Calibration of Electricity Price Models

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Abstract This paper addresses the issue of model calibration to electricity prices. The non-storability of electricity introduces new problems in terms of modeling and calibration, especially when the objective is to represent both spot prices and forward products, the latter showing a particular time interval: the delivery period. The two main approaches to model electricity prices are: (i) models on a fictitious forward curve from what we can deduce spot prices and forward products with any delivery period, and (ii) models on spot prices from what we can deduce any forward products. In this paper we study both approaches and we focus on the calibration issues. The first part of the paper studies different calibration methods for a classic Gaussian factorial model as described in Benth and Koekebakker (2008), Kiesel, Schidlmayr, and Börger (2009) and mostly based on Heath-Jarrow-Morton approach (Heath, Jarrow, and Morton, Econometrica, 1992). In this case different calibration methods can be proposed, based on spot and/or forward prices, but the main objective is to compare or validate these estimation procedures. We compare these procedures on the valuation of specific portfolios and we then stress the high impact of the calibration method. The second part concerns the calibration issues of a structural model proposed in Aïd, Campi, Langrené (2013). In particular we study the reconstruction performances of forward prices and we address the issue of model calibration in terms of determining the parameters to exactly fit the observable forward products. We propose a modification in the structural model to ensure its ability to be calibrated on all the observed forward products and we give some illustrations of calibration performances.

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1 Introduction

Modeling electricity prices is a very exciting challenge, as their behaviour is unique compared to other assets like equities or even other commodities, a fact mostly due to its non-storability. The underlying incompleteness of the recently deregulated electricity markets makes possible a vast range of models to price electricity contracts, and thus scientific literature abounds with models that try to capture the well-known stylized facts of electricity prices.

An electricity producer needs price models for different applications: electricity price prediction (in a short-term horizon), risk-management, hedging, pricing (in a mid-term horizon) and investment decisions (in a long-term horizon). The price models used must be adapted to the application of interest. In this paper we are interested in a mid-term horizon and the financial application of risk-management, pricing and the determination of hedging strategies. We position our study on the case of an electricity producer who has to manage financial risks. His portfolios are composed of physical and financial assets:

- *Production units.* The power generating plants can be represented, in a first approximation, as a basket of European spread options whose underlying assets are the spot prices of power and the fuel used to produce electricity. In the case of thermal power plants the carbon spot price is a third underlying asset. For example, a gas power plant can be represented as a basket of European options of payoff $\left(S^e_t - h^g S^g_t - h^c S^c_t - K\right)^+$, with S_t being the spot prices and the superscripts e, g and c are respectively for electricity, gas and carbon, h^g and h^c are coefficients determining the performances of the plant (the "heat rate"), and K is the fixed production cost. Of course the modelization of power plants may be more complex once we consider dynamic constraints, starting and stopping costs... But the most important point to note is that the underlying assets are the spot prices.
- *Storage assets.* Gas storage and hydraulic dams are the most common storage assets of a power producer. They are classically represented as swing options that let the option holder buy a predetermined quantity of energy at a predetermined price while having some flexibility in the amount purchased and the price paid. The underlying assets are the spot prices of power and gas for the gas storage asset.
- *Electricity supply contracts.* On the other hand, an electricity producer has contracts for supplying electricity. These contracts may have optionalities[1] that allow them to be represented as swing options, moving average options or more exotic derivatives. The underlying assets are spot prices of power and fuels, but also forward prices in the case where the sale price depends on some historical forward prices.

[1]For example, a load curve contract allows the owner to buy at a fixed price an undetermined quantity q_t of power in an interval $[q^{min}_t \ ; \ q^{max}_t]$ around a specific load curve.

In order to manage the risks of such portfolios, a price model is needed to represent both spot prices and forward products, on several commodities in the Energy market. Therefore we focus on adapted models for this objective, in particular we are not interested in models that exclusively represent spot prices (see [7] for some examples) or, on the other hand, exclusively represent forward products as it is proposed, for example, in [6, 16].

We can classify the electricity price models into two main categories depending on the element considered as the basis of modelization, but all the models aiming to represent both spot prices and forward products need to determine the "forward curve", i.e. a function $F_t(T)$ defining fictitious forward contracts delivering 1MWh of electricity at dates T during one unit of time (1 h or 1 day).

The first class of models is dedicated to directly represent the forward curve and is mostly related to classic interest rate models like in [13]. We refer to [6] and [16] for some examples and [14] to justify using interest rate models for electricity prices. The main advantage of this class of model is the ability to use the broad literature of interest rate models which leads, in general, to a lot of closed-form formulas of pricing and an easy determination of hedging strategies. However, the calibration of these models is a real issue in the case of the power market because the observed quotations are not some points of the forward curve, but a weighted average of the forward curve over different periods (the delivery periods, as detailed in Sect. 2). Therefore the relationship between the model parameters and the observed products is more complex.

The second class of models focuses on the spot price representation. The starting point is then to model power spot prices as finely as possible, using sophisticated processes, as done for example in the popular jump-diffusion model from [10]. This has also lead to a new methodology for forecasting and modeling spot prices, and we refer to [19] for a complete panel of statistical methods that are used with reduced-form models. These models may depend on several hidden factors [5] or other observable factors. In particular, structural models define a relationship between the power spot price, the fuel spot prices and other observable variables like demand and production capacities, temperature... Various structural models exist which we refer to [9] for a complete survey, underlying the fact that they differ depending on the drivers they take into account. For example, some authors decide to link the prices only to the demand, as [4] did in what is often referred to as the first structural model. The most relevant drivers are the capacity, the demand, and the prices of fuels needed to produce electricity, which are directly observable, and thus some have studied the performances of their models by confronting their simulated spot prices against historical data, as it was done in [2]. As for any spot price model, forward prices are deduced using no-arbitrage arguments, leading, for structural models, to a relationship between electricity forward prices and fundamental drivers like forward prices on fuels.

The objective of this paper is to study the calibration issues for each class of models. Firstly we study a factorial model representing the forward curve, with two Gaussian factors, as proposed in [16]. In the case of forward curve models, the calibration on initial forward products is trivial, it is sufficient to determine the appropriate initial forward curve. However the model parameters have to be estimated. In this part the proposed estimation procedures are not especially original, though the calibration on forward volatilities has not been described, to our knowledge, in previous literature. But the main objective is to highlight the practical problem of calibration due to the complex relationship between parameters and observed products, and also due to the real need to represent both spot prices and forward products.

In a second part, we study the structural model proposed in [2] in terms of forward price reconstruction and its ability to be calibrated. To our knowledge, this topic has not been treated in the literature for structural models, hence this will form the main contribution of this paper. The efficient method of calibration we propose allows to widen the application scope of such a structural model. In particular it opens up the possibility of using structural models for pricing applications.

The paper is organized as follows. Section 2 concerns the study of a 2-factor model and proposes a comparison of calibration results in a simple example of pricing application. Section 3 is focused on a structural model and its ability to represent forward prices and to be calibrated. The conclusion and some perspectives are proposed in Sect. 4.

2 Parameters Estimation for a 2-Factor Model

In this section we study the estimation problem for a very classic factorial model used to represent power prices. The exposed procedures are not original but the objective is to stress the estimation issue once the model is used to represent both spot and forward prices. The factorial representation of the power forward curve was already studied, for example in [18] and justified in [14]. The authors in [17] highlight a decomposition in two factors for modeling power prices in the Norwegian market, with a weak correlation between those two factors. And, in [16] an explicit two-factor model is proposed in the risk-neutral probability:

$$\frac{dF_t(T)}{F_t(T)} = \sigma_s(t)e^{-\alpha(T-t)}dW_t^{(s)} + \sigma_l(t)dW_t^{(l)} \tag{1}$$

where $\alpha \in \mathbb{R}_*^+$ and $\sigma_s(t)$ and $\sigma_l(t)$ are positive integrable functions. This model is very close to the well known Gabillon model [3] and exactly the same for a specific form of $\sigma_l(t)$. In all this paper the time is measured in years, the "mean-reverting" parameter α will then be measured in year^{-1}.

Because of the presence of $e^{-\alpha(T-t)}$ in the first factor we call it the "short-term factor", and the second term will be called the "long-term factor". Also the form of the short-term factor allows us to represent the specific behavior of power prices: increasing volatility when the maturity goes to zero. For simplicity we consider no correlation between the two Brownian motions but all that follows can be easily extended with a non-zero correlation.

The main advantage of using Heath-Jarrow-Morton [13] type factorial models for power prices is its ability to be calibrated on observed forward products by specifying an appropriate initial forward curve $F_0(T)$, and, because of the broad literature of this type of modeling (especially for interest rate models), an easy use for pricing applications. We note also that in the case of commodity price modeling, using HJM framework is simpler than for interest rates because no drift condition must be satisfied, except drift equals zero.

In the following we aim to highlight, however, the estimation issue when, as a power producer, both spot prices and forward products must be well represented. In particular we will expose two different estimation methods.

- The first estimation method is based on the observed forward products where the objective is to fit the volatility curve. In this context we stress the difficulty due to the forward product properties (with a delivery period) that makes its process generally non Markovian and we then propose an approximation on the diffusion process to make the parameters estimation feasible.
- The second estimation method is based on both spot prices and forward products. In this context we describe the spot price model in terms of an "observation-state equations" system to use the classic Kalman filter and estimate the short-term factor parameters. Long-term forward products are used to estimate the long-term volatility $\sigma_l(t)$.

With a simple example of pricing application we propose to stress the high impact of estimation procedures to the indicators of interest. The objective is, in this simple example, to give a performance measurement by comparing the different results to a "benchmark" value. However the objective is not to come to a conclusion on any ranking of calibration methods but only to stress their impact in the context of the power market.

2.1 Method 1: Calibration on Forward Volatilities

In this section we develop a calibration method based on forward price observations. The proposed approach aims to fit the forward price volatilities. Although this approach is classic, the main issue is due to the specificities of power forward prices. Indeed the difference between the forward prices represented in model (1) and the observed forward products (with a delivery period) makes the calibration more complex. In this context we propose some approximations to make the parameters estimation feasible. In particular we propose an approximation of forward product diffusion by a Markovian process.

Equation (1) gives the dynamics of a "unitary" forward price, i.e. a forward price of an instantaneous (or unitary) delivery period. The available observed products are defined by $F_t(T, \theta)$ as the price at time t of 1MWh delivered from T to $T + \theta$, θ being called the "delivery period". Let us consider a discretization time step

h (1 h for example). From Eq. (1) and the assumption of absence of arbitrage opportunity we can deduce the relationship between forward products and unitary forward prices:

$$F_t(T,\theta) = \frac{h}{\theta} \sum_{i=1}^{\frac{\theta}{h}-1} F_t(T+ih) \tag{2}$$

and their dynamics:

$$dF_t(T,\theta) = \frac{h}{\theta} \sum_{i=0}^{\frac{\theta}{h}-1} \left[\sigma_s(t)e^{-\alpha(T+i-t)}dW_t^{(s)} + \sigma_l(t)dW_t^{(l)}\right] F_t(T+ih) \tag{3}$$

The presence of $F_t(T+ih)$ illustrates that, in general, the SDE for $F_t(T,\theta)$ is not Markovian, as already shown in [6], which makes the calibration intricate. The approximation we propose is based on the introduction of shaping factors, defined as follows:

$$\lambda_i^{t,T,\theta} = \frac{F_t(T+ih)}{F_t(T,\theta)}, \quad \forall i = 0,\ldots,\frac{\theta}{h}-1 \tag{4}$$

These shaping factors can be interpreted as weighting factors applied in hour (or day) i of the delivery period $[T\,;\,T+\theta]$ with respect to the mean value $F_t(T,\theta)$ of forward prices over $[T\,;\,T+\theta]$. One can note that the shaping factors are normalized by definition because $\sum_{i=0}^{\frac{\theta}{h}-1} \lambda_i^{t,T,\theta} = \frac{\theta}{h}$. One can also note that these shaping factors are random and depend on the quotation date t. With the introduction of shaping factors the SDE on $F_t(T,\theta)$ can be rewritten:

$$\frac{dF_t(T,\theta)}{F_t(T,\theta)} = \sigma_s(t)e^{-\alpha(T-t)}\Psi(t,T,\theta)dW_t^{(s)} + \sigma_l(t)dW_t^{(l)} \tag{5}$$

with $\Psi(t,T,\theta) = \frac{h}{\theta}\sum_{i=0}^{\frac{\theta}{h}-1} \lambda_i^{t,T,\theta} e^{-\alpha ih}$ being the weighted average of the shaping factors over the delivery period. The fact that the dynamics on $F_t(T,\theta)$ is neither Markovian nor Gaussian is now reflected in the fact that $\Psi(t,T,\theta)$ is random and depends on time t.

Let $(t_n)_{n=0,\ldots,N}$ be a time discretization with $t_0 = 0$ and constant time step[2] $\delta t = t_{n+1} - t_n$. We consider the following approximations.

- The functions $\sigma_s(t)$ and $\sigma_l(t)$ are constant over each interval $[t_n\,;\,t_{n+1}]$.
- The shaping factors are constant with respect to time t: $\lambda_i^{t,T,\theta} = \lambda_i^{T,\theta}$.

[2]The time step δt will be related to the observed prices, therefore δt may be different from the discretization step h of the delivery period.

Therefore the function $\Psi(t,T,\theta) = \Psi(T,\theta) = \frac{h}{\theta}\sum_{i=0}^{\frac{\theta}{h}-1} \lambda_i^{T,\theta} e^{-\alpha i h}$ does not depend on time t and the return $R_n(T,\theta)$ of forward products between t_n and t_{n+1} is given by:

$$\begin{aligned} R_n(T,\theta) &= \log \frac{F_{n+1}(T,\theta)}{F_n(T,\theta)} \\ &= -\frac{1}{2}\sigma_n^2(T,\theta) + \sigma_s(t_n)e^{-\alpha(T-t_n)}\Psi(T,\theta)\sqrt{\nu(2\alpha,\delta t)}\varepsilon_n^s + \sigma_l(t_n)\sqrt{\delta t}\varepsilon_n^l \end{aligned}$$

with ε_n^s and ε_n^l two independent Gaussian random variables of zero-mean unit-variance and

$$\begin{aligned} \nu(a,\delta t) &= \frac{e^{a\delta t}-1}{a} \\ \sigma_n^2(T,\theta) &= \sigma_s^2(t_n)\Psi^2(t_n,T,\theta)e^{-2\alpha(T-t_n)}\nu(2\alpha,\delta t) + \sigma_l^2(t_n)\delta t \end{aligned}$$

Now suppose $N+1$ observations $(F_n(T,\theta))_{n=0,\ldots,N}$ at dates $(t_n)_{n=0,\ldots,N}$ and constant volatility functions $\sigma_s(t) = \sigma_s$ and $\sigma_l(t) = \sigma_l$, we can therefore compute the theoretical forward returns depending on the three parameters σ_s, σ_l and α and the corresponding volatility:

$$V_{th}^2(T,\theta,\alpha,\sigma_s,\sigma_l) = \frac{1}{N}\sum_{n=0}^{N-1} Var\left[R_n(T,\theta)\right] \tag{6}$$

In the particular case where we assume constant shaping factors $\lambda_i^{T,\theta} = 1$ we obtain:

$$V_{th}^2(T,\theta,\alpha,\sigma_s,\sigma_l) = \Phi(\Delta)\Psi^2(\theta)\sigma_s^2\nu(2\alpha,\theta) + \sigma_l^2\delta t \tag{7}$$

with $\Delta = Nh$ the quotation period, and

$$\Phi(\Delta) = \frac{1}{N}\frac{1-e^{-2\alpha\Delta}}{1-e^{-2\alpha h}} \quad \text{and} \quad \Psi(\theta) = \frac{h}{\theta}\frac{1-e^{-\alpha\theta}}{1-e^{-\alpha h}} \tag{8}$$

The calibration consists in estimating three parameters, σ_s, σ_l and α. A first solution would be to estimate them by maximizing the likelihood function, therefore to estimate parameters that fit as well as possible the observed values of forward returns. However empirical studies have shown that the parameter values are very sensitive to the choice of products considered for the estimation. Instead we propose a calibration method consisting in fitting the volatilities of the observed forward products. More precisely from the observed forward returns $R_n^{obs}(T,\theta) = \log\frac{F_{n+1}(T,\theta)}{F_n(T,\theta)}$ we can compute the empirical volatility:

$$V_{emp}^2(T,\theta) = \frac{1}{N-1}\sum_{n=1}^{N}(R_n^{obs}(T,\theta) - \overline{R}(T,\theta))^2, \quad \overline{R}(T,\theta) = \frac{1}{N}\sum_{n=1}^{N}R_n^{obs}(T,\theta) \tag{9}$$

Remark. If derivatives on forward products are available, it is possible to compute the implied (Black) volatility instead of the historical volatility.

The calibration procedure then consists in optimizing the distance between theoretical and empirical volatilities for all observed forward products:

$$(\hat{\alpha}, \hat{\sigma_s}, \hat{\sigma_l}) = \arg\min\,(\alpha, \sigma_s, \sigma_l) \sum_{(T,\theta)} \left(V_{emp}^2(T,\theta) - V_{th}^2(T,\theta,\alpha,\sigma_s,\sigma_l)\right)^2 \tag{10}$$

2.2 Method 2: Calibration on Spot Prices and Long-Term Forward Products

By integration of (1) and taking the limit $T \to t$ we obtain:

$$\log S_t = \log F_t(t) = \log F_0(t) - \frac{1}{2}\left[\sigma_s^2 \frac{1-e^{-2\alpha t}}{2\alpha} + \sigma_l^2 t\right] + \int_0^t \sigma_s e^{-\alpha(t-u)} dW_u^{(s)} + \int_0^t \sigma_l dW_u^{(l)}$$

By noting

$$X_t^s = \int_0^t \sigma_s e^{-\alpha(t-u)} dW_u^{(s)} \quad \text{and} \quad X_t^l = \int_0^t \sigma_l dW_u^{(l)} \tag{11}$$

We can rewrite the spot price dynamics as a (state - observation equations) system:

$$d(\log S_t) = \left(\frac{\partial F_0(t)}{\partial t} + \mu(t)\right) dt + dX_t^s + dX_t^l \tag{12}$$

$$dX_t^s = -\alpha X_t^s dt + \sigma_s dW_t^{(s)} \tag{13}$$

$$dX_t^l = \sigma_l dW_t^{(l)} \tag{14}$$

with $\mu(t) = \frac{1}{2}\left[\sigma_s^2 e^{-2\alpha t} + \sigma_l^2\right]$.

The drift part can be treated as a seasonality component of the spot price. Its estimation can be made by a deseasonalization step. In the following this seasonality will be represented by seven daily parameters, 12 monthly parameters and one parameter per year, which are estimated by a classic linear regression, with additional constraints of normalization for the daily and monthly parameters. After this deseasonalization step, the maximum likelihood estimation of $(\sigma_s, \sigma_l, \alpha)$ can proceed from the residual by using a Kalman filter [12, 15] to compute the likelihood. In order to also use forward products in the calibration we propose to first estimate σ_l from long-term forward products (year-ahead or season-ahead, depending on the market) with the approximation:

$$\frac{dF_t(T,\theta)}{F_t(T,\theta)} \approx \sigma_l dW_t^l, \quad \text{if } T - t >> 0 \tag{15}$$

The maximum likelihood estimation from spot prices then proceed to estimate the short-term parameters σ_s and α, with this pre-estimated σ_l.

2.3 *Results*

This section shows an illustration on the impact of the calibration methods to a simple pricing application. As already said the objective is not to give a ranking of calibration methods, but only to stress and quantify the difference of results, in terms of value for a simple portfolio, due to the choice of the calibration method.

2.3.1 Data Set

We consider a portfolio composed of a strip of European options on forward products. More precisely, we consider 24 European options on monthly forward products: product "April-2013" to product "March-2015". The date of pricing is t_0 = March 12th, 2013 and all the options are at the money. We consider two different markets: the UK power market and the French power market. The main advantage of considering options on monthly forward products is the possibility to have a "benchmark" value. Indeed, one can consider a model directly on forward monthly products, as proposed in [6], calibrated on observed empirical volatilities. let us denote by $M_t = T - t$, then the benchmark model can be written as follows:

$$\frac{dF_t(t+M_t)}{F_t(t+M_t)} = \sigma(M_t)dW_t \tag{16}$$

where $\sigma(M_t)$ is a piecewise constant function fitting exactly the empirical forward volatilities.

We consider 1 year of historical data for the calibration on forward products. The products used for calibration depend on the market.

- The products used for the calibration in the UK power market are: 1 to 4 Week-ahead, 1 to 4 Month-ahead, 1 to 4 Quarter-ahead and 1 to 6 Season-ahead.
- The products used for the calibration in the French power market are: 1 Week-ahead, 1 to 3 Month-ahead, 1 to 3 Quarter-ahead and 1 to 2 Year-ahead.

The shaping factors $\lambda_i^{T,\theta}$ are all considered equal to 1. This strong approximation is only used for the calibration purpose because it has no significant impact on the reconstructed volatilities.

Concerning the calibration on spot prices, we consider 2 years of historical data to estimate the seasonality part and 1 year of the residual signal to estimate the short term parameters σ_s and α. The long term volatility σ_l is estimated as an averaged volatility of the 1 and 2 year-ahead products for the French power market, and as an averaged volatility of the 1 to 6 Season-ahead products for the UK power market.

2.3.2 Calibration Results

Table 1 shows the estimated parameters on the UK power market with respect to the calibration method and Fig. 1 illustrates the resulting forward volatilities reconstructed by the 2-factor model in comparison to the empirical forward volatilities. We can note that the empirical volatilities do not seem to decrease monotically with the maturity. This effect can be mainly explained by the overlapping delivery period of the forward products. Concerning the reconstructed volatility curves we can observe that the long-term volatility values are similar due to the quasi-similarity of its estimation procedure. The most important point is the difference in value for the

Table 1 Estimation results on UK and French power: estimated parameters with respect to the calibration method

Parameter	Calibration on forward volatilities		Calibration on spot prices	
	UK power market	French power market	UK power market	French power market
σ_s (%)	19.1	45	84.5	302
$\alpha(Y^{-1})$	1.37	8.73	162.65	88.15
σ_l (%)	9.8	11	9.8	11

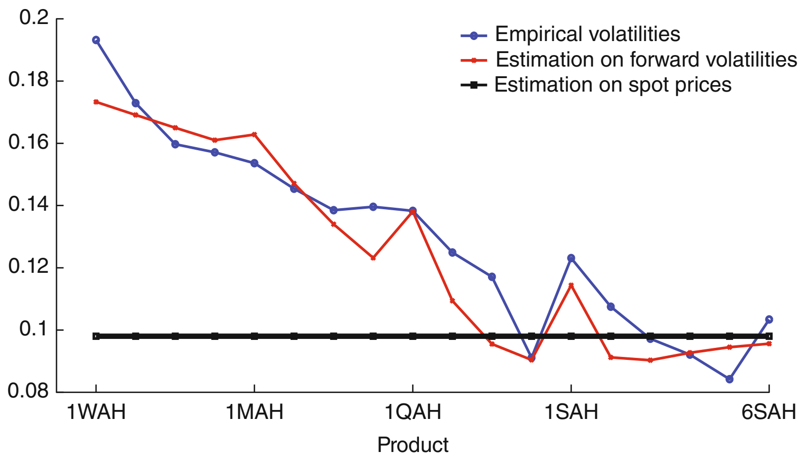

Fig. 1 Estimation results on UK power market: empirical volatility (*blue line with circles*), reconstructed volatility of the 2-factor model calibrated on forward volatilities (*red line*) and on spot prices (*black line with squares*)

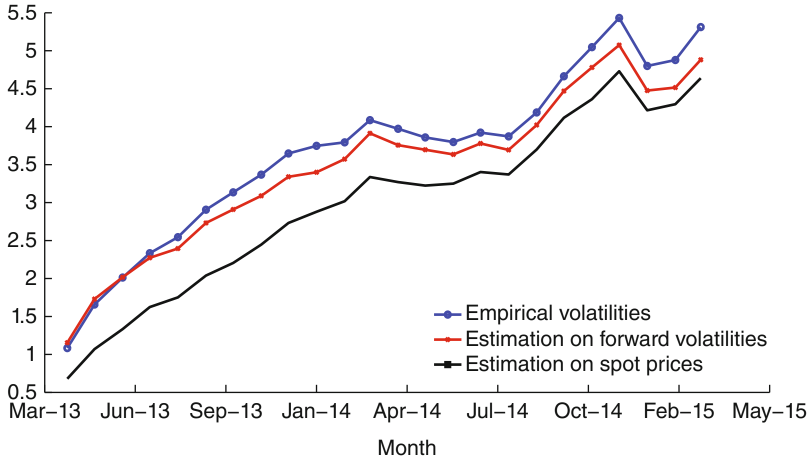

Fig. 2 European option pricing on UK power market: benchmark value (*blue line with circles*), values from the 2-factor model calibrated on forward volatilities (*red line*) and on spot prices (*black line with squares*)

short-term parameters: a factor 4 in σ_s and a factor 100 in α from the estimation on forward volatilities to the estimation on spot prices. In the case of calibration on spot prices, the parameter α can be interpreted as a "mean-reverting" coefficient driving the spot prices. Its estimated value then shows that the spot price presents highly mean-reverting behavior, this being mostly due to the presence of spikes. This high value of α leave a single constant volatility factor (the long-term factor) to fit the whole forward volatility term structure. In this case the estimated forward volatility curve, as shown in Fig. 1, is nearly completely flat and the well known Samuelson effect cannot be captured. On the other hand, because the same parameter α drives the decreasing speed of the forward volatility curve, it becomes obvious that the estimated values are completely different and depend on the calibration method.

In Fig. 2 we illustrate the impact of the estimation methods in the value of the "toy" portfolio. Compared to the benchmark value, we notice a weak error with the value computed from the two-factor model calibrated on forward volatilities. This confirms the results of [17]: two factors can be sufficient to represent forward products. However this does not take into account the need to also represent spot prices. And, as shown in Fig. 2, the resulting value, when the two-factor is used with parameters estimated on spot prices, highly underestimates the benchmark value with an error of around 20 %. Similar remarks can be made in the French power market (Table 1, Figs. 3 and 4) where the error can reach 30 % with the two-factor model calibrated on spot prices.

The main conclusion is not to reject the calibration on spot prices, because the chosen application context (pricing European options on forward products) is completely adapted to a calibration on forward volatilities. This context allowed us to build a benchmark value and then to make an objective comparison of calibration methods. Another consideration, for example with a portfolio exposed on spot

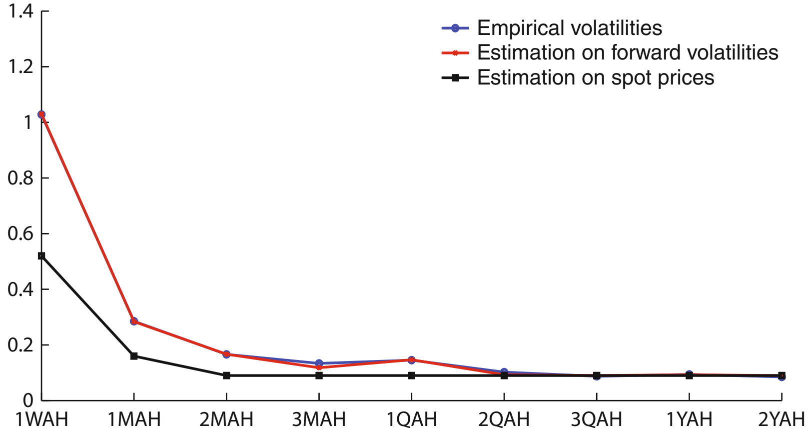

Fig. 3 Estimation results on French power market: empirical volatility (*blue line with circles*), reconstructed volatility of the 2-factor model calibrated on forward volatilities (*red line*) and on spot prices (*black line with squares*)

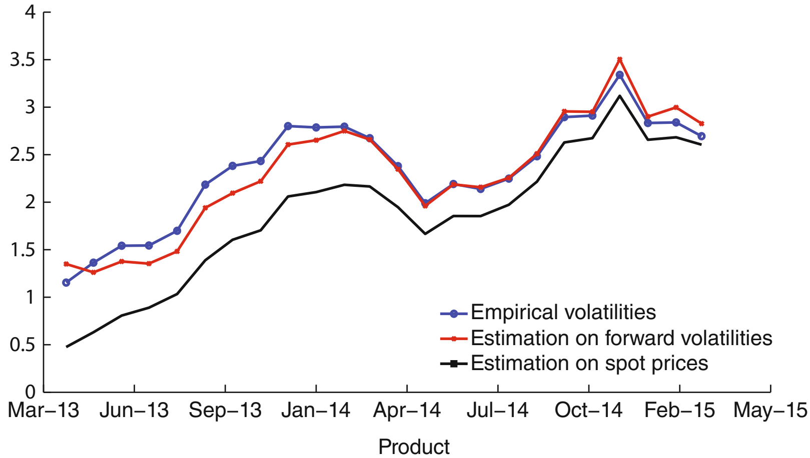

Fig. 4 European option pricing on French power market: benchmark value (*blue line with circles*), vale from the 2-factor model calibrated on forward volatilities (*red line*) and on spot prices (*black line with squares*)

prices, would have shown that the calibration on spot prices is more adapted, but in this case a benchmark value cannot be built easily and is still an open problem. The main conclusion is that, if the objective is to represent both spot forward prices, two factors are not sufficient and, before the calibration methodology, a study of more complex models is necessary. For example, adding at least a decay factor (i.e. a second mean-reverting parameter) in the long-term factor would allow us to capture both mean-reverting behaviour of spot prices and the Samuelson effect on the forward volatility curve. However this change of model will create a decreasing volatility toward 0, which may be contrary to the observed volatility curves (see, for example, Fig. 3 which shows that the forward volatility curve seems to converge toward a positive value). Another solution would be to add a third factor of the same form as the short-term factor, therefore to have two mean-reverting parameters as previously and to keep a non-zero limit value of the forward volatility level. However the issue of calibration will remain and increase with an increasing number of factors.

3 Calibration of a Structural Model

In this section we study the second class of commonly used models for power prices: the class of structural models. This approach is more recent and adapted to the stylized facts of electricity. In particular it allows to represent a strong relationship between the power spot price and the factors that explain it: the fuel spot prices, the demand and the production capacities. This section focuses on a particular structural model, proposed first in [1] and modified in [2]. By definition the model is adapted to represent power spot prices. However, to our knowledge, there is no literature on the performances of this kind of models in terms of forward price representation. This is the first objective of this section and, as a natural sequel, we address the issue of calibration on observed forward prices.

3.1 Reminder of the Model

The structural risk-neutral model we study in this paper is a modified version of the one introduced in [1], and its complete presentation can be found in [2]. In this section, we recall the approach and the main results obtained by the authors.

3.1.1 Approach and Main Results

The model is derived from the aggregation of two essential observations. On the one hand, when the market is not in a period of stress, the price of the marginal fuel of the generation system will be the dominant part of the electricity spot price. On the other hand, at times of market stress, the well-known spikes of the electricity spot prices will occur when the demand reaches the system maximal capacity. In this model, such behaviour is captured by a "scarcity" function, that will explode to form the prices spikes, thus leading to the following form for the spot price:

$$S_t = g\left(C_t^{max} - D_t\right) \sum_{i=1}^{n} h^i S_t^i \mathbf{1}_{\{D_t \in I_t^i\}}, \tag{17}$$

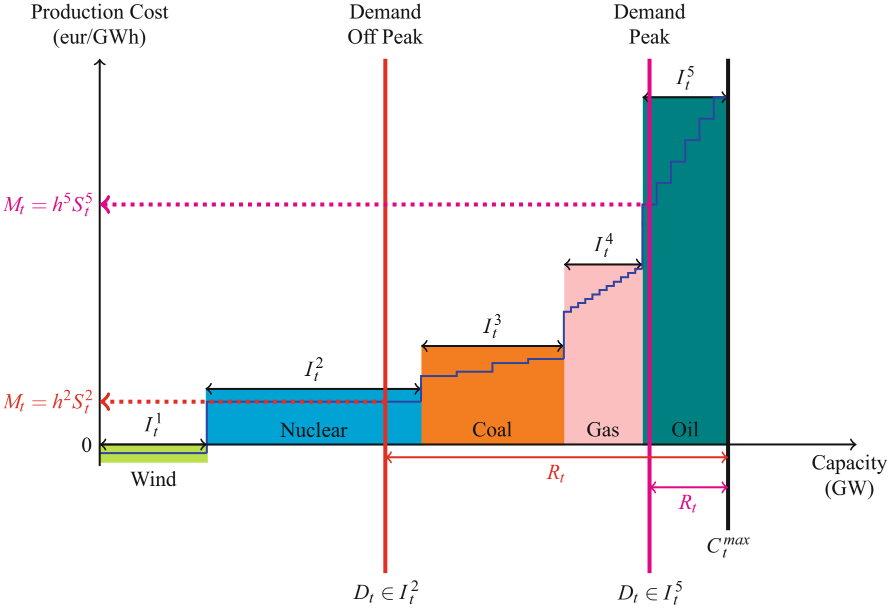

Fig. 5 Illustration scheme of electricity price construction

with D_t the demand, C_t^i, $i = 1, \ldots, n$ the capacities of the fuels used in the production system, h^i the corresponding heat rates, S_t^i the corresponding spot prices of fuels, I_t^i the capacity interval where fuel i is marginal (see Fig. 5), $C_t^{max} = \sum_{k=1}^{n} C_T^k$ the total capacity of the production system, and g the "scarcity" function:

$$g(x) = \min\left(M, \frac{\gamma}{x^\nu}\right) \mathbf{1}_{x>0} + M\mathbf{1}_{x<0}.$$

Equation (17) that defines the model can be summarized in a simpler way: when the demand is in the marginal interval of the i^{th} fuel, the spot price of electricity is equal to the cost needed to produce 1MWh of electricity from the i^{th} fuel, times the scarcity factor.

Remark. In this model the merit order of fuels is assumed to be fixed. For this study we keep the same assumption for more simplicity and also in order to keep acceptable computation time.

This spot model has been backtested, using historical data for demand, capacities, and fuel spot prices. The parameter M of the scarcity function is estimated so as to roughly match the high cap on electricity spot price, defined by the market as 3,000 €MWh.[3] Its estimated value is 30. The other parameters of the scarcity function

[3] http://www.epexspot.com/en/product-info/auction/france.

Table 2 Hourly parameters of the scarcity function g

Hour (h)	γ	ν	Hour (h)	γ	ν
0	0.44634	2.5137	12	0.73737	6.517
1	0.67993	4.0184	13	0.53414	4.1539
2	0.82051	5.2762	14	0.6109	4.6532
3	0.86753	5.0142	15	0.68395	5.0617
4	0.84629	4.7554	16	0.9543	8.9007
5	0.68841	4.2384	17	1.5229	29.843
6	0.7833	6.4935	18	1.8399	57.121
7	0.97349	12.1802	19	1.1405	14.1943
8	0.77457	7.5105	20	1.0153	9.7041
9	0.8497	8.521	21	0.56393	3.8245
10	0.72403	6.5117	22	0.55286	3.973
11	0.63956	5.6207	23	0.6688	4.6805

have been estimated to best fit the model price with the historical spot prices, for each hour of the day. The results can be seen in Table 2 and more performance illustrations can be found in [2].

The main objective is not to pursue the performance evaluation of the spot model. We intend to take a closer look at the underlying structural relationship for the forward prices, and then we start by recalling the pricing methodology adopted for this model. The presence of the demand and capacities implies an incomplete market setting, and thus an infinity of no-arbitrage prices for any derivative or, equivalently, an infinite number of risk-neutral measures. The criterion used to value an electricity derivative for this model is the Local Risk Minimization approach, which allows us to choose a risk-neutral measure $\hat{\mathbb{Q}}$. Technical details concerning the choice of this measure, which uses the Local Risk Minimization principle (introduced in [11]), can be found in the original paper. The main results are detailed in Appendix 1 and lead to a no-arbitrage price for a forward price $F_t^e(T)$ on electricity that takes the form:

$$F_t^e(T) = \mathbb{E}^{\hat{\mathbb{Q}}}[S_T|\mathscr{F}_t],$$

with $\mathscr{F}_t = \mathscr{F}_t^S \vee \mathscr{F}_t^C \vee \mathscr{F}_t^D$ the filtration representing the market information at time t, which is the filtration generated by the randomness of the fuels, the demand and the capacities, and $\hat{\mathbb{Q}}$ obtained using Local Risk Minimization, satisfying $\hat{\mathbb{Q}} = \mathbb{P}$ on $\mathscr{F}_t^C \vee \mathscr{F}_t^D$. Consequently, the forward price takes the simple form:

$$F_t^e(T) = \sum_{i=1}^{n} h^i G^i(t, T, C_t, D_t) F_t^i(T),$$

$$G^i(t, T, C_t, D_t) = \mathbb{E}\left[g(C_T^{max} - D_T)\mathbf{1}_{D_T \in I_T^i} | \mathscr{F}_t^{D,C}\right].$$

If we want to reconstruct the real forward prices that are exchanged on electricity markets, we need to average the forward price with instantaneous delivery over the delivery period $[T, T+\theta]$ leading to the final form:

$$F_t(T,\theta) = \sum_{i=1}^{n} \underbrace{\left(\frac{1}{\theta}\sum_{T'=T}^{T+\theta} G^i(t,T',C_t,D_t)\right)}_{\text{stochastic weights}} h^i F_t^i(T') \tag{18}$$

This result is the core of this study: starting from a spot model, we necessarily have a result on forward prices involving the forward prices on fuels, the demand process and the capacities processes. In the following we will study the performances of this forward relationship as well as classic studies on spot prices like for example in [1, 8]. Because this relationship shows the historical probability of demand D_t and capacities C_t^i the global model needs a specification of their dynamics.

3.1.2 Demand and Capacities Modeling

We now need to model the behaviour of the electricity demand and capacities that we take into account in the model. We follow the same model introduced in [2], and thus we decide to decompose the demand and capacities processes into two parts: a deterministic part $f_*(t)$ and a stochastic part $Z_*(t)$:

$$D_t = f_D(t) + Z_D(t),$$
$$C_t^i = f_i(t) + Z_i(t), \text{ for } i = 1,\dots,n$$

The deterministic part will model the seasonal trend of the demand or capacities, while the stochastic part will capture the randomness of these processes. We choose to slightly modify the original model for the seasonality functions. For all the processes, we will take into account the yearly seasonality, and a week trend, that will capture the trend of every hour of the week, leading to a 168 parameters. This can be interpreted as a week scheme, that will be reproduced all year long, following a yearly seasonality drift. We decide to assume:

$$f_D(t) = \mathbf{week}_D(t) + d_1 + d_2\cos\left(2\pi(t-d_3)\right),$$
$$f_i(t) = \mathbf{week}_i(t) + c_1^i + c_2^i\cos\left(2\pi(t-c_3^i)\right) \text{ for } i = 1,\dots,n$$

We keep the Ornstein-Uhlenbeck form of the stochastic parts, leading to:

$$dZ_D(t) = -\alpha_D Z_D(t)dt + \sigma_D dW_t^D, \tag{19}$$

$$dZ_i(t) = -\alpha_i Z_i(t)dt + \sigma_i dW_t^i, \text{ for } i = 1,\dots,n \tag{20}$$

3.2 Results on Reconstructed Forward Prices

3.2.1 Description of the Dataset

The whole dataset consists in 4 years of historical data from January 1st, 2009 to December 31st, 2012. Data from Demand and capacities have been retrieved from Réseau de Transport Electrique (RTE,[4]) whereas data of fuel and electricity prices come from Platts.[5] We consider a simple case with three types of production units: nuclear, gas/coal and oil power plants. This allows us to consider a fixed ranking of production cost, the cheapest cost being for nuclear plants and the most expensive being for the oil power plant. Carbon emission taxes are taken into account. Table 3 presents the datasets used in the estimation. It is interesting to note that we used day-ahead demand, because the demand, as it appears in the spot model, is used to model the spot price of electricity, which is a day-ahead price as well. Thus, in this forward framework, we need to stay consistent with the spot model, and use day-ahead demand.

3.2.2 Estimation Results for Demand and Capacities Parameters

To estimate the parameters of the demand or capacities processes, we follow the same framework: first we estimate the parameters of the seasonality function, and then we estimate the Ornstein-Uhlenbeck parameters. The seasonality estimation is done by using classic statistical tools like linear regression methods and the details can be found in Appendix 2. For the stochastic parameters, we used ordinary least squares to estimate the parameters of the Ornstein-Uhlenbeck processes, securing confidence intervals (Table 4).

Table 3 Description of the dataset

Name	Source	Data	Frequence/type	Dates covered/value
Demand	RTE	D_t	Hourly	2009–2012
Nuclear capacity		C_t^1		
Coal+gas capacity		C_t^2		
Oil capacity		C_t^3		
Fuel forwards	Platts	$F_t^i(T)$	Daily	2009–2012
Nuclear heat rate		h_1	Constant	$0.84.10^{-4}$
Coal+gas heat rate		h_2		0.45
Oil heat rate		h_3		1.5

[4] www.rte-france.com.

[5] www.platts.com.

Table 4 Demand and capacities parameters estimated from the whole dataset

Parameter	Estimate	Confidence interval	Parameter	Estimate	Confidence interval
d_1	0.37389	0.3464-0.40137	c_1^2	4.6952	4.6855-4.7049
d_2	1.967	1.9123-2.0218	c_2^2	1.3929	1.3753-1.4105
d_3	0.14137	0.14095-0.14181	c_3^2	0.19436	0.19345-0.19529
α_D	32.7958	27.2804-38.3112	α_2	21.367	16.7558-25.9782
σ_D	16.7316	16.1176-17.4223	σ_2	5.0362	4.8514-5.2441
c_1^1	7.7918	7.785-7.7985	c_1^3	49.6449	49.6182-49.6716
c_2^1	0.91707	0.90447-0.9297	c_2^3	7.0054	6.9545-7.0563
d_3^1	0.18299	0.18218-0.18382	c_3^3	0.17465	0.17428-0.17501
α_1	26.4656	21.4961-31.4351	α_3	7.842	4.8483-10.8358
σ_1	4.8794	4.7003-5.0808	σ_3	11.7357	11.305-12.2201

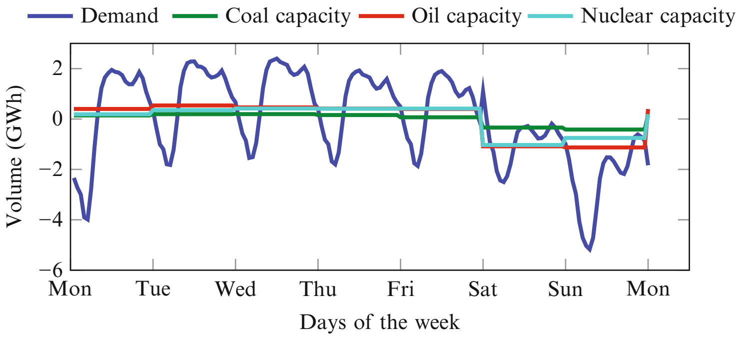

Fig. 6 Estimated **week** parameters

Figure 6 shows the estimated values of the weekly parameter for demand and capacities. It underlines the fact that no weekly seasonality can be seen for the capacities, while it is a major feature of the demand.

3.2.3 Reconstructed Forward Prices

In order to study the performances of forward price reconstruction we implemented the following algorithm: at each date t from January 1st, 2009 to December 31st, 2012,

1. Estimate parameters of demand and capacities processes from 2 years of historical data.
2. Consider the observed forward fuel prices $F_t^i(T, \theta)$ for all observable (T, θ) and assume $F_t^i(T') = F_t^i(T, \theta)$ for all $T' \in [T, \; ; \; T + \theta]$
3. Compute Eq. (18) to build the electricity forward price and compare to the observed one.

The algorithm is implemented in Matlab 2010a on a laptop.[6] The computation time is mainly due to the computation of the functions H and $\mathscr{G}$ defined in the section "Computing the Stochastic Weights" in Appendix 1. If we use the series expansion approach proposed in [2], based on the extended incomplete gamma function, the reconstruction of the 1-Year-ahead product for only one date takes 21 s. If, instead, we approximate the functions H and $\mathscr{G}$ by Monte Carlo approach with 400 samples,[7] the reconstruction of the 1-year-ahead product takes 2.3 s.

Figure 7 shows the reconstruction results for 1-month-ahead, 1-quarter-ahead and 1-year-ahead products. We can remark satisfactory reconstruction of the 1-month-ahead product, capturing level and seasonality quite well. The fundamental relationship (18), i.e. the link between electricity forward prices and an expectation of demand and capacities levels, seems dominant in the explanation of the 1-month-ahead. The 1-year-ahead reconstruction is also efficient, but we have to note a level underestimation for the period 2009 to 2010. Although no changes in demand and capacities have been observed at the end of 2010, the reconstruction results are more efficient during the period 2011 to 2012. There is no explanation at this time about this particular change of behavior and it remains an open question. Reconstruction results are less efficient for the 1-quarter-ahead product, where, in particular, the level is not well captured. The relationship (18) is then not the fundamental element that drives the price of this product. We must further investigate the comprehension of the market actors to understand this particular effect.

3.3 Calibration

In the previous section, we have seen the ability of the model when it comes to reconstructing the forward prices. But a new question arises: is it possible, with this model, to reproduce exactly the forward prices that we can observe, at a certain time? This question is extremely important on the markets, as it is very important to be able to fit a model to the real prices in order to avoid any arbitrage possibility. In this section, the aim will be to calibrate the model on the forward contracts that we can observe on the markets, which is completely different from a standard spot model calibration: indeed, if we follow an implied volatility framework, we need to find a parameter of the model that makes the price a strictly monotonous one, and thus we can obtain an implied parameter, at each calibration date, for the price. This is feasible, as the model has a lot of parameters, and that their behavior is flexible, but it does not actually give us the calibration that we are looking for. When it comes

[6] Intel(R) Core(TM) i3-2375M CPU @ 1.50 GHz.

[7] The number of samples is empirically chosen so as to obtain a difference between the series expansion and Monte Carlo computations lower than 0.1€ on the reconstructed price.

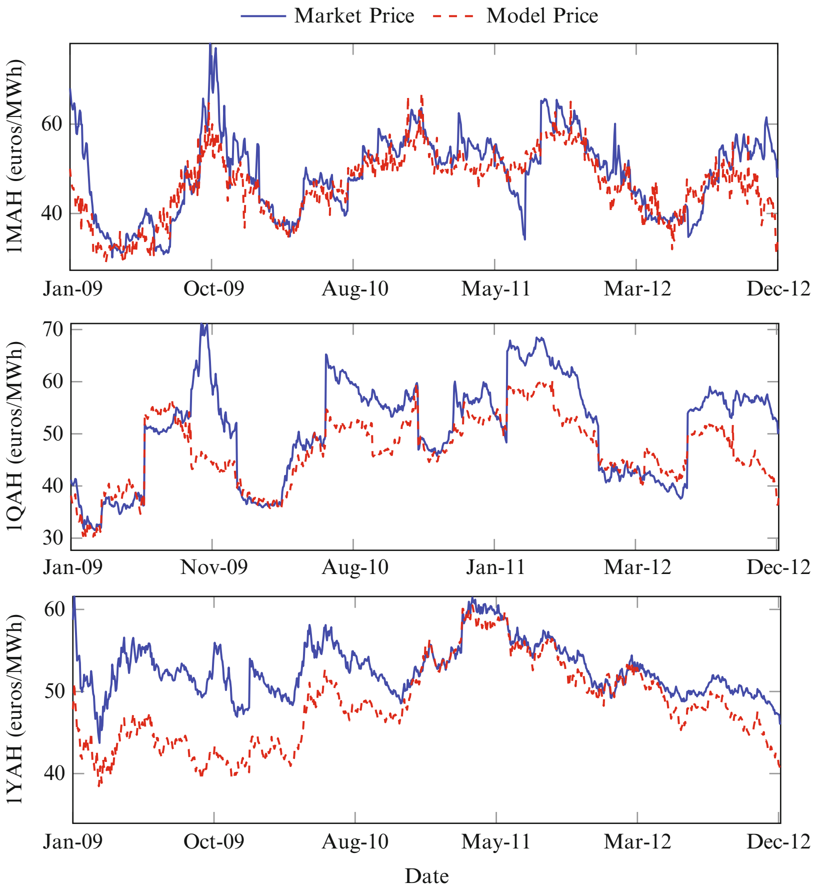

Fig. 7 Results of forward product reconstruction: observed (*solid-blue line*) and reconstructed (*dashed-red line*) 1 month-ahead (*top*), 1-quarter-ahead (*middle*) and 1-year-ahead (*bottom*) prices from January 1st, 2009 to December 31st, 2012

to calibrating on forward contracts, the problem is much more complicated, we have to calibrate the model to fit a given curve: the forward curve, that contains all the available contracts at a certain date.

3.3.1 Adjusting the Demand Parameter

In our model, we looked closely at the influence of the parameters of the demand and capacities processes, and we found the deterministic part of the demand to be a pertinent adjustment parameter, as it was already proposed in [9]. Let us recall the model for the demand process:

$$D_t = f_D(t) + Z_D(t)$$

In this case the expected value of D_T conditional to $\mathscr{F}_t$:

$$\mathbb{E}\left[D_T|\mathscr{F}_t^{D,C}\right] = f_D(T) + e^{-\alpha_D(T-t)}(D_t - f_D(t)) := m_{t,T}^D$$

The introduction of a non-zero long-term mean $\varepsilon \in \mathbb{R}$ in the process of $Z_D(t)$:

$$dZ_D(t) = \alpha_D(\varepsilon - Z_D(t))dt + \sigma_D dW_t^D$$

will only affect the mean of the demand process, conditional to $\mathscr{F}_t$:

$$\mathbb{E}\left[D_T|\mathscr{F}_t^{D,C}\right] = m_{t,T}^D + \varepsilon(1 - e^{-\alpha_D(T-t)}) \tag{21}$$

This modification may be interpreted as a change of probability for the demand process, leading to a difference between the risk-neutral and the historical probabilities. In this case the resulting pricing measure is no longer the one obtained by Local Risk Minimization.

The stochastic weights (the details of calculus are available in Appendix 1) are all affected:

$$\begin{aligned}
G^1(\varepsilon, t, T, D_t, C_t) &= H(m_{t,T}^2 + m_{t,T}^3, m_{t,T}^1 - m_{t,T}^D - \varepsilon(1 - e^{-\alpha_D(T-t)}), \sigma_2^3, \sigma_1^{1,D}) \\
G^2(\varepsilon, t, T, D_t, C_t) &= H(m_{t,T}^3, m_{t,T}^1 + m_{t,T}^2 - m_{t,T}^D - \varepsilon(1 - e^{-\alpha_D(T-t)}), \sigma_3^3, \sigma_1^{1,D}) \\
&\quad - H(m_{t,T}^2 + m_{t,T}^3, m_{t,T}^1 - m_{t,T}^D - \varepsilon(1 - e^{-\alpha_D(T-t)}), \sigma_2^3, \sigma_1^{1,D}) \\
G^3(\varepsilon, t, T, D_t, C_t) &= \mathscr{G}(m_{t,T}^1 + m_{t,T}^2 + m_{t,T}^3 - m_{t,T}^D - \varepsilon(1 - e^{-\alpha_D(T-t)}), \sigma_1^{3,D}) \\
&\quad - H(m_{t,T}^3, m_{t,T}^1 + m_{t,T}^2 - m_{t,T}^D - \varepsilon(1 - e^{-\alpha_D(T-t)}), \sigma_3^3, \sigma_1^{1,D})
\end{aligned}$$

with $m_{t,T}^i = \mathbb{E}\left[C_{T'}^i|\mathscr{F}_t^{D,C}\right]$, $i = 1,2,3$ and functions H and $\mathscr{G}$ defined in Appendix 1. We also remark that the function $\varepsilon \longmapsto F_t(T,\theta,\varepsilon)$ is strictly increasing, and that we have the following asymptotical results:

$$\lim_{\varepsilon \to +\infty} F_t(T,\theta,\varepsilon) = Mh_n F_t^n(T) \simeq 3000 \text{ eur/MWh}$$

$$\lim_{\varepsilon \to -\infty} F_t(T,\theta,\varepsilon) = 0$$

This shows that, for any contract $F_t^{obs}(T,\theta)$ observed at time t, we can find a unique value of ε able to exactly reproduce it. We can use a dichotomic algorithm to solve the equation $F_t(T,\theta,\varepsilon) = F_t^{obs}(T,\theta)$, as our interest function is strictly monotonous, but the regularity of this function in fact allows us to use the Newton-Raphson algorithm, which gives a quadratic convergence instead of a linear convergence, using the derivatives of $F_t(T,\theta,\varepsilon)$:

$$\begin{cases} \varepsilon_0 = 0 \\ \varepsilon_{n+1} = \varepsilon_n - \frac{F_t(T,\theta,\varepsilon) - F_t^{obs}(T,\theta)}{\frac{\partial F_t}{\partial \varepsilon}(T,\theta,\varepsilon)} \end{cases}$$

To compute the derivatives of $F_t(T,\theta,\varepsilon)$, we compute the derivatives of the weights:

$$\frac{\partial G^1}{\partial \varepsilon} = -(1-e^{-\alpha_D(T-t)})\left[\frac{\partial H}{\partial x_2}(m_{t,T}^2 + m_{t,T}^3, m_{t,T}^1 - m_{t,T}^D - \varepsilon(1-e^{-\alpha_D(T-t)}), \sigma_2^3, \sigma_1^{1,D})\right]$$

$$\frac{\partial G^2}{\partial \varepsilon} = -(1-e^{-\alpha_D(T-t)})\left[\frac{\partial H}{\partial x_2}(m_{t,T}^3, m_{t,T}^1 + m_{t,T}^2 - m_{t,T}^D - \varepsilon(1-e^{-\alpha_D(T-t)}), \sigma_3^3, \sigma_1^{1,D}) - \frac{\partial H}{\partial x_2}(m_{t,T}^2 + m_{t,T}^3, m_{t,T}^1 - m_{t,T}^D - \varepsilon(1-e^{-\alpha_D(T-t)}), \sigma_2^3, \sigma_1^{1,D})\right]$$

$$\frac{\partial G^3}{\partial \varepsilon} = -(1-e^{-\alpha_D(T-t)})\left[\frac{\partial \mathscr{G}}{\partial x_1}(m_{t,T}^1 + m_{t,T}^2 + m_{t,T}^3 - m_{t,T}^D - \varepsilon(1-e^{-\alpha_D(T-t)}), \sigma_1^{3,D}) - \frac{\partial H}{\partial x_2}(m_{t,T}^3, m_{t,T}^1 + m_{t,T}^2 - m_{t,T}^D - \varepsilon(1-e^{-\alpha_D(T-t)}), \sigma_3^3, \sigma_1^{1,D})\right]$$

The derivatives of the functions $\mathscr{G}$ and H can be found in the appendix of the original paper [2]. In practice we define a function $\varepsilon(T)$ added in the deterministic part of the demand process in order to calibrate the model on all observable forward products. This function will be piecewise constant, with constant parts inside the delivery periods.

3.3.2 Calibration Results

The results of the calibration are given in Figs. 8 and 9. Figure 8 shows an example of estimated $\varepsilon(T)$ at date June 28th, 2011 considering the observable baseload forward products in the French power market: 1 to 6 Month-ahead, 1 to 3 Quarter-ahead and 1 Year-ahead. This example of result shows that the bias is more important for small maturities, but is reasonable (−1.5 GW in maximum) compared to the total available capacity in France (between 80 and 130 GW). When it comes to contracts with a longer granularity, the calibration shows that the values of ε needed to fit the model given by the model, with the real prices, are small, especially for the 1YAH contract. In Fig. 9 we repeated the calibration procedure from January 1st, 2011 to December 31st, 2012 (with a weekly frequency) for three values of $\varepsilon(T)$: the ones needed to exactly retrieve the 1 Month-ahead, 1 Quarter-ahead and 1 Year-ahead products. This result confirms the previous remarks on a decreasing level of ε with an increasing maturity. It also confirms that the resulting values are acceptable (less than 3 GW) compared to the total capacity.

4 Conclusion

In this paper we exposed specific calibration issues for electricity price models. In the first part we stressed that interest rate models, currently used in practice for electricity price modeling, present additional difficulties for calibration. This is mostly

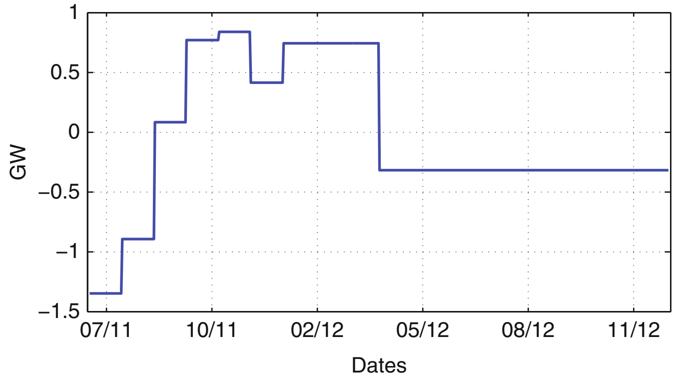

Fig. 8 Calibrated $\varepsilon(T)$ on observed baseload forward prices (1MAH → 6MAH, 1QAH → 3QAH and 1YAH in French power market) at date June 28th, 2011

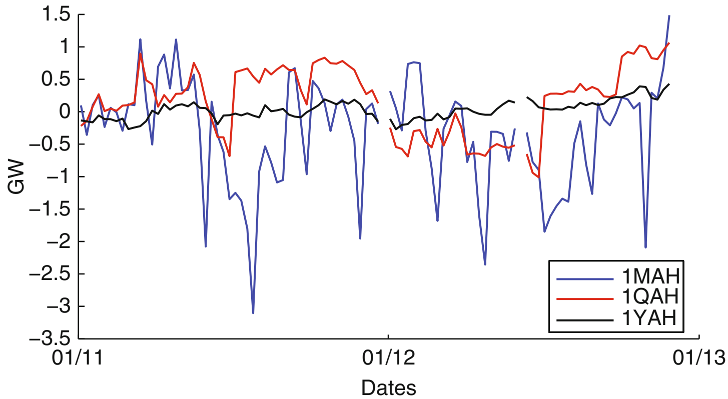

Fig. 9 Evolution of calibrated ε for 1MAH (*blue*), 1QAH (*red*) and 1YAH (*black*) from January 1st, 2011 to December 31st, 2012

due to the non storability of electricity and, hence, the presence of a delivery period in the observed forward products. In this specific context, the calibration procedures must introduce some approximation in the diffusion processes to make the parameter estimation feasible. Also, when the objective is to represent both spot prices and forward products, as an electricity producer aims to, the previous studies [17] about a sufficient number of factors to represent all the products, must be revisited.

In the second part we proposed an original study of how a structural model for electricity prices, initially dedicated to a good representation of spot prices, is able to model forward products. This kind of study is essential for practitioners and can help in the modeling choice. We proposed an easy algorithm, with a modification of the demand model in the risk-neutral probability, to calibrate the model from observed forward products. This idea has already been proposed in [9], for example, and we show how to use it in the structural model described in [2].

In further work the forward price reconstruction will be deepened by introducing a price confidence interval induced by the uncertainties on estimated parameters for demand and capacities processes. This will allow us to measure the impact of these uncertainties to reconstructed forward products. The problem of fixed merit order, as it is assumed in [2], will be addressed as a direct extension of the model. Also, following the calibration results, the future objective will be to price specific power derivatives like, for example, a power plant and study the resulting hedging strategy.

Appendix 1: Structural Model Description

In this appendix, we recall the main results concerning the model, and we detail the computation of the forward prices and stochastic weights. We refer to [1, 2] for more details and proofs.

Forward Pricing

In this section, we detail the computation of a forward contract $F_t(T,\theta)$. We first need to compute the forward price $F_t(T')$, for any $T' \in [T, T+\theta]$, and thus we need a pricing formula, i.e. we need to take a closer look at the EMM that we will use. To do so, we consider the submarket composed by the ith fuel only: assuming this submarket to be complete, there is a unique risk-neutral measure $\mathbb{Q}^i$ that is a risk-neutral measure for the ith fuel. It is shown, in the original paper, that this measure is one of the EMM for the spot price of electricity S_t. Also, the Local Risk Minimization approach, used in this structural model, leads to a Föllmer and Schweizer minimal EMM $\hat{\mathbb{Q}}$ that corresponds to zero risk premiums for the demand and the capacities. In other words, $\hat{\mathbb{Q}}$ is an EMM for the fuels, and coincides with the historical measure $\mathbb{P}$ for the demand and capacities.

These remarks, along with the mutual independence between the demand, fuel prices and capacities, lead to:

$$\begin{aligned}
F_t^e(T') = \hat{\mathbb{E}}[S_{T'}|\mathscr{F}_t] &= \hat{\mathbb{E}}\left[g\left(C_{T'}^{max} - D_{T'}\right) \sum_{i=1}^{n} h_i S_{T'}^i \mathbf{1}_{D_{T'} \in I_{T'}^i} \middle| \mathscr{F}_t \right] \\
&= \sum_{i=1}^{n} h_i \mathbb{E}\left[g\left(C_{T'}^{max} - D_{T'}\right) \mathbf{1}_{D_{T'} \in I_{T'}^i} \middle| \mathscr{F}_t \right] \mathbb{E}^i\left[S_{T'}^i | \mathscr{F}_t\right] \\
&= \sum_{i=1}^{n} h^i G^i(t, T, C_t, D_t) F_t^i(T),
\end{aligned}$$

for any $T' > t$. To obtain the real forward price with delivery period $[T, T+\theta]$, we only have to take the mean of the instantaneous prices $F_t(T')$, for any $T' \in [T, T+\theta]$. The next section will focus on the computation of the stochastic weights.

Computing the Stochastic Weights

Using the models introduced in Sect. 3.1.2 for the demand and capacities, the random variables $D_{T'}$ et $C^i_{T'}$ are Gaussian, conditional to the filtration $\mathscr{F}_t^{D,C}$, and we have:

$$
\begin{aligned}
\mathbb{E}\left[D_{T'}|\mathscr{F}_t^{D,C}\right] &= m^D_{t,T'} = f_D(T') + e^{-\alpha_D(T'-t)}(D_t - f_D(t)) \\
\mathbb{V}\text{ar}\left[D_{T'}|\mathscr{F}_t^{D,C}\right] &= \left(\sigma^i_{t,T'}\right)^2 = \frac{\sigma_D^2}{2\alpha_D}\left[1 - e^{-2\alpha_D(T'-t)}\right] \\
\mathbb{E}\left[C^i_{T'}|\mathscr{F}_t^{D,C}\right] &= m^i_{t,T'} = f_i(T') + e^{-\alpha_i(T'-t)}(C^i_t - f_i(t)) \\
\mathbb{V}\text{ar}\left[C^i_{T'}|\mathscr{F}_t^{D,C}\right] &= \left(\sigma^i_{t,T'}\right)^2 = \frac{\sigma_i^2}{2\alpha_i}\left[1 - e^{-2\alpha_i(T'-t)}\right]
\end{aligned}
\tag{22}
$$

We are trying to compute the following quantities:

$$
G^i(t, T', C_t, D_t) = \mathbb{E}\left[g(C^{max}_{T'} - D_{T'})\mathbf{1}_{D_{T'} \in I^i_{T'}}|\mathscr{F}_t^{D,C}\right]
$$

In the original paper, it is shown that:

$$
\begin{aligned}
G^1(t, T', C_t, D_t) &= H(m^n_2, m^{1,D}_1, \sigma^n_2, \sigma^{1,D}_1) \\
G^i(t, T', C_t, D_t) &= H(m^n_{i+1}, m^{i,D}_1, \sigma^n_{i+1}, \sigma^{i,D}_1) - H(m^n_i, m^{i-1,D}_1, \sigma^n_i, \sigma^{i-1,D}_1) \\
&\qquad \forall i = 2 \ldots n-1 \\
G^n(t, T', C_t, D_t) &= \mathscr{G}(m^{n,D}_1, \sigma^{n,D}_1) - H(m^n_{i+1}, m^{i,D}_1, \sigma^n_{i+1}, \sigma^{i,D}_1)
\end{aligned}
$$

with:

$$
m^n_i = \sum_{k=i}^{n} m^k_{t,T'} \qquad \left(\sigma^n_i\right)^2 = \sum_{k=i}^{n}\left(\sigma^k_{t,T'}\right)^2
$$

$$
m^{i,D}_1 = \sum_{k=1}^{i} m^k_{t,T'} - m^D_{t,T'} \qquad \left(\sigma^{i,D}_1\right)^2 = \sum_{k=1}^{i}\left(\sigma^k_{t,T'}\right)^2 + \left(\sigma^D_{t,T'}\right)^2
$$

and:

$$\mathscr{G}(m,\sigma) = \int_{\mathbb{R}} g(x)\phi_N(x,m,\sigma)dx$$
$$H(m_1,m_2,\sigma_1,\sigma_2) = \int_0^{\infty} \mathscr{G}(x+m_1,\sigma_1)\phi(x,m_2,\sigma_2)dx$$

where $x \longmapsto \phi(x,\mu,\sigma)$ the probability density function of a Gaussian random variable with mean μ and variance σ^2.

In our case, $n = 3$, these computations lead to, using the more convenient notation $m^i = m^i_{t,T'}$ and $m^D = m^D_{t,T'}$:

$$G^1(t,T',C_t,D_t) = H(m^2+m^3,m^1-m^D,\sigma_2^3,\sigma_1^{1,D})$$
$$G^2(t,T',C_t,D_t) = H(m^3,m^2+m^1-m^D,\sigma_3^3,\sigma_1^{2,D}) - H(m^2+m^3,m^1-m^D,\sigma_2^3,\sigma_1^{1,D})$$
$$G^3(t,T',C_t,D_t) = \mathscr{G}(m^2+m^3+m^1-m^D,\sigma_1^{3,D}) - H(m^3,m^2+m^1-m^D,\sigma_3^3,\sigma_1^{2,D})$$

Appendix 2: Estimation of Demand and Capacities Parameters: The Deterministic Part

The model for the deterministic part is:

$$f_D(t) = d_1^{(D)} + d_2^{(D)}\cos\left(2\pi(t-d_3^{(D)})\right) + \mathbf{week}_D(t)$$

We denote by $(Y_i) = (D_{t_i})$ the demand data, and t_i the dates corresponding with the data. To estimate the deterministic part **minus** the weekly scheme, we start with the following least-square regression:

$$Y_i = p_1 + p_2\cos(2\pi t_i) + p_3\sin(2\pi t_i) + \varepsilon_i, \text{ with } \varepsilon_i \sim \mathscr{N}(0,\sigma^2)$$

This equation becomes, using a more convenient matrix notation:

$$Y = Xp + \varepsilon,$$

with:

$$X = \begin{pmatrix} 1 & \cos(2\pi t_1) & \sin(2\pi t_1) \\ \vdots & \vdots & \vdots \\ 1 & \cos(2\pi t_n) & \sin(2\pi t_n) \end{pmatrix}$$

We can then estimate the parameters, by using the least-square estimator:

$$\hat{p} = (X^t X)^{-1} X^t Y$$

We can have access to $(1-\alpha)$-confidence intervals for the first three deterministic parameters:

$$\left[\hat{p} + t_{n-2}^{1-\alpha} S, \hat{p} - t_{n-2}^{1-\alpha} S\right]$$

With:

$$S = \sqrt{\frac{\frac{1}{n-2}\sum_{i=1}^{n} \hat{\varepsilon}_i^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2}} = \sqrt{\frac{Var(Y)}{Var(X)}}\frac{1}{\sqrt{n-2}}$$

We can then transform the previous regression into one that fits our model:

$$\begin{aligned} p_2 \cos(2\pi t_i) + p_3 \sin(2\pi t_i) &= \sqrt{p_2^2 + p_3^2}\cos(2\pi t_i + \phi) \\ &= d_2^{(D)} \cos(2\pi(t - d_3^{(D)}) \end{aligned}$$

Thus we change p into $d^{(D)}$, using :

$$\begin{aligned} d_2^{(D)} &= \sqrt{p_2^2 + p_3^2} \\ d_3^{(D)} &= \frac{1}{2\pi}\arccos\left(\frac{p_3}{d_2^{(D)}}\right) \end{aligned}$$

This transformation also allows us to compute confidence bounds for $d_1^{(D)}$, $d_2^{(D)}$ and $d_3^{(D)}$. This first part of the estimation procedure gives us estimated parameters such as:

$$\boxed{f_D(t) = d_1^{(D)} + d_2^{(D)} \cos\left(2\pi(t - d_3^{(D)}\right)}$$

We now need to estimate the weekly scheme $\mathbf{week}_D(t)$. To do so, we use a the following method, which is quite classic when it comes to estimating a weekly pattern:

- First, we compute the weekly mean of the data, and store it in a variable called $W \in \mathbb{R}^{n_w}$, if n_w is the number of weeks for which we have data.
- Then we compute the weekly residuals, which are the distance between data and the mean of the corresponding week: $R_{t_i} = D_{t_i} - W_{k_i}$, if t_i is in the weekly number k_i. The residuals are then a centered version of the demand.

- Finally, we compute the mean of these residuals, for every hour of the week (which means that we take 168 means of the corresponding residuals):

$$\mathbf{week}_D(t_i) = \frac{1}{n_w} \sum_{t_k \equiv t_i[24]} R_{t_k}$$

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