Chapter 8
Simplicial complexes

ANOTHER SETTING IN WHICH THE EULERIAN NUMBERS have arisen is in combinatorial topology. In this chapter we will put some of our previous work in the context of the study of simplicial complexes. While there is some assumed familiarity with topological concepts, no formal topological background is required for understanding this chapter.

8.1 Abstract simplicial complexes

A *simplicial complex* Δ on a vertex set V is a collection of subsets F of V, called *faces*, such that:

- if $v \in V$ then $\{v\} \in \Delta$,
- if $F \in \Delta$ and $G \subset F$, then $G \in \Delta$.

The *dimension* of a face F is $\dim F = |F| - 1$. In particular $\dim \emptyset = -1$. The dimension of the complex Δ itself, denoted by $\dim \Delta$, is the maximum of the dimensions of its faces. Maximal dimensional faces are often called *facets*.

Using the common nomenclature, vertices are zero-dimensional faces, one-dimensional faces are edges, two-dimensional faces are triangles, three-dimensional faces are tetrahedra, and so on. A k-dimensional face is called a k-simplex.

Let $\mathcal{F}_k(\Delta)$ denote the set of all k-element sets in Δ ($(k-1)$-dimensional faces), and let $\partial\mathcal{F}_k(\Delta)$ denote the *boundary* of this set, i.e., the set of all $(k-1)$-element subsets of the sets in $\mathcal{F}_k(\Delta)$. That is,

$$\partial\mathcal{F}_k(\Delta) = \bigcup_{G \in \mathcal{F}_k(\Delta)} \left\{ F \in \binom{V}{k-1} : F \subset G \right\},$$

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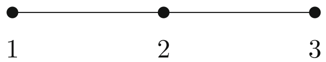
where $\binom{V}{k}$ denotes the set of all k-element subsets of the vertex set V. With this notation, the conditions for Δ to be a simplicial complex can be phrased as:

$$\partial\mathcal{F}_k(\Delta) \subseteq \mathcal{F}_{k-1}(\Delta), \qquad \text{for } k = 1, 2, \ldots, d.$$

That is, the boundary of the set of $(k-1)$-faces is contained in the set of $(k-2)$-faces. The boundary of whole complex is simply the set of all faces of Δ that are properly contained in some other face of Δ, i.e., $\partial\Delta$ is the set of all non-maximal faces of Δ.

For us, Δ is a combinatorial object, not a geometric one. However, we can construct the *geometric realization* of Δ, denoted $||\Delta||$, by creating a copy of the standard geometric k-simplex for each abstract k-simplex, and gluing faces according to inclusion of vertex sets in Δ. (The standard geometric k-simplex is the convex hull of $k+1$ standard basis vectors.) More precisely, if F and G are faces of Δ, we identify the geometric simplices $||F||$ and $||G||$ along the geometric realization of their common face: $||F \cap G||$. When we attribute a topological property to an abstract simplicial complex (e.g., Euler characteristic, homology), what we really mean is that, up to homeomorphism, the geometric realization has the property.

For example, if $V = \{1, 2, 3\}$, $\Delta = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}\}$ is a one-dimensional simplicial complex, which we can represent pictorially as:



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Similarly, the picture in Figure 8.1 encodes a two-dimensional simplicial complex. This complex has one triangle, $\{1, 3, 4\}$, six edges, $\{0, 1\}$, $\{1, 2\}$, $\{1, 3\}$, $\{1, 4\}$, $\{2, 4\}$, $\{3, 4\}$, and five vertices, $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, and $\{4\}$.

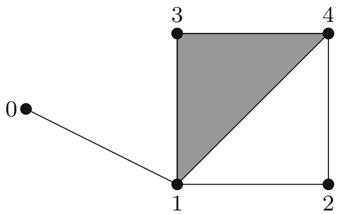
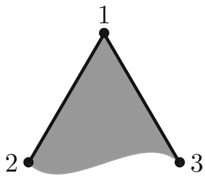


Fig. 8.1 A two-dimensional simplicial complex.

On the other hand, the set $\{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{1, 2, 3\}\}$ is *not* a simplicial complex since it is missing the edge $\{2, 3\}$. It might look like this:



Notice that part of the boundary of the 2-dimensional cell is missing from the complex.

Given a topological cell complex Δ (a bunch of open cells glued along their boundaries, i.e., a regular CW-complex) there is a quite natural partial order on its open cells given by $F \leq_{\Delta} G$ if and only if the closure of G contains the closure of F: $\overline{F} \subseteq \overline{G}$. We call this partial order the *face poset* of Δ.

In the case where Δ is a simplicial complex on vertex set V, the face poset is an order ideal in the boolean algebra 2^V defined by its maximal faces. Conversely, any order ideal of a boolean algebra defines an abstract simplicial complex, and for this reason abstract simplicial complexes are sometimes called *set systems*. For example, the face poset of the complex shown in Figure 8.1 is given in Figure 8.2.

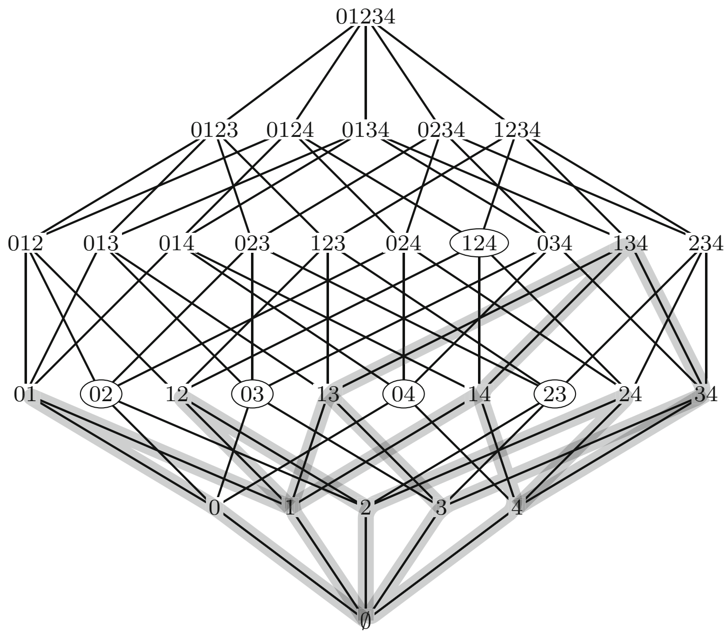


Fig. 8.2 The face poset of the complex in Figure 8.1 is the lower ideal highlighted. The minimal non-faces are circled.

Let $P(\Delta)$ denote the face poset of the complex Δ. It is self-evident that this poset is ranked by one more than dimension, which for a simplicial complex is just the cardinality of the subsets.

Observation 8.1 *The face poset of a simplicial complex Δ has the following interpretations for its rank generating function:*

$$\begin{aligned} f(P(\Delta);t) &= \sum_{F \in P(\Delta)} t^{\mathrm{rk}(F)}, \\ &= \sum_{F \in \Delta} t^{\dim F + 1}, \\ &= \sum_{F \in \Delta} t^{|F|}. \end{aligned}$$

8.2 Simple convex polytopes

Many interesting simplicial complexes arise from convex polytopes. The subject of polytopes is vast, and we will only scratch the surface here. Our purpose is only to provide a glimpse of an area in which simplicial complexes (and the tools for studying them) have natural analogues. For a more thorough treatment, see Ziegler's book [169].

There are two standard, equivalent definitions of a convex polytope. One is as the intersection of affine halfspaces (provided the intersection is bounded). The other definition is the convex hull of a finite number of points in Euclidean space. For the purposes of our discussion, take a *convex polytope* $\mathcal{P}$ to be the convex hull of a finite number of points in Euclidean space. There is a natural cell decomposition of the resulting body into vertices, edges, faces, and so on, though now k-faces need not be simplices.

First, we define the *vertices* to be those points that do not lie on a line between two other points of $\mathcal{P}$. To put it another way, any small one-dimensional neighborhood around a vertex contains points not in $\mathcal{P}$. In general, a k-dimensional face F of $\mathcal{P}$ is a maximal collection of points in $\mathcal{P}$ that are contained in a k-dimensional affine space, but for which any $(k+1)$-dimensional neighborhood of F has points outside of $\mathcal{P}$. If $\mathcal{P}$ is d-dimensional, the *facets* of $\mathcal{P}$ refer to the $(d-1)$-dimensional faces.

With this definition, the faces are closed sets, and inclusion of faces means pointwise inclusion. We can study the face poset of a polytope and count faces by dimension just as with a simplicial complex.

A *simplicial polytope* is one for which every face on the boundary of $\mathcal{P}$ is a simplex, and in this case the boundary $\partial\mathcal{P}$ is the geometric realization of a simplicial sphere.

A *simple polytope* of dimension d is one for which every vertex is adjacent to exactly d facets. In this case, the polytope is "dual" to a simplicial polytope.

That is, if $\mathcal{P}$ is a simple polytope in a vector space V there is a polytope $\mathcal{P}^*$ in the dual space V^* given by

$$\mathcal{P}^* = \{\mathbf{x} \in V^* : \langle \mathbf{x}, \mathbf{y} \rangle \leq 1 \text{ for } \mathbf{y} \in \mathcal{P}\}.$$

When $V = \mathbb{R}^d$, we have $V \cong V^*$, so we can view both polytopes as occupying the same space. The pairing between $\mathcal{P}$ and $\mathcal{P}^*$ is such that k-faces of $\mathcal{P}$ correspond to $(d-k)$-faces of $\mathcal{P}^*$. This maps $\mathcal{P}$ to the empty face, facets to vertices and so on. Since vertices of $\mathcal{P}$ are contained in d facets, this means the facets of $\mathcal{P}^*$ have d vertices, i.e., they are simplices. Thus the dual of a simple polytope is a simplicial polytope.

For example, in Figure 8.3 we see that the 3-cube is a simple polytope, but not simplicial. Its dual is the octahedron, whose boundary is a simplicial complex.

Recall that the dual of a poset $(P, \leq)$ is the poset $(P^*, \leq)$, given by $x \leq_P y$ if and only if $y \leq_{P^*} x$. Intuitively, it is the reverse of the order on P. The following is a useful fact relating the face poset of a polytope and its dual.

Proposition 8.1. *Let $P(\mathcal{P}) = (P, \leq)$ be the face poset of a polytope $\mathcal{P}$. Then $P(\mathcal{P}^*) \cong (P^*, \leq)$, that is, the face poset of dual polytope $\mathcal{P}^*$ is isomorphic to the dual poset of P.*

See Figures 8.4 and 8.5 for an illustration. Full details can be found in [169, Chapter 2].

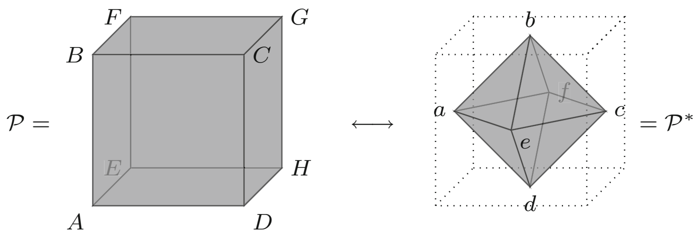


Fig. 8.3 The cube is a simple polytope. Its dual, the octahedron, is simplicial.

Other examples of simple polytopes include the permutahedra and the associahedra discussed in Chapter 5.

8.3 Boolean complexes

As mentioned, the face poset of a simplicial complex is an order ideal in 2^V. In particular, the interval $[\emptyset, F]$ in the face poset is isomorphic to 2^F, the boolean algebra on the vertices of F. A *boolean complex* is a cell complex Δ

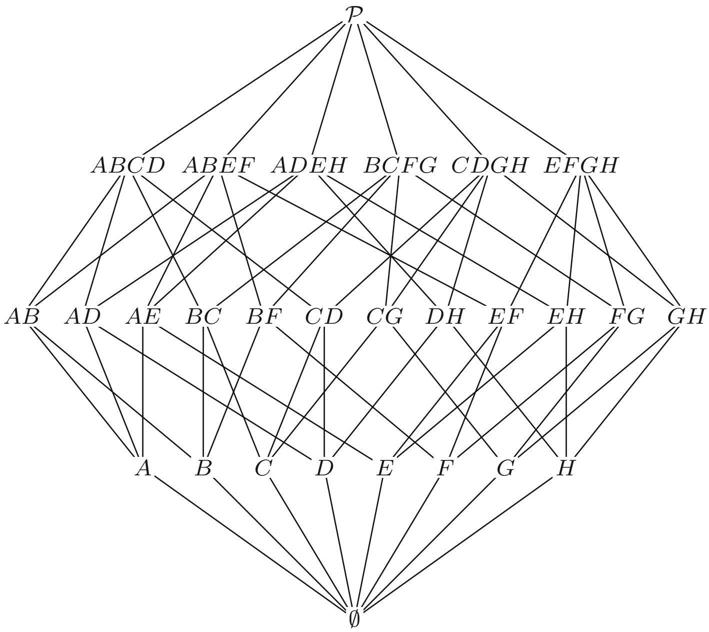


Fig. 8.4 The face poset of the cube $\mathcal{P}$ in Figure 8.3.

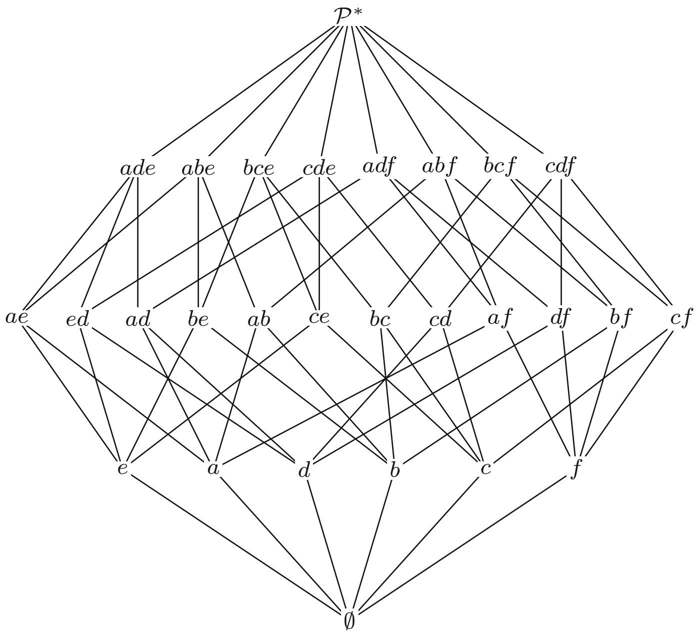


Fig. 8.5 The face poset of the octahedron $\mathcal{P}^*$ in Figure 8.3.

whose face poset requires only this weaker condition: every principal order ideal in $P(\Delta)$ is boolean. (Recall a principal order ideal is the set of elements below one particular element.) These are also sometimes called *simplicial posets*, or, because we can think of every face as a combinatorial simplex, *triangulated manifolds*.

Every simplicial complex is a boolean complex, but not conversely. Two distinct faces of a simplicial complex cannot share the same vertex set, but in a boolean complex this can happen. For example, the cell complex shown in Figure 8.6 has two triangles glued together at the corners a and b. Since there is more than one edge with vertex set $\{a, b\}$, this is not a simplicial complex. Notice that if a boolean complex is *not* simplicial, as in this example, its face poset is not a lattice, i.e., there is a collection of vertices with no least upper bound.

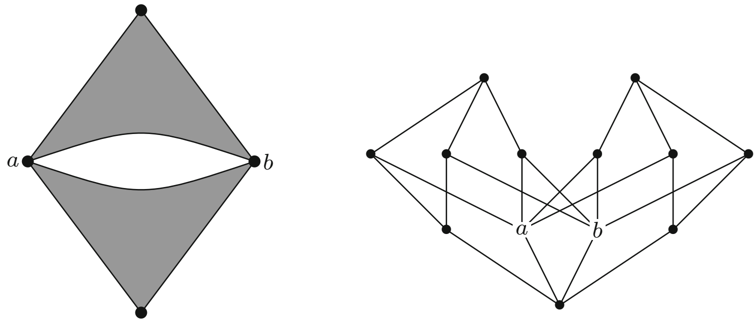


Fig. 8.6 A boolean complex and its face poset.

8.4 The order complex of a poset

Given any finite poset P, there is a natural simplicial complex associated with P called the *order complex*, denoted $\Delta(P)$. The complex $\Delta(P)$ has vertex set $V = P$, and each face of $\Delta(P)$ corresponds to a chain of elements of P:

$$a_1 <_P a_2 <_P \cdots <_P a_k \leftrightarrow F = \{a_1, a_2, \ldots, a_k\} \in \Delta(P).$$

For example, in Figure 8.7, we see a poset P and its order complex.

It is an exercise to verify that $\Delta(P)$ is indeed a simplicial complex for any finite poset P. (See Problem 8.8.) A more interesting question is to ask what complexes arise as the order complex of some poset. For example, one can show the complex shown in Figure 8.1 is *not* the order complex of any poset.

While the order complex of a poset and the face poset of a complex are not directly related, we remark that if Δ is a simplicial complex and $P = P(\Delta)$

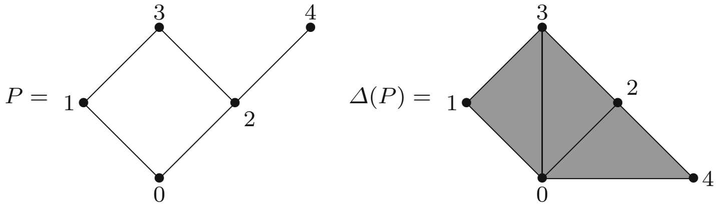


Fig. 8.7 A poset P and its order complex $\Delta(P)$.

is its face poset, then Δ and the order complex of P have the same topology (they are homeomorphic). In fact the order complex of P is known as the *barycentric subdivision* of Δ, a special construction that will be discussed further in Chapter 9.

8.5 Flag simplicial complexes

One reason why the complex of Figure 8.1 cannot be an order complex is that order complexes are part of a special family of simplicial complexes known as *flag complexes*. A flag complex is a simplicial complex whose minimal non-faces in 2^V are edges. We can see in Figure 8.2 that the simplicial complex from Figure 8.1 has the triangle $\{1, 2, 4\}$ as a minimal non-face. Hence, it is not a flag complex and cannot possibly be the order complex of a poset. We sometimes say a flag complex has no "missing faces" of dimension greater than one.

Flag complexes are completely determined by their 1-skeleton, i.e., by the graph showing only vertices and edges. Any time vertices $a_1, \ldots, a_k$ are pairwise connected in a flag complex Δ, we are guaranteed that $\{a_1, \ldots, a_k\}$ is a face of Δ. In graph theory, a collection of k pairwise connected vertices is known as a *complete graph on k vertices*, or a *k-clique*. For this reason, flag complexes are sometimes known as *clique complexes*.

To see that an order complex of a poset P is a flag complex, we first form the *comparability graph* of P, by connecting elements a and b if and only if a and b are comparable in P. Then the order complex $\Delta(P)$ is the clique complex for the comparability graph. Since the comparability graph for a chain of n elements is itself a complete graph, the order complex of an n-chain is an $(n-1)$-simplex. See Figure 8.8.

Another broad class of flag complexes are those that arise from a simplicial hyperplane arrangement. We say a linear hyperplane arrangement $\mathcal{H}$ is *simplicial* if every face gives rise to a simplex when intersected with a sphere. Rays become vertices, two-dimensional cones become edges, three-dimensional cones become triangles, and so on. See Figure 8.9. The face $(0, 0, \ldots, 0)$ at the center of the arrangement corresponds to the empty face

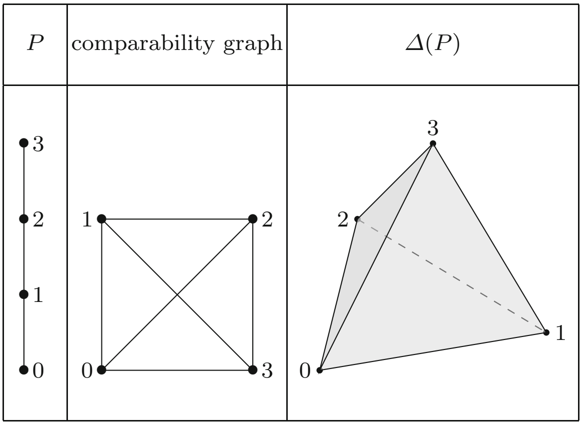


Fig. 8.8 The order complex of 4-chain is a 3-simplex.

in the simplicial complex, while the chambers in the complement of $\mathcal{H}$ correspond to facets.

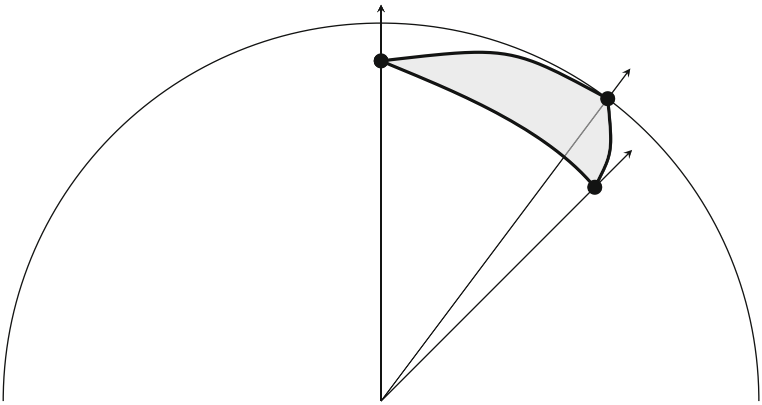


Fig. 8.9 A simplicial cone intersected with a sphere. Rays become vertices, two-dimensional cones become edges, three-dimensional cones become triangles, and so on.

Let $\Sigma = \Sigma(\mathcal{H})$ denote the cell complex obtained in this way, by intersecting $\mathcal{H}$ with a sphere. Recall we used the same notation for the poset of faces of $\mathcal{H}$ in Section 5.3—we now recognize the poset studied there as the face poset of Σ. Clearly the geometric realization of such a complex is a sphere. We want to verify that Σ is a flag simplicial complex as well.

Recall from Section 5.3 that there is a geometrically defined associative product of faces in any hyperplane arrangement, which we called the Tits product. Given two faces F and G, the product FG is the first face entered upon walking some small distance from F to G.

Proposition 5.3 states that, given any collection of faces, the faces are pairwise commuting if and only if they lie on the boundary of a common face. When applied to the rays of the hyperplane arrangement, this shows that two vertices a, b of Σ are connected with an edge if and only if they commute: $ab = ba$. If $a_1, \ldots, a_k$ are pairwise commuting vertices, then their product, taken in any order, is a face of Σ. (Indeed, Proposition 5.3 also says this product is the least upper bound for the collection $\{a_1, \ldots, a_k\}$ in the face poset.) Thus Σ is the clique complex for its one-skeleton, i.e., it is a flag complex.

Observation 8.2 *If $\mathcal{H}$ is a simplicial hyperplane arrangement, then $\Sigma(\mathcal{H})$ is a flag simplicial sphere.*

One final example we mention here is the simplicial complex dual to the associahedron. Recall from Section 5.8 that faces of the associahedron are encoded with planar rooted trees, or by a simple bijection, partial parenthesizations of a string of n symbols. The vertices of this complex are given by expressions that have only a single pair of parentheses. We say that two vertices are adjacent if and only if the pairs of parentheses are noncrossing. That is, if the positions of the parentheses from the first vertex are a and b, and the positions of the parentheses from the second vertex are c and d, then we cannot have $a < c < b < d$ or $c < a < d < b$. Every larger parenthesization can be decomposed in a natural way into a collection of mutually noncrossing vertices, and every collection of pairwise noncrossing vertices gives rise to a unique parenthesization. Thus the associahedron is the clique complex of the graph given by pairs of noncrossing vertices. See Problem 8.9.

8.6 Balanced simplicial complexes

A simplicial complex Δ with vertex set V is called *d-colorable* if there is a function $c : V \to \{1, 2, \ldots, d\}$, called a *coloring* of its vertices, such that for every face $F \in \Delta$, the restriction map $c : F \to \{1, 2, \ldots, d\}$ is one-to-one. That is, every face has distinctly colored vertices. If a $(d-1)$-dimensional complex Δ is d-colorable, we say it is a *balanced simplicial complex*.

A familiar example of a balanced simplicial complex is a bipartite graph. It is a one-dimensional complex for which two colors, say white and black, can be used to color the vertices so that every edge has one black vertex and one white vertex. See Figure 8.10. A bipartite graph can have no three pairwise connected vertices, a, b, c, since if we color a black, then one of b or c must also be black. See the graph in Figure 8.10(b). Similar reasoning

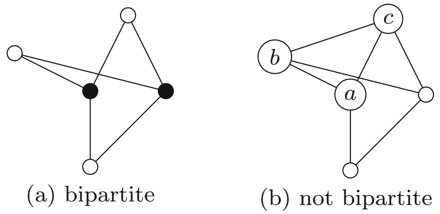


Fig. 8.10 (a) A bipartite graph and (b) a non-bipartite graph.

shows that a bipartite graph can have no cycles of odd length, and in fact this characterizes the bipartite graphs.

This fact about bipartite graphs is not important on its own, but is meant only to illustrate how special the balanced d-complexes are among all d-dimensional simplicial complexes.

As a different sort of example, we note that the order complex of a ranked poset is balanced. Indeed, if P is a ranked poset, give each element a the color $\mathrm{rk}(a)+1$. Since a chain cannot have two elements of the same rank, each face has distinctly colored vertices. A maximal chain in P corresponds to a facet of $\Delta(P)$, so the total number of colors equals the number of vertices in a maximal dimensional face. Hence when P is a ranked poset, $\Delta(P)$ is both a balanced complex and a flag complex.

Observation 8.3 *The order complex of a poset is a balanced flag complex.*

8.7 Face enumeration

The rank numbers of the face poset give an important combinatorial invariant of a simplicial complex (indeed, of any finite cell complex), which we call its *f-vector*. This vector records the number of faces of each dimension. For a simplicial complex Δ, we write

$$f(\Delta) = (f_0, f_1, \ldots),$$

with

$$f_k = |\{F \in \Delta : |F| = 1 + \dim F = k\}|.$$

The polynomial $f(P(\Delta); t) = f(\Delta; t)$ is called the *f-polynomial*, which we now write without reference to the face poset, i.e.,

$$f(\Delta; t) = \sum_{F \in \Delta} t^{|F|} = \sum_{k=0}^{1+\dim \Delta} f_k t^k.$$

So for example, with $\varDelta$ as in Figure 8.1, we have $f(\varDelta) = (1, 5, 6, 1)$ and $f(\varDelta; t) = 1 + 5t + 6t^2 + t^3$.

While the f-vector encodes purely combinatorial data, it can be used to deduce topological information. For example,

$$1 - f(\varDelta; -1) = f_1 - f_2 + f_3 - \cdots = (\text{vertices}) - (\text{edges}) + (\text{faces}) - \cdots = \chi(\varDelta),$$

is the *Euler characteristic* of $\varDelta$. For our purposes, we will find it more convenient to work with the *reduced* Euler characteristic,

$$\widetilde{\chi}(\varDelta) = -1 + \chi(\varDelta) = -f(\varDelta; -1).$$

So, for example, since an n-simplex has $f(\varDelta; t) = (1+t)^{n+1}$, and its boundary has $f(\partial\varDelta; t) = (1 + t)^{n+1} - t^{n+1}$ we have

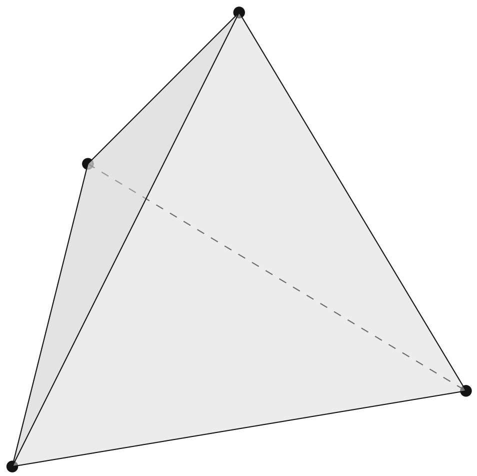
$$\widetilde{\chi}(n\text{-ball}) = 0$$

and

$$\widetilde{\chi}(n\text{-sphere}) = (-1)^n.$$

Returning to the example in Figure 8.1, we see $\widetilde{\chi}(\varDelta) = -1+5-6+1 = -1$, which we expect since $\varDelta$ can be deformed into a 1-sphere (contract the edge $\{0, 1\}$ and collapse the triangle $\{1, 3, 4\}$).

As another example, let $\varDelta$ be the boundary of the 3-simplex shown here:



Then we have $f(\varDelta) = (1, 4, 6, 4)$, and $\widetilde{\chi}(\varDelta) = -1 + 4 - 6 + 4 = 1$ since $\varDelta$ is a 2-sphere.

What characterizes an f-vector of a simplicial complex? The entries are obviously nonnegative integers, and $f_0 = 1$, but what other restrictions are there? Well, for one thing, if there are n vertices there can be at most $\binom{n}{2}$ edges, since there is at most one edge for every pair of vertices. That is,

$$f_2 \leq \binom{f_1}{2}.$$

This simple observation can be greatly generalized. It turns out there is a sharp upper bound on the number of $(k+1)$-faces expressed as a polynomial in f_k. (Likewise, there is a sharp lower bound on the number of k faces required for a given number of $(k+1)$-faces.) Collectively, these restrictions, known as the *Kruskal-Katona-Schützenberger* inequalities (or KKS inequalities), characterize the set of f-vectors of simplicial complexes. See Chapter 10. We remark that characterizing f-vectors of boolean complexes is much, much simpler. See Problem 8.7.

8.8 The *h*-vector

There is a transformation of the f-vector that can sometimes bring features of the simplicial complex into sharp focus. One way to think of this transformation is to write the f-vector in terms of right-justified copies of rows of Pascal's triangle. For example, if we consider the example of Δ shown in Figure 8.1, with $f(\Delta) = (1, 5, 6, 1)$, we have:

$$\begin{array}{r|rrrr} & (1, & 5, & 6, & 1) \\ \hline 1\times & (1, & 3, & 3, & 1) \\ 2\times & & (1, & 2, & 1) \\ -1\times & & & (1, & 1) \\ -1\times & & & & (1) \end{array}$$

The coefficients used in this expansion, read from top to bottom, make up the *h-vector* of Δ. So in this case, $h(\Delta) = (1, 2, -1, -1)$.

More generally, if Δ is $(d-1)$-dimensional (so that the f-vector is $(f_0, f_1, \ldots, f_d)$), let

$$H_d = \left[(-1)^{i+j} \binom{d-j}{i-j} \right]_{0 \leq i,j \leq d}.$$

Then we define the h-vector to be

$$h(\Delta) = H_d \cdot f(\Delta).$$

For example, with $f(\Delta) = (1, 4, 6, 4)$, we have

$$H_d = \begin{pmatrix} \binom{3}{0} & -\binom{2}{-1} & \binom{1}{-2} & -\binom{0}{-3} \\ -\binom{3}{1} & \binom{2}{0} & -\binom{1}{-1} & \binom{0}{-2} \\ \binom{3}{2} & -\binom{2}{1} & \binom{1}{0} & -\binom{0}{-1} \\ -\binom{3}{3} & \binom{2}{2} & -\binom{1}{1} & \binom{0}{0} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 3 & -2 & 1 & 0 \\ -1 & 1 & -1 & 1 \end{pmatrix},$$

and so

$$h(\Delta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 3 & -2 & 1 & 0 \\ -1 & 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 4 \\ 6 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

Notice that for any simplicial complex, $f_0 = h_0 = 1$, while

$$h_d = \sum_{i=0}^{d} (-1)^{d-i} f_i = (-1)^d f(\Delta; -1) = (-1)^d \,\widetilde{\chi}(\Delta).$$

The general formula for an entry of the h-vector is:

$$\begin{aligned} h_k &= f_k \binom{d-k}{0} - \binom{d-k+1}{1} f_{k-1} + \cdots + (-1)^i \binom{d-k+i}{i} f_{k-i} + \cdots, \\ &= \sum_{i=0}^{k} (-1)^{k-i} f_i \binom{d-i}{k-i}. \end{aligned}$$

It is easily verified that the matrix H_d has inverse

$$H_d^{-1} = \left[\binom{d-j}{i-j} \right]_{0 \le i,j \le d},$$

so there is no true loss of information when working with h-vectors instead of f-vectors, provided we know the dimension of the complex.

If we define the *h-polynomial* to be the generating function for the h-vector,

$$h(\Delta; t) = \sum_{i=0}^{d} h_i t^i,$$

we can state the linear relationship between the f-vector and the h-vector as:

$$f(\Delta; t) = \sum_{i=0}^{d} h_i t^i (1+t)^{d-i} = (1+t)^d h(\Delta; t/(1+t)), \tag{8.1}$$

and

$$h(\Delta; t) = \sum_{i=0}^{d} f_i t^i (1-t)^{d-i} = (1-t)^d f(\Delta; t/(1-t)). \tag{8.2}$$

The form of Equation (8.1) should look familiar. If $\Sigma(n)$ is the simplicial complex dual to the permutahedron (obtained by intersecting the braid arrangement with a sphere), Theorem 5.3 says

$$f(\Sigma(n); t) = (1+t)^{n-1} S_n(t/(1+t)),$$

i.e., the Eulerian polynomial is the h-polynomial of the permutahedron. Similarly, Theorem 5.4 says the Narayana polynomial is the h-polynomial of the associahedron.

8.9 The Dehn-Sommerville relations

One of the primary reasons for studying the h-vector is that it makes certain relations among the face numbers more apparent. Just as the Euler characteristic appears as the top entry in the h-vector, there are other, more subtle relationships between the entries of the f-vector that depend on the topology of Δ. The *Dehn-Sommerville relations* refer to the relations among face numbers in simplicial spheres. They were originally studied in the case of polytopes, and the idea can be applied to any triangulation of a manifold without boundary. Victor Klee called such manifolds "Eulerian." Nowadays, the face poset of such a simplicial complex is known as an *Eulerian poset*.

Define the *link* of a face $F \in \Delta$ to be

$$\mathrm{lk}(\Delta; F) = \{G \in \Delta : G \cap F = \emptyset, G \cup F \in \Delta\},$$

which we may abbreviate with $\mathrm{lk}\, F$ when Δ is understood. It turns out that if Δ triangulates a manifold, links of nonempty faces are homologous to either balls or spheres. They are balls when F lies on the boundary of Δ and spheres otherwise. In particular, if Δ triangulates a manifold without boundary (such as a sphere), then for any nonempty face F,

$$\widetilde{\chi}(\mathrm{lk}\, F) = (-1)^{\dim \mathrm{lk}\, F}.$$

We call a simplicial complex with this property an *Eulerian complex*, and throughout the rest of this section we assume Δ is an Eulerian complex. For the moment, however, we make no assumption about $\widetilde{\chi}(\mathrm{lk}\, \emptyset) = \widetilde{\chi}(\Delta)$ itself.

Now assume further that all maximal faces of Δ have the same dimension. In this case Δ is what is known as a *pure* simplicial complex. For pure complexes, the dimension of the link of a face is its codimension. That is, if Δ is $(d-1)$-dimensional, and F is a nonempty face of Δ,

$$\dim(\mathrm{lk}\, F) = \dim \Delta - \dim F,$$

and therefore,

$$\widetilde{\chi}(\mathrm{lk}\, F) = (-1)^{d-|F|}.$$

Now for any nonempty face F of Δ, define

$$\phi(F) = \sum_{F \subseteq G \in \Delta} (-1)^{|G|}.$$

Then letting $H = G - F$ denote the complement of F in G, we have $H \cap F = \emptyset$ and $H \cup F = G \in \Delta$, i.e., $H \in \operatorname{lk} F$. Conversely, if $H \in \operatorname{lk} F$, then $F \subseteq H \cup F = G$ is a face of Δ containing F. In other words, we have:

$$\begin{aligned}
\phi(F) &= \sum_{H \in \operatorname{lk} F} (-1)^{|F|+|H|}, \\
&= (-1)^{|F|} \sum_{H \in \operatorname{lk} F} (-1)^{|H|}, \\
&= (-1)^{|F|} f(\operatorname{lk} F; -1), \\
&= (-1)^{|F|} (-\widetilde{\chi}(\operatorname{lk} F)), \\
&= (-1)^{|F|} (-1)^{d-1-|F|}, \\
&= (-1)^{d-1}.
\end{aligned}$$

So if Δ is a pure Eulerian complex, $\phi(F)$ is constant for every nonempty face!

Now if we sum $\phi(F)$ over all faces of cardinality k, we get:

$$\begin{aligned}
(-1)^{d-1} f_k &= \sum_{|F|=k} \phi(F), \\
&= \sum_{|F|=k} \sum_{F \subseteq G \in \Delta} (-1)^{|G|}, \\
&= \sum_{\substack{G \in \Delta \\ k \le |G| \le d}} \sum_{\substack{F \subseteq G \\ |F|=k}} (-1)^{|G|}, \\
&= \sum_{\substack{G \in \Delta \\ k \le |G| \le d}} (-1)^{|G|} \binom{|G|}{k}, \\
&= \sum_{k \le i \le d} (-1)^i f_i \binom{i}{k}.
\end{aligned}$$

This is the first version of the Dehn-Sommerville relations.

Theorem 8.1 (Dehn-Sommerville, f-version). *For any pure Eulerian complex Δ with $f(\Delta) = (1, f_1, \ldots, f_d)$, we have, for each $k \ge 1$,*

$$(-1)^{d-1} f_k = \sum_{k \le i \le d} (-1)^i f_i \binom{i}{k}. \tag{8.3}$$

For example, with $d = 4$ we have:

$$\begin{aligned}
f_1 &= -f_1\tbinom{1}{1} + f_2\tbinom{2}{1} - f_3\tbinom{3}{1} + f_4\tbinom{4}{1}, \\
f_2 &= f_2\tbinom{2}{2} - f_3\tbinom{3}{2} + f_4\tbinom{4}{2}, \\
f_3 &= -f_3\tbinom{3}{3} + f_4\tbinom{4}{3}, \\
f_4 &= f_4\tbinom{4}{4}.
\end{aligned}$$

and with $d = 5$ we get:

$$\begin{aligned}
f_1 &= f_1\tbinom{1}{1} - f_2\tbinom{2}{1} + f_3\tbinom{3}{1} - f_4\tbinom{4}{1} + f_5\tbinom{5}{1},\\
f_2 &= \qquad\quad -f_2\tbinom{2}{2} + f_3\tbinom{3}{2} - f_4\tbinom{4}{2} + f_5\tbinom{5}{2},\\
f_3 &= \qquad\qquad\qquad\quad f_3\tbinom{3}{3} - f_4\tbinom{4}{3} + f_5\tbinom{5}{3},\\
f_4 &= \qquad\qquad\qquad\qquad\qquad -f_4\tbinom{4}{4} + f_5\tbinom{5}{4},\\
f_5 &= \qquad\qquad\qquad\qquad\qquad\qquad\qquad f_5\tbinom{5}{5}.
\end{aligned} \tag{8.4}$$

In terms of linear algebra, we are saying the f-vector is in the fixed point space of a certain linear transformation T, e.g., for $d = 4$ and $d = 5$, these transformations are

$$\begin{bmatrix} -1 & 2 & -3 & 4 \\ 0 & 1 & -3 & 6 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & -2 & 3 & -4 & 5 \\ 0 & -1 & 3 & -6 & 10 \\ 0 & 0 & 1 & -4 & 10 \\ 0 & 0 & 0 & -1 & 5 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

respectively. Computing the dimension of the fixed point space of T boils down to computing the rank of $T - I$, and the alternating ± 1 on the diagonal mean that $T - I$ has rank $\lfloor d/2 \rfloor$, e.g., for $d = 4$ and $d = 5$, $T - I$ looks like:

$$\begin{bmatrix} -2 & 2 & -3 & 4 \\ 0 & 0 & -3 & 6 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & -2 & 3 & -4 & 5 \\ 0 & -2 & 3 & -6 & 10 \\ 0 & 0 & 0 & -4 & 10 \\ 0 & 0 & 0 & -2 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

which both have rank 2.

So the f-vectors of Eulerian simplicial complexes live in a vector space of roughly half the dimension of Δ. Is there is a change of basis of the f-vector that allows us to see this fact? The answer is an emphatic "Yes!" and in the remainder of this chapter we will describe ways to do so.

As a first step, let's add a row with $\widetilde{\chi}(\Delta)$ to the top of the system of equations given by transformation T, and multiply the ith equation by $(-1)^{d-1}t^i$. With $d = 5$ this is:

$$\begin{aligned}
\widetilde{\chi}(\Delta) &= -f_0\tbinom{0}{0} + f_1\tbinom{1}{0} - f_2\tbinom{2}{0} + f_3\tbinom{3}{0} - f_4\tbinom{4}{0} + f_5\tbinom{5}{0},\\
f_1 \cdot t &= \qquad\quad \left(f_1\tbinom{1}{1} - f_2\tbinom{2}{1} + f_3\tbinom{3}{1} - f_4\tbinom{4}{1} + f_5\tbinom{5}{1}\right) t,\\
f_2 \cdot t^2 &= \qquad\qquad\quad \left(-f_2\tbinom{2}{2} + f_3\tbinom{3}{2} - f_4\tbinom{4}{2} + f_5\tbinom{5}{2}\right) t^2,\\
f_3 \cdot t^3 &= \qquad\qquad\qquad\qquad \left(f_3\tbinom{3}{3} - f_4\tbinom{4}{3} + f_5\tbinom{5}{3}\right) t^3,\\
f_4 \cdot t^4 &= \qquad\qquad\qquad\qquad\qquad \left(-f_4\tbinom{4}{4} + f_5\tbinom{5}{4}\right) t^4,\\
f_5 \cdot t^5 &= \qquad\qquad\qquad\qquad\qquad\qquad\qquad f_5\tbinom{5}{5}\, t^5.
\end{aligned}$$

Summing both sides, with the right-hand side taken column-wise, we find:

$$
\begin{aligned}
(-1)^{d-1}(f(\Delta;t)-1)+\widetilde{\chi}(\Delta) &= -f_0+f_1(1+t)-f_2(1+t)^2+f_3(1+t)^3-\cdots, \\
&= -f(\Delta;-(1+t)),
\end{aligned}
$$

or

$$
(-1)^d f(\Delta;t)-f(\Delta;-(1+t))=(-1)^d+\widetilde{\chi}(\Delta). \tag{8.5}
$$

Now we can use the transformation in Equation (8.1) for expressing the h-vector in terms of the f-vector, i.e.,

$$
f(\Delta;s)=(1+s)^d h(\Delta;s/(1+s)).
$$

Take $s=t$ and $s=-(1+t)$ respectively on the left-hand side of (8.5) to rewrite it as:

$$
(-1)^d(1+t)^d h(\Delta;t/(1+t))-(-t)^d h(\Delta;(1+t)/t)=(-1)^d+\widetilde{\chi}(\Delta),
$$

or upon multiplying both sides by $(-1)^d$,

$$
(1+t)^d h(\Delta;t/(1+t))-t^d h(\Delta;(1+t)/t)=1+(-1)^d\,\widetilde{\chi}(\Delta). \tag{8.6}
$$

Now let $x=t/(1+t)$, so that $t=x/(1-x)$. Then dividing both sides of equation (8.6) by $(1+t)^d=(t/x)^d$, i.e., multiplying by $(x/t)^d=(1-x)^d$, we get:

$$
h(\Delta;x)-x^d h(\Delta;1/x)=(1-x)^d(1+(-1)^d\,\widetilde{\chi}(\Delta)).
$$

By comparing coefficients on the left and right, we have the h-version of the Dehn-Sommerville relations.

Theorem 8.2 (Dehn-Sommerville, h-version). *Suppose Δ is a pure Eulerian complex of dimension $d-1$, with $h(\Delta)=(1,h_1,\ldots,h_d)$. Then for each $k\geq 0$,*

$$
h_k-h_{d-k}=(-1)^k\binom{d}{k}(1+(-1)^d\,\widetilde{\chi}(\Delta)).
$$

In particular, if $\widetilde{\chi}(\Delta)=(-1)^{d-1}$, the h-vector is palindromic:

$$
h_k=h_{d-k}. \tag{8.7}
$$

This is point for us. If Δ is a sphere, or any triangulated manifold with the same Euler characteristic, the h-vector is palindromic. Hence the space of h-vectors of spheres is clearly $\lfloor d/2\rfloor$-dimensional, and since the transformation $f\leftrightarrow h$ is invertible, so is the space of f-vectors of spheres.

The Dehn-Sommerville relations give a very sophisticated reason why the Eulerian numbers and Narayana numbers are palindromic: because they are the entries of the h-vector of a sphere!

There are many interesting results and open questions regarding the characterization of h-vectors of spheres, some of which are discussed further in Chapter 10.

Notes

Simplicial decomposition of topological spaces is a standard idea in algebraic topology. See Allen Hatcher's textbook for more from the point of view of topologists [88]. Two classic textbooks on polytopes include one by Branko Grünbaum [84] and another by Günter Ziegler [169].

Much of the work on connections between posets and simplicial complexes was pioneered by Richard Stanley in the 1970s and 1980s, e.g., [144–147]. See also work of Anders Björner [23], and [21] in which he connects poset theory to general CW-complexes. Chapter 4 of Stanley's textbook [154] discusses many of these results and more. Flag complexes arise naturally in graph theory, and they are of particular interest in the context of the *Charney-Davis conjecture*, stated by Ruth Charney and Mike Davis in their 1995 paper [48] and discussed further in Chapter 10.

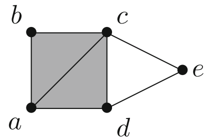
The f-vectors of abstract simplicial complexes admit a complete characterization known as the Kruskal-Katona-Schützenberger inequalities, given in Chapter 10. These are due to, independently Joseph Kruskal in 1963 [98] and Gyula Katona in 1966 [94]. We attach the name of Marcel-Paul Schützenberger because in 1959 he too described the inequalities in a technical report for MIT's Research Laboratory of Electronics [134]. However the note in which it appears is both hard to find and rather skimpy on details. Most people know of these inequalities simply as the "Kruskal-Katona" inequalities.

In [151] Stanley characterizes the f-vectors of Boolean complexes. There is also a characterization of the f-vector of a balanced simplicial complex, due to Peter Frankl, Zoltán Füredi, and Gil Kalai [75]. See Chapter 10.

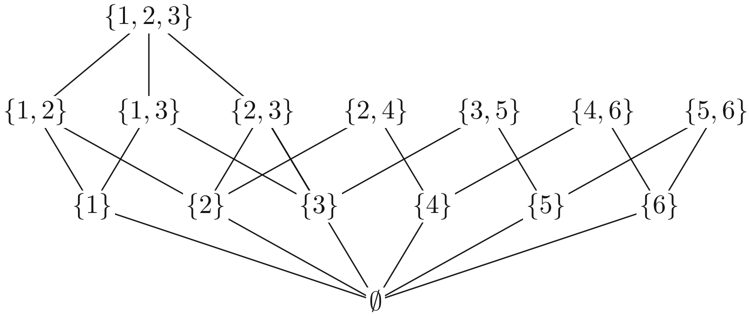
The Dehn-Sommerville relations were first stated in low dimensions by Max Dehn [55], and in 1927 by Duncan Sommerville [143]. The proof we give here is adapted from Victor Klee's 1964 paper [95]. Stanley generalizes the argument in Chapter 4 of [154] as well, where it can be phrased in terms of the Möbius function of the face poset.

Problems

8.1. 1. Draw the Hasse diagram for the face poset of the following simplicial complex:

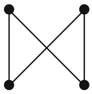
$\Delta =$  .

2. Draw a geometric realization of the abstract simplicial complex whose Hasse diagram is below:



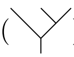
.

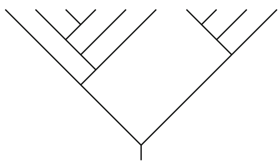
8.2. Prove that the face poset of a simplicial complex can have no "bowties" in its Hasse diagram:



i.e., no quadruple of faces, F_1, F_2, G_1, G_2, with $\dim F_1 = \dim F_2 = \dim G_1 - 1 = \dim G_2 - 1$, and such that $F_1 \subset G_1$, $F_1 \subset G_2$, $F_2 \subset G_1$, and $F_2 \subset G_2$.

8.3. Show that if a polytope $\mathcal{P}$ is a d-simplex, so is its dual, $\mathcal{P}^*$.

8.4. Recall that $\mathrm{PB}(n)$ denotes the set of planar binary trees with n internal nodes. Label the leaves from left to right by $0, 1, \ldots, n$, and then label the internal nodes $1, 2, \ldots, n$ so that node i is the one that falls between leaf $i-1$ and leaf i. Let l_i denote the number of leaves on the left branch of node i and let r_i denote the number of leaves on the right branch of node i. Let $v_i = l_i r_i$ denote the product of these two numbers, and let $v(\tau) = (v_1, \ldots, v_n)$. For example, $v($$) = (1)$, $v($$) = (2, 1)$, and if

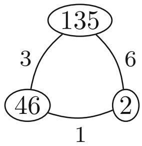
$\tau =$ 

then $v(\tau) = (5, 2, 1, 3, 4, 24, 1, 2, 3)$.

Show that the convex hull of the points $v(\tau)$, as τ runs over all planar binary trees in $\mathrm{PB}(n)$, is a geometric realization of the associahedron, whose face poset was described combinatorially in Section 5.8.

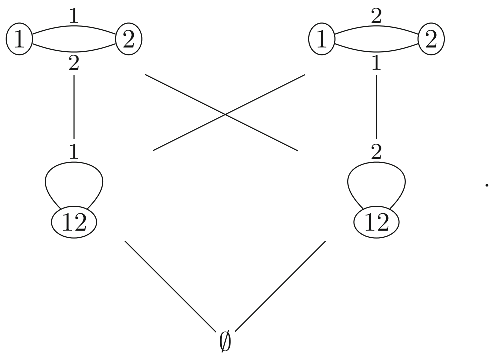
8.5. Show that the permutahedron and associahedron are simple polytopes.

8.6. A *spin necklace* is a cyclically ordered set partition of $\{1, 2, \ldots, n\}$ (drawn clockwise in a circle) together with a labeling of the edges between the blocks that respects block sizes (modulo n). That is, the difference between the edge labels on either side of a block must differ by the cardinality of the block. For example,



is a spin necklace on $\{1, 2, 3, 4, 5, 6\}$.

Let $\Sigma_T(n)$ denote the set of spin necklaces on $\{1, 2, \ldots, n\}$, together with the empty set. We partially order $\Sigma_T(n)$ by declaring that $\emptyset$ is a unique minimal element and two spin necklaces satisfy $F \leq_{\Sigma_T} G$ if and only if G is a refinement of F. For example, here is the Hasse diagram for $\Sigma_T(2)$:



Note that there are $n!$ maximal elements, corresponding to permutations, and n rank one elements (vertices), corresponding to the single block $\{1, 2, \ldots, n\}$, with a "handle" labeled by some $i = 1, 2, \ldots, n$. Show that $\Sigma_T(n)$, is a boolean complex, i.e., simplicial poset, but not a simplicial complex.

8.7. Show that $f = (f_0, f_1, \ldots, f_d)$ is the f-vector of a $(d-1)$-dimensional boolean complex (simplicial poset) if and only if $f_0 = 1$ and $f_i \geq \binom{d}{i}$ for each $i > 0$.

8.8. Show that the order complex of a finite poset is a simplicial complex.

8.9. Verify that the simplicial complex dual to the associahedron is a flag complex.

8.10. Let $\Sigma(n)$ denote the simplicial complex for the braid arrangement. Show that $\Sigma(n)$ is balanced.

8.11. Show that the f-polynomial of Δ is real-rooted if and only if its h-polynomial is real-rooted.