Chapter 14
Affine descents and the Steinberg torus (Supplemental)

14.1 Affine Weyl groups

In this section we outline some basic facts for affine Weyl groups, following standard notations. See Sections 4.3 and 4.6 of the book by James Humphreys for more details [92].

We now consider that Φ is an irreducible and *crystallographic* root system, i.e., $2\langle\alpha,\beta\rangle/\langle\beta,\beta\rangle$ is an integer for all roots α and β. These root systems are listed in Figure 11.4. The group $W = W(\Phi)$ is a finite Coxeter group, but there is an infinite Coxeter group associated with Φ as well, known as the *affine Weyl group*, and denoted $\widetilde{W}$. This is the group generated by reflections $s_{\beta,k}$ through the affine hyperplanes

$$H_{\beta,k} = \{\lambda \in V : \langle\lambda,\beta\rangle = k\},$$

where $\beta \in \Pi$ and $k \in \mathbb{Z}$.

Let $\Phi^\vee$ denote the set of *coroots*

$$\beta^\vee := 2\beta/\langle\beta,\beta\rangle,$$

with $\beta \in \Phi$. Composing two reflections $s_{\beta,k}$ corresponding to the same β corresponds to translation by a vector in $\mathbb{Z}\Phi^\vee$. Let $L = \mathbb{Z}\Phi^\vee$ denote this lattice of translations, a subgroup of $\widetilde{W}$. The affine group $\widetilde{W}$ also contains the finite group W, generated by reflections across the hyperplanes $H_{\beta,0} = H_\beta$.

The crystallographic condition guarantees that W fixes L, and we can write $\widetilde{W}$ as a semidirect product $L \rtimes W$. The product in the semidirect product is

$$(\mu, w)\cdot(\mu', w') = (\mu + w(\mu'), ww').$$

T.K. Petersen, *Eulerian Numbers*, Birkhäuser Advanced Texts Basler Lehrbücher, DOI 10.1007/978-1-4939-3091-3_14

The geometric action of $\widetilde{W}$ on V extends both the action of W by linear reflections and the action of L by translations:

$$(\mu, w) \cdot \lambda = \mu + w(\lambda),$$

for $\mu \in \mathbb{Z}\Phi^\vee$, $w \in W$, and $\lambda \in V$.

Since Φ is irreducible, there is a unique maximum in its root poset, known as the *highest root* and denoted $\widetilde{\alpha}$. The group $\widetilde{W}$ is generated by $\widetilde{S} = S \cup \{s_{\widetilde{\alpha},1}\}$. The pair $(\widetilde{W}, \widetilde{S})$ is an irreducible Coxeter system. The corresponding Coxeter graphs/Dynkin diagrams are shown in Figure 14.1. The graph for $\widetilde{W}$ differs from that of W by the addition of one node. Geometrically, the new simple root is the *lowest root* $\alpha_0 = -\widetilde{\alpha}$. If $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ denotes the set of simple roots for W, let us denote the nodes of the diagram by

$$\widetilde{\Delta} = \{\alpha_0\} \cup \Delta = \{\alpha_0, \alpha_1, \ldots, \alpha_n\}.$$

Let $\widetilde{\Sigma}$ denote the set of faces of the affine hyperplane arrangement

$$\widetilde{\mathcal{H}}(\Phi) = \{H_{\beta,k} : \beta \in \Phi, k \in \mathbb{Z}\}.$$

By adding an empty face, $\widetilde{\Sigma}$ is a simplicial complex isomorphic to the Coxeter complex for $\widetilde{W}$. The maximal faces in this arrangement are called *alcoves* (as opposed to *chambers* in the finite case).

The *fundamental alcove* is

$$A_\emptyset = C_\emptyset \cap \{\lambda \in V : \langle \lambda, \widetilde{\alpha} \rangle < 1\},$$

where $C_\emptyset$ is the fundamental chamber of the finite Coxeter arrangement. We can write the faces of the fundamental alcove as

$$A_J = \begin{cases} C_J \cap \{\lambda \in V : \langle \lambda, \widetilde{\alpha} \rangle < 1\} & \text{if } \alpha_0 \notin J, \\ C_{J-\{\alpha_0\}} \cap \{\lambda \in V : \langle \lambda, \widetilde{\alpha} \rangle = 1\} & \text{if } \alpha_0 \in J, \end{cases}$$

where J is a proper subset of $\widetilde{\Delta}$ and C_J is a face of the fundamental chamber as in Section 11.7.

In Figure 14.2 we see the affine arrangement and faces of the fundamental alcove for $\mathbf{A}_2$. The same for $\mathbf{C}_2$ is in Figure 14.3.

14.2 Faces of the affine Coxeter complex

The closure of the fundamental alcove is a fundamental domain for the action of $\widetilde{W}$ on V, and each face of $\widetilde{\Sigma}$ is of the form

$$F = \mu + w \cdot A_J,$$

where $\mu \in L$, $w \in W$, and J is a proper subset of $\widetilde{\Delta}$.

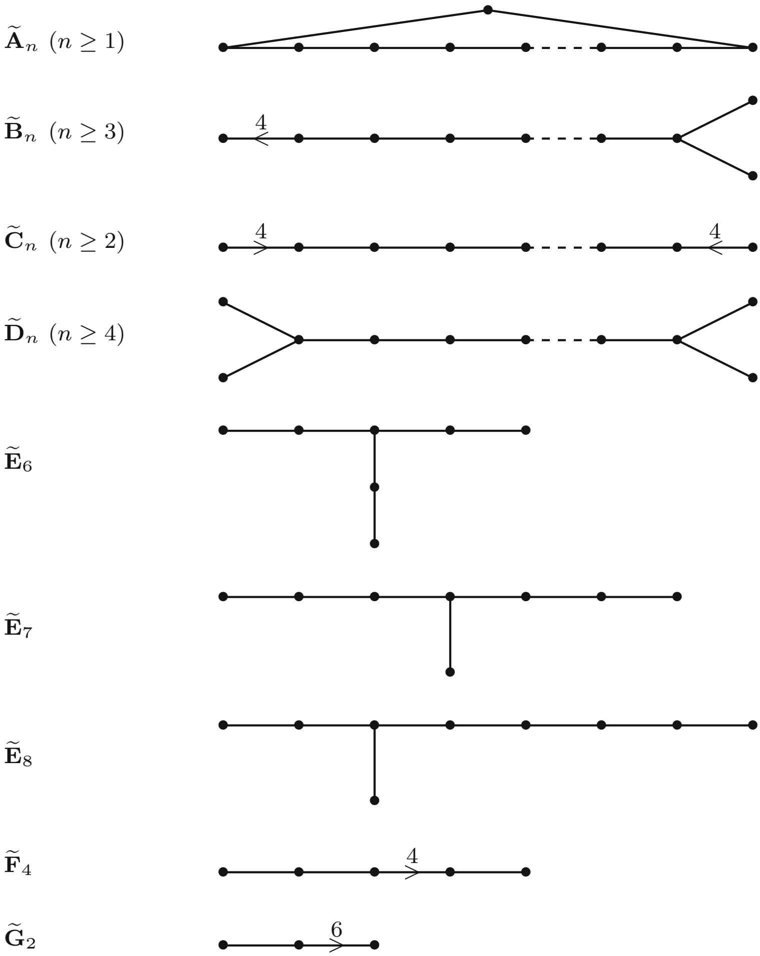
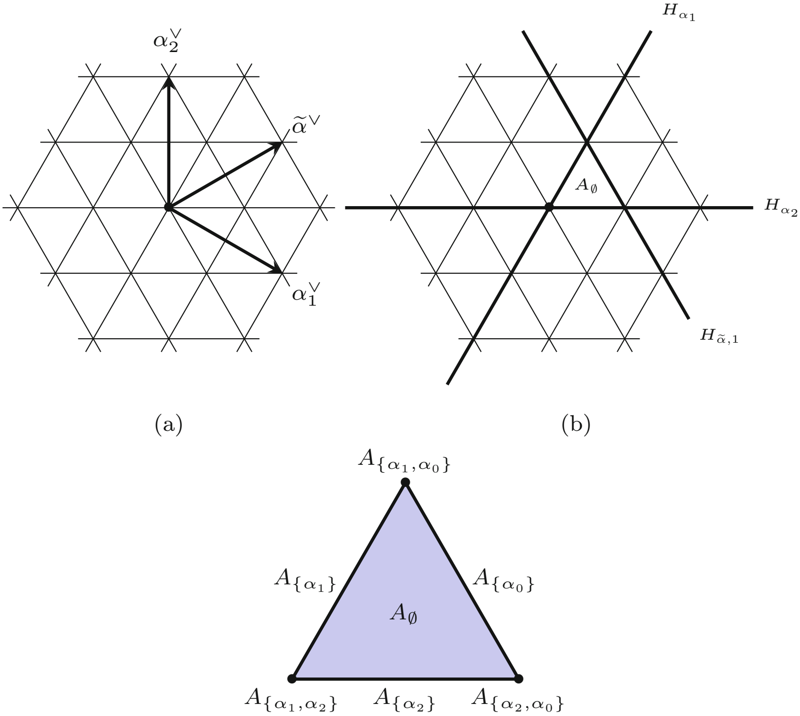


Fig. 14.1 The Dynkin diagrams for irreducible affine root systems.

The vertices of $\widetilde{\Sigma}$ are of the form $\mu + w \cdot A_{\widetilde{\Delta}-\{\alpha\}}$ for some $\alpha \in \widetilde{\Delta}$. If we assign color α to all such vertices, we obtain a balanced coloring of $\widetilde{\Sigma}$, with face $\mu + w \cdot A_J$ receiving color set $\widetilde{\Delta} - J$.

Each face F has a canonical representation, in the sense that we can identify F with a triple (μ, w, J), for $\mu \in L$, $w \in W$, and $J \subset \widetilde{\Delta}$. The uniqueness of μ is not surprising, since we can translate any face to a face in the neighborhood of the origin. The uniqueness of J follows from the fact that each face is in the orbit of a unique face of the closure of $A_\emptyset$. The finite group element w is unique up to right multiplication by the subgroup of W that fixes A_J. We can make the choice of w unique by declaring that, for any



(c)

Fig. 14.2 The affine arrangement $\widetilde{\mathcal{H}}(A_2)$. (a) Positive (co)roots. (b) Affine hyperplanes and the fundamental alcove. (c) The faces of the fundamental alcove.

$\alpha \in \widetilde{\Delta}$, if $w(\alpha) < 0$, then $\alpha \in \widetilde{\Delta} - J$. Following Paola Cellini [47], we define the *affine descent set* of an element of the finite group W to be

$$\begin{aligned}\widetilde{\mathrm{Des}}(w) &= \{\alpha \in \widetilde{\Delta} : w(\alpha) < 0\}, \\ &= \begin{cases}\mathrm{Des}(w) & \text{if } w(\alpha_0) > 0, \\ \mathrm{Des}(w) \cup \{\alpha_0\} & \text{if } w(\alpha_0) < 0.\end{cases}\end{aligned}$$

Notice that since α_0 is a negative root, this means every element $w \in W$ has at least one affine descent, including the identity. We can state the uniqueness of the representation as follows.

In the case of type A_{n-1}, we will see in Section 14.4.1 $\widetilde{\mathrm{Des}}(w)$ is the "cyclic" descent set of a permutation, i.e., the usual descent set along with a descent in zero if the last letter is larger than the first.

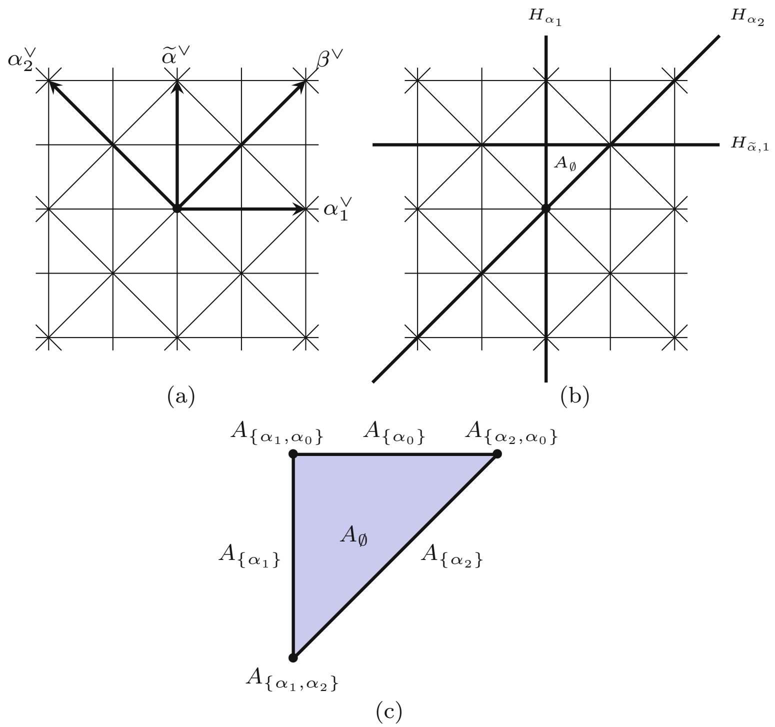


Fig. 14.3 The affine arrangement $\widetilde{\mathcal{H}}(C_2)$. (a) Positive coroots: $\alpha_1^\vee = \frac{1}{2}\alpha_1$, $\widetilde{\alpha}^\vee = \frac{1}{2}\widetilde{\alpha}$, $\alpha_2^\vee = \alpha_2$, $\beta^\vee = \beta$. (b) Affine hyperplanes and the fundamental alcove. (c) The faces of the fundamental alcove.

Proposition 14.1. *Each face $F \in \widetilde{\Sigma}$ has a unique representation*

$$F = \mu + w \cdot A_J,$$

with $\mu \in L$, $J \subset \widetilde{\Delta}$, and $w \in W$ such that $\widetilde{\mathrm{Des}}(w) \subseteq \widetilde{\Delta} - J$.

By analogy with the usual Eulerian polynomial, it now makes sense to define the *affine Eulerian polynomial* to be the generating function for *affine Eulerian numbers.* Let $\widetilde{\mathrm{des}}(w) = |\widetilde{\mathrm{Des}}(w)|$, and write

$$\widetilde{W}(t) = \sum_{w \in W} t^{\widetilde{\mathrm{des}}(w)}.$$

Before we study this polynomial and its coefficients, let us first describe a structure for which it is the h-polynomial.

14.3 The Steinberg torus

The coroot lattice L acts on V by translations and fixes the affine hyperplane arrangement $\widetilde{\mathcal{H}}$. Thus we can consider the set of L-orbits of faces of $\widetilde{\Sigma}$. The *Steinberg torus* is this quotient set of faces modulo translations, denoted by $\overline{\Sigma}$, i.e.,

$$\overline{\Sigma} = \widetilde{\Sigma}/L.$$

Geometrically, we can identify the Steinberg torus with a triangulation of the geometric torus V/L. This cell decomposition is not a simplicial complex, as different faces can share the same vertex set, but it is a boolean complex. Moreover, the balanced coloring for $\widetilde{\Sigma}$ passes through the quotient, so we inherit a balanced coloring for $\overline{\Sigma}$ as well.

The Steinberg torus is named for Robert Steinberg, who exploited the torus to help compute the length generating function for the affine Weyl group [157]. It was studied again (and named) by Kevin Dilks, John Stembridge, and the author in 2009 [60]. The presentation here largely follows [60].

Each face of $\widetilde{\Sigma}$ has in its orbit a cell in its L-orbit with $\mu = 0$, so another way to define the Steinberg torus is to identify opposite faces of the polytope

$$P_{\Phi} = \{\lambda \in V : -1 \leq \langle \lambda, \beta \rangle \leq 1 \text{ for all } \beta \in \Phi\}.$$

This polytope is the union of the closures of the alcoves $w \cdot A_{\emptyset}$, with $w \in W$. A point λ on the boundary of P_{Φ} has $\langle \lambda, \beta \rangle = -1$ for some root β. We identify λ with $\lambda' = \lambda + \beta^{\vee}$ which satisfies $\langle \lambda', \beta \rangle = 1$ and also lies on the boundary. See Figure 14.4.

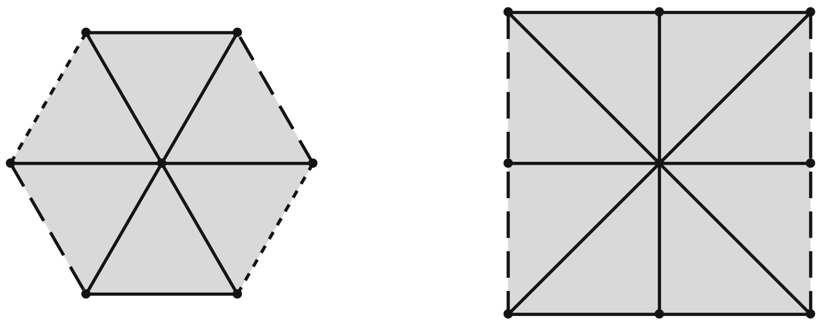


Fig. 14.4 The polytopes $P_{\mathbf{A}_2}$ and $P_{\mathbf{C}_2}$. The Steinberg tori are obtained by identifying points on the boundary.

From Proposition 14.1 we see that we can abstractly identify the faces of $\overline{\Sigma}$ with the cosets of "quasi-parabolic" subgroups of W, i.e., for any proper subset $J \subset \widetilde{\Delta}$,

$$L + w \cdot A_J \leftrightarrow wW_J = \{wv : v \in \langle s_\alpha : \alpha \in J\rangle\}.$$

As in Proposition 14.1, we can choose a unique minimal length representative w such that $\widetilde{\mathrm{Des}}(w) \subseteq \widetilde{\Delta} - J$.

Proposition 14.2. *Every face $F \in \overline{\Sigma}$ has a unique $J \subseteq \widetilde{\Delta}$ and $w \in W$ such that*

$$F = L + w \cdot A_J,$$

with $\widetilde{\mathrm{Des}}(w) \subseteq \widetilde{\Delta} - J$.

We call such subgroups W_J "quasi-parabolic" since although they are parabolic subgroups of $\widetilde{W}$, they are not necessarily parabolic subgroups of W. Such a group is always a finite Coxeter group, however, and a subgroup of W.

Just as the model of set compositions can be used to encode faces of the type A_{n-1} Coxeter complex, there is a similar combinatorial model to encode faces of the type A_{n-1} Steinberg torus, developed by Marcelo Aguiar and the author [2]. See Figure 14.5.

Also noteworthy at this point is that, unlike for Coxeter complexes, the distinction between types B_n and C_n really matters. This is because the structure of the torus is intimately linked with the root system, not merely the group. When $n \geq 3$, the polytopes P_Φ have very different boundaries, despite having the same number of maximal cells. In particular, the polytope $P_{\mathbf{C}_3}$ is a cube, while the polytope $P_{\mathbf{B}_3}$ is a rhombic dodecahedron. The identifications taking place on their boundaries lead to a different triangulated torus. In fact $\overline{\Sigma}(\mathbf{C}_3)$ and $\overline{\Sigma}(\mathbf{B}_3)$ don't even have the same number of vertices (eight and ten, respectively).

We now turn to the f- and h-vectors of the Steinberg torus. To count faces we use a similar line of reasoning as in the case of the finite Coxeter complex to count W-orbits. First, define f_J to be the number of faces of $\overline{\Sigma}$ with color set J, ignoring the empty face. Ignoring the empty face simply omits the constant term from the f-polynomial. However, omitting $f_\emptyset = 1$ has the effect of making the corresponding h-polynomial palindromic. In general, while the Dehn-Sommerville relations for a torus are not palindromic, they can be made so by ignoring the empty face. This idea was generalized to other triangulated manifolds by Isabella Novik and Ed Swartz in 2009 [113].

Now, for any nonempty subset $\emptyset \neq J \subseteq \widetilde{\Delta}$,

$$\begin{aligned} f_J &= |\{w \cdot A_{\widetilde{\Delta}-J} : w \in W\}|, \\ &= |W/W_{\widetilde{\Delta}-J}|, \\ &= |W|/|W_{\widetilde{\Delta}-J}|, \\ &= |\{w \in W : \widetilde{\mathrm{Des}}(w) \subseteq J\}|. \end{aligned}$$

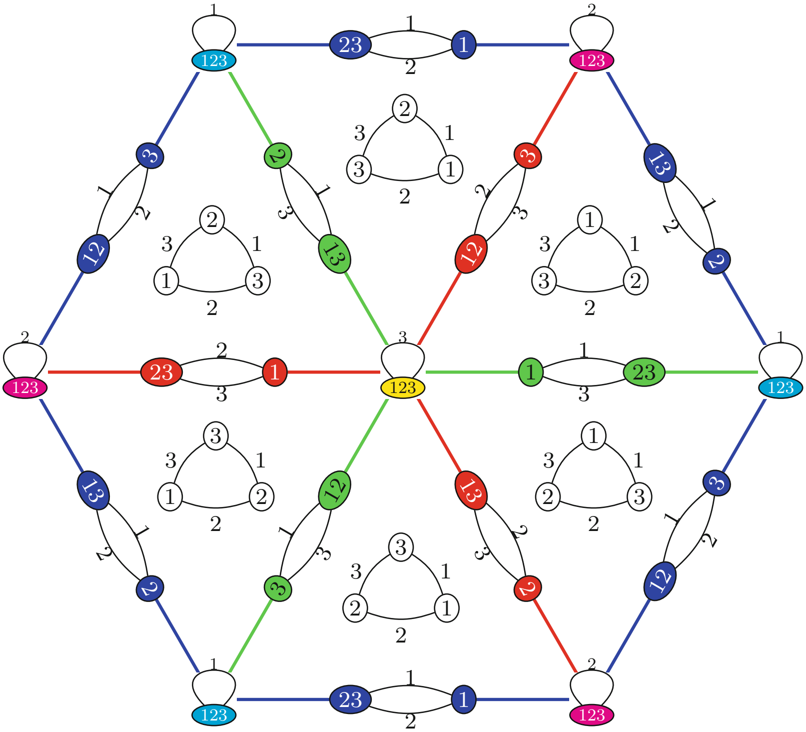


Fig. 14.5 The faces of the Steinberg torus $\overline{\Sigma}(A_2)$, with colors corresponding to W-orbits. Note the identifications along the boundary.

Define h_J to be

$$h_J = |\{w \in W : \widetilde{\mathrm{Des}}(w) = J\}|,$$

so that by inclusion-exclusion

$$h_J = \sum_{\emptyset \neq I \subseteq J} (-1)^{|J-I|} f_I.$$

Now we can express the affine Eulerian polynomial as follows:

$$\begin{aligned}
\widetilde{W}(t) &= \sum_{w \in W} t^{\widetilde{\mathrm{des}}(w)}, \\
&= \sum_{\emptyset \neq J \subseteq \widetilde{\Delta}} h_J t^{|J|}, \\
&= \sum_{\emptyset \neq I \subseteq J \subseteq \widetilde{\Delta}} (-1)^{|J-I|} f_I t^{|J|}, \\
&= \sum_{\emptyset \neq I \subseteq \widetilde{\Delta}} f_I t^{|I|} (1-t)^{n+1-|I|},
\end{aligned}$$

Using our calculation for f_J from above, we can give the following expression for the affine Eulerian polynomial.

Proposition 14.3. *The affine Eulerian polynomial has the following expression,*

$$\widetilde{W}(t) = \sum_{w \in W} t^{\widetilde{\mathrm{des}}(w)} = \sum_{\emptyset \neq I \subseteq \widetilde{\Delta}} \frac{|W|}{|W_{\widetilde{\Delta}-I}|} t^{|I|}(1-t)^{n+1-|I|}.$$

Furthermore, we can see that

$$\begin{aligned} \widetilde{W}(t) &= (1-t)^{n+1} f(\overline{\Sigma} - \{\emptyset\}; t/(1-t)), \\ &= h(\overline{\Sigma} - \{\emptyset\}; t). \end{aligned}$$

That is, the affine Eulerian polynomial is the h-polynomial of the Steinberg torus (ignoring the empty face).

14.4 Affine Eulerian numbers

We now describe the combinatorial definitions of affine descents and give some enumerative results, all of which are contained in [60]. Most generally, we can state the following fact.

Theorem 14.1. *The affine Eulerian polynomial $\widetilde{W}(t)$ is gamma-nonnegative for all finite Weyl groups W.*

It is known that $\widetilde{W}(t)$ is real-rooted in all cases except $\widetilde{D}_n$. See Section 3.5 of the paper of Carla Savage and Mirko Visontai [132].

14.4.1 Type A_{n-1}

The highest root in $\mathbf{A}_{n-1}$ is $\varepsilon_n - \varepsilon_1$, so $\alpha_0 = \varepsilon_1 - \varepsilon_n$. Thus $w \cdot \alpha_0 < 0$ whenever $w(n) > w(1)$. Therefore

$$\widetilde{\mathrm{Des}}(w) = \{0 \leq i \leq n-1 : w(i) > w(i+1)\},$$

with $w(0) = w(n)$. These are better known as "cyclic descents" since we think of the permutation w wrapping around so that we compare $w(n)$ and $w(1)$. For example, $\widetilde{\mathrm{Des}}(25413) = \{0, 2, 3\}$.

In Table 14.1 we see the affine Eulerian numbers of type A_{n-1}, i.e., the distribution of cyclic descents over the symmetric group.

Table 14.1 The affine Eulerian numbers for A_{n-1}, $0 \leq k \leq n \leq 9$.

$n \backslash k$	0	1	2	3	4	5	6	7	8
2	0	2							
3	0	3	3						
4	0	4	16	4					
5	0	5	55	55	5				
6	0	6	156	396	156	6			
7	0	7	399	2114	2114	399	7		
8	0	8	960	9528	19328	9528	960	8	
9	0	9	2223	38637	140571	140571	38637	2223	9

Cyclic descents were studied by Jason Fulman for their connections to card shuffling ("riffle shuffles with a cut") in a 2000 paper [77] and also by the author in 2005 [115]. Both papers give simple arguments for the following observation.

Observation 14.1 *For any $n \geq 2$,*

$$\widetilde{A}_n(t) = (n+1)tA_{n-1}(t),$$

where $A_{n-1}(t) = S_n(t)$ is the classical Eulerian polynomial.

Hence, $\widetilde{A}_n(t)$ is real-rooted and gamma-nonnegative from what we know in the classical case. Moreover, the following generating function is easily obtained.

Proposition 14.4. *We have the following exponential generating function for affine Eulerian polynomials:*

$$z + \sum_{n \geq 2} \widetilde{A}_{n-1}(t)\frac{z^n}{n!} = \frac{z(1-t)}{1 - te^{z(1-t)}}.$$

14.4.2 Type B_n

In type $\mathbf{B}_n$, the highest root is $\varepsilon_{n-1} + \varepsilon_n$, so $w \cdot \alpha_0 < 0$ if and only if $w(n-1) + w(n) > 0$. That is, we have a descent in 0 if $w(n-1) > -w(n)$. We have in this case,

$$\widetilde{\mathrm{Des}}(w) = \begin{cases} \mathrm{Des}(w) & \text{if } w(n-1) < -w(n), \\ \mathrm{Des}(w) \cup \{\alpha_0\} & \text{if } w(n-1) > -w(n). \end{cases}$$

For example, $\widetilde{\mathrm{Des}}(23\bar{4}5\bar{1}) = \{0, 3, 5\}$.

The type B_n affine Eulerian numbers are in Table 14.2.

Table 14.2 The affine Eulerian numbers for B_n, $0 \le k \le n \le 9$.

$n\backslash k$	0	1	2	3	4	5	6	7	8	9
2	0	4	4							
3	0	10	28	10						
4	0	24	168	168	24					
5	0	54	904	1924	904	54				
6	0	116	4452	18472	18472	4452	116			
7	0	242	20612	157294	288824	157294	20612	242		
8	0	496	91600	1227504	3841360	3841360	1227504	91600	496	
9	0	1006	396112	8989576	45616432	75788308	45616432	8989576	396112	1006

The type B_n affine Eulerian polynomial has a nonnegative gamma vector reminiscent of the type D_n Eulerian polynomials.

Proposition 14.5. *For $n \ge 2$, we have*

$$\widetilde{B}_n(t) = \sum_{u \in S_n} \phi(u)(4t)^{\mathrm{pk}(0u0)}(1+t)^{n+1-2\,\mathrm{pk}(0u0)},$$

where

$$\phi(u) = \begin{cases} 1 & \text{if } u(n-2) > u(n-1) > u(n), \\ 0 & \text{if } u(n-2) > u(n) > u(n-1), \\ 1/2 & \text{otherwise.} \end{cases}$$

Moreover, we have the following generating function.

$$2 + 2tz + \sum_{n \ge 2} \widetilde{B}_n(t)\frac{z^n}{n!} = \frac{2(1-t)(1-tze^{z(1-t)}}{1-te^{2z(1-t)}}.$$

Savage and Visontai proved in 2015 that $\widetilde{B}_n(t)$ is real-rooted [132].

14.4.3 Type C_n

In type $\mathbf{C}_n$, the highest root is $2\varepsilon_n$, and so we have a descent in α_0 if and only if $w(n) > 0$. Thus,

$$\widetilde{\mathrm{Des}}(w) = \{0 \le i \le n : w(i) > w(i+1)\},$$

with $w(0) = w(n+1) = 0$. For example, $\widetilde{\mathrm{Des}}(23\bar{4}5\bar{1}) = \{3, 5\}$. The type C_n affine Eulerian numbers are in Table 14.3.

Table 14.3 The affine Eulerian numbers for C_n, $0 \leq k \leq n \leq 9$.

$n\backslash k$	0	1	2	3	4	5	6	7	8	9
2	0	4	4							
3	0	8	32	8						
4	0	16	176	176	16					
5	0	32	832	2112	832	32				
6	0	64	3648	19328	19328	3648	64			
7	0	128	15360	152448	309248	152448	15360	128		
8	0	256	63232	1099008	3998464	3998464	1099008	63232	256	
9	0	512	257024	7479296	45175808	79969280	45175808	7479296	257024	512

The type C_n affine descent set can be thought of as a special kind of cyclic descent, and indeed we have the following connection with classical Eulerian polynomials. Just as with Observation 14.1 this observation was proved both by Fulman in [77] and the author in [115].

Observation 14.2 *For any* $n \geq 1$*,*

$$\widetilde{C}_n(t) = 2^n t A_{n-1}(t).$$

From this observation it follows that $\widetilde{C}_n(t)$ is real-rooted and gamma-nonnegative. We can express its gamma vector in terms of the classical case. Moreover, we have the following generating function.

Proposition 14.6. *We have the following exponential generating function:*

$$1 + \sum_{n \geq 1} \widetilde{C}_n(t) \frac{z^n}{n!} = \frac{1-t}{1-te^{2z(1-t)}}.$$

14.4.4 Type $\mathbf{D}_n$

The highest root for $\mathbf{D}_n$ is the same as the highest root in $\mathbf{B}_n$, with the same effect on combinatorial descents. We have an affine descent for an element $w \in D_n$ if $w(i) > w(i+1)$ for $i = 1, \ldots, n-1$ in the usual way, along with a descent at the beginning if $-w(1) > w(2)$, and another at the end if $w(n-1) > -w(n)$. For example, $\widetilde{\mathrm{Des}}(3\bar{4}2\bar{1}5) = \{0, -1, 1, 3\}$, since $w(1) > w(2)$, $w(-1) > w(2)$, $w(3) > w(4)$, and $w(4) > -w(5)$. See Table 14.4.

The type D_n affine Eulerian polynomial has a nonnegative gamma vector as well.

Table 14.4 The affine Eulerian numbers for D_n, $0 \leq k \leq n \leq 9$.

$n\backslash k$	0	1	2	3	4	5	6	7	8	9
4	0	16	80	80	16					
5	0	44	464	904	464	44				
6	0	104	2568	8848	8848	2568	104			
7	0	228	13192	79580	136560	79580	13192	228		
8	0	480	63904	665568	1850528	1850528	665568	63904	480	
9	0	988	296608	5232400	22833760	36169768	22833760	5232400	296608	988

Proposition 14.7. *For $n \geq 4$, we have*

$$\widetilde{D}_n(t) = \sum_{u \in S_n} \phi(u)\phi(\overleftarrow{u})(4t)^{\mathrm{pk}(0u0)}(1+t)^{n+1-2\,\mathrm{pk}(0u0)},$$

where $\overleftarrow{u} = u(n)\cdots u(2)u(1)$, and ϕ is the same as in Proposition 14.5. Moreover, we have the following generating function:

$$2 + 4t\frac{z^2}{2} + \sum_{n \geq 3} \widetilde{D}_n(t)\frac{z^n}{n!} = \frac{2(1-t)(1+tz^2 - 2tze^{z(1-t)})}{1-te^{2z(1-t)}}.$$

We finish by remarking that the polynomial $\widetilde{D}_n(t)$ is the only case of an affine Eulerian polynomial for which real-rootedness is not proved.