Chapter 9
Barycentric subdivision

BARYCENTRIC SUBDIVISION HAS LONG BEEN A USEFUL TOOL in geometry and topology. It is an operation that preserves topology and is well-behaved combinatorially. In this chapter we will study a transformation of Brenti and Welker that maps the f-vector of a complex to the f-vector of its barycentric subdivision.

9.1 Barycentric subdivision of a finite cell complex

The term *barycenter* refers to the center of mass of a convex polytope, and there is a straightforward notion of *barycentric subdivision* for convex polytopes which goes as follows. Place a vertex on the center of mass of each face of the polytope and connect vertices that lie in a common face. This "triangulates" the polytope in the sense that every face resulting from the subdivision is a simplex.

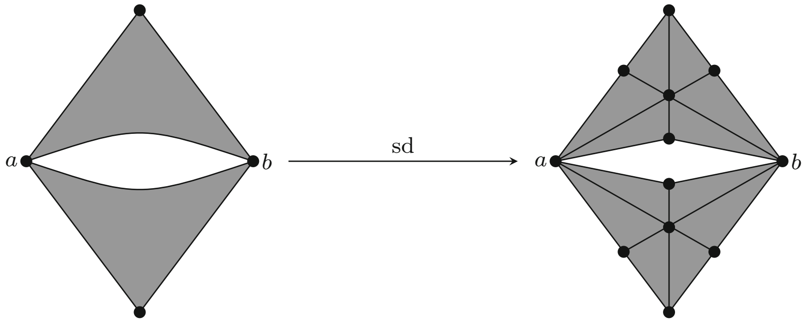

Fig. 9.1 A cell complex and its barycentric subdivision.

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Ignoring geometry, we can define the barycentric subdivision combinatorially: the barycentric subdivision of Δ is the order complex of the poset of nonempty faces of Δ. See Figure 9.1. Let $\mathrm{sd}(\Delta)$ denote this abstract simplicial complex. Then the k-faces of the complex $\mathrm{sd}(\Delta)$ are k-chains of nonempty faces of Δ, sometimes called *flags*:

$$F_1 <_\Delta F_2 <_\Delta \cdots <_\Delta F_k.$$

In particular, each nonempty face of Δ corresponds to a vertex of $\mathrm{sd}(\Delta)$.

As with general order complexes, the barycentric subdivision is a flag complex. Moreover, since face posets are ranked by dimension, $\mathrm{sd}(\Delta)$ is the order complex of a ranked poset, and hence balanced.

Our combinatorial definition of barycentric subdivision makes sense for any cell complex Δ for which there is a well-defined face poset, though we need to be a little careful about the topology. If there are cells of Δ with identifications on their boundary, i.e., a $(k-1)$-dimensional cell with fewer than k vertices, information can get lost. For example, in Figure 9.2 we see that the cell complex on a circle with just one edge and one vertex has a contractible barycentric subdivision. However, if no face has self-identifications on its boundary, e.g., if Δ is a polytope or a boolean complex, then Δ and its barycentric subdivision are homeomorphic.

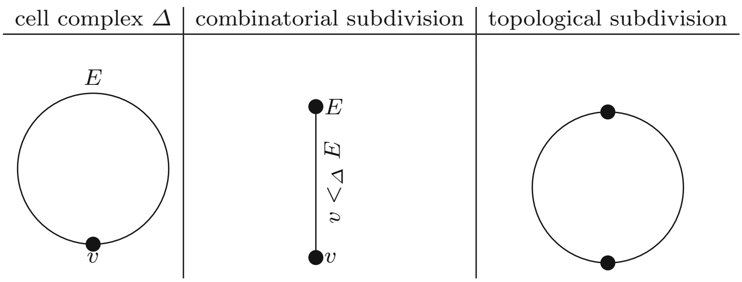


Fig. 9.2 Combinatorial barycentric subdivision can destroy topology if Δ has cells with self-identifications.

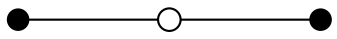
There is a topological definition of barycentric subdivision that does not have this problem. In a true cell complex, each cell "remembers where it came from" in the sense that we know how its boundary is mapped onto lower-dimensional cells. Thus, we can "unglue" the cell, deform the cell continuously into a geometric simplex, perform barycentric subdivision, and glue the subdivided cell back with the original boundary map. Doing this for each cell gives the topological definition of barycentric subdivision.

In all that follows, however, we will only consider the combinatorial definition.

9.2 The barycentric subdivision of a simplex

We will now do a careful enumeration of the faces in the barycentric subdivision of a simplex. We will denote a simplex on vertex set V by $2^V = \{F : F \subseteq V\}$.

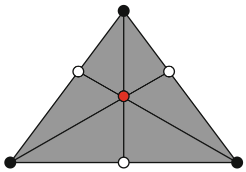
Let's do small examples first.

If $V = \{1, 2\}$, $2^V = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$ the barycentric subdivision is drawn: . We can list the flags of 2^V as:

empty face	vertices	edges
$\emptyset$	$\{1\}$	$\{1\} \subset \{1,2\}$
	$\{2\}$	$\{2\} \subset \{1,2\}$
	$\{1,2\}$	

so $f(\mathrm{sd}(2^V)) = (1, 3, 2)$ and $h(\mathrm{sd}(2^V)) = (1, 1, 0)$.

If $V = \{1, 2, 3\}$, the barycentric subdivision of a triangle is



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We color the vertices in the barycentric subdivision to recall the dimension of the corresponding face in the original complex. (This gives a balanced coloring to $\mathrm{sd}(\Delta)$.) Listing the flags we find:

empty face	vertices	edges	triangles
$\emptyset$	$\{1\}$	$\{1\} \subset \{1,2\}$	$\{1\} \subset \{1,2\} \subset \{1,2,3\}$
	$\{2\}$	$\{1\} \subset \{1,3\}$	$\{1\} \subset \{1,3\} \subset \{1,2,3\}$
	$\{3\}$	$\{2\} \subset \{1,2\}$	$\{2\} \subset \{1,2\} \subset \{1,2,3\}$
	$\{1,2\}$	$\{2\} \subset \{2,3\}$	$\{2\} \subset \{2,3\} \subset \{1,2,3\}$
	$\{1,3\}$	$\{3\} \subset \{1,3\}$	$\{3\} \subset \{1,3\} \subset \{1,2,3\}$
	$\{2,3\}$	$\{3\} \subset \{2,3\}$	$\{3\} \subset \{2,3\} \subset \{1,2,3\}$
	$\{1,2,3\}$	$\{1\} \subset \{1,2,3\}$	
		$\{2\} \subset \{1,2,3\}$	
		$\{3\} \subset \{1,2,3\}$	
		$\{1,2\} \subset \{1,2,3\}$	
		$\{1,3\} \subset \{1,2,3\}$	
		$\{2,3\} \subset \{1,2,3\}$	

and so $f(\mathrm{sd}(2^V)) = (1, 7, 12, 6)$ and $h(\mathrm{sd}(2^V)) = (1, 4, 1, 0)$.

Before moving on to larger cases, it will be a good idea to refine our bookkeeping. Notice that there is a lot of redundancy in counting flags, in that for every flag

$$S_1 \subset S_2 \subset \cdots \subset S_k \subset \{1, 2, \ldots, n\},$$

there is another flag

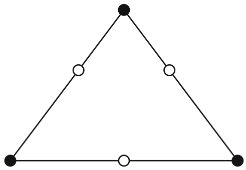
$$S_1 \subset S_2 \subset \cdots \subset S_k,$$

of one dimension lower. Thus,

$$f(\mathrm{sd}(2^V); t) = (1+t) f(\mathrm{sd}(\partial 2^V); t). \tag{9.1}$$

That is, the faces of the barycentric subdivision of $\partial 2^V$ are precisely those flags that do not contain the interior of the simplex, i.e., the face $V = \{1, 2, \ldots, n\}$.

For example, the barycentric subdivision of the boundary of the triangle is (combinatorially) a hexagon:



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corresponding to the flags below:

empty face	vertices	edges
$\emptyset$	$\{1\}$	$\{1\} \subset \{1,2\}$
	$\{2\}$	$\{1\} \subset \{1,3\}$
	$\{3\}$	$\{2\} \subset \{1,2\}$
	$\{1,2\}$	$\{2\} \subset \{2,3\}$
	$\{1,3\}$	$\{3\} \subset \{1,3\}$
	$\{2,3\}$	$\{3\} \subset \{2,3\}$

.

Thus $f(\mathrm{sd}(\partial 2^V)) = (1, 6, 6)$. We see that

$$(1+t) f(\mathrm{sd}(\partial 2^V); t) = (1+t)(1+6t+6t^2) = 1+7t+12t^2+6t^3 = f(\mathrm{sd}(2^V); t),$$

as expected.

Moreover, since

$$f(\mathrm{sd}(\partial 2^V); t) = (1+t)^{|V|-1} h(\mathrm{sd}(\partial 2^V); t/(1+t)),$$

and

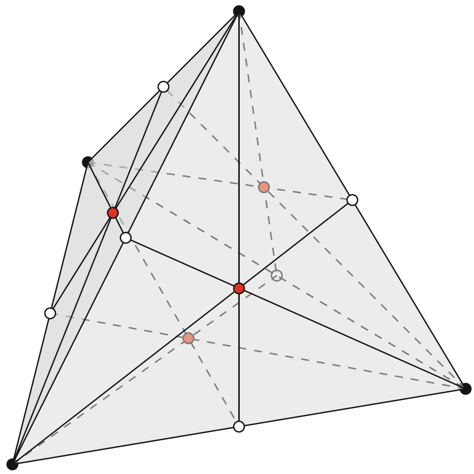
$$f(\mathrm{sd}(2^V); t) = (1+t)^{|V|} h(\mathrm{sd}(2^V); t/(1+t)),$$

Equation (9.1) gives us

$$h(\mathrm{sd}(\partial 2^V); t) = h(\mathrm{sd}(2^V); t). \tag{9.2}$$

That is, the barycentric subdivision of the boundary of the simplex has the same h-vector as the barycentric subdivision of the simplex itself. For simplicity then (since we have to keep track of fewer flags) we will restrict our attention to $\mathrm{sd}(\partial 2^V)$. Note that it is highly unusual that a simplicial complex and its boundary be related in such a way. See Problem 9.1.

For the subdivided tetrahedron we draw the boundary only:



and we get $f(\mathrm{sd}(\partial 2^V)) = (1, 14, 36, 24)$ and $h(\mathrm{sd}(\partial 2^V)) = (1, 11, 11, 1)$. Thus $f(\mathrm{sd}(2^V)) = (1, 15, 50, 60, 24)$ and $h(\mathrm{sd}(2^V)) = (1, 11, 11, 1, 0)$.

So what are the h-vectors we have computed so far?

$$(1, 1), (1, 4, 1), (1, 11, 11, 1), \ldots$$

if we throw in the vector (1) at the beginning for the trivial simplex, we have the first few rows of Table 1.3. We have Eulerian numbers!

Let us prove this connection by counting flags carefully. Throughout the remainder of Section 9.2, we will fix a finite vertex set V and let $\Delta = \mathrm{sd}(\partial 2^V)$ denote the barycentric subdivision of the boundary of the simplex with vertex set V.

This first step in computing $f(\Delta)$ is to modify our bookkeeping. Since flags are sequences of nested subsets, we can keep track only of the new additions. That is, given

$$\emptyset \subset S_1 \subset S_2 \subset \cdots \subset S_k \subset \{1, 2, \ldots, n\} = V,$$

let $A_i = S_{i+1} - S_i$, as i ranges from 0 to k, with $S_0 = \emptyset$ and $S_{k+1} = \{1, 2, \ldots, n\}$. Instead of the original flag, we can record the tuple $(A_0, A_1, \ldots, A_k)$. For example, if $V = \{1, 2, 3, 4, 5, 6, 7\}$, the flag

$$\emptyset \subset \{3, 4\} \subset \{3, 4, 6, 7\} \subset \{1, 3, 4, 6, 7\} \subset \{1, 2, 3, 4, 6, 7\} \subset V,$$

becomes the tuple

$$(\{3,4\},\{6,7\},\{1\},\{2\},\{5\}).$$

Even better, we can write

$$34|67|1|2|5,$$

if we agree to list the elements of each A_i in increasing order and drop the curly braces and commas. This is a set composition!

We enumerated set compositions in our study of the braid arrangement in Section 5.6. So the complex $\Sigma(n)$ (associated with the braid arrangement $\mathcal{H}(n)$) is isomorphic to the barycentric subdivision of the boundary of a simplex. If we restate Theorems 5.2 and 5.3, we have the following.

Theorem 9.1. *The barycentric subdivision of the boundary of the simplex 2^V, with $|V| = n$, has the following f- and h-polynomials:*

$$f(\mathrm{sd}(\partial 2^V);t) = \sum_{k=0}^{n-1}(k+1)!S(n,k+1)t^k,$$

where $S(n,k)$ is a Stirling number of the second kind, and

$$h(\mathrm{sd}(\partial 2^V);t) = \sum_{k=0}^{n-1}\left\langle {n \atop k} \right\rangle t^k,$$

where $\left\langle {n \atop k} \right\rangle$ is an Eulerian number. In other words, the Eulerian polynomial is the h-polynomial of $\mathrm{sd}(\partial 2^V)$.

9.3 Brenti and Welker's transformation

We will now use the ideas developed for the simplex to study f- and h-vectors of $\mathrm{sd}(\Delta)$, where Δ is any boolean complex. Recall from Section 8.3 that a boolean complex is a cell complex in which each face is a simplex. A simplicial complex is a boolean complex, but this family also includes complexes whose faces are not uniquely determined by their vertex sets.

As a starting point, let us consider the boolean complex and its barycentric subdivision given in Figure 9.3.

With the vertices of Δ labeled a,b,c,d, and the two edges between a and b labeled E_1 and E_2, we have the following flags of faces in $\mathrm{sd}(\Delta)$.

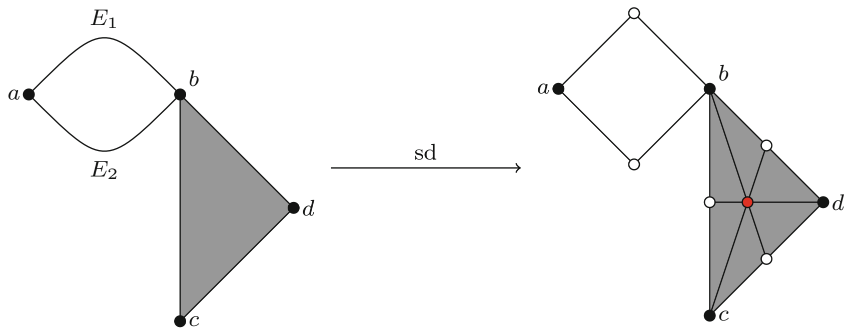


Fig. 9.3 A boolean complex and its barycentric subdivision.

empty face	vertices	edges	triangles
$\emptyset$			
	$\{a\}$		
	$\{b\}$		
	$\{c\}$		
	$\{d\}$		
	E_1	$\{a\} \subset E_1$ $\{b\} \subset E_1$	
	E_2	$\{a\} \subset E_2$ $\{b\} \subset E_2$	
	$\{b,c\}$	$\{b\} \subset \{b,c\}$ $\{c\} \subset \{b,c\}$	
	$\{b,d\}$	$\{b\} \subset \{b,d\}$ $\{d\} \subset \{b,d\}$	
	$\{c,d\}$	$\{c\} \subset \{c,d\}$ $\{d\} \subset \{c,d\}$	
	$\{b,c,d\}$	$\{b\} \subset \{b,c,d\}$ $\{c\} \subset \{b,c,d\}$ $\{d\} \subset \{b,c,d\}$ $\{b,c\} \subset \{b,c,d\}$ $\{b,d\} \subset \{b,c,d\}$ $\{c,d\} \subset \{b,c,d\}$	$\{b\} \subset \{b,c\} \subset \{b,c,d\}$ $\{b\} \subset \{b,d\} \subset \{b,c,d\}$ $\{c\} \subset \{b,c\} \subset \{b,c,d\}$ $\{c\} \subset \{c,d\} \subset \{b,c,d\}$ $\{d\} \subset \{b,d\} \subset \{b,c,d\}$ $\{d\} \subset \{c,d\} \subset \{b,c,d\}$

We have $f(\Delta) = (1,4,5,1)$ and $f(\mathrm{sd}(\Delta)) = (1,10,16,6)$. The beautiful result of Brenti and Welker gives us the means for computing $f(\mathrm{sd}(\Delta))$ as a simple linear transformation of $f(\Delta)$, which will now derive.

Notice that we have grouped the faces of $\mathrm{sd}(\Delta)$ according to the last face S_k in the flag. Within each of these groups, we can identify the flags

$$S_1 \subset S_2 \subset \cdots \subset S_k,$$

with set compositions of S_k, i.e., let $A_i = S_{i+1} - S_i$ for $i = 0, \ldots, k-1$, with $S_0 = \emptyset$. Then the composition $A = A_0|A_1|\cdots|A_{k-1}$ corresponds to the flag

$$A_0 \subset (A_0 \cup A_1) \subset \cdots \subset (A_0 \cup A_1 \cup \cdots \cup A_{k-1}).$$

For example, $b|d|c$ denotes the flag $\{b\} \subset \{b,d\} \subset \{b,c,d\}$ and $bd|c$ denotes the flag $\{b,d\} \subset \{b,c,d\}$.

We should be careful to first fix the flag we are working with, since, for example, $a|b$ could denote either the edge E_1 or the edge E_2. But once we know which maximal face S_k the flag lives in, the set compositions of that face are well defined.

For any fixed choice of face F of Δ, let $\mathrm{Comp}(F)$ denote the set of all set compositions A of the vertex set of F, i.e., all compositions $A = A_0|A_1|\cdots|A_{k-1}$ such that $A_i \cap A_j = \emptyset$ and $A_0 \cup \cdots \cup A_{k-1} = F$. These compositions represent all the flags in $\mathrm{sd}(\Delta)$ whose maximal element is F. We denote by $\mathrm{rk}(A) = k-1$ the number of bars in A, i.e., the dimension of the corresponding face of $\mathrm{sd}(\Delta)$.

If $|F| = j$, then, as we saw in the case of the simplex, there are $k!S(j,k)$ set compositions of F with k parts. These are set compositions of rank $\mathrm{rk}(A) = k-1$, and so

$$\sum_{A \in \mathrm{Comp}(F)} t^{1+\mathrm{rk}(A)} = \sum_{k \geq 0} k!S(j,k)t^k.$$

Each face $G \in \mathrm{sd}(\Delta)$ corresponds to a flag of faces of Δ, so summing over all F in Δ, we have:

$$\begin{aligned} f(\mathrm{sd}(\Delta); t) &= \sum_{G \in \mathrm{sd}(\Delta)} t^{1+\dim G}, \\ &= \sum_{F \in \Delta} \sum_{A \in \mathrm{Comp}(F)} t^{1+\mathrm{rk}(A)}, \\ &= \sum_{j \geq 0} f_j(\Delta) \cdot \sum_{k \geq 0} k!S(j,k)t^k, \\ &= \sum_{k \geq 0} \left(\sum_{j \geq 0} f_j(\Delta) k! S(j,k) \right) t^k. \end{aligned}$$

Let us state a theorem.

Theorem 9.2. *For any finite boolean complex Δ, with $\dim \Delta = d-1$, the f-vector of its barycentric subdivision is given by:*

$$f_k(\mathrm{sd}(\Delta)) = \sum_{j \geq 0} f_j(\Delta) k! S(j,k).$$

We can also describe this transformation with a matrix. Let

$$\mathfrak{B}_d = [a!S(b,a)]_{0 \leq a,b \leq d} \, .$$

Then, Theorem 9.2 says

$$f(\mathrm{sd}(\Delta)) = \mathfrak{B}_d\, f(\Delta).$$

For example, with $d = 3$, we have

$$\mathfrak{B}_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 6 \end{pmatrix}.$$

Thus, for Δ as in Figure 9.3, we have

$$f(\mathrm{sd}(\Delta)) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} 1 \\ 4 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 10 \\ 16 \\ 6 \end{pmatrix},$$

as desired.

9.4 The h-vector of sd(Δ) and j-Eulerian numbers

Recall from Section 8.8 now that the transformation from f-vector to h-vector is given by the matrix

$$H_d = \left[(-1)^{a+b} \binom{d-b}{a-b} \right]_{0 \le a,b \le d},$$

and the inverse transformation is

$$H_d^{-1} = \left[\binom{d-b}{a-b} \right]_{0 \le a,b \le d}.$$

That is,

$$h(\Delta) = H_d f(\Delta) \qquad \text{and} \qquad f(\Delta) = H_d^{-1} h(\Delta).$$

Then we can compose these operations to write $h(\mathrm{sd}(\Delta)) = H_d\, \mathfrak{B}_d\, f(\Delta)$, or

$$h(\mathrm{sd}(\Delta)) = H_d\, \mathfrak{B}_d\, H_d^{-1} h(\Delta). \tag{9.3}$$

Denote this transformation by

$$\mathfrak{E}_d = H_d\, \mathfrak{B}_d\, H_d^{-1}.$$

It turns out that this transformation is beautifully combinatorial. We see some small examples in Table 9.1.

Table 9.1 The barycenter transformation on h-vectors.

d	H_d	$\mathfrak{B}_d$	H_d^{-1}	$\mathfrak{E}_d$
0	(1)	(1)	(1)	(1)
1	$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
2	$\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$
3	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 3 & -2 & 1 & 0 \\ -1 & 1 & -1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 6 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 4 & 4 & 2 & 1 \\ 1 & 2 & 4 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
4	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -4 & 1 & 0 & 0 & 0 \\ 6 & -3 & 1 & 0 & 0 \\ -4 & 3 & -2 & 1 & 0 \\ 1 & -1 & 1 & -1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 6 & 14 \\ 0 & 0 & 0 & 6 & 36 \\ 0 & 0 & 0 & 0 & 24 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 & 0 \\ 6 & 3 & 1 & 0 & 0 \\ 4 & 3 & 2 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 11 & 8 & 4 & 2 & 1 \\ 11 & 14 & 16 & 14 & 11 \\ 1 & 2 & 4 & 8 & 11 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

There are some tantalizing properties of the matrices in Table 9.1. Notice, for example:

- the sum of all the entries in $\mathfrak{E}_d$ is $(d+1)!$,
- the sum of the entries in each column of $\mathfrak{E}_d$ is $d!$,
- the sum of the entries of row k of $\mathfrak{E}_d$, $k = 1, \ldots, d+1$, is the Eulerian number $\left\langle {d+1 \atop k} \right\rangle$.

All of these properties and more will follow from the following theorem due to Brenti and Welker.

First define the numbers

$$\left\langle {n; j \atop k} \right\rangle = |\{w \in S_n : \mathrm{des}(w) = k, w(1) = j\}|,$$

as the *j-Eulerian numbers.* These numbers refine the usual Eulerian numbers in the sense that

$$\left\langle {n \atop k} \right\rangle = \left\langle {n; 1 \atop k} \right\rangle + \left\langle {n; 2 \atop k} \right\rangle + \cdots + \left\langle {n; n \atop k} \right\rangle.$$

Similarly, define the *j-Eulerian polynomials* by

$$S_{n;j}(t) = \sum_{\substack{w \in S_n \\ w(1)=j}} t^{\mathrm{des}(w)} = \sum_{k=0}^{n-1} \left\langle {n;j \atop k} \right\rangle t^k.$$

These are the generating functions for the columns of $\mathfrak{E}_d$. For future use, let $S_{n;j} = \{w \in S_n : w(1) = j\}$ denote the set of permutations beginning with j.

Theorem 9.3. *Let $\mathfrak{E}_d$ denote the barycenter transformation for h-vectors $h = (h_0, \ldots, h_d)$. Then*

$$\mathfrak{E}_d = \left[\left\langle {d+1;j \atop k} \right\rangle \right]_{0 \le k, j-1 \le d},$$

so that if $h(t) = h(\Delta;t)$, then

$$h(\mathrm{sd}(\Delta);t) = \sum_{j=0}^{d} h_j(\Delta) S_{d+1;j+1}(t).$$

We will now prove Theorem 9.3.

First, consider an entry $T_{r,s}$, $0 \le r, s \le d$, of the matrix $T = \mathfrak{B}_d\, H_d^{-1}$. We have:

$$T_{r,s} = \sum_{b=0}^{d} \binom{d-s}{b-s} r! S(b,r) = \sum_{b=0}^{d} \binom{d-s}{d-b} r! S(b,r).$$

Since this is a positive formula, it is not too hard to come up with a combinatorial interpretation for it. Let $\mathcal{T}_{r,s}$ denote the set of all set compositions $A = A_0|A_1|\cdots|A_r$ of $\{1,2,\ldots,d+1\}$ for which $\min A_0 = s+1$. To form such a composition, we first choose $d-b$ elements from among $\{s+2,\ldots,d+1\}$ to put in A_0 along with $s+1$. This can be done in $\binom{d-s}{d-b}$ ways. To form $A_1|\cdots|A_r$ we need to create a set composition from the remaining b elements, and this can be done in $r!S(b,r)$ ways. See Figure 9.4.

$$\underbrace{1,2,\ldots,s, \quad \overset{\min A_0}{\boxed{s+1,}} \quad \overbrace{s+2,\ldots,d+1}^{\text{Choose } d-b \text{ more for } A_0}}_{\text{Form } A_1|\cdots|A_r \text{ from remaining } b \text{ elements}}$$

Fig. 9.4 Forming an element of $\mathcal{T}_{r,s}$.

Now let

$$\mathcal{T}_s = \bigcup_{r=0}^{d} \mathcal{T}_{r,s},$$

denote the set of all set compositions of $\{1,2,\ldots,d+1\}$ with $\min A_0 = s+1$. Further, let $T_s(t)$ denote the generating function counting these set compositions according to the number of bars,

$$T_s(t) = \sum_{A\in\mathcal{T}_s} t^{\mathrm{rk}(A)} = \sum_{r=0}^{d} T_{r,s}t^r.$$

In other words, $T_s(t)$ is the generating function for column s of the matrix T.

But each set composition $A = A_0|A_1|\cdots|A_r$ can be mapped to a permutation $w = w(A)$ by removing bars and writing each block in increasing order. Since $\min A_0 = s+1$, this means $w(1) = s+1$. That is, $w \in S_{d+1;s+1}$. Further, $\mathrm{Des}(w) \subseteq D$, where $D = D(A) = \{|A_0|, |A_0|+|A_1|, \ldots, |A_0|+|A_1|+\cdots+|A_{r-1}|\}$, i.e., there must be bars in A where there are descents in w. So we can write

$$\begin{aligned}
T_s(t) &= \sum_{A\in\mathcal{T}_s} t^{\mathrm{rk}(A)}, \\
&= \sum_{J\subseteq\{1,2,\ldots,d\}} \sum_{\substack{A\in\mathcal{T}_s\\ D(A)=J}} t^{|J|}, \\
&= \sum_{J\subseteq\{1,2,\ldots,d\}} \sum_{\substack{w\in S_{d+1;s+1}\\ D(w)\subseteq J}} t^{|J|}, \\
&= \sum_{w\in S_{d+1;s+1}} \sum_{\mathrm{Des}(w)\subseteq J} t^{|J|}, \\
&= \sum_{w\in S_{d+1;s+1}} t^{\mathrm{des}(w)}(1+t)^{d-\mathrm{des}(w)}, \\
&= (1+t)^d S_{d+1;s+1}(t/(1+t)).
\end{aligned}$$

Since $T_s(t)$ encodes column s of $\mathfrak{B}_d H_d^{-1}$, the polynomial $H_d T_s(t) = S_{d+1;s+1}(t)$ encodes column s of $\mathfrak{E}_d = H_d \mathfrak{B}_d H_d^{-1}$. That is, the columns of $\mathfrak{E}_d$ are encoded by the j-Eulerian polynomials, which proves Theorem 9.3.

9.5 Gamma-nonnegativity of $h(\mathrm{sd}(\Delta))$

In Theorem 9.4, we will see that if $h(\Delta)$ is nonnegative, then the polynomial $h(\mathrm{sd}(\Delta);t)$ is real-rooted. Moreover, if $h(\Delta)$ is palindromic, then $h(\mathrm{sd}(\Delta))$ is also palindromic. By Observation 4.2, this implies that $h(\mathrm{sd}(\Delta);t)$ is gamma-nonnegative as well. We can also prove this gamma-nonnegativity directly by investigating the j-Eulerian polynomials closely, as we now show.

First, we can observe that if $w(1) = 1$, there is never a descent in the first position, while if $w(1) = n$ there is always a descent in the first position. Hence the distributions of descents in $S_{n;1}$ and $S_{n;n}$ are the Eulerian distribution for S_{n-1}, i.e.,

$$S_{n;1}(t) = S_{n-1}(t) \text{ and } S_{n;n}(t) = tS_{n-1}(t).$$

In general, if we track the effect of removing the letter j from the beginning of a permutation in $S_{n;j}$, we get the following recurrence relation.

Observation 9.1 *For any* $1 \leq j \leq n$*,*

$$S_{n;j}(t) = t\sum_{k=1}^{j-1} S_{n-1;k}(t) + \sum_{k=j}^{n-1} S_{n-1;k}(t).$$

Next, notice that there are some nice symmetries in the array of j-Eulerian numbers. For example, recall the involution $w_0 : S_n \to S_n$ that maps i to $n+1-i$. This involution swaps descents for ascents, and if $w(1) = j$, then $w_0w(1) = n+1-j$. Hence, we have the following observation about symmetry.

Observation 9.2 *For any n, j, we have the following symmetries of j-Eulerian numbers:*

$$\left\langle {n;j \atop k} \right\rangle = \left\langle {n;n+1-j \atop n-1-k} \right\rangle,$$

and

$$S_{n;j}(t) = t^{n-1}S_{n;n+1-j}(1/t).$$

We now define the *palindromic j-Eulerian polynomials* by lumping together classes fixed by the involution w_0, namely all permutations beginning with either j or $n+1-j$:

$$\mathbf{S}_{n;j}(t) = \sum_{w \in S_{n;j} \cup S_{n;n+1-j}} t^{\mathrm{des}(w)}.$$

Observe that

$$\mathbf{S}_{n;j}(t) = \begin{cases} S_{n;j}(t) + S_{n,n+1-j}(t) & \text{if } j \neq (n+1)/2, \text{ and} \\ S_{n;j}(t) & \text{if } j = (n+1)/2. \end{cases}$$

By the symmetry seen in Observation 9.2, the polynomials $\mathbf{S}_{n;j}(t)$ have palindromic coefficients, and hence a gamma vector. Note the symmetry axis for $\mathbf{S}_{n;j}(t)$ is at degree $\lfloor \frac{n-1}{2} \rfloor$. If

$$\mathbf{S}_{n;j}(t) = \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \gamma_i^{(n;j)} t^i (1+t)^{n-1-2i},$$

let $\gamma^{(n;j)} = (\gamma_0^{(n;j)}, \gamma_1^{(n;j)}, \ldots, \gamma_{\lfloor (n-1)/2 \rfloor}^{(n;j)})$ denote the corresponding gamma vector.

We will develop a recursive argument for why $\gamma^{(n;j)}$ is nonnegative for all n and j. This recurrence depends on another family of gamma vectors, from the following polynomials, defined for $1 \leq j < (n+1)/2$:

$$\mathbf{S}'_{n;j}(t) = tS_{n;j}(t) + S_{n;n+1-j}(t).$$

Note that these polynomials are also palindromic by Proposition 9.2, with symmetry axis at degree $\lfloor n/2 \rfloor$. Hence $\mathbf{S}'_{n;j}(t)$ has a gamma vector, which we denote by

$$\gamma'^{(n,j)} = (\gamma_0'^{(n,j)}, \gamma_1'^{(n,j)}, \ldots, \gamma_{\lfloor n/2 \rfloor}'^{(n,j)}).$$

Note, however, that the shifted center of symmetry means we expand using the basis Γ_n for $\mathbf{S}'_{n;j}(t)$, as opposed to Γ_{n-1} for $\mathbf{S}_{n;j}(t)$.

For example,

$$\mathbf{S}_{5;1}(t) = 10t + 28t^2 + 10t^3 = 10t(1+t)^2 + 8t^2,$$

so $\gamma^{(5;1)} = (0, 10, 8)$, while

$$\mathbf{S}'_{5;1}(t) = 2t + 22t^2 + 22t^3 + 2t^4 = 2t(1+t)^3 + 16t^2(1+t),$$

so $\gamma'^{(5;1)} = (0, 2, 16)$.

Now by applying Observation 9.1 to these gamma vectors, we get the following recurrences.

Proposition 9.1. *We have the following recurrences for the* $\gamma^{(n;j)}$ *and* $\gamma'^{(n;j)}$*:*

1. *If* $j = (n+1)/2$*, then*
$$\gamma^{(n;(n+1)/2)} = \gamma'^{(n-1;1)} + \gamma'^{(n-1;2)} + \cdots + \gamma'^{(n-1;(n-1)/2)}.$$

2. *For* $j < (n+1)/2$*,*
$$\gamma^{(n;j)} = 2\sum_{k=1}^{j-1} \gamma'^{(n-1;k)} + \sum_{k=j}^{\lfloor n/2 \rfloor} \gamma^{(n-1;k)},$$
and
$$\gamma'^{(n;j)} = \sum_{k=1}^{j-1} \gamma'^{(n-1;k)} + 2\sum_{k=j}^{\lfloor n/2 \rfloor} (0, \gamma^{(n-1;k)}),$$
where for $\gamma = (\gamma_0, \gamma_1, \ldots)$*,* $(0, \gamma) = (0, \gamma_0, \gamma_1, \ldots)$*.*

Since these gamma vectors are nonnegative for small n and the recurrences are nonnegative, we get the following corollary.

Corollary 9.1. *The polynomials* $\mathbf{S}_{n;j}(t)$ *and* $\mathbf{S}'_{n;j}(t)$ *are gamma-nonnegative.*

As of this writing it is an open problem to find a combinatorial interpretation for the entries in $\gamma^{(n;j)}$ and $\gamma'^{(n;j)}$. We remark that valley-hopping clearly does not apply.

Returning now to $h(\mathrm{sd}(\Delta))$, suppose that $h(\Delta) = (h_0, h_1, \ldots, h_d)$ is nonnegative, $h_i \geq 0$, and palindromic, $h_i = h_{d-i}$. Then applying the transformation $\mathfrak{E}_d$ will give us gamma-nonnegativity.

For example, if $d = 5$ and $h(\Delta) = (h_0, h_1, h_2, h_3 = h_2, h_4 = h_1, h_5 = h_0)$, then

$$h(\mathrm{sd}(\Delta))^t = \begin{pmatrix} h_0 \\ 27h_0 + 18h_1 + 12h_2 \\ 92h_0 + 102h_1 + 108h_2 \\ 92h_0 + 102h_1 + 108h_2 \\ 27h_0 + 18h_1 + 12h_2 \\ h_0 \end{pmatrix} = h_0 \begin{pmatrix} 1 \\ 5 \\ 10 \\ 10 \\ 5 \\ 1 \end{pmatrix}$$
$$+ (22h_0 + 18h_1 + 12h_2) \begin{pmatrix} 0 \\ 1 \\ 3 \\ 3 \\ 1 \\ 0 \end{pmatrix} + (16h_0 + 48h_1 + 72h_2) \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

Equivalently,

$$h(\mathrm{sd}(\Delta); t) = h_0 \mathbf{S}_{6;1}(t) + h_1 \mathbf{S}_{6;2}(t) + h_2 \mathbf{S}_{6;3}(t),$$

or

$$\begin{aligned} \gamma(\mathrm{sd}(\Delta)) &= h_0 \gamma^{(6;1)} + h_1 \gamma^{(6;2)} + h_2 \gamma^{(6;3)}, \\ &= h_0(1, 22, 16) + h_1(0, 18, 48) + h_2(0, 12, 72). \end{aligned}$$

In general, we get the following result.

Corollary 9.2. *If Δ is a boolean complex with a palindromic h-vector $(h_0, h_1, \ldots, h_d)$, then*

$$h(\mathrm{sd}(\Delta); t) = \sum_{i=0}^{\lfloor d/2 \rfloor} h_i \mathbf{S}_{d+1;i+1}(t),$$

and

$$\gamma(\mathrm{sd}(\Delta)) = \sum_{i=0}^{\lfloor d/2 \rfloor} h_i \gamma^{(d+1;i+1)}.$$

In particular, if $h_i \geq 0$ for all i, $h(\mathrm{sd}(\Delta))$ is gamma-nonnegative.

The h-vector of a sphere is always nonnegative (this is far from obvious—see Chapter 10), and though we did not prove it, the Dehn-Sommerville relations can be applied to boolean complexes, not only simplicial complexes. Thus if Δ is a triangulated sphere, Corollary 9.2 tells us $h(\mathrm{sd}(\Delta))$ is gamma-nonnegative.

9.6 Real roots for barycentric subdivisions

Brenti and Welker asked whether $h(\mathrm{sd}(\Delta);t)$ is log-concave or real-rooted. This is not always so, but they show the following remarkable result. For any polynomial $h(t) = h_0 + h_1 t + \cdots + h_d t^d$, define the sequences of complex numbers $\{\beta_i^{(n)}\}_{n\geq 0}$ as (reciprocals of) the roots of the polynomial obtained by n applications of $\mathfrak{E}_d$ to $h(t)$:

$$\mathfrak{E}_d^n\, h(t) = \prod_{i=1}^{d}(1-\beta_i^{(n)}t).$$

So if $h(t) = h(\Delta;t)$, then $\mathfrak{E}_d^n\, h(t) = h(\mathrm{sd}^n(\Delta);t)$ is the h-polynomial of the nth barycentric subdivision of Δ.

Theorem 9.4 (Real roots). *We have the following results for real rootedness.*

1. *If $h(t) = h_0 + h_1 t + \cdots + h_d t^d$ is a nonnegative integer polynomial, then $h'(t) = \mathfrak{E}_d\, h(t)$ has only real roots.*
2. *For any $d > 1$, there are negative real numbers $\alpha_2, \ldots, \alpha_{d-1}$ such that for every $(d-1)$-dimensional boolean complex Δ, the sequence of complex roots $\beta_i^{(n)}$ associated with $h(\mathrm{sd}^n(\Delta);t)$ satisfies:*

   a. *the numbers $\beta_i^{(n)}$, $1 \leq i \leq d$, are real for n sufficiently large,*
   b. $\lim_{n\to\infty} \beta_1^{(n)} = 0$,
   c. $\lim_{n\to\infty} \beta_i^{(n)} = \alpha_i$ *for* $2 \leq i \leq d-1$,
   d. $\lim_{n\to\infty} \beta_d^{(n)} = -\infty$.

Whoa! Part (1) says that if we have any nonnegative h-polynomial, $\mathfrak{E}_d\, h(t)$ is real-rooted. If h is palindromic, then $\mathfrak{E}_d\, h(t)$ is palindromic and real rooted, which by Observation 4.2 implies that it is log-concave and gamma-nonnegative. We can prove part (1) with an interlacing argument for j-Eulerian polynomials. See Problem 9.6.

Part (2) follows from (1) by some linear algebra on $\mathfrak{E}_d$. See Problem 9.7. In short, the matrix $\mathfrak{E}_d / d!$ has largest eigenvalue 1, with multiplicity one, and we can take the corresponding eigenvector, e, to be nonnegative. Hence, there is some n such that $(\mathfrak{E}_d / d!)^n h(t)$ is close enough to $e(t)$ that all its coefficients are positive. Part (1) then says that $(\mathfrak{E}_d\,/d!)^{n+1}h(t)$ is both positive and real-rooted. Furthermore, real-rootedness holds for all subsequent applications of the transformation. We conclude $e(t)$ is real-rooted, and we call these roots $\alpha_1 = 0$, α_2, $\ldots$, α_{d-1}. The details are outlined in Problem 9.8.

One of the morals of this theorem is that while the f- and h-vectors are useful combinatorial tools, repeated barycentric subdivision "smooths out" a lot of the subtlety. All that is retained is the Euler characteristic and the dimension.

Notes

Nearly all the content in this chapter is drawn from either a 2008 paper of Francesco Brenti and Volkmar Welker [35] or a paper from 2011 by Eran Nevo, Bridget Tenner and the author [112]. Brenti and Welker's result is also studied and extended in the work of Emanuele Delucchi, Aaron Pixton, and Lucas Sabalka [56], as well as in the work of Satoshi Murai and Nevo [109].

It is worth remarking that another paper by Brenti and Welker from 2009 also involves a linear transformation of h-polynomials with a combinatorial description. See [36] and Chapter 7.

Problems

9.1. Find an example of a simplicial complex Δ for which $f(\mathrm{sd}(\Delta)) = (1, 15, 26, 12)$ and $f(\mathrm{sd}(\partial\Delta)) = (1, 11, 10)$.

9.2. Prove that the j-Eulerian polynomials, while not always palindromic, are in fact unimodal.

9.3. Define a collection of polynomials $f_1, f_2, \ldots, f_k$, to be *compatible* if every nonnegative linear combination of them,

$$c_1 f_1 + c_2 f_2 + \cdots + c_k f_k,$$

with $c_1, \ldots, c_k \geq 0$, is real-rooted. (In particular each polynomial f_i must be real-rooted.)

Prove that if $f_1, f_2, \ldots, f_k$ are pairwise compatible polynomials with positive leading coefficients, then the entire collection is compatible.

9.4. Prove that the j-Eulerian polynomials are real-rooted (and hence log-concave and unimodal).

9.5. Show that $\mathbf{S}_{n;j}(t)$ and $\mathbf{S}'_{n;j}(t)$ are real-rooted.

9.6. Prove Part 1 of Theorem 9.4.

9.7. Since $\mathfrak{B}_d$ is triangular, we can read its eigenvalues: $1, 1, 2, 6, \ldots, d!$. Since $\mathfrak{E}_d$ is similar to $\mathfrak{B}_d$, it has the same eigenvalues. Define the normalized transformations, $B_d = \mathfrak{B}_d \,/ d!$ and $E_d = \mathfrak{E}_d \,/ d!$, so that they have largest eigenvalue 1.

By the Perron-Frobenius theorem, E_d has a fixed point. Compute this fixed point for $d = 1, \ldots, 10$.

9.8. Prove the rest of Theorem 9.4. In particular, show that the fixed point $e(t)$ has only nonpositive real roots α_i as described in the theorem.