Chapter 1
Eulerian numbers

The first interesting array of numbers a typical mathematics student encounters is Pascal's triangle, shown in Table 1.1. It has many beautiful properties, some of which we will review shortly. One of the main points of this chapter is to argue that the array of Eulerian numbers is just as interesting as Pascal's triangle.

Table 1.1 Pascal's triangle of binomial coefficients $\binom{n}{k}$, $0 \le k \le n \le 9$.

$n \backslash k$	0	1	2	3	4	5	6	7	8	9
0	1									
1	1	1								
2	1	2	1							
3	1	3	3	1						
4	1	4	6	4	1					
5	1	5	10	10	5	1				
6	1	6	15	20	15	6	1			
7	1	7	21	35	35	21	7	1		
8	1	8	28	56	70	56	28	8	1	
9	1	9	36	84	126	126	84	36	9	1

1.1 Binomial coefficients

It is likely that a reader of this book is already familiar with binomial coefficients, but we will review this material to establish our point of view for future material. The approach we will take is to define $\binom{n}{k}$ to be a combinatorial quantity: namely, the number of k-element subsets of an n-element set. Therefore $\binom{n}{k} = 0$ if $n < 0$, $k < 0$, or $k > n$. (And if we aren't clear what a

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T.K. Petersen, *Eulerian Numbers*, Birkhäuser Advanced Texts Basler Lehrbücher, DOI 10.1007/978-1-4939-3091-3_1

set with a nonintegral number of elements is, we should probably set $\binom{n}{k} = 0$ unless k and n are integers.) It is immediate from this definition that these numbers have the following symmetry:

$$\binom{n}{k} = \binom{n}{n-k},$$

by the fact that a subset of an n-element set and its complement are in bijection. Further, by considering the fact that a k-element subset S of $\{1, 2, \ldots, n\}$ either has:

- $n \in S$, in which case $S - \{n\}$ is a $(k-1)$-element subset of $\{1, 2, \ldots, n-1\}$, or
- $n \notin S$, in which case S itself is a k-element subset of $\{1, 2, \ldots, n-1\}$,

we get Pascal's recurrence.

Theorem 1.1 (Pascal's recurrence). *For any $n \geq 1$, $k \geq 0$,*

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

We can get an explicit formula for $\binom{n}{k}$ by counting orderings of k-element subsets two different ways. On the one hand, we can choose a k-element subset in $\binom{n}{k}$ ways, then order the elements in $k!$ ways. On the other hand, we can order k of n things in

$$n(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!}$$

ways, since we have n choices for the first element, $n-1$ choices for the second, and so on. We get

$$k!\binom{n}{k} = \frac{n!}{(n-k)!},$$

or

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

We call the numbers $\binom{n}{k}$ *binomial coefficients* because of the following theorem.

Theorem 1.2 (Binomial theorem). *For $n \geq 0$,*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}. \tag{1.1}$$

This famous result can be proved by induction using Pascal's recurrence, but a combinatorial argument can be given by counting subsets of

$\{1, 2, \ldots, n\}$ according to cardinality. On the left-hand side, we consider each of the numbers i in $\{1, 2, \ldots, n\}$ independently as being members of a subset S (with weight x) or not (with weight y), so that

$$(x+y)^n = \sum_{S \subseteq \{1,2,\ldots,n\}} x^{|S|} y^{n-|S|}.$$

On the right-hand side, we see the coefficient of $x^k y^{n-k}$ is the number of k-element subsets, or $\binom{n}{k}$.

We can think of the binomial theorem as giving combinatorial meaning to an algebraic quantity, or we can think of the binomial theorem as a way of encoding combinatorial information algebraically. This idea is elaborated upon in the next section.

1.2 Generating functions

A *generating function* is an algebraic tool for encoding combinatorial data. For a sequence of numbers $a_0, a_1, \ldots$ the *ordinary generating function* is the series

$$\sum_{k \geq 0} a_k t^k = a_0 + a_1 t + a_2 t^2 + \cdots + a_k t^k + \cdots,$$

while the *exponential generating function* is the series

$$\sum_{k \geq 0} a_k \frac{t^k}{k!} = a_0 + a_1 t + a_2 \frac{t^2}{2} + a_3 \frac{t^3}{6} + \cdots.$$

For example, the geometric series

$$\frac{1}{1-t} = 1 + t + t^2 + \cdots, \tag{1.2}$$

can be considered the generating function for the sequence $1, 1, 1, \ldots$, while

$$e^t = 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \cdots,$$

is its exponential generating function.

Generating functions can encode finite sequences as well. For example, the binomial theorem tells us for fixed n, $(1+t)^n$ is the ordinary generating function for the binomial coefficients $\binom{n}{k}$.

While we think of generating functions as algebraic encodings of combinatorial data (a "clothesline on which the sequence hangs," in Herbert Wilf's words [166]), we can also manipulate generating functions as analytic functions. For example, if we differentiate Equation (1.2) n times, we get

$$\frac{n!}{(1-t)^{n+1}} = \sum_{k\geq 0} k(k-1)\cdots(k-n+1)t^{k-n},$$

and so after dividing by $n!$ we have another generating function for the binomial coefficients, this time taken column-wise in Pascal's triangle (Table 1.1):

$$\frac{t^i}{(1-t)^{n+1}} = \sum_{k\geq 0} \binom{k+n-i}{n} t^k. \tag{1.3}$$

1.3 Classical Eulerian numbers

Let us now meet the Eulerian numbers. As with binomial coefficients, our starting point is a combinatorial definition.

For a given positive integer n, the symmetric group S_n is the set of all permutations of $[n] := \{1, 2, \ldots, n\}$, i.e., bijections $w : [n] \to [n]$. We will usually write permutations in one-line notation: $w = w(1)w(2)\cdots w(n)$, so a typical element of S_7 is $w = 3125647$.

For any permutation $w \in S_n$, we define a *descent* to be a position i such that $w(i) > w(i+1)$, and we denote by $\mathrm{Des}(w)$ the set of descents of w,

$$\mathrm{Des}(w) = \{i : w(i) > w(i+1)\}.$$

We let $\mathrm{des}(w)$ denote the number of descents of w, i.e.,

$$\mathrm{des}(w) = |\,\mathrm{Des}(w)| = |\{i : w(i) > w(i+1)\}|. \tag{1.4}$$

For example, if $w = 3125647$, then there are descents in position 1 (since $3 > 1$) and in position 5 (since $6 > 4$). Hence, $\mathrm{des}(w) = 2$. The permutation $12\cdots n$ is the only permutation with no descents, while its reversal, $n\cdots 21$, has the maximal number, with $n-1$.

We define the *Eulerian numbers*, denoted $\left\langle {n \atop k} \right\rangle$, to be the number of permutations in S_n with k descents, i.e.,

$$\left\langle {n \atop k} \right\rangle = |\{w \in S_n : \mathrm{des}(w) = k\}|. \tag{1.5}$$

For example, we see from Table 1.2 there are 11 permutations in S_4 with two descents. Hence, $\left\langle {4 \atop 2} \right\rangle = 11$.

Table 1.3 shows the Eulerian numbers $\left\langle {n \atop k} \right\rangle$, with $1 \leq k \leq n \leq 10$. The reader should be careful not to confuse the triangle of Eulerian numbers with the sequence of *Euler numbers*: $1, 1, 2, 5, 16, 61, 272, 1385, 7936, \ldots$, though there is a connection. See the discussion at the end of Section 4.2 and Problem 4.2.

Table 1.2 The permutations in S_4 grouped according to descent number.

$\mathrm{des}(w) = 0$	$\mathrm{des}(w) = 1$	$\mathrm{des}(w) = 2$	$\mathrm{des}(w) = 3$
1234	1243	3421	4321
	1324	4231	
	1342	2431	
	1423	3241	
	2134	4312	
	2314	4132	
	2341	1432	
	2413	3142	
	3124	4213	
	3412	2143	
	4123	3214	

Table 1.3 The Eulerian numbers $\left\langle {n \atop k} \right\rangle$, $0 \le k < n \le 10$.

$n \backslash k$	0	1	2	3	4	5	6	7	8	9
1	1									
2	1	1								
3	1	4	1							
4	1	11	11	1						
5	1	26	66	26	1					
6	1	57	302	302	57	1				
7	1	120	1191	2416	1191	120	1			
8	1	247	4293	15619	15619	4293	247	1		
9	1	502	14608	88234	156190	88234	14608	502	1	
10	1	1013	47840	455192	1310354	1310354	455192	47840	1013	1

The function

$$\mathrm{des} : \bigcup_{n \ge 1} S_n \to \{0, 1, 2, 3, \ldots\}$$

is an example of a *permutation statistic*, a function from the set of permutations to the integers. Other well-known permutation statistics include the number of inversions (discussed in Section 5.1) and the number of cycles. When we count permutations according to a particular permutation statistic, this gives rise to the *distribution* of the statistic. Any statistic whose distribution gives the numbers $\left\langle {n \atop k} \right\rangle$ is called an *Eulerian statistic*.

Besides des, there are many other Eulerian permutation statistics. For example, it is easy to see the equivalence of des and the number of *ascents* defined by

$$\mathrm{asc}(w) = |\{i : w(i) < w(i+1)\}| = |[n-1] - \mathrm{Des}(w)|.$$

However, proving that two statistics are equidistributed is not always so easy. For example, the number of *excedances* of a permutation, defined as

$$\mathrm{exc}(w) = |\{i : w(i) > i\}|, \tag{1.6}$$

is an Eulerian statistic. While exc ranges from 0 to $n-1$, its relationship with des is disguised. But this disguise can be lifted with a bijection from S_n to itself that maps descents to excedances, known as the "transformation fondamentale" of Dominique Foata and Marcel-Paul Schützenberger [70]. The reader is invited to study this bijection, along with other manifestations of the Eulerian numbers, in the problems at the end of the chapter. For now, we will stick to the definition of Eulerian numbers in terms of des.

We can use the descent definition of Eulerian numbers to prove a Pascal-like linear recurrence as follows. Notice that if w is in S_n with k descents, then deleting n from w results in a permutation in S_{n-1} with k or $k-1$ descents. Conversely, we can form permutations of n with k descents from permutations of $n-1$ with k or $k-1$ descents by inserting n.

To be precise, suppose v is a permutation in S_{n-1} with $k-1$ descents. Then inserting n at the far left of v or in an ascent position of v creates a permutation $w \in S_n$ with k descents. There are $n-1-(k-1) = n-k$ such positions.

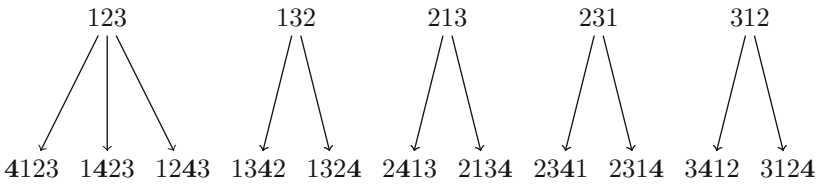
Similarly, if $v \in S_{n-1}$ already has k descents, then inserting n in a descent position of v or at the far right gives a permutation $w \in S_n$ with k descents. There are $k+1$ such positions.

We have the following result.

Theorem 1.3 (The linear recurrence). *For any k and $n > 0$,*

$$\left\langle {n \atop k} \right\rangle = (n-k)\left\langle {n-1 \atop k-1} \right\rangle + (k+1)\left\langle {n-1 \atop k} \right\rangle. \tag{1.7}$$

For example, with $n = 4$, $k = 1$, we have:



and so $\left\langle {4 \atop 1} \right\rangle = 3\left\langle {3 \atop 0} \right\rangle + 2\left\langle {3 \atop 1} \right\rangle$.

We can use the recurrence in Equation (1.7) to think of the triangle of Eulerian numbers as an edge-weighted version of Pascal's triangle. See Figure 1.1.

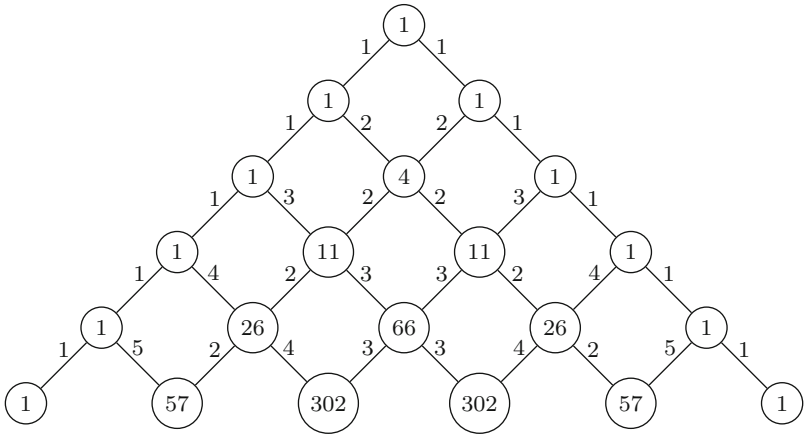


Fig. 1.1 Generating Eulerian numbers via the recurrence relation.

1.4 Eulerian polynomials

For fixed n, we define the nth *Eulerian polynomial* as the generating function for the Eulerian numbers $\left\langle {n \atop k} \right\rangle$ as follows:

$$S_n(t) = \sum_{w \in S_n} t^{\mathrm{des}(w)} = \sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle t^k. \tag{1.8}$$

For example, $S_1(t) = 1$, $S_2(t) = 1 + t$, $S_3(t) = 1 + 4t + t^2$, and $S_4(t) = 1 + 11t + 11t^2 + t^3$. We will define $S_0(t) = 1$ for convenience.

An immediate consequence of Theorem 1.3 is an identity that relates the Eulerian polynomial $S_{n+1}(t)$ to $S_n(t)$ and its derivative.

Since $S_n(t) = \sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle t^k$, we have $S_n'(t) = \sum_{k=1}^{n-1} k \left\langle {n \atop k} \right\rangle t^{k-1}$. Thus,

$$\begin{aligned}(1+nt)S_n(t) &= (1+nt)\sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle t^k, \\ &= \sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle t^k + \sum_{k=1}^{n} n \left\langle {n \atop k-1} \right\rangle t^k,\end{aligned}$$

and

$$\begin{aligned}t(1-t)S_n'(t) &= (t-t^2)\sum_{k=1}^{n-1} k \left\langle {n \atop k} \right\rangle t^{k-1}, \\ &= \sum_{k=1}^{n-1} k \left\langle {n \atop k} \right\rangle t^k - \sum_{k=1}^{n-1} (k-1) \left\langle {n \atop k-1} \right\rangle t^k.\end{aligned}$$

Adding these four sums, we get that the coefficient of t^k in $(1+nt)S_n(t)$ $+t(1-t)S_n'(t)$ is:

$$(n+1-k)\left\langle {n \atop k-1} \right\rangle + (k+1)\left\langle {n \atop k} \right\rangle,$$

which, by Theorem 1.3, equals $\left\langle {n+1 \atop k} \right\rangle$. Thus, we have the following result.

Theorem 1.4 (Linear polynomial recurrence). *For any $n \geq 0$,*

$$S_{n+1}(t) = (1+nt)S_n(t) + t(1-t)S_n'(t). \tag{1.9}$$

1.5 Two important identities

As it is equivalent to the numeric recurrence, Theorem 1.4 does not give any new combinatorial insight into Eulerian numbers. However, sometimes having a combinatorial identity rephrased like this allows us to perform algebraic and analytic operations freely, and these operations can uncover new combinatorial information that we may not have guessed at otherwise. This approach to combinatorial identities is sometimes referred to as "manipulatorics" since it often boils down to formal manipulations of formulas. While it might not be as satisfying as a direct combinatorial explanation, it is nonetheless an important skill to have as a practitioner of the combinatorial arts. Sometimes it may be the only way we know how to prove a combinatorial identity.

In general, our preference for this book will be to give direct combinatorial explanations, but in some cases the manipulatorics approach is simpler, or allows for more elegant statements of results. As practice, let us now do some manipulatorics, starting with Theorem 1.4 and finishing with a truly interesting result known as *Worpitzky's identity* (Corollary 1.2). We ask for a combinatorial proof of Worpitzky's identity in Problem 1.13.

For $n \geq 0$, define the function

$$s_n(t) = (1-t)^{n+1} \sum_{k \geq 0} (k+1)^n t^k.$$

Then $s_0 = 1$ and it is straightforward to verify the identity

$$s_{n+1}(t) = (1+nt)s_n(t) + t(1-t)s_n'(t).$$

Thus, $s_n(t) = S_n(t)$, and we have the following corollary, which we refer to as the *Carlitz identity.*

Corollary 1.1 (The Carlitz identity). *For any $n \geq 0$,*

$$\frac{S_n(t)}{(1-t)^{n+1}} = \sum_{k \geq 0} (k+1)^n t^k. \tag{1.10}$$

But using the definition of $S_n(t)$ we have

$$\frac{S_n(t)}{(1-t)^{n+1}} = \sum_{i=0}^{n-1} \left\langle {n \atop i} \right\rangle \left(\frac{t^i}{(1-t)^{n+1}} \right),$$

and recalling the expression for $t^i/(1-t)^{n+1}$ in Equation (1.3), we find:

$$\begin{aligned} \frac{S_n(t)}{(1-t)^{n+1}} &= \sum_{i=0}^{n-1} \left\langle {n \atop i} \right\rangle \sum_{k \geq 0} \binom{k+n-i}{n} t^k, \\ &= \sum_{k \geq 0} \left(\sum_{i=0}^{n-1} \left\langle {n \atop i} \right\rangle \binom{k+n-i}{n} \right) t^k. \end{aligned}$$

By comparing with the formula in Equation (1.10) we get the following wonderful identity.

Corollary 1.2 (Worpitzky's identity). *For any $n \geq 0$,*

$$(k+1)^n = \sum_{i=0}^{n-1} \left\langle {n \atop i} \right\rangle \binom{k+n-i}{n}.$$

For example, with $k = 3$, $n = 5$, we get:

$$\begin{aligned} 4^5 &= \left\langle {5 \atop 0} \right\rangle \binom{8}{5} + \left\langle {5 \atop 1} \right\rangle \binom{7}{5} + \left\langle {5 \atop 2} \right\rangle \binom{6}{5} + \left\langle {5 \atop 3} \right\rangle \binom{5}{5}, \\ &= 1 \cdot 56 + 26 \cdot 21 + 66 \cdot 6 + 26 \cdot 1, \\ &= 1024. \end{aligned}$$

Using Worpitzky's identity repeatedly, with $k \geq 0$, gives us explicit formulas for the Eulerian numbers:

$$\begin{aligned} \left\langle {n \atop 0} \right\rangle &= 1, \\ \left\langle {n \atop 1} \right\rangle &= 2^n - (n+1), \\ \left\langle {n \atop 2} \right\rangle &= 3^n - \left\langle {n \atop 1} \right\rangle \binom{n+1}{n} - \binom{n+2}{n}, \\ &= 3^n - 2^n(n+1) + (n+1)^2 - \binom{n+2}{n}, \\ &= 3^n - 2^n(n+1) + \binom{n+1}{n-1}, \\ &\vdots \end{aligned}$$

Continuing in this way, we find the following formula.

Corollary 1.3 (Alternating sum formula). *The Eulerian numbers have the following formula for any $n \geq 1$, $k \geq 0$:*

$$\begin{aligned}\left\langle {n \atop k} \right\rangle &= (k+1)^n - k^n \binom{n+1}{n} + (k-1)^n \binom{n+1}{n-1} - \ldots, \\ &\quad \ldots + (-1)^k \binom{n+1}{n+1-k}, \\ &= \sum_{i=0}^{k} (-1)^i (k+1-i)^n \binom{n+1}{n+1-i}, \\ &= \sum_{i=0}^{k} (-1)^i (k+1-i)^n \binom{n+1}{i}. \end{aligned} \tag{1.11}$$

1.6 Exponential generating function

We finish the chapter with a mixture of bijective combinatorics and manipulatorics to derive the exponential generating function for the Eulerian polynomials.

We begin with a simple, elegant way to generate permutations recursively. Recall that in deriving the linear recurrence in Theorem 1.3, we carefully examined how inserting a new largest number into a permutation affected descent numbers. This time, to form a permutation of n, we first choose which subset of the elements $\{1, 2, \ldots, n-1\}$ go to the left of n, which elements go to the right, and permute independently on the left and on the right. See Figure 1.2.

The number of descents in any permutation is one more than the sum of the descents to the left of n and the descents to the right of n. Let i denote the number of elements to the left of n. Then

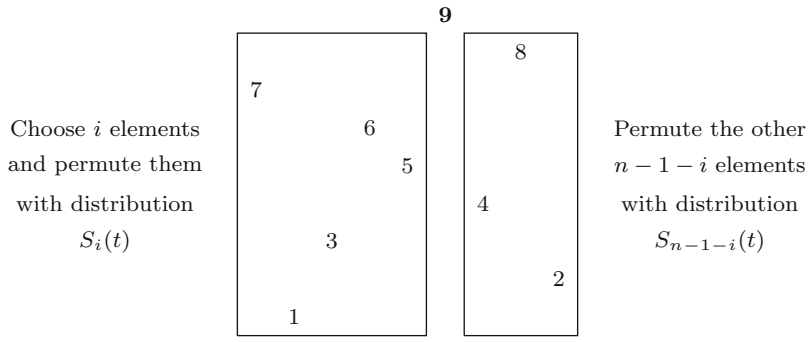


Fig. 1.2 The idea behind Equation (1.12).

$$S_i(t) \cdot t \cdot S_{n-1-i}(t)$$

is the generating function for descents of permutations with these same elements to the left of n. Of course, if there is nothing to the right of n, i.e., if $i = n-1$, then we simply get the number of descents to the left of n. Summing over all i, we have the following quadratic recurrence for Eulerian polynomials.

Theorem 1.5 (Quadratic polynomial recurrence). *For any $n > 0$,*

$$S_n(t) = S_{n-1}(t) + t \sum_{i=0}^{n-2} \binom{n-1}{i} S_i(t) S_{n-1-i}(t). \tag{1.12}$$

The recurrence in Equation (1.12) now leads to a way to find an expression for the exponential generating function

$$S(t,z) := \sum_{n \geq 0} S_n(t) \frac{z^n}{n!} = \sum_{n,k \geq 0} \left\langle {n \atop k} \right\rangle t^k \frac{z^n}{n!}.$$

Indeed, (1.12) gives:

$$\begin{aligned}
\frac{d}{dz} S(t,z) &= \sum_{n \geq 1} S_n(t) \frac{z^{n-1}}{(n-1)!}, \\
&= \sum_{n \geq 1} S_{n-1}(t) \frac{z^{n-1}}{(n-1)!} + t \sum_{n \geq 1} \sum_{i=0}^{n-2} \binom{n-1}{i} S_i(t) S_{n-1-i}(t) \frac{z^{n-1}}{(n-1)!}, \\
&= S(t,z) + t \sum_{n \geq 1} \sum_{i=0}^{n-2} S_i(t) \frac{z^i}{i!} \cdot S_{n-1-i}(t) \frac{z^{n-1-i}}{(n-1-i)!}, \\
&= S(t,z) + tS(t,z)(S(t,z) - 1)).
\end{aligned}$$

Solving the differential equation

$$f'(z) = tf^2(z) + (1-t)f(z),$$

with initial condition $f(0) = 1$ gives us the following result, originally due to Euler.

Theorem 1.6 (Exponential generating function). *We have*

$$S(t,z) = \frac{t-1}{t - e^{z(t-1)}}. \tag{1.13}$$

Those who don't want to rely on solving a differential equation to derive this formula are encouraged to see Problem 1.15.

Notes

The reader looking for more details about the generating function approach to enumerative combinatorics would do well to read Richard Stanley's classic work [154] and Herbert Wilf's book [166]. The book by Philippe Flajolet and Robert Sedgewick [66] focuses on analytic methods for extracting information from generating functions, and part A gives a nice perspective on symbolic methods for constructing generating functions. An earlier book that also contains a wealth of information about the use of generating functions in combinatorial enumeration is John Riordan's book [130].

The Eulerian numbers appear in a chapter of Euler's textbook on differential calculus [64, Part II, Caput VII, pp. 389–390]. In this chapter, Euler essentially sets himself the task of solving the differential equation for the exponential generating function we have in Equation (1.13), and when expanding its series, he finds the Eulerian numbers, and mentions the alternating sum formula given in Equation (1.11). Dominique Foata has a lovely survey in which he explains Euler's motivation and derivation [68]. Leonard Carlitz and his collaborators studied Eulerian numbers and their generalizations in several papers in the 20th century, e.g., [38, 39, 41, 43, 44], while Foata and Marcel-Paul Schützenberger wrote a then-comprehensive treatment of Eulerian numbers from a combinatorial point of view in [70]. From the 1980s onward, the number of scholarly articles on Eulerian numbers and their generalizations is too numerous to attempt to catalogue.

Most of the results in this chapter can be traced back to Carlitz or Riordan, though the Carlitz identity in Corollary 1.1 was known to Euler. We refer to it as the Carlitz identity because of a generalization of the identity obtained by Carlitz in 1975 [40], though even this generalization predates Carlitz—it can be found in Percy MacMahon's textbook from 1915/16 [106]. According to Carlitz [38], Worpitzky's identity (Corollary 1.2) dates from an 1883 paper by Julius Worpitzky [168], though Don Knuth [96, pp. 36] attributes the identity to an 1867 publication of Chinese mathematician Li Shan-Lan, and remarks that special cases for $n \leq 5$ were known to Yoshisuke Matsunaga of Japan, who died in 1744.

Problems

1.1. A *composition* of n is an ordered list of positive integers whose sum is n, denoted $\alpha = (\alpha_1, \ldots, \alpha_k)$. Show that the number of compositions of n with k parts is $\binom{n-1}{k-1}$.

1.2. How many compositions α of n have the following properties?

1. α has parts of size 1 and 2 only, e.g., for $n = 9$, $(2, 1, 1, 2, 2, 1)$ is acceptable, but not $(1, 2, 3, 1, 2)$.

2. α has only odd parts, e.g., for $n = 9$, $(3, 1, 5)$ is acceptable, but not $(1, 2, 1, 5)$.
3. α has all its parts greater than 1, except possibly the last entry, e.g., for $n = 9$, $(3, 4, 2)$ and $(3, 3, 2, 1)$ are acceptable, but not $(3, 3, 1, 2)$.

1.3. Show that the Fibonacci numbers satisfy the following identity:

$$f_n = \sum_{k \geq 0} \binom{n-k}{k}. \tag{1.14}$$

This can be quickly verified with Pascal's recurrence and careful bookkeeping, but see if you can find a combinatorial argument using one of the sets of compositions in Problem 1.2.

1.4. For $n \geq 1$, let $\phi_n = f_n/f_{n-1}$, where f_n is the nth Fibonacci number. Using the Fibonacci recurrence, find a recurrence for ϕ_n and use it to compute the limit:

$$\phi = \lim_{n \to \infty} \phi_n.$$

The number ϕ is sometimes called the *golden ratio*.

1.5. Let $f(z)$ denote the ordinary generating function for the Fibonacci numbers, i.e.,

$$f(z) = 1 + z + 2z^2 + 3z^3 + 5z^4 + 8z^5 + \cdots = \sum_{k \geq 0} f_k z^k,$$

with $f_0 = f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$.

1. Write $f(z)$ as

$$f(z) = \frac{1}{q(z)},$$

   where $q(z)$ is a quadratic polynomial. Hint: use the recurrence for the Fibonacci numbers to find an identity for $f(z)$ of the form $f(z)q(z) = 1$.
2. Use the expression found in part 1 to give a "manipulatorics" proof of Equation (1.14).
3. Factor $q(z)$ from part 1 as $q(z) = (1 - \alpha z)(1 - \beta z)$, then find numbers A and B so that

$$f(z) = \frac{A}{1 - \alpha z} + \frac{B}{1 - \beta z}.$$

4. Use part 3 to show

$$f_n = \frac{\phi^{n+1} - \overline{\phi}^{n+1}}{\sqrt{5}},$$

   where ϕ is the golden ratio found in Problem 1.4 and $\overline{\phi}$ is the other root of the polynomial $x^2 - x - 1$.

1.6. Show that the following permutation statistics are Eulerian.

1. The number of *ascents* of a permutation w, $\mathrm{asc}(w) = \{i : w(i) < w(i+1)\}$, e.g., $\mathrm{asc}(1374265) = 3$.
2. The number of (maximal, increasing) *runs* of a permutation, denoted $\mathrm{runs}(w)$, where a maximal increasing run is a substring $w(i) < w(i+1) < \cdots < w(i+r)$ such that $w(i-1)$ is not smaller than $w(i)$ and $w(i+r)$ is not smaller than $w(i+r+1)$. For example, $\mathrm{runs}(1374265) = 4$.
3. The number of *readings* of a permutation, denoted $\mathrm{read}(w)$. This is the number of times one must scan the one-line notation of w from left to right to find the numbers $1, 2, \ldots, n$ in order. For example with $w = 1374265$ we read through four times:

times read	1	3	7	4	2	6	5
1	1				2		
2		3		4			5
3						6	
4			7				

so $\mathrm{read}(1374265) = 4$.

1.7. Show that the number of *excedances*, $\mathrm{exc}(w) = \{i : w(i) > i\}$, is Eulerian.

1.8. An *inversion sequence* of length n is a vector

$$s = (s_1, \ldots, s_n),$$

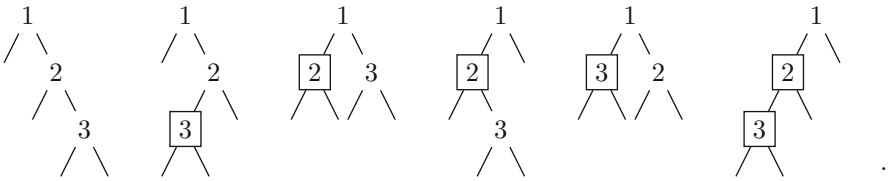
such that $0 \le s_i \le i - 1$. Show that counting inversion sequences according to ascents (with $\mathrm{asc}(s) = \{i : s_i < s_{i+1}\}$ as with permutations) gives rise to the Eulerian distribution, e.g., if $n = 3$, the inversion sequences and their ascent numbers are:

s	$\mathrm{asc}(s)$
$(0,0,0)$	0
$(0,0,1)$	1
$(0,0,2)$	1
$(0,1,0)$	1
$(0,1,1)$	1
$(0,1,2)$	2

.

1.9. An *increasing binary tree* of size n is a rooted, planar tree with n internal nodes (internal means not a leaf) such that each internal node has two children: a *left child* and a *right child*. Further, the internal nodes are labeled with $1, 2, \ldots, n$ so that any path from the root to a leaf follows increasing labels. Show that counting the number of increasing binary trees of size n

according to how many internal nodes are left children gives the Eulerian distribution. For example, with $n = 3$, we have following trees, whose internal left children are highlighted:



1.10. For any n and $k = 0, 1, \ldots, n-1$, let $\mathcal{R}_{n,k}$ denote the set of points $(x_1, \ldots, x_n)$ in the unit cube whose sum is between k and $k+1$, i.e.,

$$k \leq x_1 + \cdots + x_n \leq k + 1,$$

with $0 \leq x_i \leq 1$. What is the volume of $\mathcal{R}_{n,k}$?

1.11. The number of *cyclic descents* of a permutation $w \in S_n$ is the number of ordinary descents, plus one if $w(n) > w(1)$. We denote this statistic by $\mathrm{cdes}(w)$. For example, $\mathrm{cdes}(31524) = 3$ and $\mathrm{cdes}(43152) = 3$, whereas $\mathrm{des}(31524) = 2$ and $\mathrm{des}(43152) = 3$. Show that

$$\sum_{w \in S_n} t^{\mathrm{cdes}(w)} = ntS_{n-1}(t).$$

1.12. Give a bijective proof that

$$\left\langle {n \atop 1} \right\rangle = 2^n - n - 1.$$

1.13. Give a bijective proof of Worpitzky's identity:

$$(k+1)^n = \sum_{i=0}^{n-1} \left\langle {n \atop i} \right\rangle \binom{k+n-i}{n}.$$

Hint: interpret the left-hand side as counting the set of all integer vectors $(a_1, a_2, \ldots, a_n)$, with $0 \leq a_i \leq k$, and try to group these according to permutations of n by rearranging the sequence in weakly increasing order.

1.14. Give a combinatorial proof of the Carlitz identity in (1.10):

$$\frac{S_n(t)}{(1-t)^{n+1}} = \sum_{k \geq 0} (k+1)^n t^k.$$

Hint: try the method of "balls in boxes" as follows. Clearly $(k+1)^n$ is the number of ways to place n distinct (labeled) balls into $k+1$ boxes. Try to partition the ways to put the balls into boxes according to permutations of n, and show that for a fixed permutation w in S_n, the generating function for arrangements of ball boxes that correspond to w is:

$$\frac{t^{\mathrm{des}(w)}}{(1-t)^{n+1}}.$$

1.15. Use the Carlitz identity (Equation (1.10)) to derive Equation (1.13). Hint: start with

$$\sum_{n\geq 0} \frac{S_n(t)}{(1-t)^{n+1}} \frac{z^n}{n!}.$$