Normal Form Transformations for Capillary-Gravity Water Waves

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Abstract This paper addresses the equations of capillary-gravity waves in a two-dimensional channel of finite or infinite depth. These equations are considered in the framework of Hamiltonian systems, for which the Hamiltonian energy has a convergent Taylor expansion in canonical variables near the equilibrium solution. We give an analysis of the Birkhoff normal form transformation that eliminates third-order non-resonant terms of the Hamiltonian. We also provide an analysis of the dynamics of remaining resonant triads in certain cases, related to Wilton ripples.

1 Introduction

The system of equations for free surface water waves is known to have a formulation as a Hamiltonian partial differential equation [3, 14]. In this article, we consider the case of capillary-gravity waves in a two-dimensional channel with periodic lateral boundary conditions, with either finite of infinite depth. The Hamiltonian is analytic in natural canonical conjugate variables, and the nth term of its Taylor expansion about equilibrium is associated with n-wave interactions. We perform a Birkhoff normal form transformation to eliminate all non-resonant cubic terms from the Hamiltonian. We show that this transformation is a well-defined and continuous canonical change of variables in a neighbourhood of zero in a fixed Sobolev space, and moreover it is C^1 in the sense that its Jacobian is a bounded map on a slightly larger Sobolev space.

This work is motivated by a number of open questions in the theory of water waves. The first is the question of long-time existence of solutions for small initial data. In the case of infinite horizontal extent, there have been a number of recent

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results, including [8, 13] who give exponentially long existence times and [1, 10] who extended this to global solutions. These are results for infinite depth and zero surface tension. The question in the periodic case is open, and more difficult because of the lack of dispersive decay estimates. It is important because of the effort to derive a rigorous justification of the modulational approximation offered by the NLS equation, in the more natural setting in which the solutions are not dispersing to zero. The NLS approximation has been given a rigorous justification in the case of an infinite horizontal domain by Totz and Wu [12] and by Düll, et al. [6], following the initial analysis of Craig, et al. [4].

Another interesting aspect of a rigorous normal form transformation is that it exhibits certain classes of special solutions. In the setting of this paper, the remaining resonant terms after the third order Birkhoff normal form transformation take the form of coupled resonant triads, related to the classical Wilton ripples. In the presence of a single resonant triad, the dynamics is that of the integrable three-wave system. In cases of higher numbers of coupled triads, the dynamics are more complicated [2, 9]. The normal forms transformation in the present paper gives a rigorous justification of the behaviour of resonant triads models, over long time intervals, for the initial value problem.

A useful aspect of normal forms given through canonical transformation is that, first of all, they preserve the Hamiltonian character of the underlying equations of motion and the principle of conservation of energy. Secondly, these transformations can in principle be repeated, resulting in a normal form for higher order terms in the Hamiltonian, and eliminating non-resonant higher order nonlinearities. On a formal level, normal forms are described up to fourth order in [7].

Finally, normal form transformations play a central role in Zakharov's theory of wave turbulence, In this, nonlinear wave interactions are reduced to resonant submanifolds under canonical changes of variables [11, 15, 16]. Any effort to make a rigorous analysis of this picture of wave turbulence will need to understand the analytic properties of such transformations.

In the present paper, Sect. 2 describes the Hamiltonian for capillary-gravity water waves, and transforms it to complex symplectic coordinates. In Sect. 3, we describe the Birkhoff normal form and solve the cohomological equation. Section 4 gives the key result of the paper, namely that the time-one solution map of the Hamiltonian vector field is well-defined and continuous in an appropriately defined scale of energy spaces. This result is based on energy estimates for solutions. Moreover, the solution map of the vector field is smooth on this scale of spaces in the sense that the Jacobian of the solution and subsequent higher derivatives are bounded in slightly larger spaces of the scale. Section 5 is a study of the normal form Hamiltonian itself, truncated at fourth order, in certain specific cases of triad and multiple triad interaction. In the case of a single triad, the system reduces to two decoupled subsystems of one degree of freedom apiece, corresponding to two independent copies of the three-wave resonant system. We also consider a particular case of two coupled resonant triads giving rise to a Hamiltonian system with two degrees of freedom for which we find the stationary points and analyze their respective stability. The normal forms transformation gives a rigorous justification of the relevance of these finite dimensional dynamical systems to model the dynamics of the full water wave system, at least over long periods of time.

2 Water Waves Equations

2.1 The Classical Equations and Their Hamiltonian Formulation

The classical water problem refers to the movement of an ideal incompressible fluid in the presence of gravity and surface tension. Making the usual oceanographers assumption that the fluid is irrotational, it is described by a potential flow $u = \nabla\varphi$ satisfying

$$\Delta\varphi = 0 \,, \tag{1}$$

in the fluid domain $S(t;\eta) = \{(x,y) : x \in \mathbb{R}, -h < y < \eta(x,t)\}$, where η is the surface elevation. The boundary condition on the fixed bottom $\{y = -h\}$ of the fluid is

$$-\partial_y\varphi(x,-h) = 0 \,. \tag{2}$$

On the interface $\{y = \eta(x,t)\}$, two boundary conditions are imposed, namely

$$\begin{aligned} &\partial_t\eta = \partial_y\varphi - \partial_x\eta\,\partial_x\varphi \,, \\ &\partial_t\varphi + \frac{1}{2}|\nabla\varphi|^2 + g\eta - \sigma\partial_x\Big(\frac{\partial_x\eta}{(1+|\partial_x\eta|^2)^{1/2}}\Big) = 0 \,, \end{aligned}$$

where g is the acceleration of gravity and σ the coefficient of surface tension. We assume periodic boundary condition in horizontal direction, $\eta(x+2\pi) = \eta(x), \xi(x+2\pi) = \xi(x)$.

It is well-known that this system has a Hamiltonian formulation [14] with canonical variables $\eta(x,t)$ and $\xi(x,t) = \varphi(x,\eta(x,t),t)$, in the form

$$\partial_t \begin{pmatrix} \eta \\ \xi \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} \delta_\eta H \\ \delta_\xi H \end{pmatrix} = J\,\delta H \tag{3}$$

with the Hamiltonian being given by the expression of the total energy

$$\begin{aligned} H(\eta,\xi) &= \frac{1}{2}\int\int_{-h}^{\eta(x)} |\nabla\varphi|^2\,dydx + \int (\frac{g}{2}\eta^2 + \sigma\sqrt{1+|\partial_x\eta|^2})\,dx \\ &= \int (\frac{1}{2}\xi(x)G(\eta)\xi(x) \; + \frac{g}{2}\eta^2(x) + \sigma\sqrt{1+|\partial_x\eta|^2})\,dx \,. \end{aligned} \tag{4}$$

Here $G(\eta)$ is the Dirichlet-Neumann operator which associates to the Dirichlet data ξ on the curve $y = \eta(x)$ the normal derivative of the harmonic function φ, with a normalized factor, namely, $\partial_n \varphi \sqrt{1 + |\partial_x \eta|^2}$. The other conserved quantities are the mass $M(\eta, \xi) = \int \eta(x) dx$ and the horizontal momentum $I(\eta, \xi) = \int \xi(x) \partial_x \eta(x) dx$. Defining the Poisson bracket as

$$\{F, G\} = \int (\partial_\eta F \partial_\xi G - \partial_\xi F \partial_\eta G) dx,$$

one can check that the conserved quantities M and I Poisson-commute with the Hamiltonian H,

$$\{H, M\} = 0, \qquad \{H, I\} = 0.$$

2.2 Complex Symplectic Coordinates

The Dirichlet-Neumann operator is analytic in η, given in Taylor series by

$$G(\eta)\xi = \sum_{m=0}^{\infty} G^{(m)}(\eta),$$

with the property that each term in the Taylor expansion $G^{(m)}$ is homogeneous of degree m, $G^{(m)}(\lambda\eta) = \lambda^m G^{(m)}(\eta)$. In particular, the two first terms in the expansion are $G^{(0)} = D \tanh(hD)$, $G^{(1)} = D\eta D - G^{(0)} \eta G^{(0)}$ where $D = (1/i)\partial_x$. In turn, the Hamiltonian has an expansion in the form

$$H(\eta, \xi) = H^{(2)} + H^{(3)} + \ldots . + H^{(m)} + R^{(m+1)} \tag{5}$$

where

$$H^{(2)} = \frac{1}{2} \int_0^{2\pi} \left(\xi G^{(0)} \xi + g\eta^2 + \sigma |\partial_x \eta|^2 \right) dx,$$

$$H^{(3)} = \frac{1}{2} \int_0^{2\pi} \xi (D\eta D - G^{(0)} \eta G^{(0)}) \xi dx,$$

with similar expressions for higher order $H^{(m)}$, and where $R^{(m+1)}$ is the Taylor remainder.

We consider a periodic setting, i.e. $\eta(x + 2\pi k, t) = \eta(x, t)$ and $\xi(x + 2\pi k, t) = \xi(x, t)$, writing η and ξ as Fourier series

$$\eta(x) = \frac{1}{\sqrt{2\pi}} \sum_k \eta_k e^{ikx}, \quad \xi(x) = \frac{1}{\sqrt{2\pi}} \sum_k \xi_k e^{ikx}.$$

Since mass is conserved, we can assume, without loss of generality, that the *zero*th Fourier coefficient η_0 vanishes. Then,

$$H^{(2)} = \frac{1}{2}\sum_k k\tanh(hk)|\xi_k|^2 + (g+\sigma k^2)|\eta_k|^2),$$

$$H^{(3)} = \frac{1}{2\sqrt{2\pi}}\sum_{k_1+k_2+k_3=0}(-k_1k_3 - G^{(0)}_{k_1}G^{(0)}_{k_3})\,\xi_{k_1}\eta_{k_2}\xi_{k_3},$$

where $G^{(0)}_k = k\tanh(hk)$. Also note that the *zero*th Fourier coefficient ξ_0 of ξ does not appear in the Hamiltonian. It is convenient to introduce the complex symplectic coordinates

$$z_k = \frac{1}{\sqrt{2}}(a_k\eta_k + ia_k^{-1}\xi_k), \tag{6}$$

or equivalently,

$$\eta_k = \frac{1}{\sqrt{2}}a_k^{-1}(z_k + \bar{z}_{-k})\ , \quad \xi_k = \frac{1}{\sqrt{2}i}a_k(z_k - \bar{z}_{-k}), \tag{7}$$

with the coefficients a_k defined by

$$a_k^2 = \Big(\frac{g+\sigma k^2}{k\tanh(hk)}\Big)^{1/2}.$$

The dispersion relation

$$\omega_k^2 = (g+\sigma k^2)k\tanh(hk),$$

expresses the temporal frequencies of the normal modes of the linearized system, as given by the quadratic Hamiltonian $H^{(2)}$. A key distinction between the case of pure gravity waves and gravity—capillary waves is that the dispersion relation grows as a $3/2$ power in wavenumber k in the latter case, as compared with a $1/2$ power in the case of pure gravity waves. Using the dispersion relation, we have the identities $a_k^2\omega_k = g + \sigma k^2$ and $\omega_k/a_k^2 = k\tanh(hk) = G^{(0)}_k$.

In terms of complex symplectic coordinates, the system (3) becomes

$$\partial_t z = \frac{1}{i}\partial_{\bar{z}}H. \tag{8}$$

The quadratic part $H^{(2)}$ of the Hamiltonian takes the simple form

$$H^{(2)} = \sum_k \omega_k|z_k|^2 \tag{9}$$

while the cubic order term $H^{(3)}$ is

$$H^{(3)} = \frac{1}{8\sqrt{\pi}} \sum_{k_1+k_2+k_3=0} (k_1 k_3 + G_1 G_3) \frac{a_1 a_3}{a_2} (z_1 - \bar{z}_{-1})(z_2 + \bar{z}_{-2})(z_3 - \bar{z}_{-3}) \quad (10)$$

where, for simplicity, we have dropped the k indices and denoted $z_j = z_{k_j}$, $a_j = a_{k_j}$, and $G_k = G_k^{(0)}$. We will use this notation when there is no possible confusion.

3 Birkhoff Normal Forms

A Birkhoff normal form is a canonical change of variables up to a given order m, so that the Taylor expansion of the transformed Hamiltonian up to order m contains only resonant terms. A term in the Hamiltonian $H(z)$ is resonant at order m when

$$\sum_{j=1}^{l} \omega_{k_j} - \sum_{j=l+1}^{m} \omega_{k_j} = 0$$

and $k_1 + \dots k_l + k_{l+1} + \dots k_m = 0$. We do not include $k = 0$ in the sums because we have assumed that the zero modes of η and ξ vanish. In particular, a resonant triad takes the form

$$\omega_{k_1} - \omega_{k_2} - \omega_{k_3} = 0, \quad k_1 + k_2 + k_3 = 0, \quad k_j \neq 0. \quad (11)$$

In the presence of surface tension and gravity, there are possible non trivial resonant triads. The resulting gravity-capillary waves are known as Wilton ripples, at least this applies to the standing wave solutions. In the case of a periodic domain, generically these resonant triads do not appear, but for certain choices of parameters (g, h, σ) there can be a finite number of such triads. The maximum wave number k_j involved in a resonant triad is bounded by a constant $C = C(g, h, \sigma)$ that depends locally uniformly upon these parameters.

3.1 *Canonical Transformations*

We perform a canonical change of variables

$$\tau : \; v = (\eta, \xi) \to \; w = (\eta', \xi') \quad (12)$$

on the Hamiltonian

$$\tilde{H}(w) = H(v) = H \circ \tau^{-1}(w)$$

by the Lie method of giving τ as the time-one flow associated to a Hamiltonian K:

$$\frac{d}{ds}\psi_s = X^K(\psi_s), \text{ with } \psi_s(w)_{|s=0} = w, \ \tilde{H}(w) = H(\psi_s(w))_{|s=-1}. \tag{13}$$

This is a canonical transformation preserving the Hamiltonian character of the system. A Taylor series expansion near $s = 0$ of the new Hamiltonian $\tilde{H}$ gives

$$\tilde{H}(w) = H(\psi_s(w))_{|s=0} - \frac{d}{ds}H(\psi_s(w))_{|s=0} + \frac{1}{2}\frac{d^2}{ds^2}H(\psi_s(w))_{|s=0} - \dots \tag{14}$$

As a formal expression at least in the above equation, we have

$$\begin{aligned} H(\psi_s(w))_{|s=0} &= H(w) \\ \frac{d}{ds}H(\psi_s(w))_{|s=0} &= \int (\partial_\eta H \frac{d\eta}{ds} + \partial_\xi H \frac{d\xi}{ds})\, dx = \int (\partial_\eta H \partial_\xi K - \partial_\xi H \partial_\eta K) dx \\ &\equiv \{H, K\}\,, \end{aligned}$$

with similar formulas for higher orders of s-derivatives, thus giving the expression

$$\tilde{H}(w) = H(w) - \{K, H\}(w) + \frac{1}{2}\{K, \{K, H\}\}(w) + \dots . \tag{15}$$

3.2 *Third-Order Cohomological Equation*

The expression (15) represents an ordering of H and K in terms of powers of homogeneity with respect to the variables $z, \bar{z}$. Returning to the expansion (5) of H in terms of (η, ξ), we apply the canonical transformation associated to Hamiltonian K on each term: The transformed Hamiltonian has the form [5]

$$\begin{aligned} \tilde{H}(w) = H^{(2)}(w) &+ H^{(3)}(w) + \dots \\ &-\{K, H^{(2)}\}(w) - \{K, H^{(3)}\}(w) - \dots . \\ &+\frac{1}{2}\{K, \{K, H^{(2)}\}\}(w) + \frac{1}{2}\{K, \{K, H^{(3)}\}\}(w) + \dots \end{aligned} \tag{16}$$

If K is homogeneous of degree m, its Poisson bracket with $H^{(n)}$ (homogeneous of degree n) will be of degree $m + n - 2$. Thus if we can find $K = K^{(3)}$ homogeneous of degree 3 satisfying the relation

$$\{H^{(2)}, K^{(3)}\} + H^{(3)} = 0, \tag{17}$$

we will have eliminated the cubic terms in the transformed Hamiltonian $\tilde{H}$.

The proposition below states that it is indeed possible to solve the cohomological equation (17) explicitly, removing all cubic terms except the resonant terms of $H^{(3)}$.

Proposition 1. *The solution of the cohomological equation (17) is given by*

$$K^{(3)} = \frac{1}{\sqrt{\pi}} \sum_{k_1+k_2+k_3=0} (k_1k_3 + G_1G_3) \frac{a_1a_3}{a_2} \frac{z_1z_2z_3 - \bar{z}_{-1}\bar{z}_{-2}\bar{z}_{-3}}{\omega_1 + \omega_2 + \omega_3}$$
$$- \sum_{k_1+k_2+k_3=0} (k_1k_3 + G_1G_3) \frac{a_1a_3}{a_2} 2 \frac{z_1z_2\bar{z}_{-3} - \bar{z}_{-1}\bar{z}_{-2}z_3}{\omega_1 + \omega_2 - \omega_3}$$
$$+ \sum_{k_1+k_2+k_3=0} (k_1k_3 + G_1G_3) \frac{a_1a_3}{a_2} \frac{z_1\bar{z}_{-2}z_3 - \bar{z}_{-1}z_2\bar{z}_{-3}}{\omega_1 - \omega_2 + \omega_3} + P, \tag{18}$$

where the three sums are performed for triads (k_1, k_2, k_3), with $k_1 + k_2 + k_3 = 0$ excluding the resonant terms for which the corresponding denominator vanishes. The term P consists of the finite sum of exceptional terms. That is, it consists of the non resonant terms of $K^{(3)}$ for which (k_1, k_2, k_3) possesses a resonant triad. Generically, $P = 0$.

Proof. This equation can be solved easily in complex symplectic coordinates which diagonalize the linear operation of taking Poisson bracket with $H^{(2)}$ (the $\mathrm{ad}_{H^{(2)}}$ action). Indeed, the Poisson bracket of $H^{(2)}$ acting on monomials of the form $z_{k_1}z_{k_2}\bar{z}_{-k_3}$ is simply a multiplicative factor:

$$\{H^{(2)}, z_{k_1}z_{k_2}\bar{z}_{-k_3}\} = \frac{1}{i}(\omega_{k_1} + \omega_{k_2} - \omega_{k_3}) z_{k_1}z_{k_2}\bar{z}_{-k_3}. \tag{19}$$

We thus look for $K^{(3)}$ in the form of a linear combination of all possible monomials of degree 3 of the form similar to that above and we identify the coefficients, which is possible as long as the corresponding mutliplicative factor $(\omega_1 \pm \omega_2 \pm \omega_3)$ does not vanish. This leads to $K^{(3)}$ given in the form of (18) where, for simplicity, we have denoted $z_j = z_{k_j}$.

It is useful for the analysis to rewrite $K^{(3)}$ in terms of the (η, ξ) variables. After some algebraic manipulations, one finds

$$K^{(3)} = \frac{1}{\sqrt{2\pi}} \sum_{k_1+k_2+k_3=0} \frac{k_1k_3 + G_1G_3}{d(\omega_1, \omega_2, \omega_3)} \Big[a_1^2\omega_1(\omega_1^2 - \omega_2^2 - \omega_3^2)\eta_{k_1}\eta_{k_2}\xi_{k_3}$$
$$+ \frac{a_1^2a_3^2}{a_2^2}\omega_1\omega_2\omega_3\eta_{k_1}\xi_{k_2}\eta_{k_3} + \frac{1}{2a_2^2}\omega_2(\omega_1^2 - \omega_2^2 + \omega_3^2)\xi_{k_1}\xi_{k_2}\xi_{k_3}\Big] + P \tag{20}$$

where

$$d(\omega_1, \omega_2, \omega_3) = (\omega_1 + \omega_2 + \omega_3)(\omega_1 + \omega_2 - \omega_3)(\omega_1 - \omega_2 + \omega_3)(\omega_1 - \omega_2 - \omega_3), \tag{21}$$

and the summation is performed over all triads (k_1, k_2, k_3) such that $k_1 + k_2 + k_3 = 0$ and $d(\omega_1, \omega_2, \omega_3) \neq 0$. Finally, the term P in the RHS of (20) contains finite sums of the alternate triplets to resonant triads. Namely, it contains the terms in the first and last sums in (18) corresponding to triads for which the denominator $\omega_1 + \omega_2 - \omega_k$ of the second sum vanishes, and the terms of the first and second sums in (18) corresponding to triads for which the denominator $\omega_1 - \omega_2 - \omega_k$ of the third sum vanishes. □

Remark 1. This calculation has been performed in the case of finite depth $0 < h < \infty$. In the case $h = \infty$, the same expression holds, with the substitution $G^{0)} = |D|$.

3.3 Transformation to Third-Order Normal Form

The new coordinates $(\tilde{\eta}, \tilde{\xi})$ are obtained as the solutions at $s = -1$ of the system of equations

$$\frac{d}{ds}\begin{pmatrix}\eta\\ \xi\end{pmatrix} = \begin{pmatrix}0 & I\\ -I & 0\end{pmatrix}\begin{pmatrix}\partial_\eta K^{(3)}\\ \partial_\xi K^{(3)}\end{pmatrix} := X^{K^{(3)}} \tag{22}$$

with the (initial) condition at $s = 0$ being the original variables $(\eta, \xi)(t)$. Equivalently, in Fourier space,

$$\begin{aligned}\frac{d}{ds}\eta_{-k} &= \partial_{\xi_k} K^{(3)}\\ \frac{d}{ds}\xi_{-k} &= -\,\partial_{\eta_k} K^{(3)}\end{aligned} \tag{23}$$

with the RHS given by

$$\begin{aligned}\sqrt{2\pi}\,\partial_{\xi_k} K^{(3)} =& \Sigma'_{k_1+k_2+k=0}\Big[\frac{k_1 k + G_1 G_k}{d_{k_1 k_2 k}}(g + \sigma k_1^2)(\omega_1^2 - \omega_2^2 - \omega_k^2)\eta_{k_1}\eta_{k_2}\\ &+ \frac{k_1 k_2 + G_1 G_2}{d_{k_1 k_2 k}}(g + \sigma k_1^2)(g + \sigma k_2^2)k\tanh(hk)\eta_{k_1}\eta_{k_2}\Big]\\ &+ \Big[\frac{k_1 k + G_1 G_k}{d_{k_1 k_2 k}}k_2\tanh(hk_2)(\omega_1^2 - \omega_2^2 + \omega_k^2)\xi_{k_1}\xi_{k_2}\\ &+ \frac{k_1 k_2 + G_1 G_2}{2d_{k_1 k_2 k}}k\tanh(hk)(\omega_1^2 - \omega_k^2 + \omega_2^2)\xi_{k_1}\xi_{k_2}\Big],\end{aligned} \tag{24}$$

and

$$\sqrt{2\pi}\partial_{\eta_k} K^{(3)} = \Sigma'_{k_1+k_2+k=0}\Big[\frac{kk_2 + G_k G_2}{d_{kk_1k_2}}(g + \sigma k^2)(\omega_k^2 - \omega_1^2 - \omega_2^2)\eta_{k_1}\xi_{k_2}$$
$$+ \frac{k_1k_2 + G_1G_2}{d_{k_1kk_2}}(g + \sigma k_1^2)(\omega_1^2 - \omega_k^2 - \omega_2^2)\eta_{k_1}\xi_{k_2}$$
$$+ 2\frac{kk_1 + G_kG_1}{d_{kk_2k_1}}(g + \sigma k^2)(g + \sigma k_1^2)k_2 \tanh(hk_2)\eta_{k_1}\xi_{k_2}\Big], \qquad (25)$$

where $d_{k_1k_2k_3} = d(\omega_1, \omega_2, \omega_3)$, and the notation Σ' indicates that the summation is performed over all triads (k_1, k_2, k_3) satisfying $k_1 + k_2 + k_3 = 0$ and $d_{k_1k_2k_3} \neq 0$. In case of the presence of resonances, there is a finite number of exceptional terms. For convenience of estimates, we assume in this and the following section that we are in the generic case and there are no resonant triads.

4 Analysis of the Normal Form Transformation

Let H^r denote the Sobolev space of order r, equipped with the norm $\|f\|^2_{H^r} = \sum_k \langle k\rangle^{2r}|f_k|^2$, where $\langle\, k\,\rangle = (1 + |k|^2)^{1/2}$. Define the energy norm

$$\|(\eta, \xi)\|^2_{E^r} = \frac{1}{2}(\langle \eta, \sigma|D|^2\eta\rangle_r + \langle \xi, G_0\xi\rangle_r) \qquad (26)$$

and the energy space $E^r \simeq H^{r+1} \times H^{r+1/2}$. We denote B_R be the ball centered at the origin, of radius R of the energy space. Define a transformation $w = \tau(v) := \psi_s(v)|_{s=-1}$ given by the time-one solution map of the Hamiltonian vector field $X^{K^{(3)}}$ (22) with the auxiliary Hamiltonian $K^{(3)}$. This map is well defined and continuous in a neighbourhood of the origin $B_R \subseteq E^r$ because of the following result.

Theorem 1. *There exists $R_0 > 0$ such that, for all $R < R_0$, the canonical transformation $\tau : v \to w$ defined in (13) is continuous on $B_R \subseteq E^r$, with continuous inverse τ^{-1}, and it satisfies $\tau : B_{R/2} \to B_R$ and $\tau^{-1} : B_{R/2} \to B_R$. This transformation removes all non resonant cubic terms from the Hamiltonian. The Jacobian $\partial_{(\eta,\xi)}\tau$ of the transformation is bounded on the energy space $E^{r-1/2} \to E^{r-1/2}$.*

The proof of the existence of the mapping τ is based on an energy estimate for the vector field (23), and the result for the Jacobian follows from a similar energy estimate for the variational equation of (23). From this, the existence of the solution to (22) for $s \in [-1, 1]$ is obtained by a fixed point argument for a sequence of approximations, under the condition that the ball $B_R \subseteq E^r$ in which one takes the initial data is of sufficiently small radius.

Theorem 2. *Given* $(\eta, \xi) \in E^r$*, the vector field* $X^{K^{(3)}}$ *satisfies the following energy inequality*

$$|\langle(\eta, \xi), X^{K^{(3)}}(\eta, \xi)\rangle_{E^r}| \le C\|(\eta, \xi)\|^3_{E^r}. \tag{27}$$

The proof of Theorem 2 is given in Sects. 4.3 and 4.4, using some technical results presented in Sects. 4.1 and 4.2.

Theorem 3. *The flow* $\psi_s(\eta, \xi)$ *of the vector field* $X^{K^{(3)}}$ *satisfies the following estimates on the scale of energy spaces* E^p*,* $0 \le p \le r$*. For* $\|(\eta, \xi)\|_{E^r} \le R$*, and* $|s| < s_R$*, the Jacobian satisfies*

$$\|\big(\partial_{(\eta,\xi)}\psi_s(\eta, \xi) - I\big)\ (\tilde{\eta}, \tilde{\xi})^{\mathrm{T}}\|_{E^{r-1/2}} \ \le\ C_1\, R\, \|(\tilde{\eta}, \tilde{\xi})\|_{E^{r-1/2}}. \tag{28}$$

Higher derivatives of the flow satisfy

$$\|\partial^q_{(\eta,\xi)}\psi_s(\eta, \xi)\|_{E^{r-q/2}} \ \le\ C_{r,q}. \tag{29}$$

It follows from this result that the transformation $\tau(\eta, \xi) = \psi_{s=-1}(\eta, \xi)$ is smooth on the scale of spaces E^p, $0 \le p \le r$, in the sense that for $(\eta, \xi) \in B_R \subseteq E^r$, the derivatives $\partial^q_{(\eta,\xi)}\tau \ :\ E^{r-q/2} \to E^{r-q/2}$ are continuous. For this, we require $R \le R_0$, so that the guaranteed existence time for the flow satisfies $s_R > 1$. The proof of Theorem 3 is given in Sect. 4.5.

Proof of Theorem 1. The canonical transformation that we seek is designed as the time $s = -1$ image of the solution map for Eq. (22). The question is whether the solution exists and its regularity with respect to its initial data $(\eta, \xi)|_{s=0} \in E^r$. We will show that the solution map exists and is continuous on each ball $B_R \subset E^r$ for some interval $-s_R < s < s_R$, and that for sufficiently small R the bound $s_R > 1$. The desired transformation is $\tau = \psi_{s=-1}$ while $\tau^{-1} = \psi_{s=1}$. The vector field is not Lipschitz continuous in any reasonable Banach space, not even locally. We proceed with a strategy for the existence which is well-known from the theory of symmetric hyperbolic systems. Namely, one solves an approximate equation which has smooth solutions and takes the limit. In the case at hand,

$$\frac{d}{ds}\begin{pmatrix}\eta\\ \xi\end{pmatrix} = X^{K^{(3)}}(\eta, \xi) + \alpha\Delta\begin{pmatrix}\eta\\ \xi\end{pmatrix} \tag{30}$$

with parameter $0 < \alpha \ll 1$. Using the energy estimates of $X^{K^{(3)}}$ of Theorem 2,

$$\frac{d}{ds}\|(\eta^{(\alpha)}, \xi^{(\alpha)})\|^2_{E^r} \le C\|(\eta^{(\alpha)}, \xi^{(\alpha)})\|^3_{E^r} - \alpha\|(\eta^{(\alpha)}, \xi^{(\alpha)})\|^2_{E^{r+1}}\ . \tag{31}$$

With initial data $(\eta^{(\alpha)}, \xi^{(\alpha)}) = (\eta, \xi) \in B_R$, solutions $(\eta^{(\alpha)}(s), \xi^{(\alpha)}(s))$ exist over a time interval $(-s_R, s_R)$, uniformly in α. This gives rise to a family of continuous curves $(\eta^{(\alpha)}(s), \xi^{(\alpha)}(s)) \in C((-s_R, s_R); E^r)$ which is bounded and equicontinuous in $C((-s_R, s_R); E^0)$ whose limit points as $\alpha \to 0$ are solutions of (22). The limit is unique in E^0 and hence in all E^r as well. When one takes $R < 1/C_0$, the time of existence satisfies $s_R > 1$ and the image of B_R, $\psi_{\pm 1}(B_R) \subset B_{R/2}$. □

4.1 Lower Bound for the Denominator $d(\omega_1, \omega_2, \omega_3)$

In this section, we examine the denominator $d(\omega_1, \omega_2, \omega_3)$ defined in (21) that appears in the formulas (24) and (25) of the vector field.

Lemma 1. *The set of resonant triads (k_1, k_2, k_3), i.e. such that $k_1 + k_2 + k_3 = 0$ and $d(\omega_1, \omega_2, \omega_3) = 0$ exists only on a compact set W of $\mathbb{R}^3$.*

Proof. This is due to the change of concavity of the curve ω versus k. Assume without loss of generality that $|k_1| > |k_2|$, $k_1 > 0$ and $k_2 < 0$. For $\omega_1 - \omega_2 - \omega_3$ to vanish, one needs that the curve given by the graph of $\omega(k)$, denoted $\mathscr{C}$, starting from the origin point of coordinates O intersects the curve $\mathscr{C}'$ that starts for the point $O' = (k_2, \omega_2)$. For this to happen, the origin O' needs to be in the region where the curve $\mathscr{C}$ is concave. There are thus only a finite number of triads that are resonant. Because of the periodic boundary conditions, for generic values of the parameters g, h, σ, there are no resonances. However, there are cases in which triads resonances and multiple coupled triad resonances occur. □

In order to proceed with the estimates, we divide the plane (k_1, k_2) in 4 sectors represented in Fig. 1 and defined as follows, using that $k_1 + k_2 + k_3 = 0$,

Sector (i) = $\{(k_1, k_2),\ |k_1| < \frac{1}{5}|k_2|,\ \text{and } |k_3| \sim |k_2|\}$,
Sector (ii) = $\{(k_1, k_2),\ |k_2| < \frac{1}{5}|k_1|,\ \text{and } |k_3| \sim |k_1|\}$,
Sector (iii) = $\{(k_1, k_2),\ |k_3| < \frac{1}{5}|k_1|,\ \text{and } |k_2| \sim |k_1|\}$,
Sector (iv) = $\{(k_1, k_2),\ \text{all } |k_3|, |k_1|, |k_2| \text{ are comparable size}\}$.

Lemma 2. *The denominator $d(\omega_1, \omega_2, \omega_3)$ is bounded from below as follows:*

In region (iv), $d(\omega_1, \omega_2, \omega_3) \geq C\big(\langle k_1\rangle^{3/2} + \langle k_2\rangle^{3/2} + \langle k_3\rangle^{3/2}\big)^4$.
In regions (i), (ii) and (iii), $d(\omega_1, \omega_2, \omega_3) \geq C\langle k_1\rangle^2\langle k_2\rangle^2\langle k_3\rangle^2$.

Proof. First of all, $\omega(k) \sim \langle k\rangle^{3/2}$ for large $|k|$. Also, the expression $d(\omega_1, \omega_2, \omega_3)$ is even in k_1, k_2, k_3 and invariant after permutations of the arguments. It is thus sufficient to find for example a lower bound in sectors (iv) and (ii).

In sector (iv), $|k_3|, |k_1|, |k_2|$ are comparable size. Thus, each term in the product $d(\omega_1, \omega_2, \omega_3)$ is bounded from below by $C(\langle k_1\rangle^{3/2} + \langle k_2\rangle^{3/2} + \langle k_3\rangle^{3/2})$ and the estimate given in Lemma 2 is straightforward.

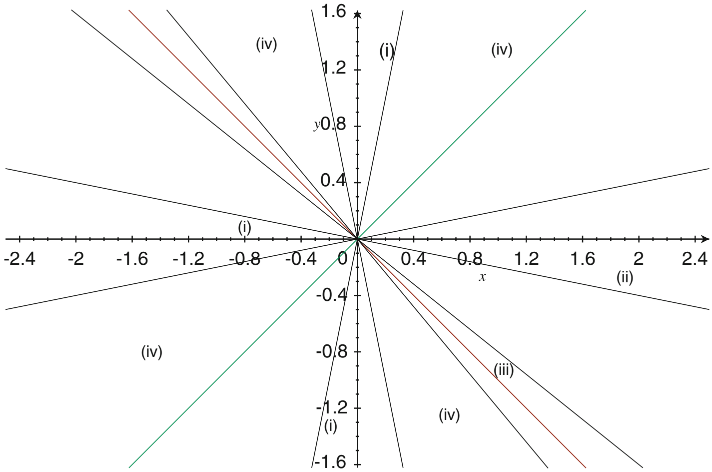

Fig. 1 Division of the plane (k_1, k_2)

In sector (ii) where $|k_2| < \frac{1}{5}|k_1|$, two of the factors appearing in the denominator are easily bounded :

$$\omega_1 + \omega_2 + \omega_3 \geq \langle k_1 \rangle^{3/2} + \langle k_2 \rangle^{3/2} + \langle k_3 \rangle^{3/2}$$

and

$$\omega_1 - \omega_2 + \omega_3 \geq \omega_3 \geq \langle k_3 \rangle^{3/2} .$$

In this sector, $\langle k_3 \rangle \sim \langle k_1 \rangle \geq 5\langle k_2 \rangle$. Thus

$$\omega_1 - \omega_2 + \omega_3 \geq C(\langle k_1 \rangle^{3/2} + \langle k_2 \rangle^{3/2} + \langle k_3 \rangle^{3/2}) .$$

It remains to bound the two other factors. Consider the factor $\omega_1 - \omega_2 - \omega_3$. In the region of (ii) where $k_1 > 0,\ k_2 > 0$, we have that $k_3 < 0$ and $|k_3| > |k_1|$. Thus, for some s_0 , $|k_1| < s_0 < |k_3|$,

$$\omega_3 - \omega_1 + \omega_2 = (|k_3| - |k_1|)\omega'(s_0) + \omega_2 \geq \langle k_2 \rangle\ C\langle k_3 \rangle^{1/2}. \tag{32}$$

However, in the region of (ii) where $k_1 > 0,\ k_2 < 0$, we have that $k_3 < 0$ and $|k_3| < |k_1|$. Thus,

$$\begin{aligned} \omega_1 - \omega_3 - \omega_2 &= \int_{|k_3|}^{|k_1|} \omega'(s)ds - \int_0^{|k_2|} \omega'(s)ds \\ &= |k_2|\ (\omega'(s_1) - \omega'(s_2))\ , \end{aligned} \tag{33}$$

for some $|k_3| < s_1 < |k_1|$ and $0 < s_2 < |k_2|$. Since $|k_2| \le |k_1|/5$, we have also $|k_2| \le |k_3|/4$, and $|\sup_{s_1} \omega'(s_1) - \inf_{s_2} \omega'(s_2)| \ge C\langle k_3\rangle^{1/2}$ for k_3 sufficiently large. The region of (ii) where $k_1 < 0$ is treated similarly as well as the last factor in the denominator $\omega_1 + \omega_2 - \omega_3$. This concludes the proof of the lower bound for the denominator $d_{k_1k_2k_3}$. □

4.2 Auxiliary Estimates

Lemma 3. *Let $r_0 > 1/2$ and $A(p,q) = \sum_{k+k_1+k_2=0} A_{kk_1k_2} p_{k_1} q_{k_2}$. Suppose that for $k + k_1 + k_2 = 0$, the kernel $A_{kk_1k_2}$ satisfies*

$$|A_{kk_1k_2}| \le C_0 \min\{\langle k_1\rangle^{-r_0}, \langle k_2\rangle^{-r_0}\}.$$

Then,

$$\|A(p,q)\|_{\ell^2} \le C\|p\|_{\ell^2} \, \|q\|_{\ell^2} \,. \tag{34}$$

Proof. We divide the bilinear form into two sums, the sum over k_1 such that $|k| \le 2|k_1|$ and the sum over k_1 such that $2|k_1| \le |k|$.

In the first sum, the argument satisfies $|k| \le 2|k_1$, therefore $|k_2| \le 3|k_1|$. We first bound the ℓ^2 -norm by the norm of the operator with kernel $A_{kk_1k_2} p_{k_1}$,

$$\begin{aligned} C_1 := \sup_k \sum_{k_2} |A_{kk_1k_2} p_{k_1}| &\le \sup_k \sum_{k_2} \frac{C_0}{\langle k_1\rangle^{r_0}} |p_{k_1}| \\ &\le \sup_k \Big(\sum_{k_2} \frac{C_0^2}{\langle k+k_2\rangle^{2r_0}}\Big)^{1/2} \sup_k \Big(\sum_{k_1} |p_{k_1}|^2\Big)^{1/2} \\ &\le C_0 \|p\|_{\ell^2}. \end{aligned} \tag{35}$$

Also,

$$\begin{aligned} C_2 := \sup_{k_2} \sum_{k} |A_{kk_1k_2} p_{k_1}| &\le \sup_{k_2} \sum_{k} \frac{C_0}{\langle k_1\rangle^{r_0}} |p_{k_1}| \\ &\le \sup_{k_2} \sum_{k} \frac{C_0}{\langle k+k_2\rangle^{r_0}} |p_{k+k_2}| \\ &\le \sup_{k_2} \Big(\sum_{k} \frac{C_0^2}{\langle k+k_2\rangle^{2r_0}}\Big)^{1/2} \sup_k \Big(\sum_{k} |p_{k+k_2}|^2\Big)^{1/2} \\ &\le C_0 \|p\|_{\ell^2}. \end{aligned} \tag{36}$$

The ℓ^2-norm of the first sum $A_1(p,q)$ is thus bounded by

$$\|A_1(p,q)\| \;\le\; \sqrt{C_1C_2}\|q\|_{\ell^2} \le\; C_0\|p\|_{\ell^2}\|q\|_{\ell^2} \;. \tag{37}$$

The second summation A_2 follows similarly, given that the conditions

$$k_1 + k_2 + k = 0 \;, \quad \text{and} \;\; 2|k_1| \le |k|$$

imply

$$2(|k| - |k_2|) \le 2|k + k_2| = 2|k_1| \le |k|,$$

therefore, $|k| \le 2|k_2|$. exchanging indices k_1 and k_2, and the role of p and q, the estimate for A_2 follows. □

This lemma also holds in the case of $k \in \mathbb{R}$ rather than $k \in \mathbb{Z}$. That is to say, for the problem posed on all of $\mathbb{R}$ rather than the periodic case.

The estimate obtained in the above lemma is however not strong enough to control all the terms appearing in estimates of Theorem 2 because the hypothesis on $A_{kk_1k_2}$ is too symmetric with respect to k_1, k_2. We sometimes need to examine the different regions separately and establish estimates accordingly.

Lemma 4. *Suppose the bilinear form $A(p,q)$ with kernel $A_{kk_1k_2}$ satisfies*

$$A(p,q) = \sum_{k+k_1+k_2=0} A_{kk_1k_2} p_{k_1} q_{k_2}$$

with

$$|A_{kk_1k_2}| \;\le\; C_0\langle k_1\rangle^{-r_0} \;, \quad r_0 > 1 \;.$$

Then

$$\|A(p,q)\|_{\ell^2} \;\le\; C|p|_{\ell^\infty}\, \|q\|_{\ell^2} \le\; C\|p\|_{\ell^2}\, \|q\|_{\ell^2} \;. \tag{38}$$

The roles of p and q in this lemma can be inverted, therefore one achieves the same conclusion (38) if $|A_{kk_1k_2}| \;\le\; C_0\langle k_2\rangle^{-r_0}$ instead.

Proof. From the hypothesis, we have that

$$|A_{kk_1k_2} p_{k_1}| \;\le\; a(k_1) = C_0\langle k_1\rangle^{-r_0} |p|_{\ell^\infty}. \tag{39}$$

For $r_0 > 1$, this is an ℓ^1-sequence. This majorant provides estimates of the norm of the linear operator with kernel $A_{kk_1k_2}p_{k_1}|_{k+k_1+k_2=0}$:

$$\sup_k \sum_{k_2} |A_{kk_1k_2}p_{k_1}| \leq \sum_{k_2} C_0 |p_{k_1}|_{\ell^\infty} \frac{1}{\langle k+k_2\rangle^{r_0}} \leq C_0' |p_{k_1}|_{\ell^\infty} \text{ when } r_0 > 1. \tag{40}$$

Similarly,

$$\sup_{k_2} \sum_k |A_{kk_1k_2}p_{k_1}| \leq C_0' |p_{k_1}|_{\ell^\infty}. \tag{41}$$

The bilinear ℓ^2 estimate follows from the simple fact that $|p|_{\ell^\infty} \leq \|p\|_{\ell^2}$ on sequence spaces. □

The analysis also encounters the Dirichlet-Neumann operator in various terms and sectors. For example, in sector (i) (where $|k_1| \ll |k_2|, |k|$) the terms that stem from the near-commutator nature of the operator is exhibited by the fact that k and k_2 have opposite signs. Therefore

$$|kk_2 + G_k G_2| \leq C\, e^{-h(|k|+|k_2|)}. \tag{42}$$

As a consequence, we have the following proposition.

Proposition 2. *Assume that the bilinear form* $B(p,q) = \sum_{k+k_1+k_2=0} B_{kk_1k_2} p_{k_1} q_{k_2}$ *has a kernel* $b(k,k_1,k_2)$ *satisfying one of conditions below :*

(a) its support is included in sector (i), and the near-commutator term is given by $kk_2 + G_k G_2$.
(b) its support is in sector (ii) and the near-commutator term is given by $kk_1 + G_k G_1$.

Then

$$\|B(p,q)\|_{\ell^2} \leq C \|p\|_{\ell^2} \|q\|_{\ell^2} . \tag{43}$$

Proposition 3. *The bilinear form* $B(p,q) = \sum_{k+k_1+k_2=0} B_{kk_1k_2} p_{k_1} q_{k_2}$ *with kernel* $B_{kk_1k_2}$ *such that*

(a) it is supported in region (iii), and
(b) for $s - \beta = r_0 > 1$ *has growth bounds of the form:*

$$|B_{kk_1k_2}| = b(k,k_1,k_2) \frac{\langle k\rangle^s}{\langle k_1\rangle^s \langle k_2\rangle^s} , \text{ with } |b(k,k_1,k_2)| \leq \langle k\rangle^\beta + \langle k_1\rangle^\beta + \langle k_2\rangle^\beta$$

gives rise to the estimate

$$\|B(p,q)\|_{\ell^2} \leq C \|p\|_{\ell^2} \|q\|_{\ell^2} . \tag{44}$$

Proof. In region (iii), we have $|k| \ll |k_1|$ and $|k_2|$. Therefore

$$|B_{kk_1k_2}| = |b(k,k_1,k_2)| \frac{1}{\langle k_1\rangle^{s/2}\langle k_2\rangle^{s/2}} \frac{\langle k\rangle^s}{\langle k_1\rangle^{s/2}\langle k_2\rangle^{s/2}} \\ \leq \frac{\langle k\rangle^\beta + \langle k_1\rangle^\beta + \langle k_2\rangle^\beta}{\langle k_1\rangle^{s/2}\langle k_2\rangle^{s/2}} \leq \frac{C}{\langle k_1\rangle^{s-\beta}} \tag{45}$$

and because $|k_1|$ and $|k_2|$ are of the same order, this satisfies the hypothesis of Lemma 3. □

Proposition 4. *The bilinear form* $B(p,q) = \sum_{k+k_1+k_2=0} B_{kk_1k_2} p_{k_1} q_{k_2}$ *with kernel* $B_{kk_1k_2}$ *such that*

(a) it is supported in region (i), and
(b) for $s - \beta \geq r_0 > 1$ *has growth bounds of the form:*

$$|B_{kk_1k_2}| = b(k,k_1,k_2)\frac{\langle k\rangle^s}{\langle k_1\rangle^s\langle k_2\rangle^s}\ , \text{ with } |b(k,k_1,k_2)| \leq \langle k_1\rangle^\beta \tag{46}$$

gives rise to the estimate

$$\|B(p,q)\|_{\ell^2} \leq C\|p\|_{\ell^2}\ \|q\|_{\ell^2}\ . \tag{47}$$

Proof. The region (i) is defined as the sector in the (k_1,k_2) plane where $|k_1| \ll |k_2|, |k|$. Therefore,

$$|B_{kk_1k_2}| = |b(k,k_1,k_2)|\ \frac{\langle k\rangle^s}{\langle k_1\rangle^s\langle k_2\rangle^s} \leq \langle k_1\rangle^{\beta-s}. \tag{48}$$

For $s - \beta = r_0 > 1$, the kernel satisfies the hypothesis of Lemma 4. □

Note: The same conclusion holds under the hypothesis that (a) the kernel $B_{kk_1k_2}$ is supported in region (ii) and (b) the estimate (46) is true with the role of k_1 and k_2 exchanged.

4.3 Energy Estimates for Vector Field $X^{K^{(3)}}$

In this section, we give energy estimates for the vector field (22), whose solutions taken at time $s = -1$ is the desired canonical transformation to the Birkhoff normal form at third order.

The evolution of energy norms are expressed as

$$\frac{d}{ds}\|(\eta,\xi)\|^2_{E^r} = 2\mathrm{Re}\Big(\langle \eta, \sigma|D|^2\partial_s\eta\rangle_r + \langle \xi, G_0\partial_s\xi\rangle_r\Big) \\ = 2\mathrm{Re}\Big(\langle \sigma|D|^2\eta, \partial_\xi K^{(3)}\rangle_r - \langle G_0\xi, \partial_\eta K^{(3)}\rangle_r\Big). \tag{49}$$

The RHS of (49) contains cancellations that are quite subtle, leading to the following estimate.

Lemma 5. *Fix $r_0 > 2$. For all $r \geq r_0$, there is a bound in the form*

$$|\langle \sigma |D|^2 \eta, \partial_\xi K^{(3)} \rangle_r - \langle G_0 \xi, \partial_\eta K^{(3)} \rangle_r| \; \leq C_{r_0} \|(\eta, \xi)\|^3_{E^r}. \tag{50}$$

Proof. The proof is somewhat computational, indeed even the expression of the LHS of (50) is rather long. For convenience, we will assume that there are no resonant triplets, namely that $[H^{(3)}] = 0$, otherwise stated as $P = 0$. The difference in (50) if it were not zero would only lead to a compact perturbation of the RHS. In this setting,

$$\begin{aligned}
&\langle \sigma |D|^2 \eta, \partial_\xi K^{(3)} \rangle_r - \langle G_0 \xi, \partial_\eta K^{(3)} \rangle_r \\
&= \Sigma'_{k_1+k_2+k_3=0} \Big[\frac{k_1 k_3 + G_1 G_3}{d_{123}} (g + \sigma k_1^2)(\omega_1^2 - \omega_2^2 - \omega_3^2) \\
&\quad + \frac{k_1 k_2 + G_1 G_2}{d_{123}} (g + \sigma k_1^2)(g + \sigma k_2^2) G_3 \Big] \sigma |k_3|^2 \; \eta_1 \eta_2 \eta_3 \langle k_3 \rangle^{2r} \\
&\quad + \Sigma'_{k_1+k_2+k_3=0} \Big[\frac{k_1 k_3 + G_1 G_3}{d_{123}} G_2 (\omega_1^2 - \omega_2^2 + \omega_3^2) \\
&\quad + \frac{k_2 k_3 + G_2 G_3}{2 d_{123}} G_1 (\omega_3^2 - \omega_1^2 + \omega_2^2) \Big] \sigma |k_1|^2 \; \eta_1 \xi_2 \xi_3 \; \langle k_1 \rangle^{2r} \\
&\quad - \Sigma'_{k_1+k_2+k_3=0} \Big[\frac{k_2 k_3 + G_2 G_3}{d_{123}} (g + \sigma k_3^2)(\omega_3^2 - \omega_1^2 - \omega_2^2) \\
&\quad\; \frac{k_1 k_2 + G_1 G_2}{d_{123}} (g + \sigma k_1^2)(\omega_1^2 - \omega_2^2 - \omega_3^2) \\
&\quad\; 2 \frac{k_1 k_3 + G_1 G_3}{d_{123}} (g + \sigma k_1^2)(g + \sigma k_3^2) G_2 \Big] \; G_3 \eta_1 \xi_2 \xi_3 \; \langle k_3 \rangle^{2r}.
\end{aligned} \tag{51}$$

The indices have been relabeled to exhibit cancellations. The notation $\Sigma'_{k_1+k_2+k_3=0}$ is as above, summation over all $k_j \in \mathbb{Z} \backslash \{0\}$, and $d_{123} = d_{k_1 k_2 k_3}$.

Estimates for the sums involving $\eta_1 \eta_2 \eta_3$ an $\eta_1 \xi_2 \xi_3$ are treated independently. We will start with the first quantity in the RHS of (51) involving $\eta_1 \eta_2 \eta_3$. Energy estimates for η in terms of E^r-norms are tantamount to ℓ^2 estimates for $p = \sigma^{1/2} |D| \langle D \rangle^r \eta$, leading to the sum

$$\begin{aligned}
&\Sigma'_{k_1+k_2+k_3=0} \Big[\frac{k_1 k_3 + G_1 G_3}{d_{123}} (g + \sigma k_1^2)(\omega_1^2 - \omega_2^2 - \omega_3^2) \\
&\qquad + \frac{k_1 k_2 + G_1 G_2}{d_{123}} (g + \sigma k_1^2)(g + \sigma k_2^2) G_3 \Big] \frac{\sigma^{1/2} |k_3| \; \langle k_3 \rangle^r}{\sigma |k_1| \langle k_1 \rangle^r |k_2| \langle k_2 \rangle^r} \; p_1 p_2 p_3.
\end{aligned} \tag{52}$$

The expression (52) has the form $\sum_{k_1+k_2+k_3} b_{k_1k_2k_3}p_1p_2p_3$ for which we write

$$|\sum_{k_1+k_2+k_3=0} b_{k_1k_2k_3}p_1p_2p_3| \leq \|\sum_{k_1+k_2+k_3=0} b_{k_1k_2k_3}p_1p_2\|_{\ell^2}\|p\|_{\ell^2}. \tag{53}$$

In order to bound the above sum, we separate the contribution of each sector (i)–(iv) of the lattice (k_1, k_2). Most of the work consists in finding appropriate bounds for the kernel $b_{k_1k_2k_3}$ in order to apply Lemma 4 and Propositions 2–4. This will be the procedure to estimate each term of (51). For the term (52), the kernel $b_{k_1k_2k_3}$ identifies to

$$\begin{aligned} b_{k_1k_2k_3} &= \frac{k_1k_3 + G_1G_3}{d_{123}}(g+\sigma k_1^2)(\omega_1^2-\omega_2^2-\omega_3^2)\frac{\sigma^{1/2}|k_3|\,\langle k_3\,\rangle^r}{\sigma|k_1|\langle k_1\rangle^r|k_2|\langle k_2\rangle^r} \\ &\quad + \frac{k_1k_2 + G_1G_2}{d_{123}}(g+\sigma k_1^2)(g+\sigma k_2^2)G_3\,\frac{\sigma^{1/2}|k_3|\,\langle k_3\,\rangle^r}{\sigma|k_1|\langle k_1\rangle^r|k_2|\langle k_2\rangle^r} \\ &=: \frac{N_a}{D_a} + \frac{N_b}{D_b} \end{aligned} \tag{54}$$

where $\frac{N_a}{D_a}$ and $\frac{N_b}{D_b}$ identify respectively to the first and second term of the RHS of (54). Next, we bound each of these terms in the different sectors.

Sector (i): where $|k_1| \ll |k_2|, |k_3|$.

The numerator N_a is bounded above by

$$|N_a| \leq \langle k_1\rangle^3\,(\langle k_1\rangle^3 + \langle k_2\rangle^3 + \langle k_3\rangle^3)\,\langle k_3\rangle^{r+1},$$

while the denominator D_a is bounded below by

$$|D_a| \geq \langle k_1\rangle^{3+r}\langle k_2\rangle^{3+r}\langle k_3\rangle^2.$$

In this sector, $\langle k_2\rangle \sim \langle k_3\rangle$, therefore

$$\frac{|N_a|}{|D_a|} \leq C_0\langle k_1\rangle^{-r}. \tag{55}$$

Taking $r \geq r_0 > 1$, the result of Lemma 4 applied to this term in the sector (i), implies the bound $\|p\|_{\ell^2}^3$.

Considering the term N_b/D_b in this sector, the analogous estimate is that

$$|N_b| \leq \langle k_1\rangle^3\langle k_2\rangle^3\langle k_3\rangle^{2+r},$$

while

$$|D_b| \geq \langle k_1\rangle^{3+r}\langle k_2\rangle^{3+r}\langle k_3\rangle^2,$$

with the conclusion that again

$$\frac{|N_b|}{|D_b|} \le C_0 \langle k_1 \rangle^{-r}. \tag{56}$$

Sector (ii): where $|k_2| \ll |k_1|, |k_3|$.

The numerator N_a is bounded above by the quantity

$$|N_a| \le |k_1 k_3 + G_1^{(0)} G_3^{(0)}| \langle k_1 \rangle^2 (\langle k_1 \rangle^3 + \langle k_2 \rangle^3 + \langle k_3 \rangle^3) \langle k_3 \rangle^{1+r}.$$

In this sector, both $|k_1|$ and $|k_3|$ are large and comparable, of opposite sign, and the commutator estimate (43) holds, which compensates any polynomial growth. Therefore

$$\frac{|N_a|}{|D_a|} \le C_0 e^{-h(|k_1|+|k_3|)/2}. \tag{57}$$

Term N_b/D_b: The numerator behaves as

$$|N_b| \le \langle k_1 \rangle^3 \langle k_2 \rangle^3 \langle k_3 \rangle^{2+r},$$

while the denominator is as above, hence

$$\frac{|N_b|}{|D_b|} \le C_0 \langle k_2 \rangle^{-r}, \tag{58}$$

again giving rise the ℓ^2-estimates in p.

In *Sector (iii)* : where $|k_3| \ll |k_1|, |k_2|$, the weights in the denominator are dominant, so that

$$|N_a| \le \langle k_1 \rangle^3 (\langle k_1 \rangle^3 + \langle k_2 \rangle^3 + \langle k_3 \rangle^3) \langle k_3 \rangle^{3+r}$$

$$|N_b| \le \langle k_1 \rangle^3 \langle k_2 \rangle^3 \langle k_3 \rangle^3,$$

while

$$|D_a|, |D_b| \ge \langle k_1 \rangle^{3+r} \langle k_2 \rangle^{3+r} \langle k_3 \rangle^2,$$

so that

$$\frac{|N_b|}{|D_b|}, \; \frac{|N_b|}{|D_b|} \le C_0 \langle k_1 \rangle^{-r/2} \langle k_2 \rangle^{-r/2}. \tag{59}$$

Finally consider sector (iv) in which all $|k_1|, |k_2|$ and $|k_3|$ are comparable. In this region,

$$|D_a|,\ |D_b| \geq \langle k_1\rangle^{3+r}\langle k_2\rangle^{3+r}\langle k_3\rangle^2,$$

which easily dominate the numerators

$$|N_a|,\ |N_b|\ \leq \langle k_1\rangle^3\ (\langle k_1\rangle^3 + \langle k_2\rangle^3 + \langle k_3\rangle^3)\ \langle k_3\rangle^{3+r},$$

giving an estimate in terms of $\|p\|^3_{\ell^2}$ as long as $r \geq r_0 > 1/2$.

We now turn to terms in the RHS of (51) that involve $\eta_1\xi_2\xi_3$. As before, we introduce the weighted quantities $p = \sigma^{1/2}|D|\langle D\rangle^r\eta$, $q = G^{(0)}\langle D\rangle^r\xi$. We rewrite the last five terms of the RHS of (51) in terms of p_1, q_2 and q_3 and symmetrize with respect to the indices 2 and 3. This leads to nine terms that we label (a_1) to (a_9) and we list below. Each term will be examined in the previously defined four sectors of the lattice (k_1, k_2). In some cases, the estimate will be similar to what we have already seen and the bound is straightforward. In other cases, the previous analysis will not be sufficient. However, by combining terms, we will be able to control them thanks to cancellations. The three first terms are

$$\begin{aligned}
&\Sigma'_{k_1+k_2+k_3=0}\ \frac{1}{2d_{123}} p_1q_2q_3\Big[\\
&(a_1)\quad (k_1k_3 + G_1G_3)G_2(\omega_1^2 - \omega_2^2 + \omega_3^2)\sigma|k_1|^2\\
&(a_2)\quad (k_1k_2 + G_1G_2)G_3(\omega_1^2 - \omega_2^2 + \omega_3^2)\sigma|k_1|^2\\
&(a_3)\quad (k_2k_3 + G_2G_3)G_2(\omega_1^2 - \omega_3^2 + \omega_2^2)\sigma|k_1|^2\Big]\frac{\langle k_1\rangle^r}{\sigma^{1/2}|k_1|\langle k_1\rangle^r G_2\langle k_2\rangle^r G_3\langle k_3\rangle^r}.
\end{aligned}$$

The next 4 terms are

$$\begin{aligned}
&-\Sigma'_{k_1+k_2+k_3=0}\ \frac{1}{2d_{123}} p_1q_2q_3\Big[\\
&(a_4)\quad (k_2k_3 + G_2G_3)(g + \sigma k_3^2)(\omega_3^2 - \omega_1^2 - \omega_2^2)G_3\frac{\langle k_3\rangle^r}{\sigma^{1/2}|k_1|\langle k_1\rangle^r G_2\langle k_2\rangle^r G_3\langle k_3\rangle^r}\\
&(a_5)\quad + (k_2k_3 + G_2G_3)(g + \sigma k_2^2)(\omega_2^2 - \omega_1^2 - \omega_3^2)G_2\frac{\langle k_2\rangle^r}{\sigma^{1/2}|k_1|\langle k_1\rangle^r G_2\langle k_2\rangle^r G_3\langle k_3\rangle^r}\\
&(a_6)\quad + (k_1k_2 + G_1G_2)(g + \sigma k_1^2)(\omega_1^2 - \omega_2^2 - \omega_3^2)G_3\frac{\langle k_3\rangle^r}{\sigma^{1/2}|k_1|\langle k_1\rangle^r G_2\langle k_2\rangle^r G_3\langle k_3\rangle^r}\\
&(a_7)\quad + (k_1k_3 + G_1G_3)(g + \sigma k_1^2)(\omega_1^2 - \omega_3^2 - \omega_2^2)G_2\frac{\langle k_2\rangle^r}{\sigma^{1/2}|k_1|\langle k_1\rangle^r G_2)\langle k_2\rangle^r G_3\langle k_3\rangle^r}\Big]
\end{aligned}$$

and finally the last two terms are

$$-\Sigma'_{k_1+k_2+k_3=0}\frac{1}{2d_{123}}p_1q_2q_3\Big[$$

$$(a_8)\quad 2(k_1k_3+G_1G_3)(g+\sigma k_1^2)(g+\sigma k_3^2)G_2G_3\frac{\langle k_3\rangle^r}{\sigma^{1/2}|k_1|\langle k_1\rangle^r G_2\langle k_2\rangle^r G_3\langle k_3\rangle^r}$$

$$(a_9)\quad +2(k_1k_2+G_1G_2)(g+\sigma k_1^2)(g+\sigma k_2^2)G_3G_2\frac{\langle k_2\rangle^r}{\sigma^{1/2}|k_1|\langle k_1\rangle^r G_2\langle k_2\rangle^r G_3\langle k_3\rangle^r}\Big].$$

One has to count the factors of k_1, k_2, k_3 for each term and each sector. The general principle is that we need at least three factors of $|k_1|$ in the numerator for sector (i) and we check that this is the case for all nine terms. In sector (ii) [resp. (iii)] , we need at least 2.5 factors of $|k_2|$ in the numerator [resp. 2.5 factors of $|k_3|$].

In Sector (ii), the terms that present a difficulty are those labelled $(a_2), (a_3), (a_4), (a_6)$. Note that all the weight can be shifted to $\langle k_1\rangle^r$ or $\langle k_3\rangle^r$ (or $\langle k_1\rangle^{r/2}\langle k_3\rangle^{r/2}$, modulo a factor of $\langle k_1\rangle - \langle k_3\rangle \sim |k_2|$). We write only the relevant terms of the sum of these four contributions, the numerator of which is

$$\begin{aligned}&\sigma(\omega_1^2-\omega_3^2)\langle k_1\rangle^{r/2}\langle k_3\rangle^{r/2}\\ &\Big[(k_1k_2+G_1G_2)G_3|k_1|^2-(k_2k_3+G_2G_3)G_1|k_1|^2\\ &\qquad+(k_2k_3+G_2G_3)G_3|k_3|^2-(k_1k_2+G_1G_2)G_3|k_1|^2\Big].\end{aligned}$$

The first and last term cancel exactly and the expression reduces to

$$\begin{aligned}&\sigma(\omega_1^2-\omega_3^2)\langle k_1\rangle^{r/2}\langle k_3\rangle^{r/2}\Big[-(k_2k_3+G_2G_3)(G_1|k_1|^2-G_3|k_3|^2)\Big]\\ &\sim O(|k_2|^3)\langle k_1\rangle^{r/2}\langle k_3\rangle^{r/2}(\langle k_1\rangle^5+\langle k_3\rangle^5).\end{aligned}\tag{60}$$

The corresponding denominator is

$$\begin{aligned}D&=d_{123}\sigma^{1/2}|k_1|\;G_2^{1/2}\langle k_2\rangle^r\;G_3^{1/2}\langle k_3\rangle^r\\ &\geq\;\langle k_1\rangle^3\langle k_2\rangle^{r+5/2}\langle k_3\rangle^{r+5/2}.\end{aligned}\tag{61}$$

Counting the powers in the numerator and denominator, we have

$$\frac{N}{D}\leq\langle k_2\rangle^{1/2-r}\langle k_1\rangle^{-1/2},\tag{62}$$

which allows us to conclude the result in this case.

In Sector (iii), the terms that are critical are $(a_1), (a_3), (a_5).(a_7)$. Proceeding as above for Sector (ii), we write the relevant terms for the sum of these four contributions. In the numerator, we have

$$(\omega_1^2 - \omega_2^2)\sigma \langle k_1 \rangle^{r/2} \langle k_2 \rangle^{r/2}$$
$$\Big[(k_1 k_3 + G_1 G_3) G_2 |k_1|^2 - (k_2 k_3 + G_2 G_3) G_1 |k_1|^2$$
$$+ (k_2 k_3 + G_2 G_3) G_2 |k_2|^2 - (k_1 k_3 + G_1 G_3) G_2 |k_1|^2\Big]. \tag{63}$$

The first and last term in the brackets cancel and the expression becomes

$$(\omega_1^2 - \omega_2^2)\sigma \langle k_1 \rangle^{r/2} \langle k_2 \rangle^{r/2} (k_2 k_3 + G_2 G_3)(-G_1 |k_1|^2 + G_2 |k_2|^2) + O(|k_3|^4)$$
$$= O(|k_3|^3).$$

The numerator N and denominator D of terms $(a_1) + (a_3) + (a_5) + (a_7)$ in sector (iii) thus satisfy

$$N \leq \langle k_1 \rangle^{r/2} \langle k_2 \rangle^{r/2} \langle k_3 \rangle^3 (\langle k_1 \rangle^5 + \langle k_2 \rangle^5) \quad ; \quad D \leq \langle k_1 \rangle^3 \langle k_2 \rangle^{r+5/2} \langle k_3 \rangle^{r+5/2},$$

leading to

$$\frac{N}{D} \leq \langle k_3 \rangle^{1/2-r} \langle k_1 \rangle^{1/2}$$

and thus the application of In sector (iv), $|k_1| \sim |k_2| \sim |k_3|$ and the bounds are straightforward. This concludes the proof of Lemma 5. □

4.4 Energy Estimates for the Variational Equation

The vector field in question is $X^{K^{(3)}}(\eta, \xi) = (\partial_\xi K^{(3)}, -\partial_\eta K^{(3)})$ where $(z = \eta, \xi) \in E^r$. Denote the variations of the orbit by $\tilde{z} = (\tilde{\eta}, \tilde{\xi}) = (\delta\eta, \delta\xi)$. These satisfy

$$\frac{d}{ds}\begin{pmatrix} \tilde{\eta} \\ \tilde{\xi} \end{pmatrix} = \partial_z X^{K^{(3)}}(\eta, \xi) \begin{pmatrix} \tilde{\eta} \\ \tilde{\xi} \end{pmatrix}$$
$$= \begin{pmatrix} \partial_\eta \partial_\xi K^{(3)} & \partial_\xi^2 K^{(3)} \\ -\partial_\eta^2 K^{(3)} & -\partial_\eta \partial_\xi K^{(3)} \end{pmatrix} \begin{pmatrix} \tilde{\eta} \\ \tilde{\xi} \end{pmatrix}. \tag{64}$$

The goal is to prove an energy estimate for solutions of (64) of the form:

Lemma 6.

$$\frac{d}{ds}\|\tilde{z}\|_{E^r}^2 \;\leq\; C_0 \|z\|_{E^{r'}} \|\tilde{z}\|_{E^r}^2 \tag{65}$$

where $r' \geq r + 1/2$.

Proof. The first task is to calculate an expression in Fourier coordinates for the RHS of (64).

$$\sqrt{2\pi}\,\partial_\eta\partial_\xi K^{(3)}\tilde{\eta}_2 = \sum_{k_1+k_2+k=0} \frac{1}{d_{12k}}\Big[(k_2k + G_2G_k)(g+\sigma k_2^2)(\omega_2^2-\omega_1^2-\omega_k^2)$$
$$(k_1k + G_1G_k)(g+\sigma k_1^2)(\omega_1^2-\omega_2^2-\omega_k^2)$$
$$2(k_1k_2 + G_1G_2)(g+\sigma k_1^2)(g+\sigma k_2^2)G_k\Big]\eta_1\tilde{\eta}_2$$

$$\sqrt{2\pi}\,\partial_\xi^2 K^{(3)}\tilde{\xi}_2 = \sum_{k_1+k_2+k=0} \frac{1}{d_{12k}}\Big[(k_2k + G_2G_k)G_1(\omega_2^2-\omega_1^2+\omega_k^2)$$
$$(k_1k + G_1G_k)G_2(\omega_1^2-\omega_2^2+\omega_k^2)$$
$$(k_1k_2 + G_1G_2)G_k(\omega_1^2-\omega_k^2+\omega_2^2)\Big]\xi_1\tilde{\xi}_2$$

$$\sqrt{2\pi}\,\partial_\eta^2 K^{(3)}\tilde{\eta}_2 = \sum_{k_1+k_2+k=0} \frac{1}{d_{12k}}\Big[2(k_2k + G_2G_k)(g+\sigma k_2^2)(g+\sigma k^2)G_1$$
$$(k_1k + G_1G_k)(g+\sigma k^2)(\omega_k^2-\omega_2^2-\omega_k^2)$$
$$(k_1k_2 + G_1G_2)(g+\sigma k_2^2)(\omega_2^2-\omega_k^2-\omega_1^2)\Big]\xi_1\tilde{\eta}_2$$

$$\frac{d}{ds}\|\tilde{z}\|_{E^r}^2 = 2\mathrm{Re}\Big[\langle\tilde{\eta},\ \sigma|D|^2(\partial_\eta\partial_\xi K^{(3)}\tilde{\eta} + \partial_\xi^2 K^{(3)}\tilde{\xi})\rangle_r$$
$$- \langle\tilde{\xi},\ G(\partial_\eta^2 K^{(3)}\tilde{\eta} + \partial_\xi\partial_\eta K^{(3)}\tilde{\xi})\rangle_r\Big]. \tag{66}$$

This expression is the sum of 12 terms that we list below and denote (a) to (ℓ).

$$(a) = \sum \frac{1}{d_{123}}\Big[2(k_1k_2 + G_1G_2)(g+\sigma k_1^2)(g+\sigma k_2^2)G_3$$
$$(b) \qquad + (k_1k_3 + G_1G_3)(g+\sigma k_1^2)(\omega_1^2-\omega_2^2-\omega_3^2)$$
$$(c) \qquad + (k_2k_3 + G_2G_3)(g+\sigma k_2^2)(\omega_2^2-\omega_1^2-\omega_3^2)\Big]\sigma k_3^2\ \eta_1\tilde{\eta}_2\tilde{\eta}_3\langle k_3\rangle^{2r}$$
$$(d) \quad + \sum \frac{1}{d_{123}}\Big[(k_1k_2 + G_1G_2)G_3(\omega_1^2-\omega_3^2+\omega_2^2)$$
$$(e) \qquad + (k_1k_3 + G_1G_3)G_2(\omega_1^2-\omega_2^2+\omega_3^2)$$

$$
\begin{aligned}
&(f) \qquad + (k_2k_3 + G_2G_3)G_1(\omega_2^2 - \omega_1^2 + \omega_3^2)\Big]\sigma k_3^2\, \xi_1\tilde{\xi}_2\tilde{\eta}_3\langle k_3\rangle^{2r} \\
&(g) \quad -\sum \frac{1}{d_{123}}\Big[2(k_1k_3 + G_1G_3)(g + \sigma k_1^2)(g + \sigma k_3^2)G_2 \\
&(h) \qquad + (k_2k_3 + G_2G_3)(g + \sigma k_3^2)(\omega_3^2 - \omega_1^2 - \omega_2^2) \\
&(i) \qquad + (k_1k_2 + G_1G_2)(g + \sigma k_1^2)(\omega_1^2 - \omega_2^2 - \omega_3^2)\Big]G_3\, \eta_1\tilde{\xi}_2\tilde{\xi}_3\langle k_3\rangle^{2r} \\
&(j) \quad -\sum \frac{1}{d_{123}}\Big[2(k_2k_3 + G_2G_3)(g + \sigma k_2^2)(g + \sigma k_3^2)G_1 \\
&(k) \qquad + (k_1k_3 + G_1G_3)(g + \sigma k_3^2)(\omega_3^2 - \omega_1^2 - \omega_2^2) \\
&(\ell) \qquad + (k_1k_2 + G_1G_2)(g + \sigma k_2^2)(\omega_2^2 - \omega_3^2 - \omega_1^2)\Big]G_3\, \xi_1\tilde{\eta}_2\tilde{\xi}_3\langle k_3\rangle^{2r}.
\end{aligned}
\tag{67}
$$

Terms $(a), (b), (c)$ of the form $\eta_1\tilde{\eta}_2\tilde{\eta}_3$ will be estimated in terms of the r'-order energy norm of $z = (\eta, \xi)$ and r-order energy norm of $\tilde{z} = (\tilde{\eta}, \tilde{\xi})$. Similarly to the energy estimates of Sect. 4.3, we introduce $p = \sigma^{1/2}|D|\langle D\rangle^{r'}\eta$ and $\tilde{p} = \sigma^{1/2}|D|\langle D\rangle^{r}\tilde{\eta}$, and write $\langle k_3\rangle^{2r}\eta_1\tilde{\eta}_2\tilde{\eta}_3 = p_1\tilde{p}_2\tilde{p}_3\frac{\langle k_3\rangle^r}{\sigma^{3/2}|k_1|\langle k_1\rangle^{r'}|k_2|\langle k_2\rangle^r|k_3|}$. We seek estimates in terms of the ℓ^2 norm of p and $\tilde{p}$., and we need to examine the corresponding kernels in order to satisfy the hypotheses of Propositions 1–3. We consider the expressions of the kernels in sectors:

In sector (i) where $|k_1| \ll |k_2| \sim |k_3|$, we need only to analyze $(a) + (b)$. Their numerators and denominators satisfy

$$|N_a| \le \langle k_1\rangle^3\langle k_2\rangle^3\langle k_3\rangle^{3+r}$$

$$|D_a| \ge \langle k_1\rangle^{3+r'}\langle k_2\rangle^{3+r}\langle k_3\rangle^3$$

$$|N_b| \le \langle k_1\rangle^3(\langle k_2\rangle^3 + \langle k_3\rangle^3)\langle k_3\rangle^{3+r}$$

$$|D_b| \ge \langle k_1\rangle^{3+r'}\langle k_2\rangle^{3+r}\langle k_3\rangle^3,$$

thus

$$\frac{N_a}{D_a}, \frac{N_b}{D_b} \le C_0\langle k_1\rangle^{-r'},$$

where we assume $r' > 1$.

In sector (ii) where $|k_2| \ll |k_1| \sim |k_3|$, terms $(a) + (c)$ are relevant.

$$|N_a| \le \langle k_1\rangle^3\langle k_2\rangle^3\langle k_3\rangle^{3+r}$$

$$|D_a| \ge \langle k_1\rangle^{3+r'}\langle k_2\rangle^{3+r}\langle k_3\rangle^3$$

thus

$$\frac{N_a}{D_a} \le C_0 \langle k_2 \rangle^{-r}. \tag{68}$$

Here we need $r' \ge r$. Also,

$$|N_c| \le \langle k_2 \rangle^3 \langle k_3 \rangle^{3+r} (\langle k_1 \rangle^3 + \langle k_3 \rangle^3)$$

$$|D_c| \ge \langle k_1 \rangle^{3+r'} \langle k_2 \rangle^{3+r} \langle k_3 \rangle^3$$

$$\frac{N_c}{D_c} \le C_0 \langle k_2 \rangle^{-r} \tag{69}$$

since $r' \ge r$. Estimates in sectors (iii) and (iv) are straightforward, using that $r' > 1$.

We turn to terms $(g), (h), (i)$ which are of the form $\eta_1 \tilde{\xi}_2 \tilde{\xi}_3$. Introducing $\tilde{q} = G^{1/2} \langle D \rangle^r \tilde{\xi}$, we write $\langle k_3 \rangle^{2r} \eta_1 \tilde{\xi}_2 \tilde{\xi}_3 = p_1 \tilde{q}_2 \tilde{q}_3 \frac{\langle k_3 \rangle^r}{\sigma^{1/2} |k_1| \langle k_1 \rangle^{r'} |G_2|^{1/2} \langle k_2 \rangle^r G_3^{1/2}}$.

In sector (i) where $|k_1| \ll |k_2| \sim |k_3|$, only $(g) + (i)$ count:

$$|N_g| \le \langle k_1 \rangle^3 \langle k_2 \rangle \langle k_3 \rangle^{4+r}$$

$$|D_g| \ge \langle k_1 \rangle^{3+r'} \langle k_2 \rangle^{2.5+r} \langle k_3 \rangle^{2.5}$$

$$|N_i| \le \langle k_1 \rangle^3 \langle k_2 \rangle^2 \langle k_3 \rangle^{3+r}$$

$$|D_i| \ge \langle k_1 \rangle^{3+r'} \langle k_2 \rangle^{2.5+r} \langle k_3 \rangle^{2.5},$$

thus in sector (i)

$$\frac{N_g}{D_g}, \ \frac{N_i}{D_i} \ \le C_0 \langle k_1 \rangle^{-r'}. \tag{70}$$

In sector (ii) where $|k_2| \ll |k_1| \sim |k_3|$, $(h) + (i)$ are relevant.

$$|N_h| \le \langle k_2 \rangle^2 \langle k_3 \rangle^{4+r} (\langle k_1 \rangle^2 + \langle k_3 \rangle^2)$$

$$|D_h| \ge \langle k_1 \rangle^{3+r'} \langle k_2 \rangle^{2.5+r} \langle k_3 \rangle^{2.5}$$

$$|N_i| \le \langle k_1 \rangle^3 \langle k_2 \rangle^2 \langle k_3 \rangle^{1+r} (\langle k_1 \rangle^2 + \langle k_3 \rangle^2)$$

$$|D_i| \ge \langle k_1 \rangle^{3+r'} \langle k_2 \rangle^{2.5+r} \langle k_3 \rangle^{2.5}.$$

We have used that $|\omega_1 - \omega_3| \le |k_2| \langle k_1 \rangle^2$. Note that this is where we need the hypothesis that $r' \ge r + 1/2$. Estimates in sectors (iii) and (iv) are straightforward.

We now consider the terms involving $\xi_1\tilde{\eta}_2\tilde{\xi}_3$, that is $((d) + (e) + (f) + (j) + (k) + (\ell)$. We need to relabel the indices of $(d)+(e)+(f)$ in order to take advantage of cancellations. We are considering the following sum:

$$
\begin{aligned}
&(d) \quad \sum \frac{1}{d_{123}}\Big[(k_1k_3 + G_1G_3)G_2(\omega_1^2 - \omega_2^2 + \omega_3^2)\\
&(e) \quad\quad + (k_1k_2 + G_1G_2)G_3(\omega_1^2 - \omega_3^2 + \omega_2^2)\\
&(f) \quad\quad + (k_2k_3 + G_2G_3)G_1(\omega_2^2 - \omega_1^2 + \omega_3^2)\Big]\sigma k_2^2\langle k_2\rangle^{2r}\\
&(j) \quad -\sum \frac{1}{d_{123}}\Big[2(k_2k_3 + G_2G_3)(g + \sigma k_2^2)(g + \sigma k_3^2)G_1\\
&(k) \quad\quad + (k_1k_3 + G_1G_3)(g + \sigma k_3^2)(\omega_3^2 - \omega_1^2 - \omega_2^2)\\
&(\ell) \quad\quad + (k_1k_2 + G_1G_2)(g + \sigma k_2^2)(\omega_2^2 - \omega_3^2 - \omega_1^2)\Big]G_3\langle k_3\rangle^{2r}\\
&\quad\quad \times \frac{1}{G_1^{1/2}\langle k_1\rangle^{r'}\sigma^{1/2}|k_2|\langle k_2\rangle^r G_3^{1/2}\langle k_3\rangle^r}\, p_1\tilde{q}_2\tilde{p}_3
\end{aligned}
\tag{71}
$$

where we are using the notation :

$$
p_1 = G_1^{1/2}\langle k_1\rangle^{r'}\xi_1, \quad \tilde{q}_2 = \sigma^{1/2}|k_2|\langle k_2\rangle^r\tilde{\eta}_2, \quad \tilde{p}_3 = G_3^{1/2}\langle k_3\rangle^r\tilde{\xi}_3,
$$

and we seek estimates in terms of the ℓ^2-norm of $p, \tilde{q}, \tilde{p}$.

The critical quantities that one must analyze depend upon sectors. In sector (iv) where $|k_1| \sim |k_2| \sim |k_3|$, all terms are tame and estimates follow, under the condition $r' > 1$. After inspection, we see that the critical terms are :

$(d) + (e) + (k) + (\ell)$ in sector (i)
$(j) + (e)$ in sector (ii)
$(d) + (f)$ in sector (iii).

Writing the principal terms of $(d) + (e) + (k) + (\ell)$ in sector (i) , we get

$$
\begin{aligned}
&\frac{\sigma}{d_{123}}(\omega_2^2 - \omega_3^2)\Big[(k_1k_3 + G_1G_3)(G_3k_3^2\langle k_3\rangle^{2r} - G_2k_2^2\langle k_2\rangle^{2r})\\
&\quad + (k_1k_2 + G_1G_2)G_3k_2^2(\langle k_2\rangle^{2r} - \langle k_3\rangle^{2r})\Big]\\
&\quad \times \frac{1}{G_1^{1/2}\langle k_1\rangle^{r'}\sigma^{1/2}|k_2|\langle k_2\rangle^r G_3^{1/2}\langle k_3\rangle^r}\, p_1\tilde{q}_2\tilde{p}_3.
\end{aligned}
$$

The numerator and denominator N and D of this expression are bounded as follows:

$$
\begin{aligned}
|N| &\le \langle k_1\rangle^3(\langle k_2\rangle + \langle k_3\rangle)^{5+2r}\\
|D| &\ge \langle k_1\rangle^{2.5+r'}\langle k_2\rangle^{2.5+r}\langle k_3\rangle^{2.5+r}
\end{aligned}
$$

leading to

$$\frac{N}{D} \leq C\langle k_1\rangle^{-r'}.$$

We can then apply Proposition 3 with $r' > 1$.

We now turn to the principal terms of $(j) + (\ell)$ in sector (ii). Proceeding as above, we get

$$|N_j| \leq \langle k_2\rangle^3 \langle k_3\rangle^{4+r} \langle k_1\rangle$$
$$|N_\ell| \leq \langle k_2\rangle^3 \langle k_1\rangle \langle k_3\rangle^{1+2r} (\langle k_1\rangle^3 + \langle k_3\rangle^3)$$
$$|D_j|,\ |D_\ell| \geq \langle k_1\rangle^{2.5+r'} \langle k_2\rangle^{3+r} \langle k_3\rangle^{2.5+r},$$

leading to the desired estimate if $r \geq r$, and $r' > 1$, Finally terms $(d) + (f)$ are bounded by:

$$|N_d| \leq \langle k_3\rangle^2 \langle k_1\rangle (\langle k_1\rangle^2 + \langle k_2\rangle^2) \langle k_2\rangle^{3+2r}$$
$$|N_f| \leq \langle k_3\rangle^2 \langle k_2\rangle^{3+2r} \langle k_1\rangle (\langle k_1\rangle^2 + \langle k_2\rangle^2)$$
$$|D_d|,\ |D_f| \geq \langle k_1\rangle^{2.5+r'} \langle k_2\rangle^{3+r} \langle k_3\rangle^{2.5+r}.$$

Again here, one can apply Proposition 3 if $r' \geq r + 1/2$. This concludes the proof of Theorem 2. □

4.5 Smoothness Estimates for the Transformation

Proof. This section presents the proof of Theorem 3. The variational equation about a solution $\psi_s(\eta, \xi)$ is given by (64), and by Lemma 6, the linearized equation has solutions which satisfy the energy estimate with a loss of $1/2$ derivative:

$$|\langle(\tilde{\eta}, \tilde{\xi}),\ \partial_{(\eta,\xi)} X^{K^{(3)}}(\eta, \xi) \left(\tilde{\eta}\ \tilde{\xi}\right)^T \rangle_{E^{r-1/2}}| \ \leq\ \|(\tilde{\eta}, \tilde{\xi})\|^2_{E^{r-1/2}} \tag{72}$$

for $\psi_s(\eta, \xi) \in B_R \subseteq E^r$. Therefore by Gronwall's lemma, for $s < s_R$

$$\|\partial_{(\eta,\xi)} \psi_s - I\|_{E^{r-1/2}}\ \leq C\, R. \tag{73}$$

Estimates of higher derivatives of the flow $\psi_s(\eta, \xi)$, $s < s_R$, follow with a similar argument. Namely, setting $z = (\eta, \xi)$, higher derivatives of the flow satisfy the inhomogeneous equations

$$\frac{\partial}{ds}(\partial_z^p \psi_s) = \partial_z X^{K^{(3)}}(\partial_z^p \psi_s) + \sum_{\substack{p_1+p_2=p \\ 1\le p_1 . p_2 < p}} C_{p_1p_2} \langle \partial_z^2 X^{K^{(3)}}(\partial_z^{p_1}\psi_s), (\partial_z^{p_2}\psi_s)\rangle. \tag{74}$$

Since $X^{K^{(3)}}$ is quadratic in its arguments, no other terms appear. Furthermore, since $p \geq 2$, $(\partial_z^p \psi_s)|_{s=0} = 0$, therefore by induction,

$$\begin{aligned} \|\partial_z^p \psi_s\|_{E^{r-p/2}} &\leq C_{rp} \sup_{|s|<s_R} \sum_{\substack{p_1+p_2=p \\ 1\le p_1 . p_2 < p}} \|\partial_z^{p_1}\psi_s\|_{E^{r-p/2+1/2}} \|\partial_z^{p_2}\psi_s\|_{E^{r-p/2+1/2}} \\ &\leq C_{rp} \end{aligned}$$

using the induction hypothesis that $\|\partial_z^{p_1}\psi_s\|_{E^{r-p/2+1/2}} \leq C_{r,p-1}$ for all $1 \leq p_1 \leq p-1$. □

5 Resonant Triads

The change of variables that we have introduced eliminates non resonant cubic terms in the Hamiltonian. In the new variables, the only third order terms that remain correspond to resonant triads. The dynamics of the resonant subsystems therefore will dominate the behavior of the full system of water waves for long periods of time. There have been a number of formal studies of the behavior of resonant triads for the system of water wave equations with surface tension, including [2, 9]. The normal forms analysis of our work gives a rigorous justification of these studies. Several special configurations of coupled resonant triads are considered in these papers, including of course the simplest triad of three interaction resonant modes which is isolated (in Fourier space) from the dynamics of the other modes, resonant or not. In this section we describe this simple configuration in the framework of a Hamiltonian system, and examine the stability of its periodic orbits. We also consider the case of two coupled resonant triads, in a different setting of coupled resonances than those of Hammack and Henderson [9]. We do not give an exhaustive analysis of all of the possible resonant cases.

5.1 Single Resonant Triad

Assume that the triads $(\pm k_1^0, \pm k_2^0, \pm k_3^0)$ are resonant; that is, $k_1^0 + k_2^0 + k_3^0 = 0$ and $\omega_1 - \omega_2 - \omega_3 = 0$, which is the standard resonant triad (modulo a possible reindexing of wave numbers). As before we use the notation that $\omega_i = \omega_{k_i^0}$. For fixed spatial period such resonances are nongeneric, but they will exist for certain values of the physical parameters. From the choice of signs we have assumed that

$|k_1^0| > |k_2^0|$, and $|k_1^0| > |k_3^0|$, and therefore we may take $k_1^0 > 0 > k_2^0, k_3^0$. Returning to the expression of the transformed Hamiltonian, and retaining only the resonant modes, the quadratic and cubic terms, the truncated Hamiltonian is given by

$$H_+ = H_+^{(2)} + H_+^{(3)} = \sum_{j=\pm 1}^{\pm 3} \omega_j z_j \bar{z}_j + C_+^{(3)} (z_1 \bar{z}_{-2} \bar{z}_{-3} + \bar{z}_{-1} z_2 z_3 + z_{-1} \bar{z}_2 \bar{z}_3 + \bar{z}_1 z_{-2} z_{-3}). \quad (75)$$

This is a system of 6 degrees of freedom, with the three modes $(z_1, \bar{z}_2, \bar{z}_3)$ decoupled from $(\bar{z}_{-1}, z_2, z_3)$. Written in terms of symplectic polar coordinates (phase and square-root of the amplitude), $z_j = \sqrt{R_j} e^{i\theta_j}$, the Hamiltonian H_+ takes the form

$$H_+ = \sum_1^3 (\omega_j R_j + \omega_{-j} R_{-j}) + 2C_+^{(3)} \Big(\sqrt{R_1 R_{-2} R_{-3}} \cos(\theta_1 - \theta_{-2} - \theta_{-3}) + \sqrt{R_{-1} R_2 R_3} \cos(\theta_{-1} - \theta_2 - \theta_3) \Big). \quad (76)$$

Only two angles appear in the expression of H_+, one for each system of three degrees of freedom. Thus there are four conserved quantities, and the system reduces to two decoupled systems each with one degree of freedom and therefore integrable, and indeed it consists of two independent copies of the well-known three-wave resonant system. Specifically, perform a change of variable in the form of a simultaneous rotation,

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_{-2} \\ \phi_{-3} \end{pmatrix} = A \begin{pmatrix} \theta_1 \\ \theta_{-2} \\ \theta_{-3} \end{pmatrix}, \quad I = \begin{pmatrix} I_1 \\ I_{-2} \\ I_{-3} \end{pmatrix} = A \begin{pmatrix} R_1 \\ R_{-2} \\ R_{-3} \end{pmatrix} \quad (77)$$

where $A = (a)_{ij}$ is a 3×3 rotation, making the natural choice to set, $\phi_1 = \frac{1}{\sqrt{3}}(\theta_1 - \theta_{-2} - \theta_{-3})$, and $\omega_1 R_1 + \omega_2 R_{-2} + \omega_3 R_{-3} = \Omega_3 I_{-3}$, where $\Omega_3 = \sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2}$. The matrix A has the form

$$A = \begin{pmatrix} 1/\sqrt{3} & -1/\sqrt{3} & -1/\sqrt{3} \\ (\omega_3 - \omega_2)/N_2 & (\omega_3 - \omega_1)/N_2 & (-\omega_2 - \omega_1)/N_2 \\ \omega_1/\Omega_3 & \omega_2/\Omega_3 & \omega_3/\Omega_3 \end{pmatrix}$$

with $N_2^2 = (\omega_3 - \omega_2)^2 + (\omega_3 - \omega_1)^2 + (\omega_2 + \omega_1)^2$. In the new variables, the first term of H_+ is written as $\Omega_3 I_3 + \Omega_3 I_{-3}$. Expressed in these action-angle variables, the Hamiltonian for the system involving $(R_1, R_{-2}, R_{-3}, \theta_1, \theta_{-2}, \theta_{-3})$ is

$$H_+ = \Omega_3 I_{-3} + 2C^{(3)} \sqrt{R_1 R_{-2} R_{-3}} \cos(\sqrt{3}\phi_1) ,$$

where $R_j = R_j(I_1)$ depend linearly upon (I_1, I_{-2}, I_{-3}). The Hamiltonian involving $(R_{-1}, R_2, R_3, \theta_{-1}, \theta_2, \theta_3)$ is similar. The equations of motion are

$$\begin{aligned} \dot{I}_1 &= -\partial_{\phi_1} H_+ \\ \dot{\phi}_1 &= \partial_{I_1} H_+ \end{aligned} \tag{78}$$

along with the two cyclic variables (ϕ_{-2}, ϕ_{-3}) and the canonically conjugate conserved quantities (I_{-2}, I_3) which are considered as parameters.

$$\begin{aligned} &\dot{I}_{-2} = -\partial_{\phi_{-2}} H_+ = 0\ , \quad \dot{I}_{-3} = -\partial_{\phi_{-3}} H_+ = 0 \\ &\dot{\phi}_{-2} = \partial_{I_{-2}} H_+\ , \quad \dot{\phi}_{-3} = \partial_{I_{-3}} H_+\ . \end{aligned} \tag{79}$$

The range of I_1 such that $R_1(I), R_{-2}(I), R_{-3}$ are all positive, which is an interval with endpoints I_1^+ and I_1^- depending parametrically on I_{-2} and I_{-3}. The endpoint I_1^- is characterized by the vanishing of R_1, while I_1^+ is defined by a zero of R_{-2} or R_{-3}, whichever vanishes first. There is one exceptional case, determined by a choice of parameters (I_{-2}, I_{-3}) such that R_{-2} and R_{-3} vanish simultaneously in I_1. In this case the factor $\sqrt{R_1 R_{-2} R_{-3}}$ vanishes linearly in I_1 at I_1^+. The system for (I_1, ϕ_1) (and for (I_{-1}, ϕ_{-1}) respectively) can be analysed through its phase plane.

Firstly, the phase plane $\{(I_1, \phi_1) : I_1^- \le I_1 \le I_1^+ , 0 \le \phi_1 < 2\pi/\sqrt{3}\}$ is identified as a sphere $\mathbb{S}^2$, with polar coordinate singularities at the endpoints $I_1 = I_1^-$ and $I_1 = I_1^+$ defining the poles. Typical orbits are time periodic, meaning that a typical orbit for the full system (78), (79) will be quasiperiodic with three basic frequencies. Lower dimensional tori are found through the stationary points of the system (78), which occur when

$$\sqrt{3}\phi_1 = 0, \pi\ , \quad \partial_{I_1}(R_1 R_{-2} R_{-3}) = (R_{-2}R_{-3} - R_1 R_{-3} - R_1 R_{-2}) / \sqrt{3 R_1 R_{-2} R_{-3}} = 0\ .$$

By inspection, except for one particular case, there are only two such stationary points per sphere, whose locations are at points $\phi_1^0 = 0, \pi/\sqrt{3}$ and $I_1 = I_1^0(I_{-2}, I_{-3})$ where $\sqrt{R_1 R_{-2} R_{-3}}$ achieves its maximum. Both are stable periodic orbits, as can be seen from the variational equation of the vector field at the stationary point in question, namely

$$\begin{aligned} J\partial^2_{I_1,\phi_1} H_+ &= \begin{pmatrix} 0 & 2C^{(3)} \partial^2_{I_1} \sqrt{R_1 R_{-2} R_{-3}} \cos(\sqrt{3}\phi_1) \\ 2C^{(3)} \sqrt{R_1 R_{-2} R_{-3}} \cos(\sqrt{3}\phi_1) & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \end{aligned}$$

where A and B have opposite signs because of the character of a maximum.

The exception occurs in the case in which the parameters (I_{-2}, I_{-3}) have been adjusted such that R_{-2} and R_{-3} vanish simultaneously in I_1. This corresponds to a stationary point of the vector field itself, rather than a polar coordinate singularity, and for the full system (78), (79) this stationary point corresponds to a basic periodic orbit of Lyapunov type that is guaranteed by the Lyapunov center theorem, associated with the highest frequency ω_1 of the resonant triad system. This periodic orbit is unstable, as the variational equation of the vector field at this point is

$$\begin{aligned} J\partial^2_{I_1,\phi_1} H_+ &= \begin{pmatrix} -2C^{(3)}\partial_{I_1}\sqrt{R_1R_{-2}R_{-3}}\sin(\sqrt{3}\phi_1) & 2C^{(3)}\partial^2_{I_1}\sqrt{R_1R_{-2}R_{-3}}\cos(\sqrt{3}\phi_1) \\ 0 & 2C^{(3)}\partial_{I_1}\sqrt{R_1R_{-2}R_{-3}}\sin(\sqrt{3}\phi_1) \end{pmatrix} \\ &= \begin{pmatrix} A & C \\ 0 & -A \end{pmatrix} \end{aligned}$$

for $A = -4C^{(3)}\sqrt{R_1(I_1^+)}\sin(\sqrt{3}\phi_1)$ real valued, which gives the Lyapunov exponent in explicit terms. There is a family of orbits homoclinic to this Lyapunov-type periodic orbit, consisting of $I_1(t)$ running down along the coordinate axis $\{\phi_1 = \pi/2\}$, through the south pole of the sphere at $I_1 = I_1^-$ and then back up the coordinate axis $\{\phi_1 = 3\pi/2\}$.

5.2 Mutliple Resonant Triads

There are numerous possibilities for multiple coupled resonant triads in this case of a non-zero coefficient of surface tension, number of which are discussed in references [2, 9]. In the present paper we will analyse one case that does not appear in these articles, of a system of two coupled resonant triads satisfying the resonance relations

$$\begin{gathered} k_1 + k_2 + k_3 = 0\,, \quad -k_1 + k_2 + k_4 = 0\,, \\ k_2\,,\ k_3 < 0 < k_1 < k_4 \\ \omega_1 - \omega_2 - \omega_3 = 0\,, \qquad \omega_4 - \omega_1 - \omega_2 = 0\,. \end{gathered}$$

This is among the simplest situations that is possible with multiple coupled triads. For further simplicity we consider only standing wave solutions, namely we impose Neumann boundary conditions on two vertical walls of the fluid domain at $\{x = 0\}$ and $\{x = \pi\}$, which has the effect that in our complex symplectic coordinates, $z_k = z_{-k}$. The water waves Hamiltonian truncated at third order becomes

$$H_+ = H_+^{(2)} + H_+^{(3)} = \sum_{j=1}^{4} \omega_j |z_j|^2 + \left[c_1^{(3)} z_1\bar{z}_2\bar{z}_3 + c_2^{(3)}\bar{z}_1\bar{z}_2 z_4 + c.c.\right]. \tag{80}$$

Introducing symplectic polar coordinates $z_j = \sqrt{R_j}e^{i\theta_j}$, we have

$$H_+^{(2)} = \sum_{j=1}^{4} \omega_j R_j \ ,$$

$$H_+^{(3)} = 2c_1^{(3)}\sqrt{R_1R_2R_3}\cos(\theta_1 - \theta_2 - \theta_3) + 2c_2^{(3)}\sqrt{R_1R_2R_4}\cos(\theta_4 - \theta_1 - \theta_2) \ .$$

Perform a symplectic change of coordinates $I = AR$ and $\Phi = A\Theta$, where

$$A = \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} \\ b_1 & b_2 & b_3 & b_4 \\ \frac{\omega_1}{\Omega_4} & \frac{\omega_2}{\Omega_4} & \frac{\omega_3}{\Omega_4} & \frac{\omega_4}{\Omega_4} \end{pmatrix}$$

where $\Omega_4^2 = \omega_1^2 + \omega_2^2 + \omega_3^2 + \omega_4^2$, and where b_j are chosen appropriately so that the matrix A is orthogonal. The Hamiltonian (80) becomes

$$H_+ = \Omega_4 I_4 + \left[2c_1^{(3)}\sqrt{R_1R_2R_3}\cos(\sqrt{3}\phi_1) + 2c_2^{(3)}\sqrt{R_1R_2R_4}\cos(\sqrt{3}\phi_2)\right] , \quad (81)$$

with $R_j = R_j(I_1, I_2; I_3, I_4)$ which are affine linear in the action variables (I_1, I_2). The two angles (ϕ_3, ϕ_4) do not appear in the Hamiltonian $H_=$ in (81), they are cyclic variables, and their canonically conjugate variables (I_3, I_4) are integrals of motion. This is a Hamiltonian system with two degrees of freedom, given by the Hamiltonian $H_+^{(3)}(I_1, I_2, \phi_1, \phi_2)$ described in (81), and posed on the manifold $M := \{(I_1, I_2, \phi_1, \phi_2) \ : \ R_j(I) \geq 0 \, , \, (\phi_1, \phi_2) \in \mathbb{T}^2\}$. Topologically, taking into account the polar coordinate singularities at the poles $R_j(I) = 0$, the manifold M is either a $\mathbb{S}^2 \times \mathbb{S}^2$ or a $\mathbb{S}^3 \times \mathbb{S}^1$, depending upon the values of the parameters (I_3, I_4).

The dynamics of a system with two degrees of freedom can be quite complex in general. We will restrict ourselves to considering special structures in this phase space, namely periodic orbits and lower dimensional tori (quasi-periodic motion with two independent frequencies), all of which can be analysed through an inspection of the vector field $X^{H_+^{(3)}}$ on the phase space M. From this point on, denote the relevant action—angle variables by $\mathrm{t}(I, \Phi) := (I_1, I_2, \phi_1, \phi_2)$; the phase space M is coordinatized by (I, Φ) except for the poles of the spheres corresponding to one or several of the polar coordinate singularities $R_j(I) = 0$. The equations of motion restricted to M are given by

$$\dot{\Phi} = \partial_I H_+^{(3)} \ , \qquad \dot{I} = -\partial_\Phi H_+^{(3)} \ . \quad (82)$$

Stationary points of (82) give rise to periodic, or more generally quasi-periodic orbits for the full system governed by the Hamiltonian (80). We first treat stationary points of (82) interior to the coordinate chart $M^0 := \{R_j(I) > 0 : \forall j\}$. Such stationary points satisfy

$$\partial_{I_j} H_+^{(3)} = \big[2c_1^{(3)} \partial_{I_j}(\sqrt{R_1 R_2 R_3}) \cos(\sqrt{3}\phi_1) + 2c_2^{(3)} \partial_{I_j}(\sqrt{R_1 R_2 R_4}) \cos(\sqrt{3}\phi_2)\big]$$
$$-\partial_{\phi_1} H_+^{(3)} = 2\sqrt{3} c_1^{(3)} \sqrt{R_1 R_2 R_3} \sin(\sqrt{3}\phi_1)$$
$$-\partial_{\phi_1} H_+^{(3)} = 2c_2^{(3)} \sqrt{R_1 R_2 R_4} \sin(\sqrt{3}\phi_2).$$

Since $\sqrt{R_1 R_2 R_3}, \sqrt{R_1 R_2 R_4} > 0$ in M^0, stationary points may only occur where $\phi_1 = 0, \pi/\sqrt{3}$ and $\phi_2 = 0, \pi/\sqrt{3}$. The case where both $\phi_1, \phi_2 = 0$ is called the *in-phase* solutions, and when $\phi_1 = 0$ but $\phi_2 = \pi/\sqrt{3}$ these are *out-of-phase* solutions. All other choices of ϕ_j reduce to these two cases, using possibly a time reversal. Thus such critical points are characterized by the condition that

$$0 = \partial_I\big(\sqrt{R_1 R_2}(c_1^{(3)} \sqrt{R_3} \pm c_2^{(3)} \sqrt{R_4})\big)$$

where the plus sign corresponds to the in-phase case and the minus sign is for the out-of-phase solutions.

Proposition 5. *For fixed parameter values $I_3 = a_3$, $I_4 = a_4$, the in-phase case has two stationary points on the manifold M , which are both stable. The resulting solutions of the full system* (80) *are generically quasi-periodic with two independent frequencies, and are geometrically distinct but related one to the other by a time reversal.*

Proof. The Hamiltonian $H_+^{(3)}$, when evaluated on the hypersurface $(\phi_1, \phi_2) = 0$ as indicated as being necessary in the paragraph above, is positive and can only vanish when either $R_1(I) = 0$ or $R_2(I) = 0$, a subset of the boundaries of the coordinate chart M^0. On the boundary sets defined by $R_3(I) = 0$ or $R_4(I) = 0$, the Hamiltonian $H_+^{(3)} > 0$ while its outward unit normal derivative is negative. Hence any maximum I^* is an interior point. Since both functions $R_1 R_2 R_3$ and $R_1 R_2 R_4$ are cubic polynomials when considered as functions of (I_1, I_2),and furthermore R_3 is independent of I_4 while R_4 is independent of I_3, there can be at most one interior maximum, and there are no other critical points. This critical point is identified as the in-phase quasi-periodic solution of the system (80).

The statement of stability of this stationary point follows from an analysis of the first variation of the vector field $J\partial_{(\Phi,I)} H_+^{(3)}$ at the critical point (Φ^*, I^*). Namely

$$\begin{pmatrix} \partial_\Phi \partial_I H_+^{(3)} & \partial_I^2 H_+^{(3)} \\ -\partial_\Phi^2 H_+^{(3)} & -\partial_\Phi \partial_I H_+^{(3)} \end{pmatrix} = \begin{pmatrix} 0 & \partial_I^2 H_+^{(3)} \\ -\partial_\Phi^2 H_+^{(3)} & 0 \end{pmatrix} \tag{83}$$

where

$$-\partial_\Phi^2 H_+^{(3)} = \begin{pmatrix} 6c_1^{(3)} \sqrt{R_1 R_2 R_3} & 0 \\ 0 & 6c_2^{(3)} \sqrt{R_1 R_2 R_4} \end{pmatrix}.$$

The eigenvalues μ of (83) are given by

$$\mu^2 = 3\big(c_1^{(3)}\sqrt{R_1R_2R_3}h_{11} + c_2^{(3)}\sqrt{R_1R_2R_4}h_{22}\big) \pm 3\sqrt{\big((c_1^{(3)}\sqrt{R_1R_2R_3}h_{11} + c_2^{(3)}\sqrt{R_1R_2R_4}h_{22})^2 - 4c_1^{(3)}c_2^{(3)}R_1R_2\sqrt{R_3R_4}\,\det(h)\big)} \tag{84}$$

where $h = (h_{j\ell}) = \partial^2_{I_{j\ell}} H_+^{(3)}(\Phi^*, I^*)$, and all other expressions are also evaluated at $(\Phi, I) = (\Phi^*, I^*)$. The constants $c_j^{(3)}$ are both positive. Because I^* is a nondegenerate maximum, $h_{11} = \partial^2_{I_1} H_+^{(3)}(\Phi^*, I^*) < 0$ and $h_{22} = \partial^2_{I_2} H_+^{(3)}(\Phi^*, I^*) < 0$, while $\det(h) > 0$. In addition the radicand of (84) also satisfies

$$\begin{aligned}&(c_1^{(3)}\sqrt{R_1R_2R_3}h_{11} + c_2^{(3)}\sqrt{R_1R_2R_4}h_{22})^2 - 4c_1^{(3)}c_2^{(3)}R_1R_2\sqrt{R_3R_4}\,\det(h)\\ &= (c_1^{(3)}\sqrt{R_1R_2R_3}h_{11} - c_2^{(3)}\sqrt{R_1R_2R_4}h_{22})^2 + 4c_1^{(3)}c_2^{(3)}R_1R_2\sqrt{R_3R_4}h_{12}^2\,,\end{aligned}$$

which is nonnegative. The radicand also satisfies

$$\begin{aligned}&(c_1^{(3)}\sqrt{R_1R_2R_3}h_{11} + c_2^{(3)}\sqrt{R_1R_2R_4}h_{22})^2 - 4c_1^{(3)}c_2^{(3)}R_1R_2\sqrt{R_3R_4}\,\det(h)\\ &< (c_1^{(3)}\sqrt{R_1R_2R_3}h_{11} + c_2^{(3)}\sqrt{R_1R_2R_4}h_{22})^2\,.\end{aligned}$$

Therefore both roots μ^2 of (84) are negative, and the eigenvalues μ of (83) arise in pure imaginary complex conjugate pairs. □

Proposition 6. *For $(I_3, I_4) = (a_3, a_4)$ fixed, the out-of-phase solutions on M are either two or four in number, and are geometrically distinct but are interchanged pairwise by time reversal of the system. The resulting solutions of the full system (80) are quasi-periodic with generically two independent frequencies, and they are all unstable.*

Proof. The Hamiltonian $H_+^{(3)}$, when evaluated on the hypersurface $\Phi^* := (\phi_1 = 0, \phi_2 = \pi/\sqrt{3})$ is either everywhere positive, or else changes sign, and in the latter case there is only one component of each sign. It is

$$H_+^{(3)}\big|_{\Phi=\Phi^*} = 2c_1^{(3)}\sqrt{R_1R_2R_3} - 2c_2^{(3)}\sqrt{R_1R_2R_4}\,. \tag{85}$$

The Hamiltonian $H_+^{(3)}$ vanishes on the boundaries of the region $M = \{(I_1, I_2) : R_j(I) > 0\}$ for which either $R_1(I) = 0$ or $R_2(I) = 0$. Since $R_3(I)$ is decreasing in I_1 and independent of I_2, while $R_4(I)$ is independent of I_1 and increasing in I_2, the boundary component on which $R_4(I) = 0$ is always nonempty, and on it $H_+^{(3)}|_{\Phi=\Phi^*} > 0$. The boundary component defined by $R_3(I) = 0$ may be empty, in which case $H_+^{(3)}|_{\Phi=\Phi^*} > 0$ throughout the region. If it is not empty, $H_+^{(3)}|_{\Phi=\Phi^*} < 0$ on this set, while its outward normal derivative is positive, giving rise to a region in which $H_+^{(3)}|_{\Phi=\Phi^*}$ is negative. Because of the monotonicity properties of $R_3(I)$ and

$R_4(I)$ there can be at most one component of each sign, and because of the affine linear character of the $R_j(I)$, the maximum and minimum critical points are unique and nondegenerate.

As mentioned above, one deduces from the monotonicity of the $R_j(I)$ that the boundary components of M^0 corresponding to $R_1(I) = 0$ and $R_4(I) = 0$ are always nonempty. On the other hand, under different choices of parameters $(I_3, I_4) = (a_3, a_4)$ it could be that there are nonempty boundary components for both $R_2(I)$ and $R_3(I)$, or it could be that one of the two is empty. In the former case, the manifold $M \simeq \mathbb{S}^2 \times \mathbb{S}^2$, while if one boundary component is empty, then $M \simeq \mathbb{S}^3 \times \mathbb{S}^1$.

The statement of instability of these orbits comes again from an inspection of the spectrum of the variational equation (83) at the stationary points (Φ^*, I^*), give in this case by

$$\begin{pmatrix} \partial_\Phi \partial_I H_+^{(3)} & \partial_I^2 H_+^{(3)} \\ -\partial_\Phi^2 H_+^{(3)} & -\partial_\Phi \partial_I H_+^{(3)} \end{pmatrix} = \begin{pmatrix} 0 & \partial_I^2 H_+^{(3)} \\ -\partial_\Phi^2 H_+^{(3)} & 0 \end{pmatrix} \tag{86}$$

where as before,

$$-\partial_\Phi^2 H_+^{(3)} = \begin{pmatrix} 6c_1^{(3)} \sqrt{R_1 R_2 R_3} & 0 \\ 0 & -6c_2^{(3)} \sqrt{R_1 R_2 R_4} \end{pmatrix}.$$

The eigenvalues of (86) are expressed by

$$\mu^2 = 3\big(c_1^{(3)} \sqrt{R_1 R_2 R_3} h_{11} - c_2^{(3)} \sqrt{R_1 R_2 R_4} h_{22}\big) \\ \pm 3\sqrt{\big((c_1^{(3)} \sqrt{R_1 R_2 R_3} h_{11} - c_2^{(3)} \sqrt{R_1 R_2 R_4} h_{22})^2 + 4c_1^{(3)} c_2^{(3)} R_1 R_2 \sqrt{R_3 R_4} \det(h)\big)} \tag{87}$$

where $h = \partial_I^2 H_+^{(3)}(\Phi^*, I^*)$ as above. For I^* a maximum critical point, $h_{11}, h_{22} < 0$ and $\det(h) > 0$, and we observe that the radicand is positive and that the radical also dominates the first term $3(c_1^{(3)} \sqrt{R_1 R_2 R_3} h_{11} - c_2^{(3)} \sqrt{R_1 R_2 R_4} h_{22})$ in absolute value. That is to say, of the roots μ^2 of (87), one is positive and one is negative, leading to the eigenvalues μ of (86) consisting of a pure imaginary eigenvalue pair and a pair of real eigenvalues, one of which is positive; this does not give rise to a fully whiskered torus for the full system, but one whose normal environment consists of two center tangent vectors as well as a one dimensional stable and a one dimensional unstable component of the tangent space of M.

In cases in which there is a nontrivial negative minimum critical point, giving rise to a second pair of stationary points (Φ^*, I^*), we have $\det(h) > 0$ while $h_{11}, h_{22} > 0$. The radicand of (87) is again positive, the radical dominates the first term, and therefore there is again a pair of real eigenvalues and a pair of pure imaginary and complex conjugate eigenvalues of the variational equation (86).

The remaining phase space orbit of interest is the stationary point at which $R_1(I)$ and $R_2(I)$ vanish simultaneously, which occurs on the boundary of the coordinate chart M^0. □

Proposition 7. *In the situation in which $R_1(I)$ and $R_2(I)$ vanish simultaneously on the boundary of the coordinate chart M^0, there is an additional quasi-periodic orbit of system (80). This orbit can be either stable or unstable, depending upon the values of the integrals of motion (I_3, I_4). But even in the stable case its variational equation has double pure imaginary roots, which have opposite Krein signature and are therefore unstable under generic perturbations.*

Proof. For this case one must work in another, and local coordinate system. Both $R_1(I) = 0 = R_2(I)$ when

$$R_1 = \frac{1}{\sqrt{3}}(I_1 - I_2) + K_1 = 0 , \quad R_2 = -\frac{1}{\sqrt{3}}(I_1 + I_2) + K_2 = 0 ,$$

where $K_1 := b_1 I_3 + \omega_1/\omega_4 I_4$ and $K_2 := b_2 I_3 + \omega_2/\omega_4 I_4$, constants set by the values of the two integrals of motion. This simultaneous zero exists whenever the subset of the boundary of M^0 defined by $\{R_2(I) = 0\}$ is nonempty, and it occurs when $I_1 = I_1^0 := -\sqrt{3}/2(K_1 - K_2)$ and $I_2 = I_2^0 := \sqrt{3}/2(K_1 + K_2)$. Using than $R_j(I) = |z_j|^2, j = 1, 2$, then the Hamiltonian $H_+^{(3)}$ can be written in terms of z_1, z_2 as

$$\begin{aligned} H_+^{(3)} &= 3c_1^{(3)}\sqrt{R_3}|z_1||z_2|\cos(\theta_1 - \theta_2 - \theta_3) \pm 3c_2^{(3)}\sqrt{R_4}|z_1||z_2|\cos(\theta_4 - \theta_1 - \theta_2) \\ &= 3c_1^{(3)}(z_1\bar{z}_2\bar{z}_3 + \bar{z}_1 z_2 z_3) \pm 3c_2^{(3)}(\bar{z}_1\bar{z}_2 z_4 + z_1 z_2 \bar{z}_4) \end{aligned}$$

where to lowest order in $(I_1 - I_1^0, I_2 - I_2^0)$, the variables (z_3, z_4) are constants (Z_3, Z_4), set by the value of the conserved quantities (I_3, I_4). The resulting variational equation in the variables (z_1, z_2) is as follows:

$$\begin{aligned} \dot{z}_1 &= i\big(3c_1^{(3)} Z_3 z_2 \pm 3c_2^{(3)} Z_4 \bar{z}_2\big) \\ \dot{z}_2 &= i\big(3c_1^{(3)} \bar{Z}_3 z_1 \pm 3c_2^{(3)} Z_4 \bar{z}_1\big) , \end{aligned} \tag{88}$$

which is written in the form

$$\begin{pmatrix} \dot{z}_1 \\ \dot{\bar{z}}_1 \\ \dot{z}_2 \\ \dot{\bar{z}}_2 \end{pmatrix} = i \begin{pmatrix} 0 & 0 & a & b \\ 0 & 0 & -\bar{b} & -\bar{a} \\ \bar{a} & b & 0 & 0 \\ -\bar{b} & -a & 0 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ \bar{z}_1 \\ z_2 \\ \bar{z}_2 \end{pmatrix} ,$$

where $a = 3c_1^{(3)} Z_3$ and $b = \pm 3c_2^{(3)} Z_4$. The eigenvalues of this matrix are $\mu = \pm i\sqrt{|a|^2 - |b|^2}$ which are all double roots; they are of course pure imaginary complex conjugates when $|a|^2 > |b|^2$, and real when otherwise, dictating the stability of the system (88). However even in the stable case the two pairs of double eigenvalues have opposite Krein signature, as can be seen by the non-positive definiteness of the Hamiltonian giving (88). Thus even the pure imaginary case is libel to instability under perturbation. □

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