Chapter 4
Gamma-nonnegativity

THE BINOMIAL DISTRIBUTION is the first probability distribution a student encounters. Among its many properties is the fact that it is palindromic and unimodal. Many combinatorial distributions, including the Eulerian and Narayana distributions, can be built out of copies of binomial distributions that are shifted to have the same center of symmetry, and this fact has many interesting consequences.

4.1 The idea of gamma-nonnegativity

We can observe that, for fixed n, the sequence of Eulerian numbers, $\left\langle {n \atop k} \right\rangle$ is *palindromic*,

$$\left\langle {n \atop k} \right\rangle = \left\langle {n \atop n-1-k} \right\rangle, \tag{4.1}$$

and *unimodal*:

$$\left\langle {n \atop 0} \right\rangle \leq \left\langle {n \atop 1} \right\rangle \leq \cdots \leq \left\langle {n \atop \lfloor (n-1)/2 \rfloor} \right\rangle \geq \cdots \geq \left\langle {n \atop n-1} \right\rangle.$$

When there is no possibility for confusion, we will call a polynomial palindromic or unimodal if its sequence of coefficients has the same property. So we say the Eulerian polynomial $S_n(t)$ is palindromic and unimodal.

The palindromicity is easy to explain combinatorially, as reversal of a permutation swaps descents and ascents. This gives a bijection between the set of permutations with k descents and permutations with k ascents, and hence $n-1-k$ descents.

Unimodality is trickier, but both these properties follow from a property that will be a major theme later in the book, called *gamma-nonnegativity*, which we now explain.

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First, observe that the sequence of binomial coefficients $\binom{n}{k}$, with n fixed, is palindromic and unimodal.

Loosely speaking, gamma-nonnegativity means a sequence of numbers can be written as a sum of rows of Pascal's triangle with the same center of symmetry. For example, rows 5 and 6 of the Eulerian triangle (Table 1.3) can be written as follows.

$$
n = 5: \qquad
\begin{array}{rccccc}
 & 1 & 26 & 66 & 26 & 1 \\ \hline
1\times & (1 & 4 & 6 & 4 & 1) \\
22\times & & (1 & 2 & 1) & \\
16\times & & & (1) & &
\end{array}
\qquad\qquad
n = 6: \qquad
\begin{array}{rcccccc}
 & 1 & 57 & 302 & 302 & 57 & 1 \\ \hline
1\times & (1 & 5 & 10 & 10 & 5 & 1) \\
52\times & & (1 & 3 & 3 & 1) & \\
136\times & & & (1 & 1) & &
\end{array}
$$

In terms of generating functions, gamma-nonnegativity means a polynomial of degree n can be written as a sum of polynomials of the form $t^j(1+t)^{n-2j}$. In the case of the Eulerian polynomials for $n = 5$ and $n = 6$ we have

$$
\begin{aligned}
S_5(t) &= (1+t)^4 + 22t(1+t)^2 + 16t^2, \\
S_6(t) &= (1+t)^5 + 52t(1+t)^3 + 136t^2(1+t).
\end{aligned}
$$

The coefficients in expansions like the ones above make up what we call the *gamma vector*. When these coefficients are nonnegative, we say the polynomial itself is *gamma-nonnegative*.

4.2 Gamma-nonnegativity for Eulerian numbers

In this section we show the Eulerian polynomials are gamma-nonnegative, a result first due to Foata and Schützenberger.

Theorem 4.1. *For any $n > 0$, there exist nonnegative integers $\gamma_{n,j}$ such that*

$$
S_n(t) = \sum_{j=0}^{\lfloor (n-1)/2 \rfloor} \gamma_{n,j} t^j (1+t)^{n-1-2j}, \tag{4.2}
$$

i.e., the Eulerian polynomials are gamma-nonnegative.

We list the entries in the gamma vectors for the Eulerian polynomials in Table 4.1.

There is a beautiful combinatorial proof of Theorem 4.1 given by Foata and Strehl, based on an action we call "valley hopping" as illustrated in Figure 4.1. Here we draw a permutation as a "mountain range," so that peaks and valleys form the upper and lower limits of the decreasing runs. By convention, we have points at infinity on the far left and far right.

Table 4.1 Entries of the gamma vector for the Eulerian polynomials, $\gamma_{n,j}$, $0 \le 2j < n \le 10$.

$n\backslash j$	0	1	2	3	4
1	1				
2	1				
3	1	2			
4	1	8			
5	1	22	16		
6	1	52	136		
7	1	114	720	272	
8	1	240	3072	3968	
9	1	494	11616	34304	7936
10	1	1004	40776	230144	176896

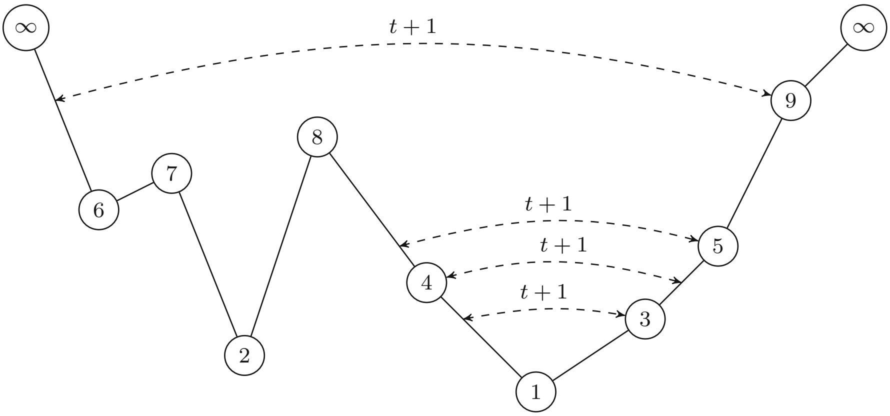


Fig. 4.1 The mountain range view of the permutation $w = 862741359$.

Formally, given $w = w(1)\cdots w(n) \in S_n$, we say a letter $w(i)$ is a *peak* if $w(i-1) < w(i) > w(i+1)$ and it is a *valley* if $w(i-1) > w(i) < w(i+1)$. Otherwise we say $w(i)$ is *free*. Using the convention that $w(0) = w(n{+}1) = \infty$, we see that w cannot begin or end with a peak.

We partition S_n into equivalence classes according to the following action on free letters. If $w(i) = j$ is free, then $H_j(w)$ denotes the permutation obtained by moving j directly across the adjacent valley(s) to the nearest mountain slope of the same height. More precisely, we have the following.

- If $w(i) = j$ lies on a downslope, i.e., $w(i-1) > w(i) > w(i+1)$, we find the smallest $k > i$ such that $w(k) < j < w(k+1)$, and

$$H_j(w) = w(1)\cdots w(i-1)w(i+1)\cdots w(k)\, j\, w(k+1)\cdots w(n),$$

- If $w(i) = j$ lies on an upslope, i.e., $w(i-1) < w(i) < w(i+1)$, we find the largest $k < i$ such that $w(k-1) > j > w(k)$, and

$$H_j(w) = w(1) \cdots w(k-1)\, j\, w(k) \cdots w(i-1)w(i+1) \cdots w(n).$$

Clearly, if j, l are free letters, $H_j^2(w) = H_l^2(w) = w$ and $H_j(H_l(w)) = H_l(H_j(w))$. Thus, for any collection of free letters $J = \{j_1, \ldots, j_k\}$, we can define the operation $H_J(w) = H_{j_1} \cdots H_{j_k}(w)$. Also, observe that $H_J(w)$ has the same set of free letters as w.

Let $\mathrm{Hop}(w)$ denote the hop-equivalence class of w. Notice that every peak of w is necessarily the larger element of a descent, for any $u \in \mathrm{Hop}(w)$, while a valley is never the larger element of a descent. If a free letter lies on an upslope of u it is not part of a descent, while if it is on a downslope it is the larger element of a descent of u. Moreover, this property is independent of the positions of the other free letters. If w has r peaks, it has $r+1$ valleys, and hence $n-1-2r$ free letters. Thus, letting $\mathrm{pk}(w)$ denote the number of peaks of w, we have:

$$\sum_{u \in \mathrm{Hop}(w)} t^{\mathrm{des}(u)} = t^{\mathrm{pk}(w)}(1+t)^{n-1-2\,\mathrm{pk}(w)}. \qquad (4.3)$$

We can choose a canonical representative for each hop-equivalence class by choosing to put each free letter on an upslope. These are precisely the permutations for which $\mathrm{pk}(w) = \mathrm{des}(w)$. We denote this set of representatives by:

$$\widehat{S}_n = \{w \in S_n : \mathrm{pk}(w) = \mathrm{des}(w)\}.$$

Thus by summing (4.3) over all $w \in \widehat{S}_n$, we get:

$$S_n(t) = \sum_{w \in \widehat{S}_n} t^{\mathrm{pk}(w)}(1+t)^{n-1-2\,\mathrm{pk}(w)}.$$

Moreover, we can now give a combinatorial interpretation to the numbers in Table 4.1.

Corollary 4.1. *For any n, j,*

$$\gamma_{n,j} = |\{w \in \widehat{S}_n : \mathrm{des}(w) = j\}|.$$

With this interpretation in hand, it is not difficult to relate the Eulerian polynomials to the generating function for the peak statistic. That is, define the *peak polynomials* $P_n(t)$ and *peak numbers* $p_{n,k}$ as follows:

$$P_n(t) = \sum_{w \in S_n} t^{\mathrm{pk}(w)} = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} p_{n,k} t^k.$$

Then we have

$$S_n(t) = \frac{(1+t)^{n-1}}{2^{n-1}} P_n\left(\frac{4t}{(1+t)^2}\right). \tag{4.4}$$

Some peak numbers are included in Table 4.2.

Table 4.2 The peak numbers, $p_{n,k}$, $0 \le 2k < n \le 10$.

$n \backslash k$	0	1	2	3	4
1	1				
2	2				
3	4	2			
4	8	16			
5	16	88	16		
6	32	416	272		
7	64	1824	2880	272	
8	128	7680	24576	7936	
9	256	31616	185856	137216	7936
10	512	128512	1304832	1841152	353792

Another consequence of Corollary 4.1 is seen when we specialize $t = -1$ in the Eulerian polynomial:

$$\begin{aligned} S_n(-1) &= \sum_{w \in \widehat{S}_n} (-1)^{\mathrm{pk}(w)} (1-1)^{n-1-2\,\mathrm{pk}(w)} \\ &= \begin{cases} (-1)^{(n-1)/2} \gamma_{n,(n-1)/2} & \text{if } n \text{ odd}, \\ 0 & \text{if } n \text{ even}. \end{cases} \end{aligned}$$

But $\gamma_{n,(n-1)/2}$ (with n odd) is the number of permutations w such that

$$w(1) < w(2) > w(3) < \cdots > w(2i-1) < w(2i) > w(2i+1) < \cdots.$$

These are known as *up-down alternating permutations* and the number of such permutations is known as the *Euler number*, denoted E_n. This definition makes sense for both even and odd values of n, and the sequence of Euler numbers begins:

$$1, 1, 2, 5, 16, 61, 272, 1385, 7936, 50521, \ldots. \tag{4.5}$$

Taking the limit as $t \to -1$ in Theorem 1.6, we get an expression for the exponential generating function for the odd-indexed Euler numbers, with alternating plus and minus signs:

$$\begin{aligned} S(-1,z)-1 &= z - 2\frac{z^3}{3!} + 16\frac{z^5}{5!} - \cdots, \\ &= \sum_{k\geq 0}(-1)^k E_{2k+1}\frac{z^{2k+1}}{(2k+1)!}, \\ &= \frac{1-e^{-2z}}{1+e^{-2z}} = \tanh z. \end{aligned}$$

The sequence $1, 2, 16, 272, 7936, \ldots$ is also known as the sequence of *tangent* numbers. Problem 4.2 investigates other properties of Euler numbers.

4.3 Gamma-nonnegativity for Narayana numbers

We will now show the Narayana polynomials $C_n(t)$ are gamma-nonnegative. Hence, the sequence of Narayana numbers $N_{n,k}$, for fixed n, is symmetric and unimodal. The reason for this is quite simple: Foata and Strehl's valley-hopping action described in Section 4.2 preserves the pattern 231. Hence, if $w \in S_n(231)$, the hop-equivalence class $\mathrm{Hop}(w)$ is composed entirely of permutations avoiding 231.

Let's make this argument rather more precise. Suppose $w \notin S_n(231)$, so that there is a triple of indices $i < j < k$ with $w(k) < w(i) < w(j)$. Then without loss of generality, we may assume $w(j)$ is a peak. (Otherwise, there is a peak $w(j')$ with $i < j' < j$ and $w(j') > w(j)$.) If neither $w(i)$ nor $w(k)$ are free letters, then clearly all members of $\mathrm{Hop}(w)$ contain 231. But even if $w(i)$ or $w(k)$ are free, the relative position of the letters $w(i), w(j), w(k)$ is preserved, since neither $w(i)$ nor $w(k)$ can hop past $w(j)$. See Figure 4.2 for an illustration.

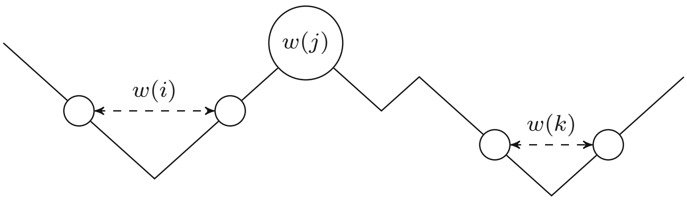


Fig. 4.2 Valley-hopping preserves the pattern 231.

Thus we have the following.

Theorem 4.2. *For any $n > 0$, there exist nonnegative integers $\widehat{\gamma}_{n,j}$ such that*

$$C_n(t) = \sum_{j=0}^{\lfloor n/2 \rfloor} \widehat{\gamma}_{n,j} t^j (1+t)^{n-1-2j}, \tag{4.6}$$

i.e., the Narayana polynomials are gamma-nonnegative.

Moreover, each hop class Hop(w) still has a unique representative for which $\mathrm{des}(w) = \mathrm{pk}(w)$, and so:

$$\widehat{\gamma}_{n,j} = |\{w \in S_n(231) : \mathrm{des}(w) = \mathrm{pk}(w) = j\}|.$$

These numbers are listed in Table 4.3.

Table 4.3 The gamma numbers $\widehat{\gamma}_{n,k}$ for the Narayana distribution, $0 \leq 2k < n \leq 10$.

$n\backslash k$	0	1	2	3	4
1	1				
2	1				
3	1	1			
4	1	3			
5	1	6	2		
6	1	10	10		
7	1	15	30	5	
8	1	21	70	35	
9	1	28	140	140	14
10	1	36	252	420	126

Of course, there is a similar connection with the peak generating function for all 231-avoiding permutations. Let

$$P_n(231;t) = \sum_{w \in S_n(231)} t^{\mathrm{pk}(w)}.$$

Then we have

$$C_n(t) = \frac{(1+t)^{n-1}}{2^{n-1}} P_n\left(231; \frac{4t}{(1+t)^2}\right). \tag{4.7}$$

For reference we include in Table 4.4 the peak numbers for 231-avoiding permutations.

4.4 Palindromicity, unimodality, and the gamma basis

We will now lay out the general definition and elementary consequences of gamma-nonnegativity.

We say a polynomial $h(t)$ is *palindromic* if its coefficients are the same when read from left to right as from right to left. To be more precise, we say h is *palindromic for n* if $h(t) = t^n h(1/t)$. Such an n is the sum of the

Table 4.4 The number of 231-avoiding permutations in S_n with k of peaks, $0 \leq 2k < n \leq 10$.

$n\backslash k$	0	1	2	3	4
1	1				
2	2				
3	4	1			
4	8	6			
5	16	24	2		
6	32	80	20		
7	64	240	120	5	
8	128	672	560	70	
9	256	1792	2240	560	14
10	512	4608	8064	3360	252

highest and lowest degrees of nonzero terms in h. In the simplest case, h has a nonzero constant term, so n is the degree of h. Here, writing

$$h(t) = h_0 + h_1 t + \cdots + h_n t^n,$$

we have $h_i = h_{n-i}$ for all i. If h has no constant term, n is greater than the degree of h, e.g., $h(t) = t^2 + t^3$ is palindromic for $n = 5$.[1]

We say a polynomial is *unimodal* if its coefficients weakly increase then weakly decrease, i.e., there is some k for which

$$h_0 \leq h_1 \leq \cdots \leq h_k \geq h_{k+1} \geq \cdots \geq h_n.$$

If $h(t)$ is palindromic for n, unimodality means that $h_0 \leq h_1 \leq \cdots \leq h_{\lfloor n/2 \rfloor}$.

As a vector space, the set of polynomials palindromic for n has dimension $\lfloor n/2 \rfloor + 1$. One natural basis for this vector space is

$$\Sigma_n = \begin{cases} \{t^j + t^{n-j}\}_{0 \leq j < n/2} & \text{if } n \text{ is odd,} \\ \{t^j + t^{n-j}\}_{0 \leq j < n/2} \cup \{t^{n/2}\} & \text{if } n \text{ is even.} \end{cases}$$

While Σ_n might be the standard basis for polynomials palindromic for n, we will now discuss a more interesting basis that we call the "gamma basis," defined as follows:

$$\Gamma_n = \{t^j (1+t)^{n-2j}\}_{0 \leq j \leq n/2}.$$

Notice that every member of Γ_n is palindromic and unimodal with the same center of symmetry at $n/2$. Hence the nonnegative span of Γ_n contains only palindromic and unimodal polynomials.

[1] In the literature the term "symmetric" is sometimes used to describe what we mean by "palindromic." This is okay in some circumstances, but there is a more common notion of "symmetric polynomial"—namely a polynomial that is fixed under permutation of its variables—so we prefer the less ambiguous term. George Andrews used another synonym for palindromic, "reciprocal polynomial," in [8] and [9].

If $h(t)$ is palindromic for n, the sequence of its coefficients in Γ_n is called the *gamma vector* of h, and the *gamma polynomial* $\gamma(h;t)$ is the generating function for the gamma vector. We have

$$h(t) = (1+t)^n \gamma(h; t/(1+t)^2) = \sum_{0 \le j \le n/2} \gamma_j t^j (1+t)^{n-2j}. \tag{4.8}$$

We say $h(t)$ is *gamma-nonnegative* if $\gamma(h;t)$ has nonnegative coefficients.

For example, if

$$h(t) = 1 + 7t + 15t^2 + 15t^3 + 7t^4 + t^5,$$

we can write

$$h(t) = (1+t)^5 + 2t(1+t)^3 - t^2(1+t),$$

and so

$$\gamma(h;t) = 1 + 2t - t^2.$$

As a vector in the space of palindromic polynomials with basis Σ_5, h is represented by $(1, 7, 15)$, whereas $\gamma = (1, 2, -1)$. We can see that palindromicity and nonnegativity of $h(t)$, and even unimodality, are not enough to guarantee gamma-nonnegativity.

The product of two gamma-nonnegative polynomials is again gamma-nonnegative, though the center of symmetry necessarily shifts. That is, if

$$g(t) = \sum_{0 \le i \le m/2} \gamma_i t^i (1+t)^{m-2i} \quad \text{and} \quad h(t) = \sum_{0 \le j \le n/2} \gamma'_j t^j (1+t)^{n-2j},$$

then

$$g(t)h(t) = \sum_{0 \le k \le (m+n)/2} \left(\sum_{i+j=k} \gamma_i \gamma'_j \right) t^k (1+t)^{m+n-2k}.$$

Thus the set of all gamma-nonnegative polynomials of bounded degree is closed under multiplication. Moreover, we see that the gamma polynomial for the product $g(t)h(t)$ is the product of the gamma polynomial for g and the gamma polynomial for h, i.e.,

$$\gamma(gh;t) = \gamma(g;t)\gamma(h;t).$$

We will record these observations for future reference.

Observation 4.1 *If h is a polynomial in the nonnegative span of Γ_n, i.e., $h(t) \in \mathbb{R}_{\ge 0}\,\Gamma_n$, then h is palindromic and unimodal, with center of symmetry $\lfloor n/2 \rfloor$. Moreover, if $g(t) \in \mathbb{R}_{\ge 0}\,\Gamma_m$, then $g(t)h(t) \in \mathbb{R}_{\ge 0}\,\Gamma_{m+n}$, and $\gamma(gh;t) = \gamma(g;t)\gamma(h;t)$.*

4.5 Computing the gamma vector

There are straightforward linear transformations that map a palindromic polynomial h to its gamma vector, implicit in Equation (4.8).

Suppose $h(t)$ is symmetric for n so that

$$h(t) = h_0 + h_1 t + \cdots + h_{\lfloor n/2 \rfloor} t^{\lfloor n/2 \rfloor} + \cdots,$$

with $h_i = h_{n-i}$. By abuse of notation, let $h = (h_0, h_1, \ldots, h_{\lfloor n/2 \rfloor})$ denote the coefficients of this polynomial in the basis Σ_n, and let $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_{\lfloor n/2 \rfloor})$ be the corresponding gamma vector. We have the following change of basis matrices:

$$G = \left[(-1)^{i-j} \left(\binom{n-i-j}{i-j} + \binom{n-i-j-1}{i-j-1} \right) \right]_{0 \leq i,j \leq n/2},$$

and

$$S = \left[\binom{n-2j}{i-j} \right]_{0 \leq i,j \leq n/2},$$

so that

$$Gh = \gamma \quad \text{and} \quad S\gamma = h.$$

While the entries in S follow immediately from Equation 4.8, the entries of G are harder to guess at. However, it is straightforward to check that S and G are inverses of one another.

For example if $n = 5$,

$$G = \begin{pmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 5 & -3 & 1 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 10 & 3 & 1 \end{pmatrix},$$

so we see that in our example of $h(t) = (1 + t^5) + 7(t + t^4) + 15(t^2 + t^3)$,

$$Gh = \begin{pmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 5 & -3 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 7 \\ 15 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \gamma,$$

and

$$S\gamma = \begin{pmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 10 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 7 \\ 15 \end{pmatrix} = h.$$

As a word of caution, we note that the palindromicity degree n is needed to recover h from γ. For example, the polynomial $h(t) = 1 + 4t + 4t^2 + t^3$ has $\gamma(h; t) = 1 + t$, but $\gamma(t) = 1 + t$ is the γ-polynomial for a whole family of symmetric polynomials, e.g.,

$$(1+t)^2\gamma(t/(1+t)^2) = 1+3t+t^2,$$
$$(1+t)^3\gamma(t/(1+t)^2) = 1+4t+4t^2+t^3,$$
$$(1+t)^4\gamma(t/(1+t)^2) = 1+5t+8t^2+5t^3+t^4,$$
$$\vdots$$

A very different way to express $\gamma(h;t)$ in terms of $h(t)$ is with the following identity of generating functions.

Proposition 4.1 (Zeilberger's lemma). *Suppose $h(t)$ is palindromic for n, with gamma polynomial $\gamma(t)$. Then we have the following identity of power series:*

$$\gamma(z) = \frac{h(zC(z)^2)}{C(z)^n}, \tag{4.9}$$

where $C(z) = \frac{1-\sqrt{1-4z}}{2z}$ is the Catalan number generating function.

To see how the identity arises, we begin with $h(t) = (1+t)^n\gamma(t/(1+t)^2)$. Now setting $z = t/(1+t)^2$, we find

$$zt^2 + (2z-1)t + z = 0.$$

Solving for t, we find

$$\begin{aligned} t &= \frac{1-2z-\sqrt{(2z-1)^2-4z^2}}{2z}, \\ &= -1 + \frac{1-\sqrt{1-4z}}{2z}, \\ &= -1 + C(z). \end{aligned}$$

The Catalan generating function also satisfies $C(z)-1 = zC(z)^2$, and Equation (4.9) now follows. See Problem 4.5 for one use of Equation (4.9).

4.6 Real roots and log-concavity

We have emphasized the importance of the palindromicity and unimodality implied by gamma-nonnegativity. There are at least two other related ideas that have been studied: *real-rootedness* and *log-concavity*. While they are somewhat ancillary to our main concern, we will briefly survey some of their properties here.

Many interesting polynomial generating functions turn out to be real-rooted, that is, these polynomials factor completely over the real numbers. The Eulerian polynomials and the Narayana polynomials, for example, have only real roots. (See Problems 4.6 and 4.7.) If a polynomial $h(t)$ is palindromic, then h is real-rooted if and only if $\gamma(h;t)$ is real-rooted. Indeed, if a

is a real root of h with $a \notin \{0,-1\}$ (the cases of $a \in \{0,-1\}$ are easily considered) then $1/a$ is also a root of h by symmetry, and

$$\gamma(h; a/(1+a)^2) = \frac{1}{(1+a)^n} h(a) = \frac{1}{(1+1/a)^n} h(1/a) = 0,$$

thus implying the real number $a/(1+a)^2$ is a (nonpositive) root of $\gamma(h;t)$. On the other hand, if $b < 0$ is a real root of $\gamma(h;t)$, then both

$$a = -1 + \frac{1+\sqrt{1-4b}}{2b} \quad \text{and} \quad \frac{1}{a} = -1 + \frac{1-\sqrt{1-4b}}{2b}$$

are roots of $h(t)$. That no other roots exist follows by considering the degrees of $h(t)$ and $\gamma(h;t)$.

It turns out that whenever a polynomial $h(t)$ has nonnegative and palindromic coefficients, having all real roots implies $h(t)$ is gamma-nonnegative, but not conversely. (Consider $h(t) = 1 + 4t + 7t^2 + 4t^3 + t^4$. It has no real roots, yet it has nonnegative γ-polynomial $\gamma(h;t) = 1 + t^2$.) In particular, nonnegative, palindromic, and real-rooted polynomials are unimodal.

To see why this is so, suppose $h(t)$ has nonnegative and symmetric coefficients, and all its roots are real. Then as mentioned earlier, its roots apart from 0 and -1 come in reciprocal pairs, $a, 1/a$. Consider

$$(t-a)(t-1/a) = (1+t)^2 - (2 + a + 1/a)t.$$

If $h(t)$ has nonnegative coefficients, then all its roots must be nonpositive. In particular, $a < 0$, and dividing the positive quantity $(a+1)^2$ by a shows

$$0 > \frac{(a+1)^2}{a} = 2 + a + 1/a.$$

Thus $(t-a)(t-1/a)$ is in the positive span of Γ_2. Since $h(t)$ can be written as a product of powers of t (in Γ_2), powers of $(1+t)$ (in Γ_1), and terms of the form $(t-a)(t-1/a)$, we have that $h(t)$ is a product of gamma-nonnegative polynomials. Since we noted in Observation 4.1 that such polynomials are closed under multiplication, $h(t)$ is gamma-nonnegative as well.

Let us collect these comments.

Observation 4.2 *If h has palindromic coefficients, then $h(t)$ is real-rooted if and only if $\gamma(h;t)$ has only real roots. Moreover, if the coefficients of $h(t)$ are nonnegative, then all the roots of h are nonpositive and $\gamma(h;t)$ has nonnegative coefficients as well. Thus if $h(t)$ is nonnegative, real-rooted, and palindromic, then it is unimodal.*

Another property related to real-rootedness and unimodality is *log-concavity.* A sequence $a_1, \ldots, a_n$ is said to be log-concave if

$$a_i^2 \geq a_{i-1}a_{i+1} \qquad \text{for all } i = 2, \ldots, n-1.$$

This immediately implies that the sequence is unimodal since if there is some j such that $a_{j-1} > a_j < a_{j+1}$, then clearly $a_j^2 < a_{j-1}a_{j+1}$. We will say a polynomial is log-concave if its sequence of coefficients is log-concave.

Log-concavity is more robust than gamma-nonnegativity in the sense that it applies perfectly well to sequences that are not palindromic, whereas the gamma vector requires palindromicity to exist. Real-rootedness also implies log-concavity (Problem 4.8), and hence unimodality, but not conversely. The polynomial $1 + 4t + 7t^2 + 4t^3 + t^4$ from before is log-concave, yet has no real roots.

Log-concave sequences are closed under multiplication, i.e., if $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ are log-concave, then so is $a_1b_1, a_2b_2, \ldots$. However, they are not closed under addition, e.g., $(0, 0, 11, 0, 0)$ and $(1, 4, 6, 4, 1)$ are both log-concave (and gamma-nonnegative), yet their sum $(1, 4, 17, 4, 1)$ is not log-concave.

We collect these comments in another observation, to compare with Observations 4.1 and 4.2.

Observation 4.3 *Suppose $h(t)$ has nonnegative coefficients. If $h(t)$ is real-rooted, then h is log-concave. In particular, h is unimodal.*

Notice, then, that if the goal is to prove unimodality of a polynomial h, real-rootedness is more than sufficient. The relationships between these three concepts: gamma-nonnegativity, log-concavity, and real-rootedness are shown in Figure 4.3. The reader is asked to find a polynomial in each distinct region of that Venn diagram in Problem 4.10.

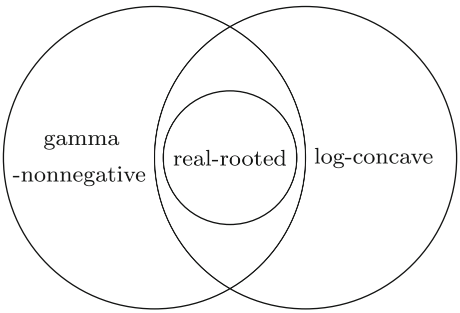


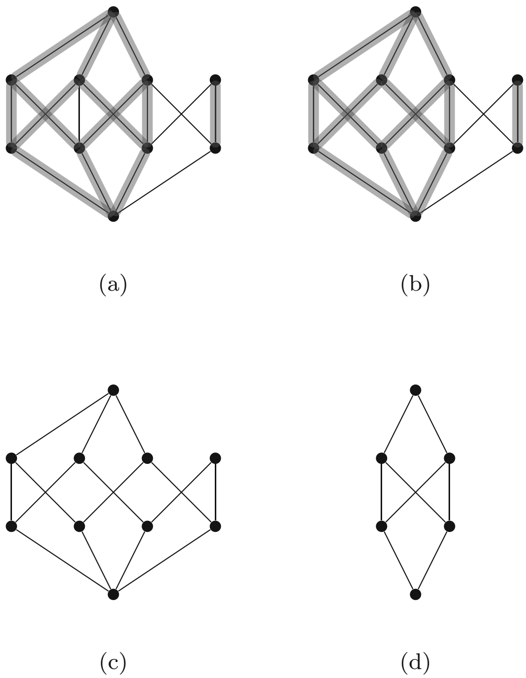
Fig. 4.3 The relationship between the notions of gamma-nonnegativity, log-concavity, and real-rootedness for palindromic polynomials with nonnegative coefficients.

4.7 Symmetric boolean decomposition

If $f(P;t)$ is the rank generating function of a poset P, the fact that $f(P;t)$ is gamma-nonnegative might only be the enumerative shadow of a deeper structural property of the poset itself, which we call *symmetric boolean decomposition*. Loosely, it means that a poset can be partitioned into a number of disjoint boolean algebras with the same center of symmetry around the middle rank of P.

This is a stronger version of a property known as a *symmetric chain decomposition* of a poset, which itself implies unimodality of the rank function $f(P;t)$. The fact that a symmetric boolean decomposition implies a symmetric chain decomposition follows once we can show that every boolean algebra has a symmetric chain decomposition. This is left to Problem 4.14. See also Problem 4.13 for more properties and consequences of symmetric chain decompositions.

Rather than giving a formal definition of symmetric boolean decomposition, let us see some examples. In Figure 4.4, posets (a) and (b) have symmetric boolean decompositions, while (c) and (d) do not.



(a) (b) (c) (d)

Fig. 4.4 Posets with and without symmetric boolean decompositions.

Formally, we say a poset P of rank n admits a *symmetric boolean decomposition* if there is a collection $\{P_1, \ldots, P_k\}$ of subposets of $(P, \leq)$ with the following properties:

- $P_i \cap P_j = \emptyset$ if $i \neq j$,
- $P_1 \cup \cdots \cup P_k = P$,
- for each $i = 1, \ldots, k$ there is a number j, $0 \leq j \leq n/2$, and a bijection $\rho_i : 2^{[n-2j]} \to P_i$ that takes cover relations to cover relations and sends elements of rank r in $2^{[n-2j]}$ to elements of rank $j + r$ in P.

That is, each induced poset $(P_i, \leq)$ has 2^{n-2j} elements (for some j) and contains a copy of the boolean algebra $2^{[n-2j]}$, plus possibly more relations.

For example, we can see in Figure 4.4 that poset (a) contains a copy of $2^{\{1,2,3\}}$ as a proper subposet, whereas in (b) the part of the partition containing it has no unnecessary cover relations. Note also the delicacy of the decomposition: the poset in (c) differs from (b) only in one cover relation.

We can also observe that just as gamma-nonnegative polynomials are closed under multiplication, so too are posets with symmetric boolean decompositions. First, we define the product of two posets $(P, \leq_P)$ and $(Q, \leq_Q)$ to be the poset on the cartesian product $P \times Q$ with partial order $(p, q) \leq_{P\times Q} (p', q')$ if and only if $p \leq_P p'$ and $q \leq_Q q'$. Then it is a straightforward matter to verify the following observation. This is left to the reader in Problem 4.16.

Observation 4.4 *Suppose $(P, \leq_P)$ and $(Q, \leq_Q)$ are posets with symmetric boolean decompositions. Then $(P \times Q, \leq_{P\times Q})$ has a symmetric boolean decomposition.*

This result is illustrated in Figure 4.5.

Two interesting examples of posets with symmetric boolean decompositions are the shard intersection order and the lattice of noncrossing partitions.

Theorem 4.3. *The shard intersection order and the lattice of noncrossing partitions admit symmetric boolean decompositions.*

This result follows from the valley-hopping argument given in Section 4.2. Indeed, the hop-equivalence classes are boolean intervals in the shard intersection order with the proper rank properties, i.e., their descents are distributed like $t^j(1+t)^{n-2j}$. Since hop-equivalence preserves the pattern 231, this gives a symmetric boolean decomposition for the shard intersection order on 231-avoiders, which we showed is isomorphic to the lattice of noncrossing partitions. How this works in S_4 is shown in Figure 4.6.

Notes

Gamma-nonnegativity of the Eulerian polynomials was observed by Dominique Foata and Marcel-Paul Schützenberger in their 1970 book [70, Théorème 5.6]. Foata and Volker Strehl [72] gave the result a combinatorial

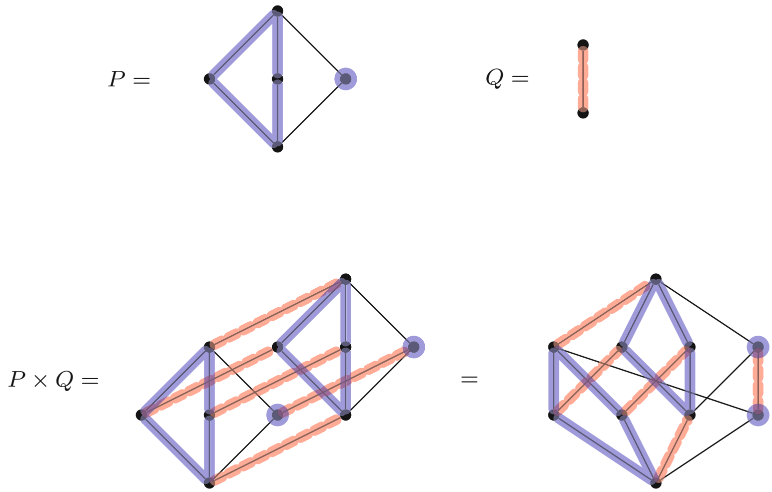


Fig. 4.5 The product of two posets with a symmetric boolean decomposition has a symmetric boolean decomposition.

proof very similar to the "valley-hopping" argument given here, which was essentially rediscovered by Louis Shapiro, Wen Jin Woan, and Seyoum Getu in 1983 [135]. In a 2008 paper, [31], Petter Brändén studies valley-hopping (what he calls the "modified Foata-Strehl" action) on a large family of polynomials that generalize the Eulerian polynomials and include the Narayana polynomials. That the Narayana polynomials are gamma-nonnegative is also implicit in the work of Rodica Simion and Daniel Ullman from 1991 [140]. See also Simion's 1994 paper [138].

We also mention that George Andrews anticipated some of the ideas in this section, proving in a 1975 paper [8] that a product of palindromic and unimodal polynomials is again palindromic and unimodal. Further, he discussed the gamma polynomial and palindromic polynomials (what he called "reciprocal polynomials") in the larger context of quadratic transformations in a 1985 paper [9].

More recent interest in gamma-nonnegativity was sparked by a 2005 paper of Światosław Gal [79]. This work showed certain questions in topology could be resolved by demonstrating gamma-nonnegativity of combinatorial invariants. Prior to Gal's work, researchers had attacked such questions via real-rootedness, but Gal showed that real-rootedness could fail yet gamma-nonnegativity still holds. (This subject is discussed further in Chapter 10.) Similar real-rootedness conjectures known as the Neggers and Stanley conjectures were disproved around the same time by Petter Brändén [28] (Stanley) and John Stembridge [158] (Neggers). Both Gal [79] and Brändén [29] showed

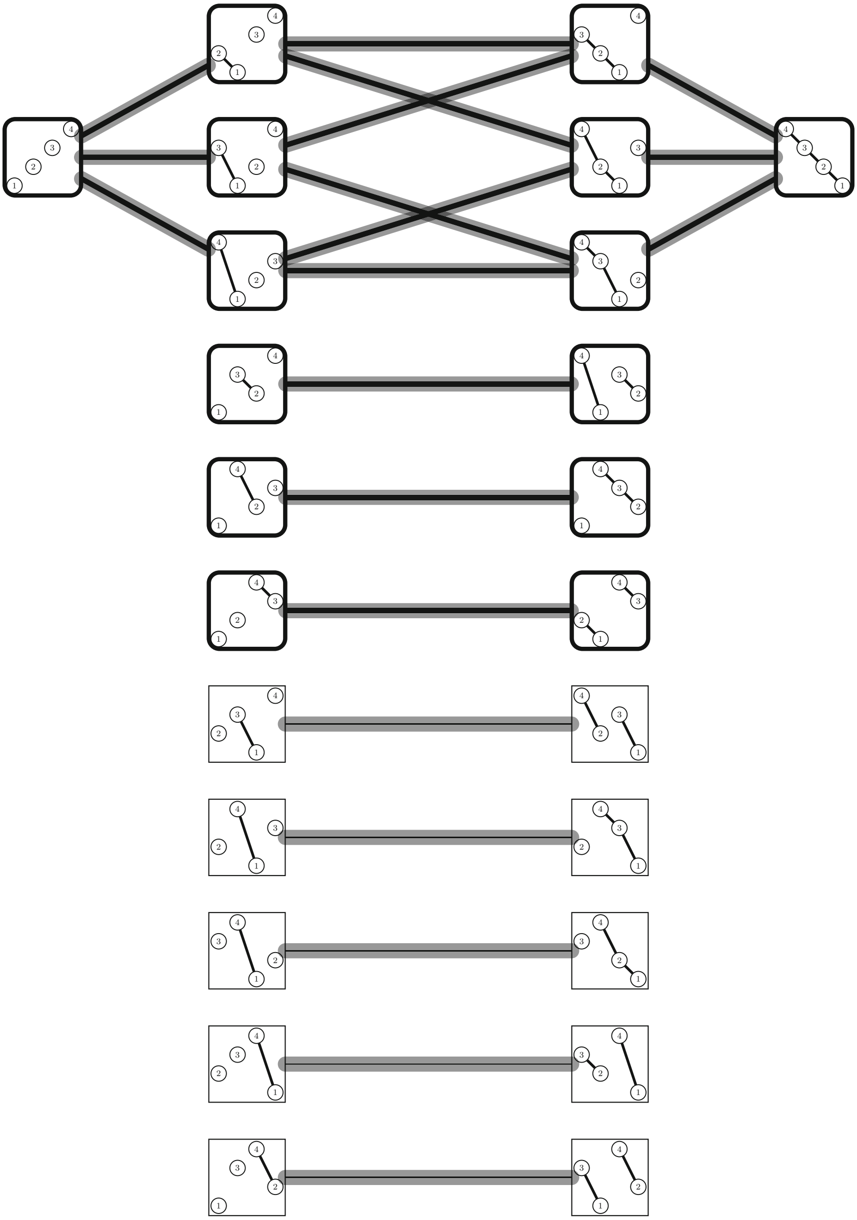


Fig. 4.6 The symmetric boolean decomposition of the shard intersection order and noncrossing partition lattice (in bold) induced by hop-equivalence classes. The Hasse diagram is drawn left to right, and edges not needed for the decomposition are omitted.

gamma-nonnegativity could be a viable replacement for real-rootedness in many of these contexts. See also the work of Victor Reiner and Volkmar Welker [128].

Some nice surveys about log-concavity, unimodality, real-rootedness, and gamma-nonnegativity include a 1989 paper by Richard Stanley [150], a 1994 paper by Francesco Brenti [33], and a 2014 survey by Brändén [32]. Only Brändén's discusses gamma-nonnegativity.

The idea of symmetric boolean decomposition first appears in Simion and Ullman's work on the lattice of noncrossing partitions, though they do not state this explicitly [140]. However, a remark about such a decomposition was later made by Simion [138, Proposition 3.4]. In 1999, while studying a generalization of the lattice of noncrossing partitions, Patricia Hersh [90] makes the definition of symmetric boolean decomposition explicit. More recently this book's author demonstrated the symmetric boolean decomposition of the shard intersection order [118].

Problems

4.1. Verify Equations (4.4) and (4.7).

4.2 (Alternating permutations). A permutation w is called *alternating* if

$$w(1) < w(2) > w(3) < \cdots \qquad \text{or} \qquad w(1) > w(2) < w(3) > \cdots .$$

In the first case, we say w is *up-down alternating*, while in the second case we say w is *down-up alternating.*

1. Let $\mathcal{E}_n$ denote the set of up-down alternating permutations of $[n]$, and let $\mathcal{E}'_n$ denote the set of down-up alternating permutations. Show $|\mathcal{E}_n| = |\mathcal{E}'_n|$.
2. Let E_n denote the cardinality of either $\mathcal{E}_n$ or $\mathcal{E}'_n$, with $E_0 := 1$. The first few values of E_n are

$$1, 1, 1, 2, 5, 16, 61, 272, 1385, 7936, 50521, \ldots .$$

Show that for $n \geq 1$,

$$2E_{n+1} = \sum_{k=0}^{n} \binom{n}{k} E_k E_{n-k}.$$

3. Show that

$$\sum_{n\geq 0} E_n \frac{z^n}{n!} = \sec z + \tan z.$$

Since $\sec z$ is an even function and $\tan z$ is an odd function, conclude that

$$\sum_{n\geq 0} E_{2n}\frac{z^{2n}}{(2n)!} = \sec z,$$

and

$$\sum_{n\geq 0} E_{2n+1}\frac{z^{2n+1}}{(2n+1)!} = \tan z.$$

4.3. A permutation $w \in S_n$ is called *min-max* if $w^{-1}(1) < w^{-1}(n)$ and *max-min* if $w^{-1}(n) < w^{-1}(1)$. Let $E_n^{\nearrow}$ denote the number of up-down alternating permutations that are min-max permutations, and let $E_n^{\nwarrow}$ denote the number of up-down alternating permutations that are max-min permutations.

Show that

$$E_n^{\nearrow} - E_n^{\nwarrow} = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ E_{n-2} & \text{if } n \text{ is even.} \end{cases}$$

4.4. The *stack-sorting* operator $\mathcal{S}$ is a recursively defined function on permutations. If w is an empty permutation $\mathcal{S}(w) := w$. If w is not empty and $\max w(i) = m$, then write $w = u{\cdot}m{\cdot}v$ for some (possibly empty) permutations u and v. Then we define $\mathcal{S}(w) = \mathcal{S}(u)\mathcal{S}(v)m$.

1. Compute $\mathcal{S}(389124576)$ and $\mathcal{S}(132549678)$.
2. Prove that $\mathcal{S}(w) = 12\cdots n$ if and only if w is 231-avoiding. We call such a permutation *stack-sortable.*
3. Show that $\mathcal{S}(w) = \mathcal{S}(w')$ if $\mathrm{Hop}(w) = \mathrm{Hop}(w')$, i.e., if w and w' are in the same valley-hopping equivalence class.
4. A permutation is called *r-stack sortable* if $\mathcal{S}^r(w) = 12\cdots n$. Show that r-stack sortability is preserved by valley hopping, and conclude that the Eulerian distribution on r-stack sortable permutations is gamma-nonnegative, i.e.,

$$\sum_{w\in S_n^r} t^{\mathrm{des}(w)} = \sum_{j\geq 0} \gamma_{r;n,j} t^j (1+t)^{n-1-2j},$$

where S_n^r denotes the set of r-stack sortable elements in S_n, and $\gamma_{r;n,j} = |\{w \in S_n^r : \mathrm{pk}(w) = \mathrm{des}(w) = j\}|$.

4.5. Let

$$h_n(t) = (1+t)(1+t+t^2)\cdots(1+t+\cdots+t^{n-1}) = \prod_{i=1}^{n} \frac{1-t^i}{1-t},$$

and let $\gamma(t)$ be the corresponding gamma polynomial. Note that $h_n(1) = n!$. What is $\gamma(-1)$?

4.6. Prove the Eulerian polynomials $S_n(t)$ are real rooted.

Hint: Let $A_n(t) = tS_n(t)$ and show that $A_n(t)$ has n real roots. We can modify Equation (1.9) to write

$$A_{n+1}(t) = t\left((n+1)A_n(t) + (1-t)A_n'(t)\right). \quad (4.10)$$

If we suppose $A_n(t)$ has n real roots, then we can use this recurrence to prove $S_{n+1}(t) = A_{n+1}(t)/t$ has n real roots as follows. Rolle's theorem shows that the roots of a polynomial $f(t)$ and its derivative $f'(t)$ are "interlacing." Show that $(n+1)A_n(t)$ and $(1-t)A_n'(t)$ have interlacing roots, and use this to show their sum has n real roots.

Moreover, show that the sequence of Eulerian polynomials forms a *Sturm sequence*, i.e., the polynomials $S_n(t)$ and $S_{n+1}(t)$ have interlacing roots.

4.7. Let $N_n(t) = tC_n(t)$ denote the Narayana polynomial multiplied by a power of t.

1. Prove the polynomials $N_n(t)$ satisfy the following recurrence:

$$(n+1)N_n(t) = (2n-1)(1+t)N_{n-1}(t) - (n-2)(1-t)^2 N_{n-2}(t). \quad (4.11)$$

   (A bijective proof would be best, but this can also be verified with generating functions using Equation (2.6).)
2. Use the recurrence in (4.11) to prove that the Narayana polynomials are real-rooted and form a Sturm sequence.

4.8 (Real roots and log-concavity). The goal of this problem is to show that a polynomial with positive coefficients and only real roots has log-concave, and hence unimodal, coefficients.

1. Show that the sequence of binomial coefficients with n fixed,

$$\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n},$$

   is log-concave, i.e., the polynomial $(1+t)^n$ is log-concave.
2. Show that the sequence of binomial coefficients with k fixed,

$$\binom{k}{k}, \binom{k+1}{k}, \ldots,$$

   is log-concave.
3. Prove that if $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ are log-concave, then the sequence $c_1, c_2, \ldots$ defined by $c_k = a_k b_k$ is log-concave.
4. Show that if $a_0, a_1, \ldots, a_n$ is a finite sequence of nonnegative numbers and the sequence $b_0, b_1, \ldots$ given by $b_k = a_k/\binom{n}{k}$ is log-concave, then $a_0, a_1, \ldots, a_n$ is itself log-concave.
5. Let $a_0, a_1, \ldots, a_n$ be a sequence of nonnegative numbers such that $f(t) = a_0 + a_1 t + \cdots + a_n t^n$ is real-rooted.

a. Write $a_k = \binom{n}{k} b_k$. Show that

$$\sum_{k=0}^{n-1} \binom{n-1}{k} b_{k+1} t^k$$

is real-rooted. (Hint: it is a multiple of the derivative of f.)

b. Show that the polynomial

$$\sum_{k=0}^{n} \binom{n}{k} b_{n-k} t^k$$

is real-rooted.

c. Use the operations indicated in parts 5a) and 5b) to show that for any $j = 1, \ldots, n-1$, the polynomial $b_{j-1} + 2b_j t + b_{j+1} t^2$ is real-rooted. Conclude that the sequence $b_0, b_1, b_2, \ldots$ is log-concave. By part 4) this proves that a real-rooted polynomial with nonnegative coefficients is log-concave, and hence unimodal.

4.9. Prove that if $f(t)$ and $g(t)$ are nonnegative and log-concave, then their product, $f(t)g(t)$, is log-concave. Hint: first prove that if $a_0, a_1, a_2, \ldots$ is a nonnegative and log-concave sequence, then $a_i a_j \leq a_{i+1} a_{j-1}$ for any $i < j-1$.

4.10. If a polynomial is real-rooted and palindromic then it is both gamma-nonnegative and log-concave, as illustrated in Figure 4.3. Find examples of polynomials with positive, palindromic integer coefficients that fit in the other regions of that Venn diagram.

1. Find a polynomial that is gamma-nonnegative but not log-concave (and hence not real-rooted).
2. Find a polynomial that is log-concave and palindromic but not gamma-nonnegative (and hence not real-rooted).
3. Find a polynomial that is log-concave and gamma-nonnegative but not real-rooted.

4.11 (Gamma-nonnegativity for involutions). An *involution* is a permutation that is its own inverse: $w = w^{-1}$. Show that the distribution of descents for involutions, i.e., the *Eulerian distribution for involutions*, is gamma-nonnegative. That is, show there exist nonnegative integers γ_j such that

$$\sum_{w=w^{-1} \in S_n} t^{\mathrm{des}(w)} = \sum_{j \geq 0} \gamma_j t^j (1+t)^{n-1-2j}.$$

4.12 (Two-dimensional gamma-nonnegativity). Let

$$S_n(s,t) = \sum_{w \in S_n} s^{\mathrm{des}(w^{-1})} t^{\mathrm{des}(w)},$$

i.e., the joint distribution of descents and inverse descents. Show that there exist nonnegative integers $\gamma_{i,j}$ such that

$$S_n(s,t) = \sum_{i,j\geq 0} \gamma_{i,j}(st)^i(s+t)^j(1+st)^{n-1-j-2i}.$$

4.13 (Symmetric chain decomposition). A *symmetric chain decomposition* of a finite ranked poset P with maximal rank n is a partition into saturated chains

$$p_0 <_P p_1 <_P \cdots <_P p_k,$$

such that $\rho(p_0)+\rho(p_k) = n$ for each chain. (Recall a "saturated" chain is one for which $\rho(p_i)+1 = \rho(p_{i+1})$ for all i.)

1. Show that if P has a symmetric chain decomposition, then its rank function,
$$f(P;t) = \sum_{p\in P} t^{\rho(p)} = \sum_{k\geq 0} f_k t^k,$$
is symmetric and unimodal.
2. Let $A \subset P$ be an antichain, i.e., a set of pairwise incomparable elements of P. Show that $|A| \leq f_{\lfloor n/2\rfloor}$.
3. Show that the product of two chains has a symmetric chain decomposition. That is, show P has a symmetric chain decomposition, where $P = [k]\times[l]$ is the set of pairs (i,j) with $1 \leq i \leq k$ and $1 \leq j \leq l$, ordered by $(i,j) \leq (i',j')$ if and only if $i \leq i'$ and $j \leq j'$.
4. Show that if P and Q are posets with symmetric chain decompositions, then their cartesian product $P\times Q$ (with partial order $(p,q) \leq_{P\times Q} (p',q')$ if and only if $p \leq_P p'$ and $q \leq_Q q'$) has a symmetric chain decomposition.

4.14 (Sperner's Theorem). Show that the boolean algebra $2^{[n]}$, i.e., the set of subsets of a finite set ordered by inclusion, has a symmetric chain decomposition. (This implies that any poset with a symmetric boolean decomposition inherits a symmetric chain decomposition.)

Conclude *Sperner's Theorem*: any collection A of subsets of $[n]$ such that no subset contains another satisfies $|A| \leq \binom{n}{\lfloor n/2\rfloor}$.

4.15 (Lattice of divisors). The lattice of positive divisors of an integer n, $\Lambda(n)$, is the set of all integers d that divide n, ordered by divisibility. If $d = p_1^{m_1}p_2^{m_2}\cdots p_k^{m_k}$ is the prime factorization of d, we define the *degree* of d to be $\deg(d) = m_1+m_2+\cdots+m_k$. The covers of $\Lambda(n)$ are given by multiplication by a single prime, $d \prec p_i d$ for some p_i. Thus $\Lambda(n)$ is ranked by degree. Let $f_k(n)$ denote the number of divisors of n of degree k.

Show that the lattice of positive divisors of an integer n has a symmetric chain decomposition, and conclude that any collection A of mutually indivisible divisors of n (i.e., if $a,b \in A$, then neither $a|b$ nor $b|a$) has cardinality at most $f_{\lfloor \deg(n)/2\rfloor}(n)$.

4.16. Verify Observation 4.4, i.e., show that if posets P and Q have a symmetric boolean decomposition, then so does their cartesian product, $P \times Q$.

4.17 (Simion and Ullmann's symmetric boolean decomposition). In [140], Rodica Simion and Daniel Ullman gave a symmetric boolean decomposition of $\mathrm{NC}(n)$ that is different from the one that follows from valley hopping. Simion and Ullman provide a certain encoding of noncrossing partitions as words on the alphabet $\{b, e, l, r\}$. Given a noncrossing partition $\pi \in \mathrm{NC}(n)$, the encoding assigns a word $w(\pi) = w = w_1 w_2 \cdots w_{n-1}$ of length $n-1$ as follows:

$$w_i = \begin{cases} b & \text{if } i \text{ and } i+1 \text{ are in different blocks} \\ & \quad \text{and } i \text{ is } \textit{not} \text{ the largest element in its block,} \\ e & \text{if } i \text{ and } i+1 \text{ are in different blocks} \\ & \quad \text{and } i+1 \text{ is } \textit{not} \text{ the smallest element in its block,} \\ l & \text{if } i \text{ and } i+1 \text{ are in different blocks,} \\ & \quad i \text{ is the largest element in its block,} \\ & \quad \text{and } i+1 \text{ is the smallest element in its block,} \\ r & \text{if } i \text{ and } i+1 \text{ are in the same block.} \end{cases}$$

We call such a word the *SU-word* for π.

For example, if $\pi = \{\{1,2,6\},\{3\},\{4,5\}\}$, we have its SU-word is $w(\pi) = rblre$. Let $B(w), E(w), L(w), R(w)$ denote the sets of positions in w containing the letters b, e, l, and r, respectively. For example $w = rblre$ has $B(w) = \{2\}$, $E(w) = \{5\}$, $L(w) = \{3\}$, and $R(w) = \{1,4\}$.

1. Show $n = |B(w)| + |E(w)| + |L(w)| + |R(w)| + 1$.
2. Show π has rank equal to $|B(w)| + |R(w)|$.
3. Show that $|B(w)| = |E(w)|$ and that these sets give a valid matching on $[n]$ by having an open parenthesis at each $b \in B(w)$ (beginning) and a closed parenthesis at $e+1$ for each e in $E(w)$ (ending).
4. Let π and π' be noncrossing partitions with SU-words w and w'. Show that if $B(w) = B(w')$, $E(w) = E(w')$, and $R(w) \subseteq R(w')$, then $\pi \leq_{\mathrm{NC}} \pi'$.
5. Use 5) to give a symmetric boolean decomposition of $\mathrm{NC}(n)$.
6. Show that this decomposition is different from the one inherited from the decomposition of the shard intersection order restricted to 231-avoiding permutations.