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Eulerian Numbers

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To Mara, Owen, Ariella, and Rebecca

Foreword

The Eulerian numbers $\left\langle {n \atop k} \right\rangle$ were originally defined by Euler in a noncombinatorial way. MacMahon proved an identity which shows that $\left\langle {n \atop k} \right\rangle$ is the number of permutations of $1, 2, \ldots, n$ with k descents, though he was apparently unaware of Euler's work. It was not until 1953 that Carlitz and Riordan showed explicitly that $\left\langle {n \atop k} \right\rangle$, as defined by Euler, has this elegant combinatorial interpretation. Who could believe that such a simple concept would have a deep and rich theory, with close connections to a vast number of other subjects? For instance, Eulerian numbers are intimately connected with counting the carries in the usual addition algorithm for positive integers! Eulerian numbers and their generalizations arise naturally in such areas as partially ordered sets, hyperplane arrangements, Coxeter groups, simplicial complexes and convex polytopes, mostly in connection with some of the most interesting examples.

Kyle Petersen has done a masterful job of organizing for the first time the bewildering variety of material on Eulerian numbers. He carefully develops all the necessary background information, so the text will be accessible to beginning graduate students and even advanced undergraduates. Readers of this book will certainly learn a lot of beautiful combinatorics. In addition, many readers will undoubtedly be enticed into pursuing further the numerous areas mentioned above connected with Eulerian numbers. It is exciting to see to what an extent the seedlings planted by Euler, MacMahon, and Carlitz-Riordan have borne fruit.

Cambridge, MA, USA — Richard Stanley
June 2015

Preface

Leonhard Euler introduced the numbers that are at the heart of this book in 1755. His motivation seems to have been to obtain a formula for the alternating sums of powers $(1^n - 2^n + 3^n - \cdots)$ in a manner analogous to what Jacques Bernoulli had done for the unsigned sums of powers. The connection with Bernoulli numbers motivated work of Julius Worpitzky in 1883 and Georg Frobenius in 1910. In the mid-twentieth century Leonard Carlitz wrote many papers surrounding Eulerian numbers and their use in number theory. We will discuss almost none of these topics in this book.

Rather, our starting point comes from later work of Carlitz and his collaborators, who began to study the Eulerian numbers as *combinatorial* quantities, following in the vein of late 19th and early 20th century combinatorialists like Simon Newcomb and Percy MacMahon. As explained by John Riordan in his 1958 textbook, a wonderful way to encounter the Eulerian numbers is as the answer to *Simon Newcomb's problem*:

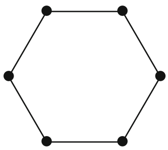
> ... a deck of cards of arbitrary specification is dealt out into a single pile so long as cards are in rising order, with like cards counted as rising, and a new pile is started whenever a non-rising card appears; with all possible arrangements of the deck, in how many ways do k piles appear?

If there are no ties among the cards (if we order the suits as well as the face values of the cards, say), then we can consider the deck of cards as a permutation, and the stacks correspond to maximal increasing runs in the permutation. The Eulerian numbers count the number of permutations of fixed size with a given number of increasing runs.

This book is not the first book written on the topic of Eulerian numbers. Dominique Foata and Marcel-Paul Schützenberger wrote "Théorie géométrique des polynômes eulériens" in 1970. This wonderful book collected and expanded upon many of the ideas surrounding the combinatorics of Eulerian numbers. Despite the title, there is little geometry (in the usual sense) in the book of Foata and Schützenberger. As they themselves explain, the title of their book comes from the fact that they use "propriétés géométriques (combinatoires) des permutations" to obtain their results. For them, "geometric" was synonymous with "combinatorial," which in this case meant a visual, almost tactile understanding of permutations and transformations of permutations.

In the decades since that book, geometry, in the usual sense, has most definitely entered the story of Eulerian numbers. For example, we now know how the Eulerian numbers arise in problems of counting integer points in polytopes, computing volumes of slices of a cube, and counting faces of simplicial complexes. Moreover, the Eulerian numbers can be understood in a larger context of finite reflection groups, known as *Coxeter groups*, where the geometry of hyperplane arrangements plays a major role.

To get a taste of the form some of these connections take, consider the following 1-dimensional simplicial complex:



It has one empty face, six vertices, and six edges. We can record this information in its *f-vector*, $(1,6,6)$, or *f-polynomial*, $1+6t+6t^2$. Next we'll rewrite the f-vector in another basis, as a linear combination of rows of Pascal's triangle (right justified):

$$\begin{array}{lr} & (1,6,6) \\ \hline 1\times & (1,2,1) \\ 4\times & (1,1) \\ 1\times & (1) \end{array}$$

The coefficients of this expansion we will put into the *h-vector*, $(1,4,1)$, or *h-polynomial*, $1+4t+t^2$.

Now let's do something completely different. List out all permutations of $\{1,2,3\}$ and count their *descents*, i.e., the number of positions i such that $w(i) > w(i+1)$:

w	des(w)
123	0
132	1
213	1
231	1
312	1
321	2

If we record the number of permutations with zero, one, and two descents in a vector, we get $(1,4,1)$. The polynomial with these coefficients is known as the Eulerian polynomial $S_3(t) = 1+4t+t^2$, which we observe is the same as the

h-polynomial of the hexagon above. This is not a coincidence! Moreover, that hexagon can be interpreted as the *Coxeter complex* of the symmetric group S_3. A big part of this book seeks to generalize and explain this example.

Another thing that was probably not apparent in 1970 but has since come to the forefront of this subject is that the Eulerian numbers have close cousins known as the *Narayana numbers*. These are named after Tadepalli Venkata Narayana, who described these numbers in a 1959 paper by counting certain types of lattice paths. The Narayana numbers possess many of the same properties as the Eulerian numbers and have many of the same geometric connections. Just as with Eulerian numbers, we can obtain the Narayana numbers by counting permutations according to descents. Here we only consider a certain subset of "pattern-avoiding" permutations that are in bijection with the paths studied by Narayana. The cardinality of this subset is given by the *Catalan numbers*. These numbers are ubiquitous in combinatorial mathematics. In fact, Richard Stanley has a book with a catalogue of objects counted by the Catalan numbers that includes over two hundred distinct entries!

This book has fourteen chapters split into three parts. Chapter 1 is a brief introduction to the classical Eulerian numbers from a modern, combinatorial point of view. Chapter 2 introduces the Catalan numbers and Narayana numbers, including a few different combinatorial models counted by these numbers. Chapter 3 discusses partially ordered sets, a topic that is central to modern enumerative combinatorics, and one that will be important for later chapters. Chapter 4 discusses a strong sort of symmetry property possessed by both the Eulerian numbers and the Narayana numbers. The first real connection to geometry comes in Chapter 5, where we discuss the geometric underpinnings for many of the later chapters. Chapter 6 is a brief diversion into refined enumeration and q-analogues for the Eulerian and Narayana numbers.

Part 2 consists of Chapters 8 and 9, with a supplementary Chapter 10*. In Chapter 8 we discuss some background from combinatorial topology, including simplicial complexes and the Dehn-Sommerville relations. Chapter 9 studies in detail how Eulerian numbers arise when counting faces of simplicial complexes.

Part 3 consists of Chapter 11, Chapter 12, and two supplementary chapters: 13* and 14*. Chapter 11 provides some background on Coxeter groups and discusses how there exist analogues of Eulerian numbers associated with any finite reflection group. Chapter 11 shows how the Narayana numbers can be similarly generalized to Coxeter groups. There are four supplementary Chapters sprinkled throughout the book, covering special topics. These are Chapters 7*, 10*, 13*, and 14*.

The book is primarily intended for a graduate student of combinatorics, or perhaps even an advanced undergraduate. Chapters 1–6, for example, would make for a good one-semester topics in combinatorics course. Very little in the way of background is assumed, particularly in the first four chapters.

Notes and literature references are included at the end of each chapter, along with some relevant problems to work on. Hints and solutions for the problems are given at the end of the book.

The writing style is meant to be expository. Rather than a “Definition-Theorem-Proof” format, I lean towards a more narrative style of writing. My hope is to focus on two main questions:

What is the truth? and Why is it true?

(Hyman Bass once told me that in any human endeavor these are the only two questions that matter.) In some cases answering these questions calls for a completely rigorous proof, but in others I find that a clearly explained “proof by generic example” does a better job of conveying the heart of the matter.

Finally, let me say this book represents my taste and knowledge in algebraic and enumerative combinatorics, omitting many interesting topics that can be related to Eulerian numbers (connections with number theory, topics in the statistics of permutations and words, theory of symmetric and quasisymmetric functions, juggling (!), card shuffling (!), and more). For me the Eulerian and Narayana numbers provide an interesting way to learn about various overlapping topics in modern combinatorial mathematics. I hope that this book can serve as an introduction to a circle of ideas that has been growing for the past few decades. Students can get a glimpse at recent developments while learning more general combinatorial techniques in a motivated way. For those in the research community, I hope the book can serve as a reference, by collecting many of these results in one place. I certainly look forward to having a copy on my shelf.

Chicago, IL, USA Kyle Petersen
June 2015

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Contents

Part I
Combinatorics

Chapter 1
Eulerian numbers

The first interesting array of numbers a typical mathematics student encounters is Pascal's triangle, shown in Table 1.1. It has many beautiful properties, some of which we will review shortly. One of the main points of this chapter is to argue that the array of Eulerian numbers is just as interesting as Pascal's triangle.

Table 1.1 Pascal's triangle of binomial coefficients $\binom{n}{k}$, $0 \leq k \leq n \leq 9$.

$n\backslash k$	0	1	2	3	4	5	6	7	8	9
0	1									
1	1	1								
2	1	2	1							
3	1	3	3	1						
4	1	4	6	4	1					
5	1	5	10	10	5	1				
6	1	6	15	20	15	6	1			
7	1	7	21	35	35	21	7	1		
8	1	8	28	56	70	56	28	8	1	
9	1	9	36	84	126	126	84	36	9	1

1.1 Binomial coefficients

It is likely that a reader of this book is already familiar with binomial coefficients, but we will review this material to establish our point of view for future material. The approach we will take is to define $\binom{n}{k}$ to be a combinatorial quantity: namely, the number of k-element subsets of an n-element set. Therefore $\binom{n}{k} = 0$ if $n < 0$, $k < 0$, or $k > n$. (And if we aren't clear what a

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set with a nonintegral number of elements is, we should probably set $\binom{n}{k} = 0$ unless k and n are integers.) It is immediate from this definition that these numbers have the following symmetry:

$$\binom{n}{k} = \binom{n}{n-k},$$

by the fact that a subset of an n-element set and its complement are in bijection. Further, by considering the fact that a k-element subset S of $\{1, 2, \ldots, n\}$ either has:

- $n \in S$, in which case $S - \{n\}$ is a $(k-1)$-element subset of $\{1, 2, \ldots, n-1\}$, or
- $n \notin S$, in which case S itself is a k-element subset of $\{1, 2, \ldots, n-1\}$,

we get Pascal's recurrence.

Theorem 1.1 (Pascal's recurrence). *For any $n \geq 1$, $k \geq 0$,*

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

We can get an explicit formula for $\binom{n}{k}$ by counting orderings of k-element subsets two different ways. On the one hand, we can choose a k-element subset in $\binom{n}{k}$ ways, then order the elements in $k!$ ways. On the other hand, we can order k of n things in

$$n(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!}$$

ways, since we have n choices for the first element, $n-1$ choices for the second, and so on. We get

$$k!\binom{n}{k} = \frac{n!}{(n-k)!},$$

or

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

We call the numbers $\binom{n}{k}$ *binomial coefficients* because of the following theorem.

Theorem 1.2 (Binomial theorem). *For $n \geq 0$,*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}. \tag{1.1}$$

This famous result can be proved by induction using Pascal's recurrence, but a combinatorial argument can be given by counting subsets of

$\{1, 2, \ldots, n\}$ according to cardinality. On the left-hand side, we consider each of the numbers i in $\{1, 2, \ldots, n\}$ independently as being members of a subset S (with weight x) or not (with weight y), so that

$$(x+y)^n = \sum_{S \subseteq \{1,2,\ldots,n\}} x^{|S|} y^{n-|S|}.$$

On the right-hand side, we see the coefficient of $x^k y^{n-k}$ is the number of k-element subsets, or $\binom{n}{k}$.

We can think of the binomial theorem as giving combinatorial meaning to an algebraic quantity, or we can think of the binomial theorem as a way of encoding combinatorial information algebraically. This idea is elaborated upon in the next section.

1.2 Generating functions

A *generating function* is an algebraic tool for encoding combinatorial data. For a sequence of numbers $a_0, a_1, \ldots$ the *ordinary generating function* is the series

$$\sum_{k \geq 0} a_k t^k = a_0 + a_1 t + a_2 t^2 + \cdots + a_k t^k + \cdots,$$

while the *exponential generating function* is the series

$$\sum_{k \geq 0} a_k \frac{t^k}{k!} = a_0 + a_1 t + a_2 \frac{t^2}{2} + a_3 \frac{t^3}{6} + \cdots.$$

For example, the geometric series

$$\frac{1}{1-t} = 1 + t + t^2 + \cdots, \tag{1.2}$$

can be considered the generating function for the sequence $1, 1, 1, \ldots$, while

$$e^t = 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \cdots,$$

is its exponential generating function.

Generating functions can encode finite sequences as well. For example, the binomial theorem tells us for fixed n, $(1+t)^n$ is the ordinary generating function for the binomial coefficients $\binom{n}{k}$.

While we think of generating functions as algebraic encodings of combinatorial data (a “clothesline on which the sequence hangs,” in Herbert Wilf’s words [166]), we can also manipulate generating functions as analytic functions. For example, if we differentiate Equation (1.2) n times, we get

$$\frac{n!}{(1-t)^{n+1}} = \sum_{k\geq 0} k(k-1)\cdots(k-n+1)t^{k-n},$$

and so after dividing by $n!$ we have another generating function for the binomial coefficients, this time taken column-wise in Pascal's triangle (Table 1.1):

$$\frac{t^i}{(1-t)^{n+1}} = \sum_{k\geq 0} \binom{k+n-i}{n} t^k. \tag{1.3}$$

1.3 Classical Eulerian numbers

Let us now meet the Eulerian numbers. As with binomial coefficients, our starting point is a combinatorial definition.

For a given positive integer n, the symmetric group S_n is the set of all permutations of $[n] := \{1, 2, \ldots, n\}$, i.e., bijections $w : [n] \to [n]$. We will usually write permutations in one-line notation: $w = w(1)w(2)\cdots w(n)$, so a typical element of S_7 is $w = 3125647$.

For any permutation $w \in S_n$, we define a *descent* to be a position i such that $w(i) > w(i+1)$, and we denote by $\mathrm{Des}(w)$ the set of descents of w,

$$\mathrm{Des}(w) = \{i : w(i) > w(i+1)\}.$$

We let $\mathrm{des}(w)$ denote the number of descents of w, i.e.,

$$\mathrm{des}(w) = |\,\mathrm{Des}(w)| = |\{i : w(i) > w(i+1)\}|. \tag{1.4}$$

For example, if $w = 3125647$, then there are descents in position 1 (since $3 > 1$) and in position 5 (since $6 > 4$). Hence, $\mathrm{des}(w) = 2$. The permutation $12\cdots n$ is the only permutation with no descents, while its reversal, $n\cdots 21$, has the maximal number, with $n-1$.

We define the *Eulerian numbers*, denoted $\left\langle {n \atop k} \right\rangle$, to be the number of permutations in S_n with k descents, i.e.,

$$\left\langle {n \atop k} \right\rangle = |\{w \in S_n : \mathrm{des}(w) = k\}|. \tag{1.5}$$

For example, we see from Table 1.2 there are 11 permutations in S_4 with two descents. Hence, $\left\langle {4 \atop 2} \right\rangle = 11$.

Table 1.3 shows the Eulerian numbers $\left\langle {n \atop k} \right\rangle$, with $1 \leq k \leq n \leq 10$. The reader should be careful not to confuse the triangle of Eulerian numbers with the sequence of *Euler numbers*: $1, 1, 2, 5, 16, 61, 272, 1385, 7936, \ldots$, though there is a connection. See the discussion at the end of Section 4.2 and Problem 4.2.

Table 1.2 The permutations in S_4 grouped according to descent number.

$\mathrm{des}(w) = 0$	$\mathrm{des}(w) = 1$	$\mathrm{des}(w) = 2$	$\mathrm{des}(w) = 3$
1234	1243	3421	4321
	1324	4231	
	1342	2431	
	1423	3241	
	2134	4312	
	2314	4132	
	2341	1432	
	2413	3142	
	3124	4213	
	3412	2143	
	4123	3214	

Table 1.3 The Eulerian numbers $\left\langle {n \atop k} \right\rangle$, $0 \le k < n \le 10$.

$n \backslash k$	0	1	2	3	4	5	6	7	8	9
1	1									
2	1	1								
3	1	4	1							
4	1	11	11	1						
5	1	26	66	26	1					
6	1	57	302	302	57	1				
7	1	120	1191	2416	1191	120	1			
8	1	247	4293	15619	15619	4293	247	1		
9	1	502	14608	88234	156190	88234	14608	502	1	
10	1	1013	47840	455192	1310354	1310354	455192	47840	1013	1

The function

$$\mathrm{des} : \bigcup_{n \ge 1} S_n \to \{0, 1, 2, 3, \ldots\}$$

is an example of a *permutation statistic*, a function from the set of permutations to the integers. Other well-known permutation statistics include the number of inversions (discussed in Section 5.1) and the number of cycles. When we count permutations according to a particular permutation statistic, this gives rise to the *distribution* of the statistic. Any statistic whose distribution gives the numbers $\left\langle {n \atop k} \right\rangle$ is called an *Eulerian statistic*.

Besides des, there are many other Eulerian permutation statistics. For example, it is easy to see the equivalence of des and the number of *ascents* defined by

$$\mathrm{asc}(w) = |\{i : w(i) < w(i+1)\}| = |[n-1] - \mathrm{Des}(w)|.$$

However, proving that two statistics are equidistributed is not always so easy. For example, the number of *excedances* of a permutation, defined as

$$\operatorname{exc}(w) = |\{i : w(i) > i\}|, \tag{1.6}$$

is an Eulerian statistic. While exc ranges from 0 to $n-1$, its relationship with des is disguised. But this disguise can be lifted with a bijection from S_n to itself that maps descents to excedances, known as the "transformation fondamentale" of Dominique Foata and Marcel-Paul Schützenberger [70]. The reader is invited to study this bijection, along with other manifestations of the Eulerian numbers, in the problems at the end of the chapter. For now, we will stick to the definition of Eulerian numbers in terms of des.

We can use the descent definition of Eulerian numbers to prove a Pascal-like linear recurrence as follows. Notice that if w is in S_n with k descents, then deleting n from w results in a permutation in S_{n-1} with k or $k-1$ descents. Conversely, we can form permutations of n with k descents from permutations of $n-1$ with k or $k-1$ descents by inserting n.

To be precise, suppose v is a permutation in S_{n-1} with $k-1$ descents. Then inserting n at the far left of v or in an ascent position of v creates a permutation $w \in S_n$ with k descents. There are $n-1-(k-1) = n-k$ such positions.

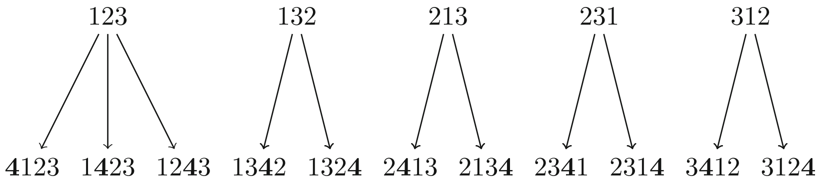
Similarly, if $v \in S_{n-1}$ already has k descents, then inserting n in a descent position of v or at the far right gives a permutation $w \in S_n$ with k descents. There are $k+1$ such positions.

We have the following result.

Theorem 1.3 (The linear recurrence). *For any k and $n > 0$,*

$$\left\langle {n \atop k} \right\rangle = (n-k)\left\langle {n-1 \atop k-1} \right\rangle + (k+1)\left\langle {n-1 \atop k} \right\rangle. \tag{1.7}$$

For example, with $n = 4$, $k = 1$, we have:



and so $\left\langle {4 \atop 1} \right\rangle = 3\left\langle {3 \atop 0} \right\rangle + 2\left\langle {3 \atop 1} \right\rangle$.

We can use the recurrence in Equation (1.7) to think of the triangle of Eulerian numbers as an edge-weighted version of Pascal's triangle. See Figure 1.1.

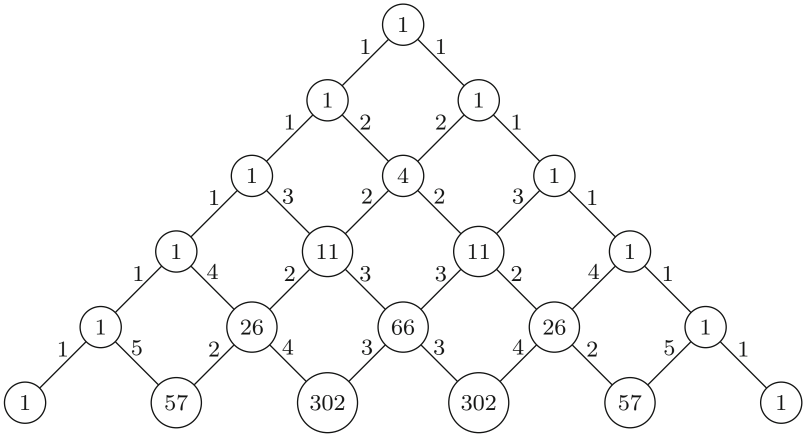


Fig. 1.1 Generating Eulerian numbers via the recurrence relation.

1.4 Eulerian polynomials

For fixed n, we define the nth *Eulerian polynomial* as the generating function for the Eulerian numbers $\left\langle {n \atop k} \right\rangle$ as follows:

$$S_n(t) = \sum_{w \in S_n} t^{\mathrm{des}(w)} = \sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle t^k. \tag{1.8}$$

For example, $S_1(t) = 1$, $S_2(t) = 1 + t$, $S_3(t) = 1 + 4t + t^2$, and $S_4(t) = 1 + 11t + 11t^2 + t^3$. We will define $S_0(t) = 1$ for convenience.

An immediate consequence of Theorem 1.3 is an identity that relates the Eulerian polynomial $S_{n+1}(t)$ to $S_n(t)$ and its derivative.

Since $S_n(t) = \sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle t^k$, we have $S_n'(t) = \sum_{k=1}^{n-1} k \left\langle {n \atop k} \right\rangle t^{k-1}$. Thus,

$$\begin{aligned}(1 + nt)S_n(t) &= (1 + nt) \sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle t^k, \\ &= \sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle t^k + \sum_{k=1}^{n} n \left\langle {n \atop k-1} \right\rangle t^k,\end{aligned}$$

and

$$\begin{aligned}t(1 - t)S_n'(t) &= (t - t^2) \sum_{k=1}^{n-1} k \left\langle {n \atop k} \right\rangle t^{k-1}, \\ &= \sum_{k=1}^{n-1} k \left\langle {n \atop k} \right\rangle t^k - \sum_{k=1}^{n-1} (k-1) \left\langle {n \atop k-1} \right\rangle t^k.\end{aligned}$$

Adding these four sums, we get that the coefficient of t^k in $(1+nt)S_n(t)$ $+t(1-t)S_n'(t)$ is:

$$(n+1-k)\left\langle {n \atop k-1} \right\rangle + (k+1)\left\langle {n \atop k} \right\rangle,$$

which, by Theorem 1.3, equals $\left\langle {n+1 \atop k} \right\rangle$. Thus, we have the following result.

Theorem 1.4 (Linear polynomial recurrence). *For any $n \geq 0$,*

$$S_{n+1}(t) = (1+nt)S_n(t) + t(1-t)S_n'(t). \tag{1.9}$$

1.5 Two important identities

As it is equivalent to the numeric recurrence, Theorem 1.4 does not give any new combinatorial insight into Eulerian numbers. However, sometimes having a combinatorial identity rephrased like this allows us to perform algebraic and analytic operations freely, and these operations can uncover new combinatorial information that we may not have guessed at otherwise. This approach to combinatorial identities is sometimes referred to as "manipulatorics" since it often boils down to formal manipulations of formulas. While it might not be as satisfying as a direct combinatorial explanation, it is nonetheless an important skill to have as a practitioner of the combinatorial arts. Sometimes it may be the only way we know how to prove a combinatorial identity.

In general, our preference for this book will be to give direct combinatorial explanations, but in some cases the manipulatorics approach is simpler, or allows for more elegant statements of results. As practice, let us now do some manipulatorics, starting with Theorem 1.4 and finishing with a truly interesting result known as *Worpitzky's identity* (Corollary 1.2). We ask for a combinatorial proof of Worpitzky's identity in Problem 1.13.

For $n \geq 0$, define the function

$$s_n(t) = (1-t)^{n+1} \sum_{k \geq 0} (k+1)^n t^k.$$

Then $s_0 = 1$ and it is straightforward to verify the identity

$$s_{n+1}(t) = (1+nt)s_n(t) + t(1-t)s_n'(t).$$

Thus, $s_n(t) = S_n(t)$, and we have the following corollary, which we refer to as the *Carlitz identity.*

Corollary 1.1 (The Carlitz identity). *For any $n \geq 0$,*

$$\frac{S_n(t)}{(1-t)^{n+1}} = \sum_{k \geq 0} (k+1)^n t^k. \tag{1.10}$$

But using the definition of $S_n(t)$ we have

$$\frac{S_n(t)}{(1-t)^{n+1}} = \sum_{i=0}^{n-1} \left\langle {n \atop i} \right\rangle \left(\frac{t^i}{(1-t)^{n+1}} \right),$$

and recalling the expression for $t^i/(1-t)^{n+1}$ in Equation (1.3), we find:

$$\begin{aligned}\frac{S_n(t)}{(1-t)^{n+1}} &= \sum_{i=0}^{n-1} \left\langle {n \atop i} \right\rangle \sum_{k\geq 0} \binom{k+n-i}{n} t^k,\\ &= \sum_{k\geq 0} \left(\sum_{i=0}^{n-1} \left\langle {n \atop i} \right\rangle \binom{k+n-i}{n} \right) t^k.\end{aligned}$$

By comparing with the formula in Equation (1.10) we get the following wonderful identity.

Corollary 1.2 (Worpitzky's identity). *For any $n \geq 0$,*

$$(k+1)^n = \sum_{i=0}^{n-1} \left\langle {n \atop i} \right\rangle \binom{k+n-i}{n}.$$

For example, with $k=3$, $n=5$, we get:

$$\begin{aligned}4^5 &= \left\langle {5 \atop 0} \right\rangle \binom{8}{5} + \left\langle {5 \atop 1} \right\rangle \binom{7}{5} + \left\langle {5 \atop 2} \right\rangle \binom{6}{5} + \left\langle {5 \atop 3} \right\rangle \binom{5}{5},\\ &= 1\cdot 56 + 26\cdot 21 + 66\cdot 6 + 26\cdot 1,\\ &= 1024.\end{aligned}$$

Using Worpitzky's identity repeatedly, with $k \geq 0$, gives us explicit formulas for the Eulerian numbers:

$$\begin{aligned}\left\langle {n \atop 0} \right\rangle &= 1,\\ \left\langle {n \atop 1} \right\rangle &= 2^n - (n+1),\\ \left\langle {n \atop 2} \right\rangle &= 3^n - \left\langle {n \atop 1} \right\rangle \binom{n+1}{n} - \binom{n+2}{n},\\ &= 3^n - 2^n(n+1) + (n+1)^2 - \binom{n+2}{n},\\ &= 3^n - 2^n(n+1) + \binom{n+1}{n-1},\\ &\ \vdots\end{aligned}$$

Continuing in this way, we find the following formula.

Corollary 1.3 (Alternating sum formula). *The Eulerian numbers have the following formula for any $n \geq 1$, $k \geq 0$:*

$$\begin{aligned}\left\langle {n \atop k} \right\rangle &= (k+1)^n - k^n\binom{n+1}{n} + (k-1)^n\binom{n+1}{n-1} - \dots, \\ &\quad \dots + (-1)^k\binom{n+1}{n+1-k}, \\ &= \sum_{i=0}^{k}(-1)^i(k+1-i)^n\binom{n+1}{n+1-i}, \\ &= \sum_{i=0}^{k}(-1)^i(k+1-i)^n\binom{n+1}{i}. \end{aligned} \tag{1.11}$$

1.6 Exponential generating function

We finish the chapter with a mixture of bijective combinatorics and manipulatorics to derive the exponential generating function for the Eulerian polynomials.

We begin with a simple, elegant way to generate permutations recursively. Recall that in deriving the linear recurrence in Theorem 1.3, we carefully examined how inserting a new largest number into a permutation affected descent numbers. This time, to form a permutation of n, we first choose which subset of the elements $\{1, 2, \dots, n-1\}$ go to the left of n, which elements go to the right, and permute independently on the left and on the right. See Figure 1.2.

The number of descents in any permutation is one more than the sum of the descents to the left of n and the descents to the right of n. Let i denote the number of elements to the left of n. Then

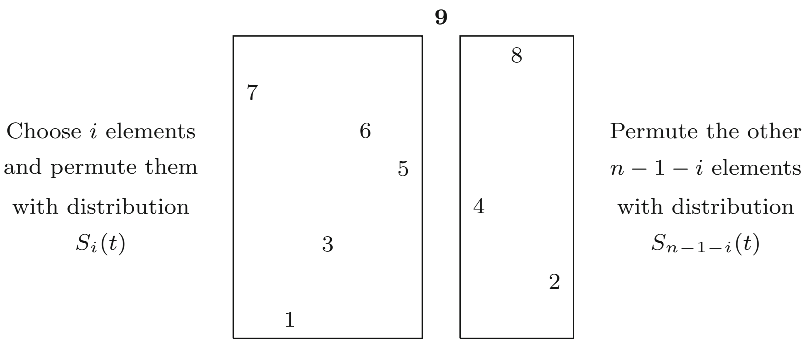


Fig. 1.2 The idea behind Equation (1.12).

$$S_i(t) \cdot t \cdot S_{n-1-i}(t)$$

is the generating function for descents of permutations with these same elements to the left of n. Of course, if there is nothing to the right of n, i.e., if $i = n-1$, then we simply get the number of descents to the left of n. Summing over all i, we have the following quadratic recurrence for Eulerian polynomials.

Theorem 1.5 (Quadratic polynomial recurrence). *For any $n > 0$,*

$$S_n(t) = S_{n-1}(t) + t \sum_{i=0}^{n-2} \binom{n-1}{i} S_i(t) S_{n-1-i}(t). \tag{1.12}$$

The recurrence in Equation (1.12) now leads to a way to find an expression for the exponential generating function

$$S(t,z) := \sum_{n \geq 0} S_n(t) \frac{z^n}{n!} = \sum_{n,k \geq 0} \left\langle {n \atop k} \right\rangle t^k \frac{z^n}{n!}.$$

Indeed, (1.12) gives:

$$\begin{aligned}
\frac{d}{dz} S(t,z) &= \sum_{n \geq 1} S_n(t) \frac{z^{n-1}}{(n-1)!}, \\
&= \sum_{n \geq 1} S_{n-1}(t) \frac{z^{n-1}}{(n-1)!} + t \sum_{n \geq 1} \sum_{i=0}^{n-2} \binom{n-1}{i} S_i(t) S_{n-1-i}(t) \frac{z^{n-1}}{(n-1)!}, \\
&= S(t,z) + t \sum_{n \geq 1} \sum_{i=0}^{n-2} S_i(t) \frac{z^i}{i!} \cdot S_{n-1-i}(t) \frac{z^{n-1-i}}{(n-1-i)!}, \\
&= S(t,z) + tS(t,z)(S(t,z) - 1)).
\end{aligned}$$

Solving the differential equation

$$f'(z) = tf^2(z) + (1-t)f(z),$$

with initial condition $f(0) = 1$ gives us the following result, originally due to Euler.

Theorem 1.6 (Exponential generating function). *We have*

$$S(t,z) = \frac{t-1}{t - e^{z(t-1)}}. \tag{1.13}$$

Those who don't want to rely on solving a differential equation to derive this formula are encouraged to see Problem 1.15.

Notes

The reader looking for more details about the generating function approach to enumerative combinatorics would do well to read Richard Stanley's classic work [154] and Herbert Wilf's book [166]. The book by Philippe Flajolet and Robert Sedgewick [66] focuses on analytic methods for extracting information from generating functions, and part A gives a nice perspective on symbolic methods for constructing generating functions. An earlier book that also contains a wealth of information about the use of generating functions in combinatorial enumeration is John Riordan's book [130].

The Eulerian numbers appear in a chapter of Euler's textbook on differential calculus [64, Part II, Caput VII, pp. 389–390]. In this chapter, Euler essentially sets himself the task of solving the differential equation for the exponential generating function we have in Equation (1.13), and when expanding its series, he finds the Eulerian numbers, and mentions the alternating sum formula given in Equation (1.11). Dominique Foata has a lovely survey in which he explains Euler's motivation and derivation [68]. Leonard Carlitz and his collaborators studied Eulerian numbers and their generalizations in several papers in the 20th century, e.g., [38, 39, 41, 43, 44], while Foata and Marcel-Paul Schützenberger wrote a then-comprehensive treatment of Eulerian numbers from a combinatorial point of view in [70]. From the 1980s onward, the number of scholarly articles on Eulerian numbers and their generalizations is too numerous to attempt to catalogue.

Most of the results in this chapter can be traced back to Carlitz or Riordan, though the Carlitz identity in Corollary 1.1 was known to Euler. We refer to it as the Carlitz identity because of a generalization of the identity obtained by Carlitz in 1975 [40], though even this generalization predates Carlitz—it can be found in Percy MacMahon's textbook from 1915/16 [106]. According to Carlitz [38], Worpitzky's identity (Corollary 1.2) dates from an 1883 paper by Julius Worpitzky [168], though Don Knuth [96, pp. 36] attributes the identity to an 1867 publication of Chinese mathematician Li Shan-Lan, and remarks that special cases for $n \leq 5$ were known to Yoshisuke Matsunaga of Japan, who died in 1744.

Problems

1.1. A *composition* of n is an ordered list of positive integers whose sum is n, denoted $\alpha = (\alpha_1, \ldots, \alpha_k)$. Show that the number of compositions of n with k parts is $\binom{n-1}{k-1}$.

1.2. How many compositions α of n have the following properties?

1. α has parts of size 1 and 2 only, e.g., for $n = 9$, $(2, 1, 1, 2, 2, 1)$ is acceptable, but not $(1, 2, 3, 1, 2)$.

2. α has only odd parts, e.g., for $n = 9$, $(3,1,5)$ is acceptable, but not $(1,2,1,5)$.
3. α has all its parts greater than 1, except possibly the last entry, e.g., for $n = 9$, $(3,4,2)$ and $(3,3,2,1)$ are acceptable, but not $(3,3,1,2)$.

1.3. Show that the Fibonacci numbers satisfy the following identity:

$$f_n = \sum_{k\geq 0} \binom{n-k}{k}. \tag{1.14}$$

This can be quickly verified with Pascal's recurrence and careful bookkeeping, but see if you can find a combinatorial argument using one of the sets of compositions in Problem 1.2.

1.4. For $n \geq 1$, let $\phi_n = f_n/f_{n-1}$, where f_n is the nth Fibonacci number. Using the Fibonacci recurrence, find a recurrence for ϕ_n and use it to compute the limit:

$$\phi = \lim_{n\to\infty} \phi_n.$$

The number ϕ is sometimes called the *golden ratio*.

1.5. Let $f(z)$ denote the ordinary generating function for the Fibonacci numbers, i.e.,

$$f(z) = 1 + z + 2z^2 + 3z^3 + 5z^4 + 8z^5 + \cdots = \sum_{k\geq 0} f_k z^k,$$

with $f_0 = f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$.

1. Write $f(z)$ as
$$f(z) = \frac{1}{q(z)},$$
where $q(z)$ is a quadratic polynomial. Hint: use the recurrence for the Fibonacci numbers to find an identity for $f(z)$ of the form $f(z)q(z) = 1$.
2. Use the expression found in part 1 to give a "manipulatorics" proof of Equation (1.14).
3. Factor $q(z)$ from part 1 as $q(z) = (1-\alpha z)(1-\beta z)$, then find numbers A and B so that
$$f(z) = \frac{A}{1-\alpha z} + \frac{B}{1-\beta z}.$$
4. Use part 3 to show
$$f_n = \frac{\phi^{n+1} - \overline{\phi}^{n+1}}{\sqrt{5}},$$
where ϕ is the golden ratio found in Problem 1.4 and $\overline{\phi}$ is the other root of the polynomial $x^2 - x - 1$.

1.6. Show that the following permutation statistics are Eulerian.

1. The number of *ascents* of a permutation w, $\mathrm{asc}(w) = \{i : w(i) < w(i+1)\}$, e.g., $\mathrm{asc}(1374265) = 3$.
2. The number of (maximal, increasing) *runs* of a permutation, denoted $\mathrm{runs}(w)$, where a maximal increasing run is a substring $w(i) < w(i+1) < \cdots < w(i+r)$ such that $w(i-1)$ is not smaller than $w(i)$ and $w(i+r)$ is not smaller than $w(i+r+1)$. For example, $\mathrm{runs}(1374265) = 4$.
3. The number of *readings* of a permutation, denoted $\mathrm{read}(w)$. This is the number of times one must scan the one-line notation of w from left to right to find the numbers $1, 2, \ldots, n$ in order. For example with $w = 1374265$ we read through four times:

times read	1	3	7	4	2	6	5
1	1				2		
2		3		4			5
3						6	
4			7				

so $\mathrm{read}(1374265) = 4$.

1.7. Show that the number of *excedances*, $\mathrm{exc}(w) = \{i : w(i) > i\}$, is Eulerian.

1.8. An *inversion sequence* of length n is a vector

$$s = (s_1, \ldots, s_n),$$

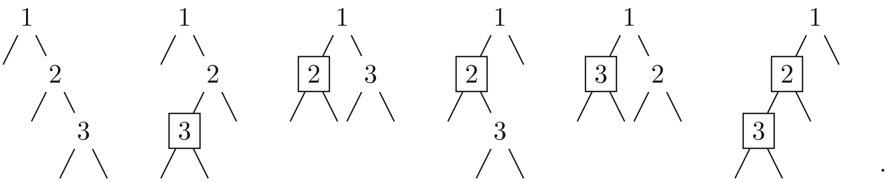
such that $0 \leq s_i \leq i - 1$. Show that counting inversion sequences according to ascents (with $\mathrm{asc}(s) = \{i : s_i < s_{i+1}\}$ as with permutations) gives rise to the Eulerian distribution, e.g., if $n = 3$, the inversion sequences and their ascent numbers are:

s	$\mathrm{asc}(s)$
$(0,0,0)$	0
$(0,0,1)$	1
$(0,0,2)$	1
$(0,1,0)$	1
$(0,1,1)$	1
$(0,1,2)$	2

.

1.9. An *increasing binary tree* of size n is a rooted, planar tree with n internal nodes (internal means not a leaf) such that each internal node has two children: a *left child* and a *right child*. Further, the internal nodes are labeled with $1, 2, \ldots, n$ so that any path from the root to a leaf follows increasing labels. Show that counting the number of increasing binary trees of size n

according to how many internal nodes are left children gives the Eulerian distribution. For example, with $n = 3$, we have following trees, whose internal left children are highlighted:



1.10. For any n and $k = 0, 1, \ldots, n-1$, let $\mathcal{R}_{n,k}$ denote the set of points $(x_1, \ldots, x_n)$ in the unit cube whose sum is between k and $k+1$, i.e.,

$$k \leq x_1 + \cdots + x_n \leq k+1,$$

with $0 \leq x_i \leq 1$. What is the volume of $\mathcal{R}_{n,k}$?

1.11. The number of *cyclic descents* of a permutation $w \in S_n$ is the number of ordinary descents, plus one if $w(n) > w(1)$. We denote this statistic by $\operatorname{cdes}(w)$. For example, $\operatorname{cdes}(31524) = 3$ and $\operatorname{cdes}(43152) = 3$, whereas $\operatorname{des}(31524) = 2$ and $\operatorname{des}(43152) = 3$. Show that

$$\sum_{w \in S_n} t^{\operatorname{cdes}(w)} = ntS_{n-1}(t).$$

1.12. Give a bijective proof that

$$\left\langle {n \atop 1} \right\rangle = 2^n - n - 1.$$

1.13. Give a bijective proof of Worpitzky's identity:

$$(k+1)^n = \sum_{i=0}^{n-1} \left\langle {n \atop i} \right\rangle \binom{k+n-i}{n}.$$

Hint: interpret the left-hand side as counting the set of all integer vectors $(a_1, a_2, \ldots, a_n)$, with $0 \leq a_i \leq k$, and try to group these according to permutations of n by rearranging the sequence in weakly increasing order.

1.14. Give a combinatorial proof of the Carlitz identity in (1.10):

$$\frac{S_n(t)}{(1-t)^{n+1}} = \sum_{k \geq 0} (k+1)^n t^k.$$

Hint: try the method of "balls in boxes" as follows. Clearly $(k+1)^n$ is the number of ways to place n distinct (labeled) balls into $k+1$ boxes. Try to partition the ways to put the balls into boxes according to permutations of n, and show that for a fixed permutation w in S_n, the generating function for arrangements of ball boxes that correspond to w is:

$$\frac{t^{\mathrm{des}(w)}}{(1-t)^{n+1}}.$$

1.15. Use the Carlitz identity (Equation (1.10)) to derive Equation (1.13). Hint: start with

$$\sum_{n\geq 0} \frac{S_n(t)}{(1-t)^{n+1}} \frac{z^n}{n!}.$$

Chapter 2
Narayana numbers

WHILE THE SEQUENCE OF FIBONACCI NUMBERS entered the public imagination a long time ago, it can be argued that the sequence introduced in this chapter is of greater importance in combinatorics today. Here we will study the *Catalan numbers*,

$$1, 1, 2, 5, 14, 42, 429, 1430, 4862, 16796, 58786, \ldots,$$

and a triangle of numbers that refine the Catalan numbers, known as the *Narayana numbers*. Throughout the book, the Narayana numbers will be shown to possess the same (or nearly the same) properties as the Eulerian numbers.

2.1 Catalan numbers

The *Catalan numbers* are denoted C_n, $n \geq 0$, and are given by the explicit formula $C_n = \frac{1}{n+1}\binom{2n}{n}$. The sequence of Catalan numbers is among the most famous sequences in mathematics. One reason for the ubiquity of the Catalan numbers may be that they satisfy the following quadratic, convolutive recurrence for $n \geq 1$:

$$C_n = \sum_{i=0}^{n-1} C_i C_{n-1-i}, \tag{2.1}$$

and this numeric recurrence is a shadow of natural structural recurrences possessed by many families of combinatorial objects.

T.K. Petersen, *Eulerian Numbers*, Birkhäuser Advanced Texts Basler Lehrbücher, DOI 10.1007/978-1-4939-3091-3_2

From (2.1), we can derive the generating function:

$$C(z) = \sum_{n \geq 0} C_n z^n,$$

as follows.

$$\begin{aligned} C(z) &= \sum_{n \geq 0} C_n z^n \\ &= 1 + z \sum_{n \geq 1} \sum_{i=0}^{n-1} C_i z^i C_{n-1-i} z^{n-1-i} \\ &= 1 + zC(z)^2. \end{aligned}$$

Therefore,

$$zC(z)^2 - C(z) + 1 = 0,$$

and we get

$$C(z) = \frac{1 - \sqrt{1-4z}}{2z}. \tag{2.2}$$

We mention the Catalan numbers because they enumerate an important subset of permutations that we will now describe. Counting these permutations according to descents gives rise to the array of *Narayana numbers*, a distribution that has many of the same properties as the Eulerian distribution.

2.2 Pattern-avoiding permutations

The permutations we will study in this chapter are *231-avoiding permutations*. These are permutations w such that there is no triple of indices $i < j < k$ such that $w(k) < w(i) < w(j)$. That is, the letters $w(i), w(j)$, and $w(k)$ are not in the same relative positions as 2, 3, and 1. If a permutation w contains such a triple, we say w *contains* the pattern 231; otherwise, we say w *avoids* the pattern 231. For example, the permutation 53412 contains the pattern 231 since $w(4) < w(2) < w(3)$ (or since $w(5) < w(2) < w(3)$), whereas the permutation 32154 avoids 231. The notion of pattern avoidance is easy to understand visually when we draw the graph of a permutation as an array of dots on a square grid. See Figure 2.1.

Let $S_n(231)$ denote the set of permutations in S_n avoiding the pattern 231. The 231-avoiding permutations, for $n \leq 5$, are listed in Table 2.1.

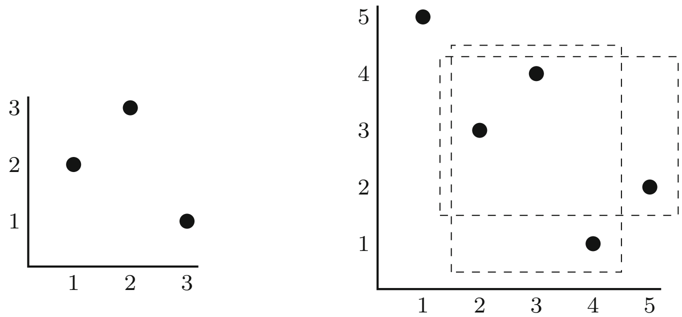


Fig. 2.1 The permutation 53412 contains the pattern 231 in several ways. Two occurrences of the pattern are indicated with dashed line boxes.

Table 2.1 The 231-avoiding permutations of n with k descents, $0 \le k < n \le 5$.

$n\backslash k$	0	1	2	3	4
1	1				
2	12	21			
3	123	213	321		
		132			
		312			
4	1234	2134	3214	4321	
		1324	2143		
		3124	1432		
		1243	4213		
		1423	4132		
		4123	4312		
5	12345	21345	32145	43215	54321
		13245	21435	32154	
		31245	14325	21543	
		12435	42135	15432	
		14235	41325	53214	
		41235	43125	52143	
		12354	21354	51432	
		12534	13254	54213	
		15234	31254	54132	
		51234	12543	54312	
			21534		
			15324		
			15243		
			15423		
			52134		
			51324		
			53124		
			51243		
			51423		
			54123		

We will now show that the 231-avoiding permutations obey a structural recurrence compatible with the numeric recurrence in (2.1). For the moment, let $c_n = |S_n(231)|$ and define $c_0 = 1$ for convenience. We will show that for $n \geq 1$:

$$c_n = \sum_{i=0}^{n-1} c_i c_{n-1-i},$$

and hence $c_n = C_n$ for all n.

First of all, suppose u is a permutation in $S_i(231)$ and v is a permutation of $\{i+1, \ldots, n-1\}$ that avoids 231. Then since every letter of u is smaller than every letter of v, the permutation

$$w = u(1) \cdots u(i)\, n\, v(1) \cdots v(n-1-i),$$

formed by inserting n between u and v, is a 231-avoiding permutation. There are c_i choices for u and c_{n-1-i} choices for v, so summing over all i, we have

$$\sum_{i=0}^{n-1} c_i c_{n-1-i} \leq c_n.$$

On the other hand, suppose $w \in S_n$ is 231-avoiding, with $w(i+1) = n$. Let $u = w(1) \cdots w(i)$ denote the word to the left of n, and let $v = w(i+2) \cdots w(n)$ denote the word to the right of n. Clearly both of these words must avoid the pattern 231. Further, if there was a letter a in u that was greater than a letter b in v, then there would be a 231 pattern formed by the letters a, n, b in w. Hence, every letter of u must be smaller than every letter of v. In other words, $u \in S_i(231)$ and v is a permutation of $\{i+1, \ldots, n-1\}$ that avoids 231. This shows

$$c_n \leq \sum_{i=0}^{n-1} c_i c_{n-1-i},$$

and so in light of our earlier discussion, the two quantities must equal each other:

$$c_n = \sum_{i=0}^{n-1} c_i c_{n-1-i}.$$

Since the c_n satisfy the same recurrence as the Catalan numbers with the same initial values, $c_n = C_n$, and we have the following combinatorial characterization of Catalan numbers. (The first of many, as we will discover later in the chapter.)

Theorem 2.1. *For $n \geq 1$,*

$$|S_n(231)| = C_n.$$

While showing that $|S_n(231)| = C_n$ recursively is fine, one would like to also have a direct combinatorial proof, e.g., by showing $(n+1)|S_n(231)| = \binom{2n}{n}$

via a bijection. This is left to Problem 2.2, though we will do something similar in Section 2.4 for another set of objects counted by Catalan numbers.

Before moving on, we remark that there is nothing particularly interesting about the pattern 231 for Theorem 2.1. It is possible to exhibit bijections between the set $S_n(231)$ and the set $S_n(p)$, where $p \in \{123, 132, 213, 312, 321\}$ is *any* pattern of length three. See Problem 2.1.

2.3 Narayana numbers

Similarly to how we defined the Eulerian numbers, we define the *Narayana number* $N_{n,k}$ to be the number of permutations in $S_n(231)$ with k descents:

$$N_{n,k} = |\{w \in S_n(231) : \mathrm{des}(w) = k\}|.$$

We have the Narayana numbers shown in Table 2.2.

Table 2.2 The Narayana numbers $N_{n,k}$, $0 \le k < n \le 10$.

$n\backslash k$	0	1	2	3	4	5	6	7	8	9
1	1									
2	1	1								
3	1	3	1							
4	1	6	6	1						
5	1	10	20	10	1					
6	1	15	50	50	15	1				
7	1	21	105	175	105	21	1			
8	1	28	196	490	490	196	28	1		
9	1	36	336	1176	1764	1176	336	36	1	
10	1	45	540	2520	5292	5292	2520	540	45	1

We will show in Section 2.4 that the Narayana numbers are given by the formula

$$N_{n,k} = \frac{1}{k+1}\binom{n}{k}\binom{n-1}{k}. \tag{2.3}$$

It is easily shown that this formula is equivalent to

$$N_{n,k} = \det\begin{pmatrix} \binom{n-1}{k} & \binom{n}{k+1} \\ \binom{n}{k} & \binom{n+1}{k+1} \end{pmatrix} = \binom{n-1}{k}\binom{n+1}{k+1} - \binom{n}{k}\binom{n}{k+1},$$

and therefore we can extract the triangle of Narayana numbers as 2×2 minors of Pascal's triangle. See Figure 2.2.

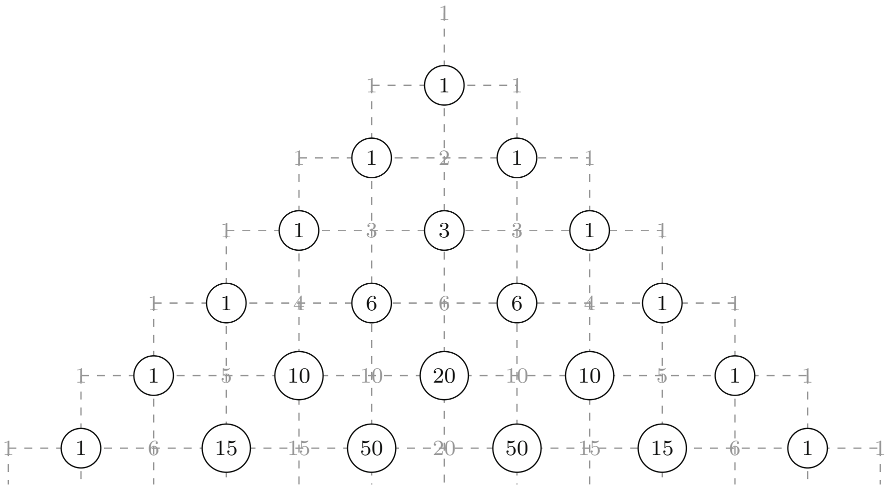


Fig. 2.2 The triangle of Narayana numbers obtained as determinants.

We will see that the generating function for Narayana numbers (with n fixed) obeys a refined Catalan recurrence. Define

$$C_n(t) = \sum_{w \in S_n(231)} t^{\mathrm{des}(w)} = \sum_{k=0}^{n-1} N_{n,k} t^k,$$

with $C_0(t) := 1$. We will refer to $C_n(t)$ as the *Narayana polynomial*.

If we follow the recursive argument that led to Theorem 2.1 while keeping track of descents, we will get a recurrence for the Narayana polynomials that refines (2.1). In that proof was an implicit bijection between elements $w \in S_n(231)$ and pairs (u, v) with $u \in S_i(231)$ (for some i) and v a permutation of $\{i+1, \ldots, n-1\}$ that avoids 231. Namely, we can write

$$w = u(1) \cdots u(i)\, n\, v(1) \cdots v(n-1-i),$$

as shown in Figure 2.3.

Since the number of descents of w is one more than the number of descents in u plus the number of descents in v, we get

$$\sum_{\substack{w \in S_n(231) \\ w(i+1)=n}} t^{\mathrm{des}(w)} = tC_i(t)C_{n-1-i}(t). \tag{2.4}$$

Of course if $i = n-1$ then v is the empty word and the number of descents of w equals only the number of descents of u. This contributes a $C_{n-1}(t)$ term to the distribution, and then summing (2.4) over all $i < n-1$ gives the following result, which is similar to Theorem 1.5 for Eulerian polynomials.

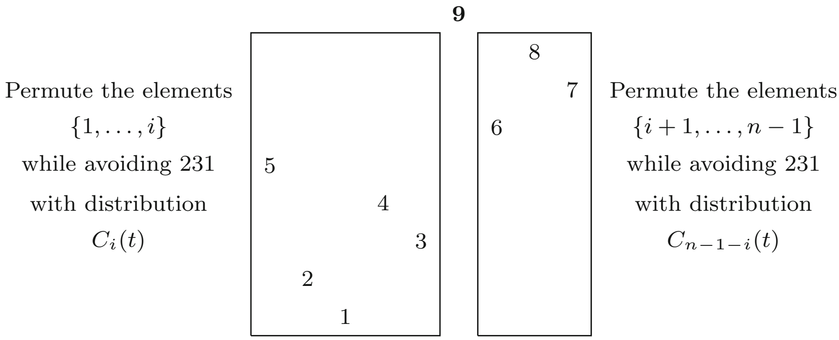


Fig. 2.3 The idea behind Equation (2.5).

Theorem 2.2. *For* $n \geq 1$*,*

$$C_n(t) = C_{n-1}(t) + t \sum_{i=0}^{n-2} C_i(t) C_{n-1-i}(t). \tag{2.5}$$

Now that we have the recurrence from Theorem 2.2 it is a straightforward matter to construct the generating function for the Narayana polynomials,

$$C(t, z) := \sum_{n \geq 0} C_n(t) z^n.$$

We have:

$$\begin{aligned}
C(t, z) &= \sum_{n \geq 0} C_n(t) z^n, \\
&= 1 + \sum_{n \geq 1} \left[C_{n-1}(t) + t \sum_{i=0}^{n-2} C_i(t) C_{n-1-i}(t) \right] z^n, \\
&= 1 + z \sum_{n \geq 1} C_{n-1} z^{n-1} + tz \sum_{n \geq 1} \sum_{i=0}^{n-2} C_i(t) z^i C_{n-1-i}(t) z^{n-1-i}, \\
&= 1 + zC(t, z) + tzC(t, z)(C(t, z) - 1).
\end{aligned}$$

From this we can conclude that $C(t, z)$ satisfies:

$$tzC(t, z)^2 - (1 + z(t - 1))C(t, z) + 1 = 0.$$

Solving for $C(t, z)$ gives:

$$C(t, z) = \frac{1 + z(t - 1) - \sqrt{1 - 2z(t + 1) + z^2(t - 1)^2}}{2tz}. \tag{2.6}$$

The 231-avoiding permutations are one combinatorial interpretation for the Catalan numbers, but there are many, many others. (See the notes at the end of the chapter.) There are three others that we will introduce and discuss now, with more deferred to the problems at the end of the chapter.

2.4 Dyck paths

A *Dyck path* of length $2n$ is a lattice path from $(0,0)$ to (n,n) consisting of n horizontal steps "East" from (i,j) to $(i+1,j)$ and n vertical steps "North" from (i,j) to $(i,j+1)$, such that all points on the path satisfy $i \leq j$, i.e., the path, when drawn in the cartesian plane, lies on or above the line $y=x$. We can either draw the picture of the path or write the list of steps the path follows as a word on the set $\{N,E\}$. For example, the path $p = NNENNEEENENENNNEEE$ would be drawn as in Figure 2.4.

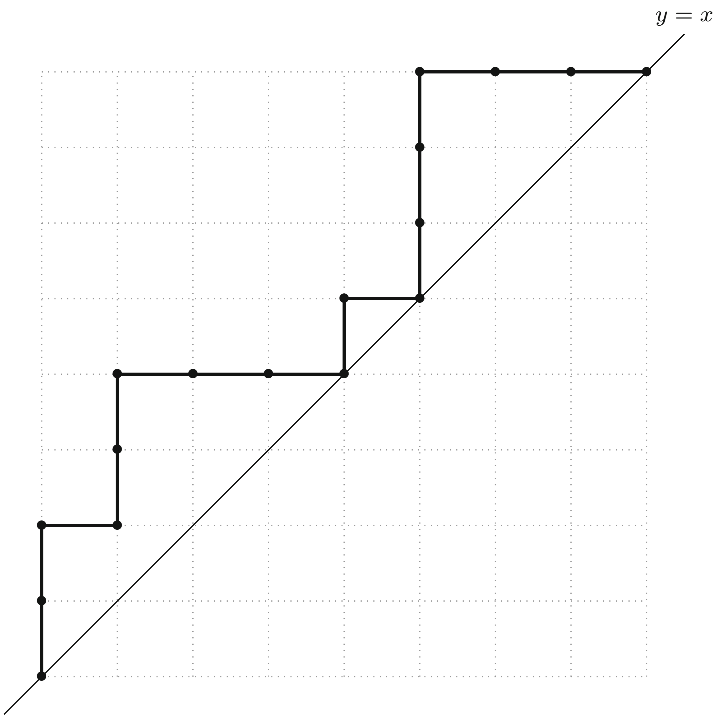


Fig. 2.4 One of the 4862 paths in Dyck(8).

Let Dyck(n) denote the set of Dyck paths of length $2n$. A *peak* of a Dyck path p is a point (i,j) such that $(i,j-1)$ and $(i+1,j)$ are on p as well.

Similarly, a *valley* of p is a point (i, j) such that $(i-1, j)$ and $(i, j+1)$ are on p. In other words, a peak corresponds to a North step followed immediately by an East step, while a valley corresponds to an East step followed immediately by a North step. The number of peaks of p is denoted $\mathrm{pk}(p)$ and the number of valleys is $\mathrm{val}(p)$. For example, the path of Figure 2.4 has four peaks, $\mathrm{pk}(p) = 4$, and three valleys, $\mathrm{val}(p) = 3$. It is easy to see that for $p \in \mathrm{Dyck}(n)$, $1 \leq \mathrm{pk}(p) \leq n$, while $0 \leq \mathrm{val}(p) = \mathrm{pk}(p) - 1 \leq n-1$. The Dyck paths for $n \leq 4$ are shown in Table 2.3, grouped according to the number of peaks in the path.

At the end of this section we will provide a bijection between Dyck paths and 231-avoiding permutations, but first we will give bijective proofs that there are Catalan-many Dyck paths and that counting Dyck paths according to the number of peaks gives rise to the Narayana numbers.

2.4.1 Counting all Dyck paths

The Catalan number C_n can be written as a difference of two binomial coefficients:

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n-1}.$$

Notice that there are a total of $\binom{2n}{n}$ lattice paths from $(0,0)$ to (n,n) since we have $2n$ steps and exactly n of them must be N steps. Similarly, we can think of $\binom{2n}{n-1}$ as counting the paths from $(0,0)$ to $(n+1, n-1)$. Thus to give a direct combinatorial proof that $|\,\mathrm{Dyck}(n)| = C_n$, we will write $\binom{2n}{n} = C_n + \binom{2n}{n-1}$ and describe a bijection

$$\left\{ \begin{array}{c} \text{lattice paths from} \\ (0,0) \text{ to } (n,n) \end{array} \right\} \longleftrightarrow \mathrm{Dyck}(n) \bigcup \left\{ \begin{array}{c} \text{lattice paths from} \\ (0,0) \text{ to } (n+1, n-1) \end{array} \right\}.$$

The idea here is called the *reflection principle*. Let p be a path from $(0,0)$ to (n,n). If p never passes below the line $y = x$, it is a Dyck path. If it does go below this line, say the *reflection point* of p is the first time the path hits the line $y = x - 1$. The *reflection* of p, $r(p)$, is the path obtained by swapping E for N on every step after the reflection point. For example, if

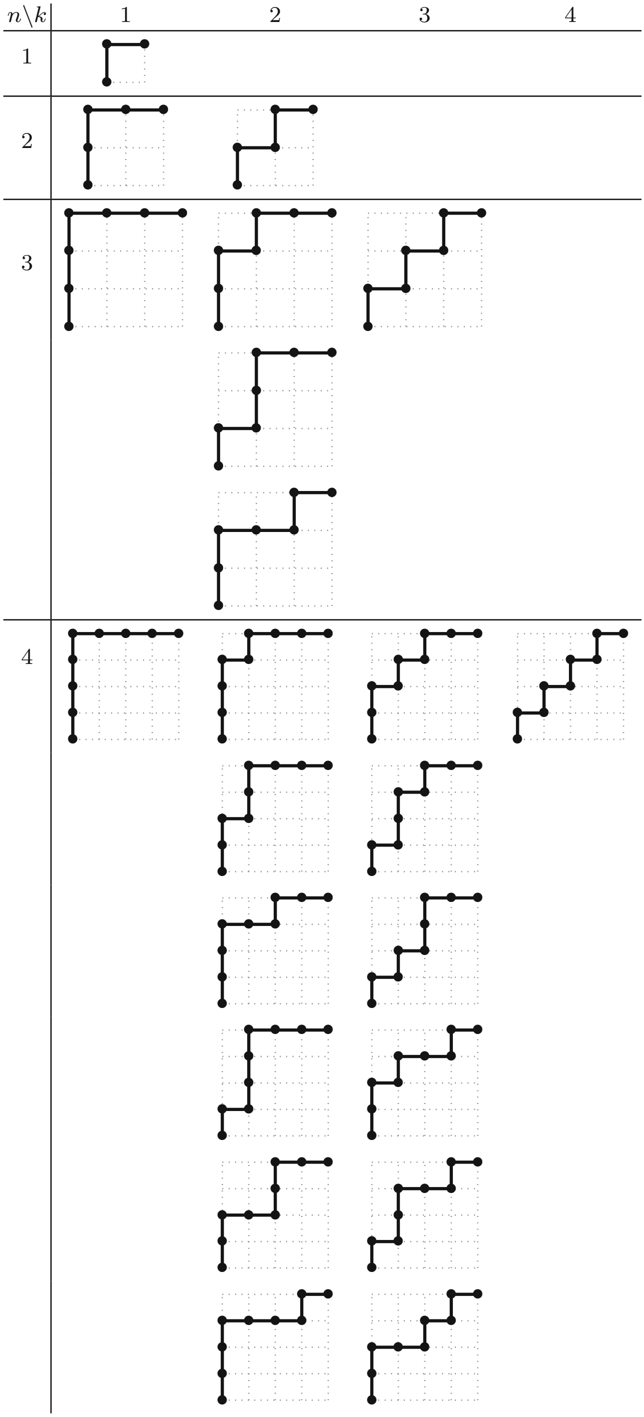
$$p = NNEEE|NNENEEENNEN,$$

then

$$r(p) = NNEEE|EENENNNEENE,$$

is its reflection. The vertical bar here is used to mark the reflection point. In terms of words on $\{N, E\}$, this is simply the first time, in reading from left to right, that we have more letters E than N. This example is drawn in Figure 2.5.

Table 2.3 The paths in Dyck(n), $n \leq 4$, grouped by number of peaks, k.



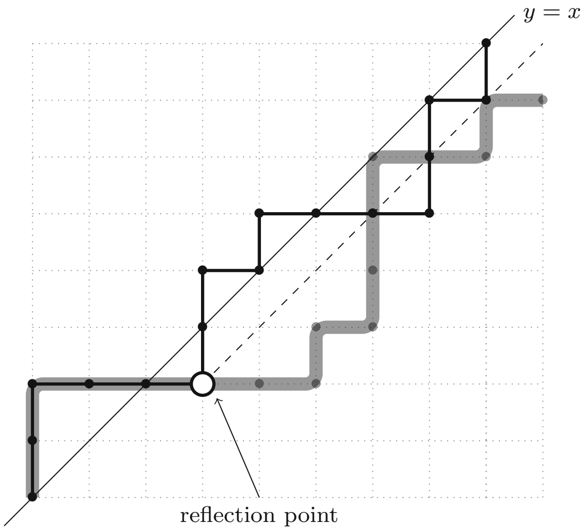


Fig. 2.5 The reflection of a lattice path.

The reflection map is easily reversed (the reflection point is well defined for both sets of paths), so the paths from $(0,0)$ to (n,n) that go below the line $y = x$ are in bijection with all the paths from $(0,0)$ to $(n+1, n-1)$. This shows

$$|\operatorname{Dyck}(n)| = \binom{2n}{n} - \binom{2n}{n-1} = C_n,$$

as desired.

2.4.2 Counting Dyck paths by peaks

Earlier we claimed the Narayana numbers are given by

$$N_{n,k} = \frac{1}{k+1}\binom{n}{k}\binom{n-1}{k}.$$

We will now show that

$$|\{p \in \operatorname{Dyck}(n) : \operatorname{pk}(p) = k+1\}| = \frac{1}{k+1}\binom{n}{k}\binom{n-1}{k}.$$

We will subsequently show that

$$|\{p \in \operatorname{Dyck}(n) : \operatorname{pk}(p) = k+1\}| = |\{w \in S_n(231) : \operatorname{des}(w) = k\}|,$$

justifying the formula for the Narayana numbers.

Our goal will be to show

$$(k+1)|\{p \in \text{Dyck}(n) : \text{pk}(p) = k+1\}| = \binom{n}{k}\binom{n-1}{k}.$$

To do so, we will exhibit a certain set $\mathcal{P}$ of $\binom{n}{k}\binom{n-1}{k}$ lattice paths and show that it can be partitioned into equivalence classes. We will then show each equivalence class has $k+1$ elements and contains exactly one path that corresponds to a Dyck path with $k+1$ peaks.

Define $\mathcal{P}$ to be the set of lattice paths from $(0,-1)$ to (n,n) that begin with a North step, end with an East step, and have exactly $k+1$ peaks, or k valleys. Each such path can be reconstructed from the coordinates of its valleys: $(x_1,y_1),(x_2,y_2),\ldots,(x_k,y_k)$. There are $\binom{n}{k}$ ways to choose the vertical coordinates: $0 \le y_1 < y_2 < \cdots < y_k \le n-1$ in such a path, and $\binom{n-1}{k}$ ways to choose the horizontal coordinates: $1 \le x_1 < x_2 < \cdots < x_k \le n-1$. Hence $|\mathcal{P}| = \binom{n}{k}\binom{n-1}{k}$. See Figure 2.6.

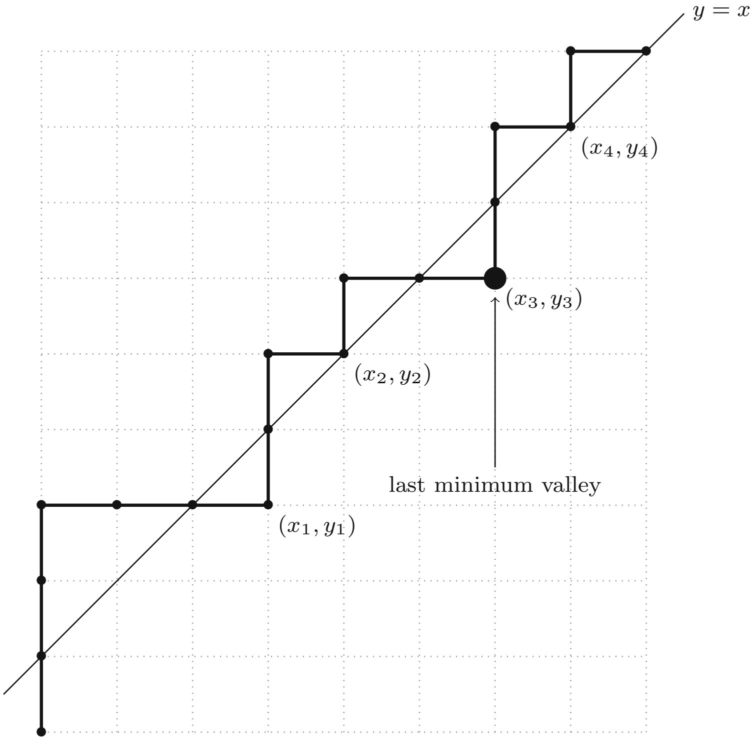


Fig. 2.6 A path from $(0,-1)$ to (n,n) with initial North step, final East step, and k valleys. Here $n = 8, k = 4$.

On the other hand, we can also characterize a path in $\mathcal{P}$ by a sequence of $k+1$ valley-less paths. In Figure 2.6, these valley-less paths are $NNNEEE$, NNE, NEE, NNE, and NE and we can write

$$p = (NNNEEE)(NEE)(NNE) \bullet (NNE)(NE).$$

An important marker in our path (indicated in the word with a $\bullet$) will be the rightmost valley (x_i, y_i) for which $y_i - x_i < 0$ is minimized. In terms of the $\{N, E\}$-word for the path, this is the rightmost position where the letters E most outnumber the letters N. If there are always more letters N than E, we put the marker on the far left.

We will now lump together these lattice paths into equivalence classes given by cyclically permuting the valley-less paths. Let $[p]$ denote the class of p. To continue our example,

$$[p] = \left\{ \begin{array}{c} (NNNEEE)(NEE)(NNE) \bullet (NNE)(NE) \\ (NE)(NNNEEE)(NEE)(NNE) \bullet (NNE) \\ \bullet(NNE)(NE)(NNNEEE)(NEE)(NNE) \\ (NNE) \bullet (NNE)(NE)(NNNEEE)(NEE) \\ (NEE)(NNE) \bullet (NNE)(NE)(NNNEEE) \end{array} \right\}.$$

Notice that the marker gets cyclically permuted along with the valley-less paths. (This is because the path from the marker onward always has more letters N than E when reading from left to right.) Hence, the marker uniquely identifies the cyclic permutation of p and the class $[p]$ must contain $k+1$ distinct paths. Moreover, there is always one path that has the marker on the far left. This path has all its valleys satisfying $y_i - x_i \geq 0$, and hence (if we ignore the initial North step) it is a Dyck path. The cyclic action is shown in pictures in Figure 2.7.

Hence, we can conclude

$$\begin{aligned} (k+1)|\{p \in \mathrm{Dyck}(n) : \mathrm{pk}(p) = k+1\}| &= |\mathcal{P}|, \\ &= \binom{n}{k}\binom{n-1}{k}, \end{aligned}$$

as desired.

2.4.3 A bijection with 231-avoiding permutations

We can construct a playful bijection between Dyck paths and 231-avoiding permutations as follows. First draw a permutation as an array of nonattacking rooks on a chessboard, i.e., if $w(i) = j$, put a rook in column i (from left to right), row j (from bottom to top). Then shade in all squares on the board that either contain a rook, or are weakly to the left and weakly above a

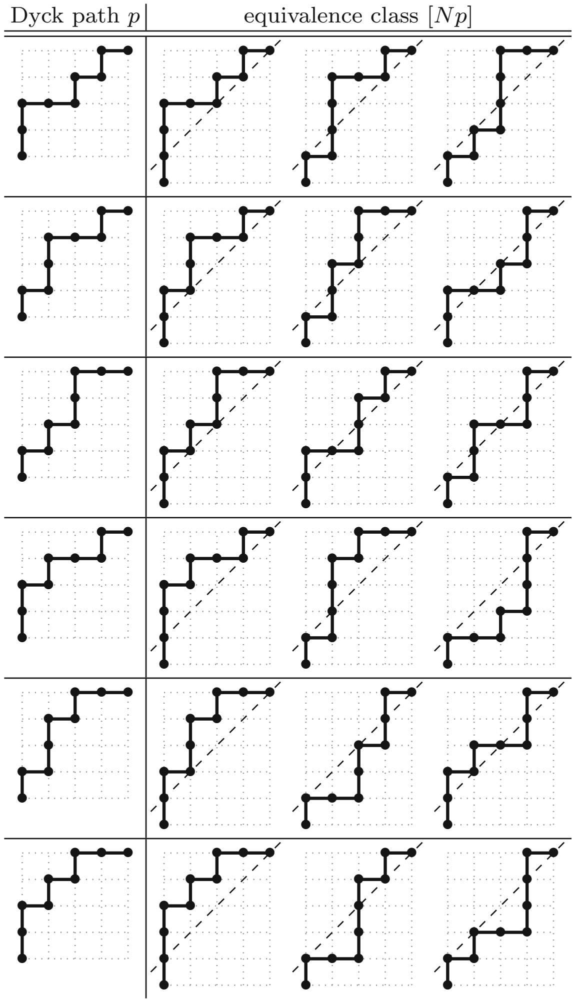


Fig. 2.7 An equivalence relation on lattice paths for $n = 4$, $k + 1 = 3$ peaks.

square with a rook. The boundary of the shaded region is a path that stays below or on the line $y = x$, so it is the mirror image of a Dyck path. Let $\psi : S_n(231) \to \mathrm{Dyck}(n)$ denote this bijection. See Figure 2.8.

The pre-image of a path p is constructed as follows. First, draw the mirror-image of path p, and place rooks, from right to left, in the lowest unoccupied row that is above the path, as shown in Figure 2.9.

From this construction, we can see that each peak of the path p (where we placed the corner rooks in ψ^{-1}) corresponds to a maximal decreasing run $w(i) > w(i+1) > \cdots > w(j)$ of $\psi^{-1}(p) = w$. The number of maximal decreasing runs is necessarily $n - \mathrm{des}(w)$, and so we have the following.

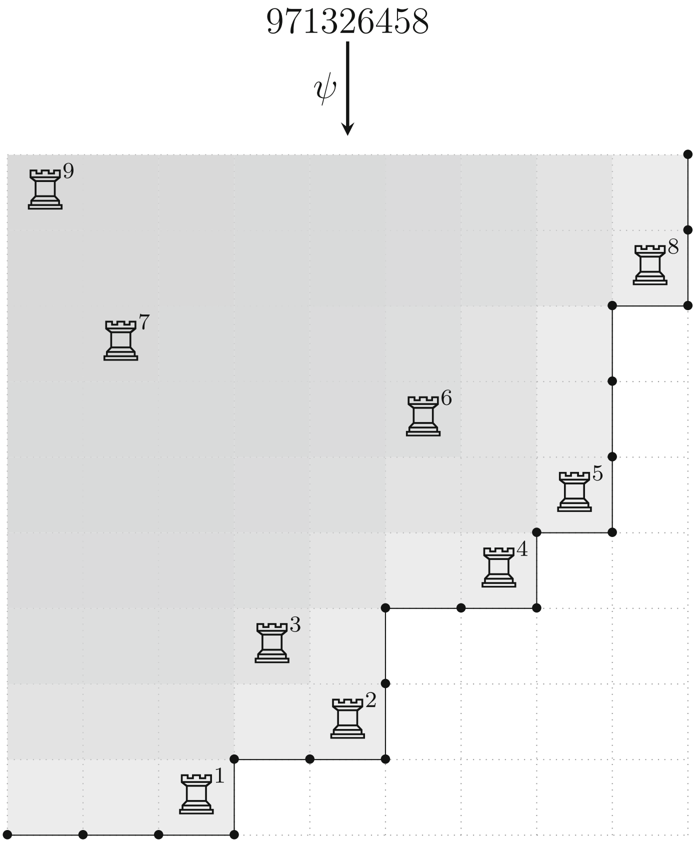


Fig. 2.8 Constructing a Dyck path from a 231-avoiding permutation.

Proposition 2.1. *For any $w \in S_n(231)$, the bijection ψ satisfies*

$$\mathrm{des}(w) = n - 1 - \mathrm{val}(p) = n - \mathrm{pk}(p).$$

Hence,

$$|\{w \in S_n(231) : \mathrm{des}(w) = k\}| = |\{p \in \mathrm{Dyck}(n) : \mathrm{pk}(p) = \mathrm{val}(p)+1 = k+1.\}|.$$

This justifies the formula for the Narayana numbers $N_{n,k} = \frac{1}{k+1}\binom{n}{k}\binom{n-1}{k}$.
We finish the chapter with brief discussion of two other popular combinatorial models counted by the Narayana numbers.

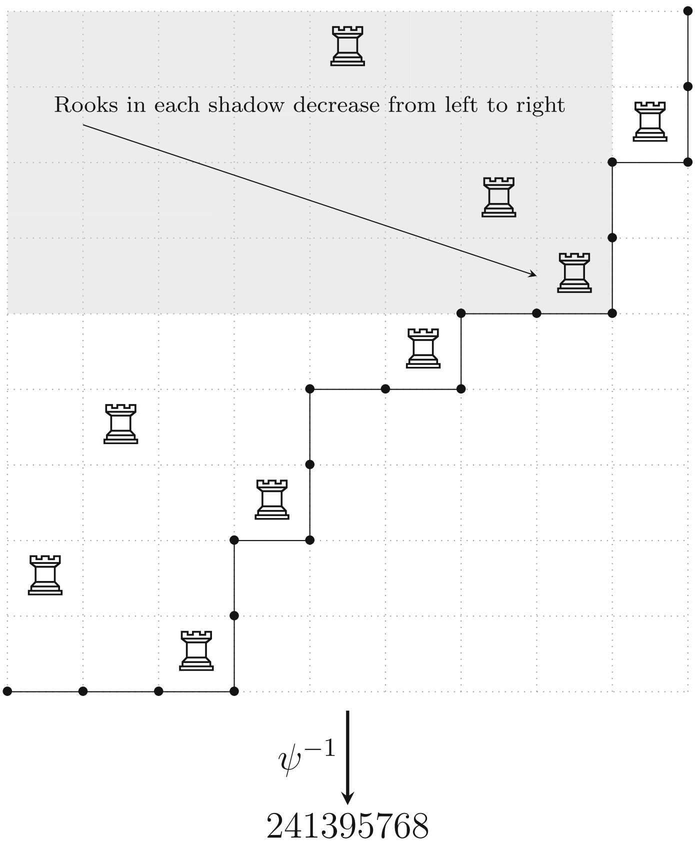


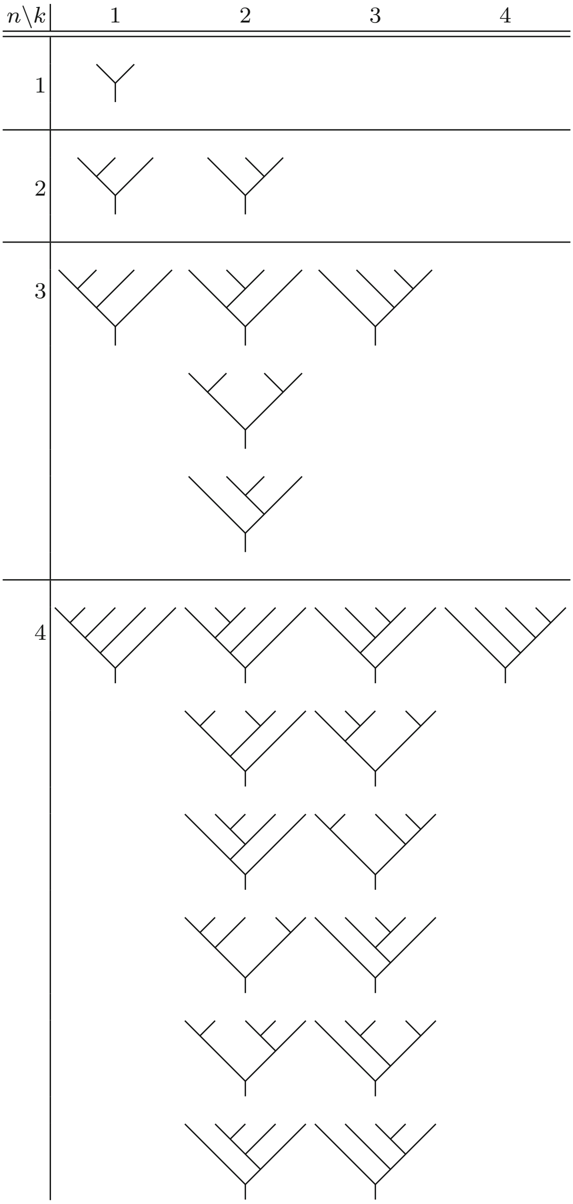
Fig. 2.9 Constructing a 231-avoiding permutation from a Dyck path.

2.5 Planar binary trees

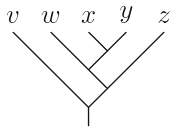
A *planar binary tree* is a rooted tree such that every interior node has precisely two successors. If there are n internal nodes, this means there are $n+1$ leaves. Let $\mathrm{PB}(n)$ denote the number of planar binary trees with n internal nodes. Table 2.4 shows the planar binary trees with at most $n = 4$ internal nodes, grouped according to the number of left-pointing leaves.

The planar binary trees are combinatorial representations for ways to evaluate an associative product of $n+1$ elements. For example, if $n = 2$, we have $((xy)z)$ and $(x(yz))$ as the two possible ways to evaluate the product xyz, and these would correspond to the following trees:

Table 2.4 Planar binary trees grouped according to the number of left-pointing leaves.



where we labeled the leaves by x, y, z to indicate the natural bijection. As a larger example,



$$\longleftrightarrow (v((w(xy))z)).$$

Planar binary trees can be shown to satisfy the Catalan recurrence (see Problem 2.3), but one can also give a direct bijection with 231-avoiding permutations that takes left-pointing leaves to descents, as suggested by the example in Figure 2.10.

Proof of the following proposition is deferred to Problem 2.4.

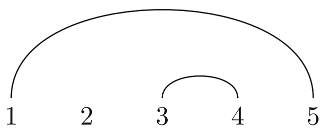
Proposition 2.2. *There is a bijection between* $\mathrm{PB}(n)$ *and* $S_n(231)$ *such that planar binary trees with* $k+1$ *left-pointing leaves are mapped to 231-avoiding permutations with* k *descents.*

In other words, the Narayana numbers count planar binary trees according to left-pointing leaves:

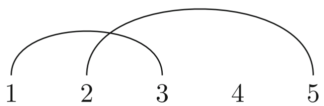
$$N_{n,k} = |\{\tau \in \mathrm{PB}(n) : \tau \text{ has } k+1 \text{ left-pointing leaves}\}|.$$

2.6 Noncrossing partitions

A *noncrossing partition* $\pi = \{R_1, R_2, \ldots, R_k\}$, is a set partition of $[n]$, such that if $\{a, c\} \subseteq R_i$ and $\{b, d\} \subseteq R_j$, with $1 \leq a < b < c < d \leq n$, then $i = j$. That is, two pairs of numbers from distinct blocks cannot be interleaved. Let $\mathrm{NC}(n)$ denote the set of all noncrossing partitions of $[n]$. We will often draw partitions as graphs with vertex set $[n]$, e.g.,



$$= \{\{1,5\},\{2\},\{3,4\}\} \in \mathrm{NC}(5),$$



$$= \{\{1,3\},\{2,5\},\{4\}\} \notin \mathrm{NC}(5).$$

Notice how the notion of a "crossing" manifests itself visually in these diagrams. So that our pictures are canonical, we will only have arcs between consecutive elements in the blocks of the partition. For example, if $i < j < k$ are in the same block, we would draw

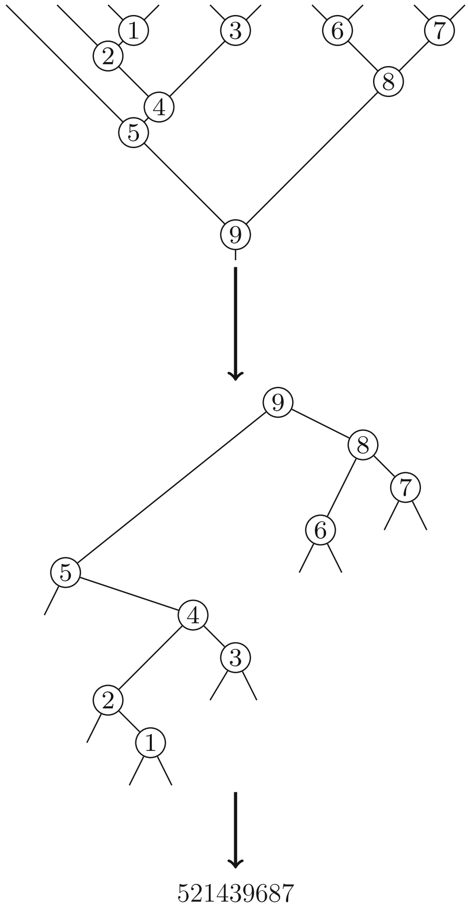
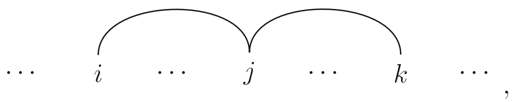


Fig. 2.10 A correspondence between planar binary trees and 231-avoiding permutations.



but not

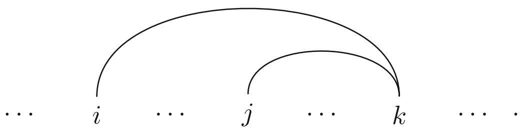
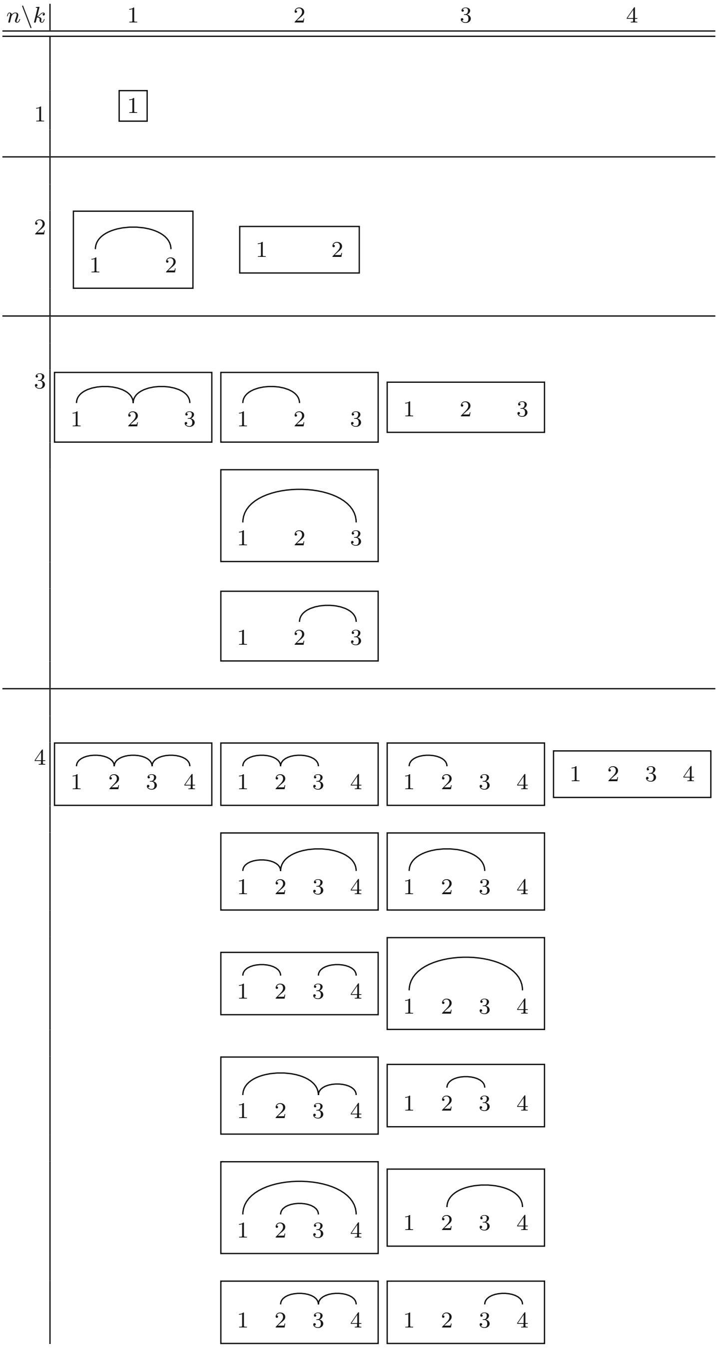


Table 2.5 shows all the noncrossing partitions on at most 4 elements, grouped according to the number of blocks.

Table 2.5 Noncrossing partitions on up to four elements, grouped according to number of blocks.



We can define a bijection $\phi : S_n(231) \to \mathrm{NC}(n)$ by mapping the decreasing runs of a permutation to blocks in a partition. See Figure 2.11.

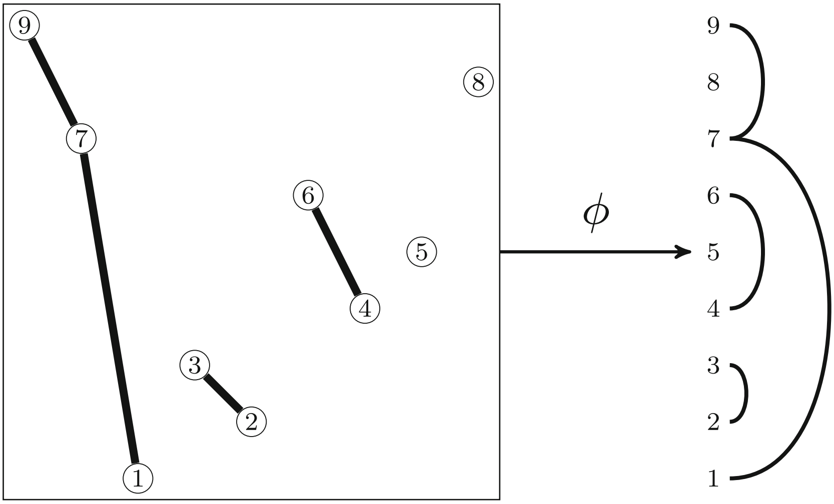


Fig. 2.11 The decreasing runs of a 231-avoiding permutation form a noncrossing partition.

Moreover, we can see that the number of decreasing runs of w, i.e., the number of blocks in π, is $n - \mathrm{des}(w)$.

Proposition 2.3. *For any $w \in S_n(231)$, the bijection ϕ satisfies*

$$\mathrm{des}(w) = n - |\phi(w)|.$$

Hence,

$$|\{w \in S_n(231) : \mathrm{des}(w) = k\}| = |\{\pi \in \mathrm{NC}(n) : |\pi| = n - k\}|.$$

In other words, the Narayana numbers count noncrossing partitions by the number of blocks:

$$N_{n,k} = |\{\pi \in \mathrm{NC}(n) : |\pi| = n - k\}|.$$

Verification of this claim is left to Problem 2.5.

Notes

Despite the name, it seems that it was Euler who first studied the Catalan numbers, which he defined as the number of ways to triangulate a convex polygon. (See Problem 2.6.) There is correspondence between Euler and Christian Goldbach from the middle of the 18th century that shows Euler knew the formula for the Catalan number generating function given in (2.2). Johann Segner was the first to publish a paper about these numbers, in which he proves the recurrence relation from (2.1). The Catalan numbers are named for Eugène Charles Catalan, a 19th century mathematician who wrote several papers about what he knew as the "Segner numbers." It was Catalan who proved that $C_n = \binom{2n}{n} - \binom{2n}{n-1}$.

Many famous mathematicians have studied the Catalan numbers, in many different guises. One can find more than two hundred different combinatorial interpretations for Catalan numbers in a book of Richard Stanley, with historical notes by Igor Pak [155]. See also [153, Problem 6.19].

The Narayana numbers are named for Tadepalli Narayana, who wrote several papers on them in the mid-twentieth century, including [110]. In this paper he essentially counts Dyck paths according to the number of peaks. Our method of counting Dyck paths can be found in the work of Robert Sulanke from 1993 [160].

Problems

2.1. Suppose p is any pattern of length three, i.e., $p \in \{123, 132, 213, 231, 312, 321\}$. Show that the Catalan numbers count the permutations of length n that avoid p.

2.2. Find a bijective proof of the fact that

$$(n+1)C_n = \binom{2n}{n}.$$

2.3. Let $b_n = |\mathrm{PB}(n)|$ denote the number of planar binary trees with n internal nodes. Show that $b_n = C_n$ by describing a structural recurrence on the trees that yields the numeric recurrence

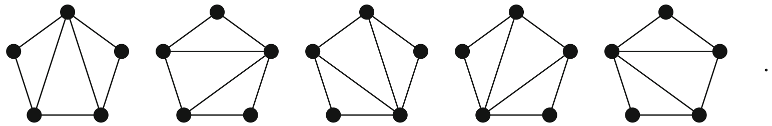
$$b_n = \sum_{i=0}^{n-1} b_i b_{n-1-i},$$

with $b_0 := 1$.

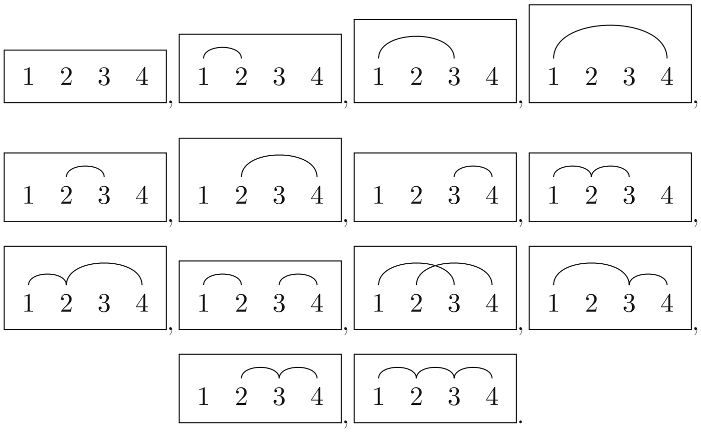
2.4. Prove Proposition 2.2. That is, construct a bijection between $\mathrm{PB}(n)$ and $S_n(231)$ such that trees with $k+1$ left-pointing leaves are mapped to 231-avoiding permutations with k descents.

2.5. Prove Proposition 2.3. That is, show that the map ϕ suggested in Figure 2.11 is indeed a bijection from $S_n(231)$ to $\mathrm{NC}(n)$ that takes decreasing runs to blocks of a noncrossing partition.

2.6 (Triangulations). Show that C_n counts the number of dissections of a convex $(n+2)$-gon into n triangles, using only lines from vertices to vertices. For example, when $n = 3$ there are five such triangulations of a pentagon:

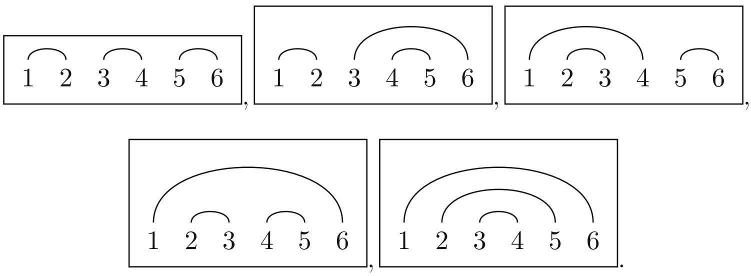


2.7 (Nonnesting partitions). Show that C_n counts the number of *nonnesting partitions* of $[n]$. A nonnesting partition is a set partition $\pi = \{R_1, \ldots, R_k\}$ such that if $\{a, d\} \subseteq R_i$ and $\{b, c\} \subseteq R_j$ with $a < b < c < d$, then $R_i = R_j$. Here are the fourteen nonnesting partitions of $\{1, 2, 3, 4\}$:



Hint: Create a bijection between noncrossing and nonnesting partitions. Conclude that counting nonnesting partitions by number of blocks gives the Narayana numbers.

2.8 (Noncrossing matchings, balanced parenthesizations). Show that C_n counts the number of *noncrossing matchings* on $[2n]$. A noncrossing matching is a noncrossing partition with all the blocks having size two. For example, here are the five noncrossing matchings on $\{1, 2, 3, 4, 5, 6\}$:



The noncrossing matchings can also be thought of as n pairs of parentheses, by mapping the beginning of an arc to a left parenthesis, "(", and mapping the end of an arc to a right parenthesis, ")". The five matchings above would then be:

$$()()(),\ ()(()),\ (())(),\ (()()),\ ((())).$$

A string of n pairs of parentheses that never has more right parentheses than left when reading from left to right is called a *balanced parenthesization.*

Refined counting: Describe a statistic for noncrossing matchings so that the distribution of this statistic gives the Narayana numbers.

2.9 (Standard Young tableaux). Show that C_n counts the number of 2 by n *standard Young tableaux.* A Young tableau is a two dimensional array of numbers that increases across rows and down columns. A standard Young tableau contains all distinct integers, from 1 to the number of entries. The fourteen 2 by 4 tableaux are:

$$\begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline 5&6&7&8\\\hline\end{array},\quad \begin{array}{|c|c|c|c|}\hline 1&2&3&5\\\hline 4&6&7&8\\\hline\end{array},\quad \begin{array}{|c|c|c|c|}\hline 1&2&4&5\\\hline 3&6&7&8\\\hline\end{array},\quad \begin{array}{|c|c|c|c|}\hline 1&2&3&6\\\hline 4&5&7&8\\\hline\end{array},$$

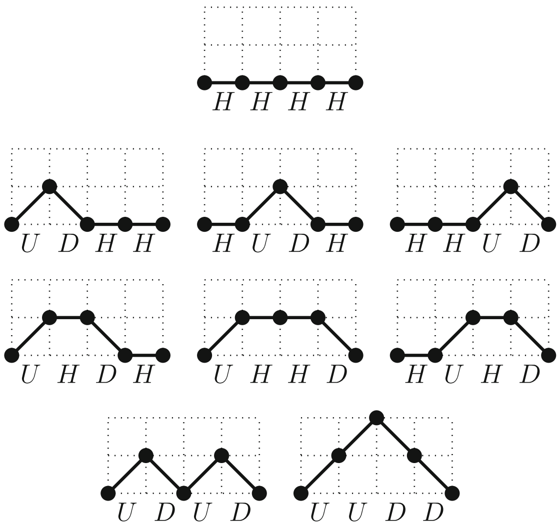
$$\begin{array}{|c|c|c|c|}\hline 1&3&4&5\\\hline 2&6&7&8\\\hline\end{array},\quad \begin{array}{|c|c|c|c|}\hline 1&2&5&6\\\hline 3&4&7&8\\\hline\end{array},\quad \begin{array}{|c|c|c|c|}\hline 1&2&3&7\\\hline 4&5&6&8\\\hline\end{array},\quad \begin{array}{|c|c|c|c|}\hline 1&2&4&6\\\hline 3&5&7&8\\\hline\end{array},$$

$$\begin{array}{|c|c|c|c|}\hline 1&3&4&6\\\hline 2&5&7&8\\\hline\end{array},\quad \begin{array}{|c|c|c|c|}\hline 1&3&5&6\\\hline 2&4&7&8\\\hline\end{array},\quad \begin{array}{|c|c|c|c|}\hline 1&2&4&7\\\hline 3&5&6&8\\\hline\end{array},\quad \begin{array}{|c|c|c|c|}\hline 1&3&4&7\\\hline 2&5&6&8\\\hline\end{array},$$

$$\begin{array}{|c|c|c|c|}\hline 1&2&5&7\\\hline 3&4&6&8\\\hline\end{array},\quad \begin{array}{|c|c|c|c|}\hline 1&3&5&7\\\hline 2&4&6&8\\\hline\end{array}.$$

Refined counting: Describe a statistic for Young tableaux so that the distribution of this statistic gives the Narayana numbers.

2.10 (Motzkin paths). A *Motzkin path* of length n is a lattice path from $(0,0)$ to (n,n) that never passes below the line $y = 0$ and uses only "up" steps from (i,j) to $(i+1,j+1)$, "down" steps from (i,j) to $(i+1,j-1)$, and "horizontal" steps from (i,j) to $(i+1,j)$. For example, here are the nine Motzkin paths of length four:



(Note that Motzkin paths that contain no horizontal steps are in bijection with Dyck paths.) Let M_n denote the number of Motzkin paths of length n, with $M_0 = 1$. Here are the first few values of M_n, sometimes called *Motzkin numbers*:

$$1, 1, 2, 4, 9, 21, 51, 127, 323, 835, 2188, 5798, \ldots.$$

Let $M(z) = \sum_{n \geq 0} M_n z^n$. Show that

$$M(z) = \frac{1 - z - \sqrt{1 - 2z - 3z^2}}{2z^2}.$$

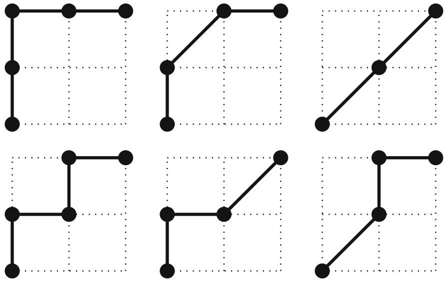
Hint: each Motzkin path is built from a Dyck path by inserting horizontal steps between the steps of the Dyck path. Use this fact to show

$$M(z) = \frac{1}{1-z} C\left(\frac{z^2}{(1-z)^2}\right),$$

where $C(z)$ is the Catalan generating function. The formula for $M(z)$ now follows from Equation (2.2).

2.11. Show that the Motzkin number M_n also counts the number of noncrossing *partial matchings* of $[n]$. In other words, M_n is the number of noncrossing partitions of $[n]$ for which the blocks have size one or two.

2.12 (Schröder paths). A *Schröder path* of size n is a lattice path from $(0, 0)$ to (n, n) that never passes below the line $y = x$ and uses only steps "North" from (i, j) to $(i, j+1)$, "East" from (i, j) to $(i+1, j)$ and "Northeast" from (i, j) to $(i+1, j+1)$. For example, here are the six Schröder paths of size 2:



(Note that Schröder paths with no northeast steps are Dyck paths.) Let R_n denote the number of Schröder paths of size n, with $R_0 = 1$. We call the number R_n a *Schröder number*. Here are the first few values for R_n:

$$1, 2, 6, 22, 90, 394, 1806, 8558, 41586, 206098, \ldots.$$

Let $R(z) = \sum_{n\geq 0} R_n z^n$. Show that

$$R(z) = \frac{1 - z - \sqrt{1 - 6z + z^2}}{2z}.$$

Hint: Just as with Motzkin paths, each Schröder path can be built from a Dyck path by inserting northeast steps between the steps of the Dyck path. Use this fact to show

$$R(z) = \frac{1}{1-z} C\left(\frac{z}{(1-z)^2}\right),$$

where $C(z)$ is the Catalan generating function. The formula then follows from Equation (2.2).

2.13 (Small Schröder numbers). Show the Schröder numbers (apart from $R_0 = 1$) are always even. You can do this by manipulating the generating function in Problem 2.12, but try to explain it combinatorially. Hint: find a bijection between the Schröder paths with a peak on the line $y = x + 1$ and those without. The number of Schröder paths with no peak on the line $y = x + 1$ are called *small Schröder numbers*, denoted r_n. Here are the first few values of the small Schröder numbers:

$$1, 1, 3, 11, 45, 197, 903, 4279, 20793, 103049, \ldots.$$

Given that $r_0 = 1$ and $r_n = R_n/2$ for $n \geq 1$, use the generating function found in Problem 2.12 to conclude that

$$\sum_{n\geq 0} r_n z^n = \frac{1 + z - \sqrt{1 - 6z + z^2}}{4z}.$$

2.14. Show that the small Schröder numbers r_n count the number of valid parenthesizations of $n+1$ symbols with at most $n-1$ pairs of parentheses. Parentheses around the entire expression are not allowed, and each pair of parentheses must enclose at least two sub-expressions. For example, here are the eleven parenthesizations of four symbols:

$$((wx)y)z \quad (w(xy))z \quad (wx)(yz) \quad w((xy)z) \quad w(x(yz))$$

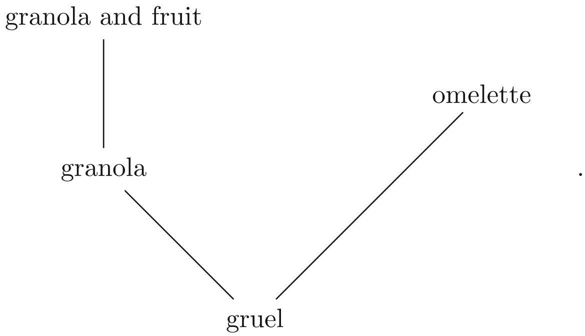
$$(wx)yz \quad (wxy)z \quad w(xy)z \quad w(xyz) \quad wx(yz)$$

$$wxyz$$

Can you interpret these parenthesizations in terms of planar rooted trees of some kind?

Chapter 3
Partially ordered sets

Apples and oranges. Sometimes things are incomparable. For breakfast, I like granola better than gruel. I like it even better when my granola has fresh fruit on top. I also like a nice omelette better than gruel. But on any given day I cannot say whether I would prefer granola (with or without fruit) or an omelette. I am only able to *partially* order my favorite breakfast foods:



Partially ordered sets are extremely important in algebraic and enumerative combinatorics, and we study and use them throughout the rest of the book.

3.1 Basic definitions and terminology

Informally, a partially ordered set P, commonly referred to as a *poset* for short, is a collection of objects with a notion of "less than" and "greater than" but for which some objects may be incomparable. To be precise, a poset is a pair $(P, \leq_P)$, such that the relation $\leq_P$ is:

T.K. Petersen, *Eulerian Numbers*, Birkhäuser Advanced Texts Basler Lehrbücher, DOI 10.1007/978-1-4939-3091-3_3

- transitive, i.e., if $a \leq_P b$ and $b \leq_P c$, then $a \leq_P c$,
- reflexive, i.e., $a \leq_P a$, and
- antisymmetric, i.e., if $a \leq_P b$ and $b \leq_P a$, then $a = b$.

While there are good reasons to study infinite partially ordered sets (especially in algebraic combinatorics), for our purposes it will suffice to assume that P is a finite set.

A *cover relation* in a poset P is a pair $x \neq y$ such that if $x \leq_P z \leq_P y$, then either $x = z$ or $y = z$. Because of transitivity, a partial order is completely determined by its cover relations. Thus a common aid for visualization is the *Hasse diagram* of P. This is a directed graph whose nodes are elements of P, with directed edges for the cover relations. For example, in Figure 3.1 we see the Hasse diagram for the collection of subsets of $\{1, 2, 3\}$ ordered by inclusion. Here, the edges are directed upwards.

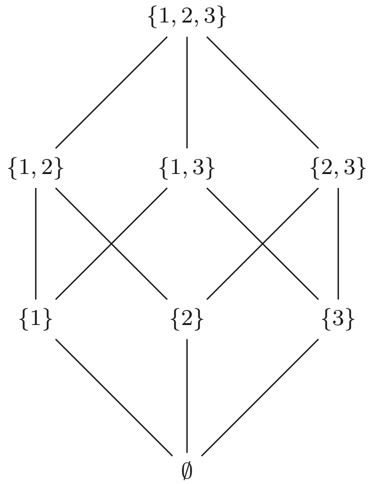
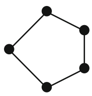
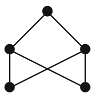


Fig. 3.1 The poset $P = 2^{\{1,2,3\}}$.

We will often discuss collections of subsets of a finite set. If S is a finite set, we write 2^S to denote the poset of subsets of S, ordered by inclusion. This poset is an example of a *lattice*. A lattice L is a poset in which every pair of elements x and y has a uniquely defined least upper bound and greatest lower bound, denoted as follows:

- (least upper bound) $x \vee y = \min\{z \in L : x \leq_L z, y \leq_L z\}$,
- (greatest lower bound) $x \wedge y = \max\{z \in L : z \leq_L x, z \leq_L y\}$.

For example, poset P below is a lattice, while Q is not:

$P =$  , $Q =$  .

Lattice theory is a broad subject with many consequences. Many of the posets discussed in this book are lattices. While we will mention this property when it exists, we will not often exploit the consequences of this fact.

An *order ideal* Q in a poset P is a sub-poset of P such that if $x \in Q$ and $y \leq_P x$, then $y \in Q$ (and $y \leq_Q x$). Notice that an order ideal is determined by its maximal elements. A *principal order ideal* is an order ideal with a unique maximum. The principal order ideal determined by an element x in P is the sub-poset of all elements in P that are less than or equal to x. In Figure 3.2 we see a poset P with two different order ideals highlighted.

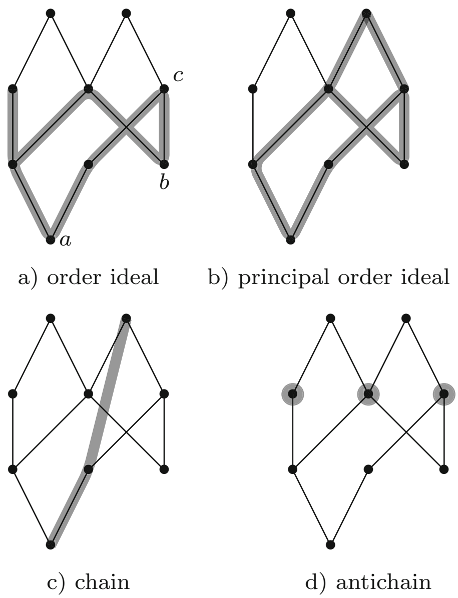
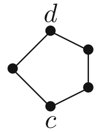


Fig. 3.2 A poset P with various subposets highlighted.

A *chain* in P is a set of elements in P such that every two elements are comparable. This is also sometimes called a *total order* or a *linear ordering*. An *antichain* in P is a subset A such that no two elements in A are comparable in P. The maximal elements in an order ideal form an antichain. See Figure 3.2.

A poset P is *ranked* if there is a function $\rho : P \to \{0, 1, 2, \ldots\}$ such that $\rho(a) = 0$ if a is a minimal element of P, and $\rho(b) = \rho(a) + 1$ if $a <_P b$ is a cover relation. For such a function to be well defined, it must be the case that all paths from a point c to a point d in the Hasse diagram traverse the same number of edges. For example, if a poset has the following Hasse diagram



we cannot define a rank function because there is a path of length two from c to d and another path of length three. Similarly, the poset P in Figure 3.2 is not ranked, since both a and b are P-minimal elements below element c, but the paths from a to c and from b to c have different lengths. On the other hand, the example of Figure 3.1 has a natural rank function given by the cardinality of the subset.

If a poset P has a rank function, then it is quite natural to define the *rank generating function*,

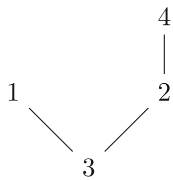
$$f(P;t) = \sum_{a \in P} t^{\rho(a)}.$$

Most of the posets encountered in this book will be ranked, and their rank generating functions often have interesting interpretations. For example, if S is a finite set and $P = 2^S$, the rank generating function is given by the binomial theorem:

$$f(2^S;t) = \sum_{A \subseteq S} t^{|A|} = (1+t)^{|S|}.$$

3.2 Labeled posets and *P*-partitions

Suppose now that P is a partial ordering on $\{1, 2, \ldots, n\}$. We will say the poset is *labeled* by $\{1, 2, \ldots, n\}$. When discussing elements of P we will use $\leq_P$ for the partial order, $\leq_{\mathbb{Z}}$ for the usual integer ordering on elements of P. For example, suppose

$P =$


Then $3 <_P 4$ and $3 <_{\mathbb{Z}} 4$, while $3 <_P 2$ and $2 <_{\mathbb{Z}} 3$.

More generally, we could consider a poset P in terms of its Hasse diagram, and consider labeling the nodes with members of any set. (To take the silly example from the start of this chapter, we could use the set { gruel, granola, granola with fruit, omelette } as our labeling set.) For the purposes of this book, we will usually label a poset P of cardinality n with the numbers $1, 2, \ldots, n$.

A *P-partition* is a function from P to the positive integers, $a : P \to \{1, 2, 3, \ldots\}$, such that:

- a is order preserving, i.e., $a(i) \le a(j)$ if $i \le_P j$, and moreover,
- $a(i) < a(j)$ if $i \le_P j$ and $i \ge_{\mathbb{Z}} j$.

Let $\mathcal{A}(P)$ denote the set of all P-partitions.

Continuing with the example above, we can characterize the P-partitions as follows:

$$\mathcal{A}\left(\begin{array}{ccc} & & 4 \\ & & \vert \\ 1 & & 2 \\ & \diagdown \; 3 \; \diagup & \end{array}\right) = \{a(1) > a(3) < a(2) \le a(4)\}.$$

We can think of permutations as totally ordered chains, with cover relations $w(i) <_w w(i+1)$. The set of *linear extensions* of a poset P, denoted $\mathcal{L}(P)$, is the set of all permutations w such that $i <_P j$ implies $i <_w j$. For example,

$$\mathcal{L}\left(\begin{array}{ccc} & & 4 \\ & & \vert \\ 1 & & 2 \\ & \diagdown \; 3 \; \diagup & \end{array}\right) = \left\{\begin{array}{c} 4 \\ \vert \\ 2 \\ \vert \\ 1 \\ \vert \\ 3 \end{array}, \begin{array}{c} 4 \\ \vert \\ 1 \\ \vert \\ 2 \\ \vert \\ 3 \end{array}, \begin{array}{c} 1 \\ \vert \\ 4 \\ \vert \\ 2 \\ \vert \\ 3 \end{array}\right\} = \{3124, 3214, 3241\}.$$

One of the key observations about P-partitions is that the set of all P-partitions splits into the disjoint union of the w-partitions for all linear extensions w of P.

Lemma 3.1 (Fundamental lemma of P-partitions). *For any finite poset P on $\{1, 2, \ldots, n\}$,*

$$\mathcal{A}(P) = \bigcup_{w \in \mathcal{L}(P)} \mathcal{A}(w),$$

and the union is disjoint.

This lemma is straightforward to verify by induction on the number of pairs of incomparable elements in P. Suppose $i <_{\mathbb{Z}} j$ is an incomparable pair of P. In linearizing, we can put i before j, in which case $a(i) \le a(j)$, or put i after j, in which case $a(i) > a(j)$. Let's carry out this decomposition for our example poset.

$$\begin{aligned} \mathcal{A}\left(\begin{array}{ccc} & & 4 \\ & & \vert \\ 1 & & 2 \\ & \diagdown \; 3 \; \diagup & \end{array}\right) &= \{a(1) > a(3) < a(2) \le a(4)\}, \\ &= \{a(3) < a(1) \le a(2) \le a(4)\} \\ &\quad \cup \{a(3) \le a(2) < a(1) \le a(4)\} \\ &\quad \cup \{a(3) \le a(2) \le a(4) < a(1)\}, \\ &= \mathcal{A}(3124) \cup \mathcal{A}(3214) \cup \mathcal{A}(3241). \end{aligned}$$

The fact that the union is disjoint is easy to see, by looking at $a(1)$ relative to $a(2)$ and $a(4)$.

The nice thing about reducing to the case of permutations is that their P-partitions are characterized by descents. That is,

$$\begin{aligned}\mathcal{A}(w) &= \{a(w(1)) \leq \cdots \leq a(w(n)) : a(w(i)) < a(w(i+1)) \text{ if } i \in \mathrm{Des}(w)\},\\ &= \{b_1 \leq b_2 \leq \cdots \leq b_n : b_i < b_{i+1} \text{ if } i \in \mathrm{Des}(w)\}.\end{aligned}$$

We now count P-partitions according to their maximum value. Define the *order polynomial*, $\Omega(P;k)$, to be the number of P-partitions $a : P \to \{1, 2, \ldots, k\}$. That is,

$$\Omega(P;k) = |\{a \in \mathcal{A}(P) : \max(a(i)) \leq k\}|.$$

The fact that this is indeed a polynomial in k will be justified shortly.

In the case of permutations, these turn out to be binomial coefficients that depend only on k and $\mathrm{des}(w)$. For example,

$$\begin{aligned}\Omega(3124;k) &= |\{1 \leq a(3) < a(1) \leq a(2) \leq a(4) \leq k\}|,\\ &= |\{1 \leq b_1 < b_2 \leq b_3 \leq b_4 \leq k\}|,\\ &= |\{1 \leq b_1 < b_2 < (b_3+1) \leq (b_4+1) \leq k+1\}|,\\ &= |\{1 \leq b_1 < b_2 < (b_3+1) < (b_4+2) \leq k+2\}|,\\ &= |\{1 \leq c_1 < c_2 < c_3 < c_4 \leq k+2\}|,\\ &= \binom{k+2}{4}.\end{aligned}$$

We are essentially choosing a set of n integers from among $k+j$ integers, where j is the number of weak inequalities among the $a(i)$. This number corresponds to the number of ascents of w, so if $w \in S_n$, j is $n-1-\mathrm{des}(w)$.

Thus, for a permutation $w \in S_n$, we have:

$$\Omega(w;k) = \binom{k+n-1-\mathrm{des}(w)}{n},$$

and Lemma 3.1 leads to the following proposition.

Proposition 3.1. *For any poset P on $\{1, 2, \ldots, n\}$,*

$$\Omega(P;k) = \sum_{w \in \mathcal{L}(P)} \binom{k+n-1-\mathrm{des}(w)}{n}.$$

Now define the order polynomial generating function to be

$$H(P;t) = \sum_{k \geq 0} \Omega(P;k) t^k,$$

and recall from Equation 1.3 that

$$\frac{t^i}{(1-t)^{n+1}} = \sum_{k\geq 0} \binom{k+n-i}{n} t^k.$$

Therefore, with a permutation w,

$$\begin{aligned} H(w;t) &= \sum_{k\geq 0} \Omega(w;k)t^k, \\ &= \sum_{k\geq 0} \binom{k+n-1-\mathrm{des}(w)}{n} t^k, \\ &= \frac{t^{\mathrm{des}(w)+1}}{(1-t)^{n+1}}. \end{aligned}$$

Summing over all linear extensions of P, we get:

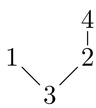
$$\begin{aligned} H(P;t) &= \sum_{k\geq 0} \Omega(P;k)t^k, \\ &= \sum_{k\geq 0} \sum_{w\in\mathcal{L}(P)} \Omega(w;k)t^k, \\ &= \sum_{w\in\mathcal{L}(P)} \left(\sum_{k\geq 0} \Omega(w;k)t^k \right), \\ &= \sum_{w\in\mathcal{L}(P)} H(w;t), \end{aligned}$$

and hence the following result.

Theorem 3.1. *For any poset P on $\{1,2,\ldots,n\}$,*

$$H(P;t) = \frac{\sum_{w\in\mathcal{L}(P)} t^{\mathrm{des}(w)+1}}{(1-t)^{n+1}}.$$

For example, with our running example of

$P =$ 

,

we have $\mathcal{L}(P) = \{3124, 3214, 3241\}$, so

$$H(P;t) = \frac{t^2+2t^3}{(1-t)^5}.$$

Theorem 3.1 yields many interesting corollaries. For example if we take P to be an antichain, i.e., the set $\{1, 2, \ldots, n\}$ with no relations, then every function $a : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, k\}$ is a P-partition and $\Omega(P; k) = k^n$. On the other hand, every permutation $w \in S_n$ is a linear extension of P, so $\mathcal{L}(P) = S_n$. Thus, we see the Eulerian polynomial emerge in the numerator for $H(P; t)$:

$$\sum_{k \geq 0} k^n t^k = \frac{\sum_{w \in S_n} t^{\mathrm{des}(w)+1}}{(1-t)^{n+1}} = \frac{tS_n(t)}{(1-t)^{n+1}}.$$

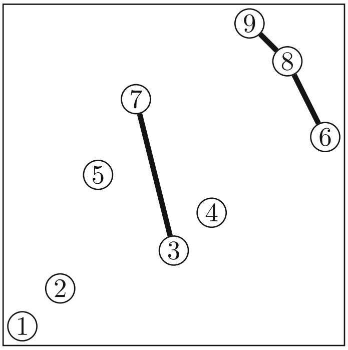
Dividing by t gives us another way to prove the Carlitz identity in Corollary 1.1. Other such results, and connections with discrete geometry, are explored in the problems at the end of the chapter.

We will now turn to some other ways the Eulerian numbers arise in the study of posets.

3.3 The shard intersection order

We will now introduce a partial order on the set of all permutations whose rank generating function is the Eulerian polynomial. Let $\mathrm{Sh}(S_n)$ denote this poset, called the *shard intersection order*. How the name arises will be discussed in Chapter 11, where a more general construction will be discussed.

To understand this partial order most simply, we first highlight the maximal decreasing runs of our permutations, e.g., if $w = 12573486$, we would write $w = 1|2|5|73|4|986$, or draw:



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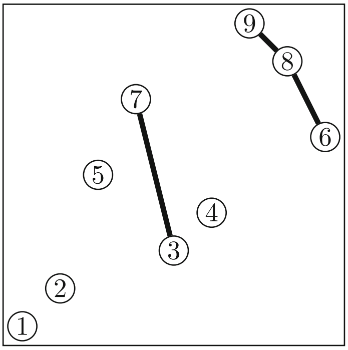
We can think of the decreasing runs of w as blocks in an ordered set partition of $\{1, 2, \ldots, n\}$. We say two blocks A, B, in such a partition are *overlapping* if there are elements $a, c \in A$ and $b \in B$ such that $a < b < c$. Continuing the example of w above, the block 73 overlaps with 5, block 4, and block 986. No other pairs of blocks in w are overlapping.

With this point of view the partial order is characterized by two things. We say $u \leq v$ in $\mathrm{Sh}(S_n)$ if:

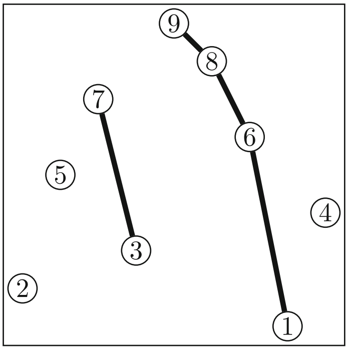
- (Refinement) u refines v as a set partition, and
- (Consistency) if i and j are in the same block in u, and k not in the same block as i and j in u with $i < k < j$, then either k is in the same block as i and j in v, or k is on the same side of i and j in both v and u.

Intuitively, v can be obtained from u by merging some blocks in u while maintaining the relative positions of the overlapping blocks.

For example, while we can merge two blocks of $1|2|5|73|4|986$ to obtain $1|2|73|54|986$, they are not comparable in the partial order since in one case 5 appears to the left of the block 73, while in the other it appears to the right. On the other hand, we have $1|2|5|73|4|986 <_{\mathrm{Sh}} 2|5|73|4|9861$, i.e.,



$<_{\mathrm{Sh}}$



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because we can obtain the permutation on the right by merging the 986 block and the 1. This new block, 9861, had to be to the right of the 73 block because the block containing the 6 was already to the right. However, there were some choices in where to place the new block relative to the 2 and the 4. These could have been placed on either side of the 9861 block, so that $w = 1|2|5|73|4|986$ is also less than the following permutations, each with the same set of decreasing runs: $2|5|73|4|9861$, $5|73|4|9861|2$, and $5|73|9861|2|4$.

We have $\mathrm{Sh}(S_3)$ shown in Figure 3.3 and $\mathrm{Sh}(S_4)$ shown in Figure 3.4 (where the partial order moves from left to right rather than bottom to top).

There is a unique minimum permutation in $\mathrm{Sh}(S_n)$ given by $1|2|\cdots|n$, with n singleton blocks, while the unique maximum element is $n\cdots 21$, with only one block. It is straightforward to verify that every cover relation $u <_{\mathrm{Sh}} v$ amounts to merging only one pair of blocks. Since the number of blocks in u is $n - \mathrm{des}(u)$, this means the poset $\mathrm{Sh}(S_n)$ is ranked, with rank function $\rho(w) = \mathrm{des}(w)$. Hence we can state the following result on the shard intersection order.

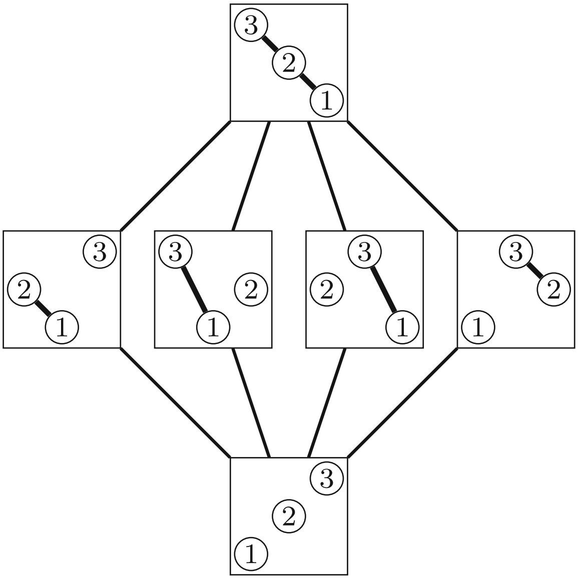
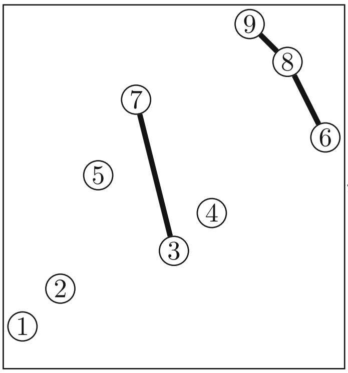
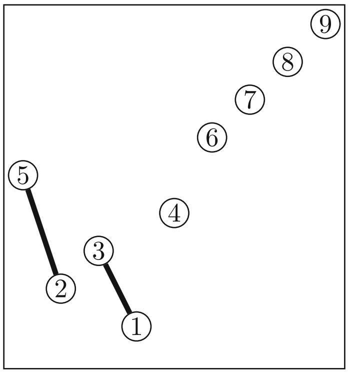


Fig. 3.3 The poset $\mathrm{Sh}(S_3)$.

Proposition 3.2. *The Eulerian polynomial $S_n(t)$ is the rank generating function for $\mathrm{Sh}(S_n)$, i.e.,*

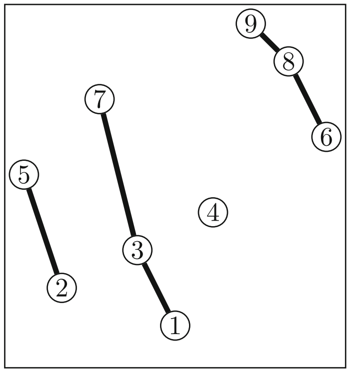
$$f(\mathrm{Sh}(S_n);t) = \sum_{w \in S_n} t^{\mathrm{des}(w)} = S_n(t).$$

It is fairly evident that this partial order is a lattice. Given elements u and v in S_n, we have some ordered blocks in u and some ordered blocks in v, e.g.,

$u =$  , $v =$  .

Their least upper bound is the permutation obtained from merging the smallest number of blocks of u needed to get a collection of blocks consistent with the blocks of v, and this can be done greedily working from left to right:

$u \vee v =$



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The greatest lower bound is similarly found.

We will now discuss a partial order for which the rank generating function is the Narayana polynomial.

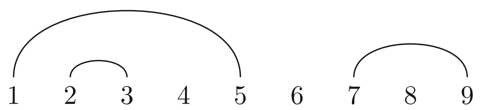
3.4 The lattice of noncrossing partitions

The most natural partial order on set partitions is refinement ordering. For convenience, however, we will consider the dual ordering of reverse refinement, or coarsening. (There are several good reasons for this. For one thing it makes this order compatible with the shard intersection order.) We defer investigation of the poset of set partitions to Problem 3.8 at the end of the chapter. Here we focus attention on the subposet consisting of the noncrossing partitions.

Recall that $\mathrm{NC}(n)$ is the set of all set partitions of $\{1, 2, \ldots, n\}$ that obey the *noncrossing* condition on their blocks. See Section 2.6. We will now discuss $\mathrm{NC}(n)$ as a partially ordered set, with $\sigma <_{\mathrm{NC}} \tau$ if σ refines τ. The unique minimum element in the poset is partition with all singleton blocks, $\{\{1\}, \{2\}, \ldots, \{n\}\}$, while the partition with only one block, $\{\{1, 2, \ldots, n\}\}$, is the unique maximum.

This partial ordering makes $\mathrm{NC}(n)$ a lattice. The greatest lower bound of partitions σ and τ is found by simply computing all intersections of the blocks in σ with blocks in τ. Clearly the noncrossing condition is preserved under intersection. The least upper bound is found by taking the union of blocks, with the proviso that if a block B of σ and a block B' of τ have nonempty intersection or if B and B' are crossing, then we merge these blocks in $\sigma \vee \tau$. For example, if

$\sigma =$



,

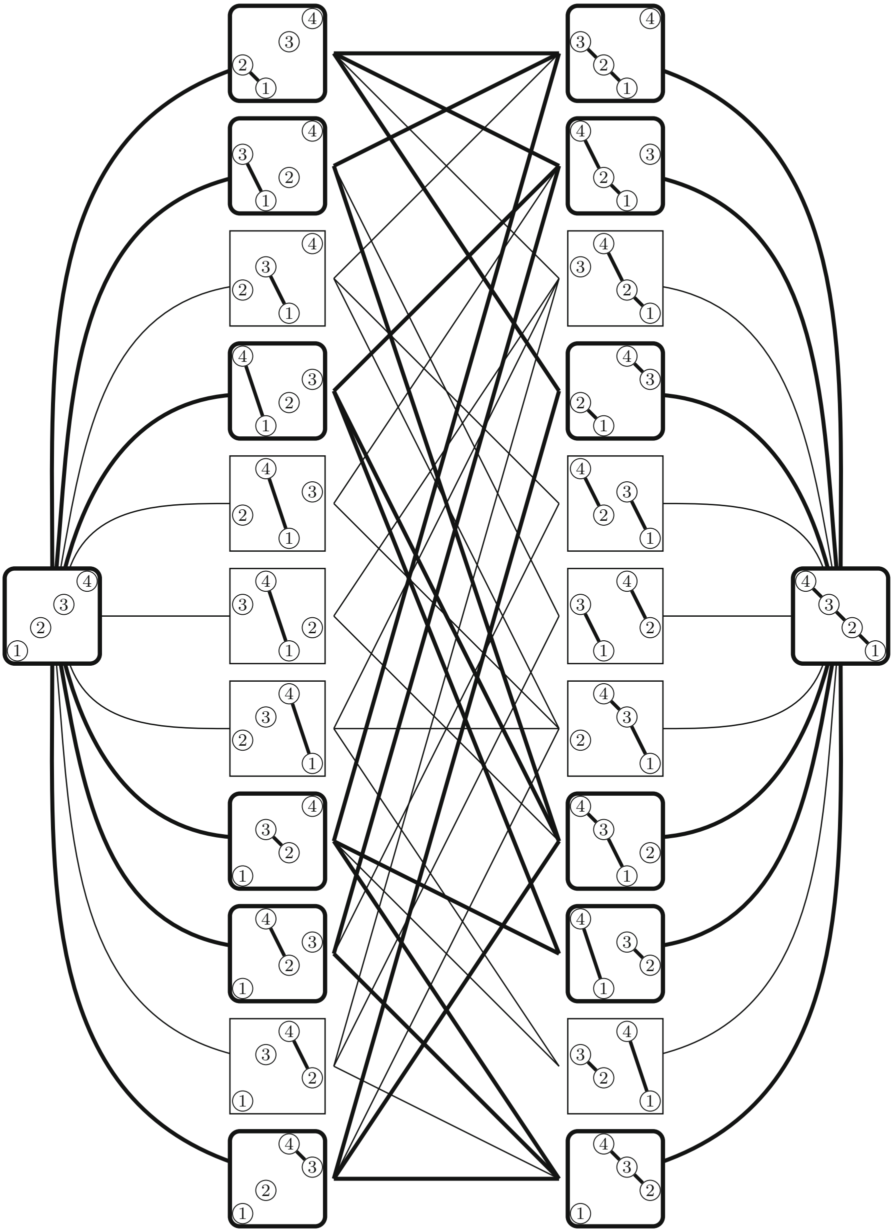
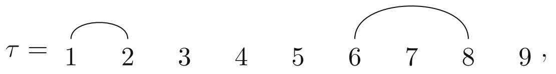
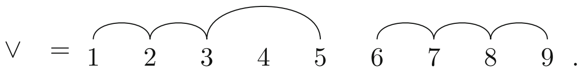


Fig. 3.4 The shard intersection poset for S_4 contains the poset of noncrossing partitions in the guise of 231-avoiding permutations.

and



then



We see the Hasse diagrams for NC(3) and NC(4) in Figures 3.5 and 3.6.

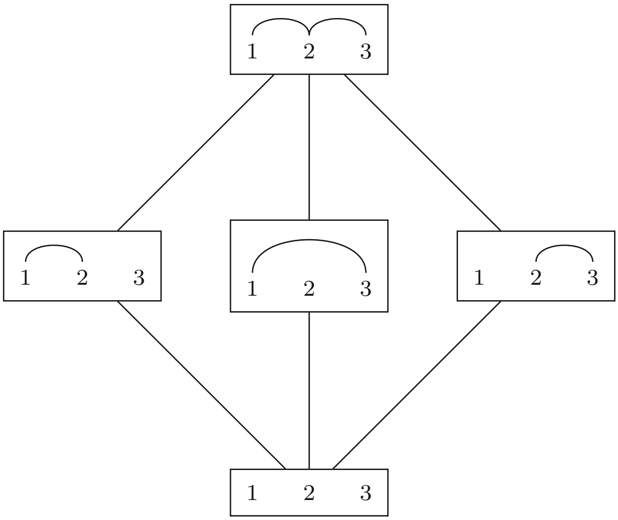


Fig. 3.5 The poset NC(3).

It is clear that NC(n) is ranked by $n-|\pi|$, i.e., n minus the number of blocks in the partition. Thanks to Proposition 2.3, this shows the rank generating function is the Narayana polynomial.

Proposition 3.3. *The rank generating function for* NC(n) *is*

$$f(\mathrm{NC}(n);t) = \sum_{\pi \in \mathrm{NC}(n)} t^{n-|\pi|} = \sum_{w \in S_n(231)} t^{\mathrm{des}(w)} = C_n(t).$$

Moreover, the bijection ϕ given in Section 2.6 between 231-avoiding permutations and noncrossing partitions can be used to show that NC(n) is isomorphic to a subposet of $\mathrm{Sh}(S_n)$, as indicated in Figure 3.4. Let $\mathrm{Sh}(S_n(231))$ denote the partial order on $\mathrm{Sh}(S_n)$ applied only to the permutations in $S_n(231)$.

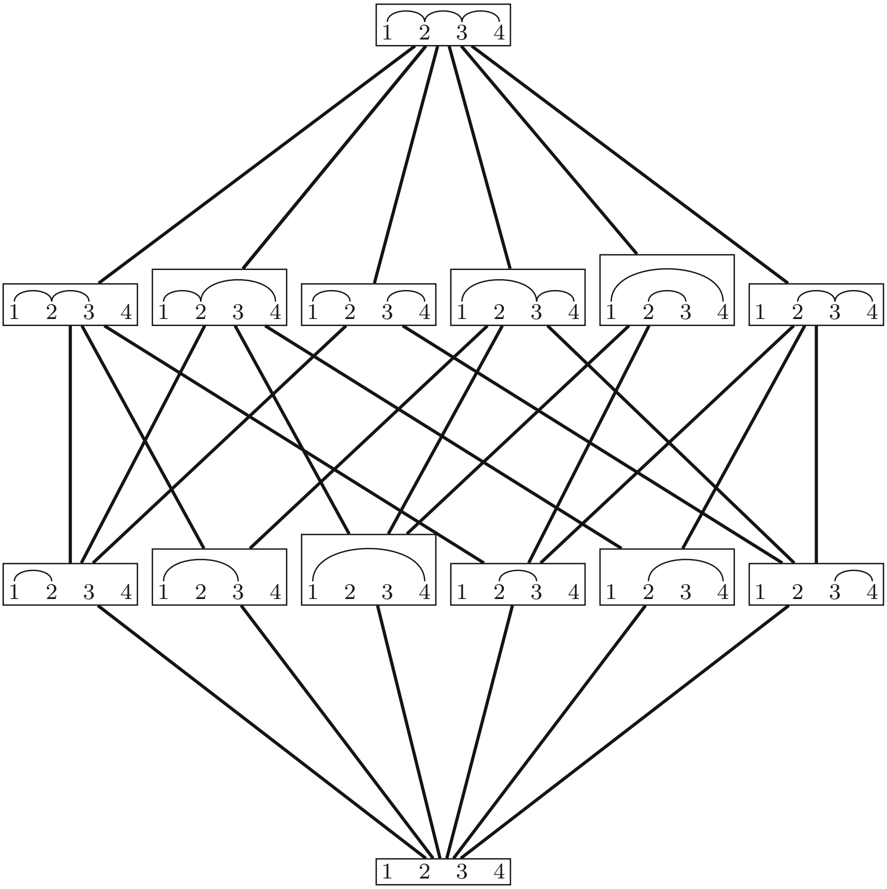
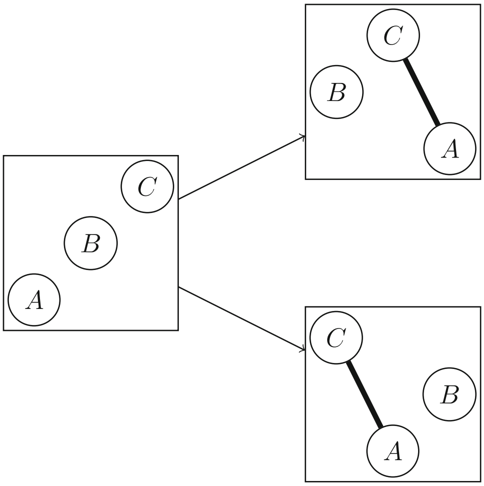


Fig. 3.6 The poset NC(4).

We can see that for merging blocks in a 231-avoiding permutation "consistency" is taken care of since there is no choice in how to write the blocks in a 231-avoiding manner. For example, suppose blocks A and C are to merge in a 231-avoiding permutation, and that block B contains elements between the maximum of A and the minimum of C. Then if we merge blocks C and A, they must go to the left of B, or else they create a 231 pattern as shown here:



Hence, every cover relation in $\mathrm{Sh}(S_n(231))$ is a cover relation in $\mathrm{NC}(n)$ and vice versa, and we make the following observation.

Proposition 3.4. *As posets,* $\mathrm{Sh}(S_n(231))$ *and* $\mathrm{NC}(n)$ *are isomorphic.*

3.5 Absolute order and Noncrossing partitions

The lattice of noncrossing partitions can be found sitting inside another partial order on permutations as we will describe in this section. In this case, it is more convenient to write elements of S_n in cycle notation. For example, instead of writing $w = 315486927$ in one-line notation, we write $w = (13582)(4)(6)(79)$, where the cycle (13582) means $w(1) = 3$, $w(3) = 5$, $w(5) = 8$, $w(8) = 2$, and $w(2) = 1$. Let $\mathrm{cyc}(w)$ denote the number of cycles of w, so that our example above has $\mathrm{cyc}(w) = 4$.

Now write $u \to v$ if there is a single transposition (ij) such that $v = u \circ (ij)$ and u has more cycles than v. For example, $(142)(3)(567) \to (142)(3675)$ since $(142)(3)(567) \circ (35) = (142)(3675)$. These directed edges can be used as the edges in the Hasse diagram of a partial order on S_n, called the *absolute order*, denoted $\mathrm{Abs}(S_n)$. See Figure 3.7 for $\mathrm{Abs}(S_4)$.

This poset is clearly ranked by n minus the number of cycles: $n - \mathrm{cyc}(w)$. (Indeed if i and j are in different cycles of u, then $v = u \circ (ij)$ has these two cycles merged into one, with all other cycles of u untouched.) Notice that the identity permutation is the unique minimum of $\mathrm{Abs}(S_n)$, while the maximal elements are n-cycles.

We can recursively compute the rank function for this poset (i.e., count permutations by the number of cycles) by considering the effect of inserting n into the cycle notation of a permutation in S_{n-1}. Let us generate S_n from

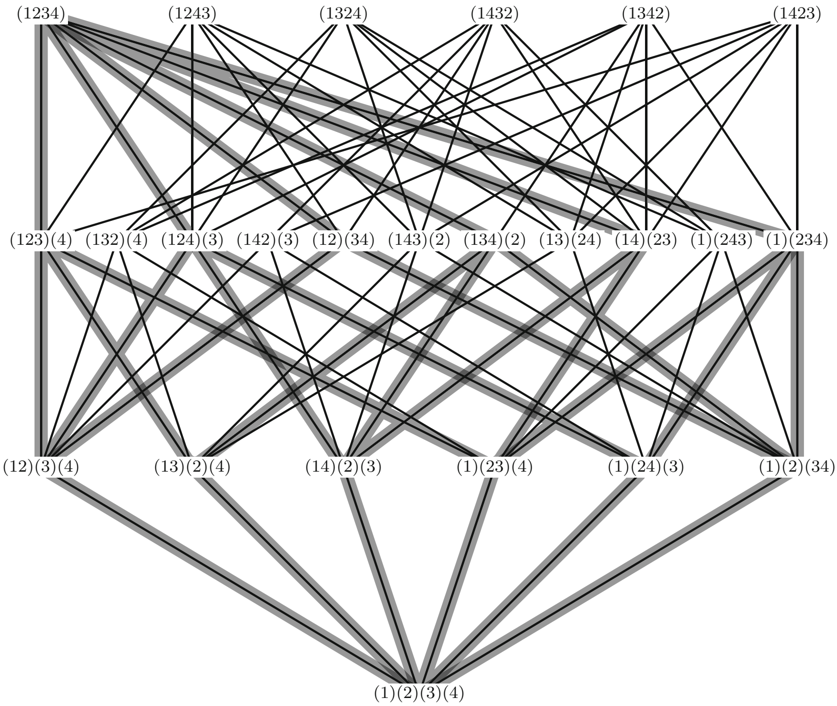


Fig. 3.7 The absolute order with noncrossing partition lattice highlighted.

S_{n-1} as follows. Fix a permutation u in S_{n-1} and form a permutation v in S_n in any of the following distinct ways.

- Let $v(i) = u(i)$ for $i = 1, \ldots, n-1$, $v(n) = n$. The rank of v in $\mathrm{Abs}(S_n)$ is $n - \mathrm{cyc}(v) = n - \mathrm{cyc}(u) - 1$, which is the rank of u in $\mathrm{Abs}(S_{n-1})$.
- For some $1 \leq j \leq n-1$, let $v(j) = n$ and $v(n) = u(j)$, while $v(i) = u(i)$ for $i \neq j$. This inserts n in the middle of a cycle of u, so the rank of v is $n - \mathrm{cyc}(v) = n - \mathrm{cyc}(u)$, which is one more than the rank of u.

This line of reasoning gives

$$f(\mathrm{Abs}(S_n); t) = (1 + (n-1)t) f(\mathrm{Abs}(S_{n-1}); t),$$

and since $f(\mathrm{Abs}(S_1); t) = 1$, we have the following observation.

Observation 3.1 *For any $n \geq 1$, the rank generating function for the absolute order on S_n is the generating function for the statistic $n - \mathrm{cyc}(w)$, and this function factors as follows:*

$$\begin{aligned} f(\mathrm{Abs}(S_n);t) &= \sum_{w\in S_n} t^{n-\mathrm{cyc}(w)}, \\ &= \prod_{i=1}^{n-1}(1+(i-1)t), \\ &= (1+t)(1+2t)\cdots(1+(n-1)t). \end{aligned}$$

We remark that the coefficients in the polynomial $(1+t)(1+2t)\cdots(1+nt)$ are known as the *Stirling numbers of the first kind*.

We can see the lattice of noncrossing partitions inside the absolute order by considering the set of all elements below any one of the n-cycles, say $(12\cdots n)$. The fact that each of these intervals is identical is left to Problem 3.11. The correspondence between this interval, which we will denote $\mathrm{Abs}((12\cdots n))$, and the lattice of noncrossing partitions is straightforward. We simply convert each cycle to a block of a partition. Conversely, given a noncrossing partition, simply write each block in increasing order and make that block a cycle. Compare Figure 3.6 with Figure 3.7.

Proposition 3.5. *As posets,* $\mathrm{Abs}((12\cdots n))$ *and* $\mathrm{NC}(n)$ *are isomorphic.*

Notes

Partially ordered sets appear in various parts of mathematics. Gian-Carlo Rota and Richard Stanley were two of the central figures in bringing the general study of posets into the mainstream of combinatorics in the second half of the 20th century, with applications in algebra and topology. Richard Stanley's book [154] introduces most of the modern results in the general theory of posets. In particular, it summarizes Stanley's theory of P-partitions. We remark that our definition differs from Stanley's. In [154] a P-partition is *order-reversing* rather than order preserving. This choice makes sense as Stanley was initially motivated by counting plane partitions, a problem discussed as far back as the book from 1915/16 by Percy MacMahon [106]. See Problem 3.6.

Nathan Reading introduced the shard intersection order in 2011 [124]. In fact, his construction can be done in any Coxeter group, as discussed in Chapter 11. A combinatorial model for this partial order in the case of the symmetric group was first described by Erin Bancroft [13], and was given the form described here by this book's author in [118].

While the shard intersection order is quite recent, the lattice of noncrossing partitions has been studied at least since Germain Kreweras [97] in 1972. Philippe Biane's 1997 paper [16] was the first to remark upon the fact that the lattice of noncrossing partitions emerges from the absolute order on the symmetric group. That $\mathrm{NC}(n)$ is isomorphic to the shard intersection order on 231-avoiding permutations is a special case of Reading's work [124]. Both

Biane's construction and Reading's construction can be generalized to give a definition of the lattice of noncrossing partitions in any Coxeter group. See Chapter 12. The relevance of the lattice of noncrossing partitions in different parts of mathematics has been surveyed by Rodica Simion [139] and by Jon McCammond [107].

Problems

3.1. Let P be the labeled poset consisting of the disjoint union of the chains $1 <_P 2 <_P \cdots <_P k$ and $k+1 <_P k+2 <_P \cdots <_P n$ for some k. Characterize the set of linear extensions of P. (Hint: for $w \in \mathcal{L}(P)$, consider $\mathrm{Des}(w^{-1})$.)

3.2. A *polytope* is the intersection of half-spaces in $\mathbb{R}^n$. For example, the standard simplex Σ_n is the intersection of the nonnegative coordinate half spaces defined by $x_i \geq 0$ and the hyperplane $\sum_{i=1}^n x_i \leq 1$, and we can define a unit cube Δ_n to be the intersection of $x_i \geq 0$ and $x_i \leq 1$ for all i.

Let $\Omega(P;k)$ denote the number of integer points in the k-fold dilation of the polytope P. (The notation is intentionally suggestive here.) For example, if

$$P = \Sigma_2 = \{(x,y) : x \geq 0, y \geq 0, x+y \leq 1\},$$

then we have $\Omega(\Sigma_2;0) = 1$, $\Omega(\Sigma_2;1) = 3$, and $\Omega(\Sigma_2;2) = 6$.

Find rational expressions for the following generating functions.

1. $\sum_{k\geq 0} \Omega(\Sigma_2;k)t^k$
2. $\sum_{k\geq 0} \Omega(\Sigma_3;k)t^k$
3. $\sum_{k\geq 0} \Omega(\Sigma_n;k)t^k$
4. $\sum_{k\geq 0} \Omega(\Delta_2;k)t^k$
5. $\sum_{k\geq 0} \Omega(\Delta_3;k)t^k$
6. $\sum_{k\geq 0} \Omega(\Delta_n;k)t^k$

3.3 (Set partitions). Let $B(n)$ denote the number of set partitions of $\{1,2,\ldots,n\}$. These are sometimes called the *Bell numbers*. The first few of them are:

$$1, 1, 2, 5, 15, 52, 203, 877, 4140, 21147, \ldots,$$

with $B(0) := 1$. This exercise will show the exponential generating function for the Bell numbers is:

$$\sum_{n\geq 0} B(n)\frac{z^n}{n!} = e^{(e^z-1)}.$$

The approach here is to refine the problem first. Let $S(n,k)$ denote the number of set partitions of $\{1,2,\ldots,n\}$ with k blocks. These numbers are known as *Stirling numbers of the second kind.*

1. Show that $S(n+1,k) = kS(n,k) + S(n,k-1)$, and use this recurrence to create a triangle of these numbers, for $1 \leq k \leq n \leq 10$.
2. Show that for fixed k,

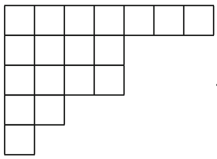
$$\sum_{n\geq 1} S(n,k)\frac{z^n}{n!} = \frac{(e^z-1)^k}{k!}.$$

3. Show that

$$1 + \sum_{n,k\geq 1} S(n,k)\frac{y^k z^n}{n!} = e^{y(e^z-1)}.$$

   and set $y = 1$ to obtain the exponential generating function for the Bell numbers.

3.4 (Integer partitions). A *partition* of an integer n is a weakly decreasing sequence of positive integers whose sum is n, i.e., $\lambda = (\lambda_1,\ldots,\lambda_k)$ is a partition of n if $\lambda_1 \geq \cdots \geq \lambda_k$ and $\sum \lambda_i = n$. For example, $\lambda = (7,4,4,2,1)$ is a partition of $n = 18$. We often draw partitions as a collection of n boxes that upper- and left-justified, so that the number of boxes in row i is λ_i. Such a picture is called a *Young diagram.* (These are also known in the literature as *Ferrers diagrams*, though Ferrers diagrams are lower-left justified.) For example, with $\lambda = (7,4,4,2,1)$ we would draw



.

Let p_n denote the number of partitions of n, with $p_0 = 1$ by convention. The first few values of p_n are:

$$1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101, 135, 176, \ldots$$

1. Show the ordinary generating function for the number of partitions of n is:

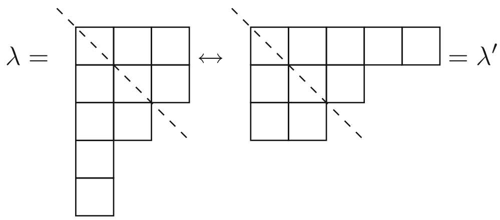
$$\sum_{n\geq 0} p_n z^n = \prod_{i\geq 1} \frac{1}{(1-z^i)}.$$

   This is called *Euler's product formula.*

2. Let $p_{n,k}$ denote the number of partitions of n into k parts. For example, $p_{5,3} = 2$ as the partitions $(3,1,1)$ and $(2,2,1)$ are the only partitions of $n = 5$ into $k = 3$ parts. Refine Euler's product formula to obtain an expression for the following generating function:

$$\sum_{n,k\geq 0} p_{n,k} t^k z^n.$$

3. The *conjugate* of a partition λ is the partition λ' with Young diagram obtained by transposing the Young diagram for λ. For example, if $\lambda = (3,3,2,1,1)$, its conjugate is $\lambda' = (5,3,2)$ as seen here:



Let p'_n denote the number of partitions that are *self-conjugate*, i.e., for which $\lambda = \lambda'$. Show that

$$\sum_{n\geq 0} p'_n z^n = (1+z)(1+z^3)(1+z^5)\cdots = \prod_{i\geq 1}(1+z^{2i-1}).$$

(Hint: show that p'_n also counts the number of partitions of n into distinct odd parts by exhibiting a bijection between partitions with distinct odd parts and self-conjugate partitions.)

4. Give a bijective proof of the following formula for the generating function for the number of partitions with exactly k parts (with k fixed):

$$\sum_{n\geq 0} p_{n,k} z^n = z^k \cdot \prod_{i=1}^{k} \frac{1}{(1-z^i)}.$$

5. Show that the number of partitions of n into distinct parts equals the number of partitions of n into odd parts. One way to prove this is to verify the identity of generating functions:

$$\prod_{i\geq 1}(1+z^i) = \prod_{i\geq 1} \frac{1}{(1-z^{2i-1})}.$$

Try to prove the claim with a bijection as well.

6. Let $\phi(z) = \prod_{i\geq 1}(1-z^i)$ denote the denominator of Euler's product formula. This is sometimes called the *Euler function* in number theory. Show that

$$\phi(z) = 1 - z - z^2 + z^5 + z^7 - z^{12} - z^{15} + z^{22} + z^{26} - \cdots .$$

Find a formula for the exponents of the nonzero terms in the series expansion of $\phi(z)$, and show the only coefficients are $1, -1$, and 0. (Hint: interpret the left-hand side as running over all partitions into distinct parts, where if the partition has an odd number of parts it gets counted with a minus sign.)

3.5. A *marked* partition of n is a partition in which a part of size i can come with i different markings. To be precise, let

$$\begin{aligned} A &= \{(a,b) : a \geq b \geq 1\}, \\ &= \{(1,1),(2,1),(2,2),(3,1),(3,2),(3,3),\ldots\}, \end{aligned}$$

and let $\alpha = ((a_1,b_1),(a_2,b_2),\ldots,(a_k,b_k))$ be a collection of points in A ordered lexicographically, i.e., $a_i > a_{i+1}$ or $a_i = a_{i+1}$ and $b_i \geq b_{i+1}$. To say α is a marked partition of n means $a_1 + \cdots + a_k = n$, ignoring the markings.

For example, $\alpha = ((3,1),(2,2),(2,1),(1,1),(1,1))$ is a marked partition of $n = 3+2+2+1+1 = 9$. Let m_n denote the number of marked partitions of n. Some small values of m_n are

$$1, 1, 3, 6, 13, 24, 48, 86, 160, 282, 500, 859, \ldots$$

with $m_0 = 1$.

Show that, by analogy with Euler's product formula for unmarked partitions, we have

$$\sum_{n\geq 0} m_n z^n = \prod_{i\geq 1} \frac{1}{(1-z^i)^i}.$$

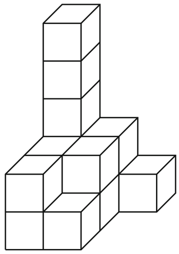
3.6. A *plane partition* is an array of nonnegative integers

$$\rho = \begin{matrix} \rho_{1,1} & \rho_{1,2} & \cdots \\ \rho_{2,1} & \rho_{2,2} & \cdots \\ \vdots & \vdots & \ddots \end{matrix} = (\rho_{i,j})_{i,j\geq 1}$$

whose rows and columns are weakly decreasing, i.e., for fixed i, $\rho_{i,j} \geq \rho_{i,j+1}$ and for fixed j, $\rho_{i,j} \geq \rho_{i+1,j}$. The *support* of a plane partition is the set of cells with positive entries. We only wish to consider those partitions with finite support.

The positive terms in a plane partition determine a Young diagram, and hence a partition, which we call the *shape of the plane partition.* One helpful way to understand plane partitions is a three-dimensional analogue of a Young diagram, where $\rho_{i,j} = r$ is represented by a stack of r boxes in position (i,j). For example, here is one plane partition of shape $3, 2, 2$, drawn both as an array of numbers and as a pile of boxes stacked in a corner:

$$\rho = \begin{array}{|c|c|c|}\hline 5 & 2 & 1 \\ \hline 2 & 2 \\ \cline{1-2} 2 & 1 \\ \cline{1-2}\end{array} =$$



.

Let $\Omega(\lambda; k)$ denote the number of plane partitions of shape λ whose largest part is at most k. Describe the generating function

$$\sum_{k \geq 0} \Omega(\lambda; k) t^k,$$

in the case where $\lambda = (3, 2, 2)$. Hint: draw a poset P on $\{1, 2, 3, 4, 5, 6, 7\}$ such that the set of P-partitions coincides with the set of plane partitions of shape $\lambda = (3, 2, 2)$.

3.7. A *bipartite* P-partition is a map $a : P \to \{1, 2, \ldots\} \times \{1, 2, \ldots\}$, where we use the lexicographic ordering on $\{1, 2, \ldots\} \times \{1, 2, \ldots\}$. Lexicographic ordering is a total order, so the order polynomial $\Omega(P; kl)$ counts bipartite P-partitions

$$a : P \to \{1, 2, \ldots, k\} \times \{1, 2, \ldots, l\}.$$

Show that

$$\binom{kl + n - 1}{n} = \sum_{w \in S_n} \binom{k + n - 1 - \operatorname{des}(w)}{n} \binom{l + n - 1 - \operatorname{des}(w^{-1})}{n}.$$

3.8 (The partition lattice). Let $\Pi(n)$ denote the set of all set partitions of $\{1, 2, \ldots, n\}$, ordered by reverse refinement.

1. Draw the Hasse diagrams for $\Pi(3)$ and $\Pi(4)$, highlighting NC(3) and NC(4) as sub-posets.
2. Show $\Pi(n)$ is a lattice.
3. Count the number of maximal chains in $\Pi(n)$, i.e., chains

$$\{\{1\}, \{2\}, \ldots, \{n\}\} \to \cdots \to \{\{1, 2, \ldots, n\}\},$$

   of length $n - 1$. For example, in $\Pi(1)$ and $\Pi(2)$ there is one such chain, while in $\Pi(3)$ there are three.

3.9 (Parking functions). A *parking function* of length n is a sequence of positive integers, $(a_1, \ldots, a_n)$, such that if $b_1 \leq \cdots \leq b_n$ is an increasing rearrangement of $a_1, \ldots, a_n$, then $b_i \leq i$. They get their name because of the following interpretation. Imagine there are n cars that want to park in n spaces on a one-way street. Denote the cars by $C_1, \ldots, C_n$, and let $a_1, \ldots, a_n$

be their preferred parking spaces. If a car finds its preferred space occupied, it will park in the next available space. All cars will be able to park if and only if $(a_1, \dots, a_n)$ is a parking function. For example, suppose there are six cars, with preferences $(1, 1, 5, 2, 2, 3)$. Then the cars will park as follows:

$$\begin{array}{r|cccccc} \text{car:} & C_1 & C_2 & C_4 & C_5 & C_3 & C_6 \\ \hline \text{space:} & 1 & 2 & 3 & 4 & 5 & 6 \end{array}.$$

On the other hand, if the preferences were $(1, 1, 6, 3, 5, 5)$, then the sixth car will be out of luck, since when it arrives it wants to park in space five or higher, and the only available space is space four:

$$\begin{array}{r|cccccc} \text{car:} & C_1 & C_2 & C_4 & & C_5 & C_3 \\ \hline \text{space:} & 1 & 2 & 3 & 4 & 5 & 6 \end{array}.$$

Let $\mathrm{PF}(n)$ denote the set of parking functions of length n. For example,

$$\mathrm{PF}(2) = \{(1, 1), (1, 2), (2, 1)\}$$

and

$$\mathrm{PF}(3) = \left\{ \begin{array}{c} (1,1,1), (1,1,2), (1,2,1), (2,1,1), \\ (1,1,3), (1,3,1), (3,1,1), (1,2,2), \\ (2,1,2), (2,2,1), (1,2,3), (1,3,2), \\ (2,1,3), (2,3,1), (3,1,2), (3,2,1) \end{array} \right\}.$$

Show that

$$|\mathrm{PF}(n)| = (n + 1)^{n-1}.$$

3.10. How many maximal chains are there in the lattice of noncrossing partitions?

3.11. Prove that the interval below any n-cycle in the absolute order on S_n is the same as any other. That is, if c and c' are two different n-cycles show that $\mathrm{Abs}(c) = \mathrm{Abs}(c')$. Hint: c and c' are conjugate to one another, i.e., there is a permutation w such that $c' = w \circ c \circ w^{-1}$. Show that conjugation by w takes cover relations to cover relations in these posets.

Chapter 4
Gamma-nonnegativity

THE BINOMIAL DISTRIBUTION is the first probability distribution a student encounters. Among its many properties is the fact that it is palindromic and unimodal. Many combinatorial distributions, including the Eulerian and Narayana distributions, can be built out of copies of binomial distributions that are shifted to have the same center of symmetry, and this fact has many interesting consequences.

4.1 The idea of gamma-nonnegativity

We can observe that, for fixed n, the sequence of Eulerian numbers, $\left\langle {n \atop k} \right\rangle$ is *palindromic*,

$$\left\langle {n \atop k} \right\rangle = \left\langle {n \atop n-1-k} \right\rangle, \tag{4.1}$$

and *unimodal*:

$$\left\langle {n \atop 0} \right\rangle \leq \left\langle {n \atop 1} \right\rangle \leq \cdots \leq \left\langle {n \atop \lfloor (n-1)/2 \rfloor} \right\rangle \geq \cdots \geq \left\langle {n \atop n-1} \right\rangle.$$

When there is no possibility for confusion, we will call a polynomial palindromic or unimodal if its sequence of coefficients has the same property. So we say the Eulerian polynomial $S_n(t)$ is palindromic and unimodal.

The palindromicity is easy to explain combinatorially, as reversal of a permutation swaps descents and ascents. This gives a bijection between the set of permutations with k descents and permutations with k ascents, and hence $n-1-k$ descents.

Unimodality is trickier, but both these properties follow from a property that will be a major theme later in the book, called *gamma-nonnegativity*, which we now explain.

T.K. Petersen, *Eulerian Numbers*, Birkhäuser Advanced Texts Basler Lehrbücher, DOI 10.1007/978-1-4939-3091-3_4

First, observe that the sequence of binomial coefficients $\binom{n}{k}$, with n fixed, is palindromic and unimodal.

Loosely speaking, gamma-nonnegativity means a sequence of numbers can be written as a sum of rows of Pascal's triangle with the same center of symmetry. For example, rows 5 and 6 of the Eulerian triangle (Table 1.3) can be written as follows.

$$n = 5: \qquad \begin{array}{rccccc} & 1 & 26 & 66 & 26 & 1 \\ \hline 1\times & (1 & 4 & 6 & 4 & 1) \\ 22\times & & (1 & 2 & 1) & \\ 16\times & & & (1) & & \end{array} \qquad\qquad n = 6: \qquad \begin{array}{rcccccc} & 1 & 57 & 302 & 302 & 57 & 1 \\ \hline 1\times & (1 & 5 & 10 & 10 & 5 & 1) \\ 52\times & & (1 & 3 & 3 & 1) & \\ 136\times & & & (1 & 1) & & \end{array}$$

In terms of generating functions, gamma-nonnegativity means a polynomial of degree n can be written as a sum of polynomials of the form $t^j(1+t)^{n-2j}$. In the case of the Eulerian polynomials for $n = 5$ and $n = 6$ we have

$$S_5(t) = (1+t)^4 + 22t(1+t)^2 + 16t^2,$$
$$S_6(t) = (1+t)^5 + 52t(1+t)^3 + 136t^2(1+t).$$

The coefficients in expansions like the ones above make up what we call the *gamma vector*. When these coefficients are nonnegative, we say the polynomial itself is *gamma-nonnegative*.

4.2 Gamma-nonnegativity for Eulerian numbers

In this section we show the Eulerian polynomials are gamma-nonnegative, a result first due to Foata and Schützenberger.

Theorem 4.1. *For any $n > 0$, there exist nonnegative integers $\gamma_{n,j}$ such that*

$$S_n(t) = \sum_{j=0}^{\lfloor (n-1)/2 \rfloor} \gamma_{n,j} t^j (1+t)^{n-1-2j}, \tag{4.2}$$

i.e., the Eulerian polynomials are gamma-nonnegative.

We list the entries in the gamma vectors for the Eulerian polynomials in Table 4.1.

There is a beautiful combinatorial proof of Theorem 4.1 given by Foata and Strehl, based on an action we call "valley hopping" as illustrated in Figure 4.1. Here we draw a permutation as a "mountain range," so that peaks and valleys form the upper and lower limits of the decreasing runs. By convention, we have points at infinity on the far left and far right.

Table 4.1 Entries of the gamma vector for the Eulerian polynomials, $\gamma_{n,j}$, $0 \le 2j < n \le 10$.

$n\backslash j$	0	1	2	3	4
1	1				
2	1				
3	1	2			
4	1	8			
5	1	22	16		
6	1	52	136		
7	1	114	720	272	
8	1	240	3072	3968	
9	1	494	11616	34304	7936
10	1	1004	40776	230144	176896

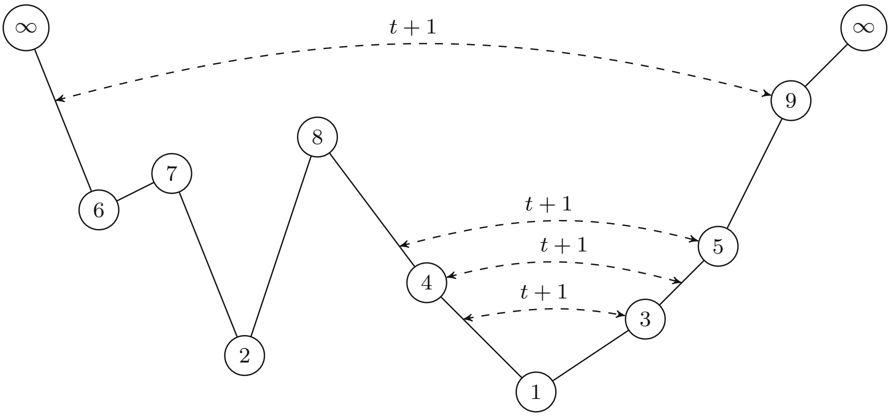


Fig. 4.1 The mountain range view of the permutation $w = 862741359$.

Formally, given $w = w(1) \cdots w(n) \in S_n$, we say a letter $w(i)$ is a *peak* if $w(i-1) < w(i) > w(i+1)$ and it is a *valley* if $w(i-1) > w(i) < w(i+1)$. Otherwise we say $w(i)$ is *free*. Using the convention that $w(0) = w(n+1) = \infty$, we see that w cannot begin or end with a peak.

We partition S_n into equivalence classes according to the following action on free letters. If $w(i) = j$ is free, then $H_j(w)$ denotes the permutation obtained by moving j directly across the adjacent valley(s) to the nearest mountain slope of the same height. More precisely, we have the following.

- If $w(i) = j$ lies on a downslope, i.e., $w(i-1) > w(i) > w(i+1)$, we find the smallest $k > i$ such that $w(k) < j < w(k+1)$, and

$$H_j(w) = w(1) \cdots w(i-1) w(i+1) \cdots w(k)\, j\, w(k+1) \cdots w(n),$$

- If $w(i) = j$ lies on an upslope, i.e., $w(i-1) < w(i) < w(i+1)$, we find the largest $k < i$ such that $w(k-1) > j > w(k)$, and

$$H_j(w) = w(1) \cdots w(k-1)\, j\, w(k) \cdots w(i-1) w(i+1) \cdots w(n).$$

Clearly, if j, l are free letters, $H_j^2(w) = H_l^2(w) = w$ and $H_j(H_l(w)) = H_l(H_j(w))$. Thus, for any collection of free letters $J = \{j_1, \ldots, j_k\}$, we can define the operation $H_J(w) = H_{j_1} \cdots H_{j_k}(w)$. Also, observe that $H_J(w)$ has the same set of free letters as w.

Let $\mathrm{Hop}(w)$ denote the hop-equivalence class of w. Notice that every peak of w is necessarily the larger element of a descent, for any $u \in \mathrm{Hop}(w)$, while a valley is never the larger element of a descent. If a free letter lies on an upslope of u it is not part of a descent, while if it is on a downslope it is the larger element of a descent of u. Moreover, this property is independent of the positions of the other free letters. If w has r peaks, it has $r + 1$ valleys, and hence $n - 1 - 2r$ free letters. Thus, letting $\mathrm{pk}(w)$ denote the number of peaks of w, we have:

$$\sum_{u \in \mathrm{Hop}(w)} t^{\mathrm{des}(u)} = t^{\mathrm{pk}(w)} (1+t)^{n-1-2\,\mathrm{pk}(w)}. \tag{4.3}$$

We can choose a canonical representative for each hop-equivalence class by choosing to put each free letter on an upslope. These are precisely the permutations for which $\mathrm{pk}(w) = \mathrm{des}(w)$. We denote this set of representatives by:

$$\widehat{S}_n = \{w \in S_n : \mathrm{pk}(w) = \mathrm{des}(w)\}.$$

Thus by summing (4.3) over all $w \in \widehat{S}_n$, we get:

$$S_n(t) = \sum_{w \in \widehat{S}_n} t^{\mathrm{pk}(w)} (1+t)^{n-1-2\,\mathrm{pk}(w)}.$$

Moreover, we can now give a combinatorial interpretation to the numbers in Table 4.1.

Corollary 4.1. *For any n, j,*

$$\gamma_{n,j} = |\{w \in \widehat{S}_n : \mathrm{des}(w) = j\}|.$$

With this interpretation in hand, it is not difficult to relate the Eulerian polynomials to the generating function for the peak statistic. That is, define the *peak polynomials* $P_n(t)$ and *peak numbers* $p_{n,k}$ as follows:

$$P_n(t) = \sum_{w \in S_n} t^{\mathrm{pk}(w)} = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} p_{n,k} t^k.$$

Then we have

$$S_n(t) = \frac{(1+t)^{n-1}}{2^{n-1}} P_n\left(\frac{4t}{(1+t)^2}\right). \tag{4.4}$$

Some peak numbers are included in Table 4.2.

Table 4.2 The peak numbers, $p_{n,k}$, $0 \le 2k < n \le 10$.

$n\backslash k$	0	1	2	3	4
1	1				
2	2				
3	4	2			
4	8	16			
5	16	88	16		
6	32	416	272		
7	64	1824	2880	272	
8	128	7680	24576	7936	
9	256	31616	185856	137216	7936
10	512	128512	1304832	1841152	353792

Another consequence of Corollary 4.1 is seen when we specialize $t = -1$ in the Eulerian polynomial:

$$\begin{aligned} S_n(-1) &= \sum_{w \in \widehat{S}_n} (-1)^{\mathrm{pk}(w)} (1-1)^{n-1-2\,\mathrm{pk}(w)} \\ &= \begin{cases} (-1)^{(n-1)/2}\gamma_{n,(n-1)/2} & \text{if } n \text{ odd}, \\ 0 & \text{if } n \text{ even}. \end{cases} \end{aligned}$$

But $\gamma_{n,(n-1)/2}$ (with n odd) is the number of permutations w such that

$$w(1) < w(2) > w(3) < \cdots > w(2i-1) < w(2i) > w(2i+1) < \cdots.$$

These are known as *up-down alternating permutations* and the number of such permutations is known as the *Euler number*, denoted E_n. This definition makes sense for both even and odd values of n, and the sequence of Euler numbers begins:

$$1, 1, 2, 5, 16, 61, 272, 1385, 7936, 50521, \ldots. \tag{4.5}$$

Taking the limit as $t \to -1$ in Theorem 1.6, we get an expression for the exponential generating function for the odd-indexed Euler numbers, with alternating plus and minus signs:

$$\begin{aligned} S(-1,z)-1 &= z - 2\frac{z^3}{3!} + 16\frac{z^5}{5!} - \cdots, \\ &= \sum_{k\geq 0} (-1)^k E_{2k+1} \frac{z^{2k+1}}{(2k+1)!}, \\ &= \frac{1-e^{-2z}}{1+e^{-2z}} = \tanh z. \end{aligned}$$

The sequence $1, 2, 16, 272, 7936, \ldots$ is also known as the sequence of *tangent* numbers. Problem 4.2 investigates other properties of Euler numbers.

4.3 Gamma-nonnegativity for Narayana numbers

We will now show the Narayana polynomials $C_n(t)$ are gamma-nonnegative. Hence, the sequence of Narayana numbers $N_{n,k}$, for fixed n, is symmetric and unimodal. The reason for this is quite simple: Foata and Strehl's valley-hopping action described in Section 4.2 preserves the pattern 231. Hence, if $w \in S_n(231)$, the hop-equivalence class $\mathrm{Hop}(w)$ is composed entirely of permutations avoiding 231.

Let's make this argument rather more precise. Suppose $w \notin S_n(231)$, so that there is a triple of indices $i < j < k$ with $w(k) < w(i) < w(j)$. Then without loss of generality, we may assume $w(j)$ is a peak. (Otherwise, there is a peak $w(j')$ with $i < j' < j$ and $w(j') > w(j)$.) If neither $w(i)$ nor $w(k)$ are free letters, then clearly all members of $\mathrm{Hop}(w)$ contain 231. But even if $w(i)$ or $w(k)$ are free, the relative position of the letters $w(i), w(j), w(k)$ is preserved, since neither $w(i)$ nor $w(k)$ can hop past $w(j)$. See Figure 4.2 for an illustration.

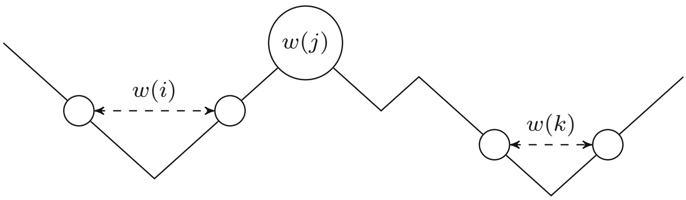


Fig. 4.2 Valley-hopping preserves the pattern 231.

Thus we have the following.

Theorem 4.2. *For any $n > 0$, there exist nonnegative integers $\widehat{\gamma}_{n,j}$ such that*

$$C_n(t) = \sum_{j=0}^{\lfloor n/2 \rfloor} \widehat{\gamma}_{n,j} t^j (1+t)^{n-1-2j}, \tag{4.6}$$

i.e., the Narayana polynomials are gamma-nonnegative.

Moreover, each hop class $\mathrm{Hop}(w)$ still has a unique representative for which $\mathrm{des}(w) = \mathrm{pk}(w)$, and so:

$$\widehat{\gamma}_{n,j} = |\{w \in S_n(231) : \mathrm{des}(w) = \mathrm{pk}(w) = j\}|.$$

These numbers are listed in Table 4.3.

Table 4.3 The gamma numbers $\widehat{\gamma}_{n,k}$ for the Narayana distribution, $0 \leq 2k < n \leq 10$.

$n\backslash k$	0	1	2	3	4
1	1				
2	1				
3	1	1			
4	1	3			
5	1	6	2		
6	1	10	10		
7	1	15	30	5	
8	1	21	70	35	
9	1	28	140	140	14
10	1	36	252	420	126

Of course, there is a similar connection with the peak generating function for all 231-avoiding permutations. Let

$$P_n(231;t) = \sum_{w \in S_n(231)} t^{\mathrm{pk}(w)}.$$

Then we have

$$C_n(t) = \frac{(1+t)^{n-1}}{2^{n-1}} P_n\left(231; \frac{4t}{(1+t)^2}\right). \tag{4.7}$$

For reference we include in Table 4.4 the peak numbers for 231-avoiding permutations.

4.4 Palindromicity, unimodality, and the gamma basis

We will now lay out the general definition and elementary consequences of gamma-nonnegativity.

We say a polynomial $h(t)$ is *palindromic* if its coefficients are the same when read from left to right as from right to left. To be more precise, we say h is *palindromic for n* if $h(t) = t^n h(1/t)$. Such an n is the sum of the

Table 4.4 The number of 231-avoiding permutations in S_n with k of peaks, $0 \leq 2k < n \leq 10$.

$n\backslash k$	0	1	2	3	4
1	1				
2	2				
3	4	1			
4	8	6			
5	16	24	2		
6	32	80	20		
7	64	240	120	5	
8	128	672	560	70	
9	256	1792	2240	560	14
10	512	4608	8064	3360	252

highest and lowest degrees of nonzero terms in h. In the simplest case, h has a nonzero constant term, so n is the degree of h. Here, writing

$$h(t) = h_0 + h_1 t + \cdots + h_n t^n,$$

we have $h_i = h_{n-i}$ for all i. If h has no constant term, n is greater than the degree of h, e.g., $h(t) = t^2 + t^3$ is palindromic for $n = 5$.[1]

We say a polynomial is *unimodal* if its coefficients weakly increase then weakly decrease, i.e., there is some k for which

$$h_0 \leq h_1 \leq \cdots \leq h_k \geq h_{k+1} \geq \cdots \geq h_n.$$

If $h(t)$ is palindromic for n, unimodality means that $h_0 \leq h_1 \leq \cdots \leq h_{\lfloor n/2 \rfloor}$.

As a vector space, the set of polynomials palindromic for n has dimension $\lfloor n/2 \rfloor + 1$. One natural basis for this vector space is

$$\Sigma_n = \begin{cases} \{t^j + t^{n-j}\}_{0 \leq j < n/2} & \text{if } n \text{ is odd,} \\ \{t^j + t^{n-j}\}_{0 \leq j < n/2} \cup \{t^{n/2}\} & \text{if } n \text{ is even.} \end{cases}$$

While Σ_n might be the standard basis for polynomials palindromic for n, we will now discuss a more interesting basis that we call the "gamma basis," defined as follows:

$$\Gamma_n = \{t^j (1+t)^{n-2j}\}_{0 \leq j \leq n/2}.$$

Notice that every member of Γ_n is palindromic and unimodal with the same center of symmetry at $n/2$. Hence the nonnegative span of Γ_n contains only palindromic and unimodal polynomials.

[1] In the literature the term "symmetric" is sometimes used to describe what we mean by "palindromic." This is okay in some circumstances, but there is a more common notion of "symmetric polynomial"—namely a polynomial that is fixed under permutation of its variables—so we prefer the less ambiguous term. George Andrews used another synonym for palindromic, "reciprocal polynomial," in [8] and [9].

If $h(t)$ is palindromic for n, the sequence of its coefficients in Γ_n is called the *gamma vector* of h, and the *gamma polynomial* $\gamma(h;t)$ is the generating function for the gamma vector. We have

$$h(t) = (1+t)^n \gamma(h; t/(1+t)^2) = \sum_{0 \le j \le n/2} \gamma_j t^j (1+t)^{n-2j}. \tag{4.8}$$

We say $h(t)$ is *gamma-nonnegative* if $\gamma(h;t)$ has nonnegative coefficients.

For example, if

$$h(t) = 1 + 7t + 15t^2 + 15t^3 + 7t^4 + t^5,$$

we can write

$$h(t) = (1+t)^5 + 2t(1+t)^3 - t^2(1+t),$$

and so

$$\gamma(h;t) = 1 + 2t - t^2.$$

As a vector in the space of palindromic polynomials with basis Σ_5, h is represented by $(1,7,15)$, whereas $\gamma = (1,2,-1)$. We can see that palindromicity and nonnegativity of $h(t)$, and even unimodality, are not enough to guarantee gamma-nonnegativity.

The product of two gamma-nonnegative polynomials is again gamma-nonnegative, though the center of symmetry necessarily shifts. That is, if

$$g(t) = \sum_{0 \le i \le m/2} \gamma_i t^i (1+t)^{m-2i} \quad \text{and} \quad h(t) = \sum_{0 \le j \le n/2} \gamma'_j t^j (1+t)^{n-2j},$$

then

$$g(t)h(t) = \sum_{0 \le k \le (m+n)/2} \left(\sum_{i+j=k} \gamma_i \gamma'_j \right) t^k (1+t)^{m+n-2k}.$$

Thus the set of all gamma-nonnegative polynomials of bounded degree is closed under multiplication. Moreover, we see that the gamma polynomial for the product $g(t)h(t)$ is the product of the gamma polynomial for g and the gamma polynomial for h, i.e.,

$$\gamma(gh;t) = \gamma(g;t)\gamma(h;t).$$

We will record these observations for future reference.

Observation 4.1 *If h is a polynomial in the nonnegative span of Γ_n, i.e., $h(t) \in \mathbb{R}_{\ge 0}\,\Gamma_n$, then h is palindromic and unimodal, with center of symmetry $\lfloor n/2 \rfloor$. Moreover, if $g(t) \in \mathbb{R}_{\ge 0}\,\Gamma_m$, then $g(t)h(t) \in \mathbb{R}_{\ge 0}\,\Gamma_{m+n}$, and $\gamma(gh;t) = \gamma(g;t)\gamma(h;t)$.*

4.5 Computing the gamma vector

There are straightforward linear transformations that map a palindromic polynomial h to its gamma vector, implicit in Equation (4.8).

Suppose $h(t)$ is symmetric for n so that

$$h(t) = h_0 + h_1 t + \cdots + h_{\lfloor n/2 \rfloor} t^{\lfloor n/2 \rfloor} + \cdots ,$$

with $h_i = h_{n-i}$. By abuse of notation, let $h = (h_0, h_1, \ldots, h_{\lfloor n/2 \rfloor})$ denote the coefficients of this polynomial in the basis Σ_n, and let $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_{\lfloor n/2 \rfloor})$ be the corresponding gamma vector. We have the following change of basis matrices:

$$G = \left[(-1)^{i-j} \left(\binom{n-i-j}{i-j} + \binom{n-i-j-1}{i-j-1} \right) \right]_{0 \leq i,j \leq n/2},$$

and

$$S = \left[\binom{n-2j}{i-j} \right]_{0 \leq i,j \leq n/2},$$

so that

$$Gh = \gamma \quad \text{and} \quad S\gamma = h.$$

While the entries in S follow immediately from Equation 4.8, the entries of G are harder to guess at. However, it is straightforward to check that S and G are inverses of one another.

For example if $n = 5$,

$$G = \begin{pmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 5 & -3 & 1 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 10 & 3 & 1 \end{pmatrix},$$

so we see that in our example of $h(t) = (1 + t^5) + 7(t + t^4) + 15(t^2 + t^3)$,

$$Gh = \begin{pmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 5 & -3 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 7 \\ 15 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \gamma,$$

and

$$S\gamma = \begin{pmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 10 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 7 \\ 15 \end{pmatrix} = h.$$

As a word of caution, we note that the palindromicity degree n is needed to recover h from γ. For example, the polynomial $h(t) = 1 + 4t + 4t^2 + t^3$ has $\gamma(h;t) = 1 + t$, but $\gamma(t) = 1 + t$ is the γ-polynomial for a whole family of symmetric polynomials, e.g.,

$$(1+t)^2\gamma(t/(1+t)^2) = 1 + 3t + t^2,$$
$$(1+t)^3\gamma(t/(1+t)^2) = 1 + 4t + 4t^2 + t^3,$$
$$(1+t)^4\gamma(t/(1+t)^2) = 1 + 5t + 8t^2 + 5t^3 + t^4,$$
$$\vdots$$

A very different way to express $\gamma(h;t)$ in terms of $h(t)$ is with the following identity of generating functions.

Proposition 4.1 (Zeilberger's lemma). *Suppose $h(t)$ is palindromic for n, with gamma polynomial $\gamma(t)$. Then we have the following identity of power series:*

$$\gamma(z) = \frac{h(zC(z)^2)}{C(z)^n}, \tag{4.9}$$

where $C(z) = \frac{1-\sqrt{1-4z}}{2z}$ is the Catalan number generating function.

To see how the identity arises, we begin with $h(t) = (1+t)^n\gamma(t/(1+t)^2)$. Now setting $z = t/(1+t)^2$, we find

$$zt^2 + (2z-1)t + z = 0.$$

Solving for t, we find

$$\begin{aligned} t &= \frac{1 - 2z - \sqrt{(2z-1)^2 - 4z^2}}{2z}, \\ &= -1 + \frac{1-\sqrt{1-4z}}{2z}, \\ &= -1 + C(z). \end{aligned}$$

The Catalan generating function also satisfies $C(z) - 1 = zC(z)^2$, and Equation (4.9) now follows. See Problem 4.5 for one use of Equation (4.9).

4.6 Real roots and log-concavity

We have emphasized the importance of the palindromicity and unimodality implied by gamma-nonnegativity. There are at least two other related ideas that have been studied: *real-rootedness* and *log-concavity*. While they are somewhat ancillary to our main concern, we will briefly survey some of their properties here.

Many interesting polynomial generating functions turn out to be real-rooted, that is, these polynomials factor completely over the real numbers. The Eulerian polynomials and the Narayana polynomials, for example, have only real roots. (See Problems 4.6 and 4.7.) If a polynomial $h(t)$ is palindromic, then h is real-rooted if and only if $\gamma(h;t)$ is real-rooted. Indeed, if a

is a real root of h with $a \notin \{0, -1\}$ (the cases of $a \in \{0, -1\}$ are easily considered) then $1/a$ is also a root of h by symmetry, and

$$\gamma(h; a/(1+a)^2) = \frac{1}{(1+a)^n} h(a) = \frac{1}{(1+1/a)^n} h(1/a) = 0,$$

thus implying the real number $a/(1+a)^2$ is a (nonpositive) root of $\gamma(h;t)$. On the other hand, if $b < 0$ is a real root of $\gamma(h;t)$, then both

$$a = -1 + \frac{1+\sqrt{1-4b}}{2b} \quad \text{and} \quad \frac{1}{a} = -1 + \frac{1-\sqrt{1-4b}}{2b}$$

are roots of $h(t)$. That no other roots exist follows by considering the degrees of $h(t)$ and $\gamma(h;t)$.

It turns out that whenever a polynomial $h(t)$ has nonnegative and palindromic coefficients, having all real roots implies $h(t)$ is gamma-nonnegative, but not conversely. (Consider $h(t) = 1 + 4t + 7t^2 + 4t^3 + t^4$. It has no real roots, yet it has nonnegative γ-polynomial $\gamma(h;t) = 1 + t^2$.) In particular, nonnegative, palindromic, and real-rooted polynomials are unimodal.

To see why this is so, suppose $h(t)$ has nonnegative and symmetric coefficients, and all its roots are real. Then as mentioned earlier, its roots apart from 0 and -1 come in reciprocal pairs, $a, 1/a$. Consider

$$(t-a)(t-1/a) = (1+t)^2 - (2+a+1/a)t.$$

If $h(t)$ has nonnegative coefficients, then all its roots must be nonpositive. In particular, $a < 0$, and dividing the positive quantity $(a+1)^2$ by a shows

$$0 > \frac{(a+1)^2}{a} = 2 + a + 1/a.$$

Thus $(t-a)(t-1/a)$ is in the positive span of Γ_2. Since $h(t)$ can be written as a product of powers of t (in Γ_2), powers of $(1+t)$ (in Γ_1), and terms of the form $(t-a)(t-1/a)$, we have that $h(t)$ is a product of gamma-nonnegative polynomials. Since we noted in Observation 4.1 that such polynomials are closed under multiplication, $h(t)$ is gamma-nonnegative as well.

Let us collect these comments.

Observation 4.2 *If h has palindromic coefficients, then $h(t)$ is real-rooted if and only if $\gamma(h;t)$ has only real roots. Moreover, if the coefficients of $h(t)$ are nonnegative, then all the roots of h are nonpositive and $\gamma(h;t)$ has nonnegative coefficients as well. Thus if $h(t)$ is nonnegative, real-rooted, and palindromic, then it is unimodal.*

Another property related to real-rootedness and unimodality is *log-concavity.* A sequence $a_1, \ldots, a_n$ is said to be log-concave if

$$a_i^2 \geq a_{i-1}a_{i+1} \qquad \text{for all } i = 2, \ldots, n-1.$$

This immediately implies that the sequence is unimodal since if there is some j such that $a_{j-1} > a_j < a_{j+1}$, then clearly $a_j^2 < a_{j-1}a_{j+1}$. We will say a polynomial is log-concave if its sequence of coefficients is log-concave.

Log-concavity is more robust than gamma-nonnegativity in the sense that it applies perfectly well to sequences that are not palindromic, whereas the gamma vector requires palindromicity to exist. Real-rootedness also implies log-concavity (Problem 4.8), and hence unimodality, but not conversely. The polynomial $1+4t+7t^2+4t^3+t^4$ from before is log-concave, yet has no real roots.

Log-concave sequences are closed under multiplication, i.e., if $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ are log-concave, then so is $a_1b_1, a_2b_2, \ldots$. However, they are not closed under addition, e.g., $(0,0,11,0,0)$ and $(1,4,6,4,1)$ are both log-concave (and gamma-nonnegative), yet their sum $(1,4,17,4,1)$ is not log-concave.

We collect these comments in another observation, to compare with Observations 4.1 and 4.2.

Observation 4.3 *Suppose $h(t)$ has nonnegative coefficients. If $h(t)$ is real-rooted, then h is log-concave. In particular, h is unimodal.*

Notice, then, that if the goal is to prove unimodality of a polynomial h, real-rootedness is more than sufficient. The relationships between these three concepts: gamma-nonnegativity, log-concavity, and real-rootedness are shown in Figure 4.3. The reader is asked to find a polynomial in each distinct region of that Venn diagram in Problem 4.10.

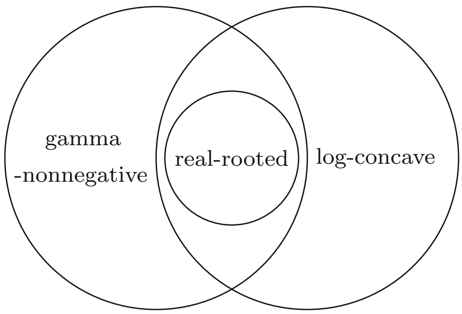


Fig. 4.3 The relationship between the notions of gamma-nonnegativity, log-concavity, and real-rootedness for palindromic polynomials with nonnegative coefficients.

4.7 Symmetric boolean decomposition

If $f(P;t)$ is the rank generating function of a poset P, the fact that $f(P;t)$ is gamma-nonnegative might only be the enumerative shadow of a deeper structural property of the poset itself, which we call *symmetric boolean decomposition*. Loosely, it means that a poset can be partitioned into a number of disjoint boolean algebras with the same center of symmetry around the middle rank of P.

This is a stronger version of a property known as a *symmetric chain decomposition* of a poset, which itself implies unimodality of the rank function $f(P;t)$. The fact that a symmetric boolean decomposition implies a symmetric chain decomposition follows once we can show that every boolean algebra has a symmetric chain decomposition. This is left to Problem 4.14. See also Problem 4.13 for more properties and consequences of symmetric chain decompositions.

Rather than giving a formal definition of symmetric boolean decomposition, let us see some examples. In Figure 4.4, posets (a) and (b) have symmetric boolean decompositions, while (c) and (d) do not.

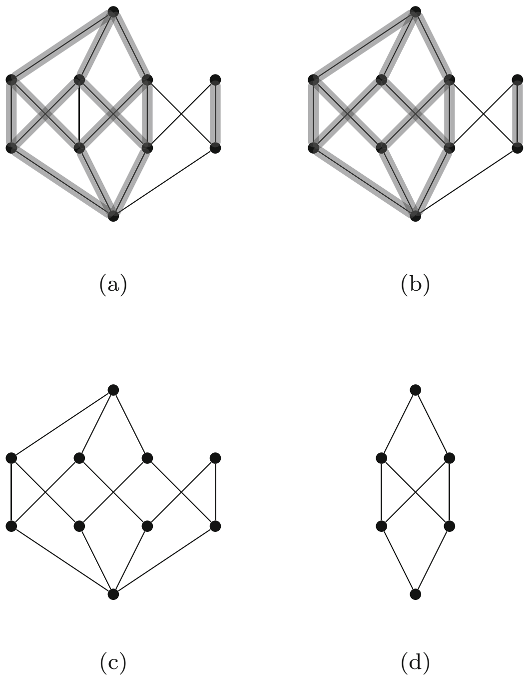


Fig. 4.4 Posets with and without symmetric boolean decompositions.

Formally, we say a poset P of rank n admits a *symmetric boolean decomposition* if there is a collection $\{P_1, \ldots, P_k\}$ of subposets of $(P, \leq)$ with the following properties:

- $P_i \cap P_j = \emptyset$ if $i \neq j$,
- $P_1 \cup \cdots \cup P_k = P$,
- for each $i = 1, \ldots, k$ there is a number j, $0 \leq j \leq n/2$, and a bijection $\rho_i : 2^{[n-2j]} \to P_i$ that takes cover relations to cover relations and sends elements of rank r in $2^{[n-2j]}$ to elements of rank $j + r$ in P.

That is, each induced poset $(P_i, \leq)$ has 2^{n-2j} elements (for some j) and contains a copy of the boolean algebra $2^{[n-2j]}$, plus possibly more relations.

For example, we can see in Figure 4.4 that poset (a) contains a copy of $2^{\{1,2,3\}}$ as a proper subposet, whereas in (b) the part of the partition containing it has no unnecessary cover relations. Note also the delicacy of the decomposition: the poset in (c) differs from (b) only in one cover relation.

We can also observe that just as gamma-nonnegative polynomials are closed under multiplication, so too are posets with symmetric boolean decompositions. First, we define the product of two posets $(P, \leq_P)$ and $(Q, \leq_Q)$ to be the poset on the cartesian product $P \times Q$ with partial order $(p, q) \leq_{P \times Q} (p', q')$ if and only if $p \leq_P p'$ and $q \leq_Q q'$. Then it is a straightforward matter to verify the following observation. This is left to the reader in Problem 4.16.

Observation 4.4 *Suppose* $(P, \leq_P)$ *and* $(Q, \leq_Q)$ *are posets with symmetric boolean decompositions. Then* $(P \times Q, \leq_{P \times Q})$ *has a symmetric boolean decomposition.*

This result is illustrated in Figure 4.5.

Two interesting examples of posets with symmetric boolean decompositions are the shard intersection order and the lattice of noncrossing partitions.

Theorem 4.3. *The shard intersection order and the lattice of noncrossing partitions admit symmetric boolean decompositions.*

This result follows from the valley-hopping argument given in Section 4.2. Indeed, the hop-equivalence classes are boolean intervals in the shard intersection order with the proper rank properties, i.e., their descents are distributed like $t^j(1 + t)^{n-2j}$. Since hop-equivalence preserves the pattern 231, this gives a symmetric boolean decomposition for the shard intersection order on 231-avoiders, which we showed is isomorphic to the lattice of noncrossing partitions. How this works in S_4 is shown in Figure 4.6.

Notes

Gamma-nonnegativity of the Eulerian polynomials was observed by Dominique Foata and Marcel-Paul Schützenberger in their 1970 book [70, Théorème 5.6]. Foata and Volker Strehl [72] gave the result a combinatorial

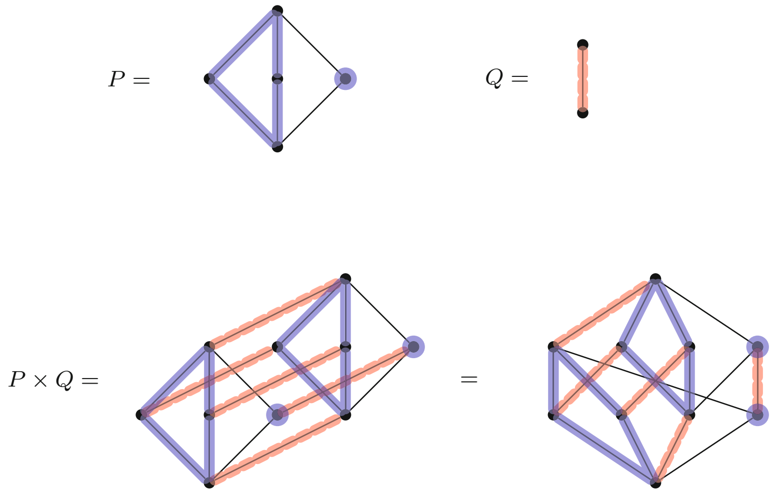


Fig. 4.5 The product of two posets with a symmetric boolean decomposition has a symmetric boolean decomposition.

proof very similar to the "valley-hopping" argument given here, which was essentially rediscovered by Louis Shapiro, Wen Jin Woan, and Seyoum Getu in 1983 [135]. In a 2008 paper, [31], Petter Brändén studies valley-hopping (what he calls the "modified Foata-Strehl" action) on a large family of polynomials that generalize the Eulerian polynomials and include the Narayana polynomials. That the Narayana polynomials are gamma-nonnegative is also implicit in the work of Rodica Simion and Daniel Ullman from 1991 [140]. See also Simion's 1994 paper [138].

We also mention that George Andrews anticipated some of the ideas in this section, proving in a 1975 paper [8] that a product of palindromic and unimodal polynomials is again palindromic and unimodal. Further, he discussed the gamma polynomial and palindromic polynomials (what he called "reciprocal polynomials") in the larger context of quadratic transformations in a 1985 paper [9].

More recent interest in gamma-nonnegativity was sparked by a 2005 paper of Światosław Gal [79]. This work showed certain questions in topology could be resolved by demonstrating gamma-nonnegativity of combinatorial invariants. Prior to Gal's work, researchers had attacked such questions via real-rootedness, but Gal showed that real-rootedness could fail yet gamma-nonnegativity still holds. (This subject is discussed further in Chapter 10.) Similar real-rootedness conjectures known as the Neggers and Stanley conjectures were disproved around the same time by Petter Brändén [28] (Stanley) and John Stembridge [158] (Neggers). Both Gal [79] and Brändén [29] showed

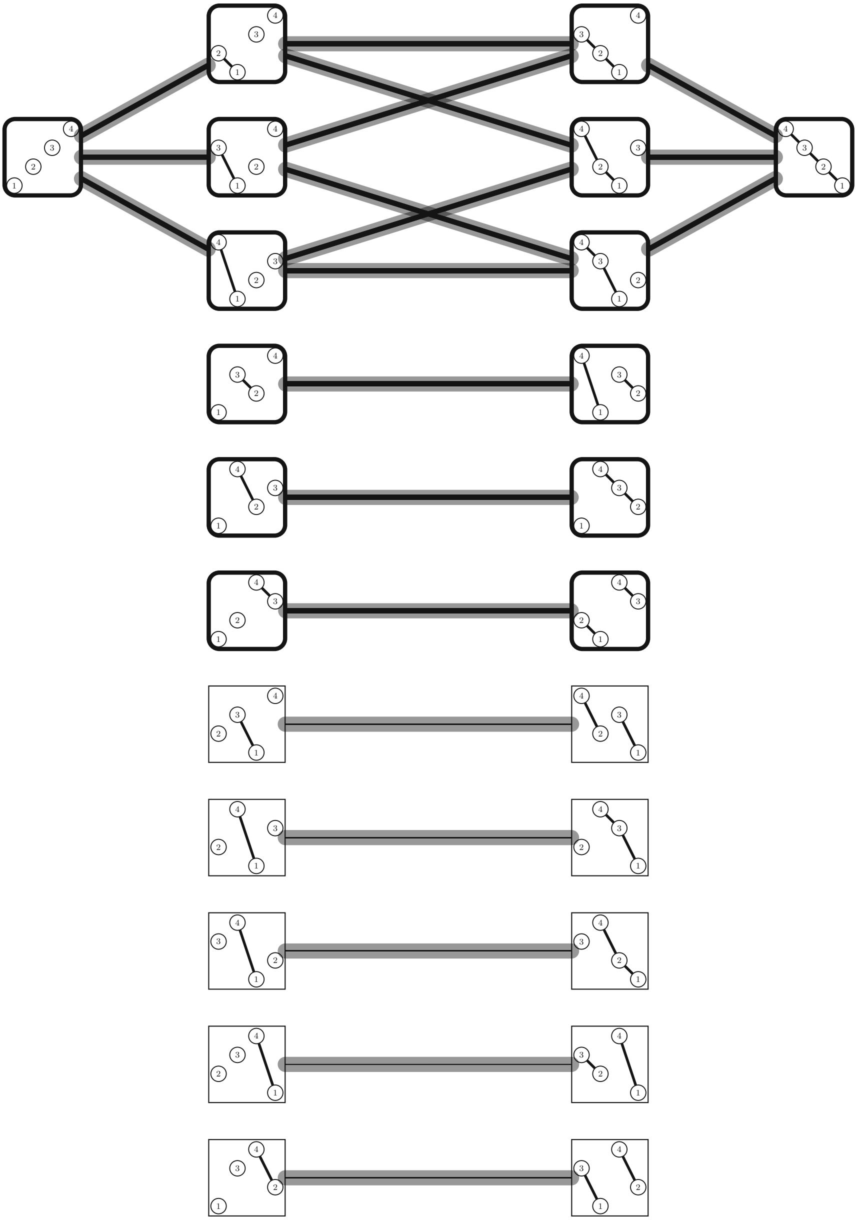


Fig. 4.6 The symmetric boolean decomposition of the shard intersection order and noncrossing partition lattice (in bold) induced by hop-equivalence classes. The Hasse diagram is drawn left to right, and edges not needed for the decomposition are omitted.

gamma-nonnegativity could be a viable replacement for real-rootedness in many of these contexts. See also the work of Victor Reiner and Volkmar Welker [128].

Some nice surveys about log-concavity, unimodality, real-rootedness, and gamma-nonnegativity include a 1989 paper by Richard Stanley [150], a 1994 paper by Francesco Brenti [33], and a 2014 survey by Brändén [32]. Only Brändén's discusses gamma-nonnegativity.

The idea of symmetric boolean decomposition first appears in Simion and Ullman's work on the lattice of noncrossing partitions, though they do not state this explicitly [140]. However, a remark about such a decomposition was later made by Simion [138, Proposition 3.4]. In 1999, while studying a generalization of the lattice of noncrossing partitions, Patricia Hersh [90] makes the definition of symmetric boolean decomposition explicit. More recently this book's author demonstrated the symmetric boolean decomposition of the shard intersection order [118].

Problems

4.1. Verify Equations (4.4) and (4.7).

4.2 (Alternating permutations). A permutation w is called *alternating* if

$$w(1) < w(2) > w(3) < \cdots \qquad \text{or} \qquad w(1) > w(2) < w(3) > \cdots .$$

In the first case, we say w is *up-down alternating*, while in the second case we say w is *down-up alternating*.

1. Let $\mathcal{E}_n$ denote the set of up-down alternating permutations of $[n]$, and let $\mathcal{E}'_n$ denote the set of down-up alternating permutations. Show $|\mathcal{E}_n| = |\mathcal{E}'_n|$.
2. Let E_n denote the cardinality of either $\mathcal{E}_n$ or $\mathcal{E}'_n$, with $E_0 := 1$. The first few values of E_n are

$$1, 1, 1, 2, 5, 16, 61, 272, 1385, 7936, 50521, \ldots.$$

Show that for $n \geq 1$,

$$2E_{n+1} = \sum_{k=0}^{n} \binom{n}{k} E_k E_{n-k}.$$

3. Show that

$$\sum_{n\geq 0} E_n \frac{z^n}{n!} = \sec z + \tan z.$$

Since $\sec z$ is an even function and $\tan z$ is an odd function, conclude that

$$\sum_{n\geq 0} E_{2n}\frac{z^{2n}}{(2n)!} = \sec z,$$

and

$$\sum_{n\geq 0} E_{2n+1}\frac{z^{2n+1}}{(2n+1)!} = \tan z.$$

4.3. A permutation $w \in S_n$ is called *min-max* if $w^{-1}(1) < w^{-1}(n)$ and *max-min* if $w^{-1}(n) < w^{-1}(1)$. Let $E_n^{\nearrow}$ denote the number of up-down alternating permutations that are min-max permutations, and let $E_n^{\nwarrow}$ denote the number of up-down alternating permutations that are max-min permutations.

Show that

$$E_n^{\nearrow} - E_n^{\nwarrow} = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ E_{n-2} & \text{if } n \text{ is even.} \end{cases}$$

4.4. The *stack-sorting* operator $\mathcal{S}$ is a recursively defined function on permutations. If w is an empty permutation $\mathcal{S}(w) := w$. If w is not empty and $\max w(i) = m$, then write $w = u{\cdot}m{\cdot}v$ for some (possibly empty) permutations u and v. Then we define $\mathcal{S}(w) = \mathcal{S}(u)\mathcal{S}(v)m$.

1. Compute $\mathcal{S}(389124576)$ and $\mathcal{S}(132549678)$.
2. Prove that $\mathcal{S}(w) = 12\cdots n$ if and only if w is 231-avoiding. We call such a permutation *stack-sortable*.
3. Show that $\mathcal{S}(w) = \mathcal{S}(w')$ if $\mathrm{Hop}(w) = \mathrm{Hop}(w')$, i.e., if w and w' are in the same valley-hopping equivalence class.
4. A permutation is called *r-stack sortable* if $\mathcal{S}^r(w) = 12\cdots n$. Show that r-stack sortability is preserved by valley hopping, and conclude that the Eulerian distribution on r-stack sortable permutations is gamma-nonnegative, i.e.,

$$\sum_{w\in S_n^r} t^{\mathrm{des}(w)} = \sum_{j\geq 0} \gamma_{r;n,j} t^j (1+t)^{n-1-2j},$$

where S_n^r denotes the set of r-stack sortable elements in S_n, and $\gamma_{r;n,j} = |\{w \in S_n^r : \mathrm{pk}(w) = \mathrm{des}(w) = j\}|$.

4.5. Let

$$h_n(t) = (1+t)(1+t+t^2)\cdots(1+t+\cdots+t^{n-1}) = \prod_{i=1}^{n} \frac{1-t^i}{1-t},$$

and let $\gamma(t)$ be the corresponding gamma polynomial. Note that $h_n(1) = n!$. What is $\gamma(-1)$?

4.6. Prove the Eulerian polynomials $S_n(t)$ are real rooted.

Hint: Let $A_n(t) = tS_n(t)$ and show that $A_n(t)$ has n real roots. We can modify Equation (1.9) to write

$$A_{n+1}(t) = t\left((n+1)A_n(t) + (1-t)A_n'(t)\right). \tag{4.10}$$

If we suppose $A_n(t)$ has n real roots, then we can use this recurrence to prove $S_{n+1}(t) = A_{n+1}(t)/t$ has n real roots as follows. Rolle's theorem shows that the roots of a polynomial $f(t)$ and its derivative $f'(t)$ are "interlacing." Show that $(n+1)A_n(t)$ and $(1-t)A_n'(t)$ have interlacing roots, and use this to show their sum has n real roots.

Moreover, show that the sequence of Eulerian polynomials forms a *Sturm sequence*, i.e., the polynomials $S_n(t)$ and $S_{n+1}(t)$ have interlacing roots.

4.7. Let $N_n(t) = tC_n(t)$ denote the Narayana polynomial multiplied by a power of t.

1. Prove the polynomials $N_n(t)$ satisfy the following recurrence:

$$(n+1)N_n(t) = (2n-1)(1+t)N_{n-1}(t) - (n-2)(1-t)^2N_{n-2}(t). \tag{4.11}$$

   (A bijective proof would be best, but this can also be verified with generating functions using Equation (2.6).)
2. Use the recurrence in (4.11) to prove that the Narayana polynomials are real-rooted and form a Sturm sequence.

4.8 (Real roots and log-concavity). The goal of this problem is to show that a polynomial with positive coefficients and only real roots has log-concave, and hence unimodal, coefficients.

1. Show that the sequence of binomial coefficients with n fixed,

$$\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n},$$

   is log-concave, i.e., the polynomial $(1+t)^n$ is log-concave.
2. Show that the sequence of binomial coefficients with k fixed,

$$\binom{k}{k}, \binom{k+1}{k}, \ldots,$$

   is log-concave.
3. Prove that if $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ are log-concave, then the sequence $c_1, c_2, \ldots$ defined by $c_k = a_k b_k$ is log-concave.
4. Show that if $a_0, a_1, \ldots, a_n$ is a finite sequence of nonnegative numbers and the sequence $b_0, b_1, \ldots$ given by $b_k = a_k/\binom{n}{k}$ is log-concave, then $a_0, a_1, \ldots, a_n$ is itself log-concave.
5. Let $a_0, a_1, \ldots, a_n$ be a sequence of nonnegative numbers such that $f(t) = a_0 + a_1 t + \cdots + a_n t^n$ is real-rooted.

a. Write $a_k = \binom{n}{k} b_k$. Show that

$$\sum_{k=0}^{n-1} \binom{n-1}{k} b_{k+1} t^k$$

is real-rooted. (Hint: it is a multiple of the derivative of f.)

b. Show that the polynomial

$$\sum_{k=0}^{n} \binom{n}{k} b_{n-k} t^k$$

is real-rooted.

c. Use the operations indicated in parts 5a) and 5b) to show that for any $j = 1, \ldots, n-1$, the polynomial $b_{j-1} + 2b_j t + b_{j+1} t^2$ is real-rooted. Conclude that the sequence $b_0, b_1, b_2, \ldots$ is log-concave. By part 4) this proves that a real-rooted polynomial with nonnegative coefficients is log-concave, and hence unimodal.

4.9. Prove that if $f(t)$ and $g(t)$ are nonnegative and log-concave, then their product, $f(t)g(t)$, is log-concave. Hint: first prove that if $a_0, a_1, a_2, \ldots$ is a nonnegative and log-concave sequence, then $a_i a_j \leq a_{i+1} a_{j-1}$ for any $i < j-1$.

4.10. If a polynomial is real-rooted and palindromic then it is both gamma-nonnegative and log-concave, as illustrated in Figure 4.3. Find examples of polynomials with positive, palindromic integer coefficients that fit in the other regions of that Venn diagram.

1. Find a polynomial that is gamma-nonnegative but not log-concave (and hence not real-rooted).
2. Find a polynomial that is log-concave and palindromic but not gamma-nonnegative (and hence not real-rooted).
3. Find a polynomial that is log-concave and gamma-nonnegative but not real-rooted.

4.11 (Gamma-nonnegativity for involutions). An *involution* is a permutation that is its own inverse: $w = w^{-1}$. Show that the distribution of descents for involutions, i.e., the *Eulerian distribution for involutions*, is gamma-nonnegative. That is, show there exist nonnegative integers γ_j such that

$$\sum_{w=w^{-1} \in S_n} t^{\mathrm{des}(w)} = \sum_{j \geq 0} \gamma_j t^j (1+t)^{n-1-2j}.$$

4.12 (Two-dimensional gamma-nonnegativity). Let

$$S_n(s,t) = \sum_{w \in S_n} s^{\mathrm{des}(w^{-1})} t^{\mathrm{des}(w)},$$

i.e., the joint distribution of descents and inverse descents. Show that there exist nonnegative integers $\gamma_{i,j}$ such that

$$S_n(s,t) = \sum_{i,j\geq 0} \gamma_{i,j}(st)^i(s+t)^j(1+st)^{n-1-j-2i}.$$

4.13 (Symmetric chain decomposition). A *symmetric chain decomposition* of a finite ranked poset P with maximal rank n is a partition into saturated chains

$$p_0 <_P p_1 <_P \cdots <_P p_k,$$

such that $\rho(p_0)+\rho(p_k) = n$ for each chain. (Recall a "saturated" chain is one for which $\rho(p_i)+1 = \rho(p_{i+1})$ for all i.)

1. Show that if P has a symmetric chain decomposition, then its rank function,
$$f(P;t) = \sum_{p\in P} t^{\rho(p)} = \sum_{k\geq 0} f_k t^k,$$
is symmetric and unimodal.
2. Let $A \subset P$ be an antichain, i.e., a set of pairwise incomparable elements of P. Show that $|A| \leq f_{\lfloor n/2\rfloor}$.
3. Show that the product of two chains has a symmetric chain decomposition. That is, show P has a symmetric chain decomposition, where $P = [k]\times[l]$ is the set of pairs (i,j) with $1 \leq i \leq k$ and $1 \leq j \leq l$, ordered by $(i,j) \leq (i',j')$ if and only if $i \leq i'$ and $j \leq j'$.
4. Show that if P and Q are posets with symmetric chain decompositions, then their cartesian product $P\times Q$ (with partial order $(p,q) \leq_{P\times Q} (p',q')$ if and only if $p \leq_P p'$ and $q \leq_Q q'$) has a symmetric chain decomposition.

4.14 (Sperner's Theorem). Show that the boolean algebra $2^{[n]}$, i.e., the set of subsets of a finite set ordered by inclusion, has a symmetric chain decomposition. (This implies that any poset with a symmetric boolean decomposition inherits a symmetric chain decomposition.)

Conclude *Sperner's Theorem*: any collection A of subsets of $[n]$ such that no subset contains another satisfies $|A| \leq \binom{n}{\lfloor n/2\rfloor}$.

4.15 (Lattice of divisors). The lattice of positive divisors of an integer n, $\Lambda(n)$, is the set of all integers d that divide n, ordered by divisibility. If $d = p_1^{m_1}p_2^{m_2}\cdots p_k^{m_k}$ is the prime factorization of d, we define the *degree* of d to be $\deg(d) = m_1+m_2+\cdots+m_k$. The covers of $\Lambda(n)$ are given by multiplication by a single prime, $d \prec p_i d$ for some p_i. Thus $\Lambda(n)$ is ranked by degree. Let $f_k(n)$ denote the number of divisors of n of degree k.

Show that the lattice of positive divisors of an integer n has a symmetric chain decomposition, and conclude that any collection A of mutually indivisible divisors of n (i.e., if $a,b\in A$, then neither $a|b$ nor $b|a$) has cardinality at most $f_{\lfloor \deg(n)/2\rfloor}(n)$.

4.16. Verify Observation 4.4, i.e., show that if posets P and Q have a symmetric boolean decomposition, then so does their cartesian product, $P \times Q$.

4.17 (Simion and Ullmann's symmetric boolean decomposition). In [140], Rodica Simion and Daniel Ullman gave a symmetric boolean decomposition of $\mathrm{NC}(n)$ that is different from the one that follows from valley hopping. Simion and Ullman provide a certain encoding of noncrossing partitions as words on the alphabet $\{b, e, l, r\}$. Given a noncrossing partition $\pi \in \mathrm{NC}(n)$, the encoding assigns a word $w(\pi) = w = w_1 w_2 \cdots w_{n-1}$ of length $n-1$ as follows:

$$w_i = \begin{cases} b & \text{if } i \text{ and } i+1 \text{ are in different blocks} \\ & \quad \text{and } i \text{ is } \textit{not} \text{ the largest element in its block,} \\ e & \text{if } i \text{ and } i+1 \text{ are in different blocks} \\ & \quad \text{and } i+1 \text{ is } \textit{not} \text{ the smallest element in its block,} \\ l & \text{if } i \text{ and } i+1 \text{ are in different blocks,} \\ & \quad i \text{ is the largest element in its block,} \\ & \quad \text{and } i+1 \text{ is the smallest element in its block,} \\ r & \text{if } i \text{ and } i+1 \text{ are in the same block.} \end{cases}$$

We call such a word the *SU-word* for π.

For example, if $\pi = \{\{1,2,6\},\{3\},\{4,5\}\}$, we have its SU-word is $w(\pi) = rblre$. Let $B(w), E(w), L(w), R(w)$ denote the sets of positions in w containing the letters b, e, l, and r, respectively. For example $w = rblre$ has $B(w) = \{2\}$, $E(w) = \{5\}$, $L(w) = \{3\}$, and $R(w) = \{1, 4\}$.

1. Show $n = |B(w)| + |E(w)| + |L(w)| + |R(w)| + 1$.
2. Show π has rank equal to $|B(w)| + |R(w)|$.
3. Show that $|B(w)| = |E(w)|$ and that these sets give a valid matching on $[n]$ by having an open parenthesis at each $b \in B(w)$ (beginning) and a closed parenthesis at $e+1$ for each e in $E(w)$ (ending).
4. Let π and π' be noncrossing partitions with SU-words w and w'. Show that if $B(w) = B(w')$, $E(w) = E(w')$, and $R(w) \subseteq R(w')$, then $\pi \leq_{\mathrm{NC}} \pi'$.
5. Use 5) to give a symmetric boolean decomposition of $\mathrm{NC}(n)$.
6. Show that this decomposition is different from the one inherited from the decomposition of the shard intersection order restricted to 231-avoiding permutations.

Chapter 5
Weak order, hyperplane arrangements, and the Tamari lattice

One of the most elegant ways in which the Eulerian numbers and the Narayana numbers arise is in the counting of faces of polytopes. These polytopes are related to certain poset structures on, respectively, the set of all permutations of $[n]$ and on the set of 231-avoiding permutations of $[n]$ (or any set of Catalan objects). These posets are known as the *weak order* and the *Tamari lattice*, respectively. Our study of the weak order leads naturally to a side trip into the realm of *hyperplane arrangements*. This geometric perspective will be useful to have in later parts of the book.

5.1 Inversions

Before we define the weak order, we should introduce a commonly known permutation statistic called *inversion number*, $\mathrm{inv}(w)$. An *inversion* is a pair $i < j$ such that $w(i) > w(j)$. The set of all inversions is denoted by

$$\mathrm{Inv}(w) = \{1 \leq i < j \leq n : w(i) > w(j)\},$$

and $\mathrm{inv}(w) = |\,\mathrm{Inv}(w)|$. For example, with $w = 5624713$, we have

$$\mathrm{Inv}(w) = \left\{ \begin{matrix} (1,3),(1,4),(1,6),(1,7),(2,3),(2,4),(2,6), \\ (2,7),(3,6),(4,6),(4,7),(5,6),(5,7) \end{matrix} \right\},$$

and so $\mathrm{inv}(w) = 13$. Notice that the set of inversions uniquely determines w. (See Problem 5.1.) Also notice that descents are simply adjacent inversions, $(i, i+1)$, and so $\mathrm{des}(w) \leq \mathrm{inv}(w)$.

It is also useful to observe that w and its inverse permutation must have the same number of inversions: if $i < j$ and $w(i) > w(j)$, then $w(j) < w(i)$ and $w^{-1}(w(j)) = j > i = w^{-1}(w(i))$. That is, $(w(j), w(i)) \in \mathrm{Inv}(w^{-1})$. For example, with $w = 5624713$, we get $w^{-1} = 6374125$ and

T.K. Petersen, *Eulerian Numbers*, Birkhäuser Advanced Texts Basler Lehrbücher, DOI 10.1007/978-1-4939-3091-3_5

$$\mathrm{Inv}(w^{-1}) = \left\{ \begin{array}{c} (2,5),(4,5),(1,5),(3,5),(2,6),(4,6),(1,6), \\ (3,6),(1,2),(1,4),(3,4),(1,7),(3,7) \end{array} \right\}.$$

In particular, $\mathrm{inv}(w) = \mathrm{inv}(w^{-1})$.

The permutation $12\cdots n$ is the only one that has no inversions, while $n\cdots 21$ has the most, with $\binom{n}{2}$ of them. Here, every pair of indices is an inversion. In Table 5.1 we see the permutations in S_4 grouped according to the number of inversions.

Table 5.1 The permutations in S_4 grouped according to number of inversions.

$\mathrm{inv}(w) =$	0	1	2	3	4	5	6
	1234	1243	1342	2341	3412	3421	4321
		1324	1423	2413	2431	4231	
		2134	2314	4123	3241	4312	
			3124	1432	4132		
			2143	3142	4213		
				3214			

Let $I_{n,k}$ denote the number of permutations in S_n with k inversions. We call these numbers *Mahonian numbers* after Percy MacMahon, who wrote a very influential book on enumerative combinatorics in the early twentieth century [106]. More about MacMahon's contribution to the study of this distribution is discussed in Chapter 6. The triangle of Mahonian numbers is shown in Table 5.2.

Table 5.2 The Mahonian numbers $I_{n,k}$, $1 \le n \le 6$, $0 \le k \le \binom{n}{2}$.

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1															
2	1	1														
3	1	2	2	1												
4	1	3	5	6	5	3	1									
5	1	4	9	15	20	22	20	15	9	4	1					
6	1	5	14	29	49	71	90	101	101	90	71	49	29	14	5	1

The generating function for permutations according to inversion number is rather straightforward to understand. Let $I_n(q)$ denote the generating function for the Mahonian numbers $I_{n,k}$, i.e.,

$$I_n(q) = \sum_{w \in S_n} q^{\mathrm{inv}(w)} = \sum_{k \ge 0} I_{n,k} q^k.$$

For any $n \geq 1$, let $[n]_q = 1 + q + q^2 + \cdots + q^{n-1} = \frac{1-q^n}{1-q}$. Then we can see that

$$I_n(q) = (1 + q + q^2 + \cdots + q^{n-1}) I_{n-1}(q) = [n] \cdot I_{n-1}(q).$$

Indeed, suppose $v \in S_{n-1}$ has k inversions. We can form n distinct permutations in S_n by adding n to the right or left of v, or by inserting n in a gap between the letters of v. Such permutations have a predictable number of inversions:

- $v(1) \cdots v(n-1)\, n$ has k inversions,
- $v(1) \cdots v(n-2)\, n\, v(n-1)$ has $k+1$ inversions,
- $v(1) \cdots v(n-3)\, n\, v(n-2) v(n-1)$ has $k+2$ inversions,

$\vdots$

- $n\, v(1) \cdots v(n-1)$ has $k+n-1$ inversions.

Since $I_1(q) = 1$, the recurrence gives the following simple formula.

Theorem 5.1 (Inversion generating function). *For any $n \geq 1$,*

$$I_n(q) = [n]_q! = \frac{\prod_{i=1}^{n}(1-q^i)}{(1-q)^n}.$$

The polynomials $I_n(q)$ are easily seen to be palindromic and unimodal. In fact, they are log-concave. (See Problem 5.4.) However, they are not gamma-nonnegative, and $I_n(q)$ has complex roots for $n \geq 3$, ($I_n(\zeta) = 0$ for any mth roots of unity with $\zeta \neq 1$, $m \leq n$).

The number of inversions in a permutation also counts the minimal number of adjacent transpositions needed to sort a permutation w into the identity permutation $12 \cdots n$. That is, let s_i denote the *ith simple transposition*, the permutation that swaps i and $i+1$ and fixes all other elements of $\{1, 2, \ldots, n\}$. Then

$$w \circ s_i = w(1) \cdots w(i-1) w(i+1) w(i) w(i+2) \cdots w(n),$$

and we have the following observation that can be proved, e.g., by induction. See Problem 5.5.

Observation 5.1 (Sorting with simple transpositions) *Inversion number equals the minimal number of simple transpositions necessary to sort a permutation. That is, if $w \in S_n$,*

$$\mathrm{inv}(w) = \min\{k : w \circ s_{i_1} \circ \cdots \circ s_{i_k} = 12 \cdots n\}.$$

For example if $w = 314526$, we have $\mathrm{inv}(w) = 4$, while the following sequence of adjacent swaps (done greedily) will sort the permutation:

$$\mathbf{31}4526 \xrightarrow{s_1} 134\mathbf{52}6 \xrightarrow{s_4} 13\mathbf{42}56 \xrightarrow{s_3} 1\mathbf{32}456 \xrightarrow{s_2} 123456.$$

We have highlighted the numbers being swapped in boldface.

5.2 The weak order

Now we present a partial order for S_n called the *weak order*, and denoted by $\mathrm{Wk}(S_n)$. We define this partial ordering as follows. For any permutations u and v in S_n,

$$u \leq_{\mathrm{Wk}} v \qquad \text{if and only if} \qquad \mathrm{Inv}(u) \subseteq \mathrm{Inv}(v).$$

Notice that if i is to the right of $(i+1)$ in w, i.e.,

$$w = w(1) \cdots (i+1) \cdots i \cdots w(n),$$

then

$$s_i \circ w = w(1) \cdots i \cdots (i+1) \cdots w(n).$$

Since i and $i+1$ are consecutive and no other letters have moved, we see that every inversion of $s_i \circ w$ is an inversion in w. Further, the only inversion of w that is not an inversion of $s_i \circ w$ is the pair corresponding to the positions of i and $i+1$, i.e., the pair $(w^{-1}(i+1), w^{-1}(i))$. For example, if $w = 1532467$ we have $\mathrm{Inv}(w) = \{(2,3),(2,4),(2,5),(3,4)\}$. If we swap the 4 and the 5, we find $s_4 \circ w = 1432567$ and $\mathrm{Inv}(s_4 \circ w) = \{(2,3),(2,4),(3,4)\}$.

In general we can say $v <_{\mathrm{Wk}} w$ is a cover relation in $\mathrm{Wk}(S_n)$ if:

- $w = s_i \circ v$ for some $i = 1, 2, \ldots, n-1$, and
- $\mathrm{inv}(w) = 1 + \mathrm{inv}(v)$.

As cover relations are given by left multiplication, this will be what we call the "left" weak order. An equally valid approach would be to define the weak order by inclusion of inversion sets for the inverse permutations, i.e., we could declare u below v if $\mathrm{Inv}(u^{-1}) \subseteq \mathrm{Inv}(v^{-1})$. In this case cover relations are given by right multiplication by simple transpositions: $w = v \circ s_i$. (See Problem 5.2.) When we need to distinguish between the two, we will write $\mathrm{Wk}^l(S_n)$ and $\mathrm{Wk}^r(S_n)$, respectively.

We will see that the left weak order is more natural from a geometric point of view, while the right weak order is often convenient for thinking about sorting. For example, Observation 5.1 essentially describes a path, in the Hasse diagram for right weak order, from a permutation w to the identity permutation. (See Problem 5.5.) The posets $\mathrm{Wk}^l(S_n)$ and $\mathrm{Wk}^r(S_n)$ are isomorphic via mapping permutations to their inverses, so when the choice is unimportant we will write $\mathrm{Wk}(S_n)$ and refer to simply "the" weak order.

Some immediate consequences of the definition are that $12 \cdots n$ is the unique minimum, $n \cdots 21$ is the unique maximum, and that $\mathrm{Wk}(S_n)$ is ranked by inversion number. We see both versions of $\mathrm{Wk}(S_3)$ in Figure 5.1 and the right version of $\mathrm{Wk}(S_4)$ in Figure 5.2.

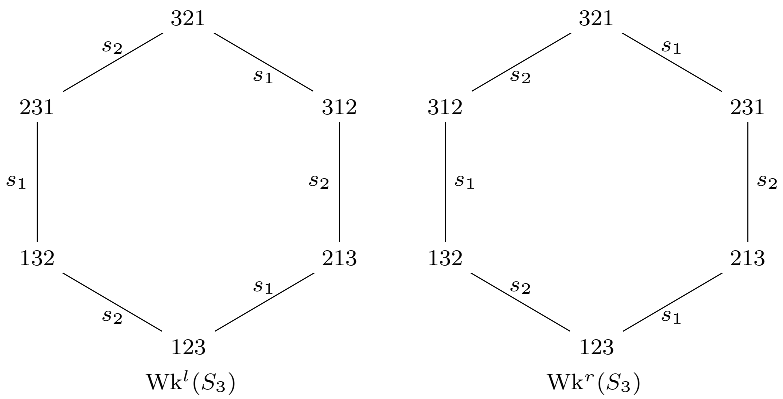


Fig. 5.1 The left and right weak orders on S_3.

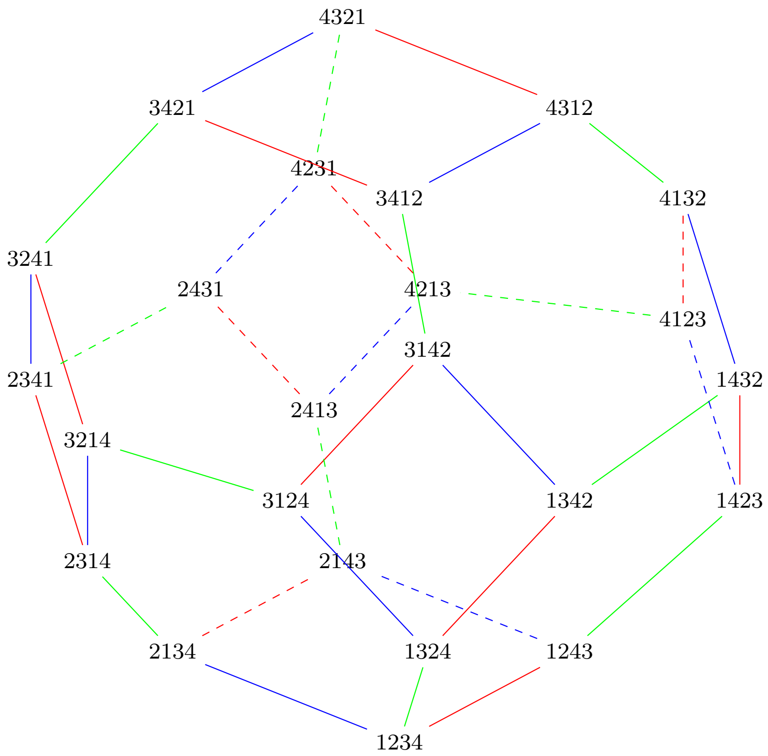


Fig. 5.2 The (right) weak order on S_4. The colored edges indicate which simple transposition is used for the cover relation.

Proposition 5.1. *The rank generating function for the weak order is the Mahonian polynomial $I_n(q)$:*

$$f(\mathrm{Wk}(S_n);q) = \sum_{w \in S_n} q^{\mathrm{inv}(w)} = [n]_q!.$$

5.3 The braid arrangement

The way we have drawn the Hasse diagrams for $\mathrm{Wk}(S_3)$ and $\mathrm{Wk}(S_4)$ suggest that there is a geometric structure related to the weak order, and indeed this is the case, as we now describe.

One way to characterize the Hasse diagram for the weak order is as the 1-skeleton of a polytope called the *permutahedron.* This is the polytope obtained by taking the convex hull of all points $(x_{w(1)}, \ldots x_{w(n)})$ where $\mathbf{x} = (x_1, \ldots, x_n)$ is some generic base point in $\mathbb{R}^n$. We will not delve further into the details of the polytopal construction, but rather describe the hyperplane arrangement dual to the polytope. This arrangement is known as the *braid arrangement* and it will be an important touchstone for us in later chapters.

Let

$$\mathcal{H}(n) = \{H_{i,j} : 1 \leq i < j \leq n\},$$

where

$$H_{i,j} = \{\mathbf{x} \in \mathbb{R}^n : x_i = x_j\}.$$

We will see that $\mathcal{H}(n)$ can be used to partition $\mathbb{R}^n$ into subsets, corresponding to various equalities and inequalities among the coordinates, that we call the *faces* of the arrangement. In fact, since the line $\ell = \mathbb{R} \cdot (1, 1, \ldots, 1)$ is contained in each hyperplane of $\mathcal{H}(n)$, we can view $\mathcal{H}(n)$ as partitioning the $(n-1)$-dimensional vector space

$$V = \mathbb{R}^n / \ell.$$

To be concrete, we will identify the quotient space $\mathbb{R}^n / \ell$ with the linear subspace

$$V^* = \{\mathbf{x} \in \mathbb{R}^n : x_1 + x_2 + \cdots + x_n = 0\}.$$

(Properly speaking these are dual spaces, but as they are both isomorphic to $\mathbb{R}^{n-1}$ we will not worry about this distinction.) The weak order arises from the braid arrangement by identifying the maximal open cones in the complement of $\mathcal{H}(n)$ with elements of S_n, and orienting the arrangement according to some transverse direction. See Figure 5.3.

Our goal for the moment is not to give another characterization of the weak order, but to describe a partial order on the set of *all* faces of $\mathcal{H}(n)$ (which corresponds to the reverse containment order on faces of the permutahedron). This will give another context in which the Eulerian numbers arise. First, let

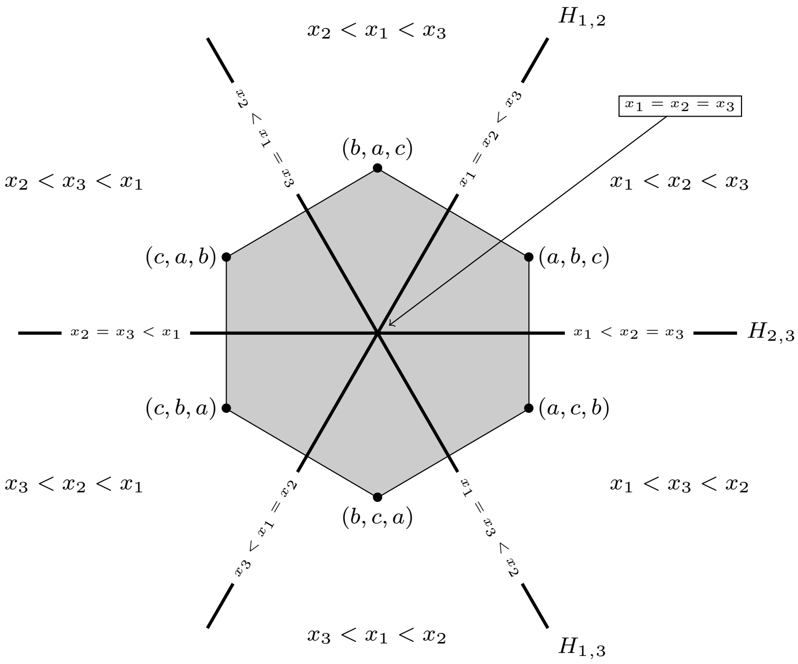


Fig. 5.3 The braid arrangement $\mathcal{H}(3)$, drawn in the plane given by $x_1 + x_2 + x_3 = 0$.

us be clear about what constitutes a "face" of $\mathcal{H}(n)$. For each hyperplane $H_{i,j} \in \mathcal{H}(n)$, we will define the *positive halfspace* to be the set of all points for which $x_j - x_i$ is positive, and the *negative halfspace* to be the set of all points for which $x_j - x_i$ is negative.

For each point $\mathbf{x} \in \mathbb{R}^n / \ell$, we define its *sign sequence* to be the vector that records, for each hyperplane $H_{i,j}$, whether $\mathbf{x}$ is in the positive halfspace, negative halfspace, or on the hyperplane itself. That is, the sign sequence is $\sigma(\mathbf{x}) = (\sigma_{i,j}(\mathbf{x}))_{1 \le i < j \le n}$, where

$$\sigma_{i,j}(\mathbf{x}) = \begin{cases} + & \text{if } x_j - x_i > 0, \\ 0 & \text{if } x_j - x_i = 0, \\ - & \text{if } x_j - x_i < 0. \end{cases}$$

For any σ, we denote the set of all points with sign sequence σ by F_σ, i.e.,

$$F_\sigma = \{\mathbf{x} \in \mathbb{R}^n / \ell : \sigma(\mathbf{x}) = \sigma\}.$$

If F_σ is nonempty, we refer to it as a *face* of the arrangement $\mathcal{H}(n)$. (Note that if none of the entries of σ are zero, F_σ actually lies in the complement of $\mathcal{H}(n)$. Nonetheless we refer to F_σ as a face of the arrangement.)

For example, $\mathbf{x} = (-.5, 0, -.5, 1)$ has $\sigma(\mathbf{x}) = (+, 0, +, -, +, +)$, and so it is a point of the face $F_\sigma = F_{(+,0,+,-,+,+)}$ in $\mathcal{H}(4)$. (We order the pairs of coordinates, (i, j), lexicographically.) Similarly,

$$F_{(0,+,+)} = \{\mathbf{x} \in \mathbb{R}^3 / \ell : x_1 = x_2 < x_3\}$$

is a face of $\mathcal{H}(3)$.

5.4 Euclidean hyperplane arrangements

So far, we have said nothing particularly special about the braid arrangement. We have used this arrangement merely as a concrete example to illustrate how we might deal with any hyperplane arrangement. In general, we may suppose V is a Euclidean vector space with an inner product $\langle \cdot, \cdot \rangle$, and that $\mathcal{H}$ is a finite collection of hyperplanes in V, i.e., suppose

$$\mathcal{H} = \{H_1, \ldots, H_N\},$$

where each H_i is a co-dimension one linear subspace of V. We can choose a collection of normal vectors for each hyperplane, say β_i for H_i, so that each hyperplane gets an explicit description:

$$H_i = \{\lambda \in V : \langle \lambda, \beta_i \rangle = 0\}.$$

This choice of β_i also divides $V - H_i$ into a *positive halfspace*:

$$H_i^+ = \{\lambda \in V : \langle \lambda, \beta_i \rangle > 0\},$$

and a *negative halfspace*:

$$H_i^- = \{\lambda \in V : \langle \lambda, \beta_i \rangle < 0\},$$

(see Figure 5.4). Notice both β_i and $-\beta_i$ give the same hyperplane, but with reversed halfspaces since $\langle \lambda, -\beta_i \rangle = -\langle \lambda, \beta_i \rangle$. Thus in order for the notion of the sign vector of a point in V to be well defined, we need to specify our choice of normal vectors, not merely the hyperplanes. In the braid arrangement, the normal vectors we chose were all vectors of the form

$$\varepsilon_j - \varepsilon_i = (0, \ldots, 0, -1, 0, \ldots, 0, 1, 0, \ldots, 0),$$

where ε_i is a standard basis vector for $\mathbb{R}^n$.

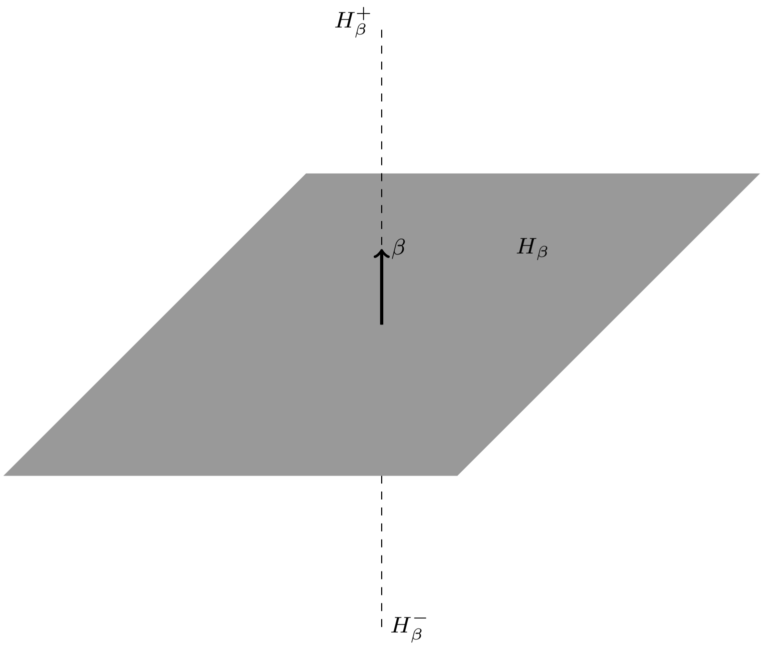


Fig. 5.4 The positive and negative halfspaces of a hyperplane.

By taking all points with common sign vector, we can define the *faces* of the arrangement to be intersections of hyperplanes and halfspaces:

$$F = \bigcap_{i=1}^{N} H_i^{\sigma_i(F)},$$

where we understand $H_i^0 = H_i$. Define the *support* of a face F to be the smallest linear subspace in which F is contained:

$$\operatorname{supp}(F) = \bigcap_{\sigma_i(F)=0} H_i.$$

This gives a convenient way to see the dimension of a face: $\dim F = \dim(\operatorname{supp}(F))$.

We now let $\Sigma = \Sigma(\mathcal{H})$ denote the set of all nonempty faces of $\mathcal{H}$. We can define a partial order on Σ by containment of their closures. That is, if F and G are faces of $\mathcal{H}$, we put $F \leq_\Sigma G$ if and only if $\overline{F} \subseteq \overline{G}$ in V. With our characterization of faces in terms of sign sequences, it turns out that this only depends on $\sigma(F)$ and $\sigma(G)$ as follows.

Proposition 5.2. *Two faces of $\mathcal{H}$ satisfy $F \leq_\Sigma G$ if and only if, for each i:*

$$\sigma_i(F) = \sigma_i(G) \quad \textit{or} \quad \sigma_i(F) = 0.$$

That is, we move up in the partial order by changing a zero entry to a nonzero entry, i.e., by stepping off of some hyperplane into a higher-dimensional cone. For example, in the braid arrangement $\Sigma(\mathcal{H}(n))$, if $\sigma(F) = (+,+,0,0,-,-)$ and $\sigma(G) = (+,+,0,+,-,-)$, then $F \leq_\Sigma G$, yet neither of these faces is comparable to the face with sign vector $(-,+,+,+,+,+)$.

We can see the partial order on faces of $\mathcal{H}(3)$ shown in Figure 5.5, where we list only the sign sequence identifying the face. Compare it with the picture of the braid arrangement in Figure 5.3.

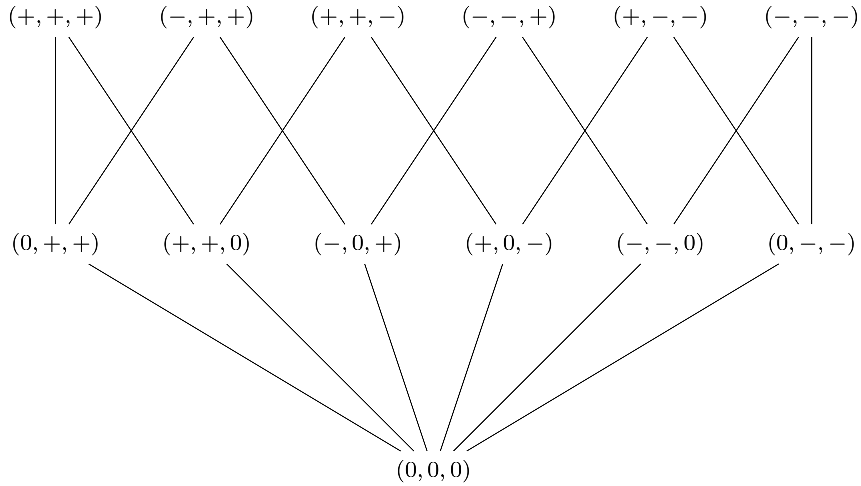


Fig. 5.5 The partial order on faces of braid arrangement $\mathcal{H}(3)$.

The face $(0, 0, \ldots, 0)$ is the unique minimum of the partial order. The maximal faces are called *chambers* and have all nonzero entries, i.e., they are the maximal-dimensional cones in the complement of $\mathcal{H}$. We can see that cover relations increase the dimension by one. Hence the rank function of the poset Σ is the dimension generating function.

Observation 5.2 *The rank generating function for the poset of faces of a hyperplane arrangement is the dimension generating function:*

$$f(\Sigma; t) = \sum_{F \in \Sigma} t^{\dim(F)}.$$

In the case of the braid arrangement, we can give an explicit combinatorial description of this rank function, as we will see in Section 5.6.

5.5 Products of faces and the weak order on chambers

We now describe further general properties of Σ. We define a product of faces as follows. Let F and G be two faces of Σ. Choose a point $\lambda \in F$ and a point $\mu \in G$, and consider the line $p(x) = (1-x)\lambda + x\mu$. For values of x in $0 < x < 1$, the line may cross various faces of Σ, and we define the *Tits product* of F and G, denoted FG, to be the first face encountered on this line segment. (The product is named for Jaques Tits, who used this product to give a geometric approach to an algebraic result of Louis Solomon. See [142, 163].) See Figure 5.6.

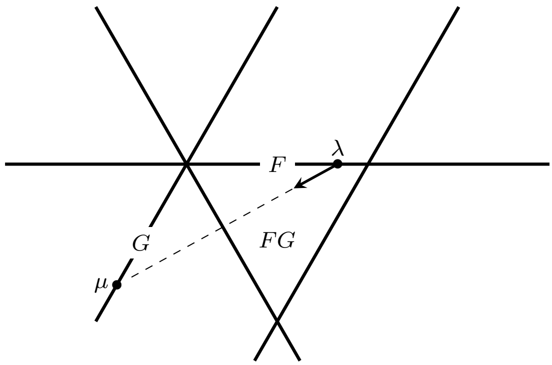


Fig. 5.6 The Tits product of two faces in a hyperplane arrangement.

Notice that

$$\langle \beta_i, p(x) \rangle = (1-x)\langle \beta_i, \lambda \rangle + x\langle \beta_i, \mu \rangle,$$

so if $0 < x < \epsilon$ is sufficiently small and $\sigma_i(F) \neq 0$

$$\sigma_i(p(x)) = \sigma_i(p(0)) = \sigma_i(F),$$

which is positive or negative according to the sign of $\langle \beta_i, \lambda \rangle$. However if $\sigma_i(F) = 0$, it is $\langle \beta_i, \mu \rangle$ whose sign is the same as the sign of $\langle \beta_i, p(x) \rangle$, i.e.,

$$\sigma_i(p(x)) = \sigma_i(G).$$

Hence the product of two faces can be given succinctly in terms of sign vectors.

Observation 5.3 *The Tits product of faces F and G is the face FG with the following sign vector:*

$$\sigma_i(FG) = \begin{cases} \sigma_i(G) & \text{if } \sigma_i(F) = 0, \\ \sigma_i(F) & \text{otherwise.} \end{cases}$$

The Tits product is associative: $(FG)H = F(GH)$, and it gives Σ the structure of a monoid, with identity $(0, 0, \ldots, 0)$. While the product is not commutative in general, commuting faces have nice properties, as we now describe.

Suppose F and G are distinct faces and $FG = GF$. This means F and G both lie strictly below $H = FG$ in Σ and moreover, H is the smallest face that contains both F and G on its boundary.

Now suppose I is another face that commutes with both F and G, i.e., $FI = IF$ and $GI = IG$. (Assume I is not equal to F, G, or H.) Then I and H form a commuting pair as well, since

$$IH = IFG = FIG = FGI = HI.$$

Thus the face $J = HI = IH$ is the smallest face containing the triple of pairwise commuting faces F, G, and I.

We can continue this line of reasoning for any number of pairwise commuting faces to obtain the following.

Proposition 5.3. *A collection of faces $F_1, \ldots, F_k$ lie on the boundary of a common face if and only if their pairwise products commute:*

$$F_iF_j = F_jF_i \text{ for all pairs } i, j.$$

Moreover, the smallest face that contains them all is the product $F_1F_2 \cdots F_k$.

We have mentioned that maximal faces in Σ are called chambers. Let $\mathcal{C}$ denote the set of chambers. The set $\mathcal{C}$ plays a special role in the study of hyperplane arrangements. For one thing $\mathcal{C}$ is an ideal in the monoid Σ, since if C is a chamber it has all nonzero entries and thus if F is any face,

$$CF = C \qquad \text{and} \qquad FC = C',$$

for some (possibly different) chamber C'.

Further, we can define a natural and important partial order on $\mathcal{C}$ as follows. By analogy with what we get in the case of the braid arrangement, we call this the *weak order* on the chambers, denoted $\mathrm{Wk}(\mathcal{C})$.

First, choose a chamber C_0 that we will call the *base chamber*, and choose the normal vectors β_i so that $\sigma(C_0) = (+, +, \ldots, +)$. Then $-C_0$ is also a chamber, with sign vector $(-, -, \ldots, -)$. We declare two chambers C_1, C_2 to be *adjacent* if the support of their intersection is a single hyperplane:

$$\mathrm{supp}(\overline{C}_1 \cap \overline{C}_2) = H,$$

and we sometimes refer to this hyperplane H as the *wall* between C_1 and C_2. It should be clear that the sign vectors of C_1 and C_2 differ only in the entry corresponding to H. Say that $\sigma_H(C_1) = +$ and $\sigma_H(C_2) = -$.

Then we partially order $\mathcal{C}$ by declaring that $C_1 <_{\mathrm{Wk}} C_2$ is a cover relation if and only if C_1 and C_2 are adjacent and $\sigma(C_1)$ has more positive entries than $\sigma(C_2)$. Define the *inversion set* of a chamber to be the set of negative entries in its sign vector:

$$\mathrm{Inv}(C) = \{i : \sigma_i(C) = -\},$$

and let $\mathrm{inv}(C) = |\,\mathrm{Inv}(C)|$ denote the number of inversions. Then taking the transitive closure of the cover relation described above leads to this simple characterization of the weak order on $\mathcal{C}$:

$$C_1 \leq_{\mathrm{Wk}} C_2 \qquad \text{if and only if} \qquad \mathrm{Inv}(C_1) \subseteq \mathrm{Inv}(C_2).$$

In Figure 5.7 we see the partial order on chambers of the braid arrangement $\mathcal{H}(3)$. This poset has minimum C_0 and maximum $-C_0$, and it is ranked by the number of minus signs in the sign vector, i.e., $\mathrm{inv}(w) = |\{i : \sigma_i(C) = -\}|$.

Observation 5.4 *The partial order on chambers has rank function*

$$f(\mathrm{Wk}(\mathcal{C}); q) = \sum_{C \in \mathcal{C}} q^{\mathrm{inv}(C)}.$$

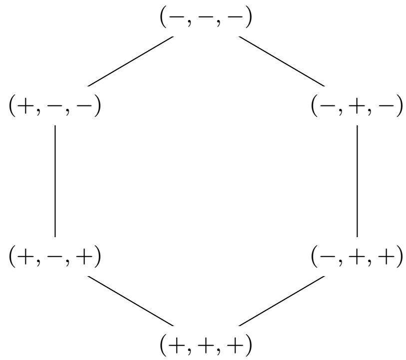


Fig. 5.7 The partial order on chambers in a hyperplane arrangement.

In the case of $\mathcal{H}(n)$, the chambers correspond to permutations and the partial order $\mathrm{Wk}(\mathcal{C})$ is isomorphic to the weak order $\mathrm{Wk}(S_n)$. Whether we get the left weak order or the right weak order depends on how we choose to label chambers. Refer to Figure 5.3.

5.6 Set compositions

We will now return to the discussion of the specific case of the braid arrangement and study its combinatorics in detail. In particular, we will enumerate all the faces of $\mathcal{H}(n)$ according to dimension.

Define a *set composition* of n to be a linearly ordered set partition of $\{1, 2, \ldots, n\}$, i.e., we say $F = B_1|B_2|\cdots|B_k$ is a set composition of n if $B_i \cap B_j = \emptyset$ for $i \neq j$ and $\bigcup B_i = \{1, 2, \ldots, n\}$. The linear ordering means, e.g., $12|34 \neq 34|12$. By convention, we will write the elements of each block B_i in increasing order. Let $\mathrm{Comp}(n)$ denote the set of all set compositions of the set $\{1, 2, \ldots, n\}$. We see the set compositions for $n = 3$ and $n = 4$ listed in Table 5.4.

The correspondence between $\mathrm{Comp}(n)$ and faces of $\mathcal{H}(n)$ is straightforward. If i and j are in the same block of set composition $F \in \mathrm{Comp}(n)$, then $x_i = x_j$, while if k is in a block to the left of l, then $x_k < x_l$. For example,

$$37|45|126 \leftrightarrow \{x_3 = x_7 < x_4 = x_5 < x_1 < x_2 < x_6\},$$

and

$$\{x_2 = x_3 < x_5 < x_1 = x_4 < x_6\} \leftrightarrow 23|5|14|6.$$

Notice the number of bars indicates the number of inequalities, and hence the dimension of the face of $\mathcal{H}(n)$.

The partial order on faces is easy to see in terms of the model of set compositions. We know that a face F is contained in a face G if, as set compositions, F is a refinement of G. In $\mathcal{H}(n)$, this amounts to breaking some equalities among the coordinates, while at the same time preserving all the inequalities of G, i.e., changing a zero to nonzero in the sign vector.

Proposition 5.4. *The poset of faces of* $\mathcal{H}(n)$ *is isomorphic to* $\mathrm{Comp}(n)$ *under refinement order.*

See Figure 5.8 for the faces of $\mathcal{H}(3)$ labeled by set compositions.

We now turn to the rank generating function for $\mathrm{Comp}(n)$, which, by Proposition 5.4, is the dimension generating function for the faces of $\mathcal{H}(n)$. For a given set composition $A = A_0|A_1|\cdots|A_k$, let $\mathrm{rk}(A) = k$ denote the number of vertical bars in A, i.e., its rank in the refinement order, i.e., the dimension of A in $\Sigma(\mathcal{H}(n))$. Then,

$$\begin{aligned} f(\mathrm{Comp}(n); t) &= \sum_{A \in \mathrm{Comp}(n)} t^{\mathrm{rk}(A)}, \\ &= \sum_{F \in \Sigma(\mathcal{H}(n))} t^{\dim(F)}, \\ &= f(\Sigma(\mathcal{H}(n)); t). \end{aligned}$$

We will let f_k denote the coefficient of t^k in $f(\mathrm{Comp}(n); t)$, i.e.,

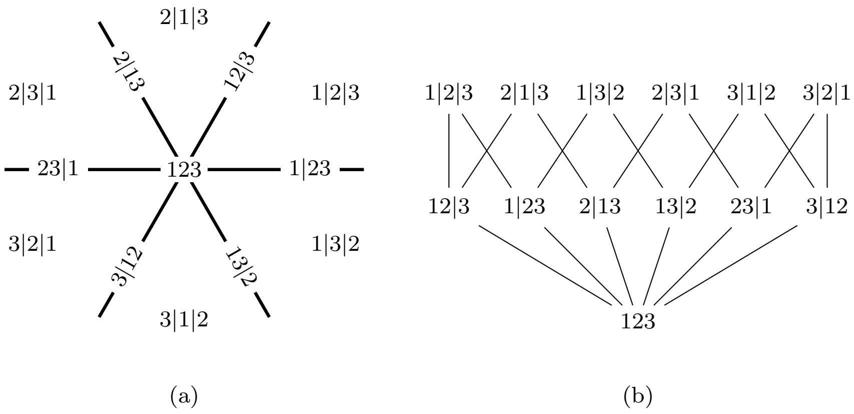


Fig. 5.8 (a) The braid arrangement $\mathcal{H}(3)$, and (b) its poset of faces, with faces now labeled by set compositions.

$$\begin{aligned} f_k &= |\{A \in \mathrm{Comp}(n) : \mathrm{rk}(A) = k\}|, \\ &= |\{\text{faces of dimension } k \text{ in } \Sigma(\mathcal{H}(n))\}|. \end{aligned}$$

For example,

$$f(\mathrm{Comp}(3); t) = 1 + 6t + 6t^2,$$

and

$$f(\mathrm{Comp}(4); t) = 1 + 14t + 36t^2 + 24t^3.$$

Table 5.3 has the number of set compositions of n with k blocks.

Table 5.3 The number of set compositions of n with k blocks.

$n\backslash k$	0	1	2	3	4	5	6	7	8	9
1	1									
2	1	2								
3	1	6	6							
4	1	14	36	24						
5	1	30	150	240	120					
6	1	62	540	1560	1800	720				
7	1	126	1806	8400	16800	15120	5040			
8	1	254	5796	40824	126000	191520	141120	40320,		
9	1	510	18150	186480	834120	1905120	2328480	1451520	362880	
10	1	1022	55980	818520	5103000	16435440	29635200	30240000	16329600	3628800

We will now find a formula for f_k in terms of n and k.

Table 5.4 Set compositions for $n = 3$ and $n = 4$, i.e., faces of $\mathcal{H}(n)$, grouped according to dimension and underlying permutation.

$w \in S_3$	dim = 0	dim = 1	dim = 2
123	123	1|23 12|3	1|2|3
132		13|2	1|3|2
213		2|13	2|1|3
231		23|1	2|3|1
312		3|12	3|1|2
321			3|2|1

$w \in S_4$	dim = 0	dim = 1	dim = 2	dim = 3
1234	1234	1|234 12|34 123|4	1|2|34 12|3|4 1|23|4	1|2|3|4
1243		124|3	1|24|3 12|4|3	1|2|4|3
1324		13|24	1|3|24 13|2|4	1|3|2|4
1342		134|2	1|34|2 13|4|2	1|3|4|2
1423		14|23	1|4|23 14|2|3	1|4|2|3
1432			14|3|2	1|4|3|2
2134		2|134	2|1|34 2|13|4	2|1|3|4
2143			2|14|3	2|1|4|3
2314		23|14	2|3|14 23|1|4	2|3|1|4
2341		234|1	2|34|1 23|4|1	2|3|4|1
2413		24|13	2|4|13 24|1|3	2|4|1|3
2431			24|3|1	2|4|3|1
3124		3|124	3|1|24 3|12|4	3|1|2|4
3142			3|14|2	3|1|4|2
3214			3|2|14	3|2|1|4
3241			3|24|1	3|2|4|1
3412		34|12	3|4|12 34|1|2	3|4|1|2
3421			34|2|1	3|4|2|1
4123		4|123	4|1|23 4|12|3	4|1|2|3
4132			4|13|2	4|1|3|2
4213			4|2|13	4|2|1|3
4231			4|23|1	4|2|3|1
4312			4|3|12	4|3|1|2
4321				4|3|2|1

Let $S(n,k)$ denote the number of ways to partition the set $\{1,2,\ldots,n\}$ into k nonempty blocks. These are the *Stirling numbers of the second kind* and are well known. (See Problem 3.3.) It is easily verified, for instance, that

$$S(n,k) = kS(n-1,k) + S(n-1,k-1),$$

since we can either add an nth element to any of k blocks in a partition counted by $S(n-1,k)$ or else create a new block containing only the new element along with any partition with $k-1$ blocks counted by $S(n-1,k-1)$. We list the number of set partitions with k blocks in Table 5.5.

Table 5.5 Stirling numbers of the second kind.

$S(n,k)$	1	2	3	4	5	6	7	8	9	10
1	1									
2	1	1								
3	1	3	1							
4	1	7	6	1						
5	1	15	25	10	1					
6	1	31	90	65	15	1				
7	1	63	301	350	140	21	1			
8	1	127	966	1701	1050	266	28	1		
9	1	255	3025	7770	6951	2646	462	36	1	
10	1	511	9330	34105	42525	22827	5880	750	45	1

Since set compositions are just ordered set partitions, the number of set compositions with k blocks is then $k!S(n,k)$. Hence, we have the following expression for $f(\mathrm{Comp}(n);t)$.

Theorem 5.2. *The number of set compositions of n with k blocks, i.e., the number of faces of dimension $k-1$ in the braid arrangement $\mathcal{H}(n)$, is*

$$f_{k-1} = k!S(n,k).$$

Thus,

$$f(\mathrm{Comp}(n);t) = \sum_{k=0}^{n-1}(k+1)!S(n,k+1)t^k.$$

Now notice that by ignoring the vertical bars, we can group set compositions according to the underlying permutation in S_n, where we recall that the numbers in each block are written in increasing order. For a set composition A, let $p(A)$ denote the permutation obtained in this way, e.g., $p(34|12|5) = 34125$. Conversely, for a permutation w, let $\mathrm{Comp}(w)$ denote the set of set compositions that map to w. That is,

$$\mathrm{Comp}(w) = \{A \in \mathrm{Comp}(n) : p(A) = w\}.$$

For example, if $w = 53214867$,

$$\begin{aligned}\mathrm{Comp}(w) = \{&5|3|2|148|67, 5|3|2|1|48|67, 5|3|2|14|8|67, 5|3|2|148|6|7,\\ &5|3|2|1|4|8|67, 5|3|2|1|48|6|7, 5|3|2|14|8|6|7, 5|3|2|1|4|8|6|7\}.\end{aligned} \tag{5.1}$$

With a moment's reflection, we can see that $\mathrm{Comp}(w)$ consists of all ways inserting bars between the letters of w such that there *must* be a bar in each descent position, since each block is written in increasing order. Hence,

$$|\,\mathrm{Comp}(w)| = 2^{n-1-\mathrm{des}(w)},$$

since there are a total of $n-1$ gaps between letters of w. Even better, we have

$$\sum_{A \in \mathrm{Comp}(w)} t^{\mathrm{rk}(A)} = t^{\mathrm{des}(w)}(1+t)^{n-1-\mathrm{des}(w)}. \tag{5.2}$$

For example, with $w = 53214867$ as in (5.1), we have $t^4(1+t)^3$, since the eight elements of $\mathrm{Comp}(w)$ are obtained from $5|3|2|148|67$ by inserting or not inserting bars between the 1 and the 4, the 4 and the 8, and the 6 and the 7:

$$\begin{array}{ccccccccccccccc} 5 & \Big| & 3 & \Big| & 2 & \Big| & 1 & \vdots & 4 & \vdots & 8 & \Big| & 6 & \vdots & 7 \\ & t & & t & & t & & (1+t) & & (1+t) & & t & & (1+t). & \end{array}$$

By summing Equation (5.2) over all permutations w in S_n, we get the following expression for $f(\mathrm{Comp}(n); t)$.

$$\begin{aligned} f(\mathrm{Comp}(n); t) &= \sum_{A \in \mathrm{Comp}(n)} t^{\mathrm{rk}(A)},\\ &= \sum_{w \in S_n} \sum_{A \in \mathrm{Comp}(w)} t^{\mathrm{rk}(A)},\\ &= \sum_{w \in S_n} t^{\mathrm{des}(w)}(1+t)^{n-1-\mathrm{des}(w)},\\ &= (1+t)^{n-1} \sum_{w \in S_n} \left(\frac{t}{1+t}\right)^{\mathrm{des}(w)}. \end{aligned}$$

Hence, we see that $f(\mathrm{Comp}(n); t)$ is just a transformation of the Eulerian polynomial!

Theorem 5.3. *The dimension generating function for the faces of the braid arrangement $\mathcal{H}(n)$ is expressed in terms of the Eulerian polynomial $S_n(t)$ as follows:*

$$f(\mathrm{Comp}(n); t) = (1+t)^{n-1} S_n(t/(1+t)).$$

5.7 The Tamari lattice

The Catalan analogue of the weak order is called the *Tamari lattice*, after Dov Tamari [162], though it was independently discovered by James Stasheff [156]. This lattice has enjoyed enormous popularity. Just as the weak order can be realized with the polytope known as the permutahedron, the Hasse diagram of the Tamari lattice is the one-skeleton of the *associahedron*. Realizing the associahedron in Euclidean space is not as simple as realizing the permutahedron. There are many different constructions and we will not attempt to describe them here. (See Problem 8.4 for a realization by Jean-Louis Loday [101]. See also the survey by Cesar Ceballos and Gunter Ziegler [46].) However, as we now describe, it is easy to see where the name of the polytope comes from.

Consider all valid parenthesizations of $wxyz$, the associative product of four elements. They are:

$$(((wx)y)z), ((wx)(yz)), ((w(xy))z), (w((xy)z)), (w(x(yz))).$$

These parenthesizations can be given a partial order by declaring that cover relations are of the form $((fg)h) < (f(gh))$ for any sub-expressions f, g, and h. Recall from Section 2.5 that parenthesizations of $n+1$ elements are naturally encoded by planar binary trees with n internal nodes and $n+1$ leaves. Then this gives us a partial order on $\mathrm{PB}(n)$.

We will take this partial order on $\mathrm{PB}(n)$ as our definition of the Tamari lattice. For example, we see PB(3) in Figure 5.9 and PB(4) in Figure 5.10.

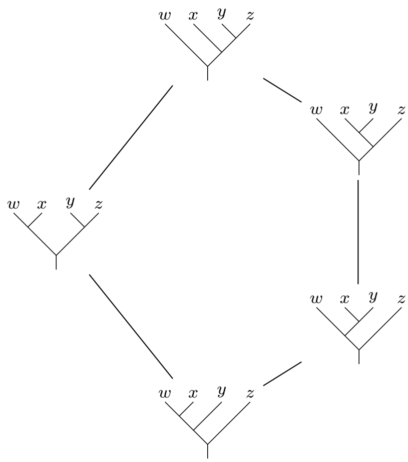


Fig. 5.9 The Tamari lattice for PB(3).

A quick observation confirms that the Tamari lattice is not ranked, so there is no rank function to discuss. The Tamari lattice can be obtained from

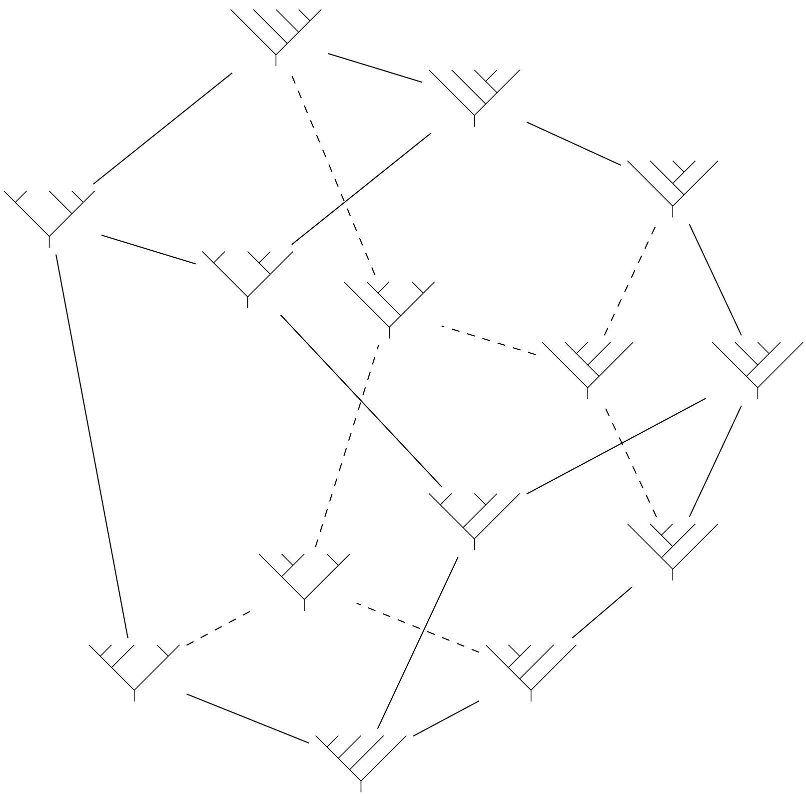


Fig. 5.10 The Tamari lattice for PB(4).

the weak order $\mathrm{Wk}(S_n)$ in a couple of ways. From a geometric standpoint, one can deform the permutahedron to get the associahedron. (In fact this is one of a family of such polytopes called "generalized permutahedra" constructed by Alexander Postnikov [121].) In terms of the dual picture of the braid arrangement, we can think of coarsening the arrangement by removing walls between certain chambers (what remains is a "fan" but no longer a hyperplane arrangement). See Figure 5.11. In purely poset-theoretic terms, the Tamari lattice can be realized by restricting the weak order to 231-avoiding permutations. For example, see Figure 5.12.

Let $\mathrm{Wk}(S_n(231))$ denote the set of 231-avoiding permutations endowed with the weak order. The bijection between $\mathrm{PB}(n)$ and $S_n(231)$ given in Section 2.5 is a poset isomorphism from the left weak order $(\mathrm{Wk}^l(S_n(231)), \leq)$ to $(\mathrm{PB}(n), \leq)$. Problem 5.11 asks for a bijection that makes the Tamari lattice isomorphic to the right weak order, as suggested by comparing Figure 5.10 and Figure 5.12.

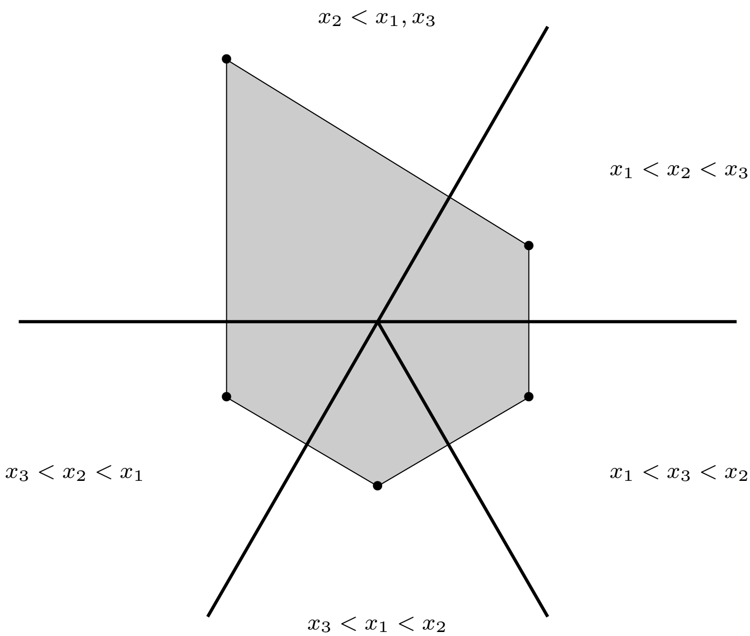


Fig. 5.11 A coarsening of the braid arrangement $\mathcal{H}(3)$ whose weak order is the Tamari lattice.

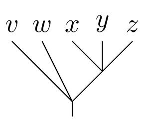
Proposition 5.5. *The Tamari lattice* $(\mathrm{PB}(n), \leq)$ *is isomorphic to* $(\mathrm{Wk}(S_n(231)), \leq)$*, the set of* 231*-avoiding permutations under the weak order.*

5.8 Rooted planar trees and faces of the associahedron

Just as we saw the Eulerian numbers emerge in counting faces of the permutahedron, we will now see the Narayana numbers crop up while enumerating the faces of the associahedron.

The combinatorial model we will develop is parenthesizations of $n+1$ elements, or rooted planar trees with $n+1$ leaves. The total number of such trees is given the small Schröder number r_n discussed in Problem 2.14.

The correspondence between parenthesizations and trees is straightforward, just as it was for the special case of planar binary trees and complete parenthesizations, e.g.,



$\longleftrightarrow (vw(xyz))$.

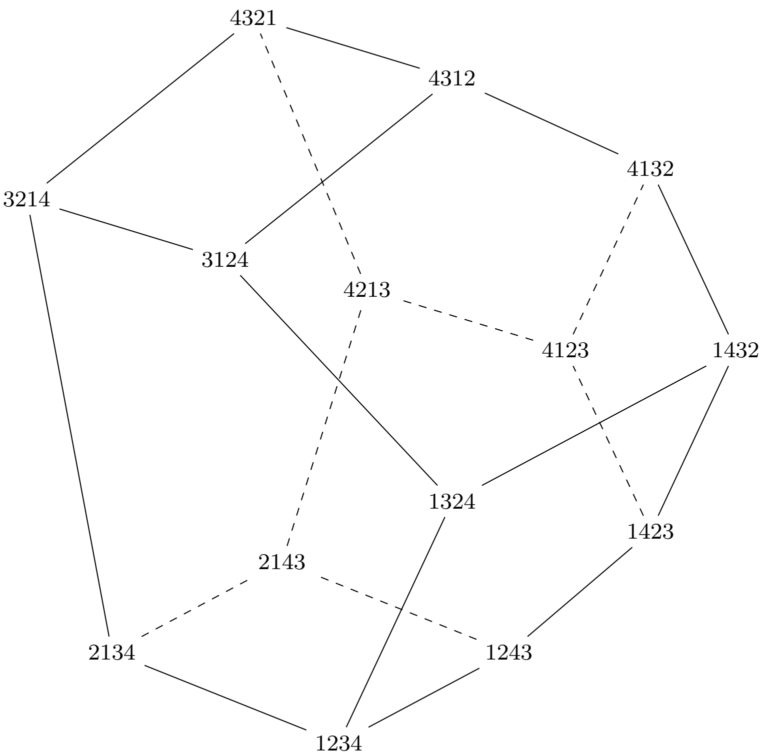
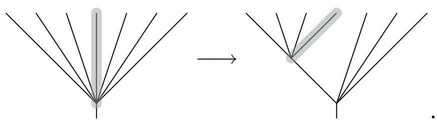
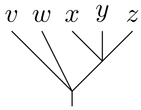


Fig. 5.12 The Tamari lattice given by the weak order on $S_4(231)$.

Let $\mathcal{P}(n)$ denote the set of rooted planar trees with $n+1$ leaves. We will define a partial order on $\mathcal{P}(n)$ that encodes the (reverse) inclusion of faces of the associahedron. See Figure 5.13(a) for the $n = 3$ case. For any pair of trees $\sigma, \tau \in \mathcal{P}(n)$, we say $\sigma \leq \tau$ if τ refines σ as a parenthesization. In terms of the pictures, this means that we can choose some internal nodes of σ with more than two branches and slide some of the branches up and away to the left or the right. By a "branch," we mean a line segment from a leaf to some internal node. (It helps to think of the branches as being anchored at the leaves.) We cannot violate planarity, so if we want to slide the fourth branch to the left, say, then branches one, two, and three must come along for the ride:

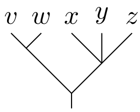


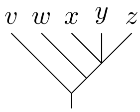
A cover relation is given by choosing only one such refinement, creating exactly one new internal node. For example, one can check that $\sigma = (vw(xyz))$, i.e.,



,

has four covers: $((vw)(xyz))$, $(v(w(xyz)))$, $(vw((xy)z))$, and $(vw(x(yz)))$, drawn as:



,


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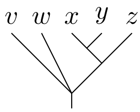
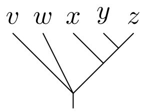


, and


.

It is easy to see that the tree with only one internal node is the unique minimum for this partial order, and that the poset is ranked by one less than the number of internal nodes. Hence, the planar binary trees, with n internal nodes, are maximal. See Figure 5.13(b).

Let $i(\sigma)$ denote the number of internal nodes of $\sigma \in \mathcal{P}(n)$. Then the rank generating function for $\mathcal{P}(n)$ (i.e., the codimension generating function for the faces of the associahedron) is

$$f(\mathcal{P}(n);t) = \sum_{\sigma \in \mathcal{P}(n)} t^{i(\sigma)-1}.$$

For example,

$$f(\mathcal{P}(3);t) = 1 + 5t + 5t^2,$$

and

$$f(\mathcal{P}(4);t) = 1 + 9t + 21t^2 + 14t^3.$$

The coefficient of t^k in $f(\mathcal{P}(n);t)$, i.e., the number of planar rooted trees with $n+1$ leaves and $k+1$ internal nodes, is given in Table 5.6.

Table 5.6 The number of planar rooted trees with $n+1$ leaves and $k+1$ internal nodes.

$n \backslash k$	0	1	2	3	4	5	6	7	8	9
1	1									
2	1	2								
3	1	5	5							
4	1	9	21	14						
5	1	14	56	84	42					
6	1	20	120	300	330	132				
7	1	27	225	825	1485	1287	429			
8	1	35	385	1925	5005	7007	5005	1430		
9	1	44	616	4004	14014	28028	32032	19448	4862	
10	1	54	936	7644	34398	91728	148512	143208	75582	16796

To any rooted tree, we can assign a canonical planar binary tree as follows. Each internal node is the endpoint for at most three kinds of branches: *left*, *right*, and *middle*. There can be several middle branches, but only one left branch and only one right branch. (We think of them from the point of view of the internal node. Planarity guarantees that these notions are well defined.) Given a tree σ, let $b(\sigma)$ denote the binary tree formed by sliding all middle

(a):

(b):

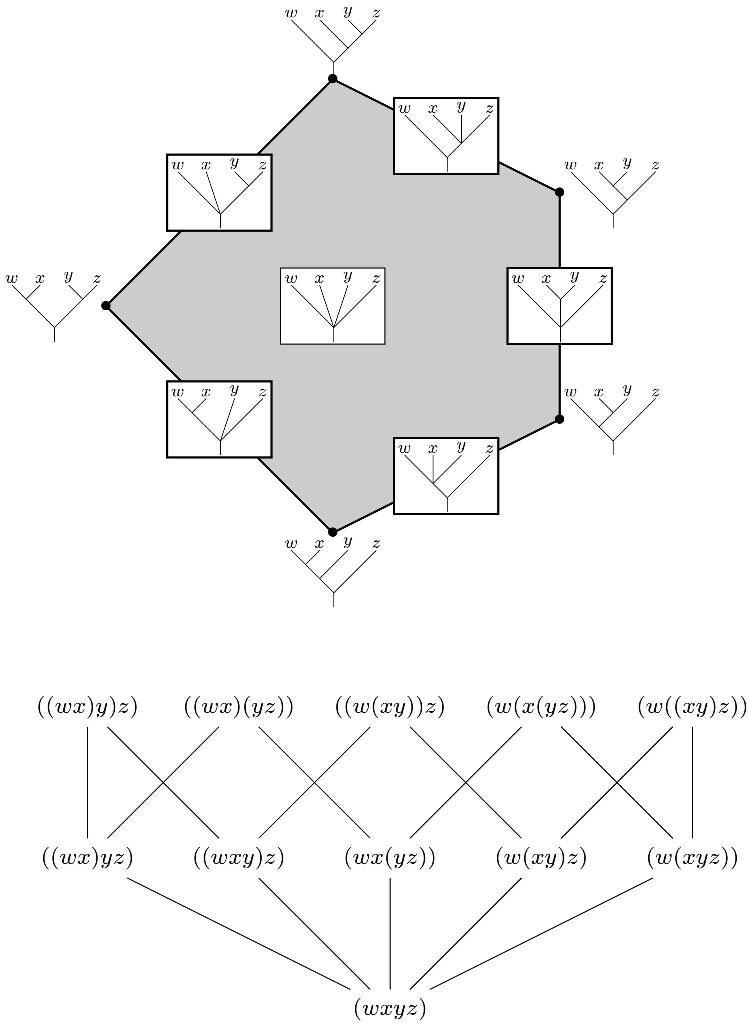
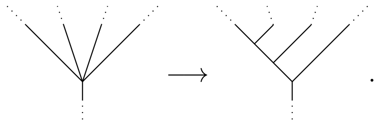


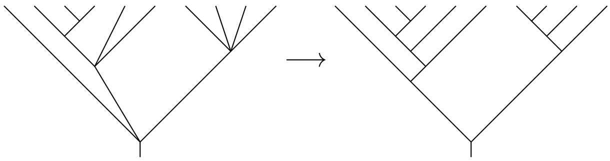
Fig. 5.13 (a) Faces of the associahedron labeled with rooted planar trees, and (b) the poset of parenthesizations.

branches one at a time to the left, so that each becomes the right branch for a new internal node:



In terms of parenthesizations, this means every substring of a parenthesization, $\cdots(fgh)\cdots$, with subexpressions f, g, and h, is refined to $\cdots((fg)h)\cdots$. Among all binary trees $\tau \in \mathrm{PB}(n)$ that are above the tree $\sigma \in \mathcal{P}(n)$ in the refinement order, this identifies the one that is minimal with respect to the Tamari lattice, since it has the fewest left-pointing leaves. See Problem 5.12.

For example,

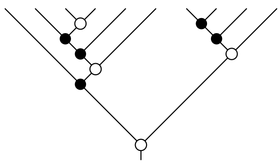
$\sigma =$  $= b(\sigma)$.

On the other hand, given a binary tree $\tau \in \mathrm{PB}(n)$, let $\mathcal{P}(\tau)$ denote the set of planar trees that project to τ, i.e.,

$$\mathcal{P}(\tau) = \{\sigma \in \mathcal{P}(n) : b(\sigma) = \tau\}.$$

In Table 5.7 we see the parenthesizations for $n = 4$ grouped according to this map.

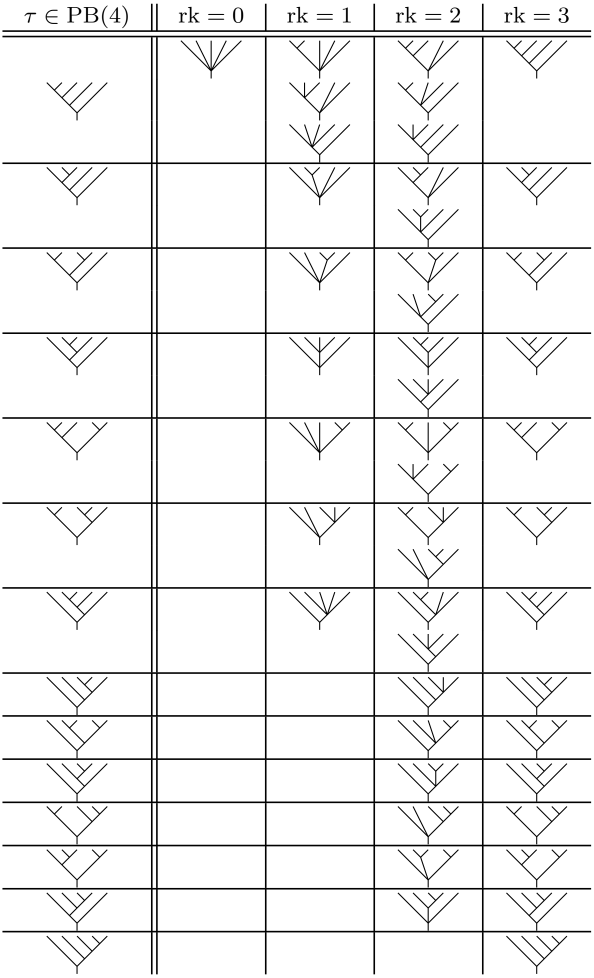
For any $\tau \in \mathrm{PB}(n)$, there is a unique internal node just above the root, which we call the *ground node*. The left branch and the right branch of the ground node are in every planar tree. We see that every other internal node lies somewhere along a left branch or a right branch whose endpoint is closer to the ground node. If an internal node lies on a left branch, it is the endpoint for a branch that terminates at a right-pointing leaf and we color it black. If it lies on a right branch, it is the endpoint for some left-pointing leaf and we color it white. See the picture below (the ground node is white):



With this color scheme, we can now see that the elements of $\mathcal{P}(\tau)$ are those elements of $\mathcal{P}(n)$ obtained by allowing the black internal nodes to merge with nodes just below them in the tree. Each black node can be chosen independently to merge or not, so the number of trees in $\mathcal{P}(\tau)$ is

$$2^{(\#\text{black nodes of } \tau)}.$$

Table 5.7 Rooted planar trees, grouped according to the map onto planar binary trees.



But we can get a more refined count than this, since each time a black node merges with the node below it, the overall number of internal nodes decreases by one. Since the white nodes remain in every tree, we have:

$$\sum_{\sigma \in \mathcal{P}(\tau)} t^{i(\sigma)} = t^{(\#\text{white nodes of } \tau)}(1+t)^{(\#\text{black nodes of } \tau)}. \tag{5.3}$$

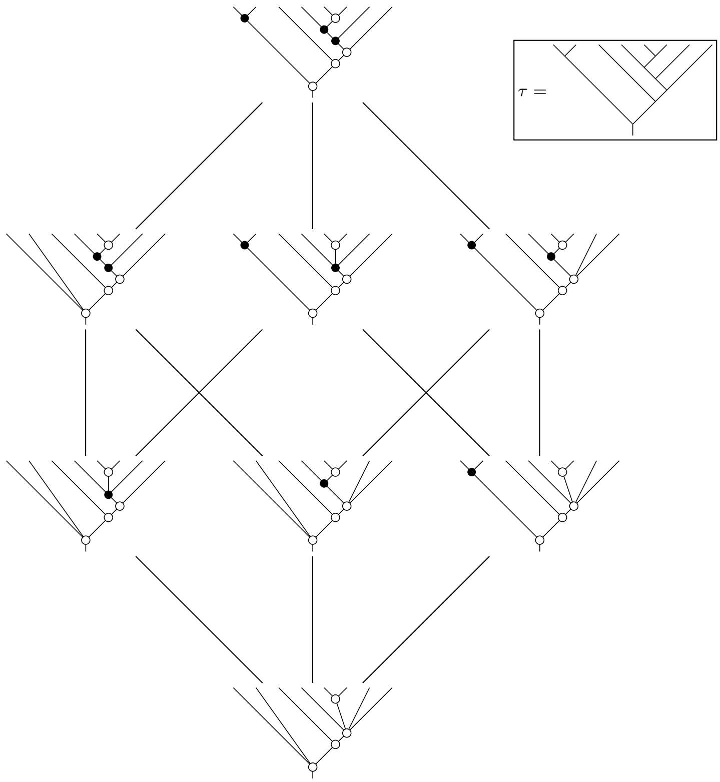


Fig. 5.14 The eight trees in $\mathcal{P}(\tau)$, ordered by refinement.

For example, in Figure 5.14 we see the trees in $\mathcal{P}(\tau)$ for a certain tree τ in PB(7) with three black nodes and four white nodes. We have drawn the trees in refinement order so that it is easy to see the distribution of internal nodes is $t^4(1+t)^3$.

If $\tau \in \mathrm{PB}(n)$, the number of white nodes is equal to the number of left leaves at the top of the tree, as each leaf can be traced back to the internal node at which its branch originates. Moreover, since there are n internal nodes, this means:

$$\begin{aligned} n &= (\#\text{black nodes in } \tau) + (\#\text{white nodes in } \tau) \\ &= (\#\text{black nodes in } \tau) + (\#\text{left leaves in } \tau). \end{aligned}$$

Hence if τ has k left leaves, it has $n-k$ black nodes. Thus Equation (5.3) becomes

$$\sum_{\sigma \in \mathcal{P}(\tau)} t^{i(\sigma)} = t^k(1+t)^{n-k}. \tag{5.4}$$

Letting $l(\tau)$ denote the number of left leaves in a planar binary tree, and summing over all trees, we have the following expression for $f(\mathcal{P}(n);t)$.

$$\begin{aligned} f(\mathcal{P}(n);t) &= \sum_{\sigma\in\mathcal{P}(n)} t^{i(\sigma)-1}, \\ &= \sum_{\tau\in\mathrm{PB}(n)} \sum_{\sigma\in\mathcal{P}(\tau)} t^{i(\sigma)-1}, \\ &= \sum_{\tau\in\mathrm{PB}(n)} t^{l(\tau)-1}(1+t)^{n-l(\tau)}, \\ &= (1+t)^{n-1} \sum_{\tau\in\mathrm{PB}(n)} \left(\frac{t}{1+t}\right)^{l(\tau)-1}. \end{aligned}$$

By Proposition 2.2, we know that counting planar binary trees by left leaves corresponds to counting 231-avoiding permutations by descents. That is, $f(\mathcal{P}(n);t)$ is a transformation of the Narayana polynomial.

Theorem 5.4. *The dimension generating function for the faces of the associahedron is expressed in terms of the Narayana polynomial $C_n(t)$ as follows:*

$$f(\mathcal{P}(n);t) = (1+t)^{n-1}C_n(t/(1+t)).$$

Compare this result with Theorem 5.3.

Notes

The distribution of inversions (the Mahonian distribution) given in Theorem 5.1 is due to Olinde Rodrigues in 1839 [131]. According to Günter Ziegler [169], the permutahedron was probably first studied by Pieter Schoute—it appears in Schoute's 1911 book [133]. The weak order on permutations has been studied since at least the early 1960s, and the connection with the permutahedron seems to have been known from the beginning. Early interest in the weak order seems to have come from computer scientists, who were interested in its help for the theory of sorting. See, for example, the paper of Georges Guilbaud and Pierre Rosenstiehl [85]. The absolute order of Section 3.5 can also be related to sorting—see Problem 5.6.

Our presentation of hyperplane arrangements and the use of sign sequences to characterize faces is mostly adapted from the book by Peter Abramenko and Ken Brown [1, Section 1.4]. The Tits product for faces of the braid arrangement (and closely related arrangements) first appears in the work of Jaques Tits from 1976 [163]. It was studied at a greater level of generality in the 1997 PhD thesis of Thomas Patrick Bidigare [17], and subsequently this concept found interesting applications to the study of random walks and card shuffling [18, 19]. See Problem 5.9.

The Tamari lattice is named for Dov Tamari, who studied it at least as early as 1962 [162], though James Stasheff also wrote a paper describing this poset in 1963 [156]. Whether the Tamari lattice could be realized as the one-skeleton a convex polytope was an open question, though it is folklore that John Milnor did as much in the 1960s. Unpublished notes of Mark Haiman from 1984 contain the first definitive construction, and Carl Lee [99] has the first construction to appear in print. Subsequently, many distinct realizations have appeared, as discussed by Cesar Ceballos and Günter Ziegler [46].

Problems

5.1. Prove that a permutation is uniquely determined by its inversion set.

5.2. Take the definition of the right weak order to be $u \leq_{\mathrm{Wk}^r} v$ if and only if $\mathrm{Inv}(u^{-1}) \subseteq \mathrm{Inv}(v^{-1})$. Prove that cover relations are given by right multiplication of simple generators: $v <_{\mathrm{Wk}^r} v \circ s_i$.

5.3. Prove the shard intersection order is a coarsening of the (right) weak order. That is, if $u \leq_{\mathrm{Sh}} v$, then $u \leq_{\mathrm{Wk}^r} v$.

5.4. Show that $I_n(q)$ is log-concave but not real-rooted.

5.5 (Length of a permutation). Define the *adjacent sorting length* of a permutation w, denoted $\ell(w)$, to be the minimal number of adjacent swaps needed to sort the permutation. That is, if $s_i = (i, i+1)$ is the transposition that swaps i and $i+1$,

$$\ell(w) = \min\{k : w \circ s_{i_1} \circ \cdots \circ s_{i_k} = 12 \cdots n\}.$$

For example, $\ell(3142) = 3$ since

$$3142 \circ s_3 \circ s_1 \circ s_2 = 3124 \circ s_1 \circ s_2 = 1324 \circ s_2 = 1234,$$

and no shorter sequence of swaps will do the same.

Show that $\ell(w) = \mathrm{inv}(w)$, i.e., the minimal number of adjacent swaps needed to sort a permutation equals the number of inversions.

Conclude that the weak order $\mathrm{Wk}(S_n)$ is ranked by adjacent sorting length, i.e.,

$$f(\mathrm{Wk}(S_n); q) = \sum_{w \in S_n} q^{\ell(w)}.$$

5.6 (Absolute length). Define the *absolute sorting length* of a permutation w, denoted $\ell'(w)$, to be the minimal number of swaps needed to sort a permutation. That is, if $t_{i,j} = (i,j)$ is the transposition that swaps i and j,

$$\ell'(w) = \min\{k : w \circ t_{i_1,j_1} \circ \cdots \circ t_{i_k,j_k} = 12\cdots n\}.$$

For example, $\ell'(31542) = 3$ since

$$31542 \circ t_{3,5} \circ t_{1,3} \circ t_{1,2} = 31245 \circ t_{1,3} \circ t_{1,2} = 21345 \circ t_{1,2} = 12345,$$

and no shorter sequence of swaps will do the same. (By default we write transpositions with $i < j$.)

Show that for $w \in S_n$, $\ell'(w) = n - \text{cyc}(w)$, where $\text{cyc}(w)$ denotes the number of cycles of w. For example $\text{cyc}(31542) = 2$, since we can write 31542 in cycle notation as $(1352)(4)$, and $n - \text{cyc}(31542) = 5 - 2 = 3 = \ell'(31542)$.

Conclude that the absolute order $\text{Abs}(S_n)$ from Section 3.5 is ranked by absolute sorting length, i.e.,

$$f(\text{Abs}(S_n); t) = \sum_{w \in S_n} t^{\ell'(w)}.$$

5.7 (Sorting index). The following is a greedy algorithm for sorting a permutation with transpositions: find the largest element that is out of place, move it to its proper place, and repeat.

More precisely, if $w(n) = n$, do nothing and move on to sort $w(1)\cdots w(n-1)$.

If $w(i) = n$, with $i < n$, apply the transposition $t_{i,n}$ to get $w' = w \circ t_{i,n}$. Then $w'(n) = n$, and we can now sort $w'(1)\cdots w'(n-1)$.

For example, here is the algorithm applied to the permutation $w = 3172546$:

$$3172546 \xrightarrow{t_{3,7}} 3162547 \xrightarrow{t_{3,6}} 3142567 \xrightarrow{t_{3,4}} 3124567 \xrightarrow{t_{1,3}} 2134567 \xrightarrow{t_{1,2}} 1234567$$

This algorithm is known as *straight selection sort.* Suppose $t_{i_1,j_1}, \ldots, t_{i_k,j_k}$ are the swaps used in straight selection sort for some w. Define the *sorting index* for w to be

$$\text{sor}(w) = \sum_{r=0}^{k} (j_r - i_r),$$

e.g., with $w = 3172546$ above, we get

$$\text{sor}(3172546) = (7-3) + (6-3) + (4-3) + (3-1) + (2-1) = 11.$$

Informally, the sorting index measures the “cost” of straight selection sort, with transpositions of elements that are far away costing more than elements closer by.

Show that

$$\sum_{w\in S_n} q^{\mathrm{sor}(w)} = [n]! = I_n(q).$$

That is, the sorting index is Mahonian. (Note, however, that it is not generally true that $\mathrm{sor}(w) = \mathrm{inv}(w)$, e.g., $\mathrm{sor}(3172546) = 11$ and $\mathrm{inv}(3172546) = 7$.)

5.8. Using the model of set compositions, let $F = B_1|\cdots|B_k$ and $G = C_1|\cdots|C_l$ be two faces of the braid arrangement $\mathcal{H}(n)$. Show that the Tits product of F and G has its blocks given by all nonempty pairwise intersections of blocks, $B_i \cap C_j$, ordered lexicographically, i.e.,

$$FG = B_1\cap C_1|\cdots|B_1\cap C_l|\cdots|B_k\cap C_1|\cdots|B_k\cap C_l,$$

ignoring empty blocks.

For example, if $F = 13|245|67$ and $G = 7|123|46|5$, then

$$FG = 13|2|4|5|7|6.$$

5.9 (Hyperplane walks and shuffling). Recall that a permutation $w \in S_n$ corresponds to the unique chamber $C(w) = w(1)|w(2)|\cdots|w(n)$, e.g., $w = 32145 \leftrightarrow C(w) = 3|2|1|4|5$.

Let R denote the set of rays in the braid arrangement, which in terms of set compositions are merely those set compositions with exactly two parts. Then $F \in R$ means $F = S|S'$, with S a proper, nonempty subset of $\{1, 2, \ldots, n\}$, and $S' = \{1, 2, \ldots, n\} - S$ the complement of S.

1. **Walking from rays.** Given permutations $u, v \in S_n$, write $u \xrightarrow{F} v$ if $FC(u) = C(v)$, i.e., if the Tits product of F with the chamber for u equals the chamber for v. Let $M(u, v) = |\{F \in R : u \xrightarrow{F} v\}|$, and let

$$\mathcal{M} = [M(u, v)]_{u,v\in S_n},$$

denote the matrix containing these numbers, with rows and columns indexed by permutations. Calculate the matrix $\mathcal{M}$ for $n = 3$ and $n = 4$.

2. **Riffle shuffles.** Given a permutation $v = v(1)\cdots v(n)$ and an integer $k = 1, \ldots, n-1$, let $P = P(v; k)$ denote the disjoint union of the two chains $v(1) <_P \cdots <_P v(k)$ and $v(k+1) <_P \cdots <_P v(n)$. A *riffle shuffle of* v is any permutation u contained in the set of linear extensions $\mathcal{L}(P(v; k))$ for some k. A particular riffle shuffle u may appear in $\mathcal{L}(P(v; k))$ for more than one k, and so denote by $N(v, u)$ the number of ways to obtain u from a riffle shuffle of v, i.e.,

$$N(v, u) = |\{k : u \in \mathcal{L}(P(v; k))\}|.$$

Let us denote the matrix of these numbers by

$$\mathcal{N} = [N(v, u)]_{v,u\in S_n}.$$

Calculate the matrix $\mathcal{N}$ for $n = 3$ and $n = 4$.

3. Show that $|M(u, v)| = |N(v, u)|$, and conclude that $\mathcal{M}$ and $\mathcal{N}$ are transposes of one another.

5.10. Suppose f is a polynomial of degree at most $n - 1$ with all real roots and $f(1) \neq 0$. Show that the transformation

$$f(t) \mapsto (1 + t)^{n-1} f(t/(1 + t)),$$

where $\deg f = n-1$, preserves real-rootedness and conclude that $f(\Sigma(\mathcal{H}(n)); t)$ is real-rooted from the real-rootedness of $S_n(t)$.

Further, show that the operation $\sum a_k t^k \mapsto \sum \frac{a_k}{k!} t^k$ preserves real-rootedness and conclude that the *Stirling polynomials*,

$$\mathrm{Stir}_n(t) = \sum S(n, k) t^k,$$

are real-rooted.

5.11. Describe a bijection $\mathrm{PB}(n) \leftrightarrow S_n(231)$ that makes the Tamari lattice isomorphic to $\mathrm{Wk}^r(S_n(231))$, the right weak order on 231-avoiding permutations.

5.12. Show that the set of planar binary trees that cover a planar tree τ form an interval in the Tamari lattice.

5.13. Recall r_n, the *small Schröder number* is the number of parenthesizations of $n + 1$ symbols, or the number of planar trees with $n + 1$ nodes.

1. Show $r_n = C_n(2)$, where $C_n(t)$ is the Narayana polynomial, e.g., $r_4 = C_4(2) = 1 + 6 \cdot 2 + 6 \cdot 2^2 + 2^3 = 45$.
2. Further, let
$$r_n(t) = \sum_{\sigma \in \mathcal{P}(n)} t^{i(\sigma)-1}.$$
Is $r_n(t)$ real-rooted? Log-concave?

Chapter 6
Refined enumeration

In this chapter we return to purely enumerative questions. Often, the way we count allows us to keep track of more than one permutation statistic without any extra effort. In particular we consider various ways to pair a statistic with an Eulerian distribution with another statistic having a Mahonian distribution. Similar ideas are explored for Catalan objects.

6.1 The idea of a q-analogue

Recall that in Theorem 5.1 we showed that if we count permutations according to the number of inversions, we get a polynomial in q that generalizes the number $n!$, i.e.,

$$\sum_{w \in S_n} q^{\mathrm{inv}(w)} = [n]_q! = (1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1}). \tag{6.1}$$

A formula such as (6.1) is sometimes called a *q-analogue*. This term means different things in different contexts. For us it will mean that we take a classic enumerative result, in this case, the fact that there are $n!$ permutations, and replace the integers in the formula with the "q-integers" $[i]_q = 1+q+\cdots+q^{i-1}$. When the symbol q is understood and there is no worry of confusion, we will write $[i]$ for brevity. Another example of a q-analogue we have already seen comes from the binomial theorem, where we can replace the formula for the number of subsets of $\{1, 2, \ldots, n\}$, 2^n, with $[2]^n = (1+q)^n$. Such a refinement replaces a simple enumeration of a set of combinatorial objects with a generating function for some statistic on that set, e.g., inversions of permutations, cardinality of subsets.

T.K. Petersen, *Eulerian Numbers*, Birkhäuser Advanced Texts Basler Lehrbücher, DOI 10.1007/978-1-4939-3091-3_6

Often the same q-analogue will have more than one combinatorial interpretation. For example, $[n]!$ also counts permutations according to *major index*. The major index of a permutation $w = w(1)\cdots w(n)$ is

$$\mathrm{maj}(w) = \sum_{i \in \mathrm{Des}(w)} i.$$

For example, $\mathrm{Des}(531264) = \{1, 2, 5\}$, so $\mathrm{maj}(531264) = 1 + 2 + 5 = 8$. We can see that major index has the same distribution as (6.1) by thinking about how major index is affected when recursively constructing a permutation.

If we have a permutation v in S_{n-1} with $\mathrm{maj}(v) = k$, then placing n at the far right clearly leaves the major index unchanged, but at first glance, other insertions have a less obvious effect on major index. For example, placing n into slot $n-1$ will raise the major index by one if $v(n-2) > v(n-1)$ was already a descent, but it will raise the major index by $n-1$ if $v(n-2) < v(n-1)$ was not a descent.

It turns out that if we want to insert n into v so that major index increases by $0, 1, \ldots, n-1$, the following insertion process will do the trick: first move from right to left placing n in descent positions, then move from left to right placing n in ascent positions. We think of the far right position as a descent position and the far left position as an ascent position. For example, consider the permutation $v = 34|17|6|258$, where we have marked the descent positions with bars for visual clarity. We have $\mathrm{maj}(v) = 2+4+5 = 11$, and here are the permutations in S_9 obtained by inserting $n = 9$ into v in all possible ways:

w	$\mathrm{maj}(w)$
$34\|17\|6\|258\mathbf{9}$	11
$34\|17\|6\mathbf{9}\|258$	12
$34\|17\mathbf{9}\|6\|258$	13
$34\mathbf{9}\|17\|6\|258$	14
$\mathbf{9}34\|17\|6\|258$	15
$3\mathbf{9}4\|17\|6\|258$	16
$34\|1\mathbf{9}7\|6\|258$	17
$34\|17\|6\|2\mathbf{9}58$	18
$34\|17\|6\|25\mathbf{9}8$	19

.

Thus we see that the distribution of major index grows by a factor of $[n] = 1+q+\cdots+q^{n-1}$ when moving from S_{n-1} to S_n. So by induction major index is a Mahonian statistic just like inversion number.

Theorem 6.1. *For any $n \geq 1$,*

$$\sum_{w \in S_n} q^{\mathrm{maj}(w)} = [n]! = \sum_{w \in S_n} q^{\mathrm{inv}(w)}.$$

Problem 6.1 asks for a direct bijective proof of this theorem.

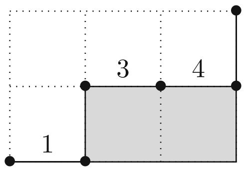
6.2 Lattice paths by area

We will now describe another common q-analogue, this one for the binomial coefficient $\binom{n}{k}$. Define

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!}, \tag{6.2}$$

where $[0]! = 1$. While it is not immediately obvious, this expression, while a priori a rational function of q, is in fact a polynomial with positive integer coefficients.

A beautiful combinatorial meaning for this polynomial is given as follows. Write $n = k+l$, and let $L(k,l)$ denote the set of lattice paths p from $(0,0)$ to (k,l) that take only steps East, from (i,j) to $(i+1,j)$, and North, from (i,j) to $(i,j+1)$. There are $n = k+l$ steps in such a path, and we can identify each path with a subset A of $\{1,2,\dots,n\}$ by identifying which steps are East. For example, the set $A = \{1,3,4\} \subset \{1,2,3,4,5\}$ corresponds to the path

$p =$ 

The area underneath such a path will be denoted $\mathrm{area}(p)$, so we can see in this example that $\mathrm{area}(p) = 2$. The claim is that $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]$ counts all such paths according to area.

For example,

$$\begin{bmatrix} 5 \\ 2 \end{bmatrix} = 1 + q + 2q^2 + 2q^3 + 2q^4 + q^5 + q^6,$$

and in Table 6.1 we see the ten lattice paths from $(0,0)$ to $(2,3)$ grouped according to area.

Let $a_{n,k}(q)$ denote the generating function for paths in $L(k,n-k)$ according to area, i.e.,

$$a_{n,k}(q) = \sum_{p \in L(k,n-k)} q^{\mathrm{area}(p)}.$$

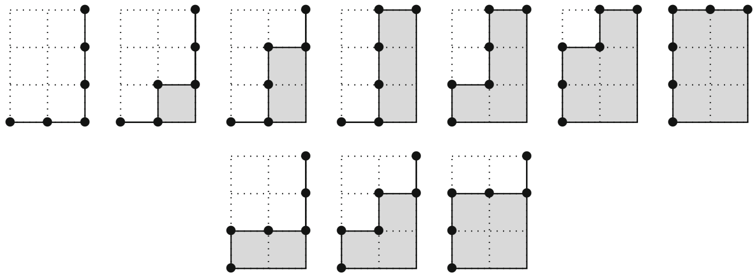
Then we have the following refinement of Pascal's recurrence,

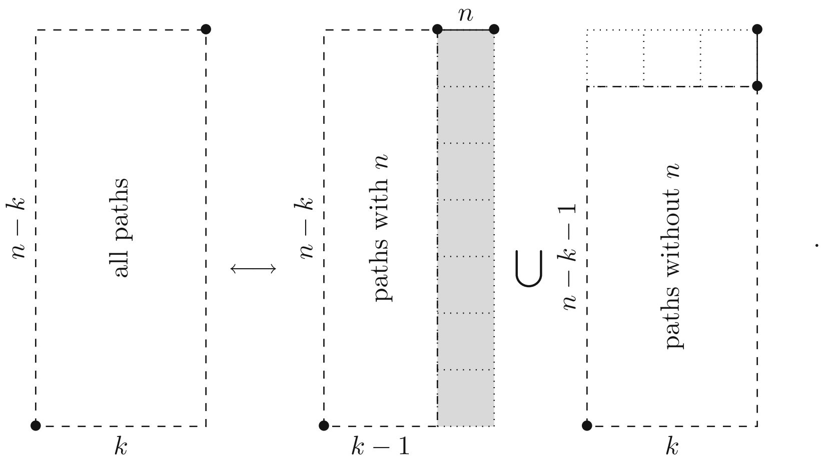
$$a_{n,k}(q) = q^{n-k} a_{n-1,k-1}(q) + a_{n-1,k}(q),$$

which can be given the following picture proof:

Table 6.1 Lattice paths counted according to area.

area(p) =	0	1	2	3	4	5	6





But by appealing to the formula in (6.2) it is easily verified that

$$\begin{bmatrix} n \\ k \end{bmatrix} = q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + \begin{bmatrix} n-1 \\ k \end{bmatrix},$$

so the q-binomial coefficients satisfy the same recurrence as the $a_{n,k}(q)$. As these functions satisfy the same recurrence (with the same boundary conditions), they must be the same.

Theorem 6.2. *For any* $n \geq k \geq 0$,

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!} = \sum_{p \in L(k,n-k)} q^{\mathrm{area}(p)}.$$

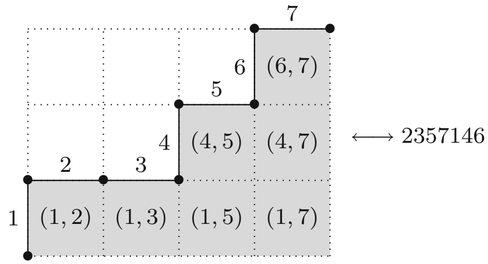


Fig. 6.1 A map between lattice paths and permutations with at most one descent.

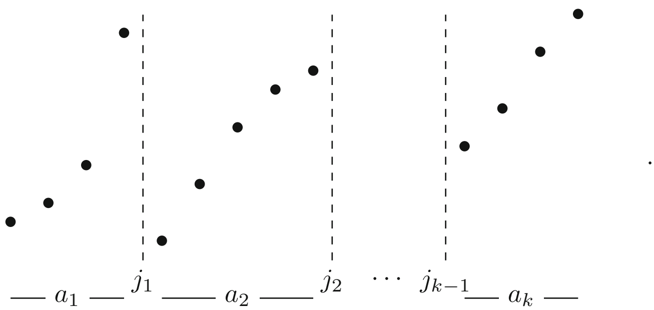
In particular, this shows that the q-binomial coefficients are polynomials with nonnegative integer coefficients.

We can create a bijection between paths in $L(k, n-k)$ and permutations w in S_n with $\mathrm{Des}(w) \subseteq \{k\}$ as follows. If A is the subset of $\{1, 2, \ldots, n\}$ corresponding to the East steps of p, then form $w = w(p)$ by writing first the elements of A in increasing order, followed by the elements of $\{1, 2, \ldots, n\} \setminus A$ in increasing order. We see an example in Figure 6.1.

It is an exercise to show that this map takes area to inversion number: $\mathrm{area}(p) = \mathrm{inv}(w)$. See Problem 6.3. Thus,

$$\begin{bmatrix} n \\ k \end{bmatrix} = \sum_{p \in L(k, n-k)} q^{\mathrm{area}(p)} = \sum_{\substack{w \in S_n \\ \mathrm{Des}(w) \subseteq \{k\}}} q^{\mathrm{inv}(w)}.$$

We can apply the same idea repeatedly to count permutations whose descent set is contained in any set $J = \{j_1, j_2, \ldots, j_{k-1}\}$. Let $a_1 = j_1$, $a_2 = j_2 - j_1$, and so on, with $a_k = n - j_{k-1}$. The structure of a permutation whose descent set is contained in J is that of k increasing runs, with the ith run having length a_i. Generically it looks like this:



The distribution for two runs is $\begin{bmatrix} n \\ a_1 \end{bmatrix}$, so to get three runs, we split up the rightmost $n - a_1$ elements into runs of length a_2 and $n - a_1 - a_2$, and so on. The total distribution is:

$$\begin{bmatrix} n \\ a_1 \end{bmatrix} \begin{bmatrix} n-a_1 \\ a_2 \end{bmatrix} \cdots \begin{bmatrix} n-a_1-a_2-\cdots-a_{k-1} \\ a_k \end{bmatrix} = \frac{[n]!}{[a_1]![a_2]!\cdots[a_k]!},$$

which, by analogy with the usual multinomial coefficients, we will denote

$$\begin{bmatrix} n \\ a_1, a_2, \ldots, a_k \end{bmatrix}.$$

We have the following result.

Theorem 6.3. *For any $J = \{j_1, \ldots, j_{k-1}\} \subseteq \{1, 2, \ldots, n-1\}$, the distribution of inversions for all w in S_n with descent set contained in J is given by the q-multinomial coefficient. That is,*

$$\sum_{\substack{w \in S_n \\ \mathrm{Des}(w) \subseteq J}} q^{\mathrm{inv}(w)} = \begin{bmatrix} n \\ a_1, a_2, \ldots, a_k \end{bmatrix},$$

where $a_1 = j_1$, $a_k = n - j_{k-1}$, and $a_i = j_i - j_{i-1}$ for $1 < i < k$.

6.3 Lattice paths by major index

We have established one combinatorial interpretation for $\begin{bmatrix} n \\ k \end{bmatrix}$ in terms of a statistic for lattice paths and connected it with inversion number for a certain collection of permutations.

Now we will show the q-binomial coefficient also counts lattice paths according to major index, and connect this with the major index distribution for another collection of permutations. Here we define the major index of a path, $\mathrm{maj}(p)$, by thinking of p as a word in $\{N, E\}$ with $E > N$. For example, if $p = NEENENE$, $\mathrm{maj}(p) = 3 + 5 = 8$. In terms of pictures, $\mathrm{maj}(p)$ is adding the positions of the valleys of p, since a valley is an East step followed by a North step. Notice that if a valley lies at the position (x, y), its position in the word is $x + y$. In Table 6.2 we see the ten paths in $L(2, 3)$ again, this time grouped according to major index.

The following can be proved with a bijection on the set of lattice paths from $(0, 0)$ to (k, l) that takes area to major index. See Problem 6.4.

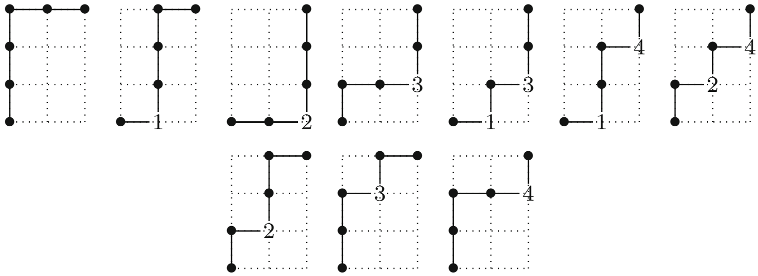
Theorem 6.4. *For any $n \geq k \geq 0$,*

$$\begin{bmatrix} n \\ k \end{bmatrix} = \sum_{p \in L(k, n-k)} q^{\mathrm{maj}(p)}.$$

Just as Theorem 6.2 could be reinterpreted as a result counting certain permutations according to inversion number, we can understand Theorem 6.4

Table 6.2 Lattice paths counted according to major index. Valleys are labeled with their position.

maj(p) =	0	1	2	3	4	5	6



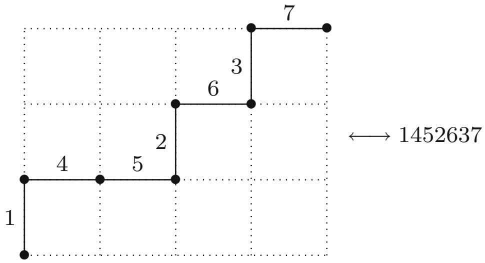


Fig. 6.2 A bijection from paths to permutations that preserves major index.

in terms of a collection of permutations counted by major index. The map from paths to permutations is again obtained by labeling the edges in the path. We first label vertical edges: $1, 2, \ldots, l$ and then we label the horizontal edges $l+1, l+2, \ldots, n$. We get the permutation w by reading the edge labels from the start of the path to the end. We see an example in Figure 6.2.

By design, all the horizontal edges have larger labels than the vertical edges, so anytime we encounter a valley: EN, we have a descent of w in that position. Hence, $\mathrm{maj}(p) = \mathrm{maj}(w)$.

But how do we characterize these permutations independent of the bijection? Define a *shuffle* of two words $u = u(1)\cdots u(k)$ and $v = v(1)\cdots v(l)$ to be the word containing the letters of u and v such that u and v appear as subwords in the proper order, and denote the set of shuffles of u and v by $u \sqcup\!\sqcup v$. In the language of posets, the shuffles of u and v are the linear extensions of the disjoint union of u and v, i.e., $u \sqcup\!\sqcup v = \mathcal{L}(u \cup v)$. Here we think of u and v as totally ordered chains and define P by $u(1) <_P u(2) <_P \cdots <_P u(k)$ and $v(1) <_P v(2) <_P \cdots <_P v(l)$. See Problem 3.1.

Table 6.3 Permutations w in $u \sqcup v$ with $u = 123$ and $v = 45$. Descent positions are marked with a bar for convenience.

$\mathrm{maj}(w)$	0	1	2	3	4	5	6
	12345	4|1235	45|123	145|23	4|15|23	4|125|3	14|25|3
			14|235	124|35	1245|3		

A moment of reflection allows us to realize that the paths in $L(l, k)$ correspond to the shuffles $u = 12 \cdots k$ and $v = (k+1) \cdots n$.

For example, the shuffles of $u = 123$ and $v = 45$ are shown in Table 6.3. Compare with the paths in Table 6.2. Hence, we have the following result.

Theorem 6.5. *The q-binomial coefficient counts shuffles according to major index, i.e.,*

$$\begin{bmatrix} n \\ k \end{bmatrix} = \sum_{w \in u \sqcup v} q^{\mathrm{maj}(w)},$$

where $u = 12 \cdots k$ and $v = (k+1) \cdots n$.

6.4 Euler-Mahonian distributions

We will now study a generating function for the joint distribution of descents and inversions.

Define

$$S_n(q,t) = \sum_{w \in S_n} q^{\mathrm{inv}(w)} t^{\mathrm{des}(w)},$$

where $S_0(q,t) = 1$ for convenience. For example, $S_1(q,t) = 1$, $S_2(q,t) = 1 + qt$, and $S_3(q,t) = 1 + 2qt + 2q^2t + q^3t^2$. For slightly larger examples, we see

$$S_4(q,t) = 1 + (3q + 4q^2 + 3q^3 + q^4)t + (q^2 + 3q^3 + 4q^4 + 3q^5)t^2 + q^6t^3,$$

and

$$\begin{aligned} S_5(q,t) = 1 &+ (4q + 6q^2 + 6q^3 + 6q^4 + 2q^5 + 2q^6)t \\ &+ (3q^2 + 9q^3 + 12q^4 + 18q^5 + 12q^6 + 9q^7 + 3q^8)t^2 \\ &+ (2q^4 + 2q^5 + 6q^6 + 6q^7 + 6q^8 + 4q^9)t^3 + q^{10}t^4. \end{aligned}$$

This is an example of what is called an *Euler-Mahonian* distribution. Any pair of statistics (s_1, s_2) where $\sum_{w \in S_n} t^{s_1(w)} = S_n(t)$ and $\sum_{w \in S_n} q^{s_2(w)} = [n]!$ is called an *Euler-Mahonian pair*. Apart from $(\mathrm{des}, \mathrm{inv})$, we will mention a result of Carlitz for $(\mathrm{des}, \mathrm{maj})$. Other interesting examples include $(\mathrm{exc}, \mathrm{inv})$ and $(\mathrm{exc}, \mathrm{maj})$. See the notes at the end of the chapter.

We can obtain a recursive formula for the distribution of the (des, inv) pair by following the proof of the quadratic recurrence for Eulerian polynomials in Theorem 1.5.

Theorem 6.6. *For any $n > 0$,*

$$S_n(q,t) = S_{n-1}(q,t) + t\sum_{i=0}^{n-2} \begin{bmatrix} n-1 \\ i \end{bmatrix} S_i(q,t)q^{n-1-i}S_{n-1-i}(q,t). \tag{6.3}$$

While this result is nice, a non-recursive way to get our hands on the Euler-Mahonian distribution is via set compositions, as studied in Section 5.6. Recall that a set composition is an ordered set partition of $\{1,2,\ldots,n\}$, and that each set composition has a natural permutation associated with it by taking all the elements of each block in increasing order. For example, the composition $A = 9|678|35|124$ is a set composition with underlying permutation $w = w(A) = 967835124$. If we say that a set composition A has inversion number equal to the inversion number of $w(A)$, we can define

$$f(\mathrm{Comp}(n);q,t) = \sum_{A\in\mathrm{Comp}(n)} q^{\mathrm{inv}(A)}t^{|A|},$$

where $\mathrm{Comp}(n)$ denotes the set of all set compositions of $\{1,2,\ldots,n\}$ and $|A|$ denotes the number of bars in A. Using Theorem 5.3, we have

$$\begin{aligned} S_n(q,t) &= (1-t)^{n-1}f(\mathrm{Comp}(n);q,t/(1-t)) \\ &= \sum_{A\in\mathrm{Comp}(n)} q^{\mathrm{inv}(w(A))}t^{|A|}(1-t)^{n-1-|A|}. \end{aligned}$$

If A has bars in positions indexed by the set J, then by construction $\mathrm{Des}(w(A)) \subseteq J$, so

$$\sum_{\substack{A\in\mathrm{Comp}(n)\\ \mathrm{bars}(A)=J}} q^{\mathrm{inv}(w(A))}t^{|A|}(1-t)^{n-1-|A|} = t^{|J|}(1-t)^{n-1-|J|}\cdot \sum_{\substack{w\in S_n\\ \mathrm{Des}(w)\subseteq J}} q^{\mathrm{inv}(w)}.$$

Summing over all sets J, we have the following formula.

$$S_n(q,t) = \sum_{J\subseteq\{1,\ldots,n-1\}} t^{|J|}(1-t)^{n-1-|J|} \sum_{\substack{w\in S_n\\ \mathrm{Des}(w)\subseteq J}} q^{\mathrm{inv}(w)}.$$

This focuses the problem on counting the distribution of inversions for permutations with particular descent sets, and we know from Theorem 6.3 exactly how to do this. If $J = \{j_1,\ldots,j_{k-1}\} \subseteq \{1,2,\ldots,n-1\}$,

$$\sum_{\substack{w \in S_n \\ \mathrm{Des}(w) \subseteq J}} q^{\mathrm{inv}(w)} = \begin{bmatrix} n \\ a_1, a_2, \ldots, a_k \end{bmatrix},$$

where $a_1 = j_1$, $a_k = n - j_{k-1}$, and $a_i = j_i - j_{i-1}$ for $1 < i < k$.

We will say a vector of positive integers $\mathbf{a} = (a_1, \ldots, a_k)$ whose entries sum to n is called a *composition* of n, denoted $\mathbf{a} \vDash n$. Its length is the number of entries, denoted $\ell(\mathbf{a}) = k$.

With this notation, we have the following formula for the Euler-Mahonian polynomials.

Theorem 6.7. *The Euler-Mahonian distribution has the following explicit formula:*

$$S_n(q,t) = \sum_{\mathbf{a} \vDash n} t^{\ell(\mathbf{a})-1}(1-t)^{n-\ell(\mathbf{a})} \begin{bmatrix} n \\ a_1, \ldots, a_{\ell(\mathbf{a})} \end{bmatrix}. \tag{6.4}$$

With the q-factorials arising in (6.3) and (6.4), it is natural to consider the following generating function:

$$S(q,t,z) := \sum_{n \geq 0} S_n(q,t) \frac{z^n}{[n]!}. \tag{6.5}$$

Clearly as $q \to 1$, this specializes to the exponential generating function for Eulerian polynomials given in Theorem 1.6.

The q-analogue of the exponential function that will arise here is

$$\exp(z;q) := \sum_{n \geq 0} \frac{z^n}{[n]!}.$$

Then if we let $f = \exp(z(1-t);q) - 1$, then

$$\begin{aligned} f^k &= \sum_{a_1,\ldots,a_k \geq 1} \frac{(z(1-t))^{a_1+\cdots+a_k}}{[a_1]! \cdots [a_k]!}, \\ &= \sum_{n \geq 1} \sum_{(a_1,\ldots,a_k) \vDash n} \frac{z^n(1-t)^n}{[a_1]! \cdots [a_k]!}, \end{aligned}$$

and hence

$$\begin{aligned} \frac{1}{1-f} &= 1 + f + f^2 + \cdots, \\ &= 1 + \sum_{n,k \geq 1} \sum_{(a_1,\ldots,a_k) \vDash n} \frac{z^n(1-t)^n}{[a_1]! \cdots [a_k]!}, \\ &= 1 + \sum_{n \geq 1} \frac{z^n(1-t)^n}{[n]!} \sum_{\mathbf{a} \vDash n} \begin{bmatrix} n \\ a_1, \ldots, a_{\ell(\mathbf{a})} \end{bmatrix}. \end{aligned}$$

Thus, if we compute the geometric series for $\frac{t}{1-t}f$, we get:

$$\frac{1}{1-(\frac{t}{1-t})f} = 1 + \sum_{n\geq 1} \frac{z^n}{[n]!} \sum_{\mathbf{a}\models n} t^{\ell(\mathbf{a})}(1-t)^{n-\ell(\mathbf{a})} \begin{bmatrix} n \\ a_1, \ldots, a_{\ell(\mathbf{a})} \end{bmatrix},$$
$$= 1 + \sum_{n\geq 1} \frac{z^n}{[n]!} \cdot tS_n(q,t),$$

using the formula in (6.4).

Since $f = \exp(z(1-t);q) - 1$, we have established that

$$1 + \sum_{n\geq 1} \frac{z^n}{[n]!} \cdot tS_n(q,t) = \frac{1-t}{1-t-t\exp(z(1-t);q)},$$

and after some simple manipulations, we obtain the following result due to Richard Stanley.

Theorem 6.8 (Stanley [147]). *The exponential generating function for the Euler-Mahonian distribution is*

$$S(q,t,z) = \frac{(1-t)\exp(z(1-t);q)}{1-t\exp(z(1-t);q)}.$$

6.5 Descents and major index

We will now turn our attention to another refinement of the Eulerian numbers. Let

$$S_n^{\mathrm{maj}}(q,t) = \sum_{w\in S_n} q^{\mathrm{maj}(w)} t^{\mathrm{des}(w)}.$$

We use the superscript “maj” to distinguish this Euler-Mahonian distribution from the one for $(\mathrm{des}, \mathrm{inv})$. Let us denote the coefficient of t^k by

$$\left\langle {n \atop k} \right\rangle^{\mathrm{maj}} = \sum_{w\in S_n, \mathrm{des}(w)=k} q^{\mathrm{maj}(w)},$$

so that

$$S_n^{\mathrm{maj}}(q,t) = \sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle^{\mathrm{maj}} t^k.$$

For example,

$$\begin{aligned}
S_3^{\mathrm{maj}}(q,t) &= 1 + (2q + 2q^2)t + q^3t^2,\\
S_4^{\mathrm{maj}}(q,t) &= 1 + (3q + 5q^2 + 3q^3)t + (3q^3 + 5q^2 + 3q^3)t^2 + q^6t^3,\\
S_5^{\mathrm{maj}}(q,t) &= 1 + (4q + 9q^2 + 9q^3 + 4q^4)t\\
&\qquad + (6q^3 + 16q^4 + 22q^5 + 16q^6 + 6q^7)t^2\\
&\qquad + (4q^6 + 9q^7 + 9q^8 + 4q^9)t^3 + q^{10}t^4.
\end{aligned}$$

Notice that, unlike (des, inv), the coefficient of t^k here is palindromic as a polynomial in q.

The same idea we used to prove Theorem 1.3 can be used to prove the following recurrence for the polynomials $\left\langle {n \atop k} \right\rangle^{\mathrm{maj}}$. We will defer a detailed proof to Problem 6.6.

Theorem 6.9. *For any k and $n > 0$,*

$$\left\langle {n \atop k} \right\rangle^{\mathrm{maj}} = [k+1]\left\langle {n-1 \atop k} \right\rangle^{\mathrm{maj}} + q^k[n-k]\left\langle {n-1 \atop k-1} \right\rangle^{\mathrm{maj}}. \tag{6.6}$$

The key observation comes from the discussion preceding Theorem 6.1 about the effect of insertion of n into a permutation in S_{n-1}.

Notice that

$$\begin{bmatrix} n+k \\ n \end{bmatrix} = \begin{bmatrix} n+k-1 \\ n-1 \end{bmatrix} + q^n \begin{bmatrix} n+k-1 \\ n \end{bmatrix},$$

from which it follows by induction on n that

$$\frac{1}{(1-t)(1-qt)\cdots(1-q^nt)} = \sum_{k\geq 0} \begin{bmatrix} n+k \\ n \end{bmatrix} t^k.$$

Manipulations of this generating function along with recurrence (6.6) can be used to prove the following generalization of Corollary 1.1 due to MacMahon, but often referred to as the *Carlitz identity*.

Corollary 6.1. *For any $n \geq 0$,*

$$\frac{S_n^{\mathrm{maj}}(q,t)}{(1-t)(1-qt)\cdots(1-q^nt)} = \sum_{k\geq 0} [k+1]^n t^k.$$

Another way to obtain Corollary 6.1 is to adapt the idea of P-partitions from Section 3.2 to obtain a q-analogue of Theorem 3.1. The key ingredient here is to define a q-analogue of the order polynomial:

$$\Omega(P;k,q) = \sum_{a \in \overline{\mathcal{A}}(P;k)} q^{a_1+\cdots+a_n},$$

where P is a labeled poset with n elements, and $\overline{\mathcal{A}}(P)$ denotes the set of *reverse* P-partitions. See Problem 6.7.

6.6 q-Catalan numbers

We will now discuss q-analogues of Catalan numbers.

The first is a q-analogue of the Catalan numbers due to MacMahon. Define

$$C_n(q) = \frac{1}{[n+1]} \begin{bmatrix} 2n \\ n \end{bmatrix}.$$

We will show that this q-analogue of the Catalan number C_n counts Dyck paths according to major index. The idea is similar to earlier proof of the formula for Catalan numbers. Using the formula for q-binomial coefficients, we see that

$$\begin{bmatrix} 2n \\ n \end{bmatrix} = q \begin{bmatrix} 2n \\ n+1 \end{bmatrix} + \frac{1}{[n+1]} \begin{bmatrix} 2n \\ n \end{bmatrix}.$$

We will show that among all lattice paths in $L(n,n)$, the paths that go below the line $y = x$, i.e., those that are not Dyck paths, have major index distributed as $q\left[\begin{smallmatrix} 2n \\ n+1 \end{smallmatrix}\right]$.

Fix a path p that passes below the line $y = x$ and let (x, y) be the first minimum valley of the path. Recall from Section 2.4.2 that a minimum valley is one for which $x - y$ is greatest. By first minimum valley we mean the minimum valley with smallest x-coordinate. Now let p' denote the path obtained by changing the East step immediately prior to (x, y) into a North step, and keeping all the remaining steps the same. See Figure 6.3.

We can see immediately that p' is a path from $(0, 0)$ to $(n-1, n+1)$ such that

$$\mathrm{maj}(p) = \mathrm{maj}(p') + 1.$$

Clearly we can do this for any path in $L(n,n)$ that passes below $y = x$, and the map is reversible. (In p' the point $(x-1, y)$ is the first minimum valley; change the North step following it into an East step. If p' does not go below the line $y = x$, then consider $(0, 0)$ to be its first minimum valley.) Hence,

$$\sum_{p \in L(n,n)-\mathrm{Dyck}(n)} q^{\mathrm{maj}(p)} = \sum_{p' \in L(n-1,n+1)} q^{\mathrm{maj}(p')+1} = q \begin{bmatrix} 2n \\ n+1 \end{bmatrix},$$

and we have established MacMahon's formula for the q-Catalan numbers.

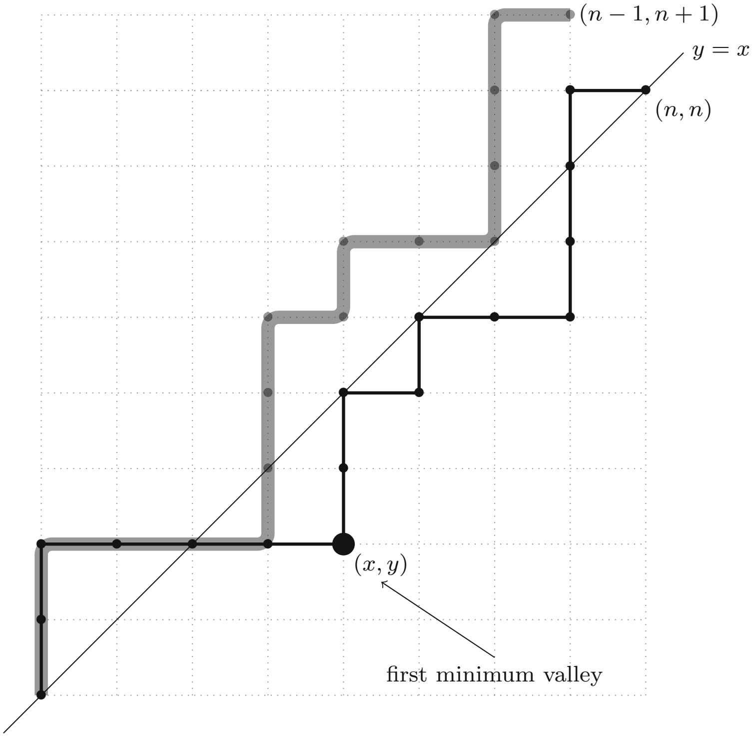


Fig. 6.3 The map $p \mapsto p'$ used for Theorem 6.10.

Theorem 6.10. *For any* $n \geq 0$,

$$\sum_{p \in \mathrm{Dyck}(n)} q^{\mathrm{maj}(p)} = \frac{1}{[n+1]} \begin{bmatrix} 2n \\ n \end{bmatrix}.$$

6.7 *q*-Narayana numbers

Taking the q-analogue of the formula for the Narayana number $N_{n,k}$ in Equation (2.3), we have

$$N_{n,k}(q) = \frac{1}{[k+1]} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} n-1 \\ k \end{bmatrix}. \tag{6.7}$$

Let $\mathrm{Dyck}(n; k)$ denote the set of Dyck paths with $2n$ steps that have k valleys (and $k + 1$ peaks). Then we have the following result, which can also be found in MacMahon's work.

Theorem 6.11. *For any $n \geq 1$ and $k \geq 0$,*

$$\sum_{p\in \mathrm{Dyck}(n;k)} q^{\mathrm{maj}(p)} = q^{k(k+1)} \cdot \frac{1}{[k+1]} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} n-1 \\ k \end{bmatrix} = q^{k(k+1)} N_{n,k}(q).$$

To prove this theorem, we will develop a recurrence for the set of paths that start at $(0, 0)$, do not pass below the line $y = x$, end at coordinate (a, b), and have k valleys. Denote this set by $\mathrm{Dyck}(a, b; k)$. In terms of words on $\{N, E\}$, these are words with b letters N, a letters E, and such that every initial subword has at least as many letters N as letters E. Hence the set is empty if $a > b$.

Let us denote the major index generating function for these paths by

$$N_{(a,b),k}(q) = \sum_{p\in \mathrm{Dyck}(a,b;k)} q^{\mathrm{maj}(p)}.$$

Theorem 6.11 applies to the special case of $a = b = n$.

It is easy to see that adding an E step to any path does not change major index, and hence along the boundary of the line $y = x$ we have

$$N_{(a,a),k}(q) = N_{(a-1,a),k}(q).$$

Now suppose $a < b$. By considering the location of the final valley of a path in $\mathrm{Dyck}(a, b; k)$, we have

$$\begin{aligned} N_{(a,b),k}(q) = N_{(a-1,b),k}(q) &+ q^{a+b-1} N_{(a-1,b-1),k-1}(q) \\ &+ \left(N_{(a,b-1),k}(q) - N_{(a-1,b-1),k}(q)\right). \end{aligned} \tag{6.8}$$

The first term accounts for paths that finish with an East step and hence whose final valley is not in column a, the second term accounts for paths whose final valley is at $(a, b - 1)$, and the third term accounts for paths whose final valley is at (a, j) for some $j < b - 1$.

To be clear about this last term, let A denote the set of paths in $\mathrm{Dyck}(a, b-1; k)$ that have their final valley at (a, j) for some $j < b - 1$, and let $B = \mathrm{Dyck}(a, b - 1; k) - A$ denote the complement of A. We have that a path is in A if and only if it ends with a North step. Thus B is the set of paths in $\mathrm{Dyck}(a, b - 1; k)$ that end in an East step. Each path p in B has the form $p = p'E$ for some path p' in $\mathrm{Dyck}(a - 1, b - 1; k)$ and $\mathrm{maj}(p) = \mathrm{maj}(p')$. This tells us that

$$\sum_{p\in B} q^{\mathrm{maj}(p)} = \sum_{p'\in \mathrm{Dyck}(a-1,b-1;k)} q^{\mathrm{maj}(p')} = N_{(a-1,b-1),k}(q),$$

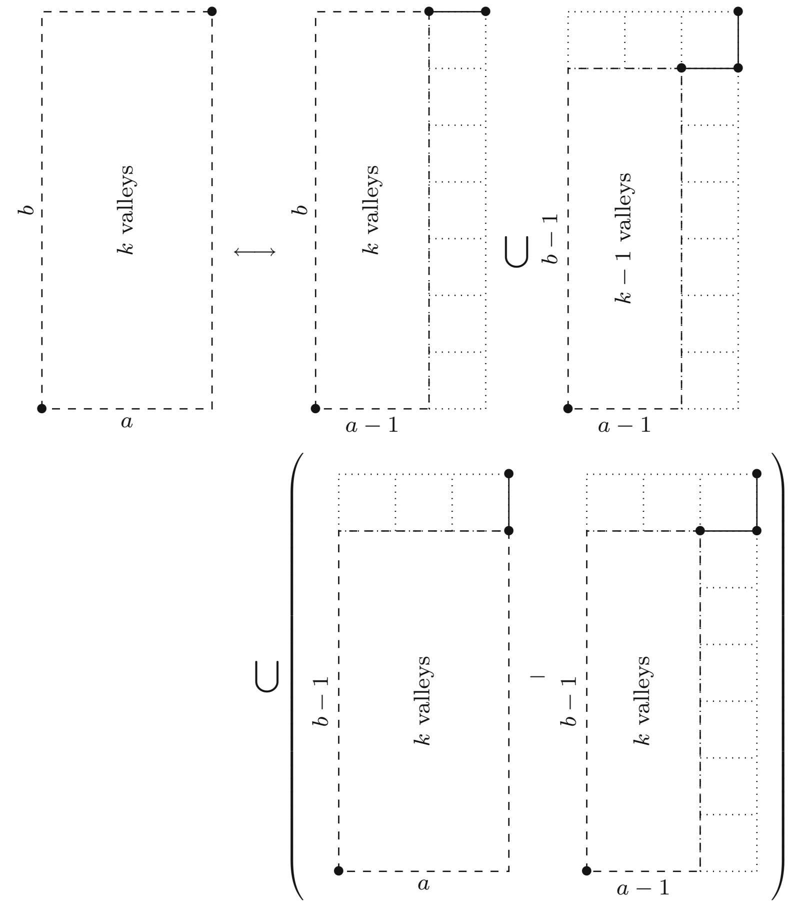


Fig. 6.4 Counting paths in $\mathrm{Dyck}(a, b; k)$ according to the location of the final valley.

and therefore

$$\sum_{p \in A} q^{\mathrm{maj}(p)} = N_{(a,b-1),k}(q) - N_{(a-1,b-1),k}(q),$$

as desired.

An illustration to accompany this argument for (6.8) is in Figure 6.4.

We have the following formula for $N_{(a,b),k}(q)$, of which Theorem 6.11 is a special case.

Proposition 6.1. *For any* $0 \le a \le b$,

$$N_{(a,b),k}(q) = q^{k^2} \left(\begin{bmatrix} a \\ k \end{bmatrix} \begin{bmatrix} b \\ k \end{bmatrix} - \begin{bmatrix} a+1 \\ k+1 \end{bmatrix} \begin{bmatrix} b-1 \\ k-1 \end{bmatrix} \right).$$

To prove Proposition 6.1 it is a straightforward matter of showing that the proposed formula satisfies the recurrence in (6.8) with the same boundary conditions. See Problem 6.8.

6.8 Dyck paths by area

Another natural q-analogue of the Catalan numbers is obtained by keeping track of the area below the lattice path. Since Dyck paths do not go below the line $y = x$, we will normalize the area statistic, so for $p \in \mathrm{Dyck}(n)$, $\mathrm{area}(p)$ counts the number of unit squares above the line $y = x$. For example, Figure 6.5 shows a path in Dyck(8) with area 7.

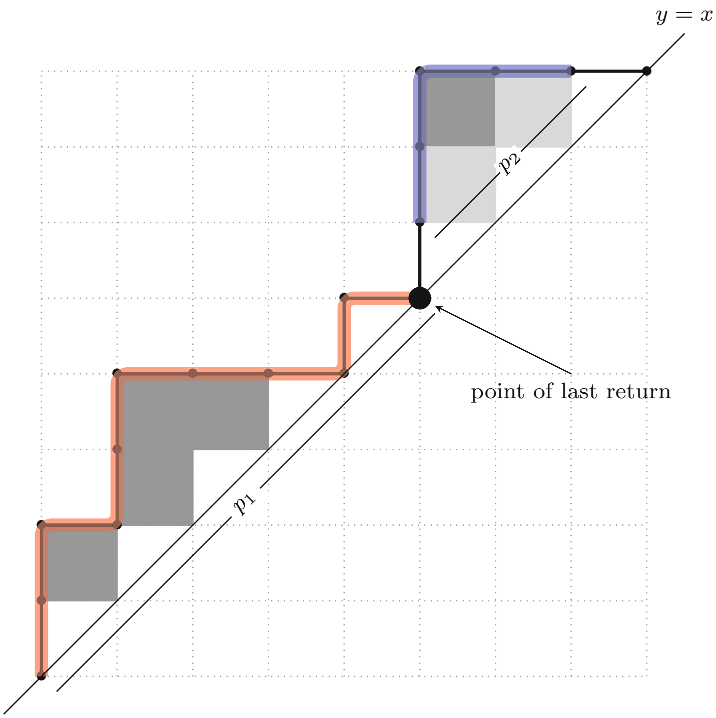


Fig. 6.5 A Dyck path with area 7, decomposed according to the point of last return into paths p_1 and p_2.

Let $C_n^{\text{area}}(q)$ denote the generating function for the area statistic on $\text{Dyck}(n)$, i.e.,

$$C_n^{\text{area}}(q) = \sum_{p \in \text{Dyck}(n)} q^{\text{area}(p)}.$$

The quadratic recurrence relation for Catalan numbers, i.e.,

$$C_n = \sum_{i=0}^{n-1} C_i C_{n-1-i},$$

can be understood for Dyck paths by decomposing a Dyck path p according to its *point of last return*, i.e., the last time the path touches the line $y = x$ before reaching (n, n). If the path never touches the line $y = x$ except at the endpoints we consider $(0, 0)$ to be the point of last return. See Figure 6.5.

Suppose $(n-1-i, n-1-i)$ is the point of last return of a Dyck path p. Then we can write $p = p_1 N p_2 E$, where p_1 is a Dyck path from $(0, 0)$ to $(n-1-i, n-1-i)$ and p_2 is a Dyck path from $(n-1-i, n-i)$ to $(n-1, n)$, i.e., a Dyck path of size i. Moreover, the area of p is

$$\text{area}(p) = \text{area}(p_1) + \text{area}(p_2) + i,$$

and we get the following quadratic recurrence for $C_n^{\text{area}}(q)$.

Proposition 6.2. *For $n \geq 1$,*

$$C_n^{\text{area}}(q) = \sum_{i=0}^{n-1} q^i C_i^{\text{area}}(q) C_{n-1-i}^{\text{area}}(q). \tag{6.9}$$

Carlitz produced an interesting formula for the generating function of the polynomials $C_n^{\text{area}}(q)$, in the form of a *continued fraction*. Let us define

$$\text{Dyck}(q, z) = \sum_{n \geq 0} C_n^{\text{area}}(q) z^n = \sum_{\text{Dyck paths } p} q^{\text{area}(p)} z^{|p|},$$

where $|p| = n$ for $p \in \text{Dyck}(n)$.

Theorem 6.12. *We have the following continued fraction expansion for* $\text{Dyck}(q, z)$*:*

$$\text{Dyck}(q, z) = \cfrac{1}{1 - \cfrac{z}{1 - \cfrac{qz}{1 - \cfrac{q^2 z}{1 - \cfrac{q^3 z}{\ddots}}}}}, \tag{6.10}$$

where for $k \geq 1$, the numerator of level k is $q^{k-1} z$.

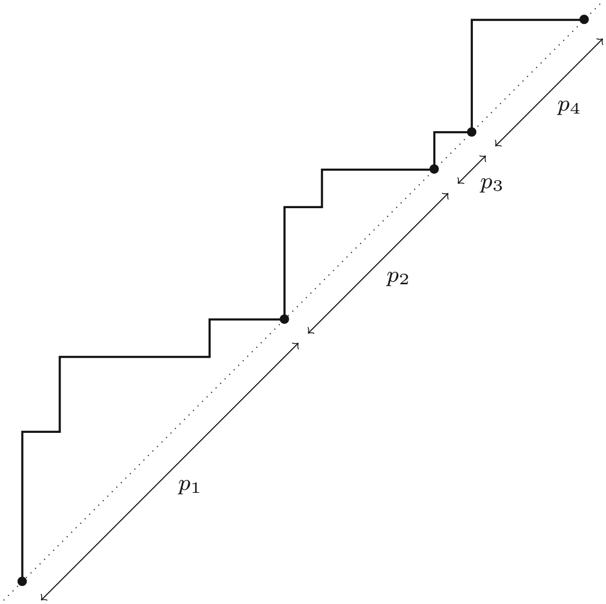


Fig. 6.6 A Dyck path decomposed into prime Dyck paths.

Theorem 6.12 result can be proved using the recurrence of (6.9), but it can also be given another interesting proof using "prime decomposition" of paths. Call a Dyck path *prime* if the only points at which it touches the line $y = x$ occur at $(0, 0)$ and (n, n). Write $\mathrm{Dyck}'(n)$ for the set of all prime Dyck paths of length n. Further, let $\mathrm{Dyck}' = \bigcup \mathrm{Dyck}'(n)$ denote the set of all prime Dyck paths and let $\mathrm{Dyck} = \bigcup \mathrm{Dyck}(n)$ denote the set of all Dyck paths.

Notice that each path $p \in \mathrm{Dyck}$ has a unique "prime decomposition" into concatenated prime paths, $p_1, p_2, \ldots$. See, for example, Figure 6.6.

Now, letting $\mathrm{Dyck}'(q, z)$ denote the generating function for prime paths by area and size, we have:

$$\begin{aligned}\mathrm{Dyck}(q, z) &= 1 + \mathrm{Dyck}'(q, z) + \big(\mathrm{Dyck}'(q, z)\big)^2 + \big(\mathrm{Dyck}'(q, z)\big)^3 + \cdots, \\ &= \frac{1}{1 - \mathrm{Dyck}'(q, z)}. \qquad (6.11)\end{aligned}$$

Further, there is a simple relationship between prime paths and arbitrary Dyck paths. In fact we have a bijection $\mathrm{Dyck}'(n) \leftrightarrow \mathrm{Dyck}(n-1)$ since a prime path in $\mathrm{Dyck}'(n)$ can be written uniquely as $p' = NpE$, where p is a Dyck path in $\mathrm{Dyck}(n-1)$. We have $\mathrm{area}(p') = \mathrm{area}(p) + (n-1)$ and $|p'| = |p| + 1$, so

$$\begin{aligned}
\operatorname{Dyck}'(q,z) &= \sum_{p'\in\operatorname{Dyck}'} q^{\operatorname{area}(p')} z^{|p'|}, \\
&= z \sum_{p\in\operatorname{Dyck}} q^{\operatorname{area}(p)} (qz)^{|p|}, \\
&= z \operatorname{Dyck}(q,qz). \qquad (6.12)
\end{aligned}$$

Thus, by applying (6.12) to (6.11) we get

$$\operatorname{Dyck}(q,z) = \frac{1}{1 - z\operatorname{Dyck}(q,qz)}, \qquad (6.13)$$

and this functional equation can be applied to the $\operatorname{Dyck}(q,qz)$ appearing in the denominator to obtain

$$\begin{aligned}
\operatorname{Dyck}(q,z) &= \frac{1}{1 - z\operatorname{Dyck}(q,qz)}, \\
&= \cfrac{1}{1 - \cfrac{z}{1 - qz\operatorname{Dyck}(q,q^2z)}}.
\end{aligned}$$

Continuing in this way, we find

$$\begin{aligned}
\operatorname{Dyck}(q,z) &= \cfrac{1}{1 - \cfrac{z}{1 - qz\operatorname{Dyck}(q,q^2z)}}, \\
&= \cfrac{1}{1 - \cfrac{z}{1 - \cfrac{qz}{1 - q^2z\operatorname{Dyck}(q,q^3z)}}}, \\
&= \cfrac{1}{1 - \cfrac{z}{1 - \cfrac{qz}{1 - \cfrac{q^2z}{1 - q^3z\operatorname{Dyck}(q,q^4z)}}}}, \\
&\vdots \\
&= \cfrac{1}{1 - \cfrac{z}{1 - \cfrac{qz}{1 - \cfrac{q^2z}{1 - \cfrac{q^3z}{1 - \cfrac{q^4z}{\ddots}}}}}},
\end{aligned}$$

as claimed in Theorem 6.12.

While a continued fraction formula can be elegant, it may seem a bit esoteric at first. However it can be useful for explicit computation as well. For example, if we wish to have the generating function for Dyck paths of size n, we can truncate the continued fraction after n levels to obtain a rational function and extract the coefficient of z^n in the usual way. With $n = 4$, this gives:

$$\begin{aligned}\cfrac{1}{1-\cfrac{z}{1-\cfrac{qz}{1-\cfrac{q^2z}{1-q^3z}}}} &= \frac{1-z(q+q^2+q^3)+z^2q^4}{1-z(1+q+q^2+q^3)+z^2(q^2+q^3+q^4)},\\ &= 1+C_1^{\text{area}}(q)z+C_2^{\text{area}}(q)z^2+C_3^{\text{area}}(q)z^3\\ &\quad +C_4^{\text{area}}(q)z^4+\sum_{|p|>4,p=p_1p_2p_3p_4} q^{\text{area}(p)}z^{|p|},\end{aligned}$$

where the final term is the sum over all paths with $|p| > 4$ and at most four prime factors. That is, truncating after the first four levels of the continued fraction produces a rational generating function for all Dyck paths with at most four prime factors, and this includes all Dyck paths of size at most four.

We can refine the Narayana numbers by keeping track of the area under a Dyck path along with its number of valleys. Recall from Proposition 2.1 that we can write the Narayana polynomial $C_n(t)$ as

$$C_n(t) = \sum_{k=0}^{n-1} N_{n,k}t^k = \sum_{p\in\text{Dyck}(n)} t^{\text{val}(p)}.$$

Recall the decomposition $p = p_1Np_2E$, with $p_1 \in \text{Dyck}(n-1-i)$ and $p_2 \in \text{Dyck}(i)$ discussed prior to Proposition 6.2 (see Figure 6.5). Notice that p has a valley at the point of last return, so we have

$$\text{val}(p) = \text{val}(p_1) + \text{val}(p_2) + 1,$$

unless $i = n-1$ and p_1 is empty, in which case $\text{val}(p) = \text{val}(p_2)$. Hence, we have a common refinement of (6.9) and (2.5).

Theorem 6.13. *For $n \geq 1$,*

$$C_n^{\text{area}}(q,t) = q^{n-1}C_{n-1}^{\text{area}}(q,t) + t\sum_{i=0}^{n-2} q^iC_i^{\text{area}}(q,t)C_{n-1-i}^{\text{area}}(q,t).$$

We can also produce a continued fraction for counting Dyck paths by area and valleys. Let

$$\text{Dyck}(q,t,z) = \sum_{p\in\text{Dyck}} q^{\text{area}(p)}t^{\text{val}(p)}z^{|p|},$$

and

$$\operatorname{Dyck}'(q,t,z) = \sum_{p' \in \operatorname{Dyck}'} q^{\operatorname{area}(p')} t^{\operatorname{val}(p')} z^{|p'|}.$$

The first important observation is

$$\operatorname{Dyck}(q,t,z) = 1 + \operatorname{Dyck}'(q,t,z) + t\big(\operatorname{Dyck}'(q,t,z)\big)^2 + t^2\big(\operatorname{Dyck}'(q,t,z)\big)^3 + \cdots,$$

so that

$$t\operatorname{Dyck}(q,t,z) = (t-1) + \frac{1}{1 - t\operatorname{Dyck}'(q,t,z)}.$$

Next, notice that the number of valleys in $p' = NpE$ is $\operatorname{val}(p') = \operatorname{val}(p)$, so

$$\operatorname{Dyck}'(q,t,z) = z\operatorname{Dyck}(q,t,qz).$$

From these observations, we can use induction to get the following result that generalizes Theorem 6.12.

Theorem 6.14. *We have the following continued fraction for Dyck paths counted by area and number of valleys:*

$$t\operatorname{Dyck}(q,t,z) = (t-1) + \cfrac{1}{1 - z(t-1) - \cfrac{z}{1 - qz(t-1) - \cfrac{qz}{1 - q^2z(t-1) - \cfrac{q^2z}{\ddots}}}}.$$

Notes

Theorem 6.1 is due to Percy MacMahon. See [106]. Since MacMahon was a major in the British army, what he called the "greater index" of a permutation, we now call the "major index."

There is a wide variety of Euler-Mahonian pairs in the literature, e.g., [40, 52, 67, 71, 80, 82, 122, 136]. The results we chose to focus on here are due to Richard Stanley [147] (Theorem 6.8) and MacMahon [106] (Corollary 6.1).

The q-analogues of the Catalan and Narayana numbers found in Theorems 6.10 and 6.11 are due to MacMahon [106], though the arguments given here are adapted from a paper by J. Fürlinger and Josef Hofbauer [78]. The generating function for area appearing in Proposition 6.2 is due to Leonard Carlitz and John Riordan [42].

The distribution for $(\operatorname{val}, \operatorname{area})$ over Dyck paths is not the most important bivariate joint distribution for Dyck paths. There is a different story of "(q,t)-Catalan numbers," which study the pairs of statistics $(\operatorname{area}, \operatorname{bounce})$ and $(\operatorname{dinv}, \operatorname{area})$. While we will not discuss bounce and dinv here, we remark that both have the same distribution as area over all Dyck paths. Further,

$$\sum q^{\text{area}(p)} t^{\text{bounce}(p)} = \sum q^{\text{bounce}(p)} t^{\text{area}(p)} = \sum q^{\text{dinv}(p)} t^{\text{area}(p)}.$$

While the second can be proved bijectively, the first is far from evident combinatorially. See Jim Haglund's manuscript for more [87].

Problems

6.1. Prove Theorem 6.1 with a bijection. That is, define a bijection $\phi : S_n \to S_n$ such that $\text{maj}(w) = \text{inv}(\phi(w))$.

6.2. Show that the q-binomial coefficients are symmetric, $\left[\begin{smallmatrix} a+b \\ a \end{smallmatrix}\right] = \left[\begin{smallmatrix} a+b \\ b \end{smallmatrix}\right]$ via a bijection, i.e., define a map $L(a,b) \to L(b,a)$ that preserves the areas of paths.

6.3. Show that the bijection illustrated in Figure 6.1 takes area to inversion number.

6.4. Prove Theorem 6.4 with a bijection $L(a,b) \to L(a,b)$ that takes a path with area k to a path with major index k.

6.5. From Problem 3.1 we know that $w \in u \sqcup\!\sqcup v$ (with $u = 12\cdots k$ and $v = (k+1)\cdots n$) if and only if $\text{Des}(w^{-1}) \subseteq \{k\}$. Thus from Theorems 6.4 and 6.2 we have

$$\begin{bmatrix} n \\ k \end{bmatrix} = \sum_{w \in u \sqcup\!\sqcup v} q^{\text{maj}(w)} = \sum_{w \in u \sqcup\!\sqcup v} q^{\text{inv}(w^{-1})}.$$

Is it true that $\text{maj}(w) = \text{inv}(w^{-1})$? If so, prove it. If not, define a bijection $\phi : u \sqcup\!\sqcup v \to u \sqcup\!\sqcup v$ such that $\text{maj}(\phi(w)) = \text{inv}(w^{-1})$.

6.6. Prove Theorem 6.9.

6.7. Prove Corollary 6.1 using the q-order polynomial and proving a refinement of Theorem 3.1. Hint: use *reverse* P-partitions, in which $a_1 \geq \cdots \geq a_n$ and $a_i > a_{i+1}$ if i is a descent.

6.8. Prove Proposition 6.1, and establish Theorem 6.11 as a special case.

Chapter 7
Cubes, Carries, and an Amazing Matrix (Supplemental)

7.1 Slicing a cube

In this supplemental chapter we will find the Eulerian numbers cropping up in some surprising places.

First, consider cutting up the n-dimensional cube $[0,1]^n$ according to the braid arrangement. For example, Figure 7.1 shows this in three dimensions.

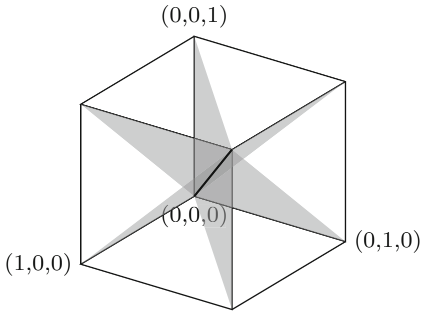


Fig. 7.1 Slicing a cube with the braid arrangement, looking down the line $x = y = z$.

Ignoring overlaps on the boundaries, each region here is a simplex of the form

$$\mathcal{S}_w = \{\mathbf{x} \in \mathbb{R}^n : 0 \leq x_{w(1)} \leq x_{w(2)} \leq \cdots \leq x_{w(n)} \leq 1\},$$

where $w \in S_n$. By symmetry, each of these regions has the same volume, and since their union has volume 1, we get

$$\mathrm{vol}(\mathcal{S}_w) = \frac{1}{n!}.$$

T.K. Petersen, *Eulerian Numbers*, Birkhäuser Advanced Texts Basler Lehrbücher, DOI 10.1007/978-1-4939-3091-3_7

Now consider slicing the cube by level sets. For fixed n, and any $k = 0, 1, \ldots, n-1$, let

$$\mathcal{R}_k = \{\mathbf{y} \in [0,1]^n : k \leq y_1 + y_2 + \cdots + y_n \leq k+1\}.$$

For three dimensions, we have illustrated these slices in Figure 7.2. The following proposition suggests how to compute the volume of these slices.

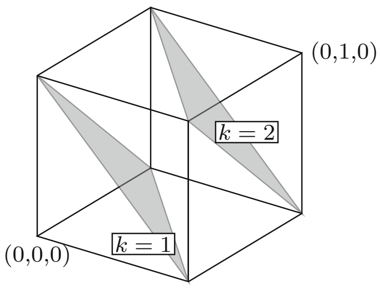


Fig. 7.2 Slicing a cube with level sets.

Proposition 7.1. *The volume of the kth slice of the n-cube is given by:*

$$\mathrm{vol}(\mathcal{R}_k) = \frac{\left\langle {n \atop k} \right\rangle}{n!},$$

where $\left\langle {n \atop k} \right\rangle$ is the number of permutations of n with k descents.

This result is mentioned in Dominique Foata's 1977 paper [67], in which he asks for a combinatorial proof. Richard Stanley provided a beautifully simple proof in a note at the end of Foata's paper, which we describe here. (This is Problem 51 in Stanley's textbook [154].)

Let

$$\mathcal{S}_k = \bigcup_{\mathrm{des}(w^{-1})=k} \mathcal{S}_w,$$

denote the union of points in the cones corresponding to permutations with k descents. We will define a map $\phi : \mathcal{S}_k \to \mathcal{R}_k$ that is "generically" a bijection, in that it is bijective for all points such that no two coordinates are equal. (Such points have measure zero and are irrelevant for the volume calculation.)

The map is given explicitly by $\phi(x_1, \ldots, x_n) = (y_1, \ldots, y_n)$ with

$$y_i = \begin{cases} x_{i+1} - x_i & \text{if } x_i < x_{i+1}, \\ 1 + x_{i+1} - x_i & \text{if } x_i > x_{i+1}, \end{cases}$$

where $x_{n+1} = 1$. If $x_i = x_{i+1}$ for some i, ϕ is undefined.

Suppose $\mathbf{x} = (x_1, \ldots, x_n)$ is a generic point in $\mathcal{S}_w$. To say that $x_i > x_{i+1}$ is to say that $i+1$ appears to the left of i in w, i.e., $w^{-1}(i+1) < w^{-1}(i)$. In other words i is a descent of w^{-1}. Notice that if $\operatorname{des}(w^{-1}) = k$, then $\sum y_i = k + 1 - x_1$. Thus ϕ maps points from $\mathcal{S}_k$ to $\mathcal{R}_k$.

For example, generic points in the region $\mathcal{S}_{631425}$ satisfy

$$0 < x_6 < x_3 < x_1 < x_4 < x_2 < x_5 < 1,$$

and these get mapped to

$$(y_1, y_2, y_3, y_4, y_5, y_6) = (x_2 - x_1, 1 + x_3 - x_2, x_4 - x_3, x_5 - x_4, 1 + x_6 - x_5, 1 - x_6).$$

The sum of the coordinates under this map is $\sum y_i = 3 - x_1$, so $2 < \sum y_i < 3$, as expected since $\operatorname{des}(w^{-1}) = 2$. Notice that on $\mathcal{S}_w$, the map ϕ is an affine transformation, given here by:

$$\mathbf{y} = \phi(\mathbf{x}) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \mathbf{x}.$$

The determinant of the linear part of this transformation has absolute value 1, so it is volume-preserving.

It remains to show that ϕ is invertible.

To reverse the map ϕ, we work from right to left, exploiting the observation that $x_i = x_{i+1} - y_i$ or $x_i = 1 + x_{i+1} - y_i$. Since $0 < x_i < 1$, only one of these expressions can be correct. By convention $y_n = 1 - x_n$, so we get started with $x_n = 1 - y_n$. Otherwise, once we have calculated x_{i+1} we get:

$$x_i = \begin{cases} x_{i+1} - y_i & \text{if } x_{i+1} > y_i, \\ 1 + x_{i+1} - y_i & \text{if } x_{i+1} < y_i. \end{cases}$$

To take an example, suppose

$$\mathbf{y} = (.3, .14, .1592, .6, .53, .58, .97).$$

Working through the coordinates one at a time we conclude that

$$\begin{aligned} x_7 &= 1 - y_7 = .03, \\ x_6 &= 1 + x_7 - y_6 = .45, \\ x_5 &= 1 + x_6 - y_5 = .92, \\ x_4 &= x_5 - y_4 = .32, \\ x_3 &= x_4 - y_3 = .1608, \end{aligned}$$

$$x_2 = x_3 - y_2 = .0208,$$
$$x_1 = 1 + x_2 - y_1 = .7208\,.$$

One can check that these coordinates define a point in the region corresponding to $w = 2734651$, and applying ϕ will take $\mathbf{x}$ back to $\mathbf{y}$.

A more succinct way to express the inverse transformation is to collect partial sums from right to left, taking only the fractional part of the partial sum as we go:

$$x_i = 1 - ((y_i + \cdots + y_n) \mod 1)\,.$$

Since the x_i must be generic, we leave this inverse map undefined whenever any subset of the y_j sums to an integer. But if the y_j are generic, this will never happen, so for the volume calculation this set has measure zero.

We have shown that ϕ is generically bijective and volume-preserving. Thus Proposition 7.1 follows.

7.2 Carries in addition

The volume calculation we just carried out turns out to have a surprising application in the problem of the distribution of "carries" in addition.

Consider adding two numbers in base ten with the usual addition algorithm. As we move from right to left we "carry" a 1 to the next column if the sum in the previous column (plus the previous carried digit) adds up to ten or more. How many carries will we expect to have?

Here is the sum of two thirty digit numbers:

$$\begin{array}{rr}
\text{carries: } & 000001\ 01011\ 11010\ 00101\ 00110\ 1001 \\
\\
 & 27182\ 81828\ 45904\ 52353\ 60287\ 47135 \\
+ & 31415\ 92653\ 58979\ 32384\ 62643\ 38328 \\
\hline
 & 58598\ 74482\ 04883\ 84738\ 22930\ 85463
\end{array}$$

We carried a one in thirteen of the thirty columns, or about forty-three percent of the time. Intuition tells us that we will carry a one about half the time, and this is indeed what will bear out.

But now consider adding three numbers. Here we can carry 0, 1, or 2. For example, here is the sum of three thirty digit numbers:

$$\begin{array}{rr}
\text{carries: } & 121011\ 11121\ 12111\ 11102\ 00001\ 0121 \\
\\
 & 57721\ 56649\ 01532\ 86060\ 65120\ 90082 \\
 & 69314\ 71805\ 59945\ 30941\ 72321\ 21458 \\
+ & 16449\ 34066\ 84822\ 64364\ 72415\ 16665 \\
\hline
 & 143485\ 62521\ 46300\ 81367\ 09857\ 28205
\end{array}$$

Of the thirty columns, seven carried zero, five carried two, and eighteen carried a one. It certainly doesn't seem that each carry is equally likely. Symmetry should suggest that carrying a zero has the same probability as carrying a two. The fact that we carry a one much more frequently is suggested by the fact that there are many more ways to obtain a number between 10 and 19 as a sum of three digits than there are ways to write a single digit number as a sum of three digits. But what exactly is the probability of getting a carry of two?

This is the problem considered by John Holte in [91]. (The title of this chapter is a nod to his fine paper.) To quote Holte's motivating question,

> What is the long-run frequency of each possible carry value when we add any number of long numbers represented in any base?

Or, when adding n random numbers in base b, what is the probability of having a carry of k? Remarkably, we will see the answer depends only on n and k, but not the base b. Let us denote the probability by $p_{n,k}$.

Theorem 7.1. *When adding n numbers in base b, the probability of having a carry of k is*

$$p_{n,k} = \frac{\left\langle {n \atop k} \right\rangle}{n!},$$

where $k = 0, 1, \ldots, n-1$.

The form this answer takes suggests that we make a connection between Holte's question and Foata's question. That is, we will show that the volume calculation in Proposition 7.1 implies Theorem 7.1.

To see the connection, suppose we are adding n numbers in base b, and that in a particular column we add digits $d_1, d_2, \ldots, d_n$, with $0 \le d_i \le b-1$. If we carried a j from the previous column, then to say that we carry k into the next column means

$$bk \le j + d_1 + d_2 + \cdots + d_n < b(k+1). \tag{7.1}$$

Now split j into n equal pieces so to write

$$j + d_1 + d_2 + \cdots + d_n = (d_1 + j/n) + (d_2 + j/n) + \cdots + (d_n + j/n).$$

Since $0 \le j \le n-1$, we have $0 \le j/n < 1$ and so $0 \le (d_i + j/n) < b$. Thus, dividing (7.1) by b, we obtain

$$k \le x_1 + x_2 + \cdots + x_n < k+1, \tag{7.2}$$

where

$$0 \le x_i = \frac{d_i + j/n}{b} < 1.$$

Let ψ denote the map from integer n-tuples to the cube $[0,1]^n$ given by $\psi(d_i) = (d_i + j/n)/b$, depending on the prior carry of j in $\{0, 1, \ldots, n-1\}$.

Thus having a carry of k corresponds to a point in the kth slice of the n-cube as discussed in Section 7.1. For fixed n and b, there are only finitely many points $(j, d_1, \ldots, d_n)$ in $[0, n-1] \times [0, b-1]^n$. Thus, the image of these points under ψ is finite as well. We want to argue that despite the discrete nature of this problem, we can use the volume calculation to obtain the result here. This can certainly be done if our points x_i are geometrically uniform in the n-cube.

If the digits d_i are uniformly random in $\{0, 1, \ldots, b-1\}$, intuition tells the points x_i are distributed roughly uniformly in the interval $[0, 1)$. While perfect uniformity won't always occur, we get something close enough to uniform. For fixed j, $d_i + j/n$ is just a slight shift away from uniform, and taking all j together splits $[0, 1)$ into n subintervals on which the x_i are identically distributed:

$$[0, 1/n) \cup [1/n, 2/n) \cup \cdots \cup [1 - 1/n, 1).$$

So whatever the probability of having a carry of j come in, this distribution is repeated in n intervals of equal size in $[0, 1)$, and this is good enough to conclude that probability is proportional to volume.

Hence we can conclude Theorem 7.1 from the geometric result: choosing n random digits in base b that results in a carry of k is equal to the probability of choosing a random point in the kth slice of the unit cube. However while [154] mentions this geometric argument, it was not the technique used by Holte. We present his argument next.

7.3 The amazing matrix

Holte's approach to the carries problem is to view the "carries process" as a Markov chain. This is natural, since carrying a k depends only on the digits in the column being added and the number j that was carried into that column.

Thus for fixed b and n, let $\pi(j, k)$ denote the probability of "carrying out" k given that we "carry in" j to a particular column. Then by (7.1),

$$\pi(j, k) = \frac{(\text{number of solutions } (d_1, \ldots, d_n) \text{ to } (7.1))}{b^n}.$$

To count the integer solutions to (7.1) is to ask for the number of integer solutions to

$$c + d_1 + d_2 + \cdots + d_n = b(k + 1) - 1 - j, \tag{7.3}$$

where $0 \leq c, d_1, d_2, \ldots, d_n \leq b - 1$. If we let $r = b(k + 1) - 1 - j$, then the number of solutions to Equation (7.3) is the coefficient of z^r in

$$(1 + z + z^2 + \cdots + z^{b-1})^{n+1} = \frac{(1-z^b)^{n+1}}{(1-z)^{n+1}}.$$

Expanding both numerator and denominator as series in z, we find

$$\begin{aligned}\frac{(1-z^b)^{n+1}}{(1-z)^{n+1}} &= \sum_{l=0}^{n+1}(-1)^l\binom{n+1}{l}z^{bl}\sum_{m\geq 0}\binom{m+n}{n}z^m,\\ &= \sum_{m,l\geq 0}(-1)^l\binom{n+1}{l}\binom{n+m}{n}z^{bl+m},\\ &= \sum_{r\geq 0}\left(\sum_{l=0}^{n+1}(-1)^l\binom{n+1}{l}\binom{n+r-bl}{n}\right)z^r.\end{aligned}$$

Given that $\binom{n+r-bl}{n} = 0$ if $r < bl$, the coefficient of z^r only ranges over $l \leq r/b = k + 1 - (j+1)/b$. We therefore have the following explicit formula for $\pi(j,k)$.

Proposition 7.2. *Suppose we are adding a list of n numbers in base b. The probability of carrying out a k from one column to the next, given that we carry in a j is*

$$\pi(j,k) = \frac{1}{b^n}\sum_{0\leq l\leq k+1-(j+1)/b}(-1)^l\binom{n+1}{l}\binom{n+b(k+1-l)-1-j}{n}.$$

The transition matrix $\Pi_n = (\pi(j,k))_{0\leq j,k\leq n-1}$ is what Holte calls the "Amazing matrix." Here are the first two matrices:

$$\Pi_2 = \frac{1}{2b}\begin{pmatrix} b+1 & b-1\\ b-1 & b+1\end{pmatrix}, \Pi_3 = \frac{1}{6b^2}\begin{pmatrix} b^2+3b+2 & 4b^2-4 & b^2-3b+2\\ b^2-1 & 4b^2+2 & b^2-1\\ b^2-3b+2 & 4b^2-4 & b^2+3b+2\end{pmatrix}.$$

It turns out that the matrix Π is diagonalizable, and its eigenvalues are $1, 1/b, 1/b^2, \ldots, 1/b^{n-1}$, though the eigenvectors are independent of b.

Let $V = V_n$ denote the matrix such that $V\Pi V^{-1} = D$, with D the diagonal matrix with the indicated eigenvalues. For example, one can check

$$\begin{pmatrix}1 & 4 & 1\\ 1 & 0 & -1\\ 1 & -2 & 1\end{pmatrix}\cdot\frac{1}{6b^2}\begin{pmatrix} b^2+3b+2 & 4b^2-4 & b^2-3b+2\\ b^2-1 & 4b^2+2 & b^2-1\\ b^2-3b+2 & 4b^2-4 & b^2+3b+2\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ 0 & 1/b & 0\\ 0 & 0 & 1/b^2\end{pmatrix}\begin{pmatrix}1 & 4 & 1\\ 1 & 0 & -1\\ 1 & -2 & 1\end{pmatrix},$$

so

$$V_3 = \begin{pmatrix} 1 & 4 & 1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{pmatrix}.$$

Let $V_n = (v(j,k))_{0 \le j,k \le n-1}$. It turns out that

$$v(j,k) = \sum_{l=0}^{k} (-1)^l \binom{n+1}{l} (k+1-l)^{n-j}.$$

The matrices V_4 and V_5 are shown here:

$$V_4 = \begin{pmatrix} 1 & 11 & 11 & 1 \\ 1 & 3 & -3 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -3 & 3 & -1 \end{pmatrix}, V_5 = \begin{pmatrix} 1 & 26 & 66 & 26 & 1 \\ 1 & 10 & 0 & -10 & -1 \\ 1 & 2 & -6 & 2 & 1 \\ 1 & -2 & 0 & 2 & -1 \\ 1 & -4 & 6 & -4 & 1 \end{pmatrix}.$$

Notice the Eulerian numbers appearing in the top row! This is because if $j = 0$,

$$v(0,k) = \sum_{l=0}^{k} (-1)^l \binom{n+1}{l} (k+1-l)^n,$$

which is the formula given in Equation (1.11) for the Eulerian number $\left\langle {n \atop k} \right\rangle$.

For fixed j, $v(j,k)$ is the coefficient of t^k in

$$\left(\sum_{l \ge 0} (-1)^l \binom{n+1}{l} t^l \right) \left(\sum_{m \ge 0} (m+1)^{n-j} t^m \right) = (1-t)^{n+1} \frac{S_{n-j}(t)}{(1-t)^{n+1-j}},$$

where the second sum is the Carlitz identity given in Equation (1.10). Thus we have a simpler way to describe the entries of V:

$$\sum_{k \ge 0} v(j,k) t^k = (1-t)^j S_{n-j}(t),$$

where $S_{n-j}(t)$ is the Eulerian polynomial.

Now let us verify that $V\Pi = DV$.

We want to show that

$$\sum_{k=0}^{n-1} v(j,k) \pi(k,l) = \frac{v(j,l)}{b^j},$$

for $0 \le j, l \le n-1$.

Using the formulas we've derived, we have

$$\begin{aligned}
&\sum_{k=0}^{n-1} v(j,k)\pi(k,l) \\
&\quad = \frac{1}{b^n}\sum_{k=0}^{n-1}\sum_{m=0}^{l+1-(k+1)/b}(-1)^m\binom{n+1}{m}\binom{n-1-k+(l+1-m)b}{n}v(j,k), \\
&\quad = \frac{1}{b^n}\sum_{m=0}^{l}(-1)^m\binom{n+1}{m}\sum_{k=0}^{(l+1-m)b-1}\binom{n-1-k+(l+1-m)b}{n}v(j,k).
\end{aligned} \tag{7.4}$$

If we let $M = (l+1-m)b-1$, we can rewrite the inner sum here as

$$\sum_{k=0}^{M}\binom{n+M-k}{n}v(j,k),$$

which we can recognize as the coefficient of t^M in

$$\left(\sum_{r\geq 0}\binom{n+r}{n}t^r\right)\left(\sum_{k\geq 0}v(j,k)t^k\right) = \frac{1}{(1-t)^{n+1}}(1-t)^j S_{n-j}(t).$$

Using the Carlitz identity once more, we find

$$\begin{aligned}
\frac{1}{(1-t)^{n+1}}(1-t)^j S_{n-j}(t) &= \frac{S_{n-j}(t)}{(1-t)^{n+1-j}}, \\
&= \sum_{M\geq 0}(M+1)^{n-j}t^M,
\end{aligned}$$

and therefore

$$\sum_{k=0}^{M}\binom{n+M-k}{n}v(j,k) = (M+1)^{n-j}.$$

Returning to Equation (7.4), we now obtain

$$\begin{aligned}
\sum_{k=0}^{n-1} v(j,k)\pi(k,l) &= \frac{1}{b^n}\sum_{m=0}^{l}(-1)^m\binom{n+1}{m}((l+1-m)b)^{n-j}, \\
&= \frac{1}{b^j}\sum_{m=0}^{l}(-1)^m\binom{n+1}{m}(l+1-m)^{n-j}, \\
&= \frac{v(j,l)}{b^j},
\end{aligned}$$

as desired.

Since the largest eigenvalue of Π is 1, the Perron-Frobenius theorem tells us the first row of V is proportional to the stable distribution for the carries process. Hence Theorem 7.1 follows.

We finish this chapter by remarking that the Amazing Matrix has reappeared in some surprising places. For instance Francesco Brenti and Volkmar Welker rediscovered the Amazing Matrix in commutative algebra [36], where Π is essentially the transformation of a Hilbert series of a graded ring to its bth "Veronese algebra." In terms of generating functions, this is the map

$$\frac{h(t)}{(1-t)^d} = \sum_{k\geq 0} a_k t^k \mapsto \sum_{k\geq 0} a_{bk} t^k = \frac{h^{\langle b\rangle}(t)}{(1-t)^d}.$$

The transformation matrix for $h \mapsto h^{\langle b\rangle}$ is (after deleting the first row and column) the Amazing Matrix.

Brenti and Welker analyze this transformation as they did for the barycentric subdivision transformation, which is discussed in Chapter 9. Since the stable distribution for the Amazing Matrix is the Eulerian distribution, they find that repeatedly applying the Veronese map takes any h-polynomial to the Eulerian polynomial in the limit. In particular, applying the map enough times yields a real-rooted h-polynomial.

In a different direction, the Amazing Matrix shows up in the analysis of card shuffling. Persi Diaconis and Jason Fulman have several papers on this topic [57–59]. A "b"-shuffle of a deck of cards is a generalization of the usual riffle shuffle, which is a b-shuffle for $b = 2$. In a b-shuffle we split the deck into b piles of sizes $c_1, \ldots, c_b$ with probability

$$\frac{\binom{n}{c_1,\ldots,c_b}}{b^n}.$$

Then we drop cards randomly from each of the piles, with probability proportional to the size of the pile. The connection between carries in addition and shuffling is most succinctly summarized by Theorem 1.1 of [58], which we quote directly here:

> The probability that the base-b carries chain goes from 0 to j in r steps is equal to the probability that the permutation in S_n obtained by performing r successive b-shuffles (started at the identity) has j descents.

The reader is encouraged to read [57] for a very friendly introduction to this story.

Part II
Combinatorial topology

Chapter 8
Simplicial complexes

ANOTHER SETTING IN WHICH THE EULERIAN NUMBERS have arisen is in combinatorial topology. In this chapter we will put some of our previous work in the context of the study of simplicial complexes. While there is some assumed familiarity with topological concepts, no formal topological background is required for understanding this chapter.

8.1 Abstract simplicial complexes

A *simplicial complex* Δ on a vertex set V is a collection of subsets F of V, called *faces*, such that:

- if $v \in V$ then $\{v\} \in \Delta$,
- if $F \in \Delta$ and $G \subset F$, then $G \in \Delta$.

The *dimension* of a face F is $\dim F = |F| - 1$. In particular $\dim \emptyset = -1$. The dimension of the complex Δ itself, denoted by $\dim \Delta$, is the maximum of the dimensions of its faces. Maximal dimensional faces are often called *facets*.

Using the common nomenclature, vertices are zero-dimensional faces, one-dimensional faces are edges, two-dimensional faces are triangles, three-dimensional faces are tetrahedra, and so on. A k-dimensional face is called a k-simplex.

Let $\mathcal{F}_k(\Delta)$ denote the set of all k-element sets in Δ ($(k-1)$-dimensional faces), and let $\partial\mathcal{F}_k(\Delta)$ denote the *boundary* of this set, i.e., the set of all $(k-1)$-element subsets of the sets in $\mathcal{F}_k(\Delta)$. That is,

$$\partial\mathcal{F}_k(\Delta) = \bigcup_{G \in \mathcal{F}_k(\Delta)} \left\{ F \in \binom{V}{k-1} : F \subset G \right\},$$

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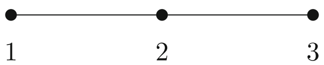
where $\binom{V}{k}$ denotes the set of all k-element subsets of the vertex set V. With this notation, the conditions for Δ to be a simplicial complex can be phrased as:

$$\partial \mathcal{F}_k(\Delta) \subseteq \mathcal{F}_{k-1}(\Delta), \qquad \text{for } k = 1, 2, \ldots, d.$$

That is, the boundary of the set of $(k-1)$-faces is contained in the set of $(k-2)$-faces. The boundary of whole complex is simply the set of all faces of Δ that are properly contained in some other face of Δ, i.e., $\partial\Delta$ is the set of all non-maximal faces of Δ.

For us, Δ is a combinatorial object, not a geometric one. However, we can construct the *geometric realization* of Δ, denoted $||\Delta||$, by creating a copy of the standard geometric k-simplex for each abstract k-simplex, and gluing faces according to inclusion of vertex sets in Δ. (The standard geometric k-simplex is the convex hull of $k+1$ standard basis vectors.) More precisely, if F and G are faces of Δ, we identify the geometric simplices $||F||$ and $||G||$ along the geometric realization of their common face: $||F \cap G||$. When we attribute a topological property to an abstract simplicial complex (e.g., Euler characteristic, homology), what we really mean is that, up to homeomorphism, the geometric realization has the property.

For example, if $V = \{1, 2, 3\}$, $\Delta = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}\}$ is a one-dimensional simplicial complex, which we can represent pictorially as:



.

Similarly, the picture in Figure 8.1 encodes a two-dimensional simplicial complex. This complex has one triangle, $\{1, 3, 4\}$, six edges, $\{0, 1\}$, $\{1, 2\}$, $\{1, 3\}$, $\{1, 4\}$, $\{2, 4\}$, $\{3, 4\}$, and five vertices, $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, and $\{4\}$.

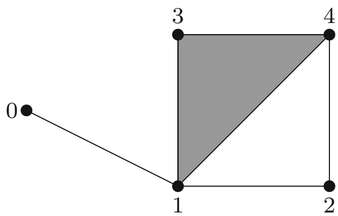
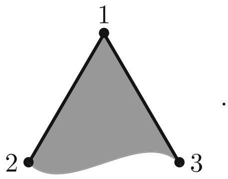


Fig. 8.1 A two-dimensional simplicial complex.

On the other hand, the set $\{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{1, 2, 3\}\}$ is *not* a simplicial complex since it is missing the edge $\{2, 3\}$. It might look like this:



Notice that part of the boundary of the 2-dimensional cell is missing from the complex.

Given a topological cell complex Δ (a bunch of open cells glued along their boundaries, i.e., a regular CW-complex) there is a quite natural partial order on its open cells given by $F \leq_\Delta G$ if and only if the closure of G contains the closure of F: $\overline{F} \subseteq \overline{G}$. We call this partial order the *face poset* of Δ.

In the case where Δ is a simplicial complex on vertex set V, the face poset is an order ideal in the boolean algebra 2^V defined by its maximal faces. Conversely, any order ideal of a boolean algebra defines an abstract simplicial complex, and for this reason abstract simplicial complexes are sometimes called *set systems*. For example, the face poset of the complex shown in Figure 8.1 is given in Figure 8.2.

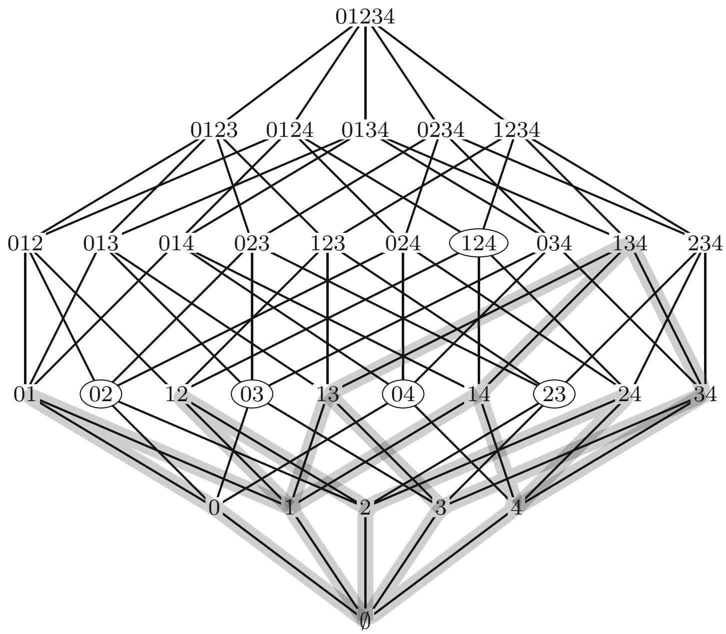


Fig. 8.2 The face poset of the complex in Figure 8.1 is the lower ideal highlighted. The minimal non-faces are circled.

Let $P(\Delta)$ denote the face poset of the complex Δ. It is self-evident that this poset is ranked by one more than dimension, which for a simplicial complex is just the cardinality of the subsets.

Observation 8.1 *The face poset of a simplicial complex Δ has the following interpretations for its rank generating function:*

$$\begin{aligned} f(P(\Delta);t) &= \sum_{F \in P(\Delta)} t^{\mathrm{rk}(F)}, \\ &= \sum_{F \in \Delta} t^{\dim F+1}, \\ &= \sum_{F \in \Delta} t^{|F|}. \end{aligned}$$

8.2 Simple convex polytopes

Many interesting simplicial complexes arise from convex polytopes. The subject of polytopes is vast, and we will only scratch the surface here. Our purpose is only to provide a glimpse of an area in which simplicial complexes (and the tools for studying them) have natural analogues. For a more thorough treatment, see Ziegler's book [169].

There are two standard, equivalent definitions of a convex polytope. One is as the intersection of affine halfspaces (provided the intersection is bounded). The other definition is the convex hull of a finite number of points in Euclidean space. For the purposes of our discussion, take a *convex polytope* $\mathcal{P}$ to be the convex hull of a finite number of points in Euclidean space. There is a natural cell decomposition of the resulting body into vertices, edges, faces, and so on, though now k-faces need not be simplices.

First, we define the *vertices* to be those points that do not lie on a line between two other points of $\mathcal{P}$. To put it another way, any small one-dimensional neighborhood around a vertex contains points not in $\mathcal{P}$. In general, a k-dimensional face F of $\mathcal{P}$ is a maximal collection of points in $\mathcal{P}$ that are contained in a k-dimensional affine space, but for which any $(k+1)$-dimensional neighborhood of F has points outside of $\mathcal{P}$. If $\mathcal{P}$ is d-dimensional, the *facets* of $\mathcal{P}$ refer to the $(d-1)$-dimensional faces.

With this definition, the faces are closed sets, and inclusion of faces means pointwise inclusion. We can study the face poset of a polytope and count faces by dimension just as with a simplicial complex.

A *simplicial polytope* is one for which every face on the boundary of $\mathcal{P}$ is a simplex, and in this case the boundary $\partial\mathcal{P}$ is the geometric realization of a simplicial sphere.

A *simple polytope* of dimension d is one for which every vertex is adjacent to exactly d facets. In this case, the polytope is "dual" to a simplicial polytope.

That is, if $\mathcal{P}$ is a simple polytope in a vector space V there is a polytope $\mathcal{P}^*$ in the dual space V^* given by

$$\mathcal{P}^* = \{\mathbf{x} \in V^* : \langle \mathbf{x}, \mathbf{y} \rangle \leq 1 \text{ for } \mathbf{y} \in \mathcal{P}\}.$$

When $V = \mathbb{R}^d$, we have $V \cong V^*$, so we can view both polytopes as occupying the same space. The pairing between $\mathcal{P}$ and $\mathcal{P}^*$ is such that k-faces of $\mathcal{P}$ correspond to $(d-k)$-faces of $\mathcal{P}^*$. This maps $\mathcal{P}$ to the empty face, facets to vertices and so on. Since vertices of $\mathcal{P}$ are contained in d facets, this means the facets of $\mathcal{P}^*$ have d vertices, i.e., they are simplices. Thus the dual of a simple polytope is a simplicial polytope.

For example, in Figure 8.3 we see that the 3-cube is a simple polytope, but not simplicial. Its dual is the octahedron, whose boundary is a simplicial complex.

Recall that the dual of a poset $(P, \leq)$ is the poset $(P^*, \leq)$, given by $x \leq_P y$ if and only if $y \leq_{P^*} x$. Intuitively, it is the reverse of the order on P. The following is a useful fact relating the face poset of a polytope and its dual.

Proposition 8.1. *Let $P(\mathcal{P}) = (P, \leq)$ be the face poset of a polytope $\mathcal{P}$. Then $P(\mathcal{P}^*) \cong (P^*, \leq)$, that is, the face poset of dual polytope $\mathcal{P}^*$ is isomorphic to the dual poset of P.*

See Figures 8.4 and 8.5 for an illustration. Full details can be found in [169, Chapter 2].

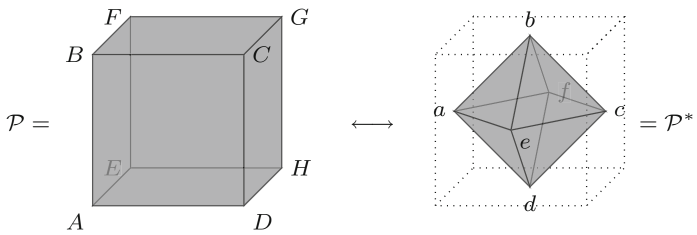


Fig. 8.3 The cube is a simple polytope. Its dual, the octahedron, is simplicial.

Other examples of simple polytopes include the permutahedra and the associahedra discussed in Chapter 5.

8.3 Boolean complexes

As mentioned, the face poset of a simplicial complex is an order ideal in 2^V. In particular, the interval $[\emptyset, F]$ in the face poset is isomorphic to 2^F, the boolean algebra on the vertices of F. A *boolean complex* is a cell complex Δ

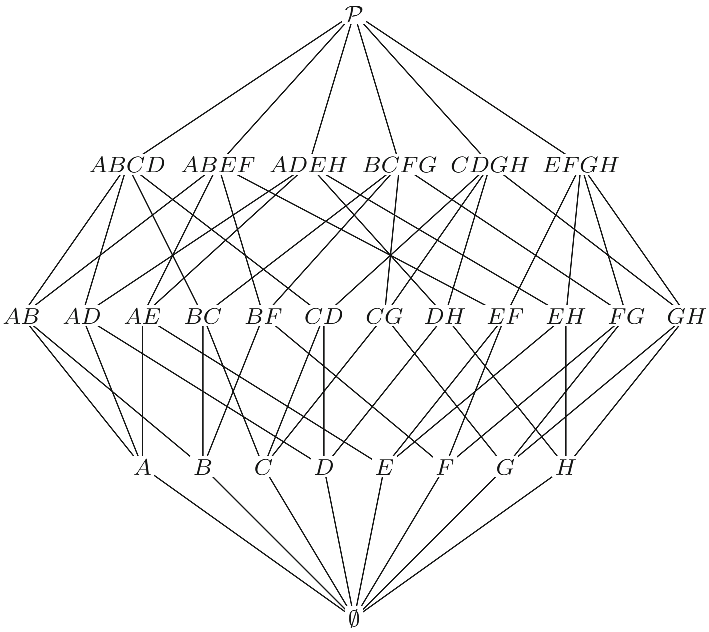


Fig. 8.4 The face poset of the cube $\mathcal{P}$ in Figure 8.3.

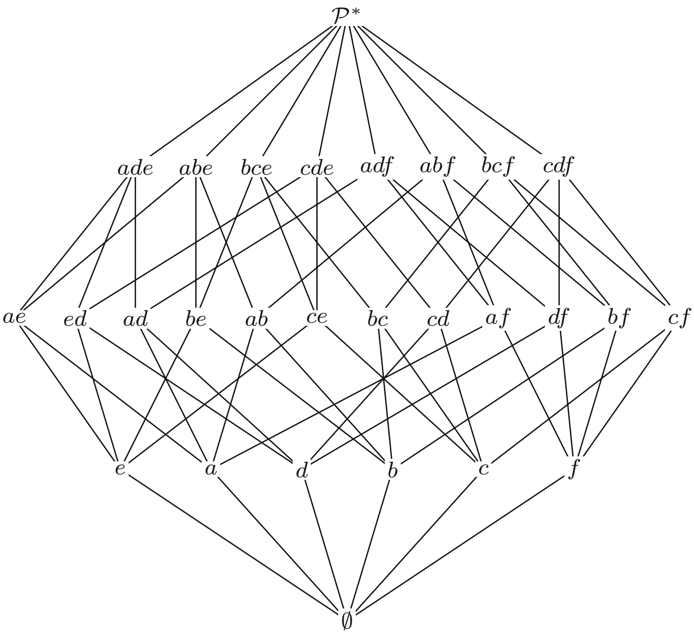


Fig. 8.5 The face poset of the octahedron $\mathcal{P}^*$ in Figure 8.3.

whose face poset requires only this weaker condition: every principal order ideal in $P(\Delta)$ is boolean. (Recall a principal order ideal is the set of elements below one particular element.) These are also sometimes called *simplicial posets*, or, because we can think of every face as a combinatorial simplex, *triangulated manifolds*.

Every simplicial complex is a boolean complex, but not conversely. Two distinct faces of a simplicial complex cannot share the same vertex set, but in a boolean complex this can happen. For example, the cell complex shown in Figure 8.6 has two triangles glued together at the corners a and b. Since there is more than one edge with vertex set $\{a, b\}$, this is not a simplicial complex. Notice that if a boolean complex is *not* simplicial, as in this example, its face poset is not a lattice, i.e., there is a collection of vertices with no least upper bound.

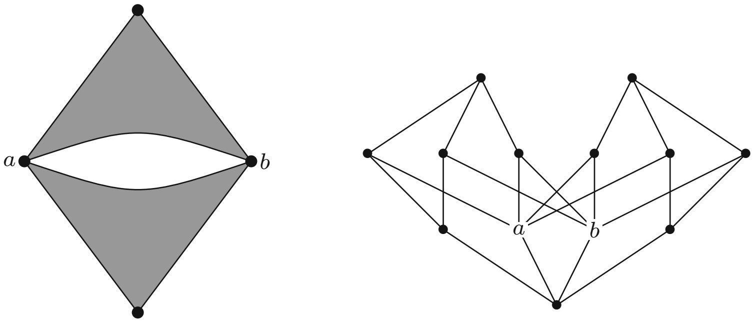


Fig. 8.6 A boolean complex and its face poset.

8.4 The order complex of a poset

Given any finite poset P, there is a natural simplicial complex associated with P called the *order complex*, denoted $\Delta(P)$. The complex $\Delta(P)$ has vertex set $V = P$, and each face of $\Delta(P)$ corresponds to a chain of elements of P:

$$a_1 <_P a_2 <_P \cdots <_P a_k \leftrightarrow F = \{a_1, a_2, \ldots, a_k\} \in \Delta(P).$$

For example, in Figure 8.7, we see a poset P and its order complex.

It is an exercise to verify that $\Delta(P)$ is indeed a simplicial complex for any finite poset P. (See Problem 8.8.) A more interesting question is to ask what complexes arise as the order complex of some poset. For example, one can show the complex shown in Figure 8.1 is *not* the order complex of any poset.

While the order complex of a poset and the face poset of a complex are not directly related, we remark that if Δ is a simplicial complex and $P = P(\Delta)$

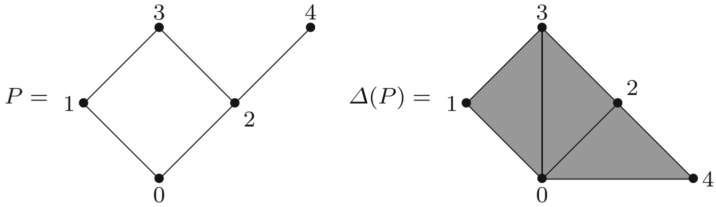


Fig. 8.7 A poset P and its order complex $\Delta(P)$.

is its face poset, then Δ and the order complex of P have the same topology (they are homeomorphic). In fact the order complex of P is known as the *barycentric subdivision* of Δ, a special construction that will be discussed further in Chapter 9.

8.5 Flag simplicial complexes

One reason why the complex of Figure 8.1 cannot be an order complex is that order complexes are part of a special family of simplicial complexes known as *flag complexes*. A flag complex is a simplicial complex whose minimal non-faces in 2^V are edges. We can see in Figure 8.2 that the simplicial complex from Figure 8.1 has the triangle $\{1,2,4\}$ as a minimal non-face. Hence, it is not a flag complex and cannot possibly be the order complex of a poset. We sometimes say a flag complex has no "missing faces" of dimension greater than one.

Flag complexes are completely determined by their 1-skeleton, i.e., by the graph showing only vertices and edges. Any time vertices $a_1, \dots, a_k$ are pairwise connected in a flag complex Δ, we are guaranteed that $\{a_1, \dots, a_k\}$ is a face of Δ. In graph theory, a collection of k pairwise connected vertices is known as a *complete graph on k vertices*, or a *k-clique*. For this reason, flag complexes are sometimes known as *clique complexes*.

To see that an order complex of a poset P is a flag complex, we first form the *comparability graph* of P, by connecting elements a and b if and only if a and b are comparable in P. Then the order complex $\Delta(P)$ is the clique complex for the comparability graph. Since the comparability graph for a chain of n elements is itself a complete graph, the order complex of an n-chain is an $(n-1)$-simplex. See Figure 8.8.

Another broad class of flag complexes are those that arise from a simplicial hyperplane arrangement. We say a linear hyperplane arrangement $\mathcal{H}$ is *simplicial* if every face gives rise to a simplex when intersected with a sphere. Rays become vertices, two-dimensional cones become edges, three-dimensional cones become triangles, and so on. See Figure 8.9. The face $(0,0,\dots,0)$ at the center of the arrangement corresponds to the empty face

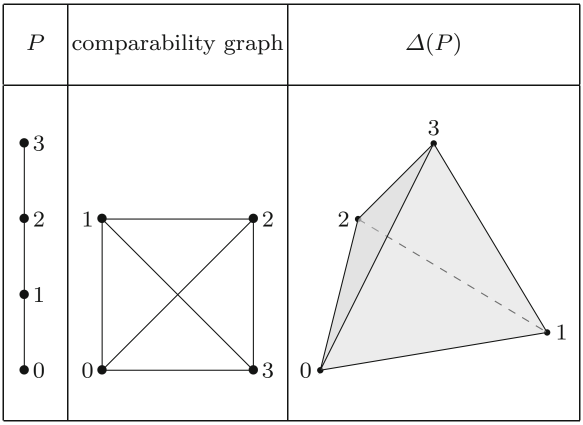


Fig. 8.8 The order complex of 4-chain is a 3-simplex.

in the simplicial complex, while the chambers in the complement of $\mathcal{H}$ correspond to facets.

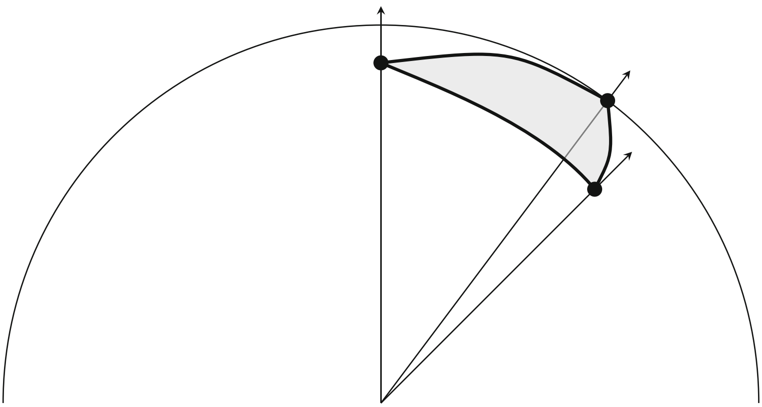


Fig. 8.9 A simplicial cone intersected with a sphere. Rays become vertices, two-dimensional cones become edges, three-dimensional cones become triangles, and so on.

Let $\Sigma = \Sigma(\mathcal{H})$ denote the cell complex obtained in this way, by intersecting $\mathcal{H}$ with a sphere. Recall we used the same notation for the poset of faces of $\mathcal{H}$ in Section 5.3—we now recognize the poset studied there as the face poset of Σ. Clearly the geometric realization of such a complex is a sphere. We want to verify that Σ is a flag simplicial complex as well.

Recall from Section 5.3 that there is a geometrically defined associative product of faces in any hyperplane arrangement, which we called the Tits product. Given two faces F and G, the product FG is the first face entered upon walking some small distance from F to G.

Proposition 5.3 states that, given any collection of faces, the faces are pairwise commuting if and only if they lie on the boundary of a common face. When applied to the rays of the hyperplane arrangement, this shows that two vertices a, b of Σ are connected with an edge if and only if they commute: $ab = ba$. If $a_1, \ldots, a_k$ are pairwise commuting vertices, then their product, taken in any order, is a face of Σ. (Indeed, Proposition 5.3 also says this product is the least upper bound for the collection $\{a_1, \ldots, a_k\}$ in the face poset.) Thus Σ is the clique complex for its one-skeleton, i.e., it is a flag complex.

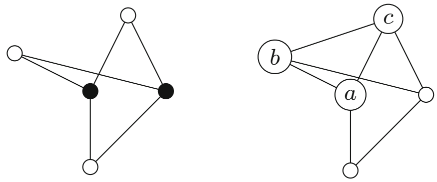
Observation 8.2 *If $\mathcal{H}$ is a simplicial hyperplane arrangement, then $\Sigma(\mathcal{H})$ is a flag simplicial sphere.*

One final example we mention here is the simplicial complex dual to the associahedron. Recall from Section 5.8 that faces of the associahedron are encoded with planar rooted trees, or by a simple bijection, partial parenthesizations of a string of n symbols. The vertices of this complex are given by expressions that have only a single pair of parentheses. We say that two vertices are adjacent if and only if the pairs of parentheses are noncrossing. That is, if the positions of the parentheses from the first vertex are a and b, and the positions of the parentheses from the second vertex are c and d, then we cannot have $a < c < b < d$ or $c < a < d < b$. Every larger parenthesization can be decomposed in a natural way into a collection of mutually noncrossing vertices, and every collection of pairwise noncrossing vertices gives rise to a unique parenthesization. Thus the associahedron is the clique complex of the graph given by pairs of noncrossing vertices. See Problem 8.9.

8.6 Balanced simplicial complexes

A simplicial complex Δ with vertex set V is called *d-colorable* if there is a function $c : V \to \{1, 2, \ldots, d\}$, called a *coloring* of its vertices, such that for every face $F \in \Delta$, the restriction map $c : F \to \{1, 2, \ldots, d\}$ is one-to-one. That is, every face has distinctly colored vertices. If a $(d-1)$-dimensional complex Δ is d-colorable, we say it is a *balanced simplicial complex*.

A familiar example of a balanced simplicial complex is a bipartite graph. It is a one-dimensional complex for which two colors, say white and black, can be used to color the vertices so that every edge has one black vertex and one white vertex. See Figure 8.10. A bipartite graph can have no three pairwise connected vertices, a, b, c, since if we color a black, then one of b or c must also be black. See the graph in Figure 8.10(b). Similar reasoning



(a) bipartite (b) not bipartite

Fig. 8.10 (a) A bipartite graph and (b) a non-bipartite graph.

shows that a bipartite graph can have no cycles of odd length, and in fact this characterizes the bipartite graphs.

This fact about bipartite graphs is not important on its own, but is meant only to illustrate how special the balanced d-complexes are among all d-dimensional simplicial complexes.

As a different sort of example, we note that the order complex of a ranked poset is balanced. Indeed, if P is a ranked poset, give each element a the color $\mathrm{rk}(a)+1$. Since a chain cannot have two elements of the same rank, each face has distinctly colored vertices. A maximal chain in P corresponds to a facet of $\Delta(P)$, so the total number of colors equals the number of vertices in a maximal dimensional face. Hence when P is a ranked poset, $\Delta(P)$ is both a balanced complex and a flag complex.

Observation 8.3 *The order complex of a poset is a balanced flag complex.*

8.7 Face enumeration

The rank numbers of the face poset give an important combinatorial invariant of a simplicial complex (indeed, of any finite cell complex), which we call its *f-vector*. This vector records the number of faces of each dimension. For a simplicial complex Δ, we write

$$f(\Delta) = (f_0, f_1, \ldots),$$

with

$$f_k = |\{F \in \Delta : |F| = 1 + \dim F = k\}|.$$

The polynomial $f(P(\Delta); t) = f(\Delta; t)$ is called the *f-polynomial*, which we now write without reference to the face poset, i.e.,

$$f(\Delta; t) = \sum_{F \in \Delta} t^{|F|} = \sum_{k=0}^{1+\dim \Delta} f_k t^k.$$

So for example, with Δ as in Figure 8.1, we have $f(\Delta) = (1, 5, 6, 1)$ and $f(\Delta; t) = 1 + 5t + 6t^2 + t^3$.

While the f-vector encodes purely combinatorial data, it can be used to deduce topological information. For example,

$$1 - f(\Delta; -1) = f_1 - f_2 + f_3 - \cdots = (\text{vertices}) - (\text{edges}) + (\text{faces}) - \cdots = \chi(\Delta),$$

is the *Euler characteristic* of Δ. For our purposes, we will find it more convenient to work with the *reduced* Euler characteristic,

$$\widetilde{\chi}(\Delta) = -1 + \chi(\Delta) = -f(\Delta; -1).$$

So, for example, since an n-simplex has $f(\Delta; t) = (1+t)^{n+1}$, and its boundary has $f(\partial\Delta; t) = (1 + t)^{n+1} - t^{n+1}$ we have

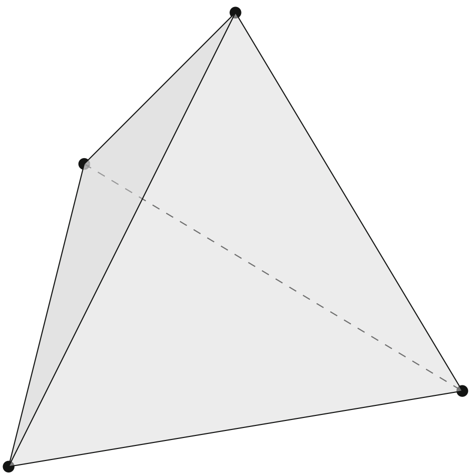
$$\widetilde{\chi}(n\text{-ball}) = 0$$

and

$$\widetilde{\chi}(n\text{-sphere}) = (-1)^n.$$

Returning to the example in Figure 8.1, we see $\widetilde{\chi}(\Delta) = -1+5-6+1 = -1$, which we expect since Δ can be deformed into a 1-sphere (contract the edge $\{0, 1\}$ and collapse the triangle $\{1, 3, 4\}$).

As another example, let Δ be the boundary of the 3-simplex shown here:



Then we have $f(\Delta) = (1, 4, 6, 4)$, and $\widetilde{\chi}(\Delta) = -1 + 4 - 6 + 4 = 1$ since Δ is a 2-sphere.

What characterizes an f-vector of a simplicial complex? The entries are obviously nonnegative integers, and $f_0 = 1$, but what other restrictions are there? Well, for one thing, if there are n vertices there can be at most $\binom{n}{2}$ edges, since there is at most one edge for every pair of vertices. That is,

$$f_2 \le \binom{f_1}{2}.$$

This simple observation can be greatly generalized. It turns out there is a sharp upper bound on the number of $(k+1)$-faces expressed as a polynomial in f_k. (Likewise, there is a sharp lower bound on the number of k faces required for a given number of $(k+1)$-faces.) Collectively, these restrictions, known as the *Kruskal-Katona-Schützenberger* inequalities (or KKS inequalities), characterize the set of f-vectors of simplicial complexes. See Chapter 10. We remark that characterizing f-vectors of boolean complexes is much, much simpler. See Problem 8.7.

8.8 The *h*-vector

There is a transformation of the f-vector that can sometimes bring features of the simplicial complex into sharp focus. One way to think of this transformation is to write the f-vector in terms of right-justified copies of rows of Pascal's triangle. For example, if we consider the example of Δ shown in Figure 8.1, with $f(\Delta) = (1,5,6,1)$, we have:

$$\begin{array}{rrrrr}
 & (1, & 5, & 6, & 1) \\ \hline
1\times & (1, & 3, & 3, & 1) \\
2\times & & (1, & 2, & 1) \\
-1\times & & & (1, & 1) \\
-1\times & & & & (1)
\end{array}$$

The coefficients used in this expansion, read from top to bottom, make up the *h-vector* of Δ. So in this case, $h(\Delta) = (1,2,-1,-1)$.

More generally, if Δ is $(d-1)$-dimensional (so that the f-vector is $(f_0, f_1, \ldots, f_d)$), let

$$H_d = \left[(-1)^{i+j} \binom{d-j}{i-j} \right]_{0 \le i,j \le d}.$$

Then we define the h-vector to be

$$h(\Delta) = H_d \cdot f(\Delta).$$

For example, with $f(\Delta) = (1,4,6,4)$, we have

$$H_d = \begin{pmatrix} \binom{3}{0} & -\binom{2}{-1} & \binom{1}{-2} & -\binom{0}{-3} \\ -\binom{3}{1} & \binom{2}{0} & -\binom{1}{-1} & \binom{0}{-2} \\ \binom{3}{2} & -\binom{2}{1} & \binom{1}{0} & -\binom{0}{-1} \\ -\binom{3}{3} & \binom{2}{2} & -\binom{1}{1} & \binom{0}{0} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 3 & -2 & 1 & 0 \\ -1 & 1 & -1 & 1 \end{pmatrix},$$

and so

$$h(\Delta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 3 & -2 & 1 & 0 \\ -1 & 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 4 \\ 6 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

Notice that for any simplicial complex, $f_0 = h_0 = 1$, while

$$h_d = \sum_{i=0}^{d} (-1)^{d-i} f_i = (-1)^d f(\Delta; -1) = (-1)^d \, \widetilde{\chi}(\Delta).$$

The general formula for an entry of the h-vector is:

$$\begin{aligned} h_k &= f_k \binom{d-k}{0} - \binom{d-k+1}{1} f_{k-1} + \cdots + (-1)^i \binom{d-k+i}{i} f_{k-i} + \cdots, \\ &= \sum_{i=0}^{k} (-1)^{k-i} f_i \binom{d-i}{k-i}. \end{aligned}$$

It is easily verified that the matrix H_d has inverse

$$H_d^{-1} = \left[\binom{d-j}{i-j} \right]_{0 \le i,j \le d},$$

so there is no true loss of information when working with h-vectors instead of f-vectors, provided we know the dimension of the complex.

If we define the *h-polynomial* to be the generating function for the h-vector,

$$h(\Delta; t) = \sum_{i=0}^{d} h_i t^i,$$

we can state the linear relationship between the f-vector and the h-vector as:

$$f(\Delta; t) = \sum_{i=0}^{d} h_i t^i (1+t)^{d-i} = (1+t)^d h(\Delta; t/(1+t)), \tag{8.1}$$

and

$$h(\Delta; t) = \sum_{i=0}^{d} f_i t^i (1-t)^{d-i} = (1-t)^d f(\Delta; t/(1-t)). \tag{8.2}$$

The form of Equation (8.1) should look familiar. If $\Sigma(n)$ is the simplicial complex dual to the permutahedron (obtained by intersecting the braid arrangement with a sphere), Theorem 5.3 says

$$f(\Sigma(n); t) = (1+t)^{n-1} S_n(t/(1+t)),$$

i.e., the Eulerian polynomial is the h-polynomial of the permutahedron. Similarly, Theorem 5.4 says the Narayana polynomial is the h-polynomial of the associahedron.

8.9 The Dehn-Sommerville relations

One of the primary reasons for studying the h-vector is that it makes certain relations among the face numbers more apparent. Just as the Euler characteristic appears as the top entry in the h-vector, there are other, more subtle relationships between the entries of the f-vector that depend on the topology of Δ. The *Dehn-Sommerville relations* refer to the relations among face numbers in simplicial spheres. They were originally studied in the case of polytopes, and the idea can be applied to any triangulation of a manifold without boundary. Victor Klee called such manifolds "Eulerian." Nowadays, the face poset of such a simplicial complex is known as an *Eulerian poset.*

Define the *link* of a face $F \in \Delta$ to be

$$\mathrm{lk}(\Delta; F) = \{G \in \Delta : G \cap F = \emptyset, G \cup F \in \Delta\},$$

which we may abbreviate with $\mathrm{lk}\, F$ when Δ is understood. It turns out that if Δ triangulates a manifold, links of nonempty faces are homologous to either balls or spheres. They are balls when F lies on the boundary of Δ and spheres otherwise. In particular, if Δ triangulates a manifold without boundary (such as a sphere), then for any nonempty face F,

$$\widetilde{\chi}(\mathrm{lk}\, F) = (-1)^{\dim \mathrm{lk}\, F}.$$

We call a simplicial complex with this property an *Eulerian complex*, and throughout the rest of this section we assume Δ is an Eulerian complex. For the moment, however, we make no assumption about $\widetilde{\chi}(\mathrm{lk}\, \emptyset) = \widetilde{\chi}(\Delta)$ itself.

Now assume further that all maximal faces of Δ have the same dimension. In this case Δ is what is known as a *pure* simplicial complex. For pure complexes, the dimension of the link of a face is its codimension. That is, if Δ is $(d-1)$-dimensional, and F is a nonempty face of Δ,

$$\dim(\mathrm{lk}\, F) = \dim \Delta - \dim F,$$

and therefore,

$$\widetilde{\chi}(\mathrm{lk}\, F) = (-1)^{d-|F|}.$$

Now for any nonempty face F of Δ, define

$$\phi(F) = \sum_{F \subseteq G \in \Delta} (-1)^{|G|}.$$

Then letting $H = G - F$ denote the complement of F in G, we have $H \cap F = \emptyset$ and $H \cup F = G \in \Delta$, i.e., $H \in \operatorname{lk} F$. Conversely, if $H \in \operatorname{lk} F$, then $F \subseteq H \cup F = G$ is a face of Δ containing F. In other words, we have:

$$\begin{aligned}
\phi(F) &= \sum_{H \in \operatorname{lk} F} (-1)^{|F|+|H|}, \\
&= (-1)^{|F|} \sum_{H \in \operatorname{lk} F} (-1)^{|H|}, \\
&= (-1)^{|F|} f(\operatorname{lk} F; -1), \\
&= (-1)^{|F|} (-\widetilde{\chi}(\operatorname{lk} F)), \\
&= (-1)^{|F|} (-1)^{d-1-|F|}, \\
&= (-1)^{d-1}.
\end{aligned}$$

So if Δ is a pure Eulerian complex, $\phi(F)$ is constant for every nonempty face!

Now if we sum $\phi(F)$ over all faces of cardinality k, we get:

$$\begin{aligned}
(-1)^{d-1} f_k &= \sum_{|F|=k} \phi(F), \\
&= \sum_{|F|=k} \sum_{F \subseteq G \in \Delta} (-1)^{|G|}, \\
&= \sum_{\substack{G \in \Delta \\ k \le |G| \le d}} \sum_{\substack{F \subseteq G \\ |F|=k}} (-1)^{|G|}, \\
&= \sum_{\substack{G \in \Delta \\ k \le |G| \le d}} (-1)^{|G|} \binom{|G|}{k}, \\
&= \sum_{k \le i \le d} (-1)^i f_i \binom{i}{k}.
\end{aligned}$$

This is the first version of the Dehn-Sommerville relations.

Theorem 8.1 (Dehn-Sommerville, f-version). *For any pure Eulerian complex Δ with $f(\Delta) = (1, f_1, \ldots, f_d)$, we have, for each $k \ge 1$,*

$$(-1)^{d-1} f_k = \sum_{k \le i \le d} (-1)^i f_i \binom{i}{k}. \tag{8.3}$$

For example, with $d = 4$ we have:

$$\begin{aligned}
f_1 &= -f_1\tbinom{1}{1} + f_2\tbinom{2}{1} - f_3\tbinom{3}{1} + f_4\tbinom{4}{1}, \\
f_2 &= \phantom{-f_1\tbinom{1}{1} + {}} f_2\tbinom{2}{2} - f_3\tbinom{3}{2} + f_4\tbinom{4}{2}, \\
f_3 &= \phantom{-f_1\tbinom{1}{1} + f_2\tbinom{2}{2}} - f_3\tbinom{3}{3} + f_4\tbinom{4}{3}, \\
f_4 &= \phantom{-f_1\tbinom{1}{1} + f_2\tbinom{2}{2} - f_3\tbinom{3}{3} + {}} f_4\tbinom{4}{4}.
\end{aligned}$$

and with $d = 5$ we get:

$$\begin{aligned}
f_1 &= f_1\tbinom{1}{1} - f_2\tbinom{2}{1} + f_3\tbinom{3}{1} - f_4\tbinom{4}{1} + f_5\tbinom{5}{1},\\
f_2 &= \qquad\quad - f_2\tbinom{2}{2} + f_3\tbinom{3}{2} - f_4\tbinom{4}{2} + f_5\tbinom{5}{2},\\
f_3 &= \qquad\qquad\qquad\quad f_3\tbinom{3}{3} - f_4\tbinom{4}{3} + f_5\tbinom{5}{3},\\
f_4 &= \qquad\qquad\qquad\qquad\qquad - f_4\tbinom{4}{4} + f_5\tbinom{5}{4},\\
f_5 &= \qquad\qquad\qquad\qquad\qquad\qquad\qquad f_5\tbinom{5}{5}.
\end{aligned} \tag{8.4}$$

In terms of linear algebra, we are saying the f-vector is in the fixed point space of a certain linear transformation T, e.g., for $d = 4$ and $d = 5$, these transformations are

$$\begin{bmatrix} -1 & 2 & -3 & 4 \\ 0 & 1 & -3 & 6 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & -2 & 3 & -4 & 5 \\ 0 & -1 & 3 & -6 & 10 \\ 0 & 0 & 1 & -4 & 10 \\ 0 & 0 & 0 & -1 & 5 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

respectively. Computing the dimension of the fixed point space of T boils down to computing the rank of $T - I$, and the alternating ± 1 on the diagonal mean that $T - I$ has rank $\lfloor d/2 \rfloor$, e.g., for $d = 4$ and $d = 5$, $T - I$ looks like:

$$\begin{bmatrix} -2 & 2 & -3 & 4 \\ 0 & 0 & -3 & 6 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & -2 & 3 & -4 & 5 \\ 0 & -2 & 3 & -6 & 10 \\ 0 & 0 & 0 & -4 & 10 \\ 0 & 0 & 0 & -2 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

which both have rank 2.

So the f-vectors of Eulerian simplicial complexes live in a vector space of roughly half the dimension of Δ. Is there is a change of basis of the f-vector that allows us to see this fact? The answer is an emphatic "Yes!" and in the remainder of this chapter we will describe ways to do so.

As a first step, let's add a row with $\widetilde{\chi}(\Delta)$ to the top of the system of equations given by transformation T, and multiply the ith equation by $(-1)^{d-1}t^i$. With $d = 5$ this is:

$$\begin{aligned}
\widetilde{\chi}(\Delta) &= -f_0\tbinom{0}{0} + f_1\tbinom{1}{0} - f_2\tbinom{2}{0} + f_3\tbinom{3}{0} - f_4\tbinom{4}{0} + f_5\tbinom{5}{0},\\
f_1 \cdot t &= \qquad\quad \left(f_1\tbinom{1}{1} - f_2\tbinom{2}{1} + f_3\tbinom{3}{1} - f_4\tbinom{4}{1} + f_5\tbinom{5}{1}\right) t,\\
f_2 \cdot t^2 &= \qquad\qquad\quad \left(-f_2\tbinom{2}{2} + f_3\tbinom{3}{2} - f_4\tbinom{4}{2} + f_5\tbinom{5}{2}\right) t^2,\\
f_3 \cdot t^3 &= \qquad\qquad\qquad\qquad \left(f_3\tbinom{3}{3} - f_4\tbinom{4}{3} + f_5\tbinom{5}{3}\right) t^3,\\
f_4 \cdot t^4 &= \qquad\qquad\qquad\qquad\qquad \left(-f_4\tbinom{4}{4} + f_5\tbinom{5}{4}\right) t^4,\\
f_5 \cdot t^5 &= \qquad\qquad\qquad\qquad\qquad\qquad\qquad f_5\tbinom{5}{5}\, t^5.
\end{aligned}$$

Summing both sides, with the right-hand side taken column-wise, we find:

$$(-1)^{d-1}(f(\Delta;t)-1)+\widetilde{\chi}(\Delta) = -f_0+f_1(1+t)-f_2(1+t)^2+f_3(1+t)^3-\cdots,$$
$$= -f(\Delta;-(1+t)),$$

or

$$(-1)^d f(\Delta;t)-f(\Delta;-(1+t)) = (-1)^d+\widetilde{\chi}(\Delta). \tag{8.5}$$

Now we can use the transformation in Equation (8.1) for expressing the h-vector in terms of the f-vector, i.e.,

$$f(\Delta;s) = (1+s)^d h(\Delta;s/(1+s)).$$

Take $s=t$ and $s=-(1+t)$ respectively on the left-hand side of (8.5) to rewrite it as:

$$(-1)^d(1+t)^d h(\Delta;t/(1+t)) - (-t)^d h(\Delta;(1+t)/t) = (-1)^d+\widetilde{\chi}(\Delta),$$

or upon multiplying both sides by $(-1)^d$,

$$(1+t)^d h(\Delta;t/(1+t)) - t^d h(\Delta;(1+t)/t) = 1+(-1)^d\,\widetilde{\chi}(\Delta). \tag{8.6}$$

Now let $x=t/(1+t)$, so that $t=x/(1-x)$. Then dividing both sides of equation (8.6) by $(1+t)^d=(t/x)^d$, i.e., multiplying by $(x/t)^d=(1-x)^d$, we get:

$$h(\Delta;x)-x^d h(\Delta;1/x) = (1-x)^d(1+(-1)^d\,\widetilde{\chi}(\Delta)).$$

By comparing coefficients on the left and right, we have the h-version of the Dehn-Sommerville relations.

Theorem 8.2 (Dehn-Sommerville, h-version). *Suppose Δ is a pure Eulerian complex of dimension $d-1$, with $h(\Delta)=(1,h_1,\ldots,h_d)$. Then for each $k\geq 0$,*

$$h_k-h_{d-k} = (-1)^k\binom{d}{k}(1+(-1)^d\,\widetilde{\chi}(\Delta)).$$

In particular, if $\widetilde{\chi}(\Delta)=(-1)^{d-1}$, the h-vector is palindromic:

$$h_k = h_{d-k}. \tag{8.7}$$

This is point for us. If Δ is a sphere, or any triangulated manifold with the same Euler characteristic, the h-vector is palindromic. Hence the space of h-vectors of spheres is clearly $\lfloor d/2\rfloor$-dimensional, and since the transformation $f\leftrightarrow h$ is invertible, so is the space of f-vectors of spheres.

The Dehn-Sommerville relations give a very sophisticated reason why the Eulerian numbers and Narayana numbers are palindromic: because they are the entries of the h-vector of a sphere!

There are many interesting results and open questions regarding the characterization of h-vectors of spheres, some of which are discussed further in Chapter 10.

Notes

Simplicial decomposition of topological spaces is a standard idea in algebraic topology. See Allen Hatcher's textbook for more from the point of view of topologists [88]. Two classic textbooks on polytopes include one by Branko Grünbaum [84] and another by Günter Ziegler [169].

Much of the work on connections between posets and simplicial complexes was pioneered by Richard Stanley in the 1970s and 1980s, e.g., [144–147]. See also work of Anders Björner [23], and [21] in which he connects poset theory to general *CW*-complexes. Chapter 4 of Stanley's textbook [154] discusses many of these results and more. Flag complexes arise naturally in graph theory, and they are of particular interest in the context of the *Charney-Davis conjecture*, stated by Ruth Charney and Mike Davis in their 1995 paper [48] and discussed further in Chapter 10.

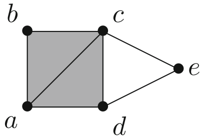
The f-vectors of abstract simplicial complexes admit a complete characterization known as the Kruskal-Katona-Schützenberger inequalities, given in Chapter 10. These are due to, independently Joseph Kruskal in 1963 [98] and Gyula Katona in 1966 [94]. We attach the name of Marcel-Paul Schützenberger because in 1959 he too described the inequalities in a technical report for MIT's Research Laboratory of Electronics [134]. However the note in which it appears is both hard to find and rather skimpy on details. Most people know of these inequalities simply as the "Kruskal-Katona" inequalities.

In [151] Stanley characterizes the f-vectors of Boolean complexes. There is also a characterization of the f-vector of a balanced simplicial complex, due to Peter Frankl, Zoltán Füredi, and Gil Kalai [75]. See Chapter 10.

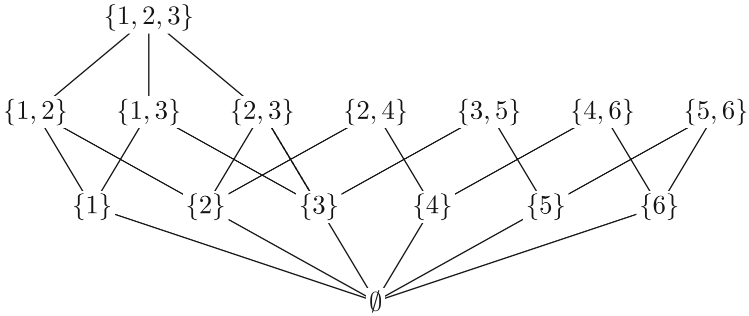
The Dehn-Sommerville relations were first stated in low dimensions by Max Dehn [55], and in 1927 by Duncan Sommerville [143]. The proof we give here is adapted from Victor Klee's 1964 paper [95]. Stanley generalizes the argument in Chapter 4 of [154] as well, where it can be phrased in terms of the Möbius function of the face poset.

Problems

8.1. 1. Draw the Hasse diagram for the face poset of the following simplicial complex:

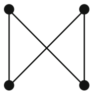
$\Delta =$  .

2. Draw a geometric realization of the abstract simplicial complex whose Hasse diagram is below:



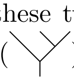
.

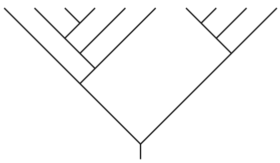
8.2. Prove that the face poset of a simplicial complex can have no "bowties" in its Hasse diagram:



i.e., no quadruple of faces, F_1, F_2, G_1, G_2, with $\dim F_1 = \dim F_2 = \dim G_1 - 1 = \dim G_2 - 1$, and such that $F_1 \subset G_1$, $F_1 \subset G_2$, $F_2 \subset G_1$, and $F_2 \subset G_2$.

8.3. Show that if a polytope $\mathcal{P}$ is a d-simplex, so is its dual, $\mathcal{P}^*$.

8.4. Recall that $\mathrm{PB}(n)$ denotes the set of planar binary trees with n internal nodes. Label the leaves from left to right by $0, 1, \ldots, n$, and then label the internal nodes $1, 2, \ldots, n$ so that node i is the one that falls between leaf $i-1$ and leaf i. Let l_i denote the number of leaves on the left branch of node i and let r_i denote the number of leaves on the right branch of node i. Let $v_i = l_i r_i$ denote the product of these two numbers, and let $v(\tau) = (v_1, \ldots, v_n)$. For example, $v($$) = (1)$, $v($$) = (2, 1)$, and if

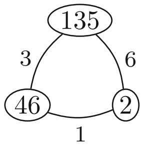
$\tau =$ 

then $v(\tau) = (5, 2, 1, 3, 4, 24, 1, 2, 3)$.

Show that the convex hull of the points $v(\tau)$, as τ runs over all planar binary trees in $\mathrm{PB}(n)$, is a geometric realization of the associahedron, whose face poset was described combinatorially in Section 5.8.

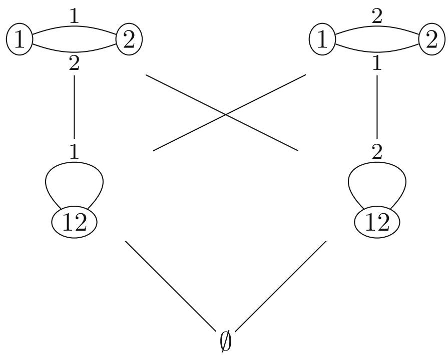
8.5. Show that the permutahedron and associahedron are simple polytopes.

8.6. A *spin necklace* is a cyclically ordered set partition of $\{1, 2, \ldots, n\}$ (drawn clockwise in a circle) together with a labeling of the edges between the blocks that respects block sizes (modulo n). That is, the difference between the edge labels on either side of a block must differ by the cardinality of the block. For example,



is a spin necklace on $\{1, 2, 3, 4, 5, 6\}$.

Let $\Sigma_T(n)$ denote the set of spin necklaces on $\{1, 2, \ldots, n\}$, together with the empty set. We partially order $\Sigma_T(n)$ by declaring that $\emptyset$ is a unique minimal element and two spin necklaces satisfy $F \leq_{\Sigma_T} G$ if and only if G is a refinement of F. For example, here is the Hasse diagram for $\Sigma_T(2)$:



Note that there are $n!$ maximal elements, corresponding to permutations, and n rank one elements (vertices), corresponding to the single block $\{1, 2, \ldots, n\}$, with a "handle" labeled by some $i = 1, 2, \ldots, n$. Show that $\Sigma_T(n)$, is a boolean complex, i.e., simplicial poset, but not a simplicial complex.

8.7. Show that $f = (f_0, f_1, \ldots, f_d)$ is the f-vector of a $(d-1)$-dimensional boolean complex (simplicial poset) if and only if $f_0 = 1$ and $f_i \geq \binom{d}{i}$ for each $i > 0$.

8.8. Show that the order complex of a finite poset is a simplicial complex.

8.9. Verify that the simplicial complex dual to the associahedron is a flag complex.

8.10. Let $\Sigma(n)$ denote the simplicial complex for the braid arrangement. Show that $\Sigma(n)$ is balanced.

8.11. Show that the f-polynomial of Δ is real-rooted if and only if its h-polynomial is real-rooted.

Chapter 9
Barycentric subdivision

BARYCENTRIC SUBDIVISION HAS LONG BEEN A USEFUL TOOL in geometry and topology. It is an operation that preserves topology and is well-behaved combinatorially. In this chapter we will study a transformation of Brenti and Welker that maps the f-vector of a complex to the f-vector of its barycentric subdivision.

9.1 Barycentric subdivision of a finite cell complex

The term *barycenter* refers to the center of mass of a convex polytope, and there is a straightforward notion of *barycentric subdivision* for convex polytopes which goes as follows. Place a vertex on the center of mass of each face of the polytope and connect vertices that lie in a common face. This "triangulates" the polytope in the sense that every face resulting from the subdivision is a simplex.

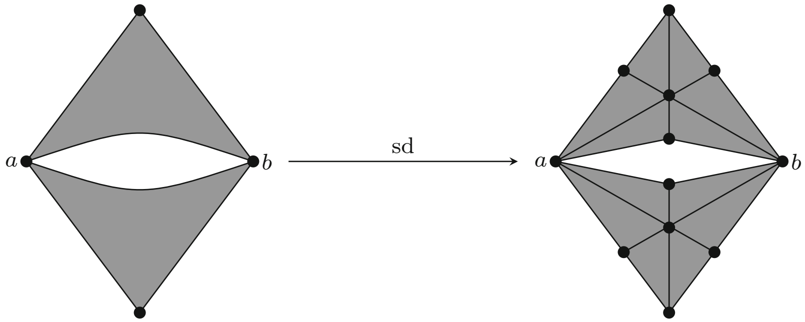


Fig. 9.1 A cell complex and its barycentric subdivision.

T.K. Petersen, *Eulerian Numbers*, Birkhäuser Advanced Texts Basler Lehrbücher, DOI 10.1007/978-1-4939-3091-3_9

Ignoring geometry, we can define the barycentric subdivision combinatorially: the barycentric subdivision of Δ is the order complex of the poset of nonempty faces of Δ. See Figure 9.1. Let $\mathrm{sd}(\Delta)$ denote this abstract simplicial complex. Then the k-faces of the complex $\mathrm{sd}(\Delta)$ are k-chains of nonempty faces of Δ, sometimes called *flags*:

$$F_1 <_\Delta F_2 <_\Delta \cdots <_\Delta F_k.$$

In particular, each nonempty face of Δ corresponds to a vertex of $\mathrm{sd}(\Delta)$.

As with general order complexes, the barycentric subdivision is a flag complex. Moreover, since face posets are ranked by dimension, $\mathrm{sd}(\Delta)$ is the order complex of a ranked poset, and hence balanced.

Our combinatorial definition of barycentric subdivision makes sense for any cell complex Δ for which there is a well-defined face poset, though we need to be a little careful about the topology. If there are cells of Δ with identifications on their boundary, i.e., a $(k-1)$-dimensional cell with fewer than k vertices, information can get lost. For example, in Figure 9.2 we see that the cell complex on a circle with just one edge and one vertex has a contractible barycentric subdivision. However, if no face has self-identifications on its boundary, e.g., if Δ is a polytope or a boolean complex, then Δ and its barycentric subdivision are homeomorphic.

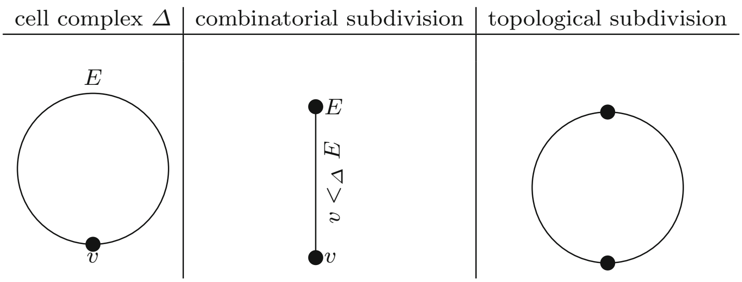


Fig. 9.2 Combinatorial barycentric subdivision can destroy topology if Δ has cells with self-identifications.

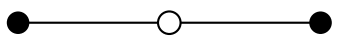
There is a topological definition of barycentric subdivision that does not have this problem. In a true cell complex, each cell "remembers where it came from" in the sense that we know how its boundary is mapped onto lower-dimensional cells. Thus, we can "unglue" the cell, deform the cell continuously into a geometric simplex, perform barycentric subdivision, and glue the subdivided cell back with the original boundary map. Doing this for each cell gives the topological definition of barycentric subdivision.

In all that follows, however, we will only consider the combinatorial definition.

9.2 The barycentric subdivision of a simplex

We will now do a careful enumeration of the faces in the barycentric subdivision of a simplex. We will denote a simplex on vertex set V by $2^V = \{F : F \subseteq V\}$.

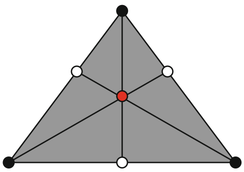
Let's do small examples first.

If $V = \{1,2\}$, $2^V = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$ the barycentric subdivision is drawn: . We can list the flags of 2^V as:

empty face	vertices	edges
$\emptyset$	$\{1\}$	$\{1\} \subset \{1,2\}$
	$\{2\}$	$\{2\} \subset \{1,2\}$
	$\{1,2\}$	

so $f(\mathrm{sd}(2^V)) = (1,3,2)$ and $h(\mathrm{sd}(2^V)) = (1,1,0)$.

If $V = \{1,2,3\}$, the barycentric subdivision of a triangle is



.

We color the vertices in the barycentric subdivision to recall the dimension of the corresponding face in the original complex. (This gives a balanced coloring to $\mathrm{sd}(\Delta)$.) Listing the flags we find:

empty face	vertices	edges	triangles
$\emptyset$	$\{1\}$	$\{1\} \subset \{1,2\}$	$\{1\} \subset \{1,2\} \subset \{1,2,3\}$
	$\{2\}$	$\{1\} \subset \{1,3\}$	$\{1\} \subset \{1,3\} \subset \{1,2,3\}$
	$\{3\}$	$\{2\} \subset \{1,2\}$	$\{2\} \subset \{1,2\} \subset \{1,2,3\}$
	$\{1,2\}$	$\{2\} \subset \{2,3\}$	$\{2\} \subset \{2,3\} \subset \{1,2,3\}$
	$\{1,3\}$	$\{3\} \subset \{1,3\}$	$\{3\} \subset \{1,3\} \subset \{1,2,3\}$
	$\{2,3\}$	$\{3\} \subset \{2,3\}$	$\{3\} \subset \{2,3\} \subset \{1,2,3\}$
	$\{1,2,3\}$	$\{1\} \subset \{1,2,3\}$	
		$\{2\} \subset \{1,2,3\}$	
		$\{3\} \subset \{1,2,3\}$	
		$\{1,2\} \subset \{1,2,3\}$	
		$\{1,3\} \subset \{1,2,3\}$	
		$\{2,3\} \subset \{1,2,3\}$	

and so $f(\mathrm{sd}(2^V)) = (1,7,12,6)$ and $h(\mathrm{sd}(2^V)) = (1,4,1,0)$.

Before moving on to larger cases, it will be a good idea to refine our bookkeeping. Notice that there is a lot of redundancy in counting flags, in that for every flag

$$S_1 \subset S_2 \subset \cdots \subset S_k \subset \{1, 2, \ldots, n\},$$

there is another flag

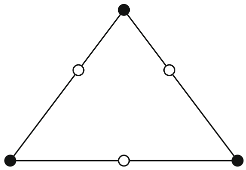
$$S_1 \subset S_2 \subset \cdots \subset S_k,$$

of one dimension lower. Thus,

$$f(\mathrm{sd}(2^V); t) = (1 + t) f(\mathrm{sd}(\partial 2^V); t). \tag{9.1}$$

That is, the faces of the barycentric subdivision of $\partial 2^V$ are precisely those flags that do not contain the interior of the simplex, i.e., the face $V = \{1, 2, \ldots, n\}$.

For example, the barycentric subdivision of the boundary of the triangle is (combinatorially) a hexagon:



,

corresponding to the flags below:

empty face	vertices	edges
$\emptyset$	$\{1\}$	$\{1\} \subset \{1,2\}$
	$\{2\}$	$\{1\} \subset \{1,3\}$
	$\{3\}$	$\{2\} \subset \{1,2\}$
	$\{1,2\}$	$\{2\} \subset \{2,3\}$
	$\{1,3\}$	$\{3\} \subset \{1,3\}$
	$\{2,3\}$	$\{3\} \subset \{2,3\}$

.

Thus $f(\mathrm{sd}(\partial 2^V)) = (1, 6, 6)$. We see that

$$(1+t) f(\mathrm{sd}(\partial 2^V); t) = (1+t)(1+6t+6t^2) = 1+7t+12t^2+6t^3 = f(\mathrm{sd}(2^V); t),$$

as expected.

Moreover, since

$$f(\mathrm{sd}(\partial 2^V); t) = (1 + t)^{|V|-1} h(\mathrm{sd}(\partial 2^V); t/(1 + t)),$$

and

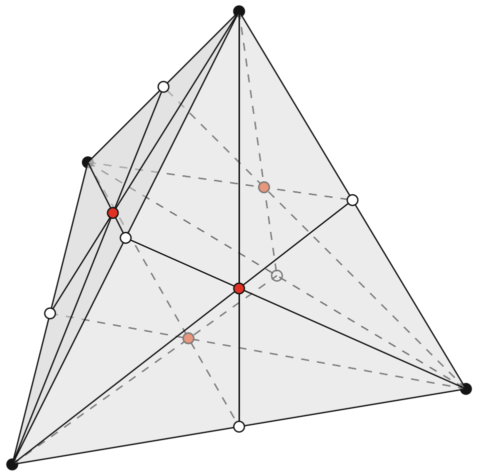
$$f(\mathrm{sd}(2^V); t) = (1 + t)^{|V|} h(\mathrm{sd}(2^V); t/(1 + t)),$$

Equation (9.1) gives us

$$h(\mathrm{sd}(\partial 2^V); t) = h(\mathrm{sd}(2^V); t). \tag{9.2}$$

That is, the barycentric subdivision of the boundary of the simplex has the same h-vector as the barycentric subdivision of the simplex itself. For simplicity then (since we have to keep track of fewer flags) we will restrict our attention to $\mathrm{sd}(\partial 2^V)$. Note that it is highly unusual that a simplicial complex and its boundary be related in such a way. See Problem 9.1.

For the subdivided tetrahedron we draw the boundary only:



and we get $f(\mathrm{sd}(\partial 2^V)) = (1, 14, 36, 24)$ and $h(\mathrm{sd}(\partial 2^V)) = (1, 11, 11, 1)$. Thus $f(\mathrm{sd}(2^V)) = (1, 15, 50, 60, 24)$ and $h(\mathrm{sd}(2^V)) = (1, 11, 11, 1, 0)$.

So what are the h-vectors we have computed so far?

$$(1, 1), (1, 4, 1), (1, 11, 11, 1), \ldots$$

if we throw in the vector (1) at the beginning for the trivial simplex, we have the first few rows of Table 1.3. We have Eulerian numbers!

Let us prove this connection by counting flags carefully. Throughout the remainder of Section 9.2, we will fix a finite vertex set V and let $\Delta = \mathrm{sd}(\partial 2^V)$ denote the barycentric subdivision of the boundary of the simplex with vertex set V.

This first step in computing $f(\Delta)$ is to modify our bookkeeping. Since flags are sequences of nested subsets, we can keep track only of the new additions. That is, given

$$\emptyset \subset S_1 \subset S_2 \subset \cdots \subset S_k \subset \{1, 2, \ldots, n\} = V,$$

let $A_i = S_{i+1} - S_i$, as i ranges from 0 to k, with $S_0 = \emptyset$ and $S_{k+1} = \{1, 2, \ldots, n\}$. Instead of the original flag, we can record the tuple $(A_0, A_1, \ldots, A_k)$. For example, if $V = \{1, 2, 3, 4, 5, 6, 7\}$, the flag

$$\emptyset \subset \{3, 4\} \subset \{3, 4, 6, 7\} \subset \{1, 3, 4, 6, 7\} \subset \{1, 2, 3, 4, 6, 7\} \subset V,$$

becomes the tuple

$$(\{3,4\},\{6,7\},\{1\},\{2\},\{5\}).$$

Even better, we can write

$$34|67|1|2|5,$$

if we agree to list the elements of each A_i in increasing order and drop the curly braces and commas. This is a set composition!

We enumerated set compositions in our study of the braid arrangement in Section 5.6. So the complex $\Sigma(n)$ (associated with the braid arrangement $\mathcal{H}(n)$) is isomorphic to the barycentric subdivision of the boundary of a simplex. If we restate Theorems 5.2 and 5.3, we have the following.

Theorem 9.1. *The barycentric subdivision of the boundary of the simplex 2^V, with $|V| = n$, has the following f- and h-polynomials:*

$$f(\mathrm{sd}(\partial 2^V);t) = \sum_{k=0}^{n-1}(k+1)!S(n,k+1)t^k,$$

where $S(n,k)$ is a Stirling number of the second kind, and

$$h(\mathrm{sd}(\partial 2^V);t) = \sum_{k=0}^{n-1}\left\langle {n \atop k} \right\rangle t^k,$$

where $\left\langle {n \atop k} \right\rangle$ is an Eulerian number. In other words, the Eulerian polynomial is the h-polynomial of $\mathrm{sd}(\partial 2^V)$.

9.3 Brenti and Welker's transformation

We will now use the ideas developed for the simplex to study f- and h-vectors of $\mathrm{sd}(\Delta)$, where Δ is any boolean complex. Recall from Section 8.3 that a boolean complex is a cell complex in which each face is a simplex. A simplicial complex is a boolean complex, but this family also includes complexes whose faces are not uniquely determined by their vertex sets.

As a starting point, let us consider the boolean complex and its barycentric subdivision given in Figure 9.3.

With the vertices of Δ labeled a, b, c, d, and the two edges between a and b labeled E_1 and E_2, we have the following flags of faces in $\mathrm{sd}(\Delta)$.

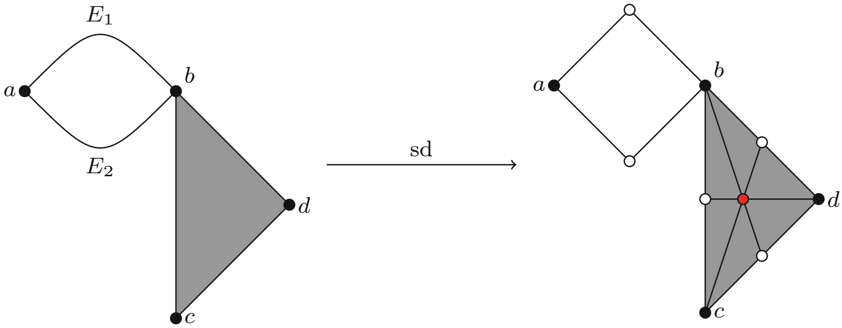


Fig. 9.3 A boolean complex and its barycentric subdivision.

empty face	vertices	edges	triangles
$\emptyset$			
	$\{a\}$		
	$\{b\}$		
	$\{c\}$		
	$\{d\}$		
	E_1	$\{a\} \subset E_1$ $\{b\} \subset E_1$	
	E_2	$\{a\} \subset E_2$ $\{b\} \subset E_2$	
	$\{b,c\}$	$\{b\} \subset \{b,c\}$ $\{c\} \subset \{b,c\}$	
	$\{b,d\}$	$\{b\} \subset \{b,d\}$ $\{d\} \subset \{b,d\}$	
	$\{c,d\}$	$\{c\} \subset \{c,d\}$ $\{d\} \subset \{c,d\}$	
	$\{b,c,d\}$	$\{b\} \subset \{b,c,d\}$ $\{c\} \subset \{b,c,d\}$ $\{d\} \subset \{b,c,d\}$ $\{b,c\} \subset \{b,c,d\}$ $\{b,d\} \subset \{b,c,d\}$ $\{c,d\} \subset \{b,c,d\}$	$\{b\} \subset \{b,c\} \subset \{b,c,d\}$ $\{b\} \subset \{b,d\} \subset \{b,c,d\}$ $\{c\} \subset \{b,c\} \subset \{b,c,d\}$ $\{c\} \subset \{c,d\} \subset \{b,c,d\}$ $\{d\} \subset \{b,d\} \subset \{b,c,d\}$ $\{d\} \subset \{c,d\} \subset \{b,c,d\}$

We have $f(\Delta) = (1,4,5,1)$ and $f(\mathrm{sd}(\Delta)) = (1,10,16,6)$. The beautiful result of Brenti and Welker gives us the means for computing $f(\mathrm{sd}(\Delta))$ as a simple linear transformation of $f(\Delta)$, which will now derive.

Notice that we have grouped the faces of $\mathrm{sd}(\Delta)$ according to the last face S_k in the flag. Within each of these groups, we can identify the flags

$$S_1 \subset S_2 \subset \cdots \subset S_k,$$

with set compositions of S_k, i.e., let $A_i = S_{i+1} - S_i$ for $i = 0, \ldots, k-1$, with $S_0 = \emptyset$. Then the composition $A = A_0|A_1|\cdots|A_{k-1}$ corresponds to the flag

$$A_0 \subset (A_0 \cup A_1) \subset \cdots \subset (A_0 \cup A_1 \cup \cdots \cup A_{k-1}).$$

For example, $b|d|c$ denotes the flag $\{b\} \subset \{b,d\} \subset \{b,c,d\}$ and $bd|c$ denotes the flag $\{b,d\} \subset \{b,c,d\}$.

We should be careful to first fix the flag we are working with, since, for example, $a|b$ could denote either the edge E_1 or the edge E_2. But once we know which maximal face S_k the flag lives in, the set compositions of that face are well defined.

For any fixed choice of face F of Δ, let $\mathrm{Comp}(F)$ denote the set of all set compositions A of the vertex set of F, i.e., all compositions $A = A_0|A_1|\cdots|A_{k-1}$ such that $A_i \cap A_j = \emptyset$ and $A_0 \cup \cdots \cup A_{k-1} = F$. These compositions represent all the flags in $\mathrm{sd}(\Delta)$ whose maximal element is F. We denote by $\mathrm{rk}(A) = k-1$ the number of bars in A, i.e., the dimension of the corresponding face of $\mathrm{sd}(\Delta)$.

If $|F| = j$, then, as we saw in the case of the simplex, there are $k!S(j,k)$ set compositions of F with k parts. These are set compositions of rank $\mathrm{rk}(A) = k-1$, and so

$$\sum_{A \in \mathrm{Comp}(F)} t^{1+\mathrm{rk}(A)} = \sum_{k \geq 0} k!S(j,k)t^k.$$

Each face $G \in \mathrm{sd}(\Delta)$ corresponds to a flag of faces of Δ, so summing over all F in Δ, we have:

$$\begin{aligned}
f(\mathrm{sd}(\Delta);t) &= \sum_{G \in \mathrm{sd}(\Delta)} t^{1+\dim G}, \\
&= \sum_{F \in \Delta} \sum_{A \in \mathrm{Comp}(F)} t^{1+\mathrm{rk}(A)}, \\
&= \sum_{j \geq 0} f_j(\Delta) \cdot \sum_{k \geq 0} k!S(j,k)t^k, \\
&= \sum_{k \geq 0} \left(\sum_{j \geq 0} f_j(\Delta) k! S(j,k) \right) t^k.
\end{aligned}$$

Let us state a theorem.

Theorem 9.2. *For any finite boolean complex Δ, with $\dim \Delta = d-1$, the f-vector of its barycentric subdivision is given by:*

$$f_k(\mathrm{sd}(\Delta)) = \sum_{j \geq 0} f_j(\Delta) k! S(j,k).$$

We can also describe this transformation with a matrix. Let

$$\mathfrak{B}_d = [a!S(b,a)]_{0 \leq a,b \leq d} \, .$$

Then, Theorem 9.2 says

$$f(\mathrm{sd}(\Delta)) = \mathfrak{B}_d\, f(\Delta).$$

For example, with $d = 3$, we have

$$\mathfrak{B}_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 6 \end{pmatrix}.$$

Thus, for Δ as in Figure 9.3, we have

$$f(\mathrm{sd}(\Delta)) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} 1 \\ 4 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 10 \\ 16 \\ 6 \end{pmatrix},$$

as desired.

9.4 The h-vector of sd(Δ) and j-Eulerian numbers

Recall from Section 8.8 now that the transformation from f-vector to h-vector is given by the matrix

$$H_d = \left[(-1)^{a+b} \binom{d-b}{a-b} \right]_{0 \le a,b \le d},$$

and the inverse transformation is

$$H_d^{-1} = \left[\binom{d-b}{a-b} \right]_{0 \le a,b \le d}.$$

That is,

$$h(\Delta) = H_d f(\Delta) \qquad \text{and} \qquad f(\Delta) = H_d^{-1} h(\Delta).$$

Then we can compose these operations to write $h(\mathrm{sd}(\Delta)) = H_d\, \mathfrak{B}_d\, f(\Delta)$, or

$$h(\mathrm{sd}(\Delta)) = H_d\, \mathfrak{B}_d\, H_d^{-1} h(\Delta). \tag{9.3}$$

Denote this transformation by

$$\mathfrak{E}_d = H_d\, \mathfrak{B}_d\, H_d^{-1}.$$

It turns out that this transformation is beautifully combinatorial. We see some small examples in Table 9.1.

Table 9.1 The barycenter transformation on h-vectors.

d	H_d	$\mathfrak{B}_d$	H_d^{-1}	$\mathfrak{E}_d$
0	(1)	(1)	(1)	(1)
1	$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
2	$\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$
3	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 3 & -2 & 1 & 0 \\ -1 & 1 & -1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 6 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 4 & 4 & 2 & 1 \\ 1 & 2 & 4 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
4	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -4 & 1 & 0 & 0 & 0 \\ 6 & -3 & 1 & 0 & 0 \\ -4 & 3 & -2 & 1 & 0 \\ 1 & -1 & 1 & -1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 6 & 14 \\ 0 & 0 & 0 & 6 & 36 \\ 0 & 0 & 0 & 0 & 24 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 & 0 \\ 6 & 3 & 1 & 0 & 0 \\ 4 & 3 & 2 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 11 & 8 & 4 & 2 & 1 \\ 11 & 14 & 16 & 14 & 11 \\ 1 & 2 & 4 & 8 & 11 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

There are some tantalizing properties of the matrices in Table 9.1. Notice, for example:

- the sum of all the entries in $\mathfrak{E}_d$ is $(d+1)!$,
- the sum of the entries in each column of $\mathfrak{E}_d$ is $d!$,
- the sum of the entries of row k of $\mathfrak{E}_d$, $k = 1, \ldots, d+1$, is the Eulerian number $\left\langle {d+1 \atop k} \right\rangle$.

All of these properties and more will follow from the following theorem due to Brenti and Welker.

First define the numbers

$$\left\langle {n; j \atop k} \right\rangle = |\{w \in S_n : \mathrm{des}(w) = k, w(1) = j\}|,$$

as the *j-Eulerian numbers*. These numbers refine the usual Eulerian numbers in the sense that

$$\left\langle {n \atop k} \right\rangle = \left\langle {n; 1 \atop k} \right\rangle + \left\langle {n; 2 \atop k} \right\rangle + \cdots + \left\langle {n; n \atop k} \right\rangle.$$

Similarly, define the *j-Eulerian polynomials* by

$$S_{n;j}(t) = \sum_{\substack{w \in S_n \\ w(1)=j}} t^{\mathrm{des}(w)} = \sum_{k=0}^{n-1} \left\langle {n;j \atop k} \right\rangle t^k.$$

These are the generating functions for the columns of $\mathfrak{E}_d$. For future use, let $S_{n;j} = \{w \in S_n : w(1) = j\}$ denote the set of permutations beginning with j.

Theorem 9.3. *Let $\mathfrak{E}_d$ denote the barycenter transformation for h-vectors $h = (h_0, \ldots, h_d)$. Then*

$$\mathfrak{E}_d = \left[\left\langle {d+1;j \atop k} \right\rangle \right]_{0 \le k, j-1 \le d},$$

so that if $h(t) = h(\Delta; t)$, then

$$h(\mathrm{sd}(\Delta); t) = \sum_{j=0}^{d} h_j(\Delta) S_{d+1;j+1}(t).$$

We will now prove Theorem 9.3.

First, consider an entry $T_{r,s}$, $0 \le r, s \le d$, of the matrix $T = \mathfrak{B}_d\, H_d^{-1}$. We have:

$$T_{r,s} = \sum_{b=0}^{d} \binom{d-s}{b-s} r! S(b,r) = \sum_{b=0}^{d} \binom{d-s}{d-b} r! S(b,r).$$

Since this is a positive formula, it is not too hard to come up with a combinatorial interpretation for it. Let $\mathcal{T}_{r,s}$ denote the set of all set compositions $A = A_0|A_1|\cdots|A_r$ of $\{1, 2, \ldots, d+1\}$ for which $\min A_0 = s+1$. To form such a composition, we first choose $d-b$ elements from among $\{s+2, \ldots, d+1\}$ to put in A_0 along with $s+1$. This can be done in $\binom{d-s}{d-b}$ ways. To form $A_1|\cdots|A_r$ we need to create a set composition from the remaining b elements, and this can be done in $r!S(b,r)$ ways. See Figure 9.4.

$$\underbrace{1, 2, \ldots, s, \quad \overset{\min A_0}{\boxed{s+1,}} \quad \overbrace{s+2, \ldots, d+1}^{\text{Choose } d-b \text{ more for } A_0}}_{\text{Form } A_1|\cdots|A_r \text{ from remaining } b \text{ elements}}$$

Fig. 9.4 Forming an element of $\mathcal{T}_{r,s}$.

Now let

$$\mathcal{T}_s = \bigcup_{r=0}^{d} \mathcal{T}_{r,s},$$

denote the set of all set compositions of $\{1,2,\ldots,d+1\}$ with $\min A_0 = s+1$. Further, let $T_s(t)$ denote the generating function counting these set compositions according to the number of bars,

$$T_s(t) = \sum_{A\in\mathcal{T}_s} t^{\mathrm{rk}(A)} = \sum_{r=0}^{d} T_{r,s}t^r.$$

In other words, $T_s(t)$ is the generating function for column s of the matrix T.

But each set composition $A = A_0|A_1|\cdots|A_r$ can be mapped to a permutation $w = w(A)$ by removing bars and writing each block in increasing order. Since $\min A_0 = s+1$, this means $w(1) = s+1$. That is, $w \in S_{d+1;s+1}$. Further, $\mathrm{Des}(w) \subseteq D$, where $D = D(A) = \{|A_0|, |A_0|+|A_1|, \ldots, |A_0|+|A_1|+\cdots+|A_{r-1}|\}$, i.e., there must be bars in A where there are descents in w. So we can write

$$\begin{aligned}
T_s(t) &= \sum_{A\in\mathcal{T}_s} t^{\mathrm{rk}(A)}, \\
&= \sum_{J\subseteq\{1,2,\ldots,d\}} \sum_{\substack{A\in\mathcal{T}_s\\ D(A)=J}} t^{|J|}, \\
&= \sum_{J\subseteq\{1,2,\ldots,d\}} \sum_{\substack{w\in S_{d+1;s+1}\\ D(w)\subseteq J}} t^{|J|}, \\
&= \sum_{w\in S_{d+1;s+1}} \sum_{\mathrm{Des}(w)\subseteq J} t^{|J|}, \\
&= \sum_{w\in S_{d+1;s+1}} t^{\mathrm{des}(w)}(1+t)^{d-\mathrm{des}(w)}, \\
&= (1+t)^d S_{d+1;s+1}(t/(1+t)).
\end{aligned}$$

Since $T_s(t)$ encodes column s of $\mathfrak{B}_d H_d^{-1}$, the polynomial $H_d T_s(t) = S_{d+1;s+1}(t)$ encodes column s of $\mathfrak{E}_d = H_d \mathfrak{B}_d H_d^{-1}$. That is, the columns of $\mathfrak{E}_d$ are encoded by the j-Eulerian polynomials, which proves Theorem 9.3.

9.5 Gamma-nonnegativity of $h(\mathrm{sd}(\Delta))$

In Theorem 9.4, we will see that if $h(\Delta)$ is nonnegative, then the polynomial $h(\mathrm{sd}(\Delta); t)$ is real-rooted. Moreover, if $h(\Delta)$ is palindromic, then $h(\mathrm{sd}(\Delta))$ is also palindromic. By Observation 4.2, this implies that $h(\mathrm{sd}(\Delta); t)$ is gamma-nonnegative as well. We can also prove this gamma-nonnegativity directly by investigating the j-Eulerian polynomials closely, as we now show.

First, we can observe that if $w(1) = 1$, there is never a descent in the first position, while if $w(1) = n$ there is always a descent in the first position. Hence the distributions of descents in $S_{n;1}$ and $S_{n;n}$ are the Eulerian distribution for S_{n-1}, i.e.,

$$S_{n;1}(t) = S_{n-1}(t) \text{ and } S_{n;n}(t) = tS_{n-1}(t).$$

In general, if we track the effect of removing the letter j from the beginning of a permutation in $S_{n;j}$, we get the following recurrence relation.

Observation 9.1 *For any $1 \leq j \leq n$,*

$$S_{n;j}(t) = t\sum_{k=1}^{j-1} S_{n-1;k}(t) + \sum_{k=j}^{n-1} S_{n-1;k}(t).$$

Next, notice that there are some nice symmetries in the array of j-Eulerian numbers. For example, recall the involution $w_0 : S_n \to S_n$ that maps i to $n+1-i$. This involution swaps descents for ascents, and if $w(1) = j$, then $w_0w(1) = n+1-j$. Hence, we have the following observation about symmetry.

Observation 9.2 *For any n, j, we have the following symmetries of j-Eulerian numbers:*

$$\left\langle {n;j \atop k} \right\rangle = \left\langle {n; n+1-j \atop n-1-k} \right\rangle,$$

and

$$S_{n;j}(t) = t^{n-1}S_{n;n+1-j}(1/t).$$

We now define the *palindromic j-Eulerian polynomials* by lumping together classes fixed by the involution w_0, namely all permutations beginning with either j or $n+1-j$:

$$\mathbf{S}_{n;j}(t) = \sum_{w \in S_{n;j} \cup S_{n;n+1-j}} t^{\mathrm{des}(w)}.$$

Observe that

$$\mathbf{S}_{n;j}(t) = \begin{cases} S_{n;j}(t) + S_{n,n+1-j}(t) & \text{if } j \neq (n+1)/2, \text{ and} \\ S_{n;j}(t) & \text{if } j = (n+1)/2. \end{cases}$$

By the symmetry seen in Observation 9.2, the polynomials $\mathbf{S}_{n;j}(t)$ have palindromic coefficients, and hence a gamma vector. Note the symmetry axis for $\mathbf{S}_{n;j}(t)$ is at degree $\lfloor \frac{n-1}{2} \rfloor$. If

$$\mathbf{S}_{n;j}(t) = \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \gamma_i^{(n;j)} t^i (1+t)^{n-1-2i},$$

let $\gamma^{(n;j)} = (\gamma_0^{(n;j)}, \gamma_1^{(n;j)}, \ldots, \gamma_{\lfloor (n-1)/2 \rfloor}^{(n;j)})$ denote the corresponding gamma vector.

We will develop a recursive argument for why $\gamma^{(n;j)}$ is nonnegative for all n and j. This recurrence depends on another family of gamma vectors, from the following polynomials, defined for $1 \le j < (n+1)/2$:

$$\mathbf{S}'_{n;j}(t) = tS_{n;j}(t) + S_{n;n+1-j}(t).$$

Note that these polynomials are also palindromic by Proposition 9.2, with symmetry axis at degree $\lfloor n/2 \rfloor$. Hence $\mathbf{S}'_{n;j}(t)$ has a gamma vector, which we denote by

$$\gamma'^{(n,j)} = (\gamma_0'^{(n,j)}, \gamma_1'^{(n,j)}, \ldots, \gamma_{\lfloor n/2 \rfloor}'^{(n,j)}).$$

Note, however, that the shifted center of symmetry means we expand using the basis Γ_n for $\mathbf{S}'_{n;j}(t)$, as opposed to Γ_{n-1} for $\mathbf{S}_{n;j}(t)$.

For example,

$$\mathbf{S}_{5;1}(t) = 10t + 28t^2 + 10t^3 = 10t(1+t)^2 + 8t^2,$$

so $\gamma^{(5;1)} = (0, 10, 8)$, while

$$\mathbf{S}'_{5;1}(t) = 2t + 22t^2 + 22t^3 + 2t^4 = 2t(1+t)^3 + 16t^2(1+t),$$

so $\gamma'^{(5;1)} = (0, 2, 16)$.

Now by applying Observation 9.1 to these gamma vectors, we get the following recurrences.

Proposition 9.1. *We have the following recurrences for the* $\gamma^{(n;j)}$ *and* $\gamma'^{(n;j)}$*:*

1. *If* $j = (n+1)/2$*, then*

$$\gamma^{(n;(n+1)/2)} = \gamma'^{(n-1;1)} + \gamma'^{(n-1;2)} + \cdots + \gamma'^{(n-1;(n-1)/2)}.$$

2. *For* $j < (n+1)/2$*,*

$$\gamma^{(n;j)} = 2\sum_{k=1}^{j-1} \gamma'^{(n-1;k)} + \sum_{k=j}^{\lfloor n/2 \rfloor} \gamma^{(n-1;k)},$$

and

$$\gamma'^{(n;j)} = \sum_{k=1}^{j-1} \gamma'^{(n-1;k)} + 2\sum_{k=j}^{\lfloor n/2 \rfloor} (0, \gamma^{(n-1;k)}),$$

where for $\gamma = (\gamma_0, \gamma_1, \ldots)$*,* $(0, \gamma) = (0, \gamma_0, \gamma_1, \ldots)$*.*

Since these gamma vectors are nonnegative for small n and the recurrences are nonnegative, we get the following corollary.

Corollary 9.1. *The polynomials* $\mathbf{S}_{n;j}(t)$ *and* $\mathbf{S}'_{n;j}(t)$ *are gamma-nonnegative.*

As of this writing it is an open problem to find a combinatorial interpretation for the entries in $\gamma^{(n;j)}$ and $\gamma'^{(n;j)}$. We remark that valley-hopping clearly does not apply.

Returning now to $h(\mathrm{sd}(\Delta))$, suppose that $h(\Delta) = (h_0, h_1, \ldots, h_d)$ is nonnegative, $h_i \geq 0$, and palindromic, $h_i = h_{d-i}$. Then applying the transformation $\mathfrak{E}_d$ will give us gamma-nonnegativity.

For example, if $d = 5$ and $h(\Delta) = (h_0, h_1, h_2, h_3 = h_2, h_4 = h_1, h_5 = h_0)$, then

$$h(\mathrm{sd}(\Delta))^t = \begin{pmatrix} h_0 \\ 27h_0 + 18h_1 + 12h_2 \\ 92h_0 + 102h_1 + 108h_2 \\ 92h_0 + 102h_1 + 108h_2 \\ 27h_0 + 18h_1 + 12h_2 \\ h_0 \end{pmatrix} = h_0 \begin{pmatrix} 1 \\ 5 \\ 10 \\ 10 \\ 5 \\ 1 \end{pmatrix}$$

$$+ (22h_0 + 18h_1 + 12h_2) \begin{pmatrix} 0 \\ 1 \\ 3 \\ 3 \\ 1 \\ 0 \end{pmatrix} + (16h_0 + 48h_1 + 72h_2) \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

Equivalently,

$$h(\mathrm{sd}(\Delta); t) = h_0 \mathbf{S}_{6;1}(t) + h_1 \mathbf{S}_{6;2}(t) + h_2 \mathbf{S}_{6;3}(t),$$

or

$$\begin{aligned} \gamma(\mathrm{sd}(\Delta)) &= h_0 \gamma^{(6;1)} + h_1 \gamma^{(6;2)} + h_2 \gamma^{(6;3)}, \\ &= h_0(1, 22, 16) + h_1(0, 18, 48) + h_2(0, 12, 72). \end{aligned}$$

In general, we get the following result.

Corollary 9.2. *If Δ is a boolean complex with a palindromic h-vector $(h_0, h_1, \ldots, h_d)$, then*

$$h(\mathrm{sd}(\Delta); t) = \sum_{i=0}^{\lfloor d/2 \rfloor} h_i \mathbf{S}_{d+1;i+1}(t),$$

and

$$\gamma(\mathrm{sd}(\Delta)) = \sum_{i=0}^{\lfloor d/2 \rfloor} h_i \gamma^{(d+1;i+1)}.$$

In particular, if $h_i \geq 0$ for all i, $h(\mathrm{sd}(\Delta))$ is gamma-nonnegative.

The h-vector of a sphere is always nonnegative (this is far from obvious—see Chapter 10), and though we did not prove it, the Dehn-Sommerville relations can be applied to boolean complexes, not only simplicial complexes. Thus if Δ is a triangulated sphere, Corollary 9.2 tells us $h(\mathrm{sd}(\Delta))$ is gamma-nonnegative.

9.6 Real roots for barycentric subdivisions

Brenti and Welker asked whether $h(\mathrm{sd}(\Delta);t)$ is log-concave or real-rooted. This is not always so, but they show the following remarkable result. For any polynomial $h(t) = h_0 + h_1 t + \cdots + h_d t^d$, define the sequences of complex numbers $\{\beta_i^{(n)}\}_{n\geq 0}$ as (reciprocals of) the roots of the polynomial obtained by n applications of $\mathfrak{E}_d$ to $h(t)$:

$$\mathfrak{E}_d^n\, h(t) = \prod_{i=1}^{d}(1-\beta_i^{(n)}t).$$

So if $h(t) = h(\Delta;t)$, then $\mathfrak{E}_d^n\, h(t) = h(\mathrm{sd}^n(\Delta);t)$ is the h-polynomial of the nth barycentric subdivision of Δ.

Theorem 9.4 (Real roots). *We have the following results for real rootedness.*

1. *If $h(t) = h_0 + h_1 t + \cdots + h_d t^d$ is a nonnegative integer polynomial, then $h'(t) = \mathfrak{E}_d\, h(t)$ has only real roots.*
2. *For any $d > 1$, there are negative real numbers $\alpha_2, \ldots, \alpha_{d-1}$ such that for every $(d-1)$-dimensional boolean complex Δ, the sequence of complex roots $\beta_i^{(n)}$ associated with $h(\mathrm{sd}^n(\Delta);t)$ satisfies:*
   a. *the numbers $\beta_i^{(n)}$, $1 \leq i \leq d$, are real for n sufficiently large,*
   b. $\lim_{n\to\infty} \beta_1^{(n)} = 0$,
   c. $\lim_{n\to\infty} \beta_i^{(n)} = \alpha_i$ *for* $2 \leq i \leq d-1$,
   d. $\lim_{n\to\infty} \beta_d^{(n)} = -\infty$.

Whoa! Part (1) says that if we have any nonnegative h-polynomial, $\mathfrak{E}_d\, h(t)$ is real-rooted. If h is palindromic, then $\mathfrak{E}_d\, h(t)$ is palindromic and real rooted, which by Observation 4.2 implies that it is log-concave and gamma-nonnegative. We can prove part (1) with an interlacing argument for j-Eulerian polynomials. See Problem 9.6.

Part (2) follows from (1) by some linear algebra on $\mathfrak{E}_d$. See Problem 9.7. In short, the matrix $\mathfrak{E}_d\,/d!$ has largest eigenvalue 1, with multiplicity one, and we can take the corresponding eigenvector, e, to be nonnegative. Hence, there is some n such that $(\mathfrak{E}_d\,/d!)^n h(t)$ is close enough to $e(t)$ that all its coefficients are positive. Part (1) then says that $(\mathfrak{E}_d/d!)^{n+1}h(t)$ is both positive and real-rooted. Furthermore, real-rootedness holds for all subsequent applications of the transformation. We conclude $e(t)$ is real-rooted, and we call these roots $\alpha_1 = 0, \alpha_2, \ldots, \alpha_{d-1}$. The details are outlined in Problem 9.8.

One of the morals of this theorem is that while the f- and h-vectors are useful combinatorial tools, repeated barycentric subdivision "smooths out" a lot of the subtlety. All that is retained is the Euler characteristic and the dimension.

Notes

Nearly all the content in this chapter is drawn from either a 2008 paper of Francesco Brenti and Volkmar Welker [35] or a paper from 2011 by Eran Nevo, Bridget Tenner and the author [112]. Brenti and Welker's result is also studied and extended in the work of Emanuele Delucchi, Aaron Pixton, and Lucas Sabalka [56], as well as in the work of Satoshi Murai and Nevo [109].

It is worth remarking that another paper by Brenti and Welker from 2009 also involves a linear transformation of h-polynomials with a combinatorial description. See [36] and Chapter 7.

Problems

9.1. Find an example of a simplicial complex Δ for which $f(\mathrm{sd}(\Delta)) = (1, 15, 26, 12)$ and $f(\mathrm{sd}(\partial\Delta)) = (1, 11, 10)$.

9.2. Prove that the j-Eulerian polynomials, while not always palindromic, are in fact unimodal.

9.3. Define a collection of polynomials $f_1, f_2, \ldots, f_k$, to be *compatible* if every nonnegative linear combination of them,

$$c_1 f_1 + c_2 f_2 + \cdots + c_k f_k,$$

with $c_1, \ldots, c_k \geq 0$, is real-rooted. (In particular each polynomial f_i must be real-rooted.)

Prove that if $f_1, f_2, \ldots, f_k$ are pairwise compatible polynomials with positive leading coefficients, then the entire collection is compatible.

9.4. Prove that the j-Eulerian polynomials are real-rooted (and hence log-concave and unimodal).

9.5. Show that $\mathbf{S}_{n;j}(t)$ and $\mathbf{S}'_{n;j}(t)$ are real-rooted.

9.6. Prove Part 1 of Theorem 9.4.

9.7. Since $\mathfrak{B}_d$ is triangular, we can read its eigenvalues: $1, 1, 2, 6, \ldots, d!$. Since $\mathfrak{E}_d$ is similar to $\mathfrak{B}_d$, it has the same eigenvalues. Define the normalized transformations, $B_d = \mathfrak{B}_d\,/d!$ and $E_d = \mathfrak{E}_d\,/d!$, so that they have largest eigenvalue 1.

By the Perron-Frobenius theorem, E_d has a fixed point. Compute this fixed point for $d = 1, \ldots, 10$.

9.8. Prove the rest of Theorem 9.4. In particular, show that the fixed point $e(t)$ has only nonpositive real roots α_i as described in the theorem.

Chapter 10
Characterizing *f*-vectors (Supplemental)

10.1 Compressed simplicial complexes

What characterizes an f-vector of a simplicial complex? The entries are obviously nonnegative integers, and $f_0 = 1$, but what other restrictions are there? Well, for one thing, if there are n vertices there can be at most $\binom{n}{2}$ edges, since there is at most one edge for every pair of vertices. That is,

$$f_2 \leq \binom{f_1}{2}.$$

This simple observation can be greatly generalized. It turns out there is a sharp upper bound on the number of $(k+1)$-faces expressed as a polynomial in f_k. (Likewise, there is a sharp lower bound on the number of k faces required for a given number of $(k+1)$-faces.) Collectively, these restrictions, known as the *Kruskal-Katona-Schützenberger* inequalities (or KKS inequalities), characterize the set of f-vectors of simplicial complexes. We remark that characterizing f-vectors of boolean complexes is much, much simpler. See Problem 8.7 (Fig. 10.1).

To explain the KKS inequalities, we will introduce the notion of the *compression* of a simplicial complex. For a simplicial complex Δ, its compression, $\mathcal{C}(\Delta)$, is a canonical simplicial complex with the same f-vector. That is, $\mathcal{C}(\Delta)$ depends only on the f-vector of Δ, so that if $f(\Delta) = f(\Delta')$, then $\mathcal{C}(\Delta) = \mathcal{C}(\Delta')$. When $\mathcal{C}(\Delta) = \Delta$, we say that Δ is a *compressed* complex. We will prove the KKS inequalities hold for compressed complexes. Once we have shown the compression of a simplicial complex is well defined, this will prove the KKS inequalities hold for all simplicial complexes.

To introduce the idea of compression, we first digress into a discussion of the reverse lexicographic, or "revlex" order on k-element sets of nonnegative

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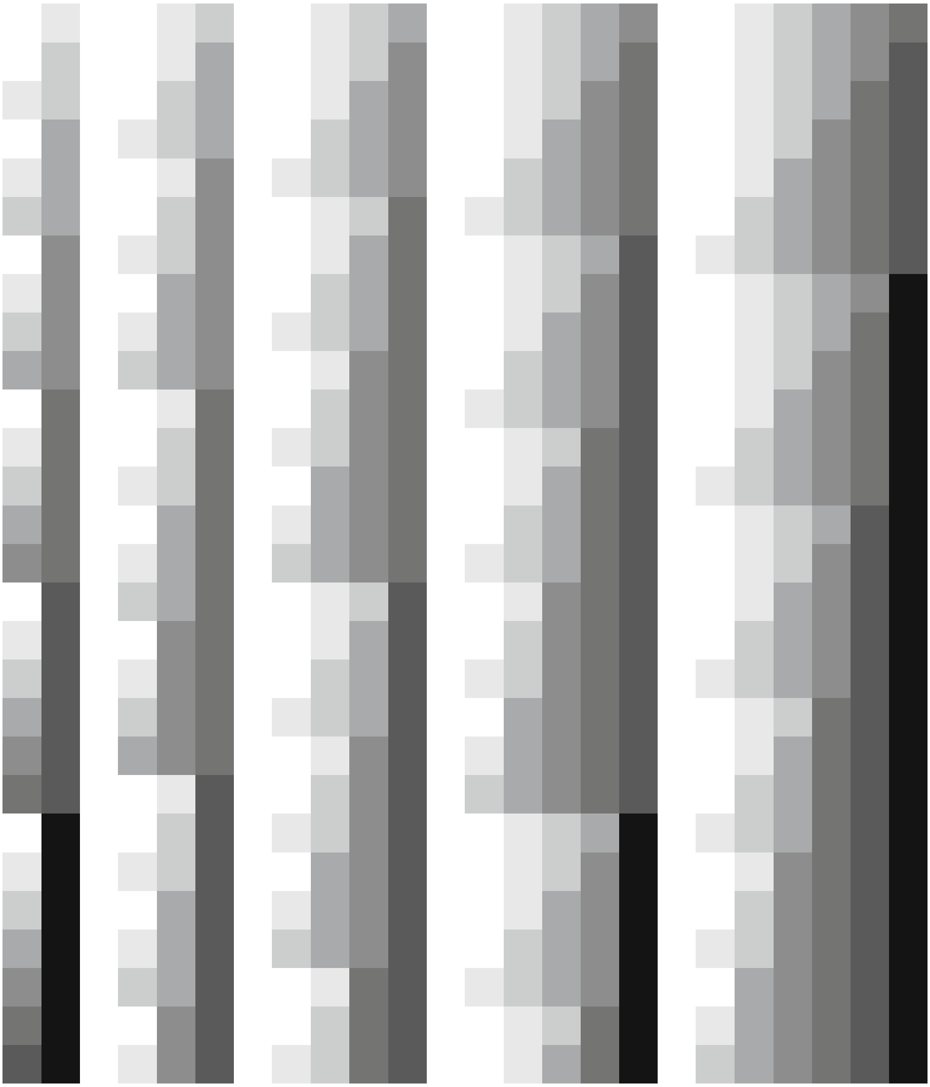


Fig. 10.1 The revlex order on k-sets.

integers. We use $\binom{\mathbb{N}}{k}$ to denote the set of all k-element sets, or "k-sets," of nonnegative integers. Let S and T be distinct sets of integers in $\binom{\mathbb{N}}{k}$. Then we write

$$S \prec T$$

if and only if the list $(s_k, \ldots, s_1)$ appears before $(t_k, \ldots, t_1)$ in lexicographic order. In other words, we can think of "revlex" as short for "lexicographic order on the elements written in reverse order." For example,

$\{2,4,5,6\} \prec \{0,1,4,7\}$ since $(6,5,4,2)$ is lexicographically earlier than $(7,4,1,0)$. In examples, we will usually write k-sets $S = \{s_1 < \cdots < s_k\}$ as words $s_k \cdots s_1$. Figure 10.2 lists the first few k-sets in revlex order, for small k, and Figure 10.1 visualizes k-sets with grayscale colors with the order increasing down columns.

Another way to think about revlex is to say $S \prec T$ if and only the largest element for which S and T differ is in T, i.e., $\max(S-T) < \max(T-S)$. To put it another way, $S \prec T$ if either:

- $\max(S) < \max(T)$, or
- $\max(S) = \max(T) = m$ and $S - \{m\} \prec T - \{m\}$ as $(k-1)$-sets.

This second characterization gives us a nice inductive understanding of revlex order.

$0 : \emptyset$
$1 : 0 \prec 1 \prec 2 \prec 3 \prec 4 \prec 5 \prec \cdots$
$2 : 10 \prec 20 \prec 21 \prec 30 \prec 31 \prec 32 \prec 40 \prec 41 \prec 42 \prec 43 \prec 50 \prec \cdots$
$3 : 210 \prec 310 \prec 320 \prec 321 \prec 410 \prec 420 \prec 421 \prec 430 \prec 431 \prec 432 \prec \cdots$
$4 : 3210 \prec 4210 \prec 4310 \prec 4320 \prec 4321 \prec 5210 \prec 5310 \prec 5320 \prec 5321 \prec \cdots$
$5 : 43210 \prec 53210 \prec 54210 \prec 54310 \prec 54320 \prec 54321 \prec 63210 \prec 64210 \prec \cdots$

Fig. 10.2 Revlex order on k-sets, $k \le 5$.

The successor of a set $S = \{s_1, \ldots, s_k\}$, with $s_1 < \cdots < s_k$, is easily described. If $s_2 > s_1 + 1$, then the k-set immediately following S is $\{s_1 + 1, s_2, \ldots, s_k\}$. Otherwise, let s_i be the smallest element of S such that $s_i + 1$ is not in S. Then the successor of S is:

$$\{0, 1, \ldots, i-2, s_i + 1, \ldots, s_k\}.$$

That is, we increase s_i by one and replace any preceding elements with the smallest possible values. For example, here are a few consecutive 6-sets in revlex order:

$$\cdots \prec 976541 \prec 976542 \prec 976543 \prec 983210 \prec 984210 \prec \cdots.$$

Let $F_k(j)$ denote the jth k-set in revlex order, and let $\mathcal{F}_k(j)$ denote the set of the first j k-sets in revlex order, i.e.,

$$\mathcal{F}_k(j) = \left\{ S \in \binom{\mathbb{N}}{k} : S \preceq F_k(j) \right\}.$$

For example, $F_4(6) = \{0,1,2,5\}$, and

$$\mathcal{F}_4(6) = \{3210, 4210, 4310, 4320, 4321, 5210\}.$$

Now let Δ be a simplicial complex with f-vector $f(\Delta) = (f_0, f_1, \ldots, f_d)$. The *compression* of Δ is defined to be the union, for $k = 0, 1, \ldots, d$, of the first f_k k-sets. In other words,

$$\mathcal{C}(\Delta) = \bigcup_{k=0}^{d} \mathcal{F}_k(f_k).$$

For example, if Δ is the complex from Figure 8.1, with $f(\Delta) = (1, 5, 6, 1)$, the associated compressed complex is

$$\mathcal{C}(\Delta) = \{\emptyset\} \cup \mathcal{F}_1(5) \cup \mathcal{F}_2(6) \cup \mathcal{F}_3(1).$$

Figure 10.3 shows a portion of revlex order, with the faces of Δ highlighted in bold, the faces of $\mathcal{C}(\Delta)$ in boxes.

$$\boxed{\emptyset}$$
$$\boxed{\mathbf{0}} \prec \boxed{\mathbf{1}} \prec \boxed{\mathbf{2}} \prec \boxed{\mathbf{3}} \prec \boxed{\mathbf{4}} \prec 5 \prec 6 \cdots$$
$$\boxed{\mathbf{10}} \prec \boxed{20} \prec \boxed{\mathbf{21}} \prec \boxed{30} \prec \boxed{\mathbf{31}} \prec \boxed{32} \prec 40 \prec \mathbf{41} \prec \mathbf{42} \prec \mathbf{43} \prec 50 \prec \cdots$$
$$\boxed{210} \prec 310 \prec 320 \prec 321 \prec 410 \prec 420 \prec 421 \prec 430 \prec \mathbf{431} \prec 432 \prec \cdots$$

Fig. 10.3 Faces of Δ (in bold) and $\mathcal{C}(\Delta)$ (boxed) shown among all k-sets in revlex order.

We can see $\mathcal{C}(\Delta)$ in Figure 10.4. Notice that while compression preserves dimension and Euler characteristic, it does not respect topology more broadly. Indeed, the complex Δ in Figure 8.1 is connected, whereas $\mathcal{C}(\Delta)$ has two connected components.

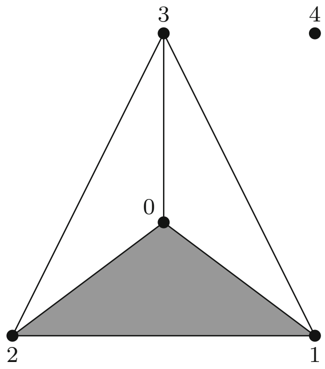


Fig. 10.4 The compression of two-dimensional simplicial complex.

To achieve our goal of characterizing the set of f-vectors of simplicial complexes, the most important theorem we will need is the following.

Theorem 10.1 (Compression of a simplicial complex). *If Δ is a simplicial complex, then its compression $\mathcal{C}(\Delta)$ is a simplicial complex.*

This theorem means that it suffices to characterize the f-vectors of compressed complexes.

The boundary of the set of k-faces of Δ is (by definition of simplicial complex) contained in the set of $(k-1)$-faces of Δ. We need to show that compression preserves this property. For this we will apply the following lemma.

Lemma 10.1. *If $\mathcal{F}$ is a collection of k-sets and $\mathcal{G}$ is a collection of $(k-1)$-sets such that*

$$\partial\mathcal{F} \subseteq \mathcal{G},$$

then

$$\partial\mathcal{C}(\mathcal{F}) \subseteq \mathcal{C}(\mathcal{G}).$$

Theorem 10.1 follows immediately, since we can relabel the vertices of Δ so that they are $\{0, 1, \ldots, n\}$. Taking $\mathcal{F} = \mathcal{F}_k(\Delta)$ and $\mathcal{G} = \mathcal{F}_{k-1}(\Delta)$, and applying the lemma for each $k = 1, 2, \ldots, d$, implies that

$$\partial\mathcal{F}_k(\mathcal{C}(\Delta)) \subseteq \mathcal{F}_{k-1}(\mathcal{C}(\Delta)).$$

In other words, $\mathcal{C}(\Delta)$ is a simplicial complex.

We will now prove Lemma 10.1. The proof is inductive and rather involved, so there is no major harm in skipping it. However, we will explore some interesting features of revlex order along the way, so there is no major harm in reading it, either.

10.2 Proof of the compression lemma

Let $\mathcal{F}$ and $\mathcal{G}$ be families of sets as in Lemma 10.1. That is, let $\mathcal{F}$ be a collection of k-sets, and let $\mathcal{G}$ be a collection of $(k-1)$-sets such that

$$\partial\mathcal{F} \subseteq \mathcal{G}.$$

We wish to show that the boundary of the compression of $\mathcal{F}$ is contained in the compression of $\mathcal{G}$. Of course it suffices to take $\mathcal{G} = \partial\mathcal{F}$, which we will assume from now on. We will proceed by induction on k and m, where m denotes the largest element appearing in a set of $\mathcal{F}$ or $\mathcal{G}$.

If $k = 1$, of course, this is trivial, while if the largest element appearing in $\mathcal{F}$ is $n = k-1$, i.e., if $\mathcal{F} = \{\{0, 1, 2, \ldots, (k-1)\}\}$, then the result follows since $\mathcal{C}(\mathcal{F}) = \mathcal{F}$ and $\mathcal{C}(\partial\mathcal{F}) = \partial\mathcal{F}$.

The main idea will be to split, for any $0 \le i \le m$, the set of k-sets into those sets containing i and those not containing i. That is,

$$\binom{\mathbb{N}}{k} = \binom{\mathbb{N}}{k}_i \bigcup \overline{\binom{\mathbb{N}}{k}}_i,$$

where

$$\binom{\mathbb{N}}{k}_i = \left\{S \in \binom{\mathbb{N}}{k} : i \in S\right\} \text{ and } \overline{\binom{\mathbb{N}}{k}}_i = \left\{S \in \binom{\mathbb{N}}{k} : i \notin S\right\}.$$

We give the sets $\binom{\mathbb{N}}{k}_i$ and $\overline{\binom{\mathbb{N}}{k}}_i$ the total ordering inherited from $\binom{\mathbb{N}}{k}$, i.e., revlex order.

For example, if $k = 4$ and $i = 2$, we get:

$$\begin{aligned} \binom{\mathbb{N}}{4}_2 &: \quad 3210 \prec 4210 \prec 4320 \prec 4321 \prec 5210 \prec 5320 \prec \cdots, \\ \overline{\binom{\mathbb{N}}{4}}_2 &: \quad 4310 \prec 5310 \prec 5410 \prec 5430 \prec 5431 \prec 6310 \prec \cdots. \end{aligned}$$

Define the *i-compression* of a family of k-sets $\mathcal{F}$ as follows. Let $a = |\mathcal{F} \cap \binom{\mathbb{N}}{k}_i|$ and let $b = |\mathcal{F} \cap \overline{\binom{\mathbb{N}}{k}}_i|$, so that $a + b = |\mathcal{F}|$. Let $\mathcal{A}$ denote the first a sets in $\binom{\mathbb{N}}{k}_i$ and let $\mathcal{B}$ denote the collection of the first b sets in $\overline{\binom{\mathbb{N}}{k}}_i$. Then the i-compression of $\mathcal{F}$ is denoted

$$\mathcal{C}_i(\mathcal{F}) = \mathcal{A} \cup \mathcal{B}.$$

For example, suppose

$$\mathcal{F} = \{54310, 63210, 64210, 65321\}.$$

Then

$$\mathcal{C}_6(\mathcal{F}) = \{63210, 64210, 64310\} \cup \{43210\},$$

(the first three 4-sets with a 6 and the first 4-set without a 6) while

$$\mathcal{C}_5(\mathcal{F}) = \{53210, 54210\} \cup \{43210, 63210\},$$

(the first two 4-sets with a 5 and the first two 4-sets without a 5).

Our first task is to show that this weaker form of compression preserves boundaries; i.e., that the i-compression of a simplicial complex is a simplicial complex.

Lemma 10.2. *Let $\mathcal{F} \subset \binom{\mathbb{N}}{k}$, let $\mathcal{G} \subset \binom{\mathbb{N}}{k-1}$, and $\partial\mathcal{F} \subseteq \mathcal{G}$. Suppose m is the largest element in a set from either of the families $\mathcal{F}$ or $\mathcal{G}$. Then for any $i \leq m$, $\partial\mathcal{C}_i(\mathcal{F}) \subseteq \mathcal{C}_i(\mathcal{G})$.*

Our approach to this lemma is to reduce to smaller cases, appealing to cases with either smaller m or smaller k. The main tool is the following map defined for any i. Let ϕ_i be defined on any set F of nonnegative integers as:

$$\phi_i(F) = \{j : i > j \in F\} \cup \{j - 1 : i < j \in F\}.$$

For example, $\phi_3(5410) = 4310$ and $\phi_4(74320) = 6320$. Notice that if $i \notin F$, $|\phi_i(F)| = |F|$, while if $i \in F$, $|\phi_i(F)| = |F| - 1$.

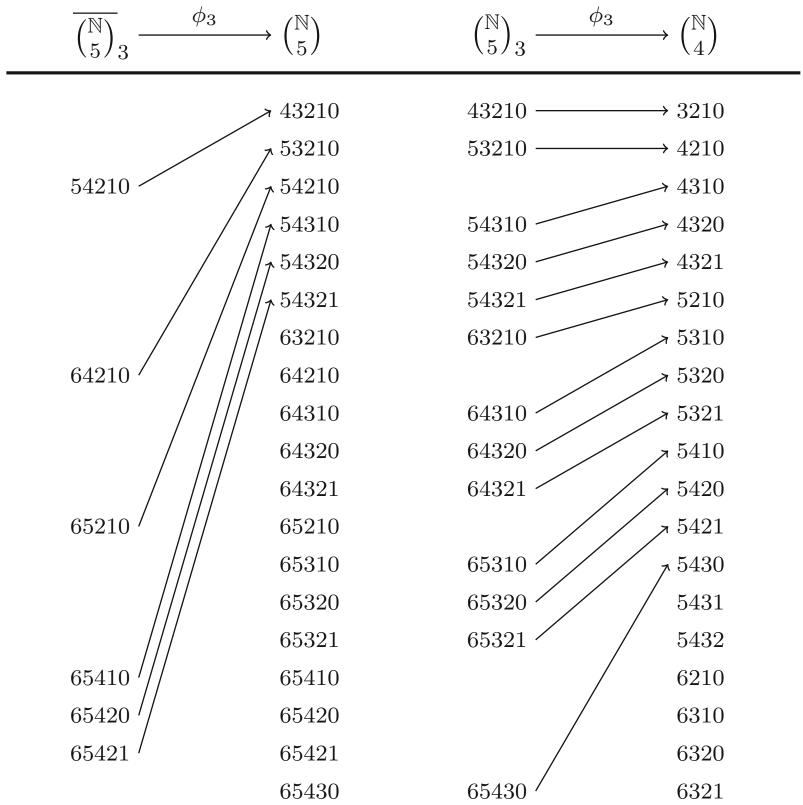


Fig. 10.5 The map ϕ_i on sets in revlex order, for $i = 3$.

The map ϕ_i is an order-preserving bijection between $\overline{\binom{\mathbb{N}}{k}}_i$ and $\binom{\mathbb{N}}{k}$. The inverse function is given by

$$\psi_i(H) = \{j : i > j \in H\} \cup \{j + 1 : i \leq j \in H\}.$$

For example, $\psi_2(43210) = 54310$. In particular, if $\mathcal{F} \subset \overline{\binom{\mathbb{N}}{k}}_i$, then the image of its i-compression, $\phi_i(\mathcal{C}_i(\mathcal{F}))$, is compressed in $\binom{\mathbb{N}}{k}$. In other words, we have the following lemma.

Lemma 10.3. *Let $i \in \mathbb{N}$, and let $k > 1$. Then for any $\mathcal{F} \subset \overline{\binom{\mathbb{N}}{k}}_i$,*

$$\phi_i(\mathcal{C}_i(\mathcal{F})) = \mathcal{C}(\phi_i(\mathcal{F})) \subset \binom{\mathbb{N}}{k}.$$

That is, the image of the i-compression is compressed.

In a similar fashion, ϕ_i is an order-preserving bijection between $\binom{\mathbb{N}}{k}_i$ and $\binom{\mathbb{N}}{k-1}$. The inverse is given by the same inverse function ψ_i above, and then taking the union with i. That is, the pre-image of $H \in \binom{\mathbb{N}}{k-1}$ is

$$\psi_i(H) \cup \{i\} = H' \in \binom{\mathbb{N}}{k}_i.$$

See Figure 10.5.

Lemma 10.4. *Let $i \in \mathbb{N}$, and let $k > 1$. Then for any $\mathcal{F} \subset \binom{\mathbb{N}}{k}_i$,*

$$\phi_i(\mathcal{C}_i(\mathcal{F})) = \mathcal{C}(\phi_i(\mathcal{F})) \subset \binom{\mathbb{N}}{k-1}.$$

That is, the image of the i-compression is compressed.

Loosely speaking, Lemmas 10.3 and 10.4 capture how ϕ_i moves us to smaller cases. Given a family $\mathcal{F} \subset \overline{\binom{\mathbb{N}}{k}}_i$, $\phi_i(\mathcal{F})$ is a family in $\binom{\mathbb{N}}{k}$ with smaller maximum, and i-compression commutes with this map. Likewise, given a family $\mathcal{F} \in \binom{\mathbb{N}}{k}_i$, $\phi_i(\mathcal{F})$ is a family of $(k-1)$-sets, and i-compression commutes with this map. It remains to show that the boundary map is compatible with the map ϕ_i as well.

Supposing $\mathcal{F} \subset \overline{\binom{\mathbb{N}}{k}}_i$ is a family of k-sets not containing i, it is easy to see that ϕ_i commutes with the boundary map:

$$\partial \phi_i(\mathcal{F}) = \phi_i(\partial \mathcal{F}). \tag{10.1}$$

Therefore Lemma 10.3 shows by induction on m that the boundary of the i-compression of $\mathcal{F}$ is contained in the i-compression of the boundary of $\mathcal{F}$.

Lemma 10.5. *Let $i \in \mathbb{N}$, and let $k > 1$. Then for any $\mathcal{F} \subset \overline{\binom{\mathbb{N}}{k}}_i$ with maximum element m, $\mathcal{F}' = \phi_i(\mathcal{F}) \subset \binom{\mathbb{N}}{k}$ with maximum element at most $m-1$. Moreover,*

$$\partial \mathcal{C}_i(\mathcal{F}) \subseteq \mathcal{C}_i(\partial \mathcal{F})$$

if and only if

$$\partial \mathcal{C}(\mathcal{F}') \subseteq \mathcal{C}(\partial \mathcal{F}').$$

Proof. Suppose $\partial\mathcal{C}_i(\mathcal{F}) \subseteq \mathcal{C}_i(\partial\mathcal{F})$. Applying ϕ_i to both sides clearly preserves the inclusion. Applying ϕ_i to the left-hand side we get:

$$\begin{aligned}
\phi_i(\partial\mathcal{C}_i(\mathcal{F})) &= \partial\phi_i(\mathcal{C}_i(\mathcal{F})), && \text{(by Equation (10.1))}\\
&= \partial\mathcal{C}(\phi_i(\mathcal{F})), && \text{(by Lemma 10.3)}\\
&= \partial\mathcal{C}(\mathcal{F}'),
\end{aligned}$$

while on the right-hand side we get:

$$\begin{aligned}
\phi_i(\mathcal{C}_i(\partial\mathcal{F})) &= \mathcal{C}(\phi_i(\partial\mathcal{F})), && \text{(by Lemma 10.3)}\\
&= \mathcal{C}(\partial\phi_i(\mathcal{F})), && \text{(by Equation (10.1))}\\
&= \mathcal{C}(\partial\mathcal{F}').
\end{aligned}$$

This proves the "only if" implication. Since ϕ_i is a bijection, all these steps can be reversed, proving the "if" statement. □

While this takes care of those k-sets not containing i, if $F \in \binom{\mathbb{N}}{k}_i$, things are not so simple. Rather, if $F = \{a_1, \dots, a_{k-1}, i\}$, we have

$$\partial F = \{F \setminus \{a_j\} : 1 \le j \le k-1\} \cup \{\{a_1, \dots, a_{k-1}\}\},$$

so that all but $\{a_1, \dots, a_{k-1}\}$ live in $\binom{\mathbb{N}}{k-1}_i$. Hence,

$$\phi_i(\partial F) = \partial\phi_i(F) \cup \{\phi_i(F)\},$$

where $\partial\phi_i(F) \subset \binom{\mathbb{N}}{k-2}$ and $\phi_i(F) \in \binom{\mathbb{N}}{k-1}$. For families $\mathcal{F} \subset \binom{\mathbb{N}}{k}_i$, we can write:

$$\partial\mathcal{F} = \bigcup_{F\in\mathcal{F}} \left(\{F \setminus \{a_j\} : 1 \le j \le k-1\} \cup \{F \setminus \{i\}\}\right),$$

and

$$\phi_i(\partial\mathcal{F}) = \partial\phi_i(\mathcal{F}) \cup \phi_i(\mathcal{F}). \tag{10.2}$$

We are now ready to prove the companion to Lemma 10.5.

Lemma 10.6. *Let $i \in \mathbb{N}$, and let $k > 1$. Then for any $\mathcal{F} \subset \binom{\mathbb{N}}{k}_i$, $\mathcal{F}' = \phi_i(\mathcal{F}) \subset \binom{\mathbb{N}}{k-1}$. Moreover,*

$$\partial\mathcal{C}_i(\mathcal{F}) \subseteq \mathcal{C}_i(\partial\mathcal{F})$$

if and only if

$$\partial\mathcal{C}(\mathcal{F}') \subseteq \mathcal{C}(\partial\mathcal{F}').$$

Proof. Suppose $\partial\mathcal{C}_i(\mathcal{F}) \subseteq \mathcal{C}_i(\partial\mathcal{F})$. Applying ϕ_i to both sides clearly preserves the inclusion.

Since $\mathcal{C}_i(\mathcal{F}) \subset \binom{\mathbb{N}}{k}_i$, on the left side we have:

$$\begin{aligned}\phi_i(\partial\mathcal{C}_i(\mathcal{F})) &= \partial\phi_i(\mathcal{C}_i(\mathcal{F})) \cup \phi_i(\mathcal{C}_i(\mathcal{F})), &&\text{(by Equation (10.2))}\\ &= \partial\mathcal{C}(\phi_i(\mathcal{F})) \cup \mathcal{C}(\phi_i(\mathcal{F})), &&\text{(by Lemma 10.4)}\\ &= \partial\mathcal{C}(\mathcal{F}') \cup \mathcal{C}(\mathcal{F}'). &&(10.3)\end{aligned}$$

Note that $\partial\mathcal{C}(\phi_i(\mathcal{F})) = \partial\mathcal{C}(\mathcal{F}') \subset \binom{\mathbb{N}}{k-2}$, while $\mathcal{C}(\phi_i(\mathcal{F})) = \mathcal{C}(\mathcal{F}') \subset \binom{\mathbb{N}}{k-1}$.

For the right-hand side, first write

$$\partial\mathcal{F} = \mathcal{A} \cup \mathcal{B},$$

with $\mathcal{A} \subset \binom{\mathbb{N}}{k-1}_i$ and $\mathcal{B} \subset \overline{\binom{\mathbb{N}}{k-1}_i}$. Specifically,

$$\mathcal{A} = \bigcup_{F\in\mathcal{F}} \{F - \{a_j\} : 1 \leq j \leq k-1\},$$

and

$$\mathcal{B} = \bigcup_{F\in\mathcal{F}} F - \{i\}.$$

Then the i-compression of $\partial\mathcal{F}$ is:

$$\mathcal{C}_i(\partial\mathcal{F}) = \mathcal{C}_i(\mathcal{A}) \cup \mathcal{C}_i(\mathcal{B}),$$

and the union is disjoint. Applying ϕ_i, we have:

$$\begin{aligned}\phi_i(\mathcal{C}_i(\partial\mathcal{F})) &= \phi_i(\mathcal{C}_i(\mathcal{A})) \cup \phi_i(\mathcal{C}_i(\mathcal{B})),\\ &= \mathcal{C}(\phi_i(\mathcal{A})) \cup \mathcal{C}(\phi_i(\mathcal{B})) \qquad \text{(by Lemmas 10.3 and 10.4)},\end{aligned}$$

where $\mathcal{C}(\phi_i(\mathcal{A})) \subset \binom{\mathbb{N}}{k-2}$ and $\mathcal{C}(\phi_i(\mathcal{B})) \subset \binom{\mathbb{N}}{k-1}$.

Notice that $|\mathcal{B}| = |\mathcal{F}|$, so $|\phi_i(\mathcal{B})| = |\phi_i(\mathcal{F})| = |\mathcal{F}'|$, and thus,

$$\mathcal{C}(\phi_i(\mathcal{B})) = \mathcal{C}(\mathcal{F}').$$

Further, it is not too difficult to see that

$$\partial\mathcal{F}' = \phi_i(\mathcal{A}),$$

since both sets consist of ϕ_i applied to the $(k-2)$-sets in the collection

$$\bigcup_{F\in\mathcal{F}} \{F - \{i, a_j\} : 1 \leq j \leq k-1\}.$$

Hence, we can write

$$\phi_i(\mathcal{C}_i(\partial\mathcal{F})) = \mathcal{C}(\partial\mathcal{F}') \cup \mathcal{C}(\mathcal{F}'),$$

and comparing with Equation (10.3), we have:

$$\partial \mathcal{C}_i(\mathcal{F}) \subseteq \mathcal{C}_i(\partial \mathcal{F}),$$

if and only if:

$$\phi_i(\partial \mathcal{C}_i(\mathcal{F})) \subseteq \phi_i(\mathcal{C}_i(\partial \mathcal{F})),$$

if and only if:

$$\partial \mathcal{C}(\mathcal{F}') \subseteq \mathcal{C}(\partial \mathcal{F}'),$$

as desired. $\square$

Taken together, the reduction Lemmas 10.5 and 10.6 prove Lemma 10.2, that i-compression preserves boundaries for arbitrary families of k-sets $\mathcal{F} \subset \binom{\mathbb{N}}{k}$. Lemma 10.5 handles $\mathcal{F} \cap \overline{\binom{\mathbb{N}}{k}_i}$ with induction on m, while Lemma 10.6 uses induction on k to address $\mathcal{F} \cap \binom{\mathbb{N}}{k}_i$.

It remains to show that i-compression can be used to obtain full compression.

First, suppose $\mathcal{F}$ is i-compressed, and $\mathcal{A} = \mathcal{F} \cap \binom{\mathbb{N}}{k}_i$, $\mathcal{B} = \mathcal{F} \cap \overline{\binom{\mathbb{N}}{k}_i}$. Let A denote the largest element of $\mathcal{A}$ in revlex order, and let B denote the largest element in $\mathcal{B}$. Suppose $A \prec B$ in revlex order (the case $B \prec A$ is similar). See Figure 10.6.

If

$$\left\{ S \preceq B : S \in \binom{\mathbb{N}}{k} \right\} = \mathcal{F},$$

then $\mathcal{F}$ is fully compressed and we are done.

Else, let C denote the (unique) element such that $C \prec B$, $C \notin \mathcal{F}$, and all k-sets less than C are in $\mathcal{F}$, i.e.,

$$C_{\prec} = \left\{ S \in \binom{\mathbb{N}}{k} : S \prec C \right\} \subset \mathcal{F}.$$

By construction, $C_{\prec}$ is the “totally compressed part” of $\mathcal{F}$. As such, $C_{\prec}$ is fixed by all subsequent partial compressions.

We can continue to compress, depending on B and C, as follows. Either:

a) $C \cap B \neq \emptyset$, in which case there is some $j \in C \cap B$ and we will apply j-compression, or
b) $C \cap B = \emptyset$ and $C \cup B \subsetneq \{0, 1, \ldots, m\}$, in which case there is some $j \leq m$ such that $j \notin C$, $j \notin B$, and we will apply j-compression, or
c) C and B are complements, $C = \{0, 1, \ldots, m\} \setminus B$, and so $m + 1 = 2k$.

In this last case, we see that when m is odd it is possible to have a family of k-sets that is i-compressed for all $i \leq m$, yet the family is not fully compressed. We will handle this special case later.

For now, suppose m is even. Then there will always be some choice of j of type a) or b), and each such j-compression strictly increases the size of the

	$\overline{\binom{\mathbb{N}}{5}}_3$		$\binom{\mathbb{N}}{5}$	$\binom{\mathbb{N}}{5}_3$	
			43210	**43210**	
		$C_{\prec} =$	**53210**	**53210**	
	54210		**54210**		$= \mathcal{A}$
			54310	**54310**	
			54320	**54320**	
			54321	**54321**	
			63210	**63210** $= A$	
$\mathcal{B} =$	**64210**		**64210**		
			64320 $= C$	64310	
			64320	64320	
			64321	64321	
	65210		**65210**		
			65310	65310	
			65320	65320	
			65321	65321	
	65410 $= B$		**65410**		
	65420		65420		
	65421		65421		
			65430	65430	

Fig. 10.6 Example of an i-compressed family, $i = 3$. The family $\mathcal{F}$ is highlighted in bold.

fully compressed part of the family. For example, in Figure 10.6, choosing $j = 6$ or $j = 4$ (both case a)) will do the trick. We continue in this way until there is no set C and the family is fully compressed.

Now suppose $m = 2k - 1$ is odd, $\mathcal{F}$ is i-compressed for all $i \leq m$, yet $\mathcal{F}$ is not fully compressed. Let B denote the largest element in $\mathcal{F}$ and let $C \prec B$ denote any element less than B, yet such that $C \notin \mathcal{F}$. Since $\mathcal{F}$ is i-compressed for all $i \leq m$, it must be that we are in case c), and so C is the complement of B in $\{0, 1, \ldots, m\}$. In particular C is uniquely determined by B.

For example, suppose

$$\begin{aligned} \mathcal{F} = \{&210, 310, 320, 321, 410, \\ &420, 421, 430, 431, 510\} \end{aligned}$$

so that the only thing missing from $\mathcal{F}$ is $C = 432$. Then $\mathcal{F}$ is i-compressed for all $i \leq 5$, yet not compressed.

Since $C \prec B$, in particular $\max C < \max B = m$. But then all the subsets of $\{0, 1, \dots, m-1\}$ are in $\partial\mathcal{F}$ already. Thus we can see that $\partial C \subset \partial\mathcal{F}$. Let $\mathcal{F}' = \mathcal{F} - \{B\} \cup \{C\}$ be the family obtained by replacing B with C. Then $\partial\mathcal{F}' \subseteq \partial\mathcal{F}$.

On the other side, suppose after some number of i-compressions we obtain a family $\mathcal{F}$ that is compressed, and a family $\mathcal{G}$ that is i-compressed for all $i \leq m$, but not fully compressed, and $\partial\mathcal{F} \subseteq \mathcal{G}$. Let B be the largest element of $\mathcal{G}$ and let C be its complement, $C \notin \mathcal{G}$. Again, we wish to swap B with C to get $\mathcal{G}' = \mathcal{G} - \{B\} \cup \{C\}$. Since $\max C < \max B = m$, in particular, $C \subseteq \{0, 1, \dots, m-1\}$. Since $C \notin \mathcal{G}$, it is not in ∂F for any $F \in \mathcal{F}$ either. Hence

$$\max\{\max F\}_{F\in\mathcal{F}} \leq m - 1,$$

so B is not on the boundary of $\mathcal{F}$ either. Hence, there is no harm in replacing B by C, and $\partial\mathcal{F} \subset \mathcal{G}'$ with $\mathcal{G}'$ compressed.

This (finally!) completes the proof of all cases of the compression lemma, Lemma 10.1. Let's move on to discussing the numeric consequences of compression.

10.3 Kruskal-Katona-Schützenberger inequalities

Recall that we denote the jth element of $\binom{\mathbb{N}}{k}$ by $F_k(j)$. It turns out the elements of $F_k(j)$ give a simple way to compute j, its position in revlex order.

For example, suppose $F_5(j) = \{0, 1, 2, 5, 7\}$. To find j, we will count the ways to form a 5-set $S \prec \{0, 1, 2, 5, 7\}$. To do so, we consider five cases (though the final three cases are empty):

- $S = \{s_1, s_2, s_3, s_4, s_5\} \subset \{0, 1, 2, 3, 4, 5, 6\}$,
- $S = \{s_1, s_2, s_3, s_4, 7\}$, with $\{s_1, s_2, s_3, s_4\} \subset \{0, 1, 2, 3, 4\}$,
- $S = \{s_1, s_2, s_3, 5, 7\}$, with $\{s_1, s_2, s_3\} \subset \{0, 1\}$,
- $S = \{s_1, s_2, 2, 5, 7\}$, with $\{s_1, s_2\} \subset \{0\}$,
- $S = \{s_1, 1, 2, 5, 7\}$, with $\{s_1\} \subset \emptyset$.

Thus we have

$$\begin{aligned} j - 1 &= (\text{ number of sets } S \prec \{0, 1, 2, 5, 7\} \text{ }), \\ &= \binom{7}{5} + \binom{5}{4} + \binom{2}{3} + \binom{1}{2} + \binom{0}{1}, \\ &= 26, \end{aligned}$$

So in this example $j = 27$.

In general, if $S \prec F_k(j) = \{a_1, a_2, \dots, a_k\}$, with $a_1 < a_2 < \cdots < a_k$, there are k cases to consider:

- $S = \{s_1, s_2, \ldots, s_k\} \subset \{0, 1, \ldots, a_k - 1\}$,
- $S = \{s_1, s_2, \ldots, s_{k-1}, a_k\}$, with $\{s_1, s_2, \ldots, s_{k-1}\} \subset \{0, 1, \ldots, a_{k-1} - 1\}$,

$\vdots$

- $S = \{s_1, s_2, a_3, \ldots a_{k-1}, a_k\}$, with $\{s_1, s_2\} \subset \{0, 1, \ldots, a_2 - 1\}$,
- $S = \{s_1, a_2, a_3, \ldots, a_{k-1}, a_k\}$, with $\{s_1\} \subset \{0, 1, \ldots, a_1 - 1\}$.

Thus,

$$j - 1 = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_1}{1}.$$

Letting a_r denote the smallest element such that $a_r + 1 \notin F_k(j)$, the successor of

$$\{a_1, \ldots, a_{r-1}, a_r, a_{r+1}, \ldots, a_k\}$$

in revlex order is

$$F_k(j+1) = \{0, 1, \ldots, r-2, a_r + 1, a_{r+1}, \ldots, a_k\}.$$

This gives us the following, more compact expression.

Proposition 10.1. *Suppose $F_k(j) = \{a_1, a_2, \ldots, a_k\}$, with $0 \leq a_1 < a_2 < \cdots < a_k$, is the jth k-set in revlex order. Then we have*

$$j = \binom{a_k}{k} + \cdots + \binom{a_{r+1}}{r+1} + \binom{a_r + 1}{r}, \tag{10.4}$$

where a_r is the smallest element such that $a_r + 1 \notin F_k(j)$.

For example, with $F_5(27) = \{0, 1, 2, 5, 7\}$ from before, we have

$$j = \binom{7}{5} + \binom{5}{4} + \binom{3}{3} = 27.$$

This way of writing the integer j as a sum of decreasing binomial coefficients: $\binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots$ as in Equation (10.4) is called the *k-binomial expansion of j*, or the *Macaulay expansion* of j. It can be computed greedily, by first finding the largest a such that $\binom{a}{k} \leq j$, then the largest b such that $\binom{b}{k-1} \leq j - \binom{a}{k}$, and so on. Thus the 93rd 4-set must be $F_4(93) = \{1, 2, 6, 8\}$, since

$$\begin{aligned} 93 &= \binom{8}{4} + 23, \\ &= \binom{8}{4} + \binom{6}{3} + 3, \\ &= \binom{8}{4} + \binom{6}{3} + \binom{3}{2}. \end{aligned}$$

The same technique we used for determining the position of a k-set in revlex order (Proposition 10.1) can be used to count the sets in the boundary of a collection of compressed k-sets, $\partial\mathcal{F}_k(j)$.

Suppose

$$S = \{s_1, \ldots, s_{k-1}\} \subset \{b_1, \ldots, b_k\} \prec \{a_1, \ldots, a_k\} = F_k(j),$$

so that S is an element in $\partial\mathcal{F}_k(j)$. Then there are $k-1$ cases (some of which may be empty):

- $S = \{s_1, \ldots, s_{k-1}\} \subset \{0, 1, \ldots, a_k - 1\}$,
- $S = \{s_1, \ldots, s_{k-2}, a_k\}$, with $\{s_1, \ldots, s_{k-2}\} \subset \{0, 1, \ldots, a_{k-1} - 1\}$,

$\vdots$

- $S = \{s_1, s_2, a_4, \ldots, a_k\}$, with $\{s_1, s_2\} \subset \{0, 1, \ldots, a_3 - 1\}$,
- $S = \{s_1, a_3, a_4, \ldots, a_k\}$, with $\{s_1\} \subset \{0, 1, \ldots, a_2 - 1\}$.

This case analysis establishes the enumeration, but it is even better. It tells us that we are counting precisely those $(k-1)$-sets $S \prec \{a_2, \ldots, a_k\}$. To put it another way, if $F \prec F_k(j)$, the greatest $(k-1)$-subset of $F_k(j)$ (in revlex order) is greater than the greatest $(k-1)$-subset of F. This will be useful later on. We have established the following, keeping in mind that $\{a_2, \ldots, a_k\}$ itself is on the boundary of $F_k(j)$.

Proposition 10.2 (Lower bounds). *Suppose $F_k(j) = \{a_1, a_2, \ldots, a_k\}$, with $0 \le a_1 < a_2 < \cdots < a_k$, is the jth k-set in revlex order. Then*

$$|\partial\mathcal{F}_k(j)| = \binom{a_k}{k-1} + \binom{a_{k-1}}{k-2} + \cdots + \binom{a_2}{1} + \binom{a_1}{0}.$$

Moreover,

$$\partial\mathcal{F}_k(j) = \{S : S \preceq \{a_2, \ldots, a_k\}\}.$$

Corollary 10.1 (KKS inequalities, lower bound version). *Suppose Δ is a simplicial complex with f-vector $(f_0, f_1, \ldots, f_d)$. Let*

$$f_k = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_1}{1},$$

be the k-binomial expansion of f_k, with $0 \le a_1 < a_2 < \cdots < a_k$. Then

$$f_{k-1} \ge \binom{a_k}{k-1} + \binom{a_{k-1}}{k-2} + \cdots + \binom{a_1}{0}.$$

For example, if we want to construct a simplicial complex with

$$f_3 = 100 = \binom{9}{3} + \binom{6}{2} + \binom{1}{1}$$

triangles, we need to have at least

$$\binom{9}{2} + \binom{6}{1} + \binom{1}{0} = 43$$

edges, which in turn means there have to be at least

$$\binom{9}{1} + \binom{6}{0} = 10$$

vertices.

Notice that because $\partial\mathcal{F}_k(j)$ does not depend on $a_1 = \min(F_k(j))$, $\partial\mathcal{F}_k(j)$ can equal $\partial\mathcal{F}_k(j')$ for $j \neq j'$. For example,

$$\partial\{F \preceq 320\} = \{10, 20, 21, 30, 31, 32\} = \partial\{F \preceq 321\}.$$

Now let's turn our thinking around and define $\mathcal{C}\mathcal{F}_k(j)$ to be the largest collection $\mathcal{F}$ of $(k+1)$-sets whose boundary $\partial\mathcal{F}$ is contained in $\mathcal{F}_k(j)$. That is,

$$\mathcal{C}\mathcal{F}_k(j) = \left\{ S \in \binom{\mathbb{N}}{k+1} : \text{ if } F \in \partial S \text{ then } F \preceq F_k(j) \right\}.$$

From the "moreover" part of Proposition 10.2, we can see that if $0 \notin F_k(j) = \{a_1, \ldots, a_k\}$, then for any $0 \leq i < a_1$,

$$\mathcal{F}_k(j) = \partial\{F \preceq \{i, a_1, \ldots, a_k\}\}.$$

Therefore in this case,

$$\begin{aligned} \mathcal{C}\mathcal{F}_k(j) &= \{F \preceq \{a_1 - 1, a_1, \ldots, a_k\}\}, \\ &= \mathcal{F}_{k+1}(j'), \end{aligned}$$

where if a_r is the smallest element such that $a_r + 1 \notin F_k(j)$,

$$j' = \binom{a_k}{k+1} + \cdots + \binom{a_{r+1}}{r+2} + \binom{a_r + 1}{r+1}.$$

In general, to find $\mathcal{C}\mathcal{F}_k(j)$, we should then find the nearest $F \preceq F_k(j)$ with $0 \notin F$ and apply this trick. To find such an F, we need to find the smallest element a_r such that $a_r \geq r + 1$. Then

$$F = \{a_r - r, \ldots, a_r - 2, a_r - 1, a_{r+1}, \ldots a_k\},$$

has $0 \notin F$. Now,

$$\begin{aligned} \mathcal{C}\mathcal{F}_k(j) &= \mathcal{C}\{F' \preceq F\}, \\ &= \{G \preceq \{a_r - r - 1, a_r - r, \ldots, a_r - 1, a_{r+1}, \ldots, a_k\}\}, \\ &= \mathcal{F}_{k+1}(j'). \end{aligned}$$

Here we can see that $a_r - 1$ is the smallest element such that $(a_r - 1) + 1 \notin F_{k+1}(j')$, so

$$j' = \binom{a_k}{k+1} + \cdots + \binom{a_{r+1}}{r+2} + \binom{a_r}{r+1},$$

exactly as in the previous case. We record this important result here.

Proposition 10.3 (Upper bounds). *Suppose $F_k(j) = \{a_1, a_2, \ldots, a_k\}$, with $0 \le a_1 < a_2 < \cdots < a_k$, is the jth k-set in revlex order. Then if a_r is the smallest element such that $a_r \ge r+1$,*

$$|\mathcal{C}\mathcal{F}_k(j)| = j' = \binom{a_k}{k+1} + \cdots + \binom{a_{r+1}}{r+2} + \binom{a_r}{r+1}.$$

Moreover,

$$\mathcal{C}\mathcal{F}_k(j) = \mathcal{F}_{k+1}(j').$$

Corollary 10.2 (KKS inequalities, upper bound version). *Suppose Δ is a simplicial complex with f-vector $(f_0, f_1, \ldots, f_d)$. Let*

$$f_k = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_1}{1},$$

be the k-binomial expansion of f_k, then

$$f_{k+1} \le \binom{a_k}{k+1} + \binom{a_{k-1}}{k} + \cdots + \binom{a_r}{r+1},$$

where r is the smallest index such that $a_r \ge r+1$.

For example there exists a simplicial complex with

$$f_3 = 1000 = \binom{19}{3} + \binom{8}{2} + \binom{3}{1}$$

triangles and

$$\binom{19}{4} + \binom{8}{3} + \binom{3}{2} = 3935$$

tetrahedra, but there is no simplicial complex with $f_3 = 1000$ and $f_4 = 3936$.

10.4 Frankl-Füredi-Kalai inequalities

There is an analogue of the KKS inequalities for balanced simplicial complexes, due to Frankl, Füredi, and Kalai. To give an idea for how things are different in a balanced world, recall that balanced 1-dimensional complexes are bipartite graphs. The greatest number of edges in a bipartite graph comes from having half the vertices colored black, half colored white, and an edge matching each black vertex with each white vertex. Thus, if there are $n = 2k$ vertices, there can be at most k^2 edges. (If $n = 2k + 1$ there are at most $k(k+1)$ edges.) Our first "FFK-inequality" is that for 2-colorable complexes, we have

$$f_2 \leq \lfloor f_1/2 \rfloor \lceil f_1/2 \rceil,$$

If the goal is to characterize f-vectors of d-colored simplicial complexes, one should study revlex order on "d-colored" k-sets. A d-colored k-set of $\mathbb{N}$ is a k-set S such that no two elements of S are congruent modulo d. Let $\binom{\mathbb{N}}{k:d}$ denote the set of d-colored k-subsets. The *color* of a set S in $\binom{\mathbb{N}}{k:d}$ is the set of remainders modulo d, i.e., $\mathrm{col}_d(S) = \{s \mod d : s \in S\}$. In Figure 10.7 we see the 4-colored k-sets listed in revlex order. Notice that $521, 531$, and 541 are not 4-colored since $1 \equiv 5 \mod 4$. Also notice there are no 4-colored sets with $k > 4$.

$0 : \emptyset$
$1 : 0 \prec 1 \prec 2 \prec 3 \prec 4 \prec 5 \prec \cdots$
$2 : 10 \prec 20 \prec 21 \prec 30 \prec 31 \prec 32 \prec 41 \prec 42 \prec 43 \prec 50 \prec 52 \prec \cdots$
$3 : 210 \prec 310 \prec 320 \prec 321 \prec 421 \prec 431 \prec 432 \prec 520 \prec 530 \prec 532 \cdots$
$4 : 3210 \prec 4321 \prec 5320 \prec 5432 \prec 6310 \prec 6431 \prec 6530 \prec 6543 \prec \cdots$
$5 : \emptyset$

Fig. 10.7 Revlex order on 4-colored k-sets, $k \leq 4$.

The notion of compression of a d-colored complex makes perfect sense, and we can define compressed d-complexes just as in the uncolored case, with the property that the compression of a d-colored simplicial complex is again a d-colored simplicial complex.

To give the characterization of revlex order for d-colored complexes, it will be helpful to have a d-colored version of binomial coefficients. That is, define

$$\binom{n}{k:d} = \left| \left\{ S \in \binom{\mathbb{N}}{k:d} : \max S < n \right\} \right|.$$

Thus if $n \leq d$, $\binom{n}{k:d} = \binom{n}{k}$ is just the usual binomial coefficient. On the other hand, if $k > d$, there is no way to choose a k-set without having two elements be congruent modulo d, so $\binom{n}{k:d} = 0$.

But when $n > d$ this number is much smaller. For example, how many 5-colored 3-sets have their maximum less than 22? We need to choose a set $\{a, b, c\}$ from $\{0, 1, 2, \ldots, 21\}$ such that each number has a distinct remainder modulo 5. Let's arrange the numbers in array of five columns first:

$$\begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 & 9 \\ 10 & 11 & 12 & 13 & 14 \\ 15 & 16 & 17 & 18 & 19 \\ 20 & 21 & & & \end{array}.$$

We see the multiples of five on the left, those numbers congruent to 1 modulo 5 in the next column, 2 modulo 5 in the third column, and so on. Every column is a congruence class. Our goal is to count the number of ways to choose four elements from this array so that no two elements lie in the same column.

There are five choices for an element in column one and five choices for column two, while if we choose an element from the third, fourth, or fifth column, we have only four choices. Thus it makes sense to refine our count according to how many elements we choose from among the first two columns. We can:

- choose two elements from the first two columns in 5^2 ways and one element from the final three columns in $\binom{3}{1} \cdot 4$ ways,
- choose one element from the first two columns in $\binom{2}{1} \cdot 5$ ways and two elements from the final three columns in $\binom{3}{2} \cdot 4^2$ ways, or
- choose no elements from the first two columns and three elements from the final three columns in $\binom{3}{3} \cdot 4^3$ ways.

In total, then, we find

$$\binom{2}{2} \cdot 5^2 \cdot \binom{3}{1} \cdot 4 + \binom{2}{1} \cdot 5 \cdot \binom{3}{2} \cdot 4^2 + \binom{2}{0} \cdot 5^0 \cdot \binom{3}{3} \cdot 4^3 = 844$$

different 5-colored 3-sets on $\{0, 1, 2, \ldots, 21\}$. Thus, we write

$$\binom{22}{3:5} = 844.$$

The reasoning we used in the example generalizes easily. For $n > d$ suppose $n = qd + r$, with $0 \le r \le d - 1$. If we draw the numbers $\{0, 1, 2, \ldots, n-1\}$ in an array with d columns, we will find the first r columns have $q + 1$ rows, while the final $d - r$ of them have q rows. Then we can write:

$$\binom{n}{k:d} = \sum_{j=0}^{k} \binom{r}{j} \binom{d-r}{k-j} (q+1)^j q^{k-j},$$

where j counts the number of elements chosen from among the first r columns.

One can show, just as in the uncolored case, that every positive integer has a unique expansion in terms of colored binomial coefficients. As with Proposition 10.1, we can connect this expansion with revlex ordering on colored k-sets.

Proposition 10.4. *Suppose $F_{k:d}(j) = \{a_1, a_2, \ldots, a_k\}$, with $0 \le a_1 < a_2 < \cdots a_k$, is the jth d-colored k-set in revlex order. Then we have*

$$j = \binom{a_k}{k:d} + \binom{a_{k-1}}{k-1:d-1} + \cdots + \binom{a_{r+1}}{r+1:d-k+r+1} + \binom{a_r+i}{r:d-k+r},$$

where r is the smallest index such that there is an element $b = a_r + i < a_{r+1}$ with $1 \leq i \leq d$ and $b \not\equiv a_i \mod d$ for any $i \neq r$.

(The strange-looking condition on $a_r + i$ merely comes from identifying the successor of $F_{k:d}(j)$ in revlex order.)

The Frankl-Füredi-Kalai result in the d-colorable case is essentially the same statement as Propositions 10.2 and 10.3, and it can be given a very similar argument after replacing the revlex order on $\binom{\mathbb{N}}{k}$ with the revlex order on $\binom{\mathbb{N}}{k:d}$. We will give the main result here without proof.

Proposition 10.5 (FFK inequalities). *Suppose Δ is a d-colorable simplicial complex with f-vector $(f_0, f_1, \ldots, f_d)$. Let*

$$f_k = \binom{a_k}{k:d} + \binom{a_{k-1}}{k-1:d-1} + \cdots + \binom{a_1}{1:d-k+1},$$

be the d-colored k-binomial expansion of f_k, with $0 \leq a_1 < a_2 < \cdots < a_k$. Then

$$f_{k-1} \geq \binom{a_k}{k-1:d} + \binom{a_{k-1}}{k-2:d-1} + \cdots + \binom{a_1}{0:d-k+1},$$

and

$$f_{k+1} \leq \binom{a_k}{k+1:d} + \binom{a_{k-1}}{k:d-1} + \cdots + \binom{a_r}{r+1:d-k+r},$$

where r is as in Proposition 10.4. Moreover, every vector satisfying these inequalities is the f-vector of a d-colorable simplicial complex.

Now that we have these inequalities, one may wonder whether the set of flag complexes, another special family of simplicial complexes, enjoys a similar characterization of its f-vectors. As of this writing, the answer is no, but we have the following partial result from Andrew Frohmader in 2008 [76].

Proposition 10.6. *The f-vector of a flag complex is an FFK-vector. There are FFK-vectors that are not realized as the f-vector of any flag complex.*

To see that this is indeed not a characterization of f-vectors for flag complexes, Frohmader points to the vector $(1, 4, 5, 1)$. It is the f-vector of a balanced complex with $d = 3$ colors. However, it cannot be the f-vector of a flag complex, since a graph with four vertices and five edges must be only one edge away from having a 4-clique, or tetrahedron. Removing one edge from a tetrahedron yields two triangles, however, and this f-vector has $f_3 = 1$.

We close this section by remarking that the KKS inequalities are in some sense the limit of the FFK inequalities as $d \to \infty$. That is, fix a vector $f = (f_0, f_1, \ldots)$. Then if d is large enough (certainly if $d > \max f_i$), f is a d-FFK vector if and only if f is a KKS vector.

10.5 Multicomplexes and M-vectors

An interesting generalization of the study of simplicial complexes is the study of *multicomplexes*. This allows not only subsets of a fixed vertex set, but *multi-subsets* of the vertex set. A multiset is a set in which the elements are allowed to have multiplicities, written $\{i_1^{m_1}, \ldots, i_k^{m_k}\}$, where m_j is the multiplicity of i_j. For example, $\{1,1,2,4\} = \{1^2, 2, 4\}$ and $\{2,5,5,5,6,6,7,7\} = \{2, 5^3, 6^2, 7^2\}$ are multisets. Every set is a multiset in which all elements have multiplicity one. The size of a multiset is the sum of the multiplicities of the distinct elements in the set, e.g., the multisets above have sizes four and eight, respectively.

If we fix a vertex set V, then a multicomplex Δ is a collection of multisets of V that are closed under containment, i.e., if G is a multiset in Δ and F is a sub-multiset of G, then F is in Δ. We use the same terminology as in the simplicial case, so that a k-multiset is called a k-face, and so on. The dimension of a face is one less than the sum of the multiplicities, e.g., $\dim(\{1,1,2,4,4,4\}) = 5$.

If $V = \{1, 2, \ldots, n\}$, let $x_1, \ldots, x_n$ be a set of commuting indeterminates. A simple way to understand a multicomplex is in terms of monomials in the x_i. (This will be described in more detail for the case of simplicial complexes in Section 10.6.) Monomials are easily identified with multisets, via

$$x_{i_1}^{m_1} \cdots x_{i_k}^{m_k} \leftrightarrow \{i_1^{m_1}, \ldots, i_k^{m_k}\}.$$

We write x^F for F a multiset, e.g., if $F = \{0,1,1,3\}$, we have $x^F = x_0 x_1^2 x_3$. With this convention, we say Δ is a multicomplex if it satisfies the following divisibility property: for any $G \in \Delta$, if x^F divides x^G, then $F \in \Delta$. From this point of view, the dimension of a face is one less than the degree of the corresponding monomial: $\dim F = \deg(x^F) - 1$.

We can identify finite multicomplexes by their facets, though multicomplexes can be infinite, even with finite vertex set. A way to identify a multicomplex in either case is by its minimal non-faces. As in the simplicial case, multicomplexes can be illustrated with a face poset. In this setting, a multicomplex Δ is a lower ideal in the poset of multisets on V. For example, the poset highlighted in Figure 10.8 is the multicomplex with facets $\{\{0,1,1\}, \{0,1,2\}, \{1,1,1\}\}$. We have shown only the first four levels of the multiset poset, but we remark it is an infinite poset unless we bound the multiplicities.

The f-vector of a multicomplex Δ is called an *M-vector*. There is a version of the KKS inequalities for multicomplexes due to Macaulay [105]. In fact, it predates the KKS inequalities by about thirty years. Macaulay's paper is from 1927, while Schützenberger's paper appeared in 1959 [134], Kruskal in 1963 [98], and Katona in 1966 [94].

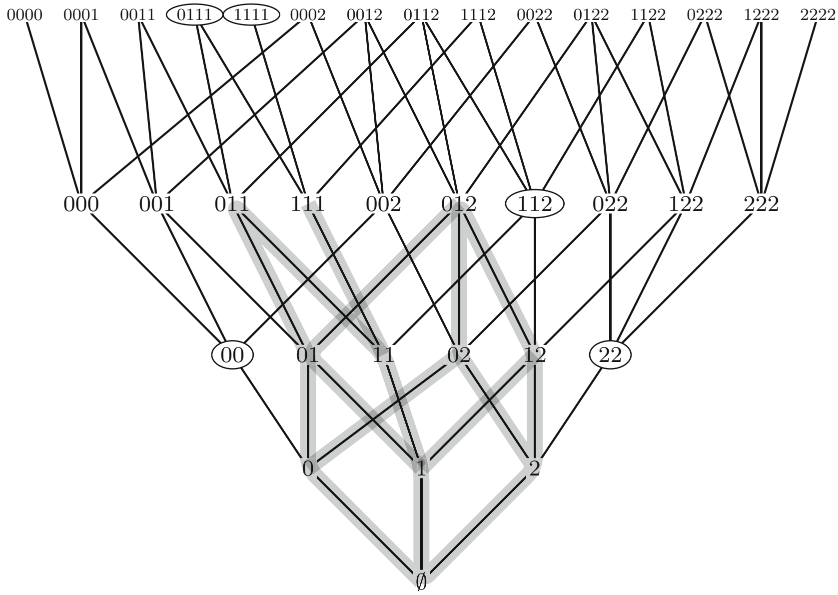


Fig. 10.8 The face poset of a multicomplex inside the first few levels of the multiset poset. The minimal non-faces are circled.

To begin the characterization, we can see if there are n vertices, then there can be as many as $\binom{n}{2} + n = \binom{n+1}{2}$ 2-faces, i.e.,

$$f_2 \leq \binom{f_1 + 1}{2}.$$

The generalization of this observation to higher dimensional faces can be stated as follows.

Proposition 10.7 (M-vector inequalities). *Suppose Δ is a multicomplex with f-vector $(f_0, f_1, \ldots, f_d)$. Let*

$$f_k = \binom{m_k}{k} + \binom{m_{k-1}}{k-1} + \cdots + \binom{m_r}{r},$$

be the k-binomial expansion of f_k, where $m_k \geq m_{k-1} \geq \cdots \geq m_r \geq r > 0$. Then,

$$f_{k-1} \geq \binom{m_k - 1}{k-1} + \binom{m_{k-1} - 1}{k-2} + \cdots + \binom{m_r - 1}{r-1},$$

and

$$f_{k-1} \leq \binom{m_k + 1}{k} + \binom{m_{k-1} + 1}{k-1} + \cdots + \binom{m_r + 1}{r}.$$

Moreover, any vector satisfying these inequalities is the f-vector of some multicomplex.

The proposition can be proved by studying revlex order for multisets. Many of the properties of compressed multicomplexes are analogous to those for simplicial complexes and the proof of Proposition 10.7 follows along similar lines as the proof of KKS inequalities. See [105].

We remark that while a simplicial complex has its face poset contained in a finite poset, the boolean algebra 2^V, a multicomplex on V has its face poset contained in an infinite poset. While there are $\binom{n}{k}$ k-sets of n elements, there are $\binom{n+k-1}{k}$ k-multisets of n elements. The rank function for 2^V is $(1+t)^n$, while the rank function for the poset of multisets is:

$$\sum_{k\geq 0}\binom{n+k-1}{k}t^k = \frac{1}{(1-t)^n}.$$

Thus, an easy observation that can be made for face numbers of multisets is the following.

Observation 10.1 *For a multicomplex with n vertices,*

$$f_i \leq \binom{n+i-1}{i}.$$

10.6 The Stanley-Reisner ring

An algebraic invariant of considerable importance is the *Stanley-Reisner ring* of a simplicial complex Δ. We will see that algebraic properties of this ring have topological implications for Δ and vice versa.

For Δ with vertex set $V = \{1, 2, \ldots, n\}$, let $x_1, x_2, \ldots, x_n$ be a set of commuting indeterminates, and let $\mathbb{k}$ be a field. We can identify monomials with multisets on V as in Section 10.5, via

$$x_{i_1}^{m_1}\cdots x_{i_k}^{m_k} \leftrightarrow \{i_1^{m_1}, \ldots, i_k^{m_k}\}.$$

We write x^F for F a multiset, e.g., if $F = \{0, 1, 1, 3\}$, we have $x^F = x_0x_1^2x_3$.

Then the Stanley-Reisner ring of Δ, denoted $\mathbb{k}[\Delta]$, is

$$\mathbb{k}[\Delta] = \mathbb{k}[x_1, \ldots, x_n]/I(\Delta),$$

where $I(\Delta)$ denotes the ideal in $\mathbb{k}[x_1, \ldots, x_n]$ generated by the minimal non-faces of Δ:

$$I(\Delta) = \langle x^F : F \subset V, F \notin \Delta\rangle.$$

For example, if Δ is the complex of Figure 8.1 whose face poset is shown in Figure 8.2, its minimal non-faces are:

$$x_0x_2, \quad x_0x_3, \quad x_0x_4, \quad x_2x_3, \quad x_1x_2x_4,$$

and so

$$\Bbbk[\Delta] = \Bbbk[x_1, \ldots, x_n]/\langle x_0x_2, x_0x_3, x_0x_4, x_2x_3, x_1x_2x_4\rangle.$$

As a vector space, $\Bbbk[\Delta]$ is the $\Bbbk$-span of all monomials whose support is a face of Δ, i.e.,

$$\Bbbk[\Delta] = \mathrm{span}\{x^F : \mathrm{supp}(F) \in \Delta\},$$

where $\mathrm{supp}(F)$ denotes the set of elements of the multiset F, ignoring multiplicities. For example $\mathrm{supp}(\{1,1,1,3,4\}) = \{1,3,4\}$.

If Δ is $(d-1)$-dimensional, suppose $\{i_1, \ldots, i_d\}$ is a maximal face. This means the elements $x_{i_1}, \ldots, x_{i_d}$ are algebraically independent in $\Bbbk[\Delta]$, and any larger collection of vertices contains a non-face. Hence $\Bbbk[\Delta]$ has dimension d as a $\Bbbk$-algebra.

Define the *fine Hilbert series* for the Stanley-Reisner ring to be the formal sum of all monomials in $\Bbbk[\Delta]$, i.e.,

$$F(\Bbbk[\Delta]; x_1, \ldots, x_n) = \sum_{\mathrm{supp}(F) \in \Delta} x^F.$$

The usual Hilbert series of $\Bbbk[\Delta]$ is obtained by setting $x_i = t$, i.e.,

$$F(\Bbbk[\Delta]; t) = \sum_{k \geq 0} \dim_k(\Bbbk[\Delta]) t^k,$$

where $\dim_k$ denotes the vector space dimension of the degree k homogeneous component of the ring.

It turns out that computing $F(\Bbbk[\Delta]; x_1, \ldots, x_n)$ (and hence $F(\Bbbk[\Delta]; t)$) is directly linked with the f- and h-vectors of Δ, and this gives a bridge connecting algebraic facts about $\Bbbk[\Delta]$ with enumerative facts about Δ.

Let us return to the example of Figure 8.1. For each face F in Δ, the sum of all monomials whose support is F is a rational function of the variables appearing in the face. For example with the edge $F = \{2,4\}$ this sum is

$$\frac{x_2x_4}{(1-x_2)(1-x_4)} = x_2x_4(1 + x_2 + x_2^2 + \cdots)(1 + x_4 + x_4^2 + \cdots).$$

Using the same idea for all the faces, the fine Hilbert series for this example is

$$\begin{aligned}F(\Bbbk[\Delta]; x_0, x_1, x_2, x_3, x_4) = 1+&\frac{x_0}{1-x_0}+\frac{x_1}{1-x_1}+\frac{x_2}{1-x_2}+\frac{x_3}{1-x_3}+\frac{x_4}{1-x_4}\\ &+\frac{x_0x_1}{(1-x_0)(1-x_1)}+\frac{x_1x_2}{(1-x_1)(1-x_2)}\\ &+\frac{x_1x_3}{(1-x_1)(1-x_3)}+\frac{x_1x_4}{(1-x_1)(1-x_4)}\\ &+\frac{x_2x_4}{(1-x_2)(1-x_4)}+\frac{x_3x_4}{(1-x_3)(1-x_4)}\\ &+\frac{x_1x_3x_4}{(1-x_1)(1-x_3)(1-x_4)}.\end{aligned}$$

Passing to the usual Hilbert series, we have:

$$\begin{aligned}F(\Bbbk[\Delta]; t) &= 1+\frac{5t}{1-t}+\frac{6t^2}{(1-t)^2}+\frac{t^3}{(1-t)^3}\\ &= \frac{1+2t-t^2-t^3}{(1-t)^3}.\end{aligned}$$

In Section 8.8 we found the f-vector of Δ is $(1, 5, 6, 1)$ and the h-vector is $(1, 2, -1, -1)$, and now we see these coefficients appearing again. This is not a coincidence.

In general for a face F of a simplicial complex Δ, we get:

$$\sum_{\operatorname{supp}(G)=F} x^G = \prod_{i\in F}\frac{x_i}{1-x_i} = \frac{\prod_{i\in F} x_i \prod_{j\notin F}(1-x_j)}{\prod_{i\in V}(1-x_i)}.$$

Hence, we can write:

$$F(\Bbbk[\Delta]; x_1, \dots, x_n) = \sum_{\operatorname{supp}(F)\in\Delta} x^F = \sum_{F\in\Delta}\frac{\prod_{i\in F} x_i \prod_{j\notin F}(1-x_j)}{\prod_{i\in V}(1-x_i)},$$

and so if we set $x_i = t$ to get the ordinary Hilbert series, we have

$$\begin{aligned}F(\Bbbk[\Delta]; t) &= \frac{\sum_{F\in\Delta} t^{|F|}(1-t)^{n-|F|}}{(1-t)^n},\\ &= \frac{\sum_{F\in\Delta} t^{|F|}(1-t)^{d-|F|}}{(1-t)^d},\\ &= \frac{h(\Delta; t)}{(1-t)^d},\end{aligned}$$

where the second equality follows because $\max|F| = d \leq n$ and the third equality follows from the transformation from f- to h-polynomials given in

Equation 8.2. The main point for us is the observation that the numerator of the Hilbert series of the Stanley-Reisner ring is the h-polynomial.

10.7 The upper bound theorem and the g-theorem

In this section we will survey some big results with minimal description of the proofs. The point is to convey some of the flavor of results in this area of mathematics as a prelude to stating some open problems.

Suppose now that Δ is a $(d-1)$-sphere with $f_1 = n$ vertices. The *upper bound theorem for spheres* states that the entries of the f-vector of Δ are bounded by the f-vector of the boundary of a *cyclic polytope*, which has $h_i = h_{d-i} = \binom{n-d+i-1}{i}$.

Theorem 10.2 (Upper Bound Theorem). *If Δ is a simplicial sphere, then for $i \leq d/2$,*

$$h_i(\Delta) \leq \binom{n-d+i-1}{i}.$$

This result was proved for simplicial convex polytopes by Peter McMullen [108] and for general triangulations of spheres by Richard Stanley [146]. Here is an outline of Stanley's approach, which was the first big application of the Stanley-Reisner ring.

- First, Stanley showed that if the ring $\Bbbk[\Delta]$ is what is known as a *Cohen-Macaulay ring*, then in particular, $h(\Delta) = (h_0, h_1, \ldots, h_d)$ is an M-vector.
- Further, a result of Reisner [129] shows that if Δ is a sphere, then $\Bbbk[\Delta]$ is Cohen-Macaulay. Hence $h(\Delta)$ is an M-vector whenever Δ is a sphere.
- Since Δ has n vertices and is $(d-1)$-dimensional, $h_1(\Delta) = n - d$. Thus $h(\Delta)$ is the f-vector of a multicomplex with $n-d$ vertices, and the Upper Bound Theorem follows from Observation 10.1.

Another big result characterizes the f-vectors of simplicial polytopes. It is known as the *g-theorem.* The "g" in g-theorem refers to the vector of first differences of the h-vector, known as the *g-vector.* That is, let $g_0 = 1$ and $g_i = h_i - h_{i-1}$ for $i = 1, 2, \ldots, \lfloor d/2 \rfloor$. For example, if $h = (1, 4, 7, 4, 1)$, then $g = (1, 3, 3)$. A simplicial polytope is a convex polytope whose boundary is a simplicial sphere, so the Dehn-Sommerville relations tell us its h-vector is palindromic. Thus in this setting the g-vector is enough to recover the h-vector, and hence the f-vector.

The characterization of g-vectors of simplicial polytopes is remarkably simple.

Theorem 10.3 (The g-theorem). *An integer vector g is the g-vector of a simplicial convex polytope if and only if g is an M-vector.*

There is a construction of Lou Billera and Carl Lee [20] that takes any M-vector g and constructs a simplicial polytope Δ with this g-vector, i.e., such that $g(\Delta) = g$. The proof that $g(\Delta)$ is an M-vector for any simplicial polytope Δ was a tour-de-force by Stanley that we will sketch here in the barest terms.

Using the fact that Δ is polytopal, we can construct a complex *toric variety* X whose cohomology is isomorphic to the following quotient of the Stanley-Reisner ring with real coefficients:

$$S = \mathbb{R}[\Delta]/\langle \theta_1, \ldots, \theta_d \rangle,$$

where the θ_i are certain degree one elements of $\mathbb{R}[\Delta]$. In particular, the Hilbert series for S is the h-polynomial of Δ, i.e., $\dim S_i = h_i$.

Working on the cohomology side, Stanley uses a result known as the *hard Lefschetz theorem* to show there is a degree one element ω (which we can take to be the sum of the vertices of Δ) such that the map $\omega : S_{i-1} \to S_i$ is an injection for all $i = 1, \ldots, \lfloor d/2 \rfloor$. This implies the Hilbert series for $S/\omega S$ has $\dim(S/\omega S)_i = h_i - h_{i-1} = g_i$. Now a result of Macaulay shows that we can associate a multicomplex to any graded algebra generated by degree one elements, and moreover the f-polynomial of this multicomplex is the Hilbert series of the algebra. Applied to $S/\omega S$, we have that the g-vector $(1, g_1, g_2, \ldots)$ is an M-vector.

We close this section by mentioning two more results regarding the characterization of f- and h-vectors, now in the context of balanced complexes. Recall that a balanced $(d-1)$-dimensional complex Δ is one for which the vertices can be assigned one of d colors so that every face of Δ has distinctly colored vertices. We say Δ is a Cohen-Macaulay complex if $\Bbbk[\Delta]$ is Cohen-Macaulay.

Theorem 10.4 (Balanced Cohen-Macaulay complexes). *An integer vector h is the h-vector of a balanced Cohen-Macaulay complex if and only if h is an FFK-vector, i.e., h is the f-vector of a balanced simplicial complex.*

There is a construction due to Anders Björner, Peter Frankl, and Stanley [22] that takes an FFK-vector f and constructs a balanced Cohen-Macaulay complex whose h-vector is f. The other implication is due to Stanley [148], and it again works with a quotient of the ring $\Bbbk[\Delta]$.

The key idea here is that we can choose θ_i to be the sum of all vertices with color i, i.e.,

$$\theta_i = \sum_{\mathrm{col}(x)=i} x.$$

With this choice the quotient

$$S = \Bbbk[\Delta]/\langle \theta_1, \ldots, \theta_d \rangle$$

has nice properties. As mentioned earlier, general arguments show we can construct a multicomplex Δ' from S whose f-vector is the h-vector of Δ. Moreover, since $xy = 0$ if x and y are vertices of the same color, $x^2 = x\theta_i = 0$

in S. So the multicomplex associated with S consists of squarefree monomials only, i.e., Δ' is a simplicial complex. If we allow Δ' to inherit the coloring from Δ, we get that Δ' is balanced, so h is an FFK-vector as well.

As shown by Reisner, Cohen-Macaulayness of $\Bbbk[\Delta]$ depends on the topology of Δ, and in particular spheres are Cohen-Macaulay. Hence, a weaker version of Theorem 10.4 could be stated for balanced spheres.

Corollary 10.3 (Balanced simplicial spheres). *The h-vector of a balanced simplicial sphere is the f-vector of a balanced simplicial complex, i.e., $h(\Delta)$ is an FFK-vector whenever Δ is a balanced sphere.*

10.8 Conjectures for flag spheres

In this section we will survey three conjectures related to *flag* simplicial spheres. In increasing order of specificity, they are: the Charney-Davis conjecture, Gal's conjecture, and the Nevo-Petersen conjecture. That is, the Nevo-Petersen conjecture implies Gal's conjecture, which implies the Charney-Davis conjecture. We will outline these conjectures and present some evidence for them now.

A certain conjecture of Heinz Hopf claims that if M is a closed $2d$-dimensional manifold of non-positive curvature, then

$$(-1)^d\chi(M) \geq 0.$$

That is, the sign of the Euler characteristic is predictable. In their 1995 paper [48], Ruth Charney and Mike Davis studied Hopf's conjecture, and came up with four new conjectures (that they labeled A, B, C, and D) that attack the larger conjecture from various points of view. Their conjecture D can be stated as follows.

Conjecture 1 (Charney-Davis) *If Δ is a flag simplicial sphere of dimension $d-1=2m-1$, then*

$$(-1)^m h(\Delta;-1) \geq 0.$$

Nearly all the examples discussed in this book provide evidence for the Charney-Davis conjecture. Any Coxeter complex is a flag sphere, as are generalized associahedra and barycentric subdivisions of boolean spheres. As we will see shortly, the Charney-Davis conjecture follows in each of these cases from the fact that the relevant h-polynomial is real-rooted. See Theorems 9.4, 11.2, and 12.3.

Since Δ is a sphere of dimension $d-1$, the Dehn-Sommerville relations tell us that $h(\Delta;t)$ is palindromic, and therefore can be written as

$$h(\Delta;t) = \sum_{j\geq 0} \gamma_j t^j (1+t)^{d-2j}.$$

Hence,

$$h(\Delta;-1) = \begin{cases} \gamma_m(-1)^m & \text{if } d = 2m, \\ 0 & \text{otherwise.} \end{cases}$$

Thus the Charney-Davis conjecture follows if $\gamma_{d/2} \geq 0$ whenever d is even.

In a 2005 paper [79], Światosław Gal went further, to claim that every entry of the gamma vector for a flag sphere is nonnegative.

Conjecture 2 (Gal's conjecture) *If Δ is a flag simplicial sphere, then $\gamma(\Delta;t)$ has nonnegative coefficients.*

Gal provided some evidence for his conjecture, showing that it was preserved under certain simple operations, such as the "join" (cartesian product) of two simplicial complexes and "edge subdivision." Starting from small examples of flag spheres (both in terms of dimension and number of cells), one can use these operations to build larger examples of flag spheres. Kalle Karu proved a result that implies Gal's conjecture for barycentric subdivisions in his 2006 paper [93].

From Observation 4.2 we can see that if the h-polynomial of a sphere is real-rooted then it is gamma-nonnegative as well. Perhaps one could conjecture that any such sphere has a real-rooted polynomial? In fact this is true for dimension at most four. However, Gal showed how to construct a flag simplicial sphere Δ of dimension five with:

$$\begin{aligned} f(\Delta;t) &= 1 + 17t + 109t^2 + 345t^3 + 575t^4 + 483t^5 + 161t^6, \\ h(\Delta;t) &= 1 + 11t + 39t^2 + 59t^3 + 39t^4 + 11t^5 + t^6, \\ \gamma(\Delta;t) &= 1 + 5t + 4t^2 + t^3. \end{aligned}$$

It is simple to verify that $\gamma(\Delta;t)$ has one real root and two complex roots. Hence, h and f are not real-rooted either. Nonetheless, Conjecture 2, and hence Conjecture 1 hold for this example.

Gal asked whether there might be some explanation for the conjectured nonnegativity of the gamma vector. This is the subject of the following conjecture from Eran Nevo and the author of this book in 2011 [111].

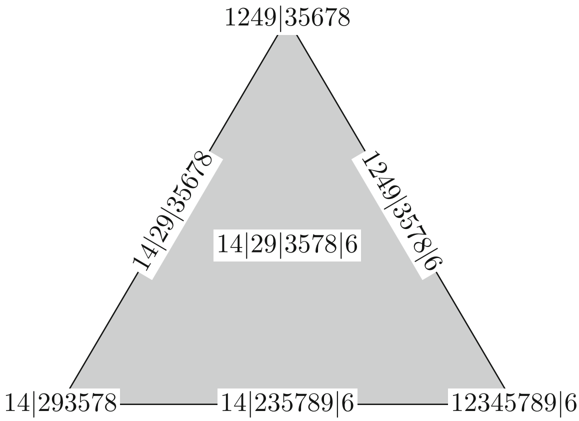
Conjecture 3 (Nevo-Petersen conjecture) *The gamma vector of a flag simplicial sphere is the f-vector of a balanced simplicial complex, i.e., $\gamma(\Delta)$ is an FFK-vector whenever Δ is a flag sphere.*

Compare this conjecture with Corollary 10.3. Conjecture 3 holds for all the examples discussed in this book. Coxeter complexes were handled by Nevo and the author in [111], as were the type A_n and B_n associahedra, along with the case of flag $(d-1)$-spheres with at most $2d+3$ vertices. Nevo, the author, and Bridget Tenner proved Conjecture 3 in the case of barycentric subdivisions in [112]. Satoshi Murai and Nevo proved the conjecture for barycentric subdivisions of polytopes in [109]. Natalie Aisbett [5] and (independently) Vadim Volodin [165] established the result for a family of simple polytopes known as flag nestohedra.

To give an idea for how such a result is proved, we will illustrate the idea used in [111] to prove it in the case of the type A_{n-1} Coxeter complex. The general idea is to take the set of combinatorial objects counted by the gamma vector and construct from them a balanced simplicial complex.

From Section 4.2, we know that in this case γ_j counts the number of permutations in S_n for which $\mathrm{pk}(w) = \mathrm{des}(w) = j$. We will draw such a permutation with bars in the descent positions, e.g., $w = 14|29|3578|6$. Notice that such a permutation is uniquely expressed as a collection of disjoint increasing runs, and that the largest element in a given block is larger than the smallest element of the following block. Moreover, to ensure that the peak set and the descent set coincide, all the blocks, apart from the last, must have at least two elements.

We now define a simplicial complex Δ whose faces are the elements of S_n whose peak set coincides with the descent set. We have $F \subset G$ if we can obtain F by removing bars from G to obtain F (rewriting elements of merged blocks in increasing order). For example, the following is a triangle in the complex:



The vertices in the complex thus correspond to elements with a single bar/descent, not appearing in position 1. We can give this complex a balanced coloring by coloring the bars as follows. Use color 1 for bars in gaps 2 and 3, use color 2 for bars in gaps 4 and 5, use color 3 for bars in gaps 6 and 7, and so on. Because our blocks have length at least two we can never have two bars of the same color in a given face. Moreover, the number of colors used equals the number of vertices in a maximal face. Thus the coloring is balanced.

This construction can be restricted to only those 231-avoiding permutations whose descent sets and peak sets coincide. (Coarsening only removes occurrences of the pattern 231.) Recall Theorem 4.2 shows the gamma vector of the associahedron of type A_{n-1} counts these permutations by descents. Thus, we also get a balanced complex whose f-vector is the gamma vector for the associahedron of type A_{n-1}.

Similar constructions can be found in Chapter 13.

Part III
Coxeter groups

Chapter 11
Coxeter groups

THE SET S_n OF PERMUTATIONS FORM A GROUP under composition, called the *symmetric group*. To this point we have hardly mentioned this fact, let alone exploited the group structure. The task of this chapter is to show how the combinatorial notions of inversion and descent can be understood as arising from the group structure. We will then generalize from the symmetric group to other finite groups with a similar structure, known as *Coxeter groups*. This gives a natural setting in which to give a more general notion of Eulerian numbers.

11.1 The symmetric group

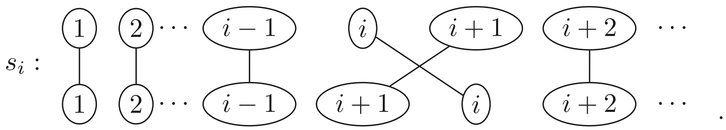
We define S_n to be the set of all bijections $w : [n] \to [n]$. This set is a group under composition. To this point, we have mainly considered elements of S_n written in one-line notation, but we now want to shift the focus a bit and write our elements as products of a very particular subset of permutations. Let s_i denote the permutation that swaps elements i and $i + 1$ while fixing all others. For example, in S_5 there are four simple transpositions, written here in one-line notation:

$$\begin{aligned} s_1 &= 21345, \\ s_2 &= 13245, \\ s_3 &= 12435, \\ s_4 &= 12354. \end{aligned}$$

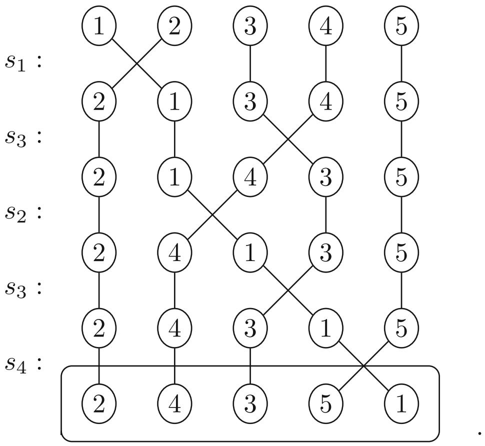
These elements are called the *simple transpositions*, and when n is understood the set of simple transpositions is denoted $S = \{s_1, \dots, s_{n-1}\}$.

We can think of permutations as acting on a collection of n labeled beads laid out in a row, with s_i swapping the beads in positions i and $i + 1$:

T.K. Petersen, *Eulerian Numbers*, Birkhäuser Advanced Texts Basler Lehrbücher, DOI 10.1007/978-1-4939-3091-3_11



This sort of picture makes it easy to compute products of simple transpositions by hand. For example, here is $s_1s_3s_2s_3s_4$ in S_5:



So we have $w = s_1s_3s_2s_3s_4$ is written 24351 in one-line notation.

Notice that our convention with the products is to write them right to left, as we would with composition of functions: s_4 is applied first, then s_3, and so on. However, for the one-line notation of the product to appear on the bottom line we apply the transpositions on the beads from left to right: s_1 first, then s_3, and so on.

Any permutation can be written as a product of simple transpositions, with the identity considered to be the empty product and written e. For example the element $w = 24351$ can be written as such a product in the following ways:

$$s_1s_3s_2s_3s_4, \quad s_3s_1s_2s_3s_4, \quad s_1s_2s_3s_2s_4, \quad s_1s_2s_3s_4s_2,$$

or even

$$s_2s_1s_3s_2s_3s_2s_3s_1s_2s_3s_2s_3s_4.$$

Of course, this last expression seems rather silly. Why write an element as a product of thirteen simple transpositions when only five will do?

Let us deduce some properties of products of simple transpositions that can help us classify such products. First of all, we notice that every simple transposition is its own inverse: $s_i^2 = e$. Also, if two simple transpositions are "far apart" then they commute: $s_is_j = s_js_i$ if $|i-j| > 1$. In the case $j = i+1$, observe that $s_is_{i+1}s_i = s_{i+1}s_is_{i+1}$ since both products have the effect of swapping i and $i+2$ while fixing all other elements. Let us collect these identities here.

Observation 11.1 (The braid relations) *The simple transpositions satisfy the following relations:*

1. $s_i^2 = e$ *for all* $i = 1, \ldots, n-1$,
2. $s_i s_j = s_j s_i$ *if* $|i-j| > 1$, *or by relation 1,* $(s_i s_j)^2 = e$,
3. $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ *for all* $i = 1, \ldots, n-2$, *or by relation 1,* $(s_i s_{i+1})^3 = e$.

Using these relations, we can see that all the expressions for $w = 24351$ above are equivalent to one another without having to convert every expression to one-line notation. For example, $s_1 s_3 s_2 s_3 s_4 = s_3 s_1 s_2 s_3 s_4$ since s_1 and s_3 commute, and $s_1 s_3 s_2 s_3 s_4 = s_1 s_2 s_3 s_2 s_4$ since $s_3 s_2 s_3 = s_2 s_3 s_2$. The reader should try to show that the longer expression above is also reducible to these expressions. Here are the first few steps in one such reduction, with the pieces to rewrite in parentheses:

$$\begin{aligned} s_2 s_1 s_3 (s_2 s_3 s_2) s_3 s_1 s_2 s_3 s_2 s_3 s_4 &= s_2 s_1 (s_3 s_3) s_2 (s_3 s_3) s_1 s_2 s_3 s_2 s_3 s_4, \\ &= (s_2 s_1 s_2) s_1 s_2 s_3 s_2 s_3 s_4, \\ &= \cdots . \end{aligned}$$

If a product of simple transpositions can be shortened in this way, we say the expression is *reducible*. Otherwise it is *reduced*.

In Table 11.1 we see the reduced expressions for each of the elements of S_4.

We define the *length* of an element $w \in S_n$, denoted $\ell(w)$, to be the minimal number of terms required to write w as a product of simple transpositions. In other words,

$$\ell(w) = \min\{k : w = s_{i_1} \cdots s_{i_k}\},$$

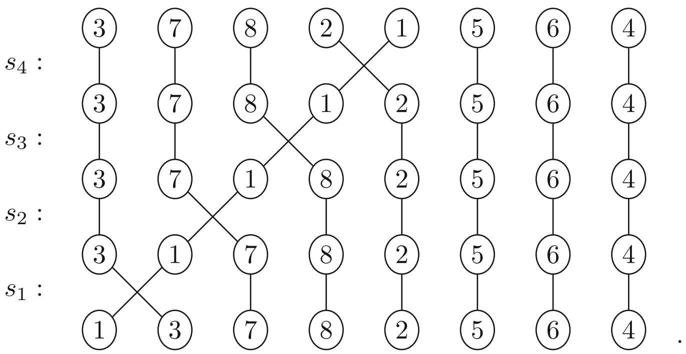
with the length of the identity equal to zero. Notice that if $w = s_{i_1} \cdots s_{i_k}$, then $w^{-1} = s_{i_k} \cdots s_{i_1}$ is the reversal of this expression. So all the reduced expressions for w^{-1} are obtained by reversing the reduced expressions for w, and in particular, $\ell(w) = \ell(w^{-1})$.

One simple way to find a reduced expression for a permutation is to sort it. An induction argument shows that the following greedy sort will produce a reduced expression for an element w in S_n. (See Problem 5.5 in Chapter 5.) We demonstrate with the example $w = 37821564$.

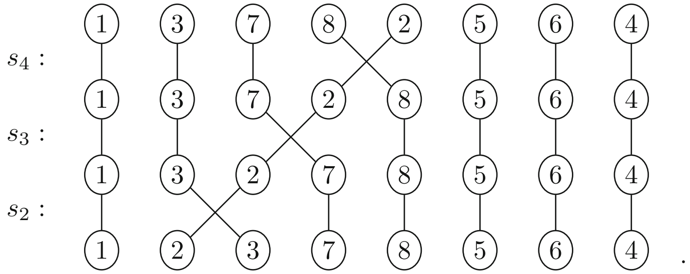
We first find the smallest element that is out of place. In this case it is 1. We apply adjacent transpositions to move it to its proper place:

Table 11.1 The reduced expressions for elements of S_4, along with length and descent statistics.

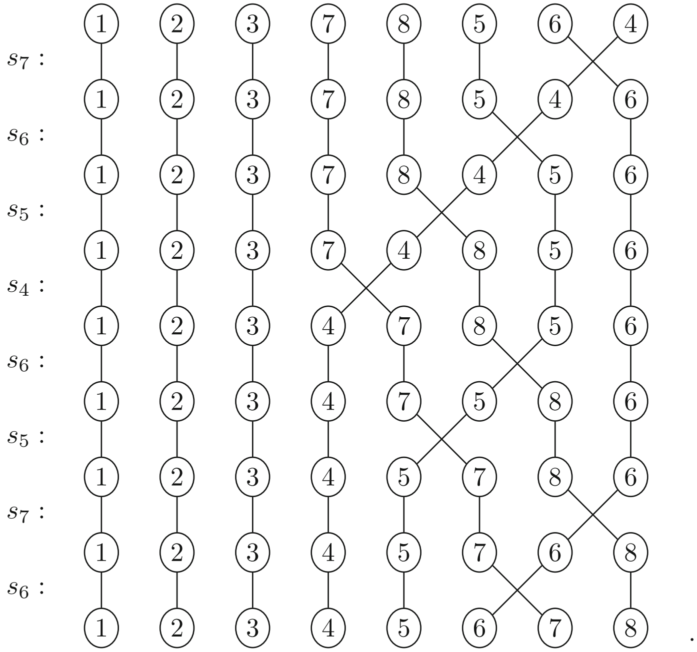
w	reduced expressions	$\ell(w) = \mathrm{inv}(w)$	$\mathrm{Des}(w)$
1234	e	0	$\emptyset$
2134	s_1	1	$\{1\}$
1324	s_2	1	$\{2\}$
1243	s_3	1	$\{3\}$
2314	s_1s_2	2	$\{2\}$
2143	s_1s_3, s_3s_1	2	$\{1,3\}$
3124	s_2s_1	2	$\{1\}$
1342	s_2s_3	2	$\{3\}$
1423	s_3s_2	2	$\{2\}$
3214	$s_1s_2s_1, s_2s_1s_2$	3	$\{1,2\}$
2341	$s_1s_2s_3$	3	$\{3\}$
2413	$s_1s_3s_2, s_3s_1s_2$	3	$\{2\}$
3142	$s_2s_1s_3, s_2s_3s_1$	3	$\{1,3\}$
1432	$s_2s_3s_2, s_3s_2s_3$	3	$\{2,3\}$
4123	$s_3s_2s_1$	3	$\{1\}$
3241	$s_1s_2s_1s_3, s_1s_2s_3s_1, s_2s_1s_2s_3$	4	$\{1,3\}$
2431	$s_1s_2s_3s_2, s_1s_3s_2s_3, s_3s_1s_2s_3$	4	$\{2,3\}$
4213	$s_1s_3s_2s_1, s_3s_1s_2s_1, s_3s_2s_1s_2$	4	$\{1,2\}$
3412	$s_2s_1s_3s_2, s_2s_3s_1s_2$	4	$\{2\}$
4132	$s_2s_3s_2s_1, s_3s_2s_1s_3, s_3s_2s_3s_1$	4	$\{1,3\}$
3421	$s_1s_2s_1s_3s_2, s_1s_2s_3s_1s_2,$ $s_2s_1s_2s_3s_2, s_2s_1s_3s_2s_3, s_2s_3s_1s_2s_3$	5	$\{2,3\}$
4231	$s_1s_2s_3s_2s_1, s_1s_3s_2s_1s_3, s_1s_3s_2s_3s_1,$ $s_3s_1s_2s_1s_3, s_3s_1s_2s_3s_1, s_3s_2s_1s_2s_3$	5	$\{1,3\}$
4312	$s_2s_1s_3s_2s_1, s_2s_3s_1s_2s_1,$ $s_2s_3s_2s_1s_2, s_3s_2s_1s_3s_2, s_3s_2s_3s_1s_2$	5	$\{1,2\}$
4321	$s_1s_2s_1s_3s_2s_1, s_1s_2s_3s_1s_2s_1, s_1s_2s_3s_2s_1s_2, s_1s_3s_2s_1s_3s_2,$ $s_1s_3s_2s_3s_1s_2, s_2s_1s_2s_3s_2s_1, s_2s_1s_3s_2s_1s_3, s_2s_1s_3s_2s_3s_1,$ $s_2s_3s_1s_2s_1s_3, s_2s_3s_1s_2s_3s_1, s_2s_3s_2s_1s_2s_3, s_3s_1s_2s_1s_3s_2,$ $s_3s_1s_2s_3s_1s_2, s_3s_2s_1s_2s_3s_2, s_3s_2s_1s_3s_2s_3, s_3s_2s_3s_1s_2s_3$	6	$\{1,2,3\}$



Now that 1 is in its proper place, we apply the same procedure again.



At this point, 1, 2, and 3 are in their proper place, so next we move 4 into place, then 5, and so on. Here are the remaining steps in the sorting:



Recording all the steps used in this way gives the expression

$$\underbrace{(s_4 s_3 s_2 s_1)}_{\text{(sort 1)}}\underbrace{(s_4 s_3 s_2)}_{\text{(sort 2)}}\underbrace{(s_7 s_6 s_5 s_4)}_{\text{(sort 4)}}\underbrace{(s_6 s_5)}_{\text{(sort 5)}}\underbrace{(s_7 s_6)}_{\text{(sort 6)}}.$$

If we call this expression u, we have that $wu = e$, so $u = w^{-1}$. Hence, to obtain a reduced expression for w, we need to reverse the order of the terms. We have

$$w = s_6 s_7 s_5 s_6 s_4 s_5 s_6 s_7 s_2 s_3 s_4 s_1 s_2 s_3 s_4,$$

and $\ell(w) = 15$.

Another way to think about this sorting process is that we are moving down in the right weak order $\mathrm{Wk}^r(S_n)$. Recall from Section 5.2 that the weak order is ranked by inversion number, and that every cover relation is of the form $u <_{\mathrm{Wk}^r} us_i$ for some i, with $\mathrm{inv}(us_i) = \mathrm{inv}(u) + 1$. Our greedy sort therefore looks at all elements covered by w and chooses to move down along the edge of the Hasse diagram labeled by the simple transposition with the smallest index. We continue in this way, decreasing the number of inversions by one with every edge traversed, finishing with the identity. Hence we have the following consequence.

Observation 11.2 (Length equals inversion number) *For any permutation w in S_n,*

$$\ell(w) = \mathrm{inv}(w),$$

and thus the weak order $\mathrm{Wk}(S_n)$ *is ranked by length:*

$$f(\mathrm{Wk}(S_n); q) = [n]! = \sum_{w \in S_n} q^{\ell(w)}.$$

Looking back upon the example of $w = 37821564$, we compute

$$\mathrm{inv}(w) = 2 + 5 + 5 + 1 + 0 + 1 + 1 + 0 = 15,$$

as expected.

There is a similar connection between descents and reduced expressions. To see the connection, let $w = w(1) \cdots w(i-1)w(i)w(i+1)w(i+2) \cdots w(n)$ be an element of S_n written in one-line notation, and consider the effect of right multiplication by s_i:

$$ws_i = w(1) \cdots w(i-1)w(i+1)w(i)w(i+2) \cdots w(n).$$

We see this action swaps the numbers in positions i and $i+1$ of w. Thus i is a descent position of w if and only if i is *not* a descent position of ws_i. In particular, i is a descent of w if and only if $\mathrm{inv}(w) > \mathrm{inv}(ws_i)$, i.e., w covers ws_i in the weak order. We have the following characterization of descents.

Observation 11.3 (Descents in terms of words) *For any permutation w in S_n,*

$$\mathrm{Des}(w) = \{1 \leq i \leq n-1 : \ell(w) > \ell(ws_i)\}.$$

In particular, the descents of w label the ways to move down from w in the (right) weak order.

A consequence of this observation is that i is a descent of w if and only if there is a reduced expression for w that has s_i as its rightmost term. For example, we can see in Table 11.1 that $w = 4312$ has five reduced expressions, and each of these end in either s_1 or s_2.

11.2 Finite Coxeter groups: generators and relations

Given the discussion of the previous section, we could have given an abstract definition of the symmetric group S_n in terms of *generators* and *relations* as follows. Fix the set of generators $S = \{s_1, \ldots, s_{n-1}\}$, and let S^* denote the set of all finite words on the alphabet S. Then S^* is a monoid under concatenation of words, with e as the identity. Define an equivalence relation on S^* by declaring that two words are equivalent if one can be transformed into the other by applying a sequence of braid relations, as listed in Observation 11.1. (However, we emphasize that now the braid relations are *definitions*, not properties deduced from what we know about the symmetric group.) Then S_n is isomorphic to S^* modulo this equivalence relation.

We use the notation $\langle S : R\rangle$ to indicate such a construction, where R is a set of words equal to the identity. Thus we claim that the symmetric group S_n has presentation

$$\left\langle S : \begin{array}{c} s_i^2, \\ (s_i s_{i+1})^3, \\ (s_i s_j)^2 \text{ for } |i-j| > 1 \end{array} \right\rangle .$$

The information in this presentation is easily captured in a graph. For example, the generators and relations presentation of S_7 can be shown with:

.

Here, the nodes correspond to the elements of S, each of which is assumed to satisfy $s_i^2 = e$. If there is no edge between two elements, they commute. An edge indicates the product of the two generators has order three.

It is really quite remarkable that a presentation with generators and relations defines a finite group. The structure of such groups is very rigid, and they admit a complete classification, which we will now describe.

First, define a *Coxeter system* to be a pair (W, S), where W is a group and S is a minimal generating set of W, subject to the following relations. (By minimal generating set we mean every element of w can be written as a product of elements of S and no proper subset of S will do the same.) For every pair s, t in S, we have

- $st = e$ if and only if $s = t$, and if $s \neq t$,
- $(st)^{m(s,t)} = (ts)^{m(s,t)} = e$ for some integer $m(s,t) > 1$.

We say such a group W is a *Coxeter group* of rank $r = |S|$.

To each Coxeter system we can associate a *Coxeter graph* whose nodes are the elements of S. If $m(s,t) > 2$, then we draw an edge between s and t, labeled with $m(s,t)$ if $m(s,t) > 3$. (The case $m(s,t) = 3$ is the most common in such pictures. A Coxeter system with $m(s,t) \leq 3$ for all s and t is called *simply laced*.) Irreducible Coxeter systems correspond to connected Coxeter graphs, which are shown in Figure 11.1.

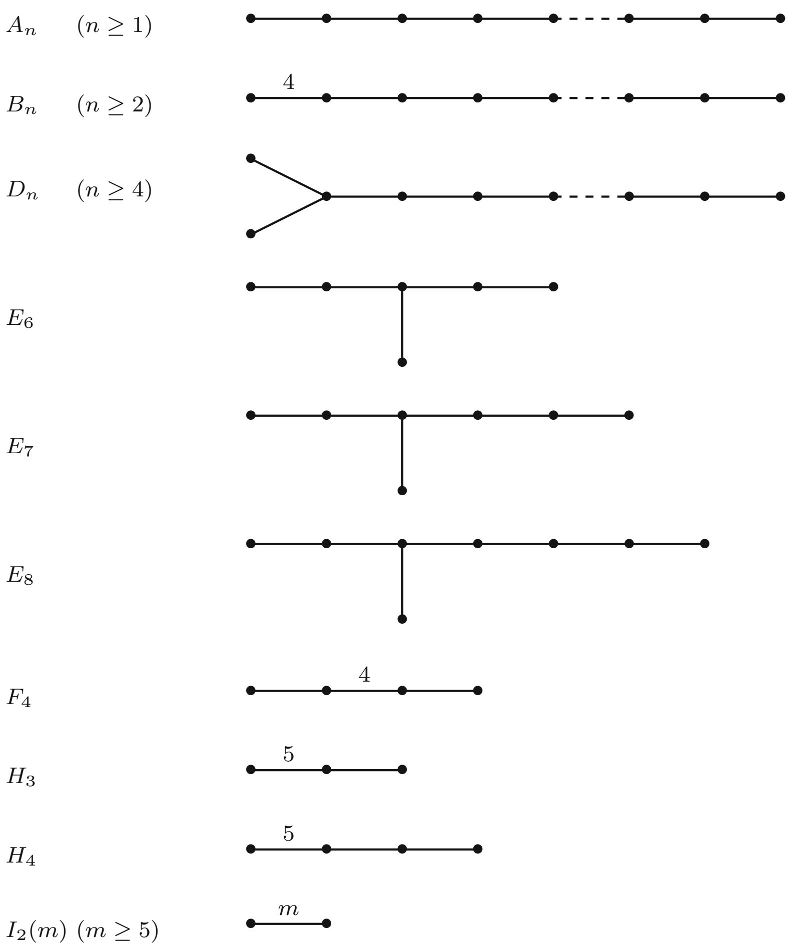
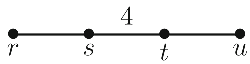


Fig. 11.1 The Coxeter graphs for irreducible finite Coxeter groups

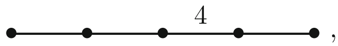
For example, the group F_4 has the following presentation. If we label the generators as:



then

$$F_4 = \left\langle \{r, s, t, u\} : \begin{array}{c} r^2, s^2, t^2, u^2, \\ (rs)^3, (st)^4, (tu)^3, \\ (rt)^2, (ru)^2, (su)^2 \end{array} \right\rangle .$$

While it may be straightforward to check that F_4 is a group (since the generators are involutions, the inverse of a word is its reversal), the fact that there are only finitely many equivalence classes of words is not trivial. To underscore how delicate the situation is, we remark that the group with five generators whose graph is obtained by adding only one generator to F_4 as follows:



(11.1)

is an infinite group. See Problem 11.3.

We will not prove the classification of finite Coxeter systems here. (See the notes at the end of the chapter for suggestions for further reading.) We will simply take it as given that such a classification exists and study the outcome. The names for the Coxeter systems in Figure 11.1 are chosen to agree with a similar classification for root systems coming from Lie theory. More will be said about this in Section 11.5.

By analogy with what we have seen for the symmetric group, we define the *length* of an element w in W to be the number of terms in a shortest expression for w as a product of elements of S. That is,

$$\ell(w) = \min\{k : w = s_1 \cdots s_k, s_i \in S\}.$$

Now we can use length to help us define the general notion of descents for elements of a Coxeter group. For any element w in the Coxeter group W, let

$$\mathrm{Des}(w) = \{s \in S : \ell(w) > \ell(ws)\},$$

and $\mathrm{des}(w) = |\,\mathrm{Des}(w)|$. We will define the *$W$-Eulerian numbers* by counting the distribution of descent numbers across the Coxeter group W. This is the topic of Section 11.4.

Before moving on, a couple of remarks are in order.

First, from the point of view of Coxeter groups, there are actually two natural definitions of the descent set. The one given here is the "right" descent set. We could equally well define the "left" descent set to be the set of w to be generators that satisfy $\ell(w) > \ell(sw)$. This alternate viewpoint is occasionally useful. The two definitions are equivalent under the involution $w \leftrightarrow w^{-1}$.

Second, we could now define the weak order for the Coxeter group W to be the transitive closure of the cover relations $w <_{\mathrm{Wk}} ws$ if $\ell(ws) = \ell(w) + 1$ (or the same for left multiplication). Rather than take this as a definition, however, we will develop a geometric story for W (beginning in Section 11.5) in which the weak order on W emerges as the weak order on the chambers of a hyperplane arrangement associated with W. This approach generalizes what we saw for the braid arrangement and $\mathrm{Wk}(S_n)$ Chapter 5.

11.3 *W*-Mahonian distribution

The distribution of length in the symmetric group is called the Mahonian distribution, with generating function:

$$\sum_{w\in S_n} q^{\mathrm{inv}(w)} = \sum_{w\in S_n} q^{\ell(w)} = [1][2]\cdots[n],$$

where $[k] = 1 + q + q^2 + \cdots + q^{k-1} = (1-q^k)/(1-q)$ is the q-analogue of k. A similar formula exists for any finite Coxeter group, and we can think of length as a general "W-Mahonian" statistic for any finite Coxeter group W. The generating function for length is also commonly known as the *Poincaré polynomial*.

The general result depends on certain constants related to W called the *degrees* of the Coxeter group. (The name refers to the degrees of the fundamental polynomial invariants for W. If these comments don't make sense, don't worry. It's not important for us.) The degrees for irreducible Coxeter groups can be found in Table 11.2. If W is reducible, $W = U \times V$, the degrees of W are the degrees of U together with the degrees of V, with repetition allowed. The remarkable thing is that the distribution of length is given by the product of the q-analogues of the degrees.

Theorem 11.1 (W-Mahonian distribution). *The distribution of length in the finite Coxeter group W of rank n is given by*

$$\sum_{w\in W} q^{\ell(w)} = [d_1][d_2]\cdots[d_n] = \frac{\prod_{i=1}^n (1-q^{d_i})}{(1-q)^n},$$

where $d_1, d_2, \ldots, d_n$ are the degrees of W.

For example, Theorem 11.1 tells us the Mahonian distribution for D_5 is:

$$\begin{aligned}\sum_{w\in D_5} q^{\ell(w)} &= \frac{(1-q^2)(1-q^4)(1-q^6)(1-q^8)(1-q^5)}{(1-q)^5},\\ &= 1 + 5q + 14q^2 + 30q^3 + 54q^4 + 85q^5 + 120q^6\\ &\quad +155q^7 + 185q^8 + 205q^9 + 212q^{10} + 205q^{11} + 185q^{12} + 155q^{13}\\ &\quad +120q^{14} + 85q^{15} + 54q^{16} + 30q^{17} + 14q^{18} + 5q^{19} + q^{20}.\end{aligned}$$

11.4 *W*-Eulerian numbers

We define the *W-Eulerian polynomial* to be the generating function for descents, denoted

Table 11.2 The degrees of the fundamental invariants for irreducible finite Coxeter groups.

Coxeter group	Degrees: $d_1, d_2, \ldots, d_n$
A_n	$2, 3, 4, \ldots, n+1$
B_n	$2, 4, 6, \ldots, 2n$
D_n	$2, 4, 6, \ldots, 2n-2, n$
E_6	$2, 5, 6, 8, 9, 12$
E_7	$2, 6, 8, 10, 12, 14, 18$
E_8	$2, 8, 12, 14, 18, 20, 24, 30$
F_4	$2, 6, 8, 12$
H_3	$2, 6, 10$
H_4	$2, 12, 20, 30$
$I_2(m)$	$2, m$

$$W(t) = \sum_{w \in W} t^{\mathrm{des}(w)} = \sum_{k=0}^{n} \left\langle {W \atop k} \right\rangle t^k,$$

where $n = |S|$ is the rank of W. The coefficients of this polynomial,

$$\left\langle {W \atop k} \right\rangle = |\{w \in W : \mathrm{des}(w) = k\}|,$$

are called the *W-Eulerian numbers.*

When $W = A_{n-1}$, these are the classical Eulerian numbers, $\left\langle {n \atop k} \right\rangle$. For a different example, the reader can try to compute

$$B_2(t) = 1 + 6t + t^2.$$

More generally, the Eulerian polynomial for the dihedral group $I_2(m)$ is $1 + 2(m-1)t + t^2$. See Problem 11.6. Type B_n Eulerian numbers for low rank can be found in Table 11.3, the type D_n Eulerian numbers of low rank are in Table 11.4, and the exceptional groups have their Eulerian distributions in Table 11.5.

Table 11.3 The Eulerian numbers $\left\langle {B_n \atop k} \right\rangle$, $0 \leq k \leq n \leq 9$.

$n\backslash k$	0	1	2	3	4	5	6	7	8	9
2	1	6	1							
3	1	23	23	1						
4	1	76	230	76	1					
5	1	237	1682	1682	237	1				
6	1	722	10543	23548	10543	722	1			
7	1	2179	60657	259723	259723	60657	2179	1		
8	1	6552	331612	2485288	4675014	2485288	331612	6552	1	
9	1	19673	1756340	21707972	69413294	69413294	21707972	1756340	19673	1

Table 11.4 The Eulerian numbers $\left\langle {D_n \atop k} \right\rangle$, $0 \leq k \leq n \leq 9$.

$n\backslash k$	0	1	2	3	4	5	6	7	8	9
4	1	44	102	44	1					
5	1	157	802	802	157	1				
6	1	530	5551	10876	5551	530	1			
7	1	1731	35121	124427	124427	35121	1731	1		
8	1	5528	208732	1265704	2201030	1265704	208732	5528	1	
9	1	17369	1187252	11816900	33427118	33427118	11816900	1187252	17369	1

Table 11.5 The W-Eulerian numbers $\left\langle {W \atop k} \right\rangle$ of exceptional type.

$W\backslash k$	0	1	2	3	4	5	6	7	8
E_6	1	1272	12183	24928	12183	1272	1		
E_7	1	17635	309969	1123915	1123915	309969	17635	1	
E_8	1	881752	28336348	169022824	300247750	169022824	28336348	881752	1
F_4	1	236	678	236	1				
H_3	1	59	59	1					
H_4	1	2636	9126	2636	1				

One of the first things we notice about the W-Eulerian numbers is their palindromicity. One general explanation for this relies on the existence of a unique element of maximal length in W. This element, called the *long element*, is denoted w_0, and it has $\mathrm{Des}(w_0) = S$. The involution $w \mapsto w_0 w$ has the effect of complementing descent sets:

$$\mathrm{Des}(w_0 w) = S - \mathrm{Des}(w).$$

See Problem 11.7. In particular, $\mathrm{des}(w_0 w) = n - \mathrm{des}(w)$. Taking these properties as given for now, we have the following.

Observation 11.4 *For any finite Coxeter group W, the W-Eulerian numbers are palindromic:*

$$\left\langle {W \atop k} \right\rangle = \left\langle {W \atop n-k} \right\rangle.$$

In the case of the symmetric group, $W = S_n$, the long element is the decreasing permutation $w_0 = n \cdots 21$. Left multiplication by w_0 has the effect of replacing i with $n+1-i$ in the one-line notation of a permutation. In this case it is clear that the descent sets of w and $w_0 w$ are complementary.

Not only are the W-Eulerian distributions palindromic, but they are also unimodal. In fact, the W-Eulerian polynomials are real-rooted and hence gamma-nonnegative for any W.

Theorem 11.2. *For any finite Coxeter group W, the W-Eulerian polynomial $W(t) = \sum_{w\in W} t^{\mathrm{des}(w)}$ is real-rooted. In particular, if W is of rank n, there exist nonnegative integers γ_j^W such that*

$$W(t) = \sum_{j=0}^{\lfloor n/2 \rfloor} \gamma_j^W t^j (1+t)^{n-2j}. \tag{11.2}$$

The first step in proving this theorem is to reduce it to the case of irreducible Coxeter groups. (As of this writing there is no case-free proof of this theorem.) Notice that if $W = U \times V$, with U and V finite Coxeter groups, i.e., if W is reducible, then all the generators for U commute with all the generators for V. (The Coxeter graphs for U and V are disjoint subgraphs of the graph for W.) Thus every element w in W can be written as $w = uv = vu$ for some unique choice of $u \in U$ and $v \in V$, and this implies that the descent set of w is the disjoint union of the descent set of u and the descent set of v. In particular,

$$\mathrm{des}(w) = \mathrm{des}(u) + \mathrm{des}(v).$$

Hence the W-Eulerian polynomial is multiplicative:

$$U(t)V(t) = \sum_{u\in U, v\in V} t^{\mathrm{des}(u)+\mathrm{des}(v)} = \sum_{w\in W} t^{\mathrm{des}(w)} = W(t).$$

Hence if $U(t)$ and $V(t)$ are real-rooted, so is $W(t)$.

Thus to prove Theorem 11.2, it suffices to prove it for W-Eulerian polynomials of irreducible Coxeter groups. As previously discussed, S_n is a Coxeter group of type A_{n-1}. This case is the subject of Problem 4.6 (see also Theorem 4.1). The W-Eulerian polynomial for the dihedral group $I_2(m)$ is $1 + 2(m-1)t + t^2 = (1+t)^2 + 2(m-2)t$, which is both real-rooted and gamma-nonnegative since $m \geq 2$. The real-rootedness for Coxeter groups of exceptional type, i.e., the groups of type E_6, E_7, E_8, F_4, H_3, and H_4, can be verified one at a time by computer. We have their W-Eulerian numbers in Table 11.5, and their corresponding gamma coefficients are listed in Table 11.6.

Table 11.6 The gamma coefficients γ_j^W of exceptional type.

$W \backslash j$	0	1	2	3	4
E_6	1	1266	7104	3104	
E_7	1	17628	221808	282176	
E_8	1	881744	23045856	63613184	17111296
F_4	1	232	208		
H_3	1	56			
H_4	1	2632	3856		

The infinite families of types B_n and D_n can be addressed in much the same way as the symmetric group, since we can represent group elements with signed permutations. (See Chapter 13.) Proofs of real-rootedness and separate proofs of gamma-nonnegativity (with combinatorial interpretations) for the W-Eulerian polynomials of type B_n and D_n are given in Chapter 13, along with exponential generating functions.

11.5 Finite reflection groups and root systems

Coxeter groups get their name from H. S. M. Coxeter, who sought a classification of regular convex polytopes via their symmetry groups. For example it is well known that the symmetric group S_n is isomorphic to the group of symmetries for an n-simplex. Similarly, the group of symmetries of an n-cube is the Coxeter group B_n. See Figure 11.2. These symmetry groups share the property that they can be generated by a set of reflections in $\mathbb{R}^n$ that satisfy the appropriate relations.

All the finite Coxeter groups can be realized in a similar way, as *finite reflection groups*. In other words, for a finite Coxeter system (W, S) we can identify the elements of S with certain reflections acting on a vector space V, such that the reflections satisfy the relations given by the Coxeter graph. Thus W can be seen as a finite subgroup of the general linear group $GL(V)$ generated by these reflections. The group W fixes a certain collection of vectors called a *root system*. Each root is orthogonal to a hyperplane in V, and the collection of all such hyperplanes is called the *Coxeter arrangement* for W. The braid arrangement discussed in Section 5.3 is the Coxeter arrangement for the symmetric group.

The hyperplane arrangement we have just described is a Euclidean arrangement as discussed in Chapter 5. Let us recall some of the details of this

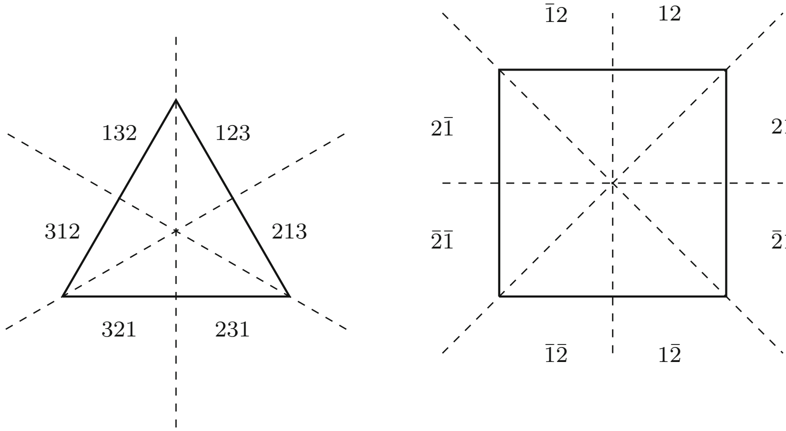


Fig. 11.2 The symmetric group S_3 is the group of symmetries of a triangle; the hyperoctahedral group B_2 is the group of symmetries of the square.

setting. First, we fix a real vector space V with an inner product $\langle \cdot, \cdot \rangle$. Given a nonzero vector β in V, let H_β denote the hyperplane orthogonal to β:

$$H_\beta = \{\lambda \in V : \langle \lambda, \beta \rangle = 0\}.$$

A hyperplane is a codimension one subspace of V. If V is a line, a hyperplane is a point. If V is a plane, a hyperplane is a line. If V is isomorphic to $\mathbb{R}^3$, a hyperplane is a plane, and so on.

Notice that any nonzero scalar multiplication of β defines the same hyperplane. We denote by s_β the orthogonal reflection through H_β. Explicitly s_β acts on a vector μ in V by

$$s_\beta(\mu) = \mu - \frac{2\langle \mu, \beta \rangle}{\langle \beta, \beta \rangle}\beta.$$

From this definition it is easy to check that s_β^2 is the identity map, and that $\langle s_\beta(\lambda), s_\beta(\mu) \rangle = \langle \lambda, \mu \rangle$ for any λ and μ in V. In particular, the action of s_β preserves distances. See Figure 11.3.

We will now let Φ denote a particular collection of vectors in V called a *root system*. A root system is defined by the following properties:

- Φ spans V,
- if $\beta \in \Phi$, then $-\beta \in \Phi$,
- for $\alpha, \beta \in \Phi$, $s_\beta(\alpha) \in \Phi$.

Further, if

- $\dfrac{2\langle \alpha, \beta \rangle}{\langle \beta, \beta \rangle}$ is an integer for all α and β in Φ,

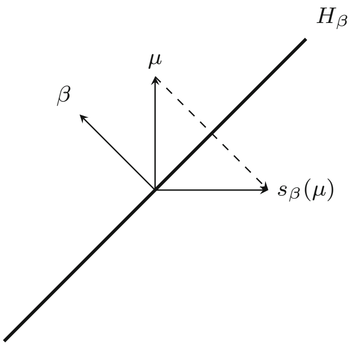


Fig. 11.3 The orthogonal reflection s_β.

we say that the root system is *crystallographic*. The classification of finite root systems follows much the same one as for finite Coxeter groups, with very similar graphical notation, which we will explain shortly. See Figure 11.4, which lists all the irreducible crystallographic root systems. The only non-crystallographic finite root systems are $\mathbf{H}_3, \mathbf{H}_4$, and $\mathbf{I}_2(m)$ with $m \notin \{2, 3, 4, 6\}$. Note that we use a boldface font for root systems and italicized font for Coxeter groups, e.g., $\mathbf{B}_2$ is a root system and B_2 is a Coxeter group.

To begin to understand the classification of root systems we choose a set of linearly independent roots called the *simple roots*, and denoted $\Delta = \{\alpha_1, \dots, \alpha_n\}$. The set Δ is chosen so that $\Phi = \Pi \cup -\Pi$, where Π is the set of all roots in the nonnegative span of Δ and $-\Pi$ is the set of all roots in the nonpositive span of Δ. The sets Π and $-\Pi$ are referred to as the *positive roots* and *negative roots*, respectively. Notice that since Φ was assumed to span V, this means Δ is a basis for V. We refer to the cardinality of Δ (i.e., the dimension of V) as the *rank* of Φ.

Now given Δ, we can construct a graph, called a *Dynkin diagram*, whose nodes correspond to the elements of Δ. Similarly to Coxeter graphs, we label the edge between α and β with the order of $s_\alpha s_\beta$. Let $m(\alpha, \beta)$ denote this power. The case $m(\alpha, \beta) = 2$ means $s_\alpha s_\beta = s_\beta s_\alpha$, and α and β are orthogonal. We draw an edge between α and β if $m(\alpha, \beta) \geq 3$. If $m(\alpha, \beta) \geq 4$ we label the edge with this power. One difference from Coxeter graphs is that in a Dynkin diagram we draw $<$ or $>$ to indicate when one root is shorter or longer than another. For example, in the diagram for $\mathbf{B}_n$, the leftmost simple root is shorter than the others, while in $\mathbf{C}_n$ it is longer.

We say a root system Φ is *reducible* if it can be partitioned as $\Phi = \Phi_1 \cup \Phi_2$ where Φ_1 and Φ_2 are root systems in their own right, lying in mutually orthogonal subspaces of V. Otherwise, we say Φ is *irreducible*. In terms of Dynkin diagrams, Φ is irreducible if and only if its Dynkin diagram is connected.

Let $W = W(\Phi) = \langle s_\beta : \beta \in \Phi \rangle$ denote the subgroup of $GL(V)$ generated by the reflections through the hyperplanes H_β. (In fact, W lies in the orthogonal subgroup $O(V)$. See Problem 11.8.) This is the *Coxeter group* generated by Φ.

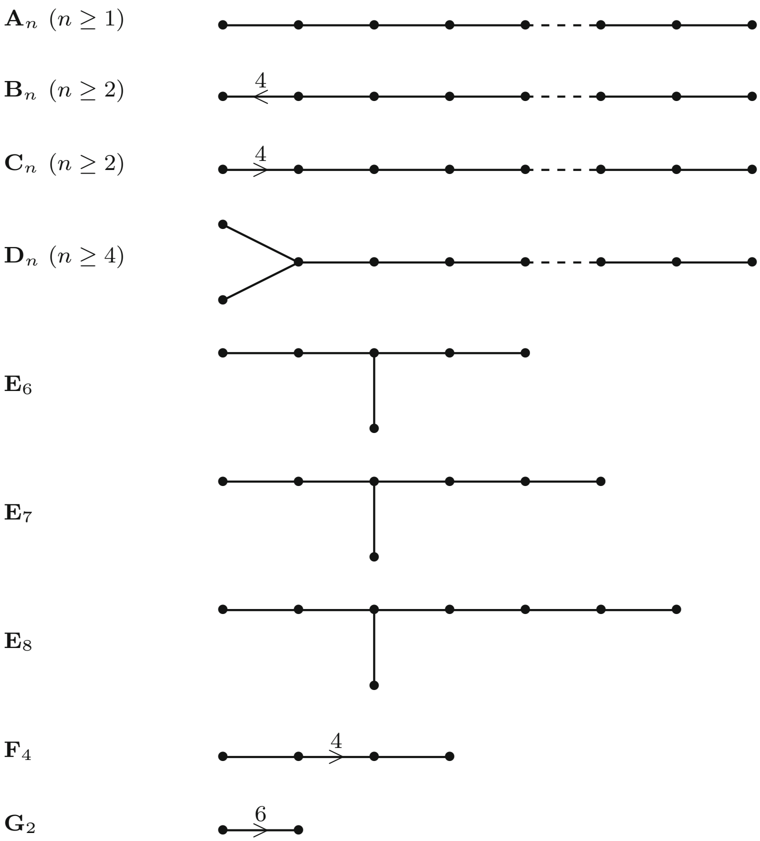


Fig. 11.4 The Dynkin diagrams for irreducible crystallographic root systems.

A special subfamily of the finite Coxeter groups are those whose root system is crystallographic. If Φ is crystallographic, W is known as the *Weyl group* of Φ.

Let $S = \{s_\alpha : \alpha \in \Delta\}$ denote the set of *simple reflections*, i.e., those reflections orthogonal to the simple roots. Then (W, S) is a Coxeter system in the sense of Section 11.2, only now we have a geometric understanding of the generating set S, and the relations satisfied by the elements of S can be verified geometrically.

We remark that every root system has a Coxeter group, and every Coxeter group has a root system, but the same Coxeter group can arise from more than one root system. For example, the root systems of type $\mathbf{B}_n$ and type $\mathbf{C}_n$ are "dual" in a certain sense, and have the same Weyl group: the hyperoctahedral group B_n. This is why there was no type C_n Coxeter graph listed in Figure 11.1.

Here are some examples of root systems.

11.5.1 Type $\mathbf{A}_{n-1}$

Let $\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n\}$ denote the standard basis elements in $\mathbb{R}^n$. The root system of type $\mathbf{A}_{n-1}$ is

$$\Phi = \{\varepsilon_i - \varepsilon_j : 1 \leq i \neq j \leq n\}.$$

Let $V_{n-1} = \{\mathbf{x} \in \mathbb{R}^n : \sum x_i = 0\}$ denote the subspace of $\mathbb{R}^n$ spanned by Φ. The hyperplane orthogonal to $\varepsilon_i - \varepsilon_j$ is

$$H_{ij} = \{\mathbf{x} \in V_{n-1} : x_i = x_j\}.$$

We let S_n act on V_{n-1} by the permuting the standard basis elements. The reflection through H_{ij} swaps ε_i and ε_j, and thus corresponds to the transposition that swaps i and j.

We take the simple roots to be $\alpha_i = \varepsilon_{i+1} - \varepsilon_i$, so that

$$\Delta = \{\alpha_i : 1 \leq i \leq n-1\}.$$

Thus, the positive roots are $\varepsilon_j - \varepsilon_i$, for $1 \leq i < j \leq n$. With our choice of simple roots, we have that the simple reflection s_i corresponds to the adjacent transposition that swaps i and $i+1$.

11.5.2 Type $\mathbf{B}_n$

Let $V = \mathbb{R}^n$ with the standard basis $\{\varepsilon_1, \ldots, \varepsilon_n\}$ as before. The root system of type $\mathbf{B}_n$ is

$$\Phi = \{\pm\varepsilon_i : 1 \leq i \leq n\} \cup \{\pm\varepsilon_i \pm \varepsilon_j : 1 \leq i \neq j \leq n\}.$$

We take the simple roots to be

$$\Delta = \{\varepsilon_1, \varepsilon_2 - \varepsilon_1, \ldots, \varepsilon_n - \varepsilon_{n-1}\},$$

i.e., the type $\mathbf{A}_{n-1}$ simple roots, $\alpha_i = \varepsilon_{i+1} - \varepsilon_i$ $(1 \leq i \leq n-1)$, together with $\alpha_0 = \varepsilon_1$. The positive roots are thus

$$\Pi = \{\varepsilon_1, \ldots, \varepsilon_n\} \cup \{\varepsilon_j \pm \varepsilon_i : 1 \leq i < j \leq n\}.$$

We can represent the Coxeter group B_n as the set of signed permutations, i.e., bijections

$$w : \{-n, \ldots, -1, 0, 1, \ldots, n\} \to \{-n, \ldots, -1, 0, 1, \ldots, n\}$$

such that $w(-i) = -w(i)$ for all i. We have these permutations acting on V by $w \cdot \varepsilon_i = \varepsilon_{w(i)}$, with the understanding that $\varepsilon_{-i} = -\varepsilon_i$.

The hyperplane orthogonal to $\beta = \varepsilon_i$ is simply a coordinate hyperplane, denoted

$$H_i = \{\mathbf{x} \in \mathbb{R}^n : x_i = 0\},$$

and the corresponding reflection swaps ε_i with $-\varepsilon_i$. We represent this with a signed permutation that swaps i with $-i$.

The hyperplane orthogonal to $\beta = \varepsilon_j - \varepsilon_i$ is denoted

$$H_{ij} = \{\mathbf{x} \in \mathbb{R}^n : x_i = x_j\}.$$

The reflection through H_{ij} swaps ε_i and ε_j. It corresponds to the signed permutation that swaps i and j (and $-i$ with $-j$).

The hyperplane orthogonal to $\beta = \varepsilon_j + \varepsilon_i$ is denoted:

$$H_{\bar{i}j} = \{\mathbf{x} \in \mathbb{R}^n : -x_i = x_j\}.$$

The reflection through $H_{\bar{i}j}$ swaps ε_i and $-\varepsilon_j$, corresponding to the signed permutation that swaps i and $-j$ (and $-i$ with j).

11.5.3 Type $\mathbf{C}_n$

The root system of type $\mathbf{C}_n$ also lives in $V = \mathbb{R}^n$. It is

$$\Phi = \{\pm 2\varepsilon_i : 1 \leq i \leq n\} \cup \{\pm\varepsilon_i \pm \varepsilon_j : 1 \leq i \neq j \leq n\}.$$

See Figure 11.5 for a comparison of the type $\mathbf{B}_2$ root system and the type $\mathbf{C}_2$ root system.

Since each of these roots are simply rescaled versions of the type $\mathbf{B}_n$ roots, the type $\mathbf{C}_n$ hyperplane arrangement and the type $\mathbf{B}_n$ hyperplane arrangement are identical. Hence, they have the same Coxeter group.

11.5.4 Type $\mathbf{D}_n$

The root system of type $\mathbf{D}_n$ is a subsystem of both the type $\mathbf{B}_n$ and type $\mathbf{C}_n$ root systems. It is

$$\Phi = \{\pm\varepsilon_i \pm \varepsilon_j : 1 \leq i \neq j \leq n\}.$$

The simple roots are

$$\Delta = \{\varepsilon_1 + \varepsilon_2, \varepsilon_2 - \varepsilon_1, \varepsilon_3 - \varepsilon_2, \ldots, \varepsilon_n - \varepsilon_{n-1}\},$$

and we label them by $\alpha_i = \varepsilon_{i+1} - \varepsilon_i$ for $i > 0$, and $\alpha_{\bar{1}} = \varepsilon_1 + \varepsilon_2$.

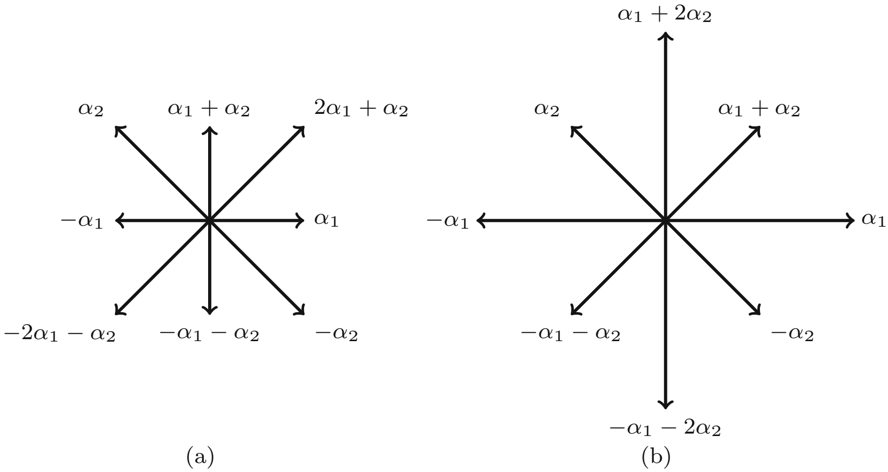


Fig. 11.5 (a) The root system of type $\mathbf{B}_2$, and (b) the root system of type $\mathbf{C}_2$.

We can represent the group D_n as the subgroup of B_n whose elements have an even number of minus signs among $\{w(1), \ldots, w(n)\}$. These elements act on the standard basis in the same way, by permutation and sign changes.

The hyperplane arrangement for $\mathbf{D}_n$ is the subarrangement of the type $\mathbf{B}_n$ arrangement consisting of the hyperplanes H_{ij} and $H_{\bar{i}j}$, but not the coordinate hyperplanes.

More about the combinatorics of the Coxeter groups of types B_n and D_n can be found in Chapter 13. We finish the discussion here with a non-crystallographic example.

11.5.5 Roots for $I_2(m)$

Let Φ denote the set of all unit vectors $\beta = (x, y)$ in the Euclidean plane such that the second coordinate y lies in the set $\{\pm\sin(\pi/2m), \pm\sin(3\pi/2m), \ldots, \}$. Then the lines orthogonal to these vectors sit equally spaced at an angle of π/m from one to the next. The group generated by reflections across these lines is the dihedral group $I_2(m)$.

This is shown in Figure 11.6 for $m = 5$, where the simple roots are $\Delta = \{\alpha_1, \alpha_2\}$, and $\Pi = \{\alpha_1, \alpha_2, \beta, \gamma, \delta\}$. We can represent all the elements of the dihedral group as products of the two reflections $s = s_{\alpha_1}$ and $t = s_{\alpha_2}$. We have labeled the cones with group elements according to how they act on any point μ in the cone labeled by 1. For example, if μ is any point in the cone labeled 1, then $st(\mu) = s(t(\mu))$ is a point in the cone labeled by st.

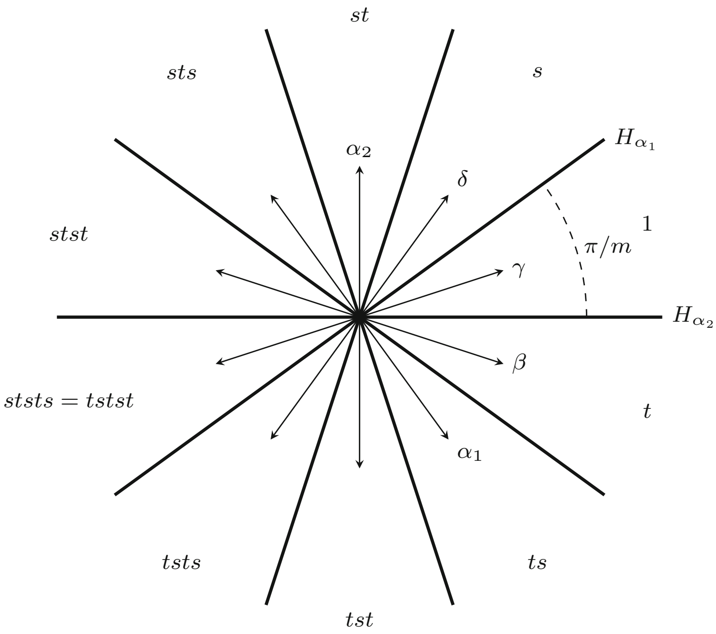


Fig. 11.6 The roots for the dihedral group $I_2(m)$ with $m = 5$.

11.6 The Coxeter arrangement and the Coxeter complex

We will now fix a root system Φ with Coxeter group W and study the corresponding hyperplane arrangement in more detail. We saw this done for the case of the $\Phi = A_{n-1}$ and $W = S_n$ in Section 5.3, where the arrangement in question was the braid arrangement. Let

$$\mathcal{H} = \mathcal{H}(\Phi) = \bigcup_{\beta \in \Pi} H_\beta,$$

where

$$H_\beta = \{\lambda \in V : \langle \lambda, \beta \rangle = 0\},$$

denotes the hyperplane orthogonal to β. We refer to $\mathcal{H}$ as the *Coxeter arrangement* of Φ. Since the corresponding group W fixes Φ, it also fixes $\mathcal{H}$.

Recall from Section 8.5 that any such hyperplane arrangement can be associated with a flag simplicial complex $\Sigma = \Sigma(\mathcal{H})$ in a natural way by identifying the face poset of $\mathcal{H}$ with the face poset of Σ. The origin in V is identified with the empty set in Σ, the rays in $\mathcal{H}$ become the vertices in Σ, and so on, until the chambers in $\mathcal{H}$ are identified with the facets in Σ. Moreover, we can realize Σ by intersecting the arrangement with a sphere. In the case that $\mathcal{H} = \mathcal{H}(\Phi)$, we call the complex $\Sigma = \Sigma(\Phi)$ the *Coxeter complex* for the root system.

From now on, we will move freely between the faces of the hyperplane arrangement $\mathcal{H}$ and the corresponding faces of the simplicial complex Σ. Let us recall some notation and terminology from the discussion of hyperplane arrangements in Section 5.4.

For each hyperplane H_β, we define the *positive halfspace*

$$H_\beta^+ = \{\lambda \in V : \langle \lambda, \beta \rangle > 0\},$$

and *negative halfspace*

$$H_\beta^- = \{\lambda \in V : \langle \lambda, \beta \rangle < 0\},$$

and let $H_\beta^0 = H_\beta$ denote the set of points on the hyperplane. The faces of the arrangement are all possible intersections of hyperplanes and halfspaces:

$$\bigcap_{\beta \in \Pi} H_\beta^{\sigma_\beta},$$

where $\sigma_\beta \in \{-, 0, +\}$. A face F is encoded by a *sign sequence* $\sigma(F) = (\sigma_\beta(F))_{\beta \in \Pi}$. For example, in Figure 11.6, the ray on the boundary between the cone labeled s and the cone labeled st is the face

$$F = H_{\alpha_1}^- \cap H_{\alpha_2}^+ \cap H_\beta^0 \cap H_\gamma^+ \cap H_\delta^+,$$

and so

$$\sigma(F) = (\sigma_{\alpha_1}(F), \sigma_{\alpha_2}(F), \sigma_\beta(F), \sigma_\gamma(F), \sigma_\delta(F)) = (-, +, 0, +, +).$$

Inclusion of faces in Σ is easily phrased in terms of sign sequences (see Proposition 5.2): $F \leq_\Sigma G$ if and only if, for all $\beta \in \Pi$,

$$\sigma_\beta(F) = \sigma_\beta(G) \text{ or } \sigma_\beta(F) = 0.$$

That is, we move up in the face poset by changing a zero entry to a nonzero entry, i.e., by stepping off of some hyperplane into a higher-dimensional cone. The maximal faces (called *chambers* in $\mathcal{H}$, and *facets* in Σ) have all their entries nonzero.

We can see the $\mathbf{B}_2$ hyperplane arrangement and corresponding sign sequences in Figure 11.7. In the figure, sign vectors are given as

$$(\sigma_{\alpha_1}, \sigma_{\alpha_2}, \sigma_{\alpha_1+\alpha_2}, \sigma_{2\alpha_1+\alpha_2}).$$

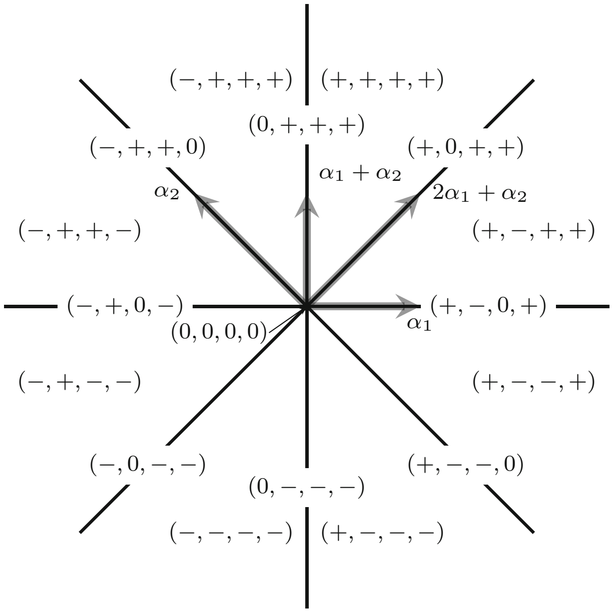


Fig. 11.7 The sign vectors for the $\mathbf{B}_2$ hyperplane arrangement. The positive roots are shown in gray.

11.7 Action of W and cosets of parabolic subgroups

What distinguishes the Coxeter arrangement from other finite Euclidean hyperplane arrangements is that we have the action of the group W to work with. We can encode faces with sign vectors when convenient, but we can also characterize faces in terms of W-orbits as follows.

Let C denote the face with sign vector $(+,+,\ldots,+)$. We call this cone the *fundamental chamber*. Every point of V is in the W-orbit of a unique point in the closure of C, so we say that $\overline{C}$ is a *fundamental domain* for the action of W on V. Let us be more explicit.

We have

$$\begin{aligned}\overline{C} &= \{\lambda \in V : \langle \lambda, \beta \rangle \geq 0 \text{ for all } \beta \in \Pi\},\\ &= \{\lambda \in V : \langle \lambda, \alpha \rangle \geq 0 \text{ for all } \alpha \in \Delta\}.\end{aligned}$$

The second equality follows since every positive root β is a nonnegative linear combination of simple roots. For sake of clarity later on, fix a labeling on the simple roots, $\Delta = \{\alpha_1, \ldots, \alpha_n\}$, and use the same labeling to index the corresponding simple reflections: $S = \{s_1, \ldots, s_n\}$, so that $s_{\alpha_j} = s_j$.

We can partition $\overline{C}$ as follows:

$$\overline{C} = \bigcup_{J \subseteq \Delta} C_J,$$

where for any subset of simple roots indexed by $J \subseteq [n] = \{1, 2, \ldots, n\}$ we define

$$C_J = \{\lambda \in V : \langle \lambda, \alpha_j \rangle = 0 \text{ if } j \in J, \langle \lambda, \alpha_j \rangle > 0 \text{ otherwise}\}.$$

Note that $C_{[n]}$ contains only the origin, while $C_\emptyset = C$ is the fundamental chamber itself. The dimension of face C_J in $\mathcal{H}$ is easily seen to be $|J^c|$, where J^c denotes the complement of J, i.e., $J^c = [n] - J$. The extreme rays of the cone $\overline{C}$ are of the form $C_{\{j\}^c}$. Let us give each such ray the color j, and define the color of face C_J to be J^c, which is easily verified to be the set of colors of the extreme rays on its boundary.

Now, because $\overline{C}$ is a fundamental domain for the action of W, each face of the Coxeter complex Σ can be expressed in the form

$$wC_J = \{w(\lambda) : \lambda \in C_J\}.$$

The coloring of faces of $\overline{C}$ extends to all faces of Σ, with the color of wC_J equal to J^c. This coloring means Σ is a *balanced* simplicial complex in the sense of Section 8.6.

Notice that the chambers of $\mathcal{H}$ are the elements in the W-orbit of $C_\emptyset$. We can thus identify chambers with elements of W:

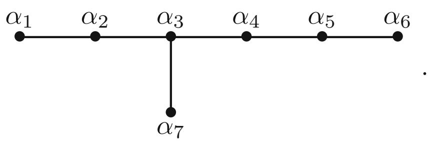
$$w \leftrightarrow wC_\emptyset.$$

In Figure 11.8 we see all the faces of $\Sigma(\mathbf{B}_2)$ labeled in this way. Compare this with Figure 11.7.

Notice that faces are sometimes fixed by elements of W. For example, the face C_{α_1} in Figure 11.8 lies in the hyperplane H_{α_1}, and hence is fixed by the reflection s_1. In general, face C_J is fixed by s_j for all $j \in J$. Therefore any product of such reflections fixes C_J as well. Define W_J, for any subset $J \subseteq [n]$, to be the subgroup of W generated by the simple reflections indexed by J:

$$W_J = \langle s_j \in S : j \in J \rangle.$$

These subgroups are Coxeter groups in their own right, which we call the *standard parabolic subgroups* of W. The Dynkin diagram for W_J is just the subgraph on the nodes indexed by J. For example, suppose we label the Dynkin diagram for $W = E_7$ as follows



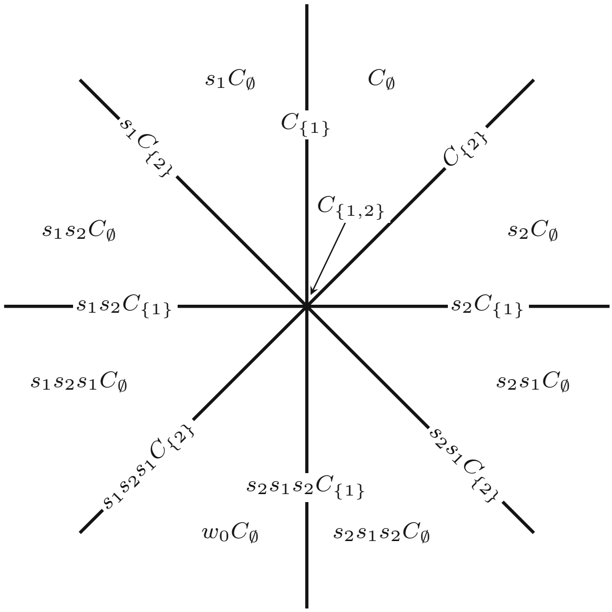
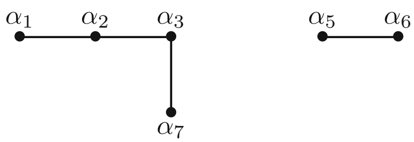


Fig. 11.8 The Coxeter arrangement of type $\mathbf{B}_2$, with labels coming from the B_2-action.

Then the parabolic subgroup $W_{\{1,2,3,5,6,7\}}$ has Dynkin diagram



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so we can see that in this case the parabolic subgroup is reducible. It is the product of two symmetric groups: $W_{\{1,2,3,5,6,7\}} \cong A_4 \times A_2$. Similarly, we find the parabolic $W_{\{2,3,4,6,7\}} \cong D_4 \times A_1$.

Consider now the W-coset $wW_J = \{wv : v \in W_J\}$. Every element of W_J fixes the face C_J, so every element $u = wv \in wW_J$ takes a point $\lambda \in C_J$ to a point $u(\lambda) = w(v(\lambda)) = w(\lambda) \in wC_J$. With this in mind, we can study faces of the Coxeter complex purely algebraically, under the identification

$$wW_J \leftrightarrow wC_J.$$

The partial order on faces of Σ corresponds quite simply to reverse inclusion of cosets:

$$vC_J \leq_\Sigma wC_K \qquad \text{if and only if} \qquad wW_K \subseteq vW_J.$$

The origin (empty face) thus corresponds to all of W and maximal cones (facets) correspond to singleton cosets: $wW_\emptyset = \{w\}$.

We remark that we could have constructed the Coxeter complex Σ this way in the first place, by defining its faces abstractly as cosets of parabolic subgroups. However, without the geometry of the underlying hyperplane arrangement, it would require a separate argument to show that the set

$$\{wW_J : w \in W, J \subseteq [n]\}$$

is isomorphic to an abstract simplicial complex with vertex set

$$\{wW_{\{j\}^c} : w \in W, j \in [n]\}.$$

See Problem 11.12.

11.8 Counting faces in the Coxeter complex

In this section we will compute the f- and h-vectors of the Coxeter complex Σ. We know from Theorem 5.3 that the classical Eulerian polynomial is the h-polynomial of the type A_{n-1} Coxeter complex, and the goal here is to show the W-Eulerian polynomial is the h-polynomial for the type W Coxeter complex.

For example, from Figure 11.8 we can see that $\Sigma(\mathbf{B}_2)$ is the boundary of an octagon. Thus we have $f(\Sigma(\mathbf{B}_2);t) = 1 + 8t + 8t^2$, and $h(\Sigma(\mathbf{B}_2);t) = W(B_2;t) = 1 + 6t + t^2$.

Recall from Section 11.6 that the Coxeter complex is a *balanced* simplicial complex. We have colored its vertices such that every face has distinctly colored vertices; in particular, F has color set J if it is of the form $F = wC_{J^c}$. For any $J \subseteq [n]$, let f_J denote the number of faces of Σ with color set J.

If a face F has color set J, then $\dim F = |J| - 1$ (thinking of the dimension in the complex Σ, not in the hyperplane arrangement $\mathcal{H}$). Hence, we can express the f-vector for Σ as:

$$f(\Sigma;t) = \sum_{F \in \Sigma} t^{\dim F + 1} = \sum_{J \subseteq [n]} f_J t^{|J|}.$$

But by construction, the J-colored faces are precisely those in the W-orbit of C_{J^c}. As discussed in Section 11.7, since C_{J^c} is stabilized by the subgroup W_{J^c}, this means we can simply count the number of cosets of the corresponding parabolic subgroups:

$$\begin{aligned} f_J &= |\{wC_{J^c} : w \in W\}|, \\ &= |W/W_{J^c}|, \\ &= |\{wW_{J^c} : w \in W\}|. \end{aligned}$$

Now consider a coset of the form wW_{J^c}. Each such coset has a unique element of minimal length. (See Problem 11.13.) Suppose u is the element of minimal length in wW_{J^c}. Then in particular, for each $i \in J^c$, us_i is in wW_{J^c} and by minimality of u, $\ell(us_i) > \ell(u)$. That is, none of the simple reflections indexed by J^c are descents of u, so if u is the minimal length element of the coset wW_{J^c},

$$\mathrm{Des}(u) \subseteq J.$$

In Section 5.6 we used the model of set compositions to capture cosets of parabolic subgroups. For example, the set composition $13|5|26|4$ would correspond to the coset

$$135264 \cdot W_{\{1,4\}} = \{135264, 315264, 135624, 315624\},$$

with color set $\{2, 3, 5\}$. The minimal length coset representative is in this case the permutation obtained by listing the elements of each block in increasing order. Right multiplication by the elements of W_{J^c} act on positions, permuting the elements within the blocks of the composition.

A consequence of this discussion is that we have translated the problem of counting faces of color J to the problem of counting the elements of W whose descent set is contained in J:

$$f_J = |\{w \in W : \mathrm{Des}(w) \subseteq J\}|.$$

This naturally partitions the faces of Σ according to the elements w of W, with each w corresponding to the collection of cosets in which it appears as the minimal length representative:

$$w \leftrightarrow \{wW_{J^c} : \mathrm{Des}(w) \subseteq J \subseteq [n]\}.$$

The dimension generating function for the set of faces corresponding to a given element w is thus $t^{\mathrm{des}(w)}(1+t)^{n-\mathrm{des}(w)}$, and we can write the f-polynomial for Σ as follows.

$$\begin{aligned} f(\Sigma; t) &= \sum_{F \in \Sigma} t^{\dim F + 1}, \\ &= \sum_{J \subseteq [n]} f_J t^{|J|}, \\ &= \sum_{J \subseteq [n]} \sum_{w \in W, \mathrm{Des}(w) \subseteq J} t^{|J|}, \end{aligned}$$

$$\begin{aligned}
&= \sum_{w\in W} \sum_{\mathrm{Des}(w)\subseteq J\subseteq[n]} t^{|J|}, \\
&= \sum_{w\in W} t^{\mathrm{des}(w)}(1+t)^{n-\mathrm{des}(w)}, \\
&= (1+t)^n \sum_{w\in W} \left(\frac{t}{1+t}\right)^{\mathrm{des}(w)}, \\
&= (1+t)^n W(t/(1+t)).
\end{aligned}$$

In other words, we have the following result.

Theorem 11.3. *For any finite Coxeter group W, the h-polynomial of Σ is the W-Eulerian polynomial.*

With this connection made, we can now give a topological argument for the palindromicity of the W-Eulerian numbers mentioned in Observation 11.4. Since the Coxeter complex Σ is a sphere, the Dehn-Sommerville relations (Section 8.9) imply its h-vector is palindromic.

11.9 The W-Euler-Mahonian distribution

The way we have counted faces here lends itself to refinement in a natural way. Define the *W-Euler-Mahonian distribution* to be the joint distribution of descents and length, with generating function

$$W(q,t) = \sum_{w\in W} q^{\ell(w)} t^{\mathrm{des}(w)}.$$

Now consider counting the elements in a coset uW_{J^c} with u the coset representative of minimal length. We have

$$\sum_{w\in uW_{J^c}} q^{\ell(w)} = \sum_{v\in W_{J^c}} q^{\ell(u)+\ell(v)} = q^{\ell(u)} W_{J^c}(q,1).$$

For fixed J, each element w in W has a unique decomposition $w = uv$ with $\mathrm{Des}(u) \subseteq J$ and $v \in W_{J^c}$. (See Problem 11.14.) Thus,

$$W(q,1) = \sum_{\mathrm{Des}(u)\subseteq J} q^{\ell(u)} W_{J^c}(q,1),$$

or

$$\sum_{\mathrm{Des}(u)\subseteq J} q^{\ell(u)} = \frac{W(q,1)}{W_{J^c}(q,1)}.$$

Let $f_J(q)$ denote this polynomial.

Then by the principle of inclusion-exclusion,

$$\sum_{\mathrm{Des}(u)=J} q^{\ell(u)} = \sum_{I\subseteq J} (-1)^{|J-I|} f_I(q).$$

Let $h_J(q)$ denote this polynomial. Then putting this into $W(q,t)$, we have:

$$\begin{aligned} W(q,t) &= \sum_{w\in W} q^{\ell(w)} t^{\mathrm{des}(w)}, \\ &= \sum_{J\subseteq[n]} h_J(q) t^{|J|}, \\ &= \sum_{I\subseteq J\subseteq[n]} (-1)^{|J-I|} f_I(q) t^{|J|}, \\ &= \sum_{I\subseteq[n]} f_I(q) t^{|I|} (1-t)^{n-|I|}. \end{aligned}$$

Putting in our expression for $f_I(q)$ gives the following result.

Proposition 11.1. *For any finite Coxeter group of rank n, the Euler-Mahonian distribution has the following recursive description:*

$$W(q,t) = \sum_{I\subseteq[n]} \frac{W(q,1)}{W_{I^c}(q,1)} t^{|I|} (1-t)^{n-|I|}.$$

In particular, if we set $q=1$ in this expression we have

$$W(t) = \sum_{I\subseteq[n]} \frac{|W|}{|W_{I^c}|} t^{|I|} (1-t)^{n-|I|}.$$

So, for example, the reader is invited to run through all parabolic subgroups of D_5 to find

$$\begin{aligned} W(t) = (1-t)^5 &+ \left(2\frac{2^4\cdot 5!}{5!} + \frac{2^4\cdot 5!}{2\cdot 2\cdot 3!} + \frac{2^4\cdot 5!}{2\cdot 4!} + \frac{2^4\cdot 5!}{2^3 4!}\right) t(1-t)^4 \\ &+ \left(4\frac{2^4\cdot 5!}{4!} + 4\frac{2^4\cdot 5!}{2\cdot 3!} + \frac{2^4\cdot 5!}{2\cdot 2\cdot 2}\right) t^2(1-t)^3 \\ &+ \left(4\frac{2^4\cdot 5!}{3!} + 6\frac{2^4\cdot 5!}{2\cdot 2}\right) t^3(1-t)^2 + 5\frac{2^4\cdot 5!}{2} t^4(1-t) + \frac{2^4\cdot 5!}{1} t^5, \\ &= 1 + 157t + 802t^2 + 802t^3 + 157t^4 + t^5. \end{aligned}$$

11.10 The weak order

Recall from Section 5.5 that there is a partial ordering on the set of chambers for any hyperplane arrangement that we call the *weak order*. Since the chambers in the Coxeter arrangement correspond to group elements, this gives us a quite natural partial ordering on the Coxeter group itself, which we will denote $\mathrm{Wk}(W)$.

In Section 5.5 we saw this partial ordering on chambers characterized with sign sequences. Recall that the *inversion set* of a chamber C is the index set for the negative entries in its sign vector:

$$\mathrm{Inv}(C) = \{\beta : \sigma_\beta(C) = -\},$$

and $\mathrm{inv}(C) = |\,\mathrm{Inv}(C)|$ denotes the number of inversions. Then for two chambers C_1 and C_2, we have

$$C_1 \leq_{\mathrm{Wk}} C_2 \qquad \text{if and only if} \qquad \mathrm{Inv}(C_1) \subseteq \mathrm{Inv}(C_2). \tag{11.3}$$

Cover relations come from crossing a wall from one chamber to another, and this has the effect of changing exactly one entry in the sign vector.

In the case of the Coxeter arrangement, the entries of the sign vector are indexed by positive roots and we can characterize the inversion sets for a chamber $wC_\emptyset$ as follows:

$$\mathrm{Inv}(wC_\emptyset) = \{\beta \in \Pi : w^{-1}(\beta) < 0\}.$$

Indeed, we can check that if λ is a point in the fundamental chamber $C_\emptyset$, then $\langle \lambda, w^{-1}(\beta)\rangle = \langle w(\lambda), \beta\rangle$, so

$$w^{-1}(\beta) < 0 \qquad \text{if and only if} \qquad \sigma_\beta(wC_\emptyset) = -.$$

By a mild abuse of notation we will define the inversion set of a group element to be the same. That is, $\mathrm{Inv}(w) = \mathrm{Inv}(wC_\emptyset)$, the set of positive roots that w^{-1} sends to negative roots.

We thus define the weak order on W, $\mathrm{Wk}(W)$, by

$$u \leq_{\mathrm{Wk}} v \qquad \text{if and only if} \qquad \mathrm{Inv}(u) \subseteq \mathrm{Inv}(v). \tag{11.4}$$

A cover relation $u <_{\mathrm{Wk}} v$ means the chambers $uC_\emptyset$ and $vC_\emptyset$ are adjacent and this implies that we can obtain v from u with right multiplication by a simple reflection, as we now explain.

Let $\lambda \in uC_\emptyset$ and $\mu \in vC_\emptyset$ so that the line segment $p(x) = (1-x)\lambda + x\mu$, with $0 \leq x \leq 1$, crosses a wall from the chamber $uC_\emptyset$ to the chamber $vC_\emptyset$. Notice that $u^{-1}(\lambda)$ lies in the fundamental chamber. Then applying u^{-1} to the entire segment $p(x)$, we have the segment

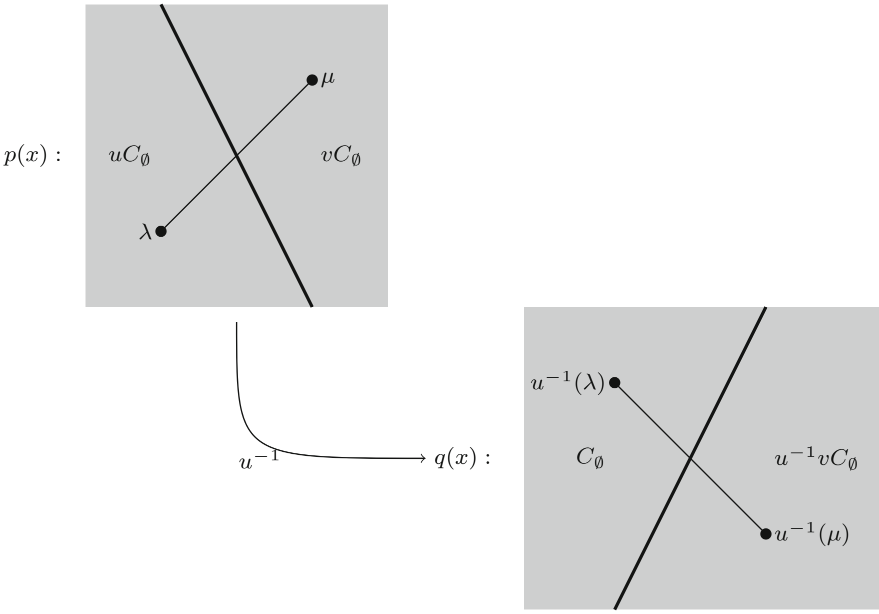


Fig. 11.9 Pulling back the line segment $p(x) = (1-x)\lambda + x\mu$ to the neighborhood of the fundamental chamber.

$$q(x) = u^{-1}p(x) = (1-x)u^{-1}(\lambda) + xu^{-1}(\mu)$$

that crosses from $C_\emptyset$ to $u^{-1}vC_\emptyset$. See Figure 11.9. But the only chambers adjacent to $C_\emptyset$ are by construction the ones corresponding to simple reflections. Thus $u^{-1}v = s$ for some simple reflection s, or $v = us$.

One consequence of this observation is that the length of an element equals its inversion number. This is easy to see by induction on length. The claim is clearly true for any simple reflection s. Now suppose it is true for all u with $\ell(u) \leq k$. If $u <_{\mathrm{Wk}} v$ is a cover relation, then $\mathrm{inv}(v) = \mathrm{inv}(u) + 1$ and $v = us$. If $\ell(u) = \mathrm{inv}(u) = k$, then $\ell(v) = k \pm 1$. But if $\ell(v) = k-1$, then our induction hypothesis says $\mathrm{inv}(v) = k-1$ as well. However, since v covers u, we already know $\mathrm{inv}(v) = k+1$. Hence $\ell(v) = k+1 = \mathrm{inv}(v)$, as desired.

Observation 11.5 *Suppose w is an element of a finite Coxeter group W. Then length equals inversion number, i.e.,*

$$\ell(w) = \mathrm{inv}(w).$$

We know that in any Euclidean hyperplane arrangement the weak order on chambers is ranked by inversion number. (See Observation 5.4). In particular the same is true for $\mathrm{Wk}(W)$, and by Observation 11.5 we can phrase the result in terms of length.

Observation 11.6 *The weak order* $\mathrm{Wk}(W)$ *is ranked by length, i.e.,*

$$f(\mathrm{Wk}(W); q) = \sum_{w \in W} q^{\ell(w)}.$$

Moving down in the weak order, then, amounts to multiplying on the right by a simple reflection in a way that reduces the length statistic: $\ell(w) > \ell(ws)$. This means that the edges in the Hasse diagram for $\mathrm{Wk}(W)$ that move downward from an element w can be labeled by the members of the descent set $\mathrm{Des}(w)$. Moreover, we can exploit this fact to show that

$$\mathrm{Des}(w) = \{s \in S : \ell(w) > \ell(ws)\} = \{s_\alpha : \alpha \in \Delta \text{ and } w(\alpha) < 0\},$$

i.e., the descent set of w corresponds to the simple roots that are taken to negative roots by w.

To see this correspondence, suppose the simple reflection s_α is a descent of v. That is, suppose $v = us_\alpha$ and that $u <_{\mathrm{Wk}} v$ is a cover relation in the weak order. Then we can write the difference of inversion sets as

$$\mathrm{Inv}(v) - \mathrm{Inv}(u) = \{\beta\},$$

for some positive root $\beta \in \Pi$. This means, in turn, that $v^{-1}(\beta) < 0$ is a negative root, i.e., if λ is a point in the fundamental chamber $C_\emptyset$ then $\langle \lambda, v^{-1}(\beta)\rangle < 0$.

Since $v^{-1} = s_\alpha u^{-1}$, we get $\langle s_\alpha(\lambda), u^{-1}(\beta)\rangle < 0$. This means the corresponding entry in the sign vector for $s_\alpha C_\emptyset$ is negative. But this chamber, being adjacent to the fundamental chamber, has only one negative entry: $\sigma_\alpha(s_\alpha C_\emptyset) = -$. Thus $u^{-1}(\beta) = \alpha$, or $u(\alpha) = \beta$.

Since β is a positive root, this means

$$v(\alpha) = u(-\alpha) = -u(\alpha) = -\beta < 0.$$

In other words, if s_α is a descent of v, then $v(\alpha) < 0$. Each of these steps is reversible, so we have the following.

Observation 11.7 *Suppose w is an element of a finite Coxeter group W. Then the descent set is indexed by those simple roots that w takes to negative roots, i.e.,*

$$\mathrm{Des}(w) = \{s_\alpha : \alpha \in \Delta, w(\alpha) < 0\} = \{s_\alpha : \alpha \in \mathrm{Inv}(w^{-1}) \cap \Delta\}.$$

Notice that throughout the discussion in this section, we are working with the "right" weak order. If we had identified $\mathrm{Inv}(w)$ with $\mathrm{Inv}(w^{-1}C_\emptyset)$ we would get the “left” weak order, where cover relations are given by left multiplication by a simple reflection. See the discussion in Section 5.2 and also Problem 11.15.

11.11 The shard intersection order

In Section 11.8 we saw how W-Eulerian numbers arose in the h-vector of the Coxeter complex. This followed from careful study of the Coxeter arrangement $\mathcal{H}$, which decomposed the ambient vector space V into a disjoint union of open cones that we called the *faces* of $\mathcal{H}$. What Nathan Reading calls *shards* arise from a different sort of decomposition of the arrangement $\mathcal{H}$. This decomposition is closely linked to the W-Eulerian numbers as well, as we now describe.

We form shards by splitting up the hyperplane arrangement $\mathcal{H}$ into closed codimension one cones, each entirely contained in some hyperplane. For example, we have a rank two arrangement in Figure 11.10. Its shards are the two lines and six half-lines. These shards appear disconnected to emphasize their locations, but in fact they all intersect at the origin.

Each shard contains walls on the boundaries of some chambers. Among these, there is unique chamber of minimal length. Call this chamber C and say C' is the chamber on the other side of the wall contained in the shard. We identify the shard with the chamber C'. (That this identification is well-defined takes some work.) It turns out that C' covers C and no other chambers in the weak order. Since covers in the weak order on W correspond to descent sets, we have an identification between shards and elements with exactly one descent:

$$\{\text{shards}\} \leftrightarrow \{w : \operatorname{des}(w) = 1\}.$$

The correspondence between shards and chambers is indicated in our labeling of the chambers in Figure 11.10. For example, shard l_2 contains the walls between two pairs of chambers. Among these, C_0 has minimal length, and so we identify the shard l_2 with the chamber C_2, which lies just across l_2 from C_0.

Since they are closed cones, the shards overlap in various ways and, remarkably, the set of intersections of the shards is in bijection with the set of chambers of $\mathcal{H}$. The lattice of intersections of shards, with partial order given by reverse containment, thus passes to a lattice structure on chambers. This partial order is a coarsening of the weak order, in the sense that if $C_1 \leq_{\mathrm{Sh}} C_2$ in the shard intersection order, then $C_1 \leq_{\mathrm{Wk}} C_2$ in the weak order. However, two elements that are comparable in the weak order may be incomparable in the shard intersection order.

Since chambers correspond to group elements, we get a partial order on W itself, denoted $(\mathrm{Sh}(W), \leq)$. The minimal element is the empty intersection

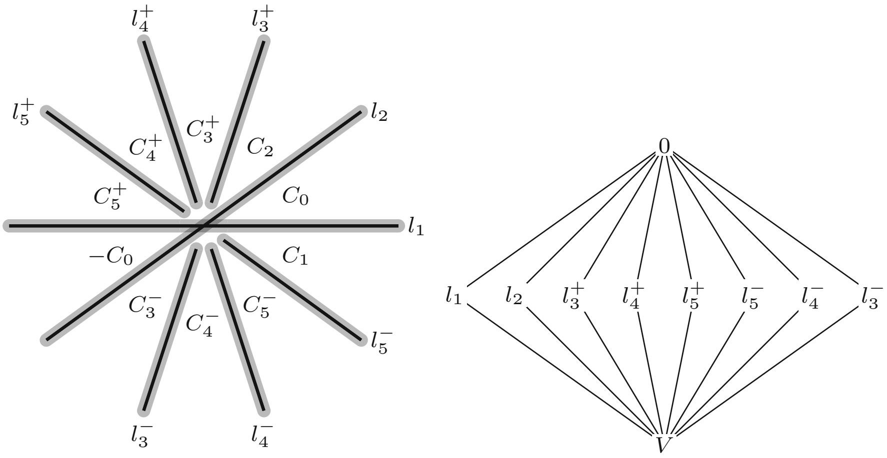


Fig. 11.10 The shards in a rank two arrangement (with fundamental chamber C_0) and the corresponding lattice of intersections.

which we identify with the identity in W and as mentioned, shards correspond to elements with a single descent. Surprisingly, the correspondence between chambers and codimension continues: an intersection of shards having codimension k corresponds to an element with k descents. Thus the rank generating function is given by the W-Eulerian polynomial.

Theorem 11.4. *The rank generating function for the shard intersection order is the W-Eulerian polynomial:*

$$f(\mathrm{Sh}(W);t) = \sum_{w \in \mathrm{Sh}(W)} t^{\mathrm{rk}(w)} = \sum_{w \in W} t^{\mathrm{des}(w)} = W(t).$$

We already saw this result for the symmetric group in Section 3.3. We give more details of the construction of the shard intersection order in Chapter 13, including combinatorial models for groups of type B_n and D_n that make Theorem 11.4 apparent.

Notes

Coxeter groups are named for Harold Scott MacDonald Coxeter whose well-known book on regular polytopes is a touchstone for many [53]. There are several good books on the general subject, including one by James Humphreys [92] and one by Anders Björner and Francesco Brenti [25]. This chapter leans heavily on both these books, as well as ideas found in the book by Peter Abramenko and Kenneth Brown [1].

As pointed out by Brenti [34], Björner essentially proved that the W-Eulerian polynomial is the h-vector of the Coxeter complex (Theorem 11.3) in 1984 [24]. Proposition 11.1 is due to Victor Reiner in 1995 [125].

Brenti conjectured the real-rootedness of W-Eulerian polynomials in 1994 [34], and this conjecture was finally resolved in a 2015 paper by Carla Savage and Mirko Visontai [132].

We have only touched the surface of Nathan Reading's work on the shard intersection order here. See his 2011 paper [124] for details.

Problems

11.1. Find all reduced expressions for the following permutations, written here in one-line notation.

1. $w = 321$
2. $w = 4321$
3. $w = 213465$
4. $w = 216345$

11.2. How many reduced expressions are there for the long element in the symmetric group S_n, i.e., $w_0 = n(n-1)\cdots 321$?

11.3. Show the group whose generators and relations are indicated in (11.1) is an infinite group.

11.4. Compute the Mahonian polynomials for B_6, D_6, E_6, and F_4.

11.5. Conclude from Theorem 11.1 that for any W, $|\Pi|$ equals the sum of the degrees minus $n = |\Delta|$, i.e.,

$$|\Pi| = d_1 + \cdots + d_n - n.$$

11.6. Compute the Eulerian polynomial for the dihedral group $I_2(m) = \langle s, t : (st)^m\rangle$.

11.7. Show that $\mathrm{Des}(w_0 w) = S - \mathrm{Des}(w)$.

11.8. Show that $\langle s_\beta(\lambda), s_\beta(\mu)\rangle = \langle \lambda, \mu\rangle$ for any reflection s_β and any pair of points λ, μ in V. This shows the group generated by such reflections contains only rigid motions, i.e., the finite reflection groups are subgroups of the orthogonal group $O(V)$.

11.9. Show that the root systems of types $\mathbf{B}_n$ and $\mathbf{C}_n$ are dual in the following sense: if $\alpha \in \mathbf{B}_n$ then $\alpha^\vee = \frac{2\alpha}{\langle \alpha, \alpha\rangle} \in \mathbf{C}_n$.

11.10. The *root poset* can be defined as the partial ordering on the positive roots given by $\alpha \leq \beta$ if and only if $\beta - \alpha$ is a positive root. Draw the Hasse diagrams for the root posets on $\mathbf{A}_5$, $\mathbf{B}_5$, $\mathbf{C}_5$, and $\mathbf{D}_5$.

11.11. 1. Describe all the standard parabolic subgroups of F_4.
2. A maximal parabolic subgroup of (W, S) is one obtained by deleting a single generator from S, i.e., of the form W_J with $J = S - \{s\}$. Find all the maximal parabolic subgroups of B_6,
3. Find all maximal parabolic subgroups of E_8.

11.12. As described at the end of Section 11.7, show that the purely algebraic description of the Coxeter complex is indeed a simplicial complex.

11.13. Show that any coset of a parabolic subgroup wW_J has a unique member of minimal length.

11.14. Fix a subset J of simple generators. Show that for any $w \in W$, there is a unique pair u, v, with $\mathrm{Des}(u) \subseteq J$ and $v \in W_{J^c} = \langle s_i : i \notin J \rangle$, such that

$$w = uv.$$

Conclude that

$$W(q, 1) = \sum_{\mathrm{Des}(u) \subseteq J} q^{\ell(u)} W_{J^c}(q, 1).$$

11.15. Compare the combinatorial definition of the inversion set for $w = 356124$ with the root-theoretic inversion set for $wC_{\emptyset}$ in $\Sigma(\mathbf{A}_5)$.

Show that Equation (11.3) gives rise to the right weak order $\mathrm{Wk}^r(S_n)$, while defining

$$u \leq_{\mathrm{Wk}} v \text{ if and only if } \{\beta \in \Pi : u(\beta) < 0\} \subseteq \{\beta \in \Pi : v(\beta) < 0\}$$

suggests the left weak order $\mathrm{Wk}^l(S_n)$.

11.16. Show the involution $w \to w_0 w$ is an involution that takes the weak order to its dual. Conclude that the weak order is isomorphic to its dual.

Chapter 12
W-Narayana numbers

JUST AS EULERIAN NUMBERS GENERALIZE to Coxeter groups, so too do the Narayana numbers. There is a great deal of interest in these generalizations. Entire books could be (and have been) dedicated to the subject. We give a brief survey of this circle of ideas in this chapter, with an emphasis on parallels with our discussion of the classical case.

12.1 Reflection length and Coxeter elements

For a given root system Φ, let $T = \{s_\beta : \beta \in \Pi\}$ denote the set of all reflections in the roots. Since $S \subseteq T$, T is clearly a generating set for the Coxeter group W, though it is usually not minimal. Moreover, we can check that, for any $\alpha \in \Delta$ and $w \in W$,

$$s_{w(\alpha)} = ws_\alpha w^{-1}.$$

(See Problem 12.1.) Every positive root is equal to $w(\alpha)$ for some w and α (the W-orbit of Δ is all of Φ), so we can therefore write $T = \{wsw^{-1} : w \in W, s \in S\}$. As words in S, the reflections are therefore palindromes, i.e., they can be written the same forwards as backwards. In the case of the symmetric group, this corresponds to the fact that any transposition is conjugate to an adjacent transposition.

We define the *reflection length* of an element w (also sometimes called the *absolute length* of w) to be the minimal number of reflections needed to express w, i.e.,

$$\ell'(w) = \min\{k : w = s_{\beta_1} \cdots s_{\beta_k}\}.$$

We originally motivated the notion of length by considering how to sort a permutation using adjacent transpositions. If we modify our sorting to allow nonadjacent transpositions as well, we have the notion of reflection length.

T.K. Petersen, *Eulerian Numbers*, Birkhäuser Advanced Texts Basler Lehrbücher, DOI 10.1007/978-1-4939-3091-3_12

Since $S \subseteq T$, the reflection length of an element w is never greater than its usual length: $\ell'(w) \leq \ell(w)$. However, while $\ell(w)$ can be as much as $|\Pi|$, reflection length has a uniform upper bound of $|\Delta|$, i.e., the rank of W.

The upper bound for $\ell(w)$ follows from the geometric interpretation of length, in terms of inversions. The upper bound for reflection length also has a geometric interpretation. Let $\mathrm{rk}(w)$ denote the codimension of the subspace of V fixed by w, i.e., suppose $U \subseteq V$ is a maximal subspace such that $w(U) = U$. Then $\mathrm{rk}\, w = n - \dim U$. Thus the identity has rank zero, reflections have rank one, and so on. In general, we have the following result.

Proposition 12.1. *Let W be a finite Coxeter group of rank n. For any $w \in W$,*

$$\ell'(w) = \mathrm{rk}(w).$$

In particular, $\ell'(w) \leq n$.

See Problem 12.2 for the proof of Proposition 12.1.

A collection of group elements achieving this upper bound are the permutations of the set S, i.e., elements obtained by taking the product of each simple reflection once, in some order. These elements are known as *Coxeter elements*. The Coxeter elements will come to play an important role in the study of "W-Catalan numbers" in Section 12.2.

For example, in the dihedral group $I_2(m)$ with generators s and t, the Coxeter elements are st and ts, which correspond to rotations of π/m and $-\pi/m$. For the symmetric group $S_n = A_{n-1}$, the Coxeter elements are the n-cycles.

All the Coxeter elements are conjugate to one another (see Problem 12.3), and hence they have the same order, denoted h. (For any Coxeter element c there is a plane in V called the *Coxeter plane* on which c acts by rotation by $2\pi/h$.) Remarkably, $h = |\Phi|/|\Delta|$—a quantity sometimes known as the *Coxeter number*—though we will not use this fact. It also turns out that h is equal to the highest degree of W, as one can verify from Table 11.2 in the irreducible cases. The number of Coxeter elements for the group W is equal to the number of orientations of the Coxeter graph of W. See Problem 12.6.

Returning to our discussion of reflection length, there is an elegant expression for the length generating function in terms of the degrees, reminiscent of Theorem 11.1. The coefficients of this generating function in the case of the symmetric group are known as the *Stirling numbers of the first kind* (see Observation 3.1) and so we refer to these numbers as *W-Stirling numbers.*

Theorem 12.1 (W-Stirling distribution). *The distribution of reflection length in the finite Coxeter group of rank n is given by*

$$\sum_{w \in W} t^{\ell'(w)} = \prod_{i=1}^{n} (1 + (d_i - 1)t),$$

where $d_1, d_2, \ldots, d_n$ are the degrees of W.

For example,

$$\sum_{w \in D_5} t^{\ell'(w)} = (1+t)(1+3t)(1+5t)(1+7t)(1+4t),$$

and for the dihedral group $I_2(m)$,

$$\sum_{w \in I_2(m)} t^{\ell'(w)} = (1+t)(1+(m-1)t) = 1 + mt + (m-1)t^2.$$

See Table 12.1 for a listing of the inversions, length, reflection length, and descents of the elements of $I_2(5)$.

Table 12.1 The elements of $I_2(5)$, their inversions and their descents.

w	$\{\beta \in \Pi : w(\beta) < 0\}$	$\ell(w)$	$\ell'(w)$	$\mathrm{Des}(w)$
1	$\emptyset$	0	0	$\emptyset$
s	$\{\alpha_1\}$	1	1	$\{s\}$
t	$\{\alpha_2\}$	1	1	$\{t\}$
st	$\{\alpha_2, \delta\}$	2	2	$\{t\}$
ts	$\{\alpha_1, \beta\}$	2	2	$\{s\}$
sts	$\{\alpha_1, \beta, \gamma\}$	3	1	$\{s\}$
tst	$\{\alpha_2, \gamma, \delta\}$	3	1	$\{t\}$
$stst$	$\{\alpha_2, \beta, \gamma, \delta\}$	4	2	$\{t\}$
$tsts$	$\{\alpha_1, \beta, \gamma, \delta\}$	4	2	$\{s\}$
$ststs = tstst$	$\{\alpha_1, \alpha_2, \beta, \gamma, \delta\}$	5	1	$\{s, t\}$

12.2 Absolute order and W-noncrossing partitions

In this section we will define a partial order on W, called the *absolute order* of W, denoted $\mathrm{Abs}(W)$. We did this in Section 3.5 for the case of the symmetric group. The motivation in that section was to identify a subinterval of $\mathrm{Abs}(S_n)$ isomorphic to the lattice of noncrossing partitions. Here we do the same for any finite Coxeter group W, thereby obtaining a notion of *W-noncrossing partitions.*

We define the absolute order of W as follows. Say that $u <_{\mathrm{Abs}} v$ is a cover relation in the absolute order if $v = ut$ and $\ell'(v) = \ell'(u) + 1$. Then $(\mathrm{Abs}(W), \leq)$ is the transitive closure of these cover relations, i.e., $u \leq_{\mathrm{Abs}} v$ if and only if $\ell'(v) = \ell'(u) + k$ and there is a sequence of reflections $t_1, \ldots, t_k$ such that $v = ut_1 \cdots t_k$. In Figure 12.1 we see two examples.

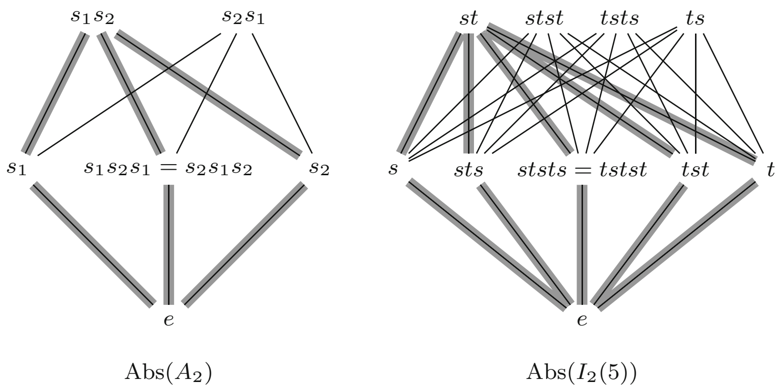


Fig. 12.1 The absolute orders $\mathrm{Abs}(A_2)$ and $\mathrm{Abs}(I_2(5))$, with the lattice of noncrossing partitions highlighted in bold.

The absolute order is obviously ranked by reflection length, so we have the following observation.

Observation 12.1 *The rank generating function for the absolute order is the W-Stirling polynomial, i.e.,*

$$\sum_{w \in \mathrm{Abs}(W)} t^{\mathrm{rk}(w)} = \sum_{w \in W} t^{\ell'(w)} = \prod_{i=1}^{n} (1 + (d_i - 1)t),$$

where $d_1, d_2, \ldots, d_n$ are the degrees of W.

Now we will consider a special subposet of $\mathrm{Abs}(W)$. Recall that the Coxeter elements are products of all the generators in $S = \{s_1, \ldots, s_n\}$, taken in some order, i.e., a Coxeter element is of the form $c = s_{i_1} \cdots s_{i_n}$ for some permutation $i_1 \cdots i_n$ of the indices $1, \ldots, n$. Let

$$[e, c] = \{w \in W : e \leq_{\mathrm{Abs}} w \leq_{\mathrm{Abs}} c\}$$

denote the interval from e to c in the absolute order, with the induced partial order. It is not too difficult to show that if c and c' are Coxeter elements, then the map $c \mapsto c' = wcw^{-1}$ is an isomorphism of posets from $[e, c]$ to $[e, c']$. (See Problem 12.5.) That is, the structure of this interval as an abstract poset does not depend on the choice of Coxeter element c.

Thus for any c, let $\mathrm{NC}(W) = [e, c]$, which we call the *W-noncrossing partition lattice*. The lattices of noncrossing partitions for A_2 and $I_2(5)$ are highlighted in Figure 12.1. We are calling the poset $\mathrm{NC}(W)$ a lattice without justification. Certainly $\mathrm{Abs}(W)$ is not a lattice, so the fact that $\mathrm{NC}(W)$ is a lattice is not immediate.

The number of elements in $\mathrm{NC}(W)$ can be computed in terms of the degrees of W, as follows.

Theorem 12.2. *For any finite Coxeter group W, with root system Φ of rank n,*

$$|\mathrm{NC}(W)| = \prod_{i=1}^{n} \frac{h + d_i}{d_i} = \frac{1}{|W|} \prod_{i=1}^{n} (h + d_i),$$

where $d_1, d_2, \ldots, d_n$ are the degrees of W, and $h = |\Phi|/n$ is the Coxeter number.

12.3 W-Catalan and W-Narayana numbers

Since we know there are Catalan-many noncrossing partitions in the classical case, $|\mathrm{NC}(A_{n-1})| = C_n = \frac{1}{n+1}\binom{2n}{n}$, it makes sense to define the *W-Catalan numbers* to be the number of W-noncrossing partitions, i.e., define

$$\mathrm{Cat}(W) = \prod_{i=1}^{n} \frac{h + d_i}{d_i}.$$

We list the W-Catalan numbers in Table 12.2.

Table 12.2 The W-Catalan numbers.

W	A_n	B_n	D_n	E_6	E_7	E_8	F_4	H_3	H_4	$I_2(m)$
$\mathrm{Cat}(W)$	$\frac{1}{n+2}\binom{2n+2}{n+1}$	$\binom{2n}{n}$	$\frac{3n-2}{n}\binom{2n-2}{n-1}$	833	4160	25080	32	280	105	$m+2$

Since the W-noncrossing partitions come with a ready-made rank function, we can define the *W-Narayana polynomial*, denoted $\mathrm{Cat}(W;t)$, to be the rank generating function for $\mathrm{NC}(W)$:

$$\mathrm{Cat}(W;t) = f(\mathrm{NC}(W);t) = \sum_{w\in \mathrm{NC}(W)} t^{\mathrm{rk}(w)} = \sum_{w\in[e,c]} t^{\ell'(w)}.$$

We define, for $k = 0,1,\ldots,n$, the *W-Narayana number*, denoted $N(W,k)$, to be the number of elements in $\mathrm{NC}(W)$ of rank k. We have

$$N(W,k) = |\{w \in [e,c] : \ell'(w) = k\}|,$$

and

$$\mathrm{Cat}(W;t) = \sum_{k=0}^{n} N(W,k)t^k.$$

For example, we see in Figure 12.1 that

$$\mathrm{Cat}(A_2;t) = 1 + 3t + t^2 \text{ and } \mathrm{Cat}(I_2(5);t) = 1 + 5t + t^2.$$

(In general, $\mathrm{Cat}(I_2(m);t) = 1+mt+t^2$ for any $m \geq 3$. See Problem 12.7.) In Table 12.3 we have the type B_n Narayana numbers, Table 12.4 contains the type D_n Narayana numbers, and the exceptional cases are listed in Table 12.5.

Table 12.3 The B_n-Narayana numbers $N(B_n,k) = \binom{n}{k}^2$, $0 \leq k \leq n \leq 9$.

$n\backslash k$	0	1	2	3	4	5	6	7	8	9
2	1	4	1							
3	1	9	9	1						
4	1	16	36	16	1					
5	1	25	100	100	25	1				
6	1	36	225	400	225	36	1			
7	1	49	441	1225	1225	441	49	1		
8	1	64	784	3136	4900	3136	784	64	1	
9	1	81	1296	7056	15876	15876	7056	1296	81	1

Table 12.4 The D_n-Narayana numbers $N(D_n,k) = \binom{n}{k}(\binom{n-1}{k} + \binom{n-2}{k-2})$, $0 \leq k \leq n \leq 9$.

$n\backslash k$	0	1	2	3	4	5	6	7	8	9
4	1	12	24	12	1					
5	1	20	70	70	20	1				
6	1	30	165	280	165	30	1			
7	1	42	336	875	875	336	42	1		
8	1	56	616	2296	3500	2296	616	56	1	
9	1	72	1044	5292	11466	11466	5292	1044	72	1

Table 12.5 The W-Narayana numbers $N(W,k)$ of exceptional type.

$W\backslash k$	0	1	2	3	4	5	6	7	8
E_6	1	36	204	351	204	36	1		
E_7	1	63	546	1470	1470	546	63	1	
E_8	1	120	1540	6120	9518	6120	1540	120	1
F_4	1	24	55	24	1				
H_3	1	15	15	1					
H_4	1	60	158	60	1				

We can see an evident symmetry in the W-Narayana numbers, i.e., for W of rank n, $N(W,k) = N(W,n-k)$. This palindromicity follows because the noncrossing partition lattice is self-dual. Proof of the following result is deferred to Problem 12.8.

Proposition 12.2. *The noncrossing partition lattice* $\mathrm{NC}(W) = [e,c]$ *is self-dual, and the map* $w \mapsto w^{-1}c$ *is an anti-isomorphism. In particular, the* W*-Narayana numbers are palindromic:*

$$N(W,k) = N(W,n-k),$$

where W *is of rank* n*.*

But, just as we saw for the W-Eulerian numbers, we can do better than symmetry. In fact, the W-Narayana polynomials are real-rooted, and hence gamma-nonnegative.

Theorem 12.3. *For any finite Coxeter group* W*, the* W*-Narayana polynomial is real-rooted. In particular, if* W *is of rank* n*, there exist nonnegative integers* $\gamma_j^{\mathrm{NC}(W)}$ *such that*

$$\mathrm{Cat}(W;t) = \sum_{w\in[e,c]} t^{\ell'(w)} = \sum_{j\geq 0} \gamma_j^{\mathrm{NC}(W)} t^j (1+t)^{n-2j}.$$

Again, proof of this result boils down to proving it in the irreducible cases. See Problem 12.9. The type A_n version of this result was the subject of Problem 4.7 (see also Theorem 4.2). The result clearly holds for dihedral groups, since

$$\mathrm{Cat}(I_2(m);t) = 1 + mt + t^2 = (1+t)^2 + (m-2)t,$$

which is both real-rooted and gamma-nonnegative since $m \geq 2$. The exceptional cases are easily checked. Their Narayana numbers are shown in Table 12.5 and their corresponding gamma numbers are shown in Table 12.6. We defer discussion of the type B_n and D_n cases to Chapter 13.

Table 12.6 The gamma coefficients $\gamma_j^{\mathrm{NC}(W)}$ of exceptional type.

$W\backslash j$	0	1	2	3	4
E_6	1	30	69	13	
E_7	1	56	245	140	
E_8	1	112	840	1024	120
F_4	1	20	9		
H_3	1	12			
H_4	1	56	40		

12.4 Coxeter-sortable elements

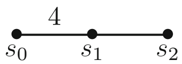
Another way to come by the W-Narayana numbers is to generalize the notion of 231-avoiding permutations. The 231-avoiding permutations are what is known as *stack-sortable* (see Problem 4.4). Work of Nathan Reading generalizes this notion as follows.

We first fix a particular choice of Coxeter element, c, and a particular reduced expression for this element, $c = s_{i_1} s_{i_2} \cdots s_{i_n}$. We write

$$c^\infty = s_{i_1} s_{i_2} \cdots s_{i_n} | s_{i_1} s_{i_2} \cdots s_{i_n} | s_{i_1} s_{i_2} \cdots s_{i_n} | \cdots .$$

This is not an element of W, but a formal product. Since every finite word in S is a subword of c^∞, clearly we can obtain any reduced expression for an element $w \in W$ as a subword of c^∞. Among all reduced expressions for w, define the *c-sorting word* for w to be the one that is lexicographically first as a subword of c^∞.

For example, label the B_3 Coxeter graph as



and consider the Coxeter element $c = s_2 s_0 s_1$. Then

$$c^\infty = s_2 s_0 s_1 | s_2 s_0 s_1 | s_2 s_0 s_1 | \cdots .$$

In Table 12.7 we see a few elements of B_3 and their c-sorting words.

We will keep the bars in a sorting word to indicate where in c^∞ we found these letters. Notice that the lexicographically first reduced expression for w in c^∞ might be a different reduced expression from the one we initially use to identify w. For example, the element $w = s_0 s_2$ first appears in c^∞ as $s_2 s_0$.

Each string of letters between the bars of the sorting word can be thought of as a subset of the letters in the word for c. Thus the sorting word for w corresponds to a sequence of such subsets. We say an element w is *c-sortable* if this sequence of subsets is decreasing under inclusion. Let $\mathrm{Sort}(c)$ denote the set of all c-sortable elements. In Table 12.7 we can see examples of sortable

Table 12.7 The c-sorting words for some elements of B_3. Here, $c = s_2 s_0 s_1$.

w	c-sorting word	c-sortable?
e	e	yes
$s_0 s_1$	$s_0 s_1$	yes
$s_0 s_2$	$s_2 s_0$	yes
$s_1 s_0$	$s_1 \vert s_0$	no
$s_0 s_2 s_1$	$s_2 s_0 s_1$	yes
$s_1 s_2 s_1$	$s_2 s_1 \vert s_2$	yes
$s_1 s_0 s_2 s_1$	$s_1 \vert s_2 s_0 s_1$	no
$s_1 s_0 s_1 s_2 s_1$	$s_1 \vert s_0 s_1 \vert s_2 s_1$	no
$s_0 s_2 s_1 s_0 s_2 s_1 s_0$	$s_2 s_0 s_1 \vert s_2 s_0 s_1 \vert s_0$	yes

and non-sortable elements, while in Table 12.8 we see all of the c-sortable elements for $c = s_2 s_0 s_1$ in B_3.

Remarkably there is a bijection between c-sortable elements and elements in the interval $[e, c]$ of the absolute order, i.e., between the sortable elements and noncrossing partitions. The way in which this works is to define an ordered set of reflections given by the c-sorting word for an element w. If $w = s_{i_1} \cdots s_{i_k}$, we let

$$t_j = s_{i_1} \cdots s_{i_{j-1}} s_{i_j} s_{i_{j-1}} \cdots s_{i_1}.$$

In our running example of $c = s_2 s_0 s_1$ in B_3, consider $w = s_2 s_0 s_1 | s_2 s_0 s_1 | s_0$. Then the reflections for this sorting word are:

$$\begin{aligned}
t_1 &= s_2, \\
t_2 &= s_2(s_0)s_2, \\
t_3 &= s_2 s_0(s_1)s_0 s_2, \\
t_4 &= s_2 s_0 s_1(s_2)s_1 s_0 s_2, \\
&= s_0 s_1 s_0, \\
t_5 &= s_2 s_0 s_1 s_2(s_0)s_2 s_1 s_0 s_2, \\
&= s_2 s_1 s_0 s_1 s_2, \\
t_6 &= s_2 s_0 s_1 s_2 s_0(s_1)s_0 s_2 s_1 s_0 s_2, \\
&= s_1 s_0 s_1 s_2 s_1 s_0 s_1, \\
t_7 &= s_2 s_0 s_1 s_2 s_0 s_1(s_0)s_1 s_0 s_2 s_1 s_0 s_2, \\
&= s_1 s_0 s_1.
\end{aligned}$$

Each of these reflections has the effect of deleting a single letter in this reduced expression for w (t_j deletes the jth letter). Thus these reflections correspond to inversions of w. In particular, some of them correspond to the descents of w, i.e., $w \geq_{\mathrm{Wk}} t_j w = ws$ for some $s \in S$. Call these the *cover reflections* for w since they correspond to the elements that w covers in the weak order.

In the example above, with $w = s_2s_0s_1s_2s_0s_1s_0$, both t_1 and t_7 are the cover reflections:

$$t_1w = s_0s_1s_2s_0s_1s_0 = ws_1,$$

and

$$t_7w = s_2s_0s_1s_2s_0s_1 = ws_0.$$

Define a map $\mathrm{nc}_c : \mathrm{Sort}(c) \to [e, c]$ from the set of c-sortable elements to members of the interval $[e, c]$ by sending w to the product of its cover reflections, taken in the order indicated above:

$$\mathrm{nc}_c(w) = t_{j_1} \cdots t_{j_{\mathrm{des}(w)}}.$$

Continuing our running example of $w = s_2s_0s_1s_2s_0s_1s_0$, we have

$$\mathrm{nc}_c(w) = t_1t_7 = (s_2)(s_1s_0s_1).$$

This map is a bijection.

Theorem 12.4 (Reading). *The map* nc_c *is a bijection between c-sortable elements and the interval* $[e, c]$*. Moreover, if* w *is a c-sortable element with* k *descents, then* $\mathrm{nc}_c(w)$ *is an element of reflection length* k*. As a consequence,*

$$N(W, k) = |\{w \in \mathrm{Sort}(c) : \mathrm{des}(w) = k\}|.$$

In Table 12.8 we see all the c-sortable elements in B_3 with $c = s_2s_0s_1$, along with the corresponding element of $[e, c]$. The reflection factorization in the second column is the one given by the map nc_c. It is minimal with respect to reflection length, but not necessarily with respect to length.

We make the comment here, without justification, that the set $\mathrm{Sort}(c)$ forms a full rank sublattice of the shard intersection order, $\mathrm{Sh}(W)$. Then the map nc_c is not only a bijection, it is a poset isomorphism $(\mathrm{Sort}(c), \leq_{\mathrm{Sh}}) \to ([e, c], \leq_{\mathrm{Abs}})$. This correspondence is one of Reading's main motivations for studying the shard intersection order.

12.5 Root posets and *W*-nonnesting partitions

Another natural way in which the W-Catalan and W-Narayana numbers arise is by counting antichains in the *root poset* for the positive roots Π. The Coxeter number, and in fact all the degrees of the fundamental invariants can be deduced from this poset. The partial order on roots is defined as follows. For any positive roots β and γ in Π, we declare that $\beta \leq \gamma$ if and only if $\gamma - \beta$ is a nonnegative linear combination of simple roots.

Table 12.8 The c-sortable elements for $c = s_2 s_0 s_1$, along with their images under the map nc_c.

w	$\mathrm{des}(w) = \ell'(\mathrm{nc}_c(w))$	$\mathrm{nc}_c(w)$
e	0	e
s_2	1	(s_2)
s_0	1	(s_0)
s_1	1	(s_1)
$s_2 s_0$	2	$(s_2)(s_2 s_0 s_2)$
$s_2 s_1$	1	$(s_2 s_1 s_2)$
$s_0 s_1$	1	$(s_0 s_1 s_0)$
$s_2 s_0 s_1$	1	$(s_2 s_0 s_1 s_0 s_2)$
$s_2 s_1 \vert s_2$	2	$(s_2)(s_1)$
$s_0 s_1 \vert s_0$	1	$(s_1 s_0 s_1)$
$s_2 s_0 s_1 \vert s_2$	2	$(s_2)(s_0 s_1 s_0)$
$s_2 s_0 s_1 \vert s_0$	1	$(s_2 s_1 s_0 s_1 s_2)$
$s_0 s_1 \vert s_0 s_1$	2	$(s_0)(s_1)$
$s_2 s_0 s_1 \vert s_2 s_0$	2	$(s_0 s_1 s_0)(s_2 s_1 s_0 s_1 s_2)$
$s_2 s_0 s_1 \vert s_0 s_1$	2	$(s_2 s_0 s_2)(s_2 s_1 s_2)$
$s_2 s_0 s_1 \vert s_2 s_0 s_1$	1	$(s_1 s_0 s_1 s_2 s_1 s_0 s_1)$
$s_2 s_0 s_1 \vert s_2 s_0 s_1 \vert s_2$	2	$(s_0 s_1 s_0)(s_2 s_1 s_2)$
$s_2 s_0 s_1 \vert s_2 s_0 s_1 \vert s_2$	2	$(s_2)(s_1 s_0 s_1)$
$s_2 s_0 s_1 \vert s_2 s_0 s_1 \vert s_2 s_0$	2	$(s_2 s_1 s_2)(s_1 s_0 s_1)$
$s_2 s_0 s_1 \vert s_2 s_0 s_1 \vert s_2 s_0 s_1$	3	$(s_2)(s_2 s_0 s_2)(s_1)$

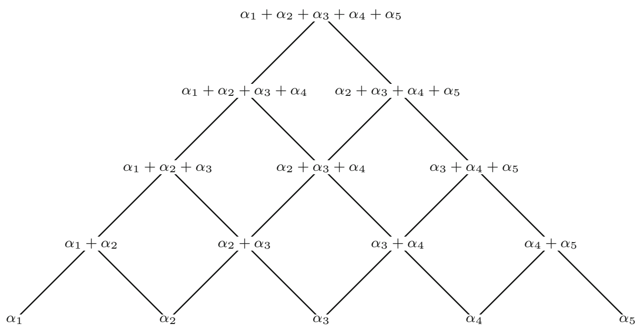


Fig. 12.2 The root poset for $\mathbf{A}_5$.

In Figures 12.2, 12.3, 12.4, and 12.5 we see the Hasse diagrams for the root posets of types $\mathbf{A}_5$, $\mathbf{B}_5$, $\mathbf{C}_5$, and $\mathbf{D}_5$.

The root poset is ranked. In the crystallographic case, the rank of $\beta = \sum_{\alpha \in \Delta} c_\alpha \alpha$ is $\mathrm{ht}(\beta) = -1 + \sum_{\alpha \in \Delta} c_\alpha$, known as the *height* of the root. For example, in $\mathbf{B}_5$, $\beta = 2\alpha_0 + \alpha_1 + \alpha_2$ has $\mathrm{ht}(\beta) = 4$ and rank 3. This statistic gives a simple way to compute the degrees of the fundamental invariants of the Coxeter group W.

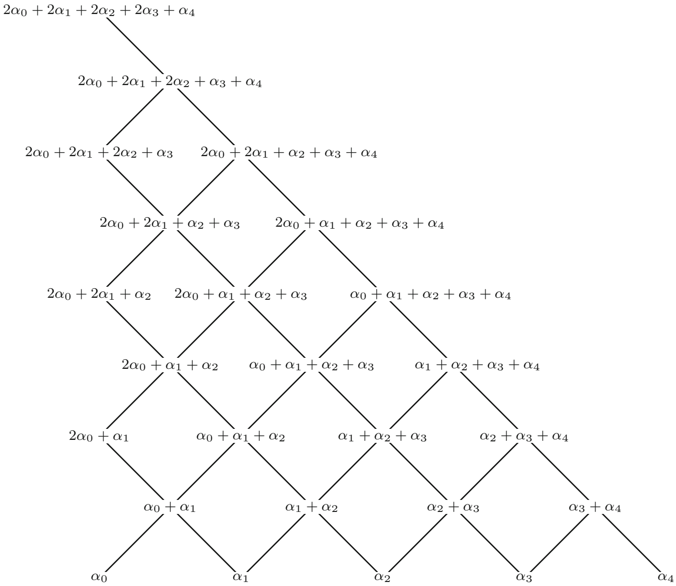


Fig. 12.3 The root poset for $\mathbf{B}_5$.

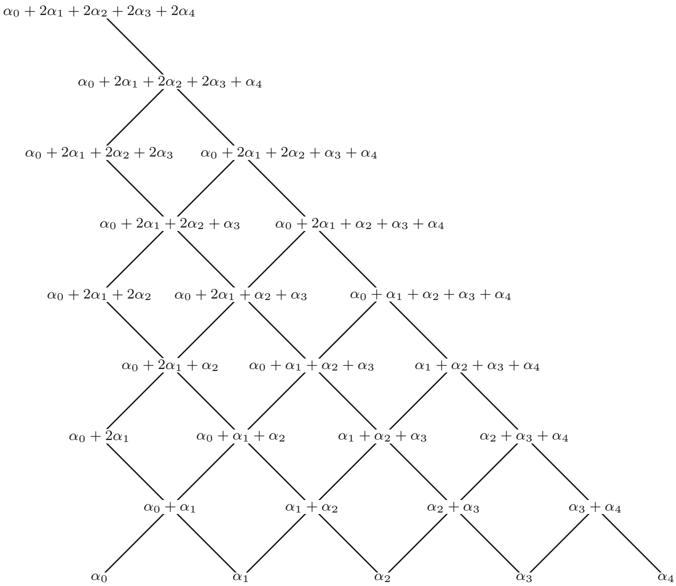


Fig. 12.4 The root poset for $\mathbf{C}_5$.

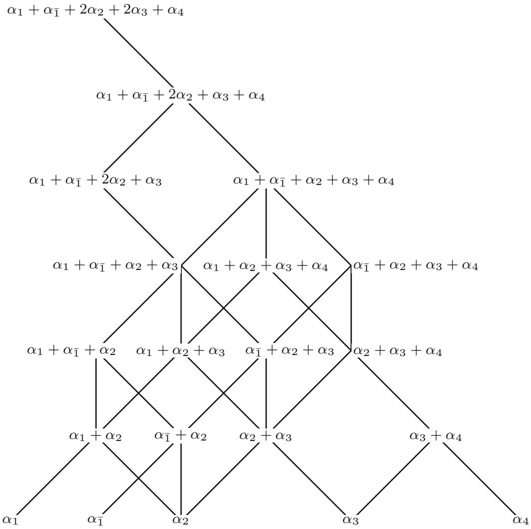


Fig. 12.5 The root poset for $\mathbf{D}_5$.

Fix a crystallographic root system Φ and let h_i denote the number of positive roots of height i. Draw an array of boxes with h_i boxes in row i (right justified). Now let e_i be the number of boxes in column i, read from left to right. These are known as the *exponents* of Φ, and remarkably, the degrees are obtained by adding one to each of the exponents: $d_i = e_i + 1$.

For example, in Figure 12.6 we have drawn the height arrays and computed the exponents for the rank 5 root systems.

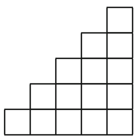
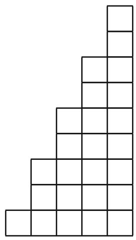
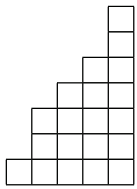
Root system:	$\mathbf{A}_5$	$\mathbf{B}_5/\mathbf{C}_5$	$\mathbf{D}_5$
Height array:			
Exponents:	$(1,2,3,4,5)$	$(1,3,5,7,9)$	$(1,3,4,5,7)$

Fig. 12.6 The height arrays for computing exponents.

The W-Catalan numbers emerge here from counting antichains A in the root poset. Recall that an antichain is a set of mutually incomparable elements. For example, the following are antichains in $\mathbf{D}_5$:

$$\{\alpha_1, \alpha_{\bar{1}}, \alpha_2 + \alpha_3, \alpha_4\}, \quad \{\alpha_1 + \alpha_{\bar{1}} + 2\alpha_2 + \alpha_3\},$$

and

$$\{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \alpha_{\bar{1}} + \alpha_2 + \alpha_3, \alpha_1 + \alpha_{\bar{1}} + \alpha_2\}.$$

These antichains are sometimes known as *W-nonnesting partitions*, and the set of all such antichains is denoted $\mathrm{NN}(W)$. The name comes from the fact that in the type $\mathbf{A}_{n-1}$ root poset, antichains can be turned into nonnesting partitions of $[n]$ as follows. Draw the numbers $1, 2, \ldots, n$ in the gaps between the simple roots. For each element of the antichain, draw diagonal lines of slope ± 1 from that root down to the numbers. In Figure 12.7 is an example of an antichain in $\mathbf{A}_7$, where we have circled the elements of the antichain and drawn the lines in gray. See Problem 12.11.

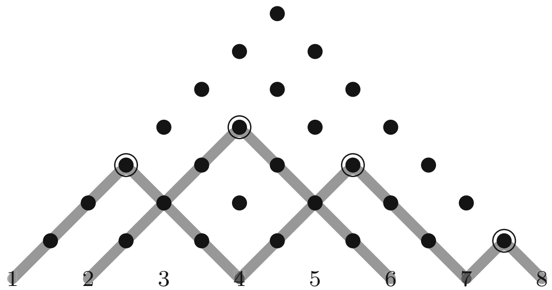


Fig. 12.7 The arc diagram for the nonnesting partition $\{\{1,4,7,8\},\{2,6\},\{3\},\{5\}\}$ corresponds to an antichain in the type $\mathbf{A}_7$ root poset.

Notice that the number of elements in the antichain equals n minus the number of blocks in the nonnesting partition. From Problem 2.7 we have a bijection between nonnesting and noncrossing partitions that preserves the number of blocks. Hence counting antichains in the $\mathbf{A}_{n-1}$ root poset according to cardinality gives the Narayana numbers. This fact generalizes as follows.

Theorem 12.5. *The number of W-nonnesting partitions, i.e., antichains A in the root poset of W, is given by the W-Catalan number:*

$$|\mathrm{NN}(W)| = \mathrm{Cat}(W).$$

Moreover, the number of antichains of cardinality k equals the W-Narayana number:

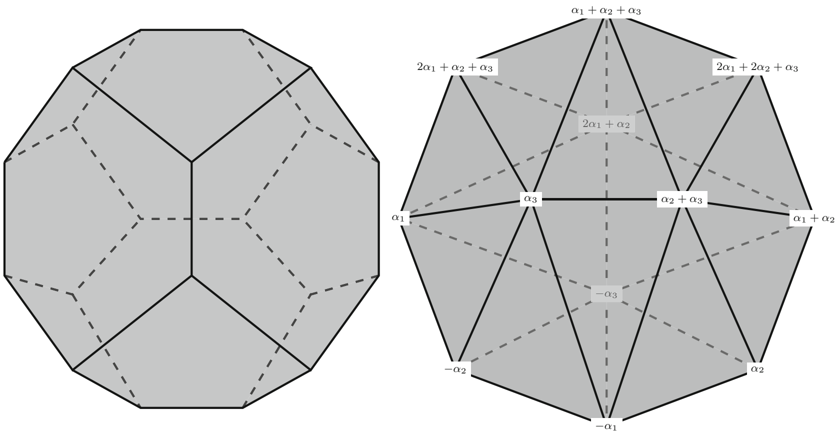
$$N(W, k) = |\{A \in \mathrm{NN}(W) : |A| = k\}|.$$

The known proofs of this theorem are case-by-case. There now exist bijections between nonnesting and noncrossing partitions of types A_n, B_n, D_n.

12.6 The *W*-associahedron

Yet another way that W-Narayana numbers arise comes from a generalization of the associahedron. Recall from Section 5.8 that the associahedron is a polytope whose one-skeleton is the Tamari lattice. This is a simple polytope, so we can also describe its dual simplicial complex. In Section 5.8 we see the faces of this complex are encoded by rooted planar trees, with Catalan-many facets encoded by the planar binary trees. Theorem 5.4 shows that the h-polynomial of the associahedron is the Narayana polynomial.

A similar story can be told for any finite Coxeter group W. That is, there is a simplicial complex whose h-polynomial is the W-Narayana polynomial. This complex is dual to a simple polytope that we will call the *W-associahedron*. We denote both the polytope and the complex by $\mathrm{Assoc}(W)$. (The context should make it clear to which object we are referring.) For example, in Figure 12.8 we see the associahedron of type B_3 and its dual simplicial complex. In general the type B_n associahedron $\mathrm{Assoc}(B_n)$ is known as the *cyclohedron*.



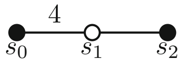
(a) (b)

Fig. 12.8 In (a) we have the B_3 associahedron, and in (b) its dual simplicial complex. The vertices of the complex are labeled with the almost positive roots in $\mathbf{B}_3$.

The construction of the W-associahedron is as follows. For the root system Φ, define the *almost positive roots* to be the set $\Phi_{\geq -1} = \Pi \cup -\Delta$, i.e., the set of positive roots together with the negatives of the simple roots. These roots will be the vertex set for a graph defined by a certain symmetric relation, $\alpha \sim \beta$, and $\mathrm{Assoc}(W)$ will be the clique complex for this graph.

We will explain the general construction through careful examination of the B_3 case.

First, we give the Coxeter graph for W a balanced coloring. (Every irreducible graph is bipartite, so this is always possible.) For example, take the following two coloring of the B_3 graph:



.

We now define two involutions $\tau_\bullet$ and $\tau_\circ$ on $\Phi_{\geq -1}$.

$$\tau_\bullet(\alpha) = \begin{cases} \alpha & \text{if } \alpha \in -\Delta \text{ and } s_\alpha \text{ is white,} \\ \prod_{\text{black nodes } i} s_i(\alpha) & \text{otherwise.} \end{cases}$$

$$\tau_\circ(\alpha) = \begin{cases} \alpha & \text{if } \alpha \in -\Delta \text{ and } s_\alpha \text{ is black,} \\ \prod_{\text{white nodes } j} s_j(\alpha) & \text{otherwise.} \end{cases}$$

Roughly, these involutions act as the product of the black nodes and the product of the white nodes, respectively. The exception comes in the case when one of the involutions acts on a negative simple root of the opposite color, in which case the involution takes the root to itself.

In our running example of B_3,

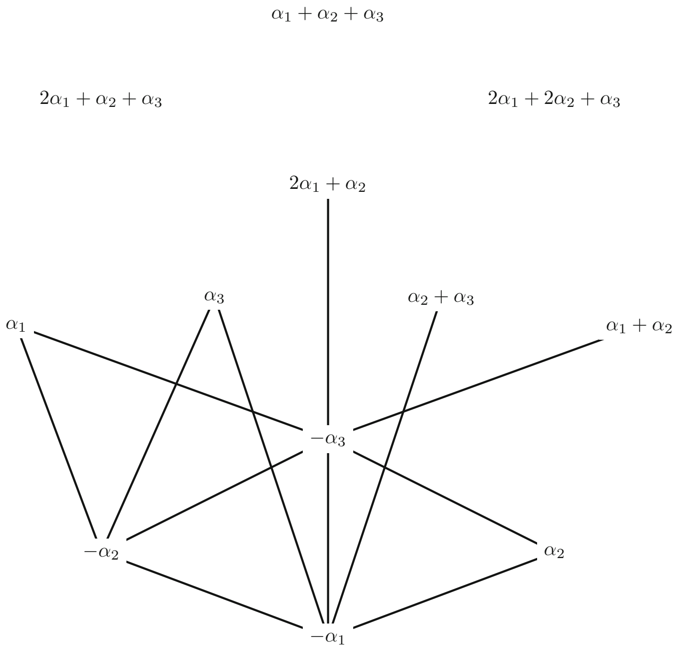
$$\tau_\bullet(\alpha) = \begin{cases} \alpha & \text{if } \alpha = -\alpha_1, \\ s_0 s_2(\alpha) & \text{otherwise,} \end{cases}$$

$$\tau_\circ(\alpha) = \begin{cases} \alpha & \text{if } \alpha = -\alpha_0 \text{ or } \alpha = -\alpha_2, \\ s_1(\alpha) & \text{otherwise.} \end{cases}$$

We can display the orbits of the group generated by $\tau_\bullet$ and $\tau_\circ$ as follows:

$$\begin{tikzcd}
-\alpha_1 \arrow[loop above, "\tau_\circ"] \arrow[r, leftrightarrow, "\tau_\bullet"] & \alpha_1 \arrow[r, leftrightarrow, "\tau_\circ"] & \alpha_1+\alpha_2 \arrow[r, leftrightarrow, "\tau_\bullet"] & \alpha_1+\alpha_2+\alpha_3 \arrow[loop above, "\tau_\circ"] \\
-\alpha_2 \arrow[loop above, "\tau_\bullet"] \arrow[r, leftrightarrow, "\tau_\circ"] & \alpha_2 \arrow[r, leftrightarrow, "\tau_\bullet"] & 2\alpha_1+\alpha_2+\alpha_3 \arrow[r, leftrightarrow, "\tau_\circ"] & 2\alpha_1+2\alpha_2+\alpha_3 \arrow[loop above, "\tau_\bullet"] \\
-\alpha_3 \arrow[loop above, "\tau_\circ"] \arrow[r, leftrightarrow, "\tau_\bullet"] & \alpha_2 \arrow[r, leftrightarrow, "\tau_\circ"] & \alpha_2+\alpha_3 \arrow[r, leftrightarrow, "\tau_\bullet"] & 2\alpha_1+\alpha_2 \arrow[loop above, "\tau_\circ"]
\end{tikzcd}$$

Now we must build the graph on $\Phi_{\geq -1}$ whose clique complex is $\mathrm{Assoc}(W)$. We do this in two stages. First for each $\alpha \in \Delta$, we connect $-\alpha$ to every $\beta \in \Phi_{\geq -1}$ such that β and α are incomparable in the root poset. In the B_3 case, this gives this much of the graph:



From here, we construct the rest of the graph by acting on edges with $\tau_\bullet$ and $\tau_\circ$. That is, if we have an edge between α and β, there must also be edges between $\tau_\bullet(\alpha)$ and $\tau_\bullet(\beta)$, and between $\tau_\circ(\alpha)$ and $\tau_\circ(\beta)$. For example, since $-\alpha_3$ and $2\alpha_1+\alpha_2$ are connected, we must also have edges between $\tau_\bullet(-\alpha_3) = \alpha_3$ and $\tau_\bullet(2\alpha_1+\alpha_2) = \alpha_2+\alpha_3$. The reader is invited to recover the rest of the edges in Figure 12.8 for themselves.

The following theorem marks the final occurrence of the W-Narayana numbers we will mention.

Theorem 12.6. *The h-polynomial of the simplicial complex dual to the W-associahedron is the W-Narayana polynomial:*

$$h(\mathrm{Assoc}(W); t) = \mathrm{Cat}(W; t).$$

Notes

The subject of "Coxeter-Catalan combinatorics" is a vast and growing subject. We have only scratched the surface here. The reader is invited to seek more information in Chapter 13 and the references.

Proposition 12.1, that reflection length equals the codimension of the fixed point space, was proved by Roger Carter in 1972 [45]. The factorization of the reflection length generating function was first described by Geoffrey Shepard and John Todd in 1954 [137] and given a case-free proof by Louis Solomon in 1963 [141].

The definition of the W-noncrossing partition lattice as an interval in the absolute order is due to Thomas Brady and Colum Watt in 2002 [27]. The connection with antichains in the root poset was remarked by Victor Reiner in his paper studying the type B_n version of noncrossing partitions [126], where he attributes the idea to Alexander Postnikov. These ideas can be generalized as described in Drew Armstrong's 2009 manuscript [10].

The real-rootedness of the W-Narayana polynomials claimed in Theorem 12.3 follows from work of Petter Brändén in 2006 [30]. The weaker property of gamma-nonnegativity was known to Rodica Simion and Daniel Ullmann in the classical case in 1991 [140], and Patricia Hersh proved it in the type B_n case in 1999 [90]. In fact, both $\mathrm{NC}(A_n)$ and $\mathrm{NC}(B_n)$ admit symmetric boolean decompositions. As of this writing it is an open question whether all W-noncrossing partition lattices admit such a decomposition. Such a decomposition is preserved under cartesian products, so the result would follow from a proof in the type D_n case. It has also been conjectured by the author that Nathan Reading's shard intersection order admits a symmetric boolean decomposition, but type A_{n-1} is the only case so far proved.

The notion of Coxeter-sortable elements, and in particular Theorem 12.4 can be found in Reading's 2007 paper [123].

The simplicial version of the W-associahedron was first studied, in the crystallographic case, by Sergey Fomin and Andrei Zelevinsky, where they arose in the theory of cluster algebras [74]. There is a wonderful survey by Fomin and Reading from 2007 that discusses the many interesting and deep connections between the associahedron and other parts of mathematics [73].

Problems

12.1. Show that every reflection is conjugate to a simple reflection.

12.2. Prove Proposition 12.1.

12.3. Prove that if c and c' are Coxeter elements, then $c' = wcw^{-1}$ for some w.

12.4. Prove Theorem 12.1 in the case of the dihedral group $I_2(m)$.

12.5. Show that if c and c' are Coxeter elements, then the intervals $[e, c]$ and $[e, c']$ are isomorphic.

12.6. Prove that each orientation of the edges of the Coxeter graph gives a unique Coxeter element c by declaring that if $s_i \to s_j$ then s_i appears to the left of s_j in any reduced expression for c.

12.7. Prove the Narayana polynomial for $I_2(m)$ is $1 + mt + t^2$.

12.8. Prove Proposition 12.2, that $[e, c]$ is self-dual, via the anti-isomorphism $w \mapsto w^{-1}c$.

12.9. Show that if $W \cong U \times V$ is reducible, then $\mathrm{NC}(W) \cong \mathrm{NC}(U) \times \mathrm{NC}(V)$. Conclude therefore that $\mathrm{Cat}(W;t) = \mathrm{Cat}(U;t)\,\mathrm{Cat}(V;t)$.

12.10. Prove that in S_n, the c-sortable permutations with $c = s_{n-1} \cdots s_2 s_1$ are precisely the 231-avoiding (i.e., stack sortable) permutations.

12.11. Show that the antichains in the root lattice of type $\mathbf{A}_{n-1}$ are in bijection with nonnesting partitions of $[n]$ (see Problem 2.7 for that definition).

Chapter 13
Combinatorics for Coxeter groups of types B_n and D_n (Supplemental)

13.1 Type B_n Eulerian numbers

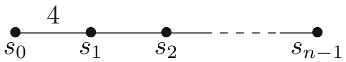
In Section 11.5.2 we saw that the group B_n is isomorphic to the set of all *signed permutations*. These are permutations

$$w : \{-n, \ldots, -1, 0, 1, \ldots, n\} \to \{-n, \ldots, -1, 0, 1, \ldots, n\},$$

such that $w(-i) = -w(i)$ for all i. Notice that this forces $w(0) = 0$ and the element w is completely determined by $w(1), \ldots, w(n)$. In one-line notation, we write $w = w(1) \cdots w(n)$ with bars to indicate negative numbers. For example, if w is determined by $w(1) = -3$, $w(2) = 4$, $w(3) = 5$, $w(4) = -1$ and $w(5) = 2$, we write $w = \bar{3}45\bar{1}2$.

Now we choose the generating set S. For $i \geq 1$ we let s_i denote the permutation that swaps i and $i+1$, and by symmetry $-i$ and $-(i+1)$. (These are analogous to the generators for the symmetric group.) We also let s_0 denote the permutation that swaps 1 and -1.

It is straightforward to verify that the elements of $S = \{s_0, s_1, \ldots, s_{n-1}\}$ satisfy the relations for a type B_n Coxeter system, with graph



.

That is, the relations for these generators satisfy

- $s_i^2 = e$,
- $(s_i s_{i+1})^3 = e$ for $i = 1, \ldots, n-2$,
- $(s_0 s_1)^4 = e$, and
- $(s_i s_j)^2 = e$ if $|i - j| > 1$.

We now wish to come up combinatorial analogues of inversion number and descent number.

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Recall from Section 11.5.2 that there are three kinds of positive type $\mathbf{B}_n$ roots. They are:

- $\varepsilon_j - \varepsilon_i$, with $1 \leq i < j \leq n$,
- $\varepsilon_j + \varepsilon_i$, with $1 \leq i < j \leq n$, and
- ε_i, with $1 \leq i \leq n$.

Thus by considering the action of B_n given by $w \cdot \varepsilon_i = \varepsilon_{w(i)}$, we can characterize precisely those roots in the inversion set, and the type B inversion number can be defined as follows:

$$\begin{aligned} \mathrm{inv}_B(w) = &|\{1 \leq i < j \leq n : w(i) > w(j)\}| \\ &+ |\{1 \leq i < j \leq n : -w(i) > w(j)\}| \\ &+ |\{1 \leq i \leq n : w(i) < 0\}|. \end{aligned}$$

By considering only the inversions corresponding to the simple roots, i.e.,

- $\varepsilon_{i+1} - \varepsilon_i$, with $1 \leq i \leq n-1$, and
- ε_1,

we see the combinatorial description of the descent set is:

$$\mathrm{Des}_B(w) = \{0 \leq i \leq n-1 : w(i) > w(i+1)\}.$$

We denote the number of descents by $\mathrm{des}_B(w) = |\mathrm{Des}_B(w)|$. Notice that the only difference from the definition for unsigned permutations is that we have a descent in position 0 if $w(1) < 0$.

With the example of $w = \bar{3}14\bar{2}5$, we have $\mathrm{inv}_B(w) = 2+3+2 = 7$ and $\mathrm{des}_B(w) = 1+1 = 2$, while if $v = 5\bar{3}\bar{2}1\bar{4}$ we have $\mathrm{inv}_B(v) = 7+6+3 = 16$ and $\mathrm{des}_B(v) = 2+0 = 2$. In Table 13.1 we have the reduced words and descent sets for the elements of B_2.

Using the model of signed permutations, it is not too difficult to deduce Theorem 11.1 in this case with an induction argument (insert n or $-n$ in all possible ways).

Theorem 13.1 (B_n-Mahonian distribution). *For any $n \geq 1$,*

$$\begin{aligned} \sum_{w \in B_n} q^{\mathrm{inv}_B(w)} &= (1+q)(1+q+q^2+q^3)\cdots(1+q+\cdots+q^{2n-1}), \\ &= [2][4]\cdots[2n]. \end{aligned}$$

We next collect a miscellany of results for the type B_n Eulerian numbers. Each fact can be proved with a bijection or else deduced from the others.

Theorem 13.2 (A miscellany for B_n-Eulerian numbers). *For any $n \geq 1$,*

$$B_n(t) = (1+t)B_{n-1}(t) + 2t\sum_{i=1}^{n-1} 2^i \binom{n-1}{i} B_{n-1-i}(t)S_i(t), \tag{13.1}$$

$$= (1+(2n-1)t)B_{n-1}(t) + 2t(1-t)B'_{n-1}(t), \tag{13.2}$$

Table 13.1 The reduced expressions for elements of B_2, along with length and descent sets.

w	reduced expressions	$\ell(w) = \mathrm{inv}_B(w)$	$\mathrm{Des}_B(w)$
12	e	0	$\emptyset$
$\bar{1}2$	s_0	1	$\{0\}$
21	s_1	1	$\{1\}$
$2\bar{1}$	$s_0 s_1$	2	$\{1\}$
$\bar{2}1$	$s_1 s_0$	2	$\{0\}$
$\bar{2}\bar{1}$	$s_0 s_1 s_0$	3	$\{0\}$
$1\bar{2}$	$s_1 s_0 s_1$	3	$\{1\}$
$\bar{1}\bar{2}$	$s_0 s_1 s_0 s_1, s_1 s_0 s_1 s_0$	4	$\{0, 1\}$

$$\frac{B_n(t)}{(1-t)^{n+1}} = \sum_{k\geq 0}(2k+1)^n t^k, \tag{13.3}$$

$$(2k+1)^n = \sum_{i=0}^{n} \left\langle {B_n \atop i} \right\rangle \binom{k+n-i}{n}, \tag{13.4}$$

$$2B_n(t^2) = (1+t)^{n+1} S_n(t) + (1-t)^{n+1} S_n(-t). \tag{13.5}$$

Equations (13.1) and (13.2) follow from straightforward bijective arguments for signed permutations. Equation (13.3) can be deduced from (13.2), though it can also be proved with a "signed" P-partition argument. (See work of Chak-On Chow [49] and the author [116].) Equations (13.4) and (13.5) follow directly from (13.3).

From the identity in (13.5), we can deduce an exponential generating function for B_n-Eulerian numbers from the classical case (Theorem 1.6). Let

$$B(t,z) := \sum_{n\geq 0} B_n(t)\frac{z^n}{n!}.$$

Recalling from Equation (1.13)

$$S(t,z) = \sum_{n\geq 0} S_n(t)\frac{z^n}{n!} = \frac{t-1}{t-e^{z(t-1)}},$$

we get

$$\begin{aligned}
2B(t^2,z) &= \sum_{n\geq 0} 2B_n(t^2)\frac{z^n}{n!},\\
&= (1+t)\sum_{n\geq 0} S_n(t)\frac{(1+t)^n z^n}{n!} + (1-t)\sum_{n\geq 0} S_n(-t)\frac{(1-t)^n z^n}{n!},\\
&= (1+t)S(t,z(1+t)) + (1-t)S(-t,z(1-t)),\\
&= \frac{2(t^2-1)e^{z(t^2-1)}}{t^2-e^{2z(t^2-1)}}.
\end{aligned}$$

Let us record this explicit formula for $B(t,z)$, first due to Francesco Brenti in 1994 [34].

Theorem 13.3. *We have*

$$B(t,z) = \frac{(t-1)e^{z(t-1)}}{t-e^{2z(t-1)}}. \tag{13.6}$$

In 1995 Victor Reiner described a generating function for the Euler-Mahonian distribution as well [125]. Recall the q-analogue of the exponential function

$$\exp(z;q) = \sum_{n\geq 0} \frac{z^n}{[n]!},$$

and define

$$\exp_B(z;q) = \sum_{n\geq 0} \frac{z^n}{B_n(q,1)} = \sum_{n\geq 0} \frac{z^n}{[2][4]\cdots[2n]}.$$

Then we have the following result. Compare it with Theorem 6.8.

Theorem 13.4. *We have the following generating function for the B_n-Euler-Mahonian polynomials:*

$$\begin{aligned}
\sum_{n\geq 0} B_n(q,t)\frac{z^n}{B_n(q,1)} &= \sum_{n\geq 0} \frac{B_n(q,t)z^n}{[2][4]\cdots[2n]},\\
&= \frac{(1-t)\exp_B(z(1-t);q)}{1-t\exp(z(1-t);q)}.
\end{aligned}$$

This result generalizes Equation 13.6 since at $q=1$, this will give $B(t,z/2)$.

13.2 Type B_n gamma-nonnegativity

Our goal here is to give a combinatorial reason why the B_n-Eulerian polynomials are gamma-nonnegative. The gamma numbers for B_n-Eulerian polynomials are given in Table 13.2. We will adapt the argument from work of John Stembridge from 2008 [159] (also found in work of Kevin Dilks, Stembridge, and the author [60]), though the combinatorial description first appeared in the author's 2007 paper [116]. As in the classical case, the idea is to use a simple combinatorial action on permutations, but the action here is quite different from valley hopping as given in Section 4.2.

Table 13.2 The gamma numbers $\gamma_j^{B_n}$, $0 \le 2j \le n \le 9$.

$n\backslash j$	0	1	2	3	4
2	1	4			
3	1	20			
4	1	72	80		
5	1	232	976		
6	1	716	7664	3904	
7	1	2172	49776	88640	
8	1	6544	292320	1217792	354560
9	1	19664	1618656	13201664	12933376

Fix an unsigned permutation $u \in S_n$, and consider the set of all signed permutations obtained by changing signs on letters of u, i.e., those $w \in B_n$ such that $|w(1)||w(2)|\cdots|w(n)| = u$. Let $B(u)$ denote the set of all 2^n such permutations. For example, if $u = 312$, the eight elements of $B(u)$, together with their type B descent numbers, are listed here:

w	$\mathrm{des}_B(w)$
312	1
$\bar{3}12$	1
$3\bar{1}2$	1
$31\bar{2}$	2
$\bar{3}\bar{1}2$	1
$\bar{3}1\bar{2}$	2
$3\bar{1}\bar{2}$	2
$\bar{3}\bar{1}\bar{2}$	2

.

The generating function for des_B over these eight elements is $4t(1+t)$. We will see that the distribution of des_B over elements in $B(u)$ is gamma-nonnegative for any u and depends only on what we call the number of *left peaks* of u.

The key idea is that, once the peaks and valleys of u have been identified, the descents of an element $w \in B(u)$ are determined independently by the sign choices of those elements not lying in a valley or at a peak.

Specifically, let $w \in B(u)$ and write $w(i) = \sigma_i u(i)$, so that $\sigma_i \in \{-,+\}$ is the sign of the ith letter of w. Then we can make the following observations:

- if $u(i-1) < u(i)$, then $w(i-1) > w(i)$ if and only if $\sigma_i = -$,
- if $u(i-1) > u(i)$, then $w(i-1) > w(i)$ if and only if $\sigma_{i-1} = +$.

To put it another way, the sign σ_j controls the descent in position $j-1$ if and only if $j-1$ is *not* a descent of u, and it controls the descent in position j if and only if j *is* a descent of u.

Consider the example of $u = 31472865$. Then there is a descent in position 0 if and only if $\sigma_1 = -$ while there is a descent in position 1 if and only if $\sigma_1 = +$. Since $u(2) = 1$ is smaller than the elements on either side of it, the sign σ_2 has no effect whatever on the descent set. With $u(3) = 4$, we find that $w(2) > w(3)$ if and only if $\sigma_3 = -$, but that σ_3 does not control whether $w(3)$ is greater than $w(4)$ (σ_4 does that). By considering the sign of each letter in turn, we have:

$$\sum_{w \in B(u)} t^{\mathrm{des}_B(w)} = (t{+}t)(1{+}1)(t{+}1)(t{+}t)(1{+}1)(t{+}t)(1{+}t)(1{+}1) = 64t^3(1{+}t)^2.$$

See Figure 13.1.

To summarize, let

$$c_j(u) = \begin{cases} 2t & \text{if } u(j-1) < u(j) > u(j+1), \\ 2 & \text{if } u(j-1) > u(j) < u(j+1), \\ 1+t & \text{otherwise}, \end{cases}$$

where $u(0) = 0$ and $u(n+1) = n+1$. Then we have

$$\sum_{w \in B(u)} t^{\mathrm{des}_B(w)} = c_1(u) \cdots c_n(u).$$

Define the *number of left peaks* of a permutation $u \in S_n$ to be

$$\mathrm{lpk}(u) = |\{1 \le i < n : u(i-1) < u(i) > u(i+1)\}|,$$

where $u(0) = 0$. In other words,

$$\mathrm{lpk}(u) = \begin{cases} \mathrm{pk}(u) & \text{if } u(1) < u(2), \\ 1 + \mathrm{pk}(u) & \text{if } u(1) > u(2). \end{cases}$$

Thus each left peak contributes a factor of $2t$, each valley contributes a factor of 2, and all other positions contribute $(1+t)$. If u has k left peaks, it has k valleys (there is no valley to the left of the first left peak), and hence $n-2k$ positions that are neither left peaks nor valleys. We have

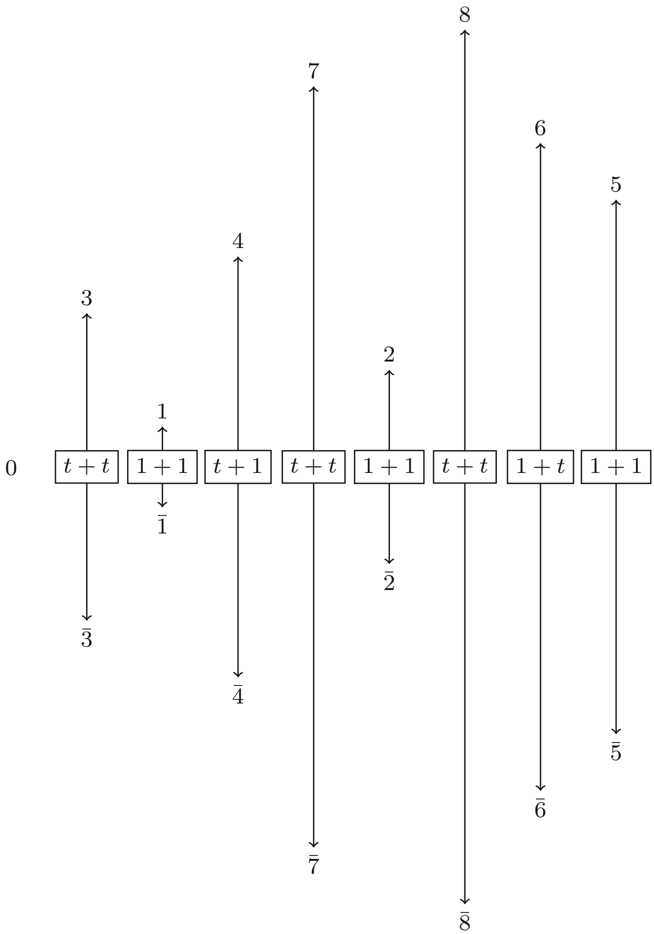


Fig. 13.1 The choices of sign independently control descents, depending on the local shape of u. Peaks have weight $t + t$, valleys have weight $1 + 1$, and other positions have weight $1 + t$.

$$\sum_{w \in B(u)} t^{\operatorname{des}_B(w)} = (4t)^{\operatorname{lpk}(u)}(1+t)^{n-2\operatorname{lpk}(u)}.$$

Summing over all $u \in S_n$ gives the desired gamma-nonnegativity result.

Theorem 13.5. *For all $n \geq 1$, there exist nonnegative integers $\gamma_j^{B_n}$ such that*

$$B_n(t) = \sum_{j=0}^{\lfloor n/2 \rfloor} \gamma_j^{B_n} t^j (1+t)^{n-2j}. \tag{13.7}$$

Moreover,

$$\gamma_j^{B_n} = 4^j \cdot |\{u \in S_n : \mathrm{lpk}(u) = j\}|.$$

This result suggests the distribution of left peaks in the symmetric group should be of interest in its own right. Define

$$\overline{P}_n(t) = \sum_{u \in S_n} t^{\mathrm{lpk}(u)} = \sum_{k=0}^{\lfloor n/2 \rfloor} \overline{p}_{n,k} t^k.$$

Then

$$B_n(t) = (1+t)^n \overline{P}_n\left(\frac{4t}{(1+t)^2}\right).$$

The left peak numbers for small n are given in Table 13.3.

Table 13.3 The left peak numbers $\overline{p}_{n,k}$, $0 \le 2k \le n \le 9$.

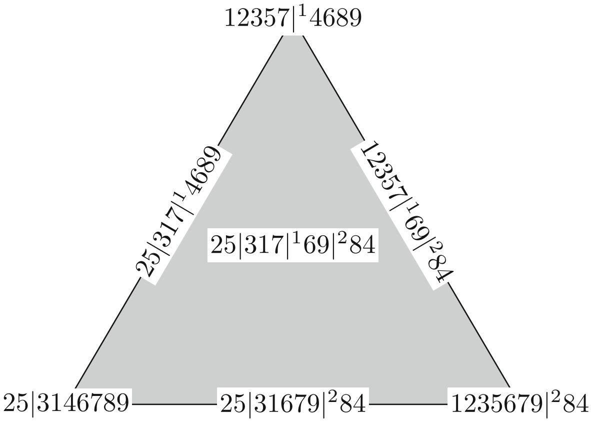
$n\backslash k$	0	1	2	3	4
1	1				
2	1	1			
3	1	5			
4	1	18	5		
5	1	58	61		
6	1	179	479	61	
7	1	543	3111	1385	
8	1	1636	18270	19028	1385
9	1	4916	101166	206276	50521

Given this combinatorial description for the gamma vector of the type B_n Coxeter complex, we can establish Conjecture 3 in the case of the B_n Coxeter complex as follows.

Define the set of *decorated permutations*, Dec_n, to be the set of all permutations $w \in S_n$ with bars in the left peak positions. The bars can come in one of four styles: $\{| = |^0, |^1, |^2, |^3\}$, and thus for each $w \in S_n$ we have $4^{\mathrm{lpk}(w)}$ decorated permutations. For example, here are three elements of Dec_9:

$$4|238|^1 76519, \quad 4|^3 238|^2 76519, \quad 25|137|^1 69|^2 84.$$

We can construct a balanced simplicial complex from Dec_n by declaring that a decorated permutation G covers F if we can remove a bar from G to obtain F. However, we don't write the merged blocks in strictly increasing order. If we remove the bar between words w_i and w_{i+1} of G, we keep the decreasing part of w_i as is, and rewrite w_{i+1}, together with the increasing part of w_i, in increasing order. (The increasing part begins with the smallest element of the word. The decreasing part can be empty.) For example, $9|76514|^2 238$ is an edge with vertices $9|76512348$ and $145679|^2 238$. Here is a triangle in the B_9 case:



We color the vertices similarly to how we did in the case of type A_{n-1}. A bar (of any style) receives color i if it occurs in position $2i$ or $2i-1$. Since peaks cannot be consecutive, this guarantees that each face has a distinctly colored vertex set.

13.3 Type D_n Eulerian numbers

The Coxeter group of type D_n is the subgroup of B_n consisting of those signed permutations with an even number of minus signs among $w(1), \ldots, w(n)$. For example, $35\bar{1}2\bar{4}$ is an element of D_n, while $35\bar{1}24$ is an element of B_n but not D_n. When we write an element of D_n as a word $w = w(1)w(2)\cdots w(n)$, we refer to it as an *even signed permutation*. This is a good model to have, but another useful model is to write w as a *forked signed permutation* as follows:

$$w = w(\bar{n})\cdots w(\bar{2})\,\begin{matrix} w(1) \\ w(\bar{1}) \end{matrix}\,w(2)\cdots w(n),$$

or simply

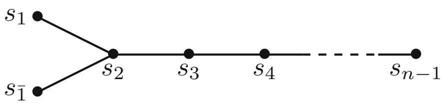
$$\begin{matrix} w(1) \\ w(\bar{1}) \end{matrix}\,w(2)\cdots w(n).$$

For example, the even signed permutation $w = 35\bar{1}2\bar{4}$ corresponds to the following forked signed permutation:

$$w = 4\bar{2}1\bar{5}\,\begin{matrix} 3 \\ \bar{3} \end{matrix}\,5\bar{1}2\bar{4}.$$

The model of forked signed permutations is particularly helpful in understanding the action of the generating set S. We let $s_1, \ldots, s_{n-1}$ be the adjacent transpositions defined as they were for A_{n-1} and B_n, but we add the generator $s_{\bar{1}}$ that swaps 1 with -2 and -1 with 2.

It is straightforward to verify that the set $S = \{s_{\bar{1}}, s_1, s_2, \ldots, s_{n-1}\}$ generate a type D_n Coxeter system, with graph:



.

In other words, the elements of S satisfy the following relations:

- $s_i^2 = e$ for $i = \bar{1}, 1, 2, \ldots, n-1$,
- $(s_{\bar{1}} s_2)^3 = e$ and $(s_i s_{i+1})^3 = e$ for $i = 1, 2, \ldots, n-2$, and
- $(s_i s_j)^2 = e$ otherwise.

The root system $\mathbf{D}_n$ is the same as the $\mathbf{B}_n$ system, except it does not contain the standard basis vectors ε_i. That is, the positive roots are:

- $\varepsilon_j - \varepsilon_i$, with $1 \le i < j \le n$, and
- $\varepsilon_j + \varepsilon_i$, with $1 \le i < j \le n$.

The group D_n acts on the standard basis vectors in the same way as B_n, so we define the type D inversion number to be:

$$\begin{aligned} \mathrm{inv}_D(w) = |\{1 \le i < j : w(i) > w(j)\}| \\ + |\{1 \le i < j : -w(i) > w(j)\}|. \end{aligned}$$

The simple roots in this case are:

- $\varepsilon_{i+1} - \varepsilon_i$, with $1 \le i \le n-1$, and
- $\varepsilon_1 + \varepsilon_2$,

so the type D descent set is:

$$\mathrm{Des}_D(w) = \{i \in \{-1, 1, 2, \ldots, n-1\} : w(i) > w(|i|+1)\}.$$

Let $\mathrm{des}_D(w) = |\mathrm{Des}_D(w)|$ denote the number of such descents. Notice that this definition is quite natural with the model of forked permutations.

We check for $w = \begin{smallmatrix}\bar{3}\\3\end{smallmatrix}5\bar{1}42$ that $\mathrm{inv}_D(w) = 4 + 2 = 6$, while $\mathrm{des}_D(w) = 2$. If $v = \begin{smallmatrix}2\\2\end{smallmatrix}\bar{4}13\bar{5}$, then $\mathrm{inv}_D(v) = 6 + 7 = 13$ and $\mathrm{des}_D(v) = 3$.

Using the model of even-signed permutations, it is not too difficult to prove Theorem 11.1 in this case with an induction argument. Take any signed permutation of length $n-1$ and add n or $-n$ in all possible ways. The only subtlety is that the parity of the number of minus signs determines whether we insert n or $-n$.

Theorem 13.6 (D_n-Mahonian distribution). *For any $n \geq 4$,*

$$\sum_{w \in D_n} q^{\mathrm{inv}_D(w)} = (1+q)\cdots(1+q+\cdots+q^{2n-3})(1+q+\cdots+q^{n-1}),$$
$$= [2][4]\cdots[2n-2][n].$$

The type D_n Eulerian polynomials have the following exponential generating function, which, like Theorem 13.3, is due to Brenti [34].

Theorem 13.7. *We have*

$$1 + tz + \sum_{n\geq 2} D_n(t)\frac{z^n}{n!} = \frac{(1-t)(e^{z(1-t)}-z)}{1-te^{2z(1-t)}}. \tag{13.8}$$

There is also an Euler-Mahonian generalization, which (as with type B_n) is due to Reiner [125]. First, define

$$\exp_D(z;q) = \sum_{n\geq 0} \frac{z^n}{D_n(q,1)} = \sum_{n\geq 0}\frac{z^n}{[2][4]\cdots[2n-2][n]}.$$

Then we have the following result. Compare it with Theorem 6.8.

Theorem 13.8. *We have the following generating function for the D_n-Euler-Mahonian polynomials:*

$$2tz + \sum_{n\geq 2} D_n(q,t)\frac{z^n}{D_n(q,1)} = \frac{(1-t)\exp_D(z(1-t);q) + t(2-tz)(\exp(z(1-t);q)-1)}{1-t\exp(z(1-t);q)}.$$

13.4 Type D_n gamma-nonnegativity

Our goal here is to give a combinatorial reason why the type D_n-Eulerian polynomials are gamma-nonnegative. The corresponding gamma numbers are shown in Table 13.4. As with type B_n, our argument is adapted from Stembridge's from 2008 paper [159] (also found in [60] and Chak-On Chow's 2008 paper [50]). It is worth remarking that real-rootedness of the type D_n Eulerian polynomials was first proved in 2015 by Carla Savage and Mirkó Visontai [132], so for some time, the combinatorial explanation was the only explanation for gamma-nonnegativity here.

To begin, notice that counting descents of type D makes sense for any signed permutation, not simply the elements of D_n. Moreover, changing the sign of $w(1)$ moves between D_n and $B_n \setminus D_n$ and leaves the type D descent set unchanged. That is, if $w = w(1)w(2)\cdots w(n)$ is an element of D_n, then $w' = \overline{w(1)}w(2)\cdots w(n)$ is an element of $B_n \setminus D_n$ such that $\mathrm{Des}_D(w) = \mathrm{Des}_D(w')$. As a consequence of this observation, we can count type D descents over all signed permutations and then divide by two to obtain the D_n-Eulerian polynomial. That is, we have

$$D_n(t) = \frac{1}{2} \sum_{w \in B_n} t^{\mathrm{des}_D(w)}.$$

Table 13.4 The gamma numbers $\gamma_j^{D_n}$, $0 \le 2j \le n \le 9$.

$n \backslash j$	0	1	2	3	4
4	1	40	16		
5	1	152	336		
6	1	524	3440	832	
7	1	1724	26480	27712	
8	1	5520	175584	480512	76032
9	1	17360	1065696	6123776	3791104

To prove gamma-nonnegativity, we will consider the same combinatorial action as we did for type B_n, only this time keeping track of the effect on type D descents. As earlier, we let $B(u)$ denote the set of all signed permutations w such that $|w(1)||w(2)| \cdots |w(n)| = u$ in S_n. We can write $w(i) = \sigma_i u(i)$ as well, and consider the effect that choosing σ_i to be positive or negative has on the descent set. For $j > 2$, σ_j impacts the descent set exactly as it does for type B. Letting $c_j(u)$ denote the contribution of the sign σ_j as before, we have, for $j \ge 3$,

$$c_j(u) = \begin{cases} 2t & \text{if } u(j-1) < u(j) > u(j+1), \\ 2 & \text{if } u(j-1) > u(j) < u(j+1), \\ 1+t & \text{otherwise.} \end{cases}$$

For $j = 1, 2$, we will be more careful. Recall that $1 \in \mathrm{Des}_D(w)$ if and only if $w(1) > w(2)$, while $-1 \in \mathrm{Des}_D(w)$ if and only if $-w(1) > w(2)$. Hence, changing the sign of $w(1)$ will not change the cardinality of $\mathrm{Des}_D(w) \cap \{-1, 1\}$. If $u(1) > u(2)$, then exactly one of -1 or 1 is a descent, regardless of the signs σ_1 and σ_2. We let

$$c_1(u) = \begin{cases} 2 & \text{if } u(1) < u(2), \\ 2t & \text{if } u(1) > u(2). \end{cases}$$

For σ_2, we have the following cases:

- if $u(1) < u(2)$, then $w(1) > w(2)$ if and only if $\sigma_2 = -$,
- if $u(1) < u(2)$, then $-w(1) > w(2)$ if and only if $\sigma_2 = -$,
- if $u(2) > u(3)$, then $w(2) > w(3)$ if and only if $\sigma_2 = +$.

Letting σ_1 and σ_2 range over all possible signs, we have:

$$c_1(u)c_2(u) = \begin{cases} 2+2t^2 & \text{if } u_1 < u_2 < u_3, \\ 2t(1+t) & \text{if } u_1 < u_2 > u_3, \\ 4t & \text{if } u_1 > u_2 < u_3, \\ 2t(1+t) & \text{if } u_1 > u_2 > u_3. \end{cases}$$

For any $u \in S_n$,

$$\sum_{w \in B(u)} t^{\mathrm{des}_D(w)} = c_1(u)c_2(u)\cdots c_n(u).$$

If $u_1 < u_2 > u_3$, $u_1 > u_2 < u_3$, or $u_1 > u_2 > u_3$, then we find:

$$\sum_{w \in B(u)} t^{\mathrm{des}_D(w)} = (4t)^{\mathrm{lpk}(u)}(1+t)^{n-2\,\mathrm{lpk}(u)}, \tag{13.9}$$

and so the contribution in each of these cases is gamma-nonnegative.

However, if $u_1 < u_2 < u_3$, the sum is not gamma-nonnegative. For example, if $u = 23541$, we find

$$\begin{aligned} \sum_{w \in B(u)} t^{\mathrm{des}_D(w)} &= (2+2t^2)(2t)(1+t)(2), \\ &= 8t + 16t^3 + 8t^5, \\ &= 8t(1+t)^4 - 32t^2(1+t)^2 + 16t^3. \end{aligned}$$

To fix this, we will lump together such permutations with some others to obtain a contribution that is overall gamma-nonnegative. For a permutation $u = u_1u_2u_3\cdots u_n$, let $u' = u_2u_1u_3\cdots u_n$. If $u_1 < u_2 < u_3$, then $u'_2 < u'_1 < u'_3$ and so $u'_1 > u'_2 < u'_3$, giving $c_1(u')c_2(u') = 4t$. Since $c_j(u) = c_j(u')$ for all $j \geq 3$, we have:

$$\begin{aligned} \sum_{w \in B(u) \cup B(u')} t^{\mathrm{des}_D(w)} &= (c_1(u)c_2(u) + c_1(u')c_2(u'))c_3(u)\cdots c_n(u), \\ &= (2+2t^2+4t)c_3(u)\cdots c_n(u), \\ &= 2(1+t)^2 c_3(u)\cdots c_n(u), \\ &= 2\cdot(4t)^{\mathrm{lpk}(u)}(1+t)^{n-2\,\mathrm{lpk}(u)}. \end{aligned} \tag{13.10}$$

Let us denote

$$\begin{aligned} S_n^{I} &= \{u \in S_n : u_1 < u_2 < u_3\}, \\ S_n^{II} &= \{u \in S_n : u_2 < u_1 < u_3\}, \end{aligned}$$

and

$$S_n^{III} = \{u \in S_n : u_1 < u_2 > u_3, u_1 > u_2 > u_3, \text{ or } u_2 < u_3 < u_1\},$$

Thus, S_n is the disjoint union $S_n^I \cup S_n^{II} \cup S_n^{III}$ and S_n^I and S_n^{II} are in bijection via $u \leftrightarrow u'$. The distribution of des_D over $B(u)$ for u in S_n^{III} is gamma-nonnegative by Equation (13.9), and the distribution of des_D over $B(u) \cup B(u')$ for u in S_n^I (with u' in $\cup S_n^{II}$) is gamma-nonnegative by (13.10).

In total,

$$\begin{aligned} D_n(t) &= \frac{1}{2} \sum_{w \in B_n} t^{\mathrm{des}_D(w)}, \\ &= \sum_{u \in S_n^I} (4t)^{\mathrm{lpk}(u)} (1+t)^{n-2\,\mathrm{lpk}(u)}, \\ &\quad + \frac{1}{2} \sum_{u \in S_n^{III}} (4t)^{\mathrm{lpk}(u)} (1+t)^{n-2\,\mathrm{lpk}(u)}. \end{aligned}$$

To put it another way, define the “indicator” function

$$\phi(u) = \begin{cases} 1 & \text{if } u_1 < u_2 < u_3, \\ 0 & \text{if } u_2 < u_1 < u_3, \\ 1/2 & \text{otherwise.} \end{cases}$$

Then,

$$D_n(t) = \sum_{u \in S_n} \phi(u) (4t)^{\mathrm{lpk}(u)} (1+t)^{n-2\,\mathrm{lpk}(u)}.$$

While this expression is not as neat and tidy as the one for type B_n, the gamma-nonnegativity is manifest. Hence we have the following result.

Theorem 13.9. *For all $n \geq 4$, there exist nonnegative integers $\gamma_j^{D_n}$ such that*

$$D_n(t) = \sum_{j=0}^{\lfloor n/2 \rfloor} \gamma_j^{D_n} t^j (1+t)^{n-2j}. \tag{13.11}$$

Moreover,

$$\begin{aligned} \gamma_j^{D_n} &= 4^j \cdot |\{u \in S_n : u_1 < u_2 < u_3, \mathrm{lpk}(u) = j\}| \\ &\quad + 2 \cdot 4^{j-1} \cdot |\{u \in S_n : u_3 \neq \max\{u_1, u_2, u_3\}, \mathrm{lpk}(u) = j\}|. \end{aligned}$$

We can use this combinatorial description to construct a balanced simplicial complex, to establish Conjecture 3 in the case of the D_n Coxeter complex.

Let Dec_n^D denote a set of decorated permutations corresponding the type D_n gamma vector, i.e.,

$$\begin{aligned}\mathrm{Dec}_n^D &= \{w \in \mathrm{Dec}_n : w_1 < w_2 < w_3\} \\ &\qquad \cup \{w \in \mathrm{Dec}_n : w_3 \neq \max\{w_1, w_2, w_3\} \text{ and } c_1 \in \{0,1\}\},\end{aligned}$$

where c_1 denotes the style of the first bar in w.

It is easy to see that Dec_n^D is a subcomplex of Dec_n, i.e., if $G \in \mathrm{Dec}_n^D$ and F is a face of Dec_n contained in G, then $F \in \mathrm{Dec}_n^D$ as well.

13.5 Combinatorial models for shard intersections

We saw the shard intersection order for type A_{n-1} in Section 3.3, though at the time we did not understand it geometrically. In Section 11.11, we saw the general definition of the shard intersection order from the geometric point of view, but we did not explore combinatorial models. In the following subsections, we make the explicit connection between the general definition of the shard intersection order and combinatorial models for types A_{n-1}, B_n, and D_n. This material is adapted from the author's paper [118], though the type A_{n-1} was studied by Erin Bancroft as well [13].

13.5.1 Type A_{n-1}

Recall the root system of type $\mathbf{A}_{n-1}$ is most naturally realized in

$$V = \{\mathbf{x} \in \mathbb{R}^n : \sum x_i = 0\},$$

where we choose the positive roots to be

$$\Pi = \{\varepsilon_j - \varepsilon_i : 1 \leq i < j \leq n\}.$$

With respect to this choice of root system, the fundamental chamber C is given by:

$$C = \{\mathbf{x} \in V : x_1 < x_2 < \cdots < x_n\},$$

and the hyperplane corresponding to a positive root $\varepsilon_j - \varepsilon_i$ is:

$$H_{ij} = \{\mathbf{x} \in V : x_i = x_j\}.$$

There are only two types of rank two subarrangements of $\mathcal{H}(\mathbf{A}_{n-1})$, shown in Figure 13.2. If we consider two hyperplanes H_{ij} and H_{kl} with $\{i,j\} \cap \{k,l\} = \emptyset$, the hyperplanes are orthogonal, and we get no cutting relations.

(This arrangement is isomorphic to $\mathcal{H}(\mathbf{A}_1 \times \mathbf{A}_1)$.) On the other hand, suppose the hyperplanes are not orthogonal. Then we get an arrangement isomorphic to $\mathcal{H}(\mathbf{A}_2)$, generated, say, by H_{ij} and H_{jk}, with $1 \leq i < j < k \leq n$. The hyperplanes H_{ij} and H_{jk} are basic (as shown in Figure 13.2), and the third hyperplane in the arrangement, H_{ik}, gets cut according to whether $x_j \leq x_i = x_k$ or $x_i = x_k \leq x_j$.

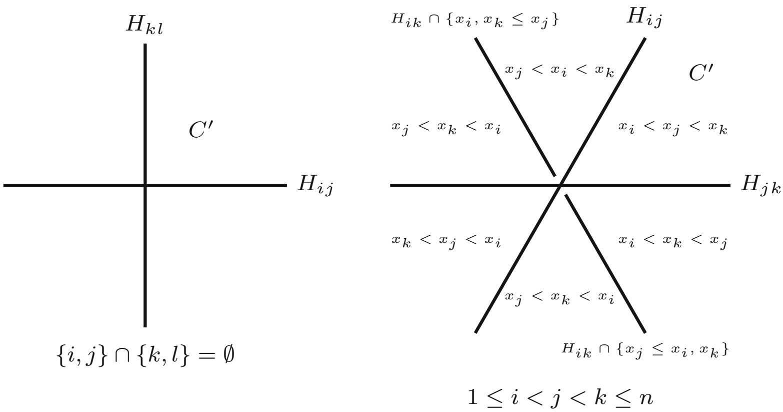


Fig. 13.2 The rank two subarrangements of $\mathcal{H}(\mathbf{A}_{n-1})$.

As there are no other possibilities for the rank two subarrangements of $\mathcal{H}(\mathbf{A}_{n-1})$, we have the following Proposition.

Proposition 13.1. *The hyperplane H_{ij} has 2^{j-i-1} shards, and hence there are*

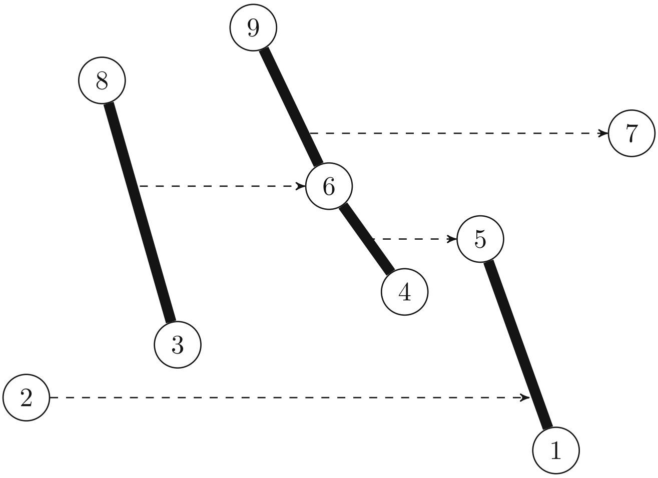
$$\sum_{1 \leq i < j \leq n} 2^{j-i-1} = 2^n - n - 1,$$

shards in $\mathcal{H}(\mathbf{A}_{n-1})$.

Notice the not surprising coincidence that the number of shards equals the Eulerian number $\left\langle {n \atop 1} \right\rangle = 2^n - n - 1$.

Now that we have identified the shards, we will describe how permutations, as drawn in Section 3.3, encode shard intersections.

Visually, we represent the permutation as an array with a mark in column i (from left to right), row j (from bottom to top) if $w(i) = j$. We group together any decreasing runs into blocks with thick lines:



(13.12)

If it is possible to draw a horizontal line to connect two decreasing runs, the block on the left is considered less than the block on the right. This gives a "pre-order" on $\{1, \ldots, n\}$ that we call a *permutation pre-order*. (It's like a partial order, but with ties allowed.) When writing a permutation in one-line notation, we simply put bars between the decreasing runs to indicate the blocks in the pre-order; relations between the blocks must be inferred. With w in (13.12), we would write $w = 2|83|964|51|7$.

For the rest of this section we will pass freely from thinking of an element $w \in S_n$ as a word and as a permutation pre-order.

Permutation pre-orders neatly encode type A_{n-1} shard intersections as follows. For any $w \in S_n$, define the *cone* of w, $C(w)$, as the set of points $\mathbf{x} \in V$ such that:

- if i and j are in the same block in w, then $x_i = x_j$,
- if $i < k < j$ and k is not in the same block as i and j, then:

  a) $x_k \leq x_i = x_j$ if k appears to the left of i in w, and
  b) $x_i = x_j \leq x_k$ if k appears to the right of i in w.

The example shown in (13.12) then corresponds to the cone of all points satisfying

$$x_8 = x_3 \geq x_9 = x_6 = x_4 \geq x_7,$$

$$x_9 = x_6 = x_4 \geq x_5 = x_1 \leq x_2, \quad \text{and}$$

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 = 0.$$

At the extremes we have $C(1|2|\cdots|n) = V$ and $C(n\cdots 21) = (0, 0, \ldots, 0)$. Notice that the dimension of $C(w)$ is equal to one less than the number of decreasing runs in w (since the sum of the coordinates is zero), and hence codimension corresponds to descent number.

Observation 13.1 *For any $w \in S_n$,*

$$\mathrm{des}(w) = n - 1 - \dim(C(w)).$$

In particular, shards correspond to elements with one descent.

Thus Proposition 13.1 gives a very roundabout way to show there are $2^n - n - 1$ permutations with one descent.

It remains to be seen that the cones so described are in fact intersections of shards.

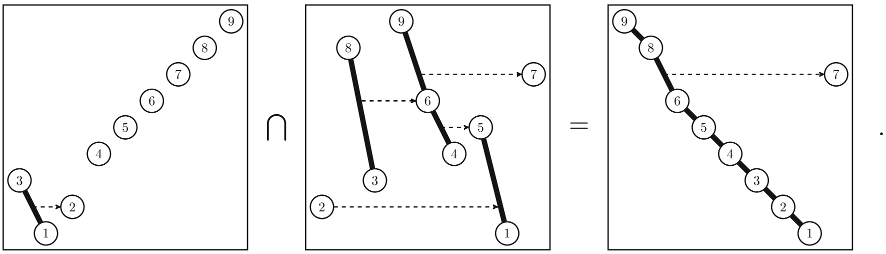
To begin, the identity permutation corresponds to the empty intersection, i.e., V, while if w has only one descent, it corresponds to a shard itself and we are done.

Now to any collection of shards $\{\sigma_1, \ldots, \sigma_r\}$, we claim we can associate an element w such that

$$\bigcap_{i=1}^{r} \sigma_i = C(w).$$

For induction, suppose the result holds for any intersection of fewer shards. In particular, $\bigcap_{i=1}^{r-1} \sigma_i = C(u)$ for some u. Let $\sigma_r = C(v)$ be a new shard with $x_a = x_b$. Then $\sigma_r \cap C(u) = C(w)$, where w is the permutation formed by merging the blocks of u containing a and b, along with any blocks between them. Moreover, if $a < k < b$ and k was left of a in u but right of a in v, then k is in the same block with a and b in w.

For example, taking the pre-order in (13.12) with the shard 31|2|4|5|6|7|8|9 we get:



We have shown that an intersection of shards corresponds to $C(w)$ for some w. Now we will show that each cone $C(w)$ corresponds to an intersection of shards.

Permutations with one descent correspond to shards themselves, so suppose w has more than one descent. The following describes a collection of shards whose intersection gives the cone $C(w)$. Given two elements in a decreasing run, say $w(i) > w(j)$ (and so $i < j$), we let σ be the shard with $x_{w(i)} = x_{w(j)}$ and such that for each k with $w(j) < k < w(i)$, we put $x_k \leq x_{w(i)}$ if $w^{-1}(k) < i$, $x_{w(i)} \leq x_k$ otherwise.

Doing this for all pairs of elements in decreasing runs yields a collection of shards σ, each of which contains $C(w)$ and such that all the conditions imposed by w are articulated by some shard.

To illustrate, let $w = 2|83|964|51|7$. Then the collection of shards we get is:

$$\begin{aligned} C(2|83|964|51|7) = {} & C(1|2|83|4|5|6|7|9) \cap C(1|2|3|4|5|8|96|7) \\ & \cap C(1|2|3|8|94|5|6|7) \cap C(1|2|3|64|5|7|8|9) \\ & \cap C(2|3|4|51|6|7|8|9). \end{aligned}$$

We remark that while this idea shows $C(w)$ is formed as an intersection of a set of shards, the set of shards we generate this way is neither necessarily minimal nor maximal. In our example, further intersecting with the shard $1|2|3|6|8|94|5|7$ would not change the result. Also, we could have removed the shard $1|2|3|8|94|5|6|7$ and still obtained $C(w)$.

We can now give a partial order on S_n by reverse inclusion of the corresponding subsets of V. That is, $u \leq v$ in the shard intersection order if and only if $C(v) \subseteq C(u)$. This definition can be stated combinatorially in terms of permutation pre-orders as was given in Section 3.3. Verification of equivalence is straightforward.

Proposition 13.2. *In terms of permutation pre-orders, $u \leq v$ in $\mathrm{Sh}(S_n)$ if and only if:*

- *(Refinement) u refines v as a set partition, and*
- *(Consistency) if i and j are in the same block in u, and $i < k < j$ (with k not in the same block as i and j in u), then either k is in the same block as i and j in v, or k is on the same side of i and j in v as in u.*

The intersection lattice is ranked by codimension, so by Observation 13.1 we can see that rank in $\mathrm{Sh}(S_n)$ corresponds to descent number:

$$\mathrm{rk}(w) = \mathrm{des}(w).$$

In other words, we have established Theorem 11.4 in the case of the symmetric group.

13.5.2 Type $\mathbf{B}_n$

The root system of type $\mathbf{B}_n$ lives in $V = \mathbb{R}^n$, with positive roots

$$\Pi = \{\varepsilon_j \pm \varepsilon_i : 1 \leq i < j \leq n\} \cup \{\varepsilon_i : 1 \leq i \leq n\}.$$

As the hyperplane arrangement for $\mathbf{C}_n$ is identical to that of $\mathbf{B}_n$, all results that follow in this section hold for the Coxeter groups of type C_n as well.

With respect to this choice of root system, the fundamental chamber C is given by:

$$C = \{\mathbf{x} \in \mathbb{R}^n : 0 < x_1 < \cdots < x_n\}.$$

The hyperplane corresponding to the positive root $\varepsilon_j - \varepsilon_i$ is:

$$H_{ij} = \{\mathbf{x} \in \mathbb{R}^n : x_i = x_j\},$$

the hyperplane corresponding to the positive root $\varepsilon_j + \varepsilon_i$ is:

$$H_{\bar{i}j} = \{\mathbf{x} \in \mathbb{R}^n : -x_i = x_j\},$$

and the hyperplane corresponding to the positive root ε_i is:

$$H_{0i} = \{\mathbf{x} \in \mathbb{R}^n : x_i = 0\}.$$

There are three possibilities for rank two subarrangements of $\mathcal{H}(\mathbf{B}_n)$. The subarrangements are either isomorphic to $\mathcal{H}(\mathbf{A}_1 \times \mathbf{A}_1)$, to $\mathcal{H}(\mathbf{A}_2)$, or to $\mathcal{H}(\mathbf{B}_2)$. The possibilities are shown in Figures 13.2 and 13.3.

The cutting relations for hyperplanes H_{ij}, with $0 \le i < j \le n$, are rather different from the cutting relations for the hyperplanes $H_{\bar{i}j}$, with $1 \le i < j \le n$. In particular there are 2^{j-i-1} shards in H_{ij} and $2^{j-i}3^{i-1}$ shards in hyperplane $H_{\bar{i}j}$.

It is easy to see that a hyperplane H_{ij}, with $0 \le i < j \le n$ is either cut according to the arrangement $\mathcal{H}(\mathbf{A}_2)$ in Figure 13.2 or, if $i = 0$, according to Figure 13.3 (a). In either case, we find the shards of H_{ij} are formed by choosing, for each k such that $i < k < j$, whether $x_k \le x_i = x_j$ or $x_i = x_j \le x_k$. In particular, there are 2^{j-i-1} shards of this hyperplane, just as we found in Proposition 13.1 for type $\mathbf{A}_{n-1}$.

The cutting relations for the hyperplane $H_{\bar{i}j}$ appear in the arrangements of Figure 13.3 (a), (b), (c), and (d).

In case (a), we have two choices. Either $0 \le x_i = -x_j$, or $x_i = -x_j \le 0$.

Now consider cases (b) and (c). Suppose, without loss of generality, that $0 \le x_i = -x_j$. Here we need to choose, for each k such that $1 \le k < i$, whether:

- $-x_k \le -x_i = x_j \le 0 \le x_i = -x_j \le x_k$,
- $-x_i = x_j \le -x_k, 0, x_k \le x_i = -x_j$, or
- $x_k \le -x_i = x_j \le 0 \le x_i = -x_j \le -x_k$.

Note that we could *not* have $0 \le x_i = -x_j \le -x_k, x_k$, as all coordinates would be forced to equal zero. Hence there are three choices for each such k, yielding a total of 3^{i-1} choices of this kind.

Finally consider case (d). Here we see we need to choose, for each k such that $i < k < j$, whether $x_k \le x_i = -x_j$ or $x_i = -x_j \le x_k$, yielding 2^{j-i-1} choices.

We have now completely described the shards of type $\mathbf{B}_n$, and moreover we have the following proposition. The formula for the sum is easily verified by induction.

Proposition 13.3. *For all* $0 \le i < j \le n$*, the hyperplane* H_{ij} *has* 2^{j-i-1} *shards. For any* $1 \le i < j \le n$*, the hyperplane* $H_{\bar{i}j}$ *has* $2^{j-i}3^{i-1}$ *shards. Therefore, there are*

$$\sum_{0 \le i < j \le n} 2^{j-i-1} + \sum_{1 \le i < j \le n} 2^{j-i}3^{i-1} = 3^n - n - 1,$$

shards of $\mathcal{H}(\mathbf{B}_n)$ *in all.*

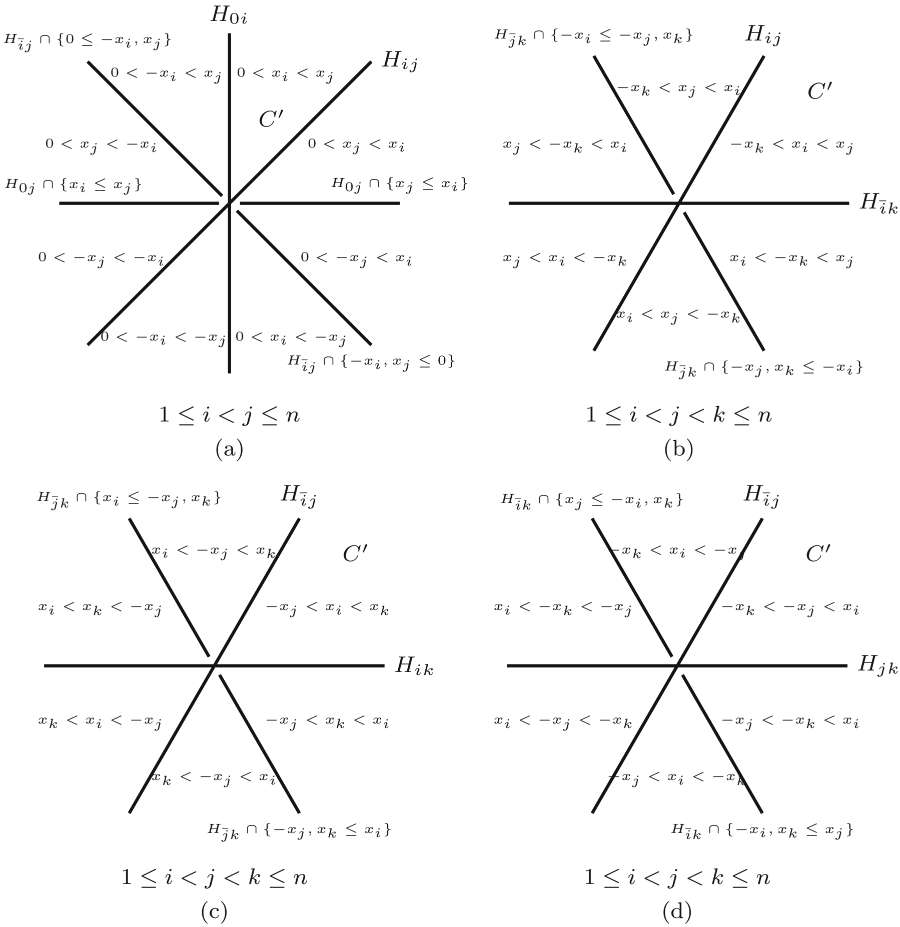
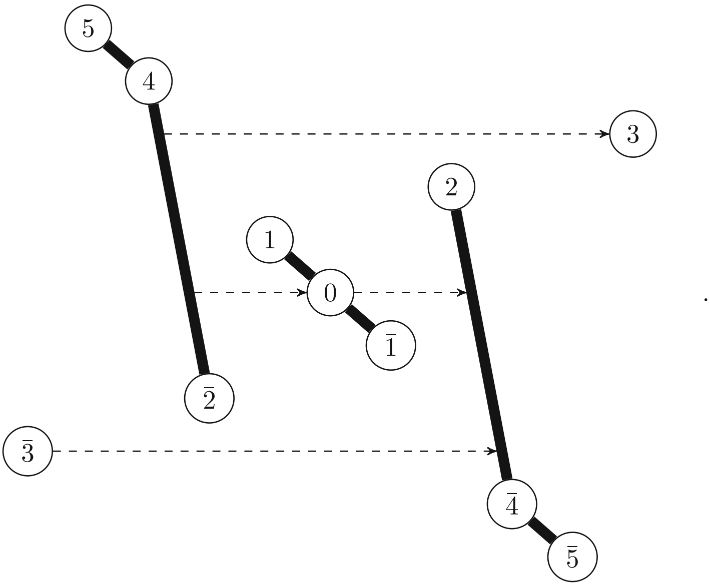


Fig. 13.3 The rank two subarrangements of $\mathcal{H}(\mathbf{B}_n)$ not pictured in Figure 13.2.

We now encode intersections of shards with signed permutations. As with the symmetric group, we will highlight the maximal decreasing runs of w, written in long form, $w = w_{\bar{n}} \cdots w_{\bar{1}} 0 w_1 \cdots w_n$, by inserting bars in ascent positions. For example, we write

$$w = \bar{3}|54\bar{2}|10\bar{1}|2\bar{4}\bar{5}|3.$$

Visually, we represent a signed permutation as an array with a mark in column i, row j ($-n \leq i, j \leq n$) if $w(i) = j$. As with the type $\mathbf{A}_{n-1}$ model, we group together decreasing runs into blocks indicated by thick lines:



(13.13)

If it is possible to draw a horizontal line to connect two decreasing runs, the block on the left is considered less than the block on the right. This gives a certain pre-order on $\{0, \pm 1, \ldots, \pm n\}$ that we will call a *signed permutation pre-order*.

Signed permutation pre-orders are in bijection with type $\mathbf{B}_n$ shard intersections. Just as with the type $\mathbf{A}_{n-1}$ model, we define a cone of points, $C(w)$, for an element $w \in B_n$ as follows:

- if i and j are in the same block in w, then we have $x_i = x_j$, with the understanding that $x_{-i} = -x_i$ and $x_0 = 0$,
- if $i < k < j$ and k is not in the same block as i and j, then:

  a) $x_k \leq x_i = x_j$ if k appears to the left of i in w, and
  b) $x_i = x_j \leq x_k$ if k appears to the right of i in w.

The example shown in (13.13) then corresponds to the set of points in $\mathbb{R}^5$ satisfying:

$$x_1 = 0 = -x_1 \leq x_2 = -x_4 = -x_5 \geq -x_3.$$

Each block has a negative counterpart, except for the block containing zero. Thus the dimension of $C(w)$ is half the number of blocks not containing zero, plus one if there is a nonzero number in the block with zero. Since the type B_n descent statistic only considers descents among $w_0 w_1 \cdots w_n$, this means codimension corresponds to the type B_n descent statistic.

Observation 13.2 *For any $w \in B_n$,*

$$\operatorname{des}_B(w) = n - \dim(C(w)).$$

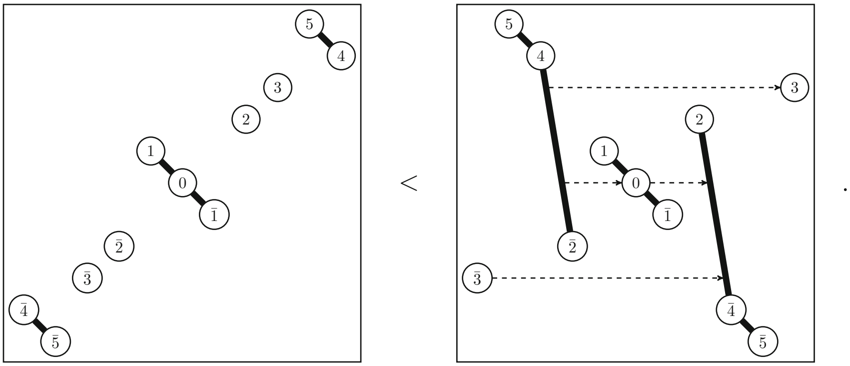
In particular, shards correspond to signed permutations with exactly one type B_n descent.

Thus Proposition 13.3 gives an indirect way to prove there are $3^n - n - 1$ signed permutations with exactly one descent, i.e., $\left\langle {B_n \atop 1} \right\rangle = 3^n - n - 1$.

We can show that the cones $C(w)$ always correspond to intersections of type B_n shards, by using an idea similar to the $\mathbf{A}_{n-1}$ case, by greedily finding a set of shards whose intersection is the desired cone.

Now we can define the shard intersection order on B_n as $u \le v$ in $(\mathrm{Sh}(B_n), \le)$ if and only if $C(v) \subseteq C(u)$. This manifests itself for signed permutation pre-orders in the same notions of "refinement" and "consistency" given in Proposition 13.2. We can use the same intuition of merging blocks to move up in the poset, taking care to act symmetrically: if i joins a block with j, then $-i$ must join a block with $-j$ and so on.

For example, $\bar{4}\bar{5}|\bar{3}|\bar{2}|10\bar{1}|2|3|54 < \bar{3}|54\bar{2}|10\bar{1}|2\bar{4}\bar{5}|3$ as shown:



In moving from the signed permutation on the left to the one on the right, we merged $\bar{2}$ with the block 54 (and hence 2 with $\bar{4}\bar{5}$). This meant that we needed to decide whether the new block would be right or left of 3 and right or left of the block containing 0. In this case, we chose $54\bar{2}$ to be left of both.

The lattice of B_n shard intersections is ranked by codimension, so by Observation 13.2 we can see that rank in $\mathrm{Sh}(B_n)$ corresponds to type B_n descent number:

$$\operatorname{rk}(w) = \operatorname{des}_B(w).$$

In other words, we have established Theorem 11.4 for the case of $W = B_n$.

13.5.3 Type $\mathbf{D}_n$

Recall the root system of type $\mathbf{D}_n$ lives in $V = \mathbb{R}^n$, with positive roots

$$\Pi = \{\varepsilon_j \pm \varepsilon_i : 1 \leq i < j \leq n\}.$$

With respect to this choice, the fundamental chamber C is given by:

$$C = \{\mathbf{x} \in \mathbb{R}^n : -x_2 < \pm x_1 < x_2 < \cdots < x_n\}.$$

Since the $\mathbf{D}_n$ roots are all the $\mathbf{B}_n$ roots save the standard basis elements, the arrangement $\mathcal{H}(\mathbf{D}_n)$ is the subarrangement of $\mathcal{H}(\mathbf{B}_n)$ generated by the hyperplanes H_{ij} and $H_{\bar{i}j}$ (but not H_{0i}).

The rank two subarrangements of $\mathcal{H}(\mathbf{D}_n)$ either look like $\mathcal{H}(\mathbf{A}_1 \times \mathbf{A}_1)$ or like $\mathcal{H}(\mathbf{A}_2)$, and we can identify all the cutting relations from the pictures in Figure 13.2 and Figure 13.3 (b), (c), and (d).

The hyperplanes H_{ij}, with $1 \leq i < j \leq n$, are once again cut according to relation in Figure 13.2. We find 2^{j-i-1} shards of this hyperplane as in $\mathbf{A}_{n-1}$ and $\mathbf{B}_n$.

Now consider a hyperplane $H_{\bar{i}j}$ with $1 \leq i < j \leq n$. The cutting relations for this hyperplane are given by parts (b), (c), and (d) of Figure 13.3. From part (d) we see that for each k such that $i < k < j$, we must choose whether $-x_k \leq x_i = -x_j$ or whether $x_i = -x_j \leq -x_k$, yielding 2^{j-i-1} choices.

The interaction between the relations in parts (b) and (c) are somewhat delicate. Since we have no hyperplanes of the form H_{0i}, we do not know explicitly whether $x_i = -x_j$ is weakly positive or negative. However, if we know that, say, $-x_i = x_j \leq \pm x_k$, we can infer that $-x_i = x_j$ is negative. Likewise, if $\pm x_k \leq -x_i = x_j$, we can infer that $-x_i = x_j$ is positive. If k is such that $1 \leq k < i$ and both x_k and $-x_k$ are on the same side of $-x_i = x_j$ we say k is in the *zero block* of the shard.

If the zero block is empty, we know that for each $k = 1, \ldots, i-1$, there are two choices:

- $x_k \leq x_i = -x_j, -x_i = x_j \leq -x_k$, or
- $-x_k \leq x_i = -x_j, -x_i = x_j \leq x_k$.

Thus there are $2^{j-i-1} \cdot 2^{i-1}$ shards of $H_{\bar{i}j}$ with an empty zero block. Note, however, that $x_i = -x_j$ and $-x_i = x_j$ are incomparable.

We will now count the remaining shards in $H_{\bar{i}j}$ according to the smallest element in the zero block.

Suppose h is the smallest element in the zero block. First of all, since the zero block is nonempty, we know whether $-x_i = x_j$ is weakly positive or weakly negative, giving two initial choices. Suppose, without loss of generality, that $\pm x_h \leq -x_i = x_j$.

Then for each $g = 1, \ldots, h-1$, there are two choices:

- $x_g \leq x_i = -x_j \leq \pm x_h \leq -x_i = x_j \leq -x_g$, or
- $-x_g \leq x_i = -x_j \leq \pm x_h \leq -x_i = x_j \leq x_g$.

For each $k = h+1, \ldots, i-1$, there are three choices:

- $x_k \leq x_i = -x_j \leq \pm x_h \leq -x_i = x_j \leq -x_k$,
- $x_i = -x_j \leq \pm x_h, \pm x_k \leq -x_i = x_j$, or
- $-x_k \leq x_i = -x_j \leq \pm x_h \leq -x_i = x_j \leq x_k$.

Hence, we find a total of $2^{j-i-1} \cdot 2 \cdot 2^{h-1} \cdot 3^{i-1-h}$ choices for a given h.

Pulling all the cases for the zero block together (empty and $h = 1, \ldots, i-1$) we find a total of:

$$\begin{aligned} 2^{j-i-1}\left(2^{i-1}+2\cdot 3^{i-2}+\cdots+2^{i-2}\cdot 3+2^{i-1}\right) &= 2^{j-i-1}(2^{i-1}+2(3^{i-1}-2^{i-1})), \\ &= 2^{j-i-1}(2\cdot 3^{i-1}-2^{i-1}), \\ &= 2^{j-i}\cdot 3^{i-1}-2^{j-2}. \end{aligned}$$

shards in $H_{\bar{i}j}$.

We have now characterized the shards of type $\mathbf{D}_n$. In particular we have the following companion to Propositions 13.1 and 13.3.

Proposition 13.4. *For all $1 \leq i < j \leq n$, the hyperplane H_{ij} has 2^{j-i-1} shards, while the hyperplane $H_{\bar{i}j}$ has $2^{j-i}\cdot 3^{i-1}-2^{j-2}$ shards. Therefore, there are*

$$\sum_{1\leq i<j\leq n} 2^{j-i-1}+2^{j-i}\cdot 3^{i-1}-2^{j-2} = 3^n - n2^{n-1} - n - 1,$$

shards of $\mathcal{H}(\mathbf{D}_n)$ in all.

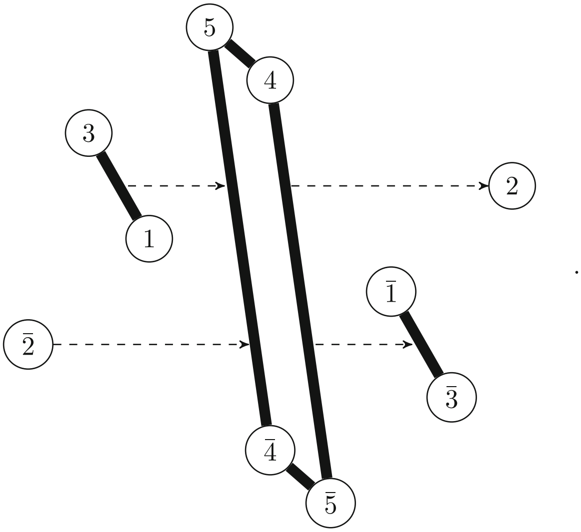
To describe shard intersections, the most helpful way to write elements $w \in D_n$ is as "forked" signed permutations, e.g.,

$$w = \bar{2}315\frac{4}{4}\bar{5}\bar{1}\bar{3}2, \tag{13.14}$$

corresponds to $\{w(1), -w(1)\} = \{4, -4\}$, $w(2) = -5$, $w(3) = -1$, $w(4) = -3$, and $w(5) = 2$. As an even signed permutation, we would write $w = \bar{4}\bar{5}\bar{1}\bar{3}2$. We choose the forked model because it is more indicative of the geometry of the corresponding chamber in the complement of $\mathcal{H}(\mathbf{D}_5)$:

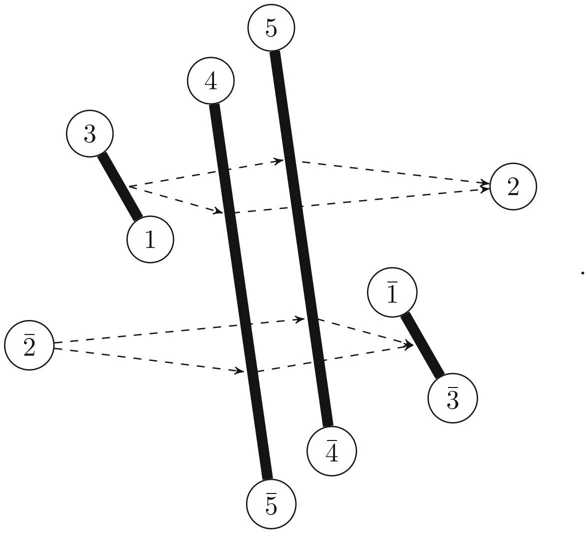
$$-x_2 < x_3 < x_1 < x_5 < \pm x_4 < -x_5 < -x_1 < -x_3 < x_2.$$

We draw $w \in D_n$ as an array with a mark in column $i \geq 0$, row j if $w(i+1) = j$. We put a mark in column $i \leq 0$, row j if $w(i-1) = j$. In effect, we draw w as if it is a type B_n element, then slide $w(i)$ one step left for i positive, one step right for i negative. Hence, $w(1)$ and $-w(1)$ appear in the same center column. Again, we draw solid lines in descent positions. For example, the element w in (13.14) is drawn as:



(13.15)

The partial order on blocks in this case is similar to earlier cases, with one caveat. Usually, if it is possible to draw a horizontal line to connect two decreasing runs, the block on the left is considered less than the block on the right. However, if $w(1)$ and $w(-1)$ are in distinct blocks, these blocks are only comparable if there is a triple $i < k < j$ with i, j in the block containing $w(1)$ and k in the block containing $w(-1)$. For example, in (13.16) the block containing $w(-1)$ and the block containing $w(1)$ are incomparable:



(13.16)

In either case, we get a pre-order on $\{-n, \ldots, -1, 1, \ldots, n\}$, which we call a *forked permutation pre-order*.

For any $w \in D_n$, we define a cone of points $C(w)$ in $\mathbb{R}^n$ just as in the A_{n-1} and B_n cases. Specifically,

- if i and j are in the same block in w, we have $x_i = x_j$, with the understanding that $x_{-i} = -x_i$,
- if $i < k < j$ and k is not in the same block as i and j, then:

  a) $x_k \leq x_i = x_j$ if k is less than i in the pre-order given by w, and
  b) $x_i = x_j \leq x_k$ if k greater than i in the pre-order given by w.

The example shown in (13.15) then corresponds to the set of points satisfying:

$$-x_2, x_1 = x_3 \leq x_4 = x_5 = -x_4 = -x_5 (= 0) \leq -x_1 = -x_3, x_2,$$

while the example shown in (13.16) corresponds to:

$$-x_2, x_1 = x_3 \leq x_4 = -x_5, x_4 = -x_5 \leq -x_1 = -x_3, x_2.$$

As with earlier cases, we can determine the dimension of $C(w)$ by the number of nonzero blocks and whether there are any coordinates equal to zero. We have the following.

Observation 13.3 *For any $w \in D_n$,*

$$\mathrm{des}_D(w) = n - \dim(C(w)).$$

In particular, shards correspond to forked signed permutations with exactly one type D_n descent.

Thus, Proposition 13.4 shows there are $3^n - n2^{n-1} - n - 1$ elements of D_n with exactly one descent.

That the cones $C(w)$ correspond to intersections of type D_n shards follows from explicit decomposition of a given cone into shards along similar lines as earlier cases. As an example, the forked permutation in (13.15) can be written as the following intersection of D_n shards (with bars drawn to indicate divisions between the blocks):

$$w = \bar{2}|31|5\frac{4}{4}\bar{5}|\bar{1}\bar{3}|2 = \bar{4}\bar{5}|\bar{3}|\bar{2}|\frac{1}{\bar{1}}|2|3|54 \cap \bar{2}|1|3|4\frac{5}{\bar{5}}\bar{4}|\bar{3}|\bar{1}|2 \cap \bar{5}|\bar{4}|\bar{2}|\bar{1}\frac{3}{\bar{3}}1|2|4|5.$$

The shard intersection order on D_n is also analogous to earlier examples. We have $u \leq v$ in $(D_n, \leq)$ if and only if $C(v) \subseteq C(u)$, and the containment of cones can be easily captured by the merging of blocks consistent with the forked pre-order.

The lattice of D_n shard intersections is ranked by codimension, so by Observation 13.3 we have see that

$$\mathrm{rk}(w) = \mathrm{des}_D(w),$$

establishing Theorem 11.4 in the case of $W = D_n$.

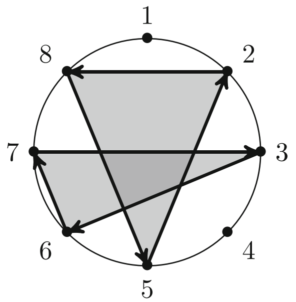
13.6 Type B_n noncrossing partitions and Narayana numbers

As mentioned in Section 12.2 we call the interval $[e, c]$ in $\mathrm{Abs}(W)$ the noncrossing partition lattice because of the case of the symmetric group. Let's revisit this connection as a lead-in to discussion of other types. See Section 3.5.

We can represent any permutation by placing n points on a circle and drawing an arc from i to j if $w(i) = j$, e.g.,

$$w = 18642735 = (1)(285)(367)(4),$$

is drawn as



Take $W = A_{n-1} = S_n$, where $c = s_1 s_2 \cdots s_n$ is the n-cycle, written $c = (12 \cdots n)$. If we draw all permutations in $[e, c]$ this way, we find all cycles are oriented clockwise and none of the arcs from different cycles cross in the interior of the disk. The correspondence with the classical noncrossing set partitions of $\{1, 2, \ldots, n\}$ becomes clear. See Figure 13.4.

Now let us turn to the type B_n model. By this stage in the book, we might guess a type B_n noncrossing partitions is some kind of "set partition with signed symmetry," and indeed this is the case. Our discussion of this model follows closely work of Victor Reiner from 1997 [126].

Just as we did with type A_{n-1}, we start with the cycle notation for a Coxeter element. Here we choose $c = s_0 s_1 \cdots s_{n-1} = (12 \cdots n\bar{1}\bar{2} \cdots \bar{n})$, so it makes sense to draw the numbers $1, 2 \ldots, n, -1, -2, \ldots, -n$ clockwise around a circle. For a signed permutation w in B_n, we draw a line from i to j if $w(i) = j$. For example,

$$w = 3\bar{2}\bar{1}4 = (13\bar{1}\bar{3})(2\bar{2})(4)(\bar{4}),$$

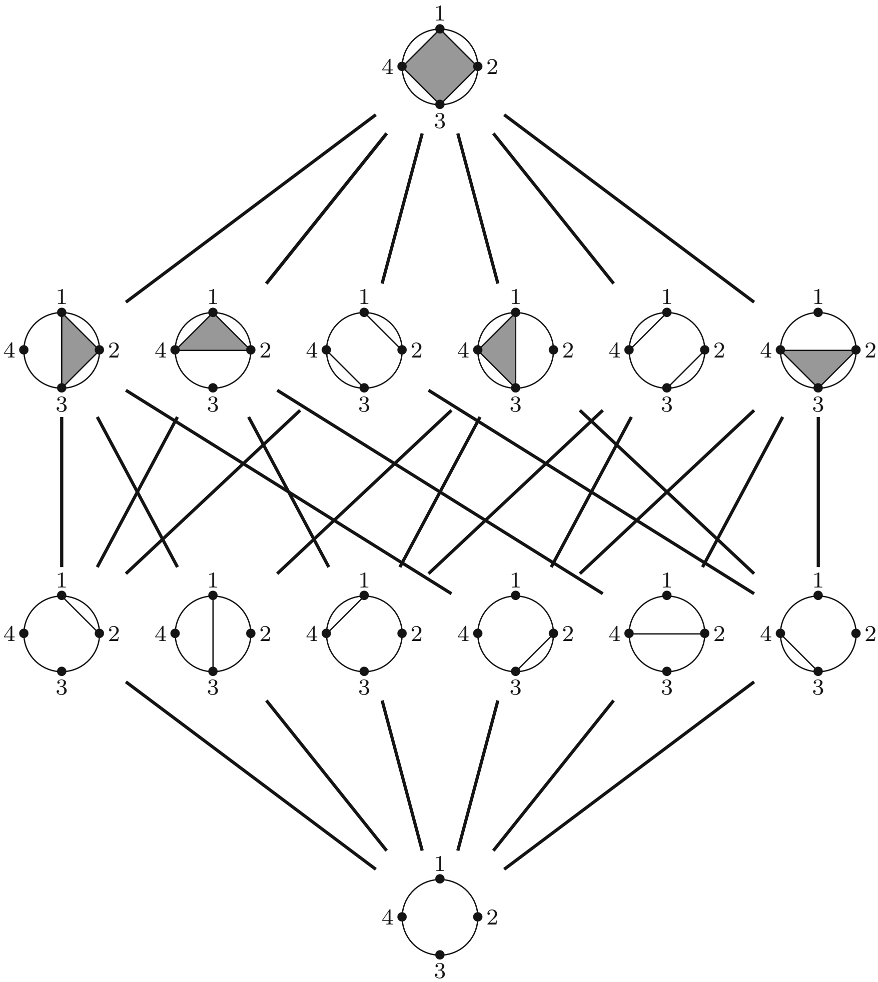
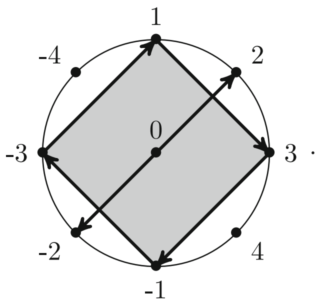


Fig. 13.4 The poset $\mathrm{NC}(A_3)$ realized as the interval below $c = (1234)$ in $\mathrm{Abs}(A_3)$.

is drawn as



.

If we draw the elements below c in $\mathrm{Abs}(B_n)$, we can see how to give the combinatorial model for B_n-noncrossing partitions. Let

$$\pi = \{J_{-k}, \dots, J_{-1}, J_0, J_1, \dots, J_k\},$$

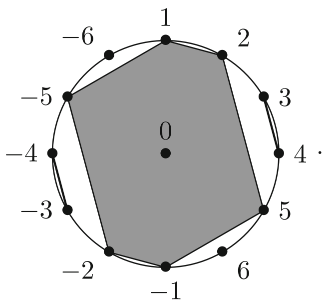
be a set partition of $\{0, \pm1, \pm2, \dots, \pm n\}$ such that:

- for each block J_i in π, the block $J_{-i} = -J_i = \{-s : s \in J_i\}$ is also a block of π, and
- if i and $-i$ are in the same block, that block is J_0, which we call the *zero block*. Note $0 \in J_0$ for every π.

We define the lattice $\mathrm{NC}(B_n)$ to be the set of all B_n-noncrossing partitions ordered by reverse refinement, so $\{\{0, 1, 2, \dots, n, -1, -2, \dots, -n\}\}$ is the maximal element and $\{\{0\}, \{1\}, \{2\}, \dots, \{n\}, \{-1\}, \{-2\}, \dots, \{-n\}\}$ is the minimal element. The rank of an element in this poset is given by half the number of nonzero blocks.

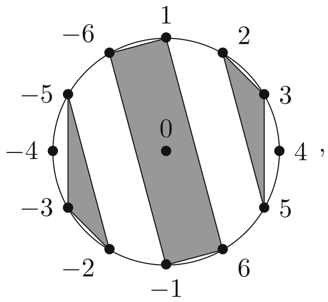
We can identify the canonical positive blocks $J_1, \dots, J_k$ as follows. Suppose a is the smallest positive number not in J_0. Then the block J containing a is J_1, and we order the nonzero blocks clockwise, until we reach $-J_1$.

For example, the partition below has positive blocks are $J_1 = \{3, 4\}$ and $J_2\{6\}$. The negative blocks are $J_{-1} = \{-3, -4\}$ and $J_{-2} = \{-6\}$, while the zero block is $J_0 = \{0, 1, 2, 5, -1, -2, -5\}$:



In Figure 13.5 we see $\mathrm{NC}(B_3)$.

According to our formula, $\mathrm{Cat}(B_n) = \binom{2n}{n}$. While this may not be obvious from the model, Reiner gives a bijection between type B_n noncrossing partitions and pairs of n-subsets with the same cardinality. Given a B_n-noncrossing partition π, let $L = L(\pi)$ denote the numbers that are the clockwise first numbers in a positive block, and let $R = R(\pi)$ denote the set of absolute values of numbers that are the clockwise last numbers in a positive block. For example, if π is the partition pictured here:

$\pi =$  ,

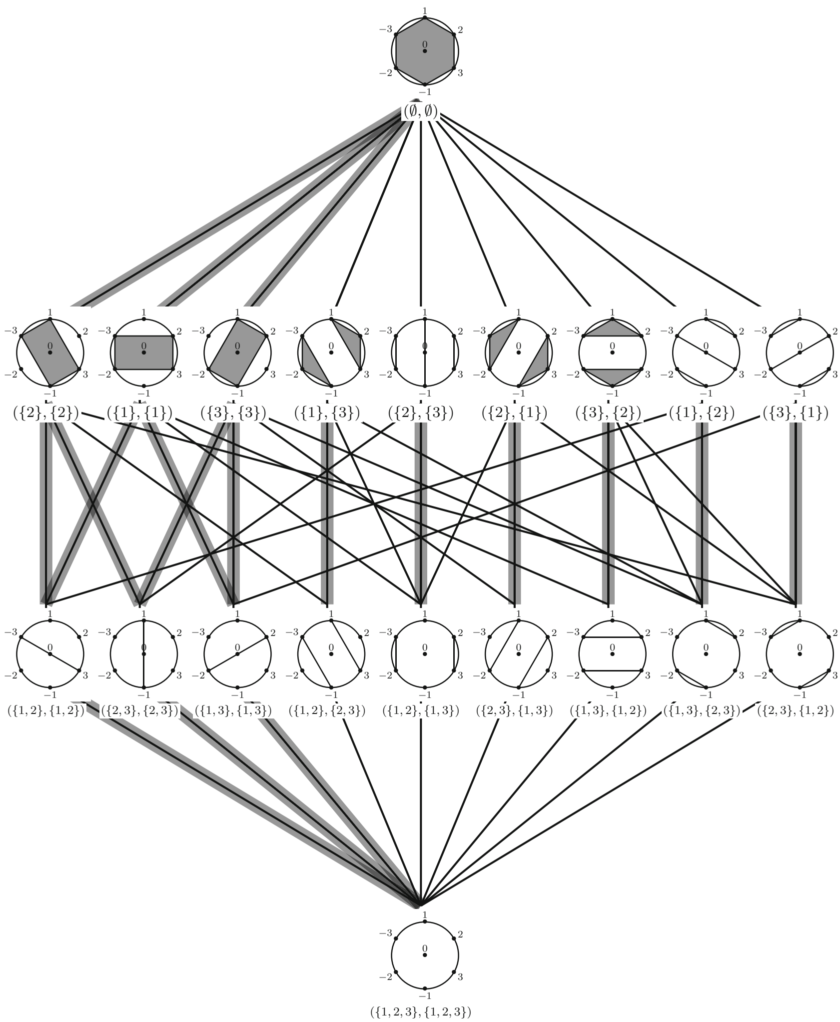
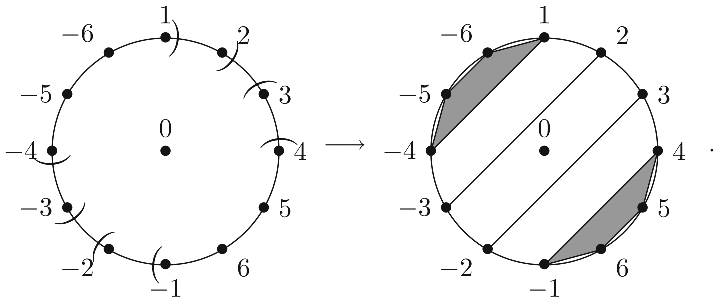


Fig. 13.5 The poset $\mathrm{NC}(B_3)$ realized as the interval below $c = (123\bar{1}\bar{2}\bar{3})$ in $\mathrm{Abs}(B_3)$. A symmetric boolean decomposition of the poset is highlighted in bold.

we have $L(\pi) = \{2,4\}$ and $R(\pi) = \{4,5\}$. Conversely, we can think of putting a left parenthesis to the left of l and $-l$ for each l in L and similarly we put a right parenthesis to the right of r and $-r$ for each r in R. Each layer the resulting parenthesization determines the blocks of the partition. For example, suppose $n = 6$ and $(L, R) = \{\{3,4\},\{1,2\}\}$. We get:



The bijection is denoted η, i.e, $\eta(\pi) = (L, R)$. The η map can be extremely helpful and we will return to it later, but we remark that it does not readily encode the poset structure of $\mathrm{NC}(B_n)$. While it is clear that if $\pi < \pi'$ in $\mathrm{NC}(B_n)$, then $L(\pi') \subset L(\pi)$ and $R(\pi') \subset R(\pi)$, the converse is not true. That is, simply because $L' \subset L$ and $R' \subset R$ it does not necessarily follow that the corresponding B_n-partitions are comparable in $\mathrm{NC}(B_n)$. For example, one can check in $\mathrm{NC}(B_5)$ that the partitions corresponding to $(L, R) = (\{2,3,4\},\{1,4,5\})$ and $(L', R') = (\{3,4\},\{1,5\})$ are incomparable.

We finish this section by remarking that since the $|L| = |R|$ equals the number of positive blocks, the rank of a B_n-noncrossing partition is $n-|L| = n-|R|$. Let $f(\mathrm{NC}(B_n);t)$ denote the rank generating function for $\mathrm{NC}(B_n)$. Then

$$f(\mathrm{NC}(B_n);t) = \sum_{\pi\in\mathrm{NC}(B_n)} t^{\mathrm{rk}(\pi)} = \sum_{\substack{L,R\subseteq\{1,\ldots,n\}\\ |L|=|R|}} t^{n-|L|} = \sum_{k=0}^{n} \binom{n}{k}^2 t^k.$$

This tells us that $\binom{n}{k}^2$ ought to be the type B_n analogue of the Narayana numbers.

Proposition 13.5. *The number of B_n-noncrossing partitions of rank $n-k$ is equal to the number of pairs (L, R) with $L, R \subseteq \{1,2,\ldots,n\}$ and $|L| = |R| = k$, i.e.,*

$$|\{\pi \in \mathrm{NC}(B_n) : \mathrm{rk}(\pi) = n-k\}| = \binom{n}{k}^2.$$

In particular, $\mathrm{NC}(B_n)$ *is rank-symmetric.*

As a consequence, there are

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n},$$

B_n-noncrossing partitions in all, confirming our formula for the type B_n Catalan numbers.

13.7 Gamma-nonnegativity for $\mathrm{Cat}(B_n; t)$

It is a not-too-difficult exercise to obtain an explicit formula for the gamma-basis coefficients of the polynomials

$$\mathrm{Cat}(B_n;t) = \sum_{k=0}^{n} \binom{n}{k}^2 t^k.$$

They are

$$\gamma_j^{\mathrm{NC}(B_n)} = \binom{n}{j, j, n-2j} = \frac{n!}{j!j!(n-2j)!}.$$

See Table 13.5. This fact was remarked upon in work of Alexander Postnikov, Reiner, and Lauren Williams from 2008 [120].

Table 13.5 The gamma numbers $\gamma_j^{\mathrm{NC}(B_n)}$, $0 \le 2j \le n \le 10$.

$n\backslash k$	0	1	2	3	4
2	1	2			
3	1	6			
4	1	12	6		
5	1	20	30		
6	1	30	90	20	
7	1	42	210	140	
8	1	56	420	560	70
9	1	72	756	1680	630

Rather than merely verify this numeric fact, we will demonstrate a symmetric boolean decomposition of $\mathrm{NC}(B_n)$. This was outlined in work of Saúl Blanco and the author in 2014 [26]. The maximal elements of the boolean pieces correspond, under Reiner's bijection η, to members of the set

$$P_n = \{(L, R) : L, R \subseteq \{1, \ldots, n\}, |L| = |R|, L \cap R = \emptyset\}.$$

Recall that each element π of $\mathrm{NC}(B_n)$ corresponds to a pair $\eta(\pi) = (L, R)$ with $L, R \subseteq \{1, \ldots, n\}$ and $|L| = |R|$. Moreover, if $\pi < \pi'$ in $\mathrm{NC}(B_n)$, then $L(\pi') \subset L(\pi)$ and $R(\pi') \subset R(\pi)$. The converse is not generally true, but if the same element i appears in both L and R, then $\{i\}$ is a singleton block of the corresponding partition. Hence, we have the following lemma.

Lemma 13.1. *Suppose $L, R \subseteq \{1, \dots, n\}$ with $|L| = |R|$, and that there is an element i in $L \cap R$. Let π be the partition corresponding to (L, R) and let π' be the partition corresponding to $(L \setminus \{i\}, R \setminus \{i\})$. Then*

$$\pi < \pi'$$

in $\mathrm{NC}(B_n)$.

For any pair of subsets with empty intersection in P_n, let

$$\mathcal{B}(L, R) = \{(L', R') : L' = L \cup A, R' = R \cup A, \text{ for some } A \subseteq \{1, \dots, n\} \setminus (L \cup R)\}.$$

Then $\mathcal{B}(L, R)$, ordered by reverse inclusion, is isomorphic to a boolean algebra on the complement of $L \cup R$. By Lemma 13.1, this partial order is consistent with the partial order on $\mathrm{NC}(B_n)$. That is, if $(L', R') \leq (L'', R'')$ in $\mathcal{B}(L, R)$, then $\eta^{-1}(L', R') \leq \eta^{-1}(L'', R'')$ in $\mathrm{NC}(B_n)$.

Hence, we can claim to have a symmetric boolean decomposition of $\mathrm{NC}(B_n)$.

Theorem 13.10. *We have a symmetric boolean decomposition of* $\mathrm{NC}(B_n)$ *given by:*

$$\bigcup_{(L,R) \in P_n} \eta^{-1} \left(\mathcal{B}(L, R) \right).$$

In particular,

$$\gamma_j^{\mathrm{NC}(B_n)} = |\{(L, R) : L, R \subseteq \{1, \dots, n\}, |L| = |R| = j, L \cap R = \emptyset\}|.$$

Figure 13.5 illustrates the theorem in the case of B_3, with the symmetric boolean decomposition highlighted in bold. We recall from Theorem 4.3 that $\mathrm{NC}(A_{n-1})$ has a symmetric boolean decomposition as well.

We finish this section by showing how to construct a simplicial complex—in fact a flag complex—whose f-vector is the gamma-vector above, thus establishing Conjecture 3 in the case of the B_n associahedron.

Let

$$P_n = \{(L, R) : L, R \subseteq \{1, \dots, n\}, |L| = |R|, L \cap R = \emptyset\}.$$

This set can be seen to form a simplicial complex with vertex set

$$V = \{(l, r) : 1 \leq l \neq r \leq n\}.$$

We declare that two vertices (l_1, r_1) and (r_1, r_2) are adjacent if and only if l_1, l_2, r_1, r_2 are all distinct and $l_1 < l_2$ if and only if $r_1 < r_2$. Then P_n is the clique complex for the adjacency graph, since we can consider every element of P_n as a list of vertices by ordering $L = \{l_1 < l_2 < \cdots\}$ and $R = \{r_1 < r_2 < \cdots\}$.

So while P_n is not itself a balanced complex, its f-vector is still an FFK-vector by Proposition 10.6.

13.8 Type D_n noncrossing partitions and Narayana numbers

Our combinatorial model for type D_n noncrossing partitions begins with choosing a particular Coxeter element,

$$c = s_{\bar{1}} s_1 s_2 \cdots s_{n-1} = (1\bar{1})(23 \cdots n\bar{2}\bar{3} \cdots \bar{n}).$$

Motivated by the geometry of the Coxeter arrangement as well as this cycle, we order $2, 3, \ldots, n, -2, -3, \ldots, -n$ clockwise around a circle and place 1 and -1 in the center of the circle, though they are not in the same point. It helps to think of 1 as slightly above the plane containing the circle, and -1 as lying slightly below, as we did with type D_n shard intersections and forked permutations.

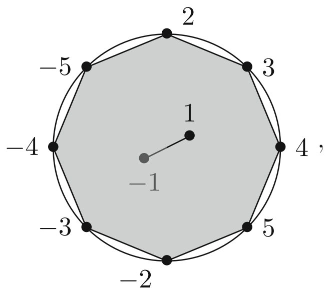
Let us now define the D_n-noncrossing partitions. Let

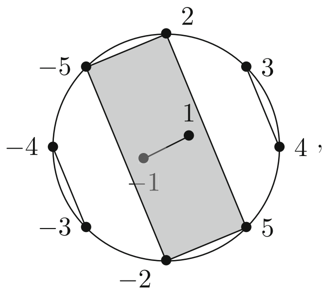
$$\pi = \{J_{-k}, \ldots, J_{-1}, J_0, J_1, \ldots, J_k\},$$

be a set partition of $\{\pm 1, \pm 2, \ldots, \pm n\}$. Then π is a D_n-signed partition if:

- each block J_i in π has its negative block $-J_i$ in π,
- there is at most one block equal to its negative, called the *zero block*, which may be empty, and
- if J_0 is not empty, it has at least four elements, and both -1 and 1 are in this block.

For example, here are some elements of $\mathrm{NC}(D_5)$:

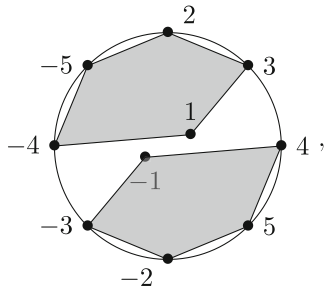
$c =$ 

, $c \cdot t_{24} =$ 

,

and

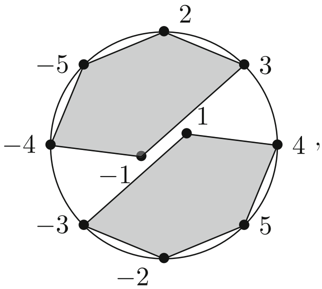
$$c \cdot t_{\bar{1}3} =$$



,

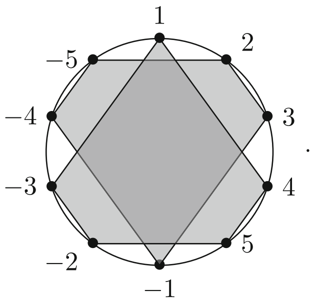
where t_{ij} denotes the reflection through hyperplane H_{ij}.

While the D_n-noncrossing partitions have some features in common with B_n-noncrossing partitions, they are not a subset of the B_n-noncrossing partitions. Notice, e.g., that the following D_n-noncrossing partition,



,

would have a crossing if drawn as a B_n-partition:



.

This model was developed by Christos Athanasiadis and Victor Reiner in 2004 [12]. Before this model was developed, Reiner came up with a plausible notion of a type D_n noncrossing partition that generalized the type A_n and B_n noncrossing partitions, but as a subposet of $\mathrm{NC}(B_n)$ [126]. However this poset is not isomorphic to the interval $[e, c]$ in the absolute order, and as the theory progressed, it was dropped in favor of the current model. Reiner's "false" type D_n noncrossing partitions are the B_n-noncrossing partitions such that $J_0 \neq \{-i, 0, i\}$ for any i, denoted $\mathrm{NC}(BD_n)$. While as a poset $\mathrm{NC}(BD_n)$

is not isomorphic to $[e, c]$, the number of such elements in each rank agrees with the same count for $\mathrm{NC}(D_n)$, i.e.,

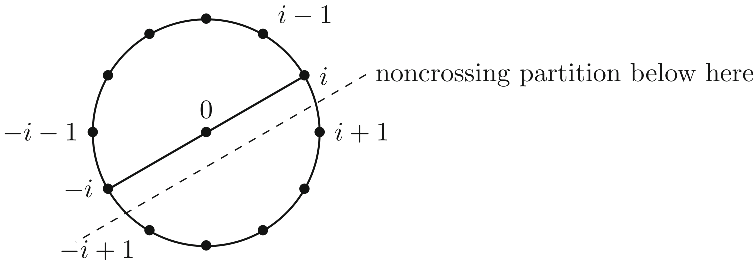
$$f(\mathrm{NC}(D_n);t) = \sum_{\pi \in \mathrm{NC}(D_n)} t^{\mathrm{rk}(\pi)} = \sum_{\pi' \in \mathrm{NC}(BD_n)} t^{\mathrm{rk}(\pi')} = f(\mathrm{NC}(BD_n);t),$$

so we can use either set for enumeration purposes. (There is no simple bijective proof of this identity at the time of this writing.) Rank enumeration in $\mathrm{NC}(BD_n)$ is relatively straightforward, as we now demonstrate. This idea follows [126].

First notice the set of all B_n-noncrossing partitions decomposes as:

$$\mathrm{NC}(B_n) = \mathrm{NC}(BD_n) \cup \bigcup_{i=1}^{n} \mathrm{NC}(B_n; i),$$

where $\mathrm{NC}(B_n; i)$ denotes the set of all B_n-noncrossing partitions with $J_0 = \{-i, 0, i\}$. Note that every element of $\mathrm{NC}(B_n; i)$ corresponds to a noncrossing partition on $\{i+1, i+2, \ldots, n, -1, -2, \ldots, -i+1\}$:



Hence the number of elements of $\mathrm{NC}(B_n; i)$ with k nonzero blocks is the number of classical noncrossing partitions in $\mathrm{NC}(n-1)$ with k blocks. This is the Narayana number

$$N_{n,k-1} = \frac{1}{n-1}\binom{n-1}{k-1}\binom{n-1}{k-2},$$

as seen from Proposition 2.3. Thus the number of elements in $\mathrm{NC}(BD_n)$ with k nonzero blocks is:

$$\begin{aligned}
|\{\pi \in \mathrm{NC}(BD_n) : \mathrm{rk}(\pi) = k\}| &= |\{\pi \in \mathrm{NC}(B_n) : \mathrm{rk}(\pi) = k\}| \\
&\quad - n \cdot |\{\pi \in \mathrm{NC}(B_n; i) : \mathrm{rk}(\pi) = k\}|, \\
&= \binom{n}{k}^2 - \frac{n}{n-1}\binom{n-1}{k-1}\binom{n-1}{k-2}, \\
&= \binom{n}{k}^2 - \binom{n}{k}\binom{n-2}{k-1}, \\
&= \binom{n}{k}\left(\binom{n-1}{k} + \binom{n-2}{k-2}\right).
\end{aligned}$$

Since $\mathrm{NC}(D_n)$ has the same rank numbers as $\mathrm{NC}(BD_n)$, we can use these numbers as the type D_n Narayana numbers.

Proposition 13.6. *The number of D_n-noncrossing partitions of with k nonzero blocks is*

$$\binom{n}{k}^2 - \frac{n}{n-1}\binom{n-1}{k-1}\binom{n-1}{k-2} = \binom{n}{k}\left(\binom{n-1}{k} + \binom{n-2}{k-2}\right).$$

Considering all such partitions without regard to rank, we get confirmation of the formula for type D_n Catalan numbers:

$$\begin{aligned}|\mathrm{NC}(D_n)| = |\mathrm{NC}(BD_n)| &= |\mathrm{NC}(B_n)| - n|\mathrm{NC}(n-1)|,\\ &= \binom{2n}{n} - \binom{2n-2}{n-1}.\end{aligned}$$

13.9 Gamma-nonnegativity for $\mathrm{Cat}(D_n; t)$

For the type D_n case, we will show that

$$\mathrm{Cat}(D_n;t) = \sum_{k=0}^{n} \binom{n}{k}\left(\binom{n-1}{k} + \binom{n-2}{k-2}\right) t^k,$$

has a nonnegative gamma vector, with

$$\gamma_j^{\mathrm{NC}(D_n)} = \frac{n-j-1}{n-1}\binom{n}{j, j, n-2j}.$$

See Table 13.6.

Table 13.6 The gamma numbers $\gamma_j^{\mathrm{NC}(D_n)}$, $0 \le 2j \le n \le 9$.

$n\backslash k$	0	1	2	3	4
4	1	8	2		
5	1	15	15		
6	1	24	54	8	
7	1	35	140	70	
8	1	48	300	320	30
9	1	63	567	1050	315

We can verify the formula by resorting again to the "false" D_n-noncrossing partitions, $\mathrm{NC}(BD_n)$. Recall the decomposition

$$\mathrm{NC}(B_n) = \mathrm{NC}(BD_n) \bigcup_{i=1}^{n} \mathrm{NC}(B_n;i),$$

where $\mathrm{NC}(B_n;i)$ consists of those B_n-noncrossing partitions with zero block $J_0 = \{-i,0,i\}$. More importantly, recall that $\mathrm{NC}(B_n;i) \cong \mathrm{NC}(A_{n-2})$, and that $\mathrm{NC}(B_n;i)$ embeds in ranks 1 to $n-1$ of $\mathrm{NC}(B_n)$. Hence,

$$\gamma_j^{\mathrm{NC}(B_n)} = \gamma_j^{\mathrm{NC}(BD_n)} + n\gamma_{j-1}^{\mathrm{NC}(A_{n-2})}.$$

Recalling the formulas for $\gamma_j^{\mathrm{NC}(B_n)}$ and $\gamma_{j-1}^{\mathrm{NC}(A_{n-2})}$, we have

$$\begin{aligned}
\gamma_j^{\mathrm{NC}(D_n)} &= \gamma_j^{\mathrm{NC}(BD_n)} \\
&= \binom{n}{j,j,n-2j} - n\left(\frac{1}{j}\binom{n-2}{j-1,j-1,n-2j}\right), \\
&= \binom{n}{j,j,n-2j} - \frac{j}{n-1}\binom{n}{j,j,n-2j}, \\
&= \frac{n-1-j}{n-1}\binom{n}{j,j,n-2j}.
\end{aligned}$$

We remark that one can show that $\mathrm{NC}(BD_n)$ admits a symmetric boolean decomposition, but it is not known whether the same is true for $\mathrm{NC}(D_n)$.

Chapter 14
Affine descents and the Steinberg torus (Supplemental)

14.1 Affine Weyl groups

In this section we outline some basic facts for affine Weyl groups, following standard notations. See Sections 4.3 and 4.6 of the book by James Humphreys for more details [92].

We now consider that Φ is an irreducible and *crystallographic* root system, i.e., $2\langle\alpha,\beta\rangle/\langle\beta,\beta\rangle$ is an integer for all roots α and β. These root systems are listed in Figure 11.4. The group $W = W(\Phi)$ is a finite Coxeter group, but there is an infinite Coxeter group associated with Φ as well, known as the *affine Weyl group*, and denoted $\widetilde{W}$. This is the group generated by reflections $s_{\beta,k}$ through the affine hyperplanes

$$H_{\beta,k} = \{\lambda \in V : \langle\lambda,\beta\rangle = k\},$$

where $\beta \in \Pi$ and $k \in \mathbb{Z}$.

Let $\Phi^\vee$ denote the set of *coroots*

$$\beta^\vee := 2\beta/\langle\beta,\beta\rangle,$$

with $\beta \in \Phi$. Composing two reflections $s_{\beta,k}$ corresponding to the same β corresponds to translation by a vector in $\mathbb{Z}\Phi^\vee$. Let $L = \mathbb{Z}\Phi^\vee$ denote this lattice of translations, a subgroup of $\widetilde{W}$. The affine group $\widetilde{W}$ also contains the finite group W, generated by reflections across the hyperplanes $H_{\beta,0} = H_\beta$.

The crystallographic condition guarantees that W fixes L, and we can write $\widetilde{W}$ as a semidirect product $L \rtimes W$. The product in the semidirect product is

$$(\mu, w)\cdot(\mu', w') = (\mu + w(\mu'), ww').$$

T.K. Petersen, *Eulerian Numbers*, Birkhäuser Advanced Texts Basler Lehrbücher, DOI 10.1007/978-1-4939-3091-3_14

The geometric action of $\widetilde{W}$ on V extends both the action of W by linear reflections and the action of L by translations:

$$(\mu, w) \cdot \lambda = \mu + w(\lambda),$$

for $\mu \in \mathbb{Z}\Phi^\vee$, $w \in W$, and $\lambda \in V$.

Since Φ is irreducible, there is a unique maximum in its root poset, known as the *highest root* and denoted $\widetilde{\alpha}$. The group $\widetilde{W}$ is generated by $\widetilde{S} = S \cup \{s_{\widetilde{\alpha},1}\}$. The pair $(\widetilde{W}, \widetilde{S})$ is an irreducible Coxeter system. The corresponding Coxeter graphs/Dynkin diagrams are shown in Figure 14.1. The graph for $\widetilde{W}$ differs from that of W by the addition of one node. Geometrically, the new simple root is the *lowest root* $\alpha_0 = -\widetilde{\alpha}$. If $\Delta = \{\alpha_1, \dots, \alpha_n\}$ denotes the set of simple roots for W, let us denote the nodes of the diagram by

$$\widetilde{\Delta} = \{\alpha_0\} \cup \Delta = \{\alpha_0, \alpha_1, \dots, \alpha_n\}.$$

Let $\widetilde{\Sigma}$ denote the set of faces of the affine hyperplane arrangement

$$\widetilde{\mathcal{H}}(\Phi) = \{H_{\beta,k} : \beta \in \Phi, k \in \mathbb{Z}\}.$$

By adding an empty face, $\widetilde{\Sigma}$ is a simplicial complex isomorphic to the Coxeter complex for $\widetilde{W}$. The maximal faces in this arrangement are called *alcoves* (as opposed to *chambers* in the finite case).

The *fundamental alcove* is

$$A_\emptyset = C_\emptyset \cap \{\lambda \in V : \langle \lambda, \widetilde{\alpha} \rangle < 1\},$$

where $C_\emptyset$ is the fundamental chamber of the finite Coxeter arrangement. We can write the faces of the fundamental alcove as

$$A_J = \begin{cases} C_J \cap \{\lambda \in V : \langle \lambda, \widetilde{\alpha} \rangle < 1\} & \text{if } \alpha_0 \notin J, \\ C_{J-\{\alpha_0\}} \cap \{\lambda \in V : \langle \lambda, \widetilde{\alpha} \rangle = 1\} & \text{if } \alpha_0 \in J, \end{cases}$$

where J is a proper subset of $\widetilde{\Delta}$ and C_J is a face of the fundamental chamber as in Section 11.7.

In Figure 14.2 we see the affine arrangement and faces of the fundamental alcove for $\mathbf{A}_2$. The same for $\mathbf{C}_2$ is in Figure 14.3.

14.2 Faces of the affine Coxeter complex

The closure of the fundamental alcove is a fundamental domain for the action of $\widetilde{W}$ on V, and each face of $\widetilde{\Sigma}$ is of the form

$$F = \mu + w \cdot A_J,$$

where $\mu \in L$, $w \in W$, and J is a proper subset of $\widetilde{\Delta}$.

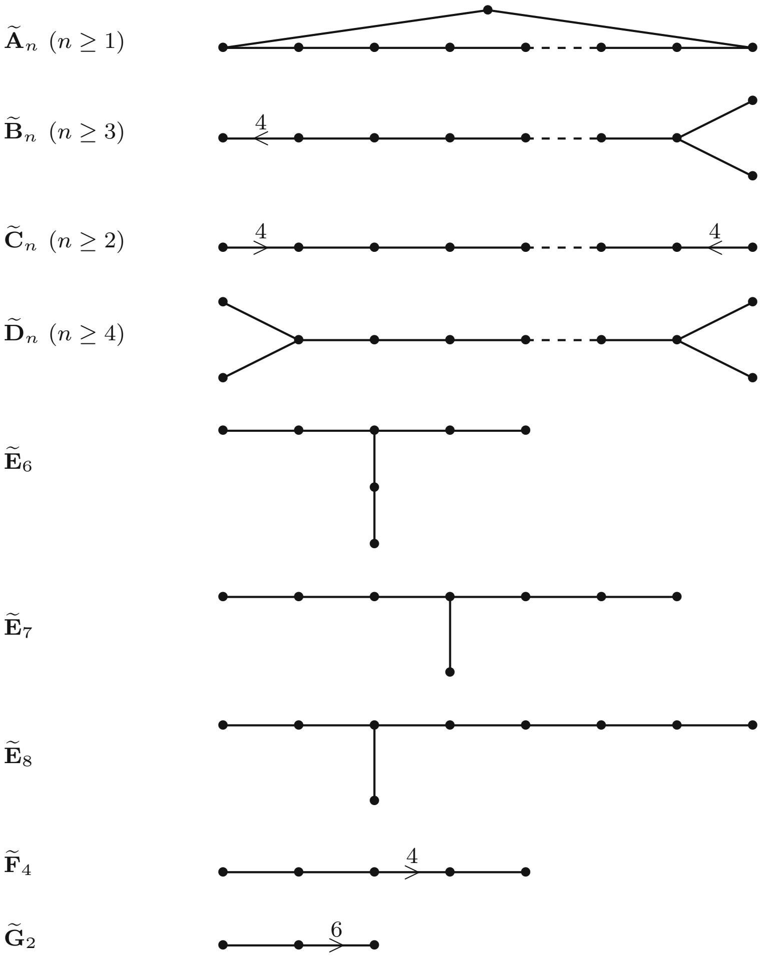


Fig. 14.1 The Dynkin diagrams for irreducible affine root systems.

The vertices of $\widetilde{\Sigma}$ are of the form $\mu + w \cdot A_{\widetilde{\Delta}-\{\alpha\}}$ for some $\alpha \in \widetilde{\Delta}$. If we assign color α to all such vertices, we obtain a balanced coloring of $\widetilde{\Sigma}$, with face $\mu + w \cdot A_J$ receiving color set $\widetilde{\Delta} - J$.

Each face F has a canonical representation, in the sense that we can identify F with a triple (μ, w, J), for $\mu \in L$, $w \in W$, and $J \subset \widetilde{\Delta}$. The uniqueness of μ is not surprising, since we can translate any face to a face in the neighborhood of the origin. The uniqueness of J follows from the fact that each face is in the orbit of a unique face of the closure of $A_\emptyset$. The finite group element w is unique up to right multiplication by the subgroup of W that fixes A_J. We can make the choice of w unique by declaring that, for any

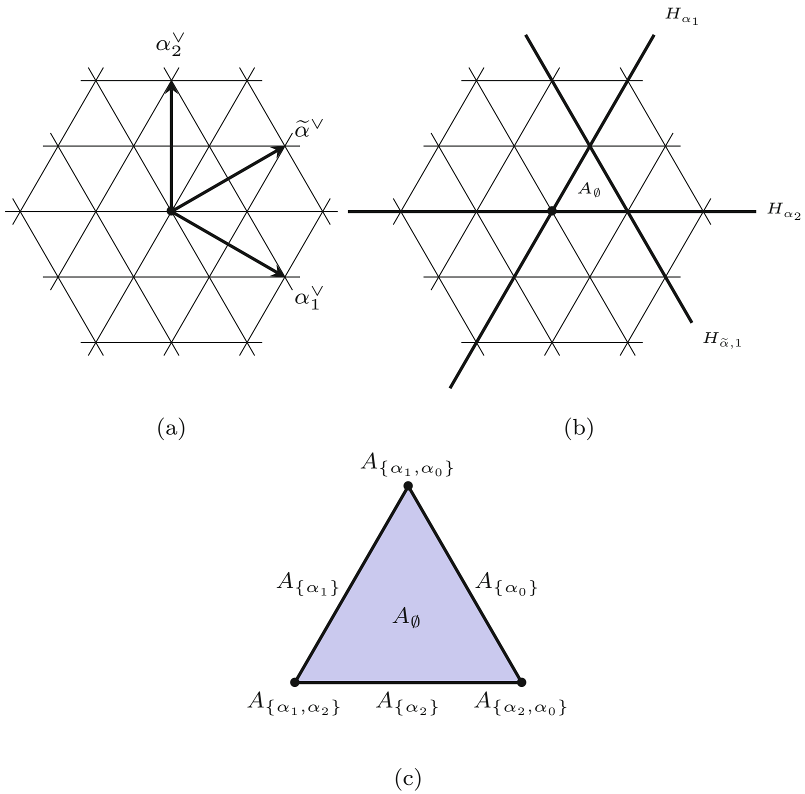


Fig. 14.2 The affine arrangement $\widetilde{\mathcal{H}}(A_2)$. (a) Positive (co)roots. (b) Affine hyperplanes and the fundamental alcove. (c) The faces of the fundamental alcove.

$\alpha \in \widetilde{\Delta}$, if $w(\alpha) < 0$, then $\alpha \in \widetilde{\Delta} - J$. Following Paola Cellini [47], we define the *affine descent set* of an element of the finite group W to be

$$\begin{aligned}\widetilde{\mathrm{Des}}(w) &= \{\alpha \in \widetilde{\Delta} : w(\alpha) < 0\}, \\ &= \begin{cases}\mathrm{Des}(w) & \text{if } w(\alpha_0) > 0, \\ \mathrm{Des}(w) \cup \{\alpha_0\} & \text{if } w(\alpha_0) < 0.\end{cases}\end{aligned}$$

Notice that since α_0 is a negative root, this means every element $w \in W$ has at least one affine descent, including the identity. We can state the uniqueness of the representation as follows.

In the case of type A_{n-1}, we will see in Section 14.4.1 $\widetilde{\mathrm{Des}}(w)$ is the "cyclic" descent set of a permutation, i.e., the usual descent set along with a descent in zero if the last letter is larger than the first.

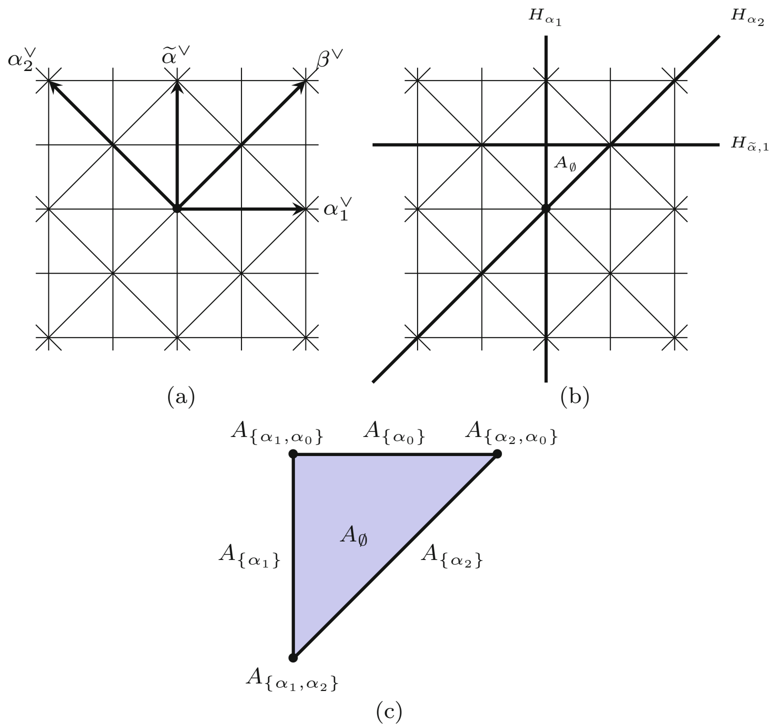


Fig. 14.3 The affine arrangement $\widetilde{\mathcal{H}}(C_2)$. (a) Positive coroots: $\alpha_1^\vee = \frac{1}{2}\alpha_1$, $\widetilde{\alpha}^\vee = \frac{1}{2}\widetilde{\alpha}$, $\alpha_2^\vee = \alpha_2$, $\beta^\vee = \beta$. (b) Affine hyperplanes and the fundamental alcove. (c) The faces of the fundamental alcove.

Proposition 14.1. *Each face $F \in \widetilde{\Sigma}$ has a unique representation*

$$F = \mu + w \cdot A_J,$$

with $\mu \in L$, $J \subset \widetilde{\Delta}$, and $w \in W$ such that $\widetilde{\mathrm{Des}}(w) \subseteq \widetilde{\Delta} - J$.

By analogy with the usual Eulerian polynomial, it now makes sense to define the *affine Eulerian polynomial* to be the generating function for *affine Eulerian numbers.* Let $\widetilde{\mathrm{des}}(w) = |\widetilde{\mathrm{Des}}(w)|$, and write

$$\widetilde{W}(t) = \sum_{w \in W} t^{\widetilde{\mathrm{des}}(w)}.$$

Before we study this polynomial and its coefficients, let us first describe a structure for which it is the h-polynomial.

14.3 The Steinberg torus

The coroot lattice L acts on V by translations and fixes the affine hyperplane arrangement $\widetilde{\mathcal{H}}$. Thus we can consider the set of L-orbits of faces of $\widetilde{\Sigma}$. The *Steinberg torus* is this quotient set of faces modulo translations, denoted by $\overline{\Sigma}$, i.e.,

$$\overline{\Sigma} = \widetilde{\Sigma}/L.$$

Geometrically, we can identify the Steinberg torus with a triangulation of the geometric torus V/L. This cell decomposition is not a simplicial complex, as different faces can share the same vertex set, but it is a boolean complex. Moreover, the balanced coloring for $\widetilde{\Sigma}$ passes through the quotient, so we inherit a balanced coloring for $\overline{\Sigma}$ as well.

The Steinberg torus is named for Robert Steinberg, who exploited the torus to help compute the length generating function for the affine Weyl group [157]. It was studied again (and named) by Kevin Dilks, John Stembridge, and the author in 2009 [60]. The presentation here largely follows [60].

Each face of $\widetilde{\Sigma}$ has in its orbit a cell in its L-orbit with $\mu = 0$, so another way to define the Steinberg torus is to identify opposite faces of the polytope

$$P_{\Phi} = \{\lambda \in V : -1 \leq \langle \lambda, \beta \rangle \leq 1 \text{ for all } \beta \in \Phi\}.$$

This polytope is the union of the closures of the alcoves $w \cdot A_{\emptyset}$, with $w \in W$. A point λ on the boundary of P_{Φ} has $\langle \lambda, \beta \rangle = -1$ for some root β. We identify λ with $\lambda' = \lambda + \beta^{\vee}$ which satisfies $\langle \lambda', \beta \rangle = 1$ and also lies on the boundary. See Figure 14.4.

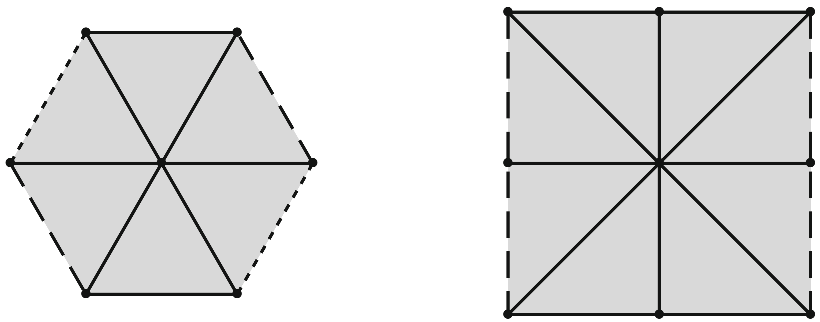


Fig. 14.4 The polytopes $P_{\mathbf{A}_2}$ and $P_{\mathbf{C}_2}$. The Steinberg tori are obtained by identifying points on the boundary.

From Proposition 14.1 we see that we can abstractly identify the faces of $\overline{\Sigma}$ with the cosets of "quasi-parabolic" subgroups of W, i.e., for any proper subset $J \subset \widetilde{\Delta}$,

$$L + w \cdot A_J \leftrightarrow wW_J = \{wv : v \in \langle s_\alpha : \alpha \in J\rangle\}.$$

As in Proposition 14.1, we can choose a unique minimal length representative w such that $\widetilde{\mathrm{Des}}(w) \subseteq \widetilde{\Delta} - J$.

Proposition 14.2. *Every face $F \in \overline{\Sigma}$ has a unique $J \subseteq \widetilde{\Delta}$ and $w \in W$ such that*

$$F = L + w \cdot A_J,$$

with $\widetilde{\mathrm{Des}}(w) \subseteq \widetilde{\Delta} - J$.

We call such subgroups W_J "quasi-parabolic" since although they are parabolic subgroups of $\widetilde{W}$, they are not necessarily parabolic subgroups of W. Such a group is always a finite Coxeter group, however, and a subgroup of W.

Just as the model of set compositions can be used to encode faces of the type A_{n-1} Coxeter complex, there is a similar combinatorial model to encode faces of the type A_{n-1} Steinberg torus, developed by Marcelo Aguiar and the author [2]. See Figure 14.5.

Also noteworthy at this point is that, unlike for Coxeter complexes, the distinction between types B_n and C_n really matters. This is because the structure of the torus is intimately linked with the root system, not merely the group. When $n \geq 3$, the polytopes P_Φ have very different boundaries, despite having the same number of maximal cells. In particular, the polytope $P_{\mathbf{C}_3}$ is a cube, while the polytope $P_{\mathbf{B}_3}$ is a rhombic dodecahedron. The identifications taking place on their boundaries lead to a different triangulated torus. In fact $\overline{\Sigma}(\mathbf{C}_3)$ and $\overline{\Sigma}(\mathbf{B}_3)$ don't even have the same number of vertices (eight and ten, respectively).

We now turn to the f- and h-vectors of the Steinberg torus. To count faces we use a similar line of reasoning as in the case of the finite Coxeter complex to count W-orbits. First, define f_J to be the number of faces of $\overline{\Sigma}$ with color set J, ignoring the empty face. Ignoring the empty face simply omits the constant term from the f-polynomial. However, omitting $f_\emptyset = 1$ has the effect of making the corresponding h-polynomial palindromic. In general, while the Dehn-Sommerville relations for a torus are not palindromic, they can be made so by ignoring the empty face. This idea was generalized to other triangulated manifolds by Isabella Novik and Ed Swartz in 2009 [113].

Now, for any nonempty subset $\emptyset \neq J \subseteq \widetilde{\Delta}$,

$$\begin{aligned} f_J &= |\{w \cdot A_{\widetilde{\Delta}-J} : w \in W\}|, \\ &= |W/W_{\widetilde{\Delta}-J}|, \\ &= |W|/|W_{\widetilde{\Delta}-J}|, \\ &= |\{w \in W : \widetilde{\mathrm{Des}}(w) \subseteq J\}|. \end{aligned}$$

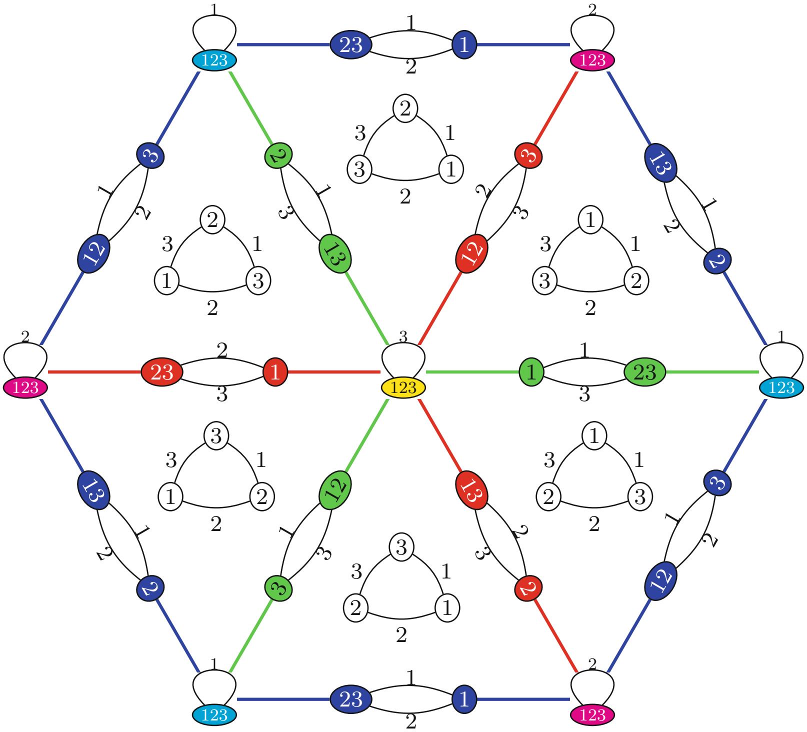


Fig. 14.5 The faces of the Steinberg torus $\overline{\Sigma}(A_2)$, with colors corresponding to W-orbits. Note the identifications along the boundary.

Define h_J to be

$$h_J = |\{w \in W : \widetilde{\mathrm{Des}}(w) = J\}|,$$

so that by inclusion-exclusion

$$h_J = \sum_{\emptyset \neq I \subseteq J} (-1)^{|J-I|} f_I.$$

Now we can express the affine Eulerian polynomial as follows:

$$\begin{aligned}
\widetilde{W}(t) &= \sum_{w \in W} t^{\widetilde{\mathrm{des}}(w)}, \\
&= \sum_{\emptyset \neq J \subseteq \widetilde{\Delta}} h_J t^{|J|}, \\
&= \sum_{\emptyset \neq I \subseteq J \subseteq \widetilde{\Delta}} (-1)^{|J-I|} f_I t^{|J|}, \\
&= \sum_{\emptyset \neq I \subseteq \widetilde{\Delta}} f_I t^{|I|} (1-t)^{n+1-|I|},
\end{aligned}$$

Using our calculation for f_J from above, we can give the following expression for the affine Eulerian polynomial.

Proposition 14.3. *The affine Eulerian polynomial has the following expression,*

$$\widetilde{W}(t) = \sum_{w \in W} t^{\widetilde{\mathrm{des}}(w)} = \sum_{\emptyset \neq I \subseteq \widetilde{\Delta}} \frac{|W|}{|W_{\widetilde{\Delta}-I}|} t^{|I|}(1-t)^{n+1-|I|}.$$

Furthermore, we can see that

$$\begin{aligned} \widetilde{W}(t) &= (1-t)^{n+1} f(\overline{\Sigma} - \{\emptyset\}; t/(1-t)), \\ &= h(\overline{\Sigma} - \{\emptyset\}; t). \end{aligned}$$

That is, the affine Eulerian polynomial is the h-polynomial of the Steinberg torus (ignoring the empty face).

14.4 Affine Eulerian numbers

We now describe the combinatorial definitions of affine descents and give some enumerative results, all of which are contained in [60]. Most generally, we can state the following fact.

Theorem 14.1. *The affine Eulerian polynomial $\widetilde{W}(t)$ is gamma-nonnegative for all finite Weyl groups W.*

It is known that $\widetilde{W}(t)$ is real-rooted in all cases except $\widetilde{D}_n$. See Section 3.5 of the paper of Carla Savage and Mirko Visontai [132].

14.4.1 Type A_{n-1}

The highest root in $\mathbf{A}_{n-1}$ is $\varepsilon_n - \varepsilon_1$, so $\alpha_0 = \varepsilon_1 - \varepsilon_n$. Thus $w \cdot \alpha_0 < 0$ whenever $w(n) > w(1)$. Therefore

$$\widetilde{\mathrm{Des}}(w) = \{0 \leq i \leq n-1 : w(i) > w(i+1)\},$$

with $w(0) = w(n)$. These are better known as "cyclic descents" since we think of the permutation w wrapping around so that we compare $w(n)$ and $w(1)$. For example, $\widetilde{\mathrm{Des}}(25413) = \{0, 2, 3\}$.

In Table 14.1 we see the affine Eulerian numbers of type A_{n-1}, i.e., the distribution of cyclic descents over the symmetric group.

Table 14.1 The affine Eulerian numbers for A_{n-1}, $0 \leq k \leq n \leq 9$.

$n\backslash k$	0	1	2	3	4	5	6	7	8
2	0	2							
3	0	3	3						
4	0	4	16	4					
5	0	5	55	55	5				
6	0	6	156	396	156	6			
7	0	7	399	2114	2114	399	7		
8	0	8	960	9528	19328	9528	960	8	
9	0	9	2223	38637	140571	140571	38637	2223	9

Cyclic descents were studied by Jason Fulman for their connections to card shuffling ("riffle shuffles with a cut") in a 2000 paper [77] and also by the author in 2005 [115]. Both papers give simple arguments for the following observation.

Observation 14.1 *For any $n \geq 2$,*

$$\widetilde{A}_n(t) = (n+1)tA_{n-1}(t),$$

where $A_{n-1}(t) = S_n(t)$ is the classical Eulerian polynomial.

Hence, $\widetilde{A}_n(t)$ is real-rooted and gamma-nonnegative from what we know in the classical case. Moreover, the following generating function is easily obtained.

Proposition 14.4. *We have the following exponential generating function for affine Eulerian polynomials:*

$$z + \sum_{n \geq 2} \widetilde{A}_{n-1}(t)\frac{z^n}{n!} = \frac{z(1-t)}{1-te^{z(1-t)}}.$$

14.4.2 Type B_n

In type $\mathbf{B}_n$, the highest root is $\varepsilon_{n-1} + \varepsilon_n$, so $w \cdot \alpha_0 < 0$ if and only if $w(n-1) + w(n) > 0$. That is, we have a descent in 0 if $w(n-1) > -w(n)$. We have in this case,

$$\widetilde{\mathrm{Des}}(w) = \begin{cases} \mathrm{Des}(w) & \text{if } w(n-1) < -w(n), \\ \mathrm{Des}(w) \cup \{\alpha_0\} & \text{if } w(n-1) > -w(n). \end{cases}$$

For example, $\widetilde{\mathrm{Des}}(23\bar{4}5\bar{1}) = \{0, 3, 5\}$.

The type B_n affine Eulerian numbers are in Table 14.2.

Table 14.2 The affine Eulerian numbers for B_n, $0 \le k \le n \le 9$.

$n\backslash k$	0	1	2	3	4	5	6	7	8	9
2	0	4	4							
3	0	10	28	10						
4	0	24	168	168	24					
5	0	54	904	1924	904	54				
6	0	116	4452	18472	18472	4452	116			
7	0	242	20612	157294	288824	157294	20612	242		
8	0	496	91600	1227504	3841360	3841360	1227504	91600	496	
9	0	1006	396112	8989576	45616432	75788308	45616432	8989576	396112	1006

The type B_n affine Eulerian polynomial has a nonnegative gamma vector reminiscent of the type D_n Eulerian polynomials.

Proposition 14.5. *For $n \ge 2$, we have*

$$\widetilde{B}_n(t) = \sum_{u \in S_n} \phi(u)(4t)^{\mathrm{pk}(0u0)}(1+t)^{n+1-2\,\mathrm{pk}(0u0)},$$

where

$$\phi(u) = \begin{cases} 1 & \text{if } u(n-2) > u(n-1) > u(n), \\ 0 & \text{if } u(n-2) > u(n) > u(n-1), \\ 1/2 & \text{otherwise.} \end{cases}$$

Moreover, we have the following generating function.

$$2 + 2tz + \sum_{n \ge 2} \widetilde{B}_n(t)\frac{z^n}{n!} = \frac{2(1-t)(1-tze^{z(1-t)})}{1-te^{2z(1-t)}}.$$

Savage and Visontai proved in 2015 that $\widetilde{B}_n(t)$ is real-rooted [132].

14.4.3 Type C_n

In type $\mathbf{C}_n$, the highest root is $2\varepsilon_n$, and so we have a descent in α_0 if and only if $w(n) > 0$. Thus,

$$\widetilde{\mathrm{Des}}(w) = \{0 \le i \le n : w(i) > w(i+1)\},$$

with $w(0) = w(n+1) = 0$. For example, $\widetilde{\mathrm{Des}}(23\bar{4}5\bar{1}) = \{3, 5\}$. The type C_n affine Eulerian numbers are in Table 14.3.

Table 14.3 The affine Eulerian numbers for C_n, $0 \le k \le n \le 9$.

$n\backslash k$	0	1	2	3	4	5	6	7	8	9
2	0	4	4							
3	0	8	32	8						
4	0	16	176	176	16					
5	0	32	832	2112	832	32				
6	0	64	3648	19328	19328	3648	64			
7	0	128	15360	152448	309248	152448	15360	128		
8	0	256	63232	1099008	3998464	3998464	1099008	63232	256	
9	0	512	257024	7479296	45175808	79969280	45175808	7479296	257024	512

The type C_n affine descent set can be thought of as a special kind of cyclic descent, and indeed we have the following connection with classical Eulerian polynomials. Just as with Observation 14.1 this observation was proved both by Fulman in [77] and the author in [115].

Observation 14.2 *For any $n \geq 1$,*

$$\widetilde{C}_n(t) = 2^n t A_{n-1}(t).$$

From this observation it follows that $\widetilde{C}_n(t)$ is real-rooted and gamma-nonnegative. We can express its gamma vector in terms of the classical case. Moreover, we have the following generating function.

Proposition 14.6. *We have the following exponential generating function:*

$$1 + \sum_{n\geq 1} \widetilde{C}_n(t)\frac{z^n}{n!} = \frac{1-t}{1-te^{2z(1-t)}}.$$

14.4.4 Type $\mathbf{D}_n$

The highest root for $\mathbf{D}_n$ is the same as the highest root in $\mathbf{B}_n$, with the same effect on combinatorial descents. We have an affine descent for an element $w \in D_n$ if $w(i) > w(i+1)$ for $i = 1, \ldots, n-1$ in the usual way, along with a descent at the beginning if $-w(1) > w(2)$, and another at the end if $w(n-1) > -w(n)$. For example, $\widetilde{\mathrm{Des}}(3\bar{4}2\bar{1}5) = \{0, -1, 1, 3\}$, since $w(1) > w(2)$, $w(-1) > w(2)$, $w(3) > w(4)$, and $w(4) > -w(5)$. See Table 14.4.

The type D_n affine Eulerian polynomial has a nonnegative gamma vector as well.

Table 14.4 The affine Eulerian numbers for D_n, $0 \le k \le n \le 9$.

$n\backslash k$	0	1	2	3	4	5	6	7	8	9
4	0	16	80	80	16					
5	0	44	464	904	464	44				
6	0	104	2568	8848	8848	2568	104			
7	0	228	13192	79580	136560	79580	13192	228		
8	0	480	63904	665568	1850528	1850528	665568	63904	480	
9	0	988	296608	5232400	22833760	36169768	22833760	5232400	296608	988

Proposition 14.7. *For $n \ge 4$, we have*

$$\widetilde{D}_n(t) = \sum_{u \in S_n} \phi(u)\phi(\overleftarrow{u})(4t)^{\mathrm{pk}(0u0)}(1+t)^{n+1-2\,\mathrm{pk}(0u0)},$$

where $\overleftarrow{u} = u(n)\cdots u(2)u(1)$, and ϕ is the same as in Proposition 14.5. Moreover, we have the following generating function:

$$2 + 4t\frac{z^2}{2} + \sum_{n \ge 3} \widetilde{D}_n(t)\frac{z^n}{n!} = \frac{2(1-t)(1+tz^2-2tze^{z(1-t)})}{1-te^{2z(1-t)}}.$$

We finish by remarking that the polynomial $\widetilde{D}_n(t)$ is the only case of an affine Eulerian polynomial for which real-rootedness is not proved.

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T.K. Petersen, *Eulerian Numbers*, Birkhäuser Advanced Texts Basler Lehrbücher, DOI 10.1007/978-1-4939-3091-3

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Hints and Solutions

Problems of Chapter 1

1.1 Compositions of n are in bijection with subsets of $\{1, 2, \ldots, n-1\}$ via the map that takes a composition α to its partial sums. That is, if $\alpha = (\alpha_1, \ldots, \alpha_k)$ is a composition of n with $k > 1$ parts, then

$$(\alpha_1, \ldots, \alpha_k) \leftrightarrow \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \cdots + \alpha_{k-1}\}.$$

For example, $(3, 1, 1, 2) \leftrightarrow \{3, 4, 5\}$. If α has only one part, i.e., if $\alpha = (n)$, then $\alpha \leftrightarrow \emptyset$.

We can visualize the bijection between compositions and subsets by picturing n stones lined up in a row, with bars placed in between the stones (at most one bar per gap):

$$\bullet\bullet\bullet|\bullet|\bullet|\bullet\bullet.$$

If we list the number of stones in each group, we get a composition of n (here, $(3, 1, 1, 2)$), while if we record the number of stones to the left of each bar, we get a subset of $n-1$ (here, $\{3, 4, 5\}$).

1.2 In each case, the answer is given by the Fibonacci numbers:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots,$$

with $f_0 = f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$, though the indexing needs to be adjusted to fit the first two cases.

1. If $n = 1$ there is only one such composition: (1), and if $n = 2$, there are two: $(1, 1)$ and (2). From here, we can build "1–2" compositions of n recursively. Let c_n denote the number of such compositions. To each of the c_{n-1} 1–2 compositions of $n-1$, we add a new part of size 1 on the right:

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$(\alpha_1, \ldots, \alpha_k) \mapsto (\alpha_1, \ldots, \alpha_k, 1)$, and to each of the c_{n-2} 1–2 compositions of $n-2$, we add a new part of size 2 on the right: $(\beta_1, \ldots, \beta_l) \mapsto (\beta_1, \ldots, \beta_l, 2)$. Since every 1–2 composition of n finishes with either a 1 or a 2, this produces all such compositions, and we get the recurrence: $c_n = c_{n-1} + c_{n-2}$, with $c_1 = 1$ and $c_2 = 2$. Thus, $c_n = f_n$, $n \geq 1$.

2. For compositions with odd parts, the smallest examples are (1) and $(1, 1)$. A recursive procedure for obtaining odd compositions of n either adds a new part of size 1 to the end of a composition of $n - 1$: $(\alpha_1, \ldots, \alpha_k) \mapsto (\alpha_1, \ldots, \alpha_k, 1)$, or else adds 2 to the final part of a composition of $n - 2$: $(\beta_1, \ldots, \beta_l) \mapsto (\beta_1, \ldots, \beta_l + 2)$. This procedure is reversible, since every odd composition of n either ends with a 1 or with an odd number greater than 1. If c_n now denotes the number of odd compositions of n, we get $c_1 = c_2 = 1$, and $c_n = c_{n-1} + c_{n-2}$ for $n \geq 3$. Thus $c_n = f_{n-1}$ for $n \geq 1$.
3. Now let c_n denote the number of compositions of n whose parts are at least 1, except possibly the last. The first examples are (1) and (2), so $c_1 = c_2 = 1$. For $n \geq 3$, we form such a composition by either adding 1 to the final part of such a composition of $n - 1$: $(\alpha_1, \ldots, \alpha_k) \mapsto (\alpha_1, \ldots, \alpha_k + 1)$, or by taking such a composition of $n - 2$ and adding both 1 to the final part and a new part of size 1: $(\beta_1, \ldots, \beta_l) \mapsto (\beta_1, \ldots, \beta_l + 1, 1)$.

1.3 Let's count 1–2 compositions according to the number of 2s we use. To create a composition α of n using only parts of size 1 and 2, we observe that:

- If there are no 2s, there is only one such composition: $\alpha = (1, 1, \ldots, 1)$.
- If there is one 2, there are $n - 2$ ones, and $n - 1$ entries of α in all. We have to choose which of the $n - 1$ entries of α will be occupied by the 2. There are thus $\binom{n-1}{1}$ such compositions.
- If there are two 2s, there are $n - 4$ ones, and $n - 2$ entries of α in all. We have to choose where the 2s go, and this can be done in $\binom{n-2}{2}$ ways.

  $\vdots$
- If there are k 2s, there are $n - 2k$ ones, and $n - k$ entries in α. We can choose where the 2s go in $\binom{n-k}{k}$ ways.

  $\vdots$

Summing over all k gives the desired formula.

1.4 We have

$$\begin{aligned}
\phi_n &= \frac{f_n}{f_{n-1}}, \\
&= \frac{f_{n-1} + f_{n-2}}{f_{n-1}}, \\
&= 1 + \frac{1}{\phi_{n-1}},
\end{aligned}$$

and taking the limit on both sides gives $\phi = 1 + 1/\phi$. Thus, $\phi^2 - \phi - 1 = 0$. Solving for ϕ, we find

$$\phi = \frac{1+\sqrt{5}}{2}.$$

1.5 Letting $f(z)$ be the generating function we seek, the recurrence relation gives us:

$$\begin{aligned} f(z) &= 1 + z\left(1 + z + 2z^2 + 3z^3 + \cdots\right) \\ &\qquad + z^2\left(1 + z + 2z^2 + 3z^3 + \cdots\right), \\ &= 1 + zf(z) + z^2 f(z). \end{aligned}$$

Solving for $f(z)$, we have

$$f(z) = \frac{1}{1 - z - z^2},$$

which answers part 1. We can also write

$$f(z) = \frac{1}{1 - z(1+z)},$$

from which part 2 follows by the binomial theorem:

$$\begin{aligned} f(z) &= 1 + z(1+z) + z^2(1+z)^2 + z^3(1+z)^3 + \cdots, \\ &= \sum_{k\geq 0} z^k \left(\sum_{j=0}^{k} \binom{k}{j} z^j\right), \\ &= \sum_{l\geq 0} \left(\sum_{j\geq 0} \binom{l-j}{j}\right) z^l. \end{aligned}$$

To address part 3, we first use the method of partial fractions to verify that:

$$f(z) = \frac{1}{(1-\alpha z)(1-\beta z)} = \frac{A}{1-\alpha z} + \frac{B}{1-\beta z},$$

with $A = \alpha/(\alpha - \beta)$ and $B = \beta/(\beta - \alpha)$. Expanding each of these terms individually yields

$$\begin{aligned} f(z) &= A\left(1 + \alpha z + \alpha^2 z^2 + \cdots\right) + B\left(1 + \beta z + \beta^2 z^2 + \cdots\right), \\ &= \sum_{k\geq 0} \left(A\alpha^k + B\beta^k\right) z^k. \end{aligned}$$

We can write the coefficient of z^k purely in terms of α and β as:

$$A\alpha^k + B\beta^k = \frac{\alpha^{k+1} - \beta^{k+1}}{\alpha - \beta}.$$

Now, factoring $1 - z - z^2$ as $(1 - \alpha z)(1 - \beta z)$ shows us

$$\alpha = \phi = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \beta = \overline{\phi} = \frac{1-\sqrt{5}}{2},$$

where ϕ is the golden ratio. So the kth Fibonacci number is

$$f_k = \frac{\phi^{k+1} - \overline{\phi}^{k+1}}{\phi - \overline{\phi}} = \frac{\phi^{k+1} - \overline{\phi}^{k+1}}{\sqrt{5}},$$

as desired.

1.6

1. If we transform a permutation by mapping $w(i)$ to $n + 1 - w(i)$, we swap ascents for descents, e.g., $351246 \mapsto 427631$. Thus the number of permutations with k descents equals the number of permutations with k ascents.
2. The descents of a permutation occur in between the runs of a permutation. This is easily seen if we put a bar between the runs of a permutation, e.g., $137|4|26|5$. Thus the number of descents is one less than the number of runs, i.e., the number of permutations with k descents equals the number of permutations with $k + 1$ runs.
3. The readings of a permutation w correspond to runs in the inverse permutation, w^{-1}. That is, in a reading of w, we look to see if 1 is to the left of 2, 2 is left of 3, and so on, i.e., if $w^{-1}(1) < w^{-1}(2) < \cdots$. We have to start a new reading whenever $w^{-1}(i) > w^{-1}(i+1)$. To take the example of $w = 1374265$, we have $w^{-1} = 1524763$ and listing the positions in which we read $1, 2, 3, \ldots$ in w correspond to the elements of the runs of w^{-1}.

times w read	1	3	7	4	2	6	5
1	1				2		
2		3		4			5
3						6	
4			7				

runs of w^{-1}	1	5	2	4	7	6	3
1	1	5					
2			2	4	7		
3						6	
4							3

Thus the number of permutations with k readings equal the number of permutations with k runs.

1.7 We can use Foata and Schützenberger's "transformation fondamentale." The idea is as follows. When writing a permutation in cycle notation, excedances correspond to ascents within cycles, e.g., in cycle notation $w = 1376245 = (1)(2375)(46)$. We can choose a canonical way to write the cycles of w so that the permutation obtained by forgetting parentheses has all its ascents coming from within cycles.

There is more than one way to do this. Our choice is to declare that *standard cycle notation* writes each cycle so that it begins with the smallest element of the cycle, and the cycles are written so that their minimal elements appear in decreasing order.

For example, we would write the cycle $1 \mapsto 7 \mapsto 6 \mapsto 2 \mapsto 1$ as (1762), and never (7621), (6217), or (2176). Writing cycles this way guarantees that the ascents of the word representing the cycle correspond precisely to excedances in the permutation. Writing (7621) in this case disguises the fact that there is an excedance in position 1.

We write the cycles so that the smallest element of each cycle is smaller than the one that follows it so that the last element of each cycle is greater than the first element of the next cycle. For example, if $w = 137482695$, its cycles are (1), (2376), (4), and (589) and putting these in decreasing order according to their smallest element yields

$$(589)(4)(2376)(1).$$

Now if we drop the parentheses, the resulting permutation, call it v, only has ascents where the cycles have ascents: $v = 589423761$.

This transformation is straightforward to reverse. Define a *left-right minimum* to be an element $w(j)$ such that $w(i) > w(j)$ for all $i < j$. If we put a bar to the left of each left-right minimum, the blocks that we form can then be transformed into cycles of a new permutation whose excedances correspond to the ascents of w. For example, if $w = 879631524$, its left-right minima are 8, 6, 3, and 1, which we mark:

$$w = |879|6|3|1524 \mapsto (879)(6)(3)(1524) = v.$$

1.8 An inversion is a pair (i, j) such that $w(i) > w(j)$. Inversion sequences are so-called because they record the number of *inversions* involving a given element of a permutation. (An inversion sequence is also known as a *Lehmer code.*) There are a few natural ways to construct a bijection between permutations and inversion sequences.

Let w be a permutation, and for each j, let s_j denote the number of elements to the left of $w(j)$ that are greater than $w(j)$, i.e.,

$$s_j = \{i < j : w(i) > w(j)\}.$$

Clearly, $0 \leq s_j \leq j - 1$, so $s(w) = (s_1, s_2, \ldots, s_n)$ is an inversion sequence. For example,

$$589423761 \mapsto (0, 0, 0, 3, 4, 2, 3, 8).$$

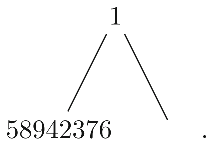
To see that the map $w \mapsto s(w)$ is a bijection, we can simply work in reverse. The value of s_n tells us what $w(n)$ must be: $w(n) = n - s_n$. Having established the value of $w(n)$, we can determine $w(n-1)$ in much the same way: it is the member of $\{1, 2, \ldots, n\} - \{w(n)\}$ that has s_{n-1} elements greater than it. Similar reasoning works for s_{n-2} and so on.

For example, if $s = s(w) = (0, 0, 1, 3, 0, 4, 1, 5, 5)$, we see that $w(9) = 9 - 5 = 4$, since it must be smaller than exactly five elements in $[9]$. Now, $w(8)$ must be smaller than five elements in $\{1, 2, \ldots, 9\} - \{4\}$, so $w(8) = 3$. At this point we know $w(7)$ is smaller than only one element of $\{1, 2, 5, 6, 7, 8, 9\}$, so $w(7) = 8$. Continuing in this manner, we find:

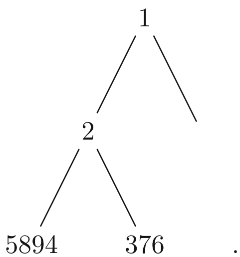
$$w = 576192834.$$

Under the map $w \mapsto s(w)$, it is easy to check that $w(i) > w(i+1)$ if and only if $s_i < s_{i+1}$, so that the number of permutations with k descents equals the number of inversion sequences with k ascents.

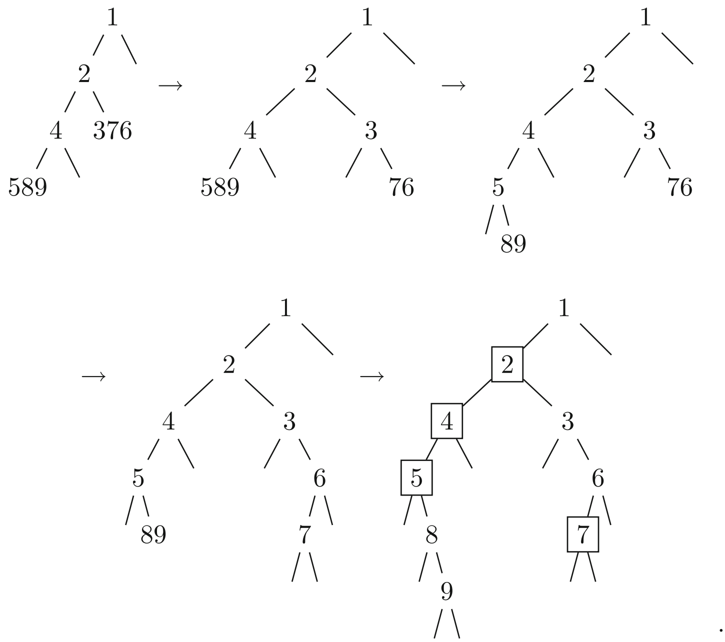
1.9 There is a bijection that takes permutations to increasing binary trees. It applies the following recursive procedure: to any permutation, identify its minimum and split the permutation into subwords, w^l and w^r, consisting of the (possibly empty) words to the left and right of the minimum. We can apply this splitting to the words w^l and w^r as well, continuing until all the leaves of the tree are labeled with empty words. The process is captured in the tree, e.g., $w = 589423761$ would split first into $w^l = 58942376$ and the empty word w^r, which we draw as:



Now applying the same decomposition to $v = 58942376$, its minimum value is 2 and we find $v^l = 5894$ is the word to the left of 2, $v^r = 376$ is the word to right of 2. At this stage, we would have:



The final steps of the procedure are indicated here:



Notice that the decomposition $w = w^l\, 1\, w^r$, shows that

$$\operatorname{des}(w) = \begin{cases} \operatorname{des}(w^l) + \operatorname{des}(w^r) + 1 & \text{if } w^l \text{ nonempty,} \\ \operatorname{des}(w^l) + \operatorname{des}(w^r) & \text{if } w^l \text{ empty.} \end{cases}$$

Moreover, if we let $c(w)$ denote the number of left internal children of the tree for w, we have the same recurrence relation:

$$c(w) = \begin{cases} c(w^l) + c(w^r) + 1 & \text{if } w^l \text{ nonempty,} \\ c(w^l) + c(w^r) & \text{if } w^l \text{ empty.} \end{cases}$$

Thus, since $c(w)$ and $\operatorname{des}(w)$ agree for small w, the number of left internal children of the tree for w equals the number of descents of w, and counting increasing binary trees according to internal left children gives rise to the Eulerian distribution.

1.10 See Chapter 7.

1.11 Let ω denote the permutation that cyclically shifts the numbers 1 to n, i.e., $\omega(i) = i + 1$ for $i \leq n - 1$, and $\omega(n) = 1$. Letting ω act on S_n by right multiplication, we partition S_n into $(n-1)!$ orbits of size n. For example with $n = 4$, we get 6 orbits:

w	$w\omega$	$w\omega^2$	$w\omega^3$
1234	2341	3412	4123
1324	3241	2413	4132
2134	1342	3421	4213
2314	3142	1423	4231
3124	1243	2431	4312
3214	2143	1432	4321

We can see that the number of cyclic descents is constant on each orbit. Moreover, each orbit has a unique member with $w(n) = n$. For this element it is clear that $\mathrm{cdes}(w) = 1 + \mathrm{des}(w) = 1 + \mathrm{des}(w')$, where $w' \in S_{n-1}$ is the permutation $w(1)\cdots w(n-1)$. Thus, each orbit contributes $nt^{\mathrm{cdes}(w)} = nt \cdot t^{\mathrm{des}(w')}$ to the cyclic descent generating function.

Summing over all orbits gives:

$$\sum_{w\in S_n} t^{\mathrm{cdes}(w)} = nt \sum_{w'\in S_{n-1}} t^{\mathrm{des}(w)} = ntS_{n-1}(t).$$

1.12 To construct a permutation with exactly one descent, we need to have two increasing runs. Suppose $w = w_1 \cdots w_n$ with

$$w_1 < \cdots < w_k > w_{k+1} < \cdots < w_n.$$

It suffices to specify the elements in the first run, so we need to choose a proper, nonempty subset of $\{1, 2, \ldots, n\}$, (so $k = 1, 2, \ldots, n-1$), such that its maximum is greater than the minimum of its complement.

The only subsets of $\{1, 2, \ldots, n\}$ that are excluded are the empty set, and sets of the form $\{1, 2, \ldots, i\}$ for $i = 1, \ldots, n$. Hence, there are $2^n - 1 - n$ ways to choose the elements of the first increasing run, yielding $\left\langle {n \atop 1} \right\rangle = 2^n - n - 1$, as desired.

1.13 The argument given here is presented in Knuth's book [96].

As suggested in the hint, we consider $(k+1)^n$ to count the number of integer vectors $(a_1, \ldots, a_n)$, with $0 \le a_i \le k$. Given such a vector, its increasing arrangement is obtained by a permutation w:

$$a_{w(1)} \le a_{w(2)} \le \cdots \le a_{w(n)}.$$

For example, if $(a_1, a_2, a_3, a_4, a_5) = (0, 1, 0, 5, 2)$, then its rearrangement is $(0, 0, 1, 2, 5)$. We can see from this example that more than one permutation can produce the sorting of a vector. In this case, since $a_1 = a_3$, we have $(0, 0, 1, 2, 5) = (a_1, a_3, a_2, a_5, a_4) = (a_3, a_1, a_2, a_5, a_4)$. In order that our choice is canonical, we will declare that if $a_{w(i)} = a_{w(i+1)}$, then $w(i) < w(i+1)$. In the example of $(0, 1, 0, 5, 2)$, we choose $w = 13254$, not 31254. Conversely, if j is a descent of w, $w(j) > w(j+1)$, then $a_{w(j)} < a_{w(j+1)}$.

So how many integer vectors correspond to a given w? Taking the example of $w = 13254$, we seek the number of integer vectors $(a_1, a_2, a_3, a_4, a_5)$ such that:

$$0 \leq a_1 \leq a_3 < a_2 \leq a_5 < a_4 \leq k,$$

or equivalently,

$$1 \leq a_1 + 1 < a_3 + 2 < a_2 + 2 < a_5 + 3 < a_4 + 3 \leq k + 3,$$

i.e., we want to choose five integers $1 \leq b_1 < b_2 < b_3 < b_4 < b_5 \leq k+3$. Thus there are $\binom{k+3}{5}$ integer vectors that correspond to this permutation w.

In general, we want the number of vectors satisfying

$$0 \leq a_{w(1)} \leq a_{w(2)} \leq \cdots \leq a_{w(n)} \leq k,$$

with $a_{w(j)} < a_{w(j+1)}$ if j is a descent of w. Letting $b_i = a_{w(i)} + 1$ plus the number of ascents to the left of position i, we can transform this counting problem into the problem of counting integer vectors $(b_1, \ldots, b_n)$ satisfying

$$1 \leq b_1 < b_2 < \cdots < b_n \leq k + 1 + (n - 1 - \operatorname{des}(w)) = k + n - \operatorname{des}(w).$$

There are $\binom{k+n-\operatorname{des}(w)}{n}$ such vectors, and hence this many vectors $(a_1, \ldots, a_n)$ associated with a given w.

We can therefore conclude:

$$\begin{aligned}(k+1)^n &= |\{(a_1, \ldots, a_n) : 0 \leq a_i \leq k\}|, \\ &= \sum_{w \in S_n} |\{(a_1, \ldots, a_n) \text{ corresponds to } w\}|, \\ &= \sum_{w \in S_n} \binom{k + n - \operatorname{des}(w)}{n}, \\ &= \sum_{i=0}^{n-1} \left\langle {n \atop i} \right\rangle \binom{k+n-i}{n},\end{aligned}$$

which is Worpitzky's identity.

1.14 Following the hint, we first remark that the generating function for all ways to put n distinct balls, labeled 1 to n, say, into boxes is

$$\sum_{k \geq 0} (k+1)^n t^k.$$

For any particular configuration of balls in boxes, there is a natural way to associate a permutation, obtained by listing the contents of the boxes from left to right (we fix a linear ordering on the boxes), and when there are two or more balls in a box, we list the balls in increasing order of their labels. We represent the boxes with a vertical bar to show the divisions between the boxes, and we call the arrangement of bars and numbers a "barred permutation."

For example, if there are seven balls, one placement of the balls into five boxes is given by the following barred permutation:

$$1|5|237||46.$$

Here, the first box contains ball 1, the second has ball 5, the third contains 2, 3, and 7, the fourth box is empty, and the fifth box has balls 4 and 6.

Note that counting arrangements of n balls in $k+1$ boxes amounts to counting barred permutations with k bars.

Let us fix a permutation w in S_n and count all the barred permutations whose underlying permutation is w. There must be at least one bar in each descent position, but other gaps can have arbitrarily many bars. That is, the weight of a gap is

$$1 + t + t^2 + \cdots = \frac{1}{1-t},$$

if there is no descent in that position, and it is

$$t + t^2 + t^3 + \cdots = \frac{t}{1-t},$$

if there is a descent. For example, if $w = 562143$ there are seven gaps in which to insert bars, and three of them have descents, so the generating function for the barred permutations corresponding to w is $t^3/(1-t)^7$, as illustrated here:

$$\begin{array}{ccccccccccccc|c} w = & 5 & & 6 & \searrow & 2 & \searrow & 1 & & 4 & \searrow & 3 & & \text{weight} \\ \hline \frac{1}{1-t} & \cdot & \frac{1}{1-t} & \cdot & \frac{t}{1-t} & \cdot & \frac{t}{1-t} & \cdot & \frac{1}{1-t} & \cdot & \frac{t}{1-t} & \cdot & \frac{1}{1-t} & = \frac{t^3}{(1-t)^7}, \end{array}$$

or in general

$$\frac{t^{\operatorname{des}(w)}}{(1-t)^{n+1}}.$$

Therefore, the generating function for putting n labeled balls into k labeled boxes is

$$\sum_{k \geq 0} (k+1)^n t^k = \sum_{w \in S_n} \frac{t^{\operatorname{des}(w)}}{(1-t)^{n+1}},$$
$$= \frac{S_n(t)}{(1-t)^{n+1}},$$

as desired.

This argument and a generalization can be found in [119].

1.15 Taking the hint, we have:

$$\begin{aligned}
\frac{1}{1-t}S(t,z/(1-t)) &= \sum_{n\geq 0}\frac{S_n(t)}{(1-t)^{n+1}}\frac{z^n}{n!},\\
&= \sum_{n\geq 0}\sum_{k\geq 0}(k+1)^n t^k\frac{z^n}{n!},\\
&= \sum_{k\geq 0}t^k\sum_{n\geq 0}\frac{(z(k+1))^n}{n!},\\
&= \sum_{k\geq 0}t^k e^{z(k+1)},\\
&= e^z\sum_{k\geq 0}(te^z)^k,\\
&= \frac{e^z}{1-te^z}.
\end{aligned}$$

Setting $u = z/(1-t)$, we have $z = u(1-t)$, and

$$\begin{aligned}
S(t,u) &= \frac{(1-t)e^{u(1-t)}}{1-te^{u(1-t)}},\\
&= \frac{1-t}{e^{-u(1-t)}-t},\\
&= \frac{t-1}{t-e^{u(t-1)}},
\end{aligned}$$

as given in Equation (1.13).

Problems of Chapter 2

2.1 Letting $S_n(p)$ denote the set of permutations avoiding the pattern p, we can see that the sets $S_n(132)$, $S_n(213)$, $S_n(312)$, and $S_n(231)$ are equinumerous by geometric symmetries. If we draw a permutation that contains the pattern 132 and flip it across a vertical line (reversing the permutation), we end up with a permutation that contains the pattern 231. If a permutation contains 231 and we flip it across a horizontal line we end up with a permutation containing 213. Flipping a permutation with 213 across a vertical line gives a permutation containing 312, and flipping this permutation horizontally brings us back to the original permutation containing 132. Similarly, flipping a permutation containing 123 across a vertical line (or a horizontal line) will give a permutation containing 321. See Figure 14.6.

Thus we have bijections showing $|S_n(132)| = |S_n(231)| = |S_n(213)| = |S_n(312)|$ and $|S_n(123)| = |S_n(321)|$. Since we know 231-avoiding permutations are counted by Catalan numbers, all that remains is to show that 123-avoiding permutations or 321-avoiding permutations are counted by Catalan numbers. There are many ways to do this, but one is to show that 321-avoiding permutations are in bijection with Dyck paths. Here is a picture of the correspondence, which can be thought of as a simple modification of the bijection between 231-avoiding permutations and Dyck paths illustrated in Figures 2.8 and 2.9. See Figure 14.7.

The details are left to the reader.

2.2 We will show $(n+1)C_n = \binom{2n}{n}$ by defining an action on all $\binom{2n}{n}$ lattice paths from $(0,0)$ to (n,n) that take North and East steps, then showing that this action defines Catalan-many equivalence classes, each containing exactly one Dyck path. This action mimicks the one presented in Section 2.4.2.

First, rather than paths from $(0,0)$ to (n,n), it will be convenient to consider paths from $(0,-1)$ to (n,n) that always begin with a north step. (See Figure 2.6.) We write such a lattice path as a word with N and E, and insert a vertical bar to the left of every occurrence of the letter N, e.g.,

$$p = |NEEE|N|NEEE|NE|N|N|N|N|NE.$$

One of these vertical bars is special: it is the place where the last minimum valley of the path occurs, and as in Section 2.4.2 we will mark it with a bullet:

$$p = |NEEE|N|NEEE|NE \bullet N|N|N|N|NE.$$

The bullet and the n bars split the word into $n+1$ blocks, and cyclically permuting these blocks gives another lattice path from $(0,-1)$ to (n,n) that starts with a north step. Further, the $\bullet$ will always be the rightmost minimal valley after permuting, since there are more letters N than E to its right.

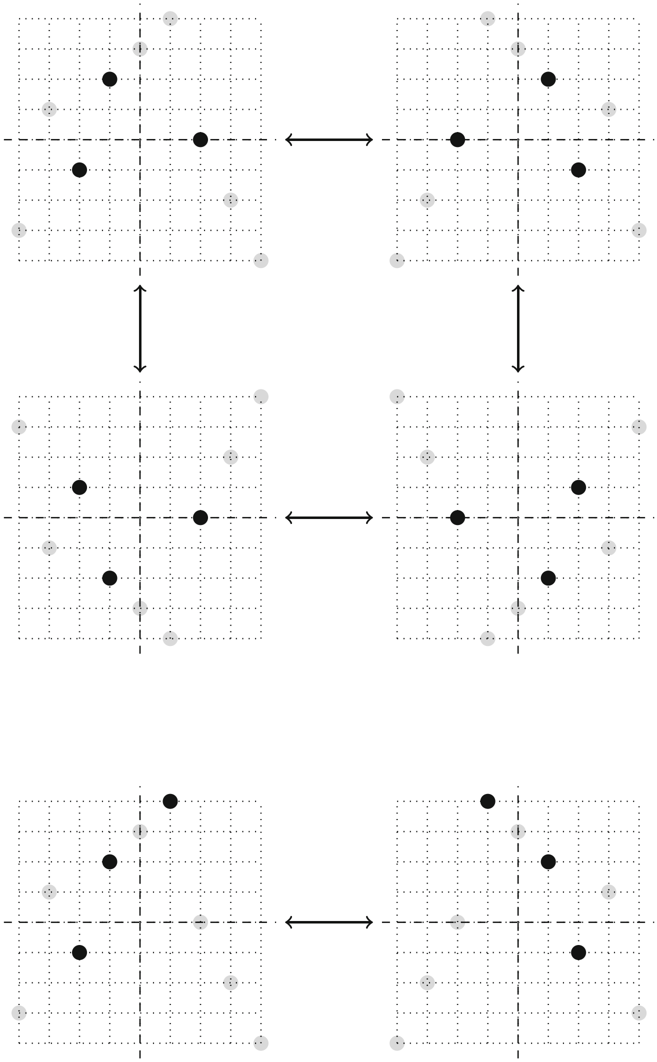


Fig. 14.6 Symmetries of patterns of length three.

Declare two paths to be equivalent if they can be obtained from one another by this cyclic action. For example, the equivalence class of our example p is the following set of nine paths:

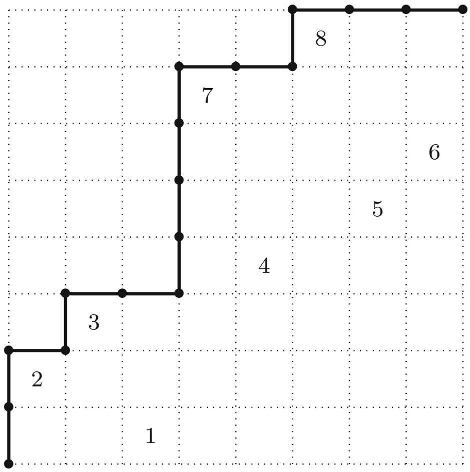


Fig. 14.7 The correspondence between Dyck paths and 321-avoiding permutations.

$$\left\{\begin{array}{c}|NEEE|N|NEEE|NE\bullet N|N|N|N|NE\\ |NE|NEEE|N|NEEE|NE\bullet N|N|N|N\\ |N|NE|NEEE|N|NEEE|NE\bullet N|N|N\\ |N|N|NE|NEEE|N|NEEE|NE\bullet N|N\\ |N|N|N|NE|NEEE|N|NEEE|NE\bullet N\\ \bullet N|N|N|N|NE|NEEE|N|NEEE|NE\\ |NE\bullet N|N|N|N|NE|NEEE|N|NEEE\\ |NEEE|NE\bullet N|N|N|N|NE|NEEE|N\\ |N|NEEE|NE\bullet N|N|N|N|NE|NEEE\end{array}\right\}.$$

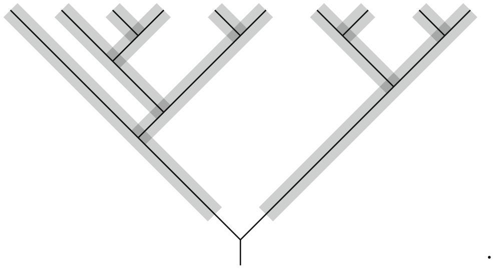
If the bullet is on the far left, that means that the path remains above the line $y = x$ after the first step. This can only happen once per equivalence class. Since each equivalence class has $n+1$ members, we have

$$\binom{2n}{n} = (n+1)C_n,$$

as desired.

2.3 Let b_n denote the number of planar binary trees with n internal nodes, and set $b_0 = 1$ (i.e., there is one tree with no internal nodes).

Any planar binary tree with n internal nodes can be partitioned at the root into a *left branch* and a *right branch*. If there are i internal nodes on the left branch, there must be $n-1-i$ internal nodes on the right branch:



Thus all trees with i internal nodes on the left branch and $n-1-i$ nodes on the right branch can be formed in $b_i b_{n-1-i}$ ways. Summing this over all i gives

$$b_n = \sum_{i=0}^{n-1} b_i b_{n-1-i}.$$

2.4 The map suggested by Figure 2.10 is a slight modification of the map from all permutations to increasing binary trees outlined in the solution to Problem 1.9.

It is straightforward to check that counting left-pointing leaves corresponds to counting left internal children.

2.5 The key thing here is to show that the inverse of the map ϕ shown in Figure 2.11 is well defined. That is, given a noncrossing partition whose blocks are made into decreasing runs, there is only one way to arrange these blocks to avoid the pattern 231.

Let these blocks be denoted by $R_1, \dots, R_k$. We claim that if we order the blocks so that

$$\min R_1 < \min R_2 < \cdots < \min R_k,$$

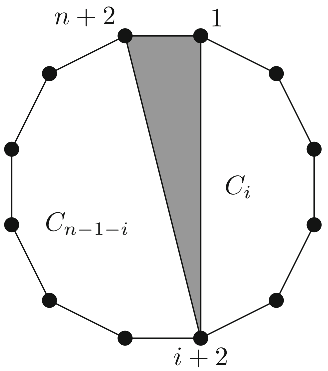
the permutation formed by concatenating the blocks in this order avoids 231. If there was a 231 pattern, the "2" would have to belong to a block R somewhere to the left of the blocks containing the "3" and the "1." Without loss of generality, we can assume that the 3 and the 1 are in the same block. Call it S. But since the minimum element of block R is less than the minimum element of block S, this means there is an element "0" in block R. That is, we have some $a, c \in R$ and $b, d \in S$ such that $a < b < c < d$. But this would imply that the blocks R and S have a crossing. Since the blocks are noncrossing, there cannot be a 231 pattern.

Now if we order the blocks in another way, there is a block R to the left of a block S such that $\min S < \min R < \max S$. But then the elements $a = \min S$, $b = \min R$, and $c = \max S$ form a 231 pattern.

2.6 We will show that triangulations of a polygon satisfy the quadratic recurrence, as the initial values clearly agree.

Label the nodes clockwise from 1 to $n+2$, and consider the triangle containing nodes $n+2$ and 1. The third node on this triangle will be labeled

$i+2$, where $i = 0, \ldots, n-1$. For each i, we partition the problem of triangulating the $(n+2)$-gon into two independent triangulation problems: we must triangulate the $(i+2)$-gon with nodes $1, \ldots, i+2$, and we must triangulate the $(n+1-i)$-gon with nodes $i+2, i+3, \ldots, n+2$:



.

If C_i counts the triangulations of $(i+2)$-gon, then C_{n-1-i} counts the triangulations of an $(n+1-i)$-gon, so summing over all i we have

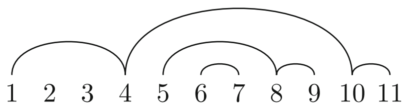
$$C_n = \sum_{i=0}^{n-1} C_i C_{n-1-i},$$

as desired.

2.7 It turns out that both noncrossing and nonnesting partitions of $[n]$ can be uniquely constructed from the list of cardinalities of its blocks and the minimal element of the block. This allows us to construct a bijection between the two sets that takes a nonnesting partition with k blocks to a noncrossing partition with k blocks.

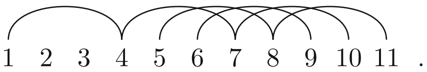
In particular, if π is a noncrossing partition that has blocks $R_1, \ldots, R_k$, with minimal elements $a_i = \min R_i$ and $a_1 < a_2 < \cdots < a_k$, then we can form a nonnesting partition π' with the same block cardinalities and restriction on its minimal elements: $\pi' = \{S_1, \ldots, S_k\}$, $|S_i| = |R_i|$, and $\min S_i = a_i = \min R_i$. This idea can be found in Athanasiadis [11] and more recently in Fink and Iriarte Giraldo [65].

Here is an example. Suppose $\pi = \{\{1,4,10,11\},\{2\},\{3\},\{5,8,9\},\{6,7\}\}$ is a noncrossing partition, drawn as:

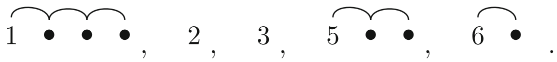


.

This partition will end up corresponding to the nonnesting partition $\pi' = \{\{1,4,7,10\},\{2\},\{3\},\{5,8,11\},\{6,9\}\}$, with arc diagram as follows:

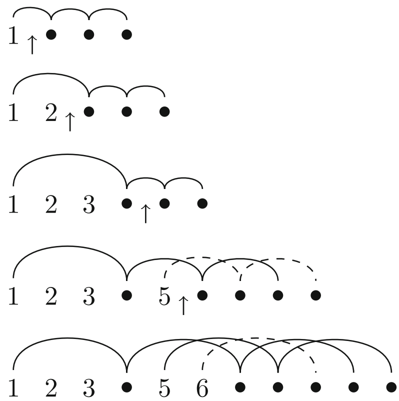


We can pull apart the arc diagrams of both π and π' into strings with only the minimal elements marked, as follows:

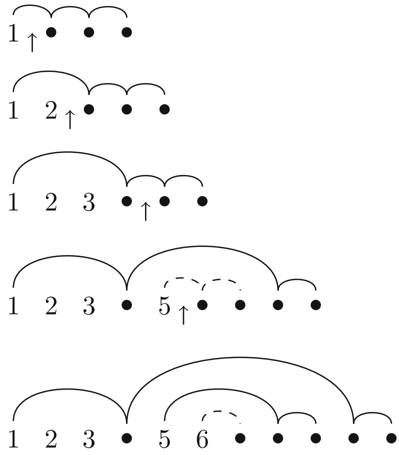


Given these marked strings we can construct a noncrossing (resp. nonnesting) partition from them in a recursive manner. Supposing we have inserted the first $i-1$ blocks in an arc diagram, there is a unique way to insert the ith block so that the marked node goes in its proper place and the remaining nodes are placed so that all arcs are noncrossing (resp. nonnesting).

For example, here are the steps in the construction of the nonnesting partition π'. We have marked with an arrow where each subsequent block must begin, and the newest arcs are shown with dashed lines.

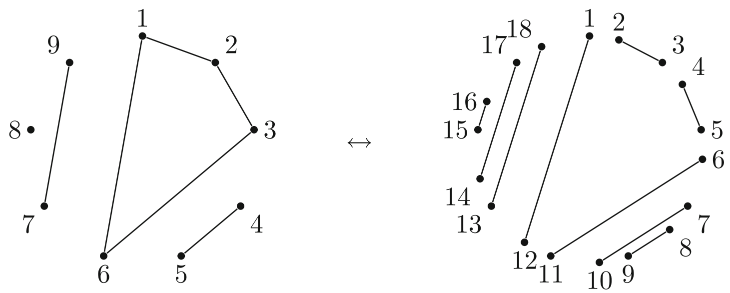


Now supposing we want to construct the noncrossing partition π from these blocks, the steps would look like this:



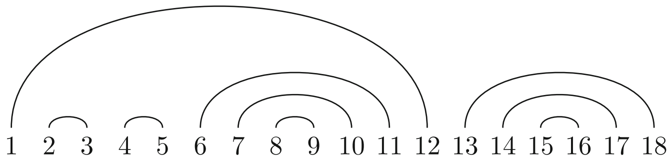
2.8 There are various ways to put noncrossing matchings in bijection with another set of Catalan objects. Here are two.

First we illustrate a bijection between noncrossing matchings on $[2n]$ and noncrossing partitions on $[n]$. After drawing the arc diagram for our noncrossing matching, identify nodes $(2k-1)$ and $2k$ for $k = 1, \ldots, n$. If $(2k-1)$ or $2k$ is matched with $(2j-1)$ or $2j$, then we put j and k in the same block. Visually, this is easiest to see if draw our nodes on a disk, e.g.,

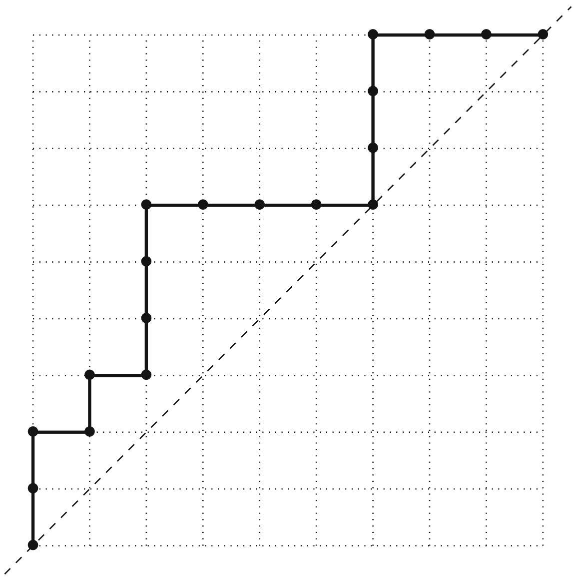


.

Another straightforward bijection can be found between noncrossing matchings and Dyck paths. If we think of our noncrossing matchings as balanced parenthesizations, we have a list of n left parentheses and n right parentheses, such that there are never more right parentheses than left in reading the string from left to right. Replacing each left parenthesis with a north step, N, and each right parenthesis with an east step, E, yields a path that never goes below the line $y = x$. For example,



corresponds to $NNENENNNEEEEENNNEEE$.



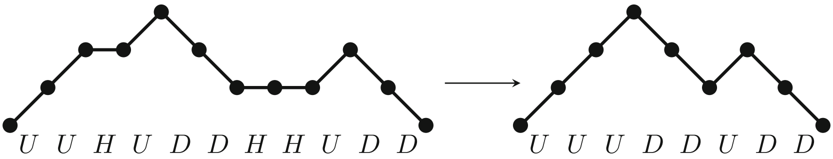
Given this correspondence with Dyck paths, we see that peaks of the Dyck path correspond to adjacent matched pairs in the noncrossing matching (e.g., $\{2,3\}$, $\{4,5\}$, $\{8,9\}$, $\{15,16\}$ in the example above). Thus counting noncrossing matchings according to the number of adjacent pairs will give the Narayana numbers.

2.9 In a standard Young tableau with two rows, we can create Dyck path by taking our ith step to the North if i is in the top row, to the East if i is in the bottom row. For example,

1	2	5	6
3	4	7	8

would correspond to the path $NNEENNEE$. We will find a peak whenever the number i appears in the top row and $i+1$ is in the bottom row. The collection of all such i is often called the *descent set* of the tableau. Since these descents correspond to peaks in the Dyck path, we get the Narayana numbers by counting two-row standard Young tableaux according to the number of descents.

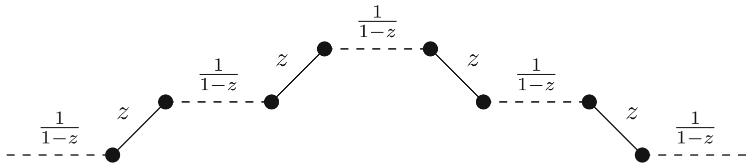
2.10 To any Motzkin path, we can form Dyck path by deleting all horizontal steps:



.

(Of course we're using U and D steps instead of N and E for the Dyck path.)

Conversely, from any Dyck path we can build infinitely many Motzkin paths by inserting horizontal steps between U and D steps. If p is a Dyck path with $2n$ steps, there are $2n+1$ gaps in which to insert H steps, e.g., the path $UUDD$ would give Motzkin paths of the form:



.

We are counting paths according to the total number of steps, so each U or D on the Dyck path gives weight z, and between the steps of the Dyck path we can insert arbitrarily many paths, with weight

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots .$$

In general, a Dyck path with $2n$ steps yields a collection of Motzkin paths with length generating function

$$\frac{z^{2n}}{(1-z)^{2n+1}} = \frac{1}{1-z} \cdot \left(\frac{z^2}{(1-z)^2} \right)^n .$$

Thus, recalling $C(z) = \frac{1-\sqrt{1-4z}}{2z}$ is the generating function for Dyck paths according to semilength n,

$$\begin{aligned} M(z) &= \sum_{n \geq 0} \sum_{p \in \mathrm{Dyck}(n)} \frac{1}{1-z} \cdot \left(\frac{z^2}{(1-z)^2} \right)^n , \\ &= \frac{1}{1-z} \cdot C(z^2/(1-z)^2), \\ &= \frac{1}{1-z} \cdot \frac{1 - \sqrt{1 - 4z^2/(1-z)^2}}{2z^2/(1-z)^2}, \\ &= \frac{1 - z - \sqrt{1 - 2z - 3z^2}}{2z^2}, \end{aligned}$$

as desired.

2.11 This bijection is straightforward. For a path encoded as a word $p = p_1 \cdots p_n$ with $p_i \in \{U, D, H\}$, we create a noncrossing partial matching as follows.

If $p_i = U$, write a left parenthesis in position i: (.

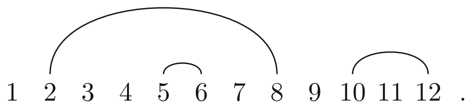
If $p_i = D$, then write a right parenthesis in position i:).

If $p_i = H$, we leave a blank space.

For example, the path $p = HUHHUDHDHUHD$ would transform as follows:

$$\begin{array}{cccccccccccc} H & U & H & H & U & D & H & D & H & U & H & D \\ \cdot & (& \cdot & \cdot & (&) & \cdot &) & \cdot & (& \cdot &) \end{array},$$

with arc diagram



The parentheses will be a noncrossing matching on the positions corresponding to the U and D steps, since this subword corresponds to a Dyck path (and we already discussed the correspondence between Dyck paths and complete noncrossing matchings in the solution to Problem 2.8).

2.12 We can construct Schröder paths from Dyck paths by inserting diagonal steps freely in the gaps between the N and E steps of the Dyck path. This is much like the way we construct Motzkin paths from Dyck paths by inserting H steps between U and D steps.

The main difference between counting Schröder paths and Motzkin paths is that each diagonal step replaces two steps: an N and an E. That is, a Schröder path from $(0, 0)$ to (n, n) has n equal the number of diagonal steps plus the number of north steps (or number of diagonals plus number or east steps). For a Dyck path of length $2n$, there are $2n+1$ gaps, and n north steps, so this means we can construct a collection of Schröder paths with generating function

$$\frac{z^n}{(1-z)^{2n+1}} = \frac{1}{1-z} \cdot \left(\frac{z}{(1-z)^2}\right)^n.$$

Recalling $C(z) = \frac{1-\sqrt{1-4z}}{2z}$ is the generating for Dyck paths according the number or north steps, we get:

$$\begin{aligned} R(z) &= \sum_{n \geq 0} \sum_{p \in \mathrm{Dyck}(n)} \frac{1}{1-z} \cdot \left(\frac{z}{(1-z)^2}\right)^n, \\ &= \frac{1}{1-z} \cdot C(z/(1-z)^2), \end{aligned}$$

$$= \frac{1}{1-z} \cdot \frac{1 - \sqrt{1 - 4z/(1-z)^2}}{2z/(1-z)^2},$$
$$= \frac{1 - z - \sqrt{1 - 6z + z^2}}{2z},$$

as desired.

2.13 Let A denote the set of Schröder paths with a peak on the line $y = x + 1$ and let B denote the set of Schröder paths without a peak on the line $y = x + 1$. We can construct a bijection from A to B as follows. Let p be a path in set A and suppose the rightmost peak on the line $y = x+1$ occurs at the point $(k, k+1)$ in the path. Then we can decompose p as $p = q \cdot NE \cdot r$, where q is a Schröder path of size k and r is a Schröder path of size $n-1-k$ that has no peak on the line $y = x + 1$.

We claim that the path p' given by $p' = N \cdot q \cdot E \cdot r$ has no peaks on the line $y = x + 1$. Indeed, the sub-path r has no such peaks by assumption, and the path $N \cdot q \cdot E$ never passes below $y = x + 1$ except at its beginning and its end, so it certainly cannot have a peak on $y = x + 1$. See Figure 14.8, where $p = (NDENENNEED)(NE)(NNENEDE)$ and $p' = (N)(NDENENNEED)(E)(NNENEDE)$.

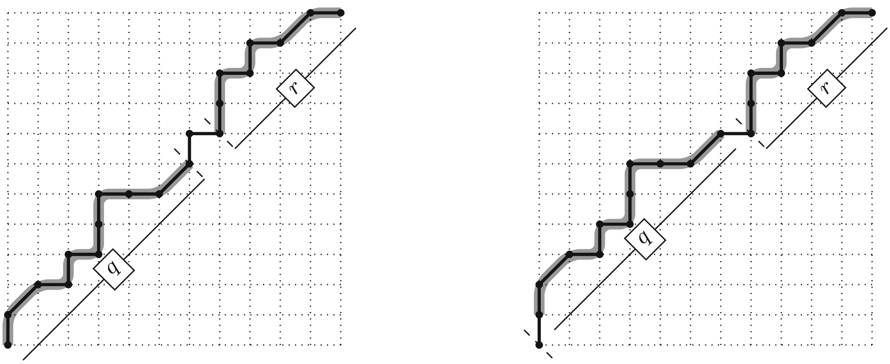


Fig. 14.8 A map between Schröder paths with and without a peak on the line $y = x + 1$.

The only exception to our rule occurs when q is the empty path, i.e., when $p = NE \cdot r$ with r having no peaks on the line $y = x + 1$. In this case, we map p to $p' = D \cdot r$, where "D" indicates a Northeast, or "diagonal," step.

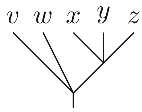
In either case, the map is straightforward to reverse. If p is a path with no peaks on the line $y = x+1$ and p begins with a diagonal step, we replace that initial diagonal step with NE. Otherwise, p can be uniquely decomposed as: $p = N \cdot q \cdot E \cdot r$, where q is a nonempty Schröder path and r is a Schröder path that has no peaks on the line $y = x + 1$. Then the path $p' = q \cdot NE \cdot r$ has at least one peak on the line $y = x + 1$.

This shows $|A| = |B|$, so $r_n = R_n/2$, for $n \geq 1$.

Now if we know $R(z) = \sum R_n z^n$, then $r(z) = \sum r_n z^n$ is easily deduced:

$$\begin{aligned} r(z) &= \sum_{n\geq 0} r_n z^n, \\ &= 1 + \frac{1}{2}\sum_{n\geq 1} R_n z^n, \\ &= 1 + \frac{1 - z - \sqrt{1-6z+z^2}}{4z} - \frac{1}{2}, \\ &= \frac{1 + z - \sqrt{1-6z+z^2}}{4z}. \end{aligned}$$

2.14 We can think of our parenthesizations as *rooted planar trees*, as indicated here:

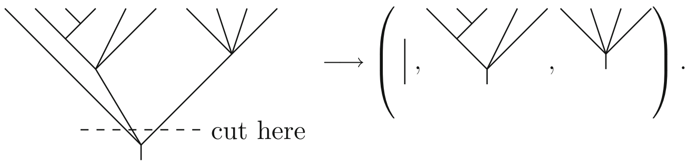


$\longleftrightarrow vw(xyz)$

.

Let t_n denote the number of rooted planar trees with n leaves, $n \geq 1$. We want to show $t_n = r_{n-1}$.

We can break down any planar tree into an ordered list of such trees by cutting it just above the root. For example,



This gives us the following recurrence, for $n \geq 2$:

$$t_n = \sum_{2\leq k\leq n} \sum_{i_1+\cdots+i_k=n} t_{i_1}\cdots t_{i_k}.$$

In other words, each tree with at least 2 leaves can be cut into an ordered list of k trees whose total number of leaves add up to n. But this sum is merely reflecting the coefficient of a k-fold product of $T(z)$ with itself.

Therefore from this recurrence we get the following formula for $T(z) = \sum_{n\geq 1} t_n z^n$:

$$\begin{aligned} T(z) &= z + \sum_{k\geq 2} T(z)^k, \\ &= z + T(z)^2 \left(\frac{1}{1 - T(z)} \right), \end{aligned}$$

and thus,

$$2T(z)^2 - (1+z)T(z) + z = 0.$$

We want $r(z) = \sum_{n\geq 0} r_n z^n = T(z)/z$, so the identity for $T(z)$ gives us:

$$2zr(z)^2 - (1+z)r(z) + 1 = 0.$$

Solving for $r(z)$ gives us:

$$r(z) = \frac{1 + z - \sqrt{1 - 6z + z^2}}{4z},$$

as we hoped to find.

Problems of Chapter 3

3.1 Let's take a small example. Let

$$P = \begin{array}{cc} 3 & \\ | & 5 \\ 2 & | \\ | & 4 \\ 1 & \end{array} .$$

Then here are the linear extensions of P:

$$\mathcal{L}(P) = \left\{ \begin{array}{cccccccccc} 5 & 5 & 3 & 5 & 3 & 3 & 5 & 3 & 3 & 3 \\ | & | & | & | & | & | & | & | & | & | \\ 4 & 3 & 5 & 3 & 5 & 2 & 3 & 5 & 2 & 2 \\ | & | & | & | & | & | & | & | & | & | \\ 3, & 4, & 4, & 2, & 2, & 5, & 2, & 2, & 5, & 1 \\ | & | & | & | & | & | & | & | & | & | \\ 2 & 2 & 2 & 4 & 4 & 4 & 1 & 1 & 1 & 5 \\ | & | & | & | & | & | & | & | & | & | \\ 1 & 1 & 1 & 1 & 1 & 1 & 4 & 4 & 4 & 4 \end{array} \right\} .$$

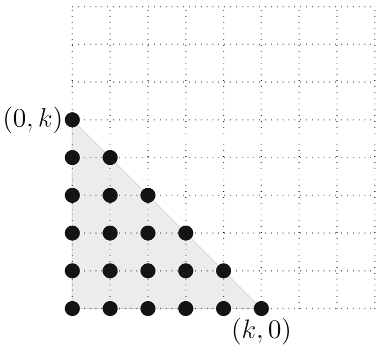
Let's write these in a table as permutations, along with the inverse of each permutation:

w	w^{-1}
12345	12345
12435	124|35
12453	125|34
14235	134|25
14253	135|24
41235	234|15
41253	235|14
41523	245|13
45123	345|12

We've marked the descent positions of the inverse permutations. Notice that these are the permutations that have at most one descent, and this descent occurs in position 3, i.e., $\mathrm{Des}(w^{-1}) \subseteq \{3\}$.

In general, if P is the disjoint union of chains $1 <_P 2 <_P \cdots <_P k$ and $k+1 <_P k+2 <_P \cdots <_P n$, then a linear extension w is characterized by the property that $w^{-1}(1) < w^{-1}(2) < \cdots < w^{-1}(k)$ and $w^{-1}(k+1) < w^{-1}(k+2) < \cdots < w^{-1}(n)$. Thus the only place where a descent can occur in such a w^{-1} is in position k. That is, $\mathrm{Des}(w^{-1}) \subseteq \{k\}$.

3.2 **(Parts 1–3)** We will show that $\Omega(\Sigma_2; k) = \binom{k+2}{2}$. This can easily be seen by examining a picture:



To see the result in a way that allows for generalization, we can see that the k-fold dilation of $\varSigma_2$ is defined by:

$$k\varSigma_2 = \{(x,y) : x \geq 0, y \geq 0, x+y \leq k\}.$$

Notice that we have $0 \leq x \leq k-y \leq k$, so that to count integer points, we want pairs of integers (a,b) satisfying

$$0 \leq a \leq b \leq k,$$

of which there are $\binom{k+2}{2}$. Thus, by Equation (1.3), we have

$$\begin{aligned}\sum_{k\geq 0} \varOmega(\varSigma_2; k)t^k &= \sum_{k\geq 0} \binom{k+2}{2} t^k, \\ &= \frac{1}{(1-t)^3}.\end{aligned}$$

When we move to $\varSigma_3$, we have

$$k\varSigma_3 = \{(x,y,z) : x \geq 0, y \geq 0, z \geq 0, x+y+z \leq k\}.$$

Notice that we can characterize this set of points with the following linear chain of inequalities:

$$0 \leq x \leq k-y-z \leq k-z \leq k,$$

and thus $\varOmega(\varSigma_3; k)$ counts integer triples (a,b,c) satisfying

$$0 \leq a \leq b \leq c \leq k,$$

of which there are $\binom{k+3}{3}$. Thus, again using Equation (1.3), we have

$$\begin{aligned}\sum_{k\geq 0} \varOmega(\varSigma_3; k)t^k &= \sum_{k\geq 0} \binom{k+3}{3} t^k, \\ &= \frac{1}{(1-t)^4}.\end{aligned}$$

In general for Σ_n, we have $\Omega(\Sigma_n;k) = \binom{k+n}{n}$, much as we found in counting P-partitions of a linear extension in Proposition 3.1. Thus,

$$\sum_{k\geq 0} \Omega(\Sigma_n;k)t^k = \frac{1}{(1-t)^{n+1}}.$$

(Parts 4–6) We will jump straight to the general case here. The k-fold dilation of the cube Δ_n has no constraints on its coordinates, apart from $0 \leq x_i \leq k$. Thus, we have $\Omega(\Delta_n;k) = (k+1)^n$. Using the Carlitz identity (1.10), we get

$$\begin{aligned}\sum_{k\geq 0} \Omega(\Delta_n;k)t^k &= \sum_{k\geq 0}(k+1)^n t^k,\\ &= \frac{S_n(t)}{(1-t)^{n+1}},\end{aligned}$$

where $S_n(t)$ is the Eulerian polynomial.

There is a general theory of counting integer points in convex polytopes, known as *Ehrhart theory*. The numerator of the rational generating function $\sum_{k\geq 0} \Omega(P;k)t^k$ is known as the h^*-*polynomial* of P. Thus, we see that $S_n(t)$ is the h^*-polynomial of a cube. For more, see the book by Matthias Beck and Sinai Robins [14].

3.3

1. To form a set partition of $\{1, 2, \ldots, n+1\}$ with k parts we can either:
   - add the number $n+1$ to an existing part of a set partition of $\{1, 2, \ldots, n\}$ with k parts, or
   - add the singleton set $\{n+1\}$ as a new part onto a set partition of $\{1, 2, \ldots, n\}$ with $k-1$ parts.

   The first case has $kS(n,k)$ options and the second case has $S(n,k-1)$ options. Thus, $S(n,k) = kS(n,k) + S(n,k-1)$, as desired. Table 5.5 contains small values for the Stirling numbers of the second kind.
2. Another way to form a set partition with k parts is to simply choose some elements for the first block, some other elements for the second block, and so on. For example, if we want only three blocks, we might choose a elements for the first block, b elements for the second block, and c elements for the third block. This can be done in

   $$\binom{n}{a}\binom{n-a}{b}\binom{n-a-b}{c} = \frac{n!}{a!(n-a)!}\frac{(n-a)!}{b!(n-a-b)!}\frac{(n-a-b)!}{c!0!} = \frac{n!}{a!b!c!}$$

   ways. But this count also considers the blocks themselves as coming in some particular order. We are counting each *unordered* set partition 3! times. Thus the number of ways to form a set partition with block sizes a, b, c is

$$\frac{1}{3!} \cdot \frac{n!}{a!b!c!}.$$

Summing over all possible positive integers a, b, c gives the total number of set partitions with three parts:

$$S(n,3) = \frac{1}{3!} \cdot \sum_{\substack{a,b,c \geq 1 \\ a+b+c=n}} \frac{n!}{a!b!c!}.$$

Generally, the number of ways to choose groups of $i_1, i_2, \ldots, i_k$ elements from an n-element set follows a similar pattern, which we denote by

$$\binom{n}{i_1, i_2, \ldots, i_k} = \frac{n!}{i_1! i_2! \cdots i_k!}.$$

Ignoring the order on the k blocks requires dividing by $k!$, and then we sum over all possible block sizes:

$$\begin{aligned} S(n,k) &= \frac{1}{k!} \cdot \sum_{\substack{i_1,\ldots,i_k \geq 1 \\ i_1+\cdots+i_k=n}} \binom{n}{i_1, i_2, \ldots, i_k}, \\ &= \frac{1}{k!} \cdot \sum_{\substack{i_1,\ldots,i_k \geq 1 \\ i_1+\cdots+i_k=n}} \frac{n!}{i_1! i_2! \cdots i_k!}. \end{aligned}$$

Thus,

$$\begin{aligned} \sum_{n \geq 1} S(n,k) \frac{z^n}{n!} &= \frac{1}{k!} \cdot \sum_{n \geq 1} \sum_{\substack{i_1,\ldots,i_k \geq 1 \\ i_1+\cdots+i_k=n}} \frac{z^n}{i_1! i_2! \cdots i_k!}, \\ &= \frac{1}{k!} \cdot \left(\sum_{i \geq 1} \frac{z^i}{i!} \right)^k, \\ &= \frac{(e^z - 1)^k}{k!}. \end{aligned}$$

3. Continuing from part 2,

$$\begin{aligned} 1 + \sum_{n,k \geq 1} S(n,k) \frac{y^k z^n}{n!} &= 1 + \sum_{k \geq 1} y^k \sum_{n \geq 1} S(n,k) \frac{z^n}{n!}, \\ &= \sum_{k \geq 0} \frac{y^k (e^z - 1)^k}{k!}, \\ &= e^{y(e^z - 1)}, \end{aligned}$$

as desired.

Since $B(n) = \sum_{k\geq 0} S(n,k)$, we get the generating function for Bell numbers by setting $y = 1$. This gives us

$$\sum_{n\geq 0} B(n)\frac{z^n}{n!} = e^{(e^z-1)}.$$

It is interesting to note that this generating function is a composition of generating functions: $e^z - 1$ is the generating function for nonempty (unordered) finite sets, while e^z is the generating function for all finite sets. When we compose, $e^{(e^z-1)}$, we get the generating function for all finite sets whose elements nonempty finite sets, i.e., set partitions. There is a more general idea about composing generating functions of combinatorial objects, sometimes known as the theory of "species." See the book by Francois Bergeron, Gilbert Labelle, and Pierre Leroux [15].

3.4 The study of partitions of an integer has a long history in Number Theory and Combinatorics, with contributions from Euler himself, Srinivasa Ramanujan, Godfrey Harold Hardy, George Andrews, and many more. A good survey of the combinatorial approach to counting partitions is Igor Pak's paper [114].

1. If we expand each term in the product as a geometric series, we have:

   $$(1+z+z^2+\cdots)(1+z^2+z^{2\cdot 2}+\cdots)\cdots(1+z^i+z^{2\cdot i}+\cdots)+\cdots.$$

   The power of a term in the expansion of this product looks like

   $$m_1\cdot 1 + m_2\cdot 2+\cdots+m_i\cdot i+\cdots,$$

   where the m_i are nonnegative integers. This corresponds to a partition λ having m_i parts of size i. For example, we obtain a z^{14} term from the product

   $$z^{0\cdot 1}\cdot z^{3\cdot 2}\cdot z^{1\cdot 3}\cdot z^{0\cdot 4}\cdot z^{1\cdot 5}\cdot z^{0\cdot 6}\cdot z^{0\cdot 7}\cdots,$$

   where $m_2 = 3$, $m_3 = 1$, $m_5 = 1$, and $m_i = 0$ otherwise. This set of multiplicities corresponds to the partition $\lambda = (5,3,2,2,2)$.
   The coefficient of z^n is therefore the number of ways to express n as a sum of positive integers, i.e., p_n.
2. We can easily augment our reasoning from part 1 to keep track of the number of parts in λ, since this is just $\sum m_i$. Now we would like each term of our generating function to have the form

   $$t^{\sum m_i} z^{\sum m_i\cdot i} = \prod_{i\geq 1}(tz^i)^{m_i}.$$

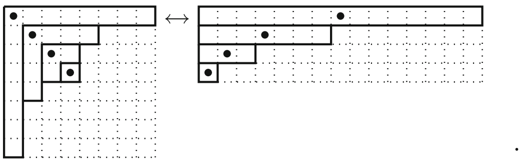
This is achieved by expanding the product of all geometric series of the form

$$1 + tz^i + (tz^i)^2 + \cdots = \frac{1}{(1 - tz^i)}.$$

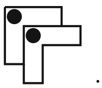
Thus,

$$\sum_{n,k \geq 0} p_{n,k} t^k z^n = \prod_{i \geq 1} \frac{1}{(1 - tz^i)}.$$

3. The product $\prod(1 + z^{2i-1})$ is clearly the generating function for partitions whose parts are distinct odd numbers. Thus we will find a bijection between self-conjugate partitions and partitions with distinct odd parts. Such a bijection can be seen visually in the following example:



That is, in any self-conjugate partition, we mark the boxes on the diagonal of the Young diagram. Now if we split the boxes into "hooks" as suggested in the illustration, each hook has the same number of boxes to the right and below the marked boxes. Hence each hook contains an odd number of boxes. Moreover, the length of two hooks cannot be the same, since that would imply the boxes of the self-conjugate Young diagram are not upper-left justified, e.g.,



In the other direction, we can take any partition with distinct odd parts and "fold" the rows into hooks and then nest those hooks to form a self-conjugate partition.

4. The conjugate of a partition with k parts is a partition whose largest part is $\lambda_1 = k$. Using the same reasoning from parts 1 and 2, the generating function for such partitions is

$$z^k \cdot \prod_{i=1}^{k} \frac{1}{(1 - z^i)}.$$

More generally, we can count partitions whose parts are restricted to come from any set S by considering

$$\prod_{s \in S} \frac{1}{(1 - z^s)}.$$

This technique can be used to answer questions like "How many ways can you make a dollar out of pennies, nickels, dimes, and quarters?" The answer to this question would be the coefficient of z^{100} in the generating function

$$\frac{1}{(1-z)(1-z^5)(1-z^{10})(1-z^{25})}.$$

5. To prove the identity of generating functions, we can plow through the following manipulations:

$$\begin{aligned}\prod_{i\geq 1}(1+z^i) &= \prod_{i\geq 1}\frac{(1-z^{2i})}{(1-z^i)},\\ &= \prod_{i\geq 1}\frac{(1-z^{2i-1})(1-z^{2i})}{(1-z^{2i-1})(1-z^i)}.\end{aligned}$$

Now, if we consider the numerator alone, it is

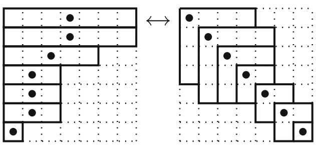
$$(1-z)(1-z^2)(1-z^3)(1-z^4)\cdots = \prod_{j\geq 1}(1-z^j).$$

If we write the denominator as two large products, we get the desired outcome:

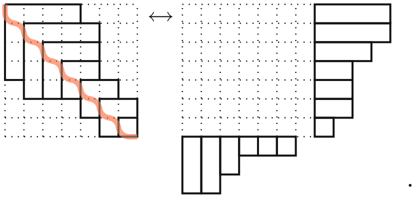
$$\begin{aligned}\prod_{i\geq 1}\frac{(1-z^{2i-1})(1-z^{2i})}{(1-z^{2i-1})(1-z^i)} &= \frac{\prod_{j\geq 1}(1-z^j)}{\prod_{i\geq 1}(1-z^{2i-1})\prod_{k\geq 1}(1-z^k)},\\ &= \prod_{i\geq 1}\frac{1}{(1-z^{2i-1})}.\end{aligned}$$

Now, to prove bijectively that the number of partitions with odd parts equals the number of partitions with distinct parts we need to be clever. There are several bijections in the literature, including the following one due to James Joseph Sylvester in the 19th century. (See [114, Section 3].) We will describe the bijection with pictures.

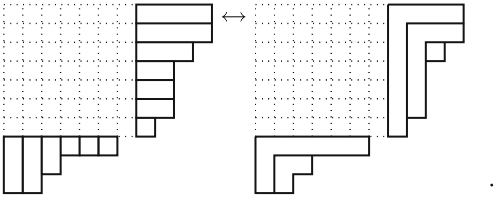
First, we take the Young diagram of our partition with odd parts and bend its rows into hooks and nest the hooks much as we did in part 3. The new array of boxes is symmetric about the diagonal, but it is not necessarily a Young diagram in its own right.



We take this symmetric array of boxes and split it into two pieces just below the diagonal as indicated here:



Now we draw each of these smaller diagrams as nested hooks:



One can check that the lengths of these hooks are all distinct, and we let them be the rows of a new Young diagram, one corresponding to a partition with distinct parts:

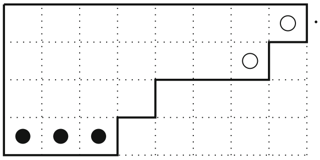
Thus in our example, we map the partition $\lambda = (7, 7, 5, 3, 3, 3, 1)$ to the partition $\lambda' = (10, 8, 7, 3, 1)$.
The inverse of this map is rather tedious. Roughly described, if μ is a partition with distinct parts $\mu_1 > \mu_2 > \cdots$, you take the rows $\mu_1, \mu_3, \ldots$ and make them hooks of the partition above the diagonal, while the rows $\mu_2, \mu_4, \ldots$ and so on are hooks below the diagonal. There is a unique way to nest the hooks so that when we push the nested hooks together we obtain a diagram that is symmetric around the diagonal.

6. The product $\phi(z) = \prod_{i \geq 1}(1 - z^i)$ counts partitions with distinct parts, but such that if λ has an odd number of parts it is counted with a minus sign. Let D_n^+ denote the set of partitions of n with an even number of distinct parts, and similarly let D_n^- denote the set of partitions with an odd number of distinct parts. Then

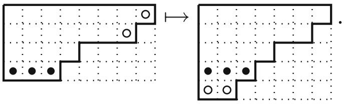
$$\phi(z) = 1 + \sum_{n \geq 1}(|D_n^+| - |D_n^-|)z^n.$$

We will show that the difference between $|D_n^+|$ and $|D_n^-|$ is at most one, and we will characterize exactly when it is nonzero. The method for doing this is to construct a sign-reversing involution on $D_n^+ \cup D_n^-$ that will pair off elements with an odd number of parts with elements having an even number of parts. This is known as *Franklin's involution*, after 19th century mathematician Fabian Franklin. See [114, Section 5].

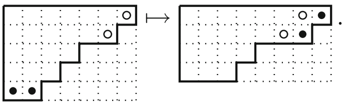
Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a partition with distinct parts. We will now define two numbers related to the Young diagram of λ. Let $s = \lambda_k$ be the smallest part of λ, and let d denote the number of boxes of λ in the 45 degree diagonal line that passes through the rightmost box of the first row of λ. For example, $\lambda = (8, 7, 4, 3)$ has $s = 3$ and $d = 2$, as shown here:



The involution does the following. If $s > d$, then take the d boxes in the rightmost diagonal and form a new bottom row with them:

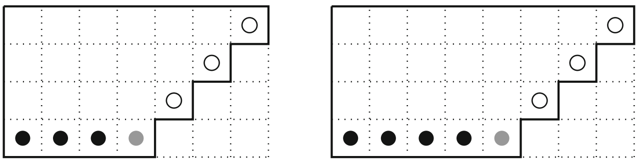


If $s \leq d$, then we take the bottom row of boxes and form a new rightmost diagonal of boxes with them:



It should be clear that this is an involution, and that the parity of the number of rows changes.

The are two exceptions to the rule that we have overlooked. If the bottom row and the rightmost diagonal have a box in common, and $s = d$ or $s = d + 1$, then the partition is a fixed point. Young diagrams such as these are illustrated here:



$(s = d)$ $(s = d + 1)$.

If n is such that neither of these situations can arise, then $|D_n^+| - |D_n^-| = 0$ and the coefficient of z^n in $\phi(z)$ is zero.

Now suppose one of these situations does occur. If $s = d$, then the total number of boxes is

$$\begin{aligned} n &= d^2 + (d-1) + (d-2) + \cdots + 1, \\ &= d^2 + d(d-1)/2, \\ &= \frac{d(3d-1)}{2}, \end{aligned}$$

while if $s = d+1$, then the total number of boxes is

$$\begin{aligned} n &= d^2 + d + (d-1) + \cdots + 1, \\ &= d^2 + d(d+1)/2, \\ &= \frac{d(3d+1)}{2}. \end{aligned}$$

Just as square numbers count the number of boxes in square arrays and triangular numbers count boxes in triangular arrays, the numbers of the form $d(3d \pm 1)/2$ are called *pentagonal numbers* since they count the boxes in these "pentagonal" arrays. (Think of it as a square with a triangle stuck on the end. That's a pentagon, right?)

We have shown the only nonzero terms in $\phi(z)$ correspond to pentagonal numbers. Moreover, for a pentagonal partition of $n = d(3d \pm 1)/2$, the number of boxes on the diagonal equals the number of rows, and hence $|D_n^+| - |D_n^-| = (-1)^d$ in this case.

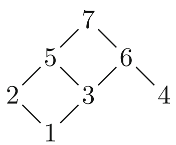
Taken together we have shown

$$\phi(z) = \prod_{i \geq 1} (1 - z^i) = 1 + \sum_{d \geq 1} (-1)^d \left(z^{\frac{d(3d-1)}{2}} + z^{\frac{d(3d+1)}{2}} \right).$$

This is known as *Euler's Pentagonal Theorem*.

3.5 This result follows exactly the same line of reasoning as part 1 of the previous problem. The only difference is that we get one copy of $1/(1 - z^i)$ for each possible marking of a part of size i.

3.6 There are several ways to do this, but the bottom line is that we think of the cells of the Young diagram of λ as being ordered from Southeast to Northwest and label each element so that the labeling is "natural," i.e., if $i \leq_P j$, then $i <_{\mathbb{Z}} j$. With $\lambda = (3, 2, 2)$, we can take the following poset:

$P =$  .

Given this poset P, the plane partitions of shape $(3, 2, 2)$ correspond to P-partitions. The number of plane partitions whose largest part is at most k is therefore the order polynomial for P, and by Theorem 3.1, we have:

$$\sum_{k\geq 0} \Omega(P;k)t^k = \frac{\sum_{w\in\mathcal{L}(P)} t^{\mathrm{des}(w)+1}}{(1-t)^8} = \frac{t+8t^2+10t^3+2t^4}{(1-t)^8}.$$

The study of plane partitions is part of what led Richard Stanley to develop the general theory of P-partitions. There are many further (harder) questions one can ask about plane partitions. For one, the generating function for all plane partitions of n turns out to coincide with the generating function for marked partitions in Problem 3.5. The formula for the number of plane partitions that fit inside an $a \times b \times c$ box also has a nice product formula. See [153, Chapter 7].

3.7 The idea of multipartite P-partitions first appears in an influential paper of Ira Gessel [81]. This identity of binomial coefficients is a corollary of that work. The general idea is that we can split apart bipartite P-partitions into a disjoint union of pairs of ordinary P-partitions.

First, we interpret the left-hand side of the equation in terms of n lexicographically ordered pairs:

$$(1,1) \leq (i_1, j_1) \leq \cdots \leq (i_n, j_n) \leq (k, l),$$

where $(a,b) \leq (c,d)$ means $a \leq c$ or $a = c$ and $b \leq d$. There are $\binom{kl+n-1}{n}$ ways to choose n pairs from among these kl pairs.

Now let us partition these collections of pairs as follows. For each subset $S \subseteq [n-1]$, let $\mathcal{A}_S$ denote the set of sequences of pairs $((i_1, j_1), \ldots, (i_n, j_n))$ such that:

- if $s \in S$, $i_s < i_{s+1}$ and $j_s > j_{s+1}$,
- if $s \notin S$, $i_s \leq i_{s+1}$ and $j_s \leq j_{s+1}$.

Thus, the first coordinates of the elements of $\mathcal{A}_S$ form an increasing sequence, with some weak and some strict inequalities, dictated by S, whereas the second coordinates either strictly decrease or weakly increase, again as determined by S.

In fact, we can think of the second coordinates as the P-partitions for a particular poset determined by S. The poset P_S is a poset on $[n]$ such that $a <_P a+1$ if $a \in S$, $a >_P a+1$ if $a \notin S$. We illustrate in the case $n = 4$ in Table 14.5.

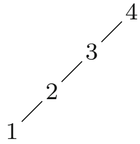
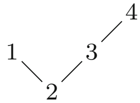
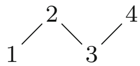
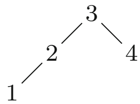
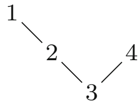
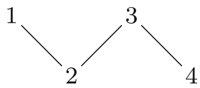
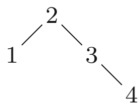
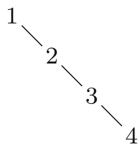
Since $1 \leq i_1 \leq \cdots \leq i_n \leq k$, this establishes is that

$$\binom{kl+n-1}{n} = \sum_{S\subseteq[n-1]} \binom{k+n-1-|S|}{n} \Omega(P_S; l).$$

Now consider the set of linear extensions of P_S. These are all permutations v such that:

- if $s \in S$, $v^{-1}(s) > v^{-1}(s+1)$, and
- if $s \notin S$, $v^{-1}(s) < v^{-1}(s+1)$.

Table 14.5 The decomposition of $(1,1) \le (i_1, j_1) \le (i_2, j_2) \le (i_3, j_3) \le (i_4, j_4) \le (k, l)$.

S	i_s	j_s	P_S
$\emptyset$	$i_1 \le i_2 \le i_3 \le i_4$	$j_1 \le j_2 \le j_3 \le j_4$	
$\{1\}$	$i_1 < i_2 \le i_3 \le i_4$	$j_1 > j_2 \le j_3 \le j_4$	
$\{2\}$	$i_1 \le i_2 < i_3 \le i_4$	$j_1 \le j_2 > j_3 \le j_4$	
$\{3\}$	$i_1 \le i_2 \le i_3 < i_4$	$j_1 \le j_2 \le j_3 > j_4$	
$\{1,2\}$	$i_1 < i_2 < i_3 \le i_4$	$j_1 > j_2 > j_3 \le j_4$	
$\{1,3\}$	$i_1 < i_2 \le i_3 < i_4$	$j_1 > j_2 \le j_3 > j_4$	
$\{2,3\}$	$i_1 \le i_2 < i_3 < i_4$	$j_1 \le j_2 > j_3 > j_4$	
$\{1,2,3\}$	$i_1 < i_2 < i_3 < i_4$	$j_1 > j_2 > j_3 > j_4$	

In other words, $\mathrm{Des}(v^{-1}) = S$, so for each linear extension v of P_S, we can write

$$\binom{k+n-1-|S|}{n} = \binom{k+n-1-\mathrm{des}(v^{-1})}{n}.$$

This means

$$\binom{k+n-1-|S|}{n}\Omega(P_S;l) = \binom{k+n-1-|S|}{n} \sum_{v\in\mathcal{L}(P_S)} \binom{l+n-1-\mathrm{des}(v)}{n},$$

or

$$\sum_{v\in\mathcal{L}(P_S)} \binom{k+n-1-\mathrm{des}(v^{-1})}{n}\binom{l+n-1-\mathrm{des}(v)}{n}.$$

Summing over all descent sets S gives the desired result.

3.8 For $n = 3$, every set partition of $[n]$ is noncrossing, so $\mathrm{NC}(3) = \Pi(3)$, and both are shown in Figure 3.5. In Figure 14.9, we see the Hasse diagram for $\Pi(4)$, with $\mathrm{NC}(4)$ highlighted in bold.

The only partition with a crossing is $\{\{1,3\},\{2,4\}\}$.

It is easily verified that $\Pi(n)$ is a lattice. The least upper bound for partitions $\{R_1,\ldots,R_k\}$ and $\{S_1,\ldots,S_l\}$ is found by drawing their arc diagrams superimposed on the same set of nodes. The greatest lower bound is found by taking the partition whose blocks are the pairwise intersections $R_i \cap S_j$ (ignoring empty intersections).

Enumeration of chains in a poset is an important subject as we will see later in the book. In the case of the partition lattice, the number of maximal chains is given by

$$\binom{n}{2}\binom{n-1}{2}\cdots\binom{2}{2} = \frac{n!(n-1)!}{2^{n-1}}.$$

This formula follows almost immediately from the observation that cover relations in the partition lattice correspond to merging a single pair of blocks, i.e., a cover is of the form:

$$\{R_1,\ldots,R_k\} \to \{R_1,\ldots,R_i\cup R_j,\ldots,R_k\},$$

for some $1 \le i < j \le k$. Thus if we have k blocks, there are $\binom{k}{2}$ ways to pick a pair of blocks to merge, independent of the structure of those blocks.

When we restrict to the lattice of noncrossing partitions, we see that counting maximal chains is a little more delicate. (The up-degree of an element depends on more than the number of blocks.)

3.9 To show there are $(n+1)^{n-1}$ parking functions, we will first count "cyclic" parking functions for n cars on $n+1$ spaces arranged in a circle.

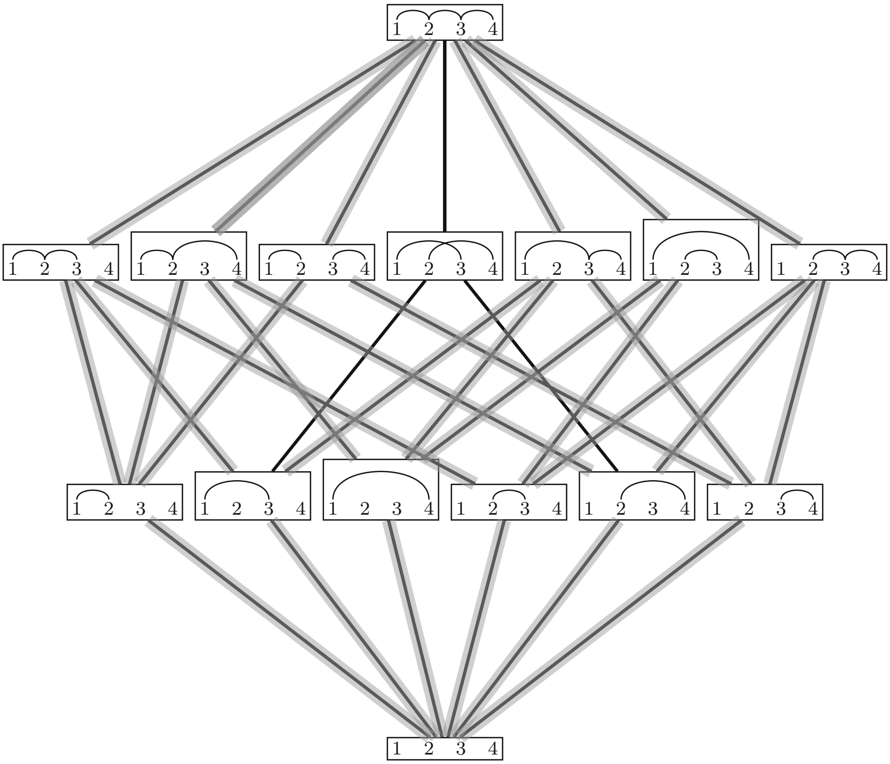


Fig. 14.9 The partition lattice $\Pi(4)$.

We allow each car to have a preference a_i from 1 to $n+1$, and now if a car finds the spot they like occupied, they can loop around back to the beginning of the spaces.

Since there are more spaces than we have cars, *every* list of preferences will result in all the cars parking. There are $(n+1)^n$ such lists.

Now observe that if we shift all of the preferences by one (modulo $n+1$), we end up with all the cars in the same spaces relative to one another. That is, if $\mathbf{a} = (a_1, \ldots, a_n) \in [n+1]^n$ leads to car C_i parking in space j, then the preference list given by shifting each preference one space will lead to car C_i parking in space $j+1 \pmod{n+1}$.

Let us denote this shift by $\mathbf{a}+1 := (a_1+1, \ldots, a_n+1) \pmod{n+1}$. If we repeat this shift $n+1$ times, we return to $\mathbf{a}$. Moreover, exactly one of $\mathbf{a}, \mathbf{a}+1, \mathbf{a}+2, \ldots, \mathbf{a}+n$ leaves space $n+1$ empty, and this vector is an ordinary parking function. For example, if $n=5$, and $\mathbf{a} = (3,5,1,4,3)$, Figure 14.10 shows the six cyclic parking functions obtained by shifting in this way. In this case, $(1,3,5,2,1)$ is the only one of the six vectors that is an ordinary parking function.

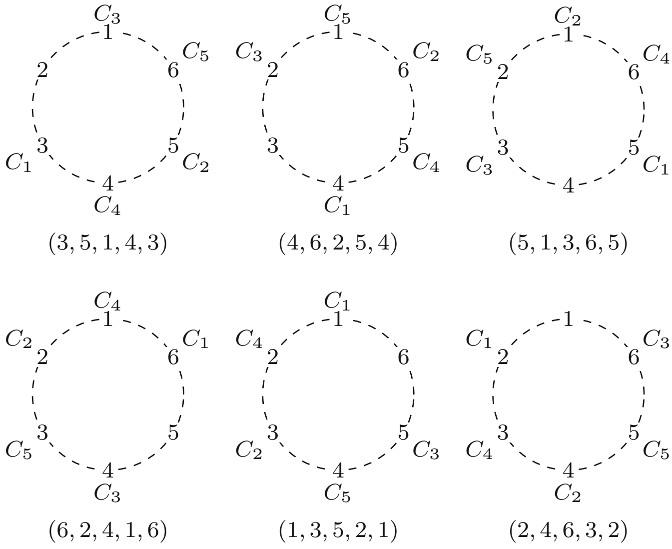


Fig. 14.10 The shifts of a cyclic parking function.

Thus each ordinary parking function gives rise to $(n+1)$ cyclic parking functions, so

$$(n+1) \cdot |\mathrm{PF}(n)| = (n+1)^n,$$

or $|\mathrm{PF}(n)| = (n+1)^{n-1}$, as desired.

3.10 We will show that there are n^{n-2} maximal chains in $\mathrm{NC}(n)$. This was proved by Germain Kreweras in the 1972 paper introducing noncrossing partitions [97, Corollary 5.2], and in 1980 Paul Edelman proved it (and a stronger result about chain enumeration) with a bijection [63, Corollary 3.3].

We will prove the formula with a bijection between maximal chains in $\mathrm{NC}(n)$ and parking functions of length $n-1$, $\mathrm{PF}(n-1)$. The result then follows from Problem 3.9. This bijection can be found in a paper of Richard Stanley [152]. The main idea is, in going up a chain in $\mathrm{NC}(n)$, to keep track of the two blocks that merge in a given cover relation.

Suppose $\pi = \{R_1, \ldots, R_k\}$ is an element of $\mathrm{NC}(n)$, and we are going to move up by merging blocks R_i and R_j. Without loss of generality, suppose we have $\min R_i < \min R_j$. Then we will label this cover relation with

$$\max\{a \in R_i : a < b \text{ for all } b \in R_j\}.$$

To any maximal chain

$$\{\{1\}, \ldots, \{n\}\} = \pi_1 <_{\mathrm{NC}} \pi_2 <_{\mathrm{NC}} \cdots <_{\mathrm{NC}} \pi_n = \{\{1, 2, \ldots, n\}\},$$

we associate the vector $\mathbf{a} = (a_1, \ldots, a_{n-1})$, where a_i is the label for the cover relation between π_i and π_{i+1}.

For example, in Figure 14.11 is a maximal chain of noncrossing partitions in NC(7) with the cover relations labeled. The corresponding parking function is $(3, 2, 4, 1, 3, 1)$.

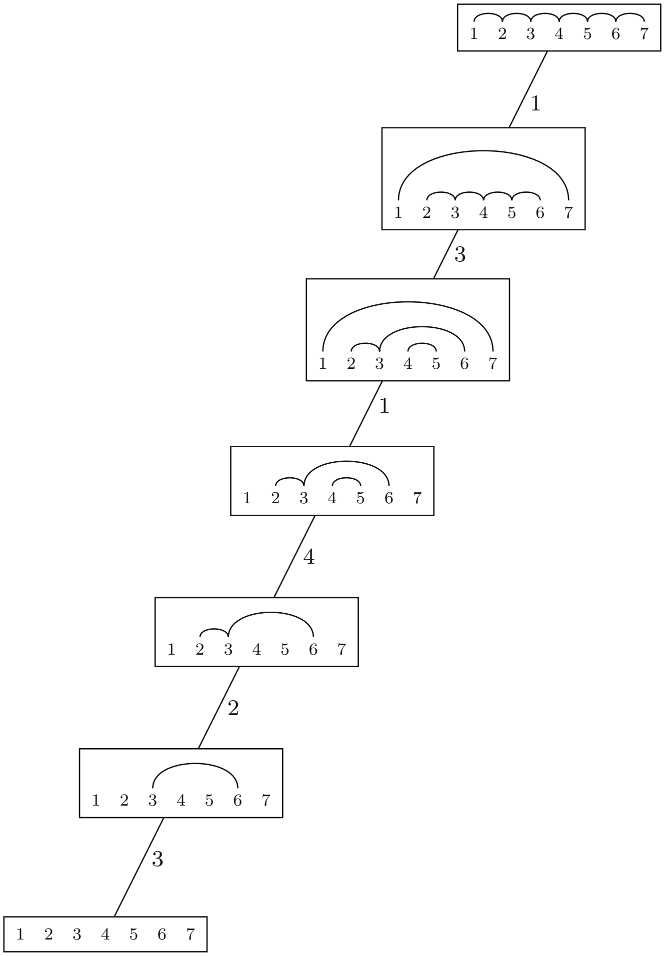


Fig. 14.11 The labeling of this chain in NC(7) corresponds to the parking function $(3, 2, 4, 1, 3, 1)$.

To see that the resulting vector is a parking function, we want to show that if $(b_1, \ldots, b_{n-1})$ is the increasing arrangement of $\mathbf{a}$, then $b_j \leq j$, i.e., for each j, there are at least $j-1$ entries of $\mathbf{a}$ that are smaller than j.

To see this, we notice that each of the elements $k = 1, \ldots, j-1$ will have to merge with one another at some point in the chain, and when their blocks merge, the label of such a merge must be smaller than j. Indeed, suppose R

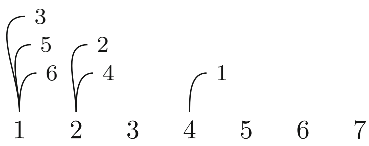
and S are two blocks with $\min R < \min S < j$. Then the label that comes from merging these blocks is

$$\max\{r \in R : r < s \text{ for all } s \in S\} < \min S < j.$$

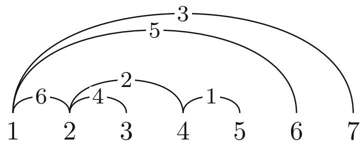
There must be at least $j-1$ such merges before the chain is complete, so there are at least $j-1$ entries in $\mathbf{a}$ that are smaller than j.

This verifies that $(a_1, \ldots, a_{n-1})$ must be an element of $\mathrm{PF}(n-1)$.

The inverse of this map is not as elegant, but it can be constructed recursively, based on the observation that, in order to avoid crossing arcs, the final time a number j appears in the parking function $\mathbf{a} = (a_1, \ldots, a_{n-1})$ its block must merge with the block containing $j+1$. To build the chain corresponding to $\mathbf{a}$, we can make a labeled arc diagram that begins by having some arcs emanating from node j, one for each occurrence of j in $\mathbf{a}$. We label these arcs from top to bottom with the positions in $\mathbf{a}$ in which j occurs. For example, if $\mathbf{a} = (4, 2, 1, 2, 1, 1)$, we would draw



These arcs can then be matched up greedily from bottom to top (avoiding crossings) to form a diagram such as this one:



The labels on these arcs now tells us the steps in forming the maximal chain, as shown in Figure 14.12.

3.11 Let's use the basic result that conjugacy classes in S_n are given by cycle type, i.e., by the number and size of the cycles. In particular, all n-cycles are conjugate to one another and all transpositions (i.e., 2-cycles) are conjugate to one another.

Now suppose $u \to v$ is a cover relation, with $u \circ t = v$ in $\mathrm{Abs}(c)$. Now suppose $c' = w \circ c \circ w^{-1}$, $u' = w \circ u \circ u^{-1}$, $v' = w \circ v \circ w^{-1}$, and $t' = w \circ t \circ w^{-1}$.

Clearly, $u' \to v'$ is a cover relation in $\mathrm{Abs}(S_n)$, since

$$u' \circ t' = (w \circ u \circ w^{-1}) \circ (w \circ t \circ w^{-1}) = w \circ u \circ t \circ w^{-1} = w \circ v \circ w^{-1} = v'.$$

It needs to be verified that $u' \to v'$ is a cover in $\mathrm{Abs}(c')$. This follows if $v' \leq_{\mathrm{Abs}} c'$.

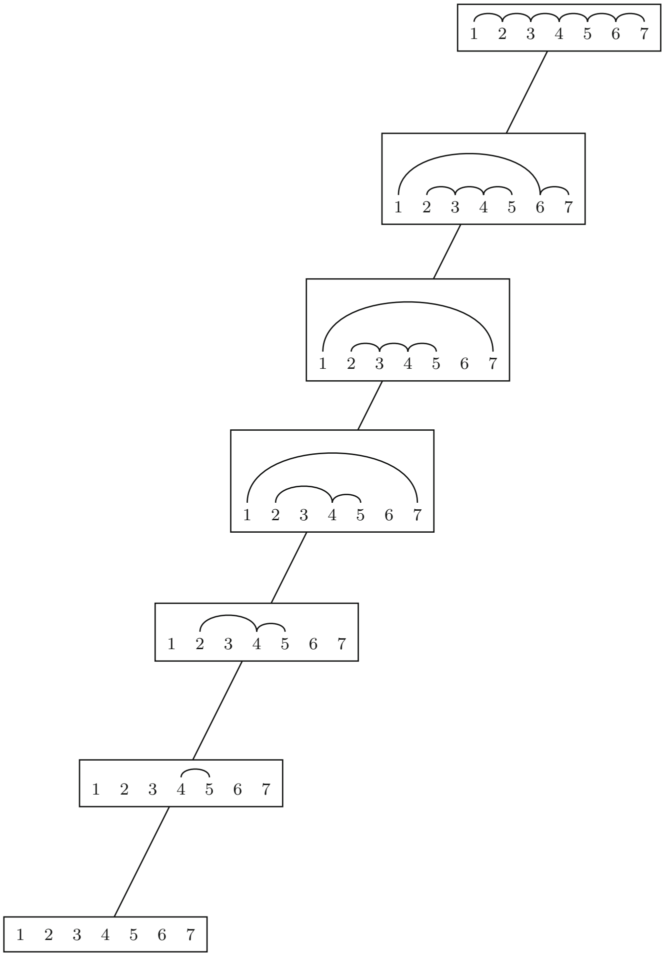


Fig. 14.12 The chain in NC(7) constructed from a parking function.

Since $v \to v_1 \to \cdots \to v_k \to c$, we can apply conjugation by w to each element in this chain and find

$$v' \to v_1' \to \cdots \to v_k' \to c'.$$

Hence $v' \leq_{\mathrm{Abs}} c'$.

Thus conjugation by w is a poset isomorphism $\mathrm{Abs}(c) \leftrightarrow \mathrm{Abs}(c')$.

Combinatorially, the effect of conjugation is to relabel the members of the elements in the blocks of a noncrossing partition.

Problems of Chapter 4

4.1 Given Corollary 4.1, we have

$$\begin{aligned} S_n(t) &= \sum_{w \in \widehat{S}_n} t^{\mathrm{pk}(w)}(1+t)^{n-1-2\,\mathrm{pk}(w)}, \\ &= (1+t)^{n-1} \sum_{w \in \widehat{S}_n} \left(\frac{t}{(1+t)^2}\right)^{\mathrm{pk}(w)}. \end{aligned}$$

Now recall that the number of elements in a hop-equivalence class depends only on the number of peaks. In particular,

$$|\operatorname{Hop}(w)| = 2^{n-1-2\,\mathrm{pk}(w)}.$$

Moreover, since the number of peaks is constant on hop-equivalence classes, we have

$$\begin{aligned} P_n(t) &= \sum_{v \in S_n} t^{\mathrm{pk}(v)}, \\ &= \sum_{w \in \widehat{S}_n} |\operatorname{Hop}(w)| t^{\mathrm{pk}(w)}, \\ &= \sum_{w \in \widehat{S}_n} 2^{n-1-2\,\mathrm{pk}(w)} t^{\mathrm{pk}(w)}, \\ &= 2^{n-1} \sum_{w \in \widehat{S}_n} \left(\frac{t}{4}\right)^{\mathrm{pk}(w)}. \end{aligned}$$

Thus,

$$\frac{(1+t)^{n-1}}{2^{n-1}} P_n\left(\frac{4t}{(1+t)^2}\right) = (1+t)^{n-1} \sum_{w \in \widehat{S}_n} \left(\frac{t}{(1+t)^2}\right)^{\mathrm{pk}(w)} = S_n(t),$$

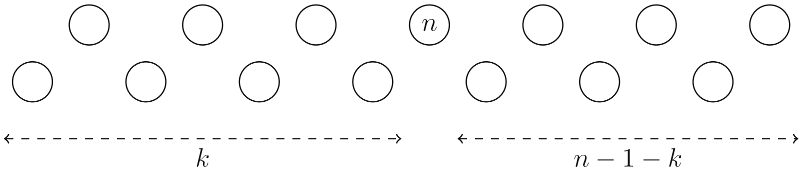
which establishes Equation (4.4). Equation (4.7) follows the same proof since hop-equivalence classes preserve the pattern 231.

4.2 Many of the enumerative results for alternating permutations are classical, due to Désiré André [6, 7].

1. We will show $|\mathcal{E}_n| = |\mathcal{E}'_n|$ with a bijection. Let ω be the involution on S_n that turns a permutation "upside down," i.e., $\omega w(i) = n+1-w(i)$. Then $i \in \operatorname{Des}(w)$ if and only if $i \notin \operatorname{Des}(\omega w)$. This clearly sends up-down alternating permutations to down-up alternating permutations. For example,

$$781924365 \leftrightarrow 32918745.$$

2. By part 1), we know the set of all alternating permutations of size n has cardinality $2E_n$.
   On the other hand, we can form the set of all alternating permutations of n as follows:



   - Choose k elements of $\{1, 2, \ldots, n-1\}$ to go to the left of n, and place the remaining elements to the right of n. This can be done in $\binom{n-1}{k}$ ways.
   - Arrange the elements to the left of n as an up-down permutation, written from right to left: $\cdots > u(3) < u(2) > u(1)$. This can be done in E_k ways.
   - Arrange the elements to the right of n as an up-down permutation written left to right: $v(1) < v(2) > v(3) < \cdots$. This can be done in E_{n-1-k} ways.

   Summing over all k yields the desired result:

$$2E_n = \sum_{k=0}^{n-1} \binom{n-1}{k} E_k E_{n-1-k},$$

   for $n \geq 2$.
3. Let $E(z) = \sum_{n \geq 0} E_n z^n / n!$ be the desired generating function. Using the result from part 2), we have:

$$\begin{aligned}
\sum_{n\geq 2} 2E_n \frac{z^{n-1}}{(n-1)!} &= \sum_{n\geq 2} \left(\sum_{k+l=n-1} \frac{E_k}{k!} \frac{E_l}{l!} \right) z^{n-1}, \\
&= \sum_{n\geq 2} \sum_{k+l=n-1} \left(E_k \frac{z^k}{k!} \right) \left(E_l \frac{z^l}{l!} \right), \\
&= \left(\sum_{k\geq 0} E_k \frac{z^k}{k!} \right) \left(\sum_{l\geq 0} E_l \frac{z^l}{l!} \right) - 1, \\
&= E(z)^2 - 1,
\end{aligned}$$

where the second to last identity comes from thinking of the index n as simply a way to keep track of the total degree in the expansion of the product. The only term missing from the product was $E_0^2 = 1$. So we have

$$E(z)^2 - 1 = 2\sum_{n\geq 2} E_n \frac{z^{n-1}}{(n-1)!}. \tag{14.1}$$

But notice

$$E'(z) = \sum_{n\geq 1} E_n \frac{z^{n-1}}{(n-1)!},$$

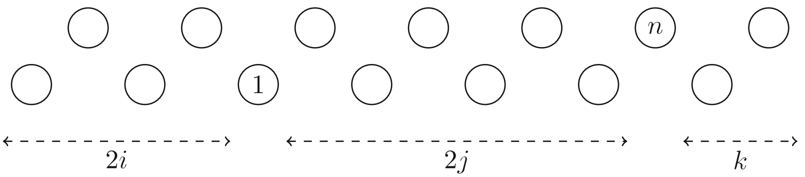
so the right-hand side of (14.1) is $2(E'(z)-1)$. Therefore we can write:

$$E(z)^2 + 1 = 2E'(z).$$

It easy to check that taking $E(z) = \sec z + \tan z$ satisfies this differential equation with initial condition $E(0) = 1$. This completes the derivation of André's generating function for up-down permutations.

4.3 This problem is taken from the paper [89] by Fiacha Heneghan and the author.

To form an up-down min-max permutation of n, we have to do three things:



1. choose the elements that go to the left of 1 and arrange them as an up-down permutation of even length,
2. choose the elements that go between 1 and n and arrange them as a down-up permutation of even length, and
3. arrange the remaining elements as an up-down permutation to the right of n.

Step 1 can be done in $\binom{n-2}{2i}E_{2i}$ ways, step 2 can be done in $\binom{n-2-2i}{2j}E_{2j}$ ways, and, letting $k = n-2-2i-2j$, step 3 can be done in E_k ways. Thus, for a fixed n, the number of min-max up-down permutations is:

$$\begin{aligned} E_n^{\nearrow} &= \sum_{\substack{i,j,k\geq 0\\ 2i+2j+k=n-2}} \binom{n-2}{2i} E_{2i} \binom{n-2-2i}{2j} E_{2j} E_k, \\ &= (n-2)! \sum_{\substack{i,j,k\geq 0\\ 2i+2j+k=n-2}} \frac{E_{2i}}{(2i)!}\frac{E_{2j}}{(2j)!}\frac{E_k}{k!}. \end{aligned} \tag{14.2}$$

Let $E^{\nearrow}(z) = \sum_{n\geq 0} E^{\nearrow}_{n+2} z^n/n!$. Then using the results of part 3 of Problem 4.2, we have

$$E^{\nearrow}(z) = \sec z \cdot \sec z \cdot (\sec z + \tan z) = \sec^3 z + \sec^2 z \tan z.$$

Nearly identical reasoning for $E^{\nwarrow}(z) = \sum_{n\geq 0} E^{\nwarrow}_{n+2} z^n/n!$ gives us

$$E^{\nwarrow}(z) = \tan z \cdot \sec z \cdot (\sec z + \tan z) = \sec^2 z \tan z + \sec z \tan^2 z.$$

Now that we have these two generating functions, we can compute their difference to find:

$$E^{\nearrow}(z) - E^{\nwarrow}(z) = \sec^3 z - \sec z \tan^2 z = \sec z \cdot (\sec^2 z - \tan^2 z) = \sec z.$$

The coefficient of $z^n/n!$ is, on the left-hand side, $E^{\nearrow}_{n+2} - E^{\nwarrow}_{n+2}$, while on the right-hand side it is 0 for odd n, E_n for even n.

It would be interesting to have a direct bijective proof of this fact.

4.4

1. We will apply the sorting operator recursively, which we can speed up with the easy observation that if a word w has no descents then $\mathcal{S}(w) = w$. We have:

$$\begin{aligned}\mathcal{S}(389124576) &= \mathcal{S}(38)\mathcal{S}(124576)9,\\ &= 38\mathcal{S}(1245)\mathcal{S}(6)9,\\ &= 38124569,\end{aligned}$$

   and

$$\begin{aligned}\mathcal{S}(132549678) &= \mathcal{S}(13254)\mathcal{S}(678)9,\\ &= \mathcal{S}(132)\mathcal{S}(4)56789,\\ &= \mathcal{S}(1)\mathcal{S}(2)3456789,\\ &= 123456789.\end{aligned}$$

2. We recall (from Chapter 2) the decomposition of a 231-avoiding permutation as $w = u(1)\cdots u(k)nv(1)\cdots v(n-1-k)$, where u is a 231-avoiding permutation of $\{1, 2, \ldots, k\}$ and v is a 231-avoiding permutation of the set $\{k+1, \ldots, n-1\}$.
   By induction, we can suppose $\mathcal{S}(u) = 12\cdots k$ and $\mathcal{S}(v) = (k+1)\cdots(n-1)$. Therefore, $\mathcal{S}(w) = \mathcal{S}(u)\mathcal{S}(v)n = 12\cdots n$.
3. This result also follows by induction on n.
   Suppose w and w' are in the same hop-equivalence class. Then all the peaks and valleys of w and peaks and valleys of w' are the same. In particular, the location of n is the same for both w and w'. Moreover, the sets of

letters to the left and the right of n in w is the same as the set of letters to the left of n in w'.
Write $w = unv$ and $w' = u'nv'$. Then $\mathrm{Hop}(u) = \mathrm{Hop}(u')$ and $\mathrm{Hop}(v) = \mathrm{Hop}(v')$. By induction, we can assume that $\mathcal{S}(u) = \mathcal{S}(u')$ and $\mathcal{S}(v) = \mathcal{S}(v')$, and the result follows.

4. By part 3), we see that the set of r-stack sortable permutations is a union of hop-equivalence classes. The result now follows.

4.5 Note that if we plug $z = -1$ into $C(z)$ we get $(\sqrt{5}-1)/2 = \phi^{-1}$, where ϕ is the golden ratio. Hence, we get

$$\begin{aligned}\gamma(-1) = \phi^{1+2+\cdots+(n-1)} \prod_{i=1}^{n} \frac{1-(-\phi^{-2})^i}{1+\phi^{-2}},\\ = \prod_{i=1}^{n} \frac{\phi^i - (-\phi^{-1})^i}{\phi+\phi^{-1}},\\ = \prod_{i=1}^{n} f_i,\end{aligned}$$

where the f_i are the Fibonacci numbers given by $f_1 = f_2 = 1$ and $f_i = f_{i-1} + f_{i-2}$.

The answer to this question is what led Zeilberger, in his personal journal with Shalosh B. Ekhad,

`http://www.math.rutgers.edu/~zeilberg/pj.html`

to come up with what we call Zeilberger's lemma in Equation (4.9). It is rather surprising that we begin with a refinement of $n!$, the product of the first n natural numbers, and through the process of passing to the gamma polynomial at $t = -1$ we end up with the product of the first n Fibonacci numbers.

Richard Stanley followed Zeilberger's proof with a more direct argument. Since by Observation 4.1 we know that gamma polynomials are multiplicative, we first investigate the gamma polynomial for each term in the product. For the term $(1 + t + t^2 + \cdots + t^i)$, we have $\gamma_i(t) = \sum_{j\geq 0}(-1)^j \binom{i-j}{j} t^j$. Since $f_i = \sum_{j\geq 0}\binom{i-j}{j}$ is another well-known identity for Fibonacci numbers, we get $\gamma_i(-1) = f_i$ and the result follows.

4.6 Suppose for induction that $A_n(t)$ has n distinct, nonpositive real roots:

$$A_n(t) = \prod_{i=1}^{n}(t-\alpha_i),$$

with

$$\alpha_n < \cdots < \alpha_2 < \alpha_1 = 0.$$

Rolle's theorem tells us that as a function of a real variable t, the derivative of $A_n(t)$ is zero somewhere between each consecutive pair of zeroes of $A_n(t)$. That is,

$$A_n'(t) = \prod_{i=1}^{n-1}(t - \beta_i),$$

with

$$\alpha_n < \beta_{n-1} < \cdots < \beta_2 < \alpha_2 < \beta_1 < \alpha_1 = 0.$$

Moreover, $(1-t)A_n'(t)$ has roots $\beta_{n-1}, \ldots, \beta_1$ and $\beta_0 = 1$.

We will show that

$$S_{n+1}(t) = (n+1)A_n(t) + (1-t)A_n'(t),$$

has n real roots, as desired.

We can check that the sign of $(1-t)A_n'(t)$, and hence the sign of $S_{n+1}(t)$, alternates at the roots of $A_n(t)$. That is,

$$\begin{aligned} S_{n+1}(\alpha_1) = S_{n+1}(0) &> 0, \\ S_{n+1}(\alpha_2) &< 0, \\ S_{n+1}(\alpha_3) &> 0, \\ &\vdots \end{aligned}$$

Thus the intermediate value theorem tells us $S_{n+1}(t)$ has real roots $\delta_1, \ldots, \delta_{n-1}$ such that:

$$\alpha_n < \delta_{n-1} < \cdots < \delta_2 < \alpha_2 < \delta_1 < \alpha_1.$$

This gives us $n-1$ distinct real roots, but we still need one more. We will find this by examining what happens as $t \to -\infty$.

Without loss of generality, suppose n is even. Then $S_{n+1}(\alpha_n) < 0$. The highest term in $(n+1)A_n(t)$ is $(n+1)t^n$, and the highest term in $(1-t)A_n'(t)$ is $-nt^n$. As $t \to -\infty$, then, $S_{n+1}(t) > 0$. Hence, there is yet another root, $\delta_n < \alpha_n$, such that $S_{n+1}(\delta_n) = 0$.

This proves that the Eulerian polynomials are real rooted. Moreover, notice that the roots of $S_n(t)$ are $\alpha_2, \ldots, \alpha_n$, and

$$\delta_n < \alpha_n < \cdots < \delta_2 < \alpha_2 < \delta_1,$$

i.e., the roots of consecutive Eulerian polynomials are interlacing. Thus the Eulerian polynomials form a Sturm sequence.

4.7

1. The recurrence shown in (4.11) is straightforward to verify with generating functions. Letting $N(t,z) = 1 + \sum_{n\geq 1} N_n(t)z^n$, we get the following formula from Equation (2.6):

$$N(t,z) = \frac{1 - z(t-1) - \sqrt{1 - 2z(t+1) + z^2(t-1)^2}}{2z}. \tag{14.3}$$

After a bit of careful bookkeeping, we can see the recurrence in (4.11) corresponds to the following identity for the generating function:

$$(1-(1+t)z)N(t,z) = 1+(t-1)z+z(2(1+t)z-(1-t)^2z^2-1)\frac{d}{dz}\left[N(t,z)\right].$$

A quick check tells us that the generating function in (14.3) does indeed satisfy this equation.
Robert Sulanke gives a bijective proof of (4.11) in [161] using "marked" Dyck paths in the style of Section 2.4.2.

2. We will prove real-rootedness by induction, using the recurrence (4.11),

$$(n+1)N_n(t) = (2n-1)(1+t)N_{n-1}(t) - (n-2)(1-t)^2N_{n-2}(t).$$

Our induction hypothesis is that, apart from the common root at $t = 0$, $N_{n-1}(t)$ and $N_{n-2}(t)$ have distinct and interlacing roots, i.e., write

$$N_{n-1}(t) = t\prod_{i=1}^{n-2}(t-\alpha_i),$$

and

$$N_{n-2}(t) = t\prod_{i=1}^{n-3}(t-\beta_i),$$

where

$$\alpha_{n-2} < \beta_{n-3} < \cdots < \beta_2 < \alpha_2 < \beta_1 < \alpha_1 < 0.$$

(This trivially holds for $N_1(t) = t$ and $N_2(t) = t(1+t)$.)
The polynomial $-(1-t)^2N_{n-2}(t)$, and hence $N_n(t)$, has the following signs at the roots of $N_{n-1}(t)$ (without loss of generality, suppose n is even).

t	α_1	α_2	$\cdots$	α_{n-3}	α_{n-2}
$-(1-t)^2N_{n-2}(t)$	$+$	$-$	$\cdots$	$+$	$-$
$N_n(t)$	$+$	$-$	$\cdots$	$+$	$-$

This tells us $N_n(t)$ has at least $n-2$ distinct roots, $\delta_0 = 0, \delta_1, \ldots, \delta_{n-3}$ such that

$$\alpha_{n-2} < \delta_{n-3} < \cdots < \delta_2 < \alpha_1 < 0 = \alpha_0 = \delta_0.$$

Further, since the largest term in $N_n(t)$ is t^n, we see that although the sign of $N_n(t)$ is negative at α_{n-2} (when n is even) the limit as $t \to -\infty$ has sign $(-1)^n$. Thus, there is another root of $N_n(t)$, say $\delta_{n-1} < \alpha_{n-2}$, moreover, since $N_n(t)$ nonnegative and symmetric coefficients, its reciprocal $1/\delta_{n-1}$ is another root of $N_n(t)$, with $\alpha_1 < 1/\delta_{n-1} < 0$.

This shows that $N_n(t)$ has n real roots. Also $N_n(t)$ and $N_{n-1}(t)$ have interlacing real roots that are distinct except for $t = 0$. Thus the Narayana polynomials, $C_n(t) = N_n(t)/t$, form a Sturm sequence.
That the techniques for proving the real-rootedness of the Eulerian and Narayana polynomials are so similar hints at a more general theory for establishing real-rootedness of a sequence of polynomials. There is a large and growing literature on this subject. Petter Brändén has a nice survey of the current state of the art [32].

4.8

1. We wish to show
$$\binom{n}{k}^2 \geq \binom{n}{k-1}\binom{n}{k+1},$$
or
$$\frac{\binom{n}{k}^2}{\binom{n}{k-1}\binom{n}{k+1}} \geq 1.$$
Using the formula for binomial coefficients, we see
$$\frac{\binom{n}{k}^2}{\binom{n}{k-1}\binom{n}{k+1}} = \frac{(n-k+1)(k+1)}{(n-k)k} > 1,$$
and the result is proved.
2. Similarly to part 1), we see, with k fixed,
$$\frac{\binom{n}{k}^2}{\binom{n+1}{k}\binom{n-1}{k}} = \frac{n^2+n-nk}{n^2+n-nk-k} \geq 1,$$
with equality only for $k = 0$.
3. Suppose $a_j^2 \geq a_{j-1}a_{j+1}$ and $b_j^2 \geq b_{j-1}b_{j+1}$. Then
$$c_j^2 = (a_jb_j)^2 \geq (a_{j-1}a_{j+1})(b_{j-1}b_{j+1}) = (a_{j-1}b_{j-1})(a_{j+1}b_{j+1}) = c_{j-1}c_{j+1}.$$
4. By part 1), $\binom{n}{k}$ forms a log-concave sequence, and $a_k = \binom{n}{k}b_k$. Thus by part 3) if sequence $\{b_k\}$ is log-concave, then so is sequence $\{a_k\}$.
5. The goal here is to show that a real-rooted polynomial with nonnegative coefficients is log-concave. The argument presented here is adapted from the introduction of Brändén's survey [32].

   a. With $f(t) = \sum a_k t^k$ and $a_k = \binom{n}{k}b_k$, we have
$$\frac{1}{n}f'(t) = \sum_{k=0}^{n} \frac{k}{n}\binom{n}{k} b_k t^{k-1} = \sum_{k=0}^{n-1} \binom{n-1}{k} b_{k+1} t^k.$$
   If f is real-rooted, the real-rootedness of this polynomial follows from Rolle's theorem.

b. By symmetry of binomial coefficients, this polynomial is simply $t^n f(1/t)$.
c. We'll call the operation of part 5a) the *normalized derivative* and the operation of part 5b) we will call *reversal*. Both of these operations preserve real-rootedness, and the goal is to isolate three consecutive terms of the polynomial f through these operations.
This can be done as follows. Apply the normalized derivative until the constant term is b_{j-1}. If $j = n-1$, we are done. If $j < n-1$, we apply reversal then again repeat the application of the normalized derivative until the constant term is b_{j+1}. An example should make the idea clear. Suppose

$$f(t) = b_0 + \binom{7}{1} b_1 t + \binom{7}{2} b_2 t^2 + \cdots + \binom{7}{6} b_6 t^6 + b_7 t^7,$$

and say that we want to show $b_4^2 \geq b_3 b_5$. Here are the steps of the process, where "d" indicates we applied the normalized derivative, and "r" indicates reversal.

$$b_0 + \tbinom{7}{1} b_1 t + \tbinom{7}{2} b_2 t^2 + \tbinom{7}{3} b_3 t^3 + \tbinom{7}{4} b_4 t^4 + \tbinom{7}{5} b_5 t^5 + \tbinom{7}{6} b_6 t^6 + b_7 t^7$$

$$\downarrow \text{(d)}$$

$$b_1 + \tbinom{6}{1} b_2 t + \tbinom{6}{2} b_3 t^2 + \tbinom{6}{3} b_4 t^3 + \tbinom{6}{4} b_5 t^4 + \tbinom{6}{5} b_6 t^5 + b_7 t^6$$

$$\downarrow \text{(d)}$$

$$b_2 + \tbinom{5}{1} b_3 t + \tbinom{5}{2} b_4 t^2 + \tbinom{5}{3} b_5 t^3 + \tbinom{5}{4} b_6 t^4 + b_7 t^5$$

$$\downarrow \text{(d)}$$

$$b_3 + \tbinom{4}{1} b_4 t + \tbinom{4}{2} b_5 t^2 + \tbinom{4}{3} b_6 t^3 + b_7 t^4$$

$$\downarrow \text{(r)}$$

$$b_7 + \tbinom{4}{1} b_6 t + \tbinom{4}{2} b_5 t^2 + \tbinom{4}{3} b_4 t^3 + b_3 t^4$$

$$\downarrow \text{(d)}$$

$$b_6 + \tbinom{3}{1} b_5 t + \tbinom{3}{2} b_4 t^2 + b_3 t^3$$

$$\downarrow \text{(d)}$$

$$b_5 + \tbinom{2}{1} b_4 t + b_3 t^2$$

We know this polynomial is real-rooted, and hence its discriminant is nonnegative, i.e.,

$$4(b_4^2 - b_3 b_5) \geq 0.$$

4.9 Following the hint, we will first prove the following.

Lemma. If $a_0, a_1, a_2, \ldots$ is a nonnegative, log-concave sequence, then for any $i < j-1$, $a_i a_j \leq a_{i+1} a_{j-1}$.

The proof follows by induction on $j-i$, with two cases. If $j-i$ is even, the base case is $j-i=2$, which is the definition of log-concavity: $a_i a_{i+2} \leq a_{i+1}^2$. If $j-i$ is odd, observe that

$$a_{i+1}^2 a_{i+2}^2 \geq (a_i a_{i+2})(a_{i+1} a_{i+3}) = a_i a_{i+3}(a_{i+1} a_{i+2}),$$

from whence it follows that

$$a_i a_{i+3} \leq a_{i+1} a_{i+2}.$$

Now suppose for induction that $a_{i+2} a_{j-2} \geq a_{i+1} a_{j-1}$ and all the terms are positive. Then

$$a_{i+1} a_{j-1} = \frac{a_{i+1}^2 a_{j-1}^2}{a_{i+1} a_{j-1}} \geq \frac{(a_i a_{i+2})(a_{j-2} a_j)}{a_{i+1} a_{j-1}} = a_i a_j \left(\frac{a_{i+2} a_{j-2})}{a_{i+1} a_{j-1}} \right) \geq a_i a_j,$$

as desired.

This proves the lemma.

Now, with the lemma in hand, let $f(t) = \sum f_i t^i$ and $g(t) = \sum g_i t^i$. Then if $h(t) = \sum h_i t^i = f(t) g(t)$, we have

$$h_n = \sum_{i+j=n} f_i g_j.$$

With some careful bookkeeping, we find

$$h_n^2 = \sum_{i=0}^{n} \sum_{j=0}^{n} f_i f_j g_{n-i} g_{n-j},$$

and

$$h_{n-1} h_{n+1} = \sum_{i=0}^{n-1} \sum_{j=0}^{n+1} f_i f_j g_{n-i-1} g_{n-j+1}.$$

Computing the difference we get:

$$
\begin{aligned}
h_n^2 - h_{n-1}h_{n+1} &= \sum_{i=0}^{n} f_i f_n g_0 g_{n-i} - \sum_{j=0}^{n-1} f_j f_{n+1} g_0 g_{n-j-1}, \\
&= f_0 f_n g_0 g_n + \sum_{j=0}^{n-1} (f_{i+1} f_n - f_i f_{n+1})\, g_0 g_{n-i-1}.
\end{aligned}
$$

By the lemma, all the differences in parentheses are nonnegative. Thus $h_n^2 \geq h_{n-1}h_{n+1}$, and $h(t)$ is log-concave, as we hoped to show.

4.10 Here are some sample polynomials:

1. $1 + 4t + 17t^3 + 4t^3 + t^4$
2. $1 + 2t + 3t^2 + 2t^3 + t^4$
3. $1 + 5t + 9t^2 + 5t^3 + t^4$

4.11 As of this writing, this is an open problem! These polynomials are not real-rooted, but it is not too difficult to show they are palindromic. See papers by Mark Dukes [61] and Victor Guo and Jiang Zeng [86]. Guo and Zeng proved unimodality, and conjectured gamma-nonnegativity.

4.12 This is an open problem! See this author's survey [119], and a related paper by Mirkó Visontai [164].

4.13

1. Every chain is obviously rank palindromic and unimodal, and within a poset P of rank n, a symmetric chain has rank generating function

   $$t^j(1 + t + \cdots + t^{n-2j}) = t^j + t^{j+1} + \cdots + t^{n-j},$$

   for some $j \leq \lfloor n/2 \rfloor$. Thus, the rank function takes the form

   $$\sum_{j=0}^{\lfloor n/2 \rfloor} g_j t^j (1 + t + \cdots + t^{n-2j}),$$

   where g_j is the number of symmetric chains in the decomposition with minimum rank equal to j. Symmetry is therefore obvious, and

   $$f_0 = g_0 \leq f_1 = g_0 + g_1 \leq f_2 = g_0 + g_1 + g_2 \leq \cdots \leq f_{\lfloor n/2 \rfloor} = g_0 + \cdots + g_{\lfloor n/2 \rfloor},$$

   so f is also unimodal.
2. Suppose $P = c_1 \cup c_2 \cup \cdots \cup c_k$ is a symmetric chain decomposition of poset P. Then any antichain A can have at most one element from each chain, and thus $|A| \leq k$. But the number of chains in a symmetric chain decomposition is equal to

   $$g_0 + g_1 + \cdots + g_{\lfloor n/2 \rfloor} = f_{\lfloor n/2 \rfloor}.$$

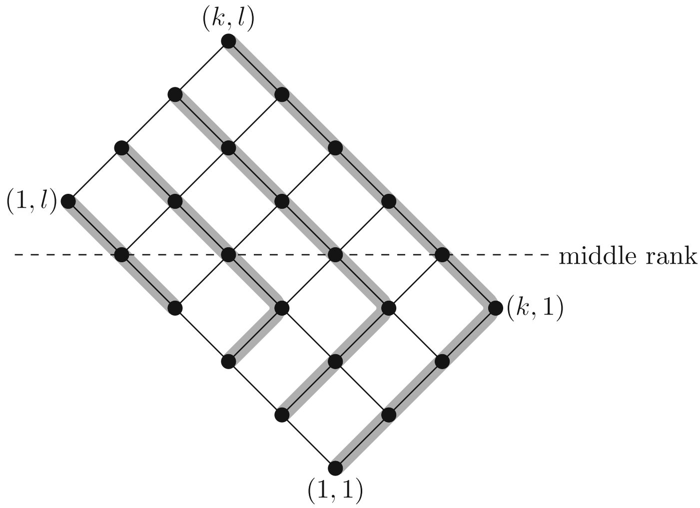
3. We can do this greedily, by taking the largest chain to be:

$$(1,1) < (2,1) < \cdots < (k,1) < (k,2) < \cdots < (k,l),$$

the second chain to be:

$$(1,2) < (2,2) < \cdots < (k-1,2) < (k-1,3) < \cdots < (k-1,l),$$

and so on. Each chain is rank symmetric since the sum of the coordinates of the minimal and maximal elements is a constant $(k+l+2)$ on each chain. The example pictured here should make idea clear.



4. Suppose P has rank m and symmetric chain decomposition $c_1 \cup c_2 \cup \cdots \cup c_r$, while Q is rank n with symmetric chain decomposition $d_1 \cup d_2 \cup \cdots \cup d_s$. Then $P \times Q$ has rank $m+n$, and we can partition this poset as

$$P \times Q = \bigcup_{\substack{1 \le i \le r \\ 1 \le j \le s}} c_i \times d_j.$$

The posets $c_i \times d_j$ are products of chains, so by part 3), they have symmetric chain decompositions. All that remains to show is that these decompositions have the same center of symmetry across all choices of i and j. This is achieved by showing that the sum of the ranks of the minimal and maximal elements in $c_i \times d_j$ is $m+n$.
Let $c_{i,0}$ and $c_{i,1}$ denote the minimal and maximal elements in chain c_i, and similarly denote the minimal and maximal elements of d_i by $d_{i,0}$ and $d_{i,1}$. The minimal element of $c_i \times d_j$ is $(c_{i,0}, d_{i,0})$ and the maximal element is $(c_{i,1}, d_{i,1})$.

By the definition of symmetric chain decomposition, we have:

$$\rho_P(c_{i,0}) + \rho_P(c_{i,1}) = m \qquad \text{and} \qquad \rho_Q(d_{j,0}) + \rho_Q(d_{j,1}) = n.$$

The rank of an element (p, q) in $P \times Q$ is the sum $\rho_P(p) + \rho_Q(q)$. Thus the sum of the minimal rank and maximal rank of $c_i \times d_j$ is:

$$\begin{aligned}
&\rho_{P\times Q}((c_{i,0}, d_{j,0})) + \rho_{P\times Q}((c_{i,1}, d_{j,1})) \\
&\quad = \rho_P(c_{i,0}) + \rho_Q(d_{j,0}) + \rho_P(c_{i,1}) + \rho_Q(d_{j,1}), \\
&\quad = \rho_P(c_{i,0}) + \rho_P(c_{i,1}) + \rho_Q(d_{j,0}) + \rho_Q(d_{j,1}), \\
&\quad = m + n,
\end{aligned}$$

as desired.

4.14 This follows by induction from Problem 4.13 once we recognize that $2^{[n]}$ is isomorphic to the product $\{0, 1\} \times 2^{[n-1]}$. Sperner's theorem is a special case of part 2 of Problem 4.13.

A more direct symmetric chain decomposition of $2^{[n]}$ is due to Martin Aigner in 1973 [4] and (independently) Curtis Greene and Daniel Kleitman in 1976 [83]. This construction also turns out to be the same as the recursive approach found in a 1951 paper by Nicolaas de Bruijn, Ca. van Ebbenhorst Tengbergen, and D. Kruyswijk [54]. (See Problem 4.15.)

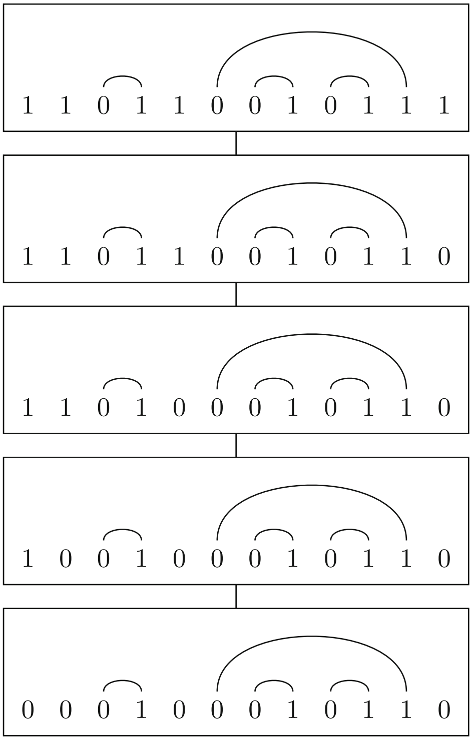
The symmetric chain decomposition works as follows. Consider subsets of $[n]$ as binary strings in $\{0, 1\}^n$, with the obvious correspondence. Now, we can consider any binary string to define a partial matching, with the 0 letters as openers and the 1 letters as closers. For example, the word 110100010110 corresponds to the following matching:

$$1 \quad 1 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 1 \quad 1 \quad 0 \ .$$

Notice that all the unmatched ones in a binary string occur to the left of the unmatched zeroes. This is the key insight that gives a natural symmetric chain decomposition.

For a given matching on n, let all the unmatched positions be zeroes. Then we can move up in a chain by converting the zeroes to ones, working from left to right. The minimal element in such a chain has rank equal to the number of matched pairs, and the maximal element has the same co-rank, i.e., (the number of ones in the minimal element of a chain) plus (the number of ones in the maximal element of the chain) equals n.

Here is the chain containing the example string used above:



.

In Figure 14.13 we see the full decomposition applied to the case $n = 4$.

4.15 The topic of this problem is studied in the 1951 paper of Nicolaas de Bruijn, Ca. van Ebbenhorst Tengbergen, and D. Kruyswijk [54].

As a first easy observation, we see that the divisibility lattice of a prime power is simply a chain: $\Lambda(p^m) = \{1 \prec p \prec p^2 \prec \cdots \prec p^m\} \cong [m+1]$, where $[n]$ denotes the set $\{1, 2, \ldots, n\}$ with the usual ordering.

A second easy observation is that if $\gcd(a, b) = 1$, then $\Lambda(ab) \cong \Lambda(a) \times \Lambda(b)$.

Putting these two observations together, we see that if n has prime decomposition $n = p_1^{m_1} \cdots p_k^{m_k}$, then

$$\begin{aligned} \Lambda(n) &\cong \Lambda(p_1^{m_1}) \times \cdots \times \Lambda(p_k^{m_k}), \\ &\cong [m_1 + 1] \times \cdots \times [m_k + 1]. \end{aligned}$$

See Figure 14.14.

The results of Problem 4.13 can now be applied.

Notice that the boolean lattice is a special case, being isomorphic to the lattice of divisors of a number whose prime factors all have multiplicity one, i.e., $2^{[k]} \cong \Lambda(p_1 \cdots p_k)$.

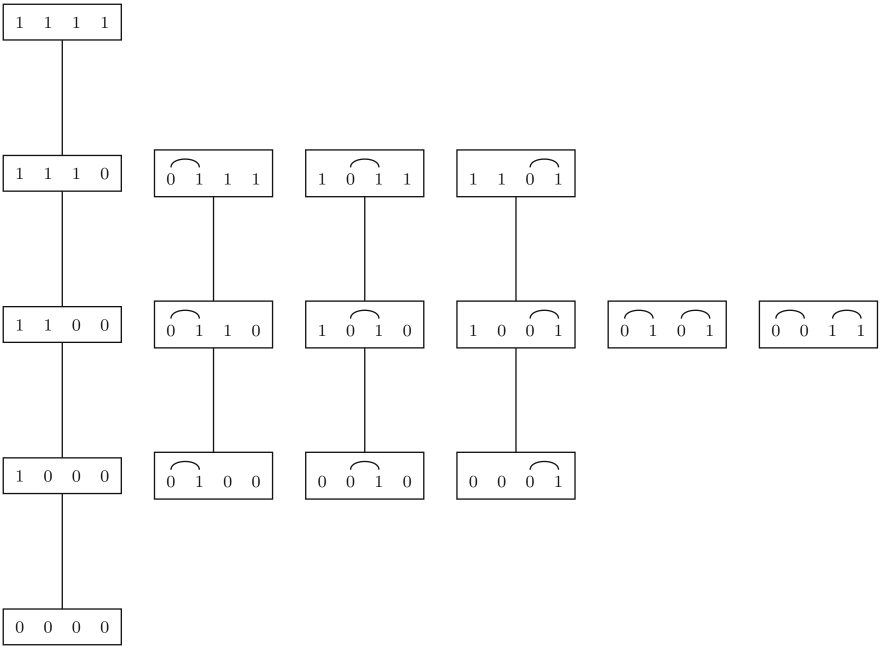


Fig. 14.13 The symmetric chain decomposition of $\{0,1\}^4$.

4.16 Similarly to part 4 of Problem 4.13, suppose P has rank m and symmetric boolean decomposition $P_1 \cup P_2 \cup \cdots \cup P_r$, while Q is rank n with symmetric boolean decomposition $Q_1 \cup Q_2 \cup \cdots \cup Q_s$.

Then $P \times Q$ has rank $m+n$, and we can partition this poset as

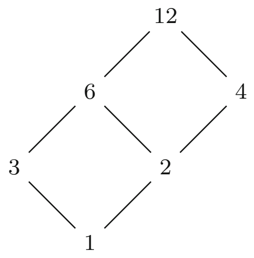
$$P \times Q = \bigcup_{\substack{1 \le i \le r \\ 1 \le j \le s}} P_i \times Q_j .$$

The product of boolean algebras of ranks k and l is a boolean algebra of rank $k+l$, and just as in part 4 of Problem 4.13, it follows that the middle ranks of all the subposets $P_i \times Q_j$, viewed inside $P \times Q$, coincide.

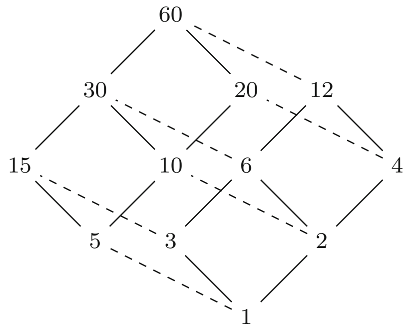
4.17 Parts 1)–4) can be found in [140, Section 2]. Part 5) is a main topic in the 2014 paper of Saúl Blanco and this book's author [26], and part 6) is in Section 5 of [118].

We remark that the existence of a symmetric boolean decomposition of $\mathrm{NC}(n)$ also follows by induction. The argument applies Problem 4.16 to each term in the poset isomorphism here:

$$\mathrm{NC}(n) \cong [2] \times \mathrm{NC}(n-1) \cup \mathrm{NC}(1) \times \mathrm{NC}(n-2) \cup \mathrm{NC}(2) \times \mathrm{NC}(n-3) \cup \cdots .$$



$$\Lambda(12) \cong \Lambda(2^2) \times \Lambda(3) \cong [3] \times [2]$$



$$\Lambda(60) \cong \Lambda(12) \times \Lambda(5) \cong [3] \times [2] \times [2]$$

Fig. 14.14 The lattice of divisors of 60 is isomorphic to $[3] \times [2] \times [2]$.

In this correspondence, $[2] \times \mathrm{NC}(n-1)$ is isomorphic to the set of all partitions with either $\{1\}$ as a singleton block or both 1 and 2 in the same block. For $i \geq 3$, $\mathrm{NC}(i-2) \times \mathrm{NC}(n-i+1)$ is isomorphic to the set of noncrossing partitions on $\{2, 3, \ldots, i-1\}$ paired with noncrossing partitions on $\{i, i+1, \ldots, n\}$. This decomposition was used by Simion and Ullman to prove that $\mathrm{NC}(n)$ has a symmetric chain decomposition.

Problems of Chapter 5

5.1 This result follows by induction with the recursive argument leading up to Theorem 5.1.

A visual correspondence between inversion sets and permutations is to use the model of permutations as arrays of nonattacking rooks on a chessboard. For each i, we put a rook in column i (from left to right) and row $w(i)$ (from bottom to top).

For each rook in the diagram consider the cells to the right of the rook in the same row. For the rook in column i, row $w(i)$, we scan the cells in row $w(i)$, column j with $j > i$. If the rook in column j is in a row below the rook in column i, i.e., $w(i) > w(j)$, we label the cell with (i, j).

For example, if $w = 971326458$, we draw

♖ 9	$(1,2)$	$(1,3)$	$(1,4)$	$(1,5)$	$(1,6)$	$(1,7)$	$(1,8)$	$(1,9)$
								♖ 8
	♖ 7	$(2,3)$	$(2,4)$	$(2,5)$	$(2,6)$	$(2,7)$	$(2,8)$	
					♖ 6	$(6,7)$	$(6,8)$	
							♖ 5	
						♖ 4		
			♖ 3	$(4,5)$				
				♖ 2				
		♖ 1						

and the inversion set is

$$\mathrm{Inv}(w) = \left\{ \begin{matrix} (1,2),(1,3),(1,4),(1,5),(1,6),(1,7),(1,8),(1,9) \\ (2,3),(2,4),(2,5),(2,6),(2,7),(2,8),(6,7),(6,8) \end{matrix} \right\}.$$

This method can be used to easily translate between a permutation, its inversion set, and its *inversion sequence* $s = (s_1, \ldots, s_n)$, given by

$$s_j = |\{i < j : w(i) > w(j)\}|,$$

since s_j is simply the number of marked cells in column j. In the example above, $s = (1, 2, 2, 3, 2, 3, 3, 1)$.

See also Problem 1.8.

5.3 Suppose $u \leq_{\mathrm{Sh}} v$. Recall the model for shards from Section 3.3, in which we think of permutations as ordered set partitions in which the blocks are given by maximal decreasing runs. The definition for $\mathrm{Sh}(S_n)$ says that $u \leq_{\mathrm{Sh}} v$ if:

- (Refinement) u refines v as a set partition, and
- (Consistency) if i and j are in the same block in u, and k is not in the same block as i and j in u with $i < k < j$, then either k is in the same block as i and j in v, or k is on the same side of i and j in both v and u.

Our goal is to show $\mathrm{Inv}(u^{-1}) \subset \mathrm{Inv}(v^{-1})$.

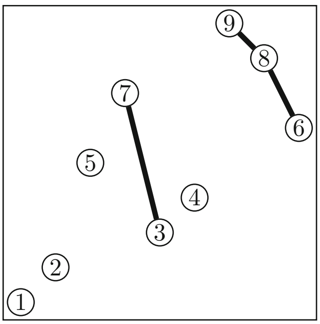
But notice the inversion set of the inverse of a permutation w is the set of all $(w(j), w(i))$ such that (i, j) is an inversion of w. That is, we need to keep track of all (k, l) such that $k < l$ and k appears to the right of l in w.

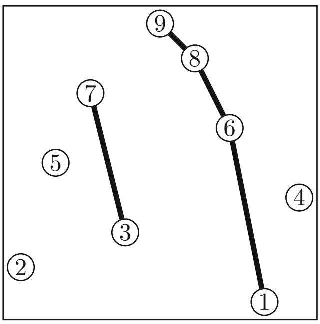
Suppose $u \leq_{\mathrm{Sh}} v$ and (k, l) is in $\mathrm{Inv}(u^{-1})$, and therefore $u \leq_{\mathrm{Wk}} v$.

If k and l are in the same block (i.e., decreasing run) in u, they must be in the same block of v by the "refinement" condition. Hence (k, l) is in $\mathrm{Inv}(v^{-1})$.

If k and l are in different blocks of u, then the "consistency" condition guarantees that (k, l) is in $\mathrm{Inv}(v^{-1})$.

This can be seen in the visual representation used in Section 3.3. For example, consider the following cover relation in $\mathrm{Sh}(S_9)$:

$u =$ 

$<_{\mathrm{Sh}}$ 

$= v.$

5.4 If $n \geq 3$, then $(1 + q + q^2)$ is a factor of $I_n(q)$, and this has roots $(-1 \pm \sqrt{-3})/2$. Thus $I_n(q)$ has complex roots. In general,

$$[k] = 1 + q + \cdots + q^{k-1} = (1 - q^k)/(1 - q),$$

so $[k]$ has roots given by all complex kth roots of unity, apart from $q = 1$.

Despite not being real rooted, $I_n(q)$ is still log-concave. This follows since each factor $[k]$ is trivially log-concave and the product of log-concave polynomials is log-concave as shown in Problem 4.9.

5.5 First we make the simple observation that for any $w \in S_n$ and any $i = 1, \ldots, n-1$, $\mathrm{inv}(w \circ s_i) = \mathrm{inv}(w) \pm 1$.

Thus we can establish that $\ell(w) = \mathrm{inv}(w)$ by exhibiting a sorting algorithm that reduces inversion number with each swap. This can be done greedily as follows.

Suppose $j \leq n$ is the largest element of w that is to the left of where it belongs, i.e., suppose $w(i) = j$ for some $i < j$, and that $w(k) = k$ for $k \geq j+1$. (If there is no such j, w is sorted and we are done.) Since $i+1 < j+1$, we know $w(i) = j > w(i+1)$. Then let $w' = w \circ s_i$ and observe that $\mathrm{inv}(w') = \mathrm{inv}(w) - 1$.

Now we repeat with the permutation w', continuing until we have no inversions.

5.6 Just as with the solution to Problem 5.5, we will use a sorting algorithm that decreases the statistic in question, $n - \mathrm{cyc}(w)$, at each step. First, however, notice that for any transposition $t_{i,j}$,

$$\mathrm{cyc}(w \circ t_{i,j}) = \mathrm{cyc}(w) \pm 1.$$

Since we have not worked with cycle structure as much as one-line notation, this assertion may require a moment of thought. Let $v = w \circ t_{i,j}$. Then if i and j are in the same cycle of w, they will be in different cycles of v. Conversely, if i and j are in different cycles of w, these two cycles will be merged in v. For example let $w = (137)(24658)$, let $v = w \circ t_{4,8}$, and let $v' = w \circ t_{3,5}$. Then in cycle notation,

$$v = (137)(24)(586) \qquad \text{and} \qquad v' = (13824657).$$

This example possesses all the necessary ingredients for the general assertion.

Now that we have seen that right multiplication by a transposition either increases cycle number by one or decreases cycle number by one, we will exhibit an algorithm that sorts permutations in a way that only increases the number of cycles (and so decreases $n - \mathrm{cyc}(w)$), until we end up with the identity: $12 \cdots n = (1)(2) \cdots (n)$.

The algorithm we will use is the "straight selection sort" algorithm described in Problem 5.7.

This algorithm finds the largest $k \leq n$ such that $w(k) \neq k$. Suppose $w(i) = k$, with $i < k$. Then we apply the transposition $t_{i,k}$ to get $w' = w \circ t_{i,k}$. We have $w'(k) = k$, i.e., k is a fixed point of w', whereas k was not a fixed point of w. That is to say, $\mathrm{cyc}(w') = \mathrm{cyc}(w) + 1$, as desired.

We now repeat with the permutation w', continuing until we have a permutation with all fixed points.

5.7 The distributions of $\mathrm{sor}(w)$ and $\mathrm{inv}(w)$ are easily seen to agree for small n. We will prove the sorting index is Mahonian by showing the following recursive formula holds:

$$\sum_{w\in S_n} q^{\mathrm{sor}(w)} = (1+q+\cdots+q^{n-1}) \sum_{u\in S_{n-1}} q^{\mathrm{sor}(u)}.$$

To see the recursive formula is true, let $S_{n;j}$ denote the set of all v in S_n such that $v(j)=n$. Clearly the distribution of sor on $S_{n;n}$ (the permutations for which n is a fixed point) is the same as the distribution of sor on S_{n-1}:

$$\sum_{v\in S_{n;n}} q^{\mathrm{sor}(v)} = \sum_{u\in S_{n-1}} q^{\mathrm{sor}(u)}.$$

Now if $w \in S_{n;j}$, with $j<n$, notice that $v = w\circ t_{j,n}$ is in $S_{n;n}$ and moreover, $\mathrm{sor}(w) = n-j+\mathrm{sor}(v)$. Thus,

$$\sum_{w\in S_{n;j}} q^{\mathrm{sor}(w)} = q^{n-j} \sum_{v\in S_{n;n}} q^{\mathrm{sor}(v)}.$$

Summing over all j, we have

$$\begin{aligned}\sum_{w\in S_n} q^{\mathrm{sor}(w)} &= \sum_{j=1}^{n} \sum_{w\in S_{n;j}} q^{\mathrm{sor}(w)},\\ &= \sum_{j=1}^{n} q^{n-j} \sum_{v\in S_{n;n}} q^{\mathrm{sor}(v)},\\ &= (1+q+\cdots+q^{n-1}) \sum_{u\in S_{n-1}} q^{\mathrm{sor}(u)},\end{aligned}$$

as desired.

The sorting index was first studied by Mark Wilson in 2010 [167] and independently by this book's author in 2011 [117], who generalized the notion to other Coxeter groups. The sorting index for permutations can also be derived from work of Dominique Foata and Guo-Niu Han in 2009 [69].

5.8 Suppose a point $\mathbf{x} = (x_1,\ldots,x_n)$ lies on face F. Then i and j are in the same block of F if and only if the sign vector has $\sigma_{ij}(F)=0$, i.e., we have $x_i = x_j$.

By Observation 5.3 regarding the Tits product, we have $\sigma_{ij}(FG) = \sigma_{ij}(F)$ if $\sigma_{ij}(F)\neq 0$. Thus, we can see that as a set composition, FG must be a refinement of F.

A given block of F signals that we have some equal coordinates: $x_{i_1} = \cdots = x_{i_k}$, and so $\sigma_{ab}(F)=0$ for $a,b\in\{i_1,\ldots,i_k\}$. By Observation 5.3 again, we have $\sigma_{ab}(FG) = \sigma_{ab}(G)$ for each such pair (a,b). This means that if a and

b are in the same block of G, they are in the same block of FG. If a and b are in different blocks of G, with, say, a to the left of b, then the same must be true for the relative positions of a and b in FG.

Thus if B is a block of F, and $G = C_1|C_2|\cdots|C_l$, then B will be refined by the linear ordering of blocks of G:

$$B \to B \cap C_1|B \cap C_2|\cdots|B \cap C_l.$$

Applying this refinement rule to each block of F in place (and ignoring empty blocks), gives the desired expression for FG.

5.9 Here are the matrices $\mathcal{M} = \mathcal{N}^T$ for $n = 3$ and $n = 4$:

u, v	123	132	213	231	312	321
123	2	1	1	1	1	0
132	1	2	1	0	1	1
213	1	1	2	1	0	1
231	1	0	1	2	1	1
312	1	1	0	1	2	1
321	0	1	1	1	1	2

u, v																								
1234	3	1	1	1	1	0	1	0	1	1	1	0	1	0	0	0	1	0	1	0	0	0	0	0
1243	1	3	1	0	1	1	0	1	1	0	1	1	1	0	0	0	0	0	1	0	0	0	1	0
1324	1	1	3	1	0	1	1	0	0	0	1	0	1	0	1	1	1	0	0	1	0	0	0	0
1342	1	0	1	3	1	1	1	0	0	0	0	0	0	1	1	0	1	1	0	1	1	0	0	0
1423	1	1	0	1	3	1	0	1	1	0	0	0	0	1	0	0	0	0	1	0	1	1	1	0
1432	0	1	1	1	1	3	0	1	0	0	0	0	0	1	1	0	0	0	0	1	1	0	1	1
2134	1	0	1	1	1	0	3	1	1	1	1	0	0	0	1	0	0	1	0	0	1	0	0	0
2143	0	1	1	0	1	1	1	3	1	0	1	1	0	0	1	0	0	0	0	0	1	0	0	1
2314	1	0	0	0	1	0	1	1	3	1	0	1	1	1	1	0	0	1	0	0	0	1	0	0
2341	1	0	0	0	0	0	1	0	1	3	1	1	1	0	0	1	1	1	1	0	0	1	0	0
2413	0	1	1	0	0	0	1	1	0	1	3	1	0	0	0	1	0	0	1	1	1	0	0	1
2431	0	1	0	0	0	0	0	1	1	1	1	3	1	0	0	1	0	0	1	0	0	1	1	1
3124	1	1	1	0	0	1	0	0	1	0	0	1	3	1	1	1	1	0	0	0	0	0	1	0
3142	1	0	0	1	1	1	0	0	1	0	0	0	1	3	1	0	1	1	0	0	0	1	1	0
3214	0	0	1	0	0	1	1	1	1	0	0	1	1	1	3	1	0	1	0	0	0	0	0	1
3241	0	0	1	0	0	0	1	0	0	1	1	1	1	0	1	3	1	1	0	1	0	0	0	1
3412	1	0	0	1	0	0	0	0	0	1	0	0	1	1	0	1	3	1	1	1	0	1	1	0
3421	0	0	0	1	0	0	1	0	0	1	0	0	0	1	1	1	1	3	0	1	1	1	0	1
4123	1	1	0	1	1	0	0	0	0	1	1	0	0	0	0	0	1	0	3	1	1	1	1	0
4132	0	1	1	1	0	1	0	0	0	0	1	0	0	0	0	1	1	0	1	3	1	0	1	1
4213	0	0	0	1	1	0	1	1	0	1	1	0	0	0	0	0	0	1	1	1	3	1	0	1
4231	0	0	0	0	1	0	0	1	1	1	0	1	0	1	0	0	0	1	1	0	1	3	1	1
4312	0	1	0	0	0	1	0	0	0	0	0	1	1	1	0	1	1	0	1	1	0	1	3	1
4321	0	0	0	0	0	1	0	1	0	0	0	1	0	1	1	1	0	1	0	1	1	1	1	3

To see that $\mathcal{M} = \mathcal{N}^T$, we will show that

$$|M(u, v)| = |N(v, u)|.$$

Consider a fixed choice of u, say $u = 82374165$, and let $F = s_1 \cdots s_k | s_{k+1} \cdots s_8$, with $s_1 < \cdots < s_k$ and $s_{k+1} < \cdots < s_8$, be any ray in the corresponding braid arrangement. Then using the notation of Problem 5.8,

$$\begin{aligned} F \cdot C(u) &= s_1 \cdots s_k | s_{k+1} \cdots s_8 \cdot 8|2|3|7|4|1|6|5, \\ &= v(1)|v(2)| \cdots |v(8) = C(v). \end{aligned}$$

The permutation v is such that $v(1) \cdots v(k)$ is the subword of u containing the letters $\{s_1, \ldots, s_k\}$, and $v(k+1) \cdots v(8)$ is the subword of u on the complement.

For example, if $F = 128|34567$, then $v = 821\,37465$.

We can see that $|M(u, v)| \neq 0$ if and only if v is a concatenation of two subwords of u, i.e., if

$$v = u(s_1) \cdots u(s_k) u(s_{k+1}) \cdots u(s_n)$$

for some complementary index sets $s_1 < \cdots < s_k$ and $s_{k+1} < \cdots < s_n$. From the point of view of v, we are saying that u can be obtained by cutting v at a point k to get subwords $v(1) \cdots v(k)$ and $v(k+1) \cdots v(n)$ such that interleaving these subwords gives u.

But this is precisely what it means to say that $u \in \mathcal{L}(P(v; k))$. Thus the number of rays F such that $u \xrightarrow{F} v$, i.e., $|M(u, v)|$, is

$$|\{k : u \in \mathcal{L}(P(v; k))\}| = N(v, u).$$

For more about the combinatorics of card shuffling from the point of view of hyperplane arrangements, in particular, computation of the eigenvalues for the matrix $\mathcal{N}$, see work of Bidigare, Hanlon, and Rockmore [18], Brown and Diaconis [37], or for a more accessible introduction, Billera, Brown, and Diaconis [19].

5.10 Suppose $\alpha \neq 1$ is a root of f, and let g be the polynomial $g(t) = (1+t)^{n-1} f(t/(1+t))$. Then let

$$\beta = \frac{\alpha}{1 - \alpha},$$

so that

$$\frac{\beta}{1 + \beta} = \alpha.$$

We have

$$g(\beta) = (1 + \beta)^{n-1} f(\alpha) = 0.$$

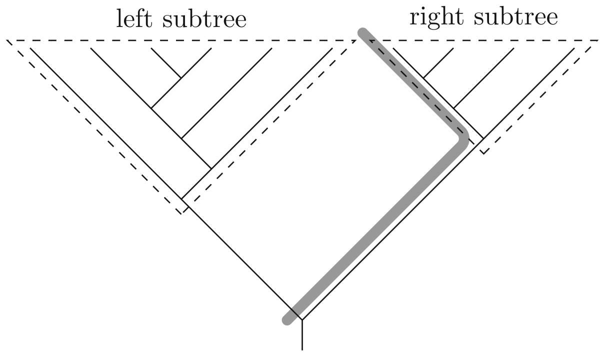
Thus every root of f corresponds to a root of g.

For the second assertion, see Brandën's discussion in Section 7.1 of [32].

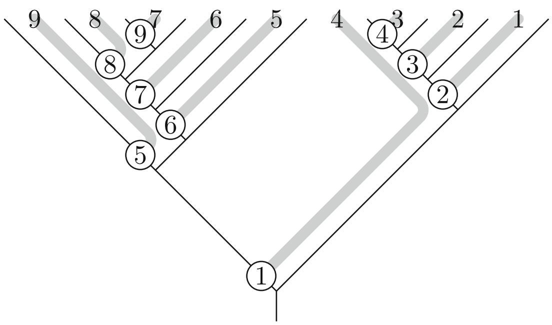
5.11 We can construct an order preserving map

$$(\mathrm{PB}(n), \leq) \to (\mathrm{Wk}^r(S_n(231)), \leq)$$

as follows. Given a tree τ we define the *core* of the tree to be path obtained by following along just to the left of the right subtree.



Label the gaps between the leaves from right to left with $1, 2, \ldots, n$. Let $w(1)$ be the value of the label of the core. Further values are found recursively by traversing subtrees from right to left, identifying the core of each as we go.

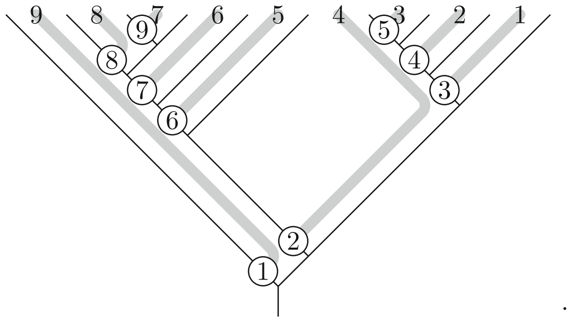


The example tree above would thus correspond to

$$w = 412395687.$$

The recursive nature of the construction makes it easy to verify that the Tamari lattice on trees is compatible with the right weak order on the corresponding permutations. Suppose $w = w^r w^l$, where $w^r = w(1) \cdots w(k)$ is the part of the permutation coming from the right subtree and w^l is the part of the permutation coming from the left subtree.

If we move up in the Tamari lattice by moving branches within either subtree, we can claim the corresponding permutations have the same relative order in $\mathrm{Wk}^r(S_n(231))$ by appealing to induction on n. Now suppose that we move up in the Tamari lattice by moving the bottom branch of the left tree onto the right branch. Continuing from the example above, we get:



The effect here is to make the first letter of w^l the first letter of the new permutation:

$$w \to w(k+1)w(1)\cdots w(k)w(k+2)\cdots w(n) = w'.$$

In the full weak order, there is a chain of permutations from w to w':

$$\begin{gathered} w = w(1)\cdots w(k-1)w(k)w(k+1)w(k+2)\cdots w(n) \\ \downarrow \\ w(1)\cdots w(k-1)w(k+1)w(k)w(k+2)\cdots w(n) \\ \downarrow \\ \vdots \\ w(1)w(k+1)w(2)\cdots w(k)w(k+2)\cdots w(n) \\ \downarrow \\ w(k+1)w(1)\cdots w(k)w(k+2)\cdots w(n) = w'. \end{gathered}$$

By construction, $w(1) < w(k+1)$ and $w(1) > w(i)$ for each $i = 2, \ldots, k$. Hence, every one but the first and the last permutation in this chain contains the pattern 231. Therefore $w < w'$ is indeed a cover relation in $\mathrm{Wk}^r(S_n(231))$.

We remark that there is a natural isomorphism between $\mathrm{Wk}^l(S_n)$ and $\mathrm{Wk}^r(S_n)$, given by mapping w to w^{-1}. However, this map does not preserve pattern avoidance. In particular, the inverse of $w = 312$ is $w^{-1} = 231$.

5.12 Any two planar binary trees that are refinements of the same planar tree can be obtained from each other by sliding branches from left to right. The maximum element in the interval has all its branches of the same level pushed to the right, and the minimum has all its branches pushed to the left.

This leads to interesting results related to the algebra of planar binary trees. See work by Maria Ronco and Jean-Louis Loday [103, 104], by Marcelo Aguiar and Frank Sottile [3], and also further articles by Loday alone [100, 102].

5.13 From Theorem 5.4 we have

$$\begin{aligned} r_n(t) &= f(\mathcal{P}(n); t), \\ &= (1+t)^{n-1} C_n(t/(1+t)), \\ &= \sum_{k=0}^{n-1} N_{n,k} t^k (1+t)^{n-1-k}. \end{aligned}$$

Substituting $t = 1$, we get

$$\begin{aligned} r_n &= N_{n,0} \cdot 2^{n-1} + N_{n,1} \cdot 2^{n-2} + \cdots + N_{n,n-2} \cdot 2 + N_{n,n-1} \cdot 1, \\ &= N_{n,n-1} \cdot 2^{n-1} + N_{n,n-2} \cdot 2^{n-2} + \cdots + N_{n,1} \cdot 2 + N_{n,0} \cdot 1, \\ &= C_n(2), \end{aligned}$$

where the second expression follows by palindromicity of the Narayana numbers.

The fact that $r_n(t)$ is real-rooted and log-concave follows from real-rootedness of the Narayana polynomial $C_n(t)$, using the same transformation as was used in Problem 5.10.

Problems of Chapter 6

6.1 We will use inversion sequences to find a bijection with both permutations counted according to inversion number as well as permutations counted according to major index.

Recall that an inversion sequence $s = (s_1, \ldots, s_n)$ is a member of the cartesian product

$$\{0\} \times \{0,1\} \times \{0,1,2\} \times \cdots \times \{0,1,\ldots,n-1\},$$

and we have a bijection between permutations and inversion sequences given by $w \mapsto (s_1, \ldots, s_n)$ given by

$$s_j = |\{i < j : w(i) > w(j)\}|.$$

Clearly $\mathrm{inv}(w) = s_1 + s_2 + \cdots + s_n$.

Now let w' be the permutation constructed from $(s_1, \ldots, s_n)$ as follows. Let $w^1 = 1$ and let w^i denote the permutation constructed from $(s_1, \ldots, s_i)$. We order the gaps of w^i from right to left according to non-ascent position, and then left to right according to non-descent positions. Place $i+1$ in gap s_{i+1}. For example, here is the permutation constructed from the inversion sequence $(0,0,1,1,0,5,6,0)$:

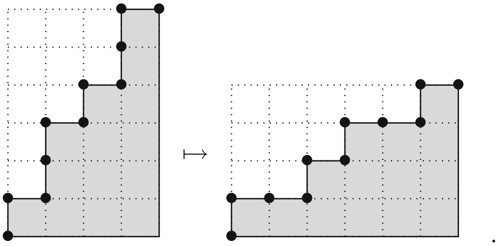
$$\begin{array}{c}
\emptyset \\
\downarrow_0 \\
\mathbf{1} \\
\downarrow_0 \\
1\mathbf{2} \\
\downarrow_1 \\
\mathbf{3}12 \\
\downarrow_1 \\
3\mathbf{4}12 \\
\downarrow_0 \\
3412\mathbf{5} \\
\downarrow_5 \\
341\mathbf{6}25 \\
\downarrow_6 \\
341\mathbf{7}625 \\
\downarrow_0 \\
3416258\mathbf{8}.
\end{array}$$

It follows by induction that the major index of the resulting permutation is the sum of the entries in the inversion sequence.

6.2 To show that $\begin{bmatrix} a+b \\ a \end{bmatrix} = \begin{bmatrix} a+b \\ b \end{bmatrix}$ with a bijection, we can create a bijection $L(a, b) \leftrightarrow L(b, a)$ that preserves area. To do this we simply read the path back to front and swap N for E. Here is an example:

$$NENNENENNE \mapsto NEENENEENE,$$

or in pictures,



By reading the path backwards we complement the area, but then we swap N for E, which again complements area, but also takes a path from $(0, 0)$ to (a, b) and creates a path from $(0, 0)$ to (b, a).

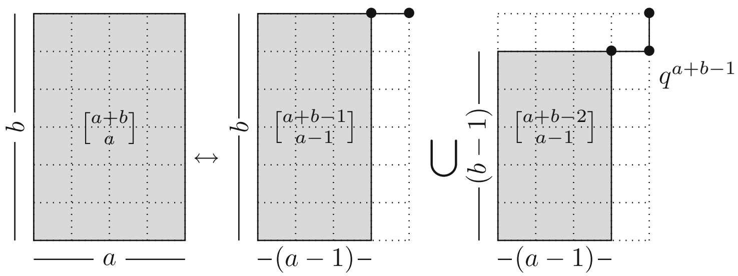
6.3 Each box (i, j) (as labeled in Figure 6.1) that sits beneath a lattice path corresponds to a unique pair of steps, with horizontal step labeled i and a vertical step labeled j. In the corresponding permutation j appears before i, so (i, j) is an inversion pair for w^{-1}. Hence $(w^{-1}(i), w^{-1}(j))$ is an inversion pair for w.

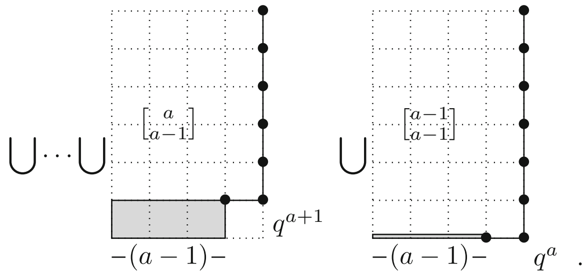
6.4 We can provide a recursively defined bijection by finding a natural recurrence for $\begin{bmatrix} a+b \\ a \end{bmatrix}$ that is compatible with both major index and area.

For example, one can verify:

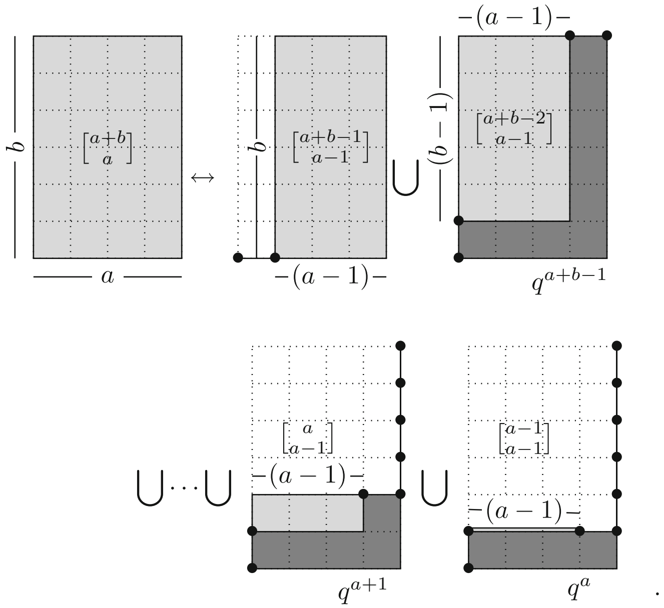
$$\begin{bmatrix} a+b \\ a \end{bmatrix} = \begin{bmatrix} a+b-1 \\ a-1 \end{bmatrix} + q^{a+b-1} \begin{bmatrix} a+b-2 \\ a-1 \end{bmatrix} + \cdots + q^{a+1} \begin{bmatrix} a \\ a-1 \end{bmatrix} + q^{a} \begin{bmatrix} a-1 \\ a-1 \end{bmatrix}.$$

In terms of major index, we have the following picture proof:





In terms of area, we have this, rather different, picture proof:



When $a = 0$ or $b = 0$ there is only one path, so the boundary paths are necessarily identified under the bijection. The remaining paths can be put in bijection by using the corresponding parts of the pictures for the recurrence.

To decompose a path according to the major index recurrence, we look to the end of the permutation. If it ends with an E step, we remove only that E. If it ends with an N, we remove all the trailing letters N and the E just prior.

For example, suppose $p = NENNNNENEE$ is a chosen path with major index 9 in $L(4,5)$. Then

$$p = NENNNNENEE \leftarrow_E (NENNNNEN)E \leftarrow_E (NENNNN)EN$$
$$\leftarrow_{EN} (N)ENNNN \leftarrow_{ENNNN} N,$$

and at this point we have a boundary path.

To find a path p' with area 9, we reverse these steps, but with the recurrence for area. In this case, a path either begins with an E, or it begins with an N and ends with an E followed by some number of N steps.

With our example, we have

$$N \to_{N\cdot ENNN} N(N)ENNN \to_{N\cdot E} N(NNENNN)E$$

$$\to_E E(NNNENNNE) \to_E E(ENNNENNNE) = p',$$

and we can see that $\mathrm{maj}(p) = 2 + 7 = 9 = \mathrm{area}(p')$.

A non-recursive description of this (or any other) bijection that takes major index to area would be nice to have.

6.5 We know that $\mathrm{inv}(w^{-1}) = \mathrm{inv}(w)$, but $\mathrm{inv}(w) \neq \mathrm{maj}(w)$ in general, so the identity cannot hold at the level of each permutation. For example, $\mathrm{maj}(3412) = 2$ while $\mathrm{inv}(3412) = 4$.

By composing the bijections indicated in Figures 6.1 (for area) and 6.2 (for major index) with the bijection in the solution to Problem 6.4, we can create a bijection that takes a permutation u with $\mathrm{Des}(u^{-1}) \subseteq \{k\}$ and major index j to a permutation v with $\mathrm{Des}(v^{-1}) \subseteq \{k\}$ and j inversions. This bijection would benefit from a more direct, non-recursive, description.

We remark that this bijection is not the same as the bijection in Problem 6.1, which typically changes the inverse descent set. For example, consider the permutation $w = 3412$, whose inverse descent set is $\{2\}$. Under the bijection of Problem 6.1 $w \mapsto w' = 4132$, whose inverse descent set is $\{2, 3\}$.

6.6 This result follows from the idea of Theorem 1.3 and the observations preceding Theorem 6.1. In particular, inserting n at the far right, and in other non-ascent positions from right to left increases major index by $0, 1, 2, \ldots$ and so on, while preserving number of descents. On the other hand, inserting n in non-descent positions from left to right increases descent number by one and major index by $k, k+1, \ldots$ and so on (provided we are inserting in a permutation with $k-1$ descents).

6.7 First, notice that

$$\begin{bmatrix} n+k \\ n \end{bmatrix} = \begin{bmatrix} n+k-1 \\ n-1 \end{bmatrix} + q^n \begin{bmatrix} n+k-1 \\ n \end{bmatrix},$$

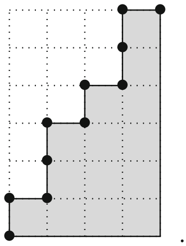
from which it follows by induction on n,

$$\frac{1}{(1-t)(1-qt)\cdots(1-q^n t)} = \sum_{k\geq 0} \begin{bmatrix} n+k \\ n \end{bmatrix} t^k.$$

Notice that we can interpret the q-binomial coefficient as

$$\begin{bmatrix} n+k \\ n \end{bmatrix} = \sum_{0\leq a_1\leq\cdots\leq a_n\leq k} q^{a_1+\cdots+a_n},$$

since for any path in $L(k,n)$, we can compute the area under the path by letting a_i denote the number of boxes under the path in row $n+1-i$, e.g., the following path has $(a_1,\ldots,a_6) = (1,1,2,3,3,4)$:



Now let $\overline{\mathcal{A}}(w)$ denote the set of all reverse P-partitions for the permutation $w \in S_n$, i.e.,

$$\overline{\mathcal{A}}(w) = \{a_1 \geq a_2 \geq \cdots \geq a_n : a_i > a_{i+1} \text{ if } i \in \mathrm{Des}(w)\}.$$

The set of reverse P-partitions for a labeled poset P is the disjoint union of the P-partitions for the linear extensions of P:

$$\overline{\mathcal{A}}(P) = \bigcup_{w \in \mathcal{L}(P)} \overline{\mathcal{A}}(w).$$

Let

$$\overline{\mathcal{A}}(w;k) = \{k \geq a_1 \geq a_2 \geq \cdots \geq a_n \geq 0 : a_i > a_{i+1} \text{ if } i \in \mathrm{Des}(w)\}.$$

Our first observation is

$$\Omega(w;k,q) = \sum_{a \in \overline{\mathcal{A}}(w;k)} q^{a_1+\cdots+a_n} = q^{\mathrm{maj}(w)} \begin{bmatrix} n+k-\mathrm{des}(w) \\ n \end{bmatrix}.$$

For example, if $\mathrm{Des}(w) = \{2,3\}$ and $w \in S_7$, we have $\overline{\mathcal{A}}(w;k)$ is the set of all vectors satisfying:

$$k \geq a_1 \geq a_2 > a_3 > a_4 \geq a_5 \geq a_6 \geq a_7 \geq 0,$$

or

$$k-2 \geq a_1 - 2 \geq a_2 - 2 \geq a_3 - 1 \geq a_4 \geq a_5 \geq a_6 \geq a_7 \geq 0,$$

or

$$k-2 \geq b_1 \geq b_2 \geq b_3 \geq b_4 \geq b_5 \geq b_6 \geq b_7 \geq 0.$$

We know the sum of $q^{b_1+\cdots+b_7}$ over all such b_i is $\left[\begin{smallmatrix}7+k-2\\7\end{smallmatrix}\right]$, and also

$$q^{a_1+\cdots+a_7} = q^5 \cdot q^{b_1+\cdots+b_7}.$$

Thus,

$$\sum_{a\in\overline{\mathcal{A}}(w;k)} q^{a_1+\cdots+a_7} = q^5 \begin{bmatrix}7+k-2\\7\end{bmatrix}.$$

For any particular permutation w, then, we have

$$\frac{q^{\mathrm{maj}(w)}t^{\mathrm{des}(w)}}{(1-t)(1-qt)\cdots(1-q^nt)} = \sum_{k\geq 0} q^{\mathrm{maj}(w)} \begin{bmatrix}n+k-\mathrm{des}(w)\\n\end{bmatrix} t^k = \sum_{k\geq 0} \Omega(w;k,q)t^k.$$

On the other hand, if P is an antichain, $\Omega(P;k,q) = [k+1]^n$, and the result follows by summing over all linear extensions, i.e., over all $w \in S_n$.

6.8 Let

$$n(a,b,k;q) = q^{k^2}\left(\begin{bmatrix}a\\k\end{bmatrix}\begin{bmatrix}b\\k\end{bmatrix} - \begin{bmatrix}a+1\\k+1\end{bmatrix}\begin{bmatrix}b-1\\k-1\end{bmatrix}\right).$$

To check the boundary conditions, verify that $n(a,a,k;q) = n(a-1,a,k;q)$. First,

$$\begin{aligned} n(a-1,a,k;q) &= q^{k^2}\left(\begin{bmatrix}a-1\\k\end{bmatrix}\begin{bmatrix}a\\k\end{bmatrix} - \begin{bmatrix}a\\k+1\end{bmatrix}\begin{bmatrix}a-1\\k-1\end{bmatrix}\right), \\ &= q^{k^2}\frac{[a]![a-1]!}{[a-k-1]![k]![a-k]![k-1]!}\left(\frac{[k+1]-[k]}{[k][k+1]}\right), \\ &= q^{k^2}\frac{[a]![a-1]!}{[a-k-1]![k]![a-k]![k-1]!}\left(\frac{q^k}{[k][k+1]}\right). \end{aligned}$$

On the other hand,

$$\begin{aligned} n(a,a,k;q) &= q^{k^2}\left(\begin{bmatrix}a\\k\end{bmatrix}\begin{bmatrix}a\\k\end{bmatrix} - \begin{bmatrix}a+1\\k+1\end{bmatrix}\begin{bmatrix}a-1\\k-1\end{bmatrix}\right), \\ &= q^{k^2}\frac{[a]![a-1]!}{[a-k-1]![k]![a-k]![k-1]!}\left(\frac{[a][k+1]-[k][a+1]}{[a-k][k][k+1]}\right), \\ &= q^{k^2}\frac{[a]![a-1]!}{[a-k-1]![k]![a-k]![k-1]!}\left(\frac{q^k}{[k][k+1]}\right). \end{aligned}$$

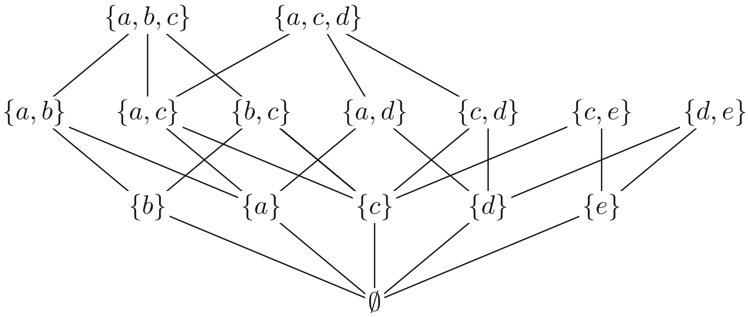
Verification of the recurrence in Equation (6.8) is equally straightforward, if tedious. It helps to use identities such as $[r+s] = [r] + q^r[s]$.

A computer can also help simplify the job.

Problems of Chapter 8

8.1

1. Here is the corresponding face poset:



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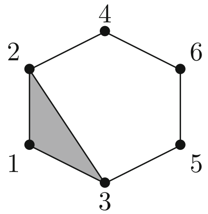
2. Here is a geometric realization. It is a triangle glued inside a hexagon:



.

8.2 This property follows immediately from the fact that the face poset is a lower ideal in the boolean lattice on the vertex set V. In particular, every pair of subsets has a least upper bound, namely their union. In the bowtie, F_1 and F_2 are both covered by G_1 and G_2. Hence both G_1 and G_2 must have vertex set $F_1 \cup F_2$. Therefore they cannot be distinct faces of a simplicial complex.

8.3 It suffices to prove this for the standard simplex, i.e., the convex hull of the standard basis elements in $\mathbb{R}^{d+1}$. Letting $\mathbf{e}_i$ denote the ith coordinate vector, we have that the k-faces of $\mathcal{P}$ are linear combinations of the form

$$t_1\mathbf{e}_{i_1} + \cdots + t_k\mathbf{e}_{i_k},$$

where $0 \leq t_i \leq 1$ and $\sum t_i = 1$. Let F denote the set of all such linear combinations. Clearly we can identify F with the subset $\{i_1, \ldots, i_k\}$. The center of each such face F is the vector given by taking $t_i = 1/k$.

The facets of $\mathcal{P}$ are faces of the form $F = \{1, 2, \ldots, d+1\} - \{i\}$ for some i. We can therefore denote the center of each facet by

$$\mathbf{e}_i^* = \frac{1}{d}\left(-e_i + \sum_{j=1}^{d+1} e_j\right).$$

For example if $d = 5$, $\mathbf{e}_3^* = (1/5, 1/5, 0, 1/5, 1/5, 1/5)$.

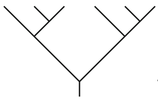
A quick linear algebra exercise shows the $\mathbf{e}_i^*$, $i = 1, 2, \ldots, d+1$ are linearly independent in $\mathbb{R}^{d+1}$, and hence their convex hull, $\mathcal{P}^*$, is a d-simplex.

8.4 This is Jean-Louis Loday's realization of the associahedron. See [101].

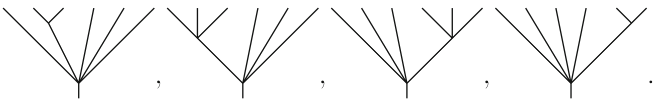
8.5 To show that a d-dimensional polytope is simple, we must show that each vertex lies on the boundary of exactly d facets. This is straightforward after we recall our combinatorial models for faces of the permutahedron and the associahedron in Chapter 5.

For the permutahedron, faces are encoded by ordered set partitions. A face F is contained in a face G if and only if G is a coarsening of F. The facets of the permutahedron have exactly two blocks, thus each vertex $w(1)|w(2)|\cdots|w(d)$ is contained in exactly $d-1$ facets. For example, the vertex $3|1|4|2|5$ is contained in $3|1245$, $13|245$, $134|25$, and $1234|5$.

For the associahedron, faces are encoded by planar trees with $d+1$ leaves, and again the containment of faces is given by coarsening. Vertices are planar binary trees, and facets are planar trees with exactly two internal nodes. Since there must always be a root node, this amounts to deciding which of the $d-1$ other nodes we keep. For example, the planar binary tree shown here:



is on the boundary of each of the four planar trees below:



8.6 The cell complex described here is the *Steinberg torus* described in Chapter 14. See [2, 60]. It is a simplicial poset since for any face G, we can identify all faces $F \leq_{\Sigma_T} G$ by the subset of the edges of G that we contract. Each set of edges of G corresponds to a distinct face that is refined by G.

While each lower interval is boolean, $\Sigma_T(n)$ is not a simplicial complex since a face is not determined by its vertex set. We can see this even from the $n = 2$ case, since both edges in that example have the same two vertices. This continues to be true for every n: each maximal face of the Steinberg torus shares the same vertex set!

8.7 This is a Theorem due to Richard Stanley [151, Theorem 2.1].

Let P be the face poset of a given boolean complex, and let B_i denote the boolean algebra on i elements. Recall that by definition of boolean complex, the interval below any element of P is isomorphic to B_i, where i is the rank of the element.

Since the complex is $(d-1)$-dimensional, P has there is at least one simplex G of maximal dimension, and the interval in P below this face is the boolean algebra on the d vertices: $[0, G] \cong B_d$. There are $\binom{d}{i}$ faces $F \subseteq G$ such that $[0, F] \cong B_i$. This implies that $f_i(P) \geq \binom{d}{i}$.

Conversely, suppose $f = (f_0, f_1, \ldots, f_d)$ satisfies $f_i \geq \binom{d}{i}$. We will show that there exists a boolean complex with this f-vector. Since the boolean algebra B_d achieves the lower bound for every i, $f_i(B_d) = \binom{d}{i}$, it suffices to show that if $f = (f_0, f_1, \ldots, f_j, \ldots, f_d)$ is the f-vector of a boolean complex, then for any $j = 1, 2, \ldots, d$, so is the vector $f' = (f_0, f_1, \ldots, f_j + 1, \ldots, f_d)$.

Suppose P is the face poset of a boolean complex with f-vector $f = (f_0, f_1, \ldots, f_j, \ldots, f_d)$. Pick any face G such that $\dim G = j - 1$, i.e., such that $[0, G] \cong B_j$. Then G covers $\binom{j}{j-1} = j$ faces, call them $F_1, F_2, \ldots, F_j$. Form a new poset $P' = P \cup \{G'\}$, where G' is a new element that also covers $F_1, F_2, \ldots, F_j$. Then $f(P') = (f_0, f_1, \ldots, f_j + 1, \ldots, f_d)$, as desired.

8.8 This is immediate from the definition of the order complex. If $x_1 <_P \cdots <_P x_k$ is a chain in P, then we can identify the chain with the members of P in the chain, $\{x_1, \ldots, x_k\}$, and clearly any subset of this set will also form a chain in P.

8.9 We can describe faces of the associahedron (and hence its dual) with planar trees, or equivalently, parenthesizations. We will say $\Delta(n)$ is the simplicial complex dual to the associahedron whose faces are valid parenthesizations of a string of $n + 1$ characters. The dimension of a face is one less than the number of pairs of parentheses. (So $\Delta(n)$ has dimension $n - 2$.) For example, the face F given by

$$F = (a((bc)de)fg)h$$

is a triangle in $\Delta(n)$.

Vertices correspond to all ways to insert a single pair of parentheses so that it encloses at least two characters (apart from a pair around the entire string). Two vertices are adjacent if and only if their pairs of parentheses are noncrossing. Let $\Delta'(n)$ denote the clique complex for this relation on pairs of parentheses.

The union of any collection of pairwise noncrossing pairs of parentheses will obviously yield a valid parenthesization, so $\Delta'(n)$ is clearly a subcomplex of $\Delta(n)$.

On the other hand, if F is any face of $\Delta(n)$, i.e., a parenthesization, we can work greedily from the inside out to find a collection of pairs of parentheses whose union is F and such that no two of the pairs are crossing. For example, with $F = (a((bc)de)fg)h$ as above, we have

$$F = a(bc)defgh \cup a(bcde)fgh \cup (abcdefg)h.$$

Thus each face is in the clique complex $\Delta'(n)$, and $\Delta(n) = \Delta'(n)$ is indeed a flag complex.

8.10 The faces of $\Sigma(n)$ are set compositions, e.g., $F = 23|156|47|8$ is a triangle in $\Sigma(8)$. The vertices correspond to set compositions with two blocks, and each vertex of a face F is obtained by removing all but one bar from F. Continuing with our example, the faces of $F = 23|156|47|8$ are

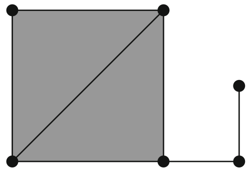
$$23|145678, 12356|478, 1234567|8.$$

We will color the vertices with colors $i = 1, 2, \ldots, n-1$ according to the number of elements in the left block, i.e., according to the position of the vertical bar. In F above, the vertices have colors 2, 5, and 7. Clearly every face has distinctly colored vertices, since there is exactly one vertex per bar. Moreover, there are $n-1$ colors and $n-1$ is the number of vertices in a maximal face.

8.11 The transformation between f- and h-polynomials is exactly the transformation described in Problem 5.10.

Problems of Chapter 9

9.1 The following simplicial complex has the desired properties:



.

9.2 Unimodality for $S_{n;j}(t)$ follows from the recurrence in Observation 9.1. Suppose for induction that $S_{n-1;j}(t)$ are unimodal and that their centers are either in the coefficient of $t^{\lfloor (n-2)/2 \rfloor}$ or $t^{1+\lfloor (n-2)/2 \rfloor}$. Moreover, suppose the center of unimodality weakly increases with j, so that in the sum

$$S_{n;j}(t) = tS_{n-1;1}(t) + \cdots + tS_{n-1;j-1}(t) + S_{n-1;j}(t) + \cdots + S_{n-1;n-1}(t),$$

there are at most two different centers of unimodality, and the centers are adjacent powers of t.

Then effectively we are adding the two sequences of coefficients of the form:

$$a_0 \le \cdots \le a_j \ge a_{j+1} \ge a_{j+2} \ge \cdots ,$$

and

$$b_0 \le \cdots \le b_j \le b_{j+1} \ge b_{j+2} \ge \cdots ,$$

yielding

$$(a_0 + b_0) \le \cdots \le (a_j + b_j) \quad \text{and} \quad (a_{j+1} + b_{j+1}) \ge (a_{j+2} + b_{j+2}) \ge \cdots ,$$

so whether $(a_j + b_j) < (a_{j+1} + b_{j+1})$ or $(a_j + b_j) \ge (a_{j+1} + b_{j+1})$, the sum is unimodal.

9.3 This is the "compatibility lemma" of Maria Chudnovsky and Paul Seymour [51, Lemma 2.2]. Its proof (found in Section 3 of that paper) uses induction on k and on the maximal degree of the polynomials.

9.4 We will use Observation 9.1 and the compatibility lemma of Problem 9.3. Our proof here is essentially Theorem 2.3 in the paper by Carla Savage and Mirko Visontai [132].

For brevity, let $f_j(t) = S_{n;j}(t)$ and let $g_j(t) = S_{n+1;j}(t)$, which by Observation 9.1 can be written as:

$$g_j(t) = t\sum_{k=1}^{j-1} f_k(t) + \sum_{k=j}^{n} f_k(t).$$

We will prove the following lemma, from which the result follows by induction. Suppose, for each $i < j$ that

(a) $f_i(t)$ and $f_j(t)$ are compatible, and
(b) $tf_i(t)$ and $f_j(t)$ are compatible.

Then for each $i < j$

(A) $g_i(t)$ and $g_j(t)$ are compatible, and
(B) $tg_i(t)$ and $g_j(t)$ are compatible.

Suppose (a) and (b) hold. To show g_i and g_j are compatible, we examine the nonnegative linear combination

$$c_i g_i(t) + c_j g_j(t) = \sum_{k=1}^{i-1}(c_i + c_j)tf_k(t) + \sum_{k=i}^{j-1}(c_i + tc_j)f_k(t) + \sum_{k=j}^{n} f_k(t). \quad (14.4)$$

To prove this polynomial is real-rooted, it suffices to prove that the polynomials in the sum are compatible. By Problem 9.3, this amounts to showing that any pair of polynomials in the following set is compatible:

$$I \cup II \cup III = \{tf_k(t)\}_{1\le k\le i-1} \cup \{(c_i + tc_j)f_k(t)\}_{i\le k\le j-1} \cup \{f_k(t)\}_{j\le k\le n}.$$

Any two polynomials within a subset (I, II, or III) are compatible by hypothesis (a), and likewise a polynomial from set I and set III are compatible by hypothesis (b). To see that a polynomial from I and a polynomial from II are compatible, let $k \le i-1$, $i \le l \le j-1$, and consider the combination

$$atf_k(t) + b(c_i + c_j t)f_l(t) = atf_k(t) + bc_i f_l(t) + bc_j tf_l(t). \quad (14.5)$$

This is therefore a nonnegative linear combination of tf_k, f_l, and tf_l. The polynomials tf_k and f_l are compatible by hypothesis (b), the polynomials tf_k and tf_l are compatible by hypothesis (a), and f_l and tf_l are obviously compatible. Hence Equation (14.5) is a nonnegative linear combination of pairwise compatible polynomials. By the compatibility lemma, this means the linear combination is real-rooted. A similar argument shows a polynomial from II and a polynomial from III are compatible. This covers all cases, showing the polynomial in Equation (14.4) is real-rooted. This establishes conclusion (A).

The proof of (B) follows from a similar sort of reasoning, where we now must consider pairs of polynomials from the set:

$$I' \cup II' \cup III' = \{(c_i t + c_j)tf_k(t)\}_{1\le k\le i-1} \cup \{tf_k(t)\}_{i\le k\le j-1} \cup \{(c_i t + c_j)f_k(t)\}_{j\le k\le n}.$$

Again, all pairs of such polynomials are compatible, given hypotheses (a) and (b).

Notice that this method gives an alternate proof that the classical Eulerian polynomial $S_n(t)$ is real-rooted, since $S_n(t) = \sum_{j\geq 1} S_{n;j}(t)$ and our lemma shows the $S_{n;j}$ are compatible.

9.5 Since $\mathbf{S}_{n;j}(t)$ is a sum of j-Eulerian polynomials, this follows directly from the lemma used in the solution to Problem 9.4.

9.6 The lemma in the solution of Problem 9.4 shows any nonnegative linear combination of j-Eulerian polynomials is real-rooted. Hence Part 1 of Theorem 9.4 follows from Theorem 9.3.

9.7 Here is the table with the entries of the eigenvector $(e_0, \ldots, e_{d-1})$ for E_d:

$d\backslash k$	0	1	2	3	4	5	6	7	8	
3	0	1	0							
4	0	1	1	0						
5	0	1	$\frac{7}{2}$	1	0					
6	0	1	$\frac{17}{2}$	$\frac{17}{2}$	1	0				
7	0	1	$\frac{586}{33}$	$\frac{5459}{132}$	$\frac{586}{33}$	1	0			
8	0	1	$\frac{1543}{45}$	$\frac{9349}{60}$	$\frac{9349}{60}$	$\frac{1543}{45}$	1	0		
9	0	1	$\frac{2780273}{43956}$	$\frac{22330265}{43956}$	$\frac{57026873}{58608}$	$\frac{22330265}{43956}$	$\frac{2780273}{43956}$	1	0	
10	0	1	$\frac{6198113}{54628}$	$\frac{371920561}{245826}$	$\frac{3260108681}{655536}$	$\frac{3260108681}{655536}$	$\frac{371920561}{245826}$	$\frac{6198113}{54628}$	1	0

It is not clear what combinatorial interpretation these polynomials might possess. When we clear the denominators, we get:

$d\backslash k$	0	1	2	3	4	5	6	7	8	
3	0	1	0							
4	0	1	1	0						
5	0	2	7	2	0					
6	0	2	17	17	2	0				
7	0	132	2344	5459	2344	132	0			
8	0	180	6172	28047	28047	6172	180	0		
9	0	175824	11121092	89321060	171080619	89321060	11121092	175824	0	
10	0	1966608	223132068	2975364488	9780326043	9780326043	2975364488	223132068	1966608	0

It would be desirable to have a combinatorial interpretation for these numbers, but no such interpretation is known as of this writing.

9.8 See Section 4 of the paper by Francesco Brenti and Volkmar Welker [35]. Approximate values for the roots of $e(t)$ are given below:

d	roots								
3	-1,	0							
4	-3.1861,	$-.3139$,	0						
5	-7.3642,	-1,	$-.1358$,	0					
6	-15.0956,	-2.1252,	$-.4705$,	-0.0662,	0				
7	-29.1205,	-3.8761,	-1,	$-.2580$,	-0.0343,	0			
8	-54.2076,	-6.5400,	-1.7664,	$-.5661$,	$-.1529$,	-0.0184,	0		
9	-98.6263,	-10.5365,	-2.8404,	-1,	$-.3521$,	-0.0949,	-0.0101,	0	
10	-176.6778,	-16.4739,	-4.3206,	-1.5830,	$-.6317$,	$-.2314$,	-0.0607,	-0.0057,	0

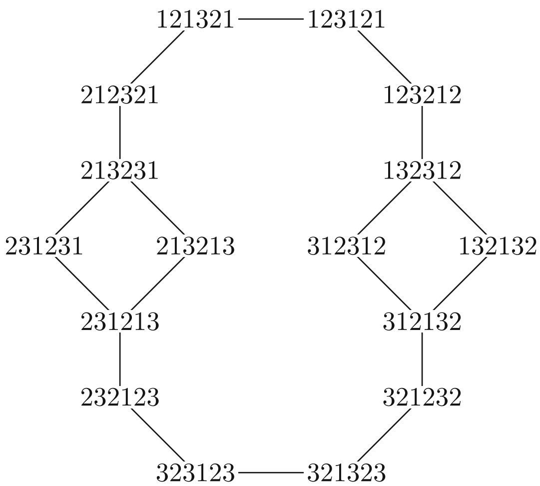
Problems of Chapter 11

11.1 Let $R(w)$ denote the set of reduced expressions for the element w in S_n. It is interesting to consider not only the set of reduced expressions, but how one can transform one reduced word to get another. We will draw an edge between two reduced expressions if one can get from one to the other by a commutation $s_i s_j = s_j s_i$ or a braid move $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$.

1.
$$R(321) = \{s_1 s_2 s_1 \leftrightarrow s_2 s_1 s_2\}$$

2. Here we will abbreviate reduced expressions by writing only the index of the generator, e.g., 1321 instead of $s_1 s_3 s_2 s_1$.

$R(4321) =$



3.
$$R(213465) = \{s_1 s_5 \leftrightarrow s_5 s_1\}$$

4.
$$R(216345) = \{s_1 s_5 s_4 s_3 \leftrightarrow s_5 s_1 s_4 s_3 \leftrightarrow s_5 s_4 s_1 s_3 \leftrightarrow s_5 s_4 s_3 s_1\}$$

Victor Reiner and Yuval Roichman studied the diameter of the graph on $R(w)$ in a 2013 paper [127]. In particular, for the symmetric group the diameter of $R(w_0)$ is $\frac{1}{24} n(n-1)(n-2)(3n-5)$.

11.2 The formula for the number of reduced expressions of w_0 is

$$\frac{\binom{n}{2}!}{\prod_{i=1}^{n-1}(2i-1)^{n-i}}.$$

For example, with $n = 4$ this yields

$$\frac{\binom{4}{2}!}{1^3 \cdot 3^2 \cdot 5} = \frac{6!}{3^2 \cdot 5} = 16,$$

as witnessed in the solution to part 2) of Problem 11.1.

The rather mysterious-looking formula is actually a special case of the *hook-length formula* that counts the number of *standard Young tableaux* of shape $\lambda = (n-1, \ldots, 3, 2, 1)$. See Chapter 7 of Richard Stanley's textbook [153] for an exposition of these ideas.

The result of this problem was proved by Stanley using symmetric functions in 1984 [149] and Paul Edelman and Curtis Greene in 1987 using bijections on Young tableaux [62].

11.3 This group is an infinite reflection group, $\widetilde{F}_4$,. See Figure 14.1 in Chapter 14.

11.4 We can use Theorem 11.1 along with Table 11.2 to get:

$$\begin{aligned} B_6(q) = {} & q^{36} + 6\,q^{35} + 20\,q^{34} + 50\,q^{33} + 104\,q^{32} + 190\,q^{31} \\ & + 315\,q^{30} + 484\,q^{29} + 699\,q^{28} + 958\,q^{27} + 1255\,q^{26} \\ & + 1580\,q^{25} + 1919\,q^{24} + 2254\,q^{23} + 2565\,q^{22} + 2832\,q^{21} \\ & + 3037\,q^{20} + 3166\,q^{19} + 3210\,q^{18} + 3166\,q^{17} + 3037\,q^{16} \\ & + 2832\,q^{15} + 2565\,q^{14} + 2254\,q^{13} + 1919\,q^{12} \\ & + 1580\,q^{11} + 1255\,q^{10} + 958\,q^{9} + 699\,q^{8} + 484\,q^{7} \\ & + 315\,q^{6} + 190\,q^{5} + 104\,q^{4} + 50\,q^{3} + 20\,q^{2} + 6\,q + 1, \end{aligned}$$

and

$$\begin{aligned} D_6(q) = {} & q^{29} + 6\,q^{28} + 20\,q^{27} + 50\,q^{26} + 104\,q^{25} + 189\,q^{24} \\ & + 309\,q^{23} + 464\,q^{22} + 649\,q^{21} + 854\,q^{20} + 1065\,q^{19} \\ & + 1265\,q^{18} + 1436\,q^{17} + 1561\,q^{16} + 1627\,q^{15} + 1627\,q^{14} \\ & + 1561\,q^{13} + 1436\,q^{12} + 1265\,q^{11} + 1065\,q^{10} + 854\,q^{9} \\ & + 649\,q^{8} + 464\,q^{7} + 309\,q^{6} + 189\,q^{5} + 104\,q^{4} \\ & + 50\,q^{3} + 20\,q^{2} + 6\,q + 1, \end{aligned}$$

and

$$\begin{aligned} E_6(q) = {} & q^{36} + 6\,q^{35} + 20\,q^{34} + 50\,q^{33} + 105\,q^{32} + 195\,q^{31} \\ & + 329\,q^{30} + 514\,q^{29} + 754\,q^{28} + 1048\,q^{27} + 1389\,q^{26} \\ & + 1765\,q^{25} + 2159\,q^{24} + 2549\,q^{23} + 2911\,q^{22} + 3222\,q^{21} \end{aligned}$$

$$+ 3461\,q^{20} + 3611\,q^{19} + 3662\,q^{18} + 3611\,q^{17} + 3461\,q^{16}$$
$$+ 3222\,q^{15} + 2911\,q^{14} + 2549\,q^{13} + 2159\,q^{12}$$
$$+ 1765\,q^{11} + 1389\,q^{10} + 1048\,q^{9} + 754\,q^{8} + 514\,q^{7}$$
$$+ 329\,q^{6} + 195\,q^{5} + 105\,q^{4} + 50\,q^{3} + 20\,q^{2} + 6\,q + 1,$$

and

$$\begin{aligned} F_4(q) = {} & q^{24} + 4\,q^{23} + 9\,q^{22} + 16\,q^{21} + 25\,q^{20} + 36\,q^{19} + 48\,q^{18} \\ & + 60\,q^{17} + 71\,q^{16} + 80\,q^{15} + 87\,q^{14} + 92\,q^{13} + 94\,q^{12} \\ & + 92\,q^{11} + 87\,q^{10} + 80\,q^{9} + 71\,q^{8} + 60\,q^{7} + 48\,q^{6} + 36\,q^{5} \\ & + 25\,q^{4} + 16\,q^{3} + 9\,q^{2} + 4\,q + 1. \end{aligned}$$

11.5 The terms appearing in Theorem 11.1 expand as

$$\frac{1-q^d}{1-q} = 1 + q + q^2 + \cdots + q^{d-1}.$$

Thus the total degree of

$$W(q) = \sum_{w \in W} q^{\ell(w)} = \prod_{i=1}^{n} \frac{(1-q^{d_i})}{(1-q)}$$

is $-n + \sum d_i$. The long element w_0 has this maximal length,

$$\ell(w_0) = -n + \sum d_i.$$

On the other hand, if $C = w_0 C_\emptyset$ is the chamber corresponding the long element, then $C = -C_\emptyset$ and therefore $\mathrm{Inv}(w_0) = \Phi$. Since we know from Observation 11.5 that number of inversions equals length, we get $|\Phi| = \ell(w_0) = -n + \sum d_i$, as desired.

11.6 The elements of the dihedral group with generators s and t are:

$$1, s, t, st, ts, sts, tst, stst, tsts, \ldots,$$

and since

$$(st)^m = (ts)^m = e,$$

we have two ways to write the long element:

$$w_0 = \underbrace{sts\cdots}_{m} = \underbrace{tst\cdots}_{m}.$$

As there are no other relations, all the expressions listed above are reduced, and the generating function for length is therefore

$$\begin{aligned} 1+2q+2q^2+2q^3+\cdots+2q^{m-1}+q^m &= (1+q)(1+q+\cdots+q^{m-1}), \\ &= \frac{(1-q^2)(1-q^m)}{(1-q)^2}. \end{aligned}$$

11.7 Recall that $w_0(\beta) < 0$ for every positive root β. Thus by Observation 11.7, we get

$$\begin{aligned} \mathrm{Des}(w_0 w) &= \{s_\alpha : \alpha \in \Delta, w_0 w(\alpha) < 0\}, \\ &= \{s_\alpha : \alpha \in \Delta, w(\alpha) = \beta > 0\}, \\ &= \{s_\alpha : \alpha \in \Delta, \alpha \notin \mathrm{Des}(w)\}, \\ &= S - \mathrm{Des}(w). \end{aligned}$$

11.8 We have

$$s_\beta(\lambda) = \lambda - \frac{2\langle\lambda,\beta\rangle}{\langle\beta,\beta\rangle}\beta,$$

and thus

$$\begin{aligned} \langle s_\beta(\lambda), s_\beta(\mu)\rangle &= \left\langle \lambda - \frac{2\langle\lambda,\beta\rangle}{\langle\beta,\beta\rangle}\beta, \mu - \frac{2\langle\mu,\beta\rangle}{\langle\beta,\beta\rangle}\beta \right\rangle, \\ &= \left\langle \lambda, \mu - \frac{2\langle\mu,\beta\rangle}{\langle\beta,\beta\rangle}\beta \right\rangle + \left\langle -\frac{2\langle\lambda,\beta\rangle}{\langle\beta,\beta\rangle}\beta, \mu - \frac{2\langle\mu,\beta\rangle}{\langle\beta,\beta\rangle}\beta \right\rangle, \\ &= \langle\lambda,\mu\rangle - \frac{2\langle\mu,\beta\rangle}{\langle\beta,\beta\rangle}\langle\lambda,\beta\rangle - \frac{2\langle\lambda,\beta\rangle}{\langle\beta,\beta\rangle}\langle\mu,\beta\rangle + \frac{4\langle\lambda,\beta\rangle\langle\mu,\beta\rangle}{\langle\beta,\beta\rangle^2}\langle\beta,\beta\rangle, \\ &= \langle\lambda,\mu\rangle. \end{aligned}$$

11.9 This is a straightforward check under the correspondence $\alpha \leftrightarrow \alpha^\vee = \frac{2\alpha}{\langle\alpha,\alpha\rangle}$:

$$\pm\varepsilon_i \leftrightarrow \pm 2\varepsilon_i$$

and

$$\pm\varepsilon_i \pm \varepsilon_j \leftrightarrow \pm\varepsilon_i \pm \varepsilon_j.$$

11.10 These are drawn in Figures 12.2, 12.3, 12.4, and 12.5.

11.11

1. Here is the Coxeter graph for F_4,

.

As there are 16 subsets of the simple generators, there are 16 parabolic subgroups. We list them here, up to isomorphism, along with multiplicities:

F_4	
B_3	(2 times)
$A_1 \times A_2$	(2 times)
A_2	(2 times)
$A_1 \times A_1$	(3 times)
B_2	
A_1	(4 times)
$\{e\}$	

2. The maximal parabolic subgroups of B_6 are:

$$\begin{array}{l} A_5 \\ A_1 \times A_4 \\ B_2 \times A_3 \\ B_3 \times A_2 \\ B_4 \times A_1 \\ B_5 \end{array}$$

3. The maximal parabolic subgroups of E_8 are:

$$\begin{array}{l} D_7 \\ A_1 \times A_6 \\ A_2 \times A_1 \times A_4 \\ A_7 \\ A_4 \times A_3 \\ D_5 \times A_2 \\ E_6 \times A_1 \\ E_7 \end{array}$$

11.12 Fix a Coxeter system (W, S), and define

$$\Sigma = \{wW_J : w \in W, J \subseteq S\},$$

We define a "face" to be wW_J, with $wW_J \leq_\Sigma vW_K$ if and only if $vW_K \subseteq wW_J$.

For notational convenience, let $F(w, J) = wW_{S-J}$. Thus $F(w, \emptyset) = wW_S = W$, regardless of the coset representative w, and $F(w, S) = wW_\emptyset = \{w\}$, a singleton coset corresponding to a maximal simplex.

With the partial ordering defined here, $F(e, S)$ is clearly a simplex with vertex set $V(e) = \{F(e, \{s\}) : s \in S\}$. Said another way, the standard parabolic subgroups form a boolean algebra under reverse inclusion.

We can act on cosets by left multiplication by an element w:

$$w \cdot uW_J = \{wv : v \in uW_J\}.$$

Thus,

$$w \cdot F(e, J) = wW_{S-J} = F(w, J),$$

and $F(e, J) \leq_{\Sigma} F(e, K)$ if and only if $F(w, J) = wW_{S-J} \subseteq wW_{S-K} = F(w, K)$. Thus every face is a simplex.

It remains to show that every face has a unique vertex set. This will follow from the following claim. If $u \neq v$, then the sets

$$V(u) = \{F(u, \{s\}) : s \in S\} \qquad \text{and} \qquad V(v) = \{F(v, \{s\}) : s \in S\}$$

are not identical.

We first consider the vertex set of the identity versus another element v of W. Suppose $s \in S$ is a descent of v. Then it follows from the braid relations for W that every reduced expression for v has an s in it. Since v^{-1} is obtained by writing any reduced expression for v in reverse, this shows v^{-1} is not a member of $W_{S-\{s\}}$. Hence, e is not in $vW_{S-\{s\}}$ and $F(v, \{s\})$ is not a vertex in $V(e)$.

In general, consider two vertex sets $V(u)$ and $V(v)$. Acting on these vertex sets by u^{-1} we see that $V(u) = V(v)$ if and only if $V(e) = V(u^{-1}v)$. But if $u \neq v$, then $u^{-1}v \neq e$, and we have already seen that these vertex sets are distinct.

To summarize, Σ is a simplicial complex with vertex set

$$V = \{F(w, \{s\}) : w \in W, s \in S\}.$$

11.13 Let $wW_J = \{wx : x \in W_J\}$. If u is an element of minimal length, then if $s \in J$, $\ell(us) > \ell(u)$ and none of the reduced expressions for u have an element of J as the rightmost letter. That implies that if $x \in W_J$, we have $\ell(ux) = \ell(u) + \ell(x)$. Thus if $v = ux$ also has minimal length, $\ell(u) = \ell(v)$, which forces $\ell(x) = 0$, which means $x = e$, which means $u = v$.

11.14 This follows immediately from Problem 11.13.

11.15 The combinatorial inversion set for $w = 356124$ is

$$\mathrm{Inv}(w) = \{(1,4),(1,5),(2,4),(2,5),(2,6),(3,4),(3,5),(3,6)\},$$

while its inverse has inversion set is

$$\mathrm{Inv}(w^{-1}) = \{(1,3),(2,3),(1,5),(2,5),(4,5),(1,6),(2,6),(4,6)\}.$$

When we think of w as labeling a chamber in the Coxeter arrangement, we obtain the following cone of points:

$$wC_{\emptyset} = \{\lambda \in V : x_3 < x_5 < x_6 < x_1 < x_2 < x_4\},$$

where $V = \mathbb{R}^6/\ell$, where ℓ is the line given by $\sum x_i = 0$. As the positive roots are $\beta_{ij} = \varepsilon_j - \varepsilon_i$ in the type $\mathbf{A}_{n-1}$ root system, we find

$$\langle \lambda, \beta_{ij} \rangle = x_j - x_i.$$

Thus $x_j - x_i > 0$ if and only if i appears to the left of j in w, i.e., $w^{-1}(i) < w^{-1}(j)$. We conclude that $\mathrm{Inv}(wC_\emptyset) \leftrightarrow \mathrm{Inv}(w^{-1})$. In this example,

$$\mathrm{Inv}(wC_\emptyset) = \{\beta_{1,3}, \beta_{2,3}, \beta_{1,5}, \beta_{2,5}, \beta_{4,5}, \beta_{1,6}, \beta_{2,6}, \beta_{4,6}\}.$$

To remove a single entry of the inversion set for the chamber, i.e., to move down by a cover relation in the weak order of Equation (11.3), we must find a root corresponding to an adjacent inversion pair in the permutation, e.g., $\beta_{1,6}$. To change the ordering of these two entries, we must multiply w on the *right* by a simple reflection; in this case by s_3, to get $ws_3 = 351624$. This is the right weak order: $\ell(ws) < \ell(w)$ implies $ws \leq_{\mathrm{Wk}} w$.

On the other hand, if we first define $\mathrm{Inv}(w) = \{\beta \in \Pi : w(\beta) < 0\}$, we are really studying the inversions of the chamber $w^{-1}C_\emptyset$. Multiplying w^{-1} on the right to move down in the chamber geometry corresponds to multiplying w on the left, so here we find the left weak order: $\ell(sw) < \ell(w)$ implies $sw \leq_{\mathrm{Wk}} w$.

11.16 This can be shown geometrically or in terms of reduced expressions. In either case, the bottom line is that $w \mapsto w_0 w$ complements inversion sets.

Problems of Chapter 12

12.1 We want to check that for any root α,

$$s_{w(\alpha)} = w s_\alpha w^{-1},$$

or $s_{w(\alpha)} w = w s_\alpha$.

Fix a vector $\lambda \in V$. We calculate the action of $s_{w(\alpha)} w$ on λ compared to the action of $w s_\alpha$ on λ. We find

$$\begin{aligned} s_{w(\alpha)} w(\lambda) &= w(\lambda) - \frac{\langle w(\alpha), w(\lambda)\rangle}{\langle w(\alpha), w(\alpha)\rangle} w(\alpha), \\ &= w\left(\lambda - \frac{\langle w(\alpha), w(\lambda)\rangle}{\langle w(\alpha), w(\alpha)\rangle}\alpha\right), \\ &= w\left(\lambda - \frac{\langle \alpha, \lambda\rangle}{\langle \alpha, \alpha\rangle}\alpha\right), \\ &= w s_\alpha, \end{aligned}$$

where the second equation follows from linearity and the third follows from orthogonality (see Problem 11.8).

12.2 This is a result due to Roger Carter [45]. The proof is as follows.

Suppose

$$w = s_{\beta_1} \cdots s_{\beta_k},$$

is a reduced expression for w that minimizes reflection length. Then clearly w fixes the intersection of the hyperplanes

$$H_{\beta_1} \cap \cdots \cap H_{\beta_k},$$

and if U denotes the maximal subspace fixed by w, we get $\dim(U) \geq n - k$.

Since U is the set of all vectors fixed by w, we can express w as a product of reflections w_{s_β} such that each root β lies in $U^\perp$. But this means w is an element of the group generated by the sub-root system spanning $U^\perp$, which has dimension at most $k \leq n$. If $k < n$, then by induction on n we are done.

Thus it suffices to consider elements w such that w has no nonzero fixed points, and show that such an element can be expressed as a product of at most n reflections.

Suppose w is such an element, i.e., $w(\lambda) \neq \lambda$ for any nonzero element $\lambda \in V$. Then in particular $w - e$ is invertible as an element of the group $GL(V)$, and for any $\beta \in \Phi$, there exists an element $\lambda \in V$ such that $(w - e)(\lambda) = \beta$. Thus $w(\lambda) = \lambda + \beta$. Since w is in the orthogonal subgroup we can write

$$\begin{aligned}\langle\lambda,\lambda\rangle &= \langle w(\lambda), w(\lambda)\rangle,\\ &= \langle\lambda+\beta,\lambda+\beta\rangle,\\ &= \langle\lambda,\lambda\rangle + 2\langle\lambda,\beta\rangle + \langle\beta,\beta\rangle,\end{aligned}$$

and so after subtracting $\langle\lambda,\lambda\rangle$ and dividing by $\langle\beta,\beta\rangle$ we find

$$\frac{2\langle\lambda,\beta\rangle}{\langle\beta,\beta\rangle} = -1.$$

But this means

$$\begin{aligned}s_\beta(\lambda) &= \lambda - \frac{2\langle\lambda,\beta\rangle}{\langle\beta,\beta\rangle}\beta,\\ &= \lambda+\beta,\\ &= w(\lambda),\end{aligned}$$

and the element $v = s_\beta w$ fixes λ:

$$v(\lambda) = s_\beta w(\lambda) = s_\beta(\lambda+\beta) = \lambda.$$

Since v has a nonzero fixed point, it sits inside a reflection group of rank less than n, and by induction $\ell'(v) \leq n-1$. But since $w = s_\beta v$, $\ell'(w) \leq n$, as desired.

12.3 One way to prove any two Coxeter elements are conjugate is to show every permutation of $s_1, \ldots, s_n$ can be obtained from the particular Coxeter element $c = s_1 s_2 \cdots s_n$ by the operations of conjugation by a simple reflection, e.g., $s_n c s_n = s_n s_1 \cdots s_{n-1}$, and by applying commutation relations $s_i s_j = s_j s_i$ for appropriate i and j. Since there is only one of each generator, braid relations are irrelevant.

For example, suppose $c' = s_6 s_4 s_5 s_1 s_2 s_3$ in some finite Coxeter group of rank six. We will abbreviate the reduced expressions in what follows by listing only the subscripts, i.e., we write $c' = 645123$.

Our first step will be to notice that the s_4 can commute with s_6, so we can first write $c' = 465123$. Next we conjugate by s_4 to get:

$$s_4 c' s_4 = 4(465123)4 = 651234.$$

Next we can conjugate by s_6 to get

$$s_6 s_4 c' s_4 s_6 = 6(651234)6 = 512346 = 561234,$$

where the last equality comes from the fact that s_6 commutes with each of s_1, s_2, s_3, and s_4. Finally, conjugating by s_5 and s_6 gives the desired result:

$$s_6 s_5 s_6 s_4 c' s_4 s_6 s_5 s_6 = 65(561234)56 = 123456 = c.$$

12.4 Since there are only two generators in $I_2(m)$, every element of odd length is a reflection. This means that every element of even length can be written as a product of at most two reflections.

The identity is the only element of reflection length zero, the m elements of odd length have reflection length one, and the remaining $m-1$ elements have reflection length two. This shows

$$\sum_{w \in I_2(m)} t^{\ell'(w)} = 1 + mt + (m-1)t^2 = (1+t)(1+(m-1)t),$$

as desired.

12.5 First, let us prove that conjugation preserves reflection length: $\ell'(u) = \ell'(wuw^{-1})$. It is clear that conjugating a single reflection, say s_β, results in another reflection: $ws_\beta w^{-1}$. Now if $u = s_{\beta_1} \cdots s_{\beta_k}$ has reflection length k, write

$$\begin{aligned} v &= wuw^{-1}, \\ &= ws_{\beta_1}s_{\beta_2}\cdots s_{\beta_k}w^{-1}, \\ &= ws_{\beta_1}w^{-1}ws_{\beta_2}w^{-1}w\cdots w^{-1}ws_{\beta_k}w^{-1}, \\ &= (ws_{\beta_1}w^{-1})(ws_{\beta_2}w^{-1})(w\cdots w^{-1})(ws_{\beta_k}w^{-1}), \end{aligned}$$

which shows $\ell'(v) \leq k$. But we have $u = w^{-1}vw$, so the same reasoning shows $k = \ell'(u) \leq \ell'(v) = k'$. Thus $k = k'$, and we see conjugation preserves reflection length, i.e., conjugation preserves ranks in the intervals $[e, c]$ and $[e, c']$.

Now we must show that conjugation takes cover relations to cover relations. As we have seen, if t is a reflection then wtw^{-1} is also a reflection. Thus a cover $u < v$ in $[e, c]$, with $v = ut$, is mapped to the cover $wuw^{-1} < wvw^{-1}$ in $[e, wcw^{-1}]$, since $wvw^{-1} = (wuw^{-1})(wtw^{-1}) = w(ut)w^{-1}$. This shows the interval $[e, wcw^{-1}]$ is isomorphic to $[e, c]$ for any w.

12.6 This follows immediately from the fact that elements not directly connected in the Coxeter graph commute.

12.7 Choose the Coxeter element $c = st$ in $I_2(m)$. Every reflection wsw^{-1} and wtw^{-1} lies below st in the absolute order:

$$st = (s)(t) = (t)(tst) = (tst)(tstst) = (sts)(s) = (ststs)(sts) = \cdots.$$

If r is a reflection that begins with an s, then $r = st(r')$, where r' is a reflection. Hence $st = rr'$. If r begins with a t, then $rst = r'$ is also a reflection and $st = rr'$ again.

We conclude that the interval $[e,c]$ contains every one of the m reflections, so the Narayana polynomial is

$$\sum_{w\in[e,c]} x^{\ell'(w)} = 1 + mx + x^2.$$

12.8 Suppose

$$u_0 = e \to u_1 = s_{\beta_1} \to u_2 = s_{\beta_1}s_{\beta_2} \to \cdots \to u_k = s_{\beta_1}\cdots s_{\beta_k} = u,$$

is a saturated chain in $[e,c]$ from e to u. If we apply the map $w \mapsto w^{-1}c$ to this chain, we get a saturated chain

$$v = v_k \to v_{k-1} \to \cdots \to v_2 \to v_1 \to v_0 = c,$$

where $v_j = u_j^{-1}c$.

To see that $v_j < v_{j+1}$ is indeed a cover relation in $[e,c]$, write

$$\begin{aligned} v_{j+1} &= u_{j+1}^{-1}c, \\ &= s_{\beta_{j+1}}u_j^{-1}c, \\ &= (u_j^{-1}cc^{-1}u_j)(s_{\beta_{j+1}}u_j^{-1}c), \\ &= u_j^{-1}c(c^{-1}u_js_{\beta_{j+1}}u_j^{-1}c), \\ &= v_jr, \end{aligned}$$

with $r = (c^{-1}u_j)s_{\beta_{j+1}}(u_j^{-1}c)$ a reflection.

12.9 This follows immediately from the fact that all the generators (and hence all the reflections) of U and V commute. Hence every element $w \in W$ can be written uniquely in the form $w = uv$ with $u \in U$ and $v \in V$, from which it follows that $\ell'(w) = \ell'(u) + \ell'(v)$. Geometrically, this makes sense since U and V are the Coxeter groups for mutually orthogonal root systems.

12.10 We will show that the stack-sortable permutations are c-sortable with the claimed choice of c. In the solution to Problem 4.4 it is showed that 231-avoiding permutations are characterized by the fact that they are sorted by the following operator $\mathcal{S}$. If w is empty, $\mathcal{S}(w) := w$. If w is not empty and $\max\{w(i)\} = m$, we write $w = u{\cdot}m{\cdot}v$ for some (possibly empty) permutations u and v. We then recursively define $\mathcal{S}(w) = \mathcal{S}(u)\mathcal{S}(v)\cdot m$. Notice that if w is sorted after one pass through the stack, this means that both u and v sorted by one application of $\mathcal{S}$, and moreover, all the letters in u are less than all the letters of v. That is, w must avoid the pattern 231.

In terms of reduced expressions for w, the fact that all the letters of u are less than the letters of v means that the simple transpositions used to produce u commute with the simple transpositions involved with $m \cdot v$. Thus it will suffice to show that elements of the form $m \cdot v$, with v a 231-avoiding permutation, are c-sortable.

Suppose w has the form $w = m \cdot v$, with $v \in S_{m-1}(231)$, and let $c = s_{m-1}s_{m-2}\cdots s_2 s_1$, so that the c-sorting word is

$$c^\infty = s_{m-1}\cdots s_2 s_1 | s_{m-1}\cdots s_2 s_1 | s_{m-1}\cdots s_2 s_1 | \cdots .$$

Applying the stack-sorting operator to w can be expressed as follows:

$$\mathcal{S}(w) = \mathcal{S}(m \cdot \mathcal{S}(v)) = \mathcal{S}(v) \cdot m.$$

As suggested above, this can be achieved in the following two steps:

- First, sort v with $\mathcal{S}$ while m sits to the left of v. By induction on m, this can be realized as a c'-sortable element for $c' = s_{m-1}s_{m-2}\cdots s_2$. (Since m is in position 1 of w, we never use s_1.) Note that a c'-sortable element is also a c-sortable element.
- Next, move m to the right of $\mathcal{S}(v)$. This is achieved by the following right multiplication by simple transpositions,

  $$(m \cdot \mathcal{S}(v)) \cdot s_1 s_2 \cdots s_{m-1} = \mathcal{S}(v) \cdot m.$$

  Note this expression is equal to the identity, since we supposed v was 231-avoiding.

  Write $s_{i_k}\cdots s_{i_1}$ for the c'-sorting word used for v above. We have shown

$$w(s_{i_1}\cdots s_{i_k}) \cdot s_1 s_2 \cdots s_{m-1} = e,$$

or

$$w = s_{m-1}\cdots s_2 s_1 | (s_{i_k}\cdots s_{i_1}).$$

(The bar is used for emphasis.) As a c'-sortable element, $s_{i_k}\cdots s_{i_1}$ is a c-sortable element that has no s_1, so this shows w is in fact c-sortable.

Let us see this idea applied to the example of $w = (13254)9(867)$ in S_9. We have placed parentheses around u and v for emphasis. First, sort $v = 867$ while keeping all other elements fixed:

$$w \to ws_7s_8 = (13254)9678.$$

Next, move the 9 to the far right:

$$(ws_7s_8)s_6s_7s_8 = (13254)6789.$$

Now we turn our attention to sorting $u = 13254 = (132)5(4)$.

$$u \to us_4 = (132)45,$$

and

$$(us_4)s_2 = 12345.$$

Notice that these generators will commute with all of the generators used above.

Putting it all together, we can write

$$(((ws_7s_8)s_6s_7s_8)s_4)s_2 = 123456789 = e.$$

Therefore

$$\begin{aligned} w &= s_2s_4s_8s_7s_6s_8s_7, \\ &= s_8s_7s_6s_4s_2|s_8s_7, \end{aligned}$$

as desired.

12.11 This is an unpublished result of Alexander Postnikov. Figure 12.7 gives the general idea. See Remark 2 of Victor Reiner's paper [126].

Index

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