Chapter 2
Narayana numbers

WHILE THE SEQUENCE OF FIBONACCI NUMBERS entered the public imagination a long time ago, it can be argued that the sequence introduced in this chapter is of greater importance in combinatorics today. Here we will study the *Catalan numbers*,

$$1, 1, 2, 5, 14, 42, 429, 1430, 4862, 16796, 58786, \ldots,$$

and a triangle of numbers that refine the Catalan numbers, known as the *Narayana numbers*. Throughout the book, the Narayana numbers will be shown to possess the same (or nearly the same) properties as the Eulerian numbers.

2.1 Catalan numbers

The *Catalan numbers* are denoted C_n, $n \geq 0$, and are given by the explicit formula $C_n = \frac{1}{n+1}\binom{2n}{n}$. The sequence of Catalan numbers is among the most famous sequences in mathematics. One reason for the ubiquity of the Catalan numbers may be that they satisfy the following quadratic, convolutive recurrence for $n \geq 1$:

$$C_n = \sum_{i=0}^{n-1} C_i C_{n-1-i}, \tag{2.1}$$

and this numeric recurrence is a shadow of natural structural recurrences possessed by many families of combinatorial objects.

T.K. Petersen, *Eulerian Numbers*, Birkhäuser Advanced Texts Basler Lehrbücher, DOI 10.1007/978-1-4939-3091-3_2

From (2.1), we can derive the generating function:

$$C(z) = \sum_{n\geq 0} C_n z^n,$$

as follows.

$$\begin{aligned} C(z) &= \sum_{n\geq 0} C_n z^n \\ &= 1 + z \sum_{n\geq 1} \sum_{i=0}^{n-1} C_i z^i C_{n-1-i} z^{n-1-i} \\ &= 1 + zC(z)^2. \end{aligned}$$

Therefore,

$$zC(z)^2 - C(z) + 1 = 0,$$

and we get

$$C(z) = \frac{1-\sqrt{1-4z}}{2z}. \tag{2.2}$$

We mention the Catalan numbers because they enumerate an important subset of permutations that we will now describe. Counting these permutations according to descents gives rise to the array of *Narayana numbers*, a distribution that has many of the same properties as the Eulerian distribution.

2.2 Pattern-avoiding permutations

The permutations we will study in this chapter are *231-avoiding permutations*. These are permutations w such that there is no triple of indices $i < j < k$ such that $w(k) < w(i) < w(j)$. That is, the letters $w(i), w(j)$, and $w(k)$ are not in the same relative positions as 2, 3, and 1. If a permutation w contains such a triple, we say w *contains* the pattern 231; otherwise, we say w *avoids* the pattern 231. For example, the permutation 53412 contains the pattern 231 since $w(4) < w(2) < w(3)$ (or since $w(5) < w(2) < w(3)$), whereas the permutation 32154 avoids 231. The notion of pattern avoidance is easy to understand visually when we draw the graph of a permutation as an array of dots on a square grid. See Figure 2.1.

Let $S_n(231)$ denote the set of permutations in S_n avoiding the pattern 231. The 231-avoiding permutations, for $n \leq 5$, are listed in Table 2.1.

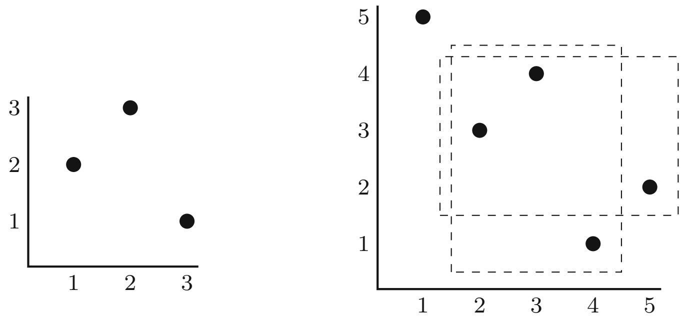


Fig. 2.1 The permutation 53412 contains the pattern 231 in several ways. Two occurrences of the pattern are indicated with dashed line boxes.

Table 2.1 The 231-avoiding permutations of n with k descents, $0 \leq k < n \leq 5$.

$n\backslash k$	0	1	2	3	4
1	1				
2	12	21			
3	123	213	321		
		132			
		312			
4	1234	2134	3214	4321	
		1324	2143		
		3124	1432		
		1243	4213		
		1423	4132		
		4123	4312		
5	12345	21345	32145	43215	54321
		13245	21435	32154	
		31245	14325	21543	
		12435	42135	15432	
		14235	41325	53214	
		41235	43125	52143	
		12354	21354	51432	
		12534	13254	54213	
		15234	31254	54132	
		51234	12543	54312	
			21534		
			15324		
			15243		
			15423		
			52134		
			51324		
			53124		
			51243		
			51423		
			54123		

We will now show that the 231-avoiding permutations obey a structural recurrence compatible with the numeric recurrence in (2.1). For the moment, let $c_n = |S_n(231)|$ and define $c_0 = 1$ for convenience. We will show that for $n \geq 1$:

$$c_n = \sum_{i=0}^{n-1} c_i c_{n-1-i},$$

and hence $c_n = C_n$ for all n.

First of all, suppose u is a permutation in $S_i(231)$ and v is a permutation of $\{i+1, \ldots, n-1\}$ that avoids 231. Then since every letter of u is smaller than every letter of v, the permutation

$$w = u(1) \cdots u(i)\, n\, v(1) \cdots v(n-1-i),$$

formed by inserting n between u and v, is a 231-avoiding permutation. There are c_i choices for u and c_{n-1-i} choices for v, so summing over all i, we have

$$\sum_{i=0}^{n-1} c_i c_{n-1-i} \leq c_n.$$

On the other hand, suppose $w \in S_n$ is 231-avoiding, with $w(i+1) = n$. Let $u = w(1) \cdots w(i)$ denote the word to the left of n, and let $v = w(i+2) \cdots w(n)$ denote the word to the right of n. Clearly both of these words must avoid the pattern 231. Further, if there was a letter a in u that was greater than a letter b in v, then there would be a 231 pattern formed by the letters a, n, b in w. Hence, every letter of u must be smaller than every letter of v. In other words, $u \in S_i(231)$ and v is a permutation of $\{i+1, \ldots, n-1\}$ that avoids 231. This shows

$$c_n \leq \sum_{i=0}^{n-1} c_i c_{n-1-i},$$

and so in light of our earlier discussion, the two quantities must equal each other:

$$c_n = \sum_{i=0}^{n-1} c_i c_{n-1-i}.$$

Since the c_n satisfy the same recurrence as the Catalan numbers with the same initial values, $c_n = C_n$, and we have the following combinatorial characterization of Catalan numbers. (The first of many, as we will discover later in the chapter.)

Theorem 2.1. *For $n \geq 1$,*

$$|S_n(231)| = C_n.$$

While showing that $|S_n(231)| = C_n$ recursively is fine, one would like to also have a direct combinatorial proof, e.g., by showing $(n+1)|S_n(231)| = \binom{2n}{n}$

via a bijection. This is left to Problem 2.2, though we will do something similar in Section 2.4 for another set of objects counted by Catalan numbers.

Before moving on, we remark that there is nothing particularly interesting about the pattern 231 for Theorem 2.1. It is possible to exhibit bijections between the set $S_n(231)$ and the set $S_n(p)$, where $p \in \{123, 132, 213, 312, 321\}$ is *any* pattern of length three. See Problem 2.1.

2.3 Narayana numbers

Similarly to how we defined the Eulerian numbers, we define the *Narayana number* $N_{n,k}$ to be the number of permutations in $S_n(231)$ with k descents:

$$N_{n,k} = |\{w \in S_n(231) : \text{des}(w) = k\}|.$$

We have the Narayana numbers shown in Table 2.2.

Table 2.2 The Narayana numbers $N_{n,k}$, $0 \le k < n \le 10$.

$n\backslash k$	0	1	2	3	4	5	6	7	8	9
1	1									
2	1	1								
3	1	3	1							
4	1	6	6	1						
5	1	10	20	10	1					
6	1	15	50	50	15	1				
7	1	21	105	175	105	21	1			
8	1	28	196	490	490	196	28	1		
9	1	36	336	1176	1764	1176	336	36	1	
10	1	45	540	2520	5292	5292	2520	540	45	1

We will show in Section 2.4 that the Narayana numbers are given by the formula

$$N_{n,k} = \frac{1}{k+1}\binom{n}{k}\binom{n-1}{k}. \tag{2.3}$$

It is easily shown that this formula is equivalent to

$$N_{n,k} = \det \begin{pmatrix} \binom{n-1}{k} & \binom{n}{k+1} \\ \binom{n}{k} & \binom{n+1}{k+1} \end{pmatrix} = \binom{n-1}{k}\binom{n+1}{k+1} - \binom{n}{k}\binom{n}{k+1},$$

and therefore we can extract the triangle of Narayana numbers as 2×2 minors of Pascal's triangle. See Figure 2.2.

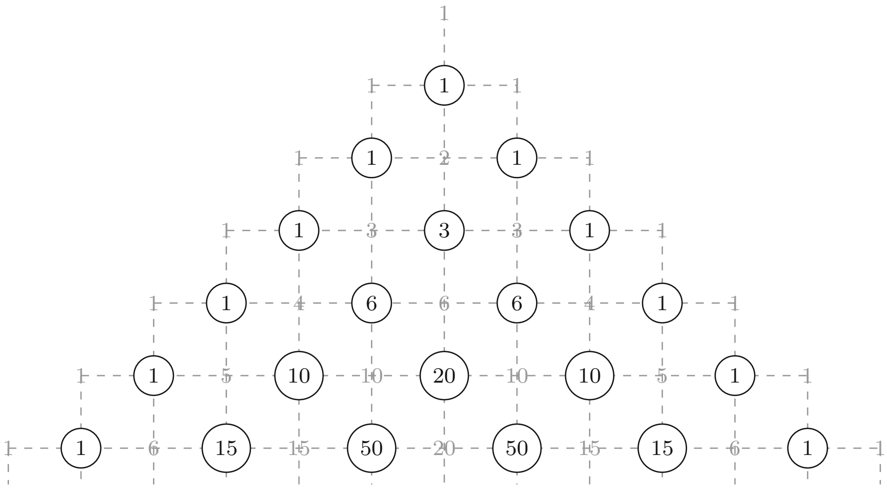


Fig. 2.2 The triangle of Narayana numbers obtained as determinants.

We will see that the generating function for Narayana numbers (with n fixed) obeys a refined Catalan recurrence. Define

$$C_n(t) = \sum_{w \in S_n(231)} t^{\mathrm{des}(w)} = \sum_{k=0}^{n-1} N_{n,k} t^k,$$

with $C_0(t) := 1$. We will refer to $C_n(t)$ as the *Narayana polynomial*.

If we follow the recursive argument that led to Theorem 2.1 while keeping track of descents, we will get a recurrence for the Narayana polynomials that refines (2.1). In that proof was an implicit bijection between elements $w \in S_n(231)$ and pairs (u, v) with $u \in S_i(231)$ (for some i) and v a permutation of $\{i+1, \ldots, n-1\}$ that avoids 231. Namely, we can write

$$w = u(1) \cdots u(i)\, n\, v(1) \cdots v(n-1-i),$$

as shown in Figure 2.3.

Since the number of descents of w is one more than the number of descents in u plus the number of descents in v, we get

$$\sum_{\substack{w \in S_n(231) \\ w(i+1)=n}} t^{\mathrm{des}(w)} = tC_i(t)C_{n-1-i}(t). \tag{2.4}$$

Of course if $i = n-1$ then v is the empty word and the number of descents of w equals only the number of descents of u. This contributes a $C_{n-1}(t)$ term to the distribution, and then summing (2.4) over all $i < n-1$ gives the following result, which is similar to Theorem 1.5 for Eulerian polynomials.

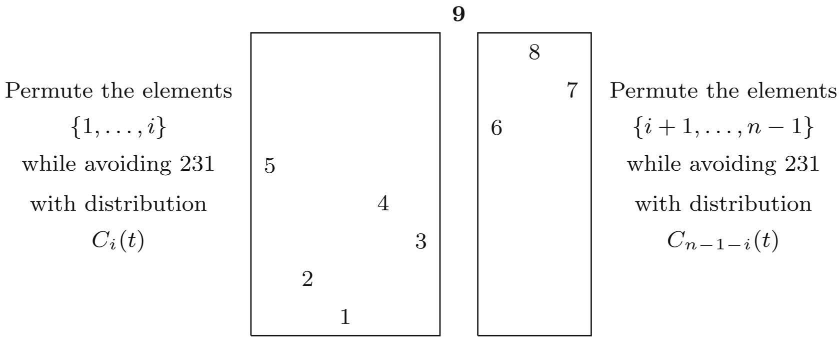


Fig. 2.3 The idea behind Equation (2.5).

Theorem 2.2. *For* $n \geq 1$,

$$C_n(t) = C_{n-1}(t) + t \sum_{i=0}^{n-2} C_i(t) C_{n-1-i}(t). \tag{2.5}$$

Now that we have the recurrence from Theorem 2.2 it is a straightforward matter to construct the generating function for the Narayana polynomials,

$$C(t, z) := \sum_{n \geq 0} C_n(t) z^n.$$

We have:

$$\begin{aligned} C(t, z) &= \sum_{n \geq 0} C_n(t) z^n, \\ &= 1 + \sum_{n \geq 1} \left[C_{n-1}(t) + t \sum_{i=0}^{n-2} C_i(t) C_{n-1-i}(t) \right] z^n, \\ &= 1 + z \sum_{n \geq 1} C_{n-1} z^{n-1} + tz \sum_{n \geq 1} \sum_{i=0}^{n-2} C_i(t) z^i C_{n-1-i}(t) z^{n-1-i}, \\ &= 1 + zC(t, z) + tzC(t, z)(C(t, z) - 1). \end{aligned}$$

From this we can conclude that $C(t, z)$ satisfies:

$$tzC(t, z)^2 - (1 + z(t - 1))C(t, z) + 1 = 0.$$

Solving for $C(t, z)$ gives:

$$C(t, z) = \frac{1 + z(t - 1) - \sqrt{1 - 2z(t + 1) + z^2(t - 1)^2}}{2tz}. \tag{2.6}$$

The 231-avoiding permutations are one combinatorial interpretation for the Catalan numbers, but there are many, many others. (See the notes at the end of the chapter.) There are three others that we will introduce and discuss now, with more deferred to the problems at the end of the chapter.

2.4 Dyck paths

A *Dyck path* of length $2n$ is a lattice path from $(0,0)$ to (n,n) consisting of n horizontal steps "East" from (i,j) to $(i+1,j)$ and n vertical steps "North" from (i,j) to $(i,j+1)$, such that all points on the path satisfy $i \leq j$, i.e., the path, when drawn in the cartesian plane, lies on or above the line $y = x$. We can either draw the picture of the path or write the list of steps the path follows as a word on the set $\{N, E\}$. For example, the path $p = NNENNEEEENENNNEEE$ would be drawn as in Figure 2.4.

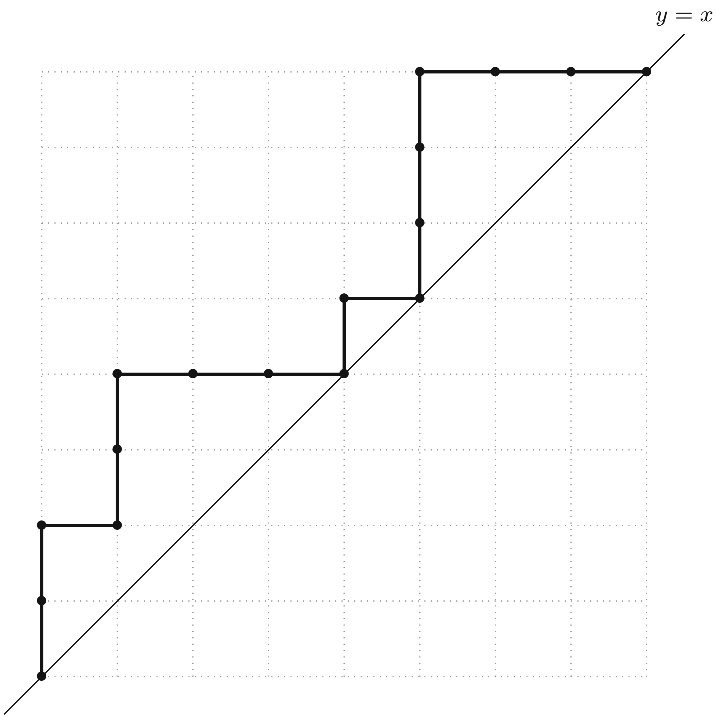


Fig. 2.4 One of the 4862 paths in Dyck(8).

Let Dyck(n) denote the set of Dyck paths of length $2n$. A *peak* of a Dyck path p is a point (i,j) such that $(i,j-1)$ and $(i+1,j)$ are on p as well.

Similarly, a *valley* of p is a point (i, j) such that $(i-1, j)$ and $(i, j+1)$ are on p. In other words, a peak corresponds to a North step followed immediately by an East step, while a valley corresponds to an East step followed immediately by a North step. The number of peaks of p is denoted $\mathrm{pk}(p)$ and the number of valleys is $\mathrm{val}(p)$. For example, the path of Figure 2.4 has four peaks, $\mathrm{pk}(p) = 4$, and three valleys, $\mathrm{val}(p) = 3$. It is easy to see that for $p \in \mathrm{Dyck}(n)$, $1 \le \mathrm{pk}(p) \le n$, while $0 \le \mathrm{val}(p) = \mathrm{pk}(p) - 1 \le n-1$. The Dyck paths for $n \le 4$ are shown in Table 2.3, grouped according to the number of peaks in the path.

At the end of this section we will provide a bijection between Dyck paths and 231-avoiding permutations, but first we will give bijective proofs that there are Catalan-many Dyck paths and that counting Dyck paths according to the number of peaks gives rise to the Narayana numbers.

2.4.1 Counting all Dyck paths

The Catalan number C_n can be written as a difference of two binomial coefficients:

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n-1}.$$

Notice that there are a total of $\binom{2n}{n}$ lattice paths from $(0,0)$ to (n,n) since we have $2n$ steps and exactly n of them must be N steps. Similarly, we can think of $\binom{2n}{n-1}$ as counting the paths from $(0,0)$ to $(n+1, n-1)$. Thus to give a direct combinatorial proof that $|\,\mathrm{Dyck}(n)| = C_n$, we will write $\binom{2n}{n} = C_n + \binom{2n}{n-1}$ and describe a bijection

$$\left\{\begin{matrix}\text{lattice paths from}\\ (0,0) \text{ to } (n,n)\end{matrix}\right\} \longleftrightarrow \mathrm{Dyck}(n) \bigcup \left\{\begin{matrix}\text{lattice paths from}\\ (0,0) \text{ to } (n+1,n-1)\end{matrix}\right\}.$$

The idea here is called the *reflection principle*. Let p be a path from $(0,0)$ to (n,n). If p never passes below the line $y = x$, it is a Dyck path. If it does go below this line, say the *reflection point* of p is the first time the path hits the line $y = x - 1$. The *reflection* of p, $r(p)$, is the path obtained by swapping E for N on every step after the reflection point. For example, if

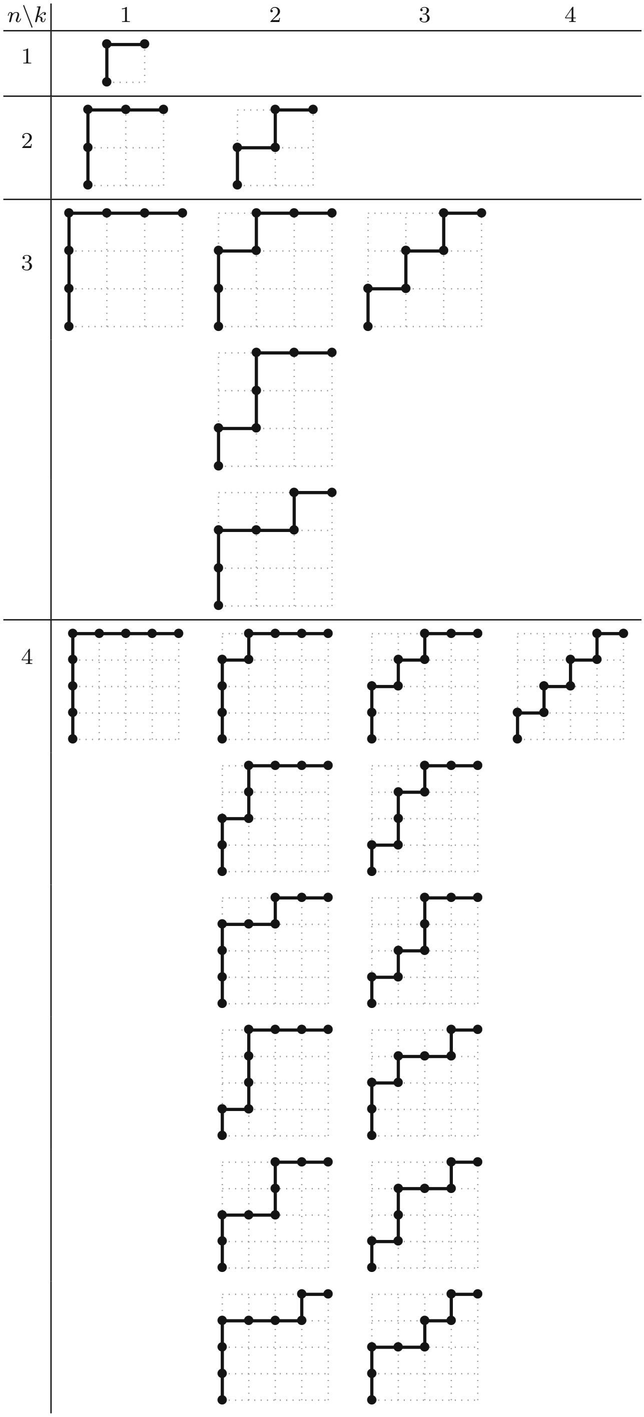
$$p = NNEEE|NNENEEENNEN,$$

then

$$r(p) = NNEEE|EENENNNEENE,$$

is its reflection. The vertical bar here is used to mark the reflection point. In terms of words on $\{N, E\}$, this is simply the first time, in reading from left to right, that we have more letters E than N. This example is drawn in Figure 2.5.

Table 2.3 The paths in Dyck(n), $n \leq 4$, grouped by number of peaks, k.



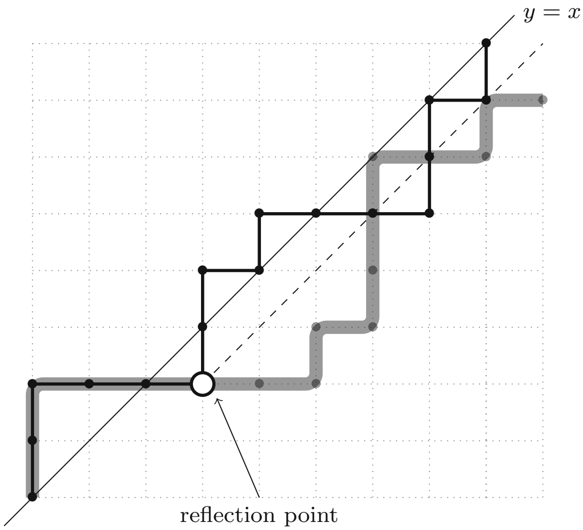


Fig. 2.5 The reflection of a lattice path.

The reflection map is easily reversed (the reflection point is well defined for both sets of paths), so the paths from $(0,0)$ to (n,n) that go below the line $y=x$ are in bijection with all the paths from $(0,0)$ to $(n+1,n-1)$. This shows

$$|\operatorname{Dyck}(n)| = \binom{2n}{n} - \binom{2n}{n-1} = C_n,$$

as desired.

2.4.2 Counting Dyck paths by peaks

Earlier we claimed the Narayana numbers are given by

$$N_{n,k} = \frac{1}{k+1}\binom{n}{k}\binom{n-1}{k}.$$

We will now show that

$$|\{p \in \operatorname{Dyck}(n) : \operatorname{pk}(p) = k+1\}| = \frac{1}{k+1}\binom{n}{k}\binom{n-1}{k}.$$

We will subsequently show that

$$|\{p \in \operatorname{Dyck}(n) : \operatorname{pk}(p) = k+1\}| = |\{w \in S_n(231) : \operatorname{des}(w) = k\}|,$$

justifying the formula for the Narayana numbers.

Our goal will be to show

$$(k+1)|\{p \in \text{Dyck}(n) : \text{pk}(p) = k+1\}| = \binom{n}{k}\binom{n-1}{k}.$$

To do so, we will exhibit a certain set $\mathcal{P}$ of $\binom{n}{k}\binom{n-1}{k}$ lattice paths and show that it can be partitioned into equivalence classes. We will then show each equivalence class has $k+1$ elements and contains exactly one path that corresponds to a Dyck path with $k+1$ peaks.

Define $\mathcal{P}$ to be the set of lattice paths from $(0,-1)$ to (n,n) that begin with a North step, end with an East step, and have exactly $k+1$ peaks, or k valleys. Each such path can be reconstructed from the coordinates of its valleys: $(x_1, y_1), (x_2, y_2), \ldots, (x_k, y_k)$. There are $\binom{n}{k}$ ways to choose the vertical coordinates: $0 \leq y_1 < y_2 < \cdots < y_k \leq n-1$ in such a path, and $\binom{n-1}{k}$ ways to choose the horizontal coordinates: $1 \leq x_1 < x_2 < \cdots < x_k \leq n-1$. Hence $|\mathcal{P}| = \binom{n}{k}\binom{n-1}{k}$. See Figure 2.6.

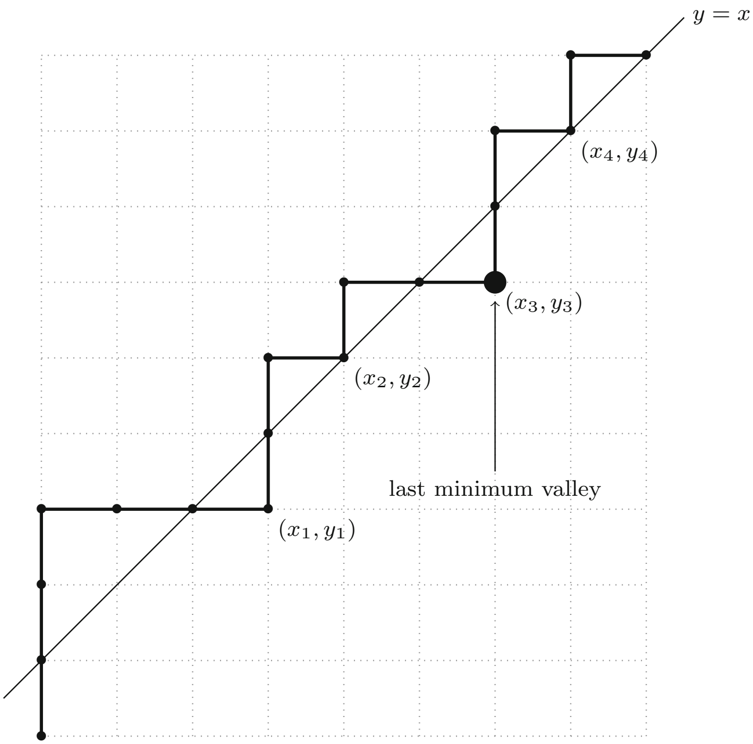


Fig. 2.6 A path from $(0,-1)$ to (n,n) with initial North step, final East step, and k valleys. Here $n = 8, k = 4$.

On the other hand, we can also characterize a path in $\mathcal{P}$ by a sequence of $k+1$ valley-less paths. In Figure 2.6, these valley-less paths are $NNNEEE$, NNE, NEE, NNE, and NE and we can write

$$p = (NNNEEE)(NEE)(NNE) \bullet (NNE)(NE).$$

An important marker in our path (indicated in the word with a $\bullet$) will be the rightmost valley (x_i, y_i) for which $y_i - x_i < 0$ is minimized. In terms of the $\{N, E\}$-word for the path, this is the rightmost position where the letters E most outnumber the letters N. If there are always more letters N than E, we put the marker on the far left.

We will now lump together these lattice paths into equivalence classes given by cyclically permuting the valley-less paths. Let $[p]$ denote the class of p. To continue our example,

$$[p] = \left\{ \begin{array}{c} (NNNEEE)(NEE)(NNE) \bullet (NNE)(NE) \\ (NE)(NNNEEE)(NEE)(NNE) \bullet (NNE) \\ \bullet (NNE)(NE)(NNNEEE)(NEE)(NNE) \\ (NNE) \bullet (NNE)(NE)(NNNEEE)(NEE) \\ (NEE)(NNE) \bullet (NNE)(NE)(NNNEEE) \end{array} \right\}.$$

Notice that the marker gets cyclically permuted along with the valley-less paths. (This is because the path from the marker onward always has more letters N than E when reading from left to right.) Hence, the marker uniquely identifies the cyclic permutation of p and the class $[p]$ must contain $k+1$ distinct paths. Moreover, there is always one path that has the marker on the far left. This path has all its valleys satisfying $y_i - x_i \geq 0$, and hence (if we ignore the initial North step) it is a Dyck path. The cyclic action is shown in pictures in Figure 2.7.

Hence, we can conclude

$$\begin{aligned} (k+1)|\{p \in \mathrm{Dyck}(n) : \mathrm{pk}(p) = k+1\}| &= |\mathcal{P}|, \\ &= \binom{n}{k}\binom{n-1}{k}, \end{aligned}$$

as desired.

2.4.3 A bijection with 231-avoiding permutations

We can construct a playful bijection between Dyck paths and 231-avoiding permutations as follows. First draw a permutation as an array of nonattacking rooks on a chessboard, i.e., if $w(i) = j$, put a rook in column i (from left to right), row j (from bottom to top). Then shade in all squares on the board that either contain a rook, or are weakly to the left and weakly above a

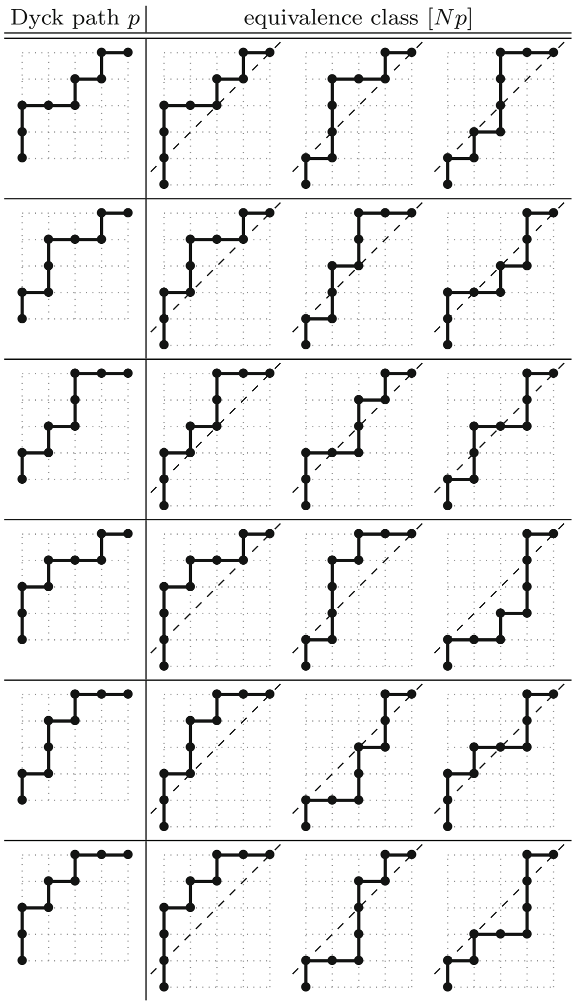


Fig. 2.7 An equivalence relation on lattice paths for $n = 4$, $k + 1 = 3$ peaks.

square with a rook. The boundary of the shaded region is a path that stays below or on the line $y = x$, so it is the mirror image of a Dyck path. Let $\psi : S_n(231) \to \text{Dyck}(n)$ denote this bijection. See Figure 2.8.

The pre-image of a path p is constructed as follows. First, draw the mirror-image of path p, and place rooks, from right to left, in the lowest unoccupied row that is above the path, as shown in Figure 2.9.

From this construction, we can see that each peak of the path p (where we placed the corner rooks in ψ^{-1}) corresponds to a maximal decreasing run $w(i) > w(i + 1) > \cdots > w(j)$ of $\psi^{-1}(p) = w$. The number of maximal decreasing runs is necessarily $n - \text{des}(w)$, and so we have the following.

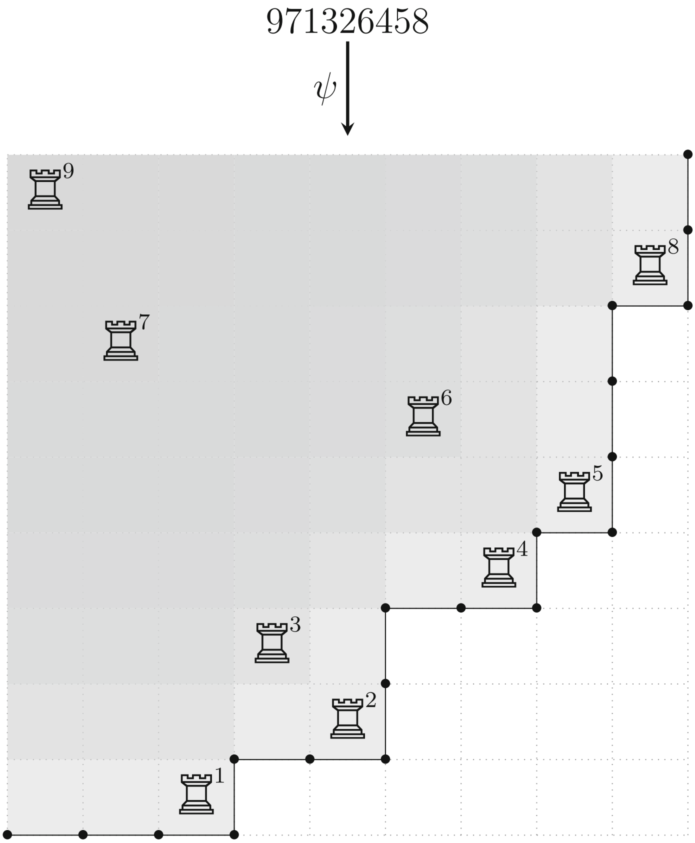


Fig. 2.8 Constructing a Dyck path from a 231-avoiding permutation.

Proposition 2.1. *For any $w \in S_n(231)$, the bijection ψ satisfies*

$$\mathrm{des}(w) = n - 1 - \mathrm{val}(p) = n - \mathrm{pk}(p).$$

Hence,

$$|\{w \in S_n(231) : \mathrm{des}(w) = k\}| = |\{p \in \mathrm{Dyck}(n) : \mathrm{pk}(p) = \mathrm{val}(p) + 1 = k + 1.\}|.$$

This justifies the formula for the Narayana numbers $N_{n,k} = \frac{1}{k+1}\binom{n}{k}\binom{n-1}{k}$.

We finish the chapter with brief discussion of two other popular combinatorial models counted by the Narayana numbers.

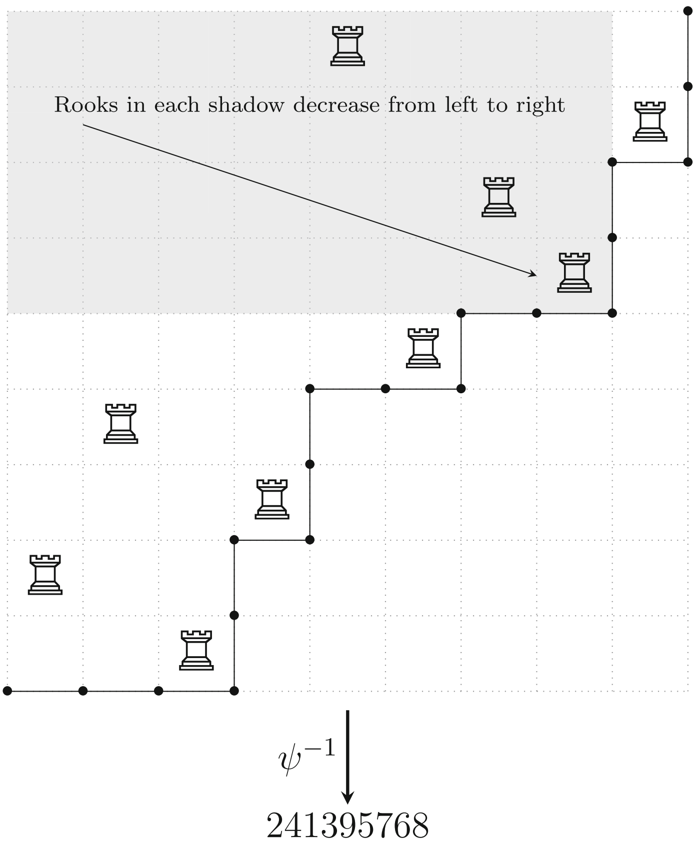
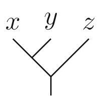


Fig. 2.9 Constructing a 231-avoiding permutation from a Dyck path.

2.5 Planar binary trees

A *planar binary tree* is a rooted tree such that every interior node has precisely two successors. If there are n internal nodes, this means there are $n+1$ leaves. Let $\mathrm{PB}(n)$ denote the number of planar binary trees with n internal nodes. Table 2.4 shows the planar binary trees with at most $n = 4$ internal nodes, grouped according to the number of left-pointing leaves.

The planar binary trees are combinatorial representations for ways to evaluate an associative product of $n+1$ elements. For example, if $n = 2$, we have $((xy)z)$ and $(x(yz))$ as the two possible ways to evaluate the product xyz, and these would correspond to the following trees:



and

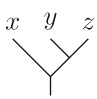


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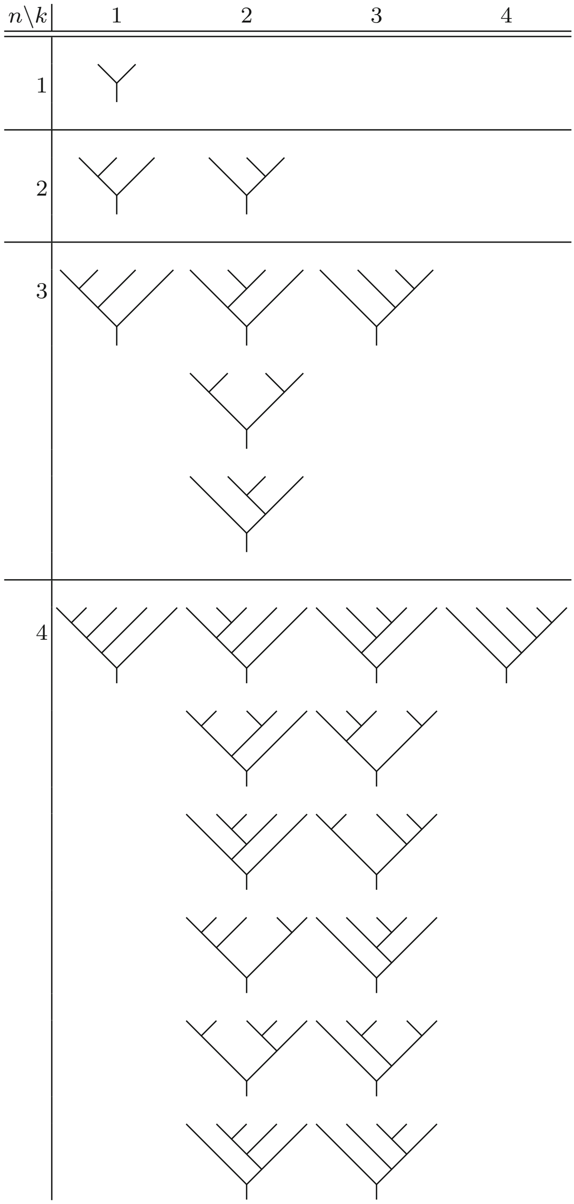
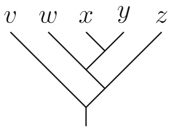
Table 2.4 Planar binary trees grouped according to the number of left-pointing leaves.



where we labeled the leaves by x, y, z to indicate the natural bijection. As a larger example,



$$\longleftrightarrow (v((w(xy))z)).$$

Planar binary trees can be shown to satisfy the Catalan recurrence (see Problem 2.3), but one can also give a direct bijection with 231-avoiding permutations that takes left-pointing leaves to descents, as suggested by the example in Figure 2.10.

Proof of the following proposition is deferred to Problem 2.4.

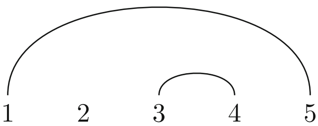
Proposition 2.2. *There is a bijection between* $\mathrm{PB}(n)$ *and* $S_n(231)$ *such that planar binary trees with* $k+1$ *left-pointing leaves are mapped to 231-avoiding permutations with* k *descents.*

In other words, the Narayana numbers count planar binary trees according to left-pointing leaves:

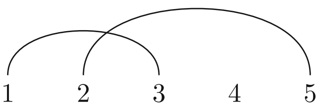
$$N_{n,k} = |\{\tau \in \mathrm{PB}(n) : \tau \text{ has } k+1 \text{ left-pointing leaves}\}|.$$

2.6 Noncrossing partitions

A *noncrossing partition* $\pi = \{R_1, R_2, \ldots, R_k\}$, is a set partition of $[n]$, such that if $\{a, c\} \subseteq R_i$ and $\{b, d\} \subseteq R_j$, with $1 \leq a < b < c < d \leq n$, then $i = j$. That is, two pairs of numbers from distinct blocks cannot be interleaved. Let $\mathrm{NC}(n)$ denote the set of all noncrossing partitions of $[n]$. We will often draw partitions as graphs with vertex set $[n]$, e.g.,



$$= \{\{1,5\},\{2\},\{3,4\}\} \in \mathrm{NC}(5),$$



$$= \{\{1,3\},\{2,5\},\{4\}\} \notin \mathrm{NC}(5).$$

Notice how the notion of a "crossing" manifests itself visually in these diagrams. So that our pictures are canonical, we will only have arcs between consecutive elements in the blocks of the partition. For example, if $i < j < k$ are in the same block, we would draw

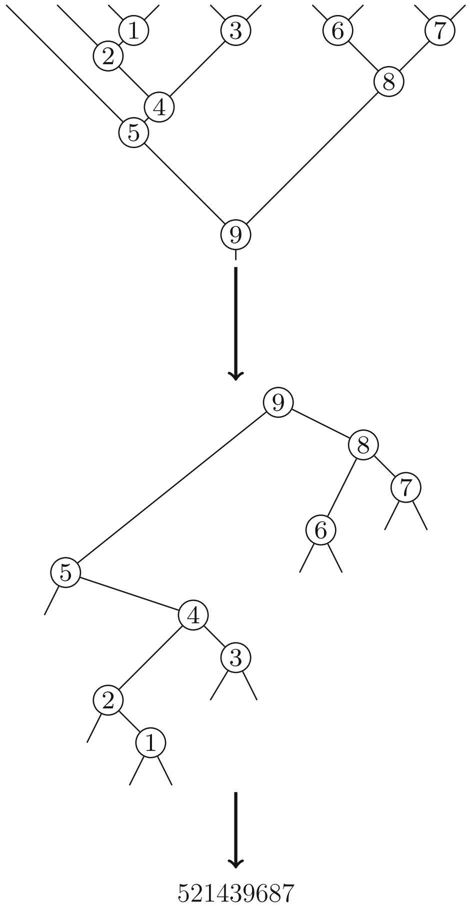
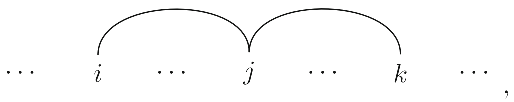


Fig. 2.10 A correspondence between planar binary trees and 231-avoiding permutations.



but not

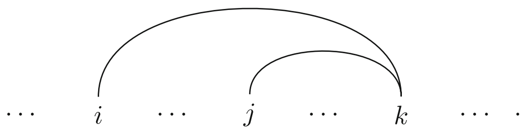
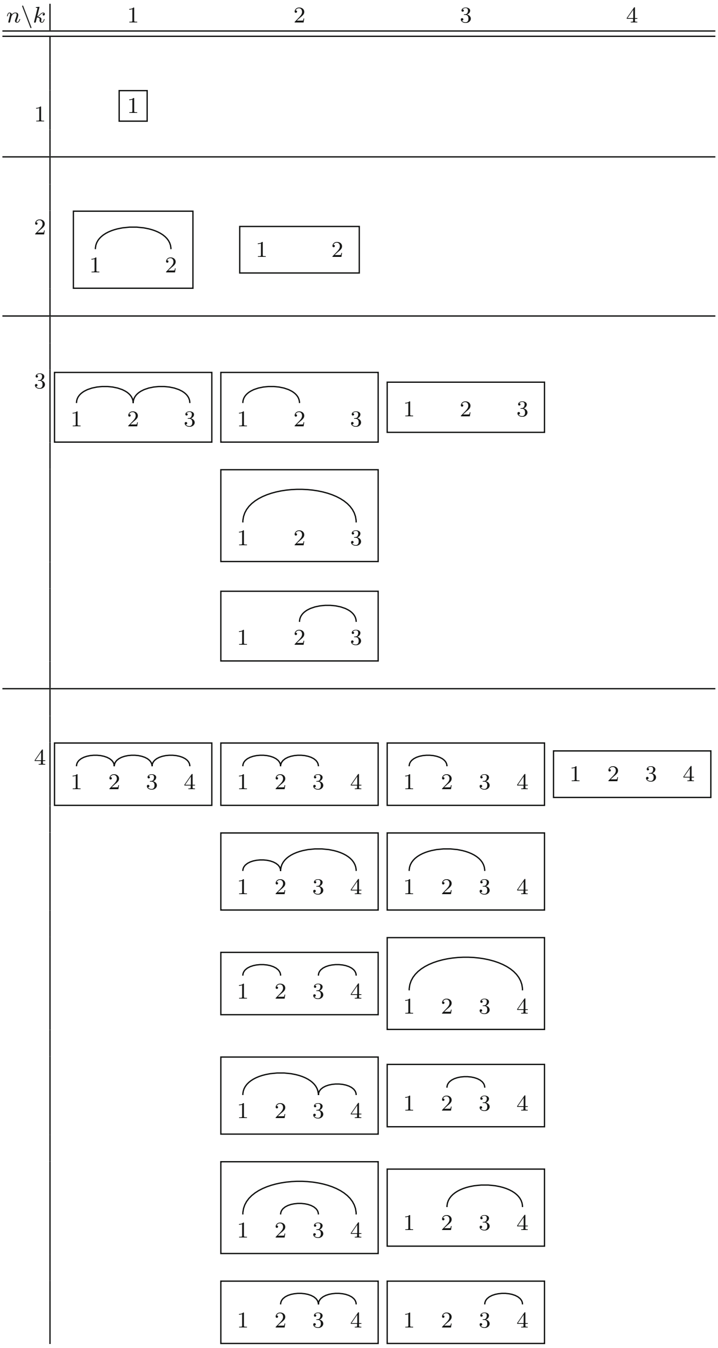


Table 2.5 shows all the noncrossing partitions on at most 4 elements, grouped according to the number of blocks.

Table 2.5 Noncrossing partitions on up to four elements, grouped according to number of blocks.



We can define a bijection $\phi : S_n(231) \to \mathrm{NC}(n)$ by mapping the decreasing runs of a permutation to blocks in a partition. See Figure 2.11.

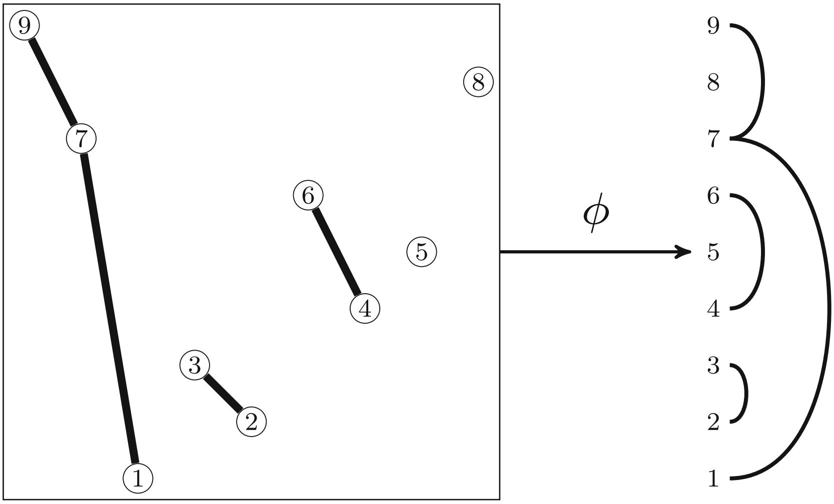


Fig. 2.11 The decreasing runs of a 231-avoiding permutation form a noncrossing partition.

Moreover, we can see that the number of decreasing runs of w, i.e., the number of blocks in π, is $n - \mathrm{des}(w)$.

Proposition 2.3. *For any $w \in S_n(231)$, the bijection ϕ satisfies*

$$\mathrm{des}(w) = n - |\phi(w)|.$$

Hence,

$$|\{w \in S_n(231) : \mathrm{des}(w) = k\}| = |\{\pi \in \mathrm{NC}(n) : |\pi| = n - k\}|.$$

In other words, the Narayana numbers count noncrossing partitions by the number of blocks:

$$N_{n,k} = |\{\pi \in \mathrm{NC}(n) : |\pi| = n - k\}|.$$

Verification of this claim is left to Problem 2.5.

Notes

Despite the name, it seems that it was Euler who first studied the Catalan numbers, which he defined as the number of ways to triangulate a convex polygon. (See Problem 2.6.) There is correspondence between Euler and Christian Goldbach from the middle of the 18th century that shows Euler knew the formula for the Catalan number generating function given in (2.2). Johann Segner was the first to publish a paper about these numbers, in which he proves the recurrence relation from (2.1). The Catalan numbers are named for Eugène Charles Catalan, a 19th century mathematician who wrote several papers about what he knew as the "Segner numbers." It was Catalan who proved that $C_n = \binom{2n}{n} - \binom{2n}{n-1}$.

Many famous mathematicians have studied the Catalan numbers, in many different guises. One can find more than two hundred different combinatorial interpretations for Catalan numbers in a book of Richard Stanley, with historical notes by Igor Pak [155]. See also [153, Problem 6.19].

The Narayana numbers are named for Tadepalli Narayana, who wrote several papers on them in the mid-twentieth century, including [110]. In this paper he essentially counts Dyck paths according to the number of peaks. Our method of counting Dyck paths can be found in the work of Robert Sulanke from 1993 [160].

Problems

2.1. Suppose p is any pattern of length three, i.e., $p \in \{123, 132, 213, 231, 312, 321\}$. Show that the Catalan numbers count the permutations of length n that avoid p.

2.2. Find a bijective proof of the fact that

$$(n+1)C_n = \binom{2n}{n}.$$

2.3. Let $b_n = |\mathrm{PB}(n)|$ denote the number of planar binary trees with n internal nodes. Show that $b_n = C_n$ by describing a structural recurrence on the trees that yields the numeric recurrence

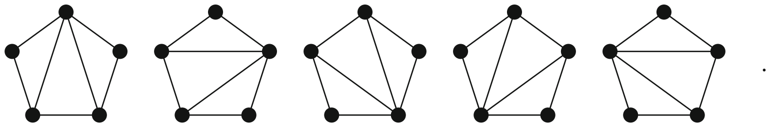
$$b_n = \sum_{i=0}^{n-1} b_i b_{n-1-i},$$

with $b_0 := 1$.

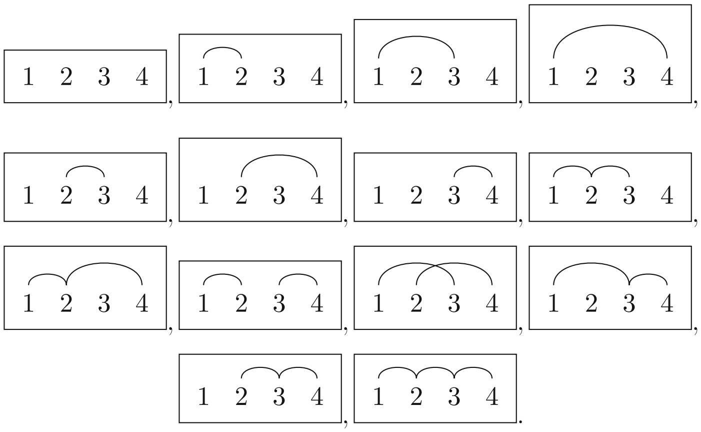
2.4. Prove Proposition 2.2. That is, construct a bijection between $\mathrm{PB}(n)$ and $S_n(231)$ such that trees with $k+1$ left-pointing leaves are mapped to 231-avoiding permutations with k descents.

2.5. Prove Proposition 2.3. That is, show that the map ϕ suggested in Figure 2.11 is indeed a bijection from $S_n(231)$ to $\mathrm{NC}(n)$ that takes decreasing runs to blocks of a noncrossing partition.

2.6 (Triangulations). Show that C_n counts the number of dissections of a convex $(n+2)$-gon into n triangles, using only lines from vertices to vertices. For example, when $n = 3$ there are five such triangulations of a pentagon:

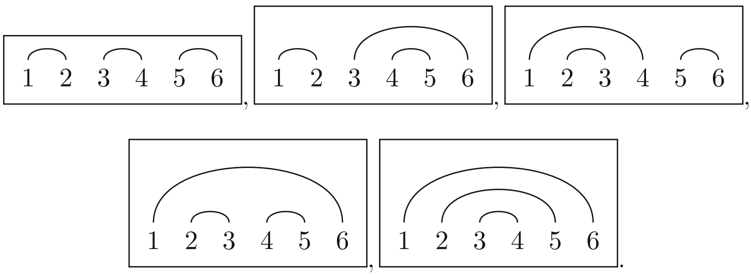


2.7 (Nonnesting partitions). Show that C_n counts the number of *nonnesting partitions* of $[n]$. A nonnesting partition is a set partition $\pi = \{R_1, \ldots, R_k\}$ such that if $\{a, d\} \subseteq R_i$ and $\{b, c\} \subseteq R_j$ with $a < b < c < d$, then $R_i = R_j$. Here are the fourteen nonnesting partitions of $\{1, 2, 3, 4\}$:



Hint: Create a bijection between noncrossing and nonnesting partitions. Conclude that counting nonnesting partitions by number of blocks gives the Narayana numbers.

2.8 (Noncrossing matchings, balanced parenthesizations). Show that C_n counts the number of *noncrossing matchings* on $[2n]$. A noncrossing matching is a noncrossing partition with all the blocks having size two. For example, here are the five noncrossing matchings on $\{1, 2, 3, 4, 5, 6\}$:



The noncrossing matchings can also be thought of as n pairs of parentheses, by mapping the beginning of an arc to a left parenthesis, "(", and mapping the end of an arc to a right parenthesis, ")". The five matchings above would then be:

$$()()(),\ ()(()),\ (())(),\ (()()),\ ((())).$$

A string of n pairs of parentheses that never has more right parentheses than left when reading from left to right is called a *balanced parenthesization*.

Refined counting: Describe a statistic for noncrossing matchings so that the distribution of this statistic gives the Narayana numbers.

2.9 (Standard Young tableaux). Show that C_n counts the number of 2 by n *standard Young tableaux*. A Young tableau is a two dimensional array of numbers that increases across rows and down columns. A standard Young tableau contains all distinct integers, from 1 to the number of entries. The fourteen 2 by 4 tableaux are:

$$\begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline 5&6&7&8\\\hline\end{array},\ \begin{array}{|c|c|c|c|}\hline 1&2&3&5\\\hline 4&6&7&8\\\hline\end{array},\ \begin{array}{|c|c|c|c|}\hline 1&2&4&5\\\hline 3&6&7&8\\\hline\end{array},\ \begin{array}{|c|c|c|c|}\hline 1&2&3&6\\\hline 4&5&7&8\\\hline\end{array},$$

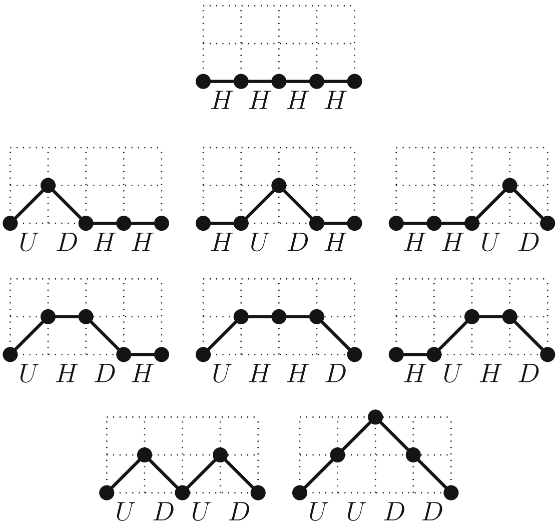
$$\begin{array}{|c|c|c|c|}\hline 1&3&4&5\\\hline 2&6&7&8\\\hline\end{array},\ \begin{array}{|c|c|c|c|}\hline 1&2&5&6\\\hline 3&4&7&8\\\hline\end{array},\ \begin{array}{|c|c|c|c|}\hline 1&2&3&7\\\hline 4&5&6&8\\\hline\end{array},\ \begin{array}{|c|c|c|c|}\hline 1&2&4&6\\\hline 3&5&7&8\\\hline\end{array},$$

$$\begin{array}{|c|c|c|c|}\hline 1&3&4&6\\\hline 2&5&7&8\\\hline\end{array},\ \begin{array}{|c|c|c|c|}\hline 1&3&5&6\\\hline 2&4&7&8\\\hline\end{array},\ \begin{array}{|c|c|c|c|}\hline 1&2&4&7\\\hline 3&5&6&8\\\hline\end{array},\ \begin{array}{|c|c|c|c|}\hline 1&3&4&7\\\hline 2&5&6&8\\\hline\end{array},$$

$$\begin{array}{|c|c|c|c|}\hline 1&2&5&7\\\hline 3&4&6&8\\\hline\end{array},\ \begin{array}{|c|c|c|c|}\hline 1&3&5&7\\\hline 2&4&6&8\\\hline\end{array}.$$

Refined counting: Describe a statistic for Young tableaux so that the distribution of this statistic gives the Narayana numbers.

2.10 (Motzkin paths). A *Motzkin path* of length n is a lattice path from $(0,0)$ to (n,n) that never passes below the line $y = 0$ and uses only "up" steps from (i,j) to $(i+1, j+1)$, "down" steps from (i,j) to $(i+1, j-1)$, and "horizontal" steps from (i,j) to $(i+1,j)$. For example, here are the nine Motzkin paths of length four:



(Note that Motzkin paths that contain no horizontal steps are in bijection with Dyck paths.) Let M_n denote the number of Motzkin paths of length n, with $M_0 = 1$. Here are the first few values of M_n, sometimes called *Motzkin numbers*:

$$1, 1, 2, 4, 9, 21, 51, 127, 323, 835, 2188, 5798, \ldots.$$

Let $M(z) = \sum_{n \geq 0} M_n z^n$. Show that

$$M(z) = \frac{1 - z - \sqrt{1 - 2z - 3z^2}}{2z^2}.$$

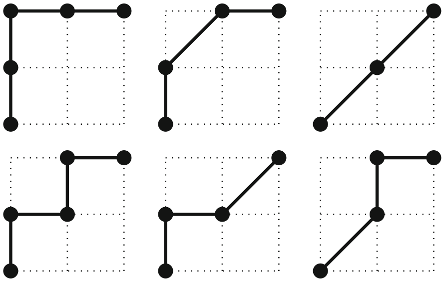
Hint: each Motzkin path is built from a Dyck path by inserting horizontal steps between the steps of the Dyck path. Use this fact to show

$$M(z) = \frac{1}{1-z} C\left(\frac{z^2}{(1-z)^2}\right),$$

where $C(z)$ is the Catalan generating function. The formula for $M(z)$ now follows from Equation (2.2).

2.11. Show that the Motzkin number M_n also counts the number of noncrossing *partial matchings* of $[n]$. In other words, M_n is the number of noncrossing partitions of $[n]$ for which the blocks have size one or two.

2.12 (Schröder paths). A *Schröder path* of size n is a lattice path from $(0, 0)$ to (n, n) that never passes below the line $y = x$ and uses only steps "North" from (i, j) to $(i, j+1)$, "East" from (i, j) to $(i+1, j)$ and "Northeast" from (i, j) to $(i + 1, j + 1)$. For example, here are the six Schröder paths of size 2:



(Note that Schröder paths with no northeast steps are Dyck paths.) Let R_n denote the number of Schröder paths of size n, with $R_0 = 1$. We call the number R_n a *Schröder number*. Here are the first few values for R_n:

$$1, 2, 6, 22, 90, 394, 1806, 8558, 41586, 206098, \ldots.$$

Let $R(z) = \sum_{n \geq 0} R_n z^n$. Show that

$$R(z) = \frac{1 - z - \sqrt{1 - 6z + z^2}}{2z}.$$

Hint: Just as with Motzkin paths, each Schröder path can be built from a Dyck path by inserting northeast steps between the steps of the Dyck path. Use this fact to show

$$R(z) = \frac{1}{1-z} C\left(\frac{z}{(1-z)^2}\right),$$

where $C(z)$ is the Catalan generating function. The formula then follows from Equation (2.2).

2.13 (Small Schröder numbers). Show the Schröder numbers (apart from $R_0 = 1$) are always even. You can do this by manipulating the generating function in Problem 2.12, but try to explain it combinatorially. Hint: find a bijection between the Schröder paths with a peak on the line $y = x + 1$ and those without. The number of Schröder paths with no peak on the line $y = x + 1$ are called *small Schröder numbers*, denoted r_n. Here are the first few values of the small Schröder numbers:

$$1, 1, 3, 11, 45, 197, 903, 4279, 20793, 103049, \ldots.$$

Given that $r_0 = 1$ and $r_n = R_n/2$ for $n \geq 1$, use the generating function found in Problem 2.12 to conclude that

$$\sum_{n \geq 0} r_n z^n = \frac{1 + z - \sqrt{1 - 6z + z^2}}{4z}.$$

2.14. Show that the small Schröder numbers r_n count the number of valid parenthesizations of $n+1$ symbols with at most $n-1$ pairs of parentheses. Parentheses around the entire expression are not allowed, and each pair of parentheses must enclose at least two sub-expressions. For example, here are the eleven parenthesizations of four symbols:

$$((wx)y)z \quad (w(xy))z \quad (wx)(yz) \quad w((xy)z) \quad w(x(yz))$$

$$(wx)yz \quad (wxy)z \quad w(xy)z \quad w(xyz) \quad wx(yz)$$

$$wxyz$$

Can you interpret these parenthesizations in terms of planar rooted trees of some kind?