# Chapter 7

# The Riemann Integral

## 7.1 Discussion: How Should Integration be Defined?

The Fundamental Theorem of Calculus is a statement about the inverse relationship between differentiation and integration. It comes in two parts, depending on whether we are differentiating an integral or integrating a derivative. Under suitable hypotheses on the functions $f$ and $F$, the Fundamental Theorem of Calculus states that

(i) $\displaystyle\int_a^b F'(x)\,dx = F(b) - F(a)$ and

(ii) if $G(x) = \displaystyle\int_a^x f(t)\,dt$, then $G'(x) = f(x)$.

Before we can undertake any type of rigorous investigation of these statements, we need to settle on a definition for $\int_a^b f$. Historically, the concept of integration was defined as the inverse process of differentiation. In other words, the integral of a function $f$ was understood to be a function $F$ that satisfied $F' = f$. Newton, Leibniz, Fermat, and the other founders of calculus then went on to explore the relationship between antiderivatives and the problem of computing areas. This approach is ultimately unsatisfying from the point of view of analysis because it results in a very limited number of functions that can be integrated. Recall that every derivative satisfies the intermediate value property (Darboux's Theorem, Theorem 5.2.7). This means that any function with a jump discontinuity cannot be a derivative. If we want to define integration via antidifferentiation, then we must accept the consequence that a function as simple as

© Springer Science+Business Media New York 2015
S. Abbott, *Understanding Analysis*, Undergraduate Texts in Mathematics, DOI 10.1007/978-1-4939-2712-8_7

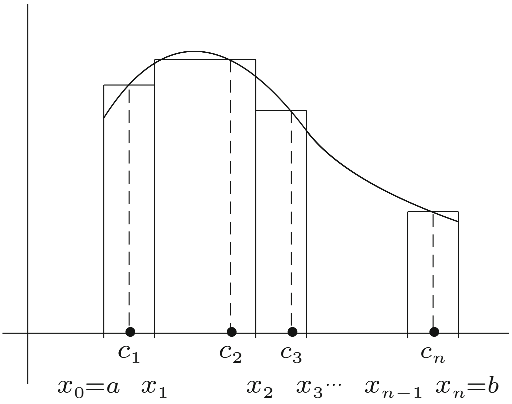



Figure 7.1: A Riemann Sum.

$$h(x) = \begin{cases} 1 & \text{for } 0 \leq x < 1 \\ 2 & \text{for } 1 \leq x \leq 2 \end{cases}$$

is not integrable on the interval $[0, 2]$.

A very interesting shift in emphasis occurred around 1850 in the work of Cauchy, and soon after in the work of Bernhard Riemann. The idea was to completely divorce integration from the derivative and instead use the notion of "area under the curve" as a starting point for building a rigorous definition of the integral. The reasons for this were complicated. As we have mentioned earlier (Section 1.2), the concept of *function* was undergoing a transformation. The traditional understanding of a function as a holistic formula such as $f(x) = x^2$ was being replaced with a more liberal interpretation, which included such bizarre constructions as Dirichlet's function discussed in Section 4.1. Serving as a catalyst to this evolution was the budding theory of Fourier series (discussed in Section 8.5), which required, among other things, the need to be able to integrate these more unruly objects.

The Riemann integral, as it is called today, is the one usually discussed in introductory calculus. Starting with a function $f$ on $[a, b]$, we partition the domain into small subintervals. On each subinterval $[x_{k-1}, x_k]$, we pick some point $c_k \in [x_{k-1}, x_k]$ and use the $y$-value $f(c_k)$ as an approximation for $f$ on $[x_{k-1}, x_k]$. Graphically speaking, the result is a row of thin rectangles constructed to approximate the area between $f$ and the $x$-axis. The area of each rectangle is $f(c_k)(x_k - x_{k-1})$, and so the total area of all of the rectangles is given by the *Riemann sum* (Fig. 7.1)

$$\sum_{k=1}^{n} f(c_k)(x_k - x_{k-1}).$$

Note that "area" here comes with the understanding that areas below the $x$-axis are assigned a negative value.

What should be evident from the graph is that the accuracy of the Riemann-sum approximation seems to improve as the rectangles get thinner. In some sense, we take the *limit* of these approximating Riemann sums as the width of the individual subintervals of the partitions tends to zero. This limit, if it exists, is Riemann's definition of $\int_a^b f$.

This brings us to a handful of questions. Creating a rigorous meaning for the limit just referred to is not too difficult. What will be of most interest to us—and was also to Riemann—is deciding what types of functions can be integrated using this procedure. Specifically, what conditions on $f$ guarantee that this limit exists?

The theory of the Riemann integral turns on the observation that smaller subintervals produce better approximations to the function $f$. On each subinterval $[x_{k-1}, x_k]$, the function $f$ is approximated by its value at some point $c_k \in [x_{k-1}, x_k]$. The quality of the approximation is directly related to the difference

$$|f(x) - f(c_k)|$$

as $x$ ranges over the subinterval. Because the subintervals can be chosen to have arbitrarily small width, this means that we want $f(x)$ to be close to $f(c_k)$ whenever $x$ is close to $c_k$. But this sounds like a discussion of continuity! We will soon see that the continuity of $f$ is intimately related to the existence of the Riemann integral $\int_a^b f$.

Is continuity sufficient to prove that the Riemann sums converge to a well-defined limit? Is it necessary, or can the Riemann integral handle a discontinuous function such as $h(x)$ mentioned earlier? Relying on the intuitive notion of area, it would seem that $\int_0^2 h = 3$, but does the Riemann integral reach this conclusion? If so, how discontinuous can a function be before it fails to be integrable? Can the Riemann integral make sense out of something as pathological as Dirichlet's function on the interval $[0, 1]$?

A function such as

$$g(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

raises another interesting question. Here is an example of a differentiable function, studied in Section 5.1, where the derivative $g'(x)$ is *not* continuous. As we explore the class of integrable functions, some attempt must be made to reunite the integral with the derivative. Having defined integration independently of differentiation, we would like to come back and investigate the conditions under which equations (i) and (ii) from the Fundamental Theorem of Calculus stated earlier hold. If we are making a wish list for the types of functions that we want to be integrable, then in light of equation (i) it seems desirable to expect this set to at least contain the set of derivatives. The fact that derivatives are not always continuous is further motivation not to content ourselves with an integral that cannot handle some discontinuities.

## 7.2 The Definition of the Riemann Integral

Although it has the benefit of some polish due to Darboux, the development of the integral presented in this chapter is closely related to the procedure just discussed. In place of Riemann sums, we will construct *upper sums* and *lower sums* (Fig. 7.2), and in place of a limit we will use a supremum and an infimum.

Throughout this section, it is assumed that we are working with a *bounded* function $f$ on a closed interval $[a,b]$, meaning that there exists an $M > 0$ such that $|f(x)| \leq M$ for all $x \in [a,b]$.

### Partitions, Upper Sums, and Lower Sums

**Definition 7.2.1.** A *partition* $P$ of $[a,b]$ is a finite set of points from $[a,b]$ that includes both $a$ and $b$. The notational convention is to always list the points of a partition $P = \{x_0, x_1, x_2, \dots, x_n\}$ in increasing order; thus,

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$

For each subinterval $[x_{k-1}, x_k]$ of $P$, let

$$m_k = \inf\{f(x) : x \in [x_{k-1}, x_k]\} \quad \text{and} \quad M_k = \sup\{f(x) : x \in [x_{k-1}, x_k]\}.$$

The *lower sum* of $f$ with respect to $P$ is given by

$$L(f,P) = \sum_{k=1}^{n} m_k(x_k - x_{k-1}).$$

Likewise, we define the *upper sum* of $f$ with respect to $P$ by

$$U(f,P) = \sum_{k=1}^{n} M_k(x_k - x_{k-1}).$$

For a particular partition $P$, it is clear that $U(f,P) \geq L(f,P)$. The fact that this same inequality holds if the upper and lower sums are computed with respect to different partitions is the content of the next two lemmas.

**Definition 7.2.2.** A partition $Q$ is a *refinement* of a partition $P$ if $Q$ contains all of the points of $P$; that is, if $P \subseteq Q$.

**Lemma 7.2.3.** *If $P \subseteq Q$, then $L(f,P) \leq L(f,Q)$, and $U(f,P) \geq U(f,Q)$.*

*Proof.* Consider what happens when we refine $P$ by adding a single point $z$ to some subinterval $[x_{k-1}, x_k]$ of $P$.

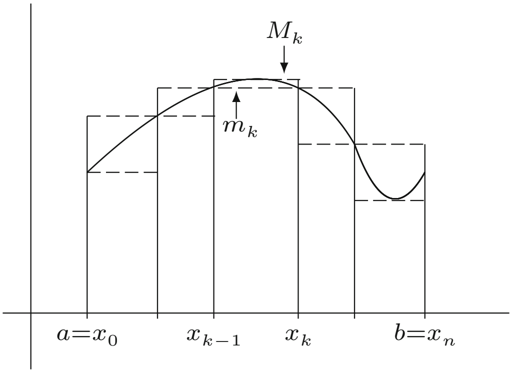
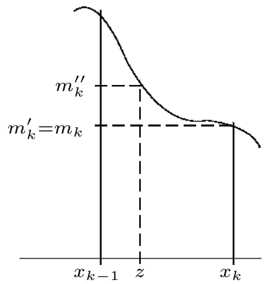



Figure 7.2: Upper and Lower Sums.




Focusing on the lower sum for a moment, we have

$$\begin{aligned} m_k(x_k - x_{k-1}) &= m_k(x_k - z) + m_k(z - x_{k-1}) \\ &\leq m_k'(x_k - z) + m_k''(z - x_{k-1}), \end{aligned}$$

where

$$m_k' = \inf\{f(x) : x \in [z, x_k]\} \quad \text{and} \quad m_k'' = \inf\{f(x) : x \in [x_{k-1}, z]\}$$

are each necessarily as large or larger than $m_k$.

By induction, we have $L(f, P) \leq L(f, Q)$, and an analogous argument holds for the upper sums. □

**Lemma 7.2.4.** *If $P_1$ and $P_2$ are any two partitions of $[a, b]$, then $L(f, P_1) \leq U(f, P_2)$.*

*Proof.* Let $Q = P_1 \cup P_2$ be the so-called *common refinement* of $P_1$ and $P_2$. Because $P_1 \subseteq Q$ and $P_2 \subseteq Q$, it follows that

$$L(f, P_1) \leq L(f, Q) \leq U(f, Q) \leq U(f, P_2). \qquad \square$$

## Integrability

Intuitively, it helps to visualize a particular upper sum as an overestimate for the value of the integral and a lower sum as an underestimate. As the partitions get more refined, the upper sums get potentially smaller while the lower sums get potentially larger. A function is *integrable* if the upper and lower sums "meet" at some common value in the middle.

Rather than taking a limit of these sums, we will instead make use of the Axiom of Completeness and consider the *infimum* of the upper sums and the *supremum* of the lower sums.

**Definition 7.2.5.** Let $\mathcal{P}$ be the collection of all possible partitions of the interval $[a, b]$. The *upper integral* of $f$ is defined to be

$$U(f) = \inf\{U(f, P) : P \in \mathcal{P}\}.$$

In a similar way, define the *lower integral* of $f$ by

$$L(f) = \sup\{L(f, P) : P \in \mathcal{P}\}.$$

The following fact is not surprising.

**Lemma 7.2.6.** *For any bounded function $f$ on $[a, b]$, it is always the case that $U(f) \geq L(f)$.*

*Proof.* Exercise 7.2.1. □

**Definition 7.2.7 (Riemann Integrability).** A bounded function $f$ defined on the interval $[a, b]$ is *Riemann-integrable* if $U(f) = L(f)$. In this case, we define $\int_a^b f$ or $\int_a^b f(x)\,dx$ to be this common value; namely,

$$\int_a^b f = U(f) = L(f).$$

The modifier "Riemann" in front of "integrable" accurately suggests that there are other ways to define the integral. In fact, our work in this chapter will expose the need for a different approach, one of which is discussed in Section 8.1. In this chapter, the Riemann integral is the only method under consideration, so it will usually be convenient to drop the modifier "Riemann" and simply refer to a function as being "integrable."

## Criteria for Integrability

To summarize the situation thus far, it is always the case for a bounded function $f$ on $[a, b]$ that

$$\sup\{L(f, P) : P \in \mathcal{P}\} = L(f) \leq U(f) = \inf\{U(f, P) : P \in \mathcal{P}\}.$$

The function $f$ is integrable if the inequality is an equality. The major thrust of our investigation of the integral is to describe, as best we can, the class of integrable functions. The preceding inequality reveals that integrability is really equivalent to the existence of partitions whose upper and lower sums are arbitrarily close together.

**Theorem 7.2.8 (Integrability Criterion).** *A bounded function $f$ is integrable on $[a,b]$ if and only if, for every $\epsilon > 0$, there exists a partition $P_\epsilon$ of $[a,b]$ such that*

$$U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon.$$

*Proof.* Let $\epsilon > 0$. If such a partition $P_\epsilon$ exists, then

$$U(f) - L(f) \leq U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon.$$

Because $\epsilon$ is arbitrary, it must be that $U(f) = L(f)$, so $f$ is integrable. (To be absolutely precise here, we could throw in a reference to Theorem 1.2.6.)

The proof of the converse statement is a familiar triangle inequality argument with parentheses in place of absolute value bars because, in each case, we know which quantity is larger. Because $U(f)$ is the greatest lower bound of the upper sums, we know that, given some $\epsilon > 0$, there must exist a partition $P_1$ such that

$$U(f, P_1) < U(f) + \frac{\epsilon}{2}.$$

Likewise, there exists a partition $P_2$ satisfying

$$L(f, P_2) > L(f) - \frac{\epsilon}{2}.$$

Now, let $P_\epsilon = P_1 \cup P_2$ be the common refinement. Keeping in mind that the integrability of $f$ means $U(f) = L(f)$, we can write

$$\begin{aligned} U(f, P_\epsilon) - L(f, P_\epsilon) &\leq U(f, P_1) - L(f, P_2) \\ &< \left(U(f) + \frac{\epsilon}{2}\right) - \left(L(f) - \frac{\epsilon}{2}\right) \\ &= \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

□

In the discussion at the beginning of this chapter, it became clear that integrability is closely tied to the concept of continuity. To make this observation more precise, let $P = \{x_0, x_1, x_2, \ldots, x_n\}$ be an arbitrary partition of $[a,b]$, and define $\Delta x_k = x_k - x_{k-1}$. Then,

$$U(f, P) - L(f, P) = \sum_{k=1}^{n} (M_k - m_k) \Delta x_k,$$

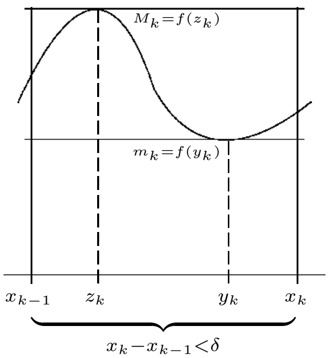
where $M_k$ and $m_k$ are the supremum and infimum of the function on the interval $[x_{k-1}, x_k]$, respectively. Our ability to control the size of $U(f,P) - L(f,P)$ hinges on the differences $M_k - m_k$, which we can interpret as the variation in the range of the function over the interval $[x_{k-1}, x_k]$. Restricting the variation of $f$ over arbitrarily small intervals in $[a,b]$ is *precisely* what it means to say that $f$ is uniformly continuous on this set.

**Theorem 7.2.9.** *If $f$ is continuous on $[a, b]$, then it is integrable.*

*Proof.* Because $f$ is continuous on a compact set, it must be bounded. It is also uniformly continuous for the same reason. This means that, given $\epsilon > 0$, there exists a $\delta > 0$ so that $|x - y| < \delta$ guarantees

$$|f(x) - f(y)| < \frac{\epsilon}{b-a}.$$

Now, let $P$ be a partition of $[a, b]$ where $\Delta x_k = x_k - x_{k-1}$ is less than $\delta$ for every subinterval of $P$.




Given a particular subinterval $[x_{k-1}, x_k]$ of $P$, we know from the Extreme Value Theorem (Theorem 4.4.2) that the supremum $M_k = f(z_k)$ for some $z_k \in [x_{k-1}, x_k]$. In addition, the infimum $m_k$ is attained at some point $y_k$ also in the interval $[x_{k-1}, x_k]$. But this means $|z_k - y_k| < \delta$, so

$$M_k - m_k = f(z_k) - f(y_k) < \frac{\epsilon}{b-a}.$$

Finally,

$$U(f, P) - L(f, P) = \sum_{k=1}^{n} (M_k - m_k)\Delta x_k < \frac{\epsilon}{b-a} \sum_{k=1}^{n} \Delta x_k = \epsilon,$$

and $f$ is integrable by the criterion given in Theorem 7.2.8. □

## Exercises

**Exercise 7.2.1.** Let $f$ be a bounded function on $[a, b]$, and let $P$ be an arbitrary partition of $[a, b]$. First, explain why $U(f) \geq L(f, P)$. Now, prove Lemma 7.2.6.

**Exercise 7.2.2.** Consider $f(x) = 1/x$ over the interval $[1, 4]$. Let $P$ be the partition consisting of the points $\{1, 3/2, 2, 4\}$.

(a) Compute $L(f,P)$, $U(f,P)$, and $U(f,P)-L(f,P)$.

(b) What happens to the value of $U(f,P)-L(f,P)$ when we add the point 3 to the partition?

(c) Find a partition $P'$ of $[1,4]$ for which $U(f,P')-L(f,P')<2/5$.

**Exercise 7.2.3 (Sequential Criterion for Integrability).** (a) Prove that a bounded function $f$ is integrable on $[a,b]$ if and only if there exists a sequence of partitions $(P_n)_{n=1}^{\infty}$ satisfying

$$\lim_{n\to\infty}[U(f,P_n)-L(f,P_n)]=0,$$

and in this case $\int_a^b f=\lim_{n\to\infty}U(f,P_n)=\lim_{n\to\infty}L(f,P_n)$.

(b) For each $n$, let $P_n$ be the partition of $[0,1]$ into $n$ equal subintervals. Find formulas for $U(f,P_n)$ and $L(f,P_n)$ if $f(x)=x$. The formula $1+2+3+\cdots+n=n(n+1)/2$ will be useful.

(c) Use the sequential criterion for integrability from (a) to show directly that $f(x)=x$ is integrable on $[0,1]$ and compute $\int_0^1 f$.

**Exercise 7.2.4.** Let $g$ be bounded on $[a,b]$ and assume there exists a partition $P$ with $L(g,P)=U(g,P)$. Describe $g$. Is it integrable? If so, what is the value of $\int_a^b g$?

**Exercise 7.2.5.** Assume that, for each $n$, $f_n$ is an integrable function on $[a,b]$. If $(f_n)\to f$ uniformly on $[a,b]$, prove that $f$ is also integrable on this set. (We will see that this conclusion does not necessarily follow if the convergence is pointwise.)

**Exercise 7.2.6.** A *tagged partition* $(P,\{c_k\})$ is one where in addition to a partition $P$ we choose a sampling point $c_k$ in each of the subintervals $[x_{k-1},x_k]$. The corresponding *Riemann sum,*

$$R(f,P)=\sum_{k=1}^{n}f(c_k)\Delta x_k,$$

is discussed in Section 7.1, where the following definition is alluded to.

**Riemann's Original Definition of the Integral**: A bounded function $f$ is *integrable* on $[a,b]$ with $\int_a^b f=A$ if for all $\epsilon>0$ there exists a $\delta>0$ such that for any tagged partition $(P,\{c_k\})$ satisfying $\Delta x_k<\delta$ for all $k$, it follows that

$$|R(f,P)-A|<\epsilon.$$

Show that if $f$ satisfies Riemann's definition above, then $f$ is integrable in the sense of Definition 7.2.7. (The full equivalence of these two characterizations of integrability is proved in Section 8.1.)

**Exercise 7.2.7.** Let $f:[a,b]\to\mathbf{R}$ be increasing on the set $[a,b]$ (i.e., $f(x)\le f(y)$ whenever $x<y$). Show that $f$ is integrable on $[a,b]$.

## 7.3 Integrating Functions with Discontinuities

The fact that continuous functions are integrable is not so much a fortunate discovery as it is evidence for a well-designed integral. Riemann's integral is a modification of Cauchy's definition of the integral, and Cauchy's definition was crafted specifically to work on continuous functions. The interesting issue is discovering just how dependent the Riemann integral is on the continuity of the integrand.

**Example 7.3.1.** Consider the function

$$f(x) = \begin{cases} 1 & \text{for } x \neq 1 \\ 0 & \text{for } x = 1 \end{cases}$$

on the interval $[0, 2]$. If $P$ is any partition of $[0, 2]$, a quick calculation reveals that $U(f, P) = 2$. The lower sum $L(f, P)$ will be less than 2 because any subinterval of $P$ that contains $x = 1$ will contribute zero to the value of the lower sum. The way to show that $f$ is integrable is to construct a partition that minimizes the effect of the discontinuity by embedding $x = 1$ into a very small subinterval.

Let $\epsilon > 0$, and consider the partition $P_\epsilon = \{0, 1 - \epsilon/3, 1 + \epsilon/3, 2\}$. Then,

$$\begin{aligned} L(f, P_\epsilon) &= 1\left(1 - \frac{\epsilon}{3}\right) + 0(\epsilon) + 1\left(1 - \frac{\epsilon}{3}\right) \\ &= 2 - \frac{2}{3}\epsilon. \end{aligned}$$

Because $U(f, P_\epsilon) = 2$, we have

$$U(f, P_\epsilon) - L(f, P_\epsilon) = \frac{2}{3}\epsilon < \epsilon.$$

We can now use Theorem 7.2.8 to conclude that $f$ is integrable.

Although the function in Example 7.3.1 is extremely simple, the method used to show it is integrable is really the same one used to prove that any bounded function with a single discontinuity is integrable. The notation in the following proof is more cumbersome, but the essence of the argument is that the misbehavior of the function at its discontinuity is isolated inside a particularly small subinterval of the partition.

**Theorem 7.3.2.** *If $f : [a, b] \to \mathbf{R}$ is bounded, and $f$ is integrable on $[c, b]$ for all $c \in (a, b)$, then $f$ is integrable on $[a, b]$. An analogous result holds at the other endpoint.*

*Proof.* Let $\epsilon > 0$. As usual, our task is to produce a partition $P$ such that $U(f, P) - L(f, P) < \epsilon$. For any partition, we can always write

$$\begin{aligned} U(f, P) - L(f, P) &= \sum_{k=1}^{n} (M_k - m_k)\Delta x_k \\ &= (M_1 - m_1)(x_1 - a) + \sum_{k=2}^{n} (M_k - m_k)\Delta x_k, \end{aligned}$$

so the first step is to choose $x_1$ close enough to $a$ so that

$$(M_1 - m_1)(x_1 - a) < \frac{\epsilon}{2}.$$

This is not too difficult. Because $f$ is bounded, we know there exists $M > 0$ satisfying $|f(x)| \leq M$ for all $x \in [a, b]$. Noting that $M_1 - m_1 \leq 2M$, let's pick $x_1$ so that

$$x_1 - a < \frac{\epsilon}{4M}.$$

Now, by hypothesis, $f$ is integrable on $[x_1, b]$, so there exists a partition $P_1$ of $[x_1, b]$ for which

$$U(f, P_1) - L(f, P_1) < \frac{\epsilon}{2}.$$

Finally, we let $P = \{a\} \cup P_1$ be a partition of $[a, b]$, from which it follows that

$$\begin{aligned} U(f, P) - L(f, P) &\leq (2M)(x_1 - a) + (U(f, P_1) - L(f, P_1)) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

□

Theorem 7.3.2 enables us to prove that a bounded function on a closed interval with a single discontinuity at an endpoint is still integrable. In the next section, we will prove that integrability on the intervals $[a, b]$ and $[b, d]$ is equivalent to integrability on $[a, d]$. This property, together with an induction argument, leads to the conclusion that any function with a *finite* number of discontinuities is still integrable. What if the number of discontinuities is infinite?

**Example 7.3.3.** Recall Dirichlet's function

$$g(x) = \begin{cases} 1 & \text{for } x \text{ rational} \\ 0 & \text{for } x \text{ irrational} \end{cases}$$

from Section 4.1. If $P$ is some partition of $[0, 1]$, then the density of the rationals in $\mathbf{R}$ implies that every subinterval of $P$ will contain a point where $g(x) = 1$. It follows that $U(g, P) = 1$. On the other hand, $L(g, P) = 0$ because the irrationals are also dense in $\mathbf{R}$. Because this is the case for every partition $P$, we see that the upper integral $U(f) = 1$ and the lower integral $L(f) = 0$. The two are not equal, so we conclude that Dirichlet's function is *not* integrable.

How discontinuous can a function be before it fails to be integrable? Before jumping to the hasty (and incorrect) conclusion that the Riemann integral fails for functions with more than a finite number of discontinuities, we should realize that Dirichlet's function is discontinuous at *every* point in $[0, 1]$. It would be useful to investigate a function where the discontinuities are infinite in number but do not necessarily make up all of $[0, 1]$. Thomae's function, also defined in Section 4.1, is one such example. The discontinuous points of this function are precisely the rational numbers in $[0, 1]$. In the exercises to follow we will see that Thomae's function *is* Riemann-integrable, raising the bar for allowable discontinuous points to include potentially infinite sets.

The conclusion of this story is contained in the doctoral dissertation of Henri Lebesgue, who presented his work in 1901. Lebesgue's elegant criterion for Riemann integrability is explored in great detail in Section 7.6. For the moment, though, we will take a short detour from questions of integrability and construct a proof of the celebrated Fundamental Theorem of Calculus.

## Exercises

**Exercise 7.3.1.** Consider the function

$$h(x) = \begin{cases} 1 & \text{for } 0 \leq x < 1 \\ 2 & \text{for } x = 1 \end{cases}$$

over the interval $[0, 1]$.

(a) Show that $L(f, P) = 1$ for every partition $P$ of $[0, 1]$.

(b) Construct a partition $P$ for which $U(f, P) < 1 + 1/10$.

(c) Given $\epsilon > 0$, construct a partition $P_\epsilon$ for which $U(f, P_\epsilon) < 1 + \epsilon$.

**Exercise 7.3.2.** Recall that Thomae's function

$$t(x) = \begin{cases} 1 & \text{if } x = 0 \\ 1/n & \text{if } x = m/n \in \mathbf{Q}\backslash\{0\} \text{ is in lowest terms with } n > 0 \\ 0 & \text{if } x \notin \mathbf{Q} \end{cases}$$

has a countable set of discontinuities occurring at precisely every rational number. Follow these steps to prove $t(x)$ is integrable on $[0, 1]$ with $\int_0^1 t = 0$.

(a) First argue that $L(t, P) = 0$ for any partition $P$ of $[0, 1]$.

(b) Let $\epsilon > 0$, and consider the set of points $D_{\epsilon/2} = \{x \in [0, 1] : t(x) \geq \epsilon/2\}$. How big is $D_{\epsilon/2}$?

(c) To complete the argument, explain how to construct a partition $P_\epsilon$ of $[0, 1]$ so that $U(t, P_\epsilon) < \epsilon$.

**Exercise 7.3.3.** Let

$$f(x) = \begin{cases} 1 & \text{if } x = 1/n \text{ for some } n \in \mathbf{N} \\ 0 & \text{otherwise.} \end{cases}$$

Show that $f$ is integrable on $[0, 1]$ and compute $\int_0^1 f$.

**Exercise 7.3.4.** Let $f$ and $g$ be functions defined on (possibly different) closed intervals, and assume the range of $f$ is contained in the domain of $g$ so that the composition $g \circ f$ is properly defined.

(a) Show, by example, that it is not the case that if $f$ and $g$ are integrable, then $g \circ f$ is integrable.

Now decide on the validity of each of the following conjectures, supplying a proof or counterexample as appropriate.

(b) If $f$ is increasing and $g$ is integrable, then $g \circ f$ is integrable.

(c) If $f$ is integrable and $g$ is increasing, then $g \circ f$ is integrable.

**Exercise 7.3.5.** Provide an example or give a reason why the request is impossible.

(a) A sequence $(f_n) \to f$ pointwise, where each $f_n$ has at most a finite number of discontinuities but $f$ is not integrable.

(b) A sequence $(g_n) \to g$ uniformly where each $g_n$ has at most a finite number of discontinuities and $g$ is not integrable.

(c) A sequence $(h_n) \to h$ uniformly where each $h_n$ is not integrable but $h$ is integrable.

**Exercise 7.3.6.** Let $\{r_1, r_2, r_3, \ldots\}$ be an enumeration of all the rationals in $[0,1]$, and define

$$g_n(x) = \begin{cases} 1 & \text{if } x = r_n \\ 0 & \text{otherwise.} \end{cases}$$

(a) Is $G(x) = \sum_{n=1}^{\infty} g_n(x)$ integrable on $[0,1]$?

(b) Is $F(x) = \sum_{n=1}^{\infty} g_n(x)/n$ integrable on $[0,1]$?

**Exercise 7.3.7.** Assume $f : [a,b] \to \mathbf{R}$ is integrable.

(a) Show that if $g$ satisfies $g(x) = f(x)$ for all but a finite number of points in $[a,b]$, then $g$ is integrable as well.

(b) Find an example to show that $g$ may fail to be integrable if it differs from $f$ at a countable number of points.

**Exercise 7.3.8.** As in Exercise 7.3.6, let $\{r_1, r_2, r_3, \ldots\}$ be an enumeration of the rationals in $[0,1]$, but this time define

$$h_n(x) = \begin{cases} 1 & \text{if } r_n < x \leq 1 \\ 0 & \text{if } 0 \leq x \leq r_n. \end{cases}$$

Show $H(x) = \sum_{n=1}^{\infty} h_n(x)/2^n$ is integrable on $[0,1]$ even though it has discontinuities at every rational point.

**Exercise 7.3.9 (Content Zero).** A set $A \subseteq [a,b]$ has *content zero* if for every $\epsilon > 0$ there exists a finite collection of open intervals $\{O_1, O_2, \ldots, O_N\}$ that contain $A$ in their union and whose lengths sum to $\epsilon$ or less. Using $|O_n|$ to refer to the length of each interval, we have

$$A \subseteq \bigcup_{n=1}^{N} O_n \quad \text{and} \quad \sum_{n=1}^{N} |O_n| \leq \epsilon.$$

(a) Let $f$ be bounded on $[a,b]$. Show that if the set of discontinuous points of $f$ has content zero, then $f$ is integrable.

(b) Show that any finite set has content zero.

(c) Content zero sets do not have to be finite. They do not have to be countable. Show that the Cantor set $C$ defined in Section 3.1 has content zero.

(d) Prove that

$$h(x) = \begin{cases} 1 & \text{if } x \in C \\ 0 & \text{if } x \notin C. \end{cases}$$

is integrable, and find the value of the integral.

## 7.4 Properties of the Integral

Before embarking on the proof of the Fundamental Theorem of Calculus, we need to verify what are probably some very familiar properties of the integral. The discussion in the previous section has already made use of the following fact.

**Theorem 7.4.1.** *Assume $f : [a,b] \to \mathbf{R}$ is bounded, and let $c \in (a,b)$. Then, $f$ is integrable on $[a,b]$ if and only if $f$ is integrable on $[a,c]$ and $[c,b]$. In this case, we have*

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

*Proof.* If $f$ is integrable on $[a,b]$, then for $\epsilon > 0$ there exists a partition $P$ such that $U(f,P) - L(f,P) < \epsilon$. Because refining a partition can only potentially bring the upper and lower sums closer together, we can simply add $c$ to $P$ if it is not already there. Then, let $P_1 = P \cap [a,c]$ be a partition of $[a,c]$, and $P_2 = P \cap [c,b]$ be a partition of $[c,b]$. It follows that

$$U(f,P_1) - L(f,P_1) < \epsilon \quad \text{and} \quad U(f,P_2) - L(f,P_2) < \epsilon,$$

implying that $f$ is integrable on $[a,c]$ and $[c,b]$.

Conversely, if we are given that $f$ is integrable on the two smaller intervals $[a,c]$ and $[c,b]$, then given an $\epsilon > 0$ we can produce partitions $P_1$ and $P_2$ of $[a,c]$ and $[c,b]$, respectively, such that

$$U(f,P_1) - L(f,P_1) < \frac{\epsilon}{2} \quad \text{and} \quad U(f,P_2) - L(f,P_2) < \frac{\epsilon}{2}.$$

Letting $P = P_1 \cup P_2$ produces a partition of $[a, b]$ for which

$$U(f, P) - L(f, P) < \epsilon.$$

Thus, $f$ is integrable on $[a, b]$.

Continuing to let $P = P_1 \cup P_2$ as earlier, we have

$$\begin{aligned} \int_a^b f \leq U(f, P) &< L(f, P) + \epsilon \\ &= L(f, P_1) + L(f, P_2) + \epsilon \\ &\leq \int_a^c f + \int_c^b f + \epsilon, \end{aligned}$$

which implies $\int_a^b f \leq \int_a^c f + \int_c^b f$. To get the other inequality, observe that

$$\begin{aligned} \int_a^c f + \int_c^b f &\leq U(f, P_1) + U(f, P_2) \\ &< L(f, P_1) + L(f, P_2) + \epsilon \\ &= L(f, P) + \epsilon \\ &\leq \int_a^b f + \epsilon. \end{aligned}$$

Because $\epsilon > 0$ is arbitrary, we must have $\int_a^c f + \int_c^b f \leq \int_a^b f$, so

$$\int_a^c f + \int_c^b f = \int_a^b f,$$

as desired. □

The proof of Theorem 7.4.1 demonstrates some of the standard techniques involved for proving facts about the Riemann integral. The next result catalogs the remainder of the basic properties of the integral that we will need in our upcoming arguments.

**Theorem 7.4.2.** *Assume $f$ and $g$ are integrable functions on the interval $[a, b]$.*

(i) *The function $f + g$ is integrable on $[a, b]$ with $\int_a^b (f + g) = \int_a^b f + \int_a^b g$.*

(ii) *For $k \in \mathbf{R}$, the function $kf$ is integrable with $\int_a^b kf = k \int_a^b f$.*

(iii) *If $m \leq f(x) \leq M$ on $[a, b]$, then $m(b - a) \leq \int_a^b f \leq M(b - a)$.*

(iv) *If $f(x) \leq g(x)$ on $[a, b]$, then $\int_a^b f \leq \int_a^b g$.*

(v) *The function $|f|$ is integrable and $|\int_a^b f| \leq \int_a^b |f|$.*

*Proof.* Properties (i) and (ii) are reminiscent of the Algebraic Limit Theorem and its many descendants (Theorems 2.3.3, 2.7.1, 4.2.4, and 5.2.4). In fact, there is a way to use the Algebraic Limit Theorem for this argument as well. An immediate corollary to Theorem 7.2.8 is that a function $f$ is integrable on $[a, b]$ if and only if there exists a sequence of partitions $(P_n)$ satisfying

$$\lim_{n\to\infty} [U(f, P_n) - L(f, P_n)] = 0, \tag{1}$$

and in this case $\int_a^b f = \lim U(f, P_n) = \lim L(f, P_n)$. (A proof for this was requested as Exercise 7.2.3.)

To prove (ii) for the case $k \geq 0$, first verify that for any partition $P$ we have

$$U(kf, P) = kU(f, P) \quad \text{and} \quad L(kf, P) = kL(f, P).$$

Exercise 1.3.5 is used here. Because $f$ is integrable, there exist partitions $(P_n)$ satisfying (1). Turning our attention to the function $(kf)$, we see that

$$\lim_{n\to\infty} [U(kf, P_n) - L(kf, P_n)] = \lim_{n\to\infty} k\,[U(f, P_n) - L(f, P_n)] = 0,$$

and the formula in (ii) follows. The case where $k < 0$ is similar except that we have

$$U(kf, P_n) = kL(f, P_n) \quad \text{and} \quad L(kf, P_n) = kU(f, P_n).$$

A proof for (i) can be constructed using similar methods and is requested in Exercise 7.4.5.

To prove (iii), observe that

$$U(f, P) \geq \int_a^b f \geq L(f, P)$$

for any partition $P$. Statement (iii) follows if we take $P$ to be the trivial partition consisting of only the endpoints $a$ and $b$.

For (iv), let $h = g - f$ and use (i), (ii), and (iii).

Because $-|f(x)| \leq f(x) \leq |f(x)|$ on $[a, b]$, statement (v) will follow from (iv) provided that we can show that $|f|$ is actually integrable. The proof of this fact is outlined in Exercise 7.4.1. □

To this point, the quantity $\int_a^b f$ is only defined in the case where $a < b$.

**Definition 7.4.3.** If $f$ is integrable on the interval $[a, b]$, define

$$\int_b^a f = -\int_a^b f.$$

Also, for $c \in [a, b]$ define

$$\int_c^c f = 0.$$

Definition 7.4.3 is a natural convention to simplify the algebra of integrals. If $f$ is an integrable function on some interval $I$, then it is straightforward to verify that the equation

$$\int_a^b f = \int_a^c f + \int_c^b f$$

from Theorem 7.4.1 remains valid for *any* three points $a, b$, and $c$ chosen in any order from $I$.

## Uniform Convergence and Integration

If $(f_n)$ is a sequence of integrable functions on $[a, b]$, and if $f_n \to f$, then we are inevitably going to want to know whether

$$\int_a^b f_n \to \int_a^b f. \tag{2}$$

This is an archetypical instance of one of the major themes of analysis: When does a mathematical manipulation such as integration respect the limiting process?

If the convergence is pointwise, then any number of things can go wrong. It is possible for each $f_n$ to be integrable but for the limit $f$ not to be integrable (Exercise 7.3.5). Even if the limit function $f$ is integrable, equation (2) may fail to hold. As an example of this, let

$$f_n(x) = \begin{cases} n & \text{if } 0 < x < 1/n \\ 0 & \text{if } x = 0 \text{ or } x \geq 1/n. \end{cases}$$

Each $f_n$ has two discontinuities on $[0, 1]$ and so is integrable with $\int_0^1 f_n = 1$. For each $x \in [0, 1]$, we have $\lim f_n(x) = 0$ so that $f_n \to 0$ pointwise on $[0, 1]$. But now observe that the limit function $f = 0$ certainly integrates to 0, and

$$0 \neq \lim_{n\to\infty} \int_0^1 f_n.$$

As a final remark on what can go wrong in (2), we should point out that it is possible to modify this example to produce a situation where $\lim \int_0^1 f_n$ does not even exist.

One way to resolve all of these problems is to add the assumption of uniform convergence.

**Theorem 7.4.4 (Integrable Limit Theorem).** *Assume that $f_n \to f$ uniformly on $[a, b]$ and that each $f_n$ is integrable. Then, $f$ is integrable and*

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b f.$$

*Proof.* The proof that $f$ is integrable was requested as Exercise 7.2.5. The properties of the integral listed in Theorem 7.4.2 allow us to assert that for any $f_n$,

$$\left|\int_a^b f_n - \int_a^b f\right| = \left|\int_a^b (f_n - f)\right| \leq \int_a^b |f_n - f|.$$

Let $\epsilon > 0$ be arbitrary. Because $f_n \to f$ uniformly, there exists an $N$ such that

$$|f_n(x) - f(x)| < \epsilon/(b-a) \quad \text{for all } n \geq N \text{ and } x \in [a, b].$$

Thus, for $n \geq N$ we see that

$$\begin{aligned}\left|\int_a^b f_n - \int_a^b f\right| &\leq \int_a^b |f_n - f| \\ &\leq \int_a^b \frac{\epsilon}{b-a} = \epsilon,\end{aligned}$$

and the result follows. □

## Exercises

**Exercise 7.4.1.** Let $f$ be a bounded function on a set $A$, and set

$$M = \sup\{f(x) : x \in A\}, \quad m = \inf\{f(x) : x \in A\},$$

$$M' = \sup\{|f(x)| : x \in A\}, \quad \text{and} \quad m' = \inf\{|f(x)| : x \in A\}.$$

(a) Show that $M - m \geq M' - m'$.

(b) Show that if $f$ is integrable on the interval $[a, b]$, then $|f|$ is also integrable on this interval.

(c) Provide the details for the argument that in this case we have $|\int_a^b f| \leq \int_a^b |f|$.

**Exercise 7.4.2.** (a) Let $g(x) = x^3$, and classify each of the following as positive, negative, or zero.

$$\text{(i)} \int_0^{-1} g + \int_0^1 g \qquad \text{(ii)} \int_1^0 g + \int_0^1 g \qquad \text{(iii)} \int_1^{-2} g + \int_0^1 g.$$

(b) Show that if $b \leq a \leq c$ and $f$ is integrable on the interval $[b, c]$, then it is still the case that $\int_a^b f = \int_a^c f + \int_c^b f$.

**Exercise 7.4.3.** Decide which of the following conjectures is true and supply a short proof. For those that are not true, give a counterexample.

(a) If $|f|$ is integrable on $[a, b]$, then $f$ is also integrable on this set.

(b) Assume $g$ is integrable and $g(x) \geq 0$ on $[a, b]$. If $g(x) > 0$ for an infinite number of points $x \in [a, b]$, then $\int_a^b g > 0$.

(c) If $g$ is continuous on $[a, b]$ and $g(x) \geq 0$ with $g(y_0) > 0$ for at least one point $y_0 \in [a, b]$, then $\int_a^b g > 0$.

**Exercise 7.4.4.** Show that if $f(x) > 0$ for all $x \in [a, b]$ and $f$ is integrable, then $\int_a^b f > 0$.

**Exercise 7.4.5.** Let $f$ and $g$ be integrable functions on $[a, b]$.

(a) Show that if $P$ is any partition of $[a, b]$, then

$$U(f + g, P) \leq U(f, P) + U(g, P).$$

Provide a specific example where the inequality is strict. What does the corresponding inequality for lower sums look like?

(b) Review the proof of Theorem 7.4.2 (ii), and provide an argument for part (i) of this theorem.

**Exercise 7.4.6.** Although not part of Theorem 7.4.2, it is true that the product of integrable functions is integrable. Provide the details for each step in the following proof of this fact:

(a) If $f$ satisfies $|f(x)| \leq M$ on $[a, b]$, show

$$|(f(x))^2 - (f(y))^2| \leq 2M|f(x) - f(y)|.$$

(b) Prove that if $f$ is integrable on $[a, b]$, then so is $f^2$.

(c) Now show that if $f$ and $g$ are integrable, then $fg$ is integrable. (Consider $(f + g)^2$.)

**Exercise 7.4.7.** Review the discussion immediately preceding Theorem 7.4.4.

(a) Produce an example of a sequence $f_n \to 0$ pointwise on $[0, 1]$ where $\lim_{n\to\infty} \int_0^1 f_n$ does not exist.

(b) Produce an example of a sequence $g_n$ with $\int_0^1 g_n \to 0$ but $g_n(x)$ does not converge to zero for any $x \in [0, 1]$. To make it more interesting, let's insist that $g_n(x) \geq 0$ for all $x$ and $n$.

**Exercise 7.4.8.** For each $n \in \mathbf{N}$, let

$$h_n(x) = \begin{cases} 1/2^n & \text{if } 1/2^n < x \leq 1 \\ 0 & \text{if } 0 \leq x \leq 1/2^n \end{cases},$$

and set $H(x) = \sum_{n=1}^{\infty} h_n(x)$. Show $H$ is integrable and compute $\int_0^1 H$.

**Exercise 7.4.9.** Let $g_n$ and $g$ be uniformly bounded on $[0,1]$, meaning that there exists a single $M > 0$ satisfying $|g(x)| \leq M$ and $|g_n(x)| \leq M$ for all $n \in \mathbf{N}$ and $x \in [0,1]$. Assume $g_n \to g$ pointwise on $[0,1]$ and uniformly on any set of the form $[0,\alpha]$, where $0 < \alpha < 1$.

If all the functions are integrable, show that $\lim_{n\to\infty} \int_0^1 g_n = \int_0^1 g$.

**Exercise 7.4.10.** Assume $g$ is integrable on $[0,1]$ and continuous at 0. Show

$$\lim_{n\to\infty} \int_0^1 g(x^n)dx = g(0).$$

**Exercise 7.4.11.** Review the original definition of integrability in Section 7.2, and in particular the definition of the upper integral $U(f)$. One reasonable suggestion might be to bypass the complications introduced in Definition 7.2.7 and simply define the integral to be the value of $U(f)$. Then *every* bounded function is integrable! Although tempting, proceeding in this way has some significant drawbacks. Show by example that several of the properties in Theorem 7.4.2 no longer hold if we replace our current definition of integrability with the proposal that $\int_a^b f = U(f)$ for every bounded function $f$.

## 7.5 The Fundamental Theorem of Calculus

The derivative and the integral have been independently defined, each in its own rigorous mathematical terms. The definition of the derivative is motivated by the problem of finding slopes of tangent lines and is given in terms of functional limits of difference quotients. The definition of the integral grows out of the desire to calculate areas under nonconstant functions and is given in terms of supremums and infimums of finite sums. The Fundamental Theorem of Calculus reveals the remarkable inverse relationship between the two processes.

The result is stated in two parts. The first is a computational statement that describes how an antiderivative can be used to evaluate an integral over a particular interval. The second statement is more theoretical in nature, expressing the fact that every continuous function is the derivative of its indefinite integral.

**Theorem 7.5.1 (Fundamental Theorem of Calculus).** (i) *If $f : [a,b] \to \mathbf{R}$ is integrable, and $F : [a,b] \to \mathbf{R}$ satisfies $F'(x) = f(x)$ for all $x \in [a,b]$, then*

$$\int_a^b f = F(b) - F(a).$$

(ii) *Let $g : [a,b] \to \mathbf{R}$ be integrable, and for $x \in [a,b]$, define*

$$G(x) = \int_a^x g.$$

*Then $G$ is continuous on $[a,b]$. If $g$ is continuous at some point $c \in [a,b]$, then $G$ is differentiable at $c$ and $G'(c) = g(c)$.*

*Proof.* (i) Let $P$ be a partition of $[a, b]$ and apply the Mean Value Theorem to $F$ on a typical subinterval $[x_{k-1}, x_k]$ of $P$. This yields a point $t_k \in (x_{k-1}, x_k)$ where

$$\begin{aligned} F(x_k) - F(x_{k-1}) &= F'(t_k)(x_k - x_{k-1}) \\ &= f(t_k)(x_k - x_{k-1}). \end{aligned}$$

Now, consider the upper and lower sums $U(f, P)$ and $L(f, P)$. Because $m_k \leq f(t_k) \leq M_k$ (where $m_k$ is the infimum on $[x_{k-1}, x_k]$ and $M_k$ is the supremum), it follows that

$$L(f, P) \leq \sum_{k=1}^{n} [F(x_k) - F(x_{k-1})] \leq U(f, P).$$

But notice that the sum in the middle telescopes so that

$$\sum_{k=1}^{n} [F(x_k) - F(x_{k-1})] = F(b) - F(a),$$

which is *independent* of the partition $P$. Thus we have

$$L(f) \leq F(b) - F(a) \leq U(f).$$

Because $L(f) = U(f) = \int_a^b f$, we conclude that $\int_a^b f = F(b) - F(a)$.

(ii) To prove the second statement, take $x > y$ in $[a, b]$ and observe that

$$\begin{aligned} |G(x) - G(y)| = \left| \int_a^x g - \int_a^y g \right| &= \left| \int_y^x g \right| \\ &\leq \int_y^x |g| \\ &\leq M(x - y), \end{aligned}$$

where $M > 0$ is a bound on $|g|$. This shows that $G$ is Lipschitz and so is uniformly continuous on $[a, b]$ (Exercise 4.4.9).

Now, let's assume that $g$ is continuous at $c \in [a, b]$. In order to show that $G'(c) = g(c)$, we rewrite the limit for $G'(c)$ as

$$\begin{aligned} \lim_{x \to c} \frac{G(x) - G(c)}{x - c} &= \lim_{x \to c} \frac{1}{x - c} \left( \int_a^x g(t)\ dt - \int_a^c g(t)\ dt \right) \\ &= \lim_{x \to c} \frac{1}{x - c} \left( \int_c^x g(t)\ dt \right). \end{aligned}$$

We would like to show that this limit equals $g(c)$. Thus, given an $\epsilon > 0$, we must produce a $\delta > 0$ such that if $|x - c| < \delta$, then

$$\left| \frac{1}{x - c} \left( \int_c^x g(t)\, dt \right) - g(c) \right| < \epsilon. \qquad (1)$$

The assumption of continuity of $g$ gives us control over the difference $|g(t)-g(c)|$. In particular, we know that there exists a $\delta > 0$ such that

$$|t-c| < \delta \quad \text{implies} \quad |g(t)-g(c)| < \epsilon.$$

To take advantage of this, we cleverly write the constant $g(c)$ as

$$g(c) = \frac{1}{x-c}\int_c^x g(c)\,dt$$

and combine the two terms in equation (1) into a single integral. Keeping in mind that $|x-c| \geq |t-c|$, we have that for all $|x-c| < \delta$,

$$\begin{aligned}\left|\frac{1}{x-c}\left(\int_c^x g(t)\,dt\right) - g(c)\right| &= \left|\frac{1}{x-c}\int_c^x (g(t)-g(c))\,dt\right| \\ &\leq \frac{1}{(x-c)}\int_c^x |g(t)-g(c)|\,dt \\ &< \frac{1}{(x-c)}\int_c^x \epsilon\,dt = \epsilon.\end{aligned}$$

□

## Exercises

**Exercise 7.5.1.** (a) Let $f(x) = |x|$ and define $F(x) = \int_{-1}^x f$. Find a piecewise algebraic formula for $F(x)$ for all $x$. Where is $F$ continuous? Where is $F$ differentiable? Where does $F'(x) = f(x)$?

(b) Repeat part (a) for the function

$$f(x) = \begin{cases} 1 & \text{if } x < 0 \\ 2 & \text{if } x \geq 0. \end{cases}$$

**Exercise 7.5.2.** Decide whether each statement is true or false, providing a short justification for each conclusion.

(a) If $g = h'$ for some $h$ on $[a,b]$, then $g$ is continuous on $[a,b]$.

(b) If $g$ is continuous on $[a,b]$, then $g = h'$ for some $h$ on $[a,b]$.

(c) If $H(x) = \int_a^x h$ is differentiable at $c \in [a,b]$, then $h$ is continuous at $c$.

**Exercise 7.5.3.** The hypothesis in Theorem 7.5.1 (i) that $F'(x) = f(x)$ for all $x \in [a,b]$ is slightly stronger than it needs to be. Carefully read the proof and state exactly what needs to be assumed with regard to the relationship between $f$ and $F$ for the proof to be valid.

**Exercise 7.5.4.** Show that if $f : [a,b] \to \mathbf{R}$ is continuous and $\int_a^x f = 0$ for all $x \in [a,b]$, then $f(x) = 0$ everywhere on $[a,b]$. Provide an example to show that this conclusion does not follow if $f$ is not continuous.

**Exercise 7.5.5.** The Fundamental Theorem of Calculus can be used to supply a shorter argument for Theorem 6.3.1 under the additional assumption that the sequence of derivatives is continuous.

Assume $f_n \to f$ pointwise and $f_n' \to g$ uniformly on $[a,b]$. Assuming each $f_n'$ is continuous, we can apply Theorem 7.5.1 (i) to get

$$\int_a^x f_n' = f_n(x) - f_n(a)$$

for all $x \in [a,b]$. Show that $g(x) = f'(x)$.

**Exercise 7.5.6 (Integration-by-parts).** (a) Assume $h(x)$ and $k(x)$ have continuous derivatives on $[a,b]$ and derive the familiar integration-by-parts formula

$$\int_a^b h(t)k'(t)dt = h(b)k(b) - h(a)k(a) - \int_a^b h'(t)k(t)dt\ .$$

(b) Explain how the result in Exercise 7.4.6 can be used to slightly weaken the hypothesis in part (a).

**Exercise 7.5.7.** Use part (ii) of Theorem 7.5.1 to construct another proof of part (i) of Theorem 7.5.1 under the stronger hypothesis that $f$ is continuous. (To get started, set $G(x) = \int_a^x f$.)

**Exercise 7.5.8 (Natural Logarithm and Euler's Constant).** Let

$$L(x) = \int_1^x \frac{1}{t}\, dt,$$

where we consider only $x > 0$.

(a) What is $L(1)$? Explain why $L$ is differentiable and find $L'(x)$.

(b) Show that $L(xy) = L(x)+L(y)$. (Think of $y$ as a constant and differentiate $g(x) = L(xy)$.)

(c) Show $L(x/y) = L(x) - L(y)$.

(d) Let

$$\gamma_n = \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) - L(n).$$

Prove that $(\gamma_n)$ converges. The constant $\gamma = \lim \gamma_n$ is called Euler's constant.

(e) Show how consideration of the sequence $\gamma_{2n} - \gamma_n$ leads to the interesting identity

$$L(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots\ .$$

**Exercise 7.5.9.** Given a function $f$ on $[a, b]$, define the *total variation* of $f$ to be

$$Vf = \sup \left\{ \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| \right\},$$

where the supremum is taken over all partitions $P$ of $[a, b]$.

(a) If $f$ is continuously differentiable ($f'$ exists as a continuous function), use the Fundamental Theorem of Calculus to show $Vf \leq \int_a^b |f'|$.

(b) Use the Mean Value Theorem to establish the reverse inequality and conclude that $Vf = \int_a^b |f'|$.

**Exercise 7.5.10 (Change-of-variable Formula).** Let $g : [a, b] \to \mathbf{R}$ be differentiable and assume $g'$ is continuous. Let $f : [c, d] \to \mathbf{R}$ be continuous, and assume that the range of $g$ is contained in $[c, d]$ so that the composition $f \circ g$ is properly defined.

(a) Why are we sure $f$ is the derivative of some function? How about $(f \circ g)g'$?

(b) Prove the change-of-variable formula

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(t)dt.$$

**Exercise 7.5.11.** Assume $f$ is integrable on $[a, b]$ and has a "jump discontinuity" at $c \in (a, b)$. This means that both one-sided limits exist as $x$ approaches $c$ from the left and from the right, but that

$$\lim_{x \to c^-} f(x) \neq \lim_{x \to c^+} f(x).$$

(This phenomenon is discussed in more detail in Section 4.6.)

(a) Show that, in this case, $F(x) = \int_a^x f$ is not differentiable at $x = c$.

(b) The discussion in Section 5.5 mentions the existence of a continuous monotone function that fails to be differentiable on a dense subset of $\mathbf{R}$. Combine the results of part (a) with Exercise 6.4.10 to show how to construct such a function.

## 7.6 Lebesgue's Criterion for Riemann Integrability

We now return to our investigation of the relationship between continuity and the Riemann integral. We have proved that continuous functions are integrable and that the integral also exists for functions with only a finite number of discontinuities. At the opposite end of the spectrum, we saw that Dirichlet's function, which is discontinuous at every point on $[0, 1]$, fails to be Riemann-integrable. The next examples show that the set of discontinuities of an integrable function can be infinite and even uncountable. (These also appear as exercises in Section 7.3.)

## Riemann-integrable Functions with Infinite Discontinuities

Recall from Section 4.1 that Thomae's function

$$t(x) = \begin{cases} 1 & \text{if } x = 0 \\ 1/n & \text{if } x = m/n \in \mathbf{Q}\backslash\{0\} \text{ is in lowest terms with } n > 0 \\ 0 & \text{if } x \notin \mathbf{Q} \end{cases}$$

is continuous on the set of irrationals and has discontinuities at every rational point. Let's prove that Thomae's function is integrable on $[0, 1]$ with $\int_0^1 t = 0$.

Let $\epsilon > 0$. The strategy, as usual, is to construct a partition $P_\epsilon$ of $[0, 1]$ for which $U(t, P_\epsilon) - L(t, P_\epsilon) < \epsilon$.

**Exercise 7.6.1.** (a) First, argue that $L(t, P) = 0$ for any partition $P$ of $[0, 1]$.

(b) Consider the set of points $D_{\epsilon/2} = \{x : t(x) \geq \epsilon/2\}$. How big is $D_{\epsilon/2}$?

(c) To complete the argument, explain how to construct a partition $P_\epsilon$ of $[0, 1]$ so that $U(t, P_\epsilon) < \epsilon$.

We first met the Cantor set $C$ in Section 3.1. We have since learned that $C$ is a compact, uncountable subset of the interval $[0, 1]$.

**Exercise 7.6.2.** Define

$$h(x) = \begin{cases} 1 & \text{if } x \in C \\ 0 & \text{if } x \notin C \end{cases}.$$

(a) Show $h$ has discontinuities at each point of $C$ and is continuous at every point of the complement of $C$. Thus, $h$ is not continuous on an uncountably infinite set.

(b) Now prove that $h$ is integrable on $[0, 1]$.

## Sets of Measure Zero

Thomae's function fails to be continuous at each rational number in $[0, 1]$. Although this set is infinite, we have seen that any infinite subset of $\mathbf{Q}$ is countable. Countably infinite sets are the smallest type of infinite set. The Cantor set is uncountable, but it is also small in a sense that we are now ready to make precise. In the introduction to Chapter 3, we presented an argument that the Cantor set has zero "length." The term "length" is awkward here because it really should only be applied to intervals or finite unions of intervals, which the Cantor set is not. There is a generalization of the concept of length to more general sets called the *measure* of a set. Of interest to our discussion are subsets that have *measure zero*.

**Definition 7.6.1.** A set $A \subseteq \mathbf{R}$ has *measure zero* if, for all $\epsilon > 0$, there exists a countable collection of open intervals $O_n$ with the property that $A$ is contained in the union of all of the intervals $O_n$ and the sum of the lengths of all of the intervals is less than or equal to $\epsilon$. More precisely, if $|O_n|$ refers to the length of the interval $O_n$, then we have

$$A \subseteq \bigcup_{n=1}^{\infty} O_n \quad \text{and} \quad \sum_{n=1}^{\infty} |O_n| \leq \epsilon.$$

**Example 7.6.2.** Consider a finite set $A = \{a_1, a_2, \ldots, a_N\}$. To show that $A$ has measure zero, let $\epsilon > 0$ be arbitrary. For each $1 \leq n \leq N$, construct the interval

$$G_n = \left(a_n - \frac{\epsilon}{2N}, a_n + \frac{\epsilon}{2N}\right).$$

Clearly, $A$ is contained in the union of these intervals, and

$$\sum_{n=1}^{N} |G_n| = \sum_{n=1}^{N} \frac{\epsilon}{N} = \epsilon.$$

**Exercise 7.6.3.** Show that any countable set has measure zero.

**Exercise 7.6.4.** Prove that the Cantor set has measure zero.

**Exercise 7.6.5.** Show that if two sets $A$ and $B$ each have measure zero, then $A \cup B$ has measure zero as well. In addition, discuss the proof of the stronger statement that the countable union of sets of measure zero also has measure zero. (This second statement is true, but a completely rigorous proof requires a result about double summations discussed in Section 2.8.)

## $\alpha$-Continuity

**Definition 7.6.3.** Let $f$ be defined on $[a, b]$, and let $\alpha > 0$. The function $f$ is *$\alpha$-continuous at $x \in [a, b]$* if there exists $\delta > 0$ such that for all $y, z \in (x-\delta, x+\delta)$ it follows that $|f(y) - f(z)| < \alpha$.

Let $f$ be a bounded function on $[a, b]$. For each $\alpha > 0$, define $D^\alpha$ to be the set of points in $[a, b]$ where the function $f$ fails to be $\alpha$-continuous; that is,

(1) $$D^\alpha = \{x \in [a, b] : f \text{ is not } \alpha\text{-continuous at } x.\}$$

The concept of $\alpha$-continuity was previously introduced in Section 4.6. Several of the ensuing exercises appeared as exercises in this section as well.

**Exercise 7.6.6.** If $\alpha < \alpha'$, show that $D^{\alpha'} \subseteq D^\alpha$.

Now, let

(2) $$D = \{x \in [a, b] : f \text{ is not continuous at } x \}.$$

**Exercise 7.6.7.** (a) Let $\alpha > 0$ be given. Show that if $f$ is continuous at $x \in [a, b]$, then it is $\alpha$-continuous at $x$ as well. Explain how it follows that $D^\alpha \subseteq D$.

(b) Show that if $f$ is not continuous at $x$, then $f$ is not $\alpha$-continuous for some $\alpha > 0$. Now, explain why this guarantees that

$$D = \bigcup_{n=1}^{\infty} D^{\alpha_n} \quad \text{where } \alpha_n = 1/n.$$

**Exercise 7.6.8.** Prove that for a fixed $\alpha > 0$, the set $D^\alpha$ is closed.

Just as with continuity, $\alpha$-continuity is defined pointwise, and just as with continuity, uniformity is going to play an important role.

For a fixed $\alpha > 0$, a function $f : A \to \mathbf{R}$ is *uniformly $\alpha$-continuous on $A$* if there exists a $\delta > 0$ such that whenever $x$ and $y$ are points in $A$ satisfying $|x - y| < \delta$, it follows that $|f(x) - f(y)| < \alpha$. By imitating the proof of Theorem 4.4.7, it is completely straightforward to show that if $f$ is $\alpha$-continuous at every point on some compact set $K$, then $f$ is uniformly $\alpha$-continuous on $K$.

## Compactness Revisited

Compactness of subsets of the real line can be described in three equivalent ways. The following theorem appears toward the end of Section 3.3.

**Theorem 7.6.4.** *Let $K \subseteq \mathbf{R}$. The following three statements are all equivalent, in the sense that if any one is true, then so are the two others.*

(i) *Every sequence contained in $K$ has a convergent subsequence that converges to a limit in $K$.*

(ii) *$K$ is closed and bounded.*

(iii) *Given a collection of open intervals $\{G_\lambda : \lambda \in \Lambda\}$ that covers $K$ (that is, $K \subseteq \bigcup_{\lambda \in \Lambda} G_\lambda$) there exists a* finite *subcollection $\{G_{\lambda_1}, G_{\lambda_2}, G_{\lambda_3}, \ldots, G_{\lambda_N}\}$ of the original set that also covers $K$.*

The equivalence of (i) and (ii) has been used throughout the core material in the text. Characterization (iii) has been less central but is essential to the upcoming argument. If the characterization of compactness in terms of open covers is not familiar, take a moment to review the second half of Section 3.3 and complete the proof that (i) and (ii) imply (iii) outlined in Exercise 3.3.9.

## Lebesgue's Theorem

We are now prepared to completely categorize the collection of Riemann-integrable functions in terms of continuity.

**Theorem 7.6.5 (Lebesgue's Theorem).** *Let $f$ be a bounded function defined on the interval $[a, b]$. Then, $f$ is Riemann-integrable if and only if the set of points where $f$ is not continuous has measure zero.*

*Proof.* Let $M > 0$ satisfy $|f(x)| \leq M$ for all $x \in [a, b]$, and let $D$ and $D^\alpha$ be defined as in the preceding equations (1) and (2). Let's first assume that $D$ has measure zero and prove that our function is integrable.

($\Leftarrow$) Let $\epsilon > 0$ and set

$$\alpha = \frac{\epsilon}{2(b-a)}.$$

**Exercise 7.6.9.** Show that there exists a *finite* collection of disjoint open intervals $\{G_1, G_2, \ldots, G_N\}$ whose union contains $D^\alpha$ and that satisfies

$$\sum_{n=1}^{N} |G_n| < \frac{\epsilon}{4M}.$$

**Exercise 7.6.10.** Let $K$ be what remains of the interval $[a, b]$ after the open intervals $G_n$ are all removed; that is, $K = [a, b] \backslash \bigcup_{n=1}^{N} G_n$. Argue that $f$ is uniformly $\alpha$-continuous on $K$.

**Exercise 7.6.11.** Finish the proof in this direction by explaining how to construct a partition $P_\epsilon$ of $[a, b]$ such that $U(f, P_\epsilon) - L(f, P_\epsilon) \leq \epsilon$. It will be helpful to break the sum

$$U(f, P_\epsilon) - L(f, P_\epsilon) = \sum_{k=1}^{n} (M_k - m_k)\Delta x_k$$

into two parts—one over those subintervals that contain points of $D^\alpha$ and the other over subintervals that do not.

($\Rightarrow$) For the other direction, assume $f$ is Riemann-integrable. We must argue that the set $D$ of discontinuities of $f$ has measure zero.

Let $\epsilon > 0$ be arbitrary, and fix $\alpha > 0$. Because $f$ is Riemann-integrable, there exists a partition $P_\epsilon$ of $[a, b]$ such that $U(f, P_\epsilon) - L(f, P_\epsilon) < \alpha\epsilon$.

**Exercise 7.6.12.** (a) Prove that $D^\alpha$ has measure zero. Point out that it is possible to choose a cover for $D^\alpha$ that consists of a finite number of open intervals.

(b) Show how this implies that $D$ has measure zero. □

Our main agenda in the remainder of this section is to employ Lebesgue's Theorem in our pursuit of a non-integrable derivative, but this elegant result has a number of other applications.

**Exercise 7.6.13.** (a) Show that if $f$ and $g$ are integrable on $[a, b]$, then so is the product $fg$. (This result was requested in Exercise 7.4.6, but notice how much easier the argument is now.)

(b) Show that if $g$ is integrable on $[a, b]$ and $f$ is continuous on the range of $g$, then the composition $f \circ g$ is integrable on $[a, b]$.

If we instead assume that $f$ is integrable and $g$ is continuous, it actually doesn't follow that the composition $f \circ g$ is an integrable function. Producing a counterexample, however, requires a few more ingredients.

## A Nonintegrable Derivative

To this point, our one example of a nonintegrable function is Dirichlet's nowhere-continuous function. We close this section with another example that has special significance. The content of the Fundamental Theorem of Calculus is that integration and differentiation are inverse processes of each other. If a function $f$ is differentiable on $[a, b]$, then part (i) of the Fundamental Theorem tells us that

$$\int_a^b f' = f(b) - f(a), \tag{3}$$

provided $f'$ is integrable. But shouldn't $f'$ be integrable just by virtue of being a derivative? A curious side-effect of staring at equation (3) for any length of time is that it starts to feel as though *every* derivative should be integrable because we have an obvious candidate for what the value of the integral ought to be. Alas, for the Riemann integral at least, reality comes up short of our expectations. What follows is the construction of a differentiable function $f$ for which equation (3) fails because $\int_a^b f'$ does not exist.

We will once again be interested in the Cantor set

$$C = \bigcap_{n=0}^{\infty} C_n,$$

defined in Section 3.1. As an initial step, let's create a function $f(x)$ that is differentiable on $[0, 1]$ and whose derivative $f'(x)$ has discontinuities at every point of $C$. The key ingredient for this construction is the function

$$g(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x > 0 \\ 0 & \text{if } x \leq 0. \end{cases}$$

**Exercise 7.6.14.** (a) Find $g'(0)$.

(b) Use the standard rules of differentiation to compute $g'(x)$ for $x \neq 0$.

(c) Explain why, for every $\delta > 0$, $g'(x)$ attains every value between 1 and $-1$ as $x$ ranges over the set $(-\delta, \delta)$. Conclude that $g'$ is not continuous at $x = 0$.

Now, we want to transport the behavior of $g$ around zero to each of the endpoints of the closed intervals that make up the sets $C_n$ used in the definition of

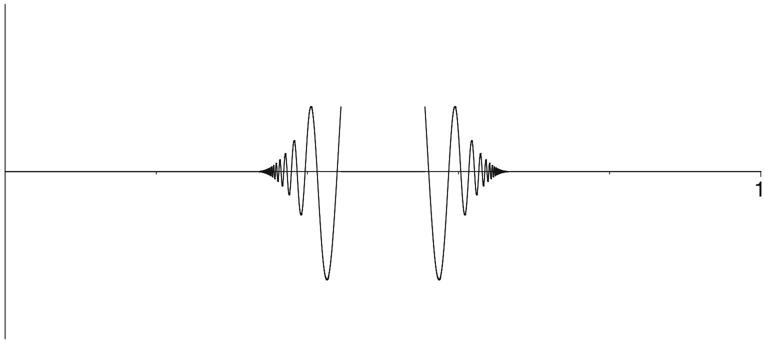



Figure 7.3: A PRELIMINARY SKETCH OF $f_1(x)$.

the Cantor set. The formulas are awkward but the basic idea is straightforward. Start by setting

$$f_0(x) = 0 \quad \text{on} \quad C_0 = [0,1].$$

To define $f_1$ on $[0,1]$, first assign

$$f_1(x) = 0 \quad \text{for all} \quad x \in C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right].$$

In the remaining open middle third, put translated "copies" of $g$ oscillating toward the two endpoints (Fig. 7.3). In terms of a formula, we have

$$f_1(x) = \begin{cases} 0 & \text{if } x \in [0,1/3] \\ g(x-1/3) & \text{if } x \text{ is just to the right of } 1/3 \\ g(-x+2/3) & \text{if } x \text{ is just to the left of } 2/3 \\ 0 & \text{if } x \in [2/3,1]. \end{cases}$$

Finally, we splice the two oscillating pieces of $f_1$ together in a way that makes $f_1$ differentiable and such that

$$|f_1(x)| \le (x-1/3)^2 \quad \text{and} \quad |f_1(x)| \le (-x+2/3)^2.$$

This splicing is no great feat, and we will skip the details so as to keep our attention focused on the two endpoints $1/3$ and $2/3$. These are the points where $f_1'(x)$ fails to be continuous.

To define $f_2(x)$, we start with $f_1(x)$ and do the same trick as before, this time in the two open intervals $(1/9, 2/9)$ and $(7/9, 8/9)$. The result (Fig. 7.4) is a differentiable function that is zero on $C_2$ and has a derivative that is not continuous on the set

$$\left\{\frac{1}{9}, \frac{2}{9}, \frac{1}{3}, \frac{2}{3}, \frac{7}{9}, \frac{8}{9}\right\}.$$

Continuing in this fashion yields a sequence of functions $f_0, f_1, f_2, \ldots$ defined on $[0,1]$.

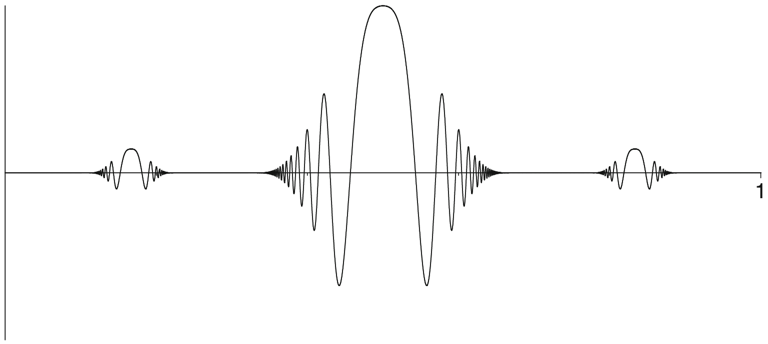



Figure 7.4: A GRAPH OF $f_2(x)$.

**Exercise 7.6.15.** (a) If $c \in C$, what is $\lim_{n\to\infty} f_n(c)$?

(b) Why does $\lim_{n\to\infty} f_n(x)$ exist for $x \notin C$?

Now, set

$$f(x) = \lim_{n\to\infty} f_n(x).$$

**Exercise 7.6.16.** (a) Explain why $f'(x)$ exists for all $x \notin C$.

(b) If $c \in C$, argue that $|f(x)| \leq (x-c)^2$ for all $x \in [0,1]$. Show how this implies $f'(c) = 0$.

(c) Give a careful argument for why $f'(x)$ fails to be continuous on $C$. Remember that $C$ contains many points besides the endpoints of the intervals that make up $C_1, C_2, C_3, \ldots$.

Let's take inventory of the situation. Our goal is to create a nonintegrable derivative. Our function $f(x)$ is differentiable, and $f'$ fails to be continuous on $C$. We are not quite done.

**Exercise 7.6.17.** Why is $f'(x)$ Riemann-integrable on $[0,1]$?

The reason the Cantor set has measure zero is that, at each stage, $2^{n-1}$ open intervals of length $1/3^n$ are removed from $C_{n-1}$. The resulting sum

$$\sum_{n=1}^{\infty} 2^{n-1}\left(\frac{1}{3^n}\right)$$

converges to one, which means that the approximating sets $C_1, C_2, C_3, \ldots$ have total lengths tending to zero. Instead of removing open intervals of length $1/3^n$ at each stage, let's see what happens when we remove intervals of length $1/3^{n+1}$.

**Exercise 7.6.18.** Show that, under these circumstances, the sum of the lengths of the intervals making up each $C_n$ no longer tends to zero as $n \to \infty$. What is this limit?

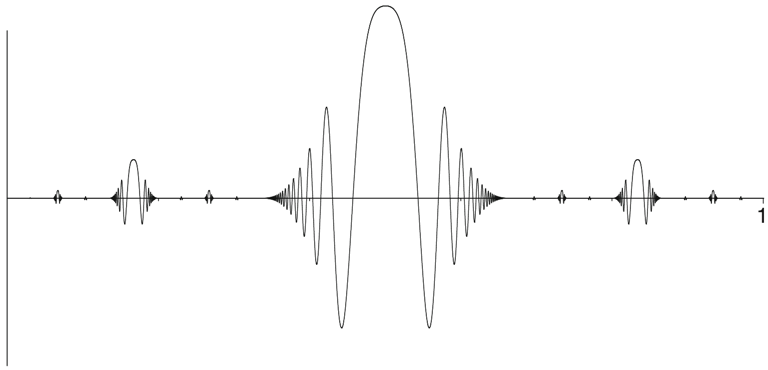



Figure 7.5: A DIFFERENTIABLE FUNCTION WITH A NON-INTEGRABLE DERIVATIVE.

If we again take the intersection $\bigcap_{n=0}^{\infty} C_n$, the result is a Cantor-type set with the same topological properties—it is closed, compact, perfect, and contains no intervals. But a consequence of the previous exercise is that it no longer has measure zero. This is just what we need to define our desired function. By repeating the preceding construction of $f(x)$ on this new Cantor-type set of *strictly positive* measure, we get a differentiable function whose derivative has too many points of discontinuity (Fig. 7.5). By Lebesgue's Theorem, this derivative cannot be integrated using the Riemann integral.

**Exercise 7.6.19.** As a final gesture, provide the example advertised in Exercise 7.6.13 of an integrable function $f$ and a continuous function $g$ where the composition $f \circ g$ is properly defined but not integrable. Exercise 4.3.12 may be useful.

## 7.7 Epilogue

Riemann's definition of the integral was a modification of Cauchy's integral, which was originally designed for the purpose of integrating continuous functions. In this goal, the Riemann integral was a complete success. For continuous functions at least, the process of integration now stood on its own rigorous footing, defined independently of differentiation. As analysis progressed, however, the dependence of integrability on continuity became problematic. The last example of Section 7.6 highlights one type of weakness: not every derivative can be integrated. Another limitation of the Riemann integral arises in association with limits of sequences of functions. To get a sense of this, let's once again consider Dirichlet's function $g(x)$ introduced in Section 4.1. Recall that $g(x) = 1$ whenever $x$ is rational, and $g(x) = 0$ at every irrational point. Focusing on the interval $[0, 1]$ for a moment, let

$$\{r_1, r_2, r_3, r_4 \ldots\}$$

be an enumeration of the countable number of rational points in this interval. Now, let $g_1(x) = 1$ if $x = r_1$ and define $g_1(x) = 0$ otherwise. Next, define $g_2(x) = 1$ if $x$ is either $r_1$ or $r_2$, and let $g_2(x) = 0$ at all other points. In general, for each $n \in \mathbf{N}$, define

$$g_n(x) = \begin{cases} 1 & \text{if } x \in \{r_1, r_2, \ldots, r_n\} \\ 0 & \text{otherwise.} \end{cases}$$

Notice that each $g_n$ has only a finite number of discontinuities and so is Riemann-integrable with $\int_0^1 g_n = 0$. But we also have $g_n \to g$ pointwise on the interval $[0, 1]$. The problem arises when we remember that Dirichlet's nowhere-continuous function is not Riemann-integrable. Thus, the equation

$$\text{(1)} \qquad \lim_{n \to \infty} \int_0^1 g_n = \int_0^1 g$$

fails to hold, not because the values on each side of the equal sign are different but because the value on the right-hand side does not exist. The content of Theorem 7.4.4 is that this equation does hold whenever we have $g_n \to g$ *uniformly.* This is a reasonable way to resolve the situation, but it is a bit unsatisfying because the deficiency in this case is not entirely with the type of convergence but lies in the strength of the Riemann integral. If we could make sense of the right-hand side via some other definition of integration, then maybe equation (1) would actually be true.

Such a definition was introduced by Henri Lebesque in 1901. Generally speaking, Lebesgue's integral is constructed using a generalization of length called the *measure* of a set. In the previous section, we studied sets of *measure zero.* In particular, we showed that the rational numbers in [0,1] (because they are countable) have measure zero. The irrational numbers in [0,1] have measure one. This should not be too surprising because we now have that the measures of these two disjoint sets add up to the length of the interval $[0, 1]$. Rather than chopping up the $x$-axis to approximate the area under the curve, Lebesgue suggested partitioning the $y$-axis. In the case of Dirichlet's function $g$, there are only two range values—zero and one. The integral, according to Lebesgue, could be defined via

$$\begin{aligned} \int_0^1 g &= 1 \cdot [\text{measure of set where } g = 1] + 0 \cdot [\text{measure of set where } g = 0] \\ &= 1 \cdot 0 + 0 \cdot 1 = 0. \end{aligned}$$

With this interpretation of $\int_0^1 g$, equation (1) is now valid!

The Lebesgue integral is presently the standard integral in advanced mathematics. The theory is taught to all graduate students, as well as to many undergraduates, and it is the integral used in most research papers where integration is required. The Lebesgue integral generalizes the Riemann integral in the sense that any function that is Riemann-integrable is Lebesgue-integrable and integrates to the same value. The real strength of the Lebesgue integral is that the class of integrable functions is much larger. Most importantly, this class includes the limits of different types of Cauchy sequences of integrable functions. This leads to a group of extremely important convergence theorems related to equation (1) with hypotheses much weaker than the uniform convergence assumed in Theorem 7.4.4.

Despite its prevalence, the Lebesgue integral does have a few drawbacks. There are functions whose *improper* Riemann integrals exist but that are not Lebesgue-integrable. Another disappointment arises from the relationship between integration and differentiation. Even with the Lebesgue integral, it is still not possible to prove

$$\int_a^b f' = f(b) - f(a)$$

without some additional assumptions on $f$. Around 1960, a new integral was proposed that can integrate a larger class of functions than either the Riemann integral or the Lebesgue integral and suffers from neither of the preceding weaknesses. Remarkably, this integral is actually a return to Riemann's original technique for defining integration, with some small modifications in how we describe the "fineness" of the partitions. An introduction to the generalized Riemann integral is the topic of Section 8.1.