1
Introduction

The purpose of this chapter is to quickly introduce enough theory so that we can present some examples that will then be used throughout the course of the book to illustrate the theory and how to use it. These examples are simple to write down in general and to understand at an elementary level, but they are also useful for the understanding of deeper parts of the theory.

Two main classes of systems considered in the book are ***holonomic systems*** and ***nonholonomic systems***. This terminology may be found in Hertz [1894]. Holonomic systems are mechanical systems that are subject to constraints that limit their possible configurations. As Hertz explains, the word holonomic (or holonomous) is comprised of the Greek words meaning "integral" (or "whole") and "law," and refers to the fact that such constraints, given as constraints on the velocity, may be integrated and re-expressed as constraints on the configuration variables. We make this idea precise as we move through the book. Examples of holonomic constraints are length constraints for simple pendula and rigidity constraints for rigid body motion.

The rolling disk and ball are archetypal nonholonomic systems: systems with *nonintegrable* constraints on their velocities. These examples have a long history going back, for example, to Vierkandt [1892] and Chaplygin [1897a]. In this chapter and the book in general we discuss both the rolling disk and ball, as well as many other nonholonomic systems such as the Chaplygin sleigh, the roller racer, and the rattleback. As pointed out in Sommerfeld [1952] a general analysis of the distinction between holonomic and nonholonomic constraints may be found as early as Voss [1885], while

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specific examples of nonholonomic systems were of course analyzed even earlier. For more on the history of nonholonomic systems, see Chapter 5.

We remark that Hertz defines a holonomic system as a system "between whose possible positions all conceivable continuous motions are also possible motions." The point is that nonholonomic constraints restrict types of motion but not position. The meaning of Hertz's statement should become clearer as the reader continues through the book.

Other examples discussed here include the free rigid body and the somewhat more complex satellite with momentum wheels. These are (holonomic) examples of free and coupled rigid body motion, respectively—the motion of bodies with nontrivial spatial extent, as opposed to the motion of point particles. The latter is illustrated by the Toda lattice, which models a set of interacting particles on the line; we shall also be interested in some associated optimal control systems.

We also describe here the Heisenberg system, which was first studied by Brockett [1981] (see also Baillieul [1975], who studied some related systems). This does not model any particular physical system, but is a prototypical example for nonlinear kinematic control problems (both optimal and nonoptimal) and can be viewed as an approximation to a number of interesting physical systems; in particular, this example is basic for understanding more sophisticated optimal reorientation and locomotion problems, such as the falling cat theorem that we shall treat later. A key point about this system (and many others in this book) is that the corresponding linear theory gives little information.

1.1 Generalized Coordinates and Newton–Euler Balance

In this and subsequent sections in this chapter we discuss some ideas from mechanics in an informal fashion. This is intended to give context to the physical examples discussed in later sections. More formal derivations of many of the ideas discussed here are given in later chapters.

Coordinates and Kinematics. The most basic goal of analytical mechanics is to provide a formalism for describing motion. This is often done in terms of a set of ***generalized coordinates***, which may be interpreted as coordinates for the system's ***configuration space***, often denoted by Q. This is a set of variables whose values uniquely specify the location in 3-space of each physical point of the mechanism. A set of generalized coordinates is minimal in the sense that no set of fewer variables suffices to determine the locations of all points on the mechanism. The number of variables in a set of generalized coordinates for a mechanical system is called the number of ***degrees of freedom*** of the system.

1.1.1 Example (A Simple Kinematic Chain). Simple ideas along this line, which will be generalized to provide the foundation of most of the models studied in this book, may be illustrated using the simple kinematic chain shown in Figure 1.1.1.

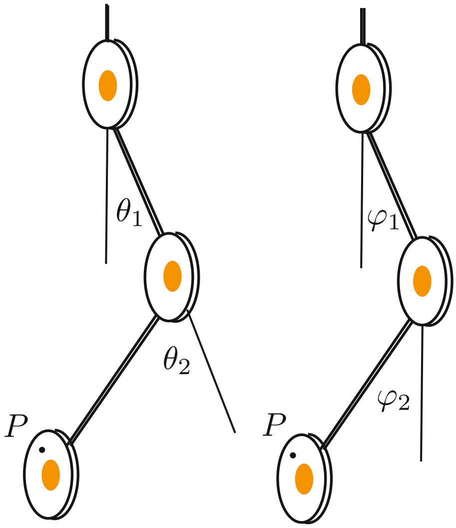

FIGURE 1.1.1. Kinematic chains.

Here there are drawn two copies of the same mechanism. This mechanism consists of planar rigid bodies connected by massless rods, and the joints are free to rotate in a fixed plane. In the first, the motion of a typical point P is described in terms of coordinate variables (θ_1, θ_2), where θ_2 is the relative angle between the two links in the chain. In Figure 1.1.1 (b), the motion of the typical point P is described in terms of coordinate variables (φ_1, φ_2), which are the (absolute) angles of the links with respect to the vertical direction.

Other choices of coordinate variables are, of course, possible. In any case, the coordinate variables serve the purpose of describing the location of typical points of the mechanism with respect to a privileged coordinate frame, which we may refer to as an ***inertial frame***. A thorough axiomatic discussion of inertial frames is beyond the scope of this book, but roughly speaking, these are frames that are "nonaccelerating relative to the distant stars." For the purposes of our discussions here it suffices to consider them as "fixed" coordinate systems.

Specifically, in this case, the inertial frame is chosen so that its origin is at the hinge point of the upper link. The y-axis is directed parallel and opposite to the gravitational field, and the x-axis is chosen so as to give the coordinate frame the standard orientation. Suppose the point P is located on the second link, as depicted. If this has coordinates (x_ℓ, y_ℓ) with respect to a local frame fixed in the second link, then the coordinates with respect to the inertial frame are given by

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} r_1 \sin\theta_1 + x_\ell \sin(\theta_1+\theta_2) + y_\ell \cos(\theta_1+\theta_2) \\ -r_1 \cos\theta_1 - x_\ell \cos(\theta_1+\theta_2) + y_\ell \sin(\theta_1+\theta_2) \end{bmatrix}, \tag{1.1.1}$$

where r_1 is the length of the first link, or equivalently by

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} r_1 \sin\varphi_1 + x_\ell \sin\varphi_2 + y_\ell \cos\varphi_2 \\ -r_1 \cos\varphi_1 - x_\ell \cos\varphi_2 + y_\ell \sin\varphi_2 \end{bmatrix}. \tag{1.1.2}$$

The mappings $(\theta_1, \theta_2) \mapsto (x, y)$ are examples of functions that associate values of the generalized coordinate variables (θ_1, θ_2) (respectively (φ_1, φ_2)) to inertial coordinates of the point P. In this example, the configuration manifold is given by $Q = S^1 \times S^1$ and is parameterized by the two angles θ_1, θ_2, which serve as generalized coordinates. One can also make the alternative choice of φ_1, φ_2 as generalized coordinates that provide a different set of coordinates on Q. ♦

Newton's Laws. The most fundamental contribution to mechanics were Newton's three laws of motion for a particle (see Newton [1650], Book I, Section 3, Propositions XI, XII, XIII) and, for example, Chorlton [1983]).

They are as follows:

(1) Every particle continues in its state of rest or of uniform velocity in a straight line unless compelled to do otherwise by a force acting on it.

(2) The rate of change of linear momentum is proportional to the impressed force and takes place in the direction of action of the force.

(3) To every action there is an equal and opposite reaction.

For a particle of constant mass m, Newton's second law can be written as:

$$m\ddot{\mathbf{x}}(t) = \mathbf{F}(t), \tag{1.1.3}$$

where $\mathbf{x} \in \mathbb{R}^3$ is the position vector of the particle and $\mathbf{F}(t)$ is the impressed force, both measured with respect to an inertial frame.

Remarks on Rigid Body Mechanics. As a preliminary to describing general rigid body mechanics, one procedure is to consider the special case of a finite number of point masses constrained so that the distance between points is constant, with each point mass experiencing internal forces of interaction (equal in magnitude and acting in opposite directions along straight lines joining the points) together with external forces. In this case, Newton's laws lead to the equations of rigid body dynamics for this special type of rigid body.

For a rigid body that is a continuum or for a system of point particles or rigid bodies mechanically linked, one may derive the equations of motion by an application of Newton's law for the motion of the center of mass and Euler's law for motion about the center of mass, i.e.,

$$I\dot{\omega}(t) = T(t), \tag{1.1.4}$$

where I is the moment of inertia of the rigid body about its center of mass, ω is the angular velocity about the center of mass, and $T(t)$ is the applied torque about the center of mass, all measured with respect to an inertial frame.

It turns out, however, that the equations of motion for the special case of a finite number of constrained point masses described above may be derived solely from Newton's laws.

The rigid body also provides a nice example of a system whose configuration space is a manifold. In fact, it is the set $Q = \mathrm{SE}(3)$ of Euclidean motions, that is, transformations of $\mathbb{R}^3$ consisting of rotations and translations. Each element of Q gives a placement of all the particles in the rigid body relative to a reference position, all in an inertial frame. We will return to the rigid body from a more advanced point of view later.

Newton–Euler Balance Laws. More generally, for a system of interconnected rigid bodies, such as the kinematic chain described earlier, one can derive the equations of motion from Newton's laws together with Euler's law giving the rate of change of angular momentum about a pivot point in terms of applied torques, as in equation (1.1.4). It is interesting to note that these equations cannot (without further assumptions) be derived from Newton's laws alone; for an illuminating discussion of these relationships, see Antman [1998].

So far, the examples mentioned are ones with holonomic constraints (the length of the pendula in the kinematic chain is assumed constrained to be constant, and the rigid body is constrained by rigidity). However, one of the purposes of this book is to study nonholonomic systems, wherein one has constraints on the velocities. Examples are systems such as rolling wheels. Even in this case, one can use Newton–Euler balance ideas to obtain the equations correctly.

For the bulk of this book, however, we will not take the point of view of Newton–Euler balance laws. One reason for this is that there is a more useful alternative given by Hamilton's principle (and the associated Euler–Lagrange equations) for holonomic systems and by the Lagrange–d'Alembert principle in the nonholonomic case. We shall briefly study these principles in the next sections and return to them in more detail later. In addition, the Hamilton principle and Lagrange–d'Alembert formalism are covariant, in the sense that they use only the intrinsic configuration manifold Q, and one may use any set of coordinates on it; in addition, there is a simple and elegant way to write the equations valid in any set of generalized coordinates. The covariant nature of the Euler–Lagrange formalism was one of the greatest discoveries of Lagrange and is the basis of the geometric approach to mechanics.

One should ask whether the Newton–Euler balance approach is equivalent to the Euler–Lagrange and Lagrange–d'Alembert approaches under general sets of hypotheses. This is a subtle question in general, which is, unfortunately, not systematically addressed in most books, including this one. However, these approaches can be shown to be equivalent in many concrete situations, such as interconnected rigid bodies and rolling rigid bodies, which we will come to later. See Jalnapurkar [1994] for one such exposition of this equivalence. We will confine ourselves to proving the equivalence in one concrete nonholonomic situation later, namely, a system called the Chaplygin sleigh; see Section 1.7.

1.2 Hamilton's Principle

In this section we give a brief introduction to the Euler–Lagrange equations of motion for holonomic systems from the point of view of variational principles. We return to this later in Chapter 3 from a more abstract point of view. The reader for whom this is familiar may, of course, skip ahead.

Let Q be the configuration space[1] of a system with (generalized) coordinates q^i, $i = 1, \ldots, n$. We are given a real-valued function $L(q^i, \dot{q}^i)$, called a ***Lagrangian***. Often we choose L to be $L = K - V$, where K is the ***kinetic energy*** of the system and $V(q)$ is the ***potential energy***.

1.2.1 Definition. ***Hamilton's principle*** *singles out particular curves $q(t)$ by the condition*

$$\delta \int_a^b L(q(t), \dot{q}(t))\, dt = 0, \tag{1.2.1}$$

where the variation is over smooth curves in Q with fixed endpoints.

To make this precise, let the ***variation*** of a trajectory $q(\cdot)$ with fixed endpoints satisfying $q(a) = q_a$ and $q(b) = q_b$ be defined to be a smooth mapping

$$(t, \epsilon) \mapsto q(t, \epsilon), \qquad a \leq t \leq b, \qquad \epsilon \in (-\delta, \delta) \subset \mathbb{R},$$

satisfying

(i) $q(t, 0) = q(t)$, $t \in [a, b]$,

(ii) $q(a, \epsilon) = q_a$, $q(b, \epsilon) = q_b$.

[1]The configuration space of a system is best thought of as a differentiable manifold, and generalized coordinates as a coordinate chart on this manifold. To enable us to introduce some examples early on, we shall treat this rather informally at first and return to a more intrinsic approach later.

Letting $\delta q(t) = (\partial/\partial\epsilon) q(t,\epsilon)|_{\epsilon=0}$ be the ***virtual displacement*** corresponding to the variation of q, we have

$$\delta q(a) = \delta q(b) = 0. \tag{1.2.2}$$

The precise meaning of Hamilton's principle is then the statement

$$\frac{d}{d\epsilon} \int_a^b L(q(t,\epsilon), \dot q(t,\epsilon))\, dt \bigg|_{\epsilon=0} = 0 \tag{1.2.3}$$

for all variations.

One can view Hamilton's principle in the following way: The quantity $\int_a^b L(q(t), \dot q(t))\, dt$ is being extremized among all curves with fixed endpoints; that is, the particular curve $q(t)$ that is sought is a *critical point* of the quantity $\int_a^b L(q(t), \dot q(t))\, dt$ thought of as a function on the space of curves with fixed endpoints. Examples show that the quantity $\int_a^b L\, dt$ being extremized in (1.2.1) need not be minimized at a solution of the Euler–Lagrange equations, just as in calculus: Critical points of functions need not be minima.[2]

A basic result of the calculus of variations is:

1.2.2 Proposition. *Hamilton's principle for a curve $q(t)$ is equivalent to the condition that $q(t)$ satisfy the* ***Euler–Lagrange equations***

$$\frac{d}{dt}\frac{\partial L}{\partial \dot q^i} - \frac{\partial L}{\partial q^i} = 0. \tag{1.2.4}$$

The idea of the proof is as follows: Let δq be a virtual displacement of the curve $q(t)$ corresponding to the variation $q(t,\epsilon)$. We may compute the variation of the integral in Definition 1.2.1 corresponding to this variation of the trajectory q by differentiating with respect to ϵ and using the chain rule. We obtain

$$\int_a^b \left(\frac{\partial L}{\partial q^i} \delta q^i + \frac{\partial L}{\partial \dot q^i} \delta \dot q^i \right) dt = 0, \tag{1.2.5}$$

where $\delta \dot q^i = \frac{d}{dt} \delta q^i$. Integrating by parts and using the boundary conditions $\delta q^i = 0$ at $t = a$ and $t = b$ yields the identity

$$\int_a^b \left(-\frac{d}{dt}\frac{\partial L}{\partial \dot q^i} + \frac{\partial L}{\partial q^i} \right) \delta q^i\, dt = 0. \tag{1.2.6}$$

Assuming a rich enough class of variations yields the result.[3]

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[2] Perhaps the simplest example of this comes up in the study of geodesics on a sphere where geodesics that "go the long way around the sphere" are critical points, but not minima. In this example, L is just the kinetic energy of a point particle on the sphere. See Gelfand and Fomin [1963] for further information.

[3] Again, further geometric insight into the notion of the variation operation is something we will return to later; for example, the equality $\delta \dot q^i = \frac{d}{dt}\delta q^i$ is self-evident from our definition of the virtual displacement and equality of mixed partials.

A critical aspect of the Euler–Lagrange equations is that they may be regarded as a way to write Newton's second law in a way that makes sense in arbitrary curvilinear and even moving coordinate systems. That is, the Euler–Lagrange formalism is *covariant*. This is of enormous benefit, not only theoretically, but for practical problems as well.

Mechanical Systems with External Forces. In the presence of external forces F_i, the equations are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = F_i \tag{1.2.7}$$

for $i = 1, \ldots, n$. Here we regard the quantities F_i as given by external agencies.[4] Note that if these forces are derivable from a potential U in the sense that $F_i = -\partial U/\partial q^i$, then these forces can be incorporated into the Lagrangian by adding $-U$ to the Lagrangian. Thus, this way of adding forces is consistent with the Euler–Lagrange equations themselves.

These equations can be derived from a variational-like principle, the ***Lagrange–d'Alembert principle*** for systems with external forces, as follows:

$$\delta \int_a^b L(q^i, \dot{q}^i)\, dt + \int_a^b F \cdot \delta q \, dt = 0, \tag{1.2.8}$$

where $F \cdot \delta q = \sum_{i=1}^n F_i \delta q^i$ is the ***virtual work*** done by the force field F with a virtual displacement δq as defined above.

A rigorous analysis of virtual work and integral laws of motion for continuum mechanics in Euclidean space may be found in Antman and Osborn [1979].

Remarks on the History of Variational Principles. The history of variational principles and the so-called principle of least action is quite complicated, and we leave most of the details to other references. Some of this history can be gleaned, for example, from Whittaker [1988] and Marsden and Ratiu [1999]. An interesting historical note is that the currently accepted notion of the "principle of least action" is regarded by some as being synonymous with "Hamilton's principle." Indeed Feynman [1989] advocates this point of view. However, both historically and factually, ***Hamilton's principle*** and the ***principle of least action*** (which should really be called the ***principle of critical action***) are slightly different. Hamilton's principle involves varying the integral of the Lagrangian

[4] In elementary books on mechanics external forces are often regarded as a given vector field, but in fact, they should be regarded as a given one-form field. Such distinctions are not important just now, but this is a crucial distinction in the geometric formulation of mechanics that will be important for us later on.

over all curves with fixed endpoint and fixed time. The principle of least action, on the other hand, involves variation of the quantity

$$\int_a^b \sum_i \dot{q}^i \frac{\partial L}{\partial \dot{q}^i}\, dt$$

over all curves with fixed energy.

The principle of critical action originated in Maupertuis's work (Maupertuis [1740]), which attempted to obtain for the corpuscular theory of light a theorem analogous to Fermat's ***principle of least time.*** Briefly put, the latter involves taking the variations of

$$\int n\, ds, \tag{1.2.9}$$

where n is the refractive index over the path of the light. This gives rise to Snel's law.[5] Maupertuis's principle was established by Euler [1744] for the case of a single particle and in more generality by Lagrange [1760].

One can expand this to obtain the Hamilton–Jacobi equation in optics, otherwise known as the *eikonal equation.*

One can observe this as follows. Since $ds^2 = d\mathbf{q}(s) \cdot d\mathbf{q}(s)$, one may rewrite the shortest path length as

$$\int_{P_1}^{P_2} n\, ds = \int_{P_1}^{P_2} n(\mathbf{q}(s)) \sqrt{\frac{d\mathbf{q}}{ds} \cdot \frac{d\mathbf{q}}{ds}}\, ds. \tag{1.2.10}$$

Taking variations leads to the ***eikonal equation***

$$\frac{d}{ds}\left(n \frac{d\mathbf{q}}{ds} \right) = \operatorname{grad} n.$$

In a homogeneous medium n is constant and thus we obtain

$$\frac{d^2\mathbf{q}}{ds^2} = 0, \tag{1.2.11}$$

implying $\mathbf{q} = s\mathbf{a} + \mathbf{b}$ for $\mathbf{a}$ and $\mathbf{b}$ constants, so the light rays travel in straight lines.

We note also that the light rays are the orthogonal to the wave fronts $S(\mathbf{q}) = \text{const}$ and thus

$$n \frac{d\mathbf{q}}{ds} = S_{\mathbf{q}}.$$

[5] A simple derivation of Snel's law from the variational point of view can be found, for example, in Feynman [1989]. This law was discovered by the Dutch mathematician and geodesist Willebord Snel van Royen. (Because his name in Latin is "Snellius" the law is often called Snell's law.)

The gradient of S is perpendicular to the wave front. The bigger the gradient the slower the front moves and hence Hamilton called the quantity

$$\mathbf{p} = \frac{\partial S}{\partial \mathbf{q}}$$

the ***vector of normal slowness*** to the front. For further details, see Born and Wolf [1980] and Holm [2008].

It is curious that Lagrange dealt with the more difficult principle of critical action already in 1760, yet Hamilton's principle, which is simpler, came only much later in Hamilton [1834, 1835].

Another bit of interesting history is that Lagrange [1788] did not derive the Lagrange equations of motion by variational methods, but he did so by requiring that simple force balance be *covariant*, that is, expressible in arbitrary generalized coordinates. For further information on the history of variational principles and the precise formulation of the principle of least action, see Marsden and Ratiu [1999].

Energy and Hamilton's Equations. If the matrix $\partial^2 L/\partial \dot{q}^i \partial \dot{q}^j$ is nonsingular, we call L a ***nondegenerate*** or ***regular*** Lagrangian, and in this case we can make (at least locally) the change of variables from $(q^i, \dot{q}^i)$ to the variables (q^i, p_i), where the momentum is defined by

$$p_i = \frac{\partial L}{\partial \dot{q}^i}.$$

This change of variables is commonly referred to as the ***Legendre transformation***. We shall see how to write it in a coordinate-free way in Chapter 3. Introducing the Hamiltonian

$$H(q^i, p_i) = \sum_{i=1}^{n} p_i \dot{q}^i - L(q^i, \dot{q}^i),$$

one checks, by a *careful* use of the chain rule, that the Euler–Lagrange equations become ***Hamilton's equations***

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q^i},$$

where $i = 1, \ldots, n$. If we think of the Hamiltonian as a function of $(q^i, \dot{q}^i)$, then we write it as $E(q^i, \dot{q}^i)$ and still refer to it as the ***energy***. If the Lagrangian is of the form kinetic minus potential, then the energy and Hamiltonian are kinetic plus potential.

If one introduces the ***Poisson bracket*** of two functions K, L of (q^i, p_i) by the definition

$$\{K, L\} = \sum_{i=1}^{n} \frac{\partial K}{\partial q^i}\frac{\partial L}{\partial p_i} - \frac{\partial L}{\partial q^i}\frac{\partial K}{\partial p_i},$$

then one checks, again using the chain rule, that Hamilton's equations may be written concisely as

$$\dot{F} = \{F, H\}$$

for all functions F. In particular, since the Poisson bracket is clearly skew symmetric in K, L, we see that $\{H, H\} = 0$, and so H has zero time derivative (conservation of energy). The corresponding statement for the energy E can be verified directly to be a consequence of the Euler–Lagrange equations (and this holds even if L is degenerate).

Exercises

◇ **1.2-1.** Consider the Lagrangian

$$L(x, y, z, \dot{x}, \dot{y}, \dot{z}) = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - mgz.$$

Compute the equations of motion in both Lagrangian and Hamiltonian form. Verify that the Hamiltonian (energy) is conserved along the flow. Are there other conserved quantities?

◇ **1.2-2.** Consider a Lagrangian of the form $L = \frac{1}{2} \sum_{k,l=1}^{n} g_{kl}(q) \dot{q}^k \dot{q}^l$, where g_{kl} is a symmetric matrix. Show that the Lagrange equation of motion are

$$\sum_s g_{rs} \ddot{q}^s + \sum_{l,m} \Gamma_{rlm} \dot{q}^l \dot{q}^m = 0$$

for suitable symbols Γ. Verify conservation of energy directly for this system.

1.3 The Lagrange–d'Alembert Principle

Holonomic and Nonholonomic Constraints. Suppose the system constraints are given by the following m equations, linear in the velocity field, where $m < n$:

$$\sum_{k=1}^{n} a_k^j(q^i) \dot{q}^k = 0, \tag{1.3.1}$$

where $j = 1, \dots, m$.

If one can find m constraints on the positions alone, that is, constraints of the form $b^j(q^i) = 0$, such that their time derivatives, namely

$$\sum_{k=1}^{n} \frac{\partial b^j}{\partial q^k} \dot{q}^k = 0,$$

determine the same constraint distribution as the constraints (1.3.1), then one says that the constraints are ***holonomic***. Otherwise, they are called

nonholonomic. For example, the length constraint on a pendulum is a holonomic constraint, whereas a constraint of rolling without slipping (which we shall discuss in the next section) is nonholonomic.

It is also sometimes useful to distinguish between constraints that are dependent or independent of time. Those that are independent of time are called ***scleronomic***, and those that depend on time are called ***rheonomic***. This terminology can also be applied to the mechanical system itself; see, e.g., Greenwood [1977]. For example, a bead on a hoop is a rheonomic system. For more details on such "moving" systems, see Marsden and Ratiu [1999].

The Frobenius theorem and differential forms, which we shall review in Chapter 2, give necessary and sufficient conditions under which a given set of constraints is integrable. We shall return to these ideas in a more geometric form in Chapter 5.

Dynamic Nonholonomic Equations of Motion. We will now sketch the derivation of the equations of motion of a nonholonomic mechanical system using Newton's laws and Lagrange's equations.[6] We omit external forces for the moment. Later on in the text we shall derive the equations of motion from other points of view.

We regard the system as being acted on by just those forces F_i, $i = 1, \ldots, n$, that have to be exerted by the constraints in order that the system satisfy the nonholonomic constraints (1.3.1). Let $F_1 \delta q^1 + F_2 \delta q^2 + \cdots + F_n \delta q^n$ be the work done by these forces when the system undergoes an arbitrary virtual displacement $(\delta q^1, \ldots, \delta q^n)$. One *assumes* that with these forces, the system is described by a holonomic system subject to the forces of constraint; therefore, the equations of motion are given by (1.2.7). To determine these forces of constraint, we make the following fundamental assumption:

> **Assumption.** In any virtual displacement consistent with the constraints, the constraint forces F_i do no work, i.e., we assume that the identity
>
> $$F_1 \delta q^1 + F_2 \delta q^2 + \cdots + F_n \delta q^n = 0$$
>
> holds for all virtual displacements δq^i satisfying the constraints (1.3.1).

Assuming that the m vectors $(a_1^1, \ldots, a_n^1)$, $(a_1^2, \ldots, a_n^2)$, $\ldots$, $(a_1^m, \ldots, a_n^m)$ are linearly independent, it follows from the same linear algebra used to prove the Lagrange multiplier theorem that the forces of constraint have the form $F_i = \lambda_1 a_i^1 + \cdots + \lambda_m a_i^m$ for $i = 1, \ldots, n$.

[6] See also, for example, Whittaker [1988] and the references therein, Ferrers [1871], Neumann [1888], and Vierkandt [1892].

In summary, the ***dynamic nonholonomic equations of motion*** are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = \sum_{j=1}^{m} \lambda_j a_i^j \,, \tag{1.3.2}$$

where $i = 1, \ldots, n$, together with the constraint equations (1.3.1). One determines the Lagrange multipliers λ_i by imposing the constraints in much the same way as one solves constrained maximum and minimum problems in calculus.

The dynamic nonholonomic equations of motion (1.3.2) are also known as the ***Lagrange–d'Alembert equations***. These equations are the correct equations for mechanical dynamical systems and in many cases (such as rolling bodies in contact) can be shown to be equivalent to Newton's law $F = ma$ with reaction forces.[7] We shall see this explicitly in the context of some simple and concrete examples shortly.

Lagrange–d'Alembert Principle. The generalization of Hamilton's principle to the nonholonomic context is as follows:

1.3.1 Definition. *The principle*

$$\delta \int_a^b L(q(t), \dot{q}(t))\, dt = 0, \tag{1.3.3}$$

where the virtual displacements δq are assumed to satisfy the constraints 1.3.1, that is,

$$\sum_{k=1}^{n} a_k^j \delta q^k = 0, \tag{1.3.4}$$

*where $j = 1, \ldots, m$, is called the **Lagrange–d'Alembert principle**.*

As with Hamilton's principle, one can check that the following propositions are true:

1.3.2 Proposition. *The Lagrange–d'Alembert principle given in Definition 1.3.1, together with the constraints (1.3.1), is equivalent to the Lagrange–d'Alembert equations of motion (1.3.2).*

This is a fundamental principle, and we shall return to it later in more detail.

Energy. We introduce the energy in the same way as with holonomic systems, namely

$$E(q^i, \dot{q}^i) = \frac{\partial L}{\partial \dot{q}^i}\dot{q}^i - L(q^i, \dot{q}^i). \tag{1.3.5}$$

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[7]See, for example, Vershik and Gershkovich [1988], Bloch and Crouch [1998a], and Jalnapurkar [1994].

1.3.3 Proposition. *Energy is conserved for nonholonomic systems; that is, for solutions of* (1.3.2) *subject to the constraints* (1.3.1)*, we have*

$$\frac{dE}{dt} = 0.$$

Proof. We begin by taking the time derivative of the energy expression (1.3.5) and using the equations of motion (1.3.2):

$$\begin{aligned}\frac{d}{dt}E(q^i, \dot{q}^i) &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\dot{q}^i - L(q^i, \dot{q}^i)\right)\\ &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right)\dot{q}^i + \frac{\partial L}{\partial \dot{q}^i}\ddot{q}^i - \frac{\partial L}{\partial q^i}\dot{q}^i - \frac{\partial L}{\partial \dot{q}^i}\ddot{q}^i\\ &= \sum_{j=1}^{m} \lambda_j a_i^j \dot{q}^i.\end{aligned}$$

But this vanishes by virtue of the constraints (1.3.1). ■

This proposition is consistent with the fact that the forces of constraint do no work. Of course, this result is under the assumptions that the Lagrangian is not explicitly time-dependent and that the constraints are time-independent.

Nonholonomic Mechanical Systems with External Forces. If external forces F^e, such as control forces, are added to the system, then one adds these forces to the right-hand side of the equations, just as we did earlier for the Lagrange equations of motion. Namely, the equations are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = \sum_{j=1}^{m} \lambda_j a_i^j + F_i^e, \tag{1.3.6}$$

where $i = 1, \ldots, n$, together with the constraint equations (1.3.1). One determines the Lagrange multipliers λ_i by imposing the constraints as before.

The corresponding Lagrange–d'Alembert principle is

$$\delta \int_a^b L(q(t), \dot{q}(t))\, dt + \int_a^b F^e \cdot \delta q\, dt = 0, \tag{1.3.7}$$

where the virtual displacements δq now are assumed to satisfy the constraints (1.3.4).

Variational Nonholonomic Equations. It is interesting to compare the dynamic nonholonomic equations, that is, the Lagrange–d'Alembert equations with the corresponding variational nonholonomic equations. The distinction between these two different systems of equations has a long and distinguished history going back to the review article of Korteweg [1899] and is discussed in a more modern context in Arnold, Kozlov, and Neishtadt [1988]. (For Kozlov's work on vakonomic systems, see, e.g., Kozlov [1983] and Kozlov [1992]).[8] The upshot of the distinction is that the Lagrange–d'Alembert equations are the correct mechanical dynamical equations, while the corresponding variational problem is asking a different question, namely one of optimal control.

Perhaps it is surprising, at least at first, that *these two procedures give different equations.* What, exactly, is the difference in the two procedures? The distinction is one of whether the constraints are imposed before or after taking variations. These two operations do not, in general, commute. We shall see this explicitly with the vertical rolling disk in the next section. *With the dynamic Lagrange–d'Alembert equations, we impose constraints only on the variations, whereas in the variational problem we impose the constraints on the velocity vectors of the class of allowable curves.*

The variational equations are obtained by *using Lagrange multipliers with the Lagrangian* rather than Lagrange multipliers with the equations, as we did earlier. Namely, we consider the modified Lagrangian

$$L(q, \dot{q}) + \sum_{k=1}^{n} \sum_{j=1}^{m} \mu_j a_k^j \dot{q}^k. \tag{1.3.8}$$

Notice that there are as many Lagrange multipliers μ_j as there are constraints, just as in the Lagrange–d'Alembert equations. Then one forms the Euler–Lagrange equations from this modified Lagrangian and determines the Lagrange multipliers, to the extent possible, from the constraints and initial conditions. We shall see explicitly how this works in the context of examples in the next section and return to the general theory later on.

Exercises

◇ **1.3-1.** Consider the Lagrangian

$$L(x, y, z, \dot{x}, \dot{y}, \dot{z}) = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right) - mgz$$

with the constraints

$$y\dot{x} - x\dot{y} = 0.$$

(a) Are these constraints holonomic or nonholonomic?
(b) Write down the dynamic nonholonomic equations.

[8] As Korteweg points out, there were many confusions and mistakes in the literature because people were using the incorrect equations, namely the variational equations, when they should have been using the Lagrange–d'Alembert equations; some of these misunderstandings persist, remarkably, to the present day. What Arnold et al. call the *vakonomic* equations, we will call the *variational nonholonomic* equations. This terminology will be useful in distinguishing the system from the *dynamic nonholonomic* equations we introduced above.

(c) Write down the variational nonholonomic equations.
(d) Are these two sets of equations the same?

◇ **1.3-2** (Rosenberg [1977])**.** Consider the Lagrangian

$$L(x, y, z, \dot{x}, \dot{y}, \dot{z}) = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$$

with the constraints

$$\dot{z} - y\dot{x} = 0\,.$$

(a) Write down the dynamic nonholonomic equations.
(b) Write down the variational nonholonomic equations.
(c) Are these two sets of equations the same?

◇ **1.3-3.** Derive a formula for dE/dt for nonholonomic systems with forces.

1.4 The Vertical Rolling Disk

Geometry and Kinematics. The vertical rolling disk is a basic and simple example of a system subject to nonholonomic constraints: a homogeneous disk rolling without slipping on a horizontal plane. In the first instance we consider the "vertical" disk, a disk that, unphysically of course, may not tilt away from the vertical; it is not difficult to generalize the situation to the "falling" disk. It is helpful to think of a coin such as a penny, since we are concerned with orientation and the roll angle (the position of Lincoln's head, for example) of the disk.[9]

Let S^1 denote the circle of radius 1 in the plane. It is parameterized by an angular variable (that is, a variable that is 2π-periodic). The configuration space for the vertical rolling disk is $Q = \mathbb{R}^2 \times S^1 \times S^1$ and is parameterized by the (generalized) coordinates $q = (x, y, \theta, \varphi)$, denoting the position of the contact point in the xy-plane, the rotation angle of the disk, and the orientation of the disk, respectively, as in Figure 1.4.1.

The variables (x, y, φ) may also be regarded as giving a translational position of the disk together with a rotational position; that is, we may regard (x, y, φ) as an element of the ***Euclidean group*** in the plane. This group, denoted by SE(2), is the group of translations and rotations in the plane, that is, the group of rigid motions in the plane. Thus, $\mathrm{SE}(2) = \mathbb{R}^2 \times S^1$ (as a set). This group and its three-dimensional counterpart in space, SE(3), play an important role throughout this book. They will be treated via their coordinate descriptions for the moment, but later on we will return to them in a more geometric and intrinsic way.

---

[9]Other references that treat this example (including the falling disk) are, for example, Vierkandt [1892], Bloch, Reyhanoglu, and McClamroch [1992], Bloch and Crouch [1995], Bloch, Krishnaprasad, Marsden, and Murray [1996], O'Reilly [1996], Cushman, Hermans, and Kemppainen [1996], and Zenkov, Bloch, and Marsden [1998].

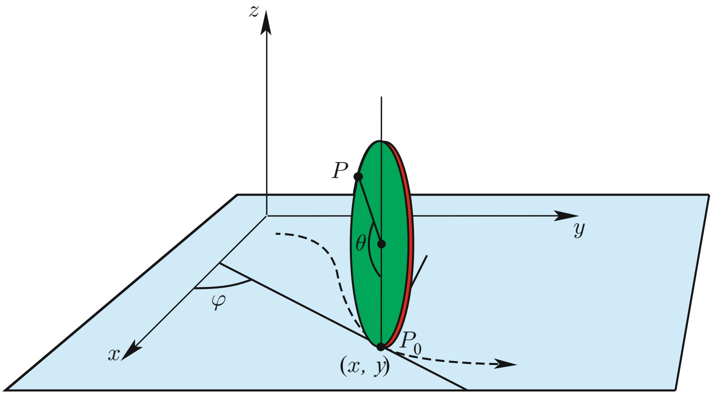

FIGURE 1.4.1. The geometry of the rolling disk.

In summary, the configuration space of the vertical rolling disk is given by $Q = \mathrm{SE}(2) \times S^1$, and this space has coordinates (generalized coordinates) given by $((x, y, \varphi), \theta)$.

The Lagrangian for the vertical rolling disk is taken to be the total kinetic energy of the system, namely

$$L(x, y, \varphi, \theta, \dot{x}, \dot{y}, \dot{\varphi}, \dot{\theta}) = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}J\dot{\varphi}^2, \tag{1.4.1}$$

where m is the mass of the disk, I is the moment of inertia of the disk about the axis perpendicular to the plane of the disk, and J is the moment of inertia about an axis in the plane of the disk (both axes passing through the disk's center).

For the derivation of kinetic energy formulas of this sort, we refer to any basic mechanics book, such as Synge and Griffiths [1950]. We shall derive such formulas from a slightly more advanced point of view in Section 3.15.

If R is the radius of the disk, the nonholonomic constraints of rolling without slipping are

$$\begin{aligned} \dot{x} &= R(\cos\varphi)\dot{\theta}\,, \\ \dot{y} &= R(\sin\varphi)\dot{\theta}\,, \end{aligned} \tag{1.4.2}$$

which state that the point P_0 fixed on the rim of the disk has zero velocity at the point of contact with the horizontal plane. Notice that these constraints have the form (1.3.1) if we write them as

$$\begin{aligned} \dot{x} - R(\cos\varphi)\dot{\theta} = 0\,, \\ \dot{y} - R(\sin\varphi)\dot{\theta} = 0\,. \end{aligned}$$

We can write these equations in the form of the equations (1.3.1), namely as the two constraint equations

$$a^1 \cdot (\dot{x}, \dot{y}, \dot{\varphi}, \dot{\theta})^T = 0\,,$$
$$a^2 \cdot (\dot{x}, \dot{y}, \dot{\varphi}, \dot{\theta})^T = 0\,,$$

where ^{T} denotes the transpose and where

$$a^1 = (1, 0, 0, -R\cos\varphi)\,, \quad a^2 = (0, 1, 0, -R\sin\varphi)\,.$$

In the notation used in (1.3.1),

$$a^1_1 = 1, \quad a^1_2 = 0, \quad a^1_3 = 0, \quad a^1_4 = -R\cos\varphi\,,$$

and similarly for a^2:

$$a^2_1 = 0, \quad a^2_2 = 1, \quad a^2_3 = 0, \quad a^2_4 = -R\sin\varphi\,.$$

We will compute the dynamical equations for this system with controls in the next section. In particular, when there are no controls, we will get the dynamical equations for the uncontrolled disk. As we shall see, these free equations can be explicitly integrated.

Dynamics of the Controlled Disk. Consider the case where we have two controls, one that can steer the disk and another that determines the roll torque. Now we shall use the general equations (1.3.6) to write down the equations for the controlled vertical rolling disk. According to these equations, we add the forces to the right-hand side of the Euler–Lagrange equations for the given Lagrangian along with Lagrange multipliers to enforce the constraints and to represent the reaction forces. In our case, L is cyclic in the configuration variables $q = (x, y, \varphi, \theta)$, and so the required dynamical equations become

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = u_\varphi f^\varphi + u_\theta f^\theta + \lambda_1 a^1 + \lambda_2 a^2, \tag{1.4.3}$$

where, from (1.4.1), we have

$$\frac{\partial L}{\partial \dot{q}} = (m\dot{x}, m\dot{y}, J\dot{\varphi}, I\dot{\theta}),$$

and where

$$f^\varphi = (0, 0, 1, 0), \quad f^\theta = (0, 0, 0, 1),$$

corresponding to assumed controls in the directions of the two angles φ and θ, respectively. Here u_φ and u_θ are control functions, so the external control forces are $F = u_\varphi f^\varphi + u_\theta f^\theta$, and the λ_i are Lagrange multipliers, chosen to ensure satisfaction of the constraints (1.4.2).

We eliminate the multipliers as follows. Consider the first two components of (1.4.3) and substitute the constraints (1.4.2) to eliminate $\dot{x}$ and $\dot{y}$ to give

$$\begin{aligned}\lambda_1 &= m\frac{d}{dt}(R\cos\varphi\,\dot{\theta})\,,\\ \lambda_2 &= m\frac{d}{dt}(R\sin\varphi\,\dot{\theta})\,.\end{aligned}$$

Substitution of these expressions for λ_1 and λ_2 into the last two components of (1.4.3) and noticing the simple identities

$$\begin{aligned}\lambda_1 a_3^1 + \lambda_2 a_3^2 &= 0\,,\\ \lambda_1 a_4^1 + \lambda_2 a_4^2 &= -mR^2\ddot{\theta}\,,\end{aligned}$$

gives the dynamic equations

$$\begin{aligned}J\ddot{\varphi} &= u_\varphi\,,\\ (I + mR^2)\ddot{\theta} &= u_\theta\,,\end{aligned} \tag{1.4.4}$$

which, together with the constraints

$$\begin{aligned}\dot{x} &= R(\cos\varphi)\dot{\theta}\,,\\ \dot{y} &= R(\sin\varphi)\dot{\theta}\,,\end{aligned} \tag{1.4.5}$$

(and some specification of the control forces), determine the dynamics of the system.

The *free equations*, in which we set $u_\varphi = u_\theta = 0$, are easily integrated. In fact, in this case, the dynamic equations (1.4.4) show that $\dot{\varphi}$ and $\dot{\theta}$ are constants; calling these constants ω and Ω, respectively, we have

$$\begin{aligned}\varphi &= \omega t + \varphi_0\,,\\ \theta &= \Omega t + \theta_0\,.\end{aligned}$$

Using these expressions in the constraint equations (1.4.5) and integrating again gives

$$\begin{aligned}x &= \frac{\Omega}{\omega}R\sin(\omega t + \varphi_0) + x_0\,,\\ y &= -\frac{\Omega}{\omega}R\cos(\omega t + \varphi_0) + y_0\,.\end{aligned}$$

Consider next the controlled case, with nonzero controls u_1, u_2. Call the variables θ and ϕ "base" or "controlled" variables and the variables x and y "fiber" variables. The distinction is that while θ and φ are controlled directly, the variables x and y are controlled indirectly via the constraints.[10]

[10]The notation "base" and "fiber" comes from the fact that the configuration space Q splits naturally into the base and fiber of a trivial fiber bundle, as we shall see later.

It is clear that the base variables are controllable in any sense we can imagine. One may ask whether the full system is controllable. Indeed it is, in a precise sense as we shall show later, by virtue of the nonholonomic nature of the constraints.

The Kinematic Controlled Disk. It is also useful to define and study a related system, the so-called kinematic controlled rolling disk. In this case we imagine we have direct control over velocities rather than forces, and accordingly, we consider the most general first-order system satisfying the constraints or lying in the "constraint distribution." In the present case of the vertically rolling disk, this system is

$$\dot{q} = u_1 X_1 + u_2 X_2, \tag{1.4.6}$$

where $X_1 = (R\cos\varphi, R\sin\varphi, 0, 1)^T$ and $X_2 = (0, 0, 1, 0)^T$ and where $\dot{q} = (\dot{x}, \dot{y}, \dot{\varphi}, \dot{\theta})^T$.

In fact, X_1 and X_2 constitute a maximal set of independent vector fields on Q satisfying the constraints, in the sense that the components of X_1 and X_2 satisfy the equations (1.4.5), as is easily checked. As we shall see, it is instructive to analyze both the control and optimal control of such systems.

The Variational Controlled System. As we indicated in the last section, the variational system is obtained by using Lagrange multipliers with the Lagrangian rather than Lagrange multipliers with the equations, as we did earlier. Namely, we consider the Lagrangian

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}J\dot{\varphi}^2 + \mu_1(\dot{x} - R\dot{\theta}\cos\varphi) + \mu_2(\dot{y} - R\dot{\theta}\sin\varphi)\,,$$

where, because of the Lagrange multipliers, we relax the constraints and take variations over all curves. In other words, we write down the Euler–Lagrange equations for this Lagrangian and determine the multipliers from the constraints and initial conditions to the extent possible.

The Euler–Lagrange equations for this Lagrangian, including external forces in the φ and θ equations, are

$$m\ddot{x} + \dot{\mu}_1 = 0\,, \tag{1.4.7}$$

$$m\ddot{y} + \dot{\mu}_2 = 0\,, \tag{1.4.8}$$

$$J\ddot{\varphi} - R\mu_1\dot{\theta}\sin\varphi + R\mu_2\dot{\theta}\cos\varphi = u_\varphi\,, \tag{1.4.9}$$

$$I\ddot{\theta} - R\frac{d}{dt}(\mu_1\cos\varphi + \mu_2\sin\varphi) = u_\theta\,. \tag{1.4.10}$$

From the constraint equations (1.4.5) and integrating equations (1.4.7) and (1.4.8) once, we have

$$\mu_1 = -mR\dot{\theta}\cos\varphi + A\,,$$

$$\mu_2 = -mR\dot{\theta}\sin\varphi + B\,,$$

where A and B are integration constants. Substituting these into equations (1.4.9) and (1.4.10) and simplifying, we obtain

$$J\ddot{\varphi} = R\dot{\theta}(A\sin\varphi - B\cos\varphi) + u_\varphi\,,$$
$$(I + mR^2)\ddot{\theta} = R\dot{\varphi}(-A\sin\varphi + B\cos\varphi) + u_\theta\,.$$

These equations, together with the constraints, define the dynamics. Notice that for nonzero A and B, *they are different from the dynamic nonholonomic (Lagrange–d'Alembert) equations.* As we have indicated, the motion determined by these equations is not that associated with physical dynamics in general, but is a model of the type of problem that is relevant to optimal control problems, as we shall see later.

Note also that the constants of motion A and B are *not determined* by the constraints or initial data. Thus in this instance there are many variational nonholonomic trajectories with a given set of initial conditions; the choice of $A = B = 0$ yields the nonholonomic (i.e., the Lagrange–d'Alembert) case. Interestingly, it is not always true that the nonholonomic trajectories are special cases of the variational nonholonomic trajectories, but it is possible to quantify when this occurs; see, e.g., Cardin and Favretti [1996].

More details on this issue may be found in Fernandez and Bloch [2008] where necessary and sufficient conditions for the equivalence of the dynamics of nonholonomic mechanics and variational nonholonomic (vakonomic) dynamics for certain initial conditions are given. In this work the notion of ***conditionally variational nonholonomic systems*** is developed. For such systems for any given initial data there exists a value of the Lagrange multiplier for the variational nonholonomic system such that the trajectories of the two types of system coincide. Similarly if the result only holds for some initial data the system is said to be ***partially conditionally variational nonholonomic***.

Exercises

◇ **1.4-1.** Write down an expression for the energy of the (dynamic nonholonomic) vertical rolling disk and compute its time rate of change under the action of the controls u_φ and u_θ.

◇ **1.4-2.** Compute the dynamic nonholonomic and variational nonholonomic equations of motion of the upright rolling penny in the presence of a linear potential of the form $V(x, y, \varphi, \theta) = \alpha x$ for a real number α. Solve the equations if possible.

1.5 The Falling Rolling Disk

A more realistic disk is of course one that is allowed to fall over (i.e., it is permitted to deviate from the vertical). This turns out to be a very instructive example to analyze. See Figure 1.5.1. As the figure indicates, we denote the coordinates of contact of the disk in the xy-plane by (x, y) and let θ, φ, and ψ denote the angle between the plane of the disk and the vertical axis, the "heading angle" of the disk, and "self-rotation" angle of the disk, respectively.[11] Note that the notation ψ for the falling rolling disk corresponds to the notation θ in the special case of the vertical rolling disk.

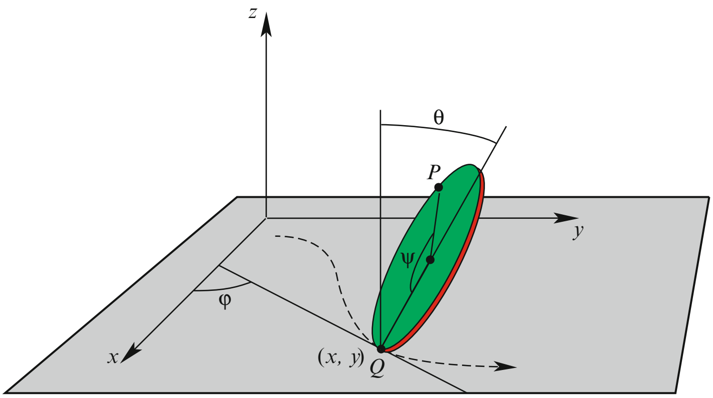

FIGURE 1.5.1. The geometry for the rolling disk.

For the moment, we just give the Lagrangian and constraints, and return to this example in Chapter 8, where we work things out in detail. While the equations of motion are straightforward to develop, as in the vertical case, they are somewhat messy, so we will defer these calculations until the later discussion. We will also show in Chapter 8 that this is a system that exhibits stability but not asymptotic stability.

Denote the mass and radius of the disk by m and R, respectively; let I be, as in the case of the vertical rolling disk, the moment of inertia about

[11] A classical reference for the rolling disk is Vierkandt [1892], who showed something very interesting: On an appropriate symmetry-reduced space, namely, the constrained velocity phase space modulo the action of the group of Euclidean motions of the plane, all orbits of the system are periodic. Modern references that treat this example are Hermans [1995], O'Reilly [1996], Cushman, Hermans, and Kemppainen [1996], and Zenkov, Bloch, and Marsden [1998].

the axis through the disk's "axle" and J the moment of inertia about any diameter. The Lagrangian is given by the kinetic minus potential energies:

$$L = \frac{m}{2}\left[(\xi - R(\dot\varphi\sin\theta + \dot\psi))^2 + \eta^2\sin^2\theta + (\eta\cos\theta + R\dot\theta)^2\right] + \frac{1}{2}\left[J(\dot\theta^2 + \dot\varphi^2\cos^2\theta) + I(\dot\varphi\sin\theta + \dot\psi)^2\right] - mgR\cos\theta,$$

where $\xi = \dot x\cos\varphi + \dot y\sin\varphi + R\dot\psi$ and $\eta = -\dot x\sin\varphi + \dot y\cos\varphi$, while the constraints are given by

$$\dot x = -\dot\psi R\cos\varphi,$$
$$\dot y = -\dot\psi R\sin\varphi.$$

Note that the constraints may also be written as $\xi = 0$, $\eta = 0$.

Unicycle with Rotor. An interesting generalization of the falling disk is the "unicycle with rotor," analyzed in Zenkov, Bloch, and Marsden [2002b], (see Figure 1.5.2).

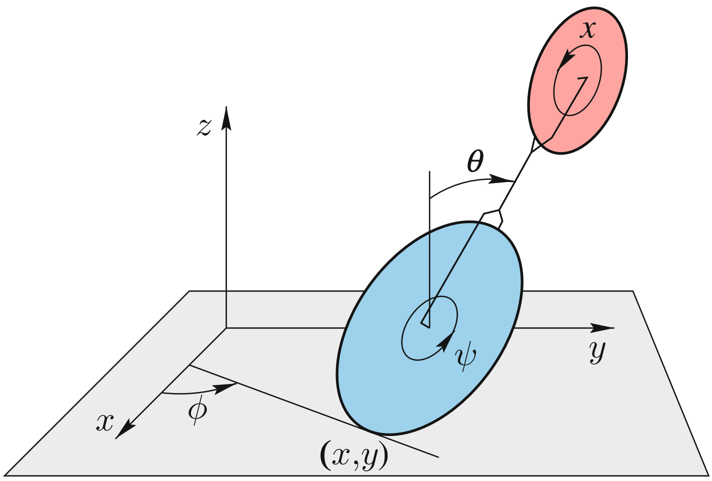

FIGURE 1.5.2. The configuration variables for the unicycle with rotor.

This is a homogeneous disk on a horizontal plane with a rotor. The rotor is free to rotate in the plane orthogonal to the disk. The rod connecting the centers of the disk and rotor keeps the direction of the radius of the disk through the contact point with the plane. We may view this system as a simple model of unicycle with rider whose arms are represented by the rotor. Stabilization is discussed in Chapter 9. A unicycle with pendulum is discussed in Zenkov, Bloch, and Marsden [2002b] and the web supplement.

The configuration space for this system is $Q = S^1 \times S^1 \times S^1 \times \mathrm{SE}(2)$, which we parameterize with coordinates $(\theta, \chi, \psi, \phi, x, y)$. As in Figure 1.5.2,

θ is the tilt of the unicycle itself, and ψ and χ are the angular positions of the wheel of the unicycle and the rotor, respectively. The variables (ϕ, x, y), regarded as a point in SE(2), represent the angular orientation of the overall system and position of the point of contact of the wheel with the ground.

Further details are given in Chapter 9.

1.6 The Knife Edge

A simple and basic example of the behavior of a system with nonholonomic constraints is a knife edge or skate on an inclined plane.[12]

To set up the problem, consider a plane slanted at an angle α from the horizontal and let (x, y) represent the position of the point of contact of the knife edge with respect to a fixed Cartesian coordinate system on the plane (see Figure 1.6.1). The angle φ represents the orientation of the knife edge with respect to the xy-axis. The knife edge is moving under the influence of gravity with the acceleration due to gravity denoted by g. It also has mass m, and the moment of inertia of the knife edge about a vertical axis through its contact point is denoted by J.

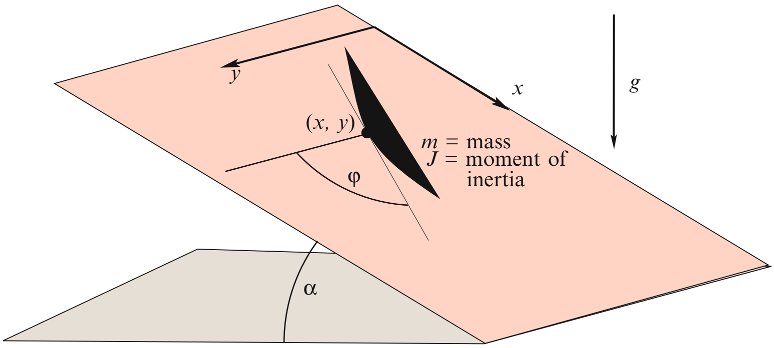

FIGURE 1.6.1. Motion of a knife edge on an inclined plane.

With this notation, the ***knife edge Lagrangian*** is taken to be

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) + \frac{1}{2}J\dot{\varphi}^2 + mgx\sin\alpha \tag{1.6.1}$$

with the constraint

$$\dot{x}\sin\varphi = \dot{y}\cos\varphi\,. \tag{1.6.2}$$

[12]This example is analyzed in, for example, Neimark and Fufaev [1972] and Arnold, Kozlov, and Neishtadt [1988].

As for the rolling penny, we will compare the mechanical nonholonomic equations and the variational equations. In contrast to the penny we cannot solve the equations explicitly in general, but this is possible for certain initial data of interest. In particular, we shall be concerned with the initial data corresponding to the knife edge spinning about a point on the plane with zero initial velocity along the plane. The question is, what is the motion of the point of contact? We shall not consider the addition of controls for the moment.

The Nonholonomic Case. The equations of motion given in general by (1.3.2) become, in this case,

$$
\begin{aligned}
m\ddot{x} &= \lambda \sin\varphi + mg\sin\alpha\,,\\
m\ddot{y} &= -\lambda\cos\varphi\,,\\
J\ddot{\varphi} &= 0\,.
\end{aligned}
$$

We assume the initial data $x(0) = \dot{x}(0) = y(0) = \dot{y}(0) = \varphi(0) = 0$ and $\dot{\varphi}(0) = \omega$. The energy is given, according to the general formula (1.3.5), by

$$
E = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) + \frac{1}{2}J\dot{\varphi}^2 - mgx\sin\alpha
$$

and is preserved along the flow. Since it is preserved, it equals its initial value

$$
E(0) = \frac{1}{2}J\omega^2\,.
$$

Hence, we have

$$
\frac{1}{2}\frac{\dot{x}^2}{\cos^2\varphi} - mgx\sin\alpha = 0\,.
$$

Solving, we obtain

$$
x = \frac{g}{2\omega^2}\sin\alpha\sin^2\omega t
$$

and, using the constraint,

$$
y = \frac{g}{2\omega^2}\sin\alpha\left(\omega t - \frac{1}{2}\sin 2\omega t\right)\,.
$$

Hence the point of contact of the knife edge undergoes a cycloid motion along the plane, but does not slide down the plane.

The Variational Nonholonomic Case. Now we consider, in contrast, the variational nonholonomic equations of motion. We consider the constrained Lagrangian

$$
L_C = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) + \frac{1}{2}J\dot{\varphi}^2 + mgx\sin\alpha - \lambda\left(\dot{x}\sin\varphi - \dot{y}\cos\varphi\right)\,.
$$

As in the general theory, define the momenta by $p_i = \partial L/\partial \dot{q}^i$, which becomes, in this case,

$$p_x = \frac{\partial L_C}{\partial \dot{x}} = m\dot{x} - \lambda \sin\varphi\,,$$
$$p_y = \frac{\partial L_C}{\partial \dot{y}} = m\dot{y} + \lambda \cos\varphi\,.$$

Now assume initial data satisfying $p_x(0) = p_y(0) = \varphi(0) = \dot{\varphi}(0) = 0$. Then from Lagrange's equations, we get

$$\dot{p_x} = mg\sin\alpha\,,$$
$$\dot{p_y} = 0\,,$$

and hence from the initial data

$$p_x = (mg\sin\alpha)t\,,$$
$$m\dot{y} + \lambda\cos\varphi = 0\,.$$

Now the equation for $\dot{\varphi}$ is

$$J\ddot{\varphi} = -\lambda\dot{x}\cos\varphi - \lambda\dot{y}\sin\varphi = -(\lambda g\sin\alpha\cos\varphi)t\,,$$

using the above expressions for p_x and p_y to solve for $\dot{x}$ and $\dot{y}$.

Again using the expressions for p_x and p_y we have

$$\lambda = p_y\cos\varphi - p_x\sin\varphi = -(mg\sin\alpha\sin\varphi)t\,.$$

Using this expression for λ gives

$$\begin{aligned}\dot{x} &= (g\sin\alpha)t + \lambda/m\sin\varphi = (g\sin\alpha)t - (g\sin\alpha\sin^2\varphi)t \\ &= (mg\sin\alpha\cos^2\varphi)t\,, \\ \dot{y} &= -\lambda/m\cos\varphi = (g\sin\alpha\sin\varphi\cos\varphi)t\,, \\ \ddot{\varphi} &= \left(\frac{m}{J}g^2\sin^2\alpha\sin\varphi\cos\varphi\right)t^2\,.\end{aligned}$$

Hence, in the variational formulation the point of contact of the knife edge slides monotonically down the plane, in contrast to the nonholonomic mechanical setting (see, e.g., Kozlov [1983]).

1.7 The Chaplygin Sleigh

One of the simplest mechanical systems that illustrates the possible "dissipative nature" of nonholonomic systems, even though they are energy-preserving, is the Chaplygin sleigh.[13]

[13]The system is discussed in the original work of Chaplygin (see the references) as well as in Neimark and Fufaev [1972] and Ruina [1998].

We now derive the equations of motion both using balance of forces as in Ruina [1998] and by the Lagrange multiplier approach, following the general theory. This system consists of a rigid body in the plane that is supported at three points, two of which slide freely without friction while the third is a knife edge, a constraint that allows no motion perpendicular to its edge.

To analyze the system, use a coordinate system Oxy fixed in the plane and a coordinate system $A\xi\eta$ fixed in the body with its origin at the point of support of the knife edge and the axis $A\xi$ through the center of mass C of the rigid body. The configuration of the body is described by the coordinates x, y and the angle θ between the moving and fixed sets of axes. Let m be the mass and I the moment of inertia about the center of mass. Let a be the distance from A to C. See Figure 1.7.1.

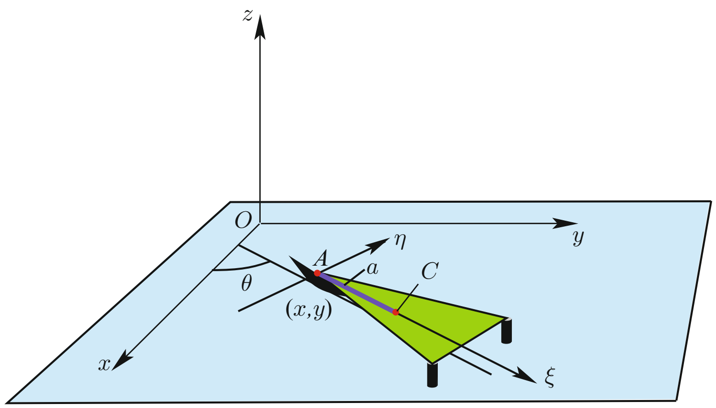

FIGURE 1.7.1. The Chaplygin sleigh is a rigid body moving on two sliding posts and one knife edge.

Denote the unit vectors along the axes $A\xi$ and $A\eta$ in the body by $\mathbf{e}_1$ and $\mathbf{e}_2$. The knife edge constraint can then be expressed as follows: The velocity at A is given by $\mathbf{v}_A = v_1\mathbf{e}_1$, where v_1 is the velocity in the direction $\mathbf{e}_1$.

The force at A is written as $R\mathbf{e}_2$; that is, the force is normal to the direction of motion at A. The position of point C is $\mathbf{r}_C = \mathbf{r}_A + a\mathbf{e}_1$, where the vectors $\mathbf{r}$ are in the fixed frame.

Since $\dot{\mathbf{e}}_1 = \dot{\theta}\mathbf{e}_2$ and $\dot{\mathbf{e}}_2 = -\dot{\theta}\mathbf{e}_1$, the velocity and acceleration of the point C are given by

$$
\begin{aligned}
\mathbf{v}_C &= v\mathbf{e}_1 + \dot{\theta}a\mathbf{e}_2\,, \\
\mathbf{a}_C &= \dot{v}\mathbf{e}_1 + v\dot{\theta}\mathbf{e}_2 + \ddot{\theta}a\mathbf{e}_2 - \dot{\theta}^2 a\mathbf{e}_1.
\end{aligned}
\tag{1.7.1}
$$

The balance of linear and angular momentum at the point A then gives

$$\begin{aligned} R\mathbf{e}_2 &= m\mathbf{a}_C\,, \\ 0 &= (\mathbf{r}_C - \mathbf{r}_A) \times (m\mathbf{a}_C) + I\ddot{\theta}\mathbf{e}_3, \end{aligned} \tag{1.7.2}$$

where $\mathbf{e}_3$ is the normal vector to the plane. Setting $\dot{\theta} = \omega$ we find that equations (1.7.1), (1.7.2) yield the equations

$$\begin{aligned} \dot{v} &= a\omega^2\,, \\ \dot{\omega} &= -\frac{ma}{I + ma^2} v\omega\,. \end{aligned} \tag{1.7.3}$$

The equations above are examples of ***momentum equations*** in nonholonomic mechanics, which we shall study in general in Chapter 5 and which will play an important role in the book. In the absence of nonholonomic constraints, this equation would yield conservation of angular momentum.

This set of equations has a family of equilibria (i.e., points at which the right-hand side vanishes) given by $\{(v, \omega) \mid v = \text{const},\, \omega = 0\}$.

Linearizing about any of these equilibria one finds that one has one zero eigenvalue together with a negative eigenvalue if $v > 0$ and a positive eigenvalue if $v < 0$. In fact, the solution curves are ellipses in the $v\omega$ plane with the positive v-axis attracting all solutions; see below. (See Figure 1.7.2.) We shall discuss this further in Section 8.6.

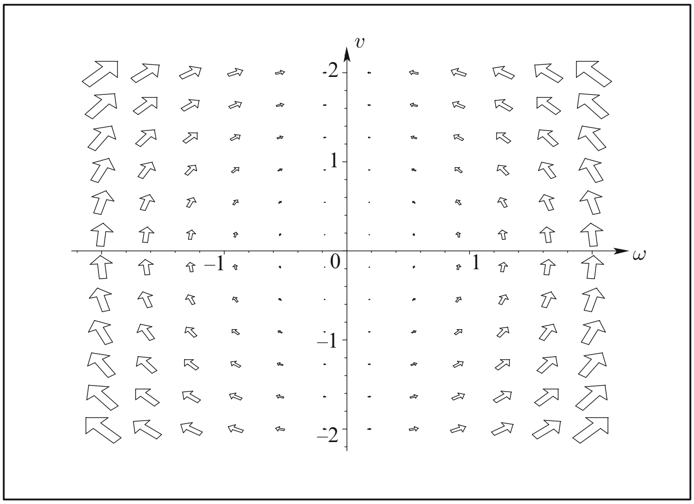

FIGURE 1.7.2. Chaplygin sleigh phase portrait.

We can also derive the equations from the Lagrangian (Lagrange–d'Alembert) point of view. The Lagrangian is given by

$$L(x_C, y_C, \theta) = \frac{1}{2}m\left(\dot{x}_C^2 + \dot{y}_C^2\right) + \frac{1}{2}I\dot{\theta}^2\,,$$

where x_C and y_C are the coordinates of the center of mass. We rewrite this in terms of the coordinates of the knife edge $x = x_C - a\cos\theta$ and $y = y_C - a\sin\theta$. Hence we may rewrite the Lagrangian as

$$\begin{aligned} L(x,y,\theta) &= \frac{1}{2}m\left(\frac{d}{dt}(x+a\cos\theta)^2 + \frac{d}{dt}(y+a\sin\theta)^2\right) + \frac{1}{2}I\dot{\theta}^2 \\ &= \frac{1}{2}m\left(\left(\dot{x} - a\sin\theta\dot{\theta}\right)^2 + \left(\dot{y} + a\cos\theta\dot{\theta}\right)^2\right) + \frac{1}{2}I\dot{\theta}^2 \\ &= \frac{1}{2}\left(m\dot{x}^2 + m\dot{y}^2 + \left(I+ma^2\right)\dot{\theta}^2 - 2ma\dot{x}\dot{\theta}\sin\theta + 2ma\dot{y}\dot{\theta}\cos\theta\right). \end{aligned} \tag{1.7.4}$$

The knife edge constraint is

$$\dot{y}\cos\theta - \dot{x}\sin\theta = 0\,. \tag{1.7.5}$$

Hence the nonholonomic equations of motion are

$$\begin{aligned} m\frac{d}{dt}\left(\dot{x} - a\sin\theta\dot{\theta}\right) &= -\lambda\sin\theta\,, \\ m\frac{d}{dt}\left(\dot{y} + a\cos\theta\dot{\theta}\right) &= \lambda\cos\theta\,, \\ \frac{d}{dt}\left(I\dot{\theta} + ma^2\dot{\theta} - ma\dot{x}\sin\theta + ma\dot{y}\cos\theta\right) & \\ -\left(-ma\dot{x}\dot{\theta}\cos\theta - ma\dot{y}\dot{\theta}\sin\theta\right) &= 0; \end{aligned}$$

that is,

$$\begin{aligned} \ddot{x} - a\cos\theta\dot{\theta}^2 - a\sin\theta\ddot{\theta} &= -\frac{\lambda\sin\theta}{m}\,, \\ \ddot{y} - a\sin\theta\dot{\theta}^2 + a\cos\theta\ddot{\theta} &= \frac{\lambda\cos\theta}{m}\,, \\ (I+ma^2)\ddot{\theta} + ma\dot{\theta}\left(\dot{x}\cos\theta + \dot{y}\sin\theta\right) &= 0\,, \end{aligned} \tag{1.7.6}$$

where in the third of equations (1.7.6) we used the constraint (1.7.5).

Now the velocity in the direction of motion is given by

$$v = \dot{x}\cos\theta + \dot{y}\sin\theta\,. \tag{1.7.7}$$

Hence the last of equations (1.7.6) becomes

$$\ddot{\theta} = \dot{\omega} = -\frac{ma}{I+ma^2}v\omega \tag{1.7.8}$$

and

$$\begin{aligned} \dot{v} &= \ddot{x}\cos\theta + \ddot{y}\sin\theta - \dot{x}\dot{\theta}\sin\theta + \dot{y}\dot{\theta}\cos\theta \\ &= a(\cos^2\theta + \sin^2\theta)\dot{\theta}^2 = a\dot{\theta}^2 = a\omega^2\,. \end{aligned} \tag{1.7.9}$$

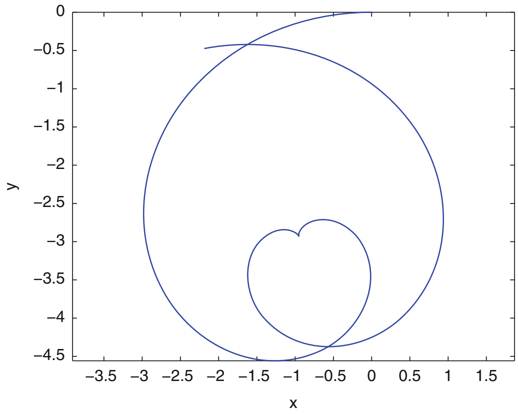

FIGURE 1.7.3. Chaplygin sleigh trajectory.

Thus we obtain our earlier sleigh equations (1.7.3).

We remark that in Ruina [1998] a piecewise holonomic version of the sleigh is discussed, where the knife edge constraint is replaced by a moving peg in a slot. This also exhibits asymptotic stability and illustrates aspects of mechanical locomotion. See also Coleman and Holmes [1999]. This phenomenon may also be seen in passive walking machines such as those of McGeer, as Ruina discusses.

Exercises

◇ **1.7-1.**

(a) Compute the nonholonomic equations of motion for the Chaplygin sleigh on an incline.

(b) **Project.** Simulate the equations on the computer and discuss the nature of the dynamics; is the sleigh stable going down the incline "forwards" or "backwards"?

◇ **1.7-2.** Compute the variational equations of motion of the Chaplygin sleigh. Say what you can about the qualitative behavior of the system.

1.8 The Heisenberg System

The Heisenberg Algebra. The Heisenberg algebra is the algebra one meets in quantum mechanics, wherein one has two operators q and p that have a nontrivial commutator, in this case a multiple of the identity. Thereby, one generates a three-dimensional Lie algebra. The system studied in this section has an associated Lie algebra with a similar structure, which is the reason the system is called the Heisenberg system. There is no intended relation to quantum mechanics per se other than this.

In Lie algebra theory this sort of a Lie algebra is of considerable interest. One refers to it as an example of a ***central extension*** because the element that one extends by (in this case a multiple of the identity) is in the center of the algebra; that is, it commutes with all elements of the algebra.

The Heisenberg system has played a significant role as an example in both nonlinear control and nonholonomic mechanics.

The Dynamic Heisenberg System. As with the previous example, the dynamic Heisenberg system comes in two forms, one associated with the Lagrange–d'Alembert principle and one with an optimal control problem. As in the previous examples, the equations in each case are different.

In the dynamic setting, we consider the following standard kinetic energy Lagrangian on Euclidean three-space $\mathbb{R}^3$:

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

subject to the constraint

$$\dot{z} = y\dot{x} - x\dot{y}. \tag{1.8.1}$$

Controls u_1 and u_2 are given in the x and y directions. Letting $q = (x, y, z)^T$, the dynamic nonholonomic control system is[14]

$$\ddot{q} = u_1 X_1 + u_2 X_2 + \lambda W\,, \tag{1.8.2}$$

where $X_1 = (1, 0, 0)^T$ and $X_2 = (0, 1, 0)^T$ and $W = (-y, x, 1)^T$. Eliminating λ we obtain the dynamic equations

$$\begin{aligned}
(1 + x^2 + y^2)\ddot{x} &= (1 + x^2)u_1 + xyu_2\,,\\
(1 + x^2 + y^2)\ddot{y} &= (1 + y^2)u_2 + xyu_1\,,\\
(1 + x^2 + y^2)\ddot{z} &= yu_1 - xu_2\,.
\end{aligned} \tag{1.8.3}$$

Optimal Control for the Heisenberg System. The control and optimal control of the corresponding kinematic problem have been quite important historically, and we shall return to them later on in the book in connection with, for example, the falling cat problem and optimal steering problems.[15]

The system may be written as

$$\dot{q} = u_1 g_1 + u_2 g_2\,, \tag{1.8.4}$$

[14]This example with controls was analyzed in Bloch and Crouch [1993]. A related nonholonomic system, but with slightly different constraints, may be found in Rosenberg [1977], Bates and Sniatycki [1993], and Bloch, Krishnaprasad, Marsden, and Murray [1996].

[15]As we mentioned earlier, this example was introduced in Brockett [1981].

where $g_1 = (1, 0, y)^T$ and $g_2 = (0, 1, -x)^T$. As in the rolling disk example, g_1 and g_2 are a maximal set of independent vector fields satisfying the constraint

$$\dot{z} = y\dot{x} - x\dot{y}\,. \tag{1.8.5}$$

Written out in full, these equations are

$$\dot{x} = u_1\,, \tag{1.8.6}$$
$$\dot{y} = u_2\,, \tag{1.8.7}$$
$$\dot{z} = yu_1 - xu_2\,. \tag{1.8.8}$$

Relationship Between the Vertical Disk and Heisenberg System. Consider the vertical disk example, but eliminate ψ from the representation of the configuration (see, e.g., Lynch, Bloch, Drakunov, Reyhanoglu, Zenkov [2011]). The system can be written as

$$\begin{aligned} \dot{x} &= vR\cos\varphi, \\ \dot{y} &= vR\sin\varphi, \\ \dot{\varphi} &= \omega, \end{aligned} \tag{1.8.9}$$

where the forward velocity and heading velocity controls are v and ω, respectively. We can define a change of coordinates $F(\varphi)$

$$\begin{bmatrix} x_1 \\ x_2 \\ z \end{bmatrix} = F(\varphi) \begin{bmatrix} x \\ y \\ \varphi \end{bmatrix},$$

where

$$F(\varphi) = \begin{bmatrix} 0 & 0 & 1 \\ \cos\varphi & \sin\varphi & 0 \\ \varphi\cos\varphi - 2\sin\varphi & \varphi\sin\varphi + 2\cos\varphi & 0 \end{bmatrix},$$

and a nonsingular state-dependent transformation of the controls

$$\begin{aligned} u_1 &= \omega, \\ u_2 &= Rv + \left(\frac{z}{2} - \frac{x_1x_2}{2}\right)\omega, \end{aligned}$$

yielding the system

$$\dot{x}_1 = u_1, \tag{1.8.10}$$
$$\dot{x}_2 = u_2, \tag{1.8.11}$$
$$\dot{z} = x_1u_2 - x_2u_1. \tag{1.8.12}$$

Aside on the Jacobi–Lie Bracket. A notion that is important in mechanics and control theory is that of the ***Jacobi–Lie bracket*** $[f, g]$ of two vector fields f and g on $\mathbb{R}^n$ that are given in components by $f = (f^1, \ldots, f^n)$ and $g = (g^1, \ldots, g^n)$. It is defined to be the vector field with components

$$[f, g]^i = \sum_{j=1}^{n} \left(f^j \frac{\partial g^i}{\partial x^j} - g^j \frac{\partial f^i}{\partial x^j} \right),$$

or in vector calculus notation

$$[f, g] = (f \cdot \nabla)\, g - (g \cdot \nabla)\, f.$$

Later on, in Chapter 2, we will define the Jacobi–Lie bracket intrinsically on manifolds. An important geometric interpretation of this bracket is as follows.

Suppose we follow the vector field g (i.e., flow along the solution of the equation $\dot{x} = g(x)$) from point $x(0) = x_0$ for t units of time, then beginning with this as initial condition, we flow along the vector field f for time t; then along the vector field $-g$, and finally along $-f$ all for t units of time. Formally, we arrive at the point

$$(\exp -tf)(\exp -tg)(\exp tf)(\exp tg)(x_0)\,, \tag{1.8.13}$$

where $(\exp tg)$ represents the flow of the vector field g for t units of time. Flows of vector fields will be described in more detail in Chapter 2.

Locally, expanding the exponential and in turn expanding each occurrence of g in the exponential in a Taylor series about x_0 we have along the flow of the equation $\dot{x} = g(x)$,

$$x(t) = x_0 + tg(x_0) + \frac{t^2}{2} g(x_0) \cdot \nabla g(x_0) + O(t^3)\,. \tag{1.8.14}$$

(Here we compute the second derivative of $x(t)$ at $t = t_0$ by differentiating $\dot{x}(t) = g(x(t))$ with respect to t.) Hence, after a short computation using additional Taylor expansions, one finds that (1.8.13) becomes

$$x_0 - t^2\, [f, g]\, (x_0) x_0 + O(t^3)\,, \tag{1.8.15}$$

where $[f, g]$ is the Lie bracket as defined above. Thus, if $[f, g]$ is not in the span of vector fields g and f, then by concatenating the flows of f and g, we obtain motion in a new independent direction.

Return to the Heisenberg System. In the Heisenberg example, one verifies that the Jacobi–Lie bracket of the vector fields g_1 and g_2 is

$$[g_1, g_2] = 2g_3\,,$$

where $g_3 = (0, 0, 1)$. In fact, the three vector fields g_1, g_2, g_3 span all of $\mathbb{R}^3$ and, as a Lie algebra, is just the Heisenberg algebra described earlier.

By general controllability theorems that we shall discuss in Chapter 4 (Chow's theorem), one knows now that one can, with suitable controls, steer trajectories between any two points in $\mathbb{R}^3$. The above geometric interpretation makes this plausible. In particular, we are interested in the following optimal steering problem (see Figure 1.8.1).

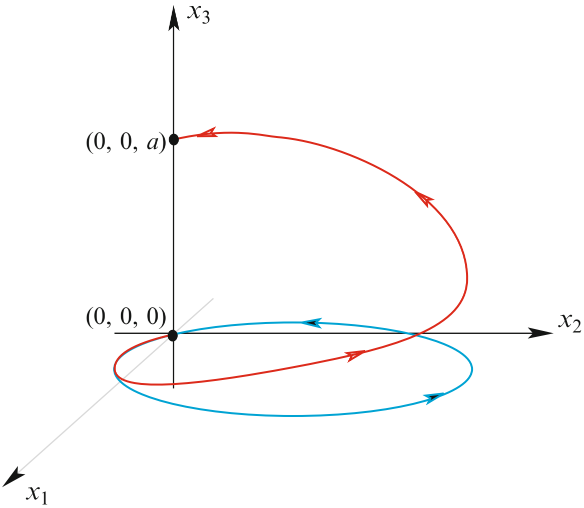

FIGURE 1.8.1. An optimal steering problem.

Optimal Steering Problem. Given a number $a > 0$, find time-dependent controls u_1, u_2 that steer the trajectory starting at $(0, 0, 0)$ at time $t = 0$ to the point $(0, 0, a)$ after a given time $T > 0$ and that among all such controls minimizes

$$\frac{1}{2}\int_0^T (u_1^2 + u_2^2)\, dt\,.$$

An equivalent formulation is the following: Minimize the integral

$$\frac{1}{2}\int_0^T (\dot{x}^2 + \dot{y}^2)\, dt$$

among all curves $q(t)$ joining $q(0) = (0, 0, 0)$ to $q(T) = (0, 0, a)$ that satisfy the constraint

$$\dot{z} = y\dot{x} - x\dot{y}\,.$$

As before, any solution must satisfy the Euler–Lagrange equations for the Lagrangian with a Lagrange multiplier inserted:

$$L\left(x, \dot{x}, y, \dot{y}, z, \dot{z}, \lambda, \dot{\lambda}\right) = \tfrac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) + \lambda\left(\dot{z} - y\dot{x} + x\dot{y}\right)\,.$$

The corresponding Euler–Lagrange equations are given by

$$\ddot{x} - 2\lambda\dot{y} = 0\,, \tag{1.8.16}$$

$$\ddot{y} + 2\lambda\dot{x} = 0\,, \tag{1.8.17}$$

$$\dot{\lambda} = 0\,. \tag{1.8.18}$$

From the third equation λ is a constant, and the first two equations state that *the particle* $(x(t), y(t))$ *moves in the plane in a constant magnetic field (pointing in the* z *direction, with charge proportional to the constant* λ*).* For more on these ideas, see Chapter 7 on optimal control.

Some remarks are in order here:

1. The fact that this optimal steering problem gives rise to an interesting mechanical system is not an accident; we shall see this in much more generality in Chapter 7 and the Internet Supplement.

2. Since particles in constant magnetic fields move in circles with constant speed, they have a sinusoidal time dependence, and hence so do the controls. This has led to the "steering by sinusoids" approach in many nonholonomic steering problems (see, for example, Murray and Sastry [1993] and Section 6.1).

Equations (1.8.16) and (1.8.17) are linear first-order equations in the velocities and are readily solved:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix} = \begin{bmatrix} \cos(2\lambda t) & \sin(2\lambda t) \\ -\sin(2\lambda t) & \cos(2\lambda t) \end{bmatrix} \begin{bmatrix} \dot{x}(0) \\ \dot{y}(0) \end{bmatrix}. \tag{1.8.19}$$

Integrating once more and using the initial conditions $x(0) = 0$, $y(0) = 0$ gives

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \frac{1}{2\lambda} \begin{bmatrix} \cos(2\lambda t) - 1 & \sin(2\lambda t) \\ -\sin(2\lambda t) & \cos(2\lambda t) - 1 \end{bmatrix} \begin{bmatrix} -\dot{y}(0) \\ \dot{x}(0) \end{bmatrix}. \tag{1.8.20}$$

The other boundary condition $x(T) = 0$, $y(T) = 0$ gives

$$\lambda = \frac{n\pi}{T}$$

for an integer n. Using this information, we find z by integration: From $\dot{z} = y\dot{x} - x\dot{y}$ and the preceding expressions we get

$$\dot{z}(t) = \frac{T}{2n\pi} \left[\dot{x}(0)^2 + \dot{y}(0)^2 - \cos\left(\frac{2n\pi t}{T}\right) (\dot{x}(0)^2 + \dot{y}(0)^2) \right].$$

Integration from 0 to T and using $z(0) = 0$ gives

$$z(T) = \frac{T^2}{2n\pi} \left[\dot{x}(0)^2 + \dot{y}(0)^2 \right].$$

Thus, to achieve the boundary condition $z(T) = a$ one must choose

$$\dot{x}(0)^2 + \dot{y}(0)^2 = \frac{2\pi n a}{T^2}.$$

One also finds that

$$\begin{aligned}\frac{1}{2}\int_0^T \left[\dot{x}(t)^2 + \dot{y}(t)^2\right] dt &= \frac{1}{2}\int_0^T \left[\dot{x}(0)^2 + \dot{y}(0)^2\right] dt \\ &= \frac{T}{2}\left[\dot{x}(0)^2 + \dot{y}(0)^2\right] \\ &= \frac{\pi n a}{T},\end{aligned}$$

so that the minimum is achieved when $n = 1$.

Summary: The solution of the optimal control problem is given by choosing initial conditions such that $\dot{x}(0)^2 + \dot{y}(0)^2 = 2\pi a/T^2$ and with the trajectory in the xy-plane given by the circle

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \frac{1}{2\lambda}\begin{bmatrix} \cos(2\pi t/T) - 1 & \sin(2\pi t/T) \\ -\sin(2\pi t/T) & \cos(2\pi t/T) - 1 \end{bmatrix}\begin{bmatrix} -\dot{y}(0) \\ \dot{x}(0) \end{bmatrix} \tag{1.8.21}$$

and with z given by

$$z(t) = \frac{ta}{T} - \frac{a}{2\pi}\sin\left(\frac{2\pi t}{T}\right).$$

Notice that any such solution can be rotated about the z axis to obtain another one.

Exercises

◇ **1.8-1.** Solve the optimal steering problem for the vertical disk problem (1.8.9) with cost function $\frac{1}{2}(v^2 + \omega^2)$.

◇ **1.8-2.** For the standard kinetic energy Lagrangian on $\mathbb{R}^3$ and constraint (1.8.1) above, write down the variational nonholonomic problem. How does this compare with the optimal steering problem?

1.9 The Rigid Body

The Free Rigid Body. A key system in mechanics is the free rigid body. There are many excellent treatments of this topic; see, for example, Whittaker [1988], Arnold [1989], and Marsden and Ratiu [1999]. We restrict ourselves here to some essentials, although we shall return to the topic in detail in the context of nonholonomic mechanics and optimal control.

The configuration space of a rigid body moving freely in space is $\mathbb{R}^3 \times \mathrm{SO}(3)$, describing the position of a coordinate frame fixed in the body and the orientation of the frame, the orientation of the frame given by an element of $\mathrm{SO}(3)$, i.e., an orthogonal 3×3 matrix with determinant 1. Since the three components of translational momentum are conserved, the body behaves as if it were rotating freely about its center of mass.[16]

Hence the phase space for the body may be taken to be $T\,\mathrm{SO}(3)$—the tangent bundle of $\mathrm{SO}(3)$—with points representing the position and velocity of the body, or in the Hamiltonian context we may choose the phase space to be the cotangent bundle $T^*\,\mathrm{SO}(3)$, with points representing the position and momentum of the body. (This example may be equally well formulated for the group $\mathrm{SO}(n)$ or indeed any compact Lie group.)

If I is the moment of inertia tensor computed with respect to a body fixed frame, which, in a *principal* body frame, we may represent by the diagonal matrix $\mathrm{diag}(I_1, I_2, I_3)$, the Lagrangian of the body is given by the kinetic energy, namely

$$L = \frac{1}{2}\Omega \cdot I\Omega, \tag{1.9.1}$$

where Ω is the vector of angular velocities computed with respect to the axes fixed in the body.

The Euler–Lagrange equations of motion may be written as the system

$$\dot{A} = A\hat{\Omega}\,, \tag{1.9.2}$$

$$I\dot{\Omega} = I\Omega \times \Omega\,, \tag{1.9.3}$$

where $A \in \mathrm{SO}(3)$ and we write

$$\hat{\Omega} \equiv \begin{pmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{pmatrix}.$$

The dynamics may be rewritten

$$I\dot{\hat{\Omega}} = [I\hat{\Omega}, \hat{\Omega}], \tag{1.9.4}$$

or, in terms of the angular momentum matrix $\hat{M} = I\hat{\Omega}$,

$$\dot{\hat{M}} = [\hat{M}, \hat{\Omega}]. \tag{1.9.5}$$

The Rolling Ball. This paragraph considers the controlled rolling inhomogeneous ball on the plane, the kinematics of which were discussed in Brockett and Dai [1992], establishing the completely nonholonomic nature

[16] This is not the case with other systems, such as a rigid body moving in a fluid; even though the system is translation-invariant, its "center of mass" need not move on a straight line, so the configuration space must be taken to be the full Euclidean group.

of the constraint distribution H. (A distribution is completely nonholonomic if the span of the iterated brackets of the vector fields lying in it has dimension equal to the dimension of the underlying manifold; see Chapter 4 for a full explanation.) See also Rojo and Bloch [2010]. The dynamics of the uncontrolled system is described, for example, in McMillan [1936] (see also Bloch, Krishnaprasad, Marsden, and Murray [1996], Jurdjevic [1993], Koon and Marsden [1997b], and Krishnaprasad, Yang and Dayawansa [1991]). We will use the coordinates x, y for the linear horizontal displacement and $P \in \mathrm{SO}(3)$ for the angular displacement of the ball. Thus P gives the orientation of the ball with respect to inertial axes $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ fixed in the plane, where the $\mathbf{e}_i$ are the standard basis vectors aligned with the x-, y-, and z-axes, respectively. See Figure 1.9.1.

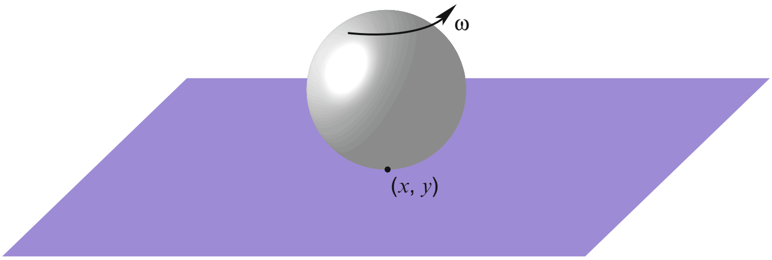

FIGURE 1.9.1. The rolling ball.

Let the ball have radius a and mass m and let $\boldsymbol{\omega} \in \mathbb{R}^3$ denote the angular velocity of the ball with respect to the inertial axes. In particular, the ball may spin freely about the z-axis, and the z-component of angular momentum is conserved. If J denotes the inertia tensor of the ball with respect to the body axes (i.e., fixed in the body), then $\mathbb{J} = P^T J P$ denotes the inertia tensor of the ball with respect to the inertial axes (i.e., fixed in space) and $\mathbb{J}\boldsymbol{\omega}$ is the angular momentum of the ball with respect to the inertial axes. The conservation law alluded to above is expressed as

$$\mathbf{e}_3^T \mathbb{J}\boldsymbol{\omega} = c\,. \tag{1.9.6}$$

The nonholonomic constraints of rolling without slipping may be expressed as

$$\begin{aligned} a\mathbf{e}_2^T \boldsymbol{\omega} + \dot{x} &= 0\,, \\ a\mathbf{e}_1^T \boldsymbol{\omega} - \dot{y} &= 0\,. \end{aligned} \tag{1.9.7}$$

We may express the kinematics for the rotating ball as $\dot{P} = \hat{\Omega} P$, where $\Omega = P\boldsymbol{\omega}$ is the angular velocity in the body frame.

Appending the constraints via Lagrange multipliers we obtain the equations of motion

$$\begin{aligned}\dot{\hat{\Omega}}P - (\widehat{J^{-1}\hat{\Omega}J\Omega})P &= a\lambda_1(\widehat{J^{-1}P\mathbf{e}_1})P + a\lambda_2(\widehat{J^{-1}P\mathbf{e}_2})P\,,\\ m\ddot{x} &= \lambda_2 + u_1\,,\\ m\ddot{y} &= -\lambda_1 + u_2\,.\end{aligned} \tag{1.9.8}$$

Using inertial coordinates $\boldsymbol{\omega} = P^T\Omega$, the system becomes

$$\begin{aligned}\dot{\boldsymbol{\omega}} &= \mathbb{J}^{-1}\hat{\boldsymbol{\omega}}\mathbb{J}\boldsymbol{\omega} + a\lambda_1\mathbb{J}^{-1}\mathbf{e}_1 + a\lambda_2\mathbb{J}^{-1}\mathbf{e}_2\,,\\ m\ddot{x} &= \lambda_2 + u_1\,,\\ m\ddot{y} &= -\lambda_1 + u_2\,,\\ \dot{P} &= P\hat{\boldsymbol{\omega}}\,.\end{aligned} \tag{1.9.9}$$

Also, from the constraints and the constants of motion we obtain the following expression for $\boldsymbol{\omega}$:

$$\boldsymbol{\omega} = \dot{x}(\alpha_2\mathbf{e}_3 - \mathbf{e}_2) + \dot{y}(\mathbf{e}_1 - \alpha_1\mathbf{e}_3) + \alpha_3\mathbf{e}_3\,,$$

where

$$\alpha_1 = \frac{\mathbf{e}_3^T\mathbb{J}\mathbf{e}_1}{a\mathbf{e}_3^T\mathbb{J}\mathbf{e}_3}\,, \quad \alpha_2 = \frac{\mathbf{e}_3^T\mathbb{J}\mathbf{e}_2}{a\mathbf{e}_3^T\mathbb{J}\mathbf{e}_3}\,, \quad \alpha_3 = \frac{c}{\mathbf{e}_3^T\mathbb{J}\mathbf{e}_3}\,. \tag{1.9.10}$$

Then the equations become

$$\begin{aligned}m\ddot{x} &= \lambda_2 + u_1\,,\\ m\ddot{y} &= -\lambda_1 + u_2\,,\\ \dot{P} &= P(\dot{x}(\alpha_2\mathbf{e}_3 - \mathbf{e}_2) + \widehat{\dot{y}(\mathbf{e}_1} - \alpha_1\mathbf{e}_3) + \alpha_3\mathbf{e}_3)\,.\end{aligned} \tag{1.9.11}$$

One can now eliminate the multipliers using the first three equations of (1.9.9) and the constraints. The resulting expressions are a little complicated in the general case (although they can be found in straightforward fashion), but become pleasingly simple in the case of a homogeneous ball, where say $J = mk^2$ (k is called the radius of gyration in the classical literature).

In the latter case, the equations of motion for ω_1 and ω_2 become simply

$$\begin{aligned}mk^2\dot{\omega}_1 &= a\lambda_1\,,\\ mk^2\dot{\omega}_2 &= a\lambda_2\,.\end{aligned} \tag{1.9.12}$$

Rewriting these equations in terms of x and y using the multipliers and substituting the resulting expressions for the λ_i into the equations of motion for x and y yields the equations

$$\begin{aligned}m\ddot{x} &= \frac{a^2}{a^2+k^2}u_1\,,\\ m\ddot{y} &= \frac{a^2}{a^2+k^2}u_2\,.\end{aligned} \tag{1.9.13}$$

A similar elimination argument works in the general nonhomogeneous case.

Note that the homogeneous ball moves under the action of external forces like a point mass located at its center but with force reduced by the ratio $a^2/(a^2+k^2)$; see also the following subsection.

A Homogeneous Ball on a Rotating Plate. A useful example is a model of a homogeneous ball on a rotating plate (see Neimark and Fufaev [1972] and Yang [1992] for the affine case and, for example, Bloch and Crouch [1992], Brockett and Dai [1992], and Jurdjevic [1993] for the linear case). As we mentioned earlier, Chaplygin [1897b, 1903] studied the motion of an *inhomogeneous* rolling ball on a fixed plane.

Let the plane rotate with constant angular velocity $\tilde{\Omega}$ about the z-axis. The configuration space of the sphere is $Q = \mathbb{R}^2 \times \mathrm{SO}(3)$, parameterized by (x, y, R), $R \in \mathrm{SO}(3)$, all measured with respect to the inertial frame. Let $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ be the angular velocity vector of the sphere measured also with respect to the inertial frame, let m be the mass of the sphere, mk^2 its inertia about any axis, and let a be its radius.

The Lagrangian of the system is

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}mk^2({\omega_1}^2 + {\omega_2}^2 + {\omega_3}^2) \tag{1.9.14}$$

with the affine nonholonomic constraints

$$\begin{aligned} \dot{x} + a\omega_2 &= -\tilde{\Omega}y, \\ \dot{y} - a\omega_1 &= \tilde{\Omega}x. \end{aligned} \tag{1.9.15}$$

Note that the Lagrangian here is a metric on Q that is bi-invariant on $\mathrm{SO}(3)$, since the ball is homogeneous. Note also that $\mathbb{R}^2 \times \mathrm{SO}(3)$ is a principal bundle over $\mathbb{R}^2$ with respect to the right $\mathrm{SO}(3)$ action on Q given by

$$(x, y, R) \mapsto (x, y, RS) \tag{1.9.16}$$

for $S \in \mathrm{SO}(3)$. The action is on the *right*, since the symmetry is a material symmetry.

A brief calculation shows that the equations of motion become

$$\begin{aligned} \ddot{x} - \frac{k^2\tilde{\Omega}}{a^2+k^2}\dot{y} &= 0, \\ \ddot{y} + \frac{k^2\tilde{\Omega}}{a^2+k^2}\dot{x} &= 0. \end{aligned} \tag{1.9.17}$$

These equations are easily integrated to show that the ball simply oscillates on the plate between two circles rather than flying off as one might expect.

Set

$$\alpha = \frac{k^2\tilde{\Omega}}{a^2+k^2}.$$

Then one can see that the equations are equivalent to

$$\dddot{x} + \alpha^2 \dot{x} = 0, \tag{1.9.18}$$

$$\dddot{y} + \alpha^2 \dot{y} = 0. \tag{1.9.19}$$

Hence

$$x = A\cos\alpha t + B\sin\alpha t + C$$

for constants A, B, C depending on the initial data, and similarly for y.

The Inverted Pendulum on a Cart. A useful classical system for testing control-theoretic ideas is the inverted pendulum on a cart, the goal being to stabilize the pendulum about the vertical using a force acting on the cart. In this book we will use this system to illustrate stabilization using the energy methods as discussed in Bloch, Marsden, and Alvarez [1997] and Bloch, Leonard, and Marsden [1997] (see Chapter 9 for further references). Here we just write down the equations of motion.

First, we compute the Lagrangian for the cart–pendulum system. Let s denote the position of the cart on the s-axis and let ϕ denote the angle of the pendulum with the upright vertical, as in Figure 1.9.2.

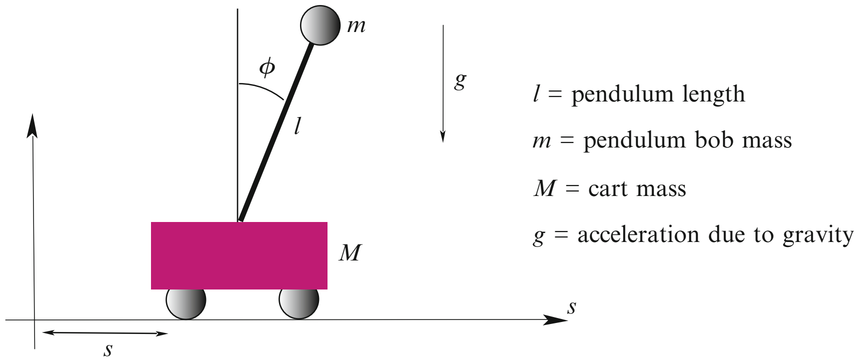

FIGURE 1.9.2. The pendulum on a cart.

Here, the configuration space is $Q = G \times S = \mathbb{R} \times S^1$ with the first factor being the cart position s, and the second factor being the pendulum angle ϕ. The velocity phase space TQ has coordinates $(s, \phi, \dot{s}, \dot{\phi})$.

The velocity of the cart relative to the lab frame is $\dot{s}$, while the velocity of the pendulum relative to the lab frame is the vector

$$v_{\text{pend}} = (\dot{s} + l\cos\phi\,\dot{\phi}, -l\sin\phi\,\dot{\phi}). \tag{1.9.20}$$

The kinetic energy of the coupled cart-pendulum system is given by

$$K\big(s, \phi, \dot{s}, \dot{\phi}\big) = \frac{1}{2}(\dot{s}, \dot{\phi})\begin{pmatrix} M+m & ml\cos\phi \\ ml\cos\phi & ml^2 \end{pmatrix}\begin{pmatrix} \dot{s} \\ \dot{\phi} \end{pmatrix}. \tag{1.9.21}$$

The Lagrangian is the kinetic minus potential energy, so we get

$$L(s, \phi, \dot{s}, \dot{\phi}) = K(s, \phi, \dot{s}, \dot{\phi}) - V(\phi), \tag{1.9.22}$$

where the potential energy is $V = mgl \cos \phi$. Note that there is a symmetry group G of the pendulum–cart system, that of translation in the s variable, so $G = \mathbb{R}$. We do not destroy this symmetry when doing stabilization in ϕ.

For convenience we rewrite the Lagrangian as

$$L(s, \phi, \dot{s}, \dot{\phi}) = \frac{1}{2}(\alpha\dot{\phi}^2 + 2\beta \cos \phi \dot{s}\dot{\phi} + \gamma \dot{s}^2) + D \cos \phi \,, \tag{1.9.23}$$

where $\alpha = ml^2, \beta = ml, \gamma = M + m$, and $D = -mgl$ are constants. Note that $\alpha\gamma - \beta^2 > 0$. The momentum conjugate to s is $p_s = \gamma\dot{s} + \beta \cos \phi \dot{\phi}$, and the momentum conjugate to ϕ is $p_\phi = \alpha\dot{\phi} + \beta \cos \phi \dot{s}$. The relative equilibrium defined by $\phi = 0$, $\dot{\phi} = 0$, and $\dot{s} = 0$ is unstable, since $D < 0$.

The equations of motion of the cart–pendulum system with a control force u acting on the cart (and no direct forces acting on the pendulum) are, since s is a cyclic variable (i.e., L is independent of s),

$$\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial \dot{s}} &= u \,, \\ \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} - \frac{\partial L}{\partial \phi} &= 0 \,, \end{aligned}$$

that is,

$$\begin{aligned} \frac{d}{dt}p_s &= \frac{d}{dt}(\gamma\dot{s} + \beta \cos \phi \dot{\theta}) = u \,, \\ \frac{d}{dt}p_\phi + \beta \sin \phi \dot{s}\dot{\phi} + D \sin \phi &= \frac{d}{dt}(\alpha\dot{\phi} + \beta \cos \phi \dot{s}) \\ &\quad + \beta \sin \phi \dot{s}\dot{\phi} + D \sin \phi = 0 \,. \end{aligned}$$

Rigid Body with a Rotor. Following the work of Krishnaprasad [1985], Bloch, Krishnaprasad, Marsden, and Alvarez [1992], and Bloch, Leonard, and Marsden[1997, 2000], we consider a rigid body with a rotor aligned along the third principal axis of the body as in Figure 1.9.3. This is a model for a satellite. The rotor spins under the influence of a torque u acting on the rotor. The configuration space is $Q = \mathrm{SO}(3) \times S^1$, with the first factor being the rigid body attitude and the second factor being the rotor angle. The Lagrangian is total kinetic energy of the system (rigid carrier plus rotor), with no potential energy.

Again, this system will be used in Section 9.2 to illustrate the energy method in analyzing stabilization and stability.

The Lagrangian for this system (see Bloch, Krishnaprasad, Marsden, and Alvarez [1992] and Bloch, Leonard, and Marsden [2001]) is

$$L = \frac{1}{2}(\lambda_1\Omega_1^2 + \lambda_2\Omega_2^2 + I_3\Omega_3^2 + J_3(\Omega_3 + \dot{\alpha})^2) \,, \tag{1.9.24}$$

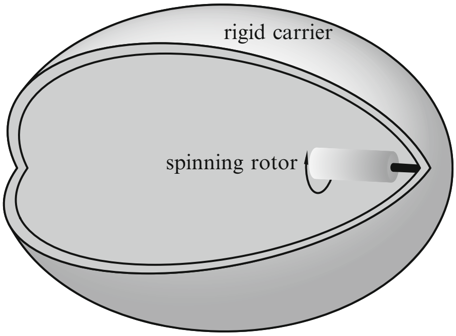

FIGURE 1.9.3. The rigid body with rotor.

where $I_1 > I_2 > I_3$ are the rigid body moments of inertia, $J_1 = J_2$ and J_3 are the rotor moments of inertia, $\lambda_i = I_i + J_i$, $\Omega = (\Omega_1, \Omega_2, \Omega_3)$ is the body angular velocity vector of the carrier, and α is the relative angle of the rotor.

The body angular momenta are determined by the Legendre transform to be

$$\begin{aligned} \Pi_1 &= \lambda_1 \Omega_1 \,, \\ \Pi_2 &= \lambda_2 \Omega_2 \,, \\ \Pi_3 &= \lambda_3 \Omega_3 + J_3 \dot{\alpha} \,, \\ l_3 &= J_3(\Omega_3 + \dot{\alpha}) \,. \end{aligned}$$

The momentum conjugate to α is l_3.

The equations of motion with a control torque u acting on the rotor are

$$\begin{aligned} \lambda_1 \dot{\Omega}_1 &= \lambda_2 \Omega_2 \Omega_3 - (\lambda_3 \Omega_3 + J_3 \dot{\alpha}) \Omega_2 \,, \\ \lambda_2 \dot{\Omega}_2 &= -\lambda_1 \Omega_1 \Omega_3 + (\lambda_3 \Omega_3 + J_3 \dot{\alpha}) \Omega_1 \,, \\ \lambda_3 \dot{\Omega}_3 + J_3 \ddot{\alpha} &= (\lambda_1 - \lambda_2) \Omega_1 \Omega_2 \,, \\ \dot{l}_3 &= u \,. \end{aligned} \tag{1.9.25}$$

The equations may also be written in Hamiltonian form:

$$\begin{aligned} \dot{\Pi}_1 &= \left(\frac{1}{I_3} - \frac{1}{\lambda_2} \right) \Pi_2 \Pi_3 - \frac{l_3 \Pi_2}{I_3} \,, \\ \dot{\Pi}_2 &= \left(\frac{1}{\lambda_1} - \frac{1}{I_3} \right) \Pi_1 \Pi_3 + \frac{l_3 \Pi_1}{I_3} \,, \\ \dot{\Pi}_3 &= \left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1} \right) \Pi_1 \Pi_2 = a_3 \Pi_1 \Pi_2 \,, \\ \dot{l}_3 &= u. \end{aligned}$$

Here $\lambda_i = I_i + J_i$.

Exercises

◇ **1.9-1.** Compute the equations of motion for the variational nonholonomic ball and compare the dynamics with the nonholonomic case.

◇ **1.9-2.** Compute the dynamics of the homogeneous ball on a *freely* rotating table. (See Weckesser [1997] and references therein.)

◇ **1.9-3.** Analyze the motion of the cart on an inclined plane making an angle of α to the horizontal. Show that with a suitable change of variable one can still find a symmetry of the motion.

1.10 The n-dimensional Rigid Body

In this section we review the classical rigid body equations in three and, more generally, in n dimensions. We shall also compare the left and right invariant equations.

For convenience we shall use the following pairing (multiple of the Killing form) on $\mathfrak{so}(n)$, the Lie algebra of $n \times n$ real skew matrices regarded as the Lie algebra of the n-dimensional proper rotation group $\mathrm{SO}(n)$:

$$\langle \xi, \eta \rangle = -\frac{1}{2}\,\mathrm{trace}(\xi\eta). \tag{1.10.1}$$

The factor of 1/2 in (1.10.1) is to make this inner product agree with the usual inner product on $\mathbb{R}^3$ when it is identified with $\mathfrak{so}(3)$ in the following standard way: associate the 3×3 skew matrix $\hat{u}$ to the vector u by $\hat{u} \cdot v = u \times v$, where $u \times v$ is the usual cross product in $\mathbb{R}^3$.

We use this inner product to identify the dual of the Lie algebra, namely $\mathfrak{so}(n)^*$, with the Lie algebra $\mathfrak{so}(n)$.

We recall from Manakov [1976] and Ratiu [1980] that the left invariant generalized rigid body equations on $\mathrm{SO}(n)$ may be written as

$$\dot{Q} = Q\Omega, \quad \dot{M} = [M, \Omega], \tag{1.10.2}$$

where $Q \in \mathrm{SO}(n)$ denotes the configuration space variable (the attitude of the body), $\Omega = Q^{-1}\dot{Q} \in \mathfrak{so}(n)$ is the body angular velocity, and

$$M := J(\Omega) = \Lambda\Omega + \Omega\Lambda \in \mathfrak{so}(n)$$

is the body angular momentum. Here $J : \mathfrak{so}(n) \to \mathfrak{so}(n)$ is the symmetric (with respect to the inner product (1.10.1)), positive definite, and hence invertible, operator defined by

$$J(\Omega) = \Lambda\Omega + \Omega\Lambda,$$

where Λ is a diagonal matrix satisfying $\Lambda_i + \Lambda_j > 0$ for all $i \neq j$. For $n = 3$ the elements of Λ_i are related to the standard diagonal moment of inertia tensor I by $I_1 = \Lambda_2 + \Lambda_3$, $I_2 = \Lambda_3 + \Lambda_1$, $I_3 = \Lambda_1 + \Lambda_2$.

The equations $\dot{M} = [M, \Omega]$ are readily checked to be the Euler–Poincaré equations on $\mathfrak{so}(n)$ for the Lagrangian $l(\Omega) = \frac{1}{2}\langle \Omega, J(\Omega)\rangle$. This corresponds to the Lagrangian on $T\,\mathrm{SO}(n)$ given by

$$L(g, \dot{g}) = \frac{1}{2}\left\langle g^{-1}\dot{g}, J(g^{-1}\dot{g})\right\rangle . \tag{1.10.3}$$

It follows from general Euler–Poincaré theory (see, for example, Marsden and Ratiu [1999]) that the equations (1.10.2) are the geodesic equations on $T\,\mathrm{SO}(n)$, left trivialized as $\mathrm{SO}(n) \times \mathfrak{s}o(n)$, relative to the left invariant metric whose expression at the identity is

$$\langle\!\langle \Omega_1, \Omega_2 \rangle\!\rangle = \langle \Omega_1, J(\Omega_2)\rangle . \tag{1.10.4}$$

According to Mishchenko and Fomenko [1978], there is a similar formalism for any semisimple Lie group and that in that context, one has integrability on the generic coadjoint orbits.

Right Invariant System. The system (1.10.2) has a right invariant counterpart. This right invariant system is given as follows. Consider the right invariant Riemannian metric on $\mathrm{SO}(n)$ whose value at the identity is given by (1.10.4). The geodesic equations of this metric on $T\,\mathrm{SO}(n)$, right trivialized as $\mathrm{SO}(n) \times \mathfrak{s}o(n)$, are given by

$$\dot{Q}_r = \Omega_r Q_r, \quad \dot{M}_r = [\Omega_r, M_r] \tag{1.10.5}$$

where in this case $\Omega_r = \dot{Q}_r Q_r^{-1}$ and $M_r = J(\Omega_r)$ where J has the same form as above.

Relating the Left and the Right Rigid Body Systems.

1.10.1 Proposition. *If $(Q(t), M(t))$ satisfies (1.10.2), then the pair $(Q_r(t), M_r(t))$, where $Q_r(t) = Q(t)^T$ and $M_r(t) = -M(t)$, satisfies (1.10.5). There is a similar converse statement.*

The proof is a straightforward verification.

The relation between the left and right systems given in this proposition is not to be confused with the right trivialized representation of the left invariant rigid body equations; that is, the left invariant system written in spatial representation. For a discussion of this distinction, see, for example, Holm, Marsden and Ratiu [1986]. One can also view the right invariant system as the *inverse* representation of the standard left invariant rigid body.

Remark. It is a remarkable fact that the dynamic rigid body equations on $\mathrm{SO}(n)$ and indeed on any semisimple Lie group are integrable (Mishchenko and Fomenko [1976]). A key observation in this regard, due to Manakov, was that one could write the generalized rigid body equations as Lax equations with parameter:

$$\frac{d}{dt}(M + \lambda\Lambda^2) = [M + \lambda\Lambda^2, \Omega + \lambda\Lambda], \tag{1.10.6}$$

where $M = J(\Omega) = \Lambda\Omega + \Omega\Lambda$, as in §2. The nontrivial coefficients of λ in the traces of the powers of $M + \lambda\Lambda^2$ then yield the right number of independent integrals in involution to prove integrability of the flow on a generic adjoint orbit of $\mathrm{SO}(n)$ (identified with the corresponding coadjoint orbit). We remark that the $SO(n)$ rigid body equations were in fact written down by F. Frahm in 1874 who also proved integrability for the case $n = 4$. In addition, F. Schottky in 1891 showed how to obtain explicit theta-function solutions in this case. For references to this work see Bogayavlenski [1994] and Fedorov and Kozlov [1995]. Moser and Veselov [1991] show that there is a corresponding formulation of the discrete rigid body equations with parameter. We shall return to this issue in Chapter 3.

1.11 The Roller Racer

We now consider a tricycle-like mechanical system called the ***roller racer***, or the ***Tennessee racer***, that is capable of locomotion by oscillating the front handlebars. This toy was studied using the methods of Bloch, Krishnaprasad, Marsden, and Murray [1996] in Tsakiris [1995] and Krishnaprasad and Tsakiris [2001] and by energy–momentum methods in Zenkov, Bloch, and Marsden [1998]. Analysis of this system may be a useful guide for modeling and studying the stability of other systems, such as aircraft landing gears and train wheels.

The roller racer is modeled as a system of two planar coupled rigid bodies (the main body and the second body) with a pair of wheels attached on each of the bodies at their centers of mass: a nonholonomic generalization of the coupled planar bodies discussed earlier. We assume that the mass and the linear momentum of the second body are negligible, but that the moment of inertia about the vertical axis is not. See Figure 1.11.1.

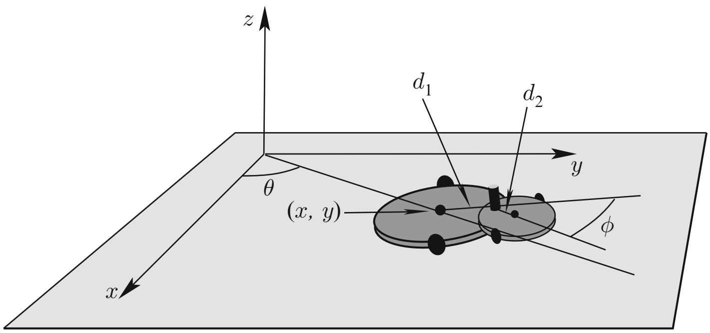

FIGURE 1.11.1. The geometry for the roller racer.

Let (x, y) be the location of the center of mass of the first body and denote the angle between the inertial reference frame and the line passing through the center of mass of the first body by θ, the angle between the bodies by ϕ, and the distances from the centers of mass to the joint by d_1 and d_2. The mass of body 1 is denoted by m, and the inertias of the two bodies are written as I_1 and I_2.

The Lagrangian and the constraints are

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I_1\dot{\theta}^2 + \frac{1}{2}I_2(\dot{\theta} + \dot{\phi})^2$$

and

$$\dot{x} = \cos\theta \left(\frac{d_1 \cos\phi + d_2}{\sin\phi}\dot{\theta} + \frac{d_2}{\sin\phi}\dot{\phi} \right);$$
$$\dot{y} = \sin\theta \left(\frac{d_1 \cos\phi + d_2}{\sin\phi}\dot{\theta} + \frac{d_2}{\sin\phi}\dot{\phi} \right).$$

The configuration space is $\mathrm{SE}(2) \times \mathrm{SO}(2)$. The Lagrangian and the constraints are invariant under the left action of $\mathrm{SE}(2)$ on the first factor of the configuration space.

We shall see later that the roller racer has a two-dimensional manifold of equilibria and that under a suitable stability condition some of these equilibria are stable modulo $\mathrm{SE}(2)$ and in addition asymptotically stable with respect to $\dot{\phi}$.

1.12 The Rattleback

A rattleback is a convex asymmetric rigid body rolling without sliding on a horizontal plane. It is known for its ability to spin in one direction and to resist spinning in the opposite direction for some parameter values, and for other values to exhibit multiple reversals. See Figure 1.12.1.

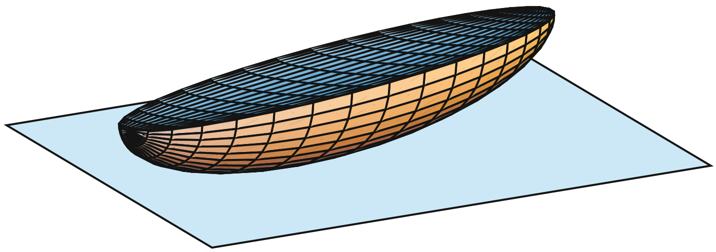

FIGURE 1.12.1. The rattleback.

Basic references on the rattleback are Walker [1896], Karapetyan [1980, 1981], Markeev [1983, 1992], Pascal [1983, 1986], and Bondi [1986]. We adopt the ideal model (with no energy dissipation and no sliding) of these references, and within that context no approximations are made. In particular, the shape need not be ellipsoidal. Walker did some initial stability and instability investigations by computing the spectrum, while Bondi extended this analysis and also used what we now recognize as the momentum equation. (See Chapter 5 for the general theory of the momentum equation and see Zenkov, Bloch, and Marsden [1998] and Section 8.5 for the explicit form of the momentum for the rattleback. A discussion of the momentum equation for the rattleback may also be found in Burdick, Goodwine and Ostrowski [1994].) Karapetyan carried out a stability analysis of the relative equilibria, while Markeev's and Pascal's main contributions were to the study of spin reversals using small-parameter and averaging techniques. Energy methods were used to analyze the problem in Zenkov, Bloch, and Marsden [1998], and we return to this in Section 8.5.

Introduce the Euler angles θ, ϕ, ψ using the principal axis body frame relative to an inertial reference frame. We use the same convention for the angles as in Arnold [1989] and Marsden and Ratiu [1999]. These angles together with two horizontal coordinates x, y of the center of mass are coordinates in the configuration space $\mathrm{SO}(3) \times \mathbb{R}^2$ of the rattleback.

The Lagrangian of the rattleback is computed to be

$$\begin{aligned} L = {} & \frac{1}{2}\left[A\cos^2\psi + B\sin^2\psi + m(\gamma_1\cos\theta - \zeta\sin\theta)^2\right]\dot\theta^2 \\ & + \frac{1}{2}\left[(A\sin^2\psi + B\cos^2\psi)\sin^2\theta + C\cos^2\theta\right]\dot\phi^2 \\ & + \frac{1}{2}\left(C + m\gamma_2^2\sin^2\theta\right)\dot\psi^2 + \frac{1}{2}m\left(\dot x^2 + \dot y^2\right) \\ & + m(\gamma_1\cos\theta - \zeta\sin\theta)\gamma_2\sin\theta\,\dot\theta\dot\psi + (A-B)\sin\theta\sin\psi\cos\psi\,\dot\theta\dot\phi \\ & + C\cos\theta\,\dot\phi\dot\psi + mg(\gamma_1\sin\theta + \zeta\cos\theta), \end{aligned}$$

where

$$\begin{aligned} A, B, C &= \text{the principal moments of inertia of the body}, \\ m &= \text{the total mass of the body}, \\ (\xi, \eta, \zeta) &= \text{coordinates of the point of contact relative to the body frame}, \\ \gamma_1 &= \xi\sin\psi + \eta\cos\psi, \\ \gamma_2 &= \xi\cos\psi - \eta\sin\psi. \end{aligned}$$

The shape of the body is encoded by the functions ξ, η, and ζ. The constraints are

$$\dot x = \alpha_1\dot\theta + \alpha_2\dot\psi + \alpha_3\dot\phi, \quad \dot y = \beta_1\dot\theta + \beta_2\dot\psi + \beta_3\dot\phi,$$

where

$$\begin{aligned}\alpha_1 &= -(\gamma_1 \sin\theta + \zeta\cos\theta)\sin\phi,\\ \alpha_2 &= \gamma_2\cos\theta\sin\phi + \gamma_1\cos\phi,\\ \alpha_3 &= \gamma_2\sin\phi + (\gamma_1\cos\theta - \zeta\sin\theta)\cos\phi,\\ \beta_k &= -\frac{\partial\alpha_k}{\partial\phi}, \quad k = 1,2,3.\end{aligned}$$

The Lagrangian and the constraints are SE(2)-invariant, where the action of an element $(a, b, \alpha) \in \mathrm{SE}(2)$ is given by

$$(x, y, \phi) \mapsto (x\cos\alpha - y\sin\alpha + a, x\sin\alpha + y\cos\alpha + b, \phi + \alpha).$$

Corresponding to this invariance, ξ, η, and ζ are functions of the variables θ and ψ only.

1.13 The Toda Lattice

An important and beautiful mechanical system that describes the interaction of particles on the line (i.e., in one dimension) is the Toda lattice. We shall describe the nonperiodic finite Toda lattice following the treatment of Moser [1975].

This is a key example in integrable systems theory. Later on, in Chapter 8, we shall compare the behavior of this system to certain nonholonomic systems. In the Internet Supplement we also consider the Toda lattice from the point of view of optimal control theory.

The model consists of n particles moving freely on the x-axis and interacting under an exponential potential. Denoting the position of the kth particle by x_k, the Hamiltonian is given by

$$H(x, y) = \frac{1}{2}\sum_{k=1}^{n} y_k^2 + \sum_{k=1}^{n-1} e^{(x_k - x_{k+1})}.$$

The associated Hamiltonian equations are

$$\begin{aligned}\dot{x}_k &= \frac{\partial H}{\partial y_k} = y_k\,,\\ \dot{y}_k &= -\frac{\partial H}{\partial x_k} = e^{x_{k-1}-x_k} - e^{x_k - x_{k+1}}\,,\end{aligned} \tag{1.13.1}$$

where we use the convention $e^{x_0 - x_1} = e^{x_n - x_{n+1}} = 0$, which corresponds to formally setting $x_0 = -\infty$ and $x_{n+1} = +\infty$.

This system of equations has an extraordinarily rich structure. Part of this is revealed by Flaschka's (Flaschka [1974]) change of variables given by

$$a_k = \frac{1}{2}e^{(x_k - x_{k+1})/2} \quad \text{and} \quad b_k = -\frac{1}{2}y_k\,. \tag{1.13.2}$$

In these new variables, the equations of motion then become

$$\begin{aligned}\dot{a}_k &= a_k(b_{k+1} - b_k)\,, \quad k = 1, \ldots, n-1\,,\\ \dot{b}_k &= 2(a_k^2 - a_{k-1}^2)\,, \quad k = 1, \ldots, n\,,\end{aligned}$$

with the boundary conditions $a_0 = a_n = 0$. This system may be written in the following matrix form (called the ***Lax pair representation***):

$$\frac{d}{dt}L = [B, L] = BL - LB, \tag{1.13.3}$$

where

$$L = \begin{pmatrix} b_1 & a_1 & 0 & \cdots & 0 \\ a_1 & b_2 & a_2 & \cdots & 0 \\ & & \ddots & & \\ & & & b_{n-1} & a_{n-1} \\ 0 & & & a_{n-1} & b_n \end{pmatrix}, \quad B = \begin{pmatrix} 0 & a_1 & 0 & \cdots & 0 \\ -a_1 & 0 & a_2 & \cdots & 0 \\ & & \ddots & & \\ & & & 0 & a_{n-1} \\ 0 & & & -a_{n-1} & 0 \end{pmatrix}.$$

If $O(t)$ is the orthogonal matrix solving the equation

$$\frac{d}{dt}O = BO\,, \quad O(0) = \text{ Identity}\,,$$

then from (1.13.3) we have

$$\frac{d}{dt}(O^{-1}LO) = 0\,.$$

Thus, $O^{-1}LO = L(0)$; i.e., $L(t)$ is related to $L(0)$ by a similarity transformation, and thus the eigenvalues of L, which are real and distinct, are preserved along the flow. This is enough to show that in fact this system is explicitly solvable or integrable.

Discussion. There is, however, much more structure in this example. For instance, if N is the matrix $\operatorname{diag}[1, 2, \ldots, n]$, the Toda flow (1.13.3) may be written in the following ***double bracket*** form:

$$\dot{L} = [L, [L, N]]\,. \tag{1.13.4}$$

This was shown in Bloch [1990] and analyzed further in Bloch, Brockett, and Ratiu [1990], Bloch, Brockett, and Ratiu [1992], and Bloch, Flaschka, and Ratiu [1990]. This double bracket equation restricted to a level set of the integrals described above is in fact the gradient flow of the function $\mathrm{Tr}LN$ with respect to the so-called normal metric; see Bloch, Brockett, and Ratiu [1990]. Double bracket flows are derived in Brockett [1994].

From this observation it is easy to show that the flow tends asymptotically to a diagonal matrix with the eigenvalues of $L(0)$ on the diagonal and ordered according to magnitude, recovering the observation of Moser, Symes [1982], and Deift, Nanda, and Tomei [1983].

A very important feature of the tridiagonal aperiodic Toda lattice flow is that it can be solved explicitly as follows: Let the initial data be given by $L(0) = L_0$. Given a matrix A, use the Gram–Schmidt process on the columns of A to factorize A as $A = k(A)u(A)$, where $k(A)$ is orthogonal and $u(A)$ is upper triangular. Then the explicit solution of the Toda flow is given by

$$L(t) = k(\exp(tL_0))L_0k^T(\exp(tL_0)) . \tag{1.13.5}$$

The reader can check this explicitly or refer, for example, to Symes [1980, 1982].

Four-Dimensional Toda. Here we simulate the Toda lattice in four dimensions (see Bloch [2000]). The Hamiltonian is

$$H(a,b) = a_1^2 + a_2^2 + b_1^2 + b_2^2 + b_1b_2 , \tag{1.13.6}$$

and one has the equations of motion

$$\begin{aligned} \dot{a}_1 &= -a_1(b_1 - b_2) \quad & \dot{b}_1 &= 2a_1^2 , \\ \dot{a}_2 &= -a_2(b_1 + 2b_2) \quad & \dot{b}_2 &= -2(a_1^2 - a_2^2) \end{aligned} \tag{1.13.7}$$

(setting $b_1 + b_2 + b_3 = 0$, for convenience, which we may do since the trace is preserved along the flow). In particular, Trace LN is, in this case, equal to b_2 and can be checked to decrease along the flow.

Figure 1.13.1 exhibits the asymptotic behavior of the Toda flow. We will return to this property in Chapter 8.

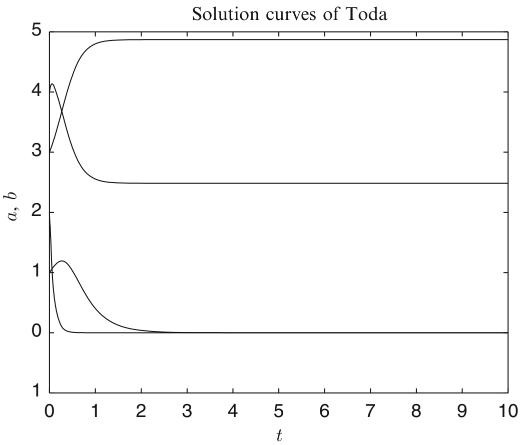

FIGURE 1.13.1. Asymptotic behavior of the solutions of the four-dimensional Toda lattice.

Exercises

◇ **1.13-1.** Show that Trace L^k for all k is conserved along the flow of the Toda lattice

◇ **1.13-2.** Characterize all the equilibria for the Toda flow (allowing a_i to take the value 0). Hint: Use the double bracket form of the equations.