http://www.logicaldecisions.com. Logical Decisions for Windows (LDW) is decision support software for structuring and analyzing MADM problems. Based on MAUT, LDW offers five methods for assessing weights in value judgments, ranging from the *smarter* method, through *tradeoff* method, to AHP. The user interface is considered a significant attraction, with a graphical, point and click way to adjust weights. The results can be displayed in various ways, and one can compare pairs of alternatives to see their major differences. Interactive graphical sensitivity analysis displays are available. Logical Decisions offer a windows-based single user version (LDW for Windows), a group version (LDW for groups), and a portfolio version (LDW Portfolio). A 30 days free trial version of LDW is available and a free student version is also available with the book *Value-Added Decision Making for Managers* [11].

29.3.37 MakeItRational

http://makeitrational.com. MakeItRational organizes the process of multi-criteria evaluation by breaking it up into multiple judgments. MakeItRational is based on AHP and supports pair-wise comparisons of criteria. Evaluation results are represented in four types of charts: alternatives ranking, alternatives comparison, criteria weights, and sensitivity analysis. Desktop versions of MakeItRational are offered for Windows and Mac, as is an on-line version. A free demo version, which doesn't allow saving data, is available.

29.3.38 Markex (Market Expert)

http://www.ergasya.tuc.gr/software.html. Markex [44] is a multi-criteria decision support system for analyzing consumer behavior and market shares. The system uses consumer-based methodology [70] to support various stages in the product development process. The database of consumer survey results is analyzed to build different models for forecasting, data analysis, multi-criteria analysis, and branch choice. Specifically, Markex applies UTASTAR, an improved algorithm based on original UTA method, to model the multi-criteria consumer preferences. In addition, Markex employs three partial expert systems to support financial evaluation of the involved enterprises, selection of brand choice models, and selection of data analysis models. The software system is Windows-based, though the speed of the computer is critical in the solution of linear programs, calculation of utilities in the UTASTAR model, and representation of different models.

29.3.39 MindDecider

http://www.minddecider.com. MindDecider uses the concepts of mind mapping, MCDA, and AHP. A simple graphic interface allows fast click menu options to access decision constructs and then drag-and-drop onto a project canvas. User preferences can be modeled through utility functions and pair-wise comparisons. Uncertainty can be incorporated using fuzzy calculations feature. MindDecider is Windows-based and offers a personal version and a team version. Currently, the commercial version of MindDecider works only on the Microsoft.NET 2.0 framework. Mono versions for MaxOS and Android exist as beta versions. Users need 512MB free RAM space and up to 64MB free hard disk space to run MindDecider. Demo versions are available.

29.3.40 MINORA

http://www.ergasya.tuc.gr/software.html. MINORA (Multicriteria Interactive Ordinal Regression) [71] is an interactive decision support system based on the UTA method [34]. The interaction takes the form of an analysis of inconsistencies between the decision maker's rankings and those derived from utility measures. The method stops when an acceptable compromise is determined. The result is an additive utility function, which is used to rank the set of alternatives.

29.3.41 M-MACBETH and WISED

http://www.m-macbeth.com/en/m-home.html. M-MACBETH software deploys the MACBETH (*Measuring Attractiveness by a Categorical Based Evaluation Technique*) method, which is an interactive approach that requires only qualitative judgments about differences of values to help DMs quantify the relative attractiveness of options [2]. The user's qualitative preference judgment is captured through an interactive questioning procedure that compares two elements at a time. Judgmental disagreement or hesitation is also allowed. Using mathematical programming, the consistency of judgment is automatically verified and a numerical scale is generated based on seven semantic categories: *no*, *very weak*, *weak*, *moderate*, *strong*, *very strong*, and *extreme* difference of attractiveness. Weighting scales for decision criteria are generated in a similar manner, and an overall score for each option is calculated by weighted sum. The software provides some powerful tools like sensitivity analysis, structuring criteria in a value tree, robustness analyses of the final ranking, and profile comparison. M-MACBETH is desktop-based, with a minimum of 800×600px screen resolution running on a PC with Windows 2000 or earlier. A free demo version with a feature restriction of five criteria/options is available for download. Licensing options range from academic, to professional, and corporate versions with different pricing. An online tool called WISED is available as a new implementation of the MACBETH methodology with added online collaboration (both for evaluators and for the suppliers/representatives of the options under evaluation). It has a user-friendly layout, which makes it easier to undertake the tasks of scoring and weighting. WISED is available online as software as a service (SaaS) or installed on a companies' server.

29.3.42 modeFrontier

http://www.esteco.com/home/mode_frontier/mode_frontier.html. The name modeFrontier is in reference to the *Pareto frontier*, providing a boundary for "best" solutions. modeFrontier is multi-objective optimization software that allows easy coupling to any computer aided engineering (CAE) tool. The algorithms used in modeFrontier include linear and non-linear multi-objective optimization, Hurwicz algorithm [33], and Savage method [66]. The software also includes a MORDO (*Multiobjective Robust Design Optimization*) module [60] to support robust design analysis to check system sensitivity to any variation of the input parameters. MADA methods, including Hurwica, Savage, and soon with AHP, are also supported. According to the developers, the software supports hundreds of design alternatives and dozens of decision criteria. modeFrontier supports both Windows and Linux environments.

29.3.43 MOIRA and MOIRA Plus

MOIRA (***MO****del-Based Computerized System for Management Support to* ***I****dentify Optimal Remedial Strategies for* ***R****estoring Radionuclide Contaminated* ***A****quatic Ecosystems and Drainage Areas*) is a project financed by the European Commission. Both MOIRA DSS [56] and MOIRA-PLUS [46] are designed to help DMs to select countermeasure strategies for different kinds of aquatic ecosystems and contamination scenarios. Both systems include an evaluation module based on an additive multi-attribute value model to assess different alternatives. The utility assessment methods, probability equivalent method (PE) and certainty equivalent method (CE) [26], are implemented jointly to assess component value functions. The evaluation module also provides multi-parametric sensitivity analyses with respect to both weights and value. MOIRA-PLUS includes some functionality improvements based on the testing and assessment of MOIRA in various project. The improvements include prediction for the migration of heavy metals and improved software interfaces. Both versions are windows-based.

29.3.44 NAIADE

http://www.aiaccproject.org/meetings/Trieste_02/trieste_cd/Software/Software.htm. NAIADE (***N****ovel* ***A****pproach to* ***I****mprecise* ***A****ssessment and* ***D****ecision* ***E****nvironments*) is a discrete multi-criteria method [49] which provides an impact or evaluation matrix that may include either crisp, stochastic, or fuzzy measurements of the performance of an alternative with respect to an evaluation criterion. A peculiarity of NAIADE is the use of conflict analysis procedures integrated with the multi-criteria results. NAIADE can give rankings of the alternatives with respect to the evaluation criteria (leading to a technical compromise solution), indications of the distance of the positions of the various interest groups (possibly leading to convergence of interests or to coalition formation), and rankings of the alternatives with respect to the actors' impacts or preferences (leading to a social compromise solution). NAIADE runs on Windows-based systems.

29.3.45 OnBalance

http://www.quartzstar.com. OnBalance is based on a simple weighting approach: each decision option is scored against each decision criterion, and each decision criterion is given a weight. It then computes an overall weight for each option. Multiple hierarchies, called trees, using different weights, can be created to allow for different perspectives. Thus the approach appears to be similar to AHP, but no information is given as to how the overall weights are calculated. The package is designed to be easy to use by anyone, without much technical understanding. The interface of OnBalance is specifically designed for group decision-making and weight sets feature can be created to capture multiple stakeholders' different opinions. The current version OnBalance3 is free to download as charity-ware. OnBalance is desktop-based.

29.3.46 Optimus

http://www.sigmetrix.com/optimus.htm. Optimus is process integration and design optimization software, bundling a collection of design exploration and optimization methods. A single main window graphical user interface provides all the functionality. The numerical simulation methods of Optimus are based on gradient-based local algorithms or genetic global algorithms, both for single or multiple objectives with continuous and/or discrete design variables. Optimus includes mechanical variation effects in multi-objective performance optimization, multi-physics simulation and optimization, design robustness optimization, and manufacturing cost optimization. Optimus is desktop-based and a demo is available by request.

29.3.47 ParadisEO-MOEO

http://paradiseo.gforge.inria.fr. ParadisEO is a software framework for metaheuristics, and the MOEO (*metaheuristics for multiobjective optimization*) module implements evolutionary multi-objective optimization techniques [10, 37]. It is white-box, object-oriented, C++, portable across both Unix-like and Windows systems. ParadisEO is based on Evolving Objects (EO), a template-based ANSI-C++ compliant evolutionary computation library. There is conceptually no restriction on problem size, however, classical Pareto-based metaheuristics usually solve problems with up to five objectives. As an open source framework, ParadisEO is compatible with Windows, Unix-like, and MacOS environments. It also supports parallel and distributed architectures. The related source code is maintained and regularly updated by the developers.

29.3.48 Pareto Front Viewer

http://www.ccas.ru/mmes/mmeda/soft/third.htm. Pareto Front Viewer (PFV) [40] is software for interactive Pareto frontier visualization for nonlinear models in the case of two to eight criteria. PFV can be combined with any Pareto frontier approximation technique. PFV is windows-based and a demo version (PFV 1.2), as well as the Manual, is downloadable. The demo version is restricted to 5 criteria and 1000 criteria points.

29.3.49 Prime Decisions

http://www.sal.tkk.fi/en/resources/downloadables/prime. PRIME Decisions [62] emphasizes its ability to use incomplete preference information. It relies on the PRIME method that uses interval valued ratio statements of preference. These lead to linear constraints for a series of linear programming problems. Solving the linear programs leads to dominance structures. There is an "elicitation tour" to guide the decision maker. Because of the large number of linear programs that must be solved, the approach is best suited to problems with relatively few non-dominated alternatives. The software runs on Windows platform and is downloadable for academic use.

29.3.50 Priority Mapper

http://www.infoharvest.com/ihroot/gis/index.asp. Priority Mapper is an extension of ESRI's ArcMap, which integrates priority analysis with *geographical information systems* (GIS). It is targeted at managers and executives to realistically prioritize actions related to geographically distributed assets and resources. The output is in the form of visual representations of the prioritizations and recommended alternatives. The target operating platform is Windows. Due to a bug in Microsoft's installer for SQL, the beta launch of Priority Mapper was delayed.

29.3.51 Prism's Group Decision Support System

http://www.prismdecision.com/solutions/decision-support. Prism's Group Decision Support System provides group multi-criteria decision support. The software is based on a simple weighted criteria scoring approach for MCDA problems. After developing a set of possible solutions and agreeing to a set of decision criteria, the group members weigh each criterion using a pair-wise comparison analysis. The criteria weights, solution set, and criteria set consist of a multiple criteria decision matrix. The group members assess each solution against each criterion and vote on a 1 to 9 scale. In case of disagreement, a revote is taken after group discussion. After all cells are voted, the raw worth (sum of the 1 to 9 votes) and the weighted worth for each solution are displayed.

29.3.52 PROBE

PROBE (Portfolio Robustness Evaluation) is a decision support system developed to aid a decision-maker in the task of selecting a robust portfolio of projects in the presence of limited resources, multiple criteria, different project interactions, and several types of uncertainty [43]. PROBE identifies all efficient portfolios, either convex or non-convex, depicts them in a cost versus benefit graph within a given portfolio cost range, and allows performing in-depth interactive analysis of the robustness of selecting a proposed portfolio. PROBE integrates two main architectural components: a multi-criteria decision analysis component and a portfolio decision analysis component. The multi-criteria component allows the user to structure the benefit criteria in the form of a value tree, input data for the costs of the projects and their benefit scores on each bottom-level criterion of the value tree, and weights for the criteria at each level of the value tree. A hierarchical value model is used for aggregation evaluation. The portfolio component uses optimization to find all the efficient portfolios for the given project costs and aggregated benefit value scores for a user-defined portfolio cost range. The modeling of uncertainty is also supported.

29.3.53 RGDB

http://www.ccas.ru/mmes/mmeda/rgdb/index.htm. RGDB (*Reasonable Goal for Database*) is a prototype version of a Web application server that can support easy selection of large databases for preferred items, such as preferable goods and services, suspicious data, efficient investment strategies, etc. It is a Web implementation of the RGM/IDM (*Reasonable Goals Method/Interactive Decision Maps*) technique [41] using Java applets. From the same research group as FGM (see Sect. 29.3.21), RGM uses IDM to support the identification of goals. However, the identified goals might not be feasible, and thus a reasonable goal is identified and feasible decisions (based on users' preferences) that are in line with the goal are selected. When applying RGM for databases, users can select preferable rows from thousands or even millions of rows by simply clicking a preferable criterion point (a preferable goal) on a picture and then receiving one or more rows that are in line with the identified goal. The prototype RGDB server supports up to 5 attributes and up to 2000 alternatives. Five different versions of the applet are available: (1) the simplest applet for beginners, (2) the applet for negotiation support, (3) the applet with an additional matrix of decision maps, (4) the applet for negotiation support with matrix of decision maps, and (5) the applet with a structured procedure of Pareto frontier exploration. Internet Explorer and Java 1.3 are needed to use the RGDB application server.

29.3.54 RICH Decisions

http://www.rich.tkk.fi/index.html. RICH Decisions is a web-based free decision support software based on the RICH (*Rank Inclusion in Criteria Hierarchies*) method [65] which admits incomplete ordinal preference information in hierarchical weighting models. It allows the DM to state such preference information by specifying pairs of two sets, possibly of different size, of which the first consists of attributes and the second of importance rankings that are attained by the attributes in the first set (e.g., a set of three attributes of which one has the highest importance ranking, or a singleton set consisting of one attribute which is the second or third most important). Taken together, these pairs define the set of feasible attribute weights. RICH Decisions has a graphical user interface for structuring alternatives and attributes in both flat and multi-level value trees. Scores can be elicited by assessing all alternatives with regard to a given attribute or by assessing a given alternative across all attributes. Based on the elicited score and weight information, RICH Decisions derives decision recommendations by checking dominance relations and by applying decision rules. Results such as value intervals and dominance relations are shown graphically. The software supports up to 29 alternatives. The computations can be time-consuming if there are more than ten attributes. RICH Decisions is a Java-applet, which requires a Java-enabled browser. For security reasons, only models can be saved on the server.

29.3.55 Rubis (Plug-in)

http://www.decision-deck.org/d2/plugins.html. Developed as a plug-in for Decision Desktop Software/d2 (see Sect. 29.3.11), Rubis, a bipolar-valued concordance based decision aiding method [4], is a progressive decision aiding tool to help a DM determine a single best decision alternative. The methodology focuses on pair-wise comparison of alternatives, which lead to the bipolar-value outranking digraph.

29.3.56 SANNA 2009

http://nb.vse.cz/~jablon/sanna.htm. SANNA 2009 is a Excel add-in for multi-criteria decision support. It is freeware that contains a support tool for estimation of weights using several methods including pair-wise comparisons and incorporates basic MCDA methods including WSA, TOPSIS, ELECTRE I and III, PROMETHEE I and II, ORESTE, and MAPPAC. It can solve problems up to 180 alternatives and 50 criteria.

29.3.57 MC-SDSS for ArcGIS

http://arcscripts.esri.com/details.asp?dbid=16980. MC-SDSS (*multiple criteria spatial decision support system*) is a .NET extension of ArcGIS desktop to solve optimization tasks (based on spatial data) using SAW (*simple additive weighting*) and TOPSIS (*technique for order preference by similarity to ideal solution*) scoring methods.

29.3.58 SOLVEX

http://www.ccas.ru/pma/product.htm. SOLVEX is a Fortran library of more than 20 numerical algorithms for solving unconstrained, nonlinear constrained, global minimization, and multi-criteria optimization problems [55]. The MOO algorithms cover additive convolution, Chebyshev convolution, goal programming, and epsilon approximation. Two versions, SOLVEX Windows and SOLVEX DOC are available for download.

29.3.59 TransparentChoice

http://www.transparentchoice.com. TransparentChoice is a Web-based application for collaborative decision-making, based on AHP. The software is built for providing the following "must-have" features for AHP: intuitive way to build and visualize hierarchy; option to reduce the number of pairwise comparisons; consistency checking of pairwise comparison results and resolving inconsistencies; collaborative decision making and voting; and sensitivity analysis. In TransparentChoice, each decision starts by creating a project for a specific decision goal, followed by defining alternatives, criteria, and custom scales. The collaboration is supported through the User Tab, allowing multiple users' decision inputs. Once all decision inputs (alternatives, criteria, and scales) are captured, each user is can evaluate each alternative using pairwise comparison, and collective votes are organized by reviewing input with assigned voting strengths to individuals and groups. The results for final decision are presented in graphic format, including criteria priorities and alternatives ranking. A free 30-day trial version is available.

29.3.60 Triptych

http://www.stat-design.com/Software/Triptych.html. Triptych is an Excel-based tool suite that asserts to capture the voice of customers and translate it to design requirements in product development. The software includes different worksheets implementing different MCDA methods, among which are AHP, Pugh, TOPSIS (*Technique for Order Preference by Similarity to Ideal Solution*), and the SDI Method. The AHP worksheet can support an AHP matrix with up to 200×200 item and includes a Consistency evaluation. The TOPSIS worksheet can support a TOPSIS matrix with up to 200 criteria and 200 options. The Pugh, TOPSIS, and SDI Method worksheets can support a matrix with up to 200 criteria and 200 options. Both qualitative and quantitative options are supported in the TOPSIS worksheet. A 10-day free trial is available.

29.3.61 TRIMAP

http://www.inescc.pt/ingles/produtos.php. TRIMAP [14] is an interactive approach that explores the Pareto optimal set for three-criterion linear programming models. The aim is to aid the decision maker in eliminating parts of the Pareto optimal solution set that are judged to be of less value. The limitation to three objectives permits graphical displays that facilitate the decision maker's information processing. The procedure does not converge to a particular solution, but the decision maker can stop the process when sufficient information has been learned about the Pareto optimal solutions. A demo is available.

29.3.62 UTA Plus

http://www.lamsade.dauphine.fr/spip.php?rubrique69. UTA Plus is the latest Windows implementation of the UTA method, originally proposed in 1982 [34]. The method can be used to solve multi-criteria choice and ranking problems on a finite set of alternatives. It constructs an additive utility function from a weak preference order defined by the user on a subset of reference alternatives. Constructing the utility function, based on a principle of ordinal regression, requires solving a small LP-problem. The software proposes marginal utility functions in piece-wise linear form based on the given weak order, and then allows the user to interactively modify the marginal utility functions, helped by a graphical user interface. Software and user manual are available for download.

29.3.63 Very Good Choice

http://www.verygoodchoice-addin.com. Very Good Choice (VGC) is an excel add-in for supporting both multi-alternative ranking and sorting problems. Based on the ELECTRE family of outranking methods, VGC allows users to determine alternatives and qualitative criteria and weights, and then score the alternatives. Ranked alternatives, including non-distinguishable alternatives (alternatives with the same rank), are presented in an ordered table. All the data about the decision process can be stored in an XML format. A free version is available for download.

29.3.64 VIP Analysis

http://www.uc.pt/en/feuc/ldias/software/vipa. VIP (*Variable Interdependent Parameters*) Analysis [20] was proposed to support the selection of the most preferred alternative from a list, considering the impacts of each alternative on multiple evaluation criteria. While the approach uses a basic additive aggregation value function, it permits the decision maker to provide imprecise parameters for the criteria importance (scaling weights). In the authors' words, they propose "a methodology of analysis based on the progressive reduction of the number of alternatives, introducing a concept of tolerance that lets the decision makers use some of the approaches in a more flexible manner." Several output options exist depending on the size of the problem and the nature of the input data (including value range, maximum regret for each alternative, and dominance relations). The software supports a thousand alternatives and up to 49 criteria. The Windows-based software is distributed for free upon request. A tutorial is available for download.

29.3.65 Visual Market/2

http://www.ccas.ru/mmes/mmeda/soft/second.htm. Visual Market/2 is a Windows-based implementation of the RGM/IDM technique for visualization of large databases (including GIS), similar to RGDB (Sect. 29.3.52). In addition to returning a small number of items that correspond to the identified goal, auxiliary data filtering and pseudo-decision trees are also provided. The software supports a maximum of 12,000 alternatives and up to 7 decision criteria. It was developed for Windows XP; a new version for Windows 7 and Windows 8 is under development. A demo of Visual Market/2 version 2.1 and a manual are available for download.

29.3.66 VISA

http://www.visadecisions.com. VISA (*Visual Interactive Sensitivity Analysis*) is based on an approach described Belton and Stewart [3]. Applying a linear, multi-attribute value function, it has been available in a Windows version since 1994, emphasizing a friendly graphical interface for adjusting the criteria hierarchy and other components of the model. For example, an interactive value tree can be structured to show all criteria on the main decision. Users can interactively provide input of weights and scores using bar charts, thermometer scales, or numerical input. The weights and scores can be adjusted by dragging the computer mouse, and the effects can be seen immediately on several output windows. VISA version 8 is available as *Standard* (a stand-alone desktop application), *Education* (Windows stand-alone campus license and free 3 month student licenses), and *Multi-user*. A 30-day free trial is also available.

29.3.67 VisualUTA

http://idss.cs.put.poznan.pl/site/visualuta.html. VisualUTA is developed by LDSS (*Laboratory of Intelligent Decision Support Systems*) at Poznan University of Technology, Poland, the same developer as for 4wMka2 (Sect. 29.3.2) and JAMM (Sect. 29.3.35). It is the first implementation of the UTA-GMS method [31] for multiple criteria ranking of alternatives. The method is interactive, with progressive pair-wise comparisons. The software is free for downloading.

29.3.68 WINGDSS

http://www.oplab.sztaki.hu/wingdss_en.htm. WINGDSS [15] is a group decision support system for multiple attribute problems. WINGDSS provides a final score for every alternative and thus a complete ranking. Voting powers are assigned to each decision maker for each criterion. Both subjective and factual criteria can be used. Sensitivity analysis permits studying the effect of the variations of parameters such as individual preferences, voting powers, and scores. It includes an attribute tree editor, data from the editor, and dynamic linkage to external databases. WINGDSS is Windows-based.

29.3.69 WINPRE

http://www.sal.tkk.fi/en/resources/downloadables/winpre. WINPRE [64] is a MCDA tool available from the *Systems Analysis Laboratory* in Finland, the group that also offers PRIME Decisions (Sect. 29.3.49) and HIPRE 3+ family (Sect. 29.3.25). WINPRE relies on a method called PAIRS (*Preference Assessment by Imprecise Ratio Statements*) that permits the decision maker to state a range of numbers to indicate preferences among alternatives. These preference statements result in linear constraints that lead to a feasible region for each criterion that is consistent with the decision maker's judgments. The software is available free for academic use.

29.4 Concluding Remarks

Increases in computing power have been at the heart of the substantial growths in applications of MCDA [74]. In 2005, Weistroffer et al. provided a comprehensive survey of MCDA software, but many of the software packages presented in that survey have been discontinued or are no longer supported. More recently, Poles et al. [54] reviewed MOO software available since 1999, focusing on the tools and features that advisable MOO software should contain. An early empirical evaluation of five MCDA software packages and a comparison of their usefulness to a basic spreadsheet package was conducted by Zapatero et al. [78]. Taking a different angle, Seixedo and Tereso [68] constructed an AHP-based MCDA software application for selecting MCDA software and presented the MCDA tools using a similar approach to Weistroffer et al. [75]. Mustajoki and Marttunen [50] recently did a comparison of some MCDA software with a specific focus on applicability to environmental impact assessment.

An updated review of the current state of MCDA software provides insights of not only what has been improved or not changed in MCDA software application development, but also what will be interesting for the future. Several findings from the previous software review Weistroffer et al. [75] are still valid. First, a large majority of commercially marketed packages deal primarily with MADM problem models and use relatively simple algorithmic approaches. For example, many commercial software packages adopt MAUT and/or AHP methods, where AHP and SMART are frequently implemented together. Second, the large variety of sophisticated MCDM methods proposed in the literature have mostly been implemented only on an ad hoc basis to solve a specific problem situation, or as experimental software to demonstrate the salient features of the proposed method. There are still relatively few commercial MOO software packages, though many MOO methods have been proposed in the literature. The available MOO commercial packages are mostly either integrated solver engines (e.g. Analytica Optimizer), or integrated in application-specific software solutions (e.g. iSight, modeFrontier).

Changes in MCDA software are also evident. First, MCDA has begun to penetrate many new areas of research and applications. For example, MCDA methods have been applied in new engineering applications, such as ESY for nuclear emergencies and iSight in 3D simulation design. Another example is spatial planning and management, where MCDA software packages are designed for integration with GIS, such as MC-SDSS for ArcGIS, Priority Mapper, and Visual PROMETHEE PROMap, and engineering applications. Second, MCDA software solutions have moved towards web-based and service-oriented platforms, facilitated by increasing computing power and improved Internet technology. Third, it is interesting to see MCDA applications, such as ParadisEO-MOEO and Decision Lab 2000, that have adopted an open source philosophy, an approach that has already become a major part of general, mainstream information technology development. Open architecture provides greater opportunities for implementation of state-of-the-art MCDA methods and continuous software enhancements by open source developers. It also allows the flexibility to adapt specific MCDA methods for particular business problems. However, the learning curves for open source solutions are quite steep and open source development may require sophisticated understanding of MCDA principles and methods. Nevertheless, we expect to see more open source initiatives in MCDA software development in the future. Another area for potentially more future MCDA software development is mobile MCDA applications. Currently, MCDA mobile applications seem to be designed only for personal decision-making. We did not include these in our survey, but some examples of such applications include Mobile Decision Maker by Broad Research Software (http://mobiledecisionmaker.com), decision buddy (http://www.decisionbuddyapp.com), and Decisionaker by lemonway (http://www.lemonway.com/index.php/products/14-ios-application/58-decisionaker-support-page).

References

1. Agrell, P.J., Steuer, R.E.: ACADEA—a decision support system for faculty performance reviews. J. Multi-Criteria Decis. Anal. **9**(5), 191–204 (2000)
2. Bana e Costa, C.A., Vansnick, J.-C.: The MACBETH approach: basic ideas, software, and an application. In: Meskens, N., Roubens, M. (eds.) Advances in Decision Analysis, pp. 131–157. Kluwer Academic Publishers, Dordrecht (1999)
3. Belton, V., Stewart, T.J.: Multiple Criteria Decision Analysis. Kluwer Academic Publishers, Dordrecht (2002)
4. Bisdorff, R., Meyer, P., Roubens, M.: RUBIS: a bipolar-valued outranking method for the choice problem. 4OR: Q. J. Oper. Res. **6**(2), 143–165 (2008)
5. Brans, J.-P., Mareschal, B.: The PROMCALC & GAIA decision support system for multicriteria decision aid. Decis. Support. Syst. **12**(4–5), 297–310 (1994)
6. Brans, J.P.: L'ingénierie de la décision: élaboration d'instruments d'aide à la décision. La méthode PROMETHEE. L'aide à la décision: Nature, Instruments et Perspectives d'Avenir, pp. 183–213. Presses de l'Université Laval, Québec (1982)
7. Brans, J.P., Vincke, P.: A preference ranking organisation method: the PROMETHEE method for MCDM. Manag. Sci. **31**(6), 647–656 (1985)
8. Brans, J.P., Vincke, P., Mareschal, B.: How to select and how to rank projects: the PROMETHEE method. Eur. J. Oper. Res. **24**(2), 228–238 (1986)
9. Buede, D.: Decision analysis software: aiding the development of insight. OR/MS Today **20**(2), 52–60 (1993)
10. Cahon, S., Melab, N., Talbi, E.G.: ParadisEO: a framework for the reusable design of parallel and distributed metaheuristics. J. Heuristics **10**(3), 357–380 (2004)
11. Chelst, K., Canbolat, Y.B.: Value-Added Decision Making for Managers. CRC Press, Boca Raton (2012)
12. Chen, J.Q., Lee, S.M.: An exploratory cognitive DSS for strategic decision making. Decis. Support Syst. **36**(2), 147–160 (2003)
13. Cho, K.T.: Multicriteria decision methods: an attempt to evaluate and unify. Math. Comput. Model. **37**(9–10), 1099–1119 (2003)
14. Clímaco, J., Antunes, C.H.: Implementation of a user friendly software package – a guided tour of TRIMAP. Math. Comput. Model. **12**(10–11), 1299–1309 (1989)
15. Csáki, P., Rapcsák, T., Turchányi, P., Vermes, M.: R and D for group decision aid in Hungary by WINGDSS, a Microsoft Windows based Group Decision Support System. Decis. Support Syst. **14**(3), 205–217 (1995)

16. Danielson, M., Ekenberg, L.: A framework for analysing decisions under risk. Eur. J. Oper. Res. **104**(3), 474–484 (1998)
17. Danielson, M., Ekenberg, L.: Computing upper and lower bounds in interval decision trees. Eur. J. Oper. Res. **181**(2), 808–816 (2007)
18. Dempster, A.P.: Upper and lower probabilities induced by a multi-valued mapping. Ann. Math. Stat. **38**, 325–339 (1967)
19. Dias, L., Clímaco, J.: ELECTRE TRI for groups with imprecise information on parameter values. Group Decis. Negot. **9**(5), 355–377 (2000)
20. Dias, L.C., Clímaco, J.N.: Additive aggregation with variable interdependent parameters: the VIP analysis software. J. Oper. Res. Soc. **51**(9), 1070–1082 (2000)
21. Dias, L.C., Mousseau, V., Figueira, J., Clímaco, J., Silva, C.G.: IRIS 1.0 software. Newsletter of the European Working Group "Multicriteria Aid for Decisions" **3**(5), 4–6 (2002)
22. Druzdzel, M.J.: SMILE: Structural Modeling, Inference, and Learning Engine and GeNIe: a development environment for graphical decision-theoretic models. In: Proceedings of the Sixteenth National Conference on Artificial Intelligence and the Eleventh Innovative Applications of Artificial Intelligence Conference, American Association for Artificial Intelligence, Orlando, pp. 902–903 (1999)
23. Dyer, J.S., Fishburn, P.C., Steuer, R.E., Wallenius, J., Zionts, S.: Multiple criteria decision making, multiattribute utility theory: the next ten years. Manag. Sci. **38**(5), 645–654 (1992)
24. Edwards, W.: How to use multiattribute utility measurement for social decision making. IEEE Trans. Syst. Man Cybern. **7**(5), 326–340 (1977)
25. Edwards, W., Barron, F.H.: SMARTS and SMARTER: improved simple methods for multiattribute utility measurement. Organ. Behav. Hum. Decis. Process. **60**(3), 306–325 (1994)
26. Farquhar, P.H.: Utility assessment methods. Manag. Sci. **30**(11), 1283–1300 (1984)
27. Figueira, J., Mousseau, V., Roy, B.: ELECTRE methods. In: Figueira, J., Greco, S., Ehrgott, M. (eds.) Multiple Criteria Decision Analysis: State of the Art Surveys, pp. 133–162. Springer, Boston (2005)
28. Geldermann, J., Zhang, K.: Software review: "Decision Lab 2000". J. Multi-Criteria Decis. Anal. **10**(6), 317–323 (2001)
29. Greco, S., Matarazzo, B., Slowinski, R.: The Use of Rough Sets and Fuzzy Sets in MCDM. Kluwer Academic Publishers, Dordrecht (1999)
30. Greco, S., Matarazzo, B., Slowinski, R.: Rough sets theory for multicriteria decision analysis. Eur. J. Oper. Res. **129**(1), 1–47 (2001)
31. Greco, S., Mousseau, V., Słowiński, R.: Ordinal regression revisited: multiple criteria ranking using a set of additive value functions. Eur. J. Oper. Res. **191**(2), 416–436 (2008)
32. Hansen, P., Ombler, F.: A new method for scoring additive multi-attribute value models using pairwise rankings of alternatives. J. Multi-Criteria Decis. Anal. **15**(3–4), 87–107 (2008)
33. Hurwicz, L.: Optimality criteria for decision making under ignorance. Cowles Commission Discussion Paper, Statistics 370 (1951)
34. Jacquet-Lagrèze, E., Siskos, J.: Assessing a set of additive utility functions for multicriteria decision-making: the UTA method. Eur. J. Oper. Res. **10**(2), 151–164 (1982)
35. Keeney, R.L., Raiffa, H.: Decisions with Multiple Objectives. Wiley, New York (1976)
36. Kelly, G.: The Psychology of Personal Constructs. Norton, New York (1955)
37. Liefooghe, A., Basseur, M., Jourdan, L., Talbi, E.-G.: ParadisEO-MOEO: a framework for evolutionary multi-objective optimization. In: Obayashi, S., Deb, K., Poloni, C., Hiroyasu, T., Murata, T. (eds.) Evolutionary Multi-Criterion Optimization, Vol 4403 of Lecure Notes in Computer Science, pp. 386–400. Springer, Berlin (2007)
38. Limayem, M., DeSanctis, G.: Providing decisional guidance for multicriteria decision making in groups. Inf. Syst. Res. **11**(4), 386–401 (2000)
39. Löhne, A.: Vector Optimization with Infimum and Supremum. Springer, Berlin (2011)
40. Lotov, A.V., Berezkin, V.E., Kamenev, G.K.: Approximation and visualization of the Pareto frontier for non-convex multi-criteria problems. Dokl. Math. **66**(2), 260–262 (2002)
41. Lotov, A.V., Bushenkov, V.A., Chernov, A.V., Gusev, D.V., Kamenev, G.K.: INTERNET, GIS, and interactive decision maps. J. Geogr. Inf. Decis. Anal. **1**(2), 118–149 (1997)

42. Lotov, A.V., Bushenkov, V.A., Kamenev, G.K.: Feasible Goals Method – Search for Smart Decisions. Computing Centre RAS, Moscow (2001)
43. Lourenço, J.C., Morton, A., Bana e Costa, C.A.: PROBE – a multicriteria decision support system for portfolio robustness evaluation. Decis. Support Syst. **54**(1), 534–555 (2012)
44. Matsatsinis, N.F., Siskos, Y.: MARKEX: an intelligent decision support system for product development decisions. Eur. J. Oper. Res. **113**(2), 336–354 (1999)
45. Miettinen, K., Mäkelä, M.M.: Synchronous approach in interactive multiobjective optimization. Eur. J. Oper. Res. **170**(3), 909–922 (2006)
46. Monte, L., Brittain, J.E., Gallego, E., Håkanson, L., Hofman, D., Jiménez, A.: MOIRA-PLUS: a decision support system for the management of complex fresh water ecosystems contaminated by radionuclides and heavy metals. Comput. Geosci. **35**(5), 880–896 (2009)
47. Mousseau, V., Figueira, J., Dias, L., Gomes da Silva, C., Clímaco, J.: Resolving inconsistencies among constraints on the parameters of an MCDA model. Eur. J. Oper. Res. **147**(1), 72–93 (2003)
48. Mousseau, V., Slowinski, R., Zielniewicz, P.: A user-oriented implementation of the ELECTRE-TRI method integrating preference elicitation support. Comput. Oper. Res. **27**(7–8), 757–777 (2000)
49. Munda, G.: Multicriteria Evaluation in a Fuzzy Environment. Physica-Verlag, Heidelberg (1995)
50. Mustajoki, J., Marttunen, M.: Comparison of multi-criteria decision analytical software – searching for ideas for developing a new EIA-specific multi-criteria software. IMPERIA Project Report, University of Jyväskylä, Finland (2013)
51. Newman, J.W.: Management Applications of Decision Theory. Harper & Row, New York (1971)
52. Papamichail, K.N., French, S.: Decision support in nuclear emergencies. J. Hazard. Mater. **71**(1–3), 321–342 (2000)
53. Patchak, W.M.: Software survey: decision analysis. OR/MS Today **39**(5), 39–49 (2012)
54. Poles, S., Vassileva, M., Sasaki, D.: Multiobjective optimization software. In: Branke, J., Deb, K., Miettinen, K., Słowinski, R. (eds.) Multiobjective Optimization. pp. 329–348. Springer, Berlin (2008)
55. Potapov, M.A., Kabanov, P.N.: SOLVEX – system for solving nonlinear, global and multicriteria problems. In: Proceedings 3rd IFIP WG-7.6 Working Conference on Optimization-Based Computer-Aided Modelling and Design, Prague, pp. 343–347 (1995)
56. Ríos Insua, D., Gallego, E., Mateos, A., Ríos-Insua, S.: MOIRA: a decision support system for decision making on aquatic ecosystems contaminated by radioactive fallout. Ann. Oper. Res. **95**(1), 341–364 (2000)
57. Roy, B.: How outranking relation helps multiple criteria decision making. In: Cochrane, J., Zeleny, M. (eds.) Topics in Multiple Criteria Decision Making, pp. 179–201. University of South Carolina Press, Columbia (1973)
58. Roy, B.: ELECTRE III: Un algorithme de classements fondé sur une représentation floue des préférences en présence de critères multiples. Cahiers du Centre d'Etudes de Recherche Opérationnelle (Belgique) **20**(1), 3–24 (1978)
59. Roy, B.: The outranking approach and the foundations of ELECTRE methods. Theor. Decis. **31**(1), 49–73 (1991)
60. Parashar, S., Clarich, A., Geremia, P., Otani, A.: Reverse Multi-Objective Robust Design Optimization (R-MORDO) using chaos collocation based robustness quantification for engine calibration. In: 13th AIAA/ISSMO Multidisciplinary Analysis Optimization Conference, Fort Worth, p. 9038 (2010)
61. Saaty, T.L.: The Analytic Hierarchy Process. McGraw-Hill, New York (1980)
62. Salo, A., Hämäläinen, R.P.: Preference ratio in multiattribute evaluation (PRIME) –elicitation and decision procedures under incomplete information. IEEE Trans. Syst. Man Cybern. Part A **31**(6), 533–545 (2001)
63. Salo, A., Hämäläinen, R.P.: Processing interval judgments in the analytic hierarchy process. In: Goicocchea, A., Duckstein, L., Zionts, S. (eds.) Multiple Criteria Decision Making, pp. 359–371. Springer, New York (1991)

64. Salo, A., Hämäläinen, R.P.: Preference programming through approximate ratio comparisons. Eur. J. Oper. Res. **82**(3), 458–475 (1995)
65. Salo, A., Punkka, A.: Rank inclusion in criteria hierarchies. Eur. J. Oper. Res. **163**(2), 338–356 (2005)
66. Savage, L.J.: The theory of statistical decision. J. Am. Stat. Assoc. **46**(253), 55–67 (1951)
67. Savage, L.J.: Foundations of Statistics, 2nd edn. Dover, New York (1972)
68. Seixedo, C., Tereso, A.: A multicriteria decision aid software application for selecting MCDM software using AHP. In: Proceedings of the 2nd International Conference on Engineering Optimization, Lisbon (2010)
69. Shafer, G.: A Mathematical Theory of Evidence. Princeton University Press, Princeton (1976)
70. Siskos, Y., Matsatsinis, N.F.: A DSS for market analysis and new product design. J. Decis. Syst. **2**(1), 35–63 (1993)
71. Siskos, Y., Spyridakos, A., Yannacopoulos, D.: MINORA: a multicriteria decision aiding system for discrete alternatives. J. Inf. Sci. Technol. **2**(2), 136–149 (1993)
72. Talašová, J., Holeček, P.: The FuzzME software package. In: Proceedings of the Joint International Fuzzy Systems Association World Congress and the European Society of Fuzzy Logic and Technology Conference, Lisbon (2009)
73. Von Winterfelt, D., Edwards, W.: Decision Analysis and Behavioral Research. Cambridge University Press, Cambridge (1986)
74. Wallenius, J., Dyer, J.S., Fishburn, P.C., Steuer, R.E., Zionts, S., Deb, K.: Multiple criteria decision making, multiattribute utility theory: recent accomplishments and what lies ahead. Manag. Sci. **54**(7), 1336–1349 (2008)
75. Weistroffer, H.R., Smith, C.H., Narula, S.C.: Multiple criteria decision support software. In: Figueira, J., Greco, S., Ehrgott, M. (eds.), Multiple Criteria Decision Analysis: State of the Art Surveys, Springer, Berlin (2005)
76. Yang, J.B.: Rule and utility based evidential reasoning approach for multiattribute decision analysis under uncertainties. Eur. J. Oper. Res. **131**(1), 31–61 (2001)
77. Yang, J.B., Singh, M.G.: An evidential reasoning approach for multiple-attribute decision making with uncertainty. IEEE Trans. Syst. Man Cybern. **24**(1), 1–18 (1994)
78. Zapatero, E.G., Smith, C.H., Weistroffer, H.R.: Evaluating multiple-attribute decision support systems. J. Multi-Criteria Decis. Anal. **6**, 201–214 (1997)

Index

S. Greco et al. (eds.), *Multiple Criteria Decision Analysis*, International Series in Operations Research & Management Science 233,
DOI 10.1007/978-1-4939-3094-4

L

M

N

O

P

Q

R

S

T

U